Approximation Algorithms

Lecture 11: MaxSat via Randomized Rounding

Part I: Maximum Satisfiability (MaxSat)

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Is a given formula in conjunctive normal form satisfiable?
E.g., $(x_1 \vee \overline{x_2} \vee x_3) \wedge (x_2 \vee \overline{x_3} \vee x_4) \wedge (x_1 \vee \overline{x_4})$.

Approximation Algorithms

Lecture 11: MaxSat via Randomized Rounding

Part II: A Simple Randomized Algorithm

A Simple Randomized Algorithm

Theorem. Independently setting each variable to 1 (`true`) with probability $1/2$ provides an expected -approximation for $\textsc{MaxSat}$.

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Part III:
Derandomization by Conditional Expectation

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Consider a partial assignment $x_1 = b_1, \ldots, x_i = b_i$
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Consider a partial assignment $x_1 = b_1, \ldots, x_i = b_i$
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The conditional expectation is simply the sum of the
contributions from each clause. $\square$

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Global optimization?

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Part IV:
Randomized Rounding

An ILP

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$z_j \in \{0, 1\}$, for $j = 1, \ldots, m$

An ILP

Let $C_j = \bigvee_{i \in P_j} x_i \vee \bigvee_{i \in N_j} \bar{x_i}$ for $j = 1, \ldots, m$.

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$$\begin{array}{lll} \textbf{maximize} & \displaystyle\sum_{j=1}^{m} w_j z_j & \\ \textbf{subject to} & \displaystyle\sum_{i \in P_j} \quad + \sum_{i \in N_j} & \text{for } j = 1, \ldots, m \\ & y_i \in \{0, 1\}, & \text{for } i = 1, \ldots, n \\ & z_j \in \{0, 1\}, & \text{for } j = 1, \ldots, m \end{array}$$

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\textbf{maximize} \quad & \sum_{j=1}^{m} w_j z_j \\
\textbf{subject to} \quad & \sum_{i \in P_j} y_i + \sum_{i \in N_j} (1 - y_i) && \text{for } j = 1, \ldots, m \\
& y_i \in \{0, 1\}, && \text{for } i = 1, \ldots, n \\
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\end{aligned}$$

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\textbf{maximize} & \displaystyle\sum_{j=1}^{m} w_j z_j & \\
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An ILP . . . and Its Relaxation

Let $C_j = \bigvee_{i \in P_j} x_i \vee \bigvee_{i \in N_j} \bar{x_i}$ for $j = 1, \ldots, m$.

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Theorem. Let (y^*, z^*) be an optimal solution to the LP-relaxation.

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Theorem. Let (y^*, z^*) be an optimal solution to the LP-relaxation. Independently setting each variable x_i to 1 with probability y_i^* provides a (   )-approximation for MaxSat.

An ILP ... and Its Relaxation

Let $C_j = \bigvee_{i \in P_j} x_i \vee \bigvee_{i \in N_j} \bar{x_i}$ for $j = 1, \dots, m$.

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\end{array}$$

Theorem. Let (y^*, z^*) be an optimal solution to the LP-relaxation. Independently setting each variable x_i to 1 with probability y_i^* provides a $(1 - 1/e)$-approximation for MaxSat.

An ILP . . . and Its Relaxation

Let $C_j = \bigvee_{i \in P_j} x_i \vee \bigvee_{i \in N_j} \bar{x_i}$ for $j = 1, \dots, m$.

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Approximation Algorithms

Lecture 11:
MaxSat via Randomized Rounding

Part V:
Randomized Rounding – Proof

Mathematical Toolkit

Let f be a function that is concave on $[0, 1]$

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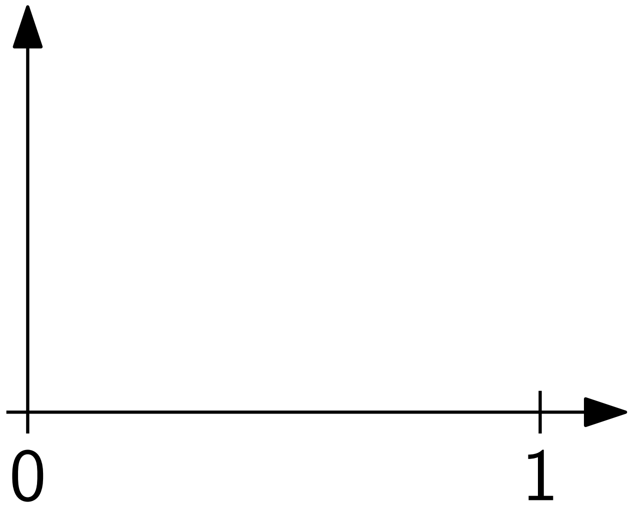

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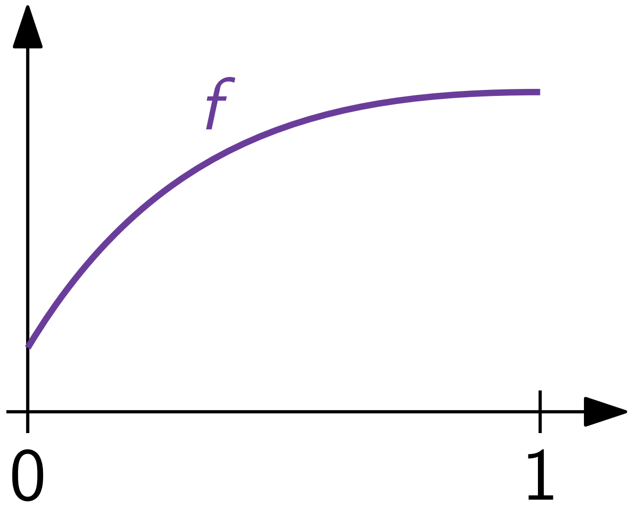
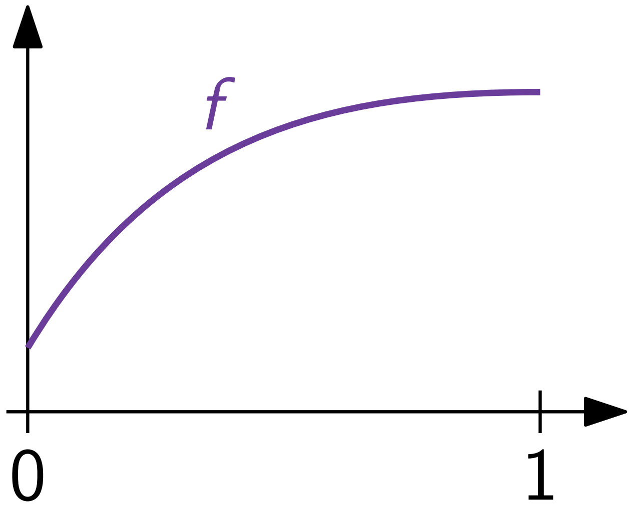


Mathematical Toolkit

Let f be a function that is concave on $[0, 1]$ (i.e. $f''(x) \leq 0$ on $[0, 1]$) with $f(0) = a$



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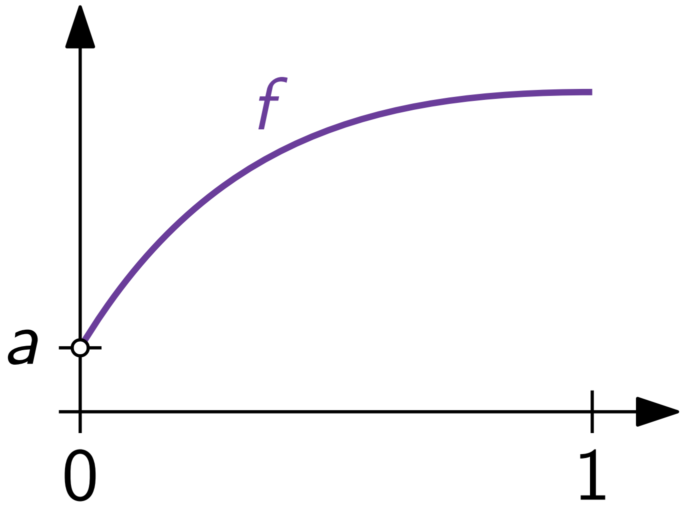

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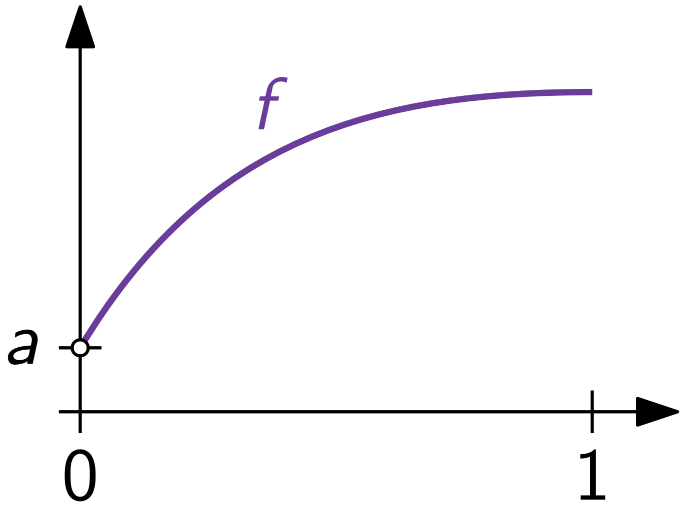

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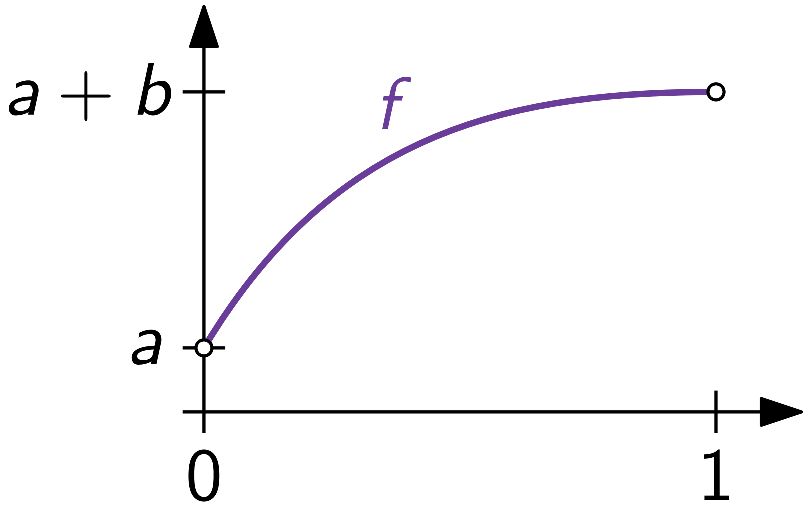

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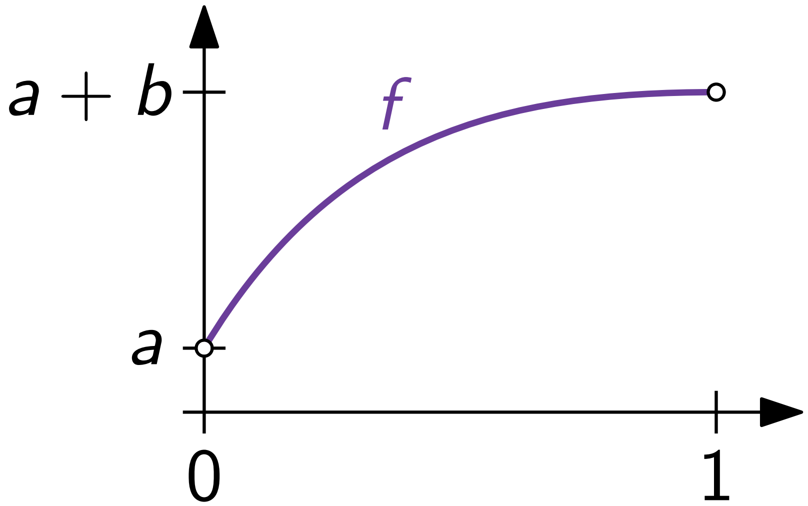
$\Rightarrow f(x) \geq bx + a$ for $x \in [0,1]$.



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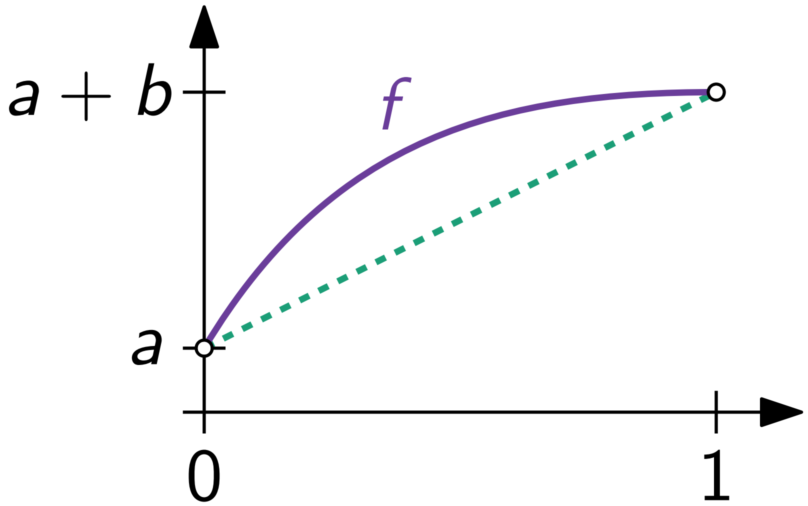

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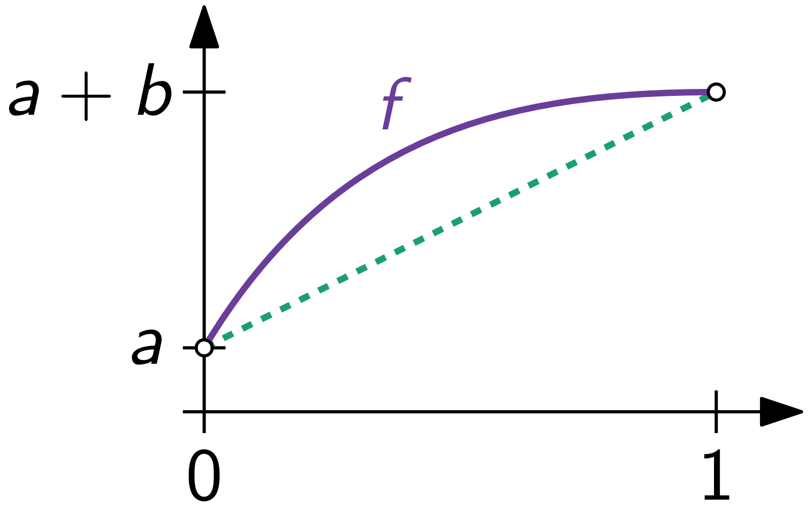

Arithmetic–Geometric Mean Inequality (AGMI):

Mathematical Toolkit

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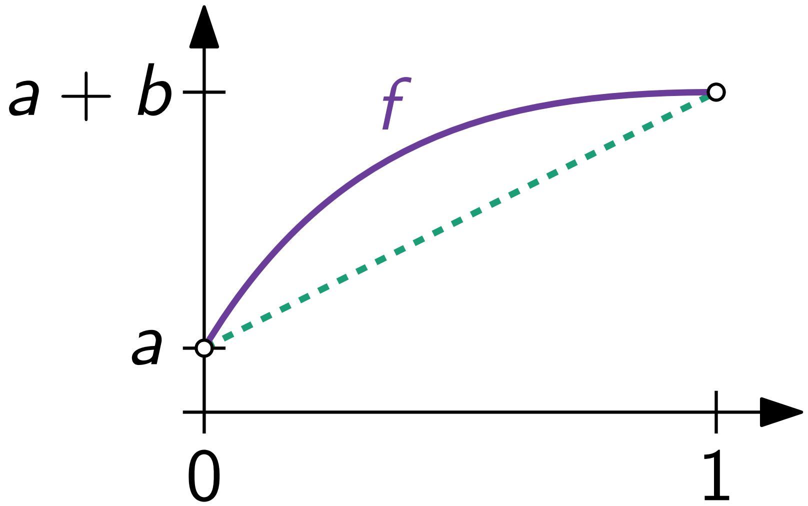

Arithmetic–Geometric Mean Inequality (AGMI):

For all non-negative numbers $a_1, \ldots, a_k$:

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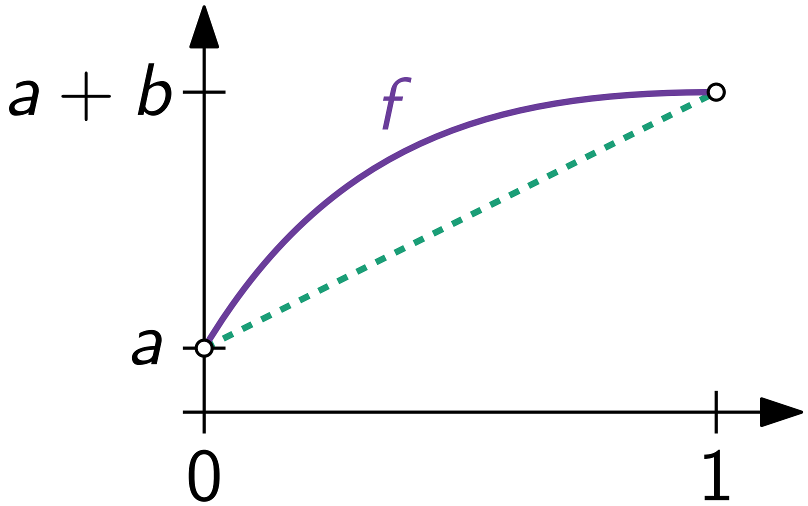

Arithmetic–Geometric Mean Inequality (AGMI):

For all non-negative numbers $a_1, \ldots, a_k$:

$$\leq \quad \frac{1}{k}\left(\sum_{i=1}^{k} a_i\right)$$

Mathematical Toolkit

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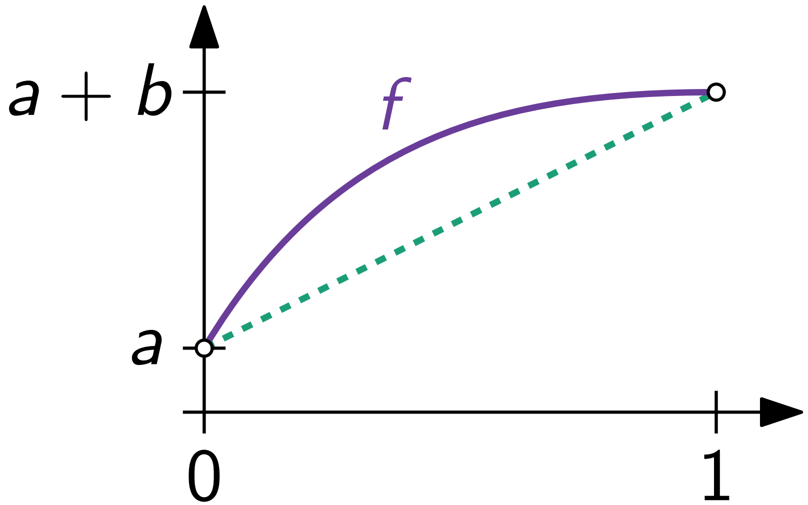

Arithmetic–Geometric Mean Inequality (AGMI):

For all non-negative numbers $a_1, \ldots, a_k$:

$$\left(\prod_{i=1}^{k} a_i\right)^{1/k} \leq \frac{1}{k}\left(\sum_{i=1}^{k} a_i\right)$$

Randomized Rounding (Proof)

Consider a fixed clause C_j of length l_j. Then we have:

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AGMI

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AGMI

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AGMI

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by LP constraints

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AGMI

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$$\leq \left(1 - \frac{z_j^*}{l_j}\right)^{l_j}$$

Randomized Rounding (Proof)

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Therefore

$$\mathbf{E}[W] = \sum_{j=1}^{m} \mathbf{Pr}[C_j \text{ satisfied}] \cdot w_j$$

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$$

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$$= \left(1 - \frac{1}{e}\right) \mathrm{OPT}_{\mathrm{LP}}$$

$$\geq \left(1 - \frac{1}{e}\right) \mathrm{OPT} \qquad \square$$

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&\geq \left(1 - \frac{1}{e}\right) \mathrm{OPT}
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$$

□

Theorem. The previous algorithm can be derandomized by the method of conditional expectation.

Approximation Algorithms

Lecture 11:
MaxSat via Randomized Rounding

Part VI:
Combining the Algorithms

Take the better of the two solutions!

Theorem. The better solution among the randomized algorithm and the randomized LP-rounding algorithm provides a -approximation for MaxSat.

Take the better of the two solutions!

Theorem. The better solution among the randomized algorithm and the randomized LP-rounding algorithm provides a $3/4$-approximation for MaxSat.

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Proof.

We use another probabilistic argument.
With probability $1/2$, choose the solution of the first algorithm; otherwise the solution of the second algorithm.

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Theorem. The better solution among the randomized algorithm and the randomized LP-rounding algorithm provides a $3/4$-approximation for MaxSat.

Proof.

We use another probabilistic argument.
With probability $1/2$, choose the solution of the first algorithm; otherwise the solution of the second algorithm.

The better solution is at least as good as the expectation of the above randomized algorithm.

Take the better of the two solutions!

The probability that clause C_j is satisfied is at least:

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$$\frac{1}{2}\left[\underbrace{\left(1-\left(1-\frac{1}{l_j}\right)^{l_j}\right) z_j^*}_{\text{LP-rounding}} + \qquad\qquad \right].$$

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The probability that clause C_j is satisfied is at least:

$$\frac{1}{2}\left[\underbrace{\left(1-\left(1-\frac{1}{l_j}\right)^{l_j}\right) z_j^*}_{\text{LP-rounding}} + \underbrace{\left(1-2^{-l_j}\right)}_{\text{rand. alg.}}\right].$$

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For $l_j \in \{1, 2\}$, a simple calculation yields exactly $\frac{3}{4} z_j^*$.

For $l_j \geq 3$, $1-(1-1/l_j)^{l_j} \geq$ and $1-2^{-l_j} \geq$

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Thus, we have at least:

$$\frac{1}{2}\left[\left(1-\frac{1}{e}\right)+\frac{7}{8}\right] z_j^* \approx$$

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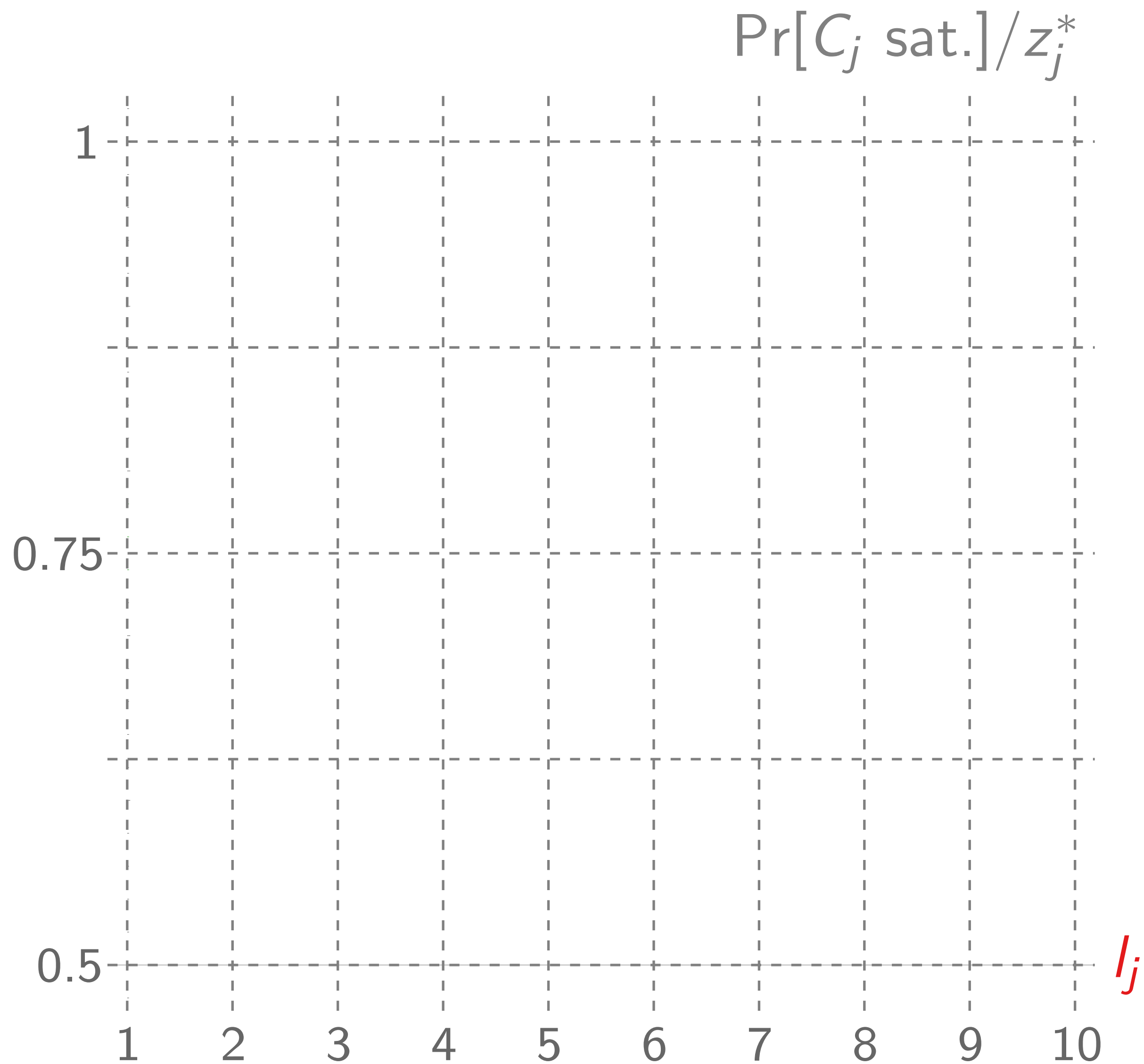
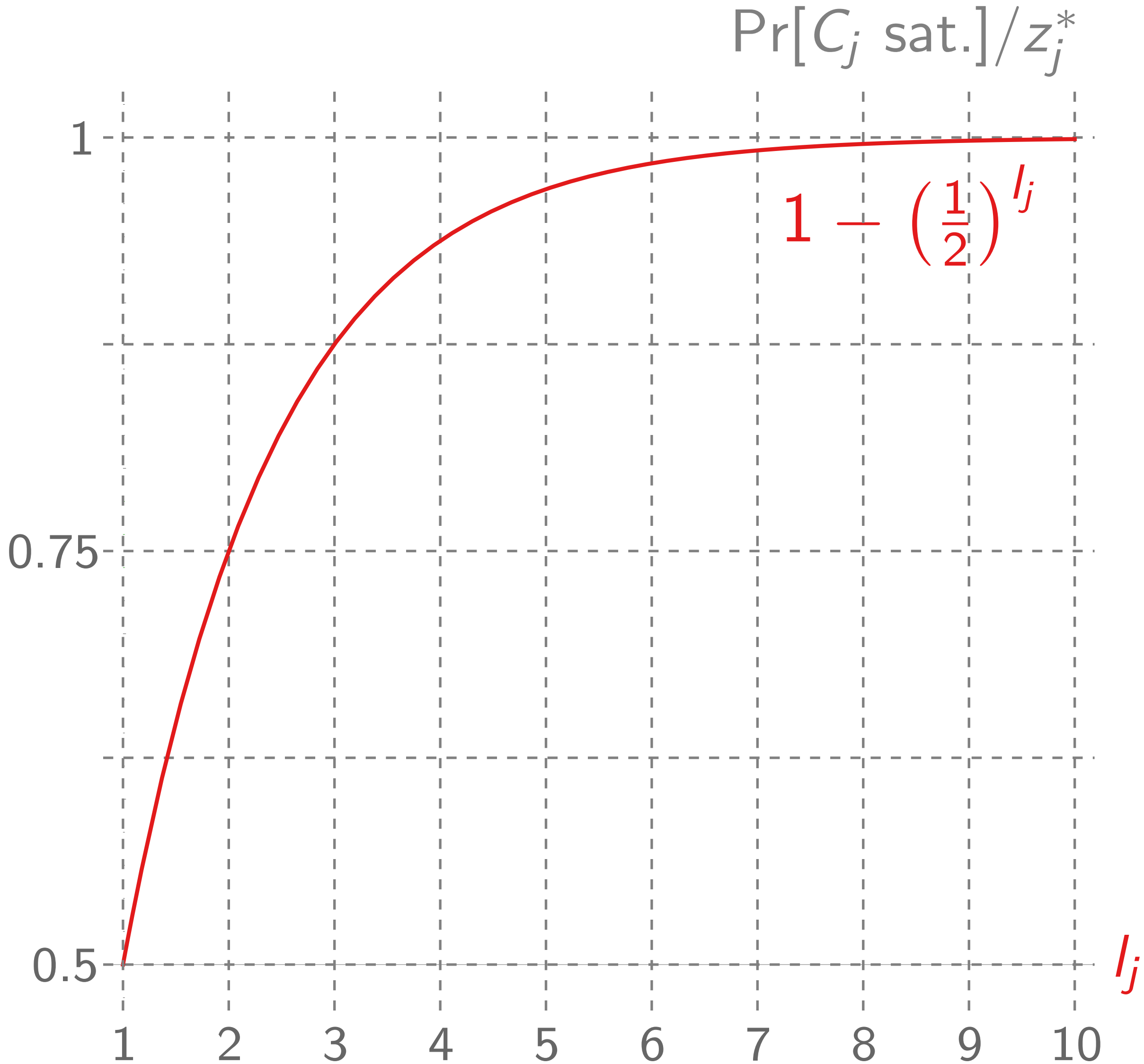
□

Visualization and Derandomization

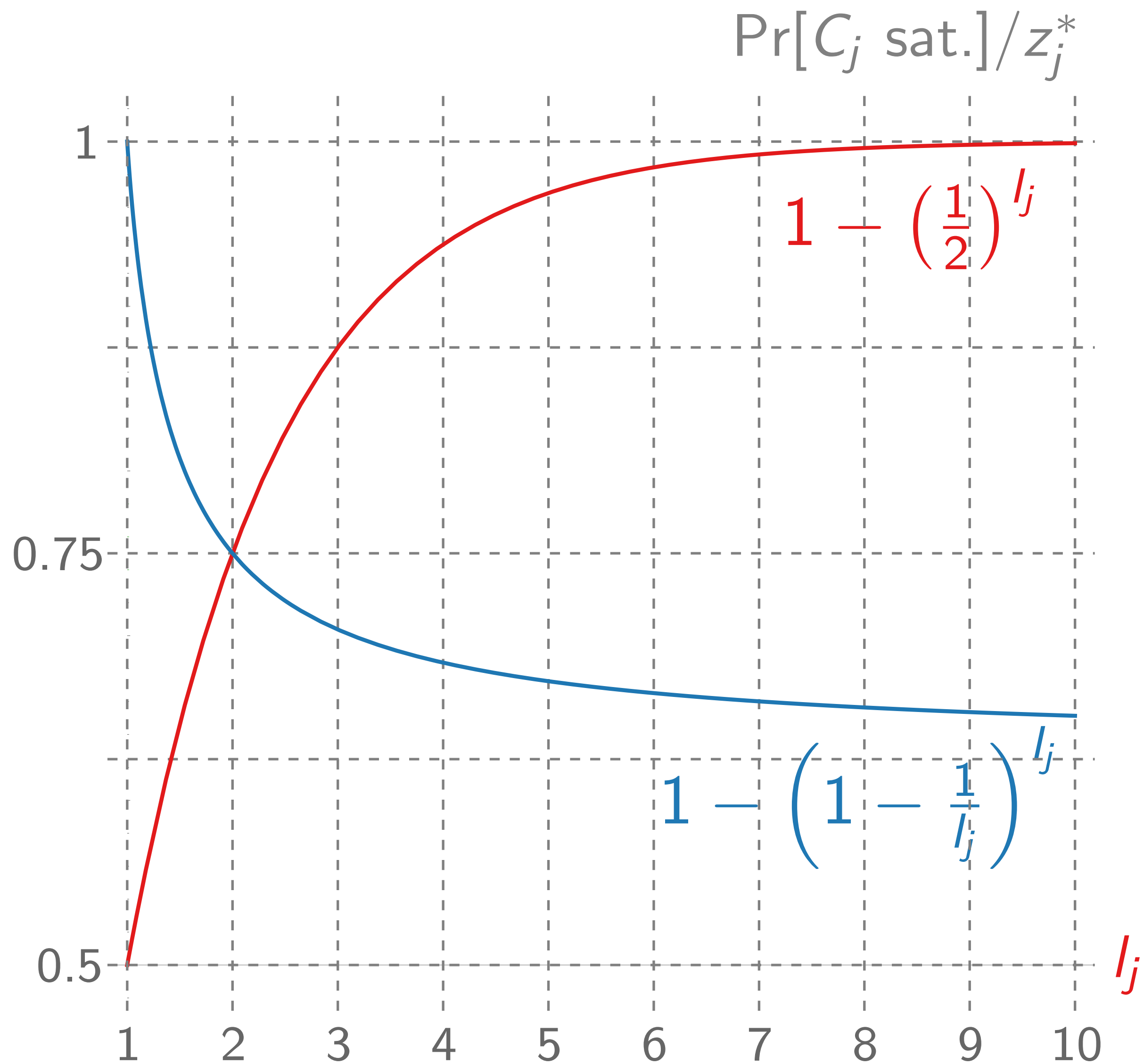
Visualization and Derandomization



Visualization and Derandomization

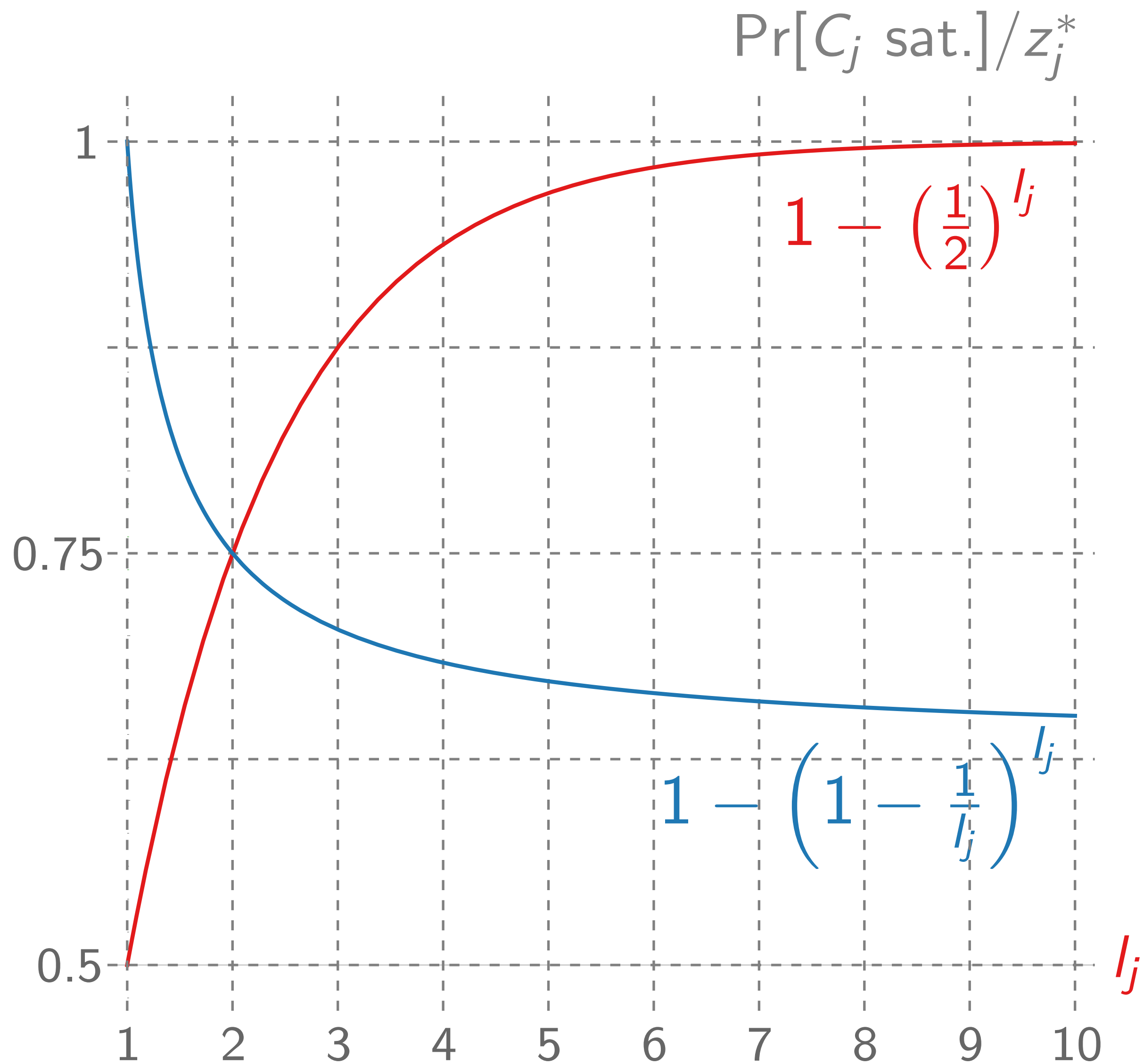


Visualization and Derandomization



Visualization and Derandomization

- Randomized alg. is better for large values of l_j.



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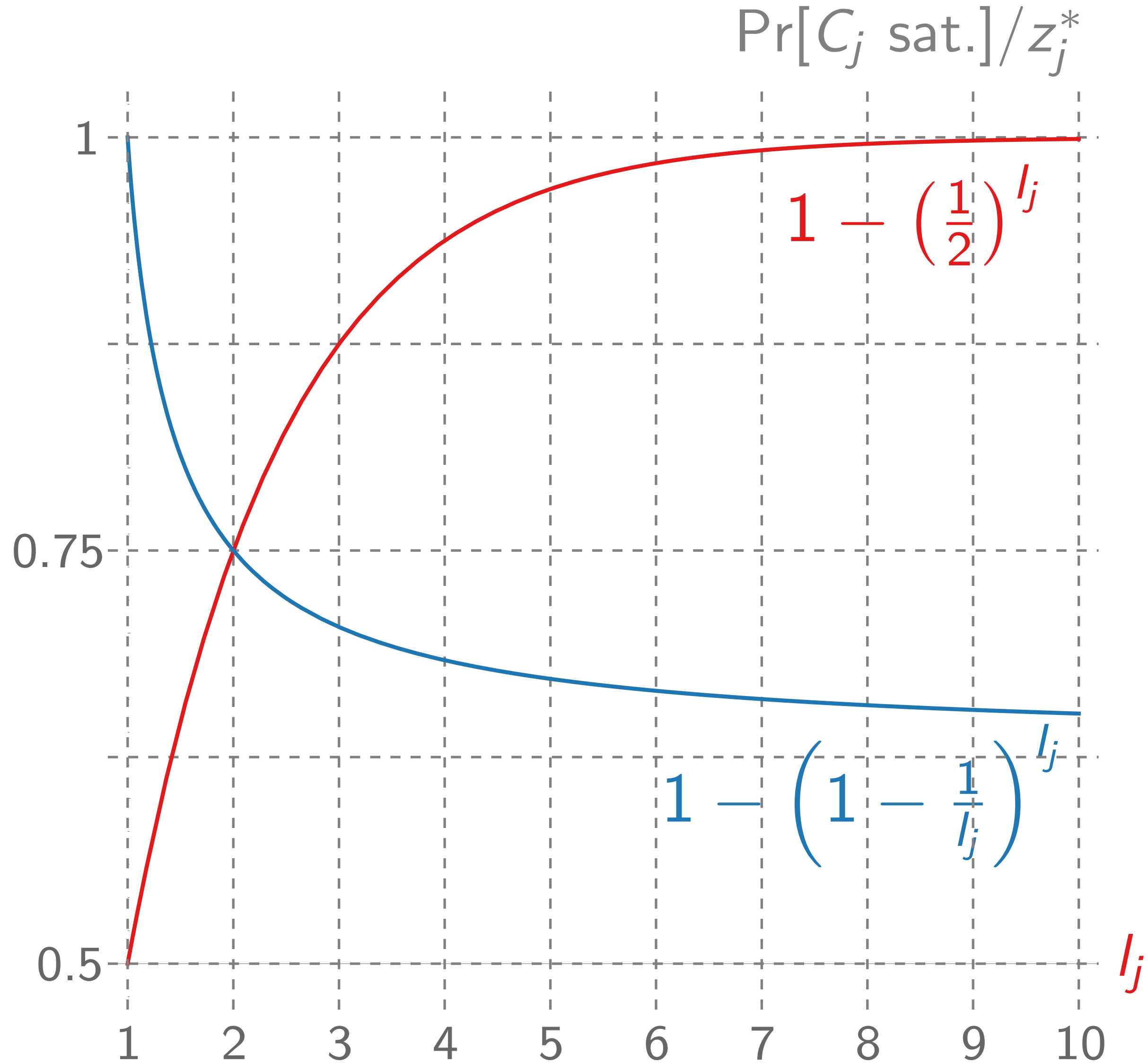

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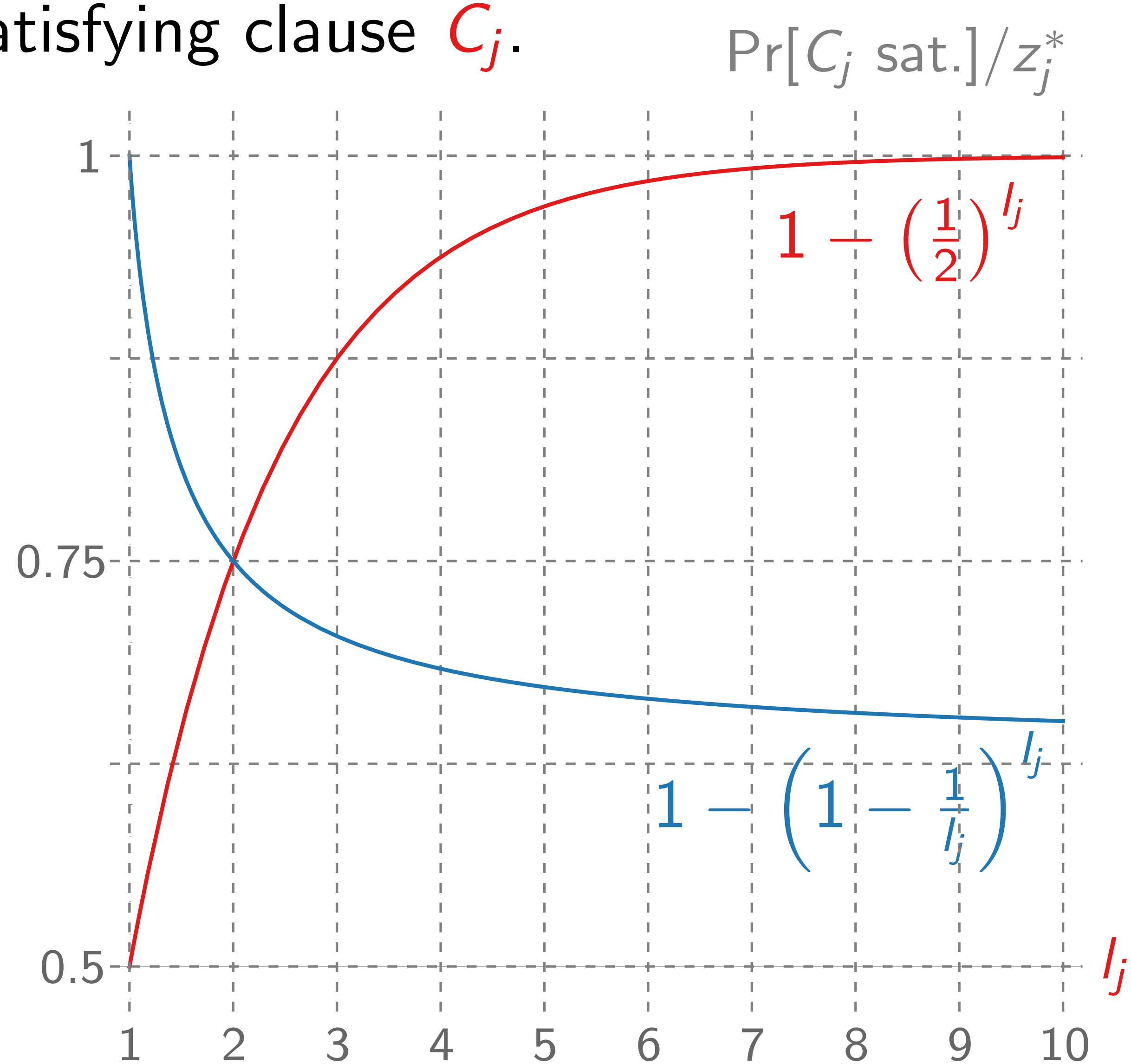
$\Rightarrow$ higher probability of satisfying clause C_j.



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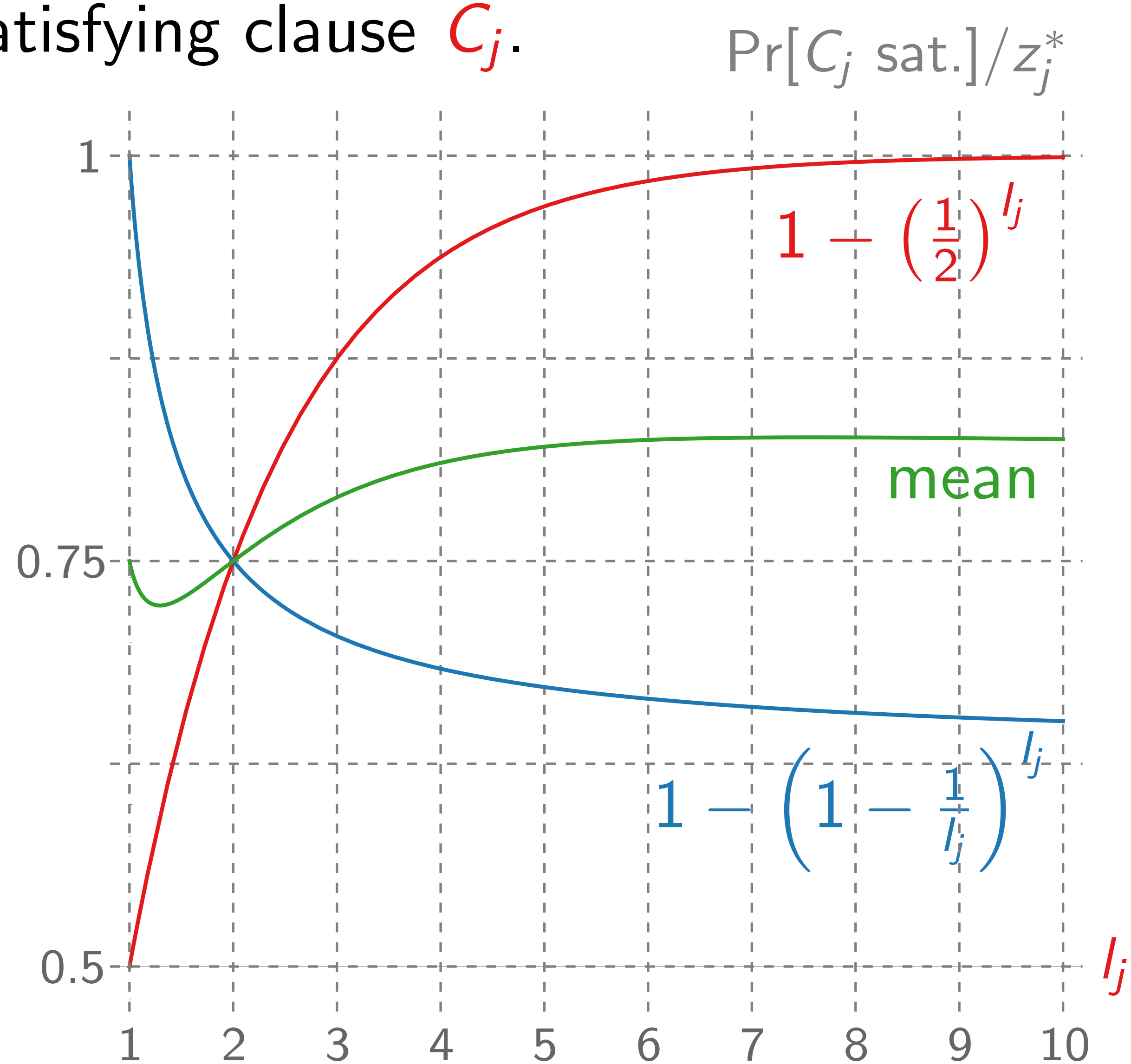

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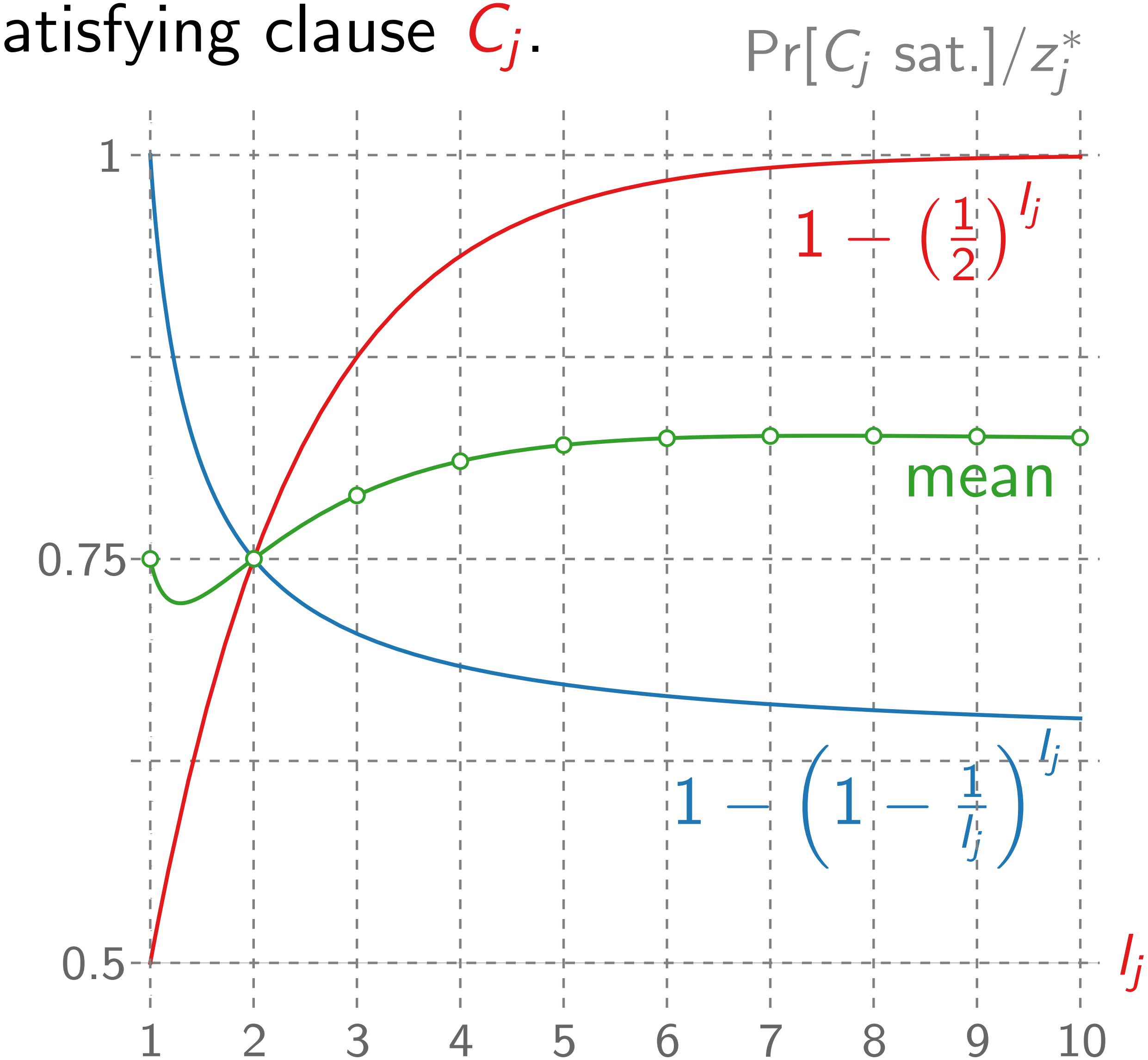
The mean of the two solutions is at least $3/4$ for *integer* l_j.



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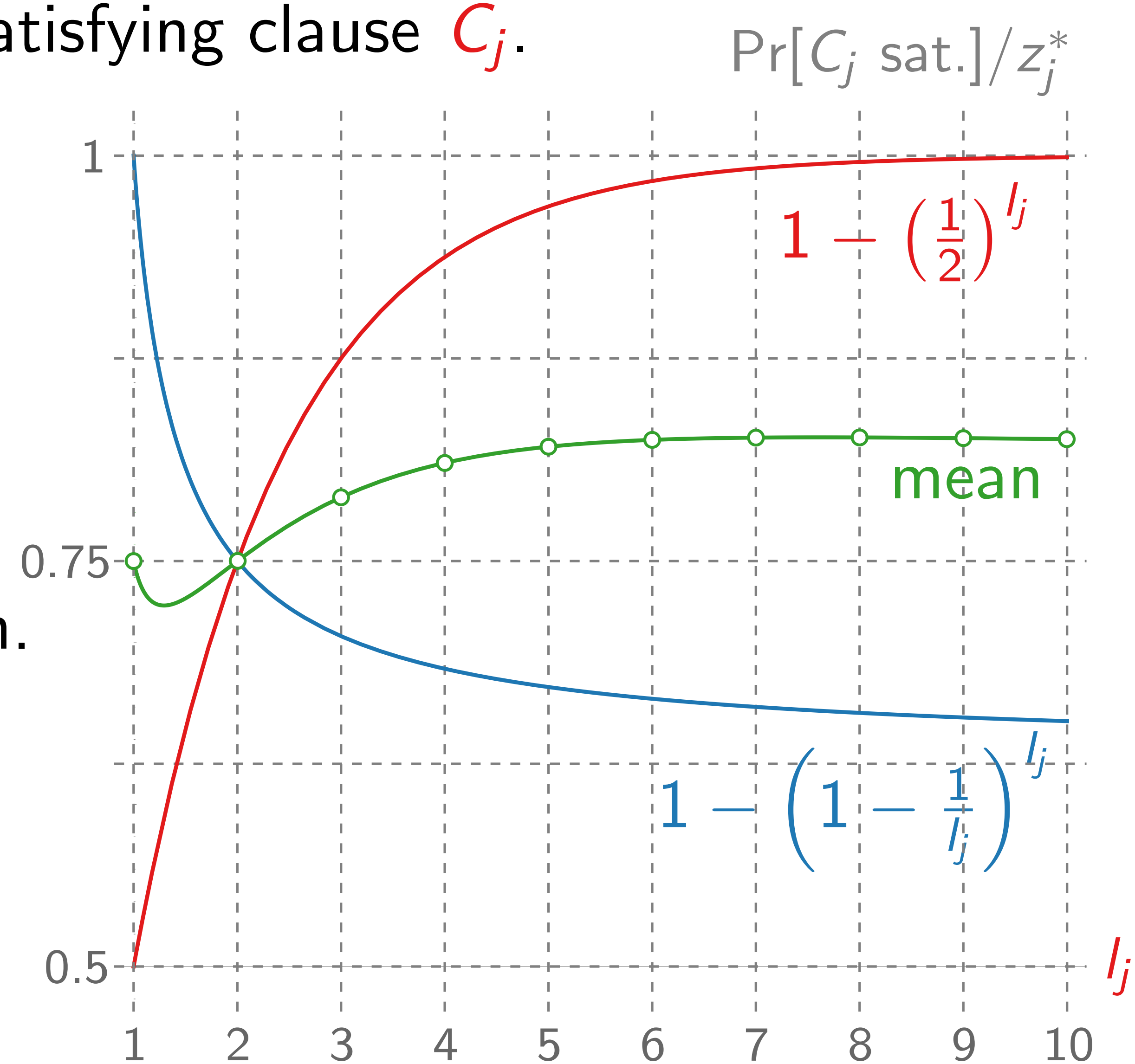

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$\Rightarrow$ higher probability of satisfying clause C_j.

The mean of the two solutions is at least $3/4$ for *integer* l_j.

The maximum is at least as large as the mean.

This algorithm, too, can be derandomized by conditional expectation.