Approximation Algorithms

Lecture 7: Scheduling Jobs on Parallel Machines

Part I: ILP & Parametric Pruning

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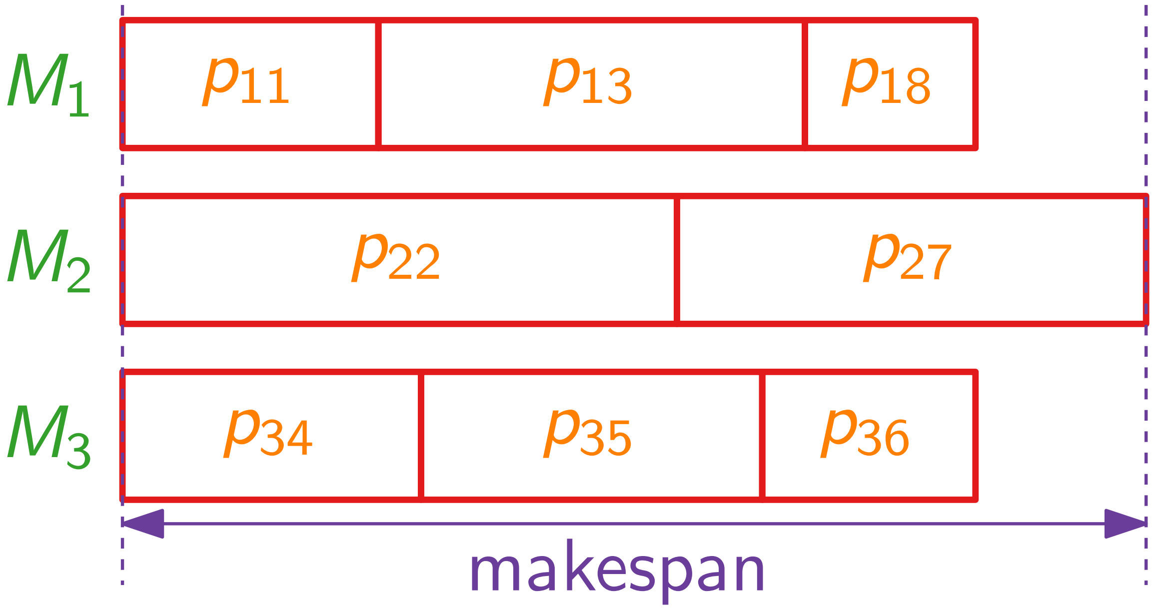
Scheduling on Parallel Machines

Given: A set $\mathcal{J}$ of **jobs**,
a set $\mathcal{M}$ of **machines**, and
for each $M_i \in \mathcal{M}$ and $J_j \in \mathcal{J}$
the **processing time** $p_{ij} \in \mathbb{N}^+$ of J_j on M_i.

Task: A **schedule** $\sigma \colon \mathcal{J} \to \mathcal{M}$ of the jobs on the machines that minimizes the total time to completion (**makespan**), i.e., minimizes the maximum time a machine is in use.



$\mathcal{J} = \{J_1, J_2, \dots, J_8\}$

$\mathcal{M} = \{M_1, M_2, M_3\}$

$(p_{ij})_{M_i \in \mathcal{M}, J_j \in \mathcal{J}}$

Formulation as ILP

$$
\begin{aligned}
\textbf{minimize} \quad & t \\
\textbf{subject to} \quad & \sum_{M_i \in \mathcal{M}} x_{ij} = 1, && J_j \in \mathcal{J} \\
& \sum_{J_j \in \mathcal{J}} x_{ij} p_{ij} \leq t, && M_i \in \mathcal{M} \\
& x_{ij} \in \{0, 1\}, && M_i \in \mathcal{M}, J_j \in \mathcal{J}
\end{aligned}
$$

Task: Prove that the integrality gap is unbounded!

Solution: m machines and one job with processing time m
$\Rightarrow \mathrm{OPT} = m$ and $\mathrm{OPT}_{\mathrm{frac}} = 1$.

Parametric Pruning

Strengthen the ILP $\rightarrow$ implicit (non-linear) constraint:
If $p_{ij} > t$, then set $x_{ij} = 0$.

Introduce new parameter $T \in \mathbb{N}$ as a lower bound on OPT.

Define $S_T := \{\,(i,j)\colon M_i \in \mathcal{M},\, J_j \in \mathcal{J},\, p_{ij} \le T\,\}$.

Define the "pruned" relaxation LP(T):

$$\begin{aligned} \sum_{i\colon (i,j)\in S_T} x_{ij} &= 1, & J_j &\in \mathcal{J} \\ \sum_{j\colon (i,j)\in S_T} x_{ij} p_{ij} &\le T, & M_i &\in \mathcal{M} \\ x_{ij} &\ge 0, & (i,j) &\in S_T \end{aligned}$$

Note:

LP(T) has no objective function; we just need to check whether a feasible solution exists.

But why does this LP give a good integrality gap?

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Part II:
Properties of Extreme-Point Solutions

Properties of Extreme Point Solutions

Use binary search to find the smallest T so that LP(T) has a solution. Let T^* be this value of T.

What are the bounds for our search?

Observe: $T^* \leq \text{OPT}$

Idea: Round an extreme-point solution of LP(T^*) to a schedule whose makespan is at most $2T^*$.

LP(T):

$$\sum_{i\colon (i,j)\in S_T} x_{ij} = 1, \quad J_j \in \mathcal{J}$$

$$\sum_{j\colon (i,j)\in S_T} x_{ij} p_{ij} \leq T, \quad M_i \in \mathcal{M}$$

$$x_{ij} \geq 0, \quad (i,j) \in S_T$$

Lemma 1.
Every extreme-point solution of LP(T) has at most $|\mathcal{J}| + |\mathcal{M}|$ positive variables.

Lemma 2.
Every extreme-point solution of LP(T) sets at least $|\mathcal{J}| - |\mathcal{M}|$ jobs integrally.

Lemma 1

$$\sum_{i:\,(i,j)\in S_T} x_{ij} = 1, \quad J_j \in \mathcal{J}$$

$$\sum_{j:\,(i,j)\in S_T} x_{ij} p_{ij} \le T, \quad M_i \in \mathcal{M}$$

$$x_{ij} \ge 0, \quad (i,j) \in S_T$$

Lemma 1.
Every extreme-point solution of LP(T) has at most $|\mathcal{J}| + |\mathcal{M}|$ positive variables.

Proof. $L(T)$: $|S_T|$ variables

extreme-point solution: $|S_T|$ inequalities tight

at most $|\mathcal{J}|$ inequalities

at most $|\mathcal{M}|$ inequalities

$\Rightarrow$ At least $|S_T| - |\mathcal{J}| - |\mathcal{M}|$ variables are 0.

$\Rightarrow$ At most $|\mathcal{J}| + |\mathcal{M}|$ variables are positive. □

Lemma 2

$$\sum_{i:\,(i,j)\in S_T} x_{ij} = 1, \quad J_j \in \mathcal{J}$$
$$\sum_{j:\,(i,j)\in S_T} x_{ij} p_{ij} \le T, \quad M_i \in \mathcal{M}$$
$$x_{ij} \ge 0, \quad (i,j) \in S_T$$

Lemma 2.
Every extreme-point solution of LP(T) sets at least $|\mathcal{J}| - |\mathcal{M}|$ jobs integrally.

Proof. Let x be an extreme-point solution of LP(T).
Assume x has α integral jobs und β fractional jobs.
$\Rightarrow \alpha + \beta = |\mathcal{J}|$
Each fractional job runs on at least two machines.
$\Rightarrow$ For each such job, at least two variables are pos.
$\Rightarrow \alpha + 2\beta \le |\mathcal{J}| + |\mathcal{M}|$ (Lemma 1)
$\Rightarrow \beta \le |\mathcal{M}|$ and $\alpha \ge |\mathcal{J}| - |\mathcal{M}|$ $\square$

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Part III:
An Algorithm

Extreme Point Solutions of LP(T)

Definition: Bipartite graph $G = (\mathcal{M} \cup \mathcal{J}, E)$ with $(i,j) \in E \Leftrightarrow x_{ij} \neq 0$ (in extreme-point sol.).

Jobs can be assigned *integrally* or *fractionally*.
$(\exists M_i \in \mathcal{M} : 0 < x_{ij} < 1)$

Let $F \subseteq \mathcal{J}$ be the set of fractionally assigned jobs.
Let $H := G[\mathcal{M} \cup F]$.

Observe: (i,j) is an edge in $H \Leftrightarrow 0 < x_{ij} < 1$

A matching in H is called *F-perfect* if it matches every vertex in F.

Main step: Show that H always has an F-perfect matching.

And why is this useful ...?

Algorithm

Assign job J_j to machine M_i that minimizes p_{ij}.
Let τ be the makespan of this schedule.

Do a binary search in the interval $[\frac{\tau}{|\mathcal{M}|}, \tau]$ to find the smallest value T^* of $T \in \mathbb{Z}^+$ s.t. LP(T) has a feasible solution.

Find an extreme-point solution x for LP(T^*).

Assign all integrally set jobs to machines as in x.

Construct the graph H and find an F-perfect matching P in it (see Lemma 4 later, F is set of fractionally assigned jobs)

Assign the fractional jobs to machines using P.

Theorem. This is a factor-2 approximation algorithm (assuming that we have an F-perfect matching).

Approximation Factor

$$\sum_{i\colon (i,j)\in S_T} x_{ij} = 1, \quad J_j \in \mathcal{J}$$

$$\sum_{j\colon (i,j)\in S_T} x_{ij} p_{ij} \le T, \quad M_i \in \mathcal{M}$$

$$x_{ij} \ge 0, \quad (i,j) \in S_T$$

Theorem. This is a factor-2 approximation algorithm (assuming that we have an F-perfect matching).

Proof. $T^* \le \mathrm{OPT}$.
Let x be an extreme-point solution for $LP(T^*)$
Fractional solution: Makespan $\le T^*$.
$\Rightarrow$ Restriction to integral jobs has makespan $\le T^*$.
For each edge $(i,j) \in S_{T^*}$, it holds that $p_{ij} \le T^*$.
Matching: at most one extra job per maschine.
$\Rightarrow$ total makespan $\le 2T^* \le 2\,\mathrm{OPT}$ □

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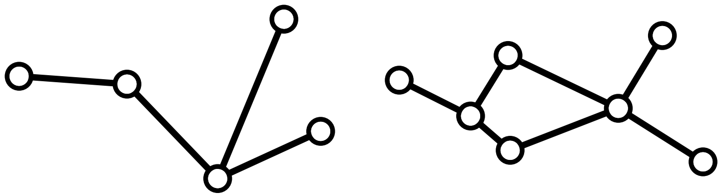
Part IV:
Pseudo-Trees and -Forests

Pseudo-Trees and -Forests



Pseudo-tree: a connected graph with at most as many edges as vertices.
(A pseudo-tree is either a tree or a tree plus a single edge.)

Pseudo-forest: a collection of disjoint pseudo-trees.

Lemma 3.
The bipartite graph $G = (\mathcal{M} \cup \mathcal{J}, E)$ is a pseudo-forest.

Extreme-point solutions have $\le |\mathcal{M}| + |\mathcal{J}|$ positive variables (Lemma 1).

Each conn. component C of G corresponds to an extreme-point solution.

(Suppose not. Then the solution that corresponds to C is the convex combination of other solutions. But this contradicts the definition of G.)

$\Rightarrow$ C has at most as many edges (pos. var.) as vertices (jobs+machines).

Lemma 4. The graph H has an F-perfect matching.

In G, every vertex in $\mathcal{J} \setminus F$ is a leaf. $\overset{\text{remove leaves}}{\Rightarrow}$ H is a pseudo-forest, too.
Vertices in F have minimum degree 2. $\Rightarrow$ The leaves in H are machines.
After iteratively matching all leaves, only *even* cycles remain. (H is bipartite :-)

Scheduling on Parallel Machines

Theorem. There is an LP-based 2-approximation algorithm for the problem of scheduling jobs on unrelated parallel machines.

Tight? Yes!

Instance I_m:

m machines and $m^2 - m + 1$ jobs

Job J_1 has processing time m on every machine,

all other jobs have processing time 1 on every machine.

Optimum: one machine gets J_1, and all others spread evenly.

$\Rightarrow$ Makespan $= m$.

Algorithm:

LP(T) has no feasible solution for any $T < m$.

Extreme-point solution:

Assign $1/m$ of J_1 and $m - 1$ other jobs to each machine.

$\Rightarrow$ Makespan $2m - 1$.

Scheduling on Parallel Machines

Theorem. There is an LP-based 2-approximation algorithm for the problem of scheduling jobs on unrelated parallel machines.

Can we do better?

No better approximation algorithm is known.

The problem cannot be approximated within factor $< 3/2$ (unless P=NP). [Lenstra, Shmoys & Tardos '90]

For a constant number of machines, for every $\varepsilon > 0$ there is a factor-$(1 + \varepsilon)$ approximation algorithm. [Horowitz & Sahni '76]

For uniform machines, for every $\varepsilon > 0$ there is a factor-$(1 + \varepsilon)$ approximation algorithm. [Hochbaum & Shmoys '87]

(Machines may have different speeds, but process jobs uniformly.)