Approximation Algorithms

Lecture 7:
Scheduling Jobs on Parallel Machines

Part I:
ILP & Parametric Pruning

Alexander Wolff

Winter 2023/24

Scheduling on Parallel Machines

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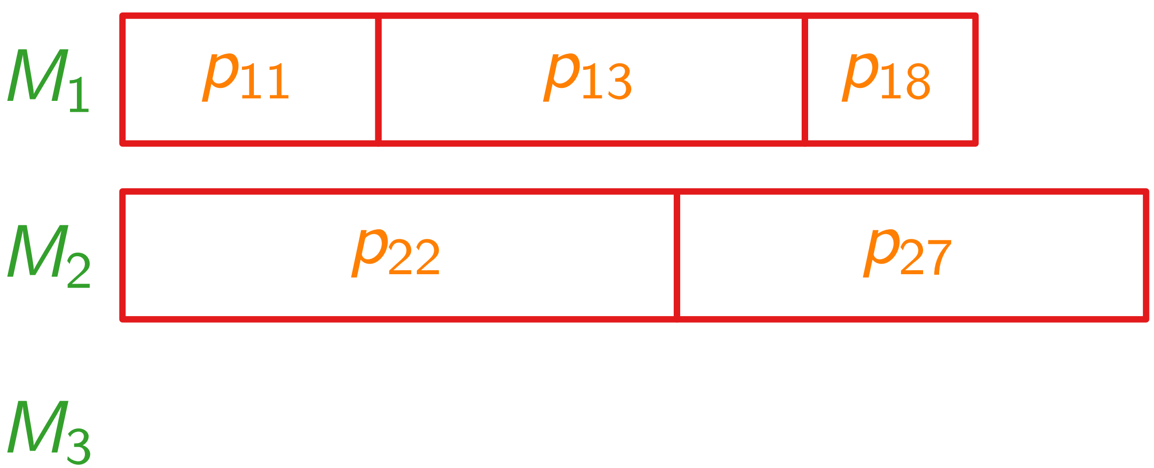

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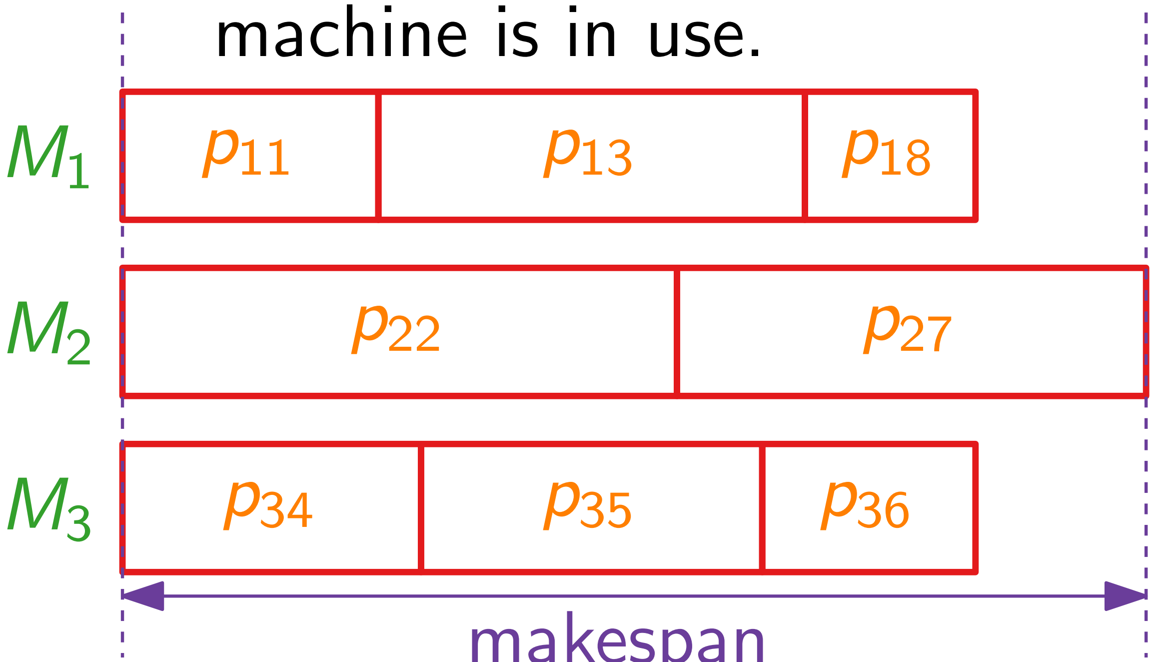

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minimize t

subject to

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$$x_{ij} \in \{0, 1\}, \qquad M_i \in \mathcal{M}, J_j \in \mathcal{J}$$

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Solution: m machines and one job with processing time m
$\Rightarrow \mathrm{OPT} = m$ and $\mathrm{OPT}_{\mathrm{frac}} = 1$.

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LP(T) has no objective function; we just need to check whether a feasible solution exists.

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But why does this LP give a good integrality gap?

Approximation Algorithms

Lecture 7:
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Part II:
Properties of Extreme-Point Solutions

Properties of Extreme Point Solutions

Use binary search to find the smallest T so that LP(T) has a solution.

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Every extreme-point solution of LP(T) has at most $|\mathcal{J}| + |\mathcal{M}|$ positive variables.

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Every extreme-point solution of LP(T) has at most $|\mathcal{J}| + |\mathcal{M}|$ positive variables.

Lemma 2.
Every extreme-point solution of LP(T) sets at least $|\mathcal{J}| - |\mathcal{M}|$ jobs integrally.

Lemma 1

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Proof. $L(T)$: $|S_T|$ variables

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Proof. $L(T)$: $|S_T|$ variables

extreme-point solution: $|S_T|$ inequalities tight

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Every extreme-point solution of LP(T) has at most $|\mathcal{J}| + |\mathcal{M}|$ positive variables.

Proof.

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extreme-point solution: $|S_T|$ inequalities tight

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$\Rightarrow \beta \leq |\mathcal{M}|$ and $\alpha \geq |\mathcal{J}| - |\mathcal{M}|$ $\square$

Approximation Algorithms

Lecture 7:
Scheduling Jobs on Parallel Machines

Part III:
An Algorithm

Extreme Point Solutions of LP(T)

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Theorem. This is a factor-2 approximation algorithm (assuming that we have an F-perfect matching).

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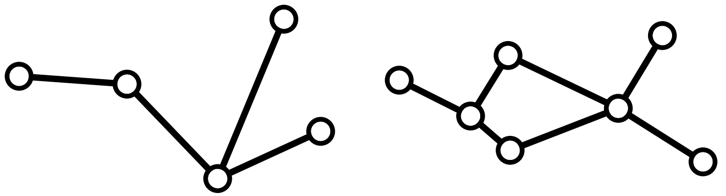
Approximation Algorithms

Lecture 7:
Scheduling Jobs on Parallel Machines

Part IV:
Pseudo-Trees and -Forests

Pseudo-Trees and -Forests



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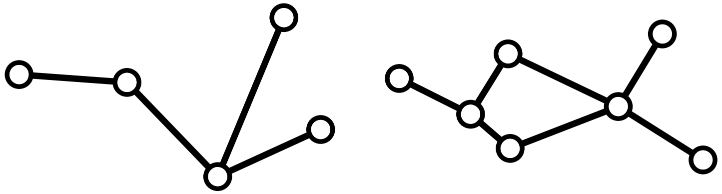
Pseudo-Trees and -Forests



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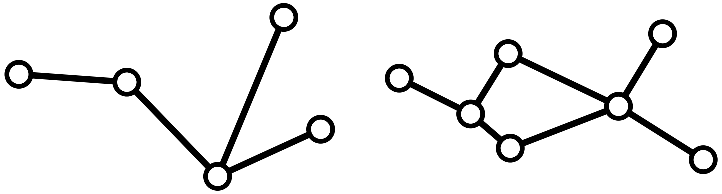

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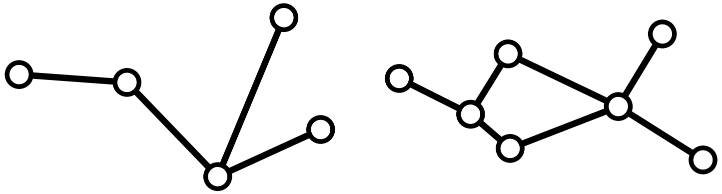

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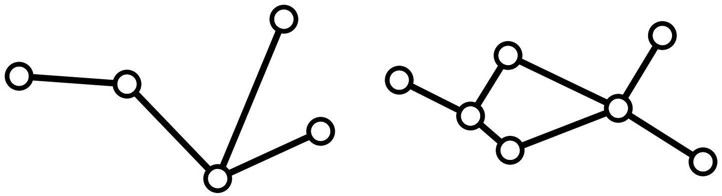

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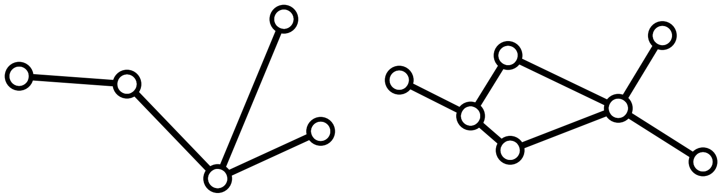

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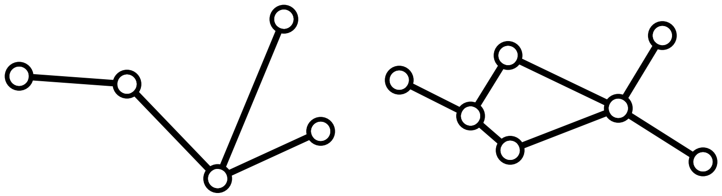

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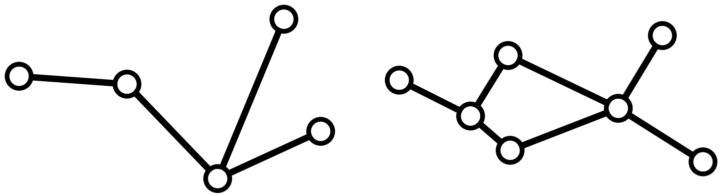

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$\Rightarrow$ C has at most as many edges (pos. var.) as vertices (jobs+machines).

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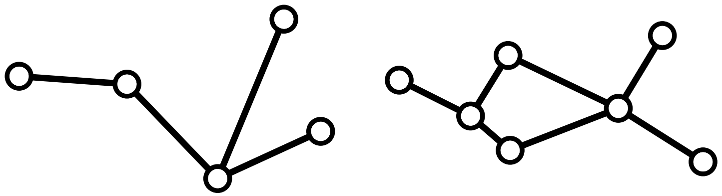

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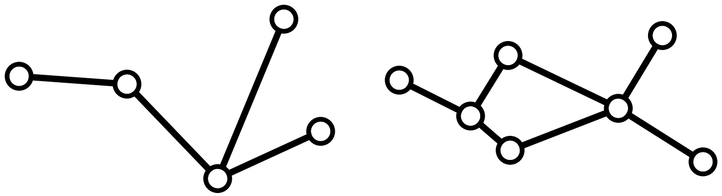

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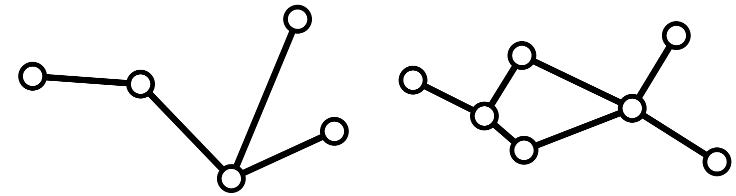

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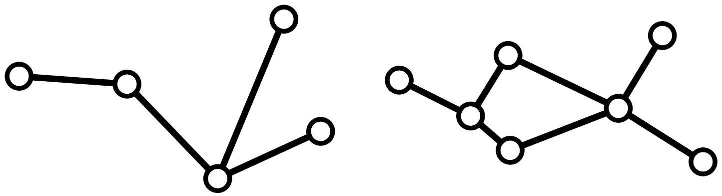

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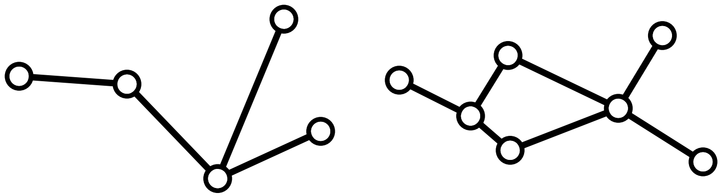

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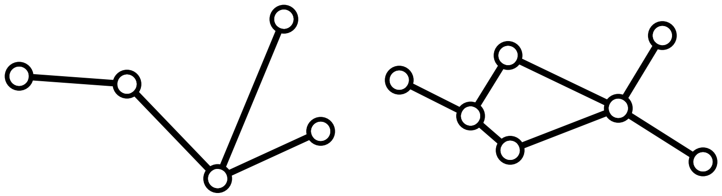

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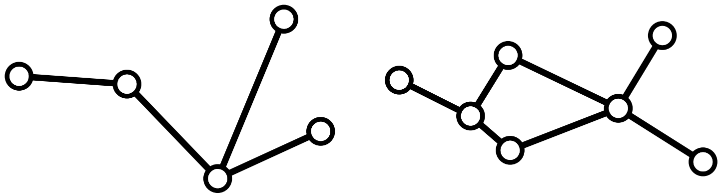

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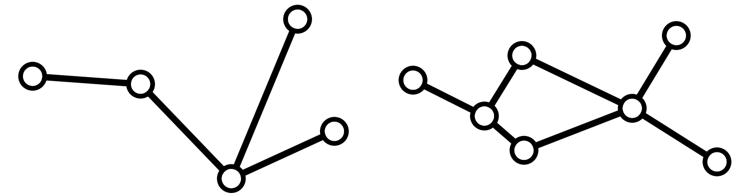

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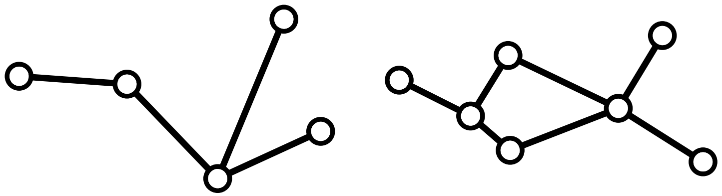

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