Approximation Algorithms

Lecture 6: k-Center via Parametric Pruning

Part I: Metric k-Center

Alexander Wolff

Winter 2023/24

Metric k-Center

Given: A graph $G = (V, E)$

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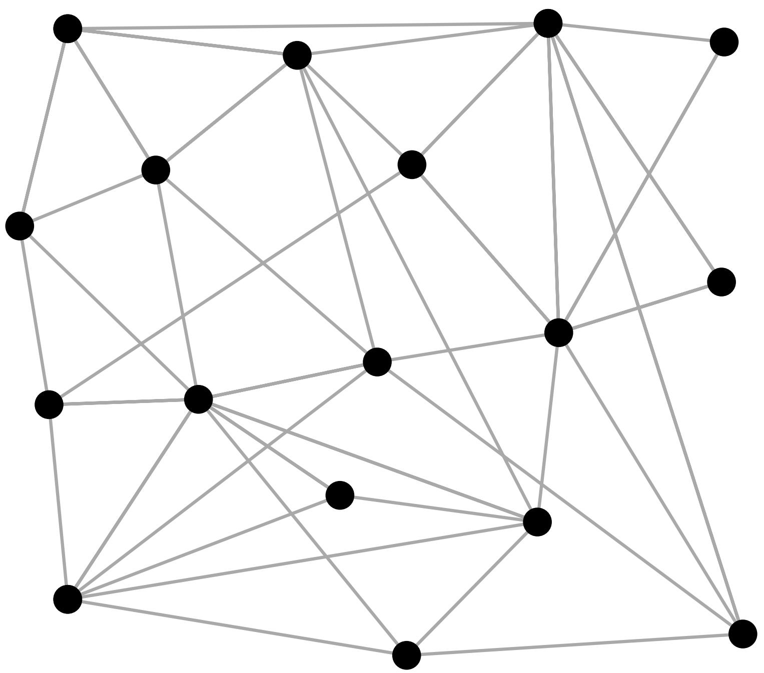

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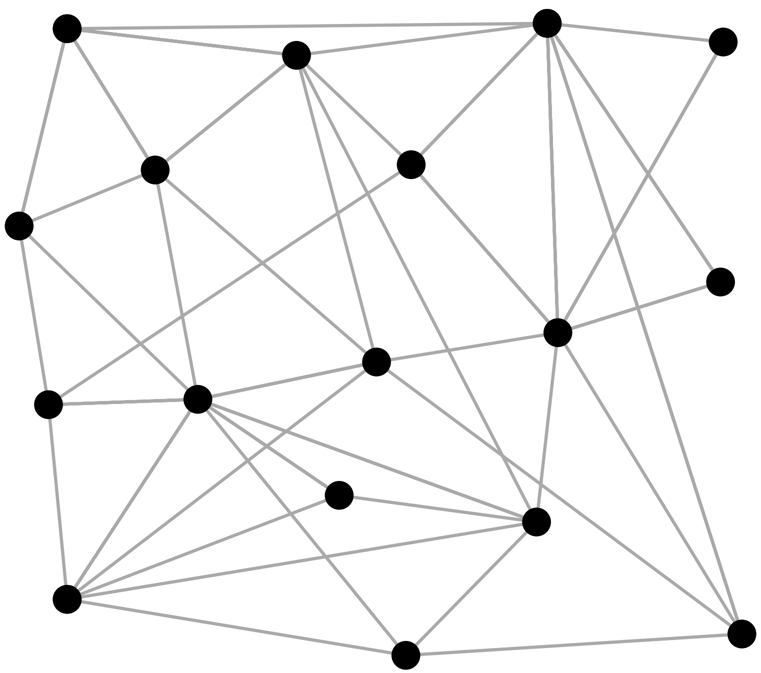
Given: A complete graph $G = (V, E)$ with edge costs $c \colon E \to \mathbb{Q}_{\geq 0}$ satisfying the triangle inequality



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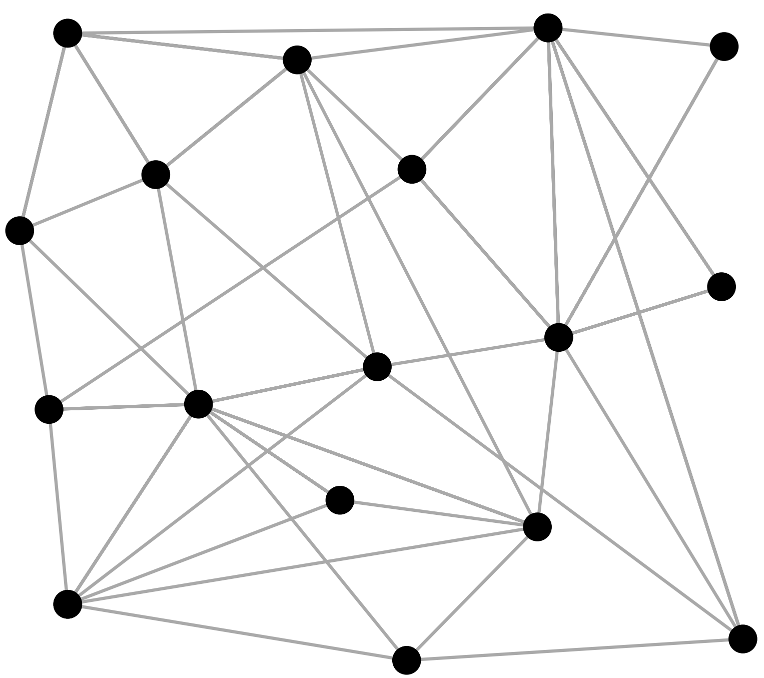

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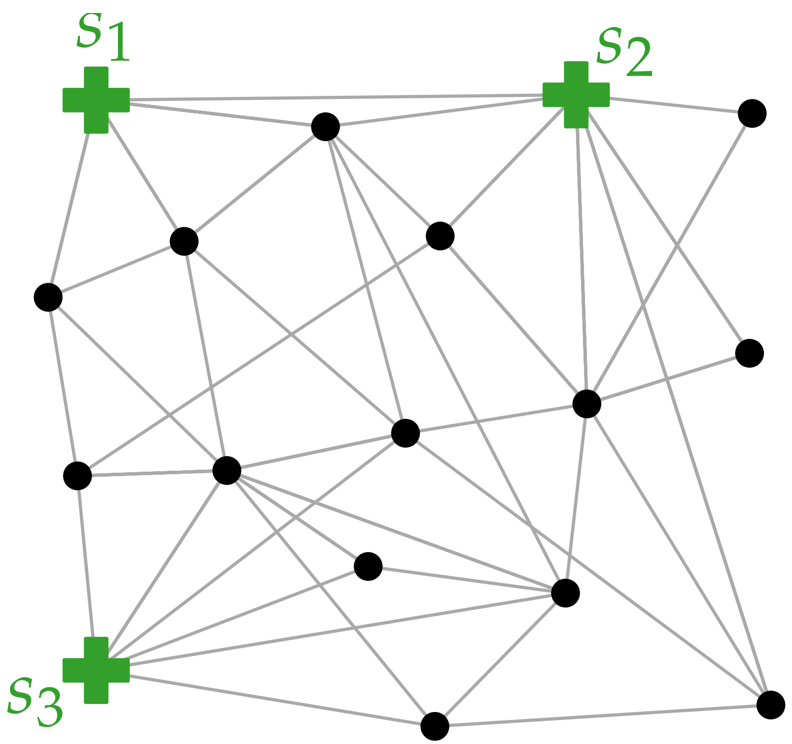

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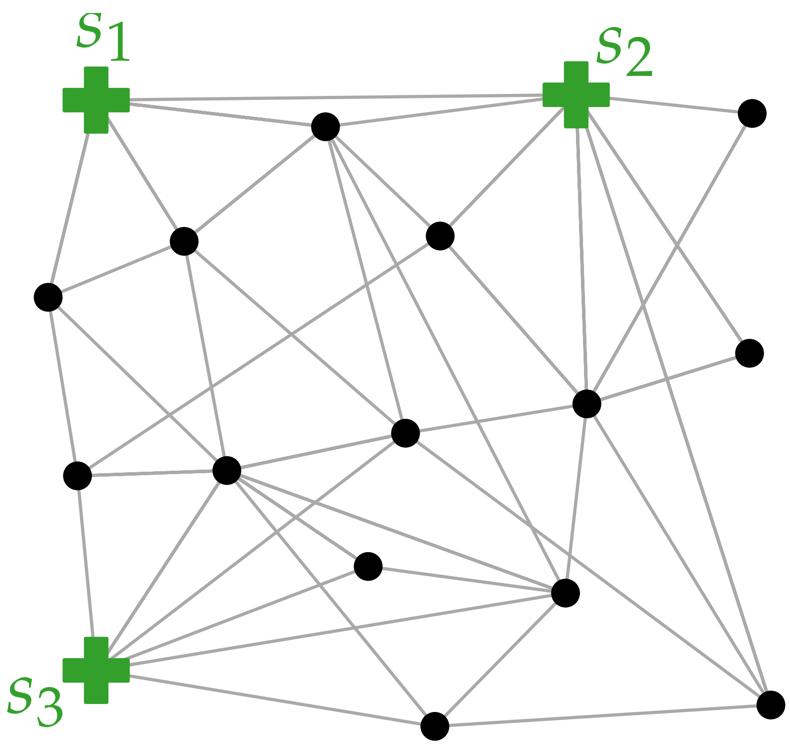
Given: A complete graph $G = (V, E)$ with edge costs $c \colon E \to \mathbb{Q}_{\geq 0}$ satisfying the triangle inequality

For each vertex set $S \subseteq V$, $c(v, S)$ is the cost of the cheapest edge from v to a vertex in S.



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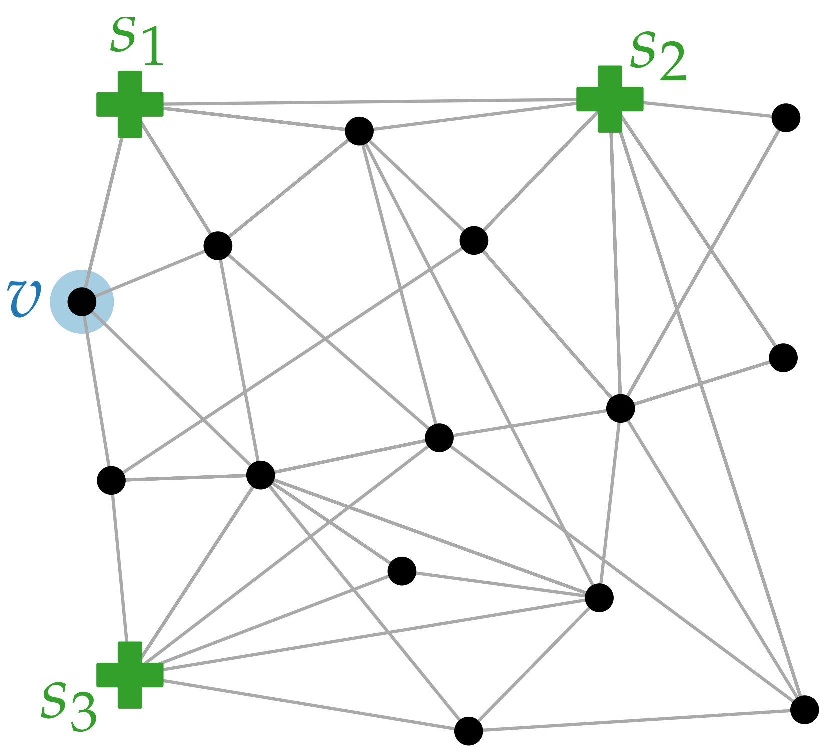

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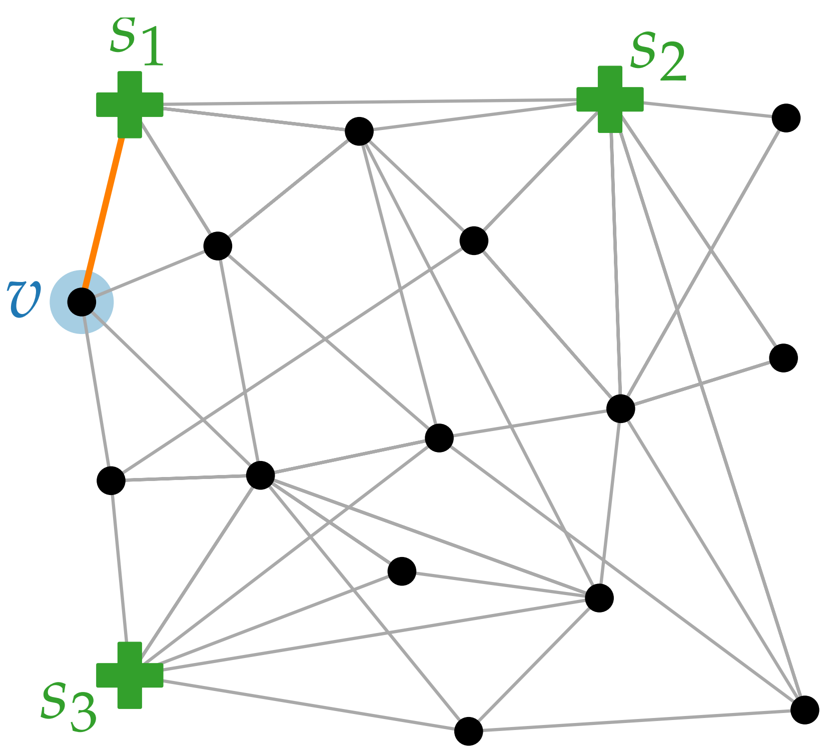

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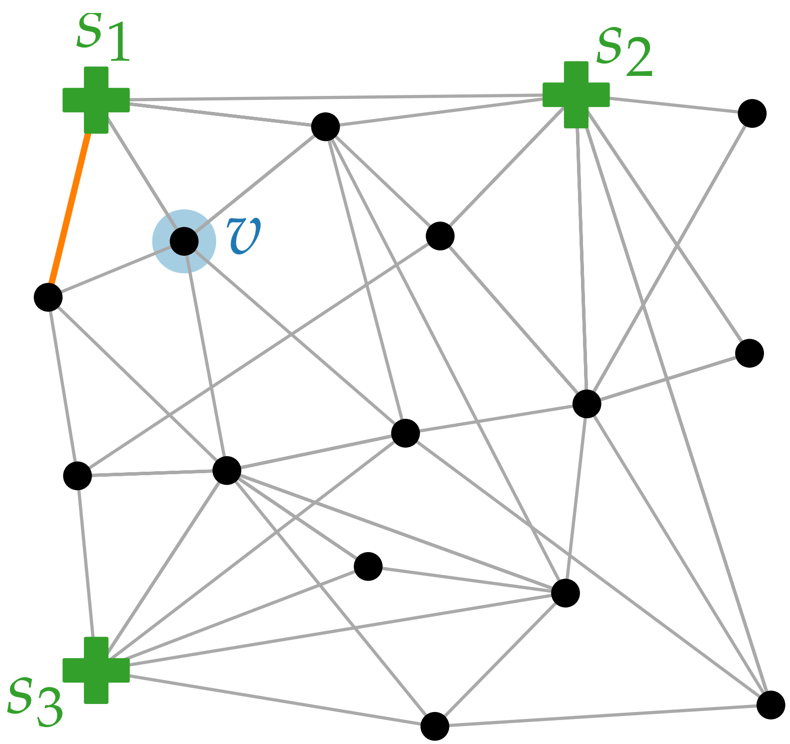

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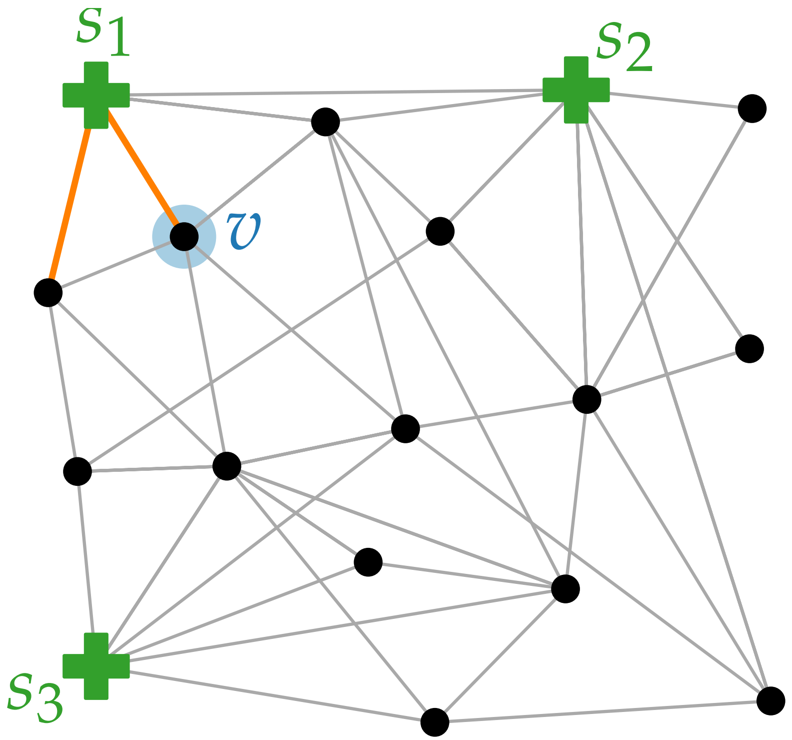

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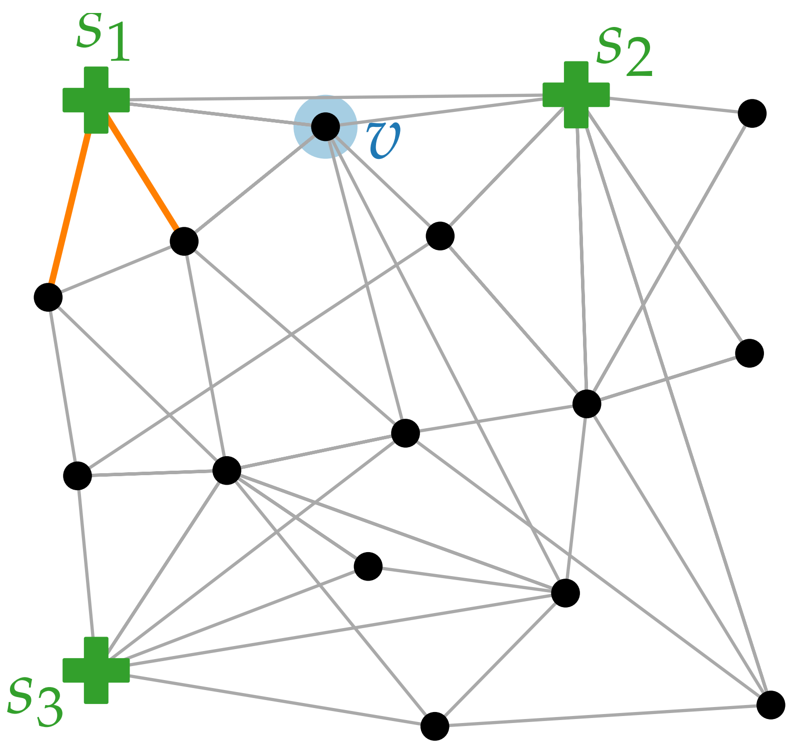

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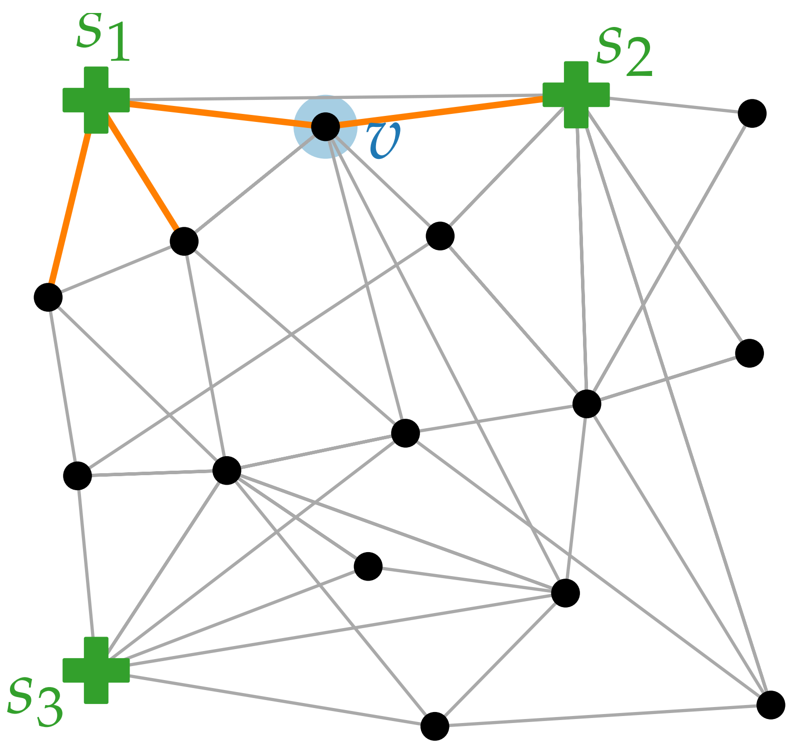

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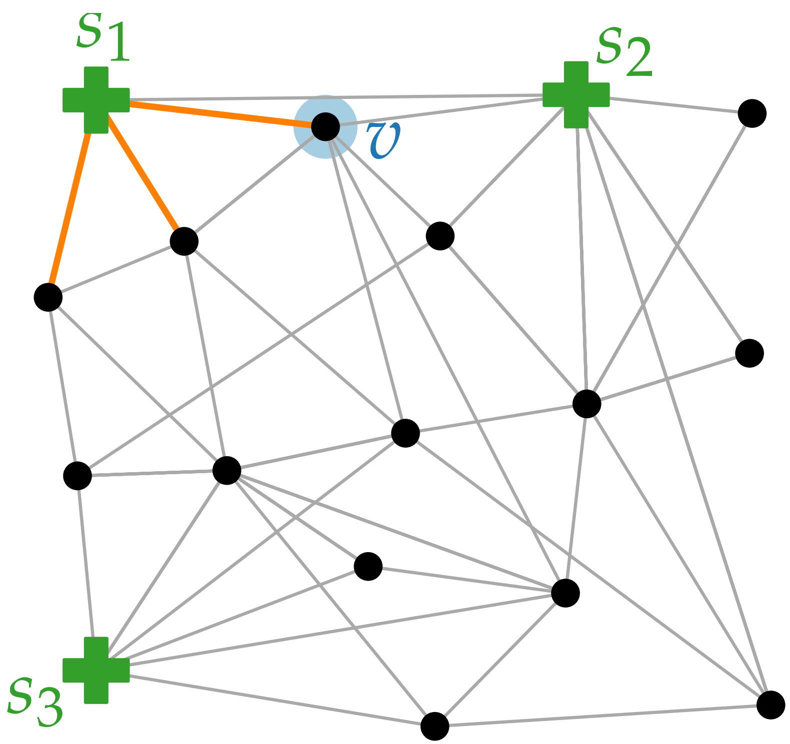

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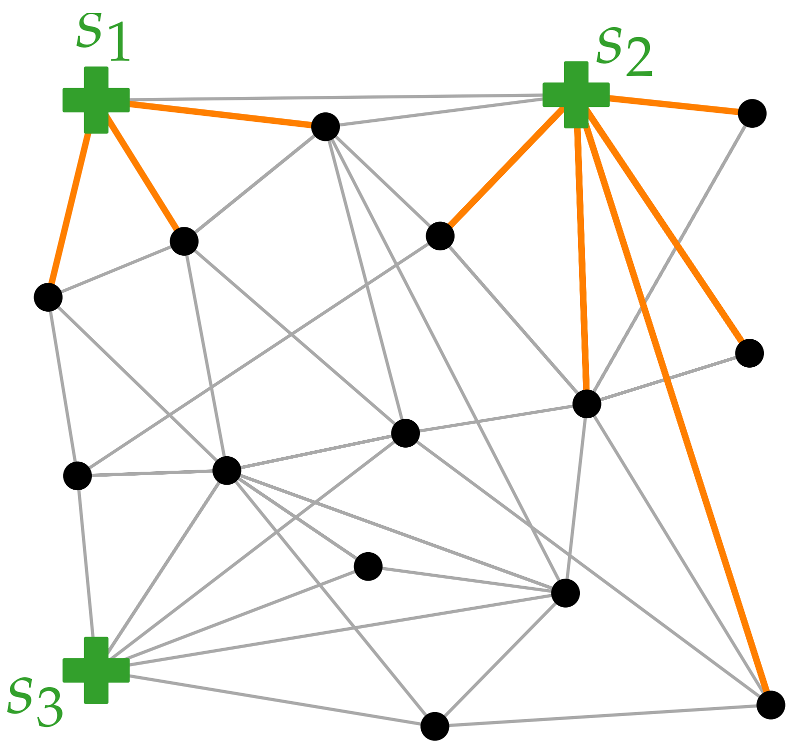

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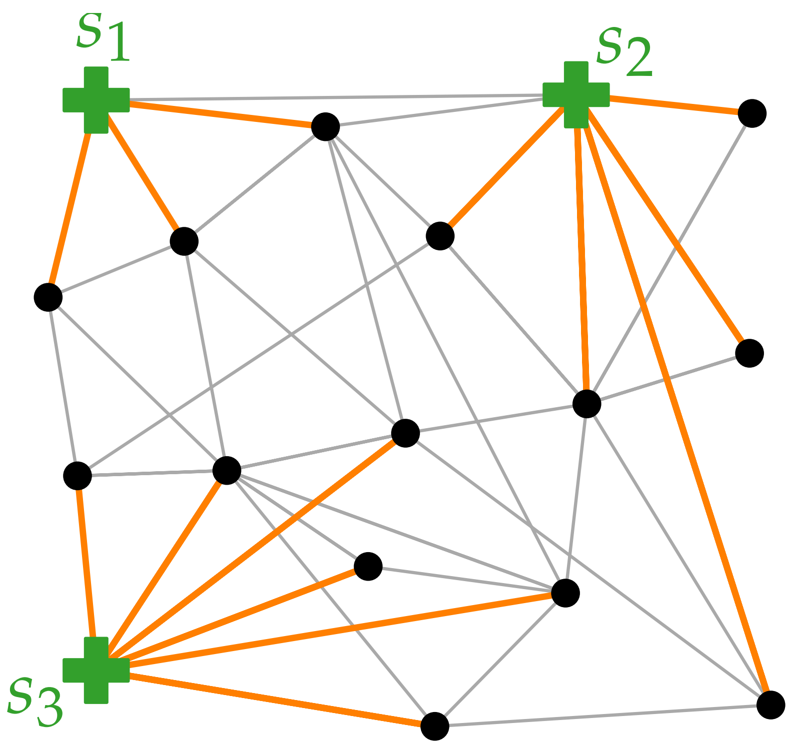

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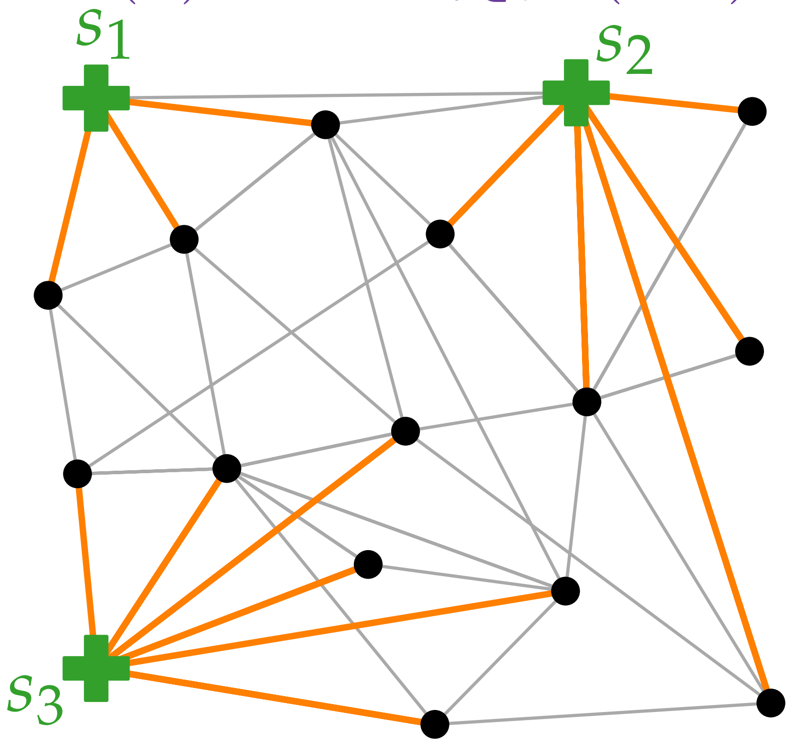

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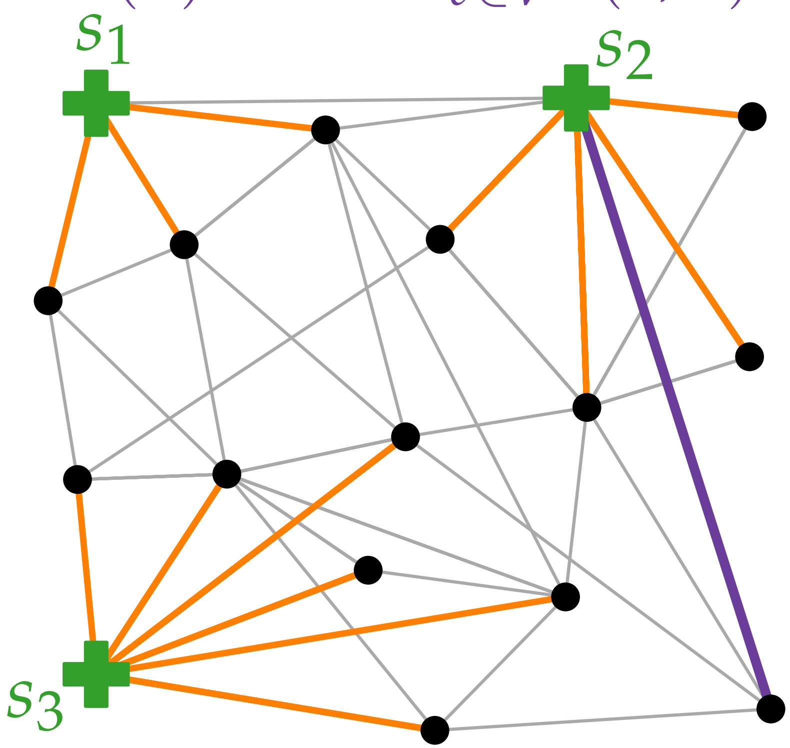

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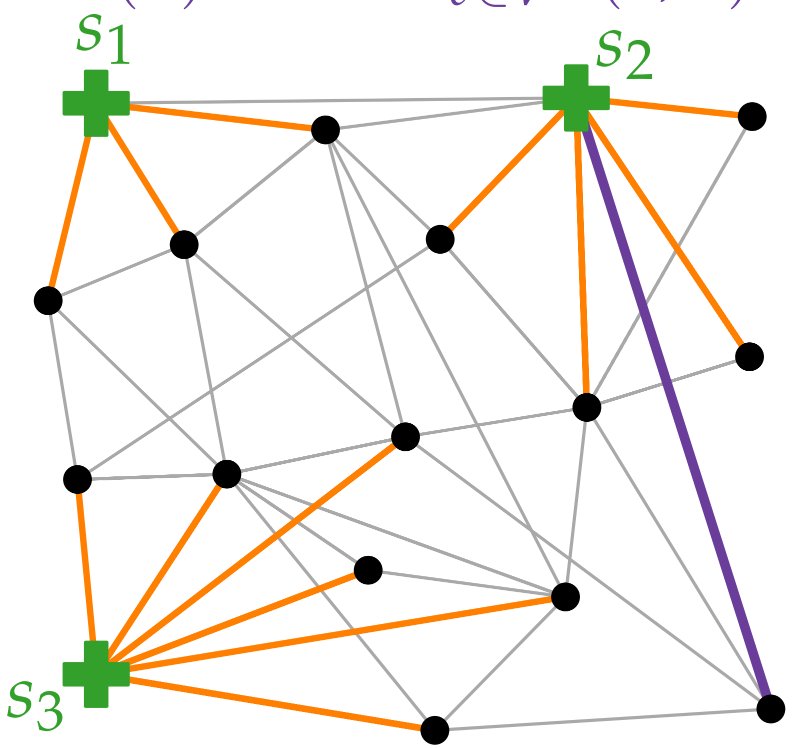

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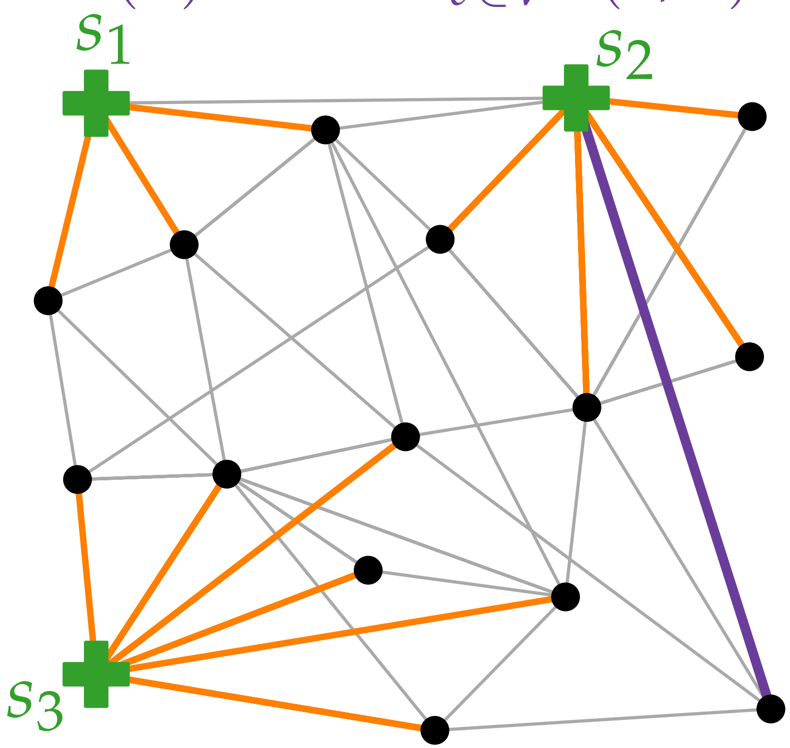

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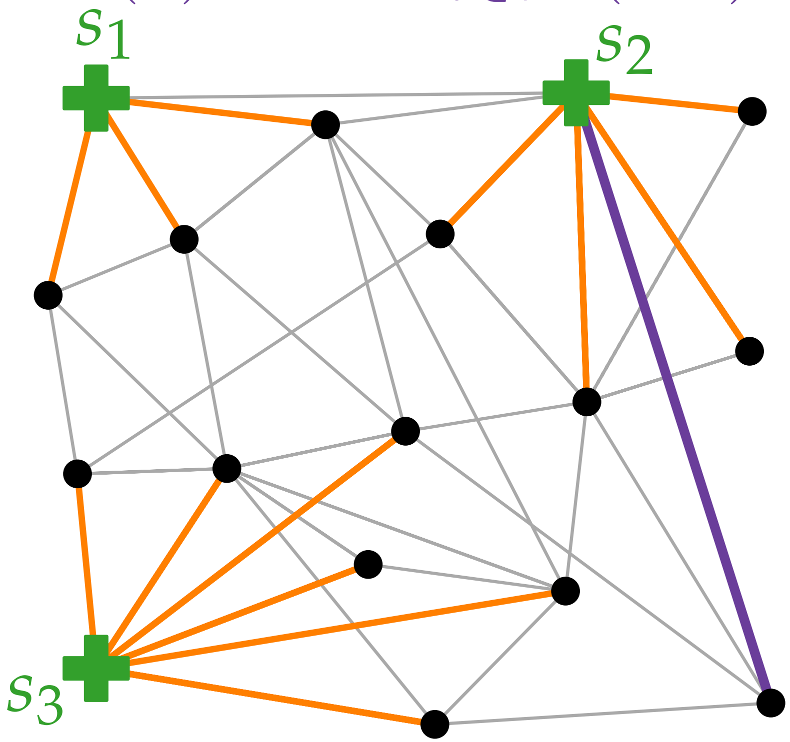

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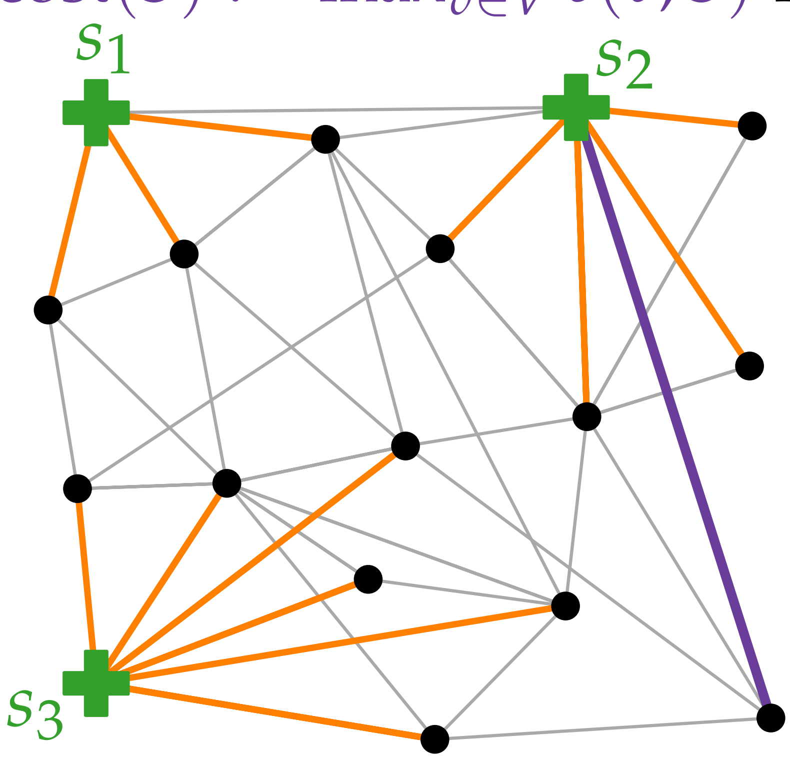

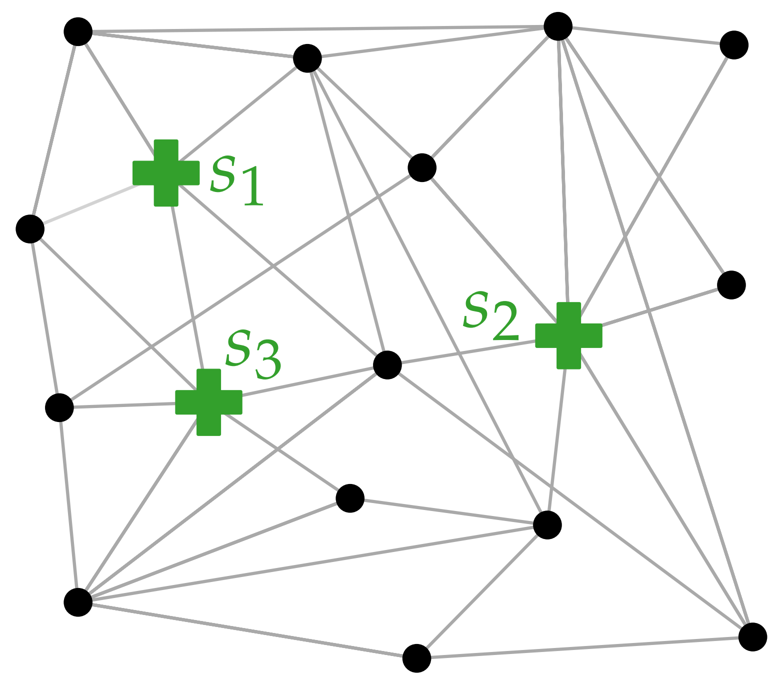

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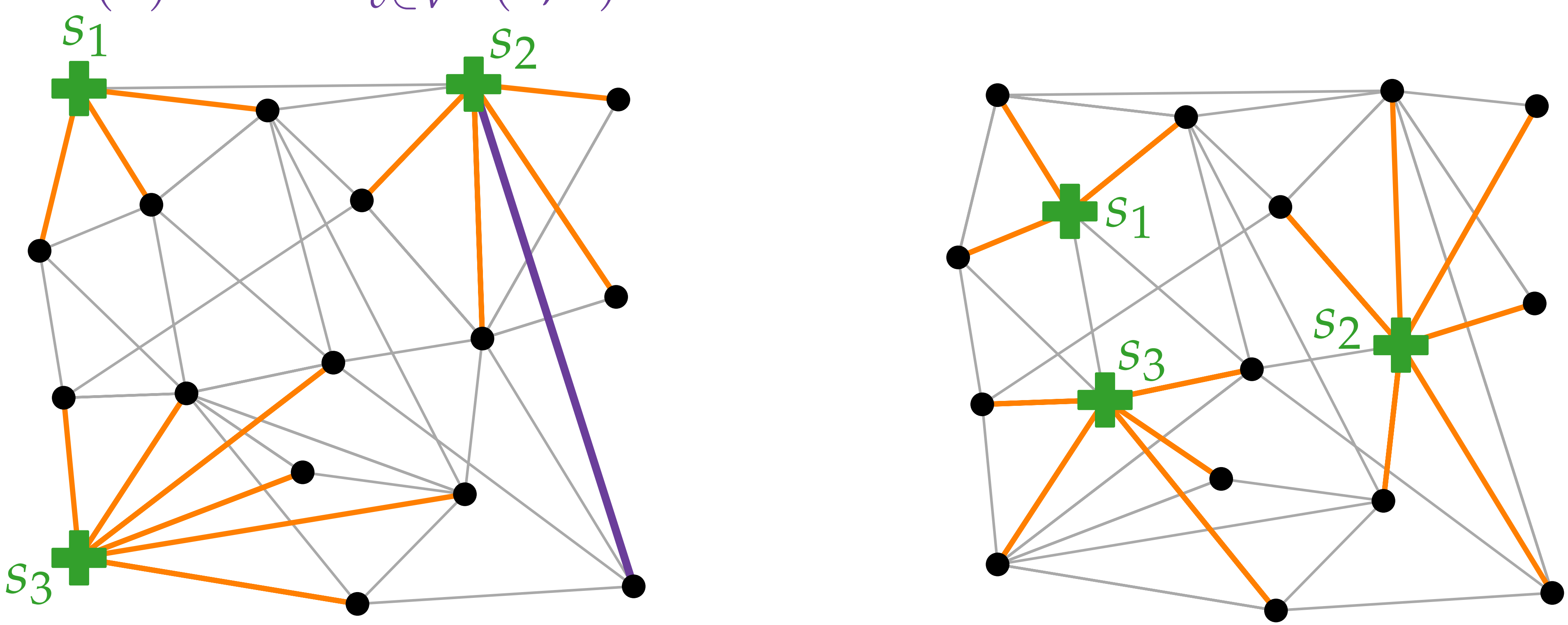

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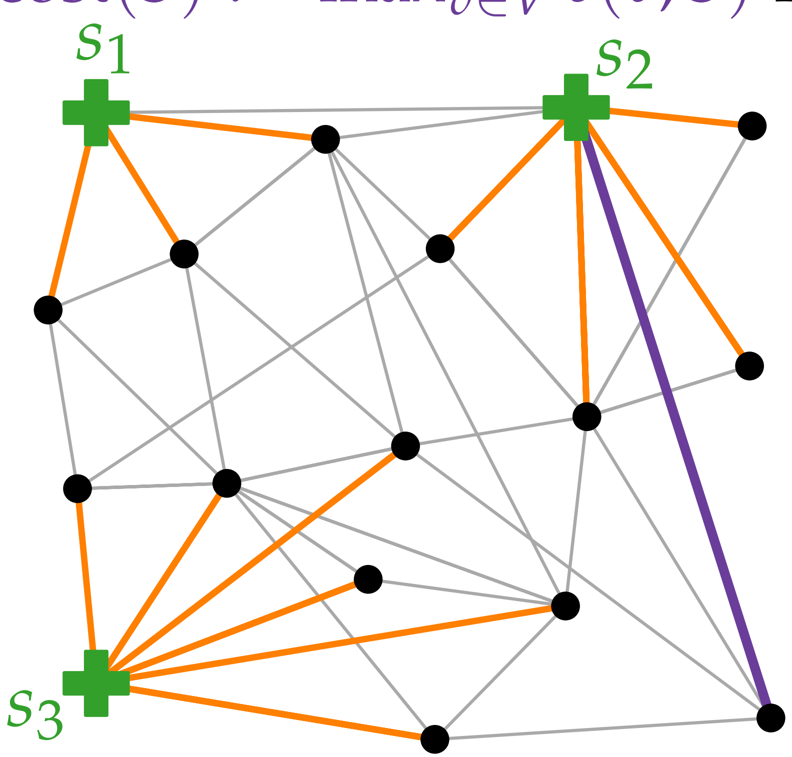
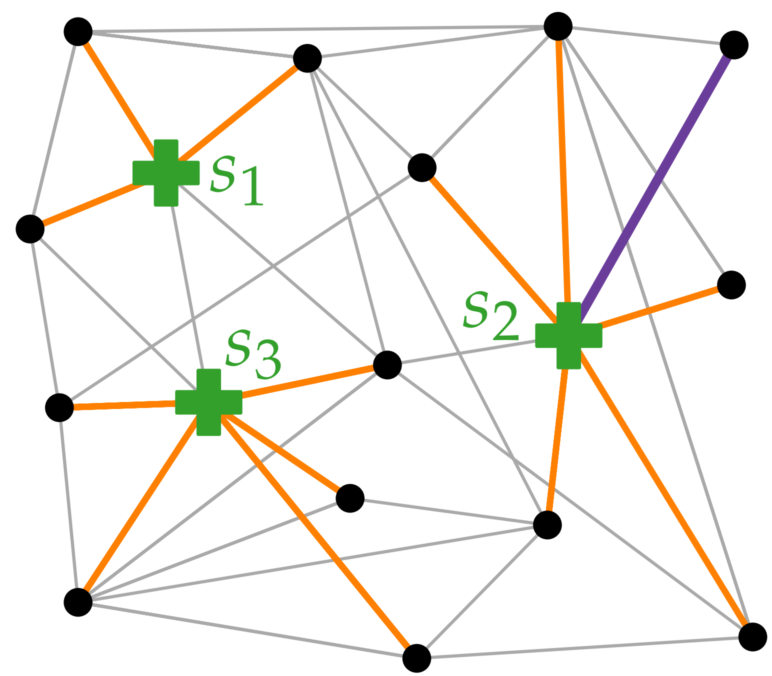


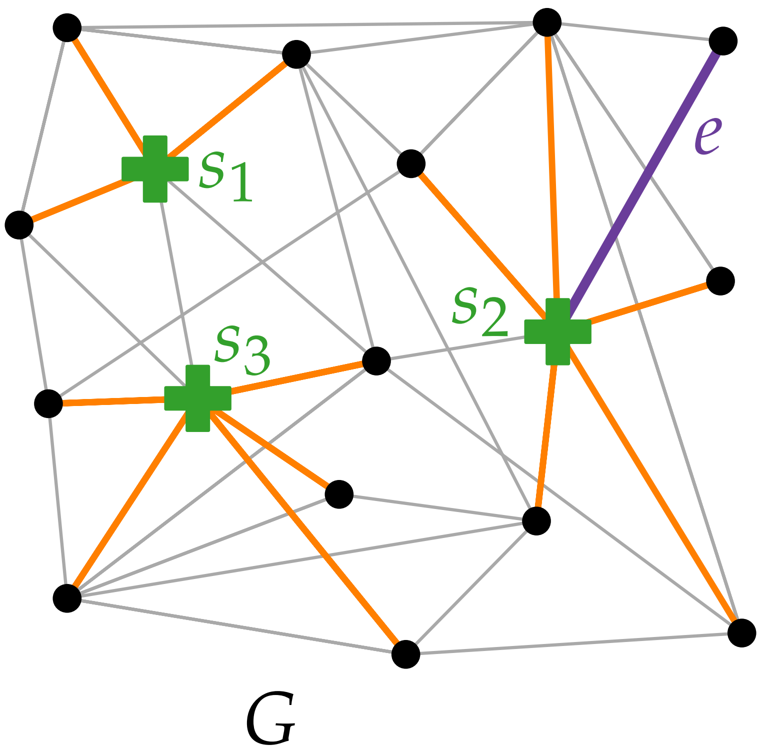


Approximation Algorithms

Lecture 6:
k-Center via Parametric Pruning

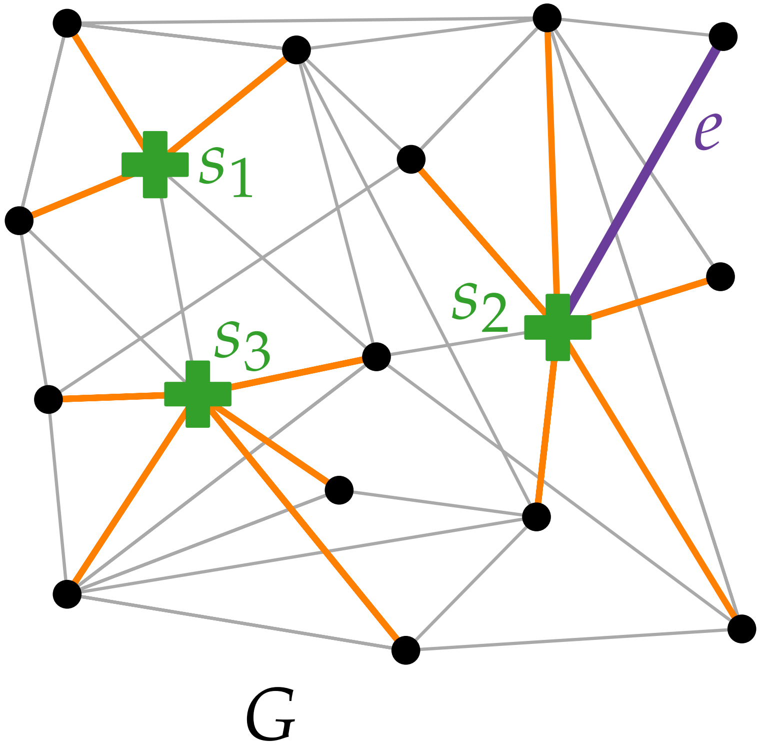
Part II:
Parametric Pruning

Parametric Pruning



Parametric Pruning

Let $E = \{e_1, \ldots, e_m\}$ with $c(e_1) \leq \cdots \leq c(e_m)$.



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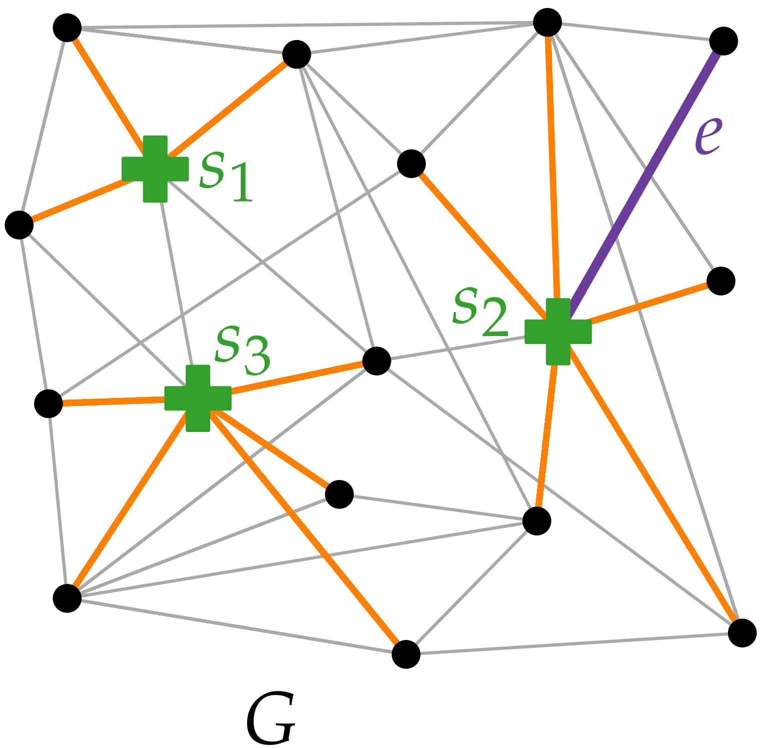
Suppose we know that $\mathrm{OPT} = c(e_j)$.



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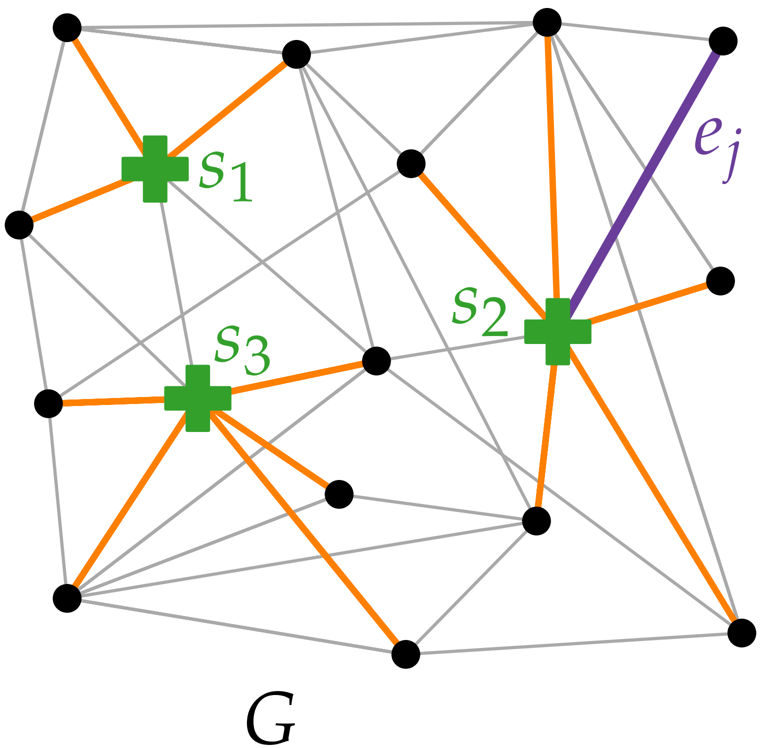

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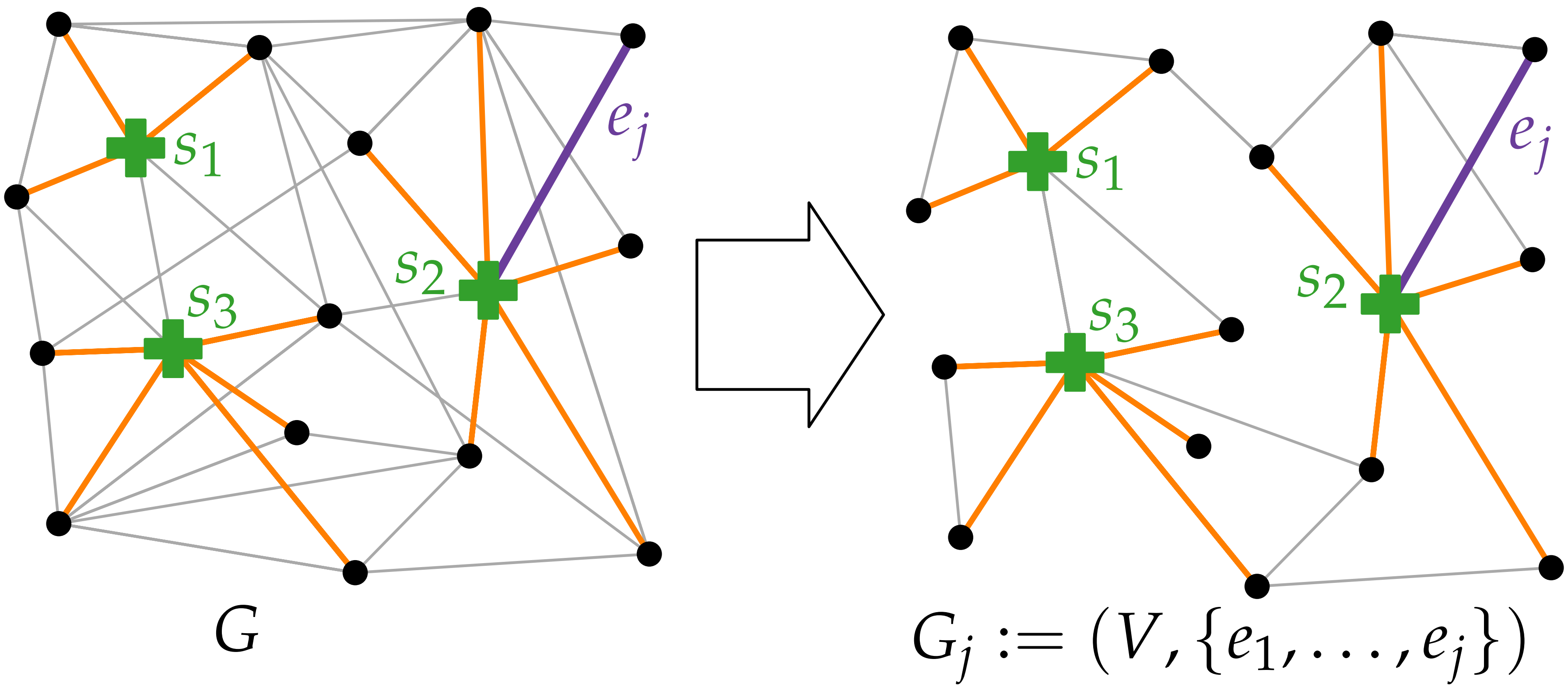

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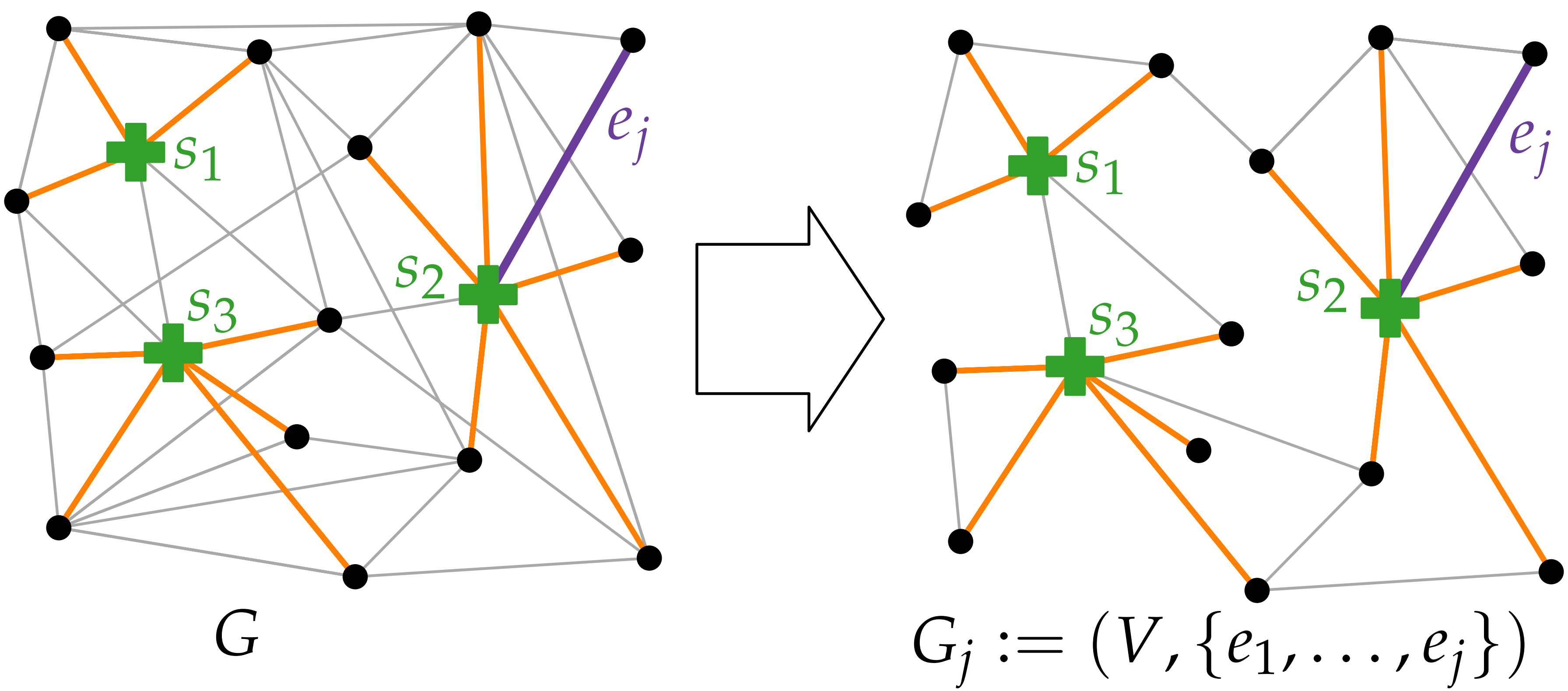

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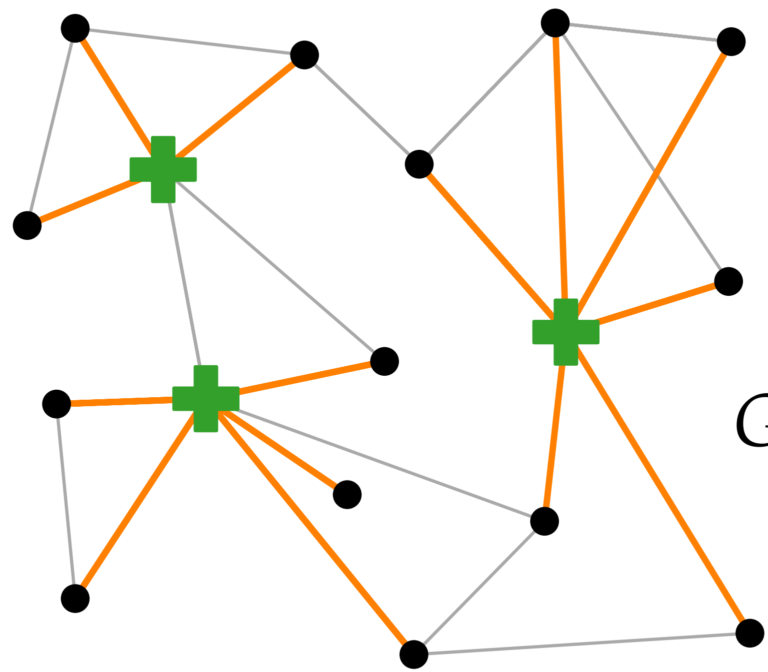
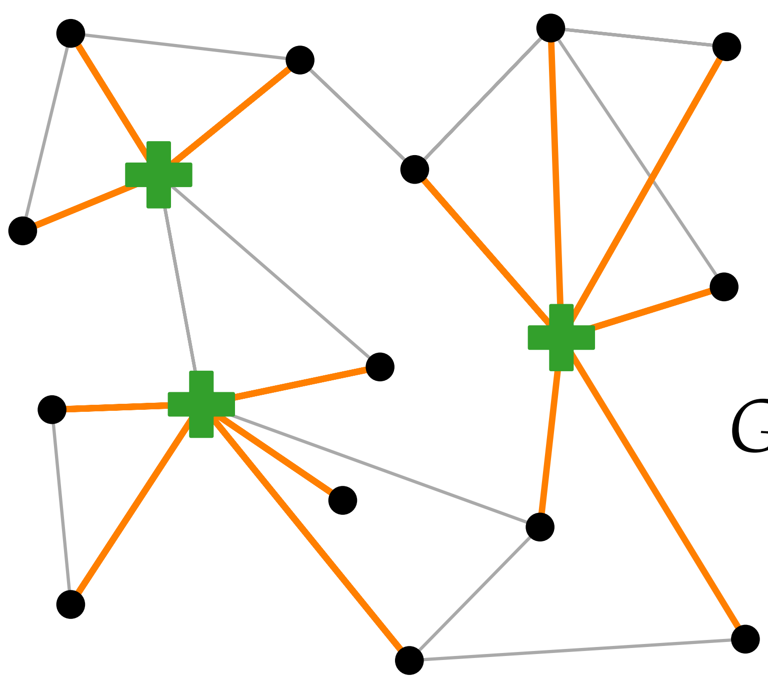
Def.



$$G_j := (V, \{e_1, \ldots, e_j\})$$

... try each G_j.

Def. A vertex set D of a graph H is **dominating** if each vertex is either in D or adjacent to a vertex in D.



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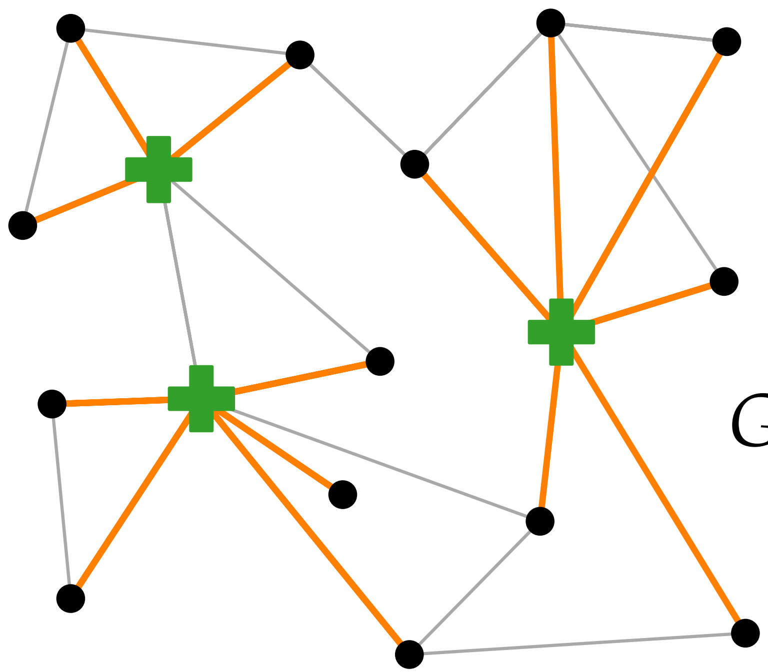

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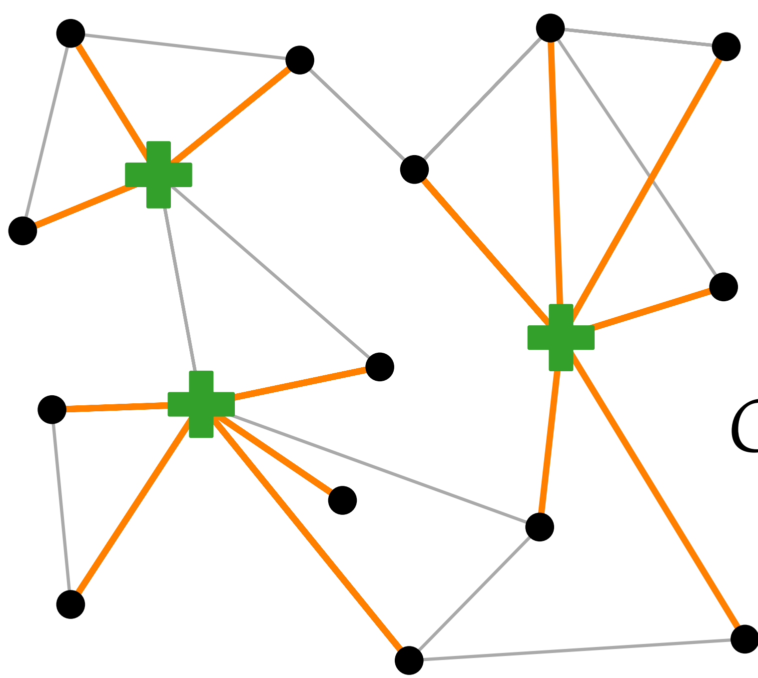

$$\text{dom}(G_j) \leq k$$

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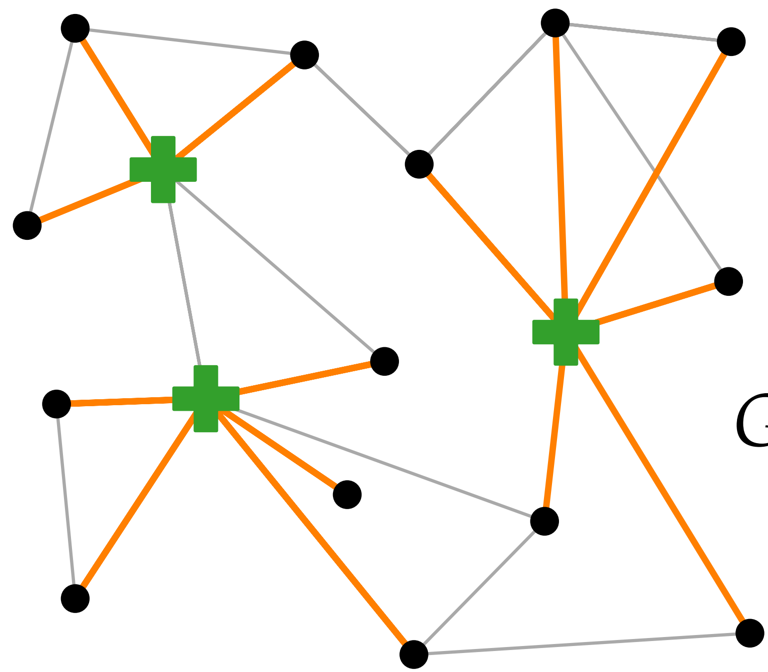

$\mathrm{dom}(G_j) \leq k$

$G_j := (V, \{e_1, \ldots, e_j\})$

… but computing $\mathrm{dom}(H)$ is NP-hard.

Approximation Algorithms

Lecture 6:
k-Center via Parametric Pruning

Part III:
Square of a Graph

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Idea: Find a small dominating set in a "coarsened" G_j.

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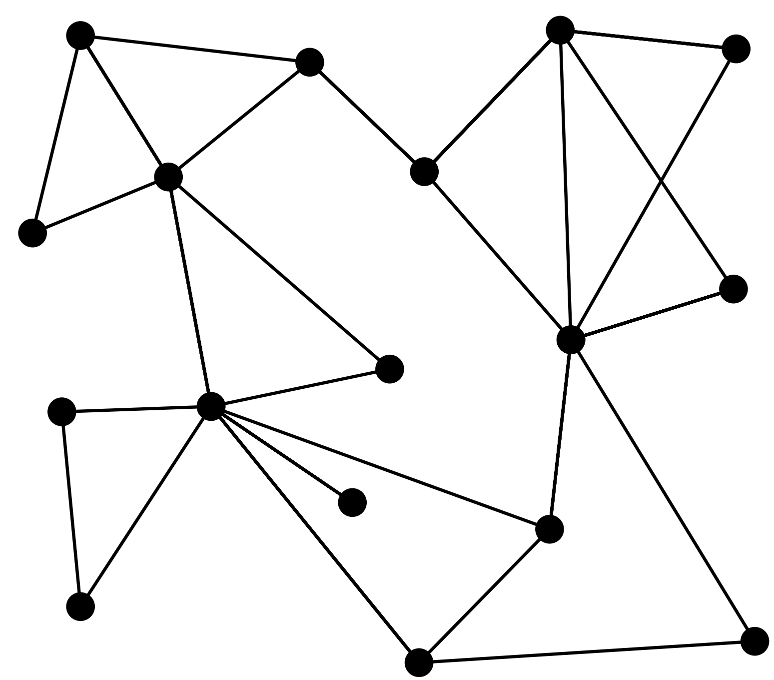
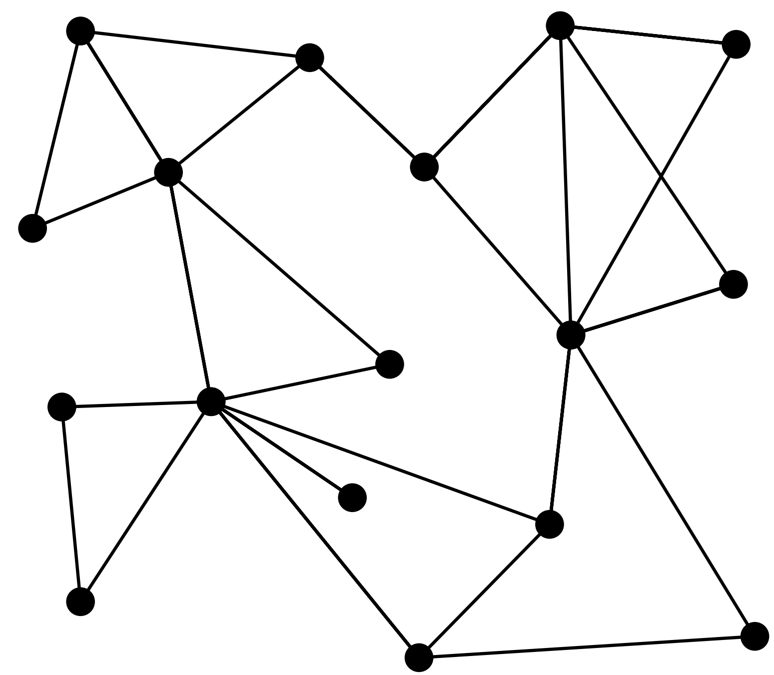


G_j

Square of a Graph

Idea: Find a small dominating set in a "coarsened" G_j.

Def. The **square** H^2 of a graph H has the same vertex set as H. Two vertices $u \neq v$ are adjacent in H^2 iff they are within distance at most **two** in H.



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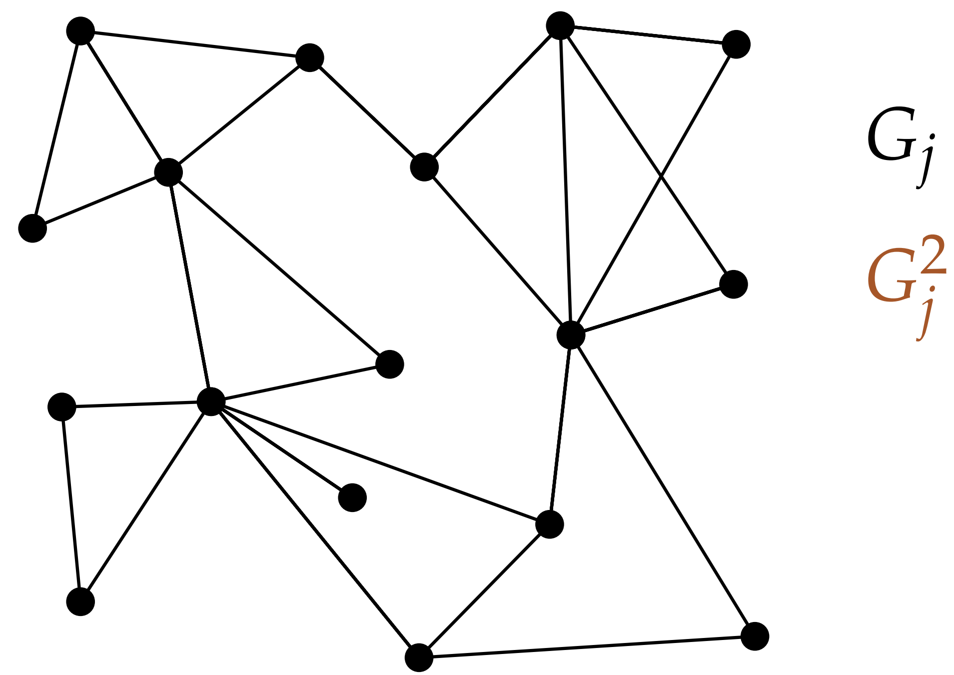

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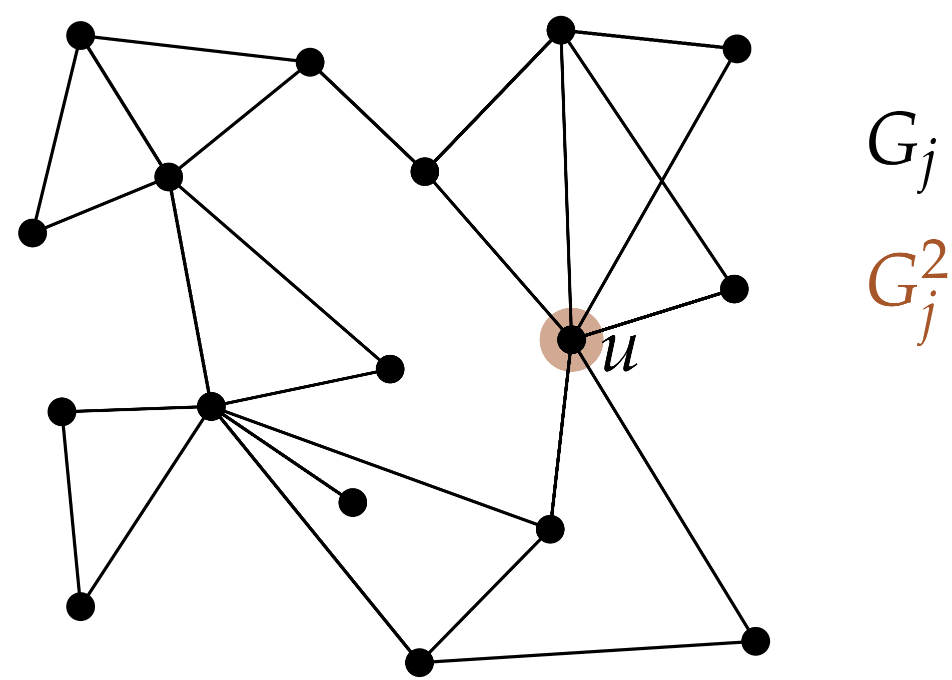

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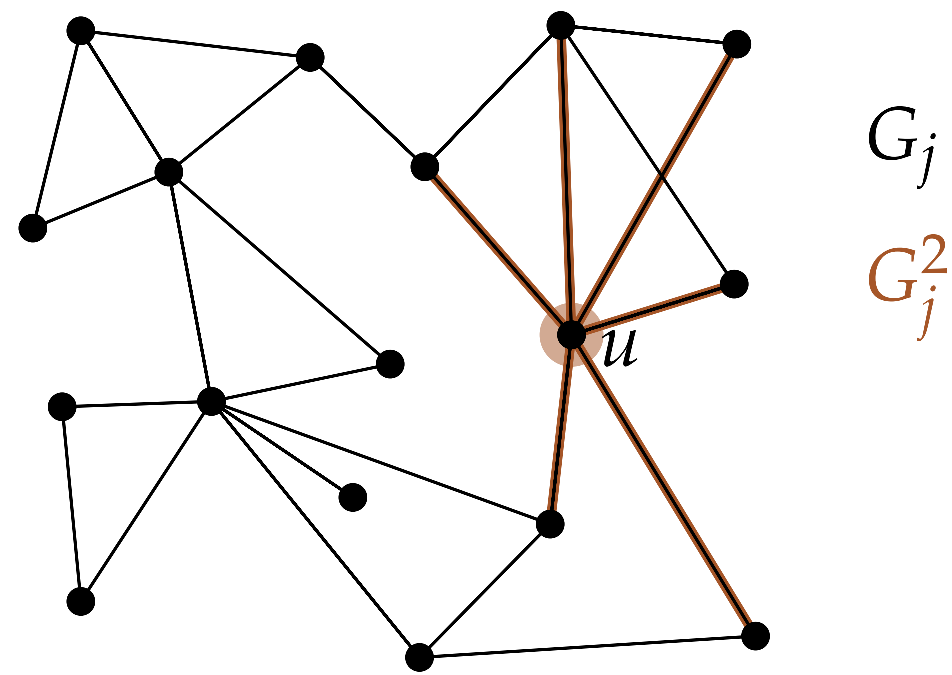

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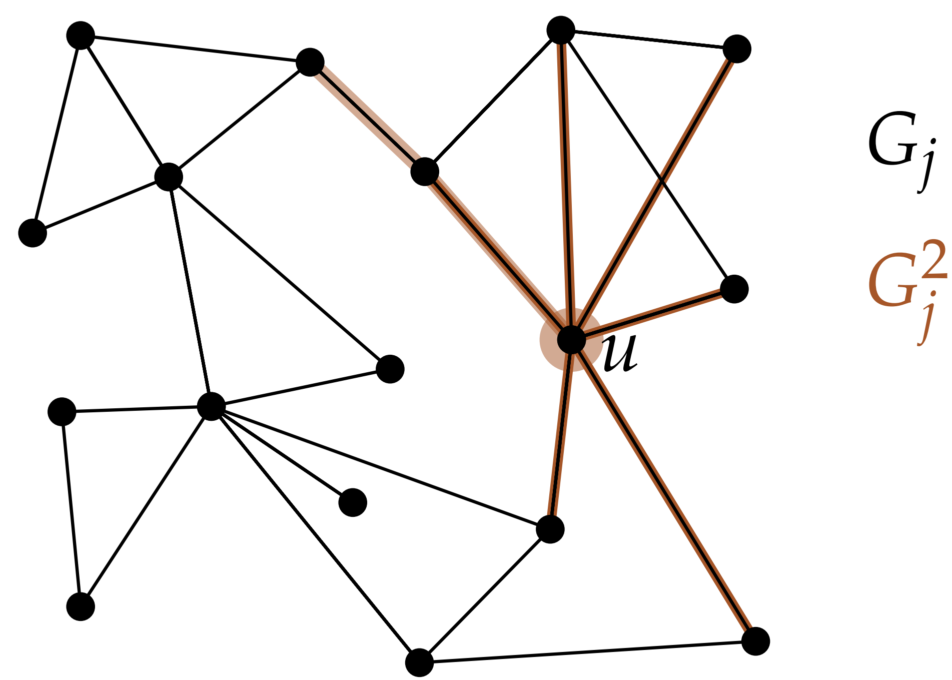

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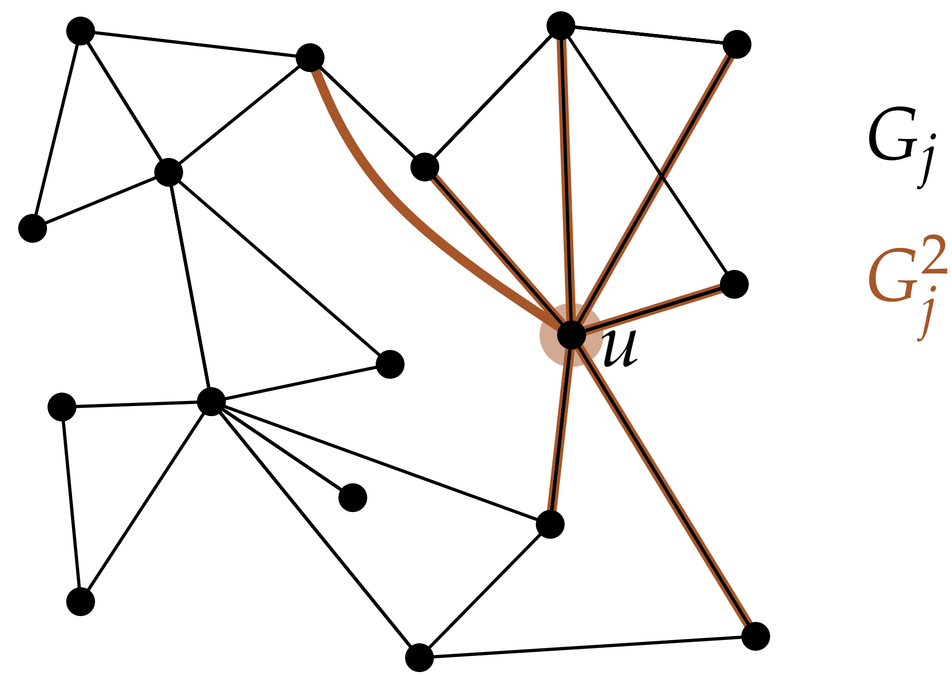

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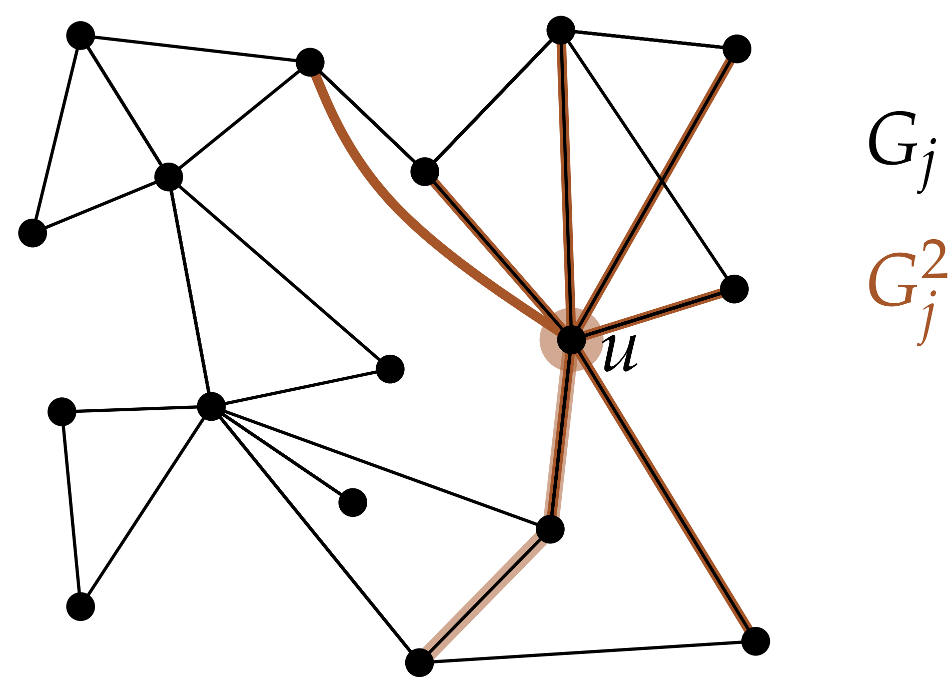

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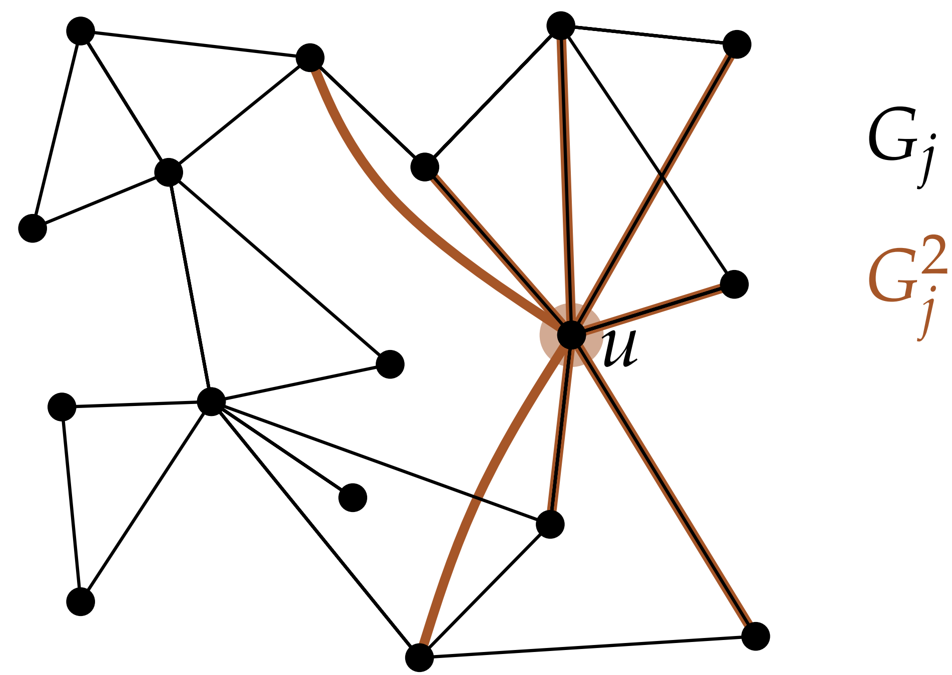

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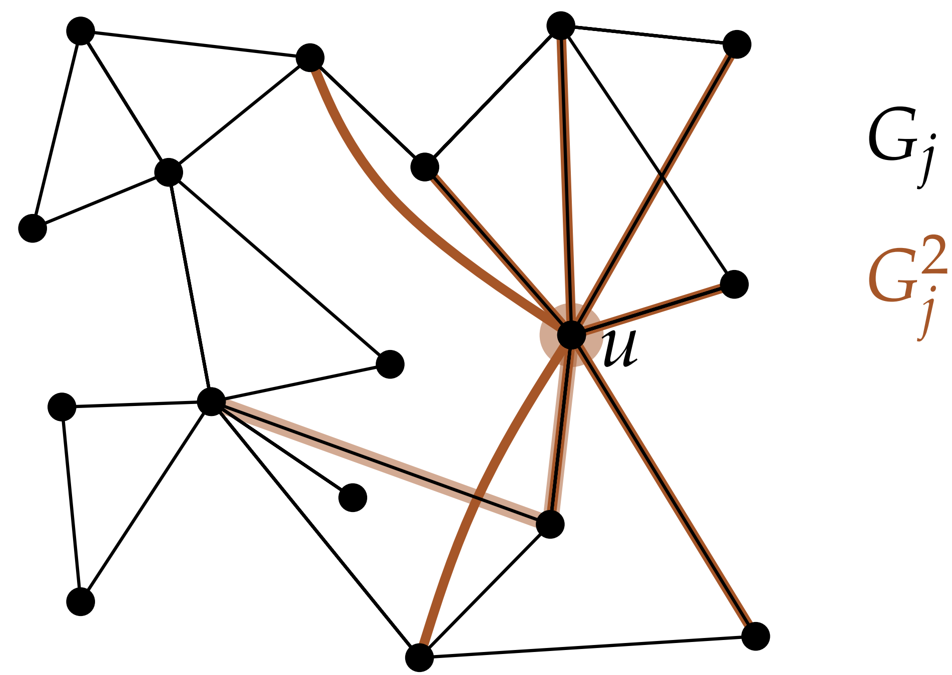

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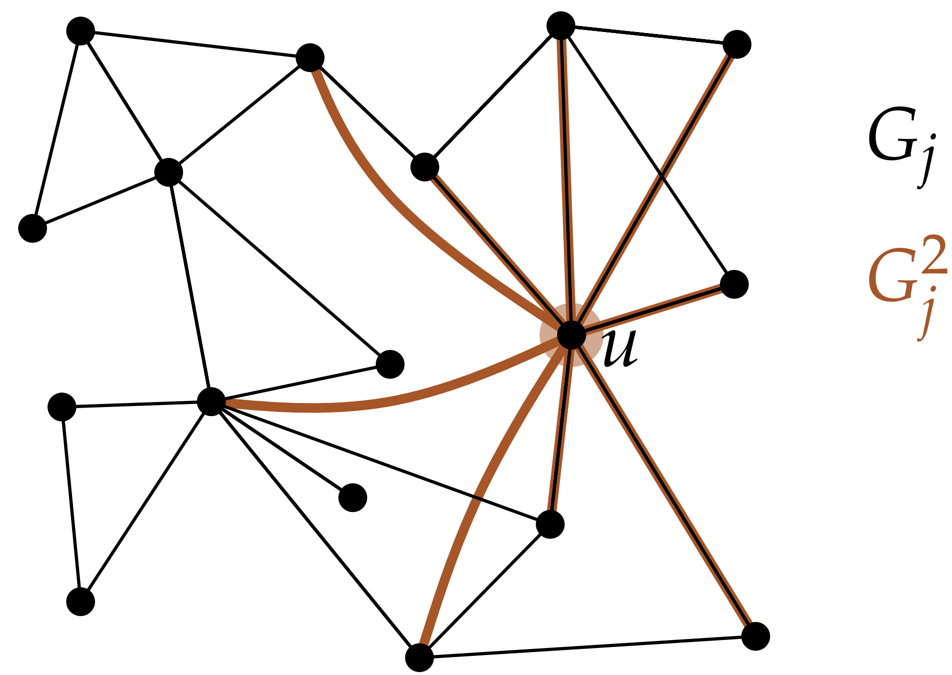

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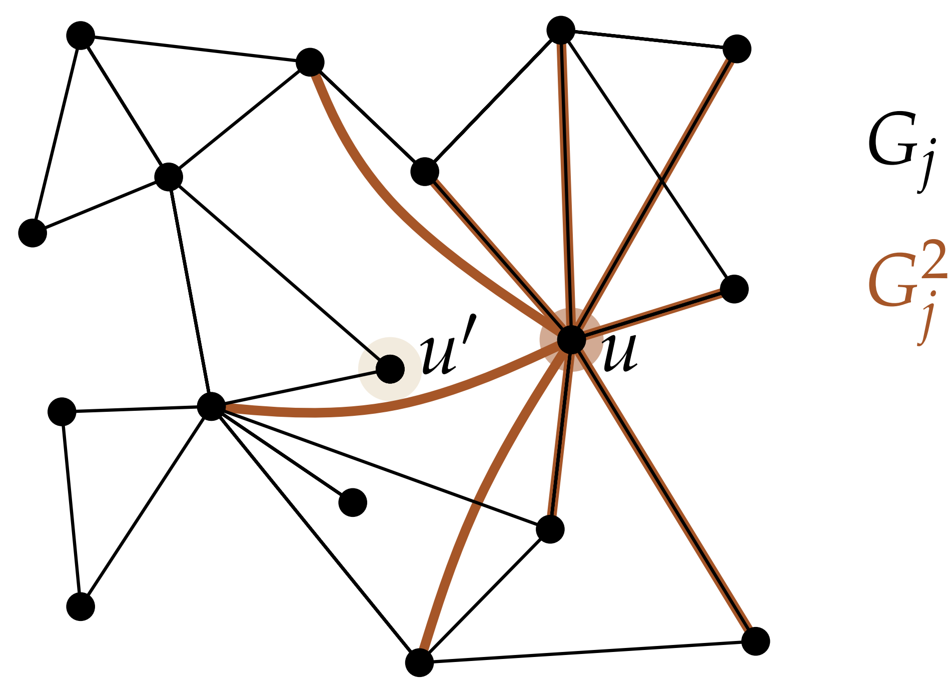

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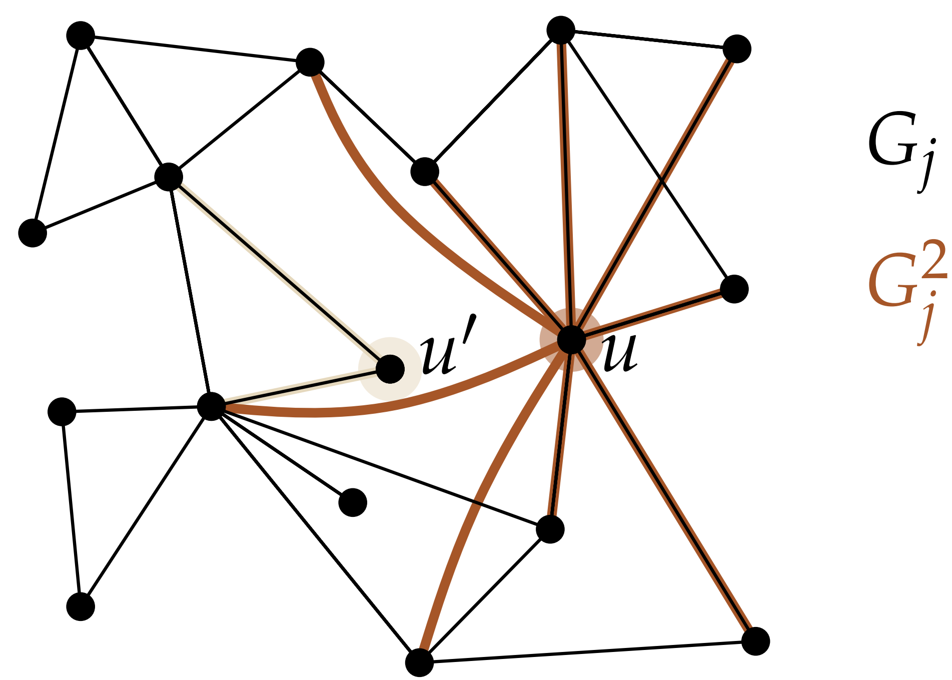

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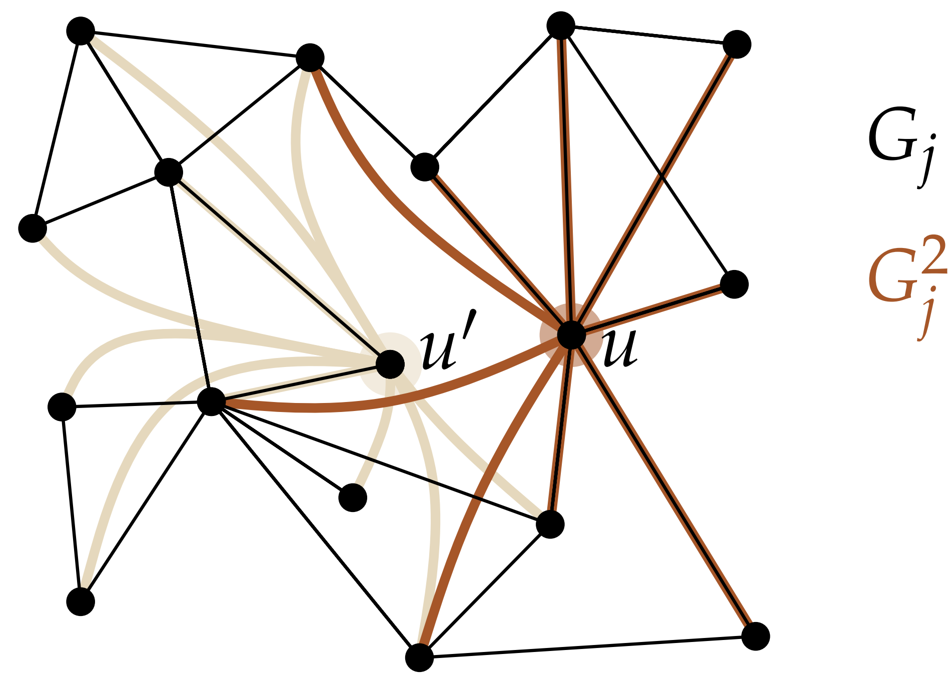

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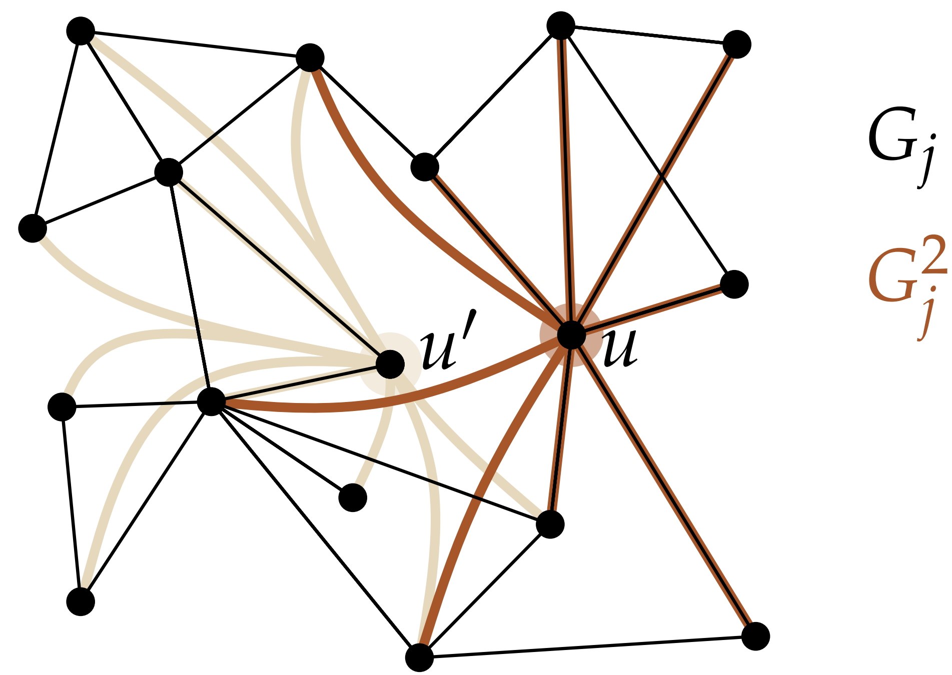

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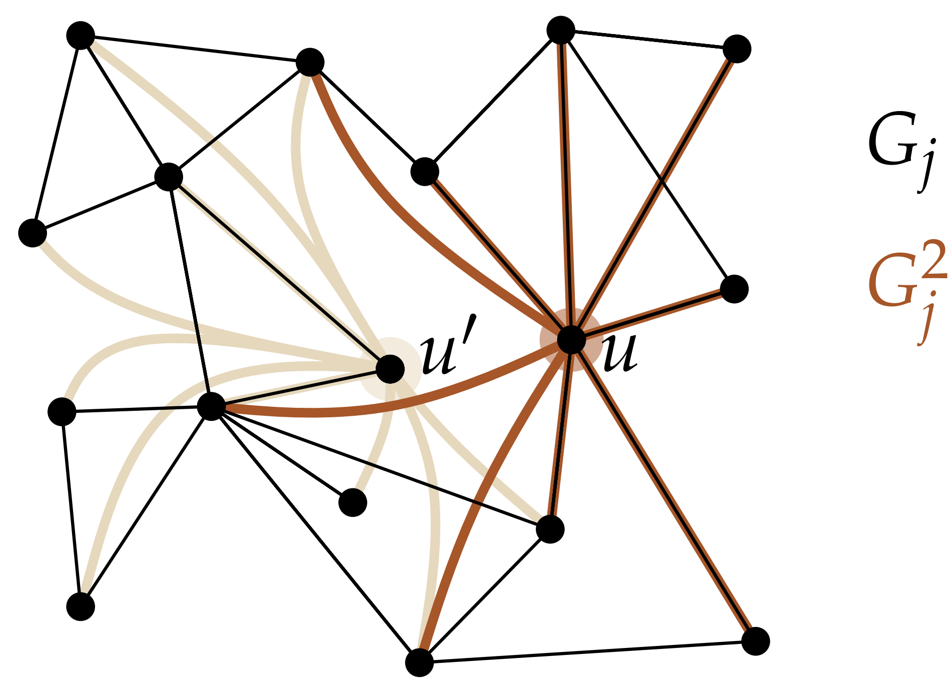

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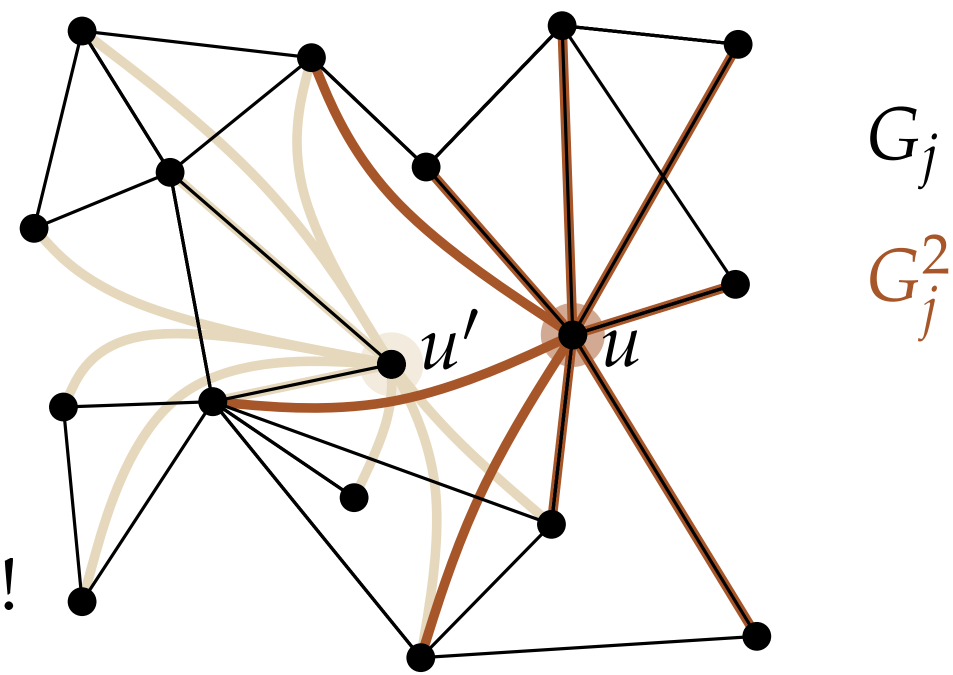

Why? $\max_{e \in E(G_j)} c(e) = \text{OPT}$!

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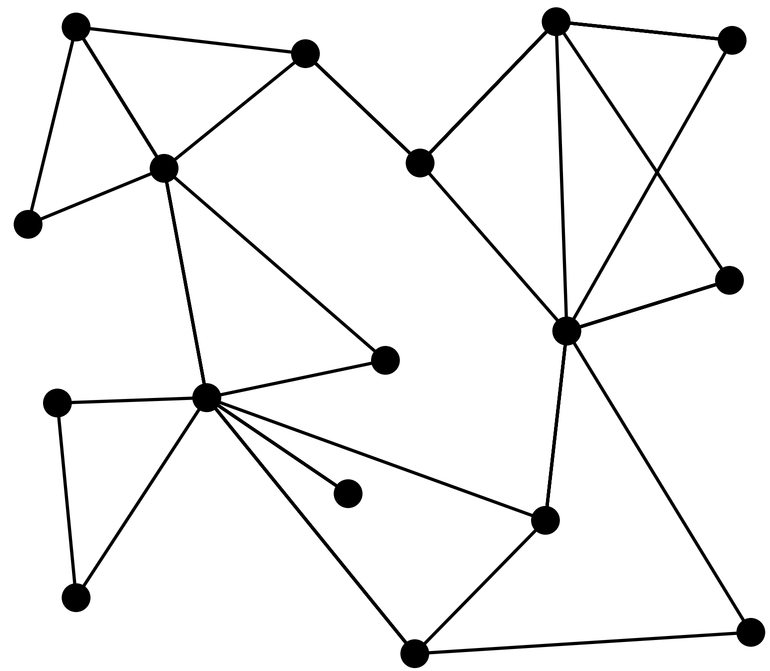

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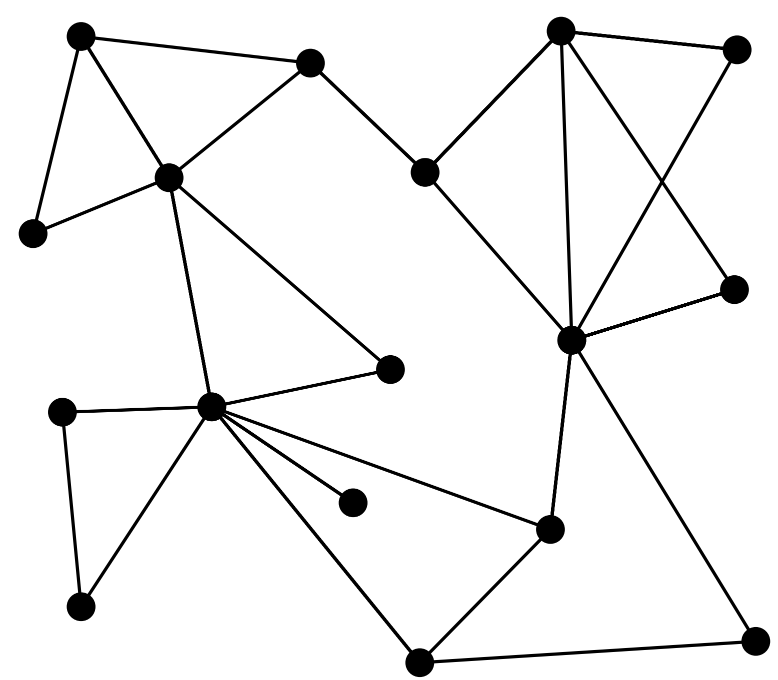

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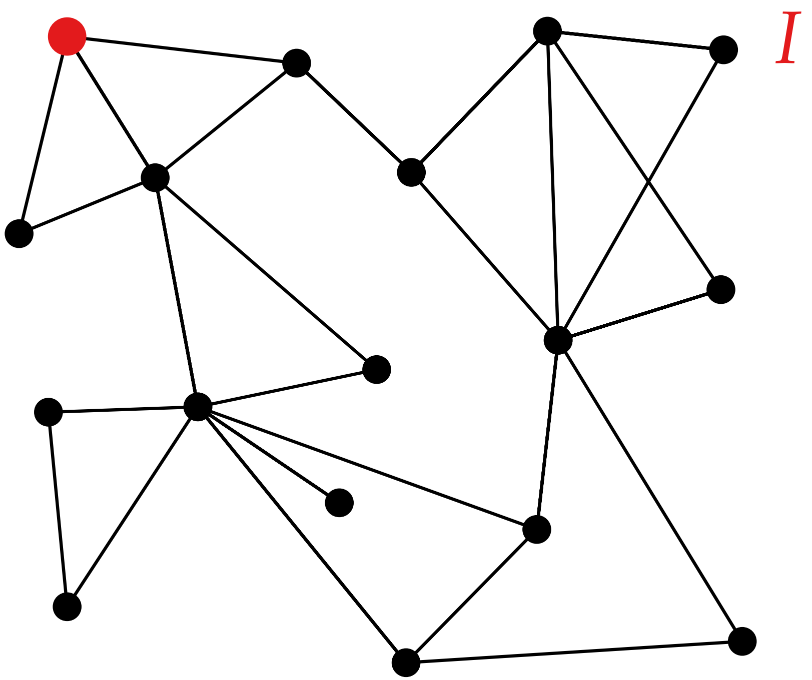

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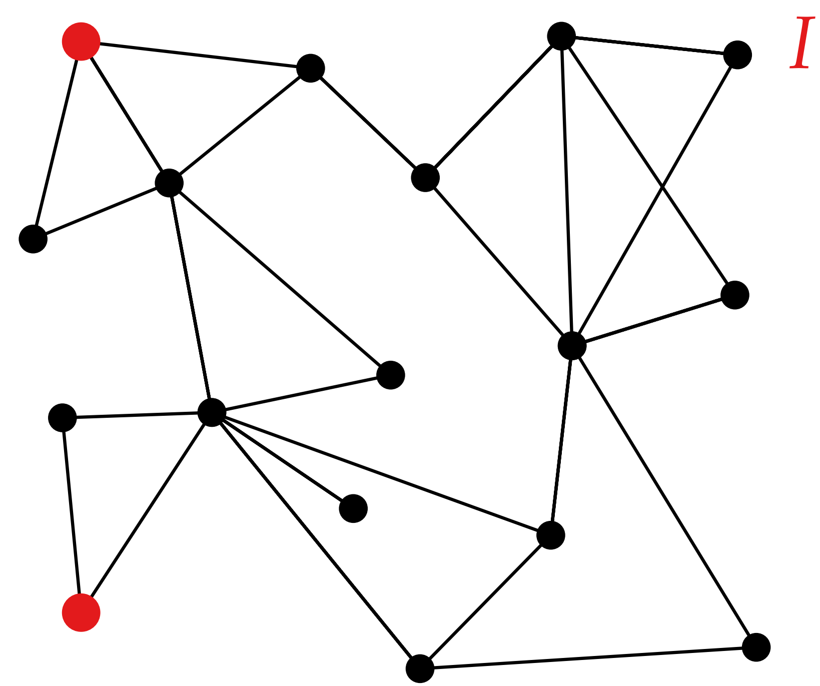

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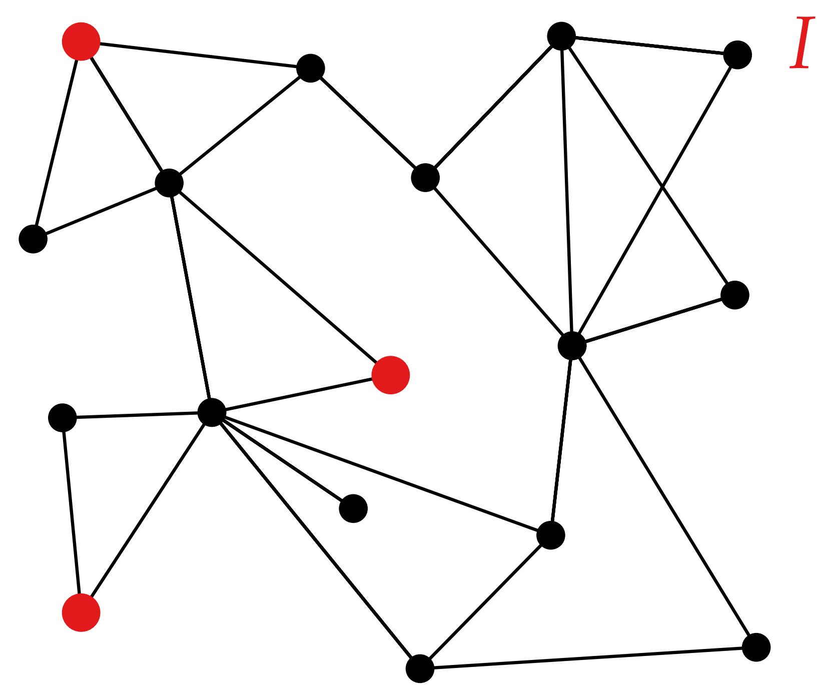

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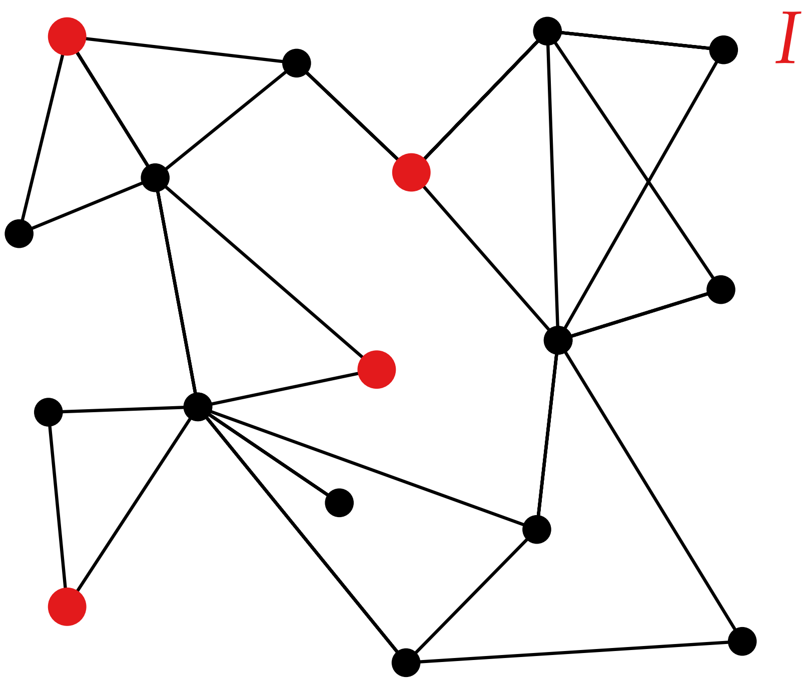

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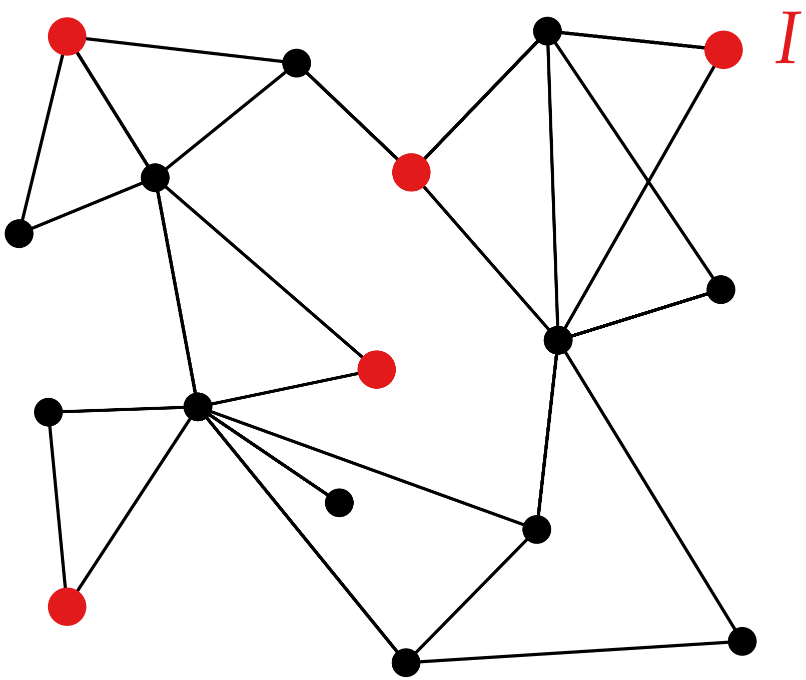

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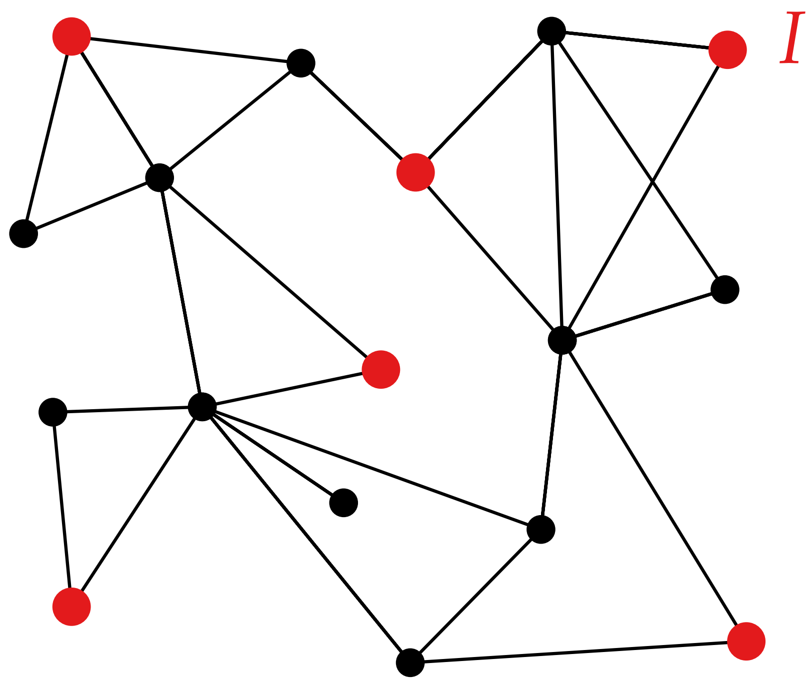

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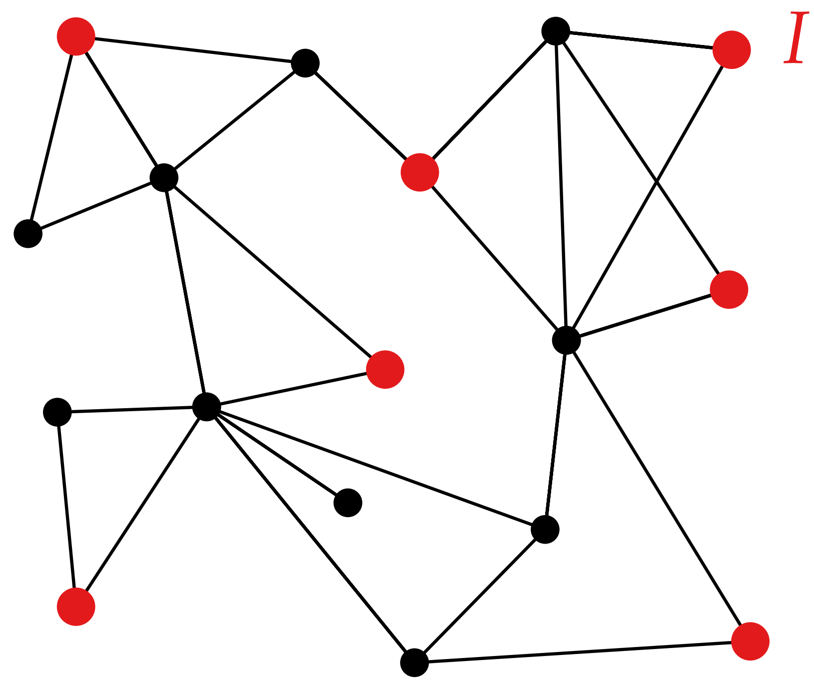

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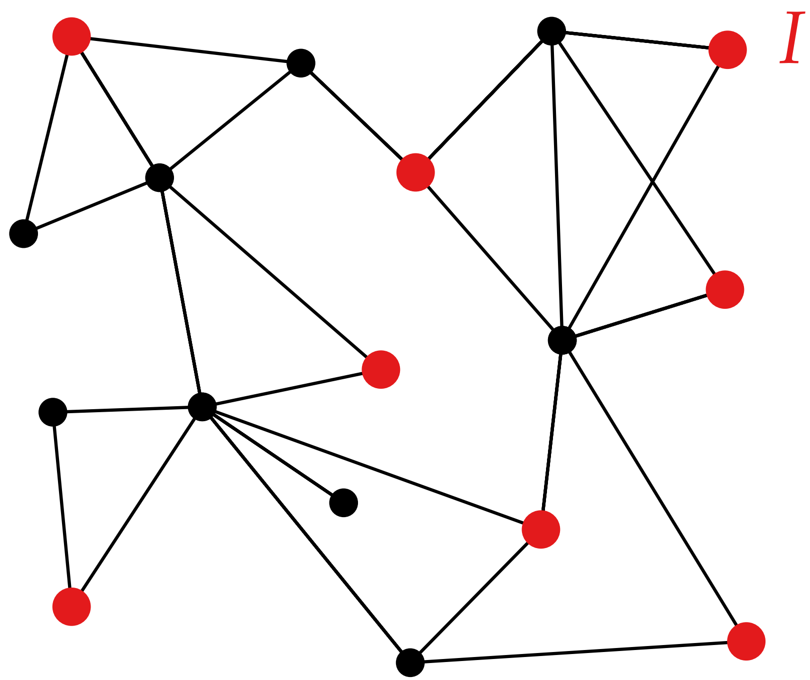

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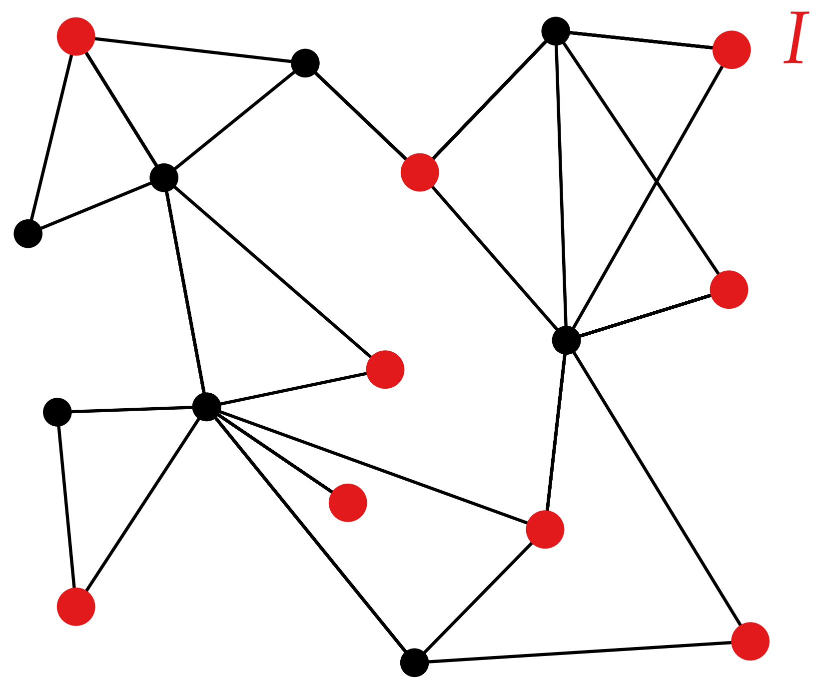

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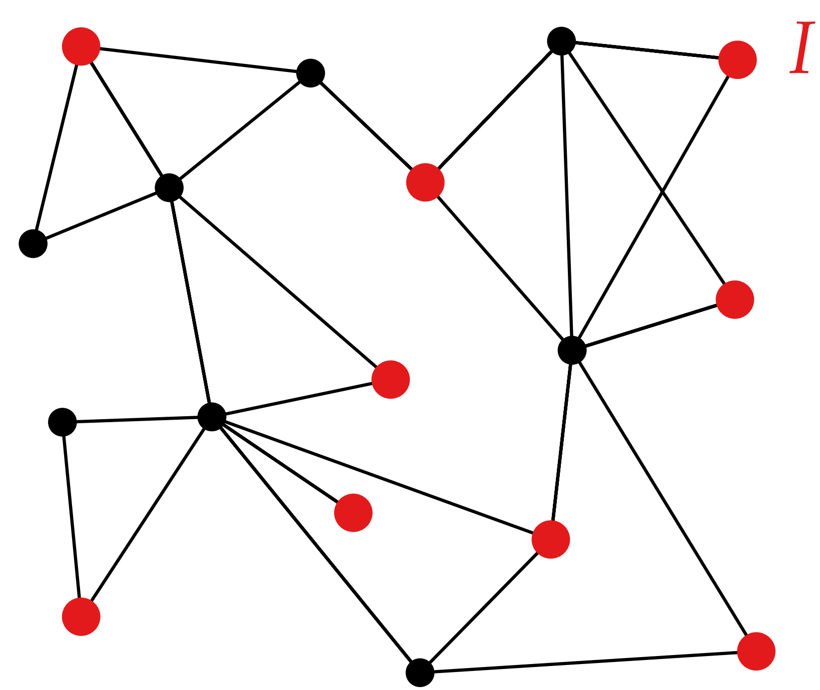

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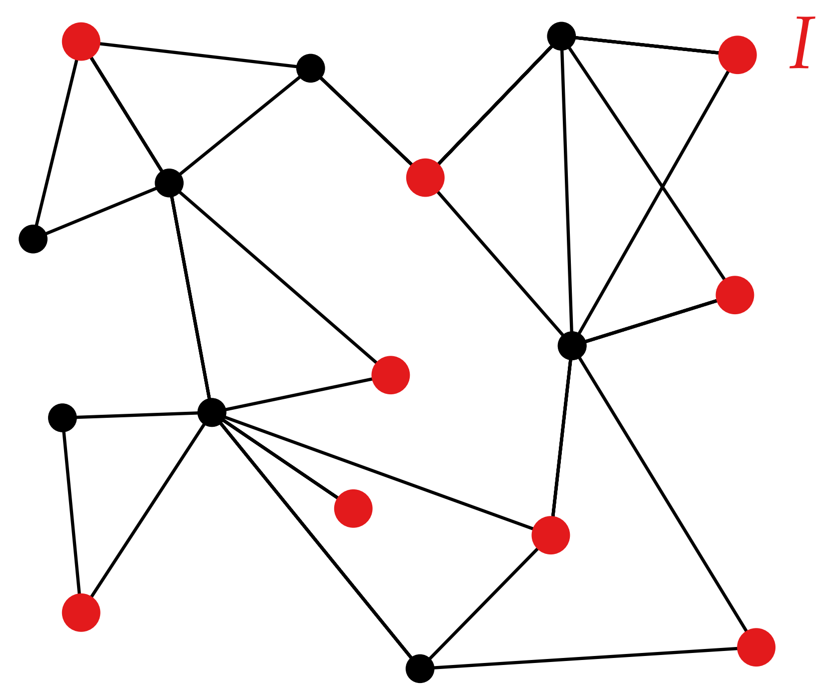

Obs. Maximal independent sets are dominating sets :-)

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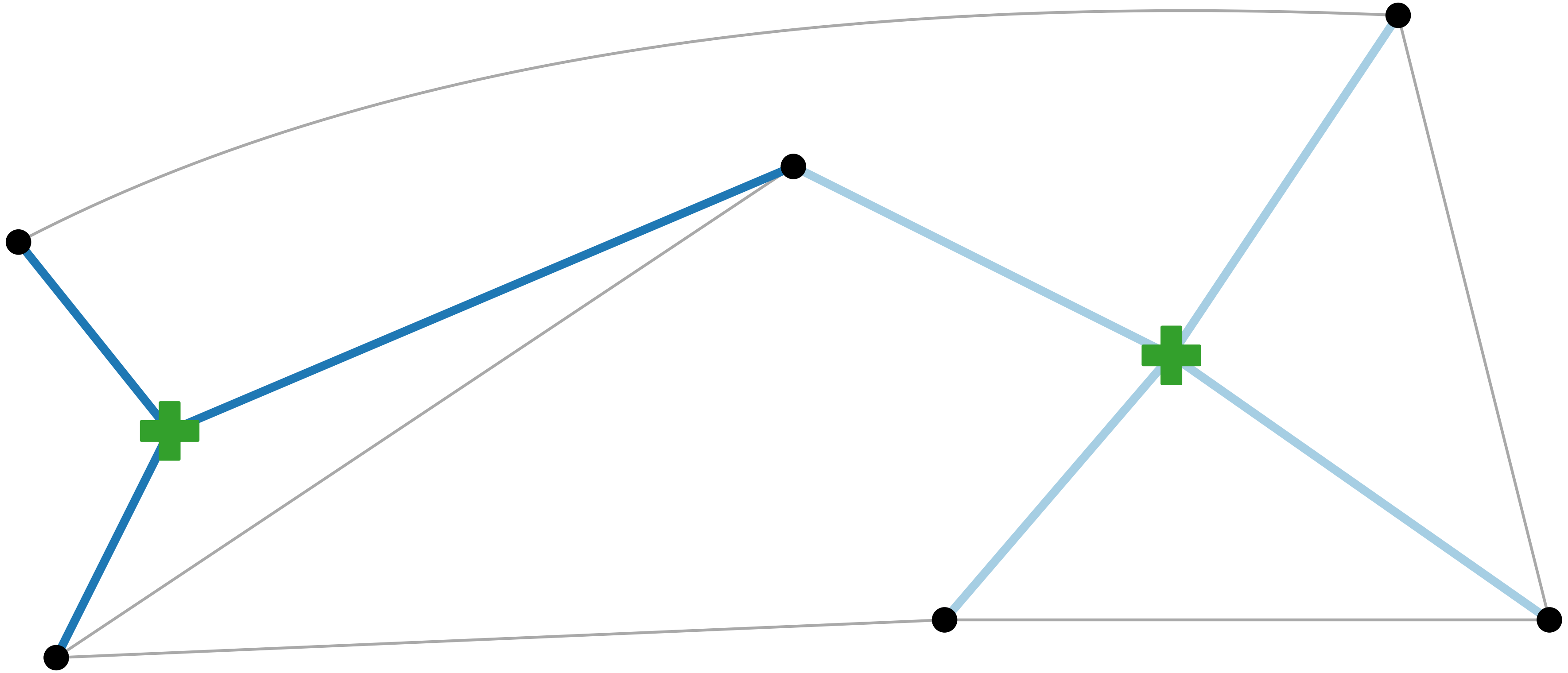

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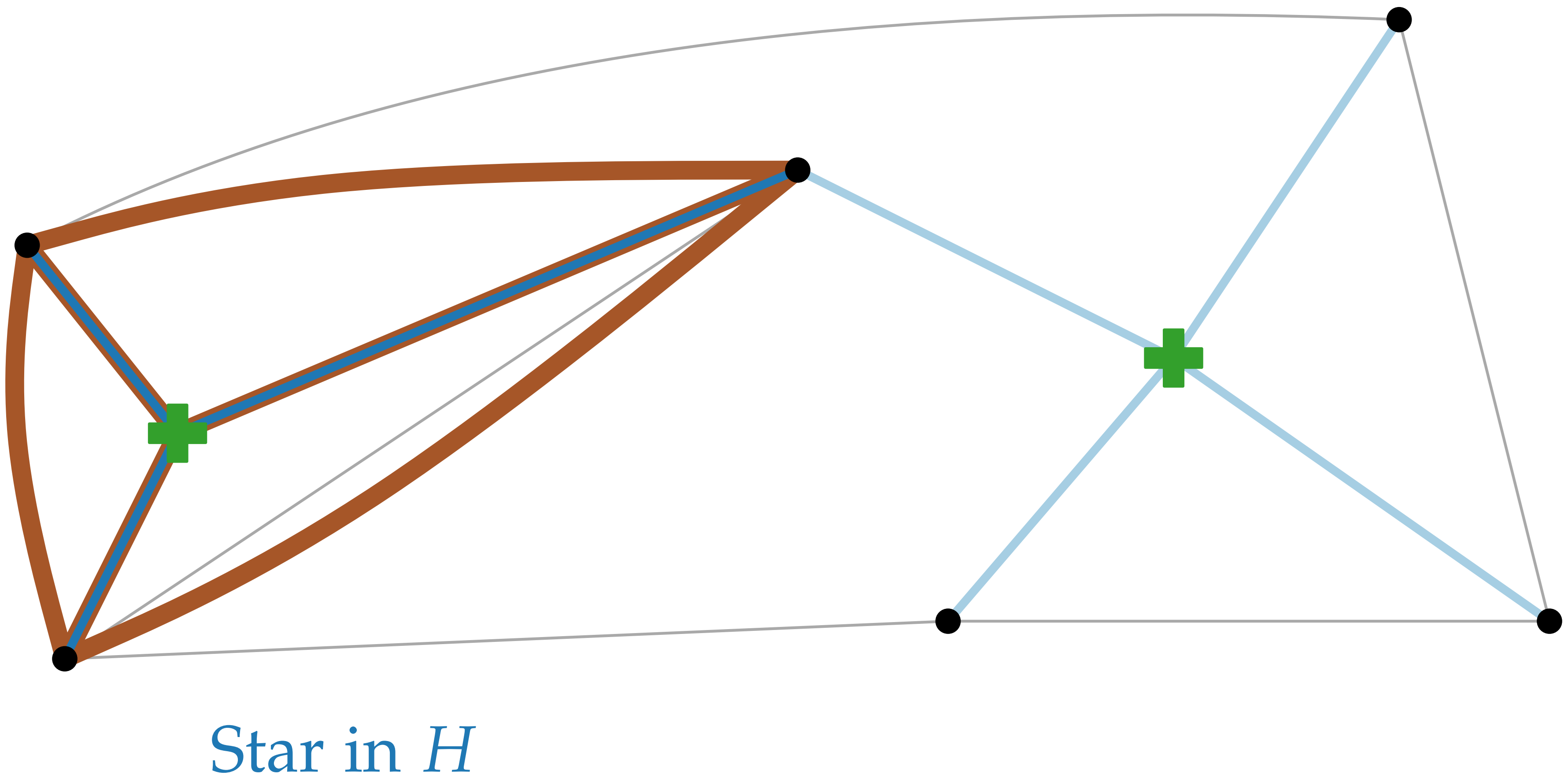

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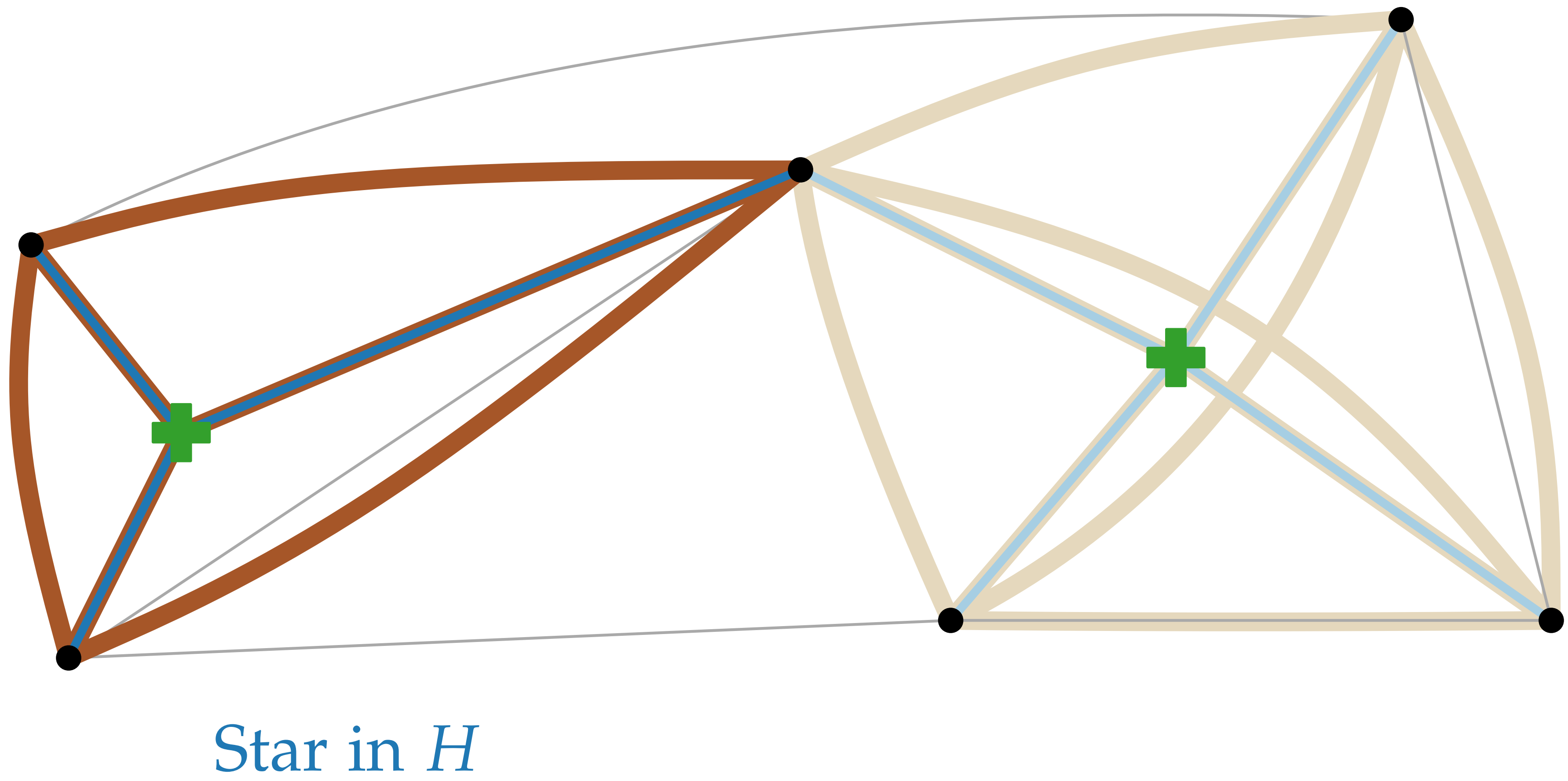

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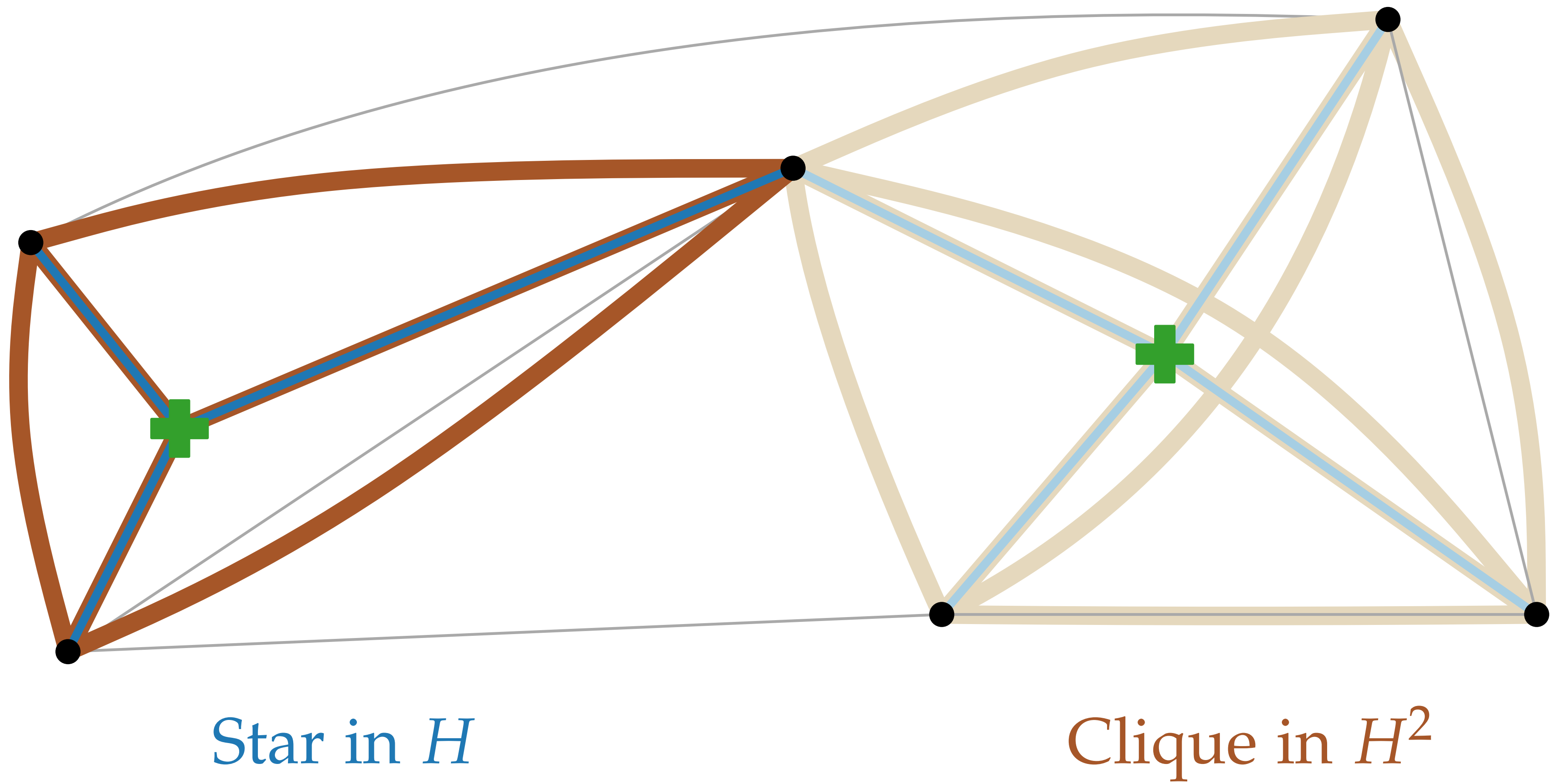

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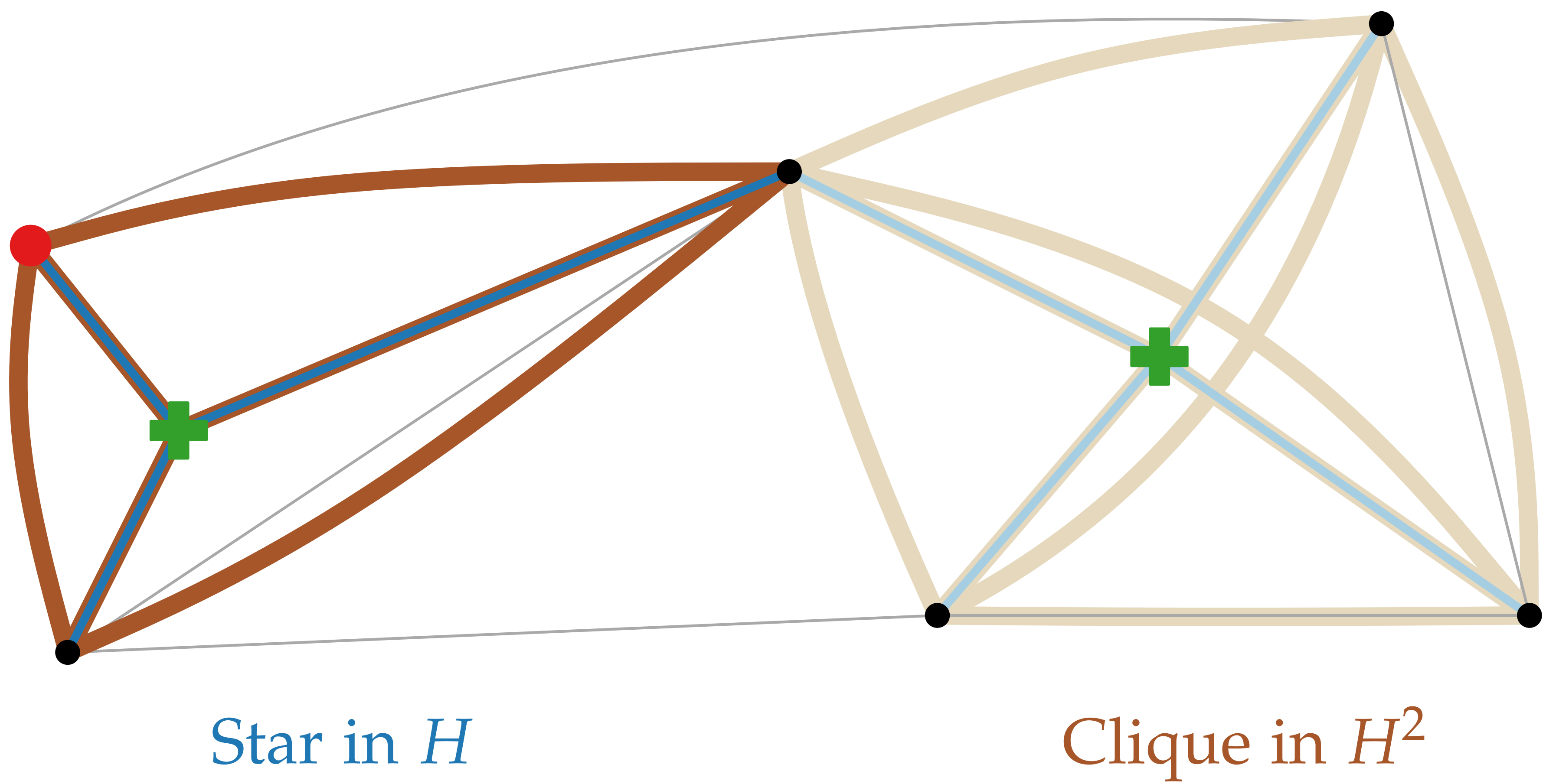

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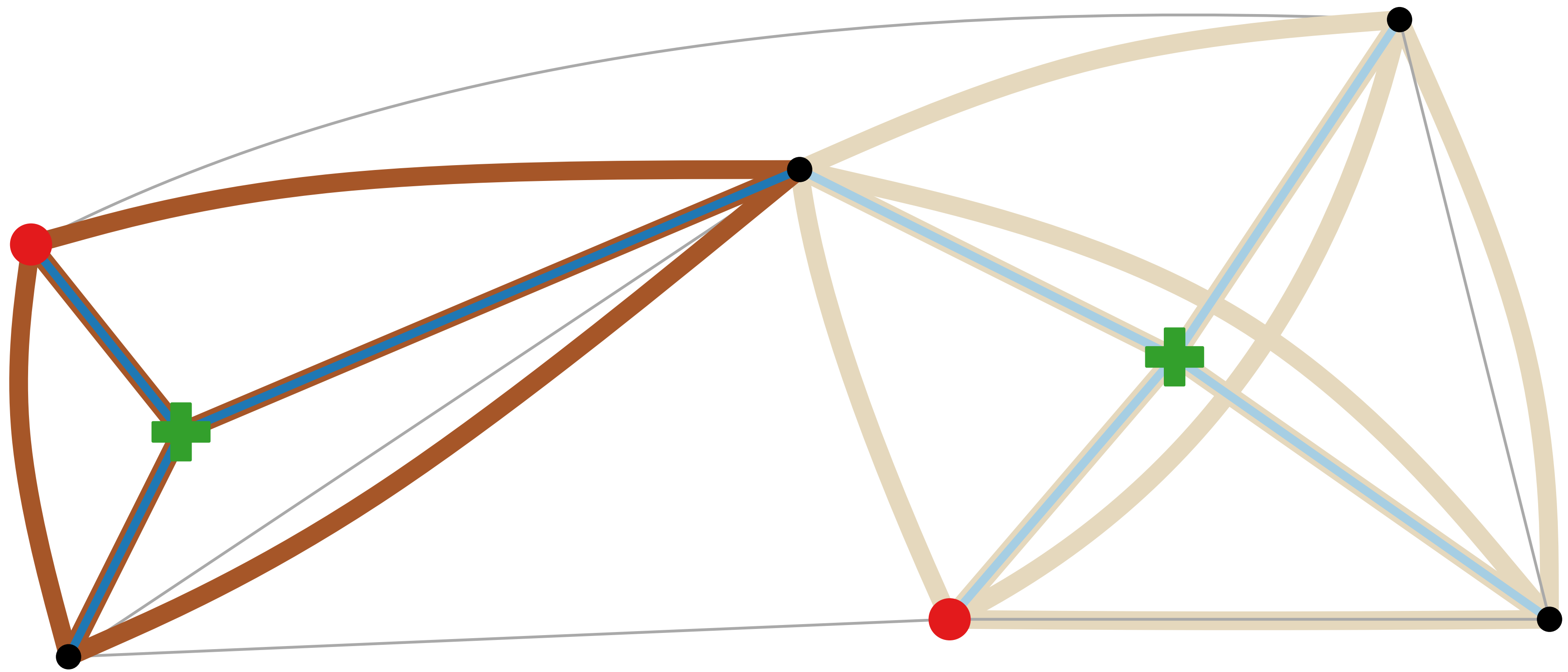

Approximation Algorithms

Lecture 6:
k-Center via Parametric Pruning

Part IV:
Factor-2 Approximation for Metric-k-Center

Factor-2 Approx. for Metric k-Center

Metric-k-Center($G = (V, E; c), k$)

Sort the edges of G by cost: $c(e_1) \leq \cdots \leq c(e_m)$

Factor-2 Approx. for Metric k-CENTER

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- Find a maximal independent set I_j in G_j^2
- **if** $|I_j| \leq k$ **then**
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Lemma. For j provided by the algorithm, it holds that $c(e_j) \leq \text{OPT}$.

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Lemma. For j provided by the algorithm, it holds that $c(e_j) \leq \mathrm{OPT}$.

Theorem. The above algorithm is a factor-2 approximation algorithm for the metric k-CENTER problem.

Can we do better . . . ?

Can we do better ...?

What about a tight example?

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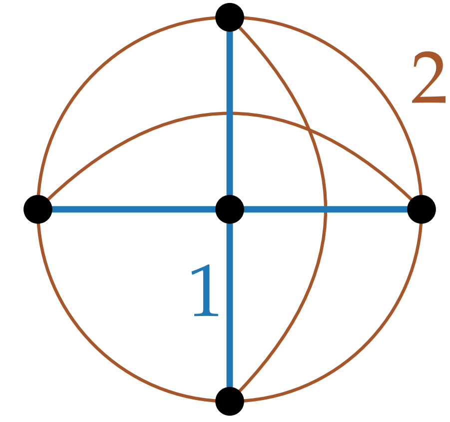

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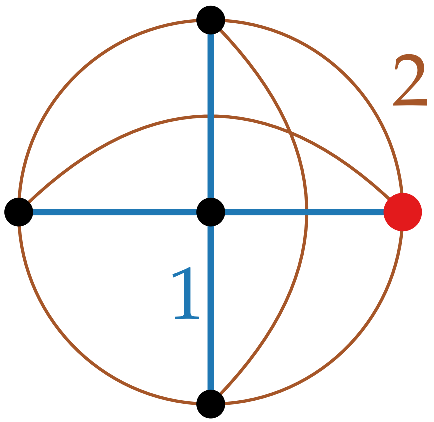

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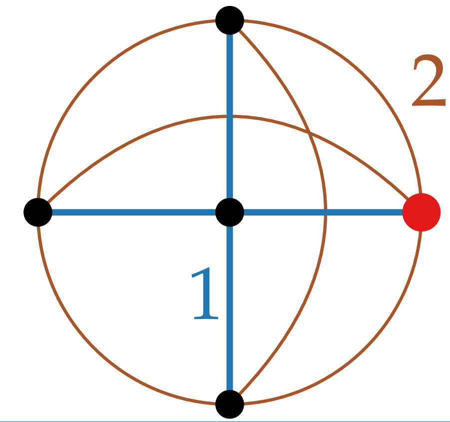

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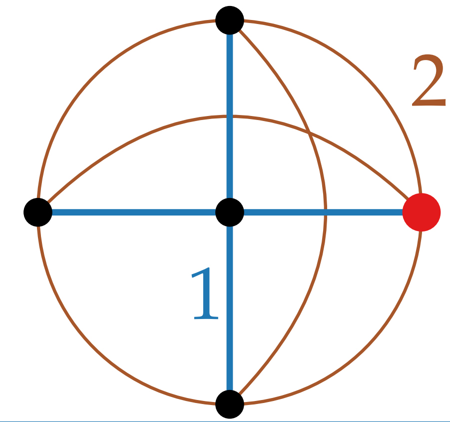

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Theorem. Assuming P $\neq$ NP, there is no factor-$(2-\varepsilon)$ approximation algorithm for the metric k-CENTER problem, for any $\varepsilon > 0$.

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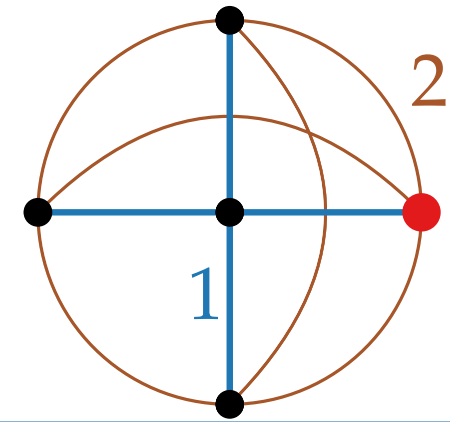

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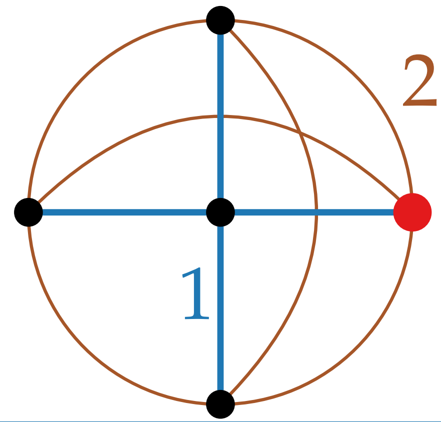

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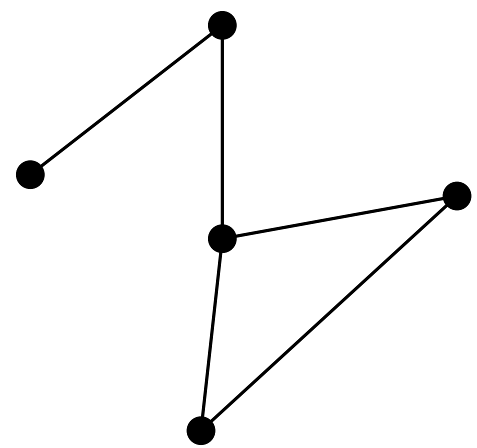

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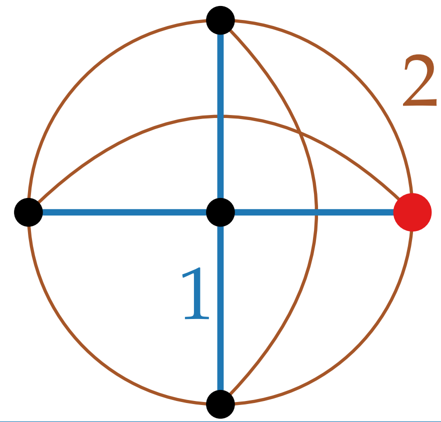

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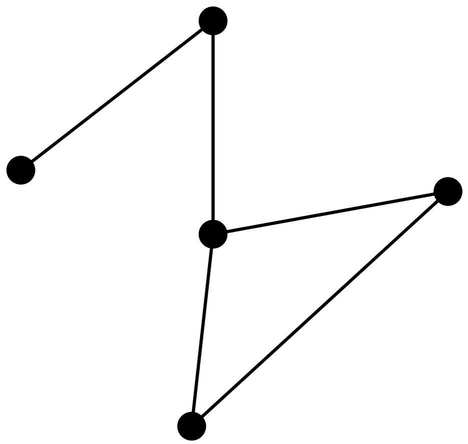

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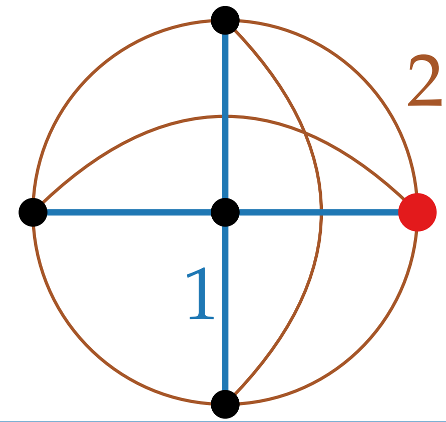

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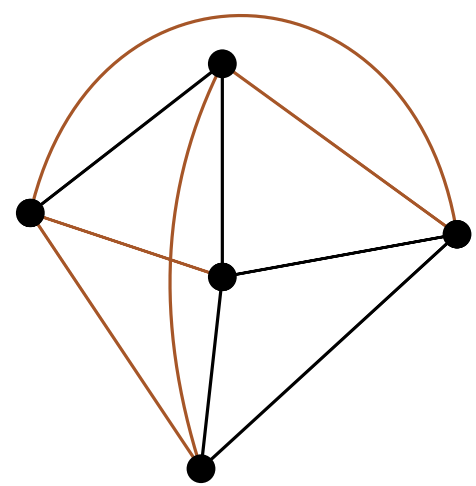

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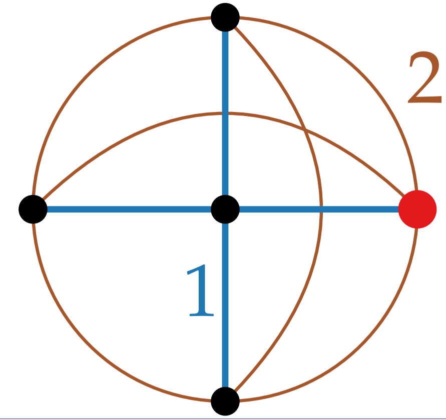

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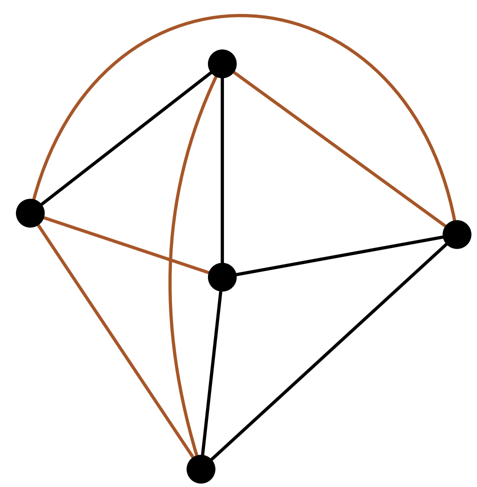

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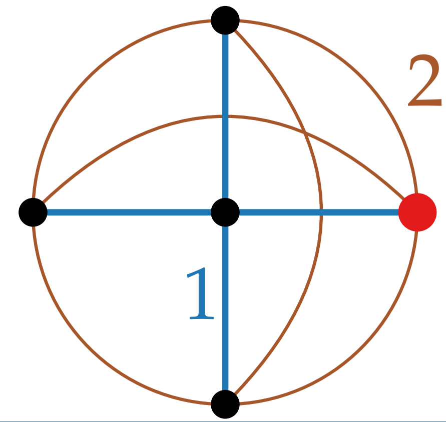

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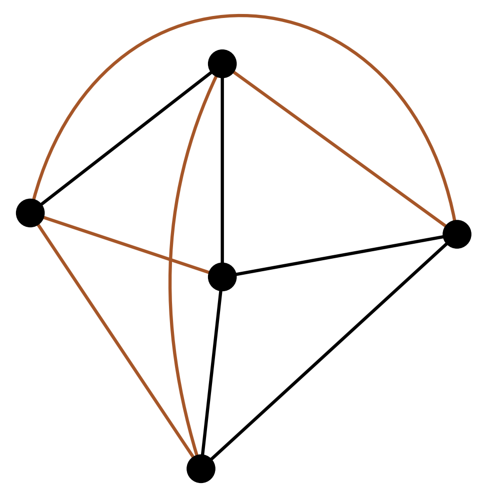

Let S be a metric k-center of G'.

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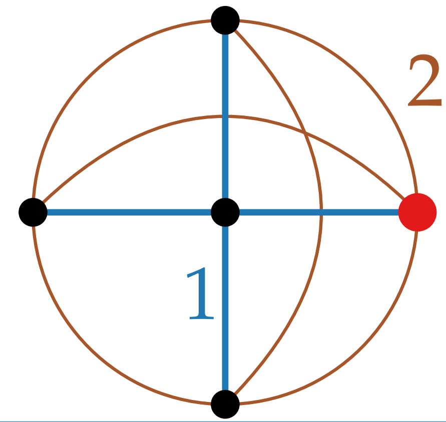

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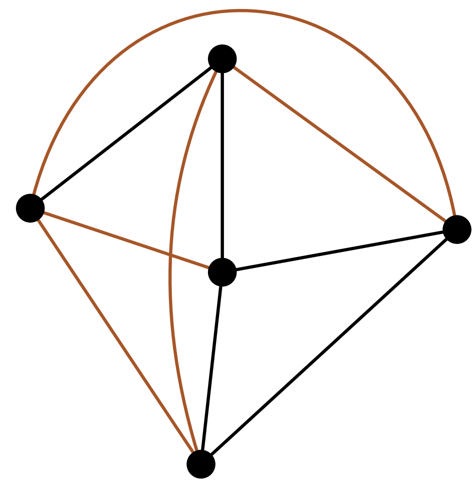

Let S be a metric k-center of G'.
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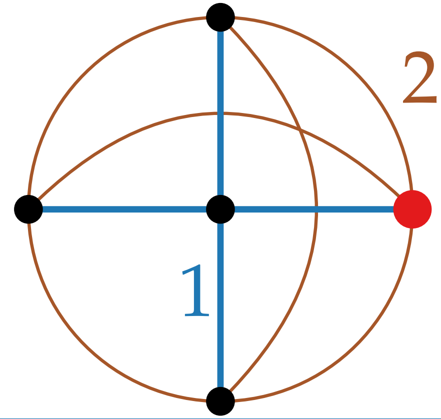
Can we do better . . . ?

What about a tight example?



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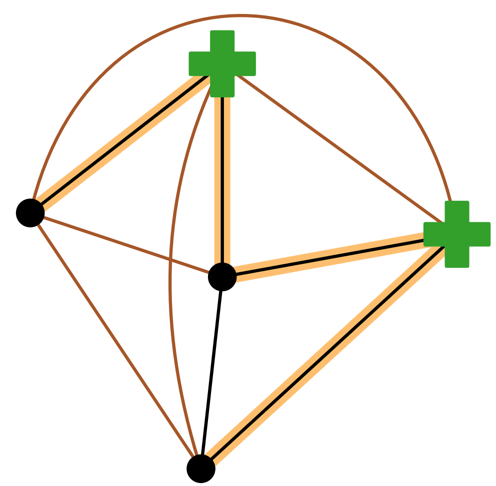

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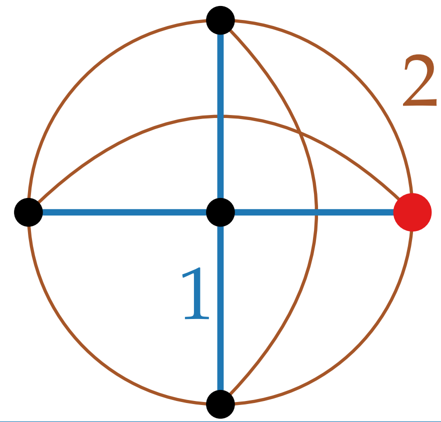

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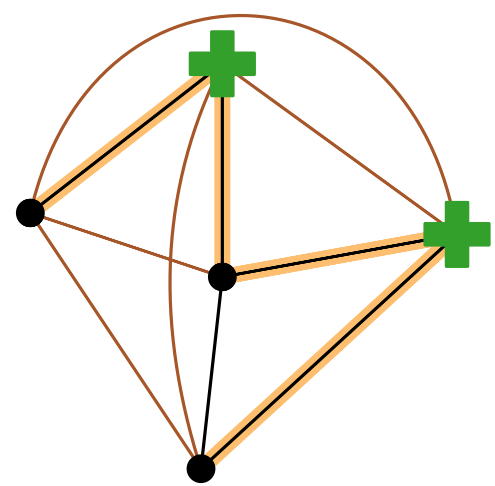

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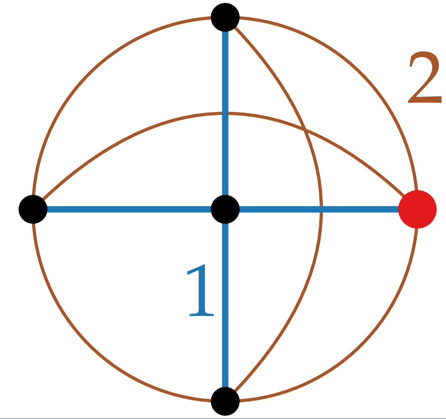

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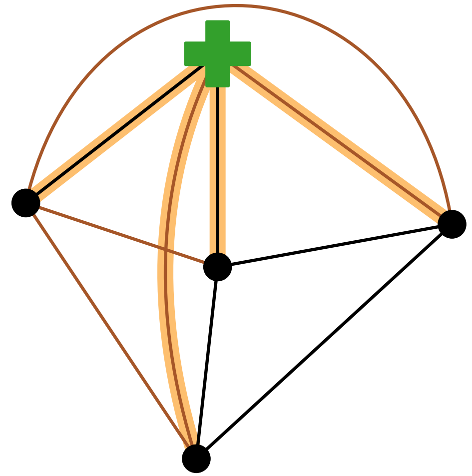

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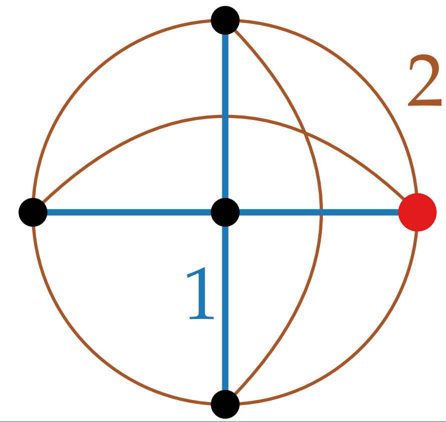

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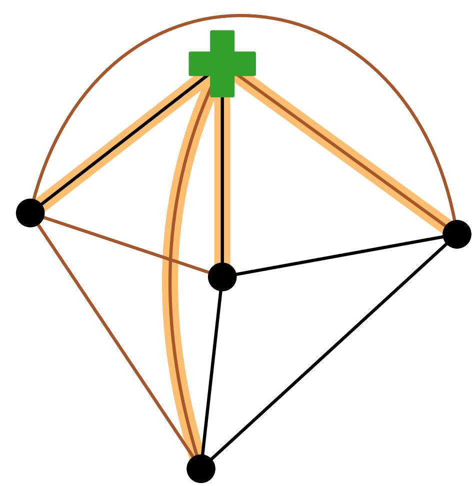

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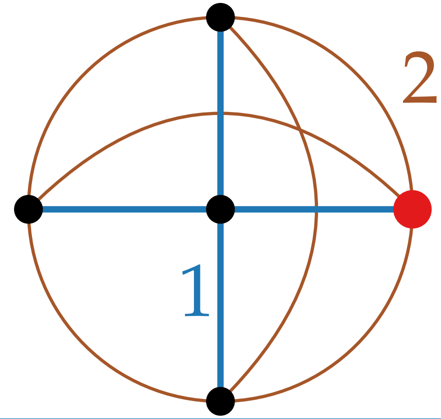

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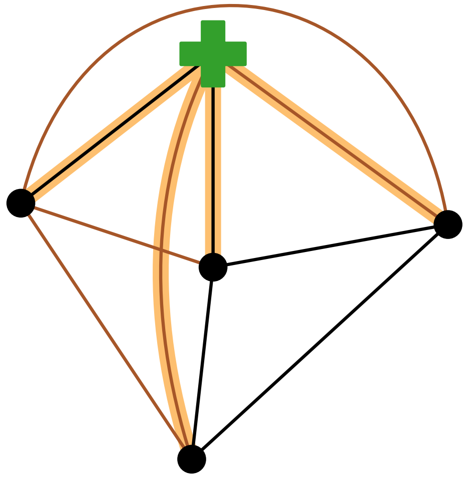
$\triangle$-inequality holds



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Approximation Algorithms

Lecture 6:
k-Center via Parametric Pruning

Part V:
Metric-Weighted-Center

Metric-k-Center

Given: A complete graph $G = (V, E)$ with metric edge costs $c \colon E \to \mathbb{Q}_{\geq 0}$ and a natural number $k \leq |V|$.

METRIC-~~*k*~~-CENTER

WEIGHTED

Given: A complete graph $G = (V, E)$ with metric edge costs $c\colon E \to \mathbb{Q}_{\geq 0}$ and a natural number $k \leq |V|$.

For $S \subseteq V$, $c(v, S)$ is the cost of the cheapest edge from v to a vertex in S.

Find: A k-element vertex set S such that $\text{cost}(S) := \max_{v \in V} c(v, S)$ is minimized.

Metric-~~k~~-Center

Weighted

Given: A complete graph $G = (V, E)$ with metric edge costs $c\colon E \to \mathbb{Q}_{\geq 0}$ and ~~a natural number $k \leq |V|$.~~ , vertex weights $w\colon V \to \mathbb{Q}_{\geq 0}$ and a budget $W \in \mathbb{Q}_+$

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Find: A k-element vertex set S such that $\text{cost}(S) := \max_{v \in V} c(v, S)$ is minimized.

Metric-~~k~~-Center

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For $S \subseteq V$, $c(v, S)$ is the cost of the cheapest edge from v to a vertex in S.

Find: A ~~k-element vertex set S~~ vertex set S of weight at most W such that $\text{cost}(S) := \max_{v \in V} c(v, S)$ is minimized.

Algorithm for the Weighted Version

Algorithm Metric- -Center

Sort the edges of G by cost : $c(e_1) \leq \ldots \leq c(e_m)$

for $j = 1, \ldots, m$ **do**

Construct G_j^2

Find a maximal independent set I_j in G_j^2

if $|I_j| \leq k$ **then**

return I_j

Algorithm for the Weighted Version

Algorithm Metric-Weighted-Center

Sort the edges of G by cost : $c(e_1) \leq \ldots \leq c(e_m)$

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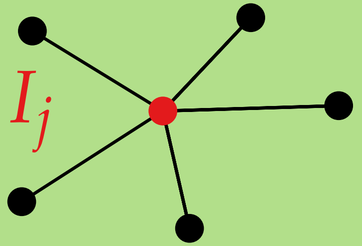
$u \in I_j$



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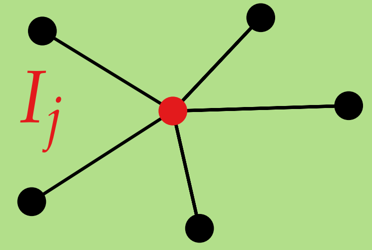

$s_j(u) :=$ lightest node in $N_{G_j}(u) \cup \{u\}$

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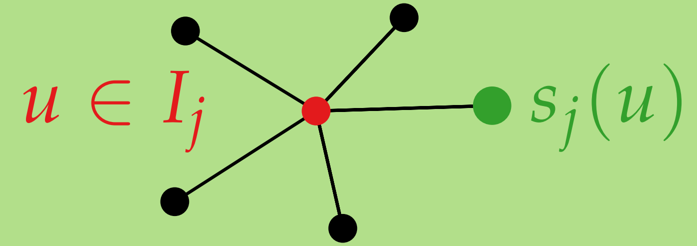

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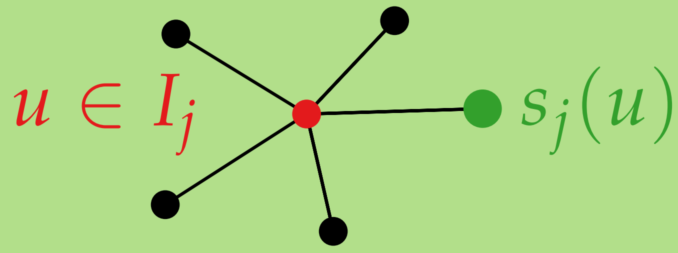

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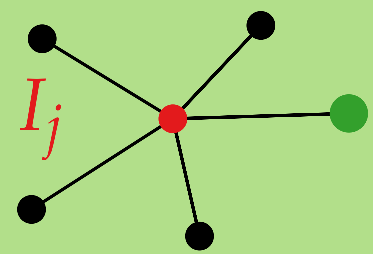
- **if** ~~$|I_j| \leq k$~~ $w(S_j) \leq W$ **then**
  - **return** I_j

$u \in I_j$



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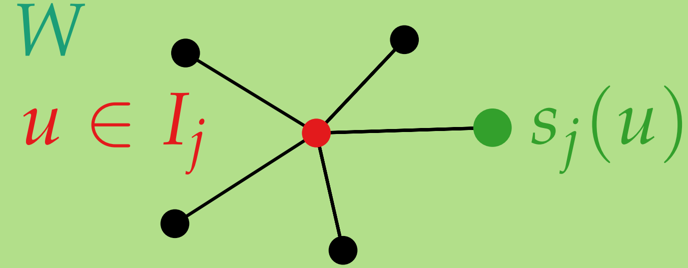

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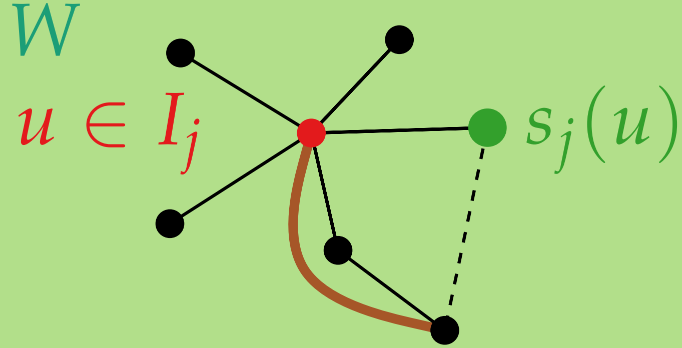

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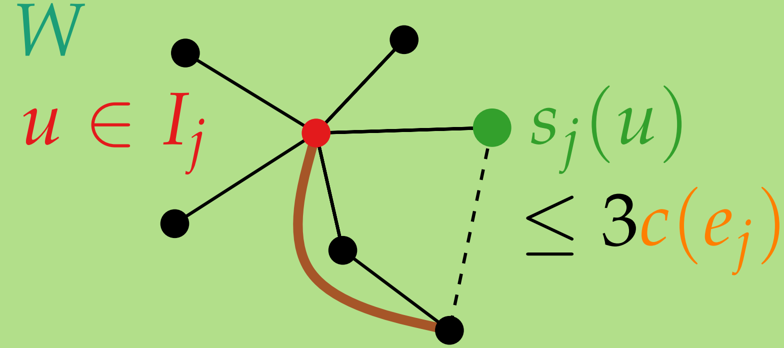
  if ~~$|I_j| \leq k$~~ **then** $w(S_j) \leq W$

    return ~~I_j~~ S_j



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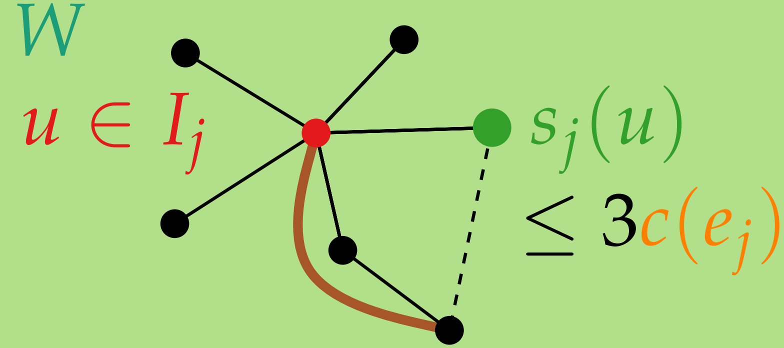

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Theorem. The above is a factor-3 approximation algorithm for Metric-Weighted-Center.

Tight Example... ?

Here, we need to have a budget W,
and edge costs satisfying the triangle inequality.

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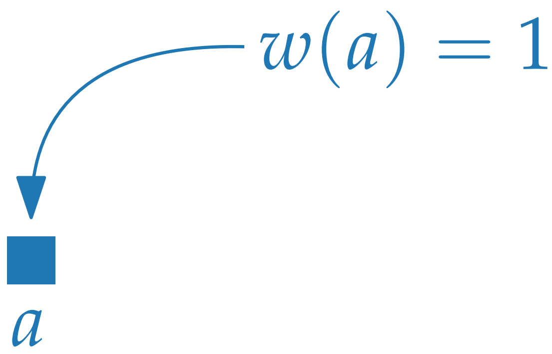

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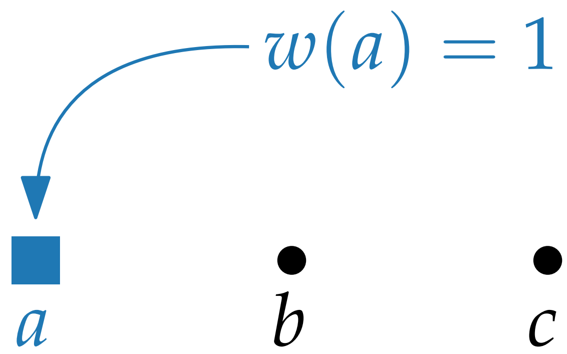

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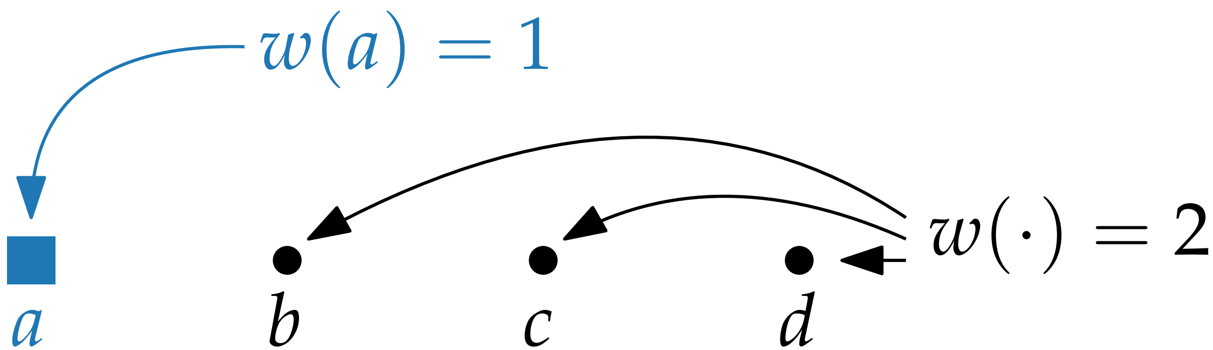

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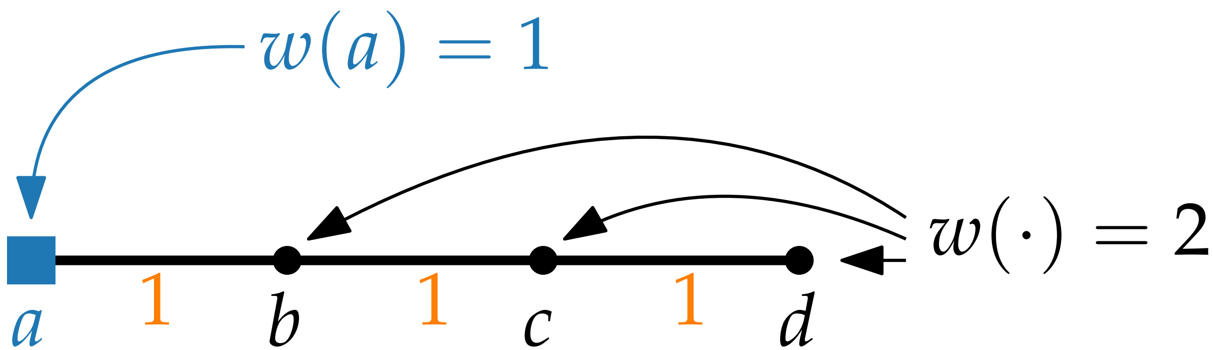

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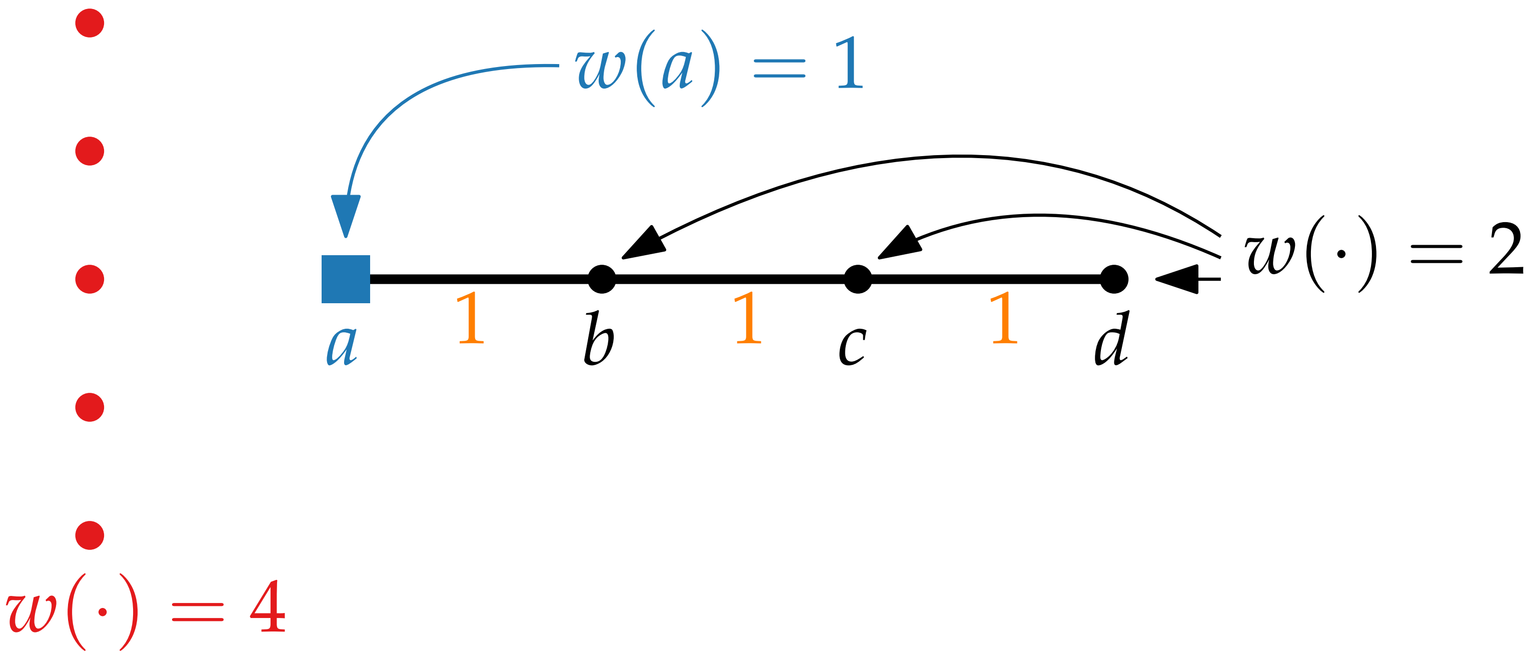

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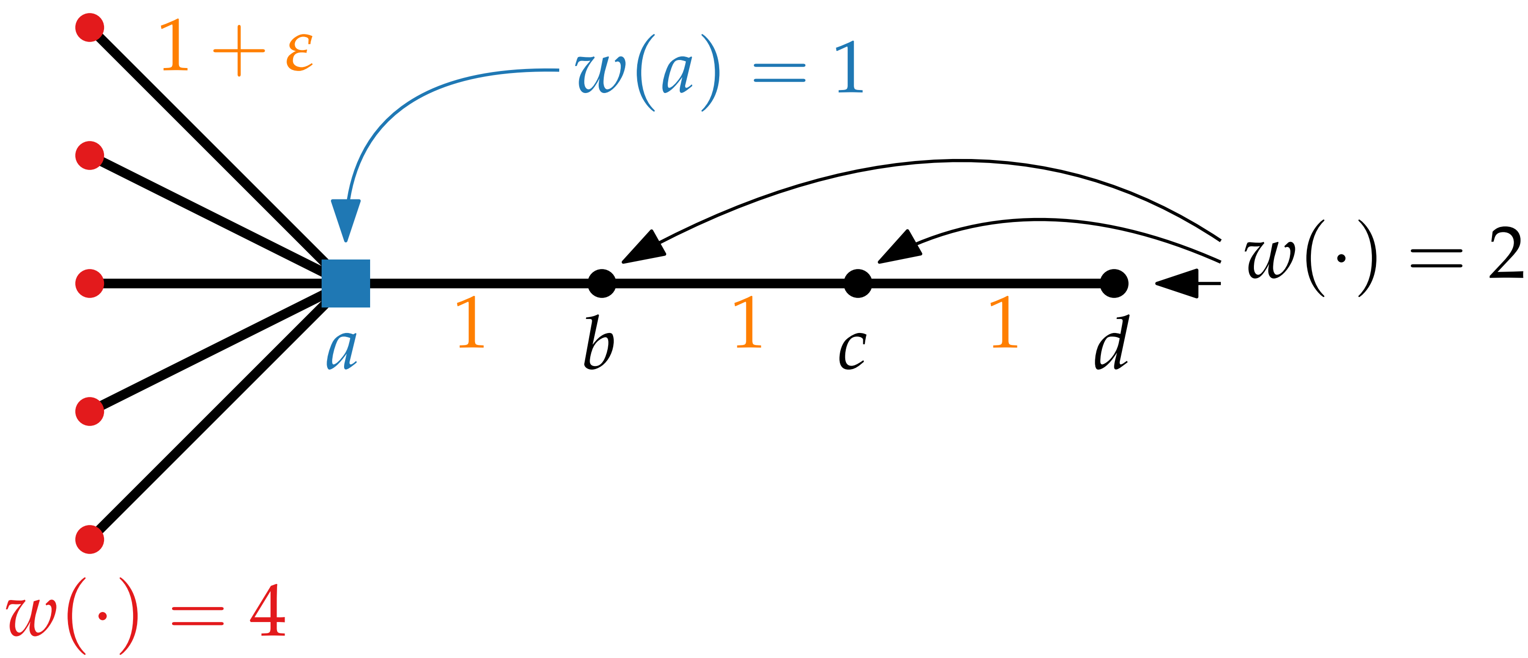

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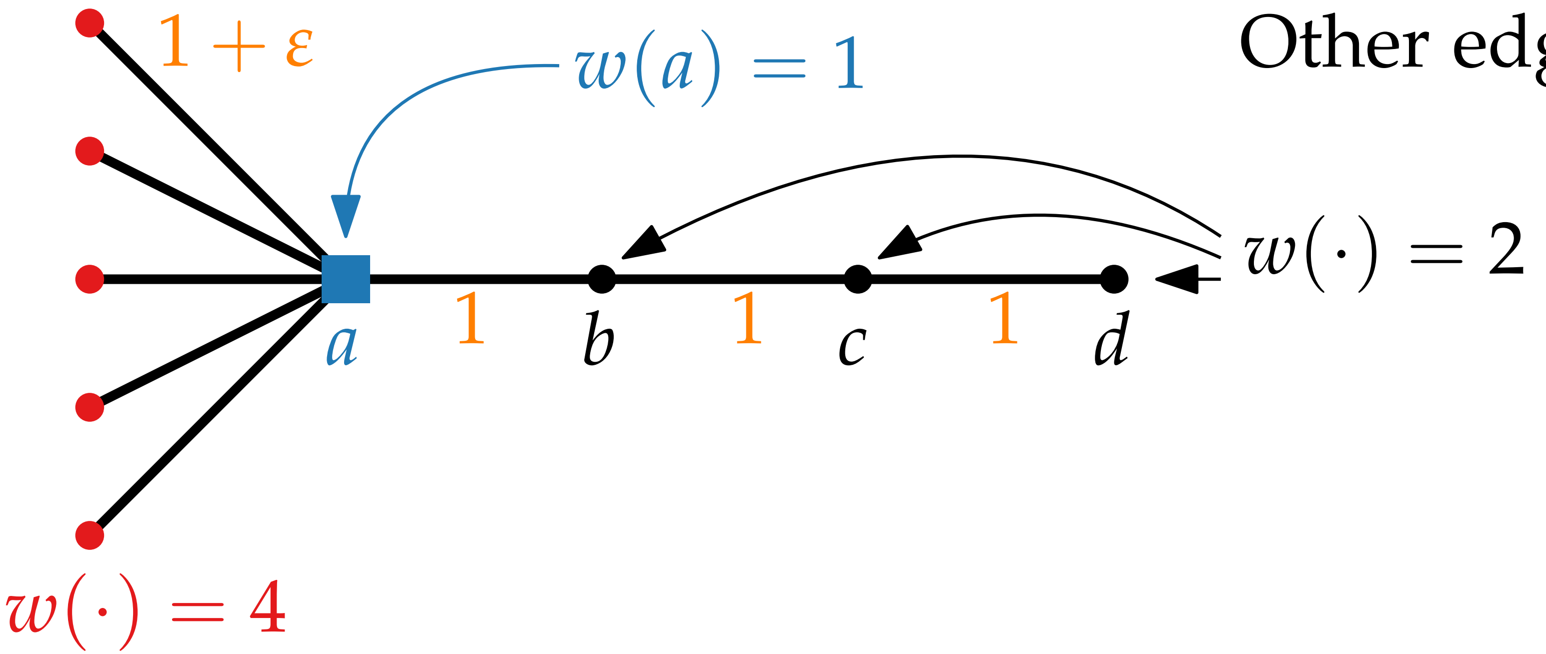
Other edge costs?



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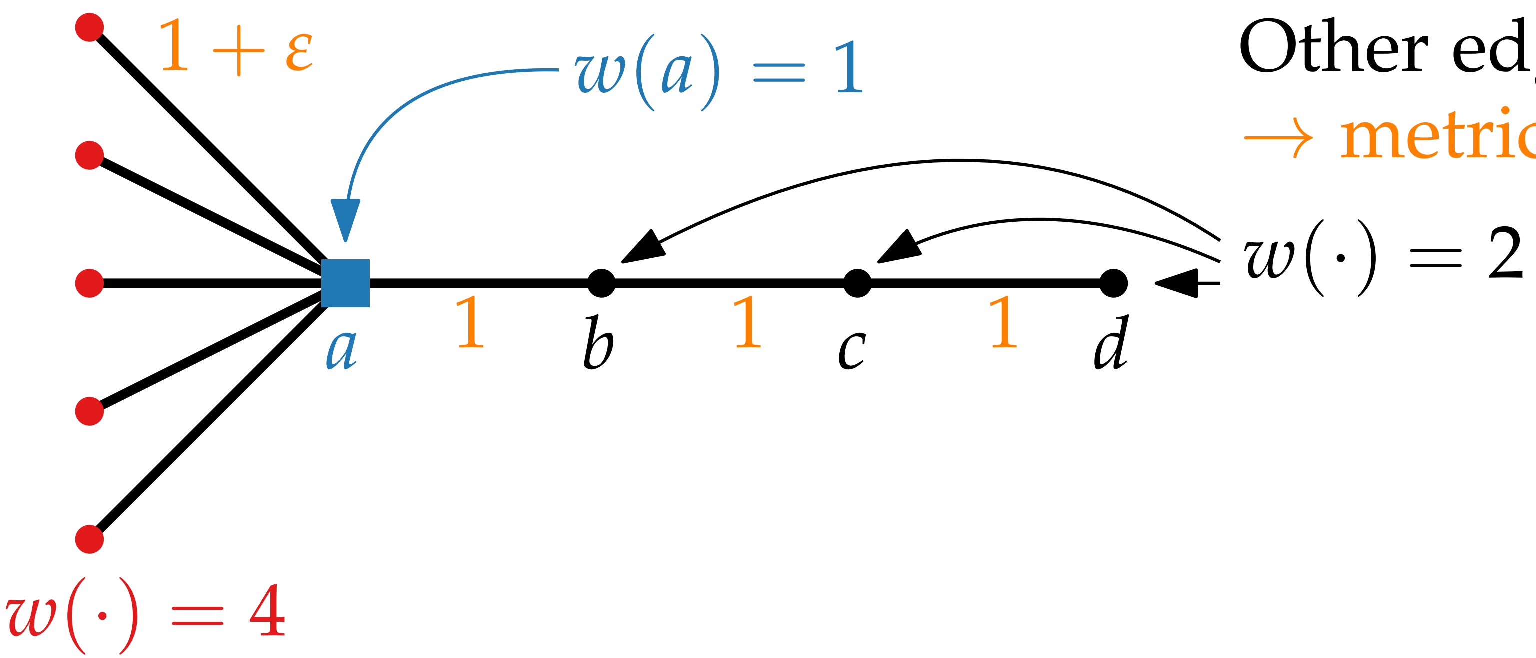
$\rightarrow$ metric completion!



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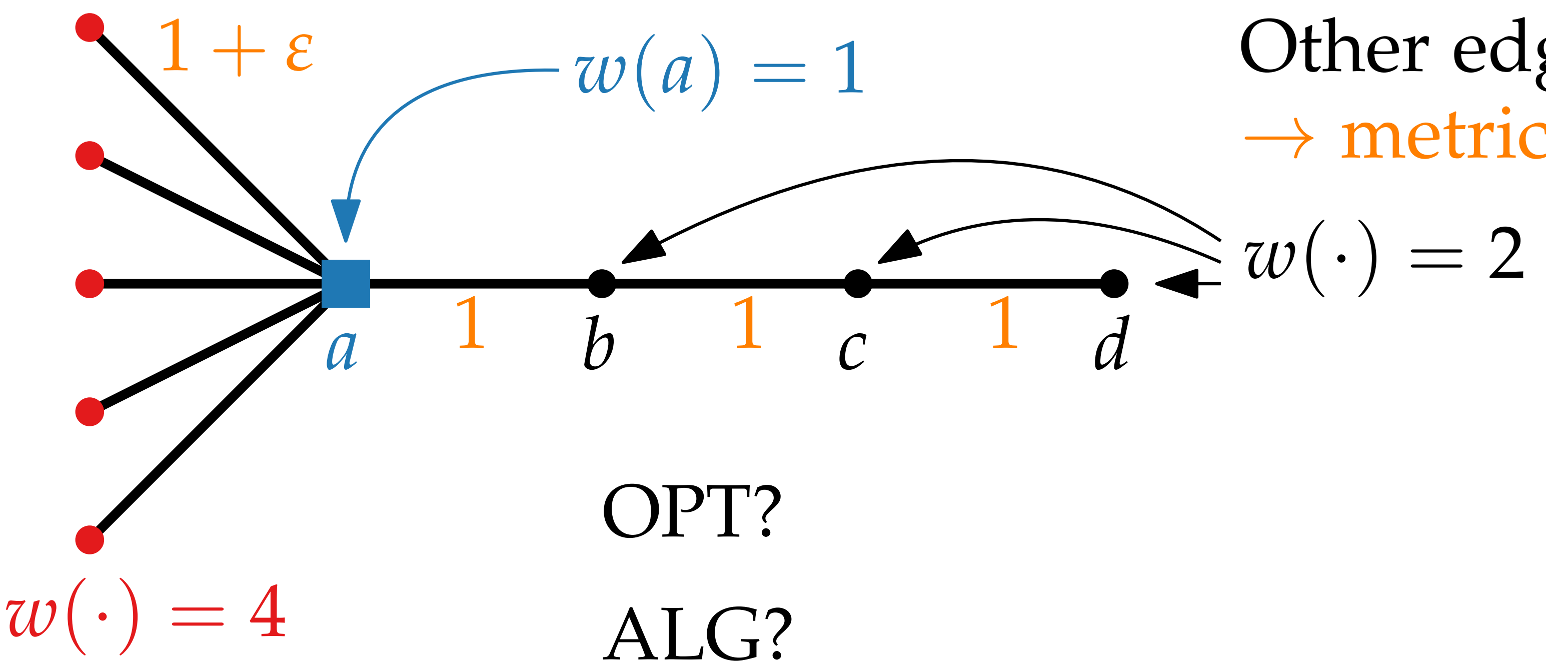

OPT?

ALG?

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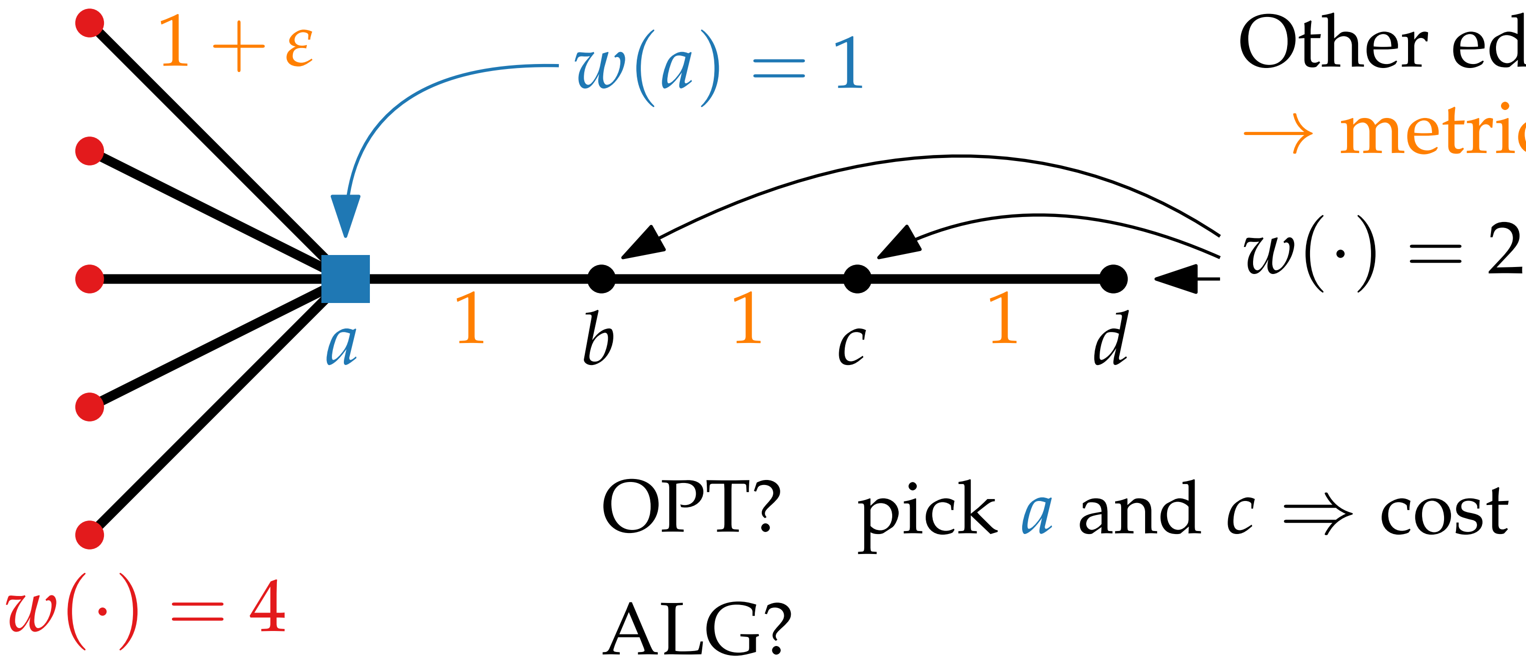

OPT? pick a and $c \Rightarrow$ cost $1 + \varepsilon$.

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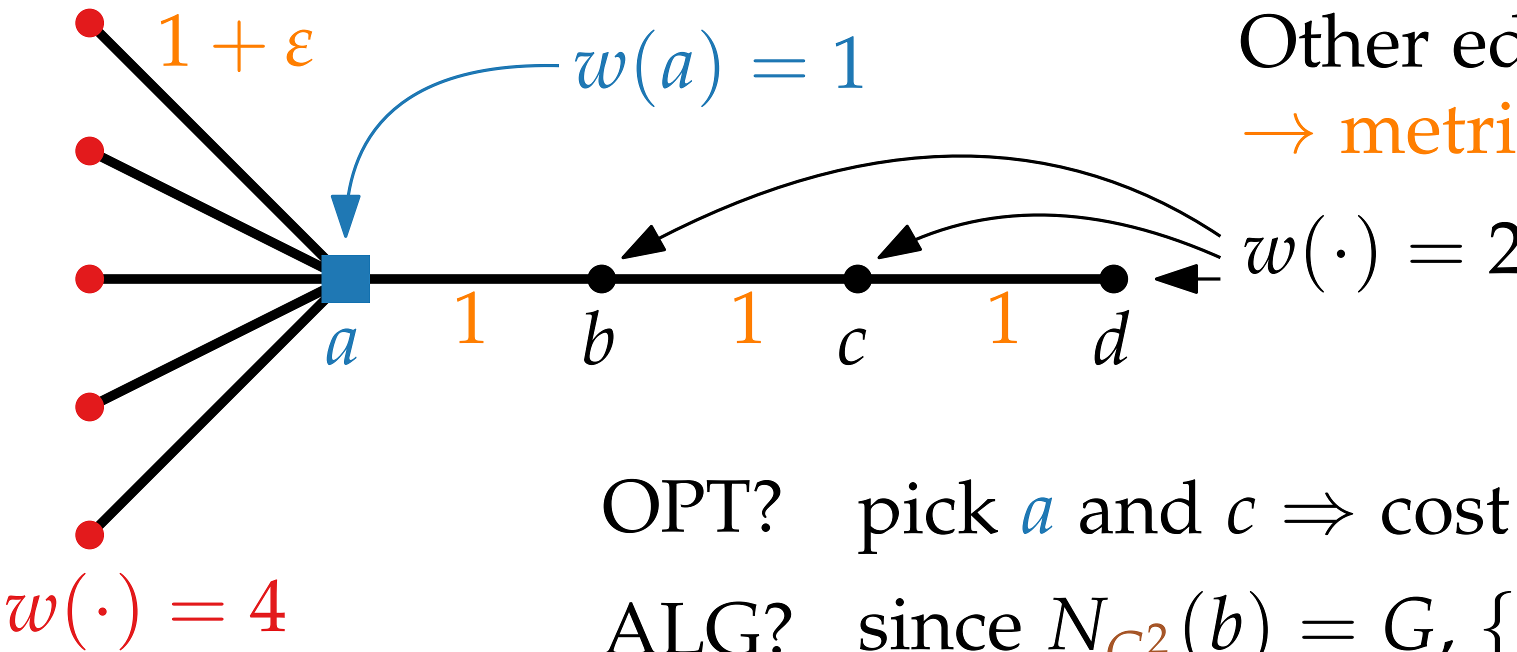

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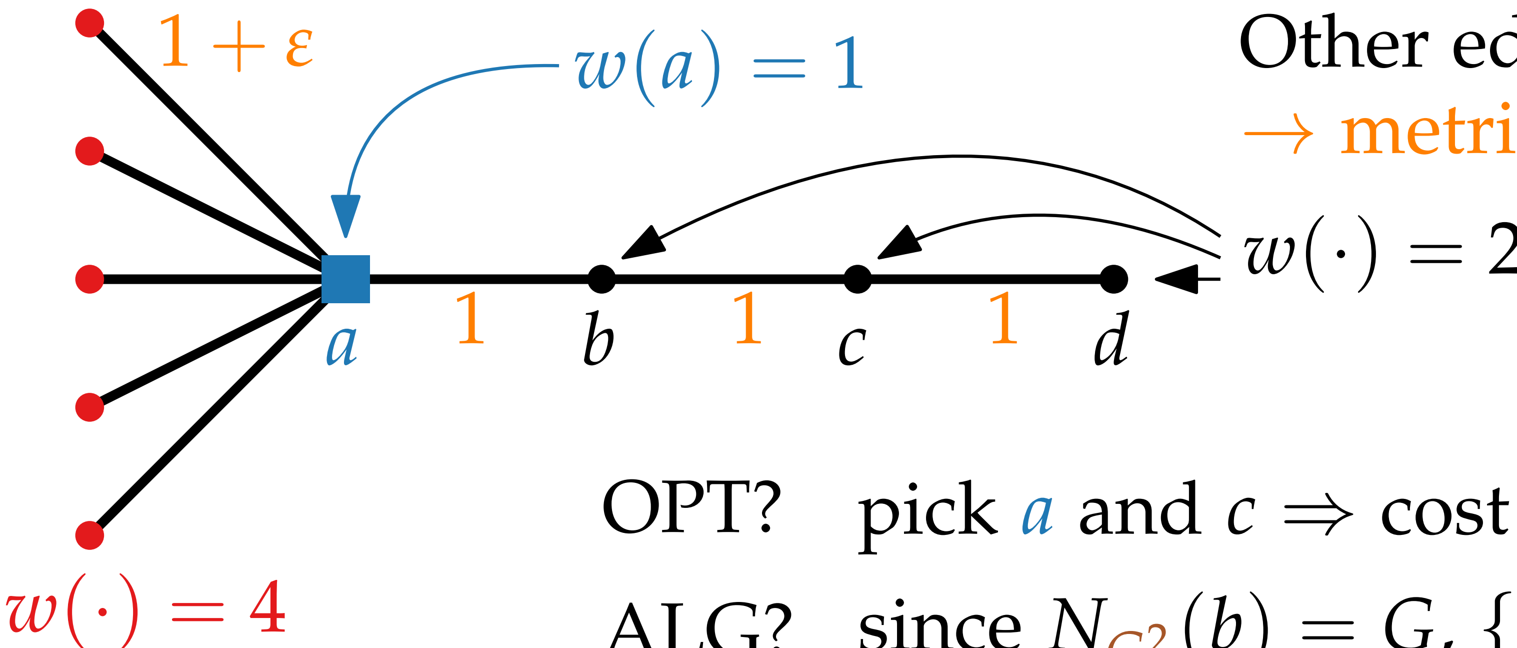

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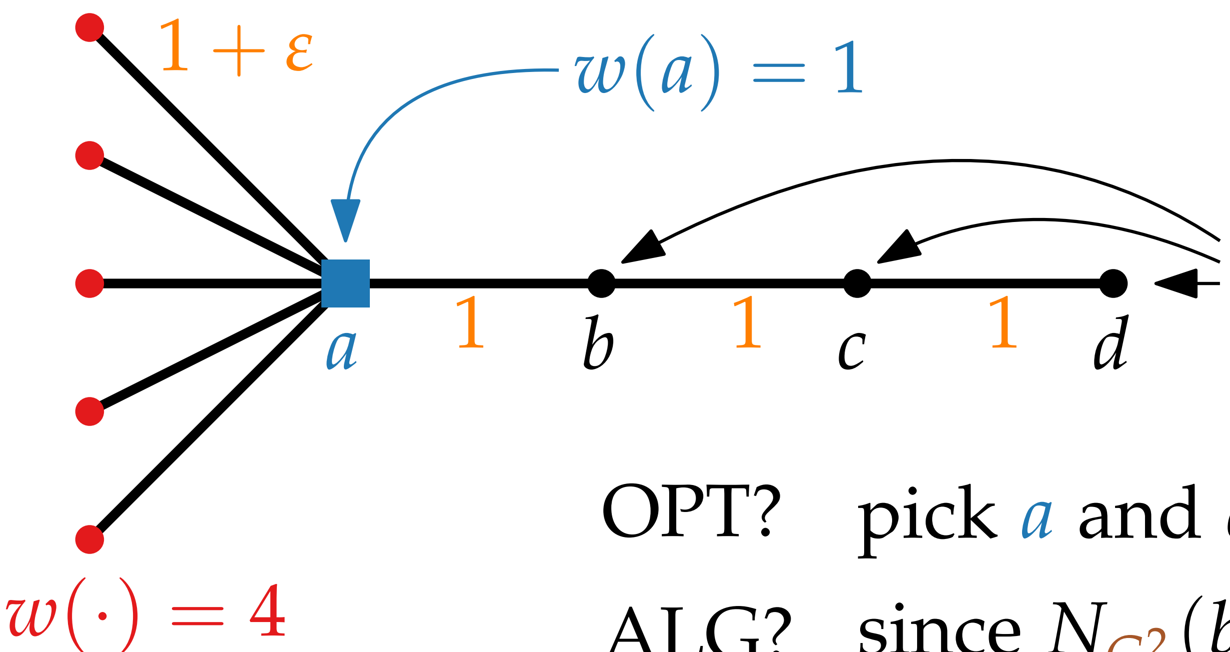

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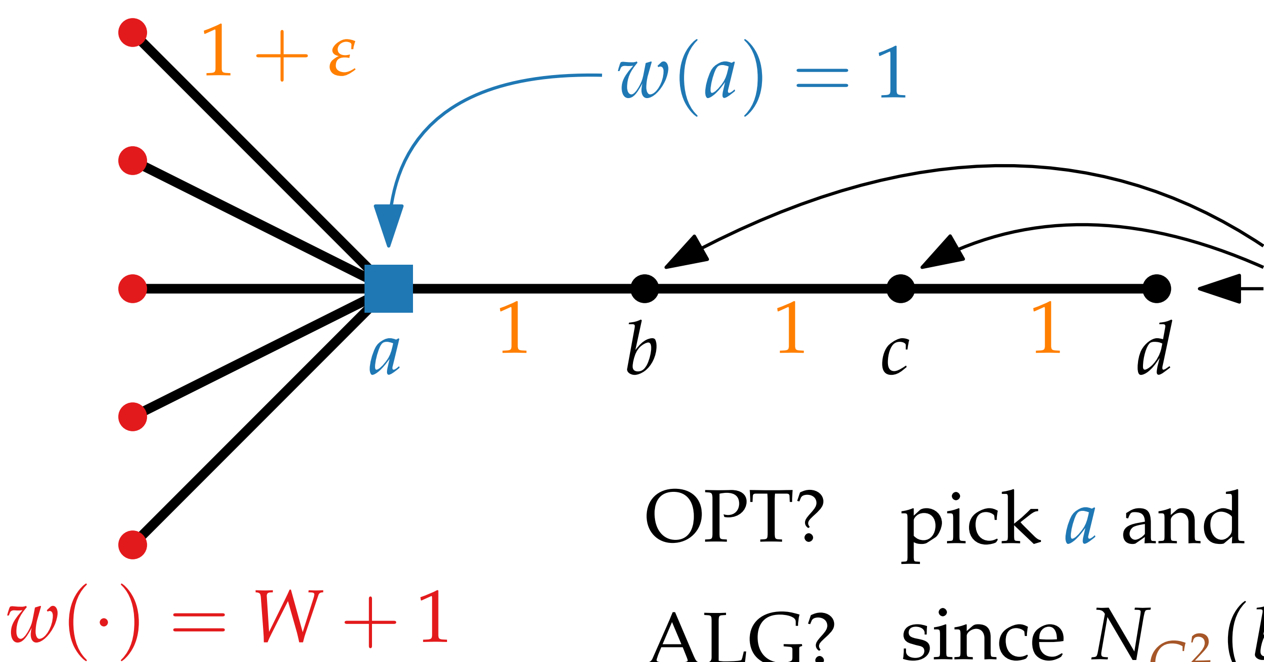

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