Approximation Algorithms

Lecture 5: LP-based Approximation Algorithms for SetCover

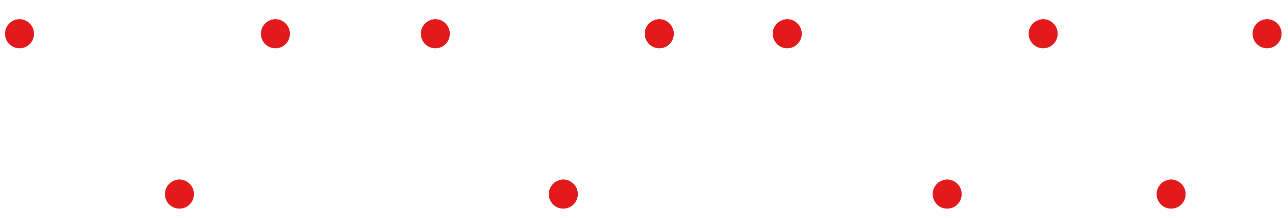
Part I: SetCover as an ILP

Alexander Wolff

Winter 2024/25

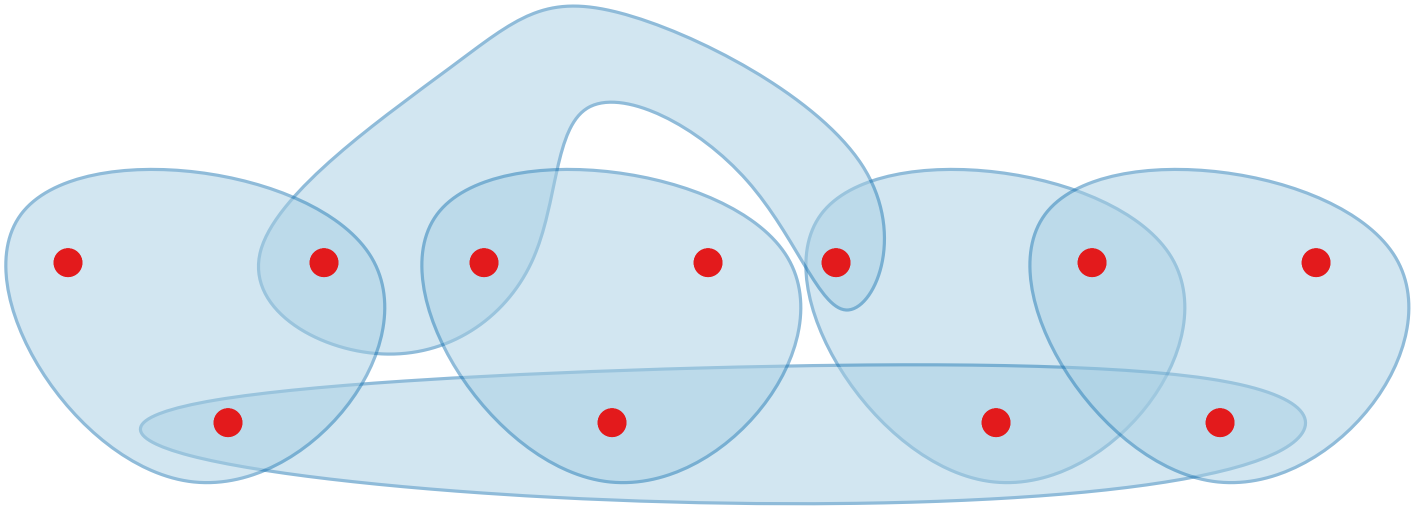
SetCover as an ILP

Ground set U



SetCover as an ILP

Ground set U
Family $\mathcal{S} \subseteq 2^U$ with $\bigcup \mathcal{S} = U$

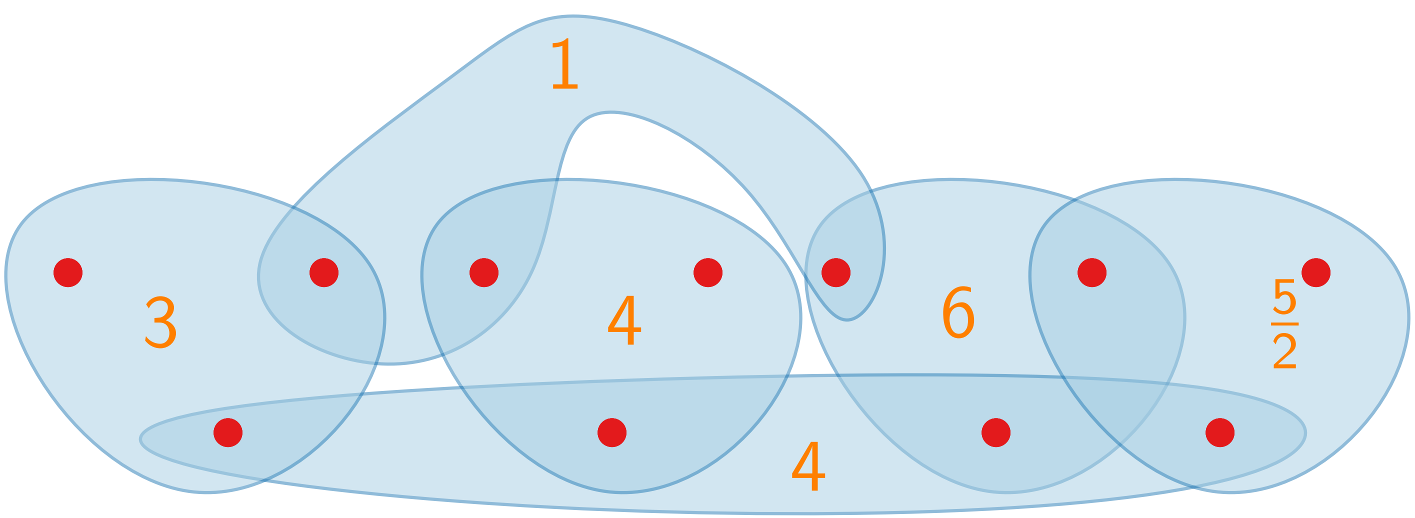


SetCover as an ILP

Ground set U

Family $\mathcal{S} \subseteq 2^U$ with $\bigcup \mathcal{S} = U$

Costs $c \colon \mathcal{S} \to \mathbb{Q}^+$



SetCover as an ILP

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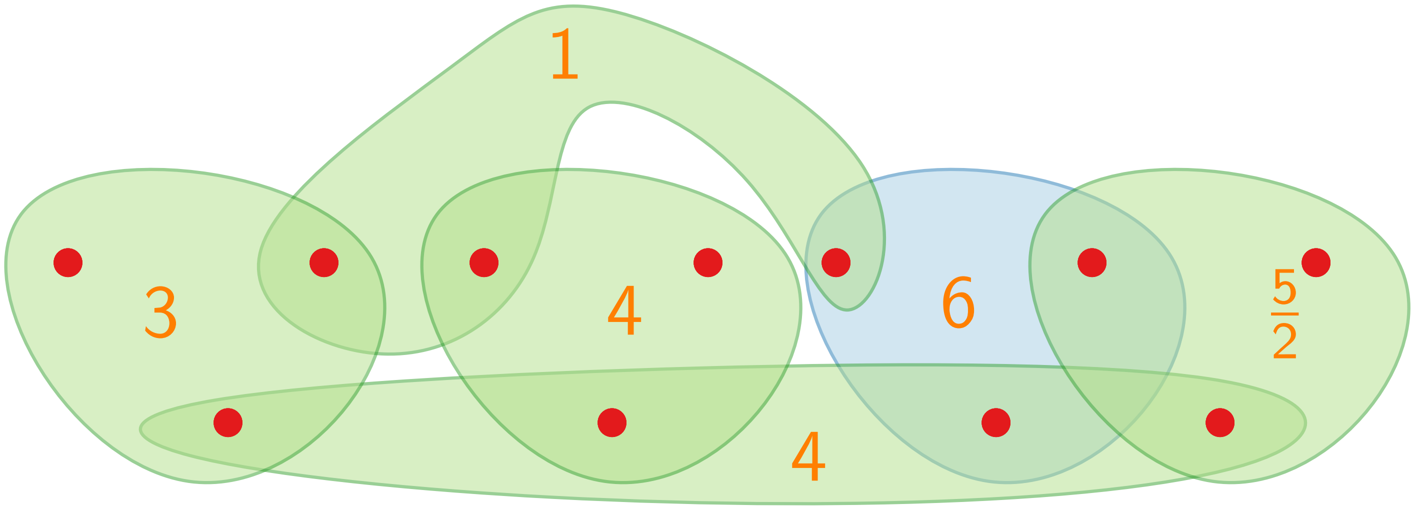


Find cover $\mathcal{S}' \subseteq \mathcal{S}$ of U with minimum cost.

SetCover as an ILP

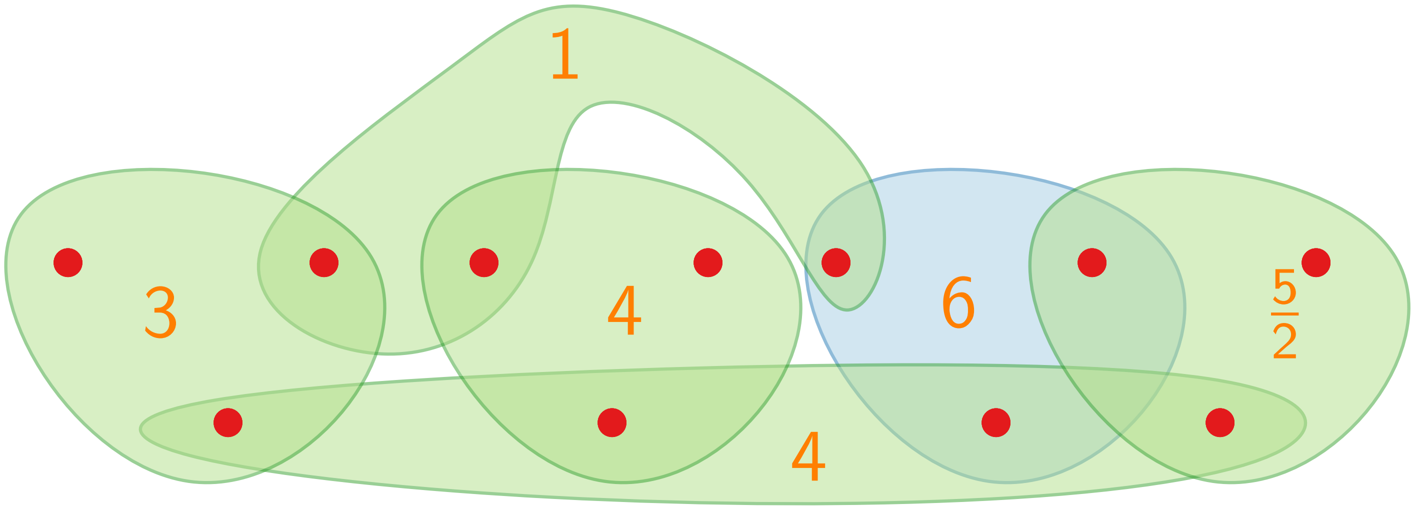
minimize

subject to

Ground set U

Family $\mathcal{S} \subseteq 2^U$ with $\bigcup \mathcal{S} = U$

Costs $c\colon \mathcal{S} \to \mathbb{Q}^+$



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x_S $\forall S \in \mathcal{S}$

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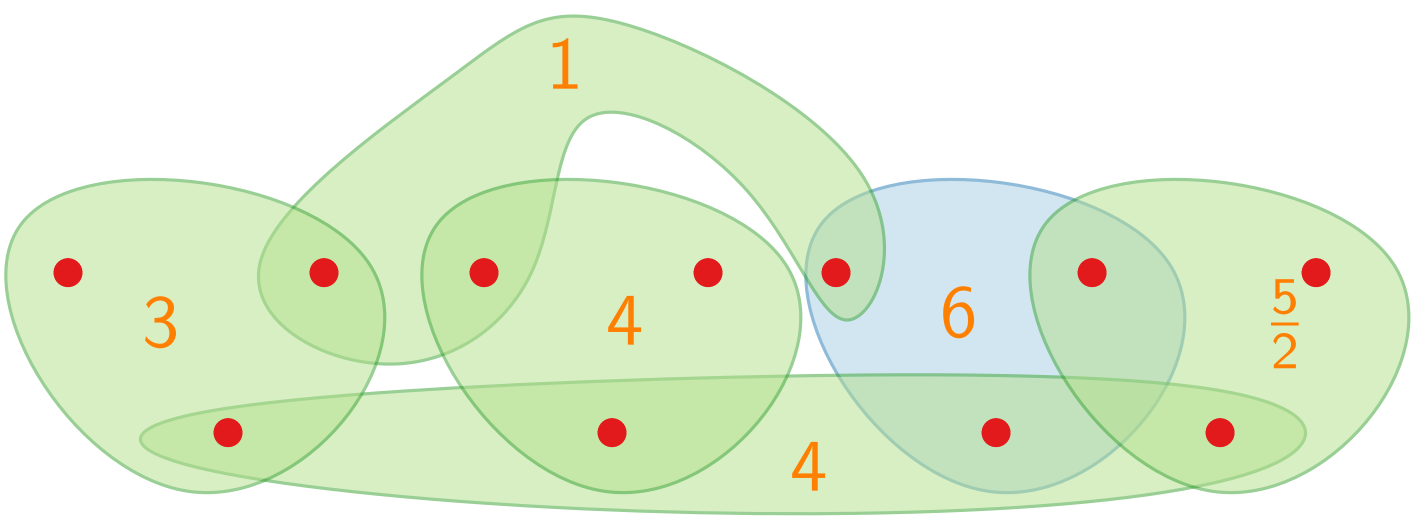


Find cover $\mathcal{S}' \subseteq \mathcal{S}$ of U with minimum cost.

SetCover as an ILP

minimize

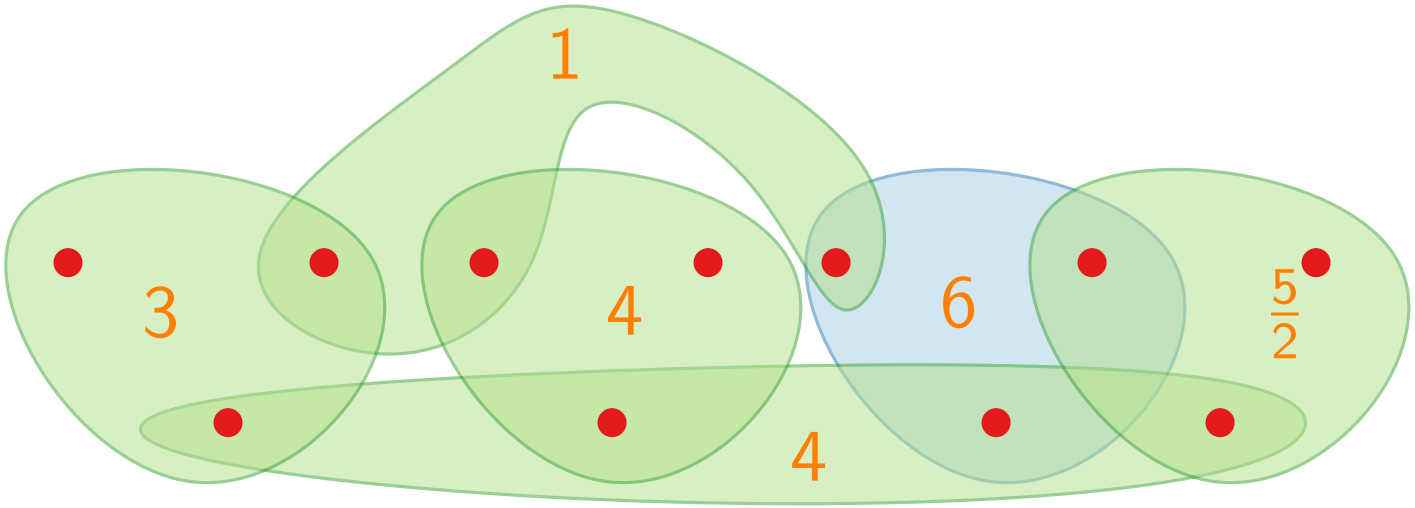
subject to

$x_S \in \{0, 1\} \quad \forall S \in \mathcal{S}$

Ground set U

Family $\mathcal{S} \subseteq 2^U$ with $\bigcup \mathcal{S} = U$

Costs $c \colon \mathcal{S} \to \mathbb{Q}^+$



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SetCover as an ILP

$$\begin{array}{ll} \textbf{minimize} & \sum_{S \in \mathcal{S}} c_S x_S \\ \textbf{subject to} & \\ & x_S \in \{0,1\} \quad \forall S \in \mathcal{S} \end{array}$$

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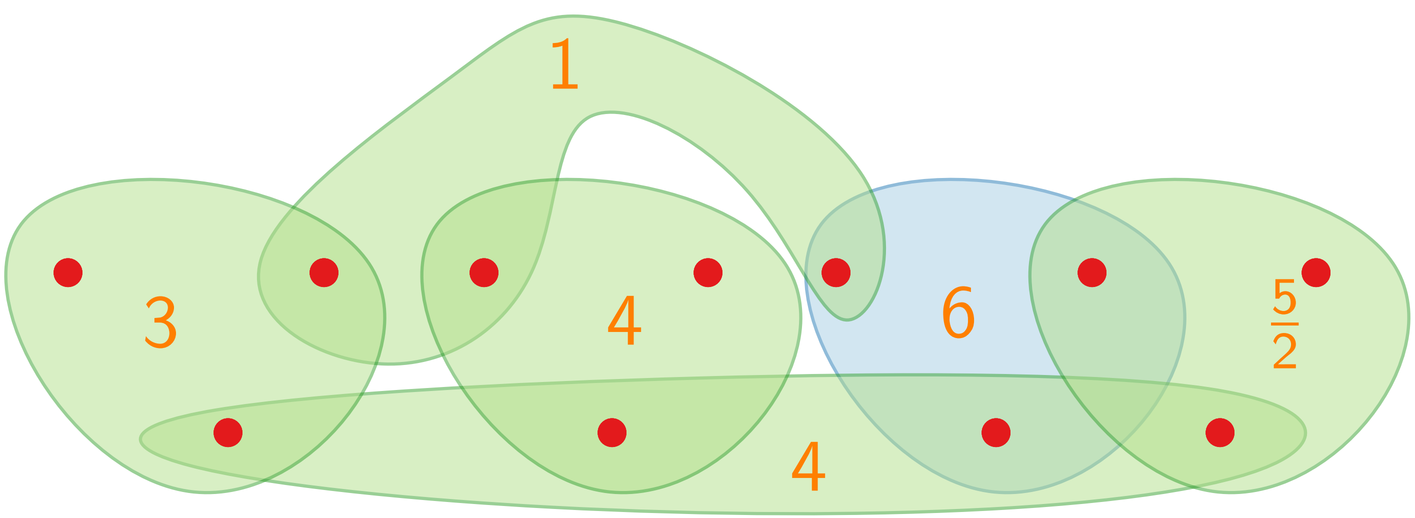

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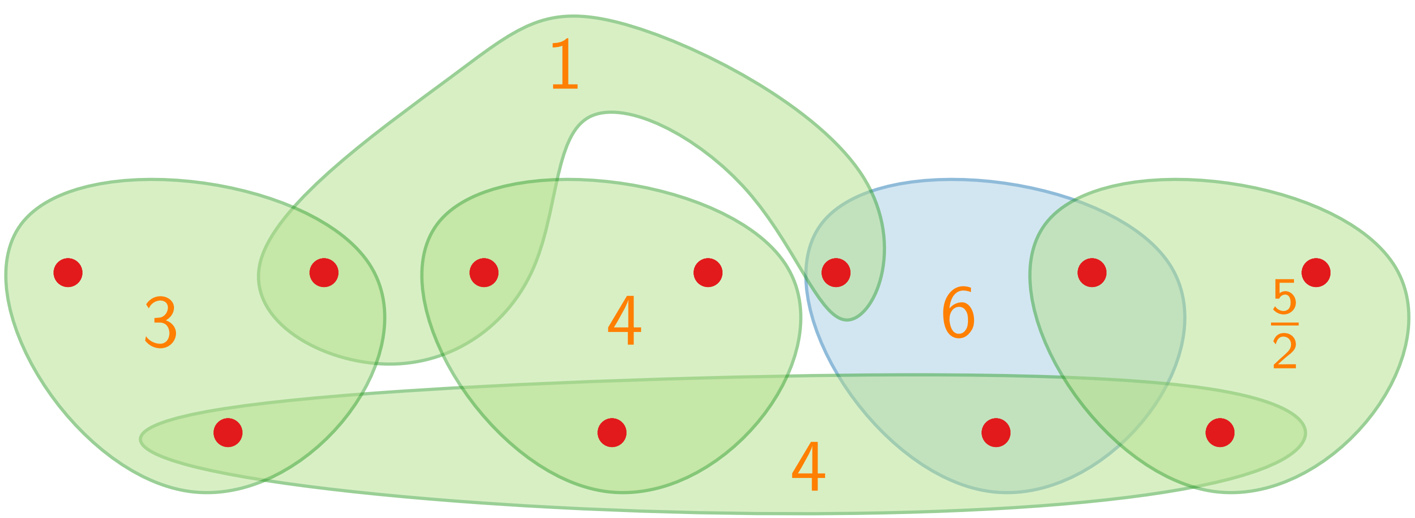


Find cover $\mathcal{S}' \subseteq \mathcal{S}$ of U with minimum cost.

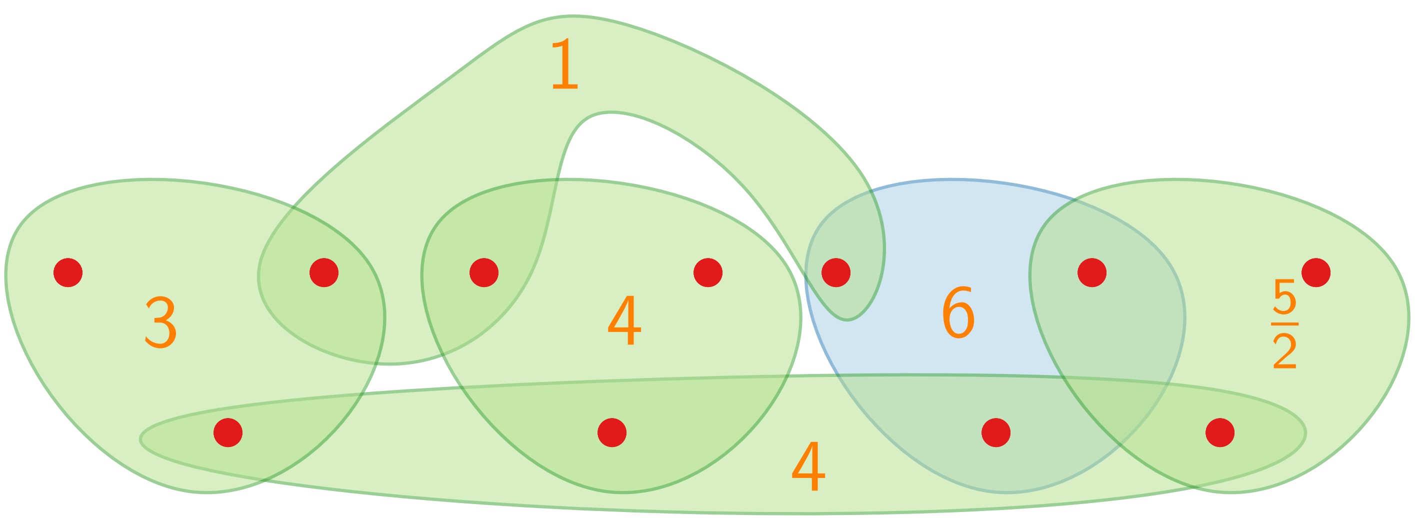
SetCover as an ILP

$$\begin{aligned} &\textbf{minimize} && \sum_{S \in \mathcal{S}} c_S x_S \\ &\textbf{subject to} && \sum_{S \ni u} x_S \geq 1 && \forall u \in U \\ & && x_S \in \{0,1\} && \forall S \in \mathcal{S} \end{aligned}$$

Ground set U

Family $\mathcal{S} \subseteq 2^U$ with $\bigcup \mathcal{S} = U$

Costs $c \colon \mathcal{S} \to \mathbb{Q}^+$



Find cover $\mathcal{S}' \subseteq \mathcal{S}$ of U with minimum cost.

Approximation Algorithms

Lecture 5:
LP-based Approximation Algorithms for SetCover

Part II:
LP-Rounding

Technique I) LP-Rounding

Consider a minimization problem Π in ILP form.

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Consider a minimization problem Π in ILP form.

Compute a solution for the LP-relaxation.

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Round to obtain an integer solution for Π.

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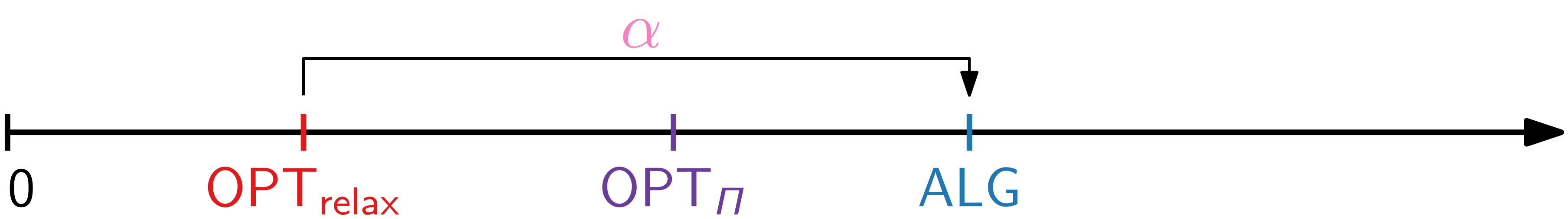
Compute a solution for the LP-relaxation.

Round to obtain an integer solution for Π.

Difficulty: Ensure the **feasiblity** of the solution.

Technique I) LP-Rounding



Consider a minimization problem Π in ILP form.

Compute a solution for the LP-relaxation.

Round to obtain an integer solution for Π.

Difficulty: Ensure the **feasiblity** of the solution.

Approximation factor: $\mathrm{ALG}/\mathrm{OPT}_{\Pi} \leq \mathrm{ALG}/\mathrm{OPT}_{\mathrm{relax}}$.

SetCover – LP-Relaxation

$$\begin{array}{lll} \textbf{minimize} & \displaystyle\sum_{S \in \mathcal{S}} c_S x_S & \\ \textbf{subject to} & \displaystyle\sum_{S \ni u} x_S \geq 1 & \forall u \in U \\ & x_S \geq 0 & \forall S \in \mathcal{S} \end{array}$$

SetCover – LP-Relaxation

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Optimal?

SetCover – LP-Relaxation

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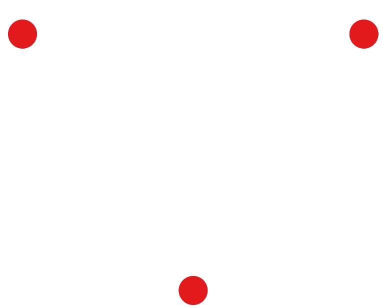

SetCover – LP-Relaxation

$$
\begin{aligned}
\textbf{minimize} \quad & \sum_{S \in \mathcal{S}} c_S x_S \\
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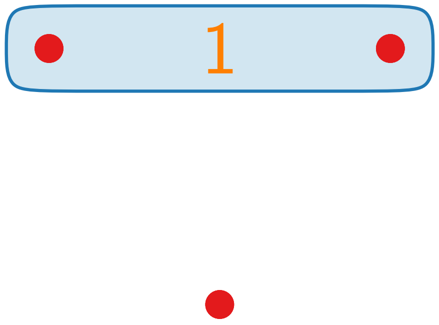
& x_S \geq 0 \quad \forall S \in \mathcal{S}
\end{aligned}
$$

Optimal?



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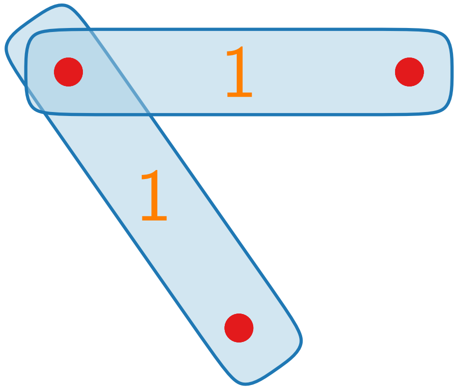
Optimal?



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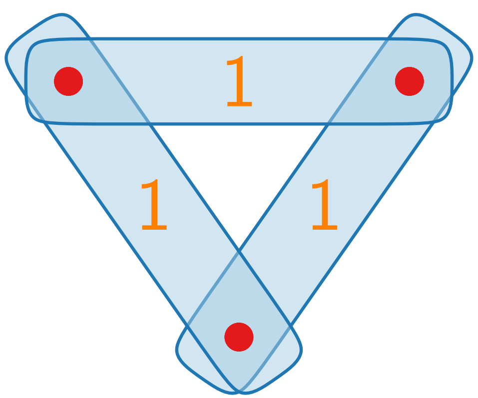

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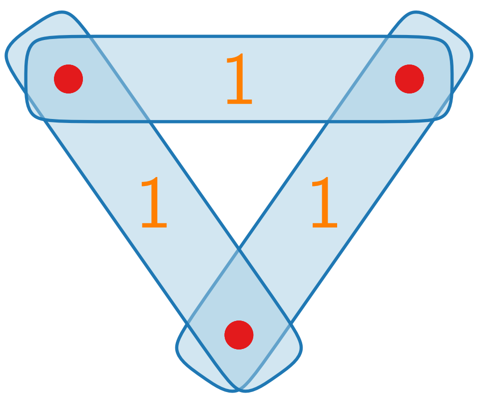

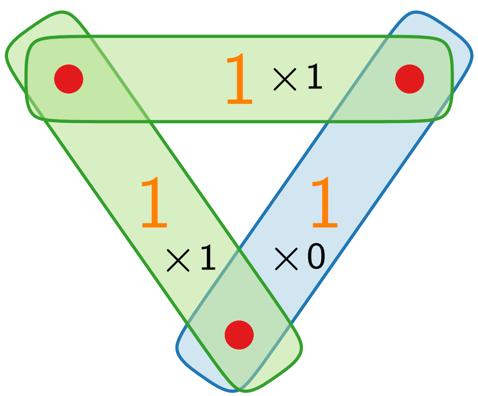

integer: 2

SetCover – LP-Relaxation

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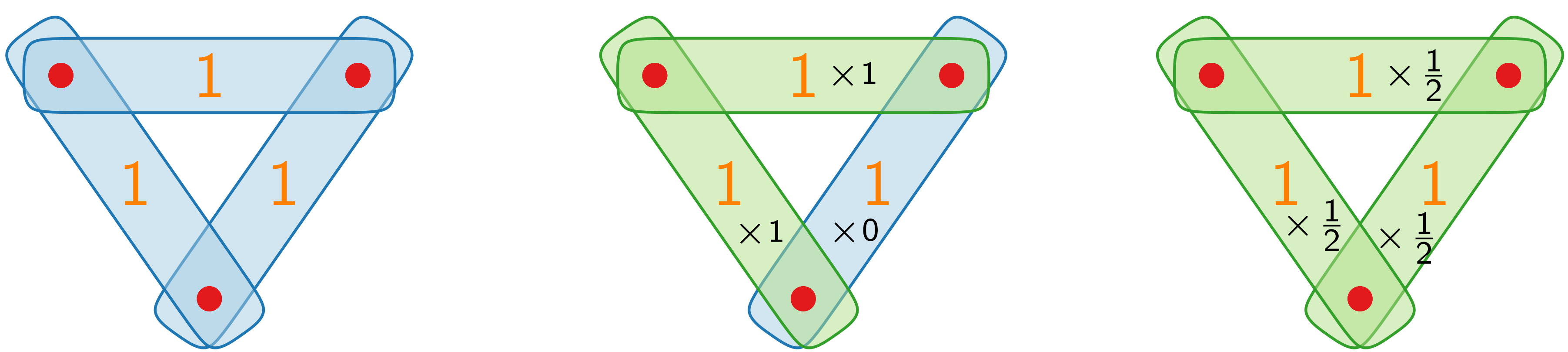

integer: 2

fractional: $\frac{3}{2}$

LP-Rounding: Approach I

$$\begin{array}{lll} \textbf{minimize} & \displaystyle\sum_{S \in \mathcal{S}} c_S x_S & \\ \textbf{subject to} & \displaystyle\sum_{S \ni u} x_S \geq 1 & \forall u \in U \\ & x_S \geq 0 & \forall S \in \mathcal{S} \end{array}$$

LP-Rounding-One($U, \mathcal{S}, c$)

LP-Rounding: Approach I

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Compute optimal solution x for LP-relaxation.

Round each x_S with $x_S > 0$ to 1.

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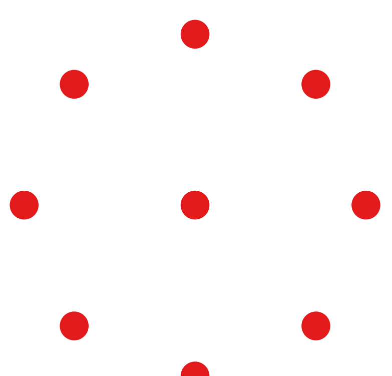

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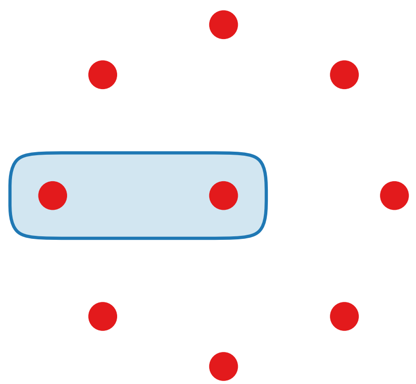

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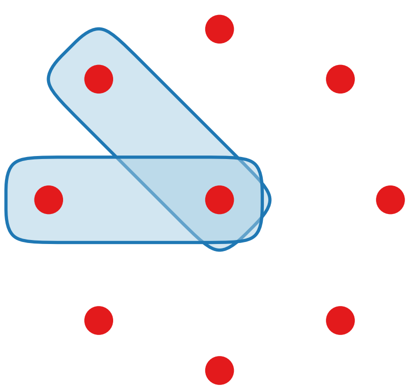

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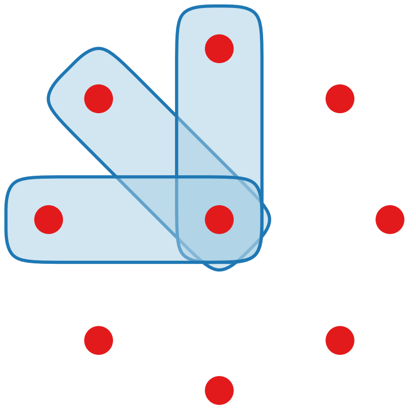

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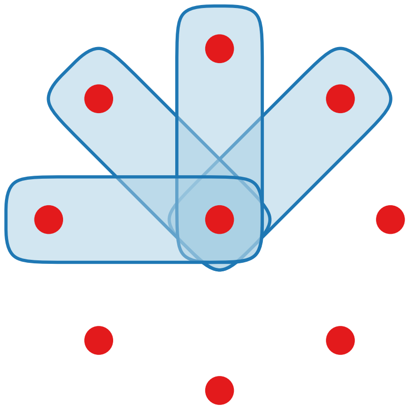

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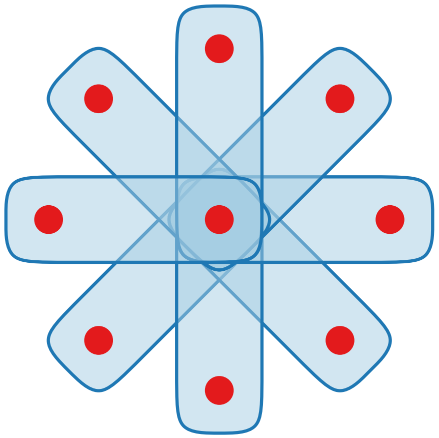

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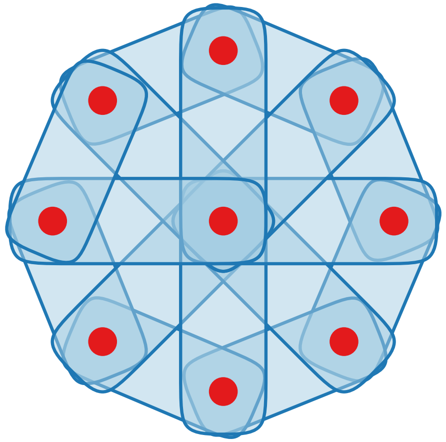

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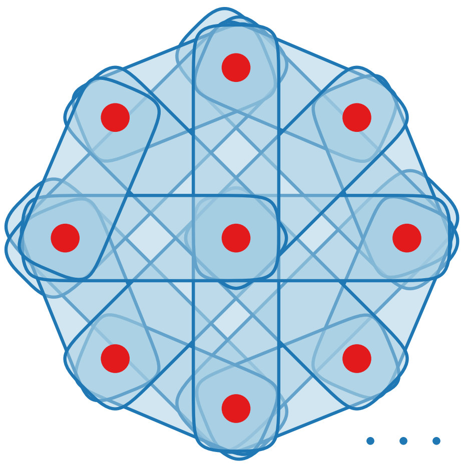

LP-Rounding: Approach I

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LP-Rounding-One($U, \mathcal{S}, c$)

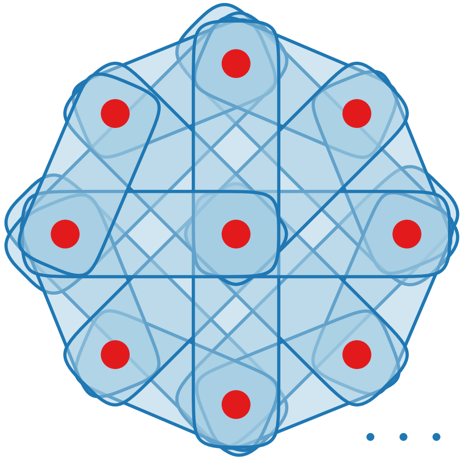
Compute optimal solution x for LP-relaxation.

Round each x_S with $x_S > 0$ to 1.

– Generates a feasible solution.

– Scaling factor arbitrarily large.

Use frequency f



LP-Rounding: Approach II

$$
\begin{array}{lll}
\textbf{minimize} & \displaystyle\sum_{S \in \mathcal{S}} c_S x_S & \\
\textbf{subject to} & \displaystyle\sum_{S \ni u} x_S \geq 1 & \forall u \in U \\
 & x_S \geq 0 & \forall S \in \mathcal{S}
\end{array}
$$

LP-Rounding-Two($U, \mathcal{S}, c$)

Compute optimal solution x for LP-relaxation.

Round each x_S with $x_S \geq$ to 1; remaining to 0.

Let f be the frequency of (i.e., the number of sets containing) the most frequent element.

LP-Rounding: Approach II

$$\begin{aligned} \textbf{minimize} \quad & \sum_{S \in \mathcal{S}} c_S x_S \\ \textbf{subject to} \quad & \sum_{S \ni u} x_S \geq 1 \quad \forall u \in U \\ & x_S \geq 0 \quad \forall S \in \mathcal{S} \end{aligned}$$

LP-Rounding-Two($U, \mathcal{S}, c$)

Compute optimal solution x for LP-relaxation.
Round each x_S with $x_S \geq 1/f$ to 1; remaining to 0.

Let f be the frequency of (i.e., the number of sets containing) the most frequent element.

LP-Rounding: Approach II

$$\begin{array}{lll} \textbf{minimize} & \displaystyle\sum_{S \in \mathcal{S}} c_S x_S & \\ \textbf{subject to} & \displaystyle\sum_{S \ni u} x_S \geq 1 & \forall u \in U \\ & x_S \geq 0 & \forall S \in \mathcal{S} \end{array}$$

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Compute optimal solution x for LP-relaxation.

Round each x_S with $x_S \geq 1/f$ to 1; remaining to 0.

Let f be the frequency of (i.e., the number of sets containing) the most frequent element.

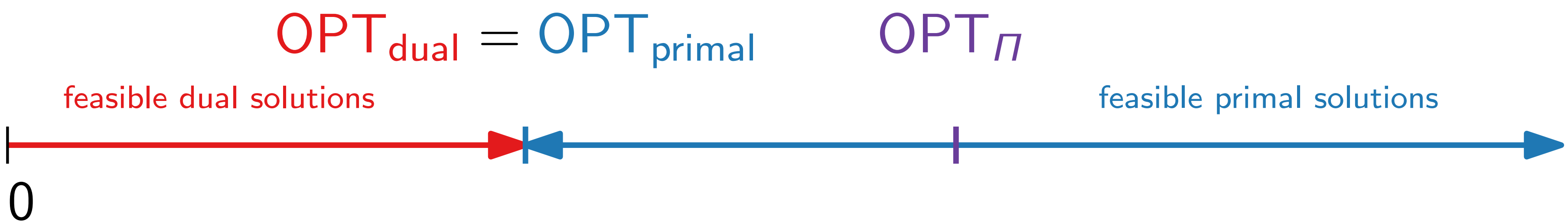
Theorem. LP-Rounding-Two is a factor-f approximation algorithm for SetCover.

Approximation Algorithms

Lecture 5:
LP-based Approximation Algorithms for SetCover

Part III:
The Primal-Dual Schema

Technique II) Primal–Dual Approach



Consider a minimization problem Π in ILP form.

Technique II) Primal–Dual Approach

Consider a minimization problem Π in ILP form.

- Start with (trivial) feasible dual solution and infeasible primal solution (e.g., all variables $= 0$).

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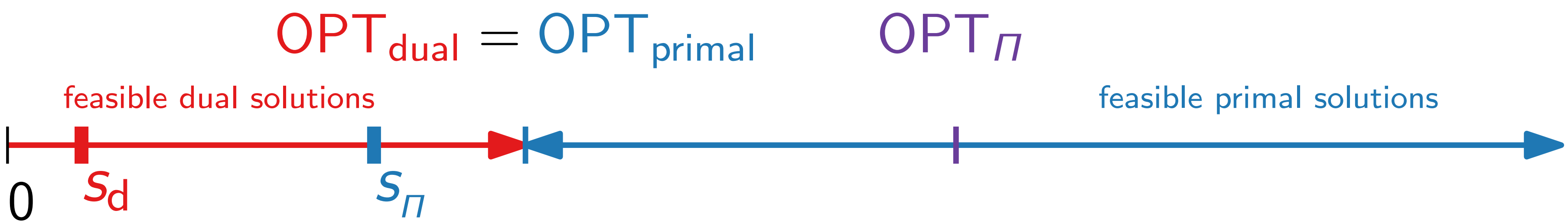
Technique II) Primal–Dual Approach



Consider a minimization problem Π in ILP form.

- Start with (trivial) feasible dual solution and infeasible primal solution (e.g., all variables $= 0$).
- Compute dual solution s_{d} and integral primal solution s_{Π} for Π iteratively:
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Technique II) Primal–Dual Approach

Consider a minimization problem Π in ILP form.

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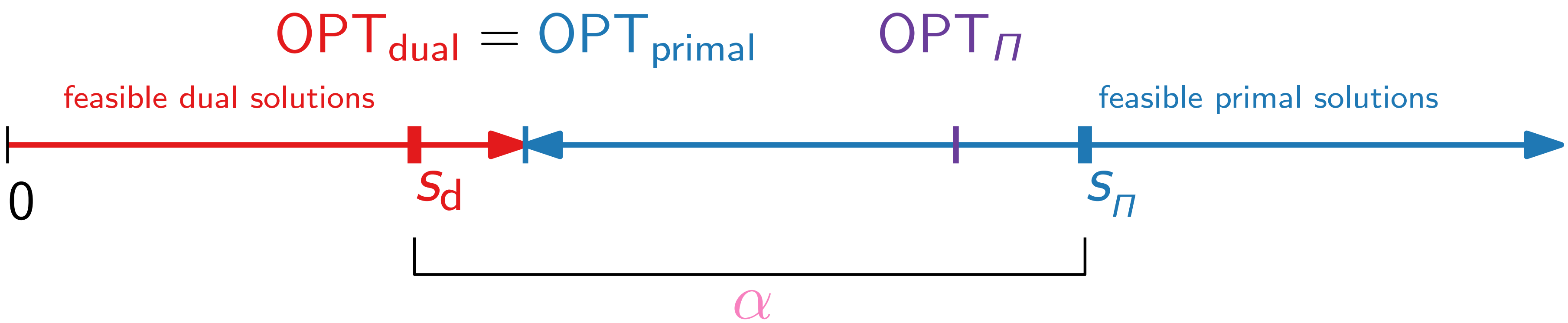

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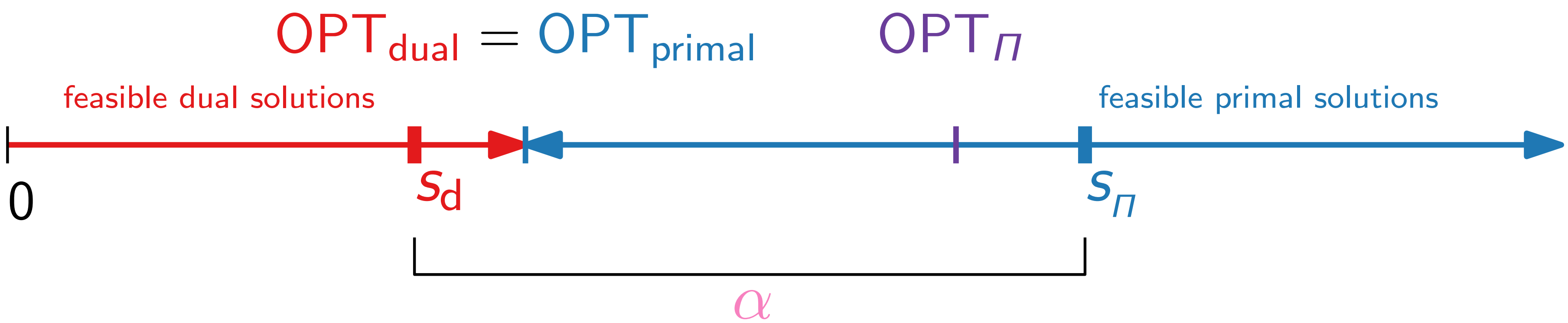

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Approximation factor $\leq \mathrm{obj}(s_{\Pi})/\mathrm{obj}(s_{\mathrm{d}})$

Advantage: Don't need LP-"machinery"; possibly faster, more flexible.

SetCover – Dual LP

$$\begin{array}{lll} \textbf{minimize} & \displaystyle\sum_{S \in \mathcal{S}} c_S x_S & \\ \textbf{subject to} & \displaystyle\sum_{S \ni u} x_S \geq 1 & \forall u \in U \\ & x_S \geq 0 & \forall S \in \mathcal{S} \end{array}$$

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maximize

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Complementary Slackness

$$\begin{array}{llcl} \textbf{minimize} & c^\intercal x & & \\ \textbf{subject to} & Ax & \geq & b \\ & x & \geq & 0 \end{array}$$

$$\begin{array}{llcl} \textbf{maximize} & b^\intercal y & & \\ \textbf{subject to} & A^\intercal y & \leq & c \\ & y & \geq & 0 \end{array}$$

Theorem. Let $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_m)$ be valid solutions for the primal and dual program, respectively.
Then x and y are optimal $\Leftrightarrow$ following conditions are met:

Primal CS:
For each $j = 1, \ldots, n$: $\quad x_j = 0 \;\text{ or }\; \sum_{i=1}^{m} a_{ij} y_i = c_j$

Dual CS:
For each $i = 1, \ldots, m$: $\quad y_i = 0 \;\text{ or }\; \sum_{j=1}^{n} a_{ij} x_j = b_i$

Relaxing Complementary Slackness

minimize	$c^\intercal x$		
subject to	Ax	$\geq$	b
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~~**Primal CS**~~: Relaxed Primal CS

For each $j = 1, \ldots, n$: $x_j = 0$ or $\cancel{\sum_{i=1}^m a_{ij} y_i = c_j}$

$$c_j/\alpha \leq \sum_{i=1}^m a_{ij} y_i \leq c_j$$

Dual CS:

For each $i = 1, \ldots, m$: $y_i = 0$ or $\sum_{j=1}^n a_{ij} x_j = b_i$

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For each $i = 1, \ldots, m$: $y_i = 0$ or $\cancel{\sum_{j=1}^n a_{ij} x_j = b_i}$

$$b_i \leq \sum_{j=1}^n a_{ij} x_j \leq \beta \cdot b_i$$

$$\Leftrightarrow \sum_{j=1}^{n} c_j x_j = \sum_{i=1}^{m} b_i y_i$$

Relaxing Complementary Slackness

minimize	$c^\intercal x$		
subject to	Ax	$\geq$	b
	x	$\geq$	0

maximize	$b^\intercal y$		
subject to	$A^\intercal y$	$\leq$	c
	y	$\geq$	0

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$$\cancel{\Leftrightarrow \sum_{j=1}^n c_j x_j = \sum_{i=1}^m b_i y_i} \quad \Rightarrow \sum_{j=1}^n c_j x_j \leq \alpha\beta \sum_{i=1}^m b_i y_i \leq \alpha\beta \cdot \mathrm{OPT}_{\mathrm{LP}}$$

Primal–Dual Schema

Start with a feasible dual and infeasible primal solution (often trivial).

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"Improve" the feasibility of the primal solution...

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Do so until the relaxed CS conditions are met.

Maintain that the primal solution is integer-valued.

The feasibility of the primal solution and the relaxed CS conditions provide an approximation ratio.

Relaxed CS for SetCover

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Relaxed CS for SetCover

minimize $\sum_{S \in \mathcal{S}} c_S x_S$

subject to $\sum_{S \ni u} x_S \geq 1 \quad \forall u \in U$

$x_S \geq 0 \quad \forall S \in \mathcal{S}$

maximize $\sum_{u \in U} y_u$

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(Unrelaxed) primal CS:

Relaxed CS for SetCover

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critical set

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only chooses critical sets

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Relaxed CS for SetCover

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(Unrelaxed) primal CS: $x_S \neq 0 \Rightarrow \sum_{u \in S} y_u = c_S$

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Relaxed dual CS: $y_u \neq 0 \Rightarrow 1 \leq \sum_{S \ni u} x_S \leq f$

Relaxed CS for SetCover

minimize $\sum_{S \in \mathcal{S}} c_S x_S$	**maximize** $\sum_{u \in U} y_u$
subject to $\sum_{S \ni u} x_S \geq 1 \quad \forall u \in U$	**subject to** $\sum_{u \in S} y_u \leq c_S \quad \forall S \in \mathcal{S}$
$x_S \geq 0 \quad \forall S \in \mathcal{S}$	$y_u \geq 0 \quad \forall u \in U$

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trivial for binary x

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Primal–Dual Schema for SetCover

PrimalDualSetCover($U, \mathcal{S}, c$)

$x \leftarrow 0$, $y \leftarrow 0$

repeat

until all elements are covered.

return x

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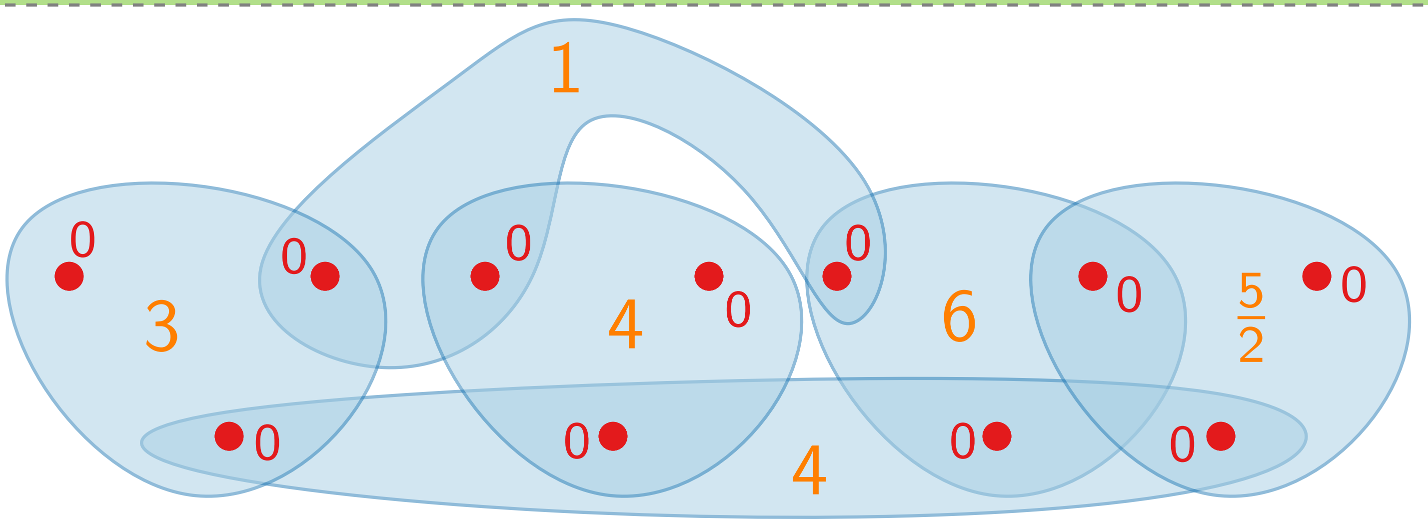

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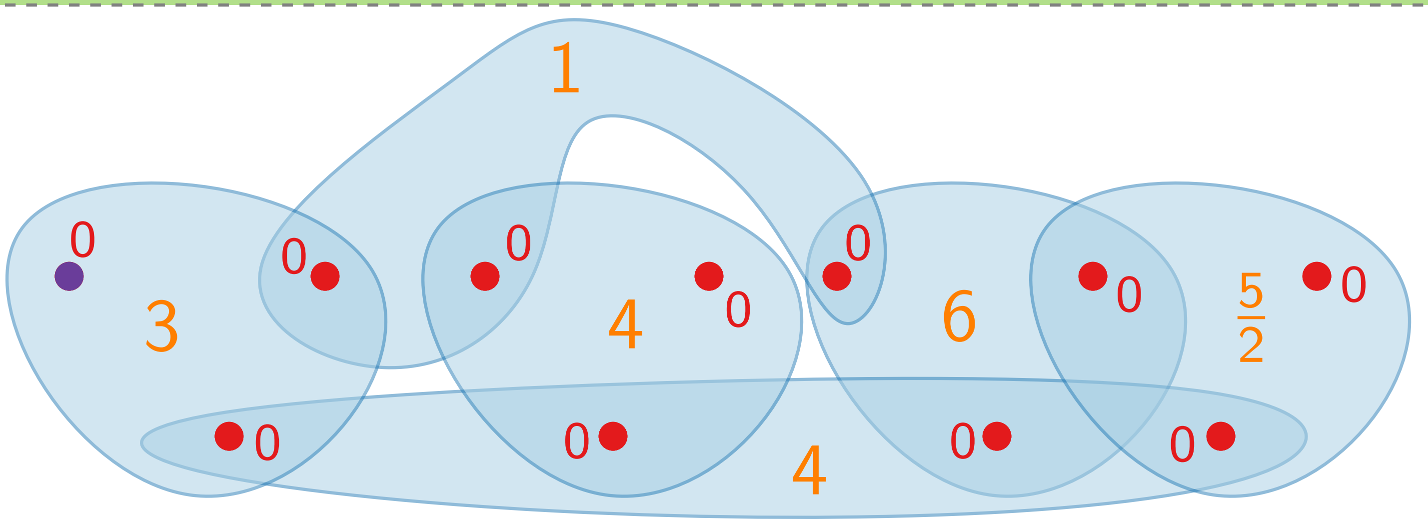

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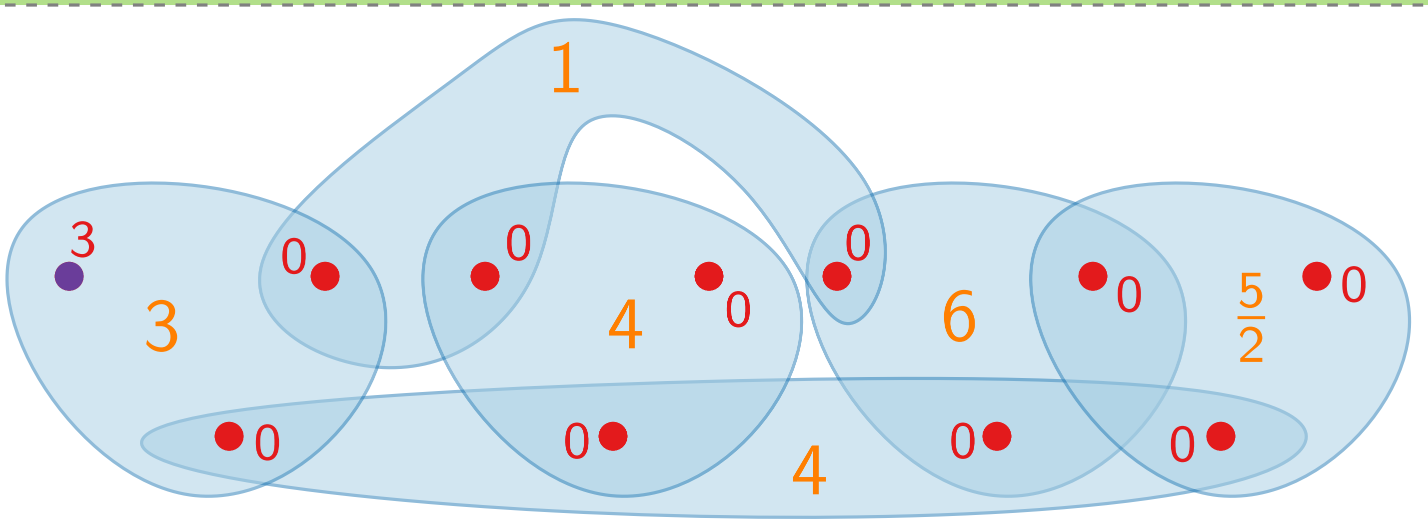

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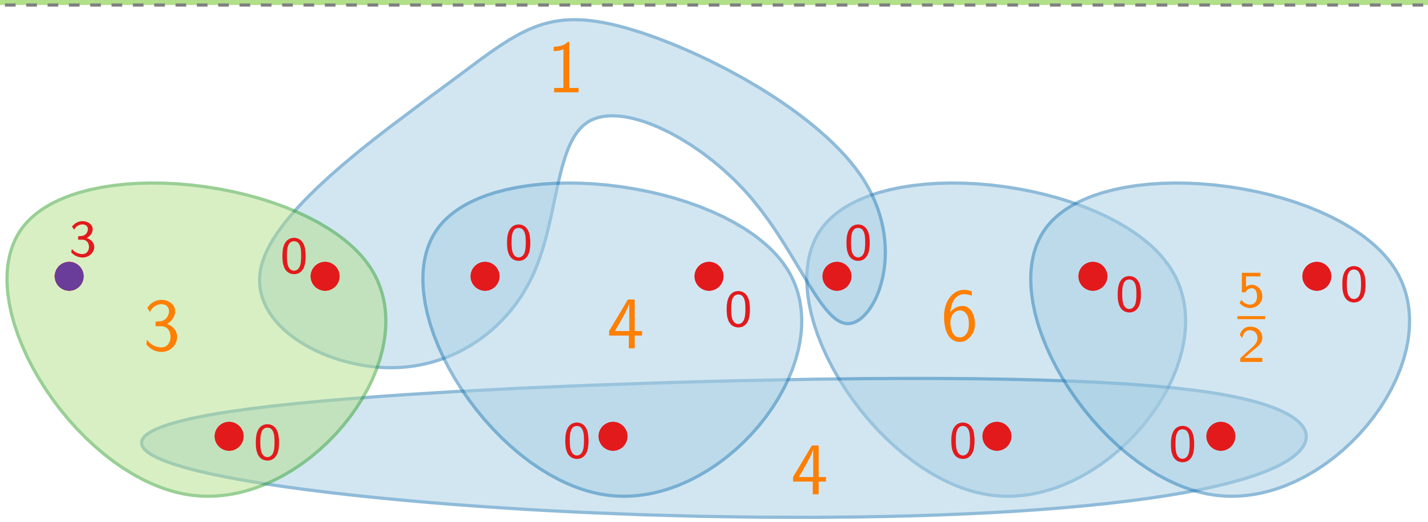

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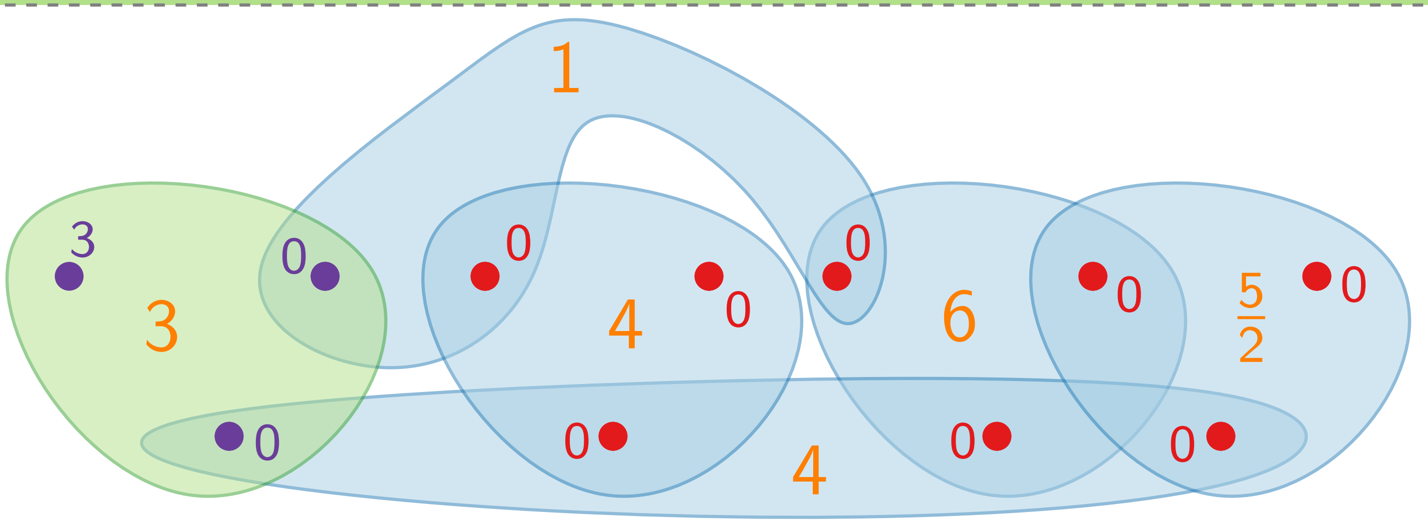

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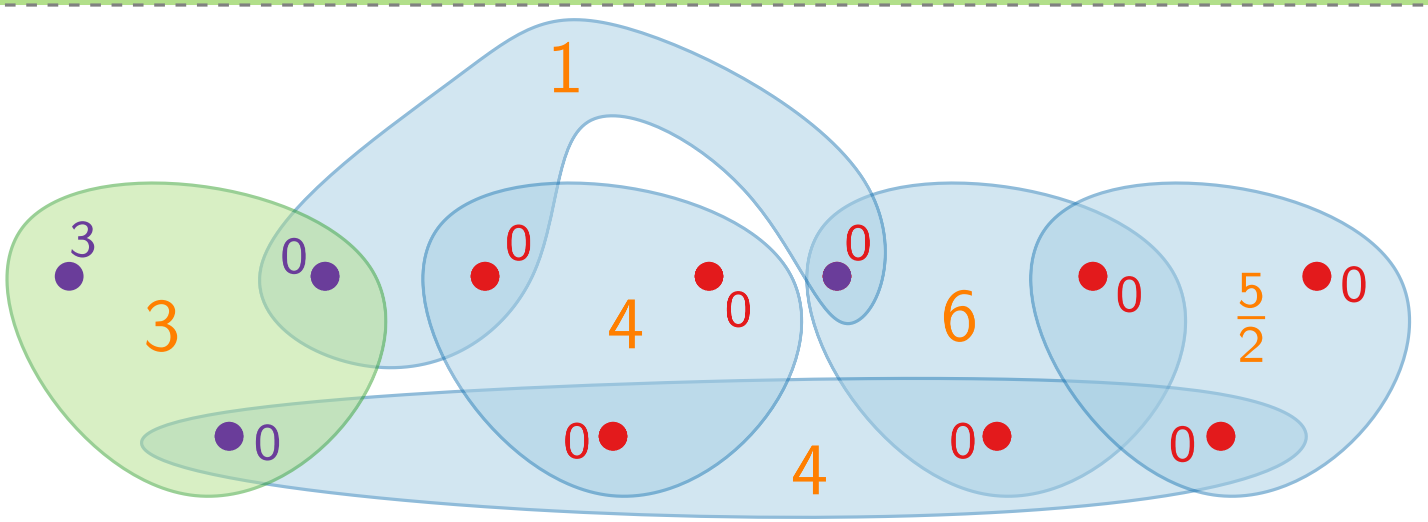

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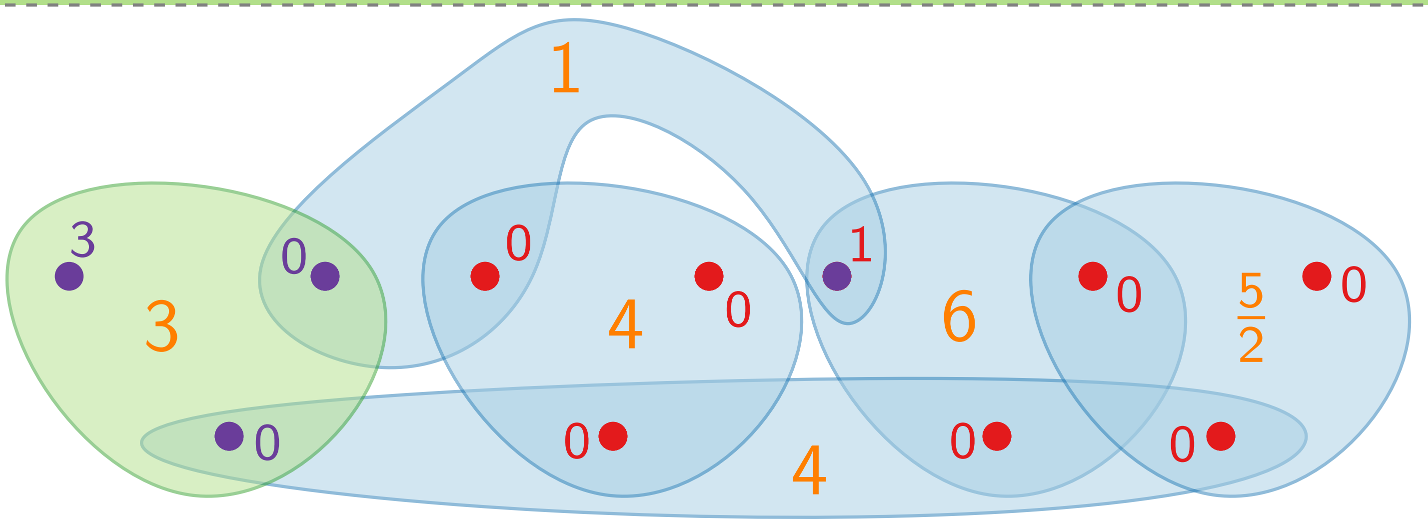

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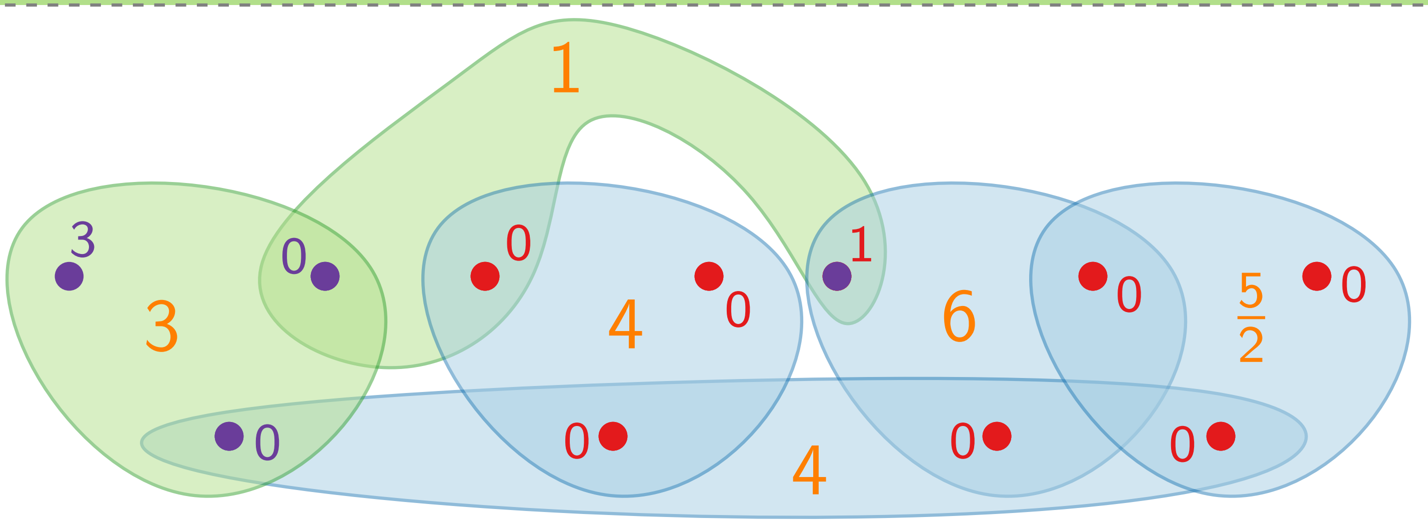

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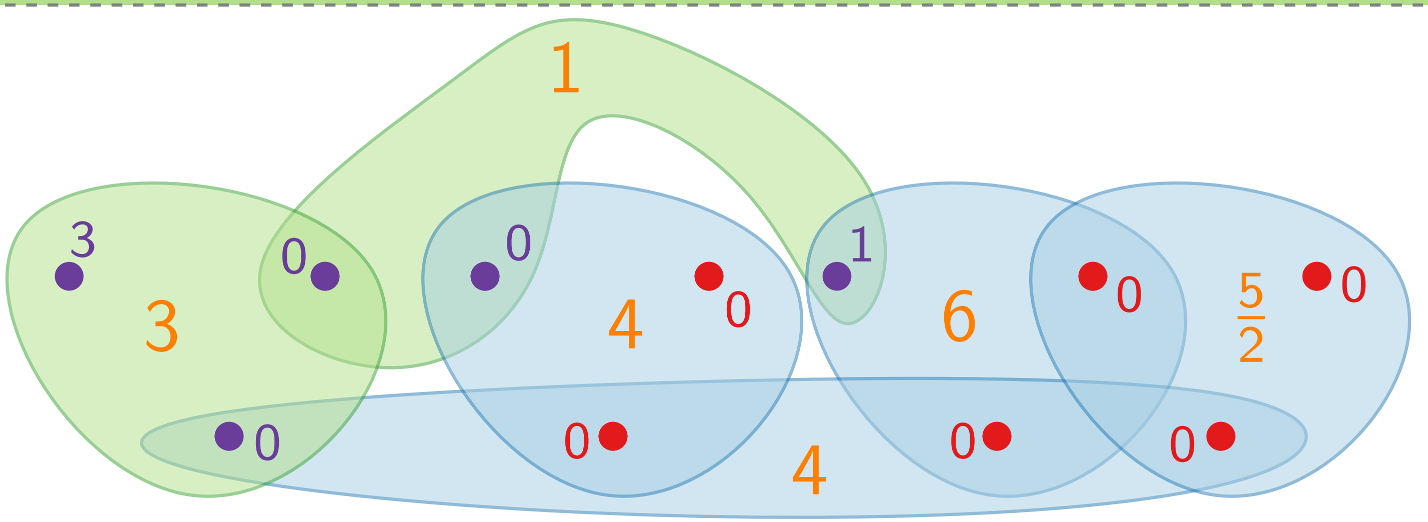

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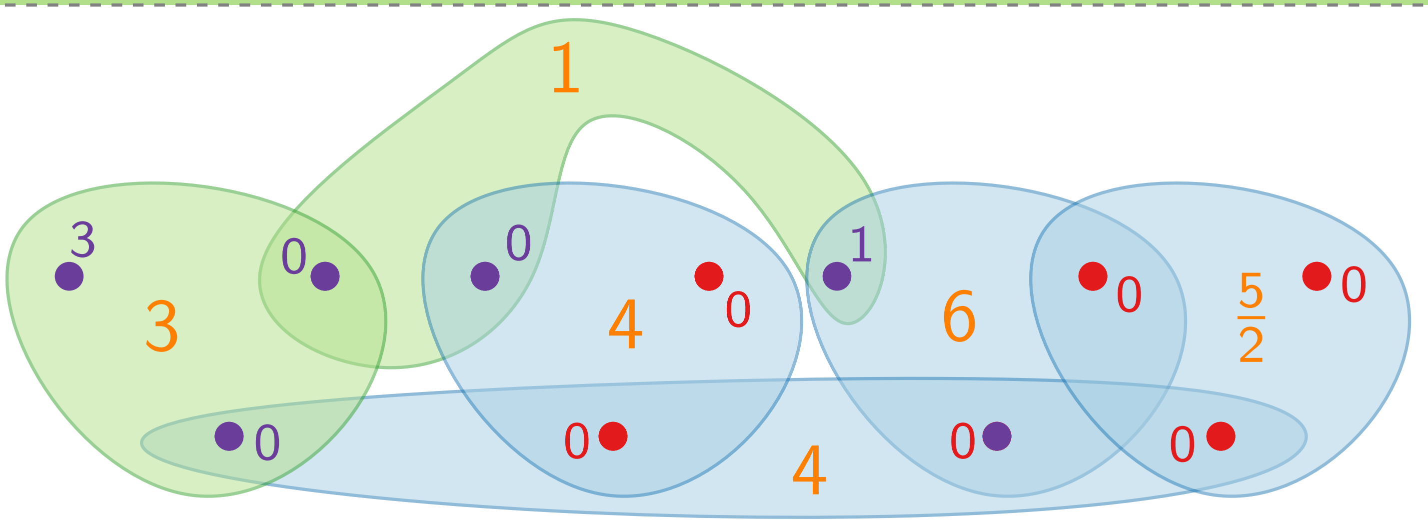

Primal–Dual Schema for SetCover

PrimalDualSetCover($U, \mathcal{S}, c$)

$x \leftarrow 0, y \leftarrow 0$
repeat
  Select an uncovered element u.
  Increase y_u until a set S is critical ($\sum_{u' \in S} y_{u'} = c_S$).
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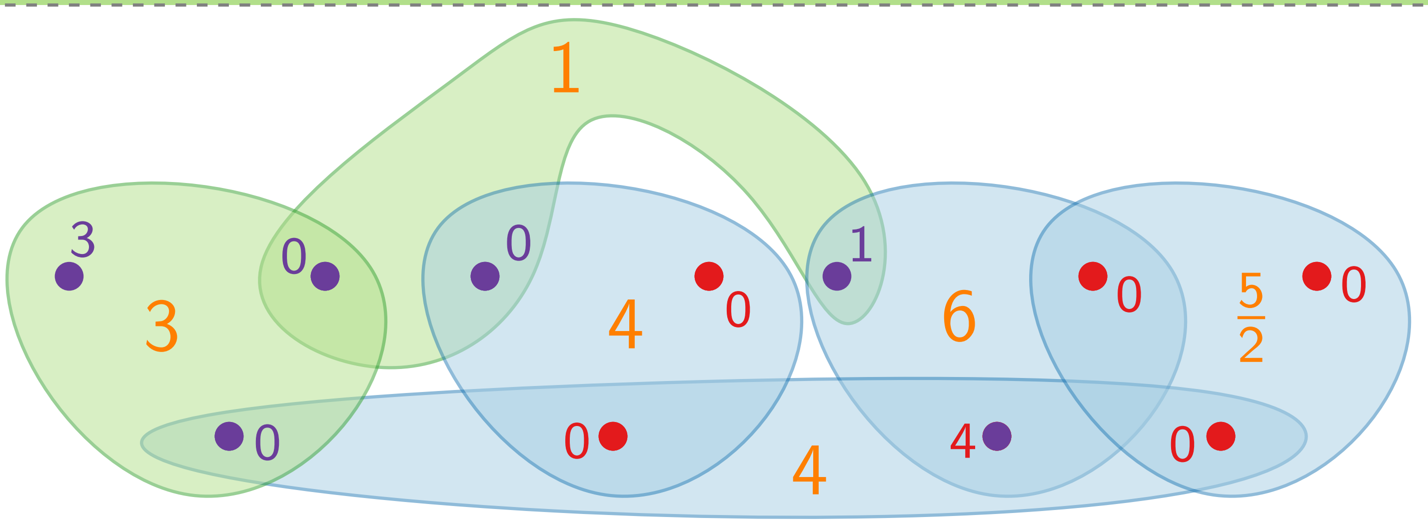

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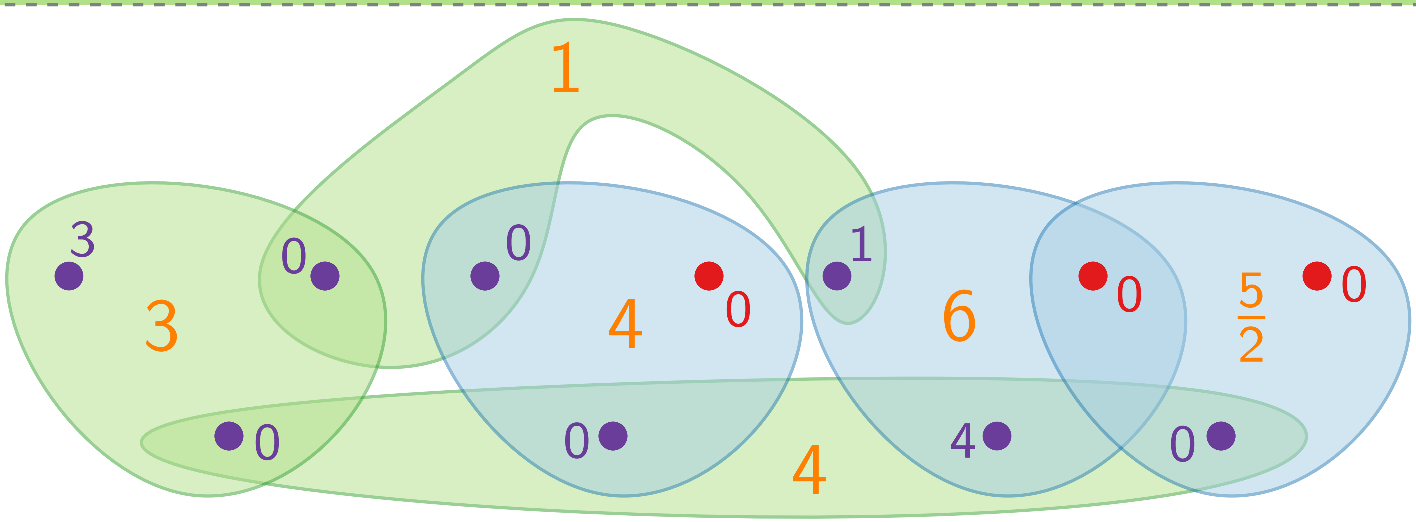

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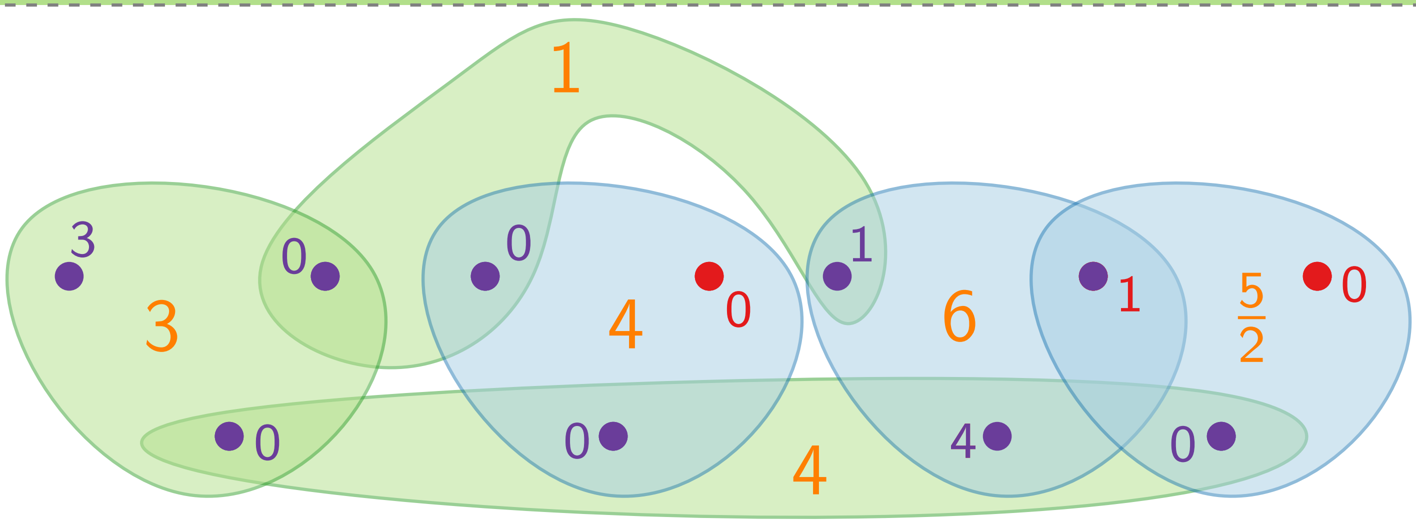

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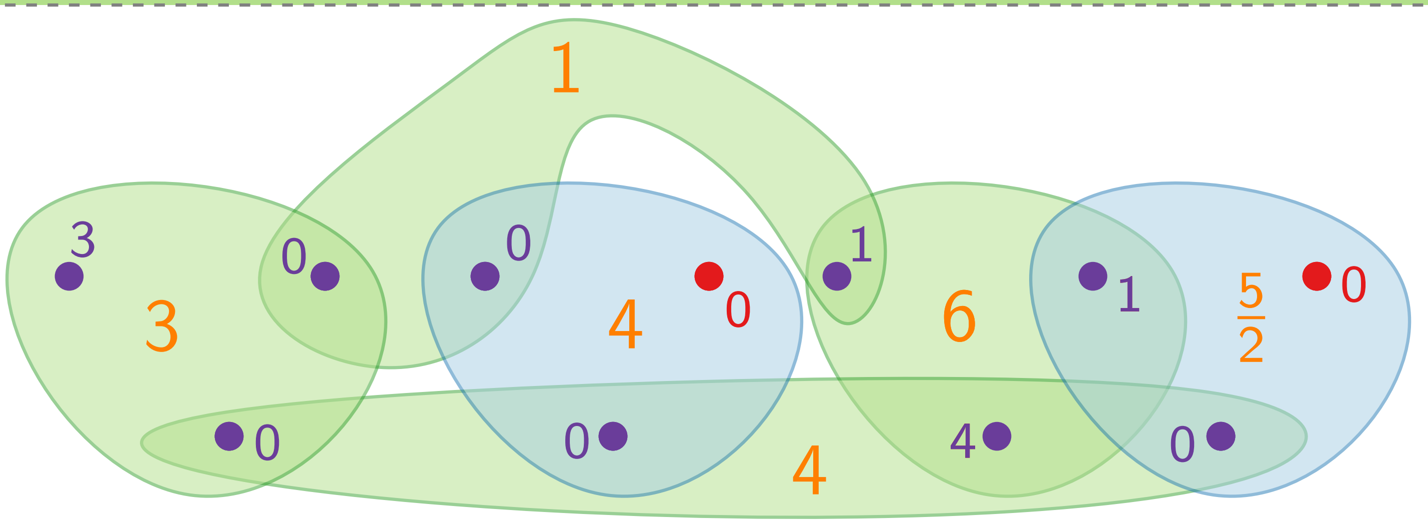

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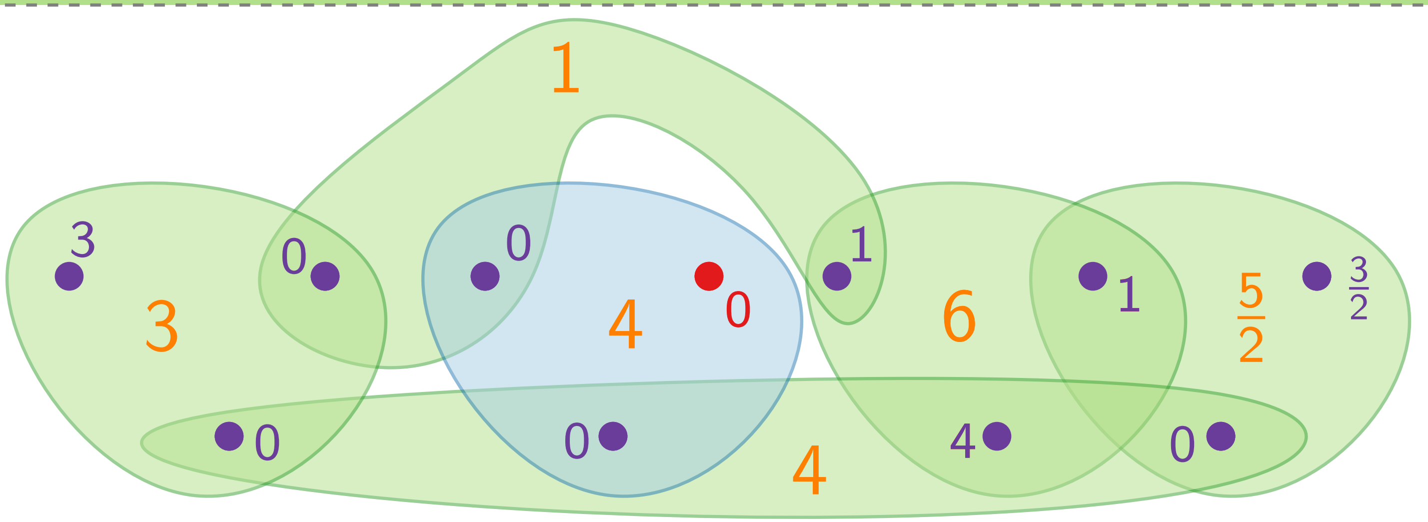

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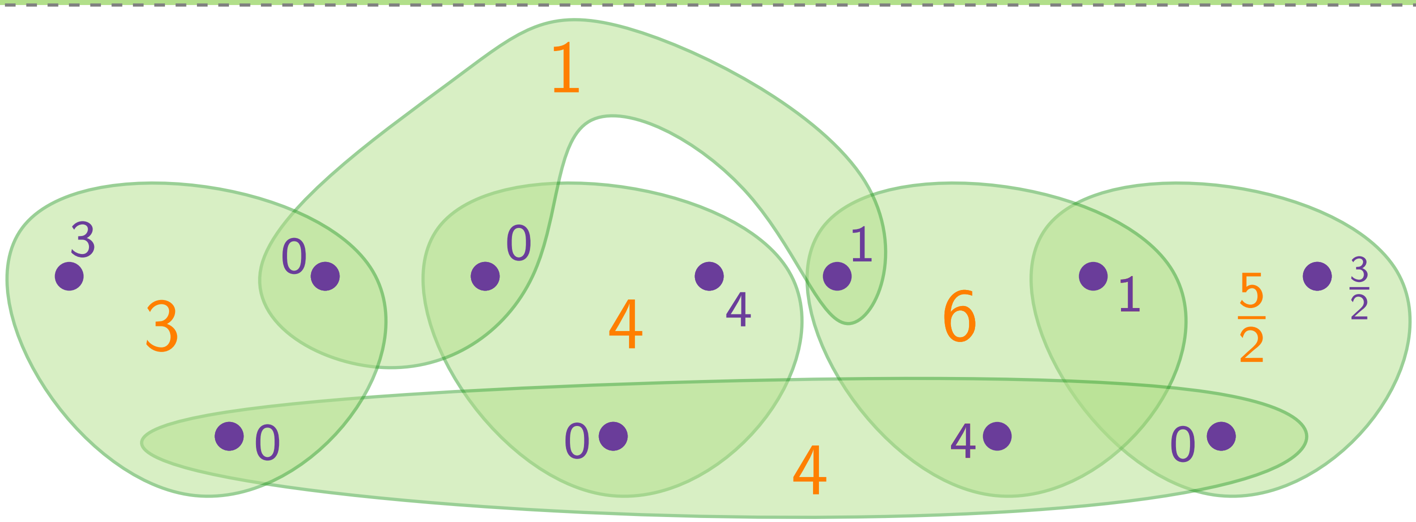

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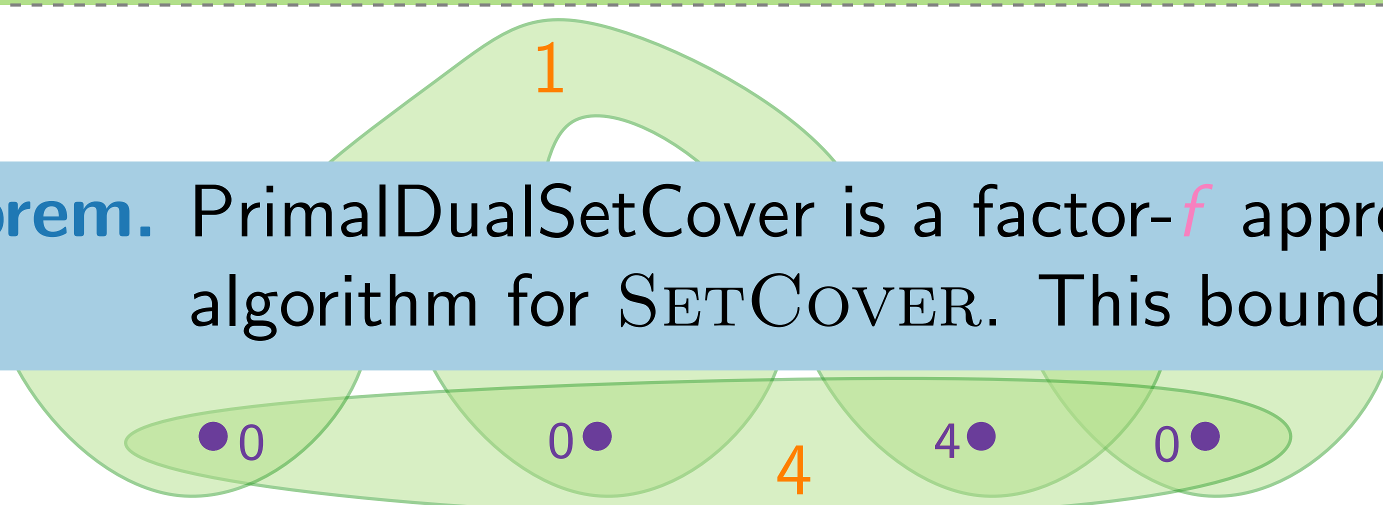

Theorem. PrimalDualSetCover is a factor-f approximation algorithm for SetCover. This bound is tight.

Tight Example

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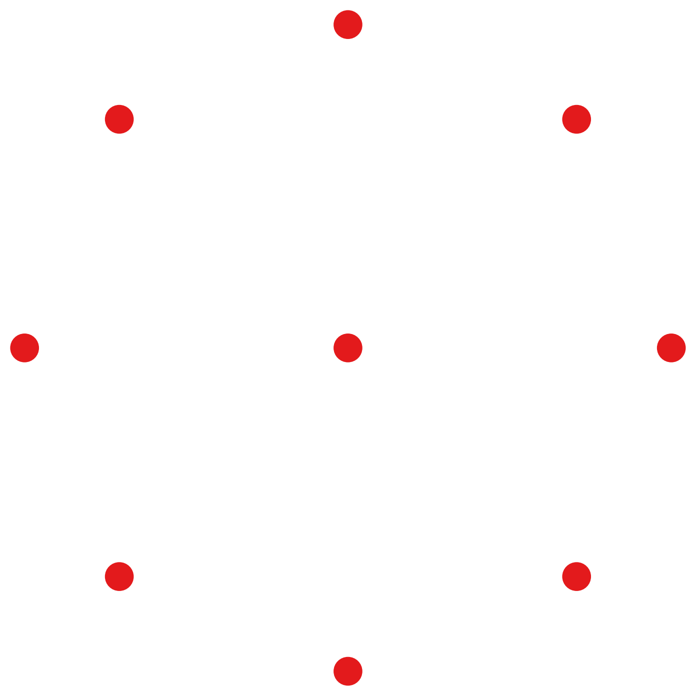

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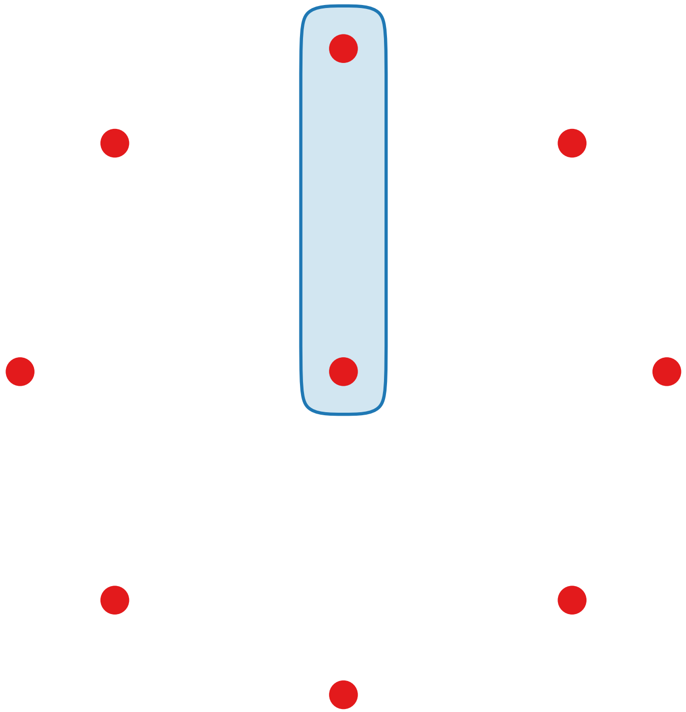

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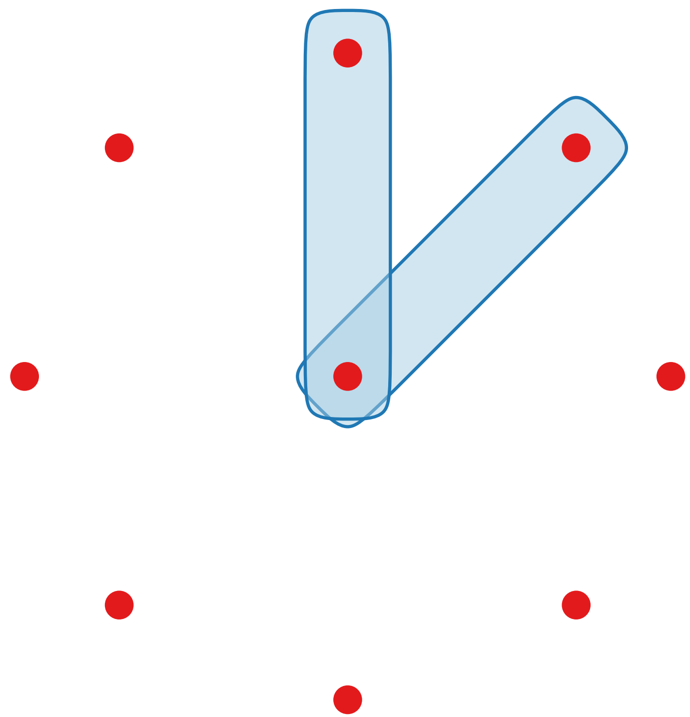

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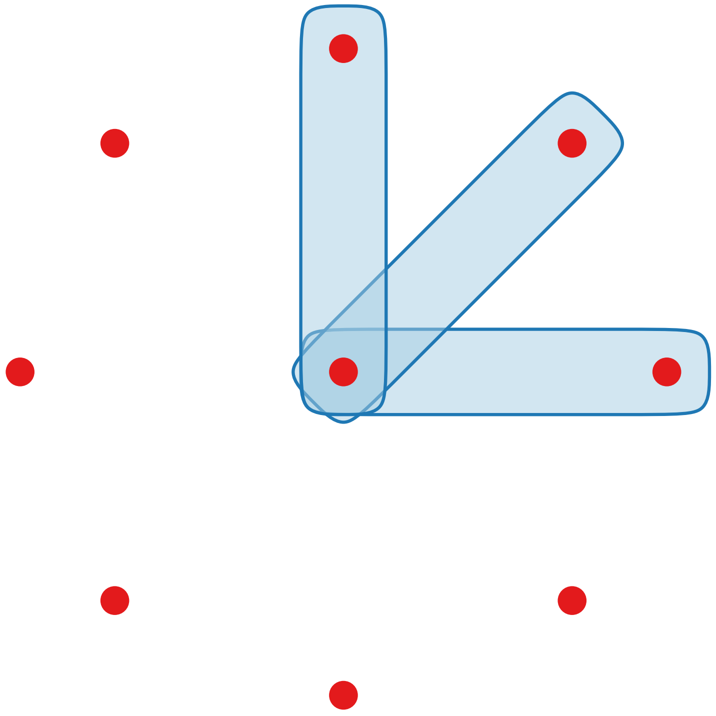

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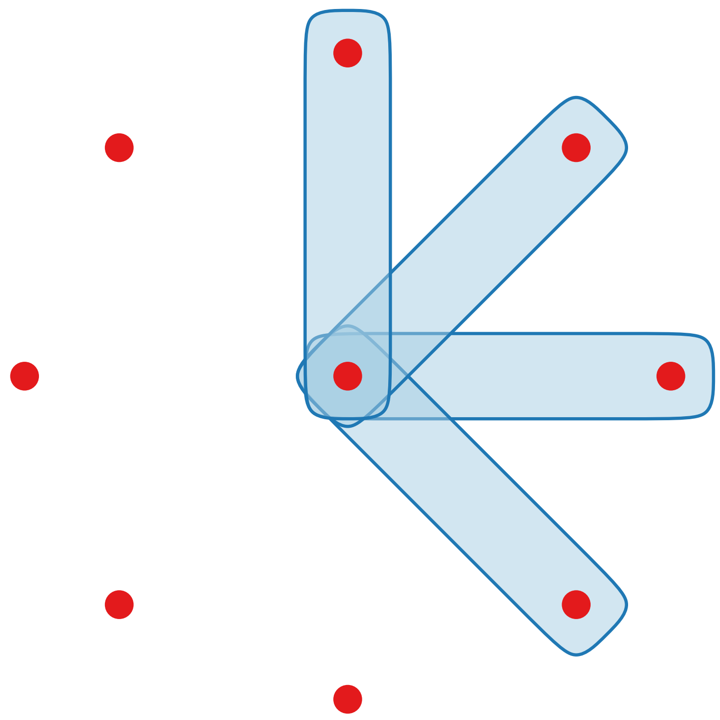

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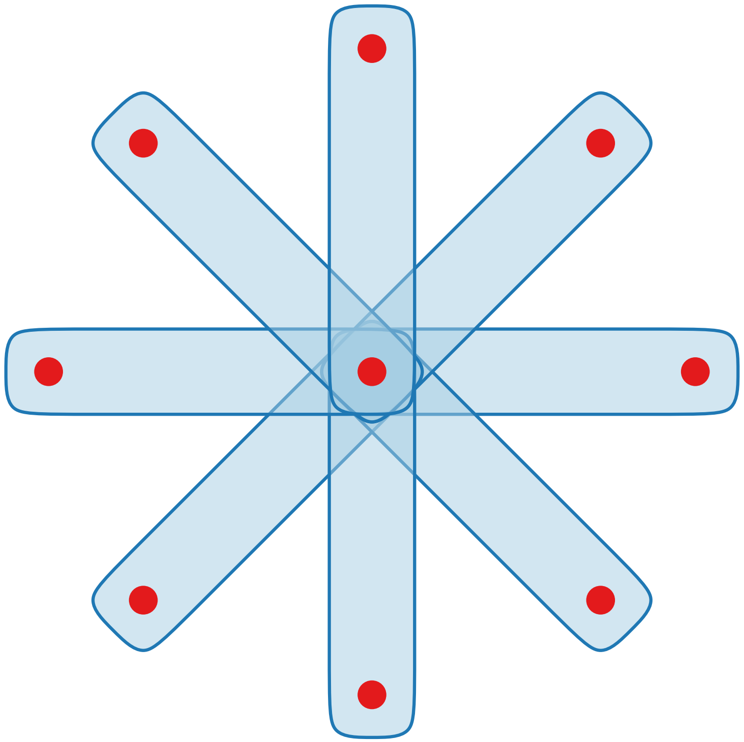

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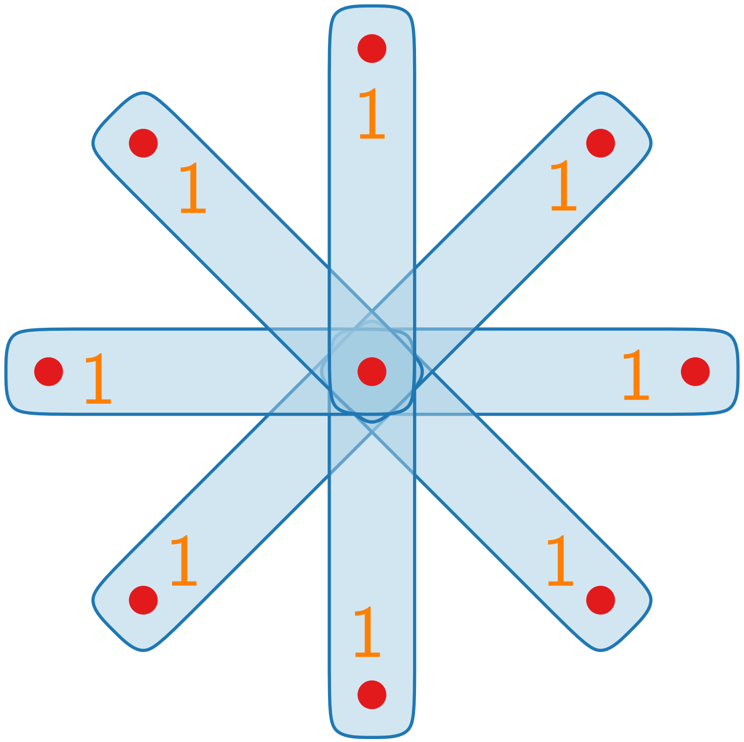

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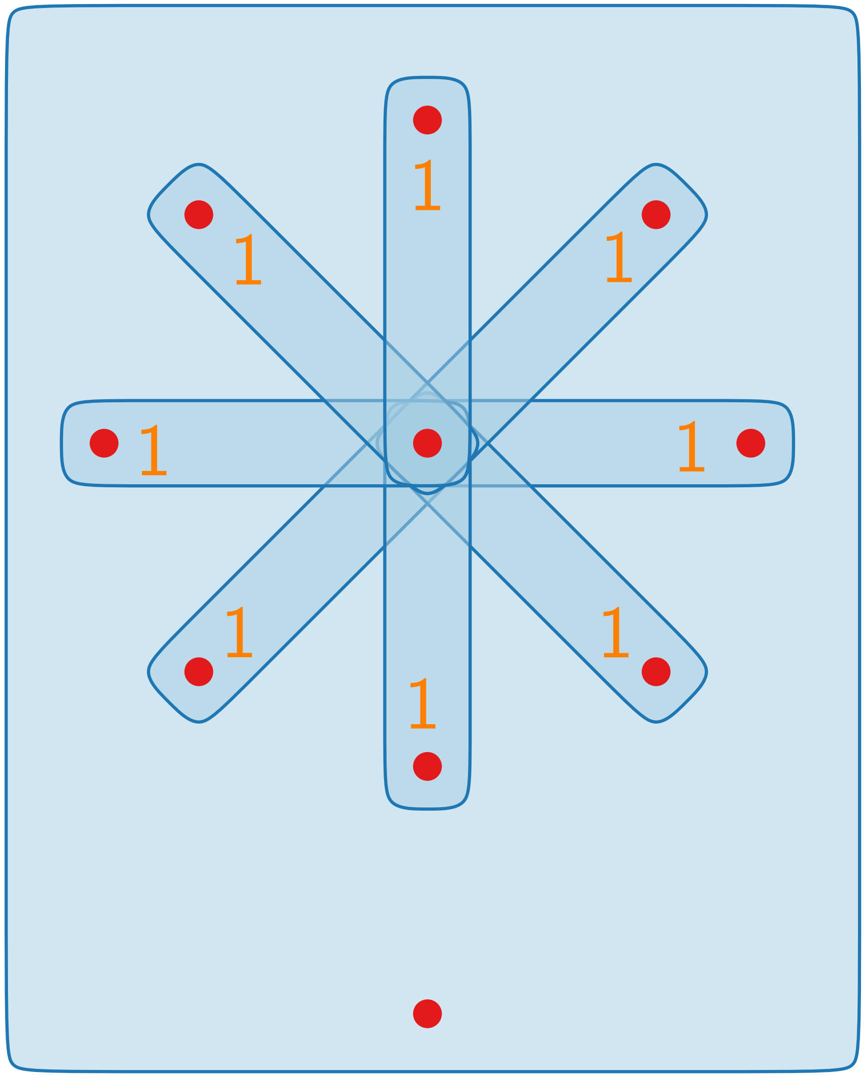

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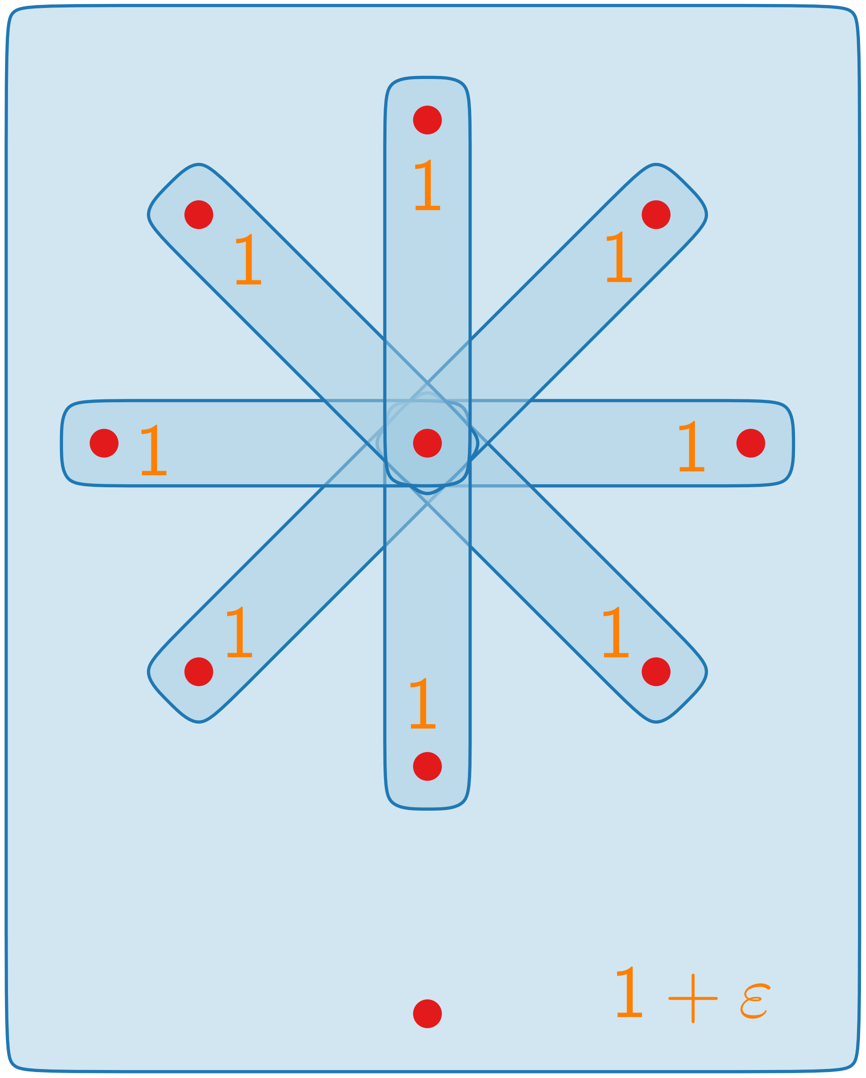

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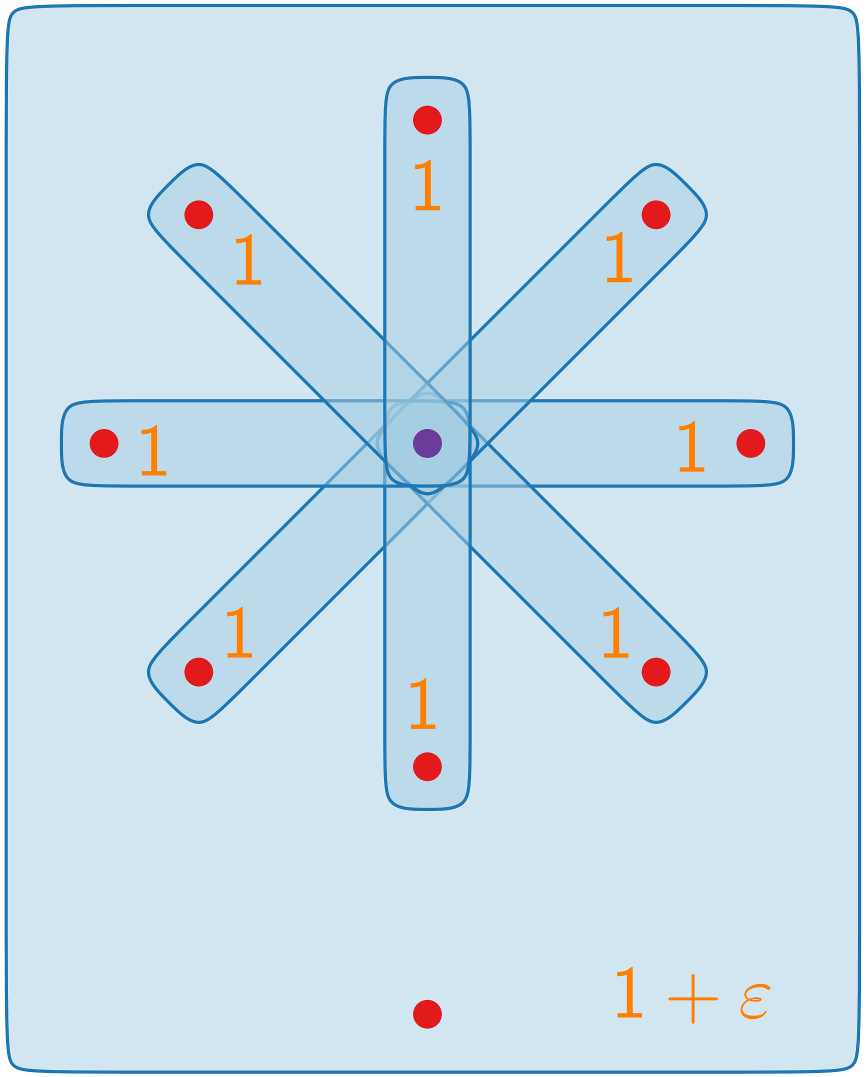

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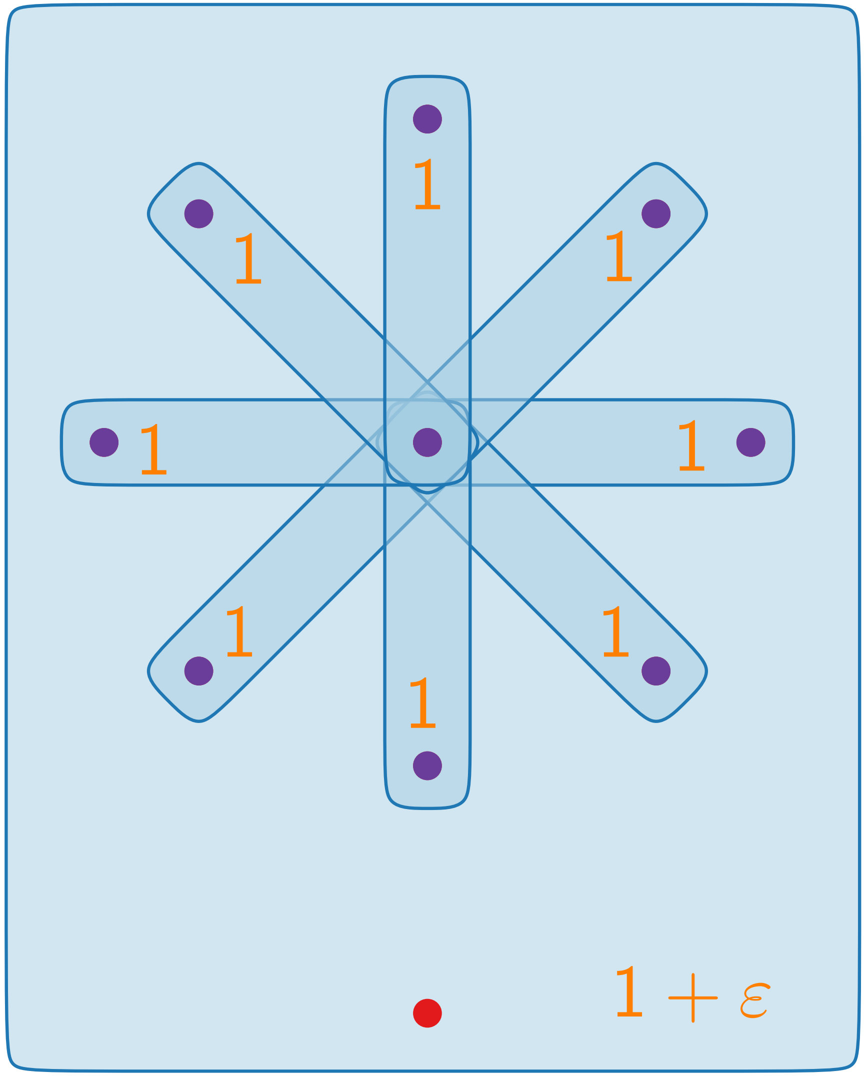

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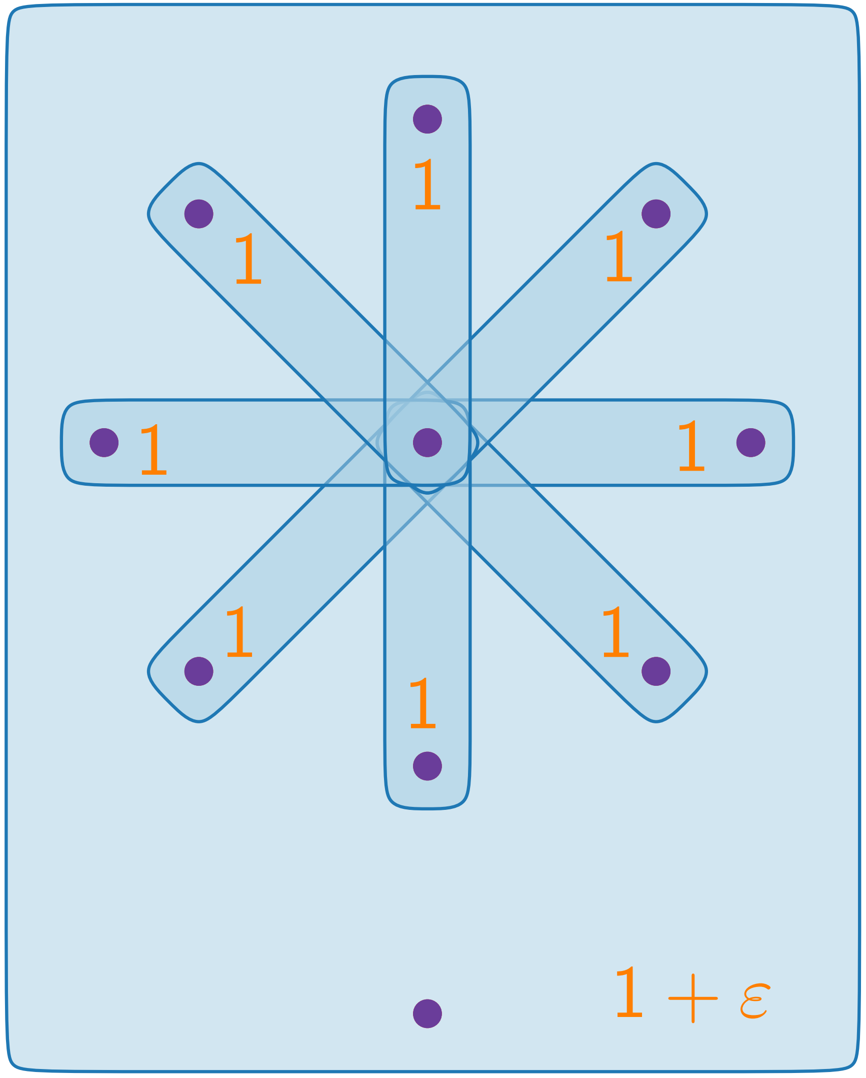
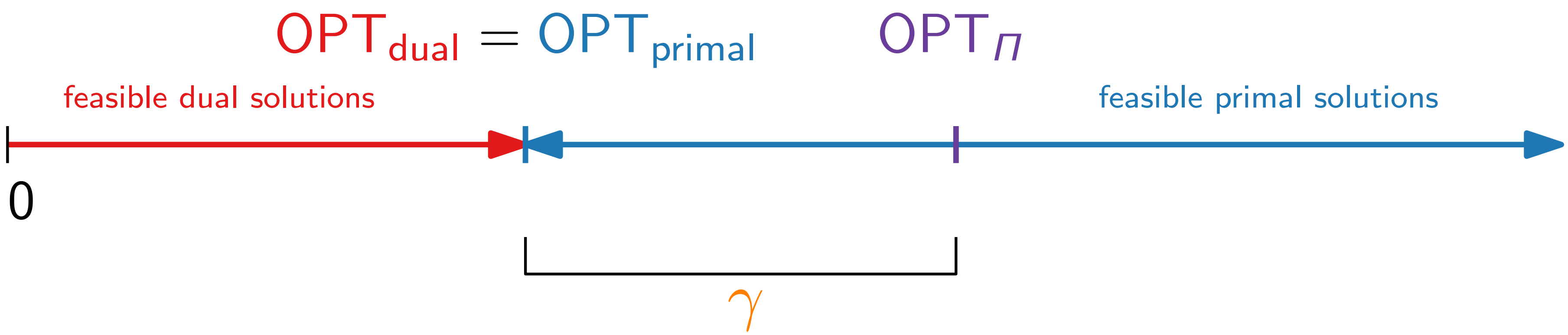
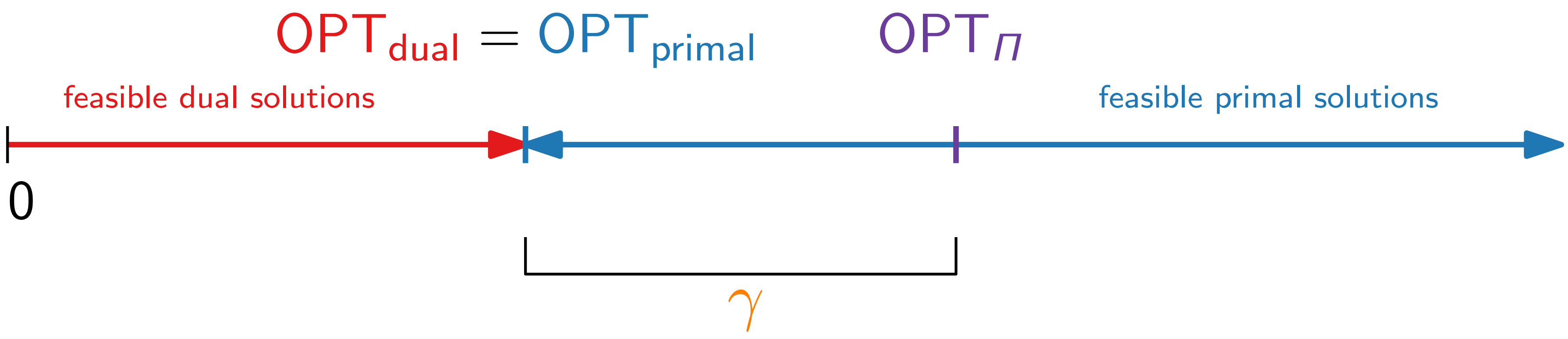


Integrality Gap



Consider a minimization problem Π in ILP form.

Integrality Gap



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Dual methods (without outside help) are limited by the *integrality gap* of the LP-relaxation:

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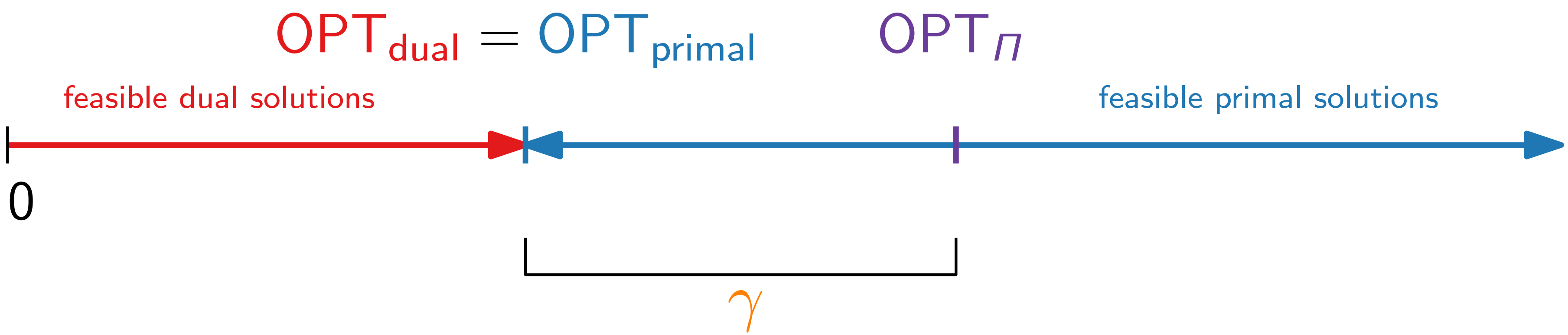

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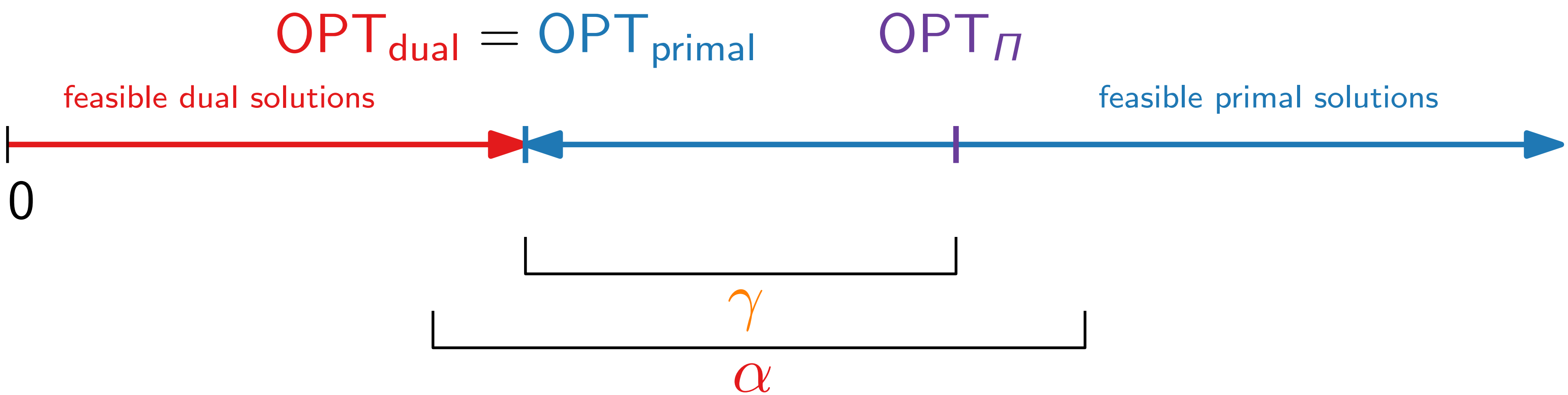

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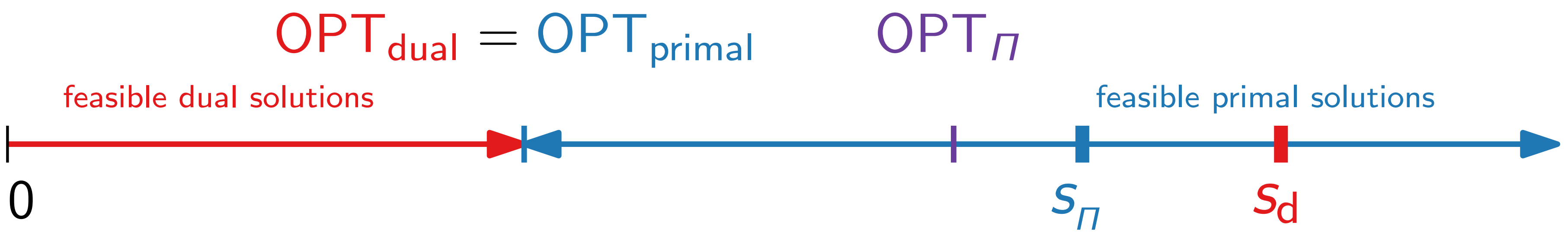
Approximation Algorithms

Lecture 5:
LP-based Approximation Algorithms for SetCover

Part IV:
Dual Fitting

Technique III) Dual Fitting



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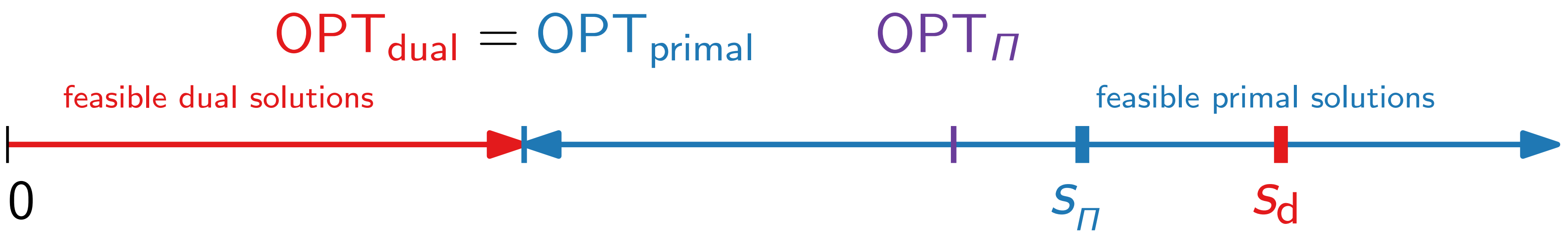

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Combinatorial algorithm (e.g., greedy) computes feasible primal solution s_{Π} and infeasible dual solution s_{d} that completely "pays" for s_{Π},

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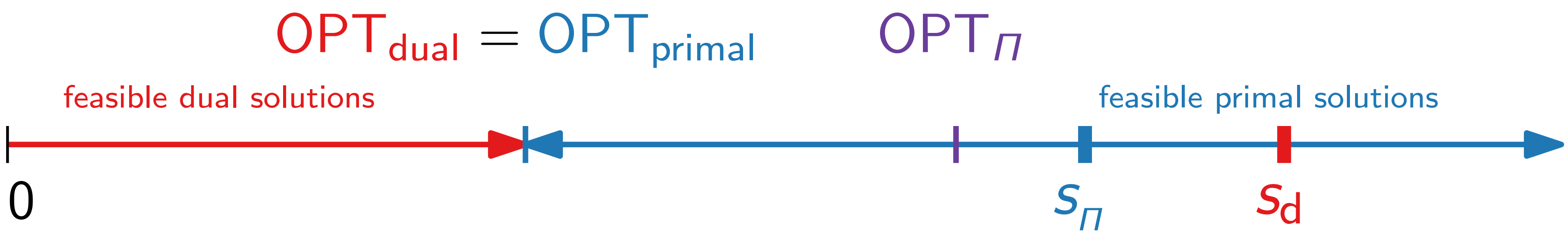

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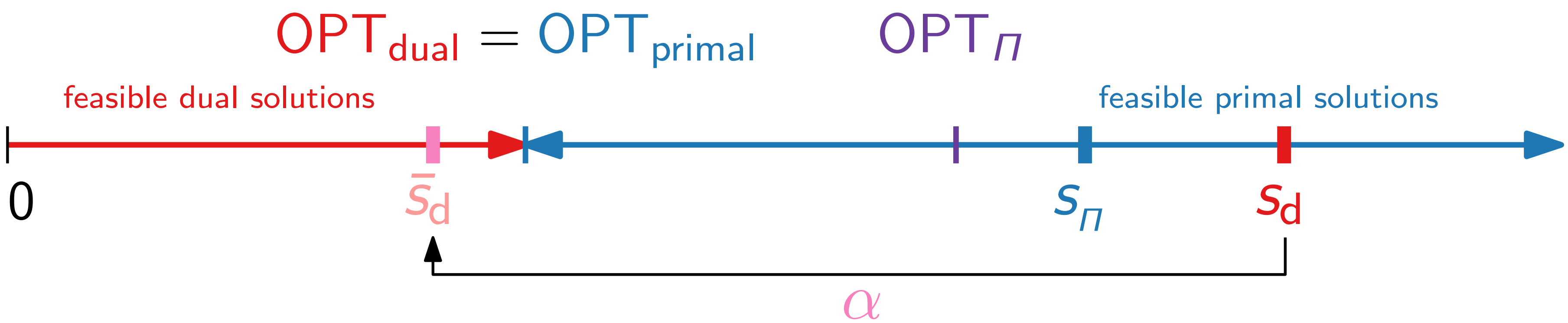

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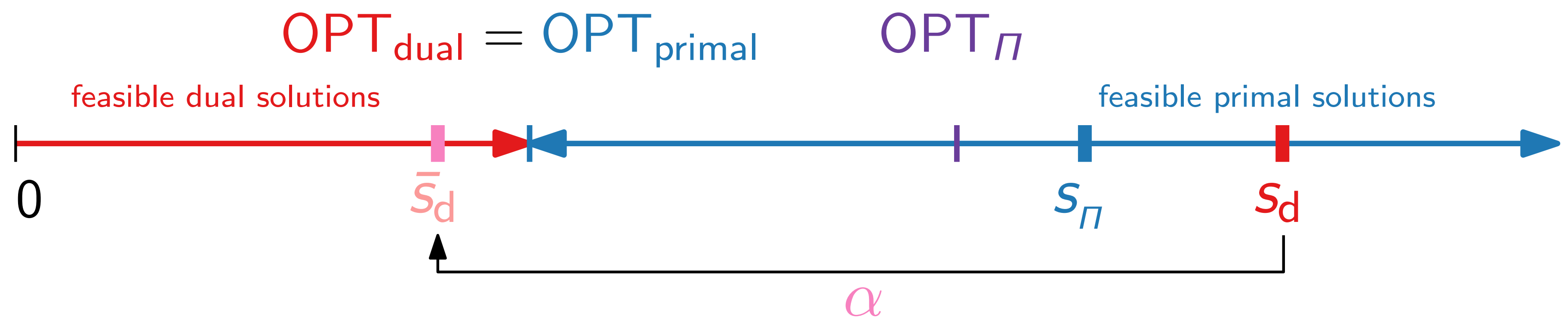

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$$\Rightarrow \qquad obj(\bar{s}_d) \leq OPT_{dual} \leq OPT_{\Pi}$$

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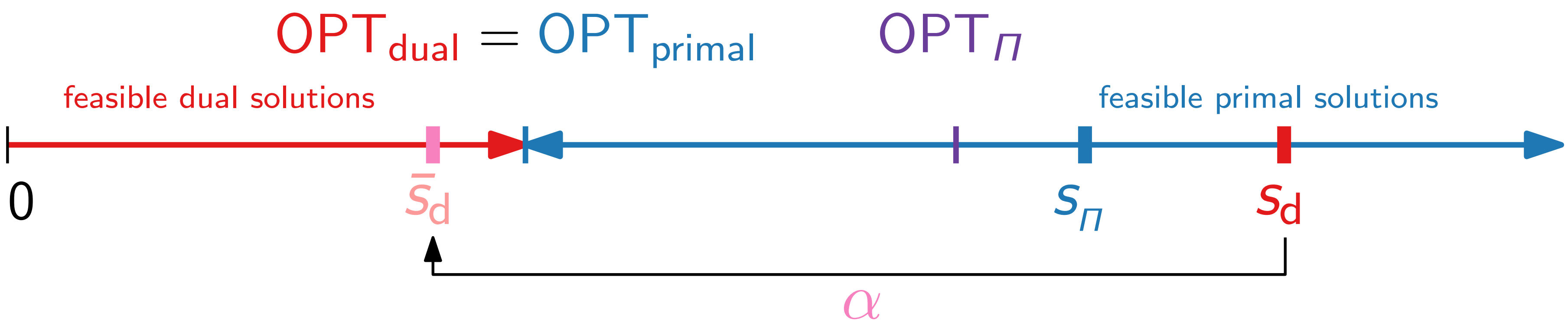

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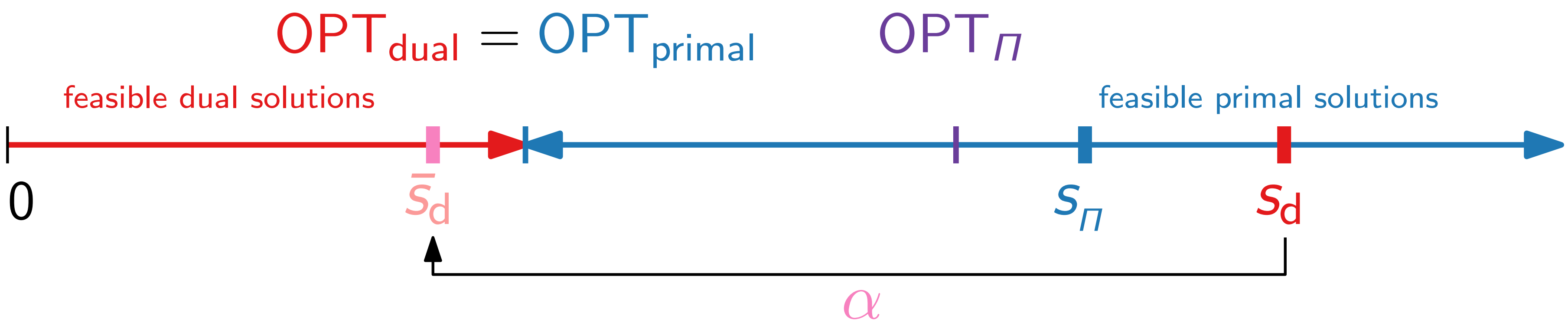

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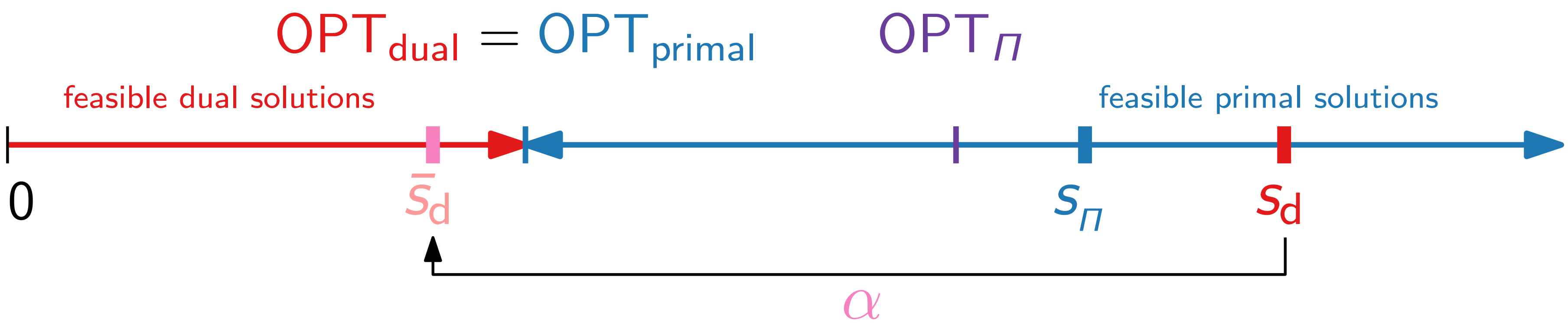

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$\Rightarrow$ Scaling factor α is approximation factor :-)

Dual Fitting for SetCover

Combinatorial (greedy) algorithm (see Lecture #2):

GreedySetCover(universe U, $\mathcal{S} \subseteq 2^U$, costs $c \colon \mathcal{S} \to \mathbb{Q}_{\geq 0}$)

$C \leftarrow \emptyset$
$\mathcal{S}' \leftarrow \emptyset$
while $C \neq U$ **do**
  $S \leftarrow$ set from $\mathcal{S}$ that minimizes $\frac{c(S)}{|S \setminus C|}$
  foreach $u \in S \setminus C$ **do**
    price$(u) \leftarrow \frac{c(S)}{|S \setminus C|}$
  $C \leftarrow C \cup S$
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return $\mathcal{S}'$ // Cover of U

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Reminder: $\sum_{u \in U}$ price(u) ...

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Reminder: $\sum_{u \in U} \text{price}(u)$ completely pays for $\mathcal{S}'$.

New: LP-based Analysis

Observation. For each $u \in U$, $\text{price}(u)$ is a dual variable

$$
\begin{aligned}
\textbf{maximize} \quad & \sum_{u \in U} y_u \\
\textbf{subject to} \quad & \sum_{u \in S} y_u \le c_S \quad && \forall S \in \mathcal{S} \\
& y_u \ge 0 && \forall u \in U
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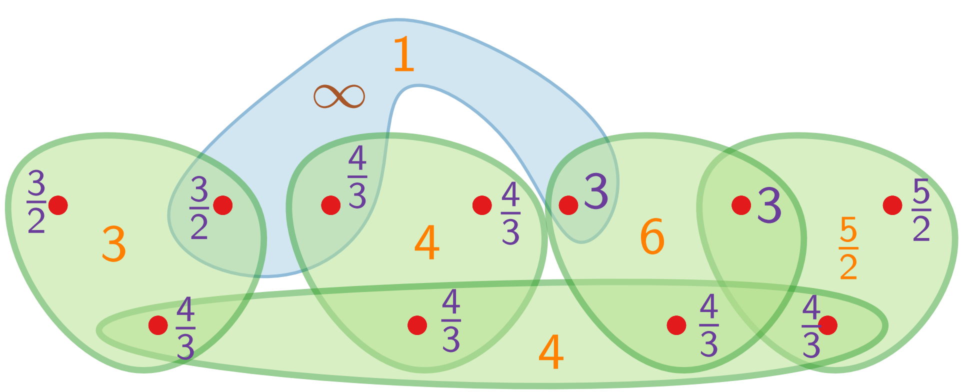

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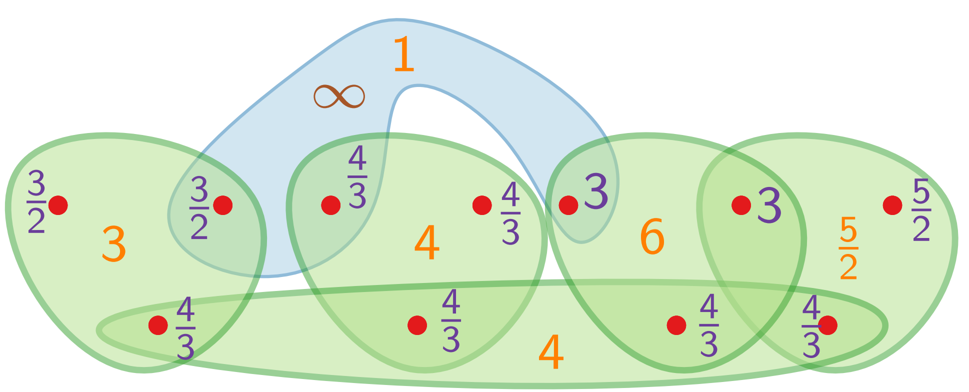

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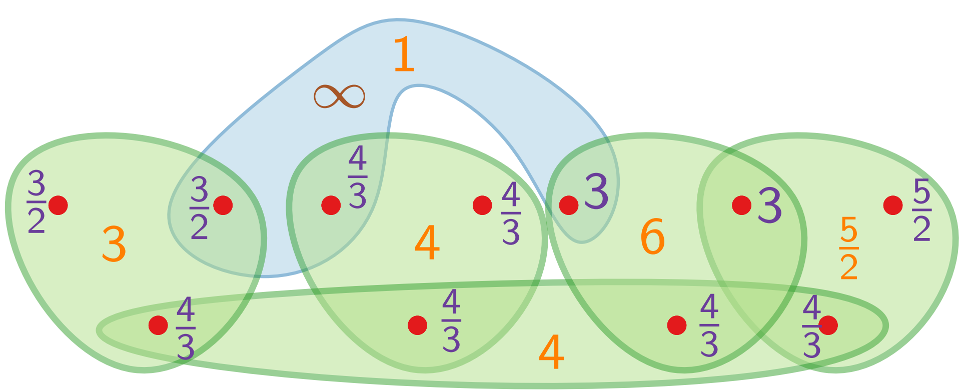
Scale dual variables such that no set is overpacked.



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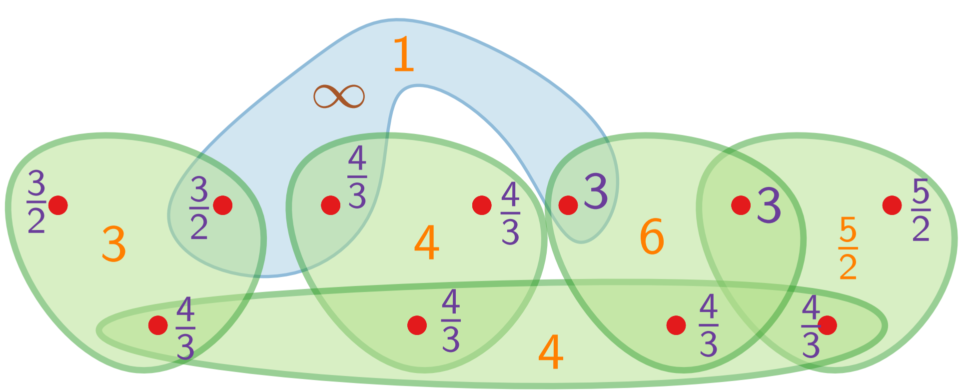

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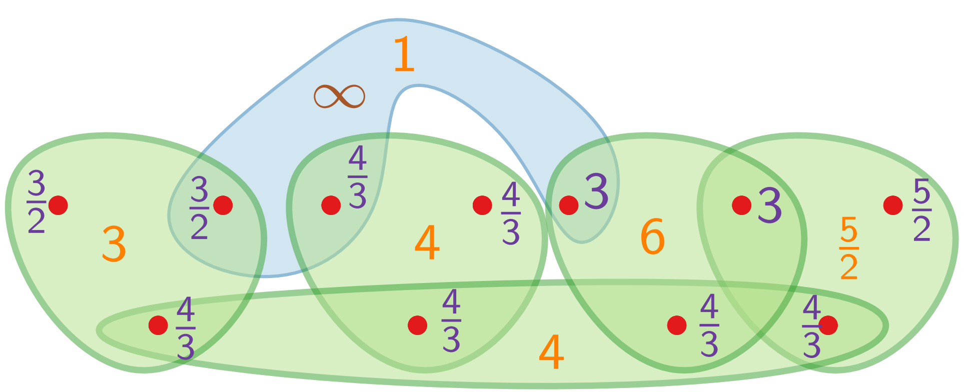

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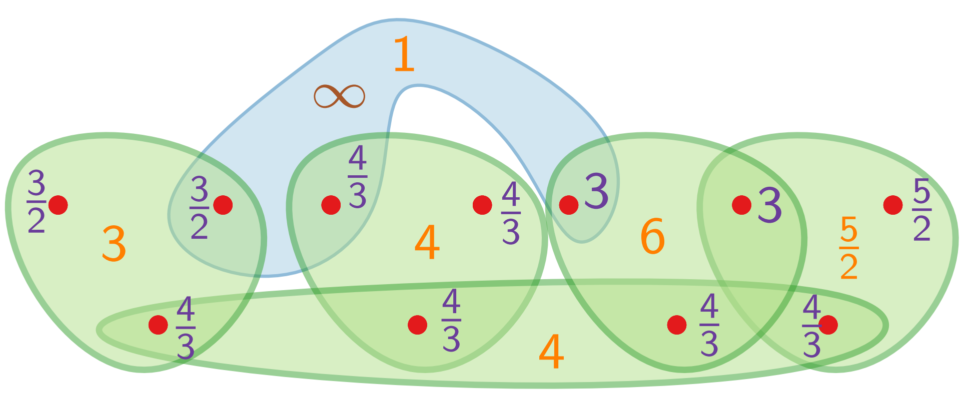

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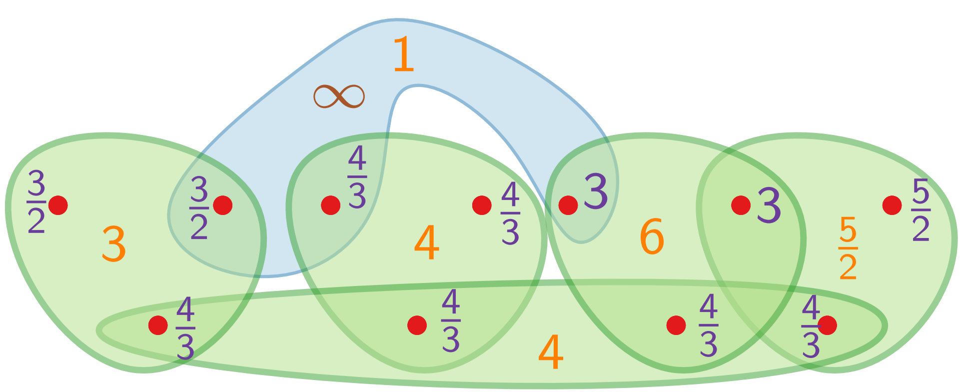

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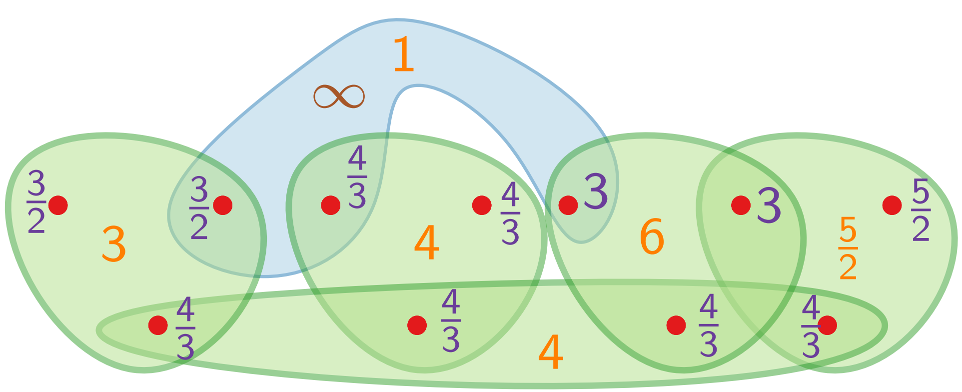

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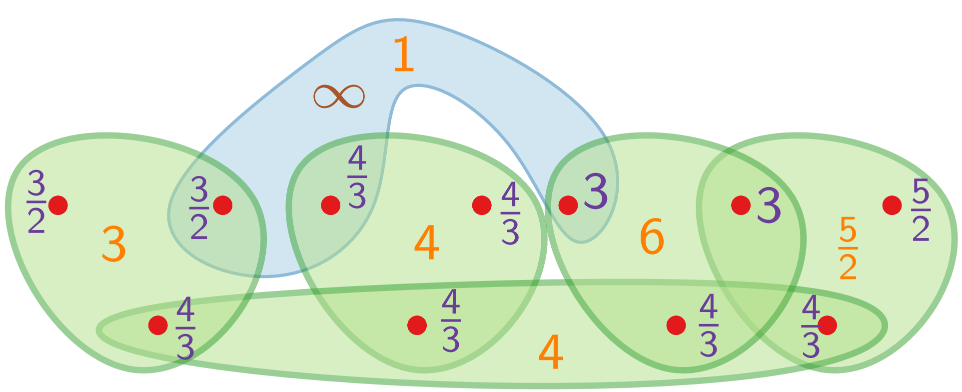

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