Approximation Algorithms

Lecture 4:
Linear Programming and LP-Duality

Part I:
Introduction to Linear Programming

Alexander Wolff

Winter 2024/25

Maximizing Profits

You're the boss of a small company that produces two products P_1 and P_2. For the production of x_1 units of P_1 and x_2 units of P_2, your profit in € is:

$$G(x_1, x_2) = 30x_1 + 50x_2$$

Three machines M_A, M_B and M_C produce the required components A, B and C for the products. The components are used in different quantities for the products, and each machine requires some time for the production.

$$\begin{aligned} M_A: &\quad 4x_1 + 11x_2 \leq 880 \\ M_B: &\quad x_1 + x_2 \leq 150 \\ M_C: &\quad x_2 \leq 60 \end{aligned}$$

Which choice of (x_1, x_2) maximizes the profit?

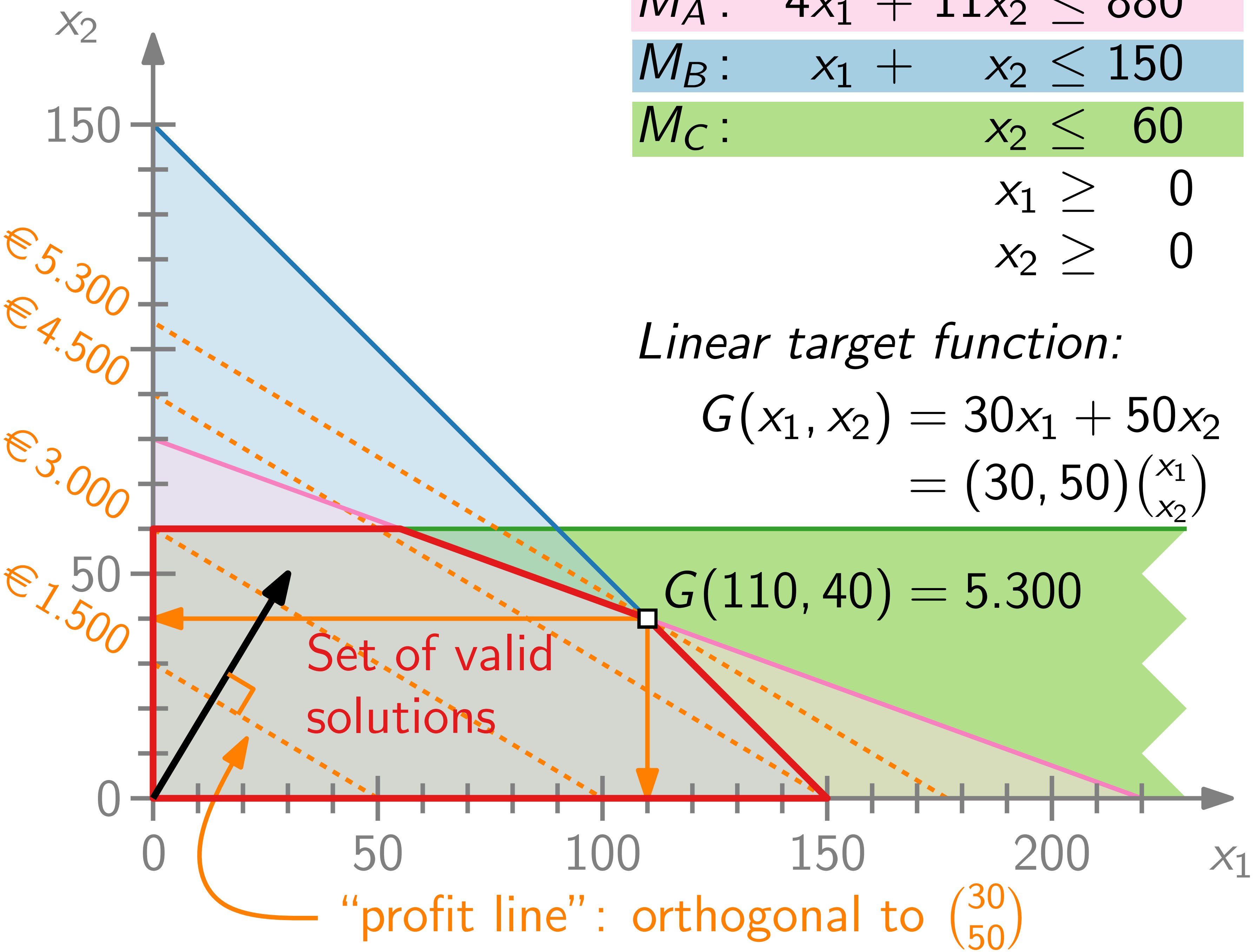
Solution

Linear constraints:

$$\begin{aligned} M_A: \quad 4x_1 + 11x_2 &\leq 880 \\ M_B: \quad x_1 + x_2 &\leq 150 \\ M_C: \quad x_2 &\leq 60 \\ x_1 &\geq 0 \\ x_2 &\geq 0 \end{aligned}$$

Linear target function:

$$\begin{aligned} G(x_1, x_2) &= 30x_1 + 50x_2 \\ &= (30, 50)\binom{x_1}{x_2} \end{aligned}$$



Approximation Algorithms

Lecture 4:
Linear Programming and LP-Duality

Part II:
Upper Bounds for LPs

Motivation: Upper and Lower Bounds

Consider an NP-hard minimization problem.

Decision Problem:
Is a given U an **upper bound** on OPT?

A feasible sol. S provides efficiently verifiable "yes"-certificate.

Lower bounds / "no"-certificates?
$\leadsto$ probably not! (conjecture: NP $\neq$ coNP)

For an approximation algorithm, we need a lower bound $L \geq \mathrm{OPT}/\alpha$ (i.e., an approximate "no"-certificate)!

Examples:

- Vertex Cover: lower bound by matchings
- TSP: lower bound by MST or by cycle cover

Linear Programming

Optimize (i.e., minimize or maximize) a linear (*objective*) function subject to linear inequalities (*constraints*).

$$\begin{array}{llcl} \textbf{minimize} & c^{\mathsf{T}} x & & \\ \textbf{subject to} & Ax & \geq & b \\ & x & \geq & 0 \end{array} \qquad \textit{standard form}$$

Example. $c = \begin{pmatrix} 7 \\ 1 \\ 5 \end{pmatrix}$ $A = \begin{pmatrix} 1 & -1 & 3 \\ 5 & 2 & -1 \end{pmatrix}$ $b = \begin{pmatrix} 10 \\ 6 \end{pmatrix}$

$$\begin{array}{lrcrcrcr} \textbf{minimize} & 7x_1 & + & x_2 & + & 5x_3 & & \\ \textbf{subject to} & x_1 & - & x_2 & + & 3x_3 & \geq & 10 \\ & 5x_1 & + & 2x_2 & - & x_3 & \geq & 6 \\ & & & & & x_1, x_2, x_3 & \geq & 0 \end{array}$$

Linear Programming – Upper Bounds

Optimize (i.e., minimize or maximize) a linear (*objective*) function subject to linear inequalities (*constraints*).

$$\begin{array}{lrcrcrcrr}
\textbf{minimize} & 7x_1\ 14 & + & x_2\ 1 & + & 5x_3\ 15 & = & & 30 \\
\textbf{subject to} & x_1\ 2 & - & x_2\ 1 & + & 3x_3\ 9 & \geq & 10 & 10 \\
 & 5x_1\ 10 & + & 2x_2\ 2 & - & x_3\ 3 & \geq & 6 & 9 \\
 & & & & & x_1, x_2, x_3 & \geq & 0 &
\end{array}$$

Valid solution?

$x = (2, 1, 3)$

$\Rightarrow$ obj$(x) = 30$ is upper bound for OPT

Approximation Algorithms

Lecture 4:
Linear Programming and LP-Duality

Part III:
Lower Bounds for LPs

Linear Programming – Lower Bounds

Optimize (i.e., minimize or maximize) a linear (*objective*) function subject to linear inequalities (*constraints*).

$$
\begin{array}{lrcrcrcr}
\textbf{minimize} & 7x_1 & + & x_2 & + & 5x_3 & & \\
\textbf{subject to} & 2\cdot\ x_1 & - & 2\cdot\ x_2 & + & 2\cdot 3x_3 & \geq & 2\cdot 10 \\
 & 5x_1 & + & 2x_2 & - & x_3 & \geq & 6 \\
 & & & & & x_1, x_2, x_3 & \geq & 0
\end{array}
$$

$$7x_1 + x_2 + 5x_3 \geq x_1 - x_2 + 3x_3 \quad \Rightarrow \mathrm{OPT} \geq 10$$

$$
\begin{aligned}
7x_1 + x_2 + 5x_3 &\geq (x_1 - x_2 + 3x_3) + (5x_1 + 2x_2 - x_3) \\
&\geq 10 + 6 \qquad \Rightarrow \mathrm{OPT} \geq 16
\end{aligned}
$$

$$
\begin{aligned}
7x_1 + x_2 + 5x_3 &\geq 2 \cdot (x_1 - x_2 + 3x_3) + (5x_1 + 2x_2 - x_3) \\
&\geq 2 \cdot 10 + 6 \qquad \Rightarrow \mathrm{OPT} \geq 26
\end{aligned}
$$

Linear Programming – Lower Bounds

$$
\begin{array}{lrcrcrcl}
\textbf{minimize} & 7x_1 & + & x_2 & + & 5x_3 & & \\
\textbf{subject to}\; y_1(& x_1 & - & x_2 & + & 3x_3\,) & \geq & 10\; y_1 \\
y_2(& 5x_1 & + & 2x_2 & - & x_3\,) & \geq & 6\; y_2 \\
 & & & & & x_1, x_2, x_3 & \geq & 0
\end{array}
$$

$$
\begin{aligned}
7x_1 + x_2 + 5x_3 &\geq y_1 \cdot (x_1 - x_2 + 3x_3) + y_2 \cdot (5x_1 + 2x_2 - x_3) \\
&\geq y_1 \cdot 10 + y_2 \cdot 6 \;\Rightarrow\; \mathrm{OPT} \geq 10y_1 + 6y_2
\end{aligned}
$$

$10y_1 + 6y_2$ is lower bound for OPT

Linear Programming – Lower Bounds

$$
\begin{array}{rrcrcrl}
\textbf{minimize} & 7x_1 & + & x_2 & + & 5x_3 & \\
 & \rotatebox{90}{$\le$} & & \rotatebox{90}{$\le$} & & \rotatebox{90}{$\le$} & \\
\textbf{subject to}\ y_1(& x_1 & - & x_2 & + & 3x_3\ ) & \geq\ 10\ y_1 \\
 & + & & + & & + & \\
y_2(& 5x_1 & + & 2x_2 & - & x_3\ ) & \geq\ 6\ y_2 \\
 & & & & & x_1, x_2, x_3 & \geq\ 0
\end{array}
$$

$$
\begin{aligned}
7x_1 + x_2 + 5x_3 &\geq y_1 \cdot (x_1 - x_2 + 3x_3) + y_2 \cdot (5x_1 + 2x_2 - x_3) \\
&\geq y_1 \cdot 10 + y_2 \cdot 6 \Rightarrow \text{OPT} \geq 10y_1 + 6y_2
\end{aligned}
$$

maximize

Bounds for y_1, y_2:

$$
\begin{array}{rcrcl}
y_1 & + & 5y_2 & \leq & 7 \\
-y_1 & + & 2y_2 & \leq & 1 \\
3y_1 & - & y_2 & \leq & 5 \\
 & & y_1, y_2 & \geq & 0
\end{array}
$$

Primal–Dual

primal program

$$\begin{array}{llll} \textbf{minimize} & c^\intercal x & & \\ \textbf{subject to} & Ax & \geq & b \\ & x & \geq & 0 \end{array}$$

dual program

$$\begin{array}{llll} \textbf{maximize} & b^\intercal y & & \\ \textbf{subject to} & A^\intercal y & \leq & c \\ & y & \geq & 0 \end{array}$$

dual of the dual program

$$\begin{array}{llll} \textbf{minimize} & c^\intercal x & & \\ \textbf{subject to} & Ax & \geq & b \\ & x & \geq & 0 \end{array}$$

Approximation Algorithms

Lecture 4:
Linear Programming and LP-Duality

Part IV:
LP-Duality and Complementary Slackness

LP-Duality

$$\begin{array}{lrcl} \textbf{minimize} & c^\intercal x & & \\ \textbf{subject to} & Ax & \geq & b \\ & x & \geq & 0 \end{array}$$

Primal

$$\begin{array}{lrcl} \textbf{maximize} & b^\intercal y & & \\ \textbf{subject to} & A^\intercal y & \leq & c \\ & y & \geq & 0 \end{array}$$

Dual

Theorem. The primal program has a finite optimum $\Leftrightarrow$ the dual program has a finite optimum. Moreover, if $x^* = (x_1^*, \ldots, x_n^*)$ and $y^* = (y_1^*, \ldots, y_m^*)$ are *optimal* solutions for the primal and dual program, respectively, then

$$\sum_{j=1}^{n} c_j x_j^* = \sum_{i=1}^{m} b_i y_i^* .$$

Weak LP-Duality

$$\begin{array}{ll} \textbf{minimize} & c^\intercal x \\ \textbf{subject to} & \boxed{Ax \geq b} \\ & \boxed{x \geq 0} \end{array} \qquad \begin{array}{ll} \textbf{maximize} & b^\intercal y \\ \textbf{subject to} & \boxed{A^\intercal y \leq c} \\ & \boxed{y \geq 0} \end{array}$$

Theorem. If $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_m)$ are *valid* solutions for the primal and dual program, resp., then

$$\sum_{j=1}^{n} c_j x_j \geq \sum_{i=1}^{m} b_i y_i \, .$$

Proof.

$$\sum_{j=1}^{n} \boxed{c_j} x_j \overset{\geq}{\geq} \sum_{j=1}^{n} \boxed{\left(\sum_{i=1}^{m} a_{ij} y_i \right)} x_j = \sum_{i=1}^{m} \left(\boxed{\sum_{j=1}^{n} a_{ij} x_j} \right) y_i \overset{\geq}{\geq} \sum_{i=1}^{m} \boxed{b_i} y_i \, .$$

Complementary Slackness

$$\begin{array}{ll} \textbf{minimize} & c^\intercal x \\ \textbf{subject to} & \boxed{Ax \geq b} \\ & \boxed{x \geq 0} \end{array} \qquad \begin{array}{ll} \textbf{maximize} & b^\intercal y \\ \textbf{subject to} & \boxed{A^\intercal y \leq c} \\ & \boxed{y \geq 0} \end{array}$$

Theorem. Let $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_m)$ be valid solutions for the primal and dual program, respectively. Then x and y are optimal if and only if the following conditions are met:

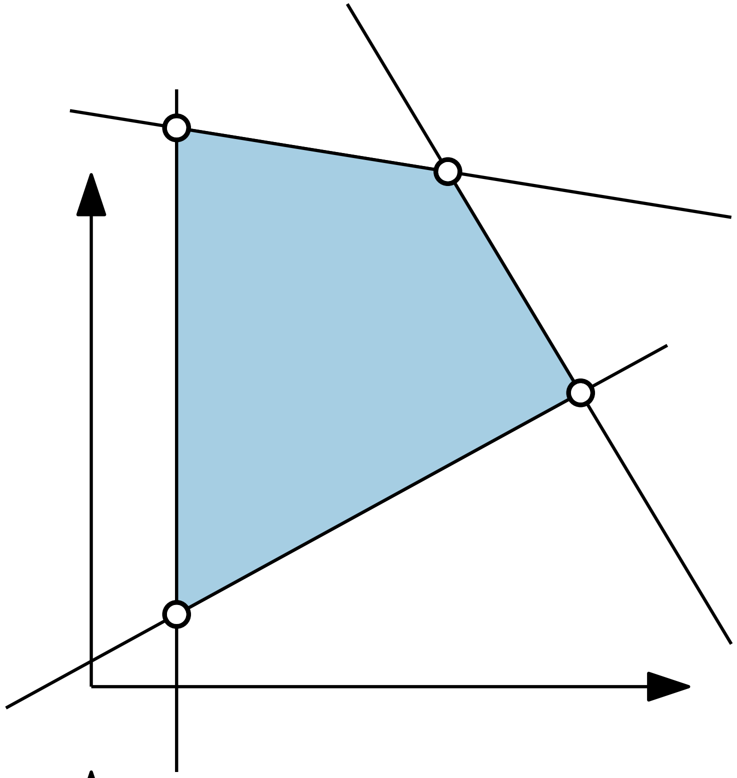
Primal CS:
For each $j = 1, \ldots, n$: $\quad x_j = 0 \quad$ or $\quad \sum_{i=1}^m a_{ij} y_i = c_j$

Dual CS:
For each $i = 1, \ldots, m$: $\quad y_i = 0 \quad$ or $\quad \sum_{j=1}^n a_{ij} x_j = b_i$

Proof. Follows from LP-duality: For every summond...

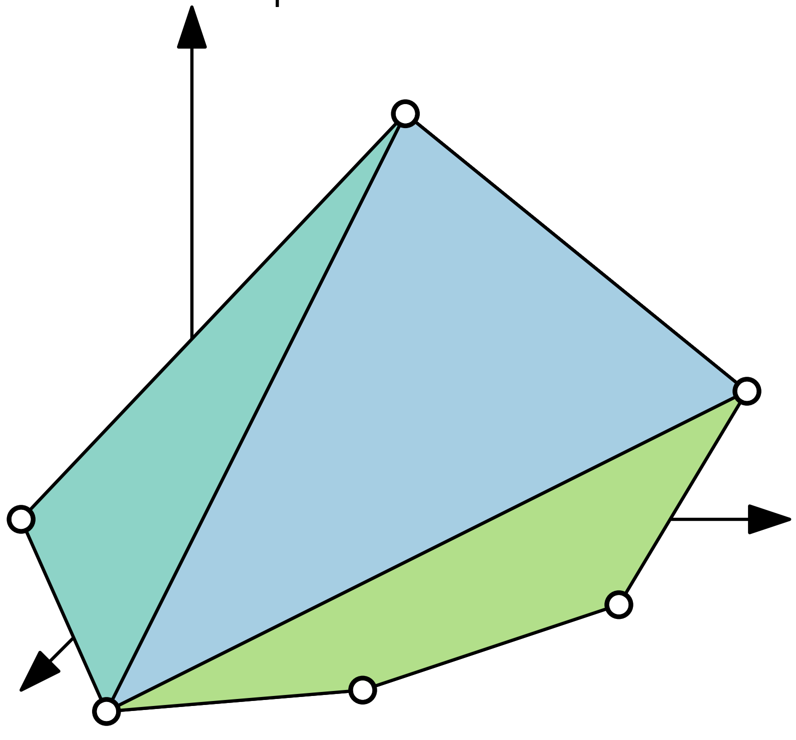
$$\sum_{j=1}^{n} \boxed{c_j x_j} \geq \sum_{j=1}^{n} \boxed{\left(\sum_{i=1}^{m} a_{ij} y_i \right) x_j} = \sum_{i=1}^{m} \boxed{\left(\sum_{j=1}^{n} a_{ij} x_j \right) y_i} \geq \sum_{i=1}^{m} \boxed{b_i y_i}.$$

LPs and Convex Polytopes



The feasible solutions of an LP with n variables form a **convex polytope** in $\mathbb{R}^n$ (intersection of halfspaces).

Corners of the polytope are called **extreme point solutions** $\Leftrightarrow$ n linearly independent inequalities (constraints) are satisfied with equality.



If an optimal solution exists, some extreme point is also optimal.

Integer Linear Programs (ILPs)

$$\begin{array}{lrcl} \textbf{minimize} & c^\intercal x & & \\ \textbf{subject to} & Ax & \geq & b \\ & x & \geq & 0 \end{array}$$

$$\begin{array}{lrcl} \textbf{minimize} & c^\intercal x & & \\ \textbf{subject to} & Ax & \geq & b \\ & x & \in & \mathbb{N} \end{array}$$

Many NP-optimization problems can be formulated as ILPs; thus ILPs are NP-hard to solve.

LP-relaxation provides a lower bound: $\mathrm{OPT}_{\mathrm{LP}} \leq \mathrm{OPT}_{\mathrm{ILP}}$

Approximation Algorithms

Lecture 4:
Linear Programming and LP-Duality

Part V:
Min–Max Relationships

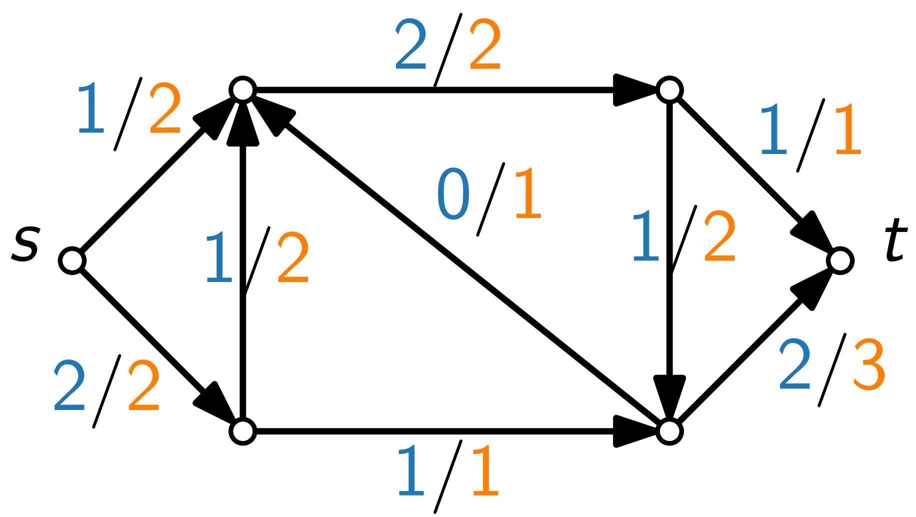
Max-Flow Problem

Given: A directed graph G with edge capacities $c\colon E(G) \to \mathbb{Q}_+$ and two special vertices: the source s and sink t.

Find: A maximum s–t flow (i.e., non-negative edge weights f) such that

- $f(u,v) \leq c(u,v)$ for each edge $(u,v) \in E(G)$,
- $\sum_{u\colon (u,v)\in E} f(u,v) = \sum_{z\colon (v,z)\in E} f(v,z)$ for each $v \in V(G) \setminus \{s,t\}$.

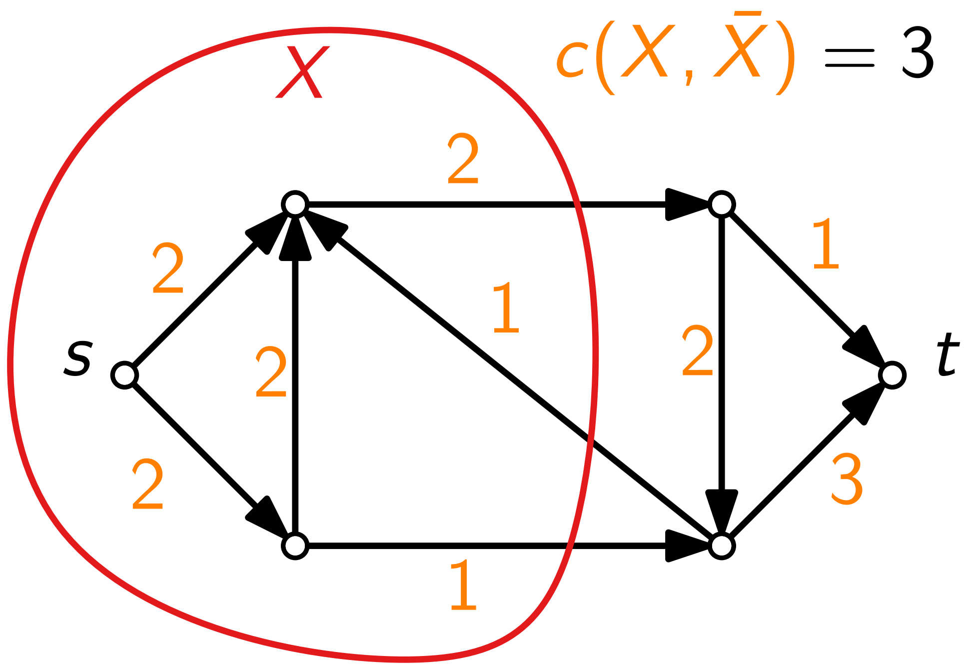
The **flow value** is the inflow to t minus the outflow from t.



Min-Cut Problem

Given: A directed graph G with edge capacities $c\colon E(G) \to \mathbb{Q}_+$ and two special vertices: the source s and sink t.

Find: An s–t cut, i.e., a vertex set X with $s \in X$ and $t \in \overline{X}$ such that the total weight $c(X, \overline{X})$ of the edges from X to $\overline{X}$ is minimum.



Max-Flow-Min-Cut Theorem

Theorem. The value of a maximum s–t flow and the weight of a minimum s–t cut are the same.

Proof. Special case of LP-Duality . . .

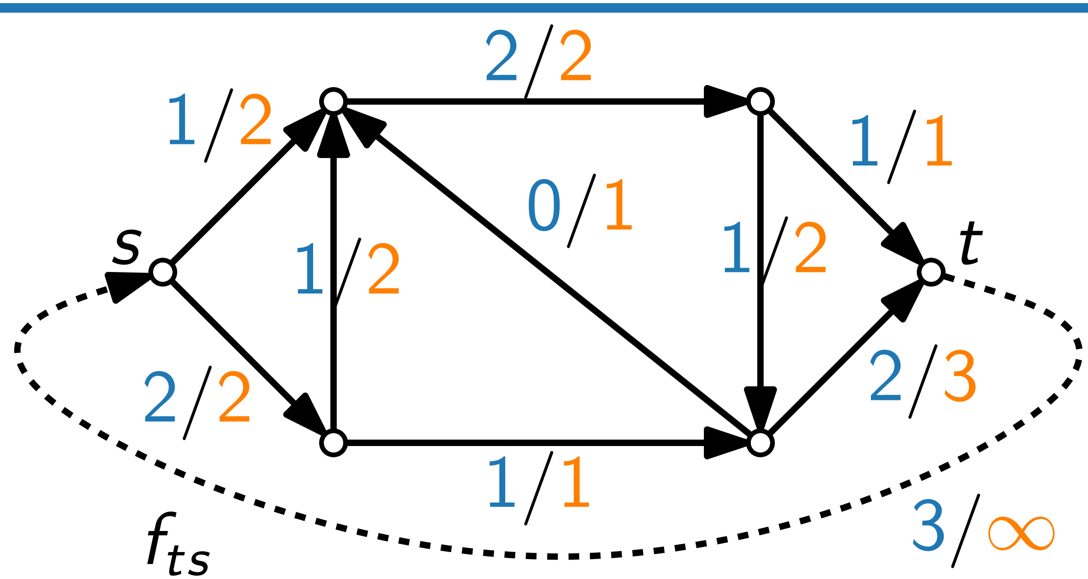
maximize f_{ts}

subject to

$$f_{uv} \leq c_{uv} \quad \forall (u,v) \neq (t,s)$$

$$\sum_{u\colon (u,v)\in E(G)} f_{uv} - \sum_{z\colon (v,z)\in E(G)} f_{vz} \leq 0 \quad \forall v \in V(G)$$

$$f_{uv} \geq 0 \quad \forall (u,v) \in E(G)$$



Max-Flow-Min-Cut Theorem

Theorem. The value of a maximum s–t flow and the weight of a minimum s–t cut are the same.

Proof. Special case of LP-Duality …

$$
\begin{array}{llll}
\textbf{maximize} & f_{ts} & & \\
\textbf{subject to} & f_{uv} \le c_{uv} & \forall (u,v) \ne (t,s) & d_{uv} \\
\displaystyle\sum_{u\colon (u,v)\in E(G)} f_{uv} - \sum_{z\colon (v,z)\in E(G)} & f_{vz} \le 0 & \forall v \in V(G) & p_v \\
& f_{uv} \ge 0 & \forall (u,v) \in E(G) &
\end{array}
$$

maximize $c^\intercal x \quad = \sum_{(u,v)\in E(G)} (0 \cdot f_{uv}) + 1 \cdot f_{ts} \quad \Rightarrow c^\intercal = (0, \dots, 0, 1)$

Which constraints contain f_{uv} for $(u,v) \ne (t,s)$? $\quad d_{uv},\ p_u,\ p_v$

$$\Rightarrow d_{uv} - p_u + p_v \ge 0$$

Which constraints contain f_{ts}? $\quad p_s,\ p_t$

$$\Rightarrow p_s - p_t \ge 1$$

Max-Flow-Min-Cut Theorem

Theorem. The value of a maximum s–t flow and the weight of a minimum s–t cut are the same.

Proof. Special case of LP-Duality . . .

$$\begin{array}{llrlll}
\textbf{maximize} & f_{ts} & & & & \\
\textbf{subject to} & & f_{uv} & \leq c_{uv} & \forall (u,v) \neq (t,s) & d_{uv} \\
& \displaystyle\sum_{u\colon (u,v)\in E(G)} f_{uv} - \sum_{z\colon (v,z)\in E(G)} & f_{vz} & \leq 0 & \forall v \in V(G) & p_v \\
& & f_{uv} & \geq 0 & \forall (u,v) \in E(G) &
\end{array}$$

$$\begin{array}{llrll}
\textbf{minimize} & \displaystyle\sum_{(u,v)\in E(G)\setminus\{(t,s)\}} c_{uv} \cdot d_{uv} & & & \\
\textbf{subject to} & & d_{uv} - p_u + p_v & \geq 0 & \forall (u,v) \in E(G) \setminus \{(t,s)\} \\
& & p_s - p_t & \geq 1 & \\
& & d_{uv} & \geq 0 & \forall (u,v) \in E(G) \\
& & p_u & \geq 0 & \forall u \in V(G)
\end{array}$$

Approximation Algorithms

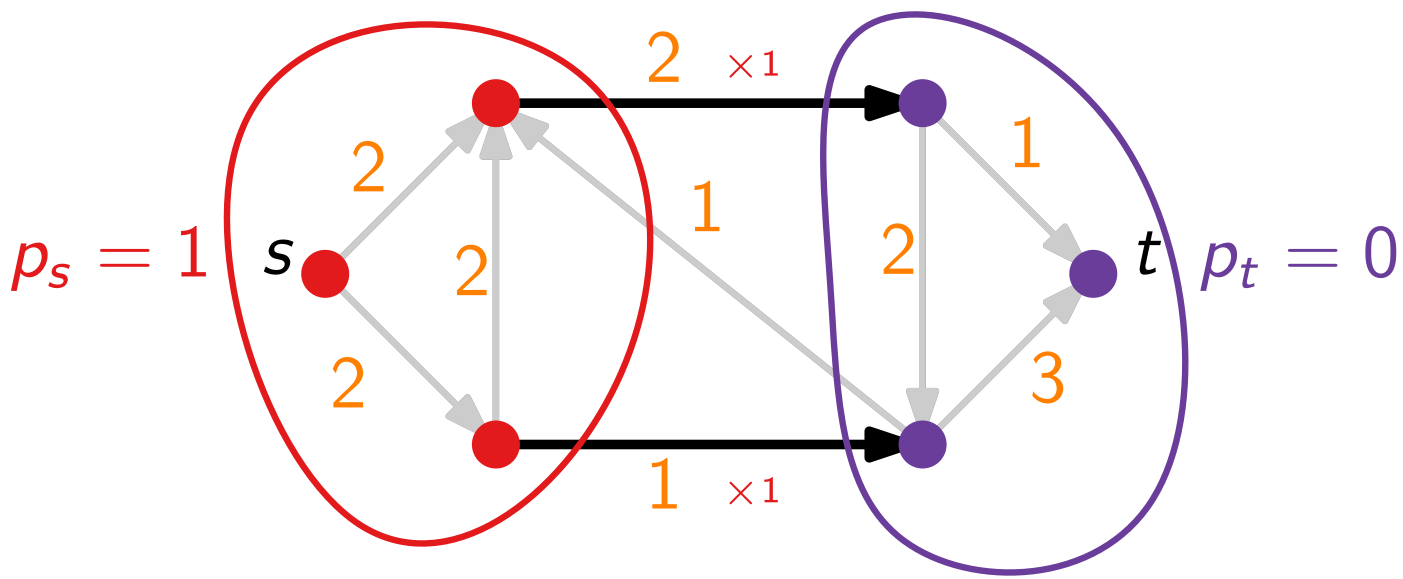
Lecture 4:
Linear Programming and LP-Duality

Part VI:
Dual LP of Max Flow

Dual LP – Interpretation as ILP

$$
\begin{array}{lrll}
\textbf{minimize} & \displaystyle\sum_{(u,v)\in E(G)\setminus\{(t,s)\}} c_{uv}\cdot d_{uv} & & \\
\textbf{subject to} & d_{uv} - p_u + p_v \geq 0 & & \forall (u,v) \neq (t,s) \\
& p_s - p_t \geq 1 & & \\
& d_{uv} \cancel{\geq 0} \in \{0,1\} & & \forall (u,v) \in E(G) \\
& p_u \cancel{\geq 0} \in \{0,1\} & & \forall u \in V(G)
\end{array}
$$

equivalent to Min-Cut!



Dual LP – Fractional Cuts

LP-relaxation of the ILP

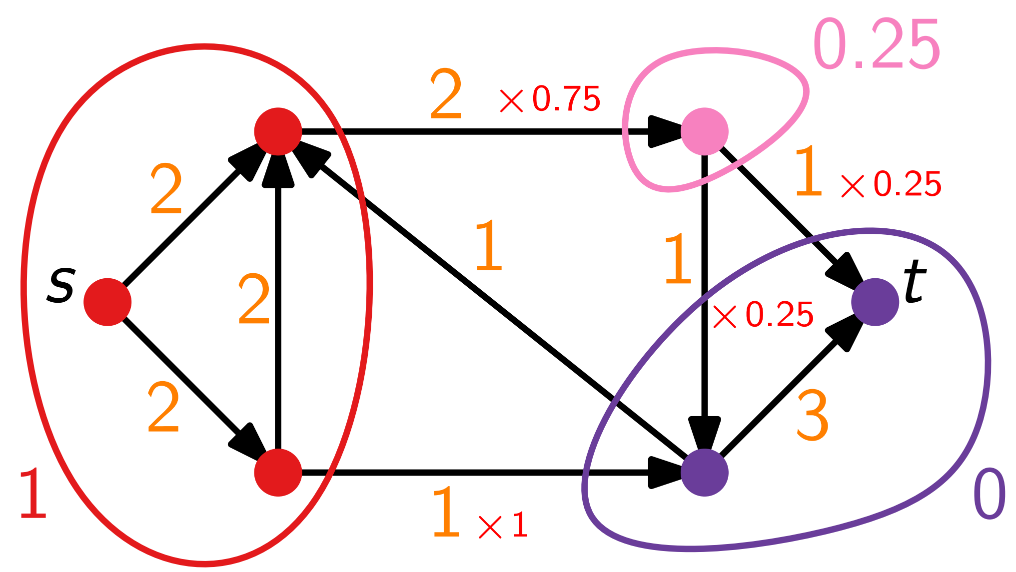
$$\begin{array}{lrll} \textbf{minimize} & \displaystyle\sum_{(u,v)\in E(G)\setminus\{(t,s)\}} c_{uv}\cdot d_{uv} & & \\ \textbf{subject to} & d_{uv}-p_u+p_v \ge 0 & \forall (u,v)\in E(G)\setminus\{(t,s)\} \\ & p_s-p_t \ge 1 & \\ & d_{uv} \ge 0 & \forall (u,v)\in E(G) \\ & p_u \ge 0 & \forall u\in V(G) \end{array}$$

Moreover, all extreme-point solutions of this polytope are **integral**! (HW)

Note that every s–t path $s=v_0,\dots,v_k=t$ has length ≥ 1 w.r.t. d:

$$\sum_{i=0}^{k-1} d_{i,i+1} \ge \sum_{i=0}^{k-1}(p_i-p_{i+1})$$

$$= p_s - p_t \ge 1$$



Dual LP – Complementary Slackness

$$\begin{array}{lrcll}
\textbf{maximize} & f_{ts} & & & \\
\textbf{subject to} & f_{uv} & \le & c_{uv} & \forall (u,v) \neq (t,s) \\
& \displaystyle\sum_{u:\,(u,v)\in E(G)} f_{uv} - \sum_{z:\,(v,z)\in E(G)} f_{vz} & \le & 0 & \forall v \in V(G) \\
& f_{uv} & \ge & 0 & \forall (u,v) \in E(G)
\end{array}$$

$$\begin{array}{lrcl}
\textbf{minimize} & \displaystyle\sum_{(u,v)\in E(G)\setminus\{(t,s)\}} c_{uv} \cdot d_{uv} & & \\
\textbf{subject to} & d_{uv} - p_u + p_v & \ge & 0 \\
& p_s - p_t & \ge & 1 \\
& d_{uv} & \ge & 0 \\
& p_u & \ge & 0
\end{array}$$

Primal CS:
$\forall j$: $x_j = 0$ or $\sum_{i=1}^{m} a_{ij} y_i = c_j$

Dual CS:
$\forall i$: $y_i = 0$ or $\sum_{j=1}^{n} a_{ij} x_j = b_i$

For a max flow and min cut:

- For each forward edge (u, v) of the cut: $f_{uv} = c_{uv}$.
  ($d_{uv} = 1$, so by dual CS: $f_{uv} = c_{uv}$.)
- For each backward edge (u, v) of the cut: $f_{uv} = 0$.
  (Otherwise, by primal CS: $d_{uv} - 0 + 1 = 0$.)