Approximation Algorithms

Lecture 4: Linear Programming and LP-Duality

Part I: Introduction to Linear Programming

Alexander Wolff

Winter 2024/25

Maximizing Profits

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Which choice of (x_1, x_2) maximizes the profit?

Solution

Linear constraints:

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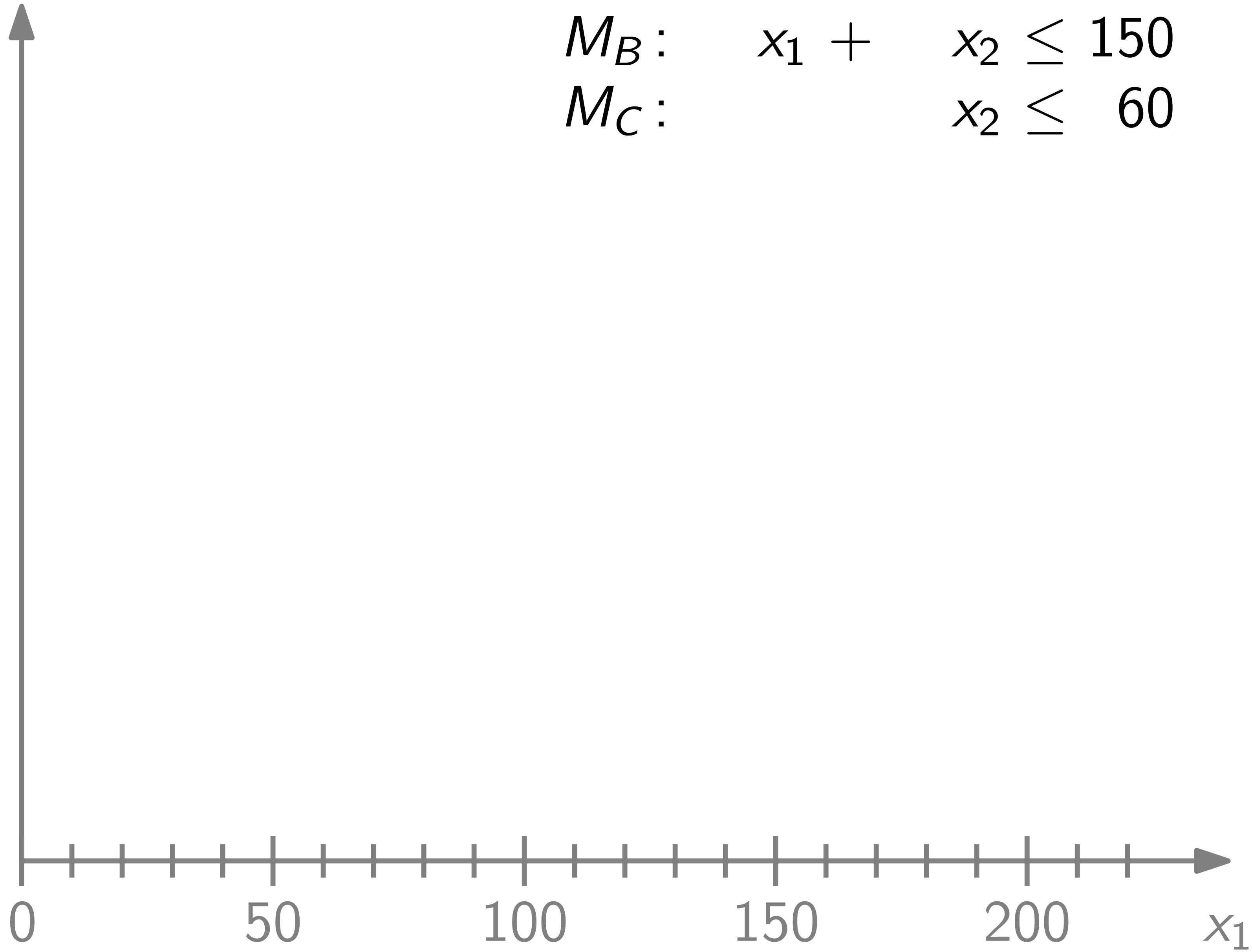

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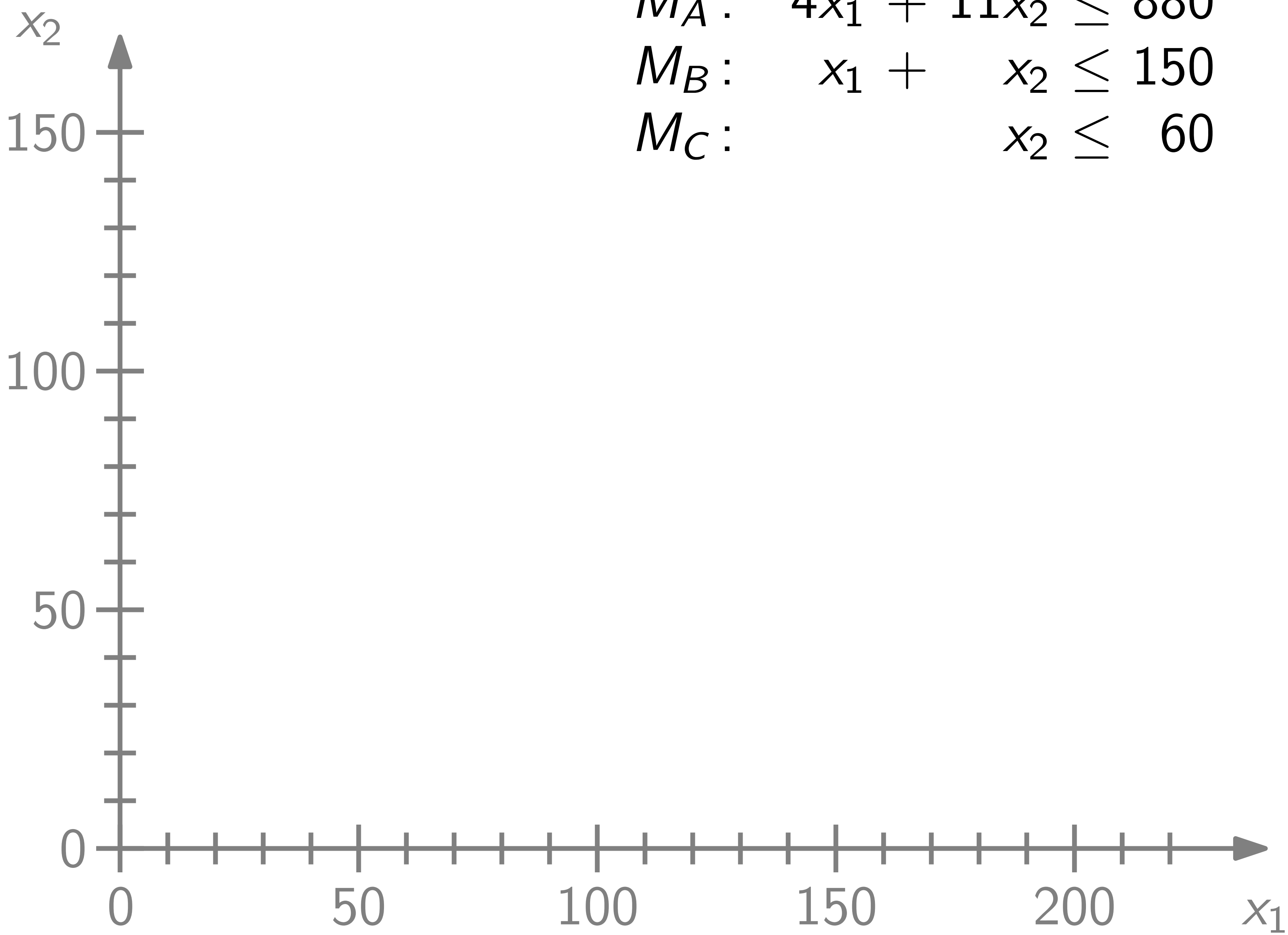

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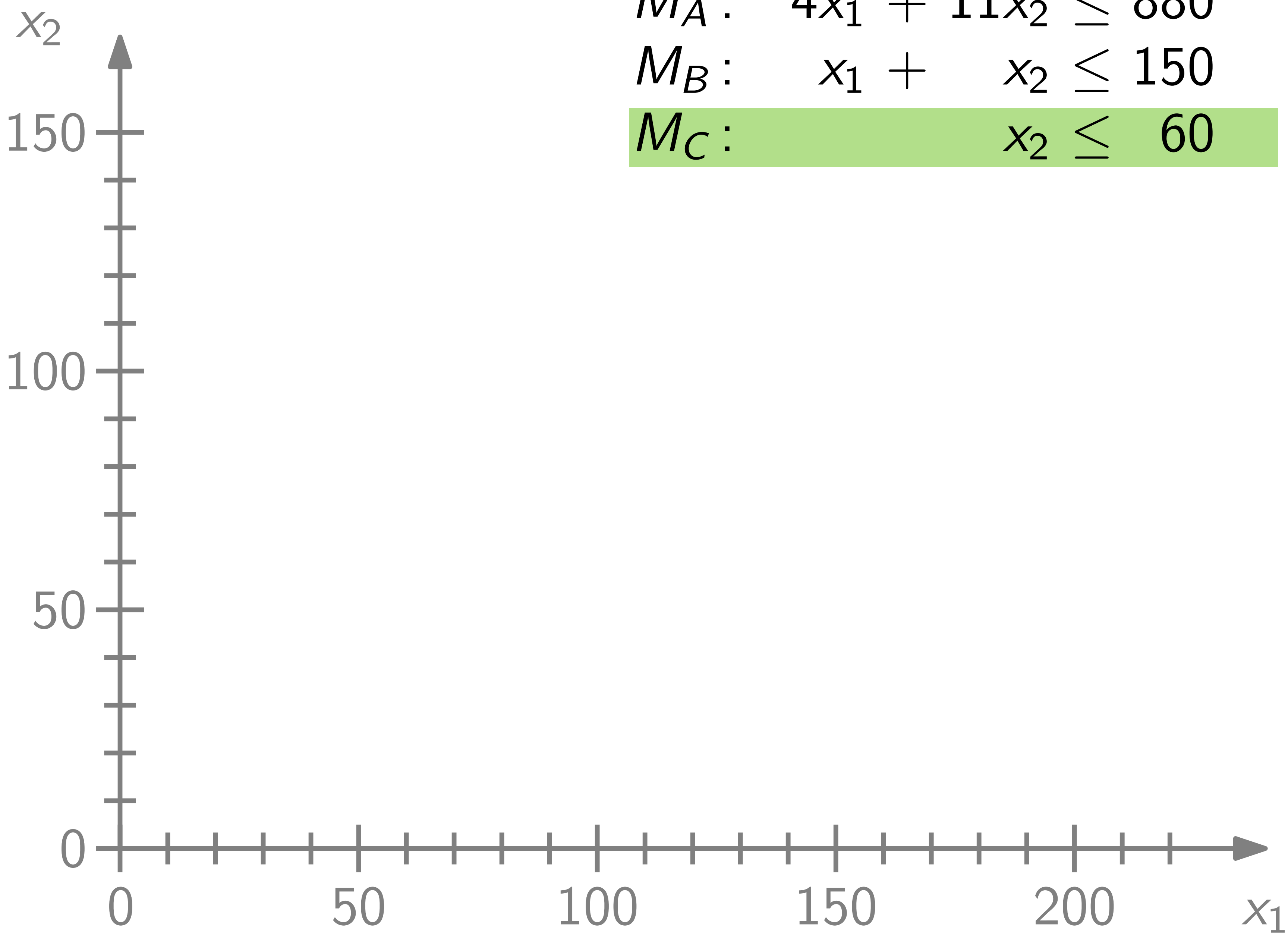

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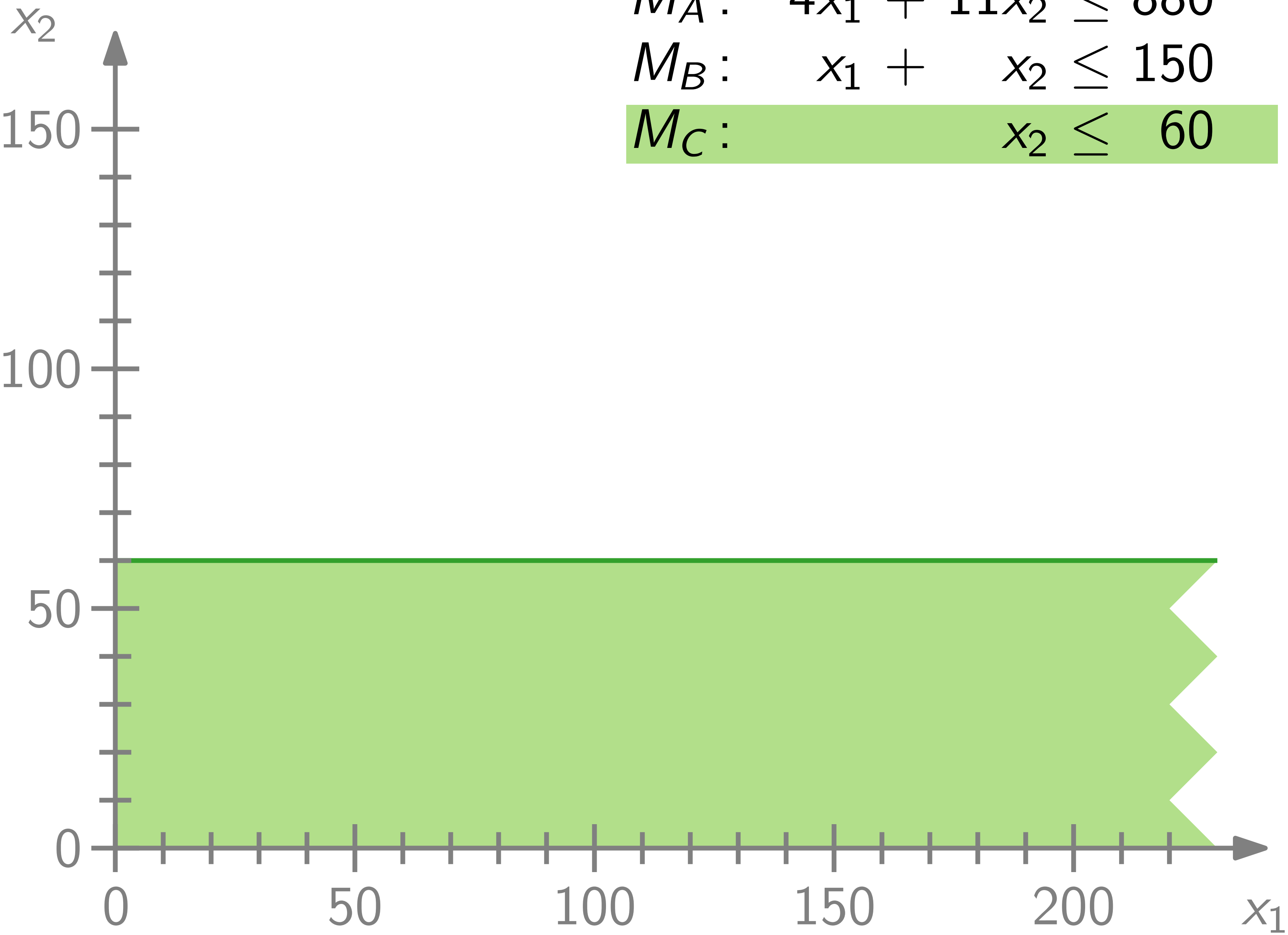

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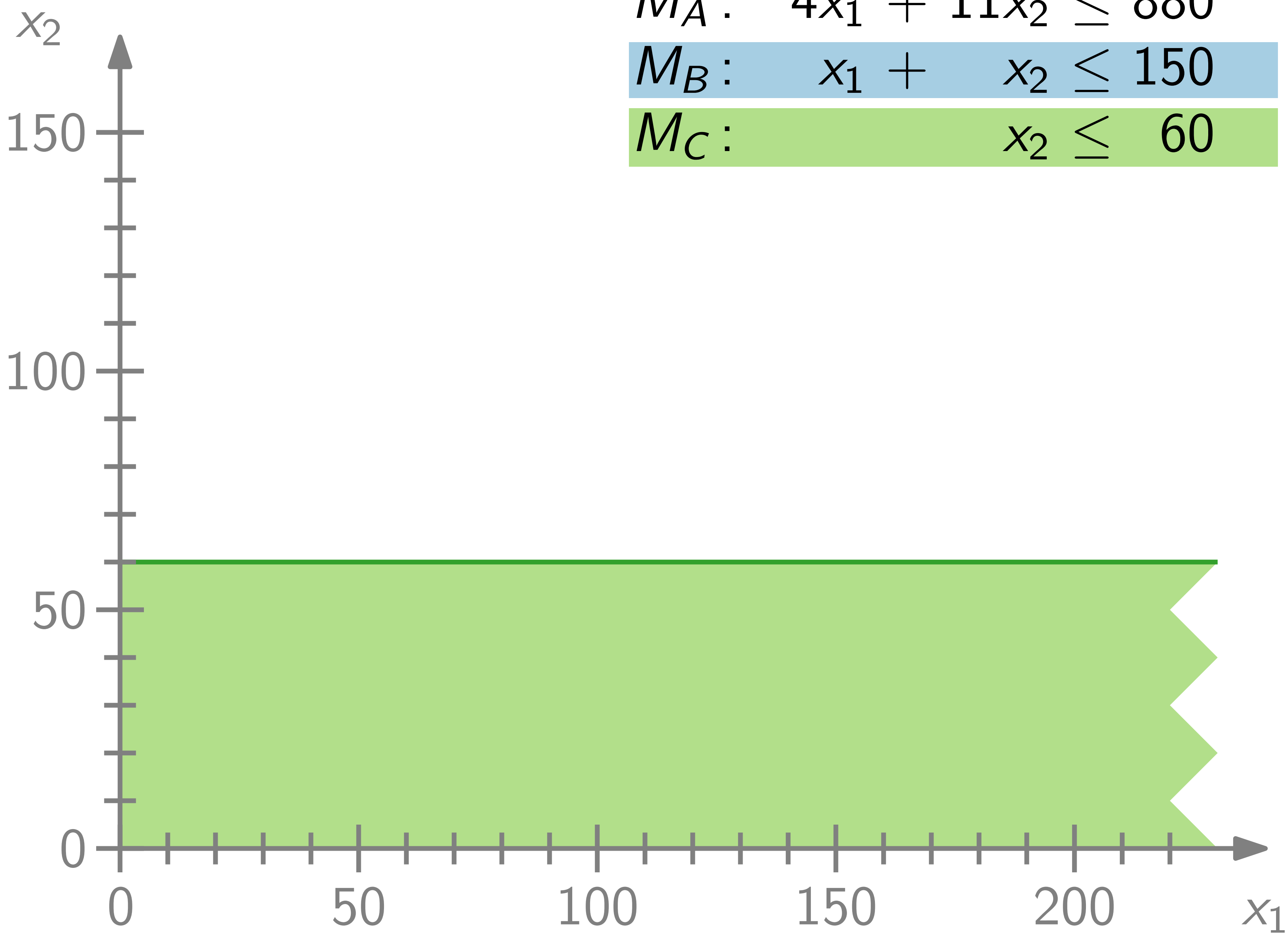

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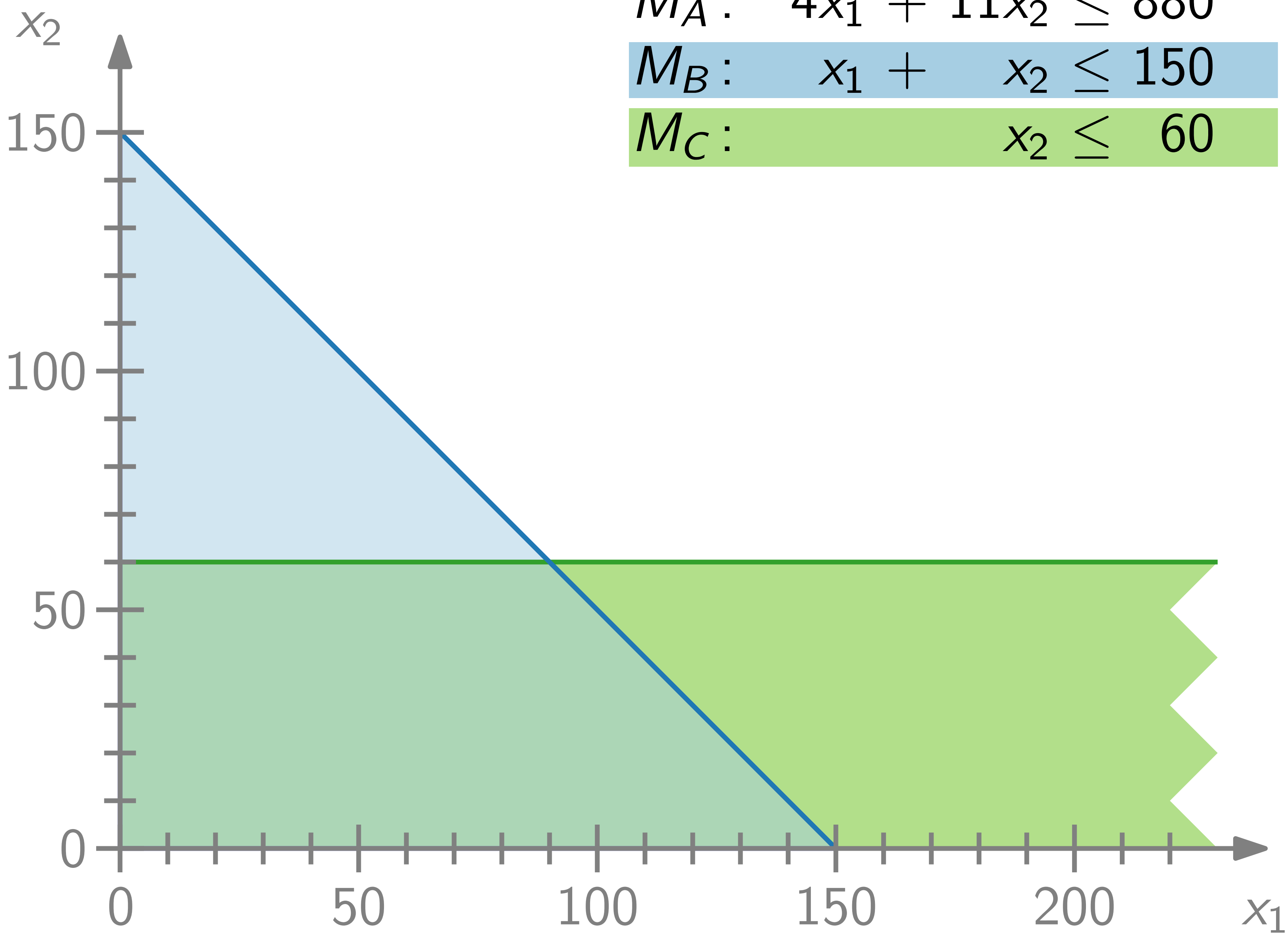

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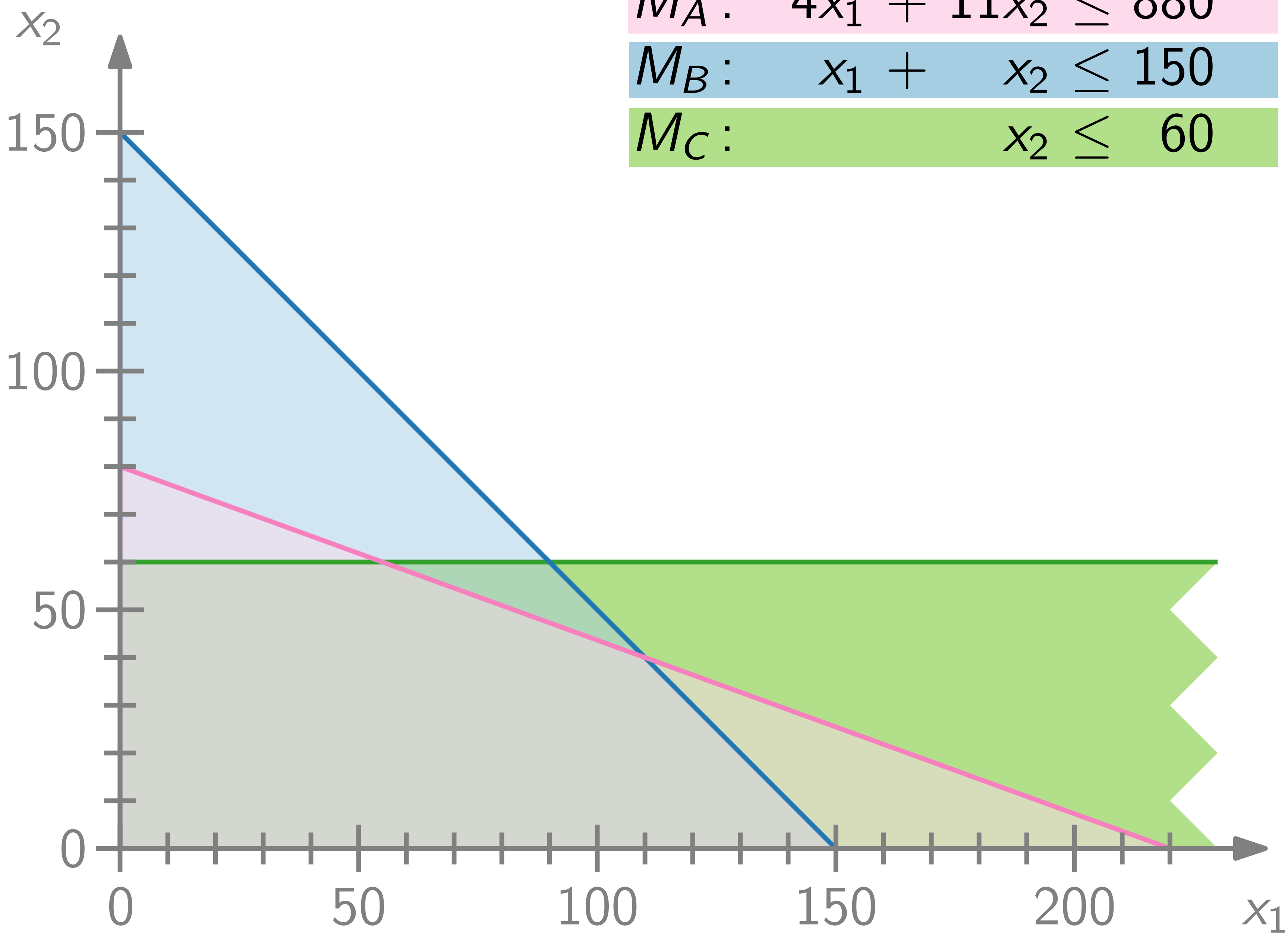

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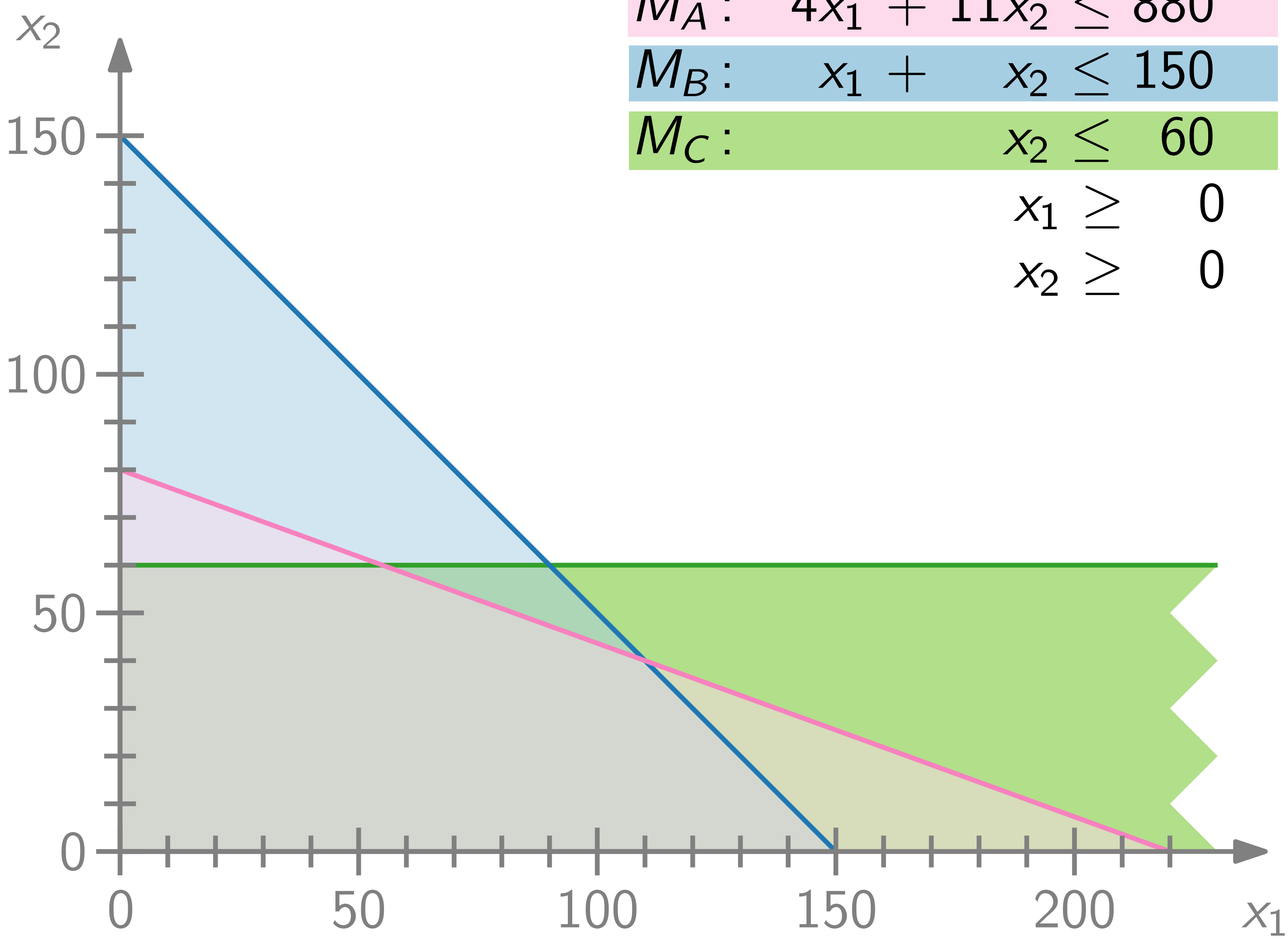

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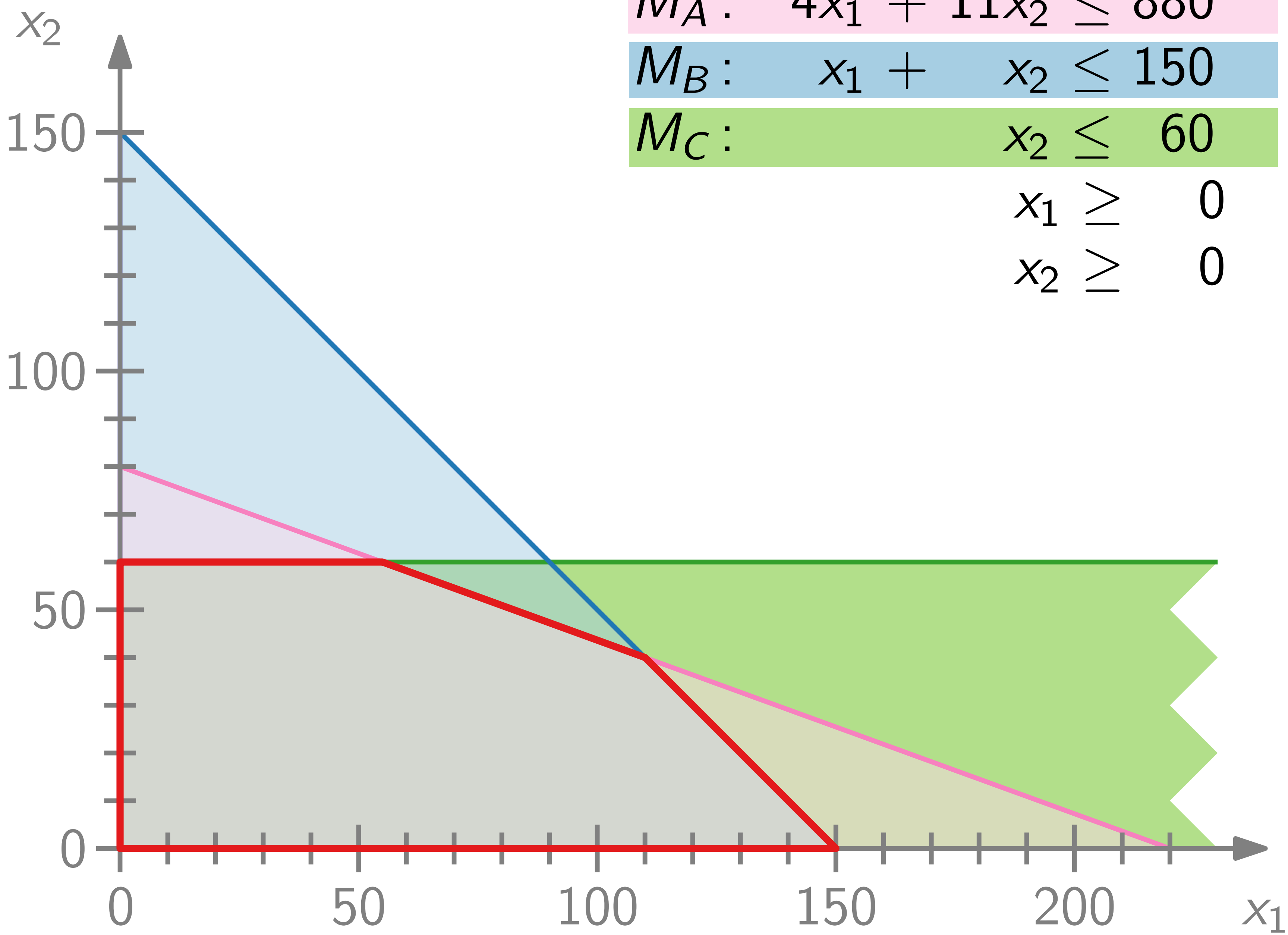

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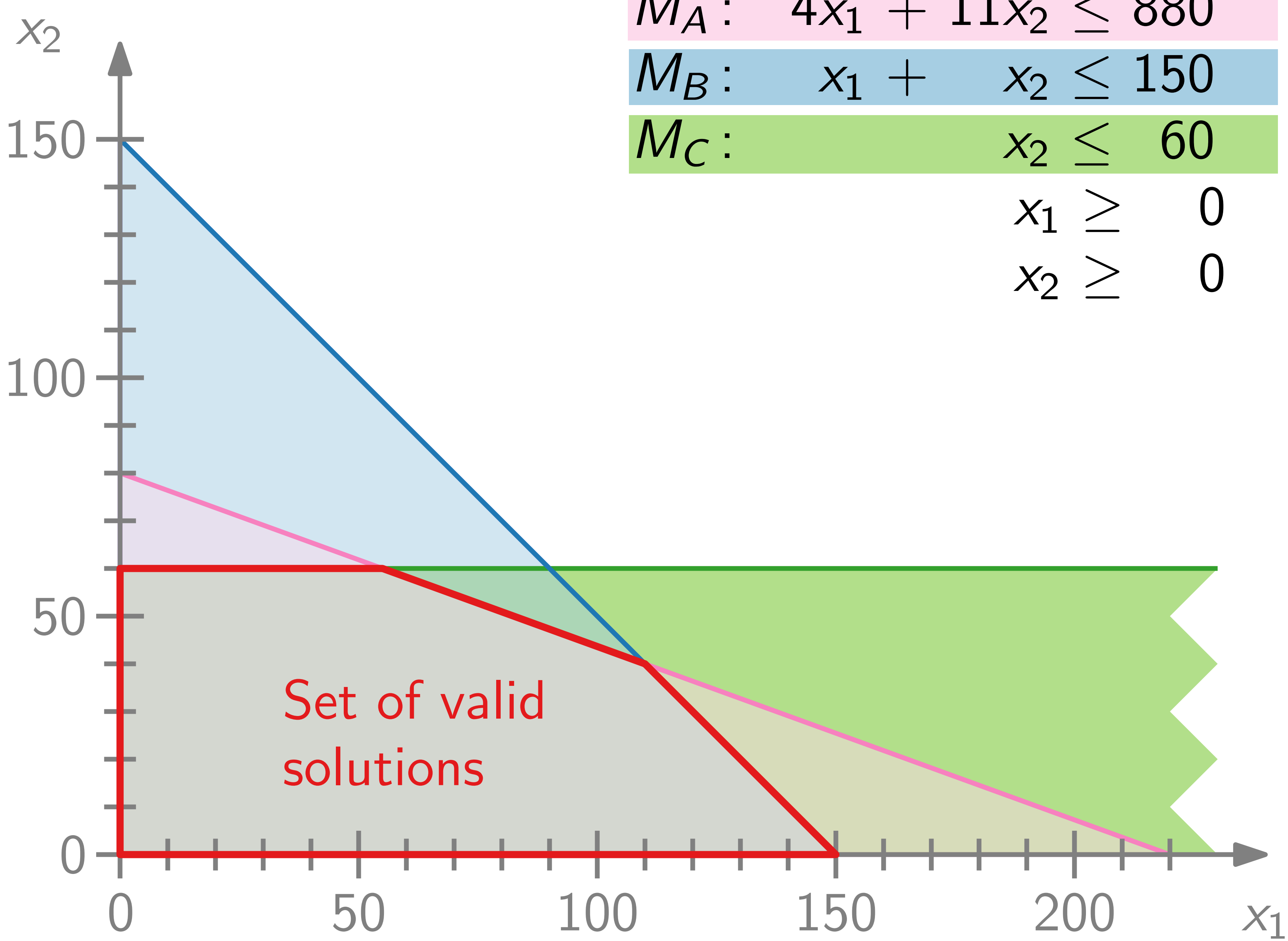

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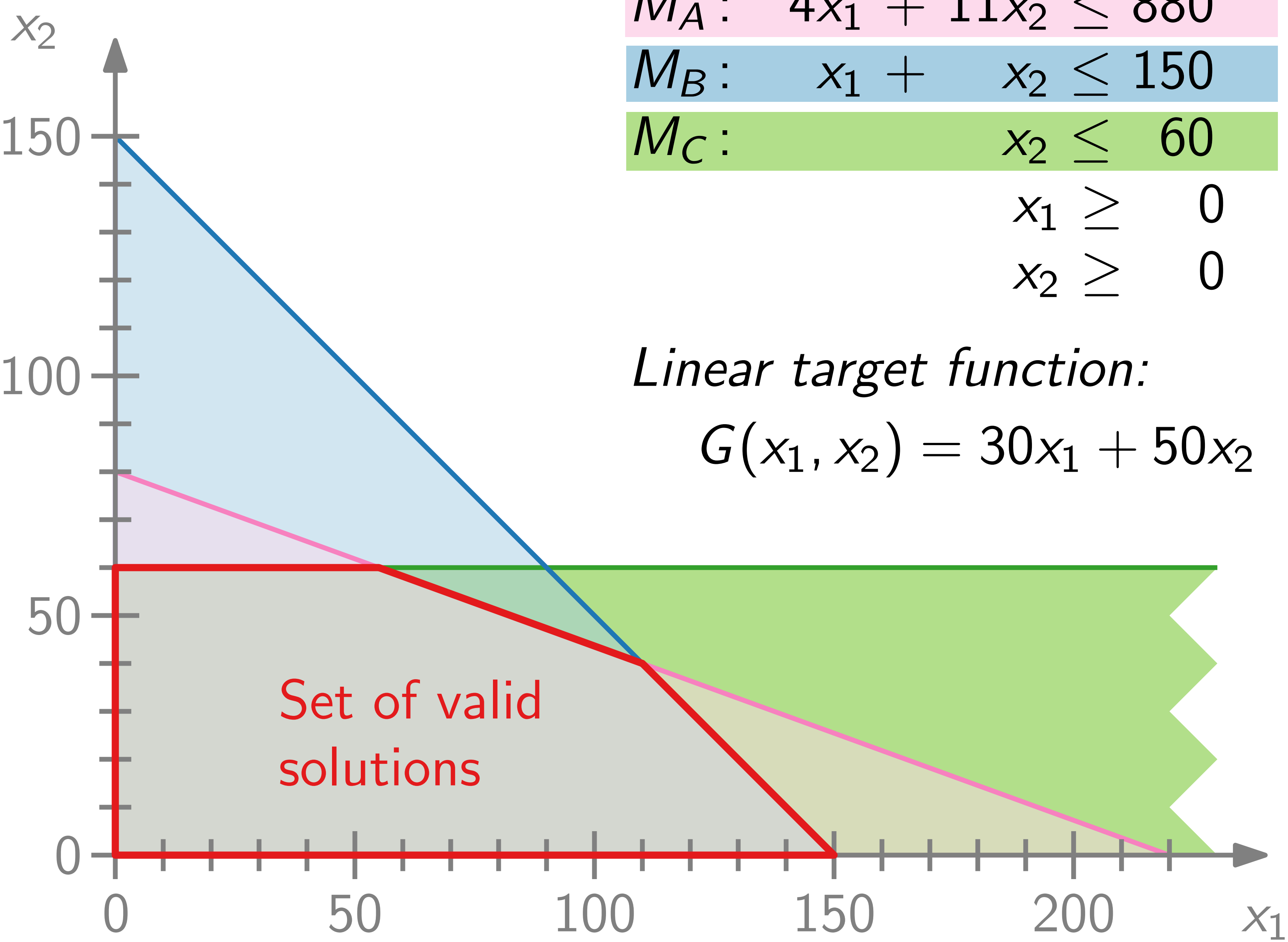

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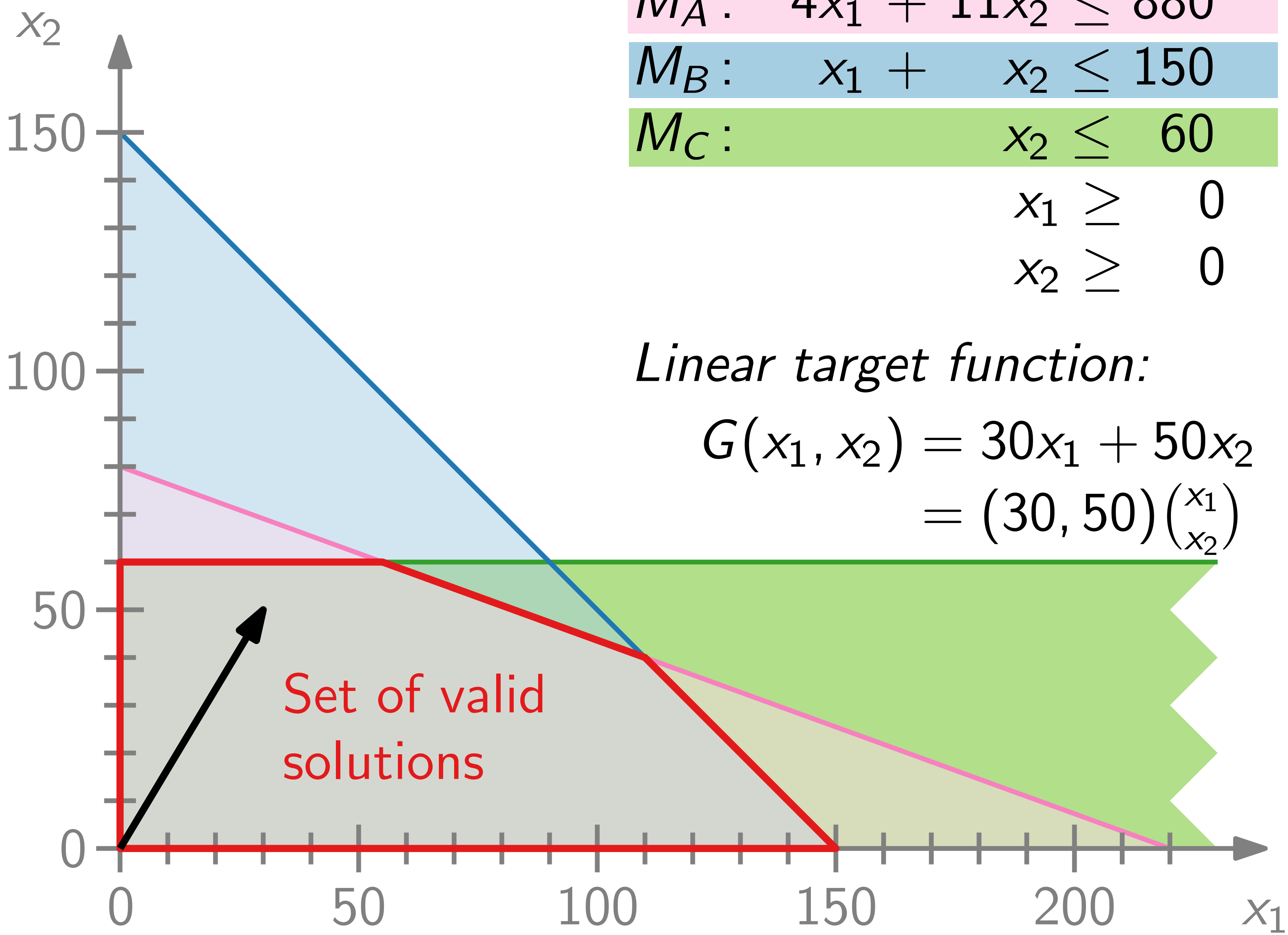

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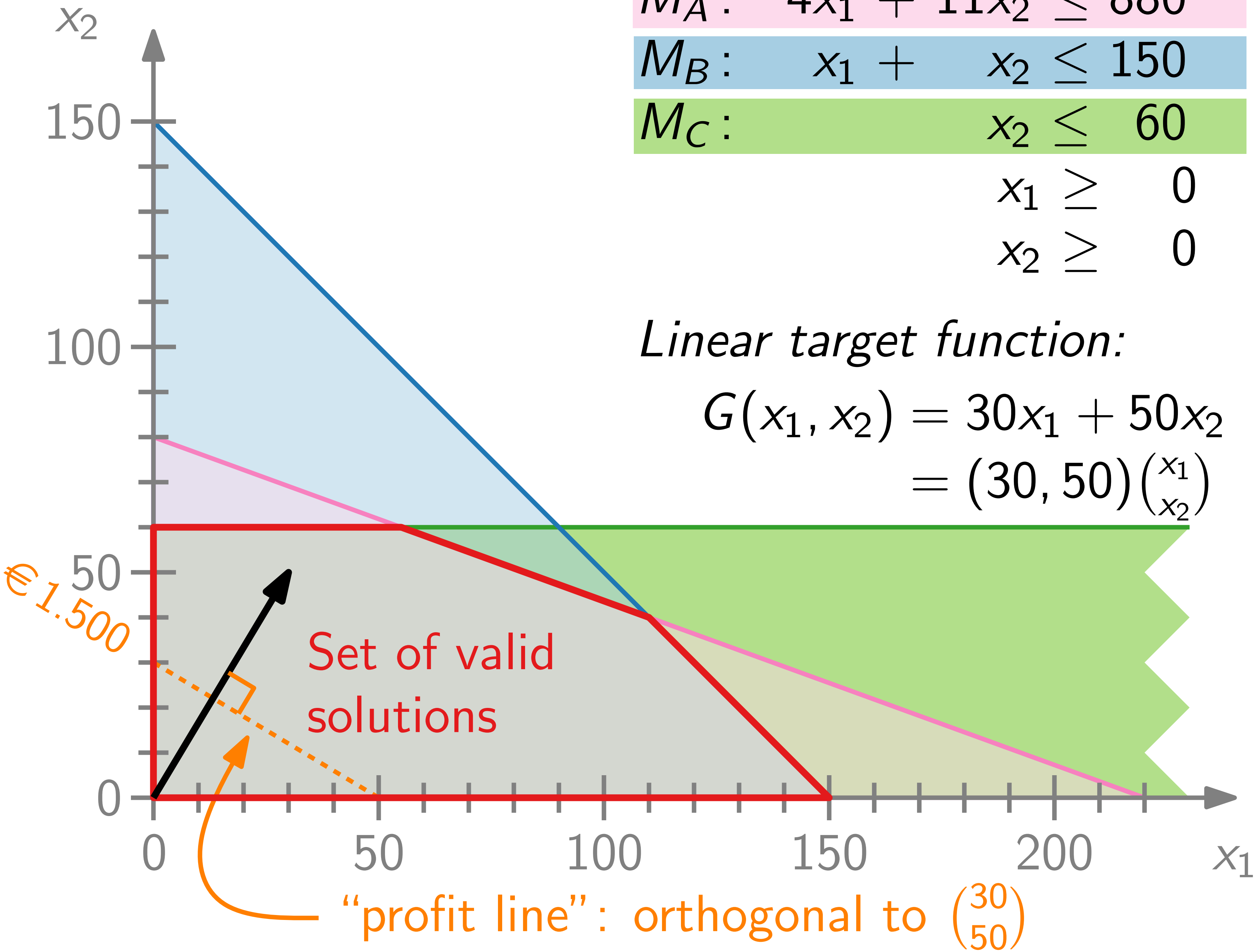

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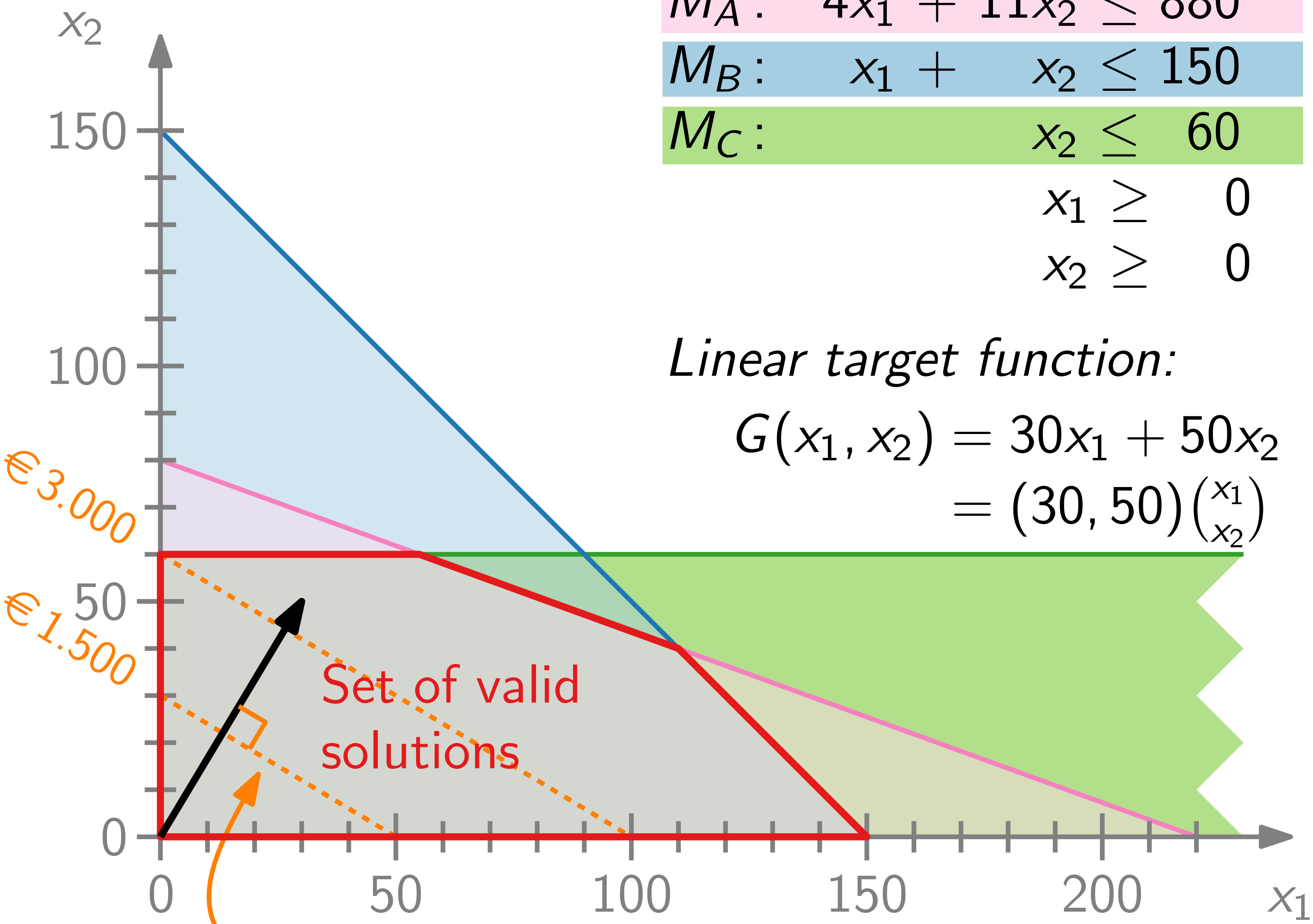

“profit line”: orthogonal to $\binom{30}{50}$

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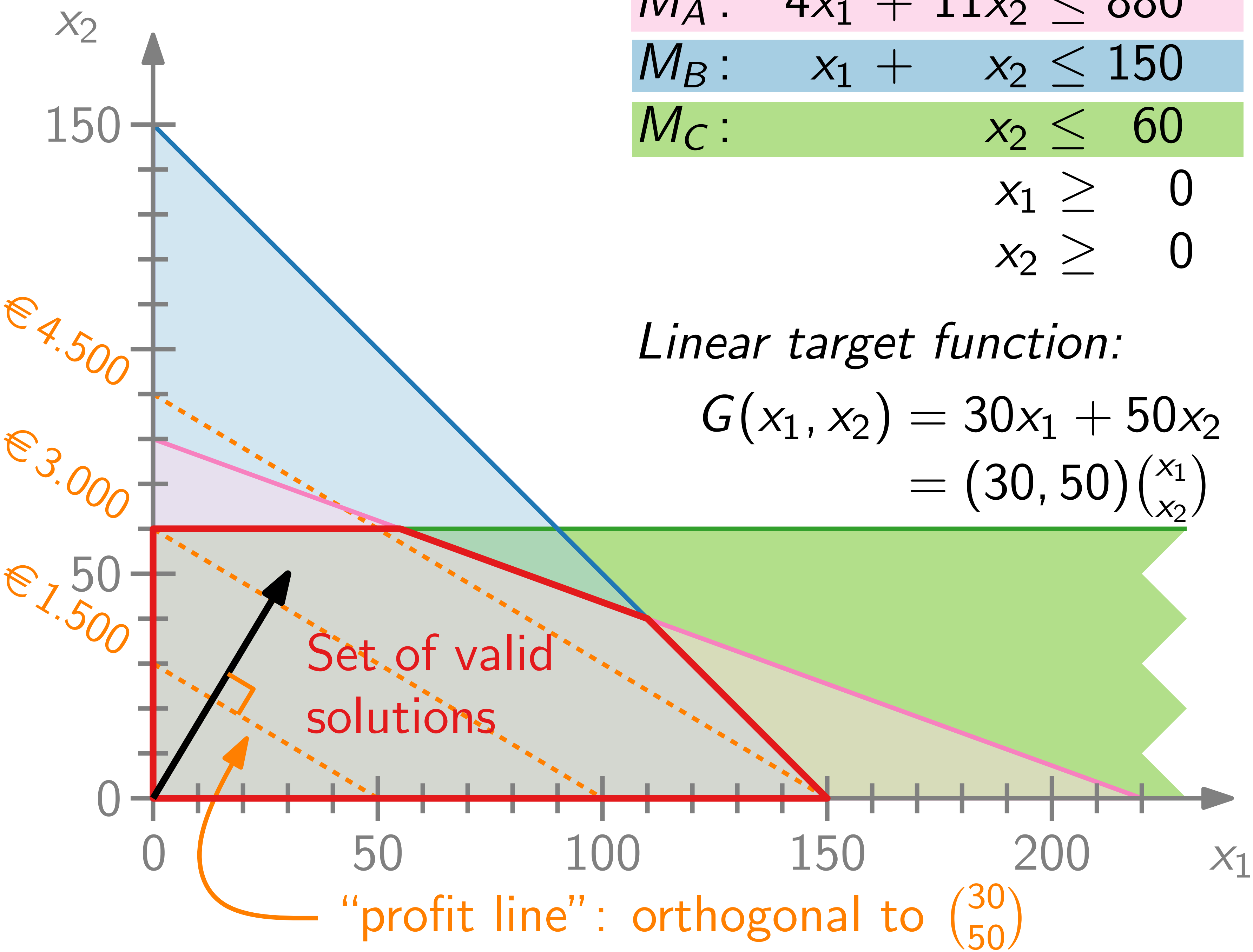

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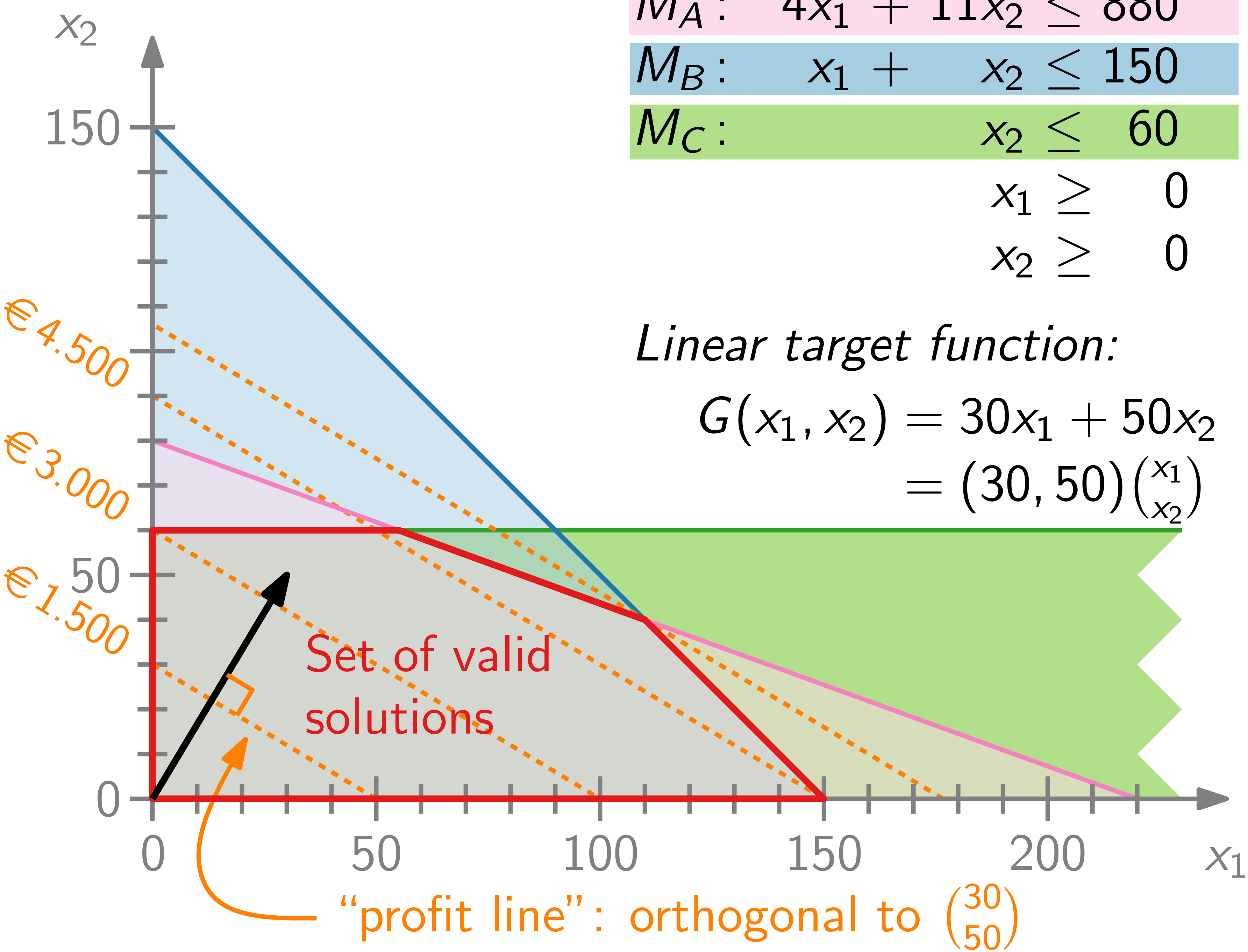

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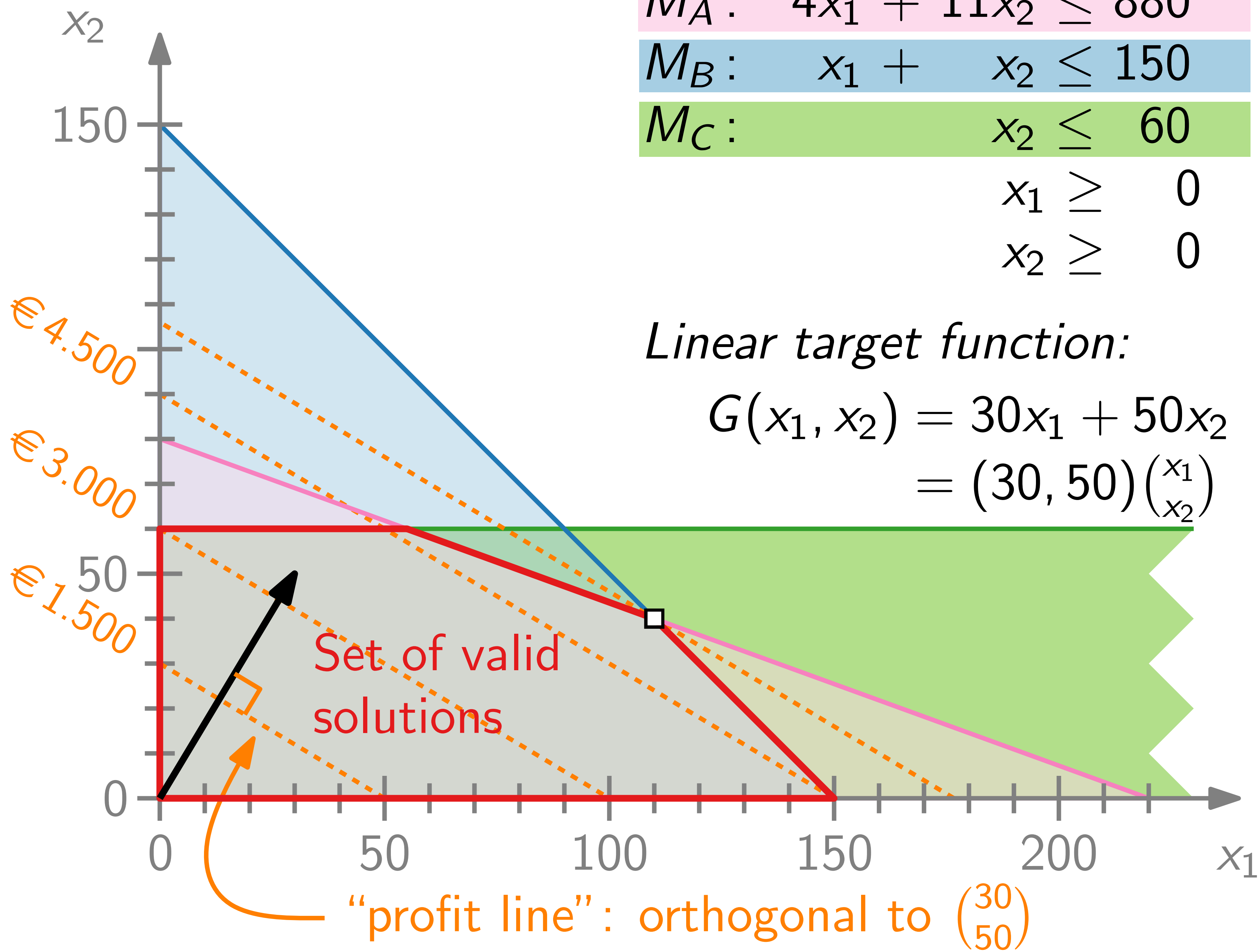

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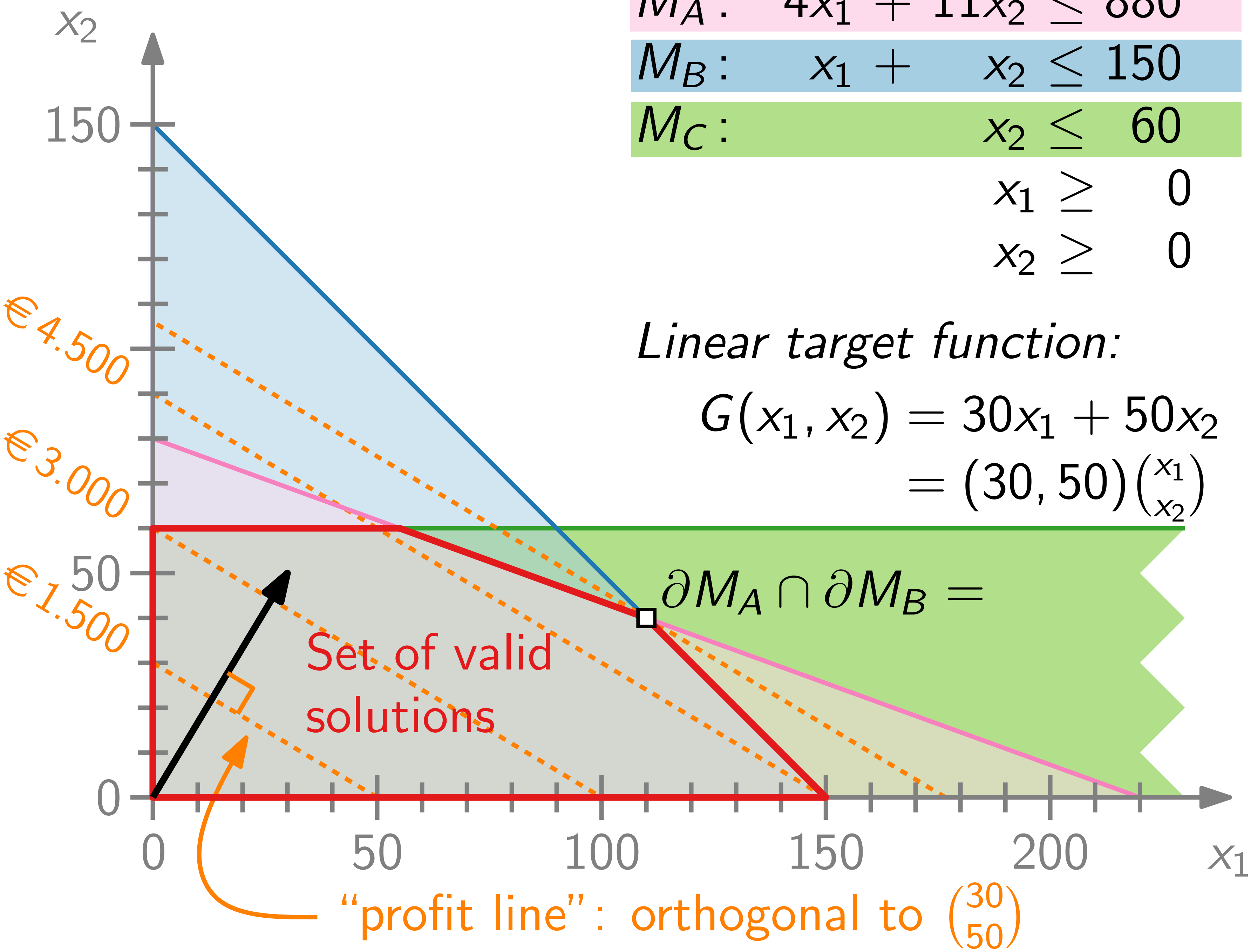

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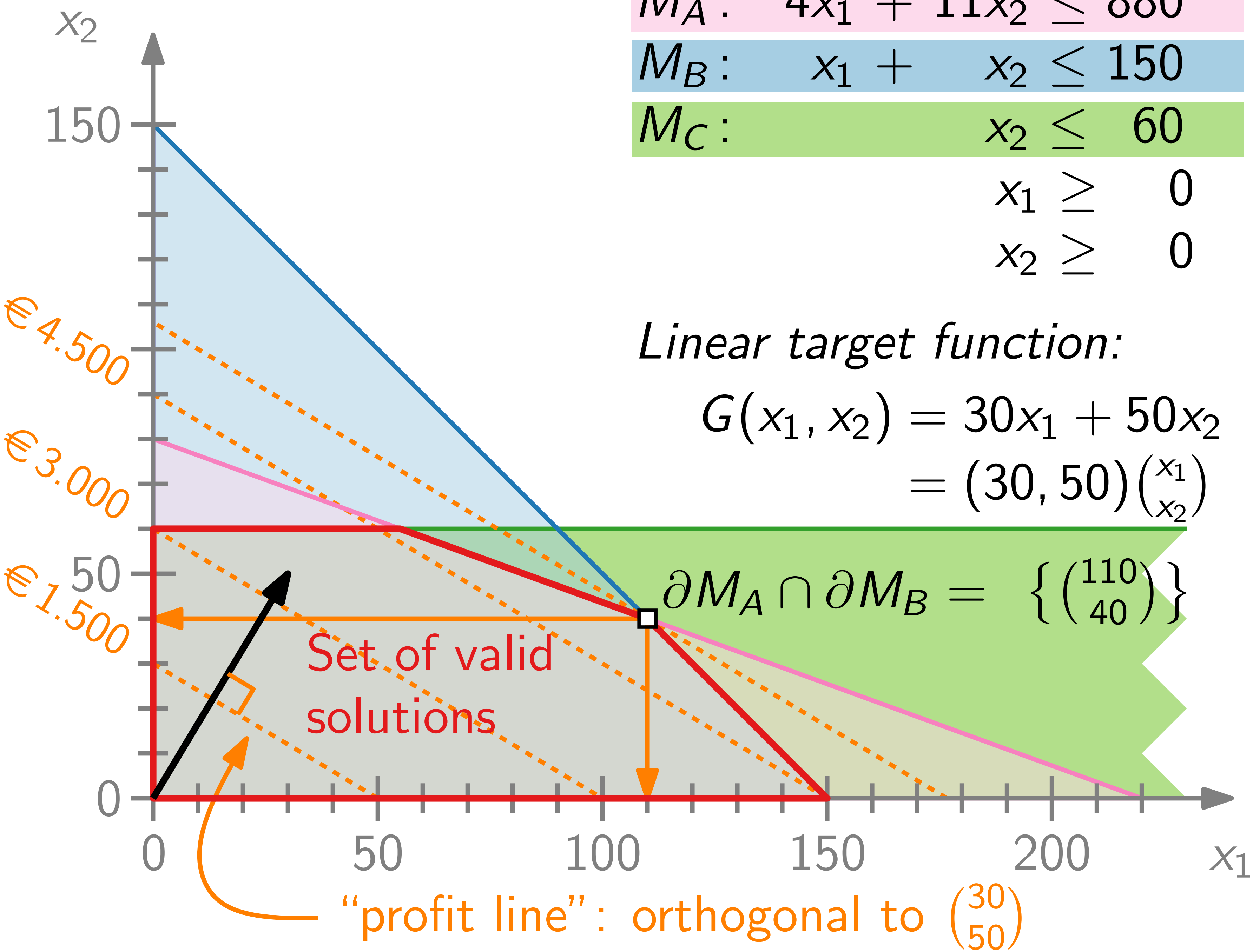

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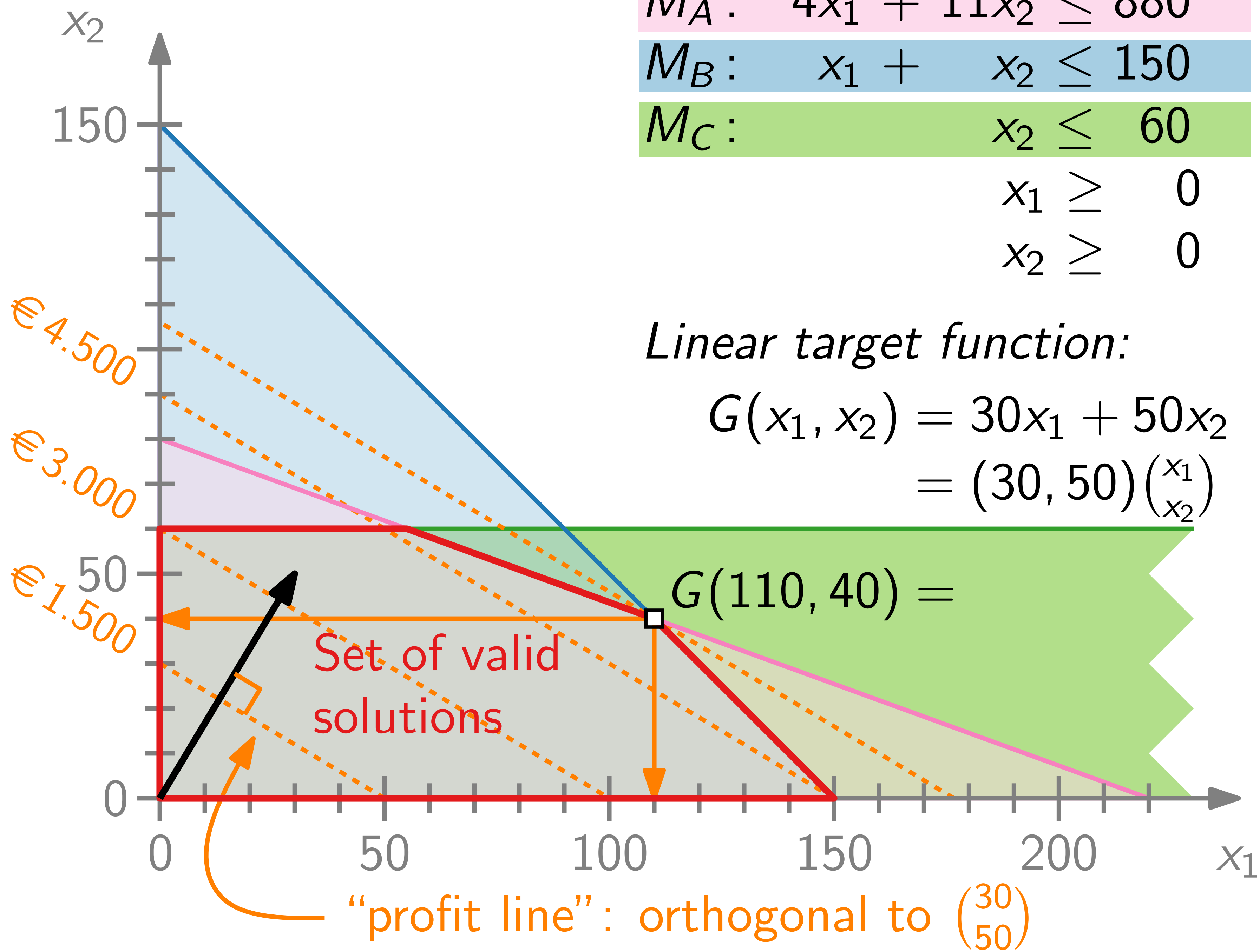

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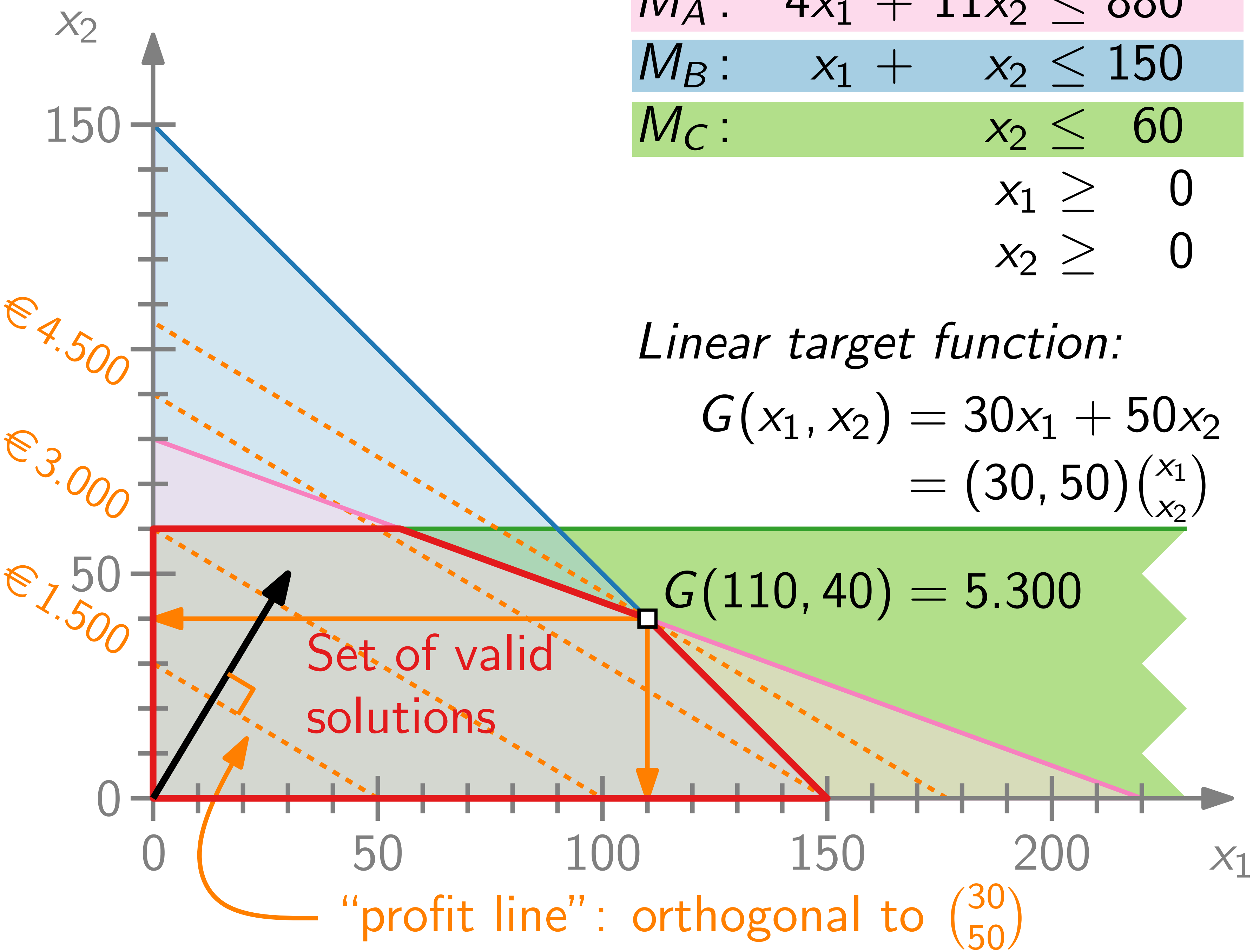

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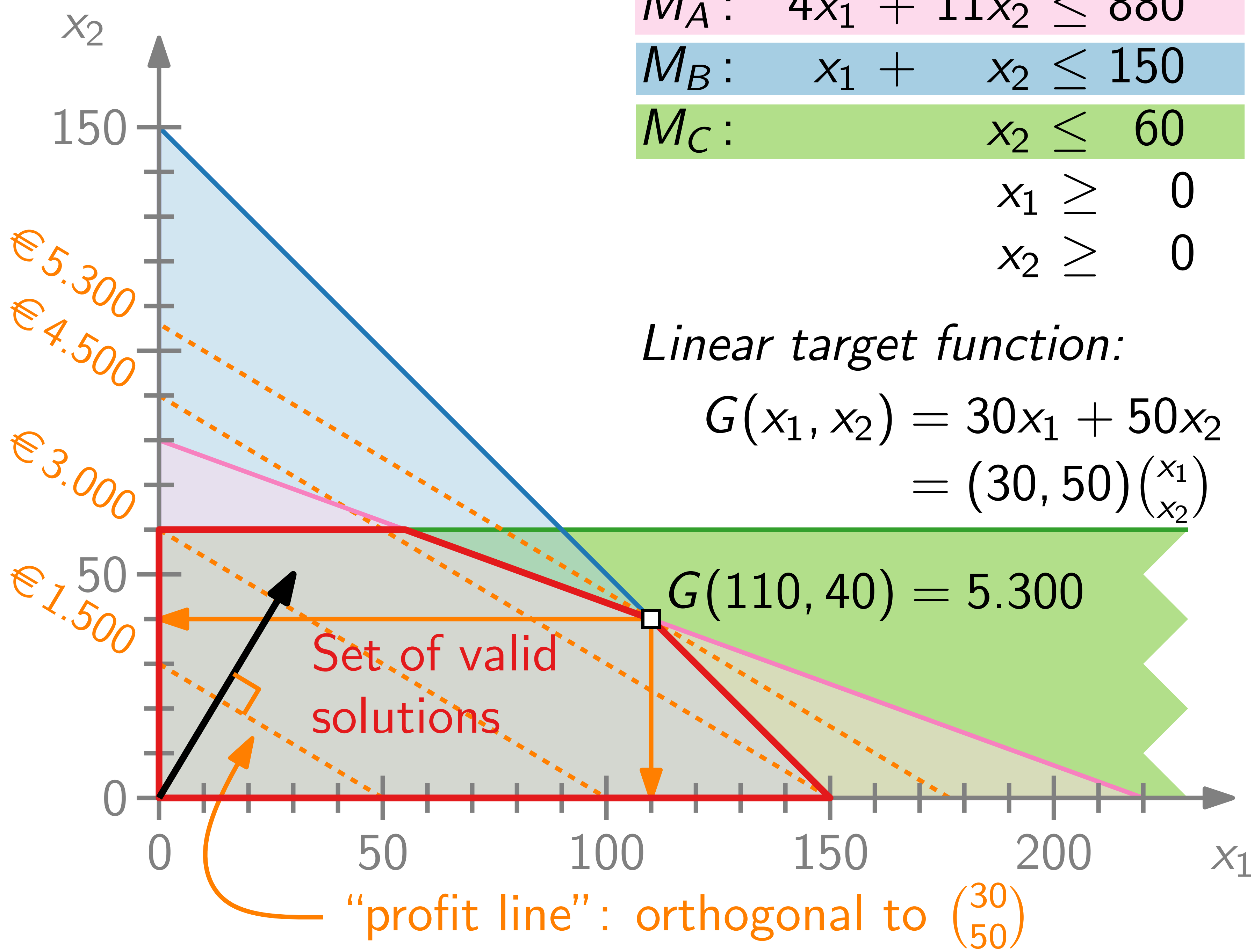

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Part II:
Upper Bounds for LPs

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Examples:

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- TSP: lower bound by MST or by cycle cover

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$$\begin{array}{lrcrcrcr} \textbf{minimize} & 7x_1 & + & x_2 & + & 5x_3 & & \\ \textbf{subject to} & x_1 & - & x_2 & + & 3x_3 & \geq & 10 \\ & 5x_1 & + & 2x_2 & - & x_3 & \geq & 6 \\ & & & & & x_1, x_2, x_3 & \geq & 0 \end{array}$$

Linear Programming – Upper Bounds

Optimize (i.e., minimize or maximize) a linear (*objective*) function subject to linear inequalities (*constraints*).

$$\boxed{\begin{array}{lrcrcrcr} \textbf{minimize} & 7x_1 & + & x_2 & + & 5x_3 & & \\ \textbf{subject to} & x_1 & - & x_2 & + & 3x_3 & \geq & 10 \\ & 5x_1 & + & 2x_2 & - & x_3 & \geq & 6 \\ & & & & & x_1, x_2, x_3 & \geq & 0 \end{array}}$$

Linear Programming – Upper Bounds

Optimize (i.e., minimize or maximize) a linear (*objective*) function subject to linear inequalities (*constraints*).

$$
\begin{array}{lrcrcrcr}
\textbf{minimize} & 7x_1 & + & x_2 & + & 5x_3 & & \\
\textbf{subject to} & x_1 & - & x_2 & + & 3x_3 & \geq & 10 \\
 & 5x_1 & + & 2x_2 & - & x_3 & \geq & 6 \\
 & & & & x_1, & x_2, x_3 & \geq & 0
\end{array}
$$

Valid solution?

Linear Programming – Upper Bounds

Optimize (i.e., minimize or maximize) a linear (*objective*) function subject to linear inequalities (*constraints*).

$$
\begin{array}{lrcrcrcr}
\textbf{minimize} & 7x_1 & + & x_2 & + & 5x_3 & & \\
\textbf{subject to} & x_1 & - & x_2 & + & 3x_3 & \geq & 10 \\
 & 5x_1 & + & 2x_2 & - & x_3 & \geq & 6 \\
 & & & & & x_1, x_2, x_3 & \geq & 0
\end{array}
$$

Valid solution?

$x = (2, 1, 3)$

Linear Programming – Upper Bounds

Optimize (i.e., minimize or maximize) a linear (*objective*) function subject to linear inequalities (*constraints*).

$$
\begin{array}{lrcrcrcr}
\textbf{minimize} & 7x_1 & + & x_2 & + & 5x_3 & & \\
\textbf{subject to} & x_1 \;2 & - & x_2 & + & 3x_3 & \geq & 10 \\
 & 5x_1 & + & 2x_2 & - & x_3 & \geq & 6 \\
 & & & & & x_1, x_2, x_3 & \geq & 0
\end{array}
$$

Valid solution?

$x = (2, 1, 3)$

Linear Programming – Upper Bounds

Optimize (i.e., minimize or maximize) a linear (*objective*) function subject to linear inequalities (*constraints*).

$$
\begin{array}{lrcrcrcr}
\textbf{minimize} & 7x_1 & + & x_2 & + & 5x_3 & & \\
\textbf{subject to} & x_1 \; 2 & - & x_2 \; 1 & + & 3x_3 & \geq & 10 \\
 & 5x_1 & + & 2x_2 & - & x_3 & \geq & 6 \\
 & & & & & x_1, x_2, x_3 & \geq & 0
\end{array}
$$

Valid solution?

$x = (2, 1, 3)$

Linear Programming – Upper Bounds

Optimize (i.e., minimize or maximize) a linear (*objective*) function subject to linear inequalities (*constraints*).

$$
\begin{array}{lrcrcrcr}
\textbf{minimize} & 7x_1 & + & x_2 & + & 5x_3 & & \\
\textbf{subject to} & x_1 \; 2 & - & x_2 \; 1 & + & 3x_3 \; 9 & \geq & 10 \\
 & 5x_1 & + & 2x_2 & - & x_3 & \geq & 6 \\
 & & & & & x_1, x_2, x_3 & \geq & 0
\end{array}
$$

Valid solution?

$x = (2, 1, 3)$

Linear Programming – Upper Bounds

Optimize (i.e., minimize or maximize) a linear (*objective*) function subject to linear inequalities (*constraints*).

$$
\begin{array}{lrcrcrcrl}
\textbf{minimize} & 7x_1 & + & x_2 & + & 5x_3 & & & \\
\textbf{subject to} & x_1 \; {\color{green}2} & - & x_2 \; {\color{green}1} & + & 3x_3 \; {\color{green}9} & \geq & 10 & {\color{green}10} \\
 & 5x_1 & + & 2x_2 & - & x_3 & \geq & 6 & \\
 & & & & & x_1, x_2, x_3 & \geq & 0 &
\end{array}
$$

Valid solution?

$x = (2, 1, 3)$

Linear Programming – Upper Bounds

Optimize (i.e., minimize or maximize) a linear (*objective*) function subject to linear inequalities (*constraints*).

$$
\begin{array}{llcccccccc}
\textbf{minimize} & 7x_1 & & + & x_2 & & + & 5x_3 & & & & \\
\textbf{subject to} & x_1 & 2 & - & x_2 & 1 & + & 3x_3 & 9 & \geq & 10 & 10 \\
 & 5x_1 & 10 & + & 2x_2 & 2 & - & x_3 & 3 & \geq & 6 & 9 \\
 & & & & & & & x_1, x_2, x_3 & & \geq & 0 &
\end{array}
$$

Valid solution?

$x = (2, 1, 3)$

Linear Programming – Upper Bounds

Optimize (i.e., minimize or maximize) a linear (*objective*) function subject to linear inequalities (*constraints*).

$$
\begin{array}{lrrrrrrrrr}
\textbf{minimize} & 7x_1 & 14 & + & x_2 & & + & 5x_3 & & & & \\
\textbf{subject to} & x_1 & 2 & - & x_2 & 1 & + & 3x_3 & 9 & \geq & 10 & 10 \\
& 5x_1 & 10 & + & 2x_2 & 2 & - & x_3 & 3 & \geq & 6 & 9 \\
& & & & & & & x_1, x_2, x_3 & & \geq & 0 &
\end{array}
$$

Valid solution?

$x = (2, 1, 3)$

Linear Programming – Upper Bounds

Optimize (i.e., minimize or maximize) a linear (*objective*) function subject to linear inequalities (*constraints*).

$$
\begin{array}{llllllll}
\textbf{minimize} & 7x_1\ 14 & + & x_2\ 1 & + & 5x_3 & & \\
\textbf{subject to} & x_1\ 2 & - & x_2\ 1 & + & 3x_3\ 9 & \geq & 10\ 10 \\
 & 5x_1\ 10 & + & 2x_2\ 2 & - & x_3\ 3 & \geq & 6\ 9 \\
 & & & & & x_1, x_2, x_3 & \geq & 0
\end{array}
$$

Valid solution?

$x = (2, 1, 3)$

Linear Programming – Upper Bounds

Optimize (i.e., minimize or maximize) a linear (*objective*) function subject to linear inequalities (*constraints*).

$$
\begin{array}{lrcrcrcrr}
\textbf{minimize} & 7x_1 \; 14 & + & x_2 \; 1 & + & 5x_3 \; 15 & & & \\
\textbf{subject to} & x_1 \; 2 & - & x_2 \; 1 & + & 3x_3 \; 9 & \geq & 10 & 10 \\
 & 5x_1 \; 10 & + & 2x_2 \; 2 & - & x_3 \; 3 & \geq & 6 & 9 \\
 & & & & & x_1, x_2, x_3 & \geq & 0 &
\end{array}
$$

Valid solution?

$x = (2, 1, 3)$

Linear Programming – Upper Bounds

Optimize (i.e., minimize or maximize) a linear (*objective*) function subject to linear inequalities (*constraints*).

$$
\begin{array}{lrrrrrrrrrr}
\textbf{minimize} & 7x_1 & 14 & + & x_2 & 1 & + & 5x_3 & 15 & = & & 30 \\
\textbf{subject to} & x_1 & 2 & - & x_2 & 1 & + & 3x_3 & 9 & \geq & 10 & 10 \\
 & 5x_1 & 10 & + & 2x_2 & 2 & - & x_3 & 3 & \geq & 6 & 9 \\
 & & & & & & & x_1, x_2, x_3 & & \geq & 0 &
\end{array}
$$

Valid solution?

$x = (2, 1, 3)$

Linear Programming – Upper Bounds

Optimize (i.e., minimize or maximize) a linear (*objective*) function subject to linear inequalities (*constraints*).

$$
\begin{array}{lrcrcrcrr}
\textbf{minimize} & 7x_1\;14 & + & x_2\;1 & + & 5x_3\;15 & = & & 30 \\
\textbf{subject to} & x_1\;2 & - & x_2\;1 & + & 3x_3\;9 & \geq & 10 & 10 \\
 & 5x_1\;10 & + & 2x_2\;2 & - & x_3\;3 & \geq & 6 & 9 \\
 & & & & & x_1, x_2, x_3 & \geq & 0 &
\end{array}
$$

Valid solution?

$x = (2, 1, 3)$

$\Rightarrow$ obj$(x) = 30$ is upper bound for OPT

Approximation Algorithms

Lecture 4:
Linear Programming and LP-Duality

Part III:
Lower Bounds for LPs

Linear Programming – Lower Bounds

Optimize (i.e., minimize or maximize) a linear (*objective*) function subject to linear inequalities (*constraints*).

$$
\begin{array}{lrcrcrcr}
\textbf{minimize} & 7x_1 & + & x_2 & + & 5x_3 & & \\
\textbf{subject to} & x_1 & - & x_2 & + & 3x_3 & \geq & 10 \\
 & 5x_1 & + & 2x_2 & - & x_3 & \geq & 6 \\
 & & & & & x_1, x_2, x_3 & \geq & 0
\end{array}
$$

Linear Programming – Lower Bounds

Optimize (i.e., minimize or maximize) a linear (*objective*) function subject to linear inequalities (*constraints*).

$$
\begin{array}{lrcrcrcr}
\textbf{minimize} & 7x_1 & + & x_2 & + & 5x_3 & & \\
 & \rotatebox{-90}{$\geq$} & & & & & & \\
\textbf{subject to} & x_1 & - & x_2 & + & 3x_3 & \geq & 10 \\
 & 5x_1 & + & 2x_2 & - & x_3 & \geq & 6 \\
 & & & & & x_1, x_2, x_3 & \geq & 0
\end{array}
$$

Linear Programming – Lower Bounds

Optimize (i.e., minimize or maximize) a linear (*objective*) function subject to linear inequalities (*constraints*).

$$
\begin{array}{lrcrcrcr}
\textbf{minimize} & 7x_1 & + & x_2 & + & 5x_3 & & \\
\textbf{subject to} & x_1 & - & x_2 & + & 3x_3 & \geq & 10 \\
 & 5x_1 & + & 2x_2 & - & x_3 & \geq & 6 \\
 & & & & & x_1, x_2, x_3 & \geq & 0
\end{array}
$$

Linear Programming – Lower Bounds

Optimize (i.e., minimize or maximize) a linear (*objective*) function subject to linear inequalities (*constraints*).

$$
\begin{array}{lrcrcrcr}
\textbf{minimize} & 7x_1 & + & x_2 & + & 5x_3 & & \\
 & \rotatebox{-90}{$\geq$} & & \rotatebox{-90}{$\geq$} & & \rotatebox{-90}{$\geq$} & & \\
\textbf{subject to} & x_1 & - & x_2 & + & 3x_3 & \geq & 10 \\
 & 5x_1 & + & 2x_2 & - & x_3 & \geq & 6 \\
 & & & & & x_1, x_2, x_3 & \geq & 0
\end{array}
$$

Linear Programming – Lower Bounds

Optimize (i.e., minimize or maximize) a linear (*objective*) function subject to linear inequalities (*constraints*).

$$
\begin{array}{lrcrcrcr}
\textbf{minimize} & 7x_1 & + & x_2 & + & 5x_3 & & \\
 & \rotatebox{-90}{$\geq$} & & \rotatebox{-90}{$\geq$} & & \rotatebox{-90}{$\geq$} & & \\
\textbf{subject to} & x_1 & - & x_2 & + & 3x_3 & \geq & 10 \\
 & 5x_1 & + & 2x_2 & - & x_3 & \geq & 6 \\
 & & & & & x_1, x_2, x_3 & \geq & 0
\end{array}
$$

$$7x_1 + x_2 + 5x_3 \quad \geq x_1 - x_2 + 3x_3$$

Linear Programming – Lower Bounds

Optimize (i.e., minimize or maximize) a linear (*objective*) function subject to linear inequalities (*constraints*).

$$
\begin{array}{lrcrcrcl}
\textbf{minimize} & 7x_1 & + & x_2 & + & 5x_3 & & \\
 & \rotatebox{90}{$\le$} & & \rotatebox{90}{$\le$} & & \rotatebox{90}{$\le$} & & \\
\textbf{subject to} & x_1 & - & x_2 & + & 3x_3 & \geq & 10 \\
 & 5x_1 & + & 2x_2 & - & x_3 & \geq & 6 \\
 & & & & & x_1, x_2, x_3 & \geq & 0
\end{array}
$$

$$7x_1 + x_2 + 5x_3 \ \geq x_1 - x_2 + 3x_3 \ \Rightarrow \mathrm{OPT} \geq 10$$

Linear Programming – Lower Bounds

Optimize (i.e., minimize or maximize) a linear (*objective*) function subject to linear inequalities (*constraints*).

$$
\begin{array}{llcrcrcrcr}
\textbf{minimize} & & 7x_1 & + & x_2 & + & 5x_3 & & \\
\textbf{subject to} & & x_1 & - & x_2 & + & 3x_3 & \geq & 10 \\
 & & 5x_1 & + & 2x_2 & - & x_3 & \geq & 6 \\
 & & & & & & x_1, x_2, x_3 & \geq & 0
\end{array}
$$

$$7x_1 + x_2 + 5x_3 \geq x_1 - x_2 + 3x_3 \Rightarrow \mathrm{OPT} \geq 10$$

Linear Programming – Lower Bounds

Optimize (i.e., minimize or maximize) a linear (*objective*) function subject to linear inequalities (*constraints*).

$$
\begin{array}{lrcrcrcr}
\textbf{minimize} & 7x_1 & + & x_2 & + & 5x_3 & & \\
 & \rotatebox{90}{$\leq$} & & \rotatebox{90}{$\leq$} & & \rotatebox{90}{$\leq$} & & \\
\textbf{subject to} & x_1 & - & x_2 & + & 3x_3 & \geq & 10 \\
 & + & & + & & & & \\
 & 5x_1 & + & 2x_2 & - & x_3 & \geq & 6 \\
 & & & & & x_1, x_2, x_3 & \geq & 0
\end{array}
$$

$$7x_1 + x_2 + 5x_3 \geq x_1 - x_2 + 3x_3 \Rightarrow \text{OPT} \geq 10$$

Linear Programming – Lower Bounds

Optimize (i.e., minimize or maximize) a linear (*objective*) function subject to linear inequalities (*constraints*).

$$
\begin{array}{lrcrcrcr}
\textbf{minimize} & \underset{\geq}{7x_1} & + & \underset{\geq}{x_2} & + & \underset{\geq}{5x_3} & & \\
\textbf{subject to} & \underset{+}{x_1} & - & \underset{+}{x_2} & + & \underset{+}{3x_3} & \geq & 10 \\
 & 5x_1 & + & 2x_2 & - & x_3 & \geq & 6 \\
 & & & & & x_1, x_2, x_3 & \geq & 0
\end{array}
$$

$$7x_1 + x_2 + 5x_3 \quad \geq x_1 - x_2 + 3x_3 \quad \Rightarrow \text{OPT} \geq 10$$

Linear Programming – Lower Bounds

Optimize (i.e., minimize or maximize) a linear (*objective*) function subject to linear inequalities (*constraints*).

$$
\begin{array}{lrcrcrcr}
\textbf{minimize} & 7x_1 & + & x_2 & + & 5x_3 & & \\
 & \geq & & \geq & & \geq & & \\
\textbf{subject to} & x_1 & - & x_2 & + & 3x_3 & \geq & 10 \\
 & + & & + & & + & & \\
 & 5x_1 & + & 2x_2 & - & x_3 & \geq & 6 \\
 & & & & & x_1, x_2, x_3 & \geq & 0
\end{array}
$$

$$7x_1 + x_2 + 5x_3 \geq x_1 - x_2 + 3x_3 \Rightarrow \mathrm{OPT} \geq 10$$

$$
\begin{aligned}
7x_1 + x_2 + 5x_3 &\geq (x_1 - x_2 + 3x_3) + (5x_1 + 2x_2 - x_3) \\
&\geq 10 + 6
\end{aligned}
$$

Linear Programming – Lower Bounds

Optimize (i.e., minimize or maximize) a linear (*objective*) function subject to linear inequalities (*constraints*).

$$
\begin{array}{lrcrcrcr}
\textbf{minimize} & 7x_1 & + & x_2 & + & 5x_3 & & \\
\textbf{subject to} & x_1 & - & x_2 & + & 3x_3 & \geq & 10 \\
 & 5x_1 & + & 2x_2 & - & x_3 & \geq & 6 \\
 & & & & & x_1, x_2, x_3 & \geq & 0
\end{array}
$$

$$7x_1 + x_2 + 5x_3 \geq x_1 - x_2 + 3x_3 \quad \Rightarrow \text{OPT} \geq 10$$

$$
\begin{aligned}
7x_1 + x_2 + 5x_3 &\geq (x_1 - x_2 + 3x_3) + (5x_1 + 2x_2 - x_3) \\
&\geq 10 + 6 \qquad\qquad \Rightarrow \text{OPT} \geq 16
\end{aligned}
$$

Linear Programming – Lower Bounds

Optimize (i.e., minimize or maximize) a linear (*objective*) function subject to linear inequalities (*constraints*).

$$
\begin{array}{lrcrcrcr}
\textbf{minimize} & 7x_1 & + & x_2 & + & 5x_3 & & \\
 & \rotatebox{90}{$\leq$} & & \rotatebox{90}{$\leq$} & & \rotatebox{90}{$\leq$} & & \\
\textbf{subject to} & 2\cdot\; x_1 & - & 2\cdot x_2 & + & 2\cdot 3x_3 & \geq & 2\cdot 10 \\
 & + & & + & & + & & \\
 & 5x_1 & + & 2x_2 & - & x_3 & \geq & 6 \\
 & & & & & x_1, x_2, x_3 & \geq & 0
\end{array}
$$

$$7x_1 + x_2 + 5x_3 \geq x_1 - x_2 + 3x_3 \quad \Rightarrow \mathrm{OPT} \geq 10$$

$$
\begin{aligned}
7x_1 + x_2 + 5x_3 &\geq (x_1 - x_2 + 3x_3) + (5x_1 + 2x_2 - x_3) \\
&\geq 10 + 6 \qquad \Rightarrow \mathrm{OPT} \geq 16
\end{aligned}
$$

Linear Programming – Lower Bounds

Optimize (i.e., minimize or maximize) a linear (*objective*) function subject to linear inequalities (*constraints*).

$$
\begin{array}{lrcrcrcr}
\textbf{minimize} & 7x_1 & + & x_2 & + & 5x_3 & & \\
 & \rotatebox{90}{$\le$} & & \rotatebox{90}{$\le$} & & \rotatebox{90}{$\le$} & & \\
\textbf{subject to} & 2\cdot x_1 & - & 2\cdot x_2 & + & 2\cdot 3x_3 & \geq & 2\cdot 10 \\
 & + & & + & & + & & \\
 & 5x_1 & + & 2x_2 & - & x_3 & \geq & 6 \\
 & & & & & x_1, x_2, x_3 & \geq & 0
\end{array}
$$

$$7x_1 + x_2 + 5x_3 \geq x_1 - x_2 + 3x_3 \quad \Rightarrow \mathrm{OPT} \geq 10$$

$$
\begin{aligned}
7x_1 + x_2 + 5x_3 &\geq (x_1 - x_2 + 3x_3) + (5x_1 + 2x_2 - x_3) \\
&\geq 10 + 6 \quad \Rightarrow \mathrm{OPT} \geq 16
\end{aligned}
$$

$$
\begin{aligned}
7x_1 + x_2 + 5x_3 &\geq 2 \cdot (x_1 - x_2 + 3x_3) + (5x_1 + 2x_2 - x_3) \\
&\geq 2 \cdot 10 + 6
\end{aligned}
$$

Linear Programming – Lower Bounds

Optimize (i.e., minimize or maximize) a linear (*objective*) function subject to linear inequalities (*constraints*).

$$\begin{array}{lrcrcrcr}
\textbf{minimize} & 7x_1 & + & x_2 & + & 5x_3 & & \\
 & \rotatebox{90}{$\leq$} & & \rotatebox{90}{$\leq$} & & \rotatebox{90}{$\leq$} & & \\
\textbf{subject to} & 2\cdot\, x_1 & - & 2\cdot\, x_2 & + & 2\cdot 3x_3 & \geq & 2\cdot 10 \\
 & + & & + & & + & & \\
 & 5x_1 & + & 2x_2 & - & x_3 & \geq & 6 \\
 & & & & & x_1, x_2, x_3 & \geq & 0
\end{array}$$

$$7x_1 + x_2 + 5x_3 \geq x_1 - x_2 + 3x_3 \quad \Rightarrow \mathrm{OPT} \geq 10$$

$$\begin{aligned}
7x_1 + x_2 + 5x_3 &\geq (x_1 - x_2 + 3x_3) + (5x_1 + 2x_2 - x_3) \\
&\geq 10 + 6 \quad \Rightarrow \mathrm{OPT} \geq 16
\end{aligned}$$

$$\begin{aligned}
7x_1 + x_2 + 5x_3 &\geq 2 \cdot (x_1 - x_2 + 3x_3) + (5x_1 + 2x_2 - x_3) \\
&\geq 2 \cdot 10 + 6 \quad \Rightarrow \mathrm{OPT} \geq 26
\end{aligned}$$

Linear Programming – Lower Bounds

$$\begin{array}{lrcrcrcr}
\textbf{minimize} & 7x_1 & + & x_2 & + & 5x_3 & & \\
\textbf{subject to} & {\scriptstyle 2\cdot}\, x_1 & - & {\scriptstyle 2\cdot}\, x_2 & + & {\scriptstyle 2\cdot} 3x_3 & \geq & {\scriptstyle 2\cdot} 10 \\
 & 5x_1 & + & 2x_2 & - & x_3 & \geq & 6 \\
 & & & & & x_1, x_2, x_3 & \geq & 0
\end{array}$$

$$\begin{aligned}
7x_1 + x_2 + 5x_3 &\geq 2 \cdot (x_1 - x_2 + 3x_3) + 1 \cdot (5x_1 + 2x_2 - x_3) \\
&\geq 2 \cdot 10 + 1 \cdot 6 \quad \Rightarrow \text{OPT} \geq 2 \cdot 10 + 1 \cdot 6
\end{aligned}$$

Linear Programming – Lower Bounds

$$
\begin{array}{lrcrcrcr}
\textbf{minimize} & 7x_1 & + & x_2 & + & 5x_3 & & \\
\textbf{subject to}\; y_1(& x_1 & - & x_2 & + & 3x_3\,) & \geq & 10\; y_1 \\
 & 5x_1 & + & 2x_2 & - & x_3 & \geq & 6 \\
 & & & & & x_1, x_2, x_3 & \geq & 0
\end{array}
$$

$$
\begin{aligned}
7x_1 + x_2 + 5x_3 &\geq 2 \cdot (x_1 - x_2 + 3x_3) + 1 \cdot (5x_1 + 2x_2 - x_3) \\
&\geq 2 \cdot 10 + 1 \cdot 6 \qquad \Rightarrow \text{OPT} \geq 2 \cdot 10 + 1 \cdot 6
\end{aligned}
$$

Linear Programming – Lower Bounds

$$
\begin{array}{rrcrcrcrl}
\textbf{minimize} & 7x_1 & + & x_2 & + & 5x_3 & & & \\
\textbf{subject to}\; y_1(& x_1 & - & x_2 & + & 3x_3 &) & \geq & 10\; y_1 \\
y_2(& 5x_1 & + & 2x_2 & - & x_3 &) & \geq & 6\; y_2 \\
 & & & & & x_1, x_2, x_3 & & \geq & 0
\end{array}
$$

$$
\begin{aligned}
7x_1 + x_2 + 5x_3 \quad & \geq 2 \cdot (x_1 - x_2 + 3x_3) + 1 \cdot (5x_1 + 2x_2 - x_3) \\
& \geq 2 \cdot 10 + 1 \cdot 6 \qquad \Rightarrow \text{OPT} \geq 2 \cdot 10 + 1 \cdot 6
\end{aligned}
$$

Linear Programming – Lower Bounds

$$
\begin{array}{lrcrcrcrr}
\textbf{minimize} & 7x_1 & + & x_2 & + & 5x_3 & & & \\
\textbf{subject to}\; y_1(& x_1 & - & x_2 & + & 3x_3\;) & \geq & 10 & y_1 \\
y_2(& 5x_1 & + & 2x_2 & - & x_3\;) & \geq & 6 & y_2 \\
 & & & & & x_1, x_2, x_3 & \geq & 0 &
\end{array}
$$

$$
\begin{aligned}
7x_1 + x_2 + 5x_3 &\geq y_1 \cdot (x_1 - x_2 + 3x_3) + y_2 \cdot (5x_1 + 2x_2 - x_3) \\
&\geq y_1 \cdot 10 + y_2 \cdot 6 \Rightarrow \text{OPT} \geq 10y_1 + 6y_2
\end{aligned}
$$

Linear Programming – Lower Bounds

$$
\begin{array}{rrcrcrcl}
\textbf{minimize} & 7x_1 & + & x_2 & + & 5x_3 & & \\
\textbf{subject to}\; y_1(& x_1 & - & x_2 & + & 3x_3\,) & \geq & 10\; y_1 \\
y_2(& 5x_1 & + & 2x_2 & - & x_3\,) & \geq & 6\; y_2 \\
 & & & & & x_1, x_2, x_3 & \geq & 0
\end{array}
$$

$$
\begin{aligned}
7x_1 + x_2 + 5x_3 &\geq y_1 \cdot (x_1 - x_2 + 3x_3) + y_2 \cdot (5x_1 + 2x_2 - x_3) \\
&\geq y_1 \cdot 10 + y_2 \cdot 6 \Rightarrow \mathrm{OPT} \geq 10y_1 + 6y_2
\end{aligned}
$$

$10y_1 + 6y_2$ is lower bound for OPT

Linear Programming – Lower Bounds

$$\begin{array}{rrcrcrcl}
\textbf{minimize} & 7x_1 & + & x_2 & + & 5x_3 & & \\
\textbf{subject to}\; y_1(& x_1 & - & x_2 & + & 3x_3\,) & \geq & 10\;y_1 \\
y_2(& 5x_1 & + & 2x_2 & - & x_3\,) & \geq & 6\;y_2 \\
 & & & & & x_1, x_2, x_3 & \geq & 0
\end{array}$$

$$\begin{aligned}
7x_1 + x_2 + 5x_3 &\geq y_1 \cdot (x_1 - x_2 + 3x_3) + y_2 \cdot (5x_1 + 2x_2 - x_3) \\
&\geq y_1 \cdot 10 + y_2 \cdot 6 \Rightarrow \mathrm{OPT} \geq 10y_1 + 6y_2
\end{aligned}$$

Bounds for y_1, y_2:

Linear Programming – Lower Bounds

$$
\begin{array}{llcccccccl}
\textbf{minimize} & & 7x_1 & + & x_2 & + & 5x_3 & & & \\
& & \text{\rotatebox{90}{$\le$}} & & \text{\rotatebox{90}{$\le$}} & & \text{\rotatebox{90}{$\le$}} & & & \\
\textbf{subject to} & y_1(& x_1 & - & x_2 & + & 3x_3 &) & \geq & 10 \; y_1 \\
& & + & & + & & + & & & \\
& y_2(& 5x_1 & + & 2x_2 & - & x_3 &) & \geq & 6 \; y_2 \\
& & & & & & x_1, x_2, x_3 & & \geq & 0
\end{array}
$$

$$
\begin{aligned}
7x_1 + x_2 + 5x_3 \quad & \geq y_1 \cdot (x_1 - x_2 + 3x_3) + y_2 \cdot (5x_1 + 2x_2 - x_3) \\
& \geq y_1 \cdot 10 + y_2 \cdot 6 \Rightarrow \mathrm{OPT} \geq 10y_1 + 6y_2
\end{aligned}
$$

Bounds for y_1, y_2:

Linear Programming – Lower Bounds

$$\begin{array}{lrcrcrcl}
\textbf{minimize} & 7x_1 & + & x_2 & + & 5x_3 & & \\
 & \text{\rotatebox{90}{$\le$}} & & \text{\rotatebox{90}{$\le$}} & & \text{\rotatebox{90}{$\le$}} & & \\
\textbf{subject to}\ y_1(& x_1 & - & x_2 & + & 3x_3\ ) & \geq & 10\ y_1 \\
 & + & & + & & + & & \\
y_2(& 5x_1 & + & 2x_2 & - & x_3\ ) & \geq & 6\ y_2 \\
 & & & & & x_1, x_2, x_3 & \geq & 0
\end{array}$$

$$\begin{aligned}
7x_1 + x_2 + 5x_3 &\geq y_1 \cdot (x_1 - x_2 + 3x_3) + y_2 \cdot (5x_1 + 2x_2 - x_3) \\
&\geq y_1 \cdot 10 + y_2 \cdot 6 \Rightarrow \mathrm{OPT} \geq 10y_1 + 6y_2
\end{aligned}$$

$$y_1 \quad + \quad 5y_2 \quad \leq \quad 7$$

Bounds for y_1, y_2:

Linear Programming – Lower Bounds

$$\begin{array}{llcccccccc}
\textbf{minimize} & & 7x_1 & + & x_2 & + & 5x_3 & & & \\
& & \text{\textbar}\vee & & \text{\textbar}\vee & & \text{\textbar}\vee & & & \\
\textbf{subject to} & y_1(& x_1 & - & x_2 & + & 3x_3 &) & \geq & 10\ y_1 \\
& & + & & + & & + & & & \\
& y_2(& 5x_1 & + & 2x_2 & - & x_3 &) & \geq & 6\ y_2 \\
& & & & & & x_1, x_2, x_3 & & \geq & 0
\end{array}$$

$$\begin{aligned}
7x_1 + x_2 + 5x_3 &\geq y_1 \cdot (x_1 - x_2 + 3x_3) + y_2 \cdot (5x_1 + 2x_2 - x_3) \\
&\geq y_1 \cdot 10 + y_2 \cdot 6 \Rightarrow \mathrm{OPT} \geq 10y_1 + 6y_2
\end{aligned}$$

Bounds for y_1, y_2:

$$\begin{array}{rcrcl}
y_1 & + & 5y_2 & \leq & 7 \\
-y_1 & + & 2y_2 & \leq & 1
\end{array}$$

Linear Programming – Lower Bounds

$$
\begin{array}{rrcrcrcl}
\textbf{minimize} & 7x_1 & + & x_2 & + & 5x_3 & & \\
 & \rotatebox{90}{$\le$} & & \rotatebox{90}{$\le$} & & \rotatebox{90}{$\le$} & & \\
\textbf{subject to}\ y_1(& x_1 & - & x_2 & + & 3x_3\ ) & \geq & 10\ y_1 \\
 & + & & + & & + & & \\
y_2(& 5x_1 & + & 2x_2 & - & x_3\ ) & \geq & 6\ y_2 \\
 & & & & & x_1, x_2, x_3 & \geq & 0
\end{array}
$$

$$
\begin{aligned}
7x_1 + x_2 + 5x_3 &\geq y_1 \cdot (x_1 - x_2 + 3x_3) + y_2 \cdot (5x_1 + 2x_2 - x_3) \\
&\geq y_1 \cdot 10 + y_2 \cdot 6 \Rightarrow \mathrm{OPT} \geq 10y_1 + 6y_2
\end{aligned}
$$

Bounds for y_1, y_2:

$$
\begin{array}{rcrcl}
y_1 & + & 5y_2 & \leq & 7 \\
-y_1 & + & 2y_2 & \leq & 1 \\
3y_1 & - & y_2 & \leq & 5
\end{array}
$$

Linear Programming – Lower Bounds

$$
\begin{array}{llrcrcrcl}
\textbf{minimize} & & 7x_1 & + & x_2 & + & 5x_3 & & \\
& & \rotatebox{90}{$\geq$} & & \rotatebox{90}{$\geq$} & & \rotatebox{90}{$\geq$} & & \\
\textbf{subject to} & y_1(& x_1 & - & x_2 & + & 3x_3\,) & \geq & 10\ y_1 \\
& & + & & + & & + & & \\
& y_2(& 5x_1 & + & 2x_2 & - & x_3\,) & \geq & 6\ y_2 \\
& & & & & & x_1, x_2, x_3 & \geq & 0
\end{array}
$$

$$
\begin{aligned}
7x_1 + x_2 + 5x_3 &\geq y_1 \cdot (x_1 - x_2 + 3x_3) + y_2 \cdot (5x_1 + 2x_2 - x_3) \\
&\geq y_1 \cdot 10 + y_2 \cdot 6 \Rightarrow \mathrm{OPT} \geq 10y_1 + 6y_2
\end{aligned}
$$

maximize

Bounds for y_1, y_2:

$$
\begin{array}{rcrcl}
y_1 & + & 5y_2 & \leq & 7 \\
-y_1 & + & 2y_2 & \leq & 1 \\
3y_1 & - & y_2 & \leq & 5 \\
& & y_1, y_2 & \geq & 0
\end{array}
$$

Linear Programming – Lower Bounds

$$\begin{array}{lrcrcrcrl}
\textbf{minimize} & 7x_1 & + & x_2 & + & 5x_3 & & & \\
 & \text{IV} & & \text{IV} & & \text{IV} & & & \\
\textbf{subject to} \; y_1(& x_1 & - & x_2 & + & 3x_3 &) & \geq & 10 \; y_1 \\
 & + & & + & & + & & & \\
y_2(& 5x_1 & + & 2x_2 & - & x_3 &) & \geq & 6 \; y_2 \\
 & & & & & x_1, x_2, x_3 & & \geq & 0
\end{array}$$

$$\begin{aligned}
7x_1 + x_2 + 5x_3 \quad & \geq y_1 \cdot (x_1 - x_2 + 3x_3) + y_2 \cdot (5x_1 + 2x_2 - x_3) \\
& \geq y_1 \cdot 10 + y_2 \cdot 6 \Rightarrow \mathrm{OPT} \geq 10y_1 + 6y_2
\end{aligned}$$

$$\begin{array}{lrcrcl}
\textbf{maximize} & 10y_1 & + & 6y_2 & & \\
\textbf{subject to} & y_1 & + & 5y_2 & \leq & 7 \\
 & -y_1 & + & 2y_2 & \leq & 1 \\
 & 3y_1 & - & y_2 & \leq & 5 \\
 & & & y_1, y_2 & \geq & 0
\end{array}$$

Linear Programming – Lower Bounds

Primal

$$\begin{array}{lrcrcrcr}
\textbf{minimize} & 7x_1 & + & x_2 & + & 5x_3 & & \\
\textbf{subject to} & x_1 & - & x_2 & + & 3x_3 & \geq & 10 \\
 & 5x_1 & + & 2x_2 & - & x_3 & \geq & 6 \\
 & & & & & x_1, x_2, x_3 & \geq & 0
\end{array}$$

$$7x_1 + x_2 + 5x_3 \geq y_1 \cdot (x_1 - x_2 + 3x_3) + y_2 \cdot (5x_1 + 2x_2 - x_3)$$

$$\geq y_1 \cdot 10 + y_2 \cdot 6 \Rightarrow \text{OPT} \geq 10y_1 + 6y_2$$

$$\begin{array}{lrcrcr}
\textbf{maximize} & 10y_1 & + & 6y_2 & & \\
\textbf{subject to} & y_1 & + & 5y_2 & \leq & 7 \\
 & -y_1 & + & 2y_2 & \leq & 1 \\
 & 3y_1 & - & y_2 & \leq & 5 \\
 & & & y_1, y_2 & \geq & 0
\end{array}$$

Linear Programming – Lower Bounds

$$\begin{array}{lrcrcrcr}
\textbf{minimize} & 7x_1 & + & x_2 & + & 5x_3 & & \\
\textbf{subject to} & x_1 & - & x_2 & + & 3x_3 & \geq & 10 \\
 & 5x_1 & + & 2x_2 & - & x_3 & \geq & 6 \\
 & & & & & x_1, x_2, x_3 & \geq & 0
\end{array}$$

Primal

$$7x_1 + x_2 + 5x_3 \geq y_1 \cdot (x_1 - x_2 + 3x_3) + y_2 \cdot (5x_1 + 2x_2 - x_3)$$

$$\geq y_1 \cdot 10 + y_2 \cdot 6 \Rightarrow \text{OPT} \geq 10y_1 + 6y_2$$

$$\begin{array}{lrcrcr}
\textbf{maximize} & 10y_1 & + & 6y_2 & & \\
\textbf{subject to} & y_1 & + & 5y_2 & \leq & 7 \\
 & -y_1 & + & 2y_2 & \leq & 1 \\
 & 3y_1 & - & y_2 & \leq & 5 \\
 & & & y_1, y_2 & \geq & 0
\end{array}$$

Dual

Linear Programming – Lower Bounds

$$
\begin{array}{lrcrcrcr}
\textbf{minimize} & 7x_1 & + & x_2 & + & 5x_3 & & \\
\textbf{subject to} & x_1 & - & x_2 & + & 3x_3 & \geq & 10 \\
 & 5x_1 & + & 2x_2 & - & x_3 & \geq & 6 \\
 & & & & & x_1, x_2, x_3 & \geq & 0
\end{array}
\qquad \text{Primal}
$$

$$
\begin{aligned}
7x_1 + x_2 + 5x_3 &\geq y_1 \cdot (x_1 - x_2 + 3x_3) + y_2 \cdot (5x_1 + 2x_2 - x_3) \\
&\geq y_1 \cdot 10 + y_2 \cdot 6 \Rightarrow \mathrm{OPT} \geq 10y_1 + 6y_2
\end{aligned}
$$

$$
\begin{array}{lrcrcr}
\textbf{maximize} & 10y_1 & + & 6y_2 & & \\
\textbf{subject to} & y_1 & + & 5y_2 & \leq & 7 \\
 & -y_1 & + & 2y_2 & \leq & 1 \\
 & 3y_1 & - & y_2 & \leq & 5 \\
 & & & y_1, y_2 & \geq & 0
\end{array}
\qquad \text{Dual}
$$

Any feasible solution to the **dual** program provides a lower bound for the optimum of the **primal** program.

Linear Programming – Lower Bounds

$$\begin{array}{lrcrcrcr} \textbf{minimize} & 7x_1 & + & x_2 & + & 5x_3 & & \\ \textbf{subject to} & x_1 & - & x_2 & + & 3x_3 & \geq & 10 \\ & 5x_1 & + & 2x_2 & - & x_3 & \geq & 6 \\ & & & & & x_1, x_2, x_3 & \geq & 0 \end{array}$$

Primal

$$\begin{aligned} 7x_1 + x_2 + 5x_3 &\geq y_1 \cdot (x_1 - x_2 + 3x_3) + y_2 \cdot (5x_1 + 2x_2 - x_3) \\ &\geq y_1 \cdot 10 + y_2 \cdot 6 \Rightarrow \mathrm{OPT} \geq 10y_1 + 6y_2 \end{aligned}$$

$$\begin{array}{lrcrcr} \textbf{maximize} & 10y_1 & + & 6y_2 & & \\ \textbf{subject to} & y_1 & + & 5y_2 & \leq & 7 \\ & -y_1 & + & 2y_2 & \leq & 1 \\ & 3y_1 & - & y_2 & \leq & 5 \\ & & & y_1, y_2 & \geq & 0 \end{array}$$

Dual

Any feasible solution to the **dual** program provides a lower bound for the optimum of the **primal** program.

Both $x = (\frac{7}{4}, 0, \frac{11}{4})$ and $y = (2, 1)$ provide objective value 26.

Linear Programming – Lower Bounds

$$
\begin{array}{lrcrcrcr}
\textbf{minimize} & 7x_1 & + & x_2 & + & 5x_3 & & \\
\textbf{subject to} & x_1 & - & x_2 & + & 3x_3 & \geq & 10 \\
 & 5x_1 & + & 2x_2 & - & x_3 & \geq & 6 \\
 & & & & & x_1, x_2, x_3 & \geq & 0
\end{array}
\qquad \text{Primal}
$$

$$
\begin{aligned}
7x_1 + x_2 + 5x_3 &\geq y_1 \cdot (x_1 - x_2 + 3x_3) + y_2 \cdot (5x_1 + 2x_2 - x_3) \\
&\geq y_1 \cdot 10 + y_2 \cdot 6 \Rightarrow \text{OPT} \geq 10y_1 + 6y_2
\end{aligned}
$$

$$
\begin{array}{lrcrcr}
\textbf{maximize} & 10y_1 & + & 6y_2 & & \\
\textbf{subject to} & y_1 & + & 5y_2 & \leq & 7 \\
 & -y_1 & + & 2y_2 & \leq & 1 \\
 & 3y_1 & - & y_2 & \leq & 5 \\
 & & & y_1, y_2 & \geq & 0
\end{array}
\qquad \text{Dual}
$$

Any feasible solution to the **dual** program provides a lower bound for the optimum of the **primal** program.

Both $x = (\frac{7}{4}, 0, \frac{11}{4})$ and $y = (2, 1)$ provide objective value 26. // OPT

Primal–Dual

primal program

$$\begin{array}{lrcl} \textbf{minimize} & c^\intercal x & & \\ \textbf{subject to} & Ax & \geq & b \\ & x & \geq & 0 \end{array}$$

Primal–Dual

primal program

$$\begin{array}{lrcl} \textbf{minimize} & c^{\intercal} x & & \\ \textbf{subject to} & Ax & \geq & b \\ & x & \geq & 0 \end{array}$$

dual program

$$\begin{array}{lrcl} \textbf{maximize} & b^{\intercal} y & & \\ \textbf{subject to} & A^{\intercal} y & \leq & c \\ & y & \geq & 0 \end{array}$$

Primal–Dual

primal program

$$\begin{array}{lrcl} \textbf{minimize} & c^\intercal x & & \\ \textbf{subject to} & Ax & \geq & b \\ & x & \geq & 0 \end{array}$$

dual program

$$\begin{array}{lrcl} \textbf{maximize} & b^\intercal y & & \\ \textbf{subject to} & A^\intercal y & \leq & c \\ & y & \geq & 0 \end{array}$$

dual of the dual program

Primal–Dual

primal program

$$\begin{array}{lrcl} \textbf{minimize} & c^\intercal x & & \\ \textbf{subject to} & Ax & \geq & b \\ & x & \geq & 0 \end{array}$$

dual program

$$\begin{array}{lrcl} \textbf{maximize} & b^\intercal y & & \\ \textbf{subject to} & A^\intercal y & \leq & c \\ & y & \geq & 0 \end{array}$$

dual of the dual program

$$\begin{array}{lrcl} \textbf{minimize} & c^\intercal x & & \\ \textbf{subject to} & Ax & \geq & b \\ & x & \geq & 0 \end{array}$$

Approximation Algorithms

Lecture 4:
Linear Programming and LP-Duality

Part IV:
LP-Duality and Complementary Slackness

LP-Duality

Primal

$$\begin{array}{lrcl} \textbf{minimize} & c^{\mathsf{T}} x & & \\ \textbf{subject to} & Ax & \geq & b \\ & x & \geq & 0 \end{array}$$

Dual

$$\begin{array}{lrcl} \textbf{maximize} & b^{\mathsf{T}} y & & \\ \textbf{subject to} & A^{\mathsf{T}} y & \leq & c \\ & y & \geq & 0 \end{array}$$

LP-Duality

Primal

$$\begin{array}{lrcl} \textbf{minimize} & c^\intercal x & & \\ \textbf{subject to} & Ax & \geq & b \\ & x & \geq & 0 \end{array}$$

Dual

$$\begin{array}{lrcl} \textbf{maximize} & b^\intercal y & & \\ \textbf{subject to} & A^\intercal y & \leq & c \\ & y & \geq & 0 \end{array}$$

Theorem. The primal program has a finite optimum
$\Leftrightarrow$ the dual program has a finite optimum.

LP-Duality

$$\begin{array}{llll} \textbf{minimize} & c^\intercal x & & \\ \textbf{subject to} & Ax & \geq & b \\ & x & \geq & 0 \end{array}$$

Primal

$$\begin{array}{llll} \textbf{maximize} & b^\intercal y & & \\ \textbf{subject to} & A^\intercal y & \leq & c \\ & y & \geq & 0 \end{array}$$

Dual

Theorem. The primal program has a finite optimum $\Leftrightarrow$ the dual program has a finite optimum. Moreover, if $x^* = (x_1^*, \ldots, x_n^*)$ and $y^* = (y_1^*, \ldots, y_m^*)$ are *optimal* solutions for the primal and dual program, respectively, then

LP-Duality

$$\begin{array}{lrcl} \textbf{minimize} & c^\intercal x & & \\ \textbf{subject to} & Ax & \geq & b \\ & x & \geq & 0 \end{array}$$

Primal

$$\begin{array}{lrcl} \textbf{maximize} & b^\intercal y & & \\ \textbf{subject to} & A^\intercal y & \leq & c \\ & y & \geq & 0 \end{array}$$

Dual

Theorem. The primal program has a finite optimum $\Leftrightarrow$ the dual program has a finite optimum. Moreover, if $x^* = (x_1^*, \ldots, x_n^*)$ and $y^* = (y_1^*, \ldots, y_m^*)$ are *optimal* solutions for the primal and dual program, respectively, then

$$\sum_{j=1}^{n} c_j x_j^* = \sum_{i=1}^{m} b_i y_i^* .$$

Weak LP-Duality

minimize	$c^\intercal x$		
subject to	Ax	$\geq$	b
	x	$\geq$	0

maximize	$b^\intercal y$		
subject to	$A^\intercal y$	$\leq$	c
	y	$\geq$	0

Theorem. If $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_m)$ are *valid* solutions for the primal and dual program, resp., then

Weak LP-Duality

$$\begin{array}{llcl} \textbf{minimize} & c^\intercal x & & \\ \textbf{subject to} & Ax & \geq & b \\ & x & \geq & 0 \end{array}$$

$$\begin{array}{llcl} \textbf{maximize} & b^\intercal y & & \\ \textbf{subject to} & A^\intercal y & \leq & c \\ & y & \geq & 0 \end{array}$$

Theorem. If $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_m)$ are *valid* solutions for the primal and dual program, resp., then

$$\sum_{j=1}^{n} c_j x_j \geq \sum_{i=1}^{m} b_i y_i \,.$$

Weak LP-Duality

$$\begin{array}{llcl} \textbf{minimize} & c^\intercal x & & \\ \textbf{subject to} & Ax & \geq & b \\ & x & \geq & 0 \end{array}$$

$$\begin{array}{llcl} \textbf{maximize} & b^\intercal y & & \\ \textbf{subject to} & A^\intercal y & \leq & c \\ & y & \geq & 0 \end{array}$$

Theorem. If $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_m)$ are *valid* solutions for the primal and dual program, resp., then

$$\sum_{j=1}^{n} c_j x_j \geq \sum_{i=1}^{m} b_i y_i \,.$$

Proof.

Weak LP-Duality

$$\begin{array}{llcl} \textbf{minimize} & c^\intercal x & & \\ \textbf{subject to} & Ax & \geq & b \\ & x & \geq & 0 \end{array}$$

$$\begin{array}{llcl} \textbf{maximize} & b^\intercal y & & \\ \textbf{subject to} & A^\intercal y & \leq & c \\ & y & \geq & 0 \end{array}$$

Theorem. If $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_m)$ are *valid* solutions for the primal and dual program, resp., then

$$\sum_{j=1}^{n} c_j x_j \geq \sum_{i=1}^{m} b_i y_i \, .$$

Proof.

$$\sum_{j=1}^{n} c_j x_j \geq \qquad \sum_{i=1}^{m} b_i y_i \, .$$

Weak LP-Duality

minimize	$c^\intercal x$		
subject to	Ax	$\geq$	b
	x	$\geq$	0

maximize	$b^\intercal y$		
subject to	$A^\intercal y$	$\leq$	c
	y	$\geq$	0

Theorem. If $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_m)$ are *valid* solutions for the primal and dual program, resp., then

$$\sum_{j=1}^{n} c_j x_j \geq \sum_{i=1}^{m} b_i y_i \, .$$

Proof.

$$\sum_{j=1}^{n} \boxed{c_j} x_j \geq \qquad\qquad \sum_{i=1}^{m} b_i y_i \, .$$

Weak LP-Duality

$$\begin{array}{lrcl} \textbf{minimize} & c^\intercal x & & \\ \textbf{subject to} & Ax & \geq & b \\ & x & \geq & 0 \end{array}$$

$$\begin{array}{lrcl} \textbf{maximize} & b^\intercal y & & \\ \textbf{subject to} & \boxed{A^\intercal y \leq c} & & \\ & y & \geq & 0 \end{array}$$

Theorem. If $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_m)$ are *valid* solutions for the primal and dual program, resp., then

$$\sum_{j=1}^{n} c_j x_j \geq \sum_{i=1}^{m} b_i y_i \,.$$

Proof.

$$\sum_{j=1}^{n} \boxed{c_j} x_j \geq \qquad\qquad \sum_{i=1}^{m} b_i y_i \,.$$

Weak LP-Duality

minimize	$c^\intercal x$		
subject to	Ax	$\geq$	b
	$\boxed{x \quad \geq \quad 0}$		

maximize	$b^\intercal y$		
subject to	$\boxed{A^\intercal y \quad \leq \quad c}$		
	y	$\geq$	0

Theorem. If $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_m)$ are *valid* solutions for the primal and dual program, resp., then

$$\sum_{j=1}^{n} c_j x_j \geq \sum_{i=1}^{m} b_i y_i \, .$$

Proof.

$$\sum_{j=1}^{n} \boxed{c_j} x_j \geq \qquad\qquad \sum_{i=1}^{m} b_i y_i \, .$$

Weak LP-Duality

minimize	$c^\intercal x$		
subject to	Ax	$\geq$	b
	x	$\geq$	0

maximize	$b^\intercal y$		
subject to	$A^\intercal y$	$\leq$	c
	y	$\geq$	0

Theorem. If $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_m)$ are *valid* solutions for the primal and dual program, resp., then

$$\sum_{j=1}^{n} c_j x_j \geq \sum_{i=1}^{m} b_i y_i \,.$$

Proof.

$$\sum_{j=1}^{n} \boxed{c_j} x_j \geq \sum_{j=1}^{n} \boxed{\left(\sum_{i=1}^{m} a_{ij} y_i \right)} x_j = \qquad \geq \sum_{i=1}^{m} b_i y_i \,.$$

Weak LP-Duality

$$\begin{array}{lrcl} \textbf{minimize} & c^\intercal x & & \\ \textbf{subject to} & Ax & \geq & b \\ & \boxed{x} & \boxed{\geq} & \boxed{0} \end{array}$$

$$\begin{array}{lrcl} \textbf{maximize} & b^\intercal y & & \\ \textbf{subject to} & \boxed{A^\intercal y} & \boxed{\leq} & \boxed{c} \\ & y & \geq & 0 \end{array}$$

Theorem. If $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_m)$ are *valid* solutions for the primal and dual program, resp., then

$$\sum_{j=1}^{n} c_j x_j \geq \sum_{i=1}^{m} b_i y_i \,.$$

Proof.

$$\sum_{j=1}^{n} \boxed{c_j} x_j \geq \sum_{j=1}^{n} \boxed{\left(\sum_{i=1}^{m} a_{ij} y_i \right)} x_j = \sum_{j=1}^{n} \left(\sum_{i=1}^{m} a_{ij} y_i x_j \right) \geq \sum_{i=1}^{m} b_i y_i \,.$$

Weak LP-Duality

$$\begin{array}{lrcl} \textbf{minimize} & c^\intercal x & & \\ \textbf{subject to} & Ax & \geq & b \\ & \boxed{x \geq 0} & & \end{array} \qquad \begin{array}{lrcl} \textbf{maximize} & b^\intercal y & & \\ \textbf{subject to} & \boxed{A^\intercal y \leq c} & & \\ & y & \geq & 0 \end{array}$$

Theorem. If $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_m)$ are *valid* solutions for the primal and dual program, resp., then

$$\sum_{j=1}^{n} c_j x_j \geq \sum_{i=1}^{m} b_i y_i \,.$$

Proof.

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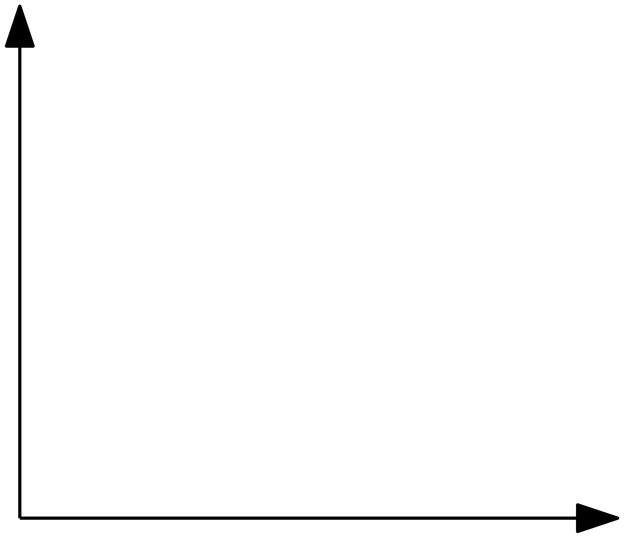

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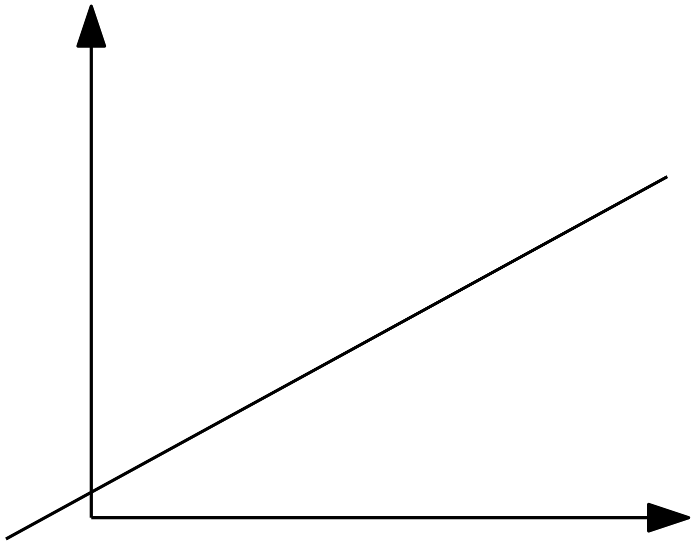

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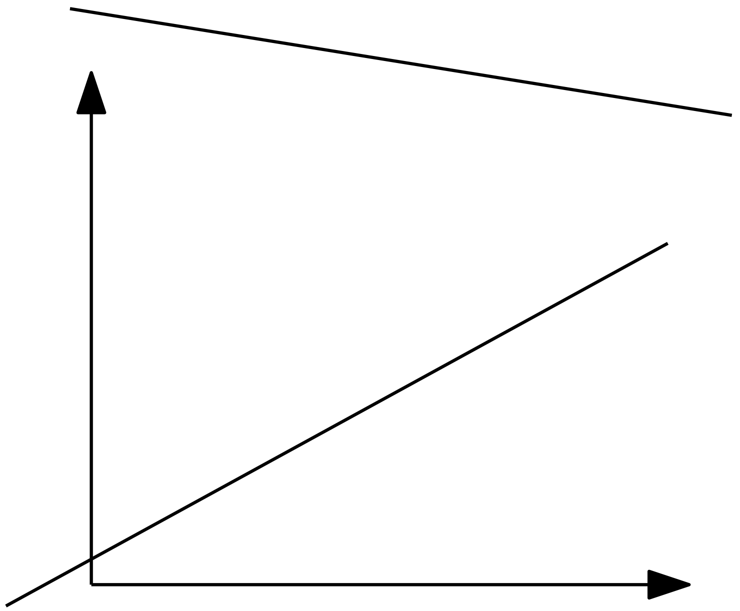

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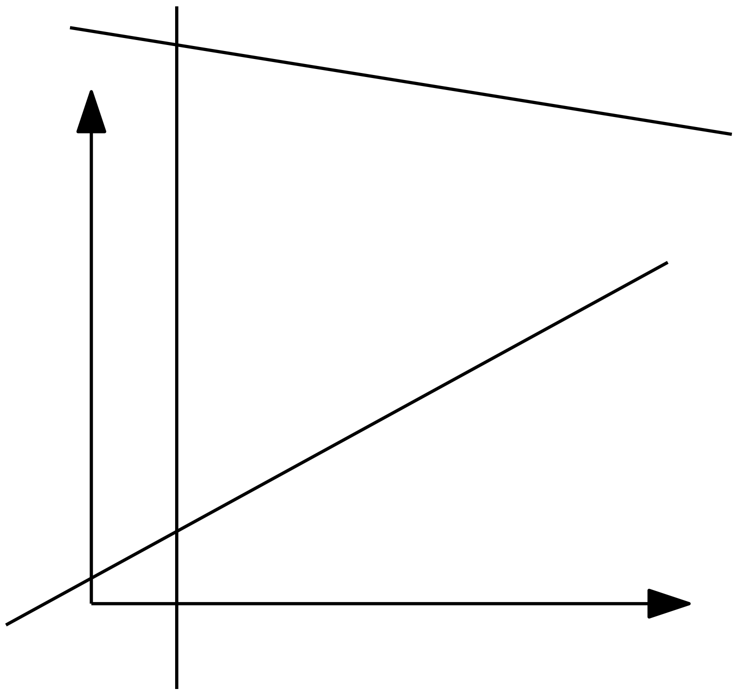

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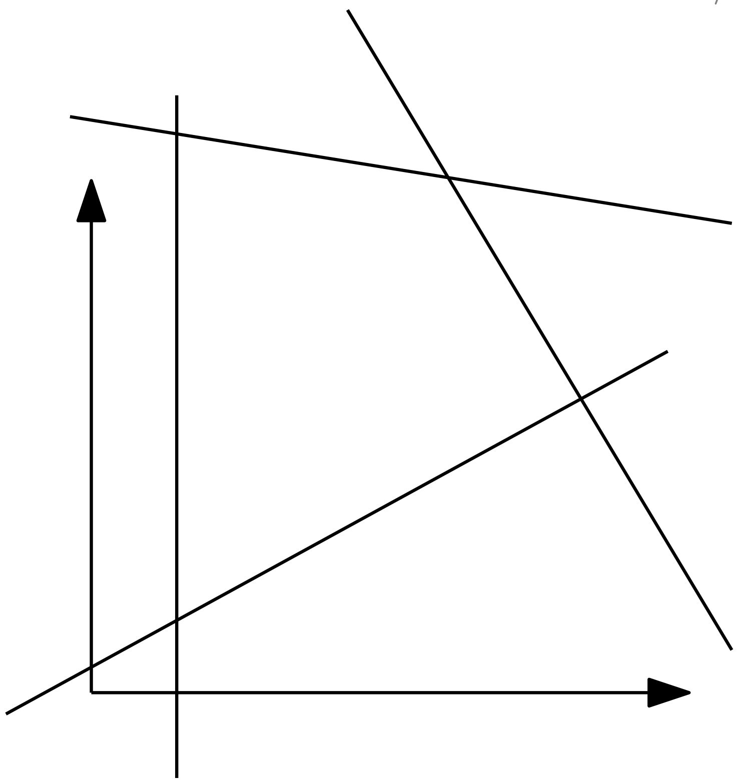

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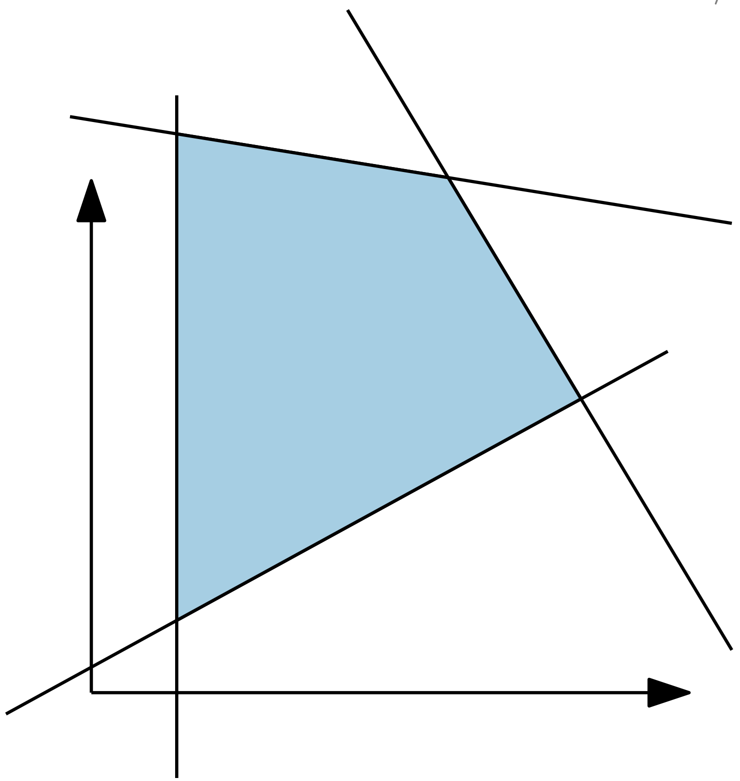

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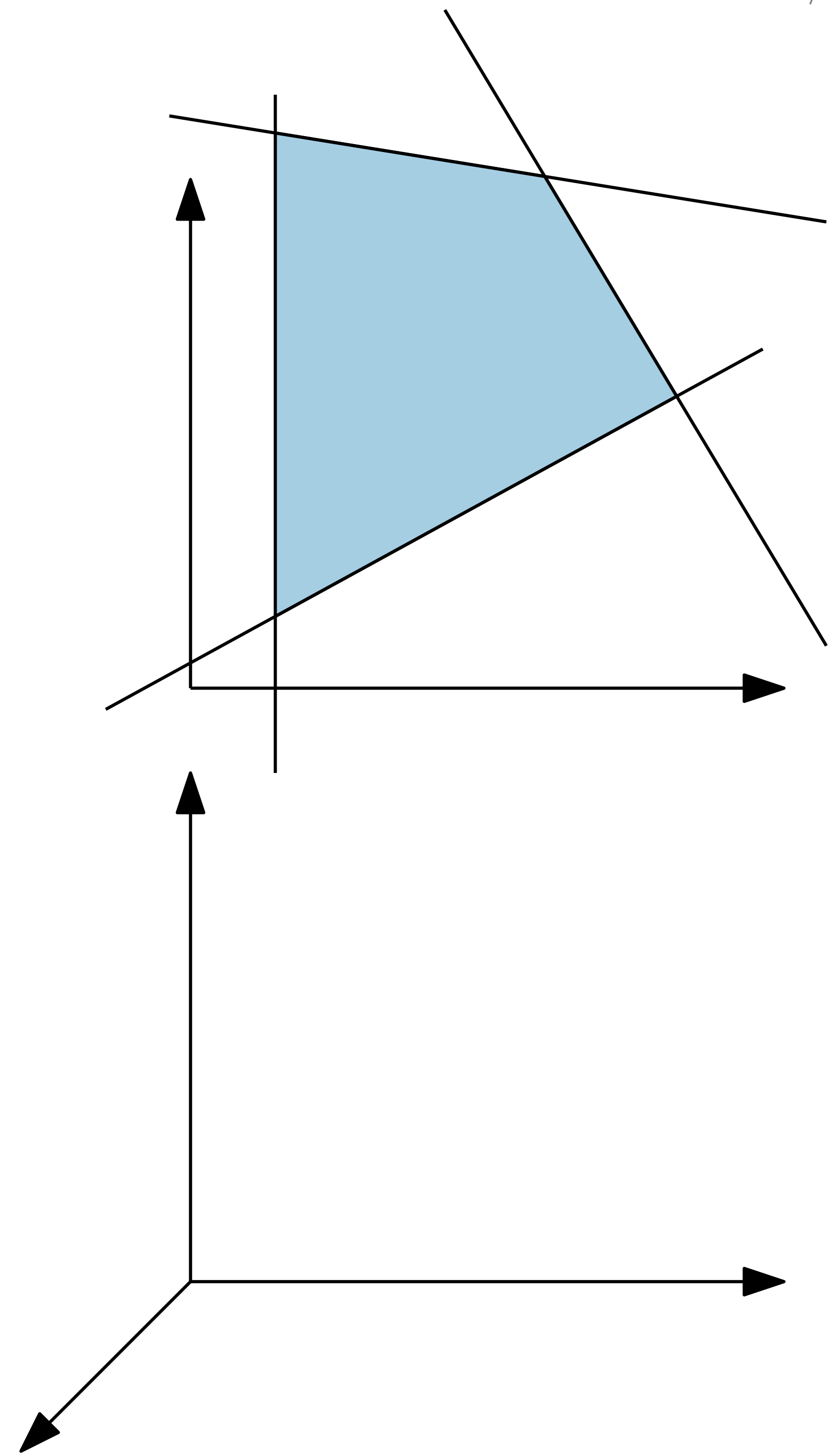

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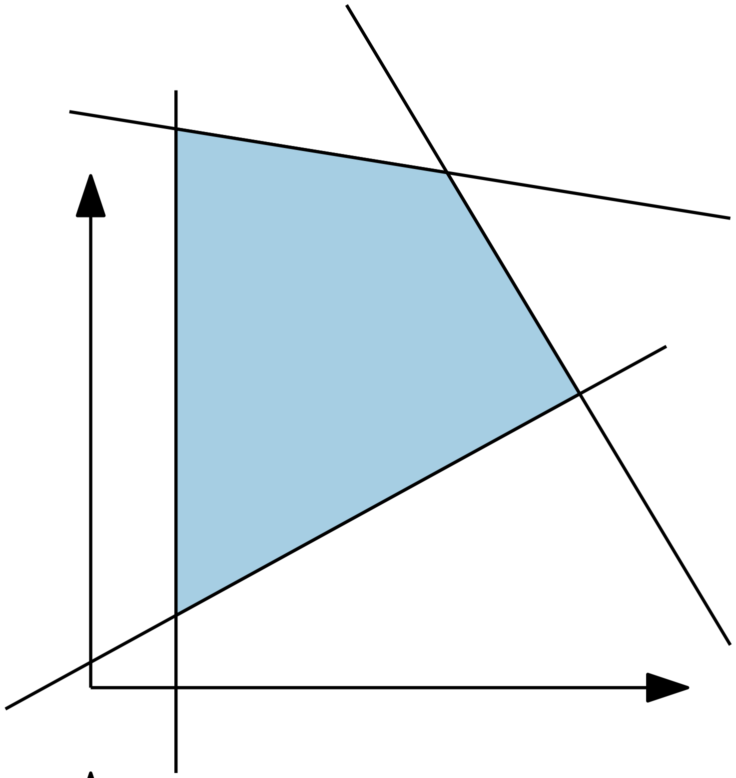

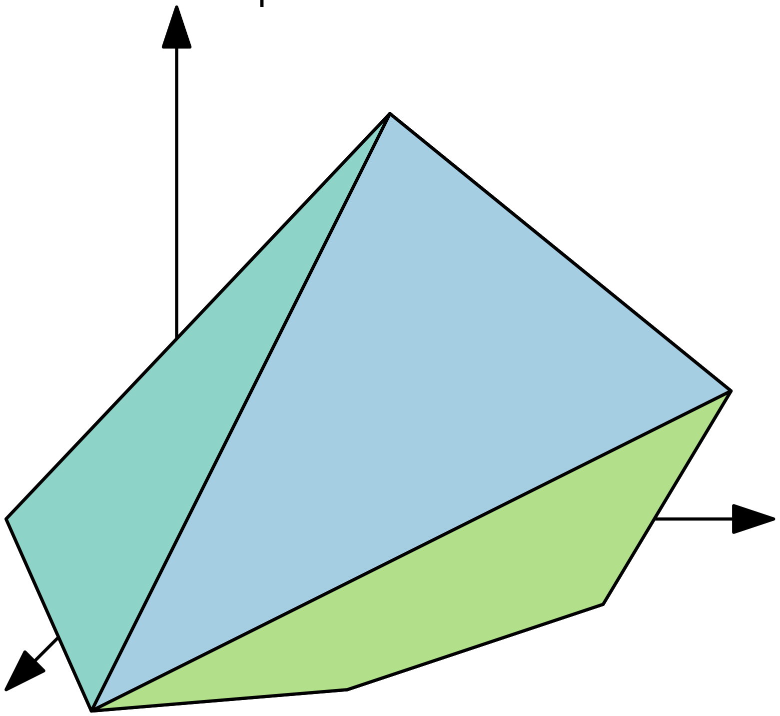

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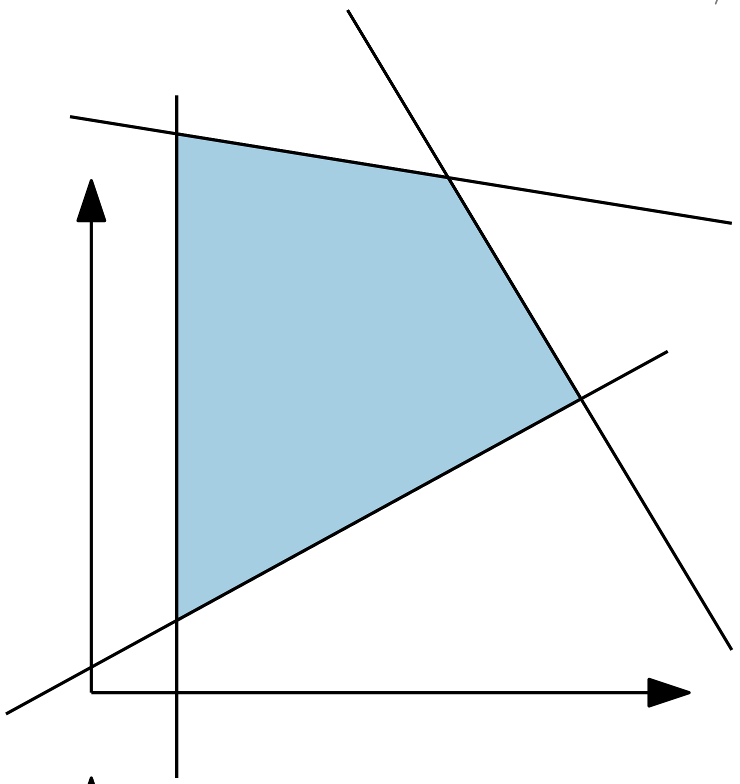

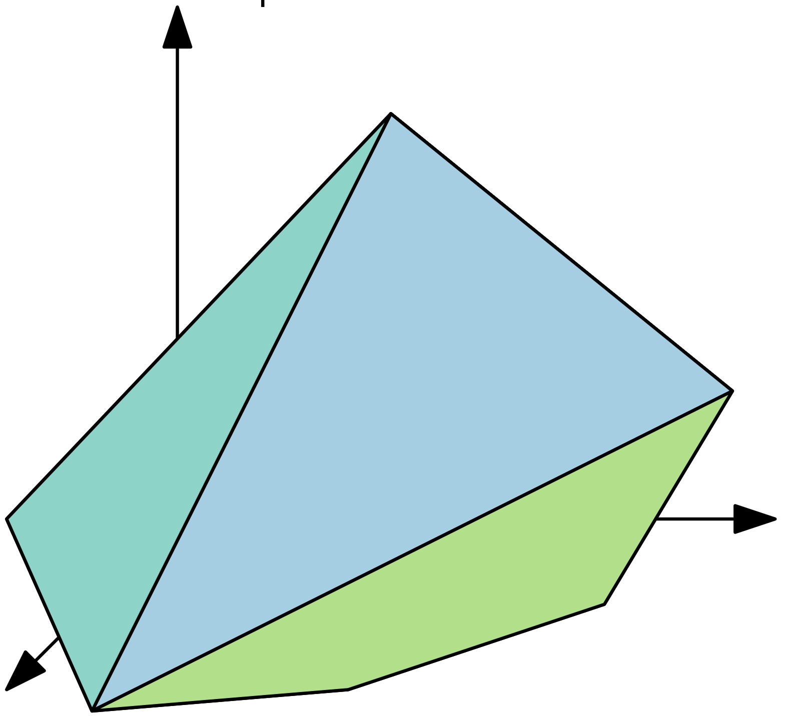

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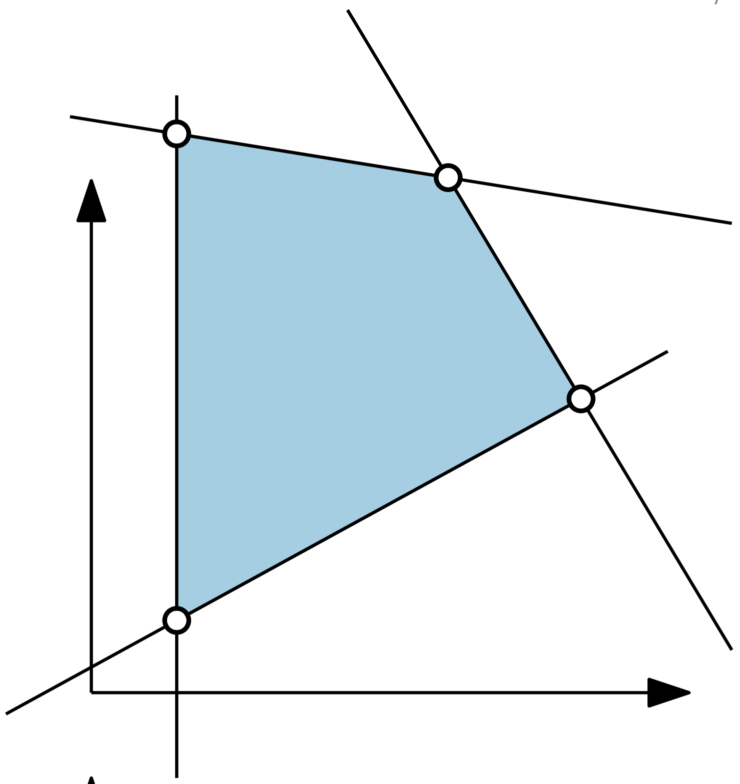

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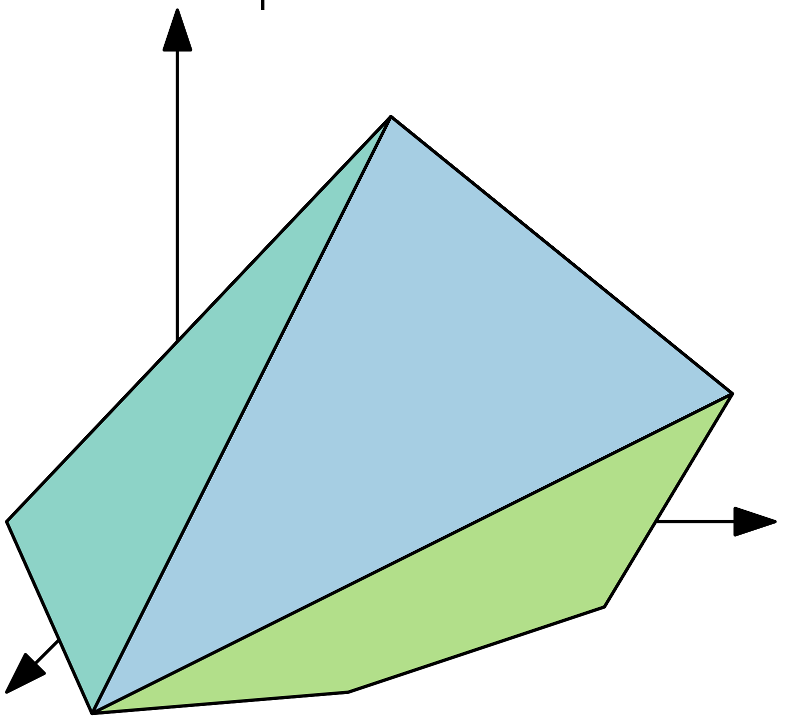

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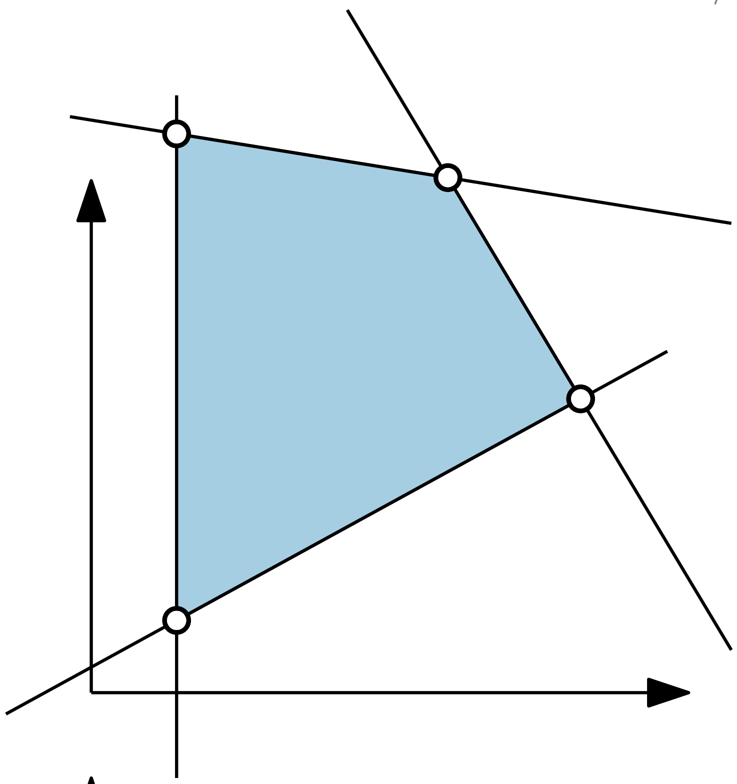

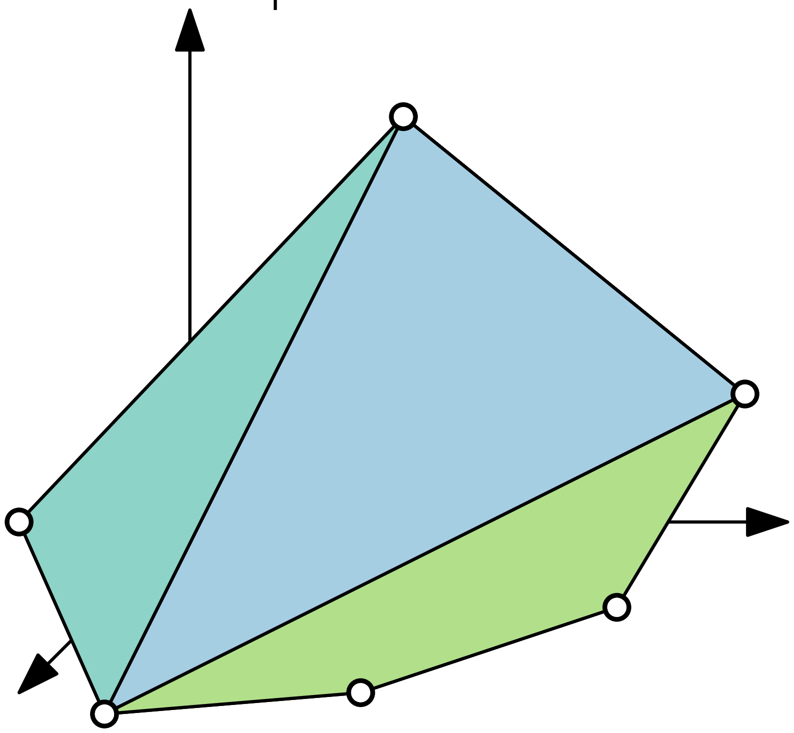

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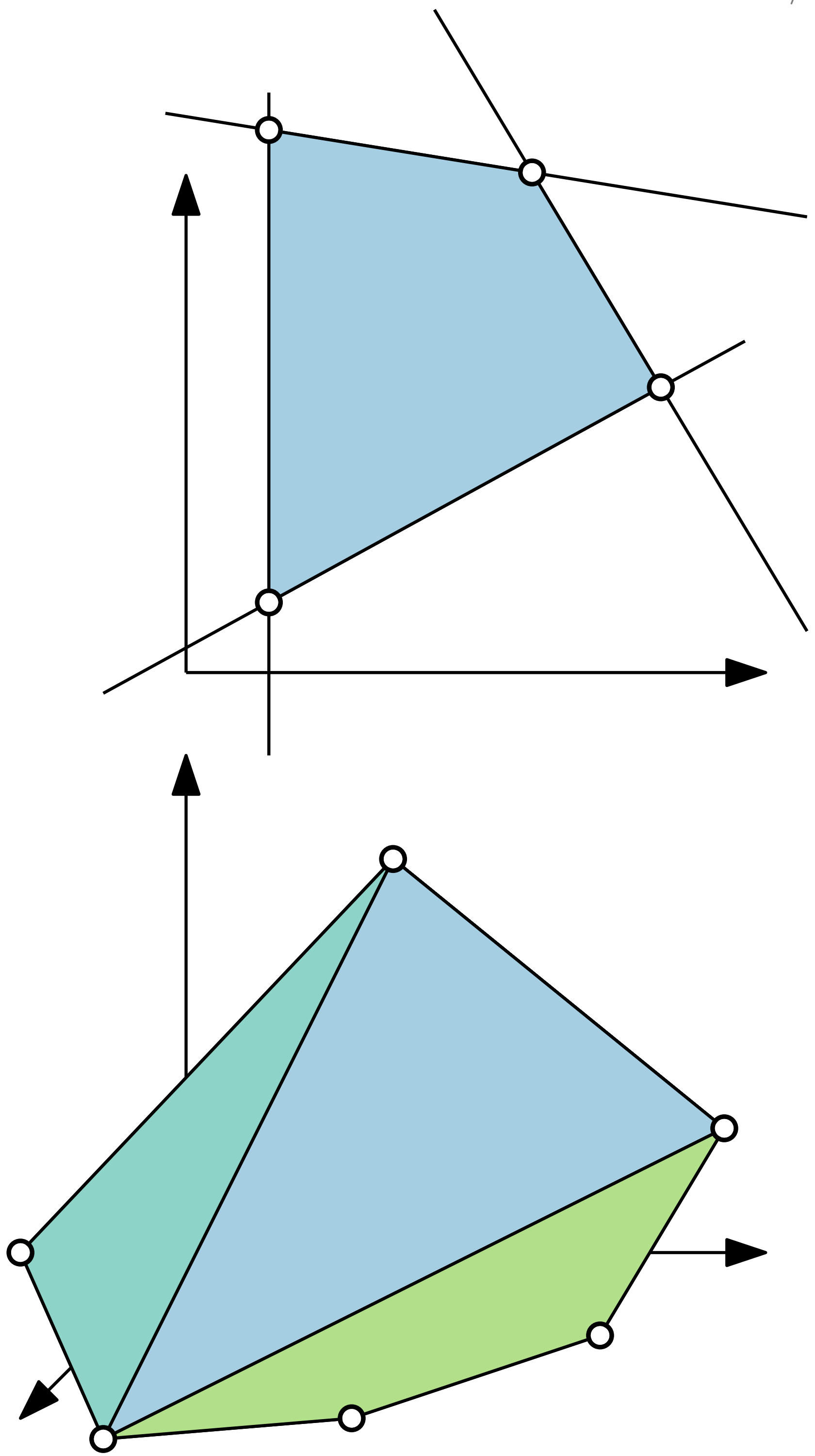

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LP-relaxation provides a lower bound: $\mathrm{OPT}_{\mathrm{LP}} \leq \mathrm{OPT}_{\mathrm{ILP}}$

Approximation Algorithms

Lecture 4:
Linear Programming and LP-Duality

Part V:
Min–Max Relationships

Max-Flow Problem

Given: A directed graph G with edge capacities $c\colon E(G) \to \mathbb{Q}_+$ and two special vertices: the source s and sink t.

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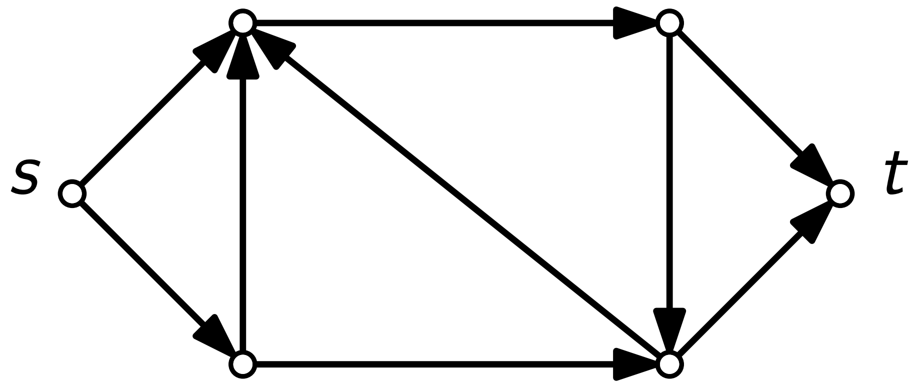

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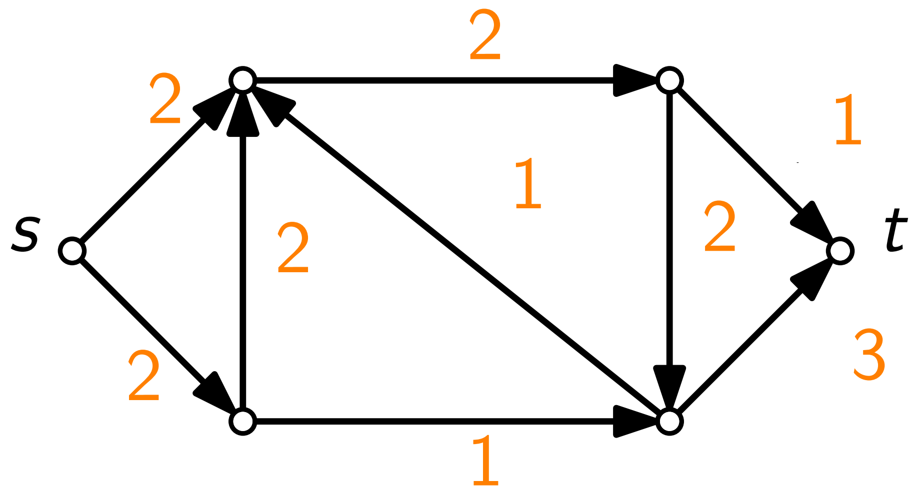

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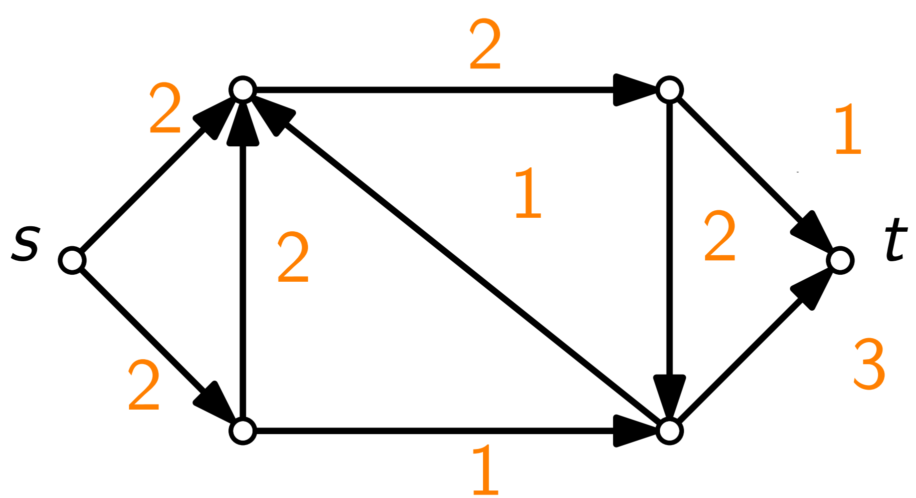
Find: A maximum s–t flow (i.e., non-negative edge weights f) such that

- $f(u, v) \leq c(u, v)$ for each edge $(u, v) \in E(G)$,



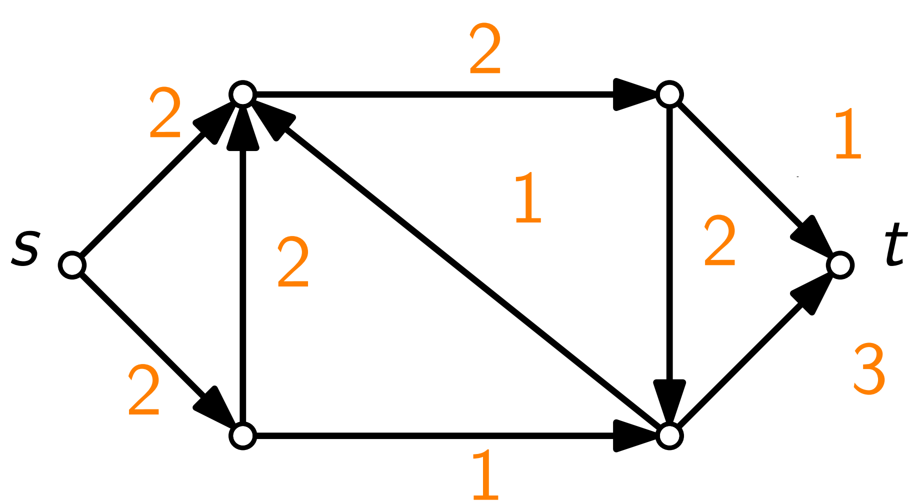
Max-Flow Problem

Given: A directed graph G with edge capacities $c\colon E(G) \to \mathbb{Q}_+$ and two special vertices: the source s and sink t.

Find: A maximum s–t flow (i.e., non-negative edge weights f) such that

- $f(u,v) \leq c(u,v)$ for each edge $(u,v) \in E(G)$,
- $\sum_{u\colon (u,v)\in E} f(u,v) = \sum_{z\colon (v,z)\in E} f(v,z)$ for each $v \in V(G) \setminus \{s,t\}$.

The **flow value** is the inflow to t minus the outflow from t.



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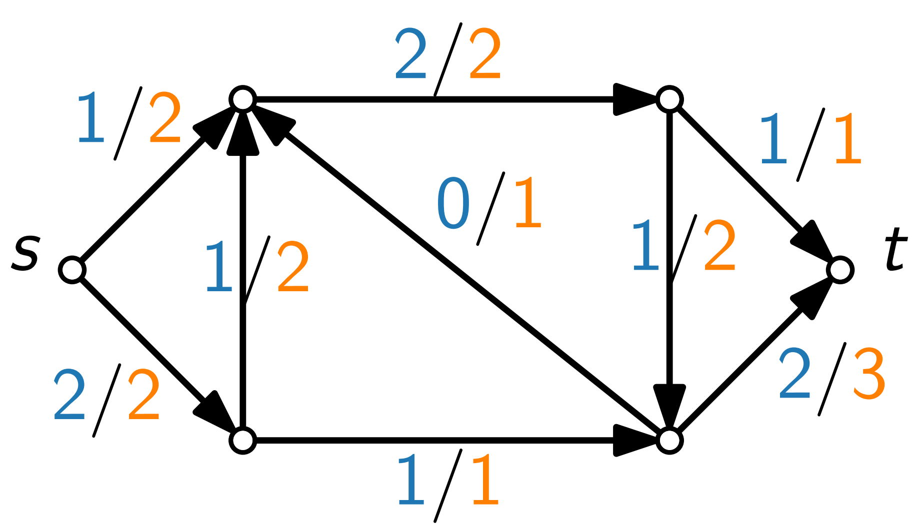
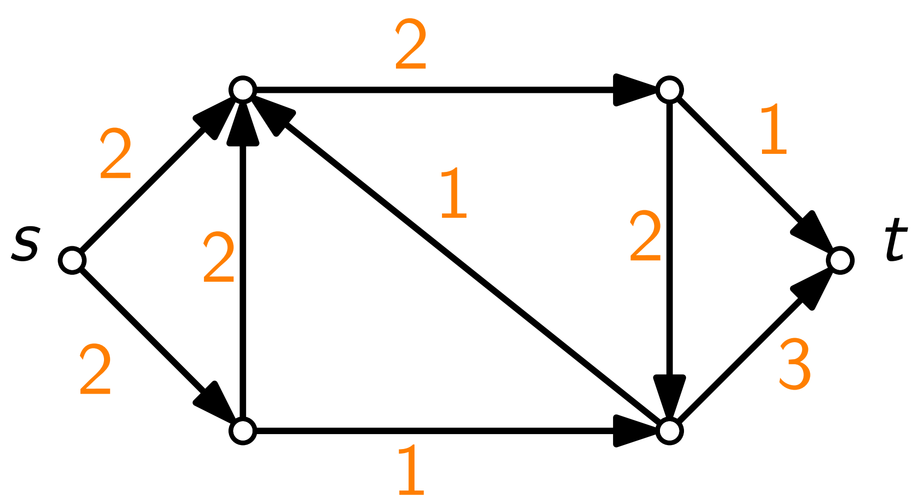


Min-Cut Problem

Given: A directed graph G with edge capacities $c\colon E(G) \to \mathbb{Q}_+$ and two special vertices: the source s and sink t.

Find: An s–t cut, i.e., a vertex set X with $s \in X$ and $t \in \overline{X}$ such that the total weight $c(X, \overline{X})$ of the edges from X to $\overline{X}$ is minimum.



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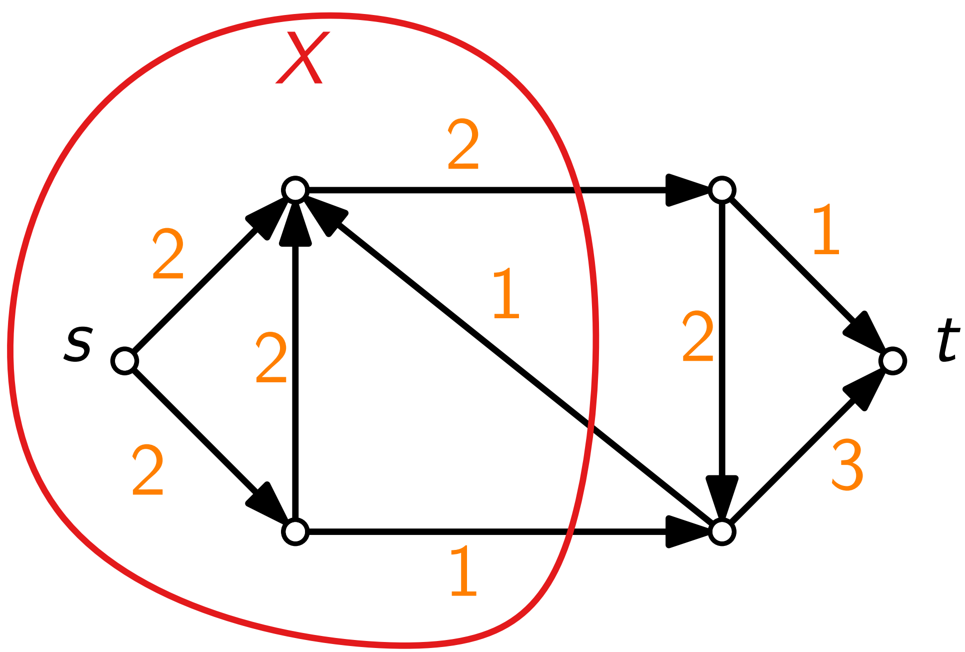

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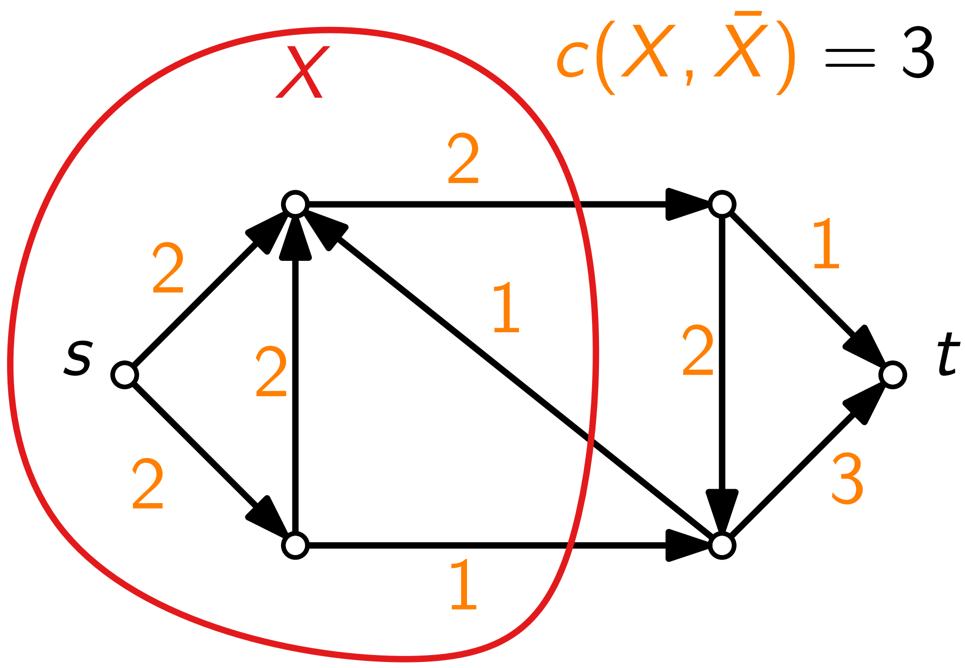

Max-Flow-Min-Cut Theorem

Theorem. The value of a maximum s–t flow and the weight of a minimum s–t cut are the same.

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Proof. Special case of LP-Duality . . .

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maximize
subject to

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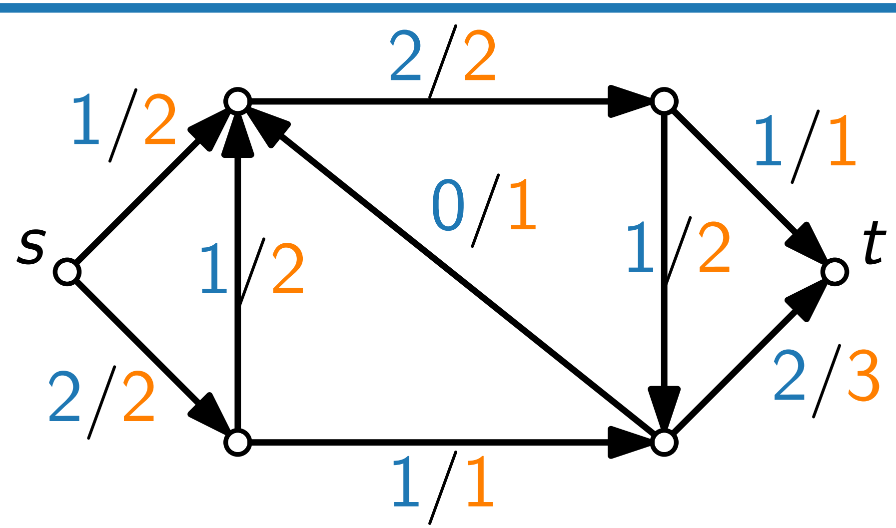

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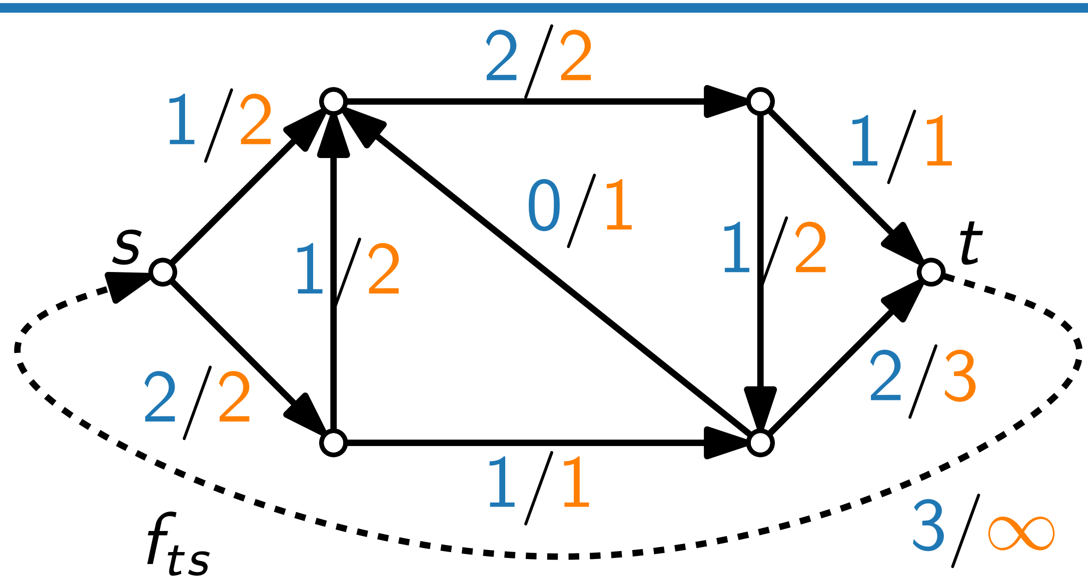

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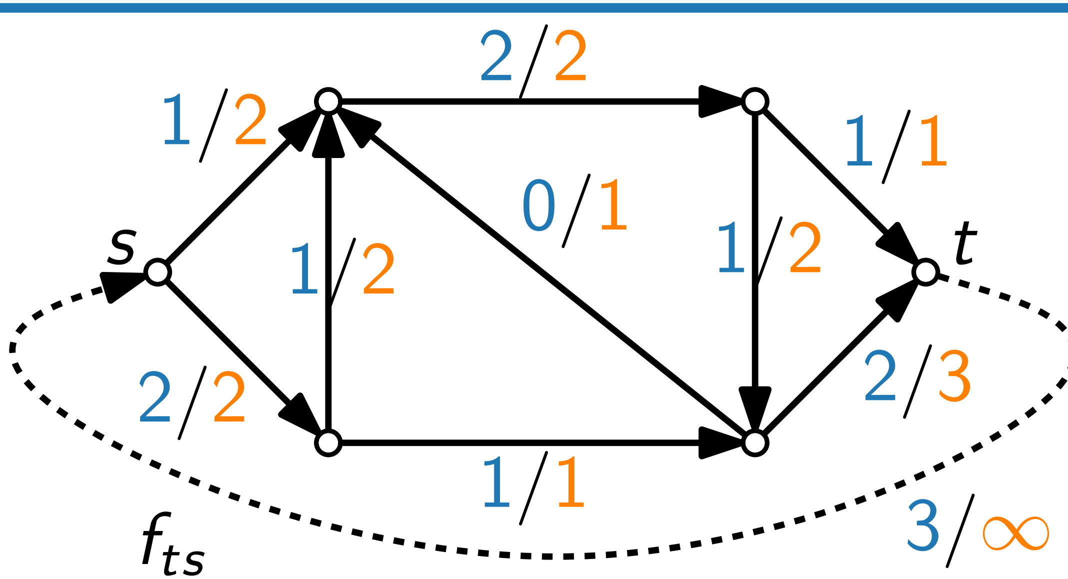

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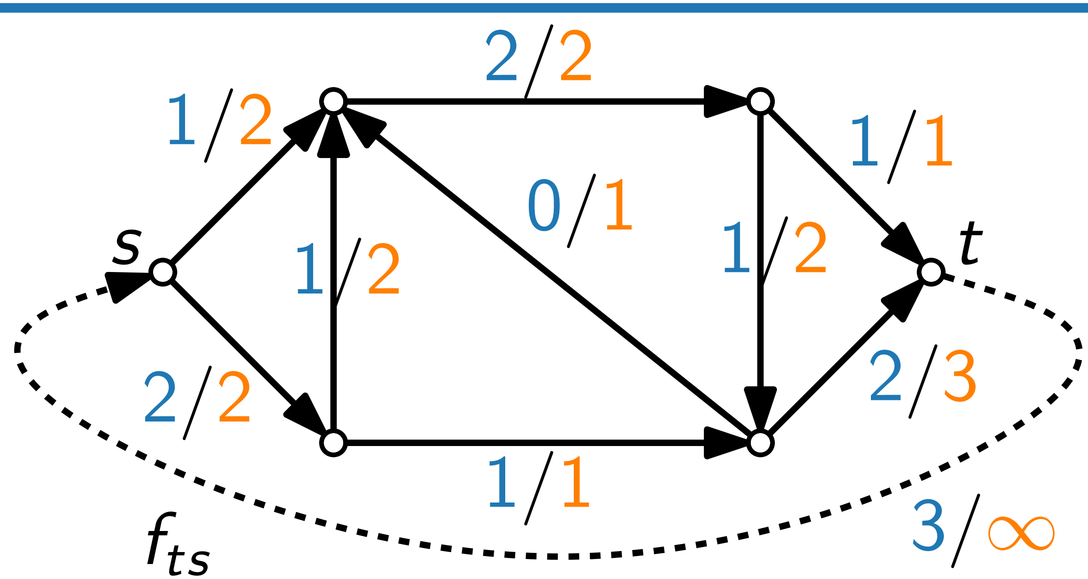
$$\sum_{u\colon (u,v)\in E(G)} f_{uv} - \sum_{z\colon (v,z)\in E(G)} f_{vz} \leq 0 \quad \forall v \in V(G)$$



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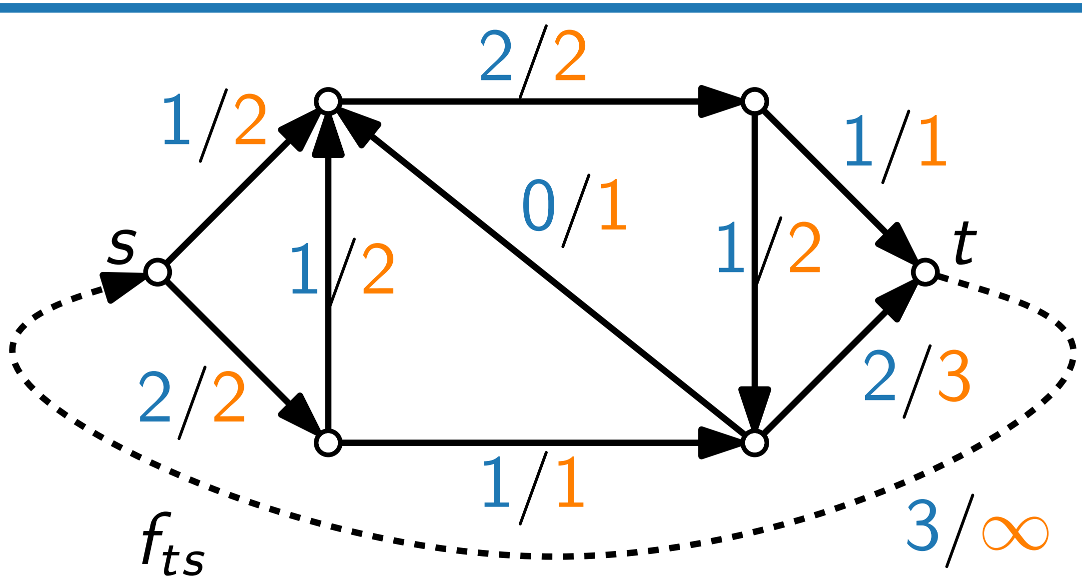

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maximize $c^{\mathsf{T}} x$

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Which constraints contain f_{uv} for $(u,v) \neq (t,s)$? d_{uv}, p_u, p_v

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	$\sum_{u\colon (u,v)\in E(G)} f_{uv} - \sum_{z\colon (v,z)\in E(G)} f_{vz} \leq 0$		$\forall v \in V(G)$	p_v
		$f_{uv} \geq 0$	$\forall (u,v) \in E(G)$	

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\end{array}
$$

$$
\begin{array}{lrcll}
\textbf{subject to} & d_{uv} - p_u + p_v & \geq & 0 & \forall (u,v) \in E(G) \setminus \{(t,s)\} \\
& p_s - p_t & \geq & 1 & \\
& d_{uv} & \geq & 0 & \forall (u,v) \in E(G) \\
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$$

$$
\begin{array}{llll}
\textbf{minimize} & \displaystyle\sum_{(u,v)\in E(G)\setminus\{(t,s)\}} c_{uv} \cdot d_{uv} & & \\
\textbf{subject to} & d_{uv} - p_u + p_v & \ge 0 & \forall (u,v) \in E(G) \setminus \{(t,s)\} \\
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Approximation Algorithms

Lecture 4:
Linear Programming and LP-Duality

Part VI:
Dual LP of Max Flow

Dual LP – Interpretation as ILP

$$
\begin{array}{lrll}
\textbf{minimize} & \displaystyle\sum_{(u,v)\in E(G)\setminus\{(t,s)\}} c_{uv}\cdot d_{uv} & & \\
\textbf{subject to} & d_{uv} - p_u + p_v & \geq 0 & \forall (u,v) \neq (t,s) \\
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\textbf{subject to} & d_{uv}-p_u+p_v \geq 0 & & \forall (u,v)\neq(t,s) \\
 & p_s-p_t \geq 1 & & \\
 & d_{uv} \cancel{\geq 0} \in \{0,1\} & & \forall (u,v)\in E(G) \\
 & p_u \cancel{\geq 0} \in \{0,1\} & & \forall u\in V(G)
\end{array}
$$

Dual LP – Interpretation as ILP

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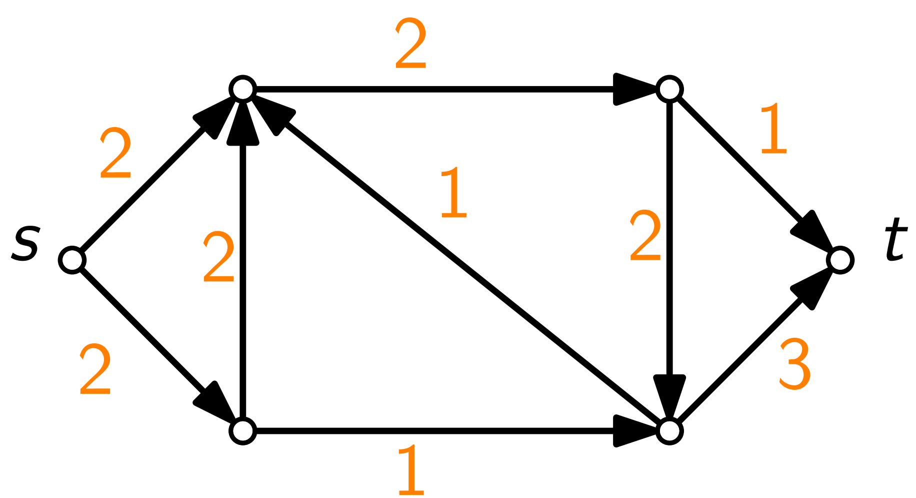
\end{array}$$



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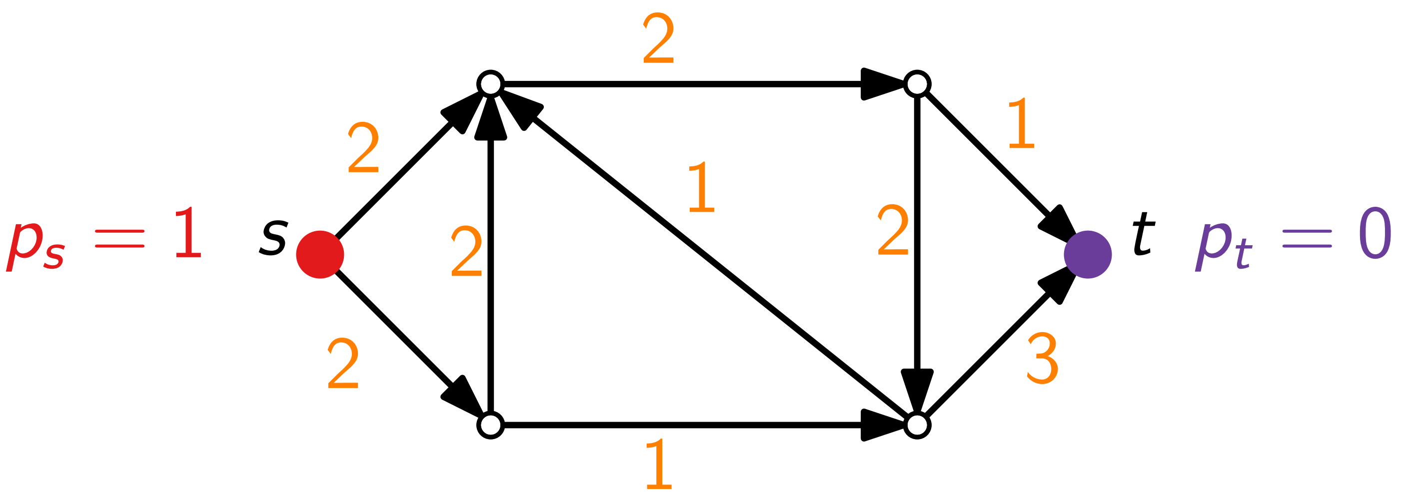
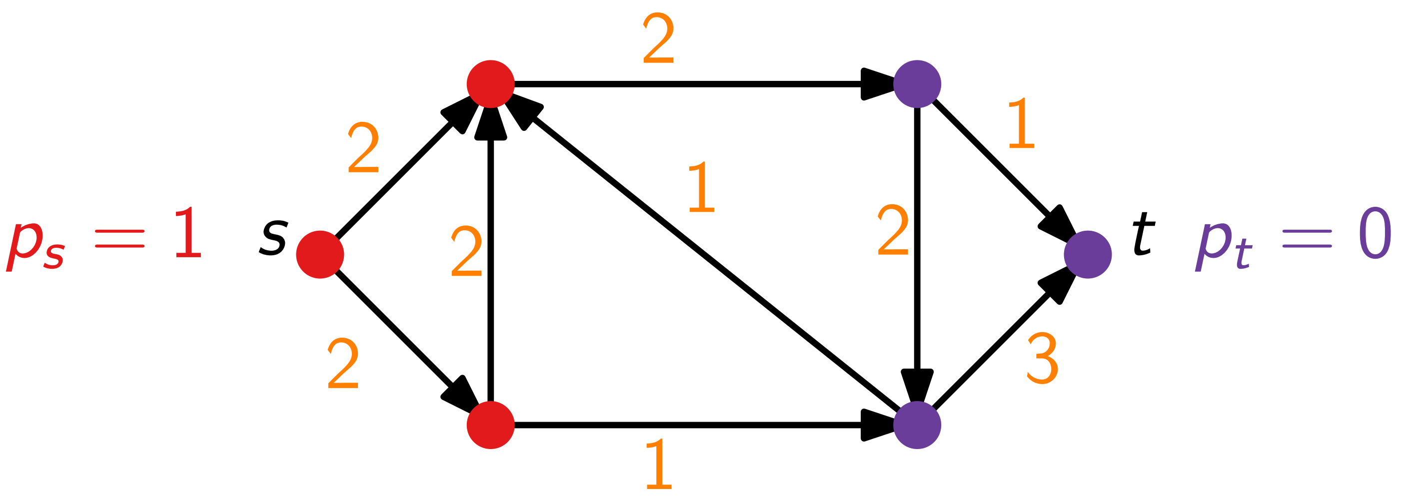
\textbf{subject to} & d_{uv} - p_u + p_v \geq 0 & & \forall (u,v)\neq(t,s) \\
& p_s - p_t \geq 1 & & \\
& d_{uv}\ \cancel{\geq 0} \in \{0,1\} & & \forall (u,v)\in E(G) \\
& p_u\ \cancel{\geq 0} \in \{0,1\} & & \forall u\in V(G)
\end{array}
$$



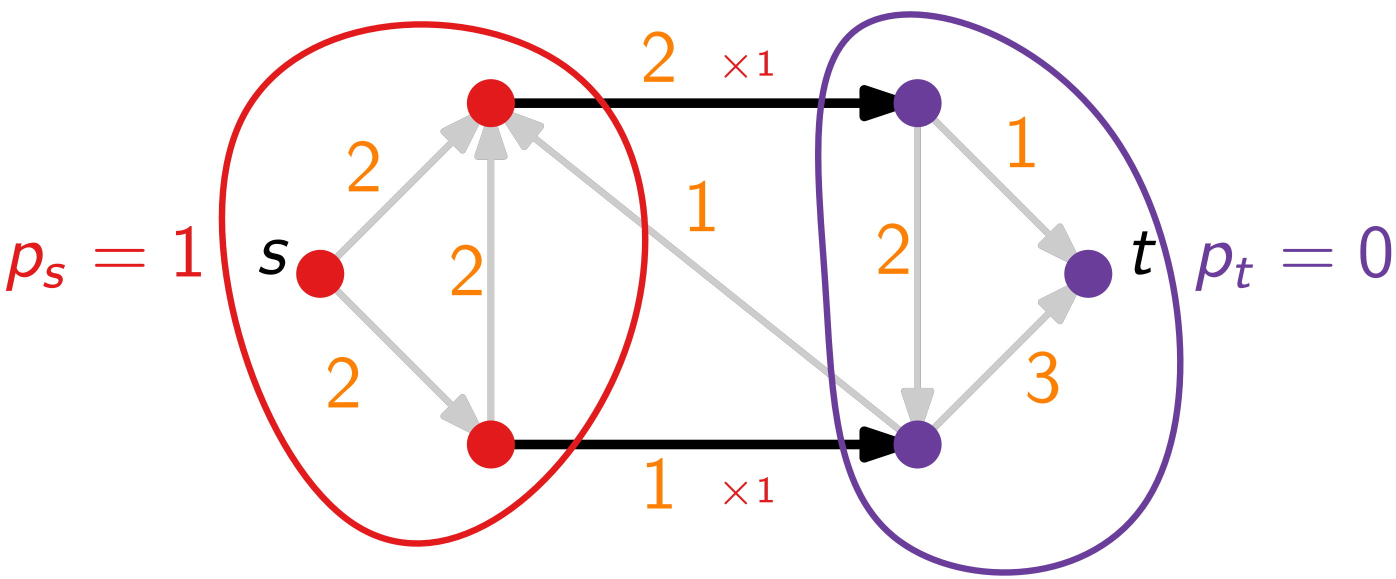
Dual LP – Interpretation as ILP

$$
\begin{array}{llll}
\textbf{minimize} & \displaystyle\sum_{(u,v)\in E(G)\setminus\{(t,s)\}} c_{uv}\cdot d_{uv} & & \\
\textbf{subject to} & d_{uv} - p_u + p_v \geq 0 & & \forall (u,v) \neq (t,s) \\
& p_s - p_t \geq 1 & & \\
& d_{uv} \cancel{\geq 0} \in \{0,1\} & & \forall (u,v) \in E(G) \\
& p_u \cancel{\geq 0} \in \{0,1\} & & \forall u \in V(G)
\end{array}
$$



Dual LP – Interpretation as ILP

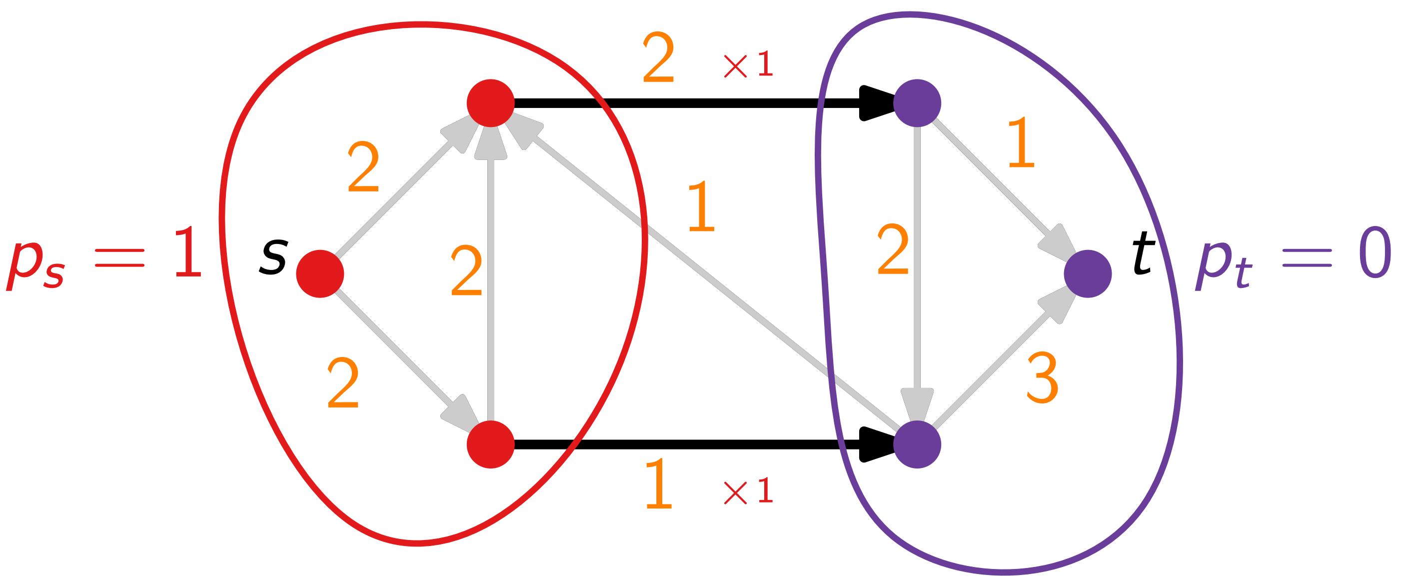
$$
\begin{aligned}
\textbf{minimize} \quad & \sum_{(u,v)\in E(G)\setminus\{(t,s)\}} c_{uv} \cdot d_{uv} \\
\textbf{subject to} \quad & d_{uv} - p_u + p_v \geq 0 \qquad \forall (u,v) \neq (t,s) \\
& p_s - p_t \geq 1 \\
& d_{uv} \cancel{\geq 0} \in \{0,1\} \quad \forall (u,v) \in E(G) \\
& p_u \cancel{\geq 0} \in \{0,1\} \quad \forall u \in V(G)
\end{aligned}
$$



Dual LP – Interpretation as ILP

$$
\begin{array}{lrll}
\textbf{minimize} & \displaystyle\sum_{(u,v)\in E(G)\setminus\{(t,s)\}} c_{uv}\cdot d_{uv} & & \\
\textbf{subject to} & d_{uv} - p_u + p_v \geq 0 & & \forall (u,v) \neq (t,s) \\
 & p_s - p_t \geq 1 & & \\
 & d_{uv} \cancel{\geq 0} \in \{0,1\} & & \forall (u,v) \in E(G) \\
 & p_u \cancel{\geq 0} \in \{0,1\} & & \forall u \in V(G)
\end{array}
$$

equivalent to Min-Cut!



Dual LP – Fractional Cuts

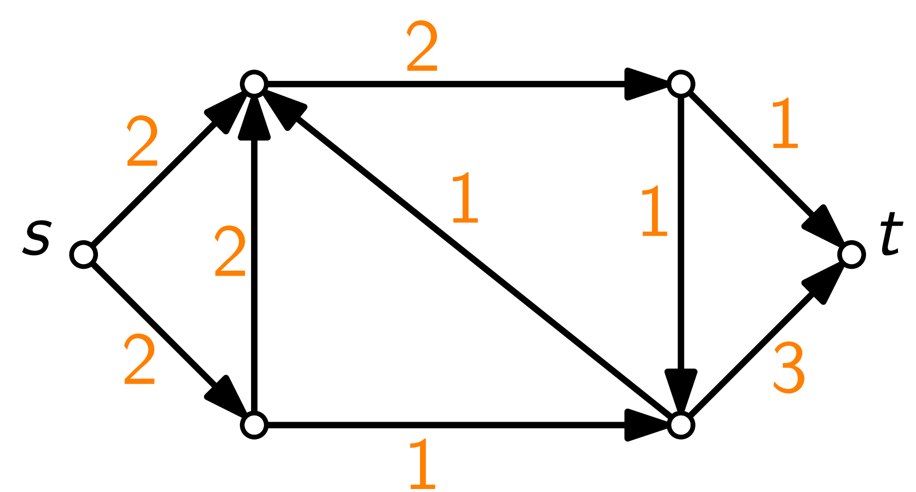
LP-relaxation of the ILP

$$
\begin{aligned}
\textbf{minimize} \quad & \sum_{(u,v)\in E(G)\setminus\{(t,s)\}} c_{uv} \cdot d_{uv} \\
\textbf{subject to} \quad & d_{uv} - p_u + p_v \geq 0 \quad \forall (u,v) \in E(G) \setminus \{(t,s)\} \\
& p_s - p_t \geq 1 \\
& d_{uv} \geq 0 \quad \forall (u,v) \in E(G) \\
& p_u \geq 0 \quad \forall u \in V(G)
\end{aligned}
$$

Dual LP – Fractional Cuts

LP-relaxation of the ILP

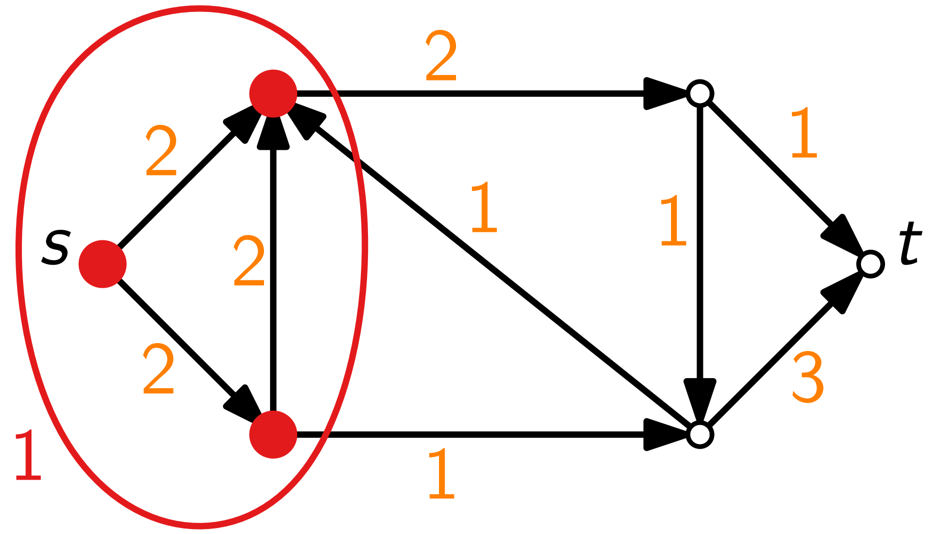
$$
\begin{aligned}
\textbf{minimize} \quad & \sum_{(u,v)\in E(G)\setminus\{(t,s)\}} c_{uv} \cdot d_{uv} \\
\textbf{subject to} \quad & d_{uv} - p_u + p_v \geq 0 \quad \forall (u,v) \in E(G) \setminus \{(t,s)\} \\
& p_s - p_t \geq 1 \\
& d_{uv} \geq 0 \quad \forall (u,v) \in E(G) \\
& p_u \geq 0 \quad \forall u \in V(G)
\end{aligned}
$$



Dual LP – Fractional Cuts

LP-relaxation of the ILP

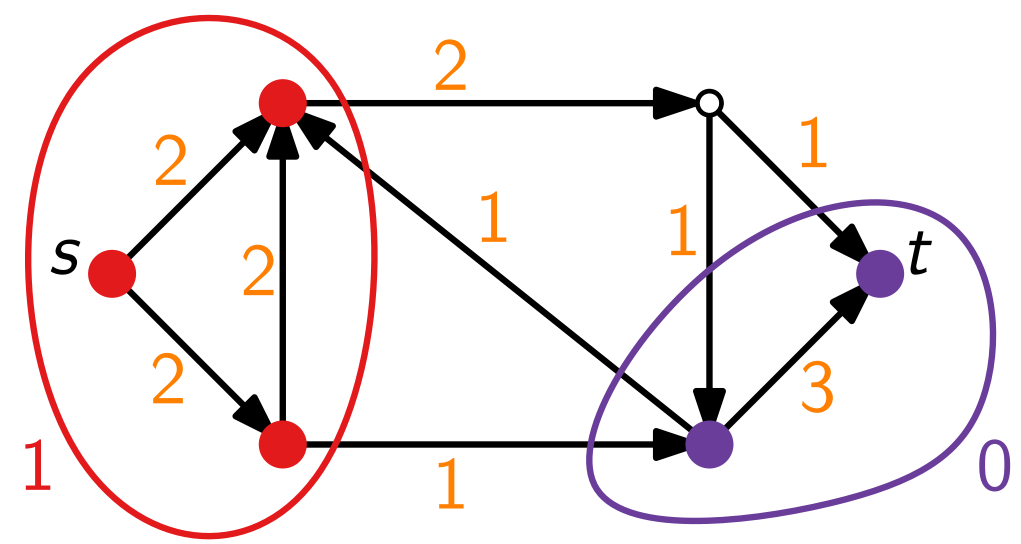
$$\begin{aligned}
\textbf{minimize} \quad & \sum_{(u,v)\in E(G)\setminus\{(t,s)\}} c_{uv} \cdot d_{uv} \\
\textbf{subject to} \quad & d_{uv} - p_u + p_v \geq 0 \quad \forall (u,v) \in E(G) \setminus \{(t,s)\} \\
& p_s - p_t \geq 1 \\
& d_{uv} \geq 0 \quad \forall (u,v) \in E(G) \\
& p_u \geq 0 \quad \forall u \in V(G)
\end{aligned}$$



Dual LP – Fractional Cuts

LP-relaxation of the ILP

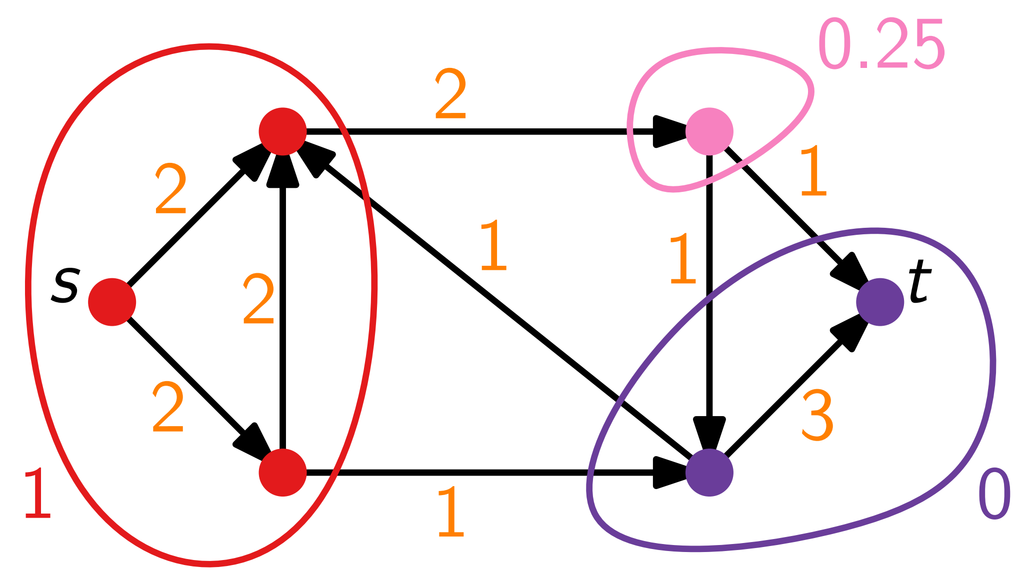
$$
\begin{array}{llll}
\textbf{minimize} & \displaystyle\sum_{(u,v)\in E(G)\setminus\{(t,s)\}} c_{uv}\cdot d_{uv} & & \\
\textbf{subject to} & d_{uv}-p_u+p_v & \geq 0 & \forall (u,v)\in E(G)\setminus\{(t,s)\} \\
 & p_s-p_t & \geq 1 & \\
 & d_{uv} & \geq 0 & \forall (u,v)\in E(G) \\
 & p_u & \geq 0 & \forall u\in V(G)
\end{array}
$$



Dual LP – Fractional Cuts

LP-relaxation of the ILP

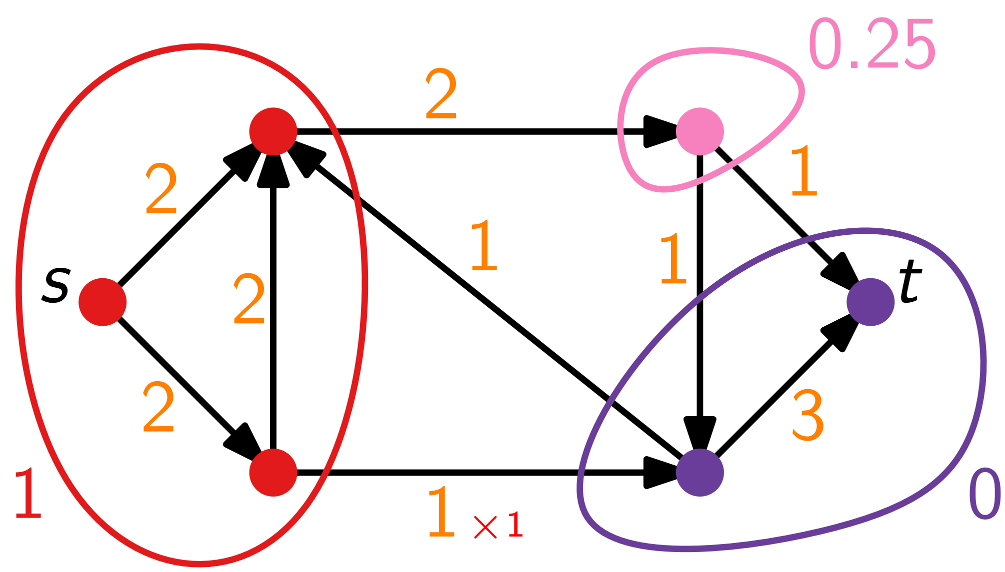
$$
\begin{array}{lrll}
\textbf{minimize} & \displaystyle\sum_{(u,v)\in E(G)\setminus\{(t,s)\}} c_{uv}\cdot d_{uv} & & \\
\textbf{subject to} & d_{uv} - p_u + p_v \geq 0 & \forall (u,v)\in E(G)\setminus\{(t,s)\} \\
 & p_s - p_t \geq 1 & \\
 & d_{uv} \geq 0 & \forall (u,v)\in E(G) \\
 & p_u \geq 0 & \forall u\in V(G)
\end{array}
$$



Dual LP – Fractional Cuts

LP-relaxation of the ILP

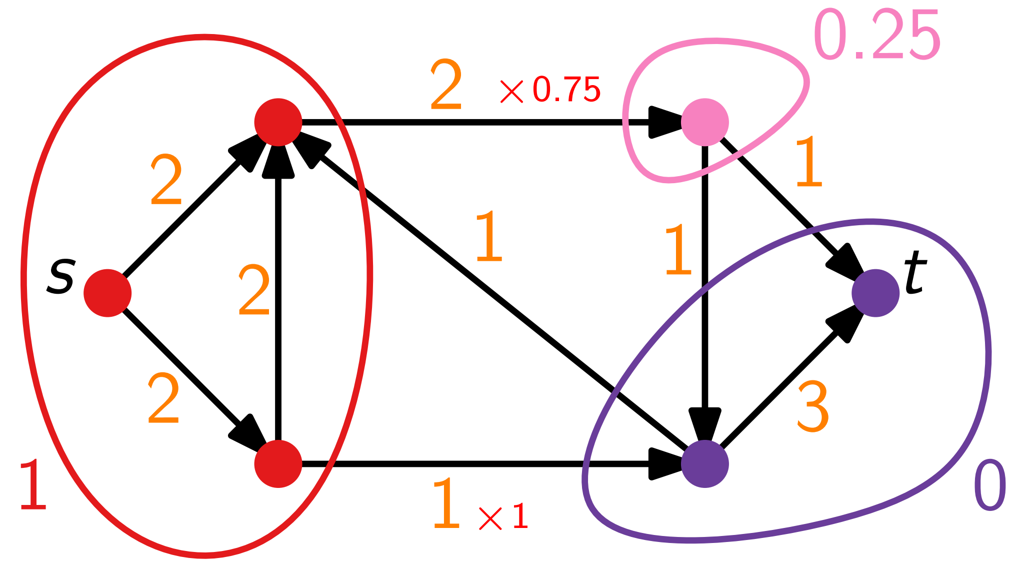
$$\begin{aligned}
\textbf{minimize} \quad & \sum_{(u,v)\in E(G)\setminus\{(t,s)\}} c_{uv}\cdot d_{uv} \\
\textbf{subject to} \quad & d_{uv} - p_u + p_v \geq 0 \quad \forall (u,v)\in E(G)\setminus\{(t,s)\} \\
& p_s - p_t \geq 1 \\
& d_{uv} \geq 0 \quad \forall (u,v)\in E(G) \\
& p_u \geq 0 \quad \forall u\in V(G)
\end{aligned}$$



Dual LP – Fractional Cuts

LP-relaxation of the ILP

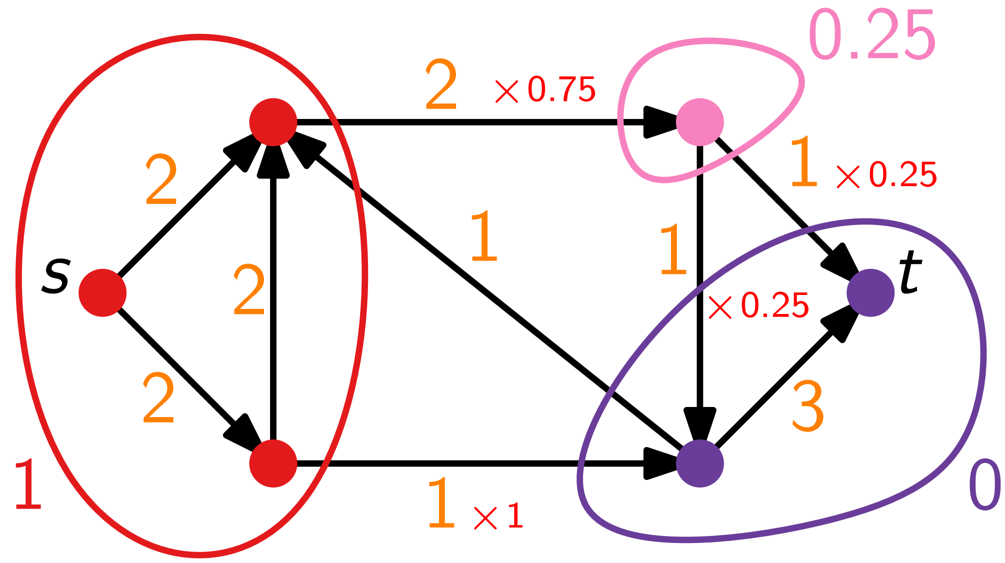
$$\begin{aligned}
\textbf{minimize} \quad & \sum_{(u,v)\in E(G)\setminus\{(t,s)\}} c_{uv}\cdot d_{uv} \\
\textbf{subject to} \quad & d_{uv} - p_u + p_v \geq 0 \quad \forall (u,v)\in E(G)\setminus\{(t,s)\} \\
& p_s - p_t \geq 1 \\
& d_{uv} \geq 0 \quad \forall (u,v)\in E(G) \\
& p_u \geq 0 \quad \forall u \in V(G)
\end{aligned}$$



Dual LP – Fractional Cuts

LP-relaxation of the ILP

$$\begin{aligned}
\textbf{minimize} \quad & \sum_{(u,v)\in E(G)\setminus\{(t,s)\}} c_{uv} \cdot d_{uv} \\
\textbf{subject to} \quad & d_{uv} - p_u + p_v \geq 0 \quad \forall (u,v) \in E(G) \setminus \{(t,s)\} \\
& p_s - p_t \geq 1 \\
& d_{uv} \geq 0 \quad \forall (u,v) \in E(G) \\
& p_u \geq 0 \quad \forall u \in V(G)
\end{aligned}$$

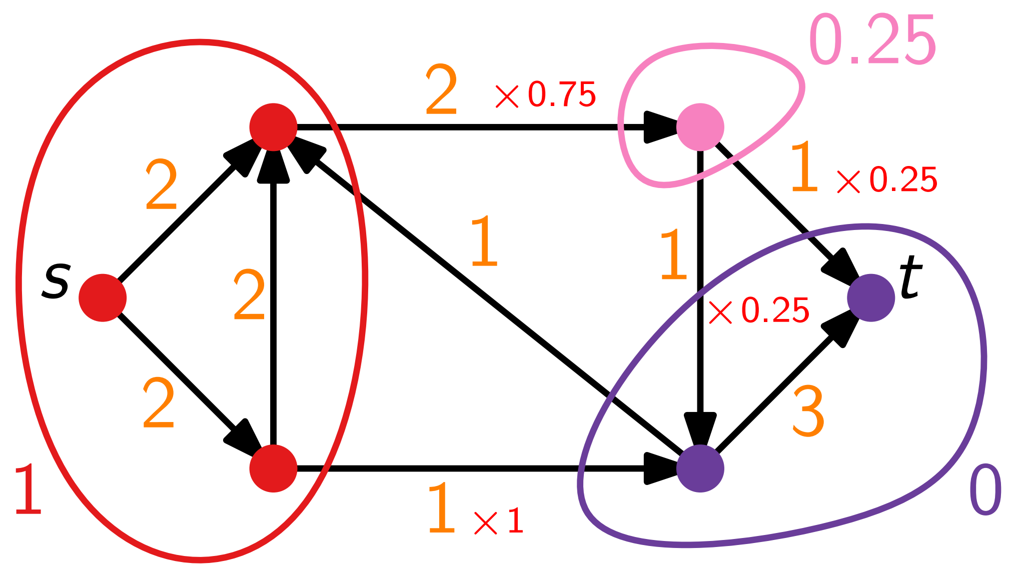


Dual LP – Fractional Cuts

LP-relaxation of the ILP

$$\begin{array}{lrll}
\textbf{minimize} & \displaystyle\sum_{(u,v)\in E(G)\setminus\{(t,s)\}} c_{uv} \cdot d_{uv} & & \\
\textbf{subject to} & d_{uv} - p_u + p_v & \geq 0 & \forall (u,v) \in E(G) \setminus \{(t,s)\} \\
& p_s - p_t & \geq 1 & \\
& d_{uv} & \geq 0 & \forall (u,v) \in E(G) \\
& p_u & \geq 0 & \forall u \in V(G)
\end{array}$$

Note that every s–t path $s = v_0, \ldots, v_k = t$ has length ≥ 1 w.r.t. d:



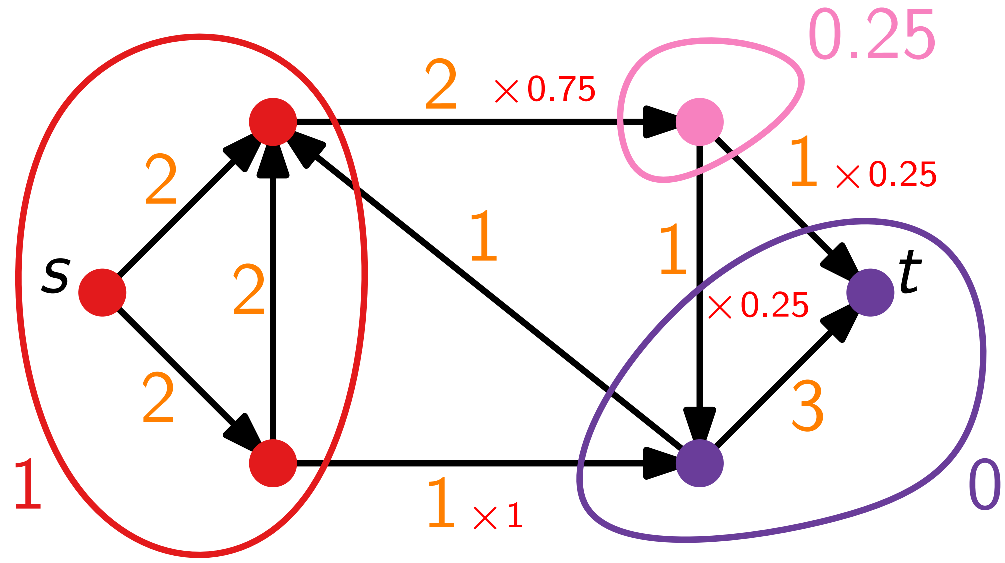
Dual LP – Fractional Cuts

LP-relaxation of the ILP

$$
\begin{array}{lrll}
\textbf{minimize} & \displaystyle\sum_{(u,v)\in E(G)\setminus\{(t,s)\}} c_{uv}\cdot d_{uv} & & \\
\textbf{subject to} & d_{uv} - p_u + p_v \geq 0 & \forall (u,v)\in E(G)\setminus\{(t,s)\} \\
 & p_s - p_t \geq 1 & \\
 & d_{uv} \geq 0 & \forall (u,v)\in E(G) \\
 & p_u \geq 0 & \forall u\in V(G)
\end{array}
$$

Note that every s–t path $s = v_0, \ldots, v_k = t$ has length ≥ 1 w.r.t. d:

$$\sum_{i=0}^{k-1} d_{i,i+1} \geq$$



Dual LP – Fractional Cuts

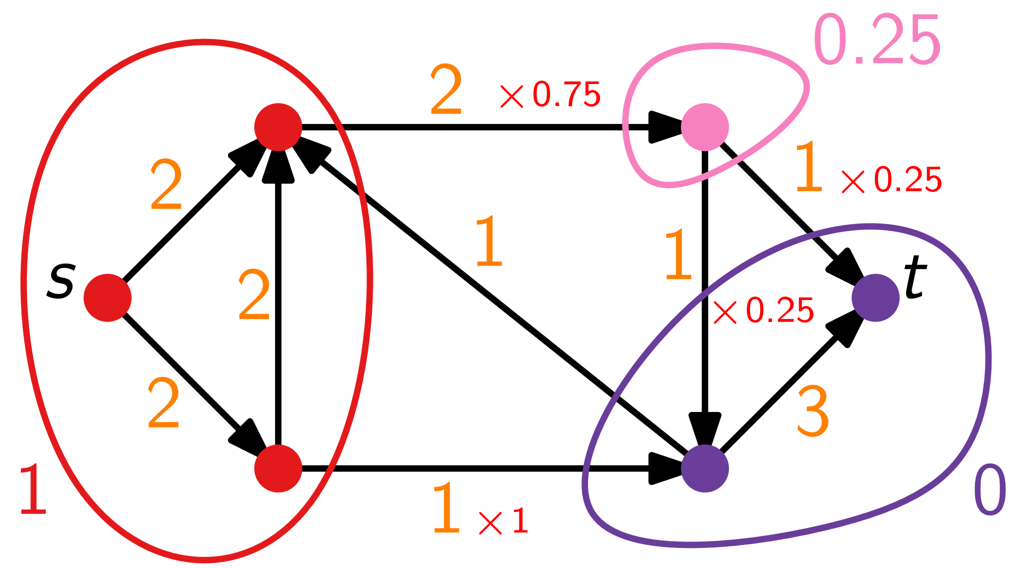
LP-relaxation of the ILP

$$\begin{array}{lrll}
\textbf{minimize} & \displaystyle\sum_{(u,v)\in E(G)\setminus\{(t,s)\}} c_{uv}\cdot d_{uv} & & \\
\textbf{subject to} & d_{uv} - p_u + p_v \geq 0 & \forall (u,v)\in E(G)\setminus\{(t,s)\} \\
& p_s - p_t \geq 1 & \\
& d_{uv} \geq 0 & \forall (u,v)\in E(G) \\
& p_u \geq 0 & \forall u \in V(G)
\end{array}$$

Note that every s–t path $s = v_0, \ldots, v_k = t$ has length ≥ 1 w.r.t. d:

$$\sum_{i=0}^{k-1} d_{i,i+1} \geq \sum_{i=0}^{k-1} (p_i - p_{i+1})$$

$$=$$



Dual LP – Fractional Cuts

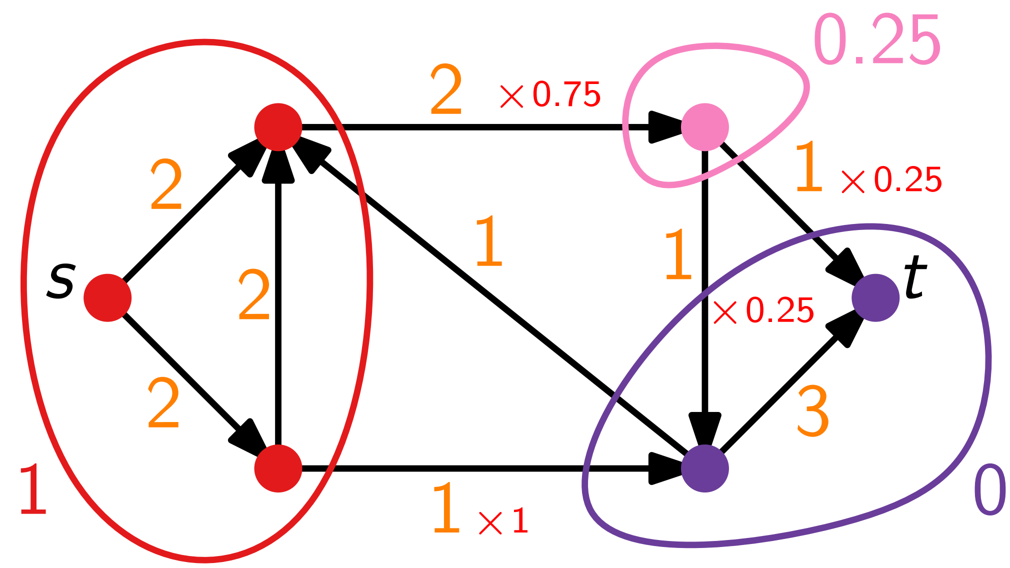
LP-relaxation of the ILP

$$\begin{aligned}
\textbf{minimize} \quad & \sum_{(u,v)\in E(G)\setminus\{(t,s)\}} c_{uv} \cdot d_{uv} \\
\textbf{subject to} \quad & d_{uv} - p_u + p_v \geq 0 \quad \forall (u,v) \in E(G) \setminus \{(t,s)\} \\
& p_s - p_t \geq 1 \\
& d_{uv} \geq 0 \quad \forall (u,v) \in E(G) \\
& p_u \geq 0 \quad \forall u \in V(G)
\end{aligned}$$

Note that every s–t path $s = v_0, \ldots, v_k = t$ has length ≥ 1 w.r.t. d:

$$\sum_{i=0}^{k-1} d_{i,i+1} \geq \sum_{i=0}^{k-1} (p_i - p_{i+1})$$

$$= p_s - p_t \geq 1$$



Dual LP – Fractional Cuts

LP-relaxation of the ILP

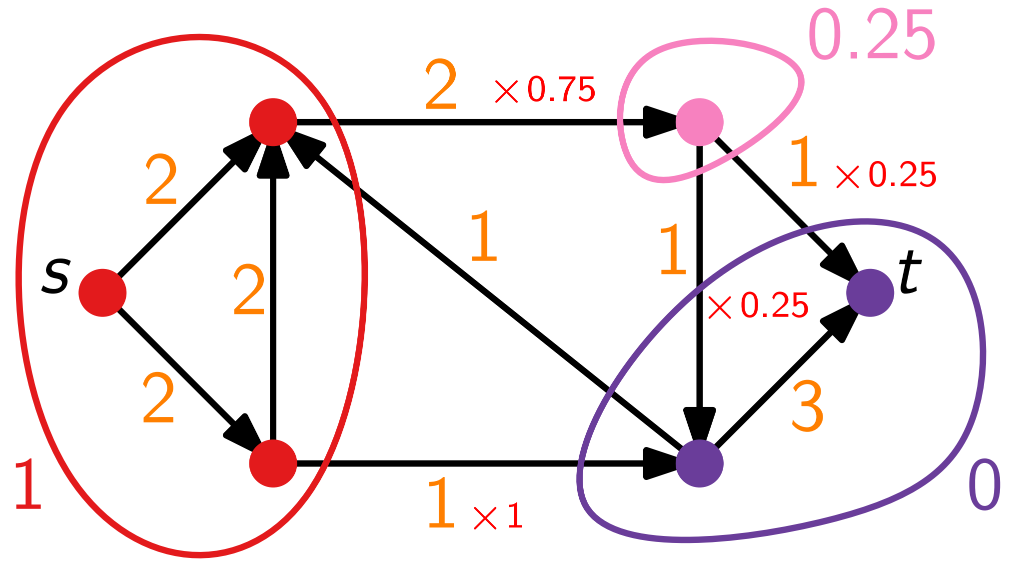
$$\begin{array}{lrll} \textbf{minimize} & \displaystyle\sum_{(u,v)\in E(G)\setminus\{(t,s)\}} c_{uv} \cdot d_{uv} & & \\ \textbf{subject to} & d_{uv} - p_u + p_v \geq 0 & \forall (u,v) \in E(G) \setminus \{(t,s)\} \\ & p_s - p_t \geq 1 & \\ & d_{uv} \geq 0 & \forall (u,v) \in E(G) \\ & p_u \geq 0 & \forall u \in V(G) \end{array}$$

Moreover, all extreme-point solutions of this polytope are **integral**! (HW)

Note that every s–t path $s = v_0, \ldots, v_k = t$ has length ≥ 1 w.r.t. d:

$$\sum_{i=0}^{k-1} d_{i,i+1} \geq \sum_{i=0}^{k-1} (p_i - p_{i+1})$$

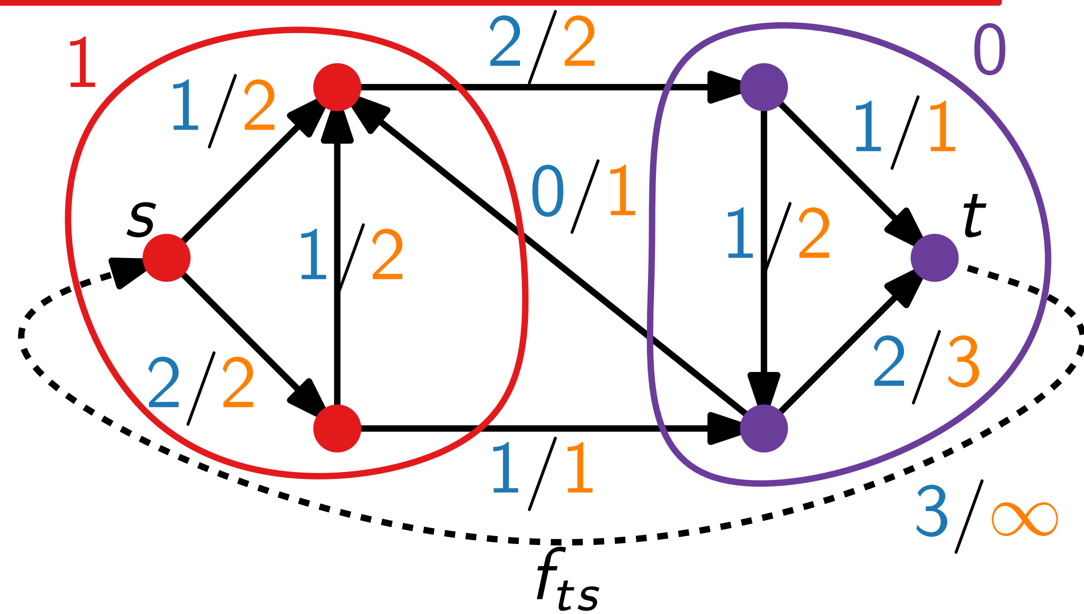
$$= p_s - p_t \geq 1$$



Dual LP – Complementary Slackness

$$\begin{array}{lrll} \textbf{maximize} & f_{ts} & & \\ \textbf{subject to} & f_{uv} & \leq c_{uv} & \forall (u,v) \neq (t,s) \\ & \displaystyle\sum_{u:\ (u,v)\in E(G)} f_{uv} - \sum_{z:\ (v,z)\in E(G)} f_{vz} & \leq 0 & \forall v \in V(G) \\ & f_{uv} & \geq 0 & \forall (u,v) \in E(G) \end{array}$$

$$\begin{array}{lrll} \textbf{minimize} & \displaystyle\sum_{(u,v)\in E(G)\setminus\{(t,s)\}} c_{uv} \cdot d_{uv} & & \\ \textbf{subject to} & d_{uv} - p_u + p_v & \geq 0 & \forall (u,v) \in E(G) \setminus \{(t,s)\} \\ & p_s - p_t & \geq 1 & \\ & d_{uv} & \geq 0 & \forall (u,v) \in E(G) \\ & p_u & \geq 0 & \forall u \in V(G) \end{array}$$

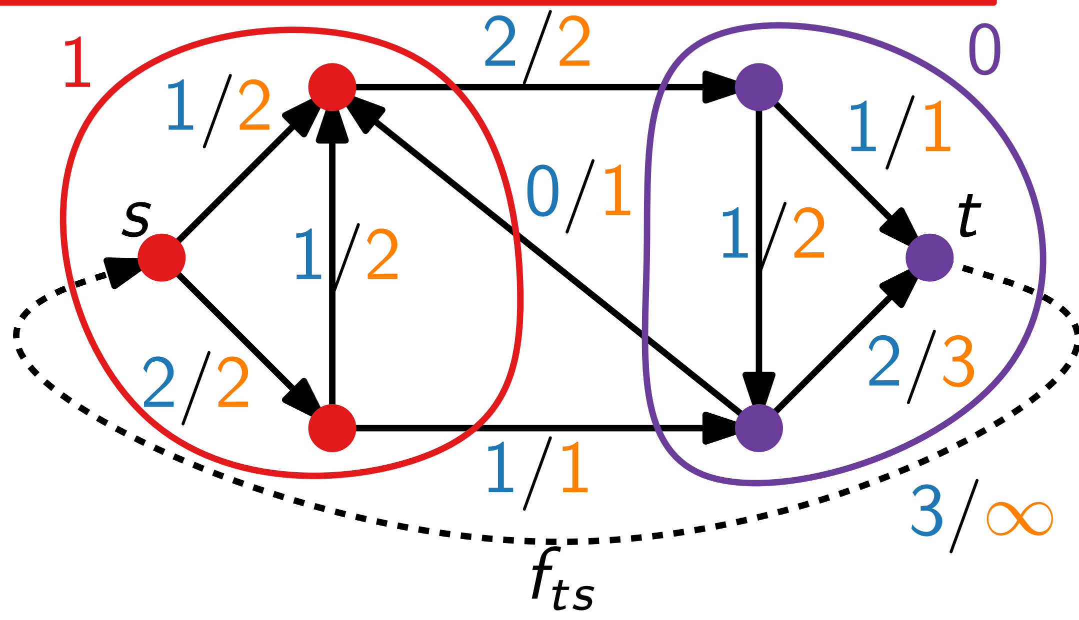


Dual LP – Complementary Slackness

$$
\begin{array}{lrll}
\textbf{maximize} & f_{ts} & & \\
\textbf{subject to} & f_{uv} & \le c_{uv} & \forall (u,v) \neq (t,s) \\
& \displaystyle\sum_{u\colon (u,v)\in E(G)} f_{uv} - \sum_{z\colon (v,z)\in E(G)} f_{vz} & \le 0 & \forall v \in V(G) \\
& f_{uv} & \ge 0 & \forall (u,v) \in E(G)
\end{array}
$$

$$
\begin{array}{lrll}
\textbf{minimize} & \displaystyle\sum_{(u,v)\in E(G)\setminus\{(t,s)\}} c_{uv} \cdot d_{uv} & & \\
\textbf{subject to} & d_{uv} - p_u + p_v & \ge 0 & \forall (u,v) \in E(G) \setminus \{(t,s)\} \\
& p_s - p_t & \ge 1 & \\
& d_{uv} & \ge 0 & \forall (u,v) \in E(G) \\
& p_u & \ge 0 & \forall u \in V(G)
\end{array}
$$

For a max flow and min cut:



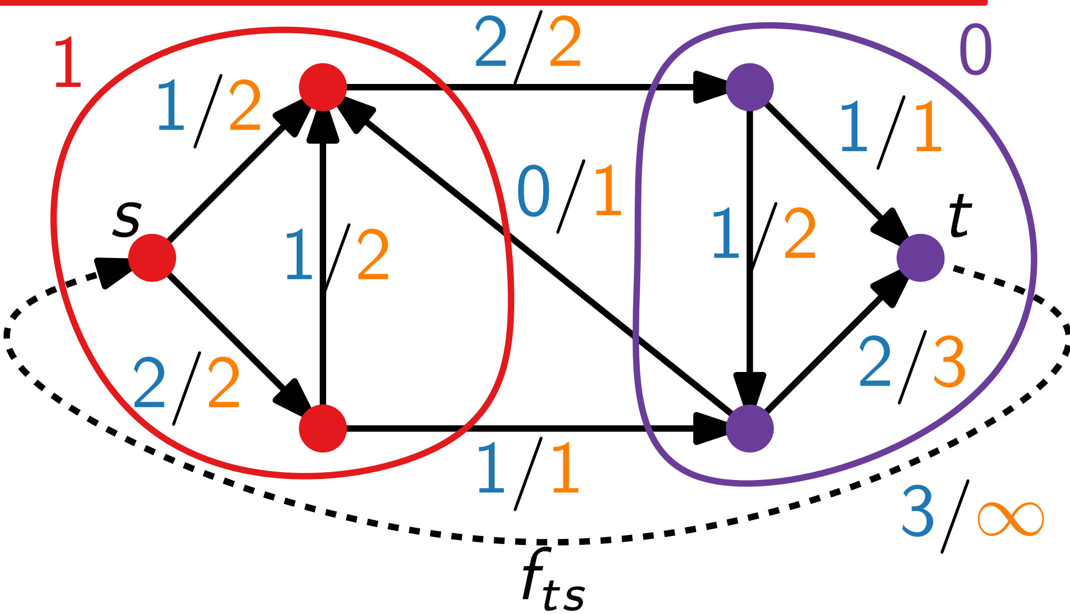
Dual LP – Complementary Slackness

$$
\begin{aligned}
&\textbf{maximize} \quad f_{ts} \\
&\textbf{subject to} \quad && f_{uv} \le c_{uv} && \forall (u,v) \neq (t,s) \\
& \sum_{u\colon (u,v)\in E(G)} f_{uv} - \sum_{z\colon (v,z)\in E(G)} && f_{vz} \le 0 && \forall v \in V(G) \\
& && f_{uv} \ge 0 && \forall (u,v) \in E(G)
\end{aligned}
$$

$$
\begin{aligned}
&\textbf{minimize} \quad \sum_{(u,v)\in E(G)\setminus\{(t,s)\}} c_{uv} \cdot d_{uv} \\
&\textbf{subject to} \quad & d_{uv} - p_u + p_v &\ge 0 && \forall (u,v) \in E(G) \setminus \{(t,s)\} \\
& & p_s - p_t &\ge 1 \\
& & d_{uv} &\ge 0 && \forall (u,v) \in E(G) \\
& & p_u &\ge 0 && \forall u \in V(G)
\end{aligned}
$$

For a max flow and min cut:

- For each forward edge (u,v) of the cut:



Dual LP – Complementary Slackness

maximize f_{ts}

subject to

$$
\begin{aligned}
f_{uv} &\le c_{uv} && \forall (u,v) \neq (t,s)\\
\sum_{u\colon (u,v)\in E(G)} f_{uv} - \sum_{z\colon (v,z)\in E(G)} f_{vz} &\le 0 && \forall v \in V(G)\\
f_{uv} &\ge 0 && \forall (u,v) \in E(G)
\end{aligned}
$$

minimize $\displaystyle\sum_{(u,v)\in E(G)\setminus\{(t,s)\}} c_{uv} \cdot d_{uv}$

subject to

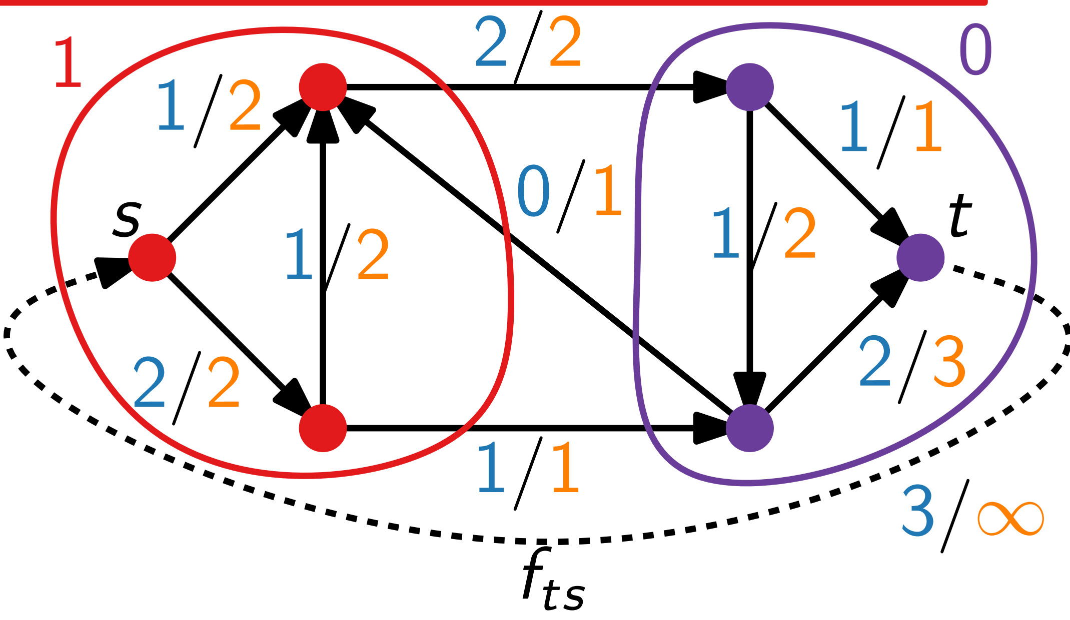
$$
\begin{aligned}
d_{uv} - p_u + p_v &\ge 0\\
p_s - p_t &\ge 1\\
d_{uv} &\ge 0\\
p_u &\ge 0
\end{aligned}
$$

Primal CS:
$\forall j$: $x_j = 0$ or $\sum_{i=1}^{m} a_{ij} y_i = c_j$

Dual CS:
$\forall i$: $y_i = 0$ or $\sum_{j=1}^{n} a_{ij} x_j = b_i$

For a max flow and min cut:

- For each forward edge (u, v) of the cut:



Dual LP – Complementary Slackness

$$
\begin{array}{lrcll}
\textbf{maximize} & f_{ts} & & & \\
\textbf{subject to} & & f_{uv} & \le c_{uv} & \forall (u,v) \neq (t,s) \\
& \displaystyle\sum_{u:\,(u,v)\in E(G)} f_{uv} - \sum_{z:\,(v,z)\in E(G)} & f_{vz} & \le 0 & \forall v \in V(G) \\
& & f_{uv} & \ge 0 & \forall (u,v) \in E(G)
\end{array}
$$

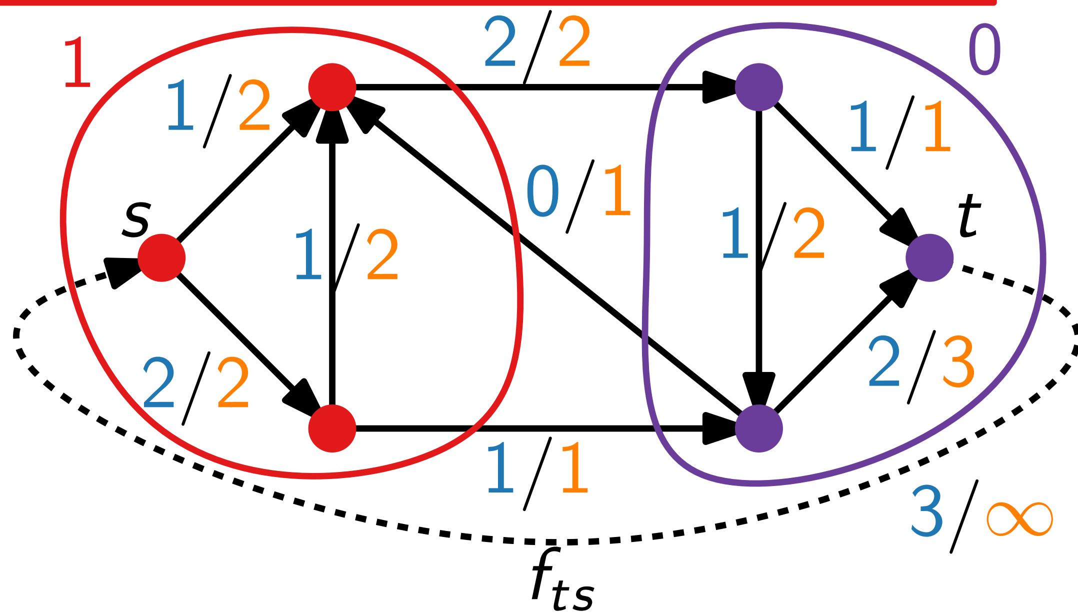
$$
\begin{array}{lrl}
\textbf{minimize} & \displaystyle\sum_{(u,v)\in E(G)\setminus\{(t,s)\}} c_{uv} \cdot d_{uv} & \\
\textbf{subject to} & d_{uv} - p_u + p_v & \ge 0 \\
& p_s - p_t & \ge 1 \\
& d_{uv} & \ge 0 \\
& p_u & \ge 0
\end{array}
$$

Primal CS:
$\forall j$: $x_j = 0$ or $\sum_{i=1}^{m} a_{ij} y_i = c_j$

Dual CS:
$\forall i$: $y_i = 0$ or $\sum_{j=1}^{n} a_{ij} x_j = b_i$

For a max flow and min cut:

- For each forward edge (u,v) of the cut: $f_{uv} = c_{uv}$.



Dual LP – Complementary Slackness

maximize f_{ts}

subject to

$$f_{uv} \le c_{uv} \quad \forall (u,v) \neq (t,s)$$

$$\sum_{u\colon (u,v)\in E(G)} f_{uv} - \sum_{z\colon (v,z)\in E(G)} f_{vz} \le 0 \quad \forall v \in V(G)$$

$$f_{uv} \ge 0 \quad \forall (u,v) \in E(G)$$

minimize $\sum_{(u,v)\in E(G)\setminus\{(t,s)\}} c_{uv} \cdot d_{uv}$

subject to

$$d_{uv} - p_u + p_v \ge 0$$

$$p_s - p_t \ge 1$$

$$d_{uv} \ge 0$$

$$p_u \ge 0$$

Primal CS:

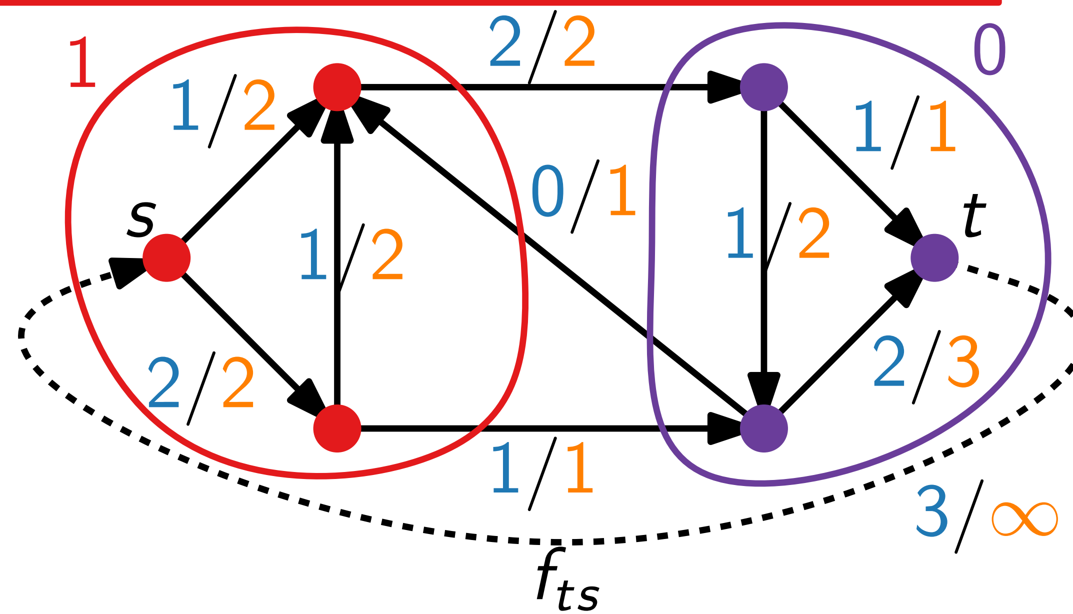
$\forall j$: $x_j = 0$ or $\sum_{i=1}^{m} a_{ij} y_i = c_j$

Dual CS:

$\forall i$: $y_i = 0$ or $\sum_{j=1}^{n} a_{ij} x_j = b_i$

For a max flow and min cut:

- For each forward edge (u,v) of the cut: $f_{uv} = c_{uv}$.
  ($d_{uv} = 1$, so by dual CS: $f_{uv} = c_{uv}$.)



Dual LP – Complementary Slackness

maximize f_{ts}

subject to

$$f_{uv} \leq c_{uv} \quad \forall (u,v) \neq (t,s)$$

$$\sum_{u\colon (u,v)\in E(G)} f_{uv} - \sum_{z\colon (v,z)\in E(G)} f_{vz} \leq 0 \quad \forall v \in V(G)$$

$$f_{uv} \geq 0 \quad \forall (u,v) \in E(G)$$

minimize $\sum_{(u,v)\in E(G)\setminus\{(t,s)\}} c_{uv} \cdot d_{uv}$

subject to

$$d_{uv} - p_u + p_v \geq 0$$

$$p_s - p_t \geq 1$$

$$d_{uv} \geq 0$$

$$p_u \geq 0$$

Primal CS:

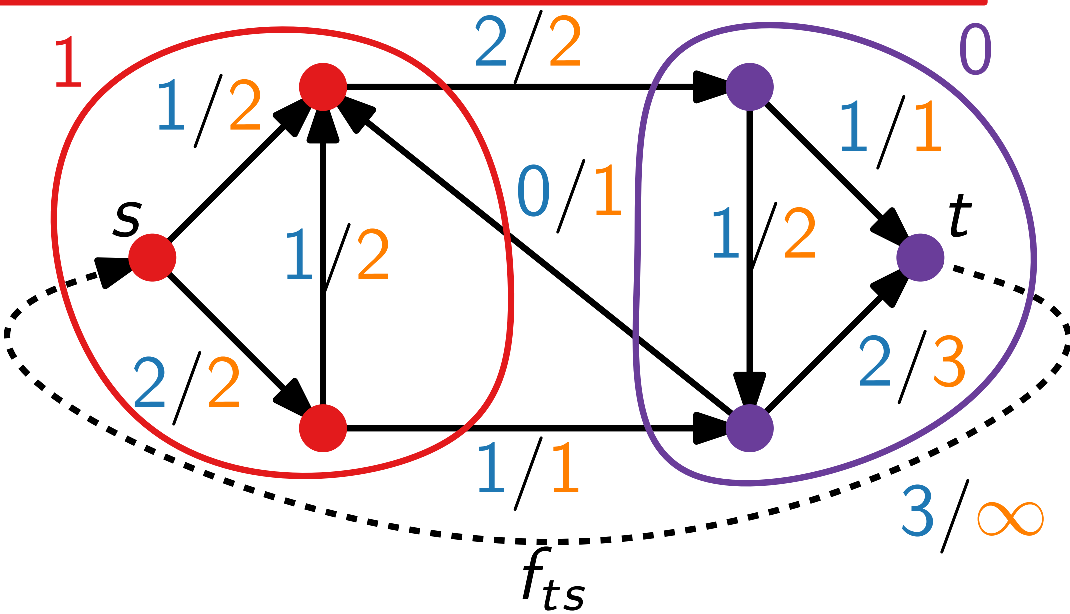
$\forall j$: $x_j = 0$ or $\sum_{i=1}^{m} a_{ij} y_i = c_j$

Dual CS:

$\forall i$: $y_i = 0$ or $\sum_{j=1}^{n} a_{ij} x_j = b_i$

For a max flow and min cut:

- For each forward edge (u,v) of the cut: $f_{uv} = c_{uv}$.
  ($d_{uv} = 1$, so by dual CS: $f_{uv} = c_{uv}$.)
- For each backward edge (u,v) of the cut:



Dual LP – Complementary Slackness

maximize f_{ts}

subject to

$$
\begin{aligned}
f_{uv} &\leq c_{uv} && \forall (u,v) \neq (t,s) \\
\sum_{u\colon (u,v)\in E(G)} f_{uv} - \sum_{z\colon (v,z)\in E(G)} f_{vz} &\leq 0 && \forall v \in V(G) \\
f_{uv} &\geq 0 && \forall (u,v) \in E(G)
\end{aligned}
$$

minimize $\displaystyle\sum_{(u,v)\in E(G)\setminus\{(t,s)\}} c_{uv} \cdot d_{uv}$

subject to

$$
\begin{aligned}
d_{uv} - p_u + p_v &\geq 0 \\
p_s - p_t &\geq 1 \\
d_{uv} &\geq 0 \\
p_u &\geq 0
\end{aligned}
$$

Primal CS:
$\forall j$: $x_j = 0$ or $\sum_{i=1}^{m} a_{ij} y_i = c_j$

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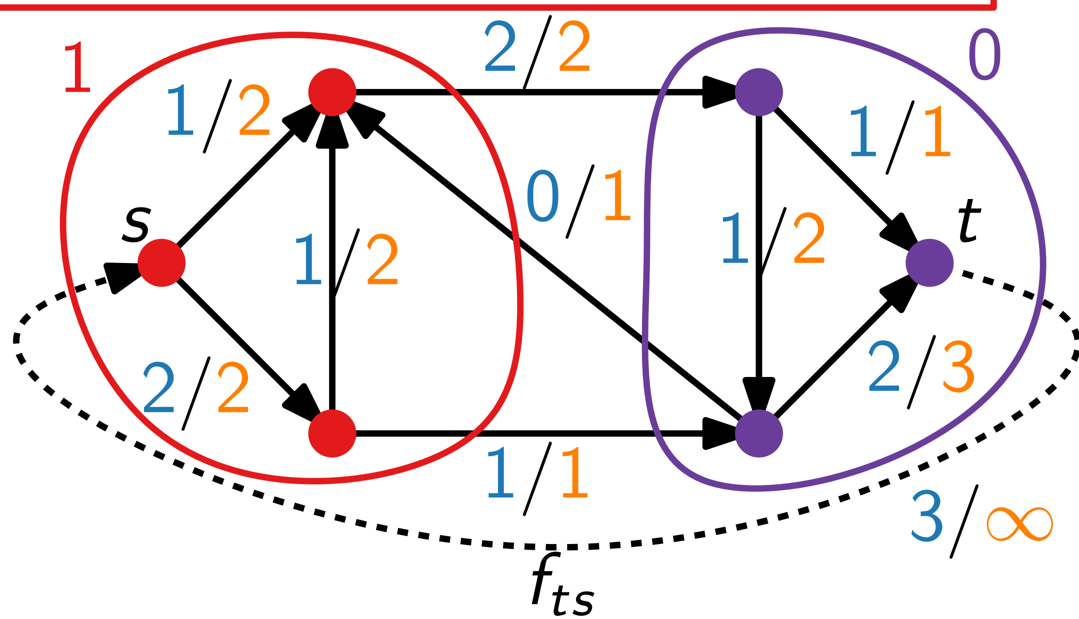
$\forall i$: $y_i = 0$ or $\sum_{j=1}^{n} a_{ij} x_j = b_i$

For a max flow and min cut:

- For each forward edge (u,v) of the cut: $f_{uv} = c_{uv}$.
  ($d_{uv} = 1$, so by dual CS: $f_{uv} = c_{uv}$.)
- For each backward edge (u,v) of the cut: $f_{uv} = 0$.



Dual LP – Complementary Slackness

maximize f_{ts}

subject to

$$f_{uv} \le c_{uv} \quad \forall (u,v) \neq (t,s)$$

$$\sum_{u:\,(u,v)\in E(G)} f_{uv} - \sum_{z:\,(v,z)\in E(G)} f_{vz} \le 0 \quad \forall v \in V(G)$$

$$f_{uv} \ge 0 \quad \forall (u,v) \in E(G)$$

minimize $\sum_{(u,v)\in E(G)\setminus\{(t,s)\}} c_{uv} \cdot d_{uv}$

subject to

$$d_{uv} - p_u + p_v \ge 0$$
$$p_s - p_t \ge 1$$
$$d_{uv} \ge 0$$
$$p_u \ge 0$$

Primal CS:
$\forall j$: $x_j = 0$ or $\sum_{i=1}^m a_{ij} y_i = c_j$

Dual CS:
$\forall i$: $y_i = 0$ or $\sum_{j=1}^n a_{ij} x_j = b_i$

For a max flow and min cut:

- For each forward edge (u,v) of the cut: $f_{uv} = c_{uv}$.
  ($d_{uv} = 1$, so by dual CS: $f_{uv} = c_{uv}$.)
- For each backward edge (u,v) of the cut: $f_{uv} = 0$.
  (Otherwise, by primal CS: $d_{uv} - 0 + 1 = 0$.)