# Approximation Algorithms

## Lecture 2:
## SetCover and ShortestSuperString

### Part I:
### SetCover

*Alexander Wolff*

*Winter 2024/25*

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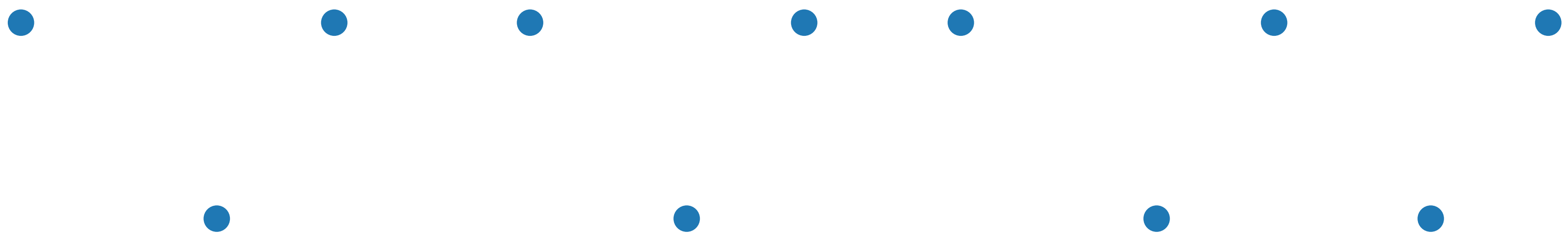

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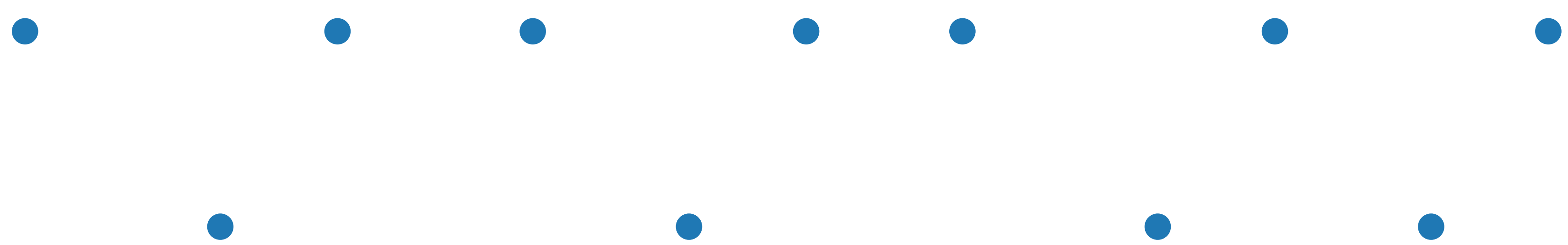

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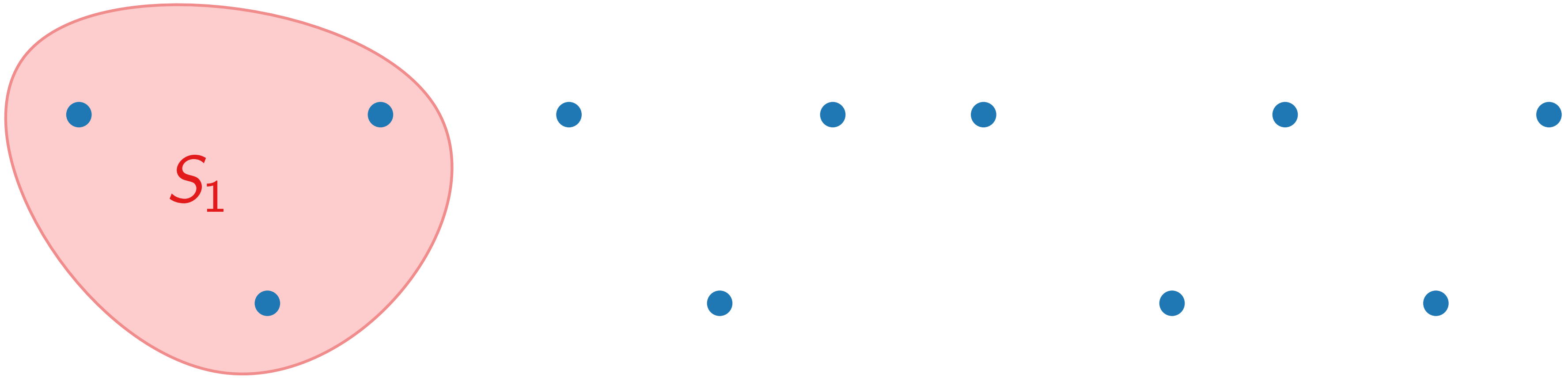

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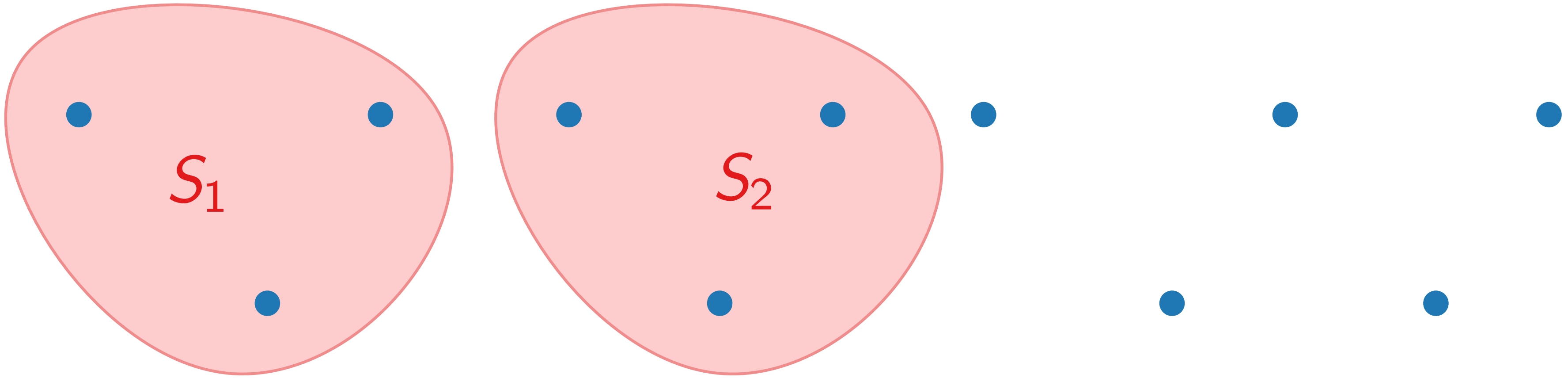

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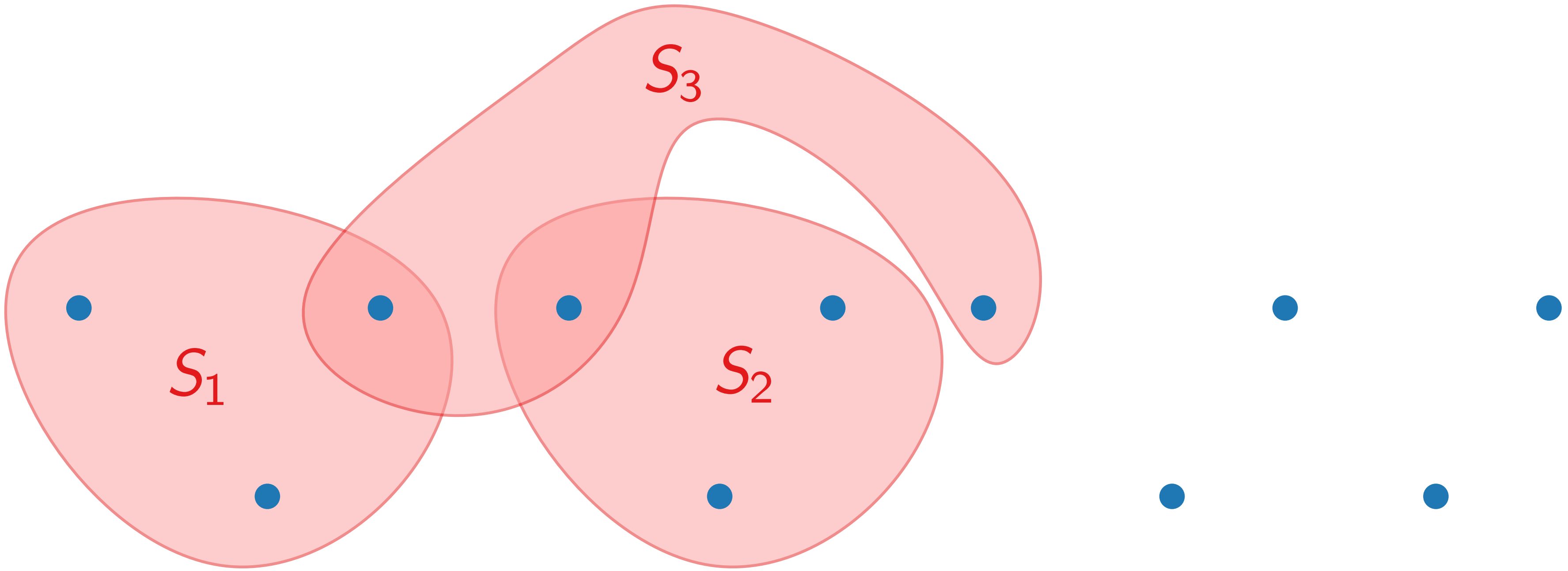

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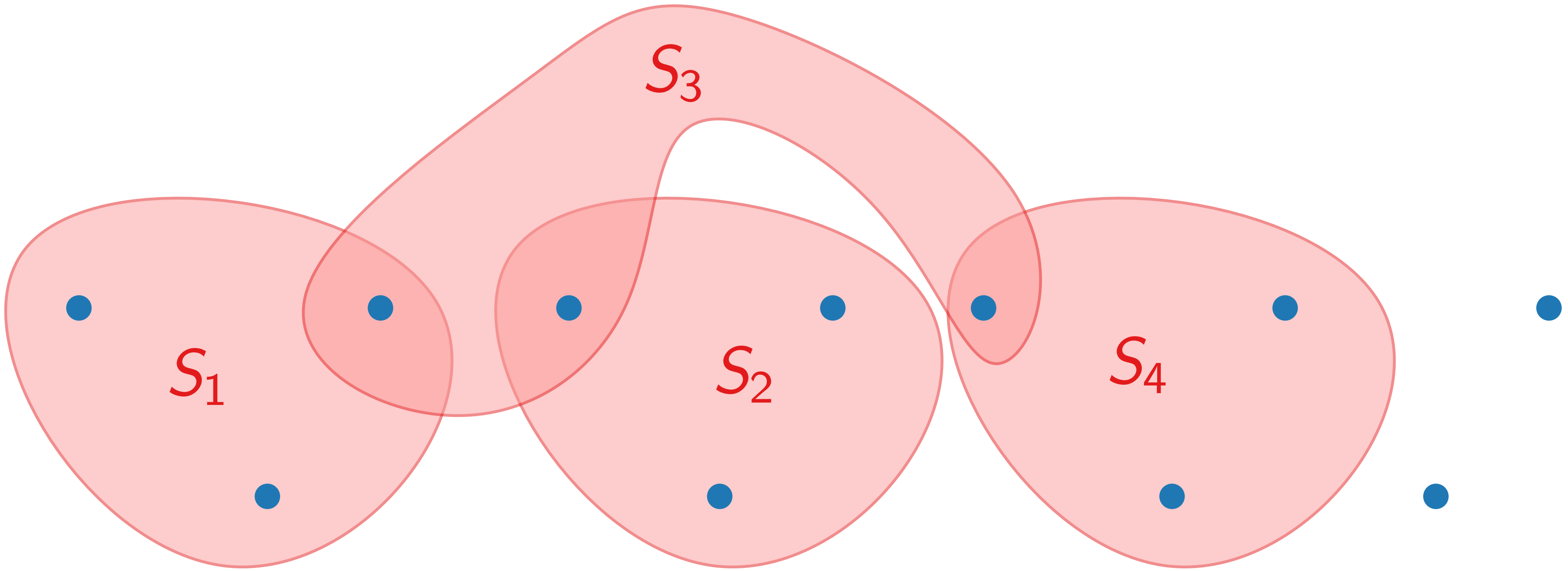

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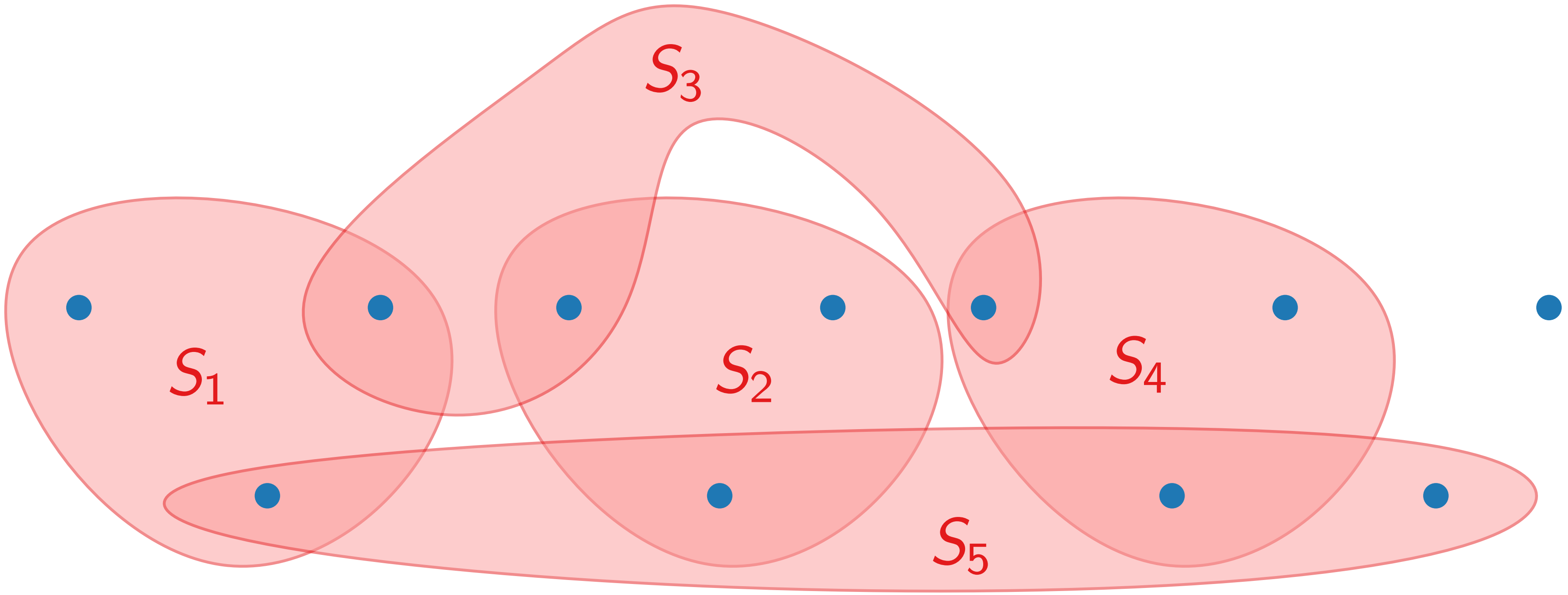

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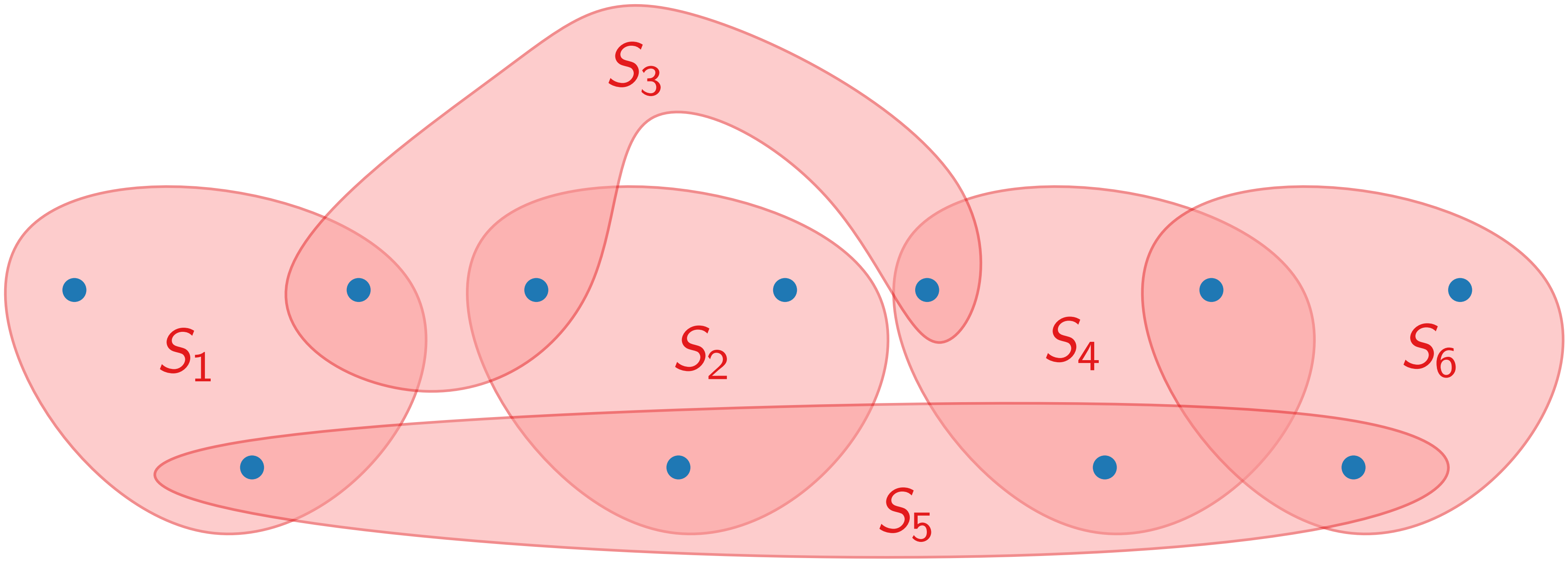

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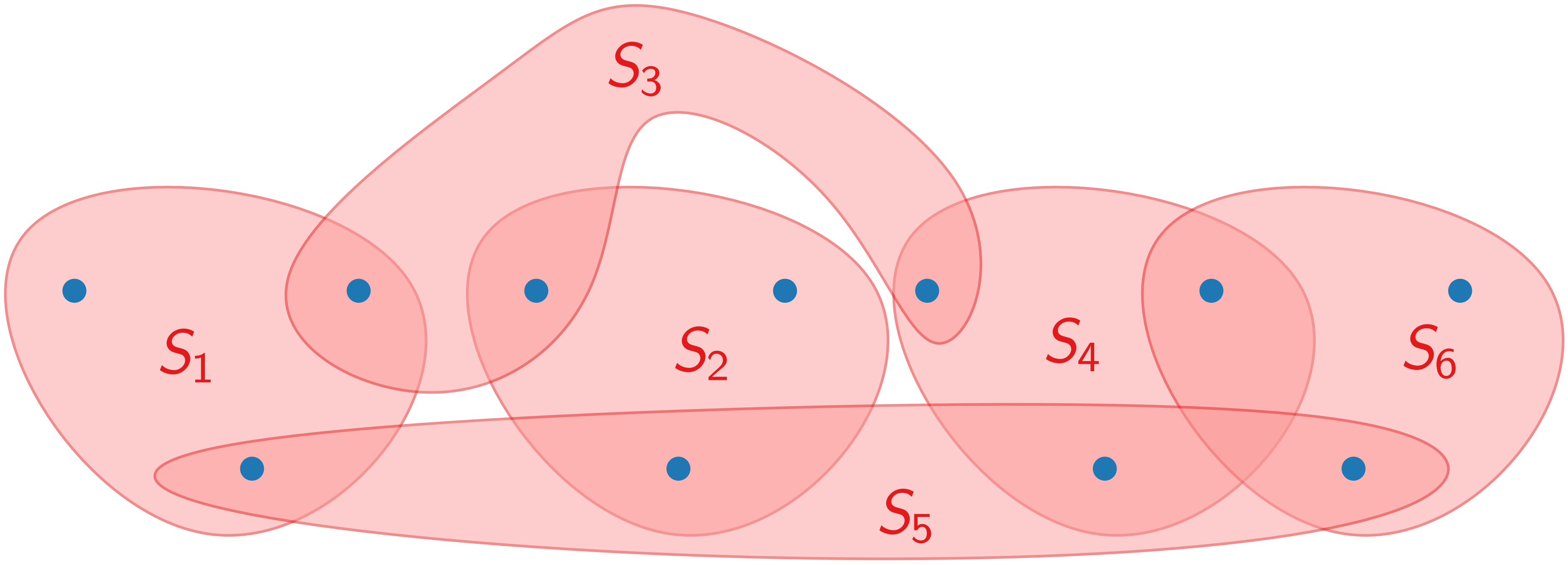

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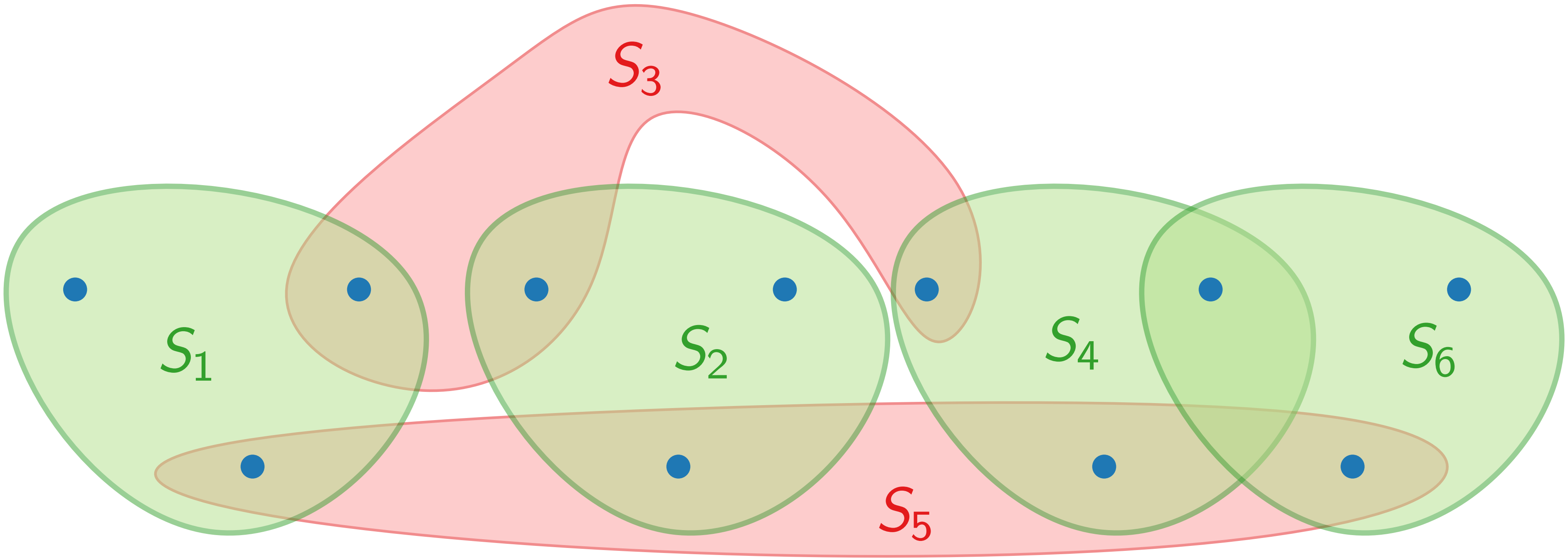

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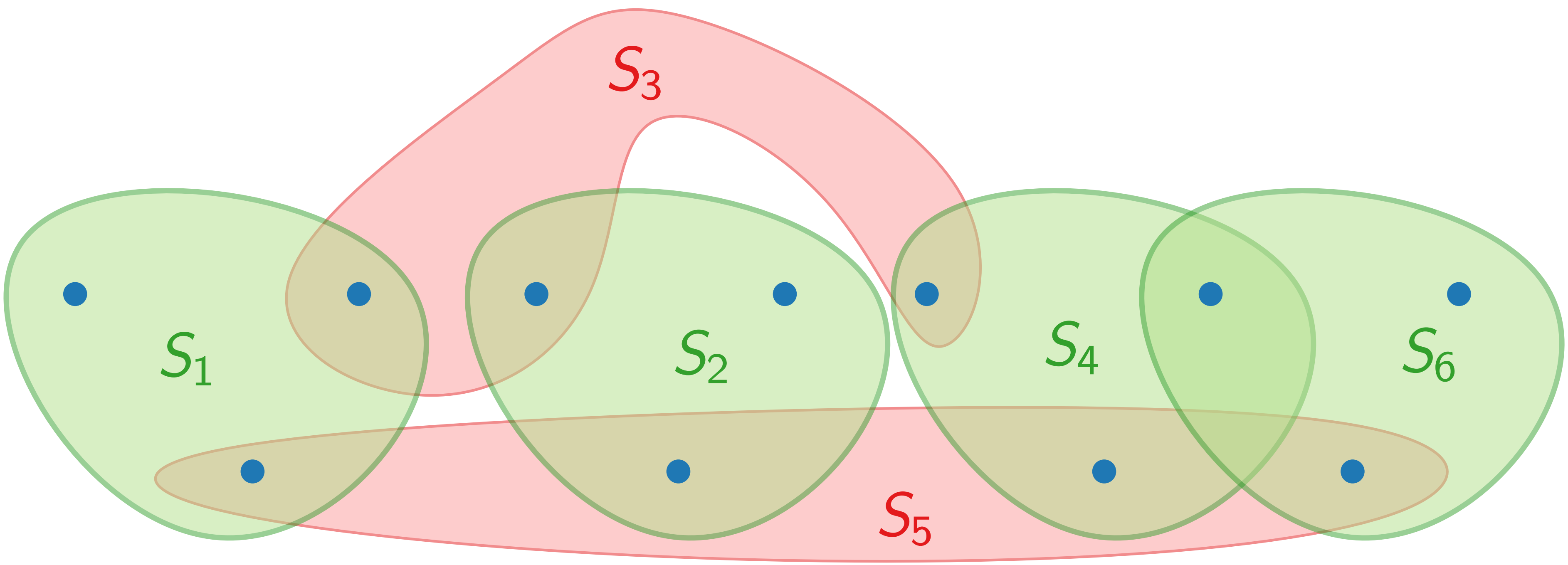

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Each $S \in \mathcal{S}$ has cost $c(S) > 0$.

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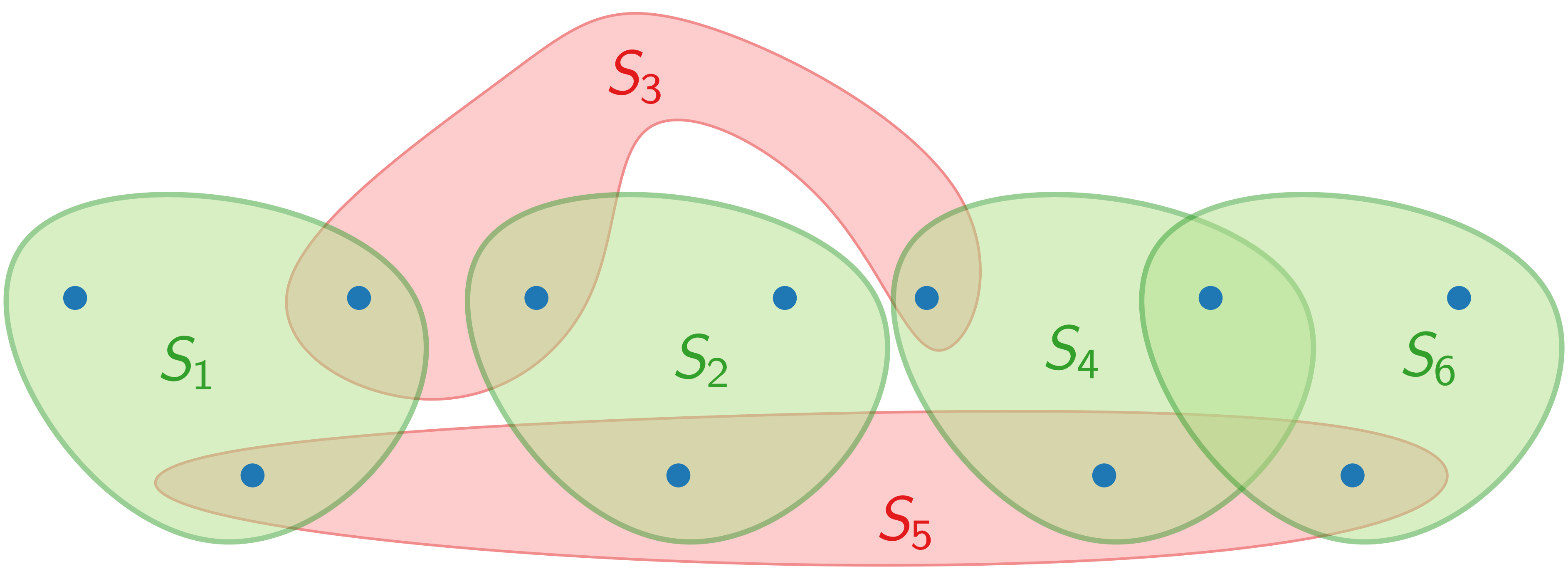

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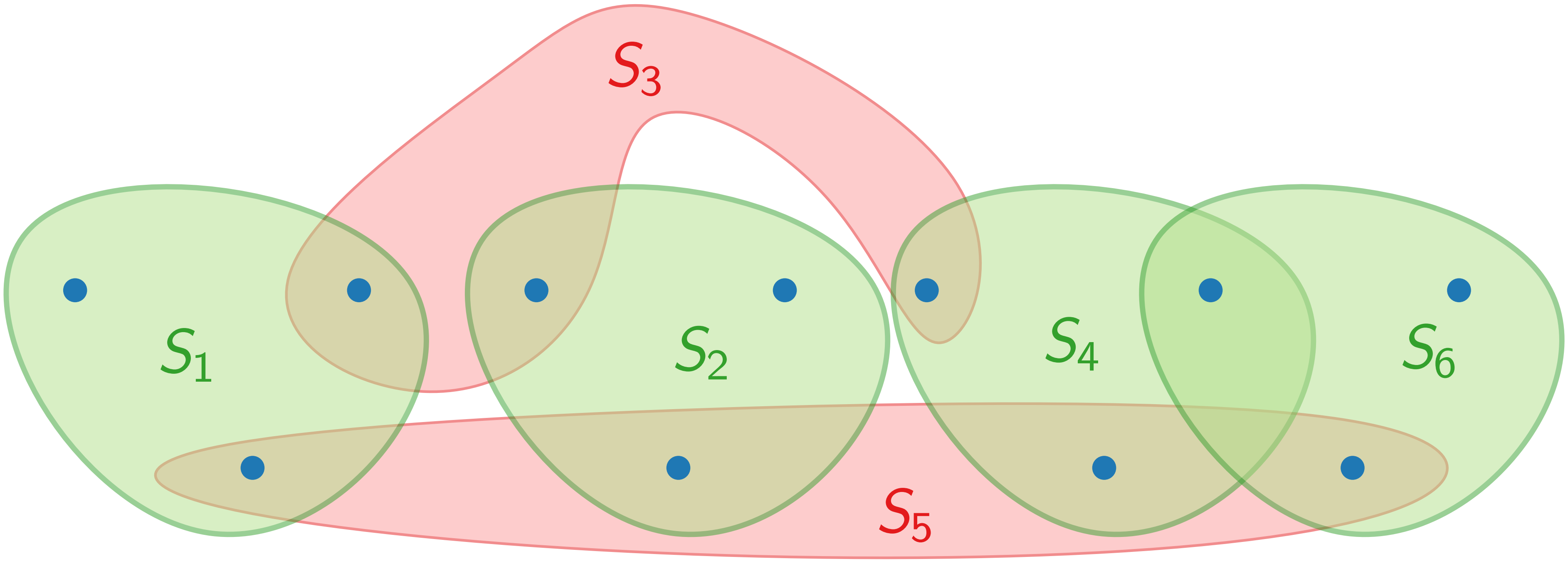

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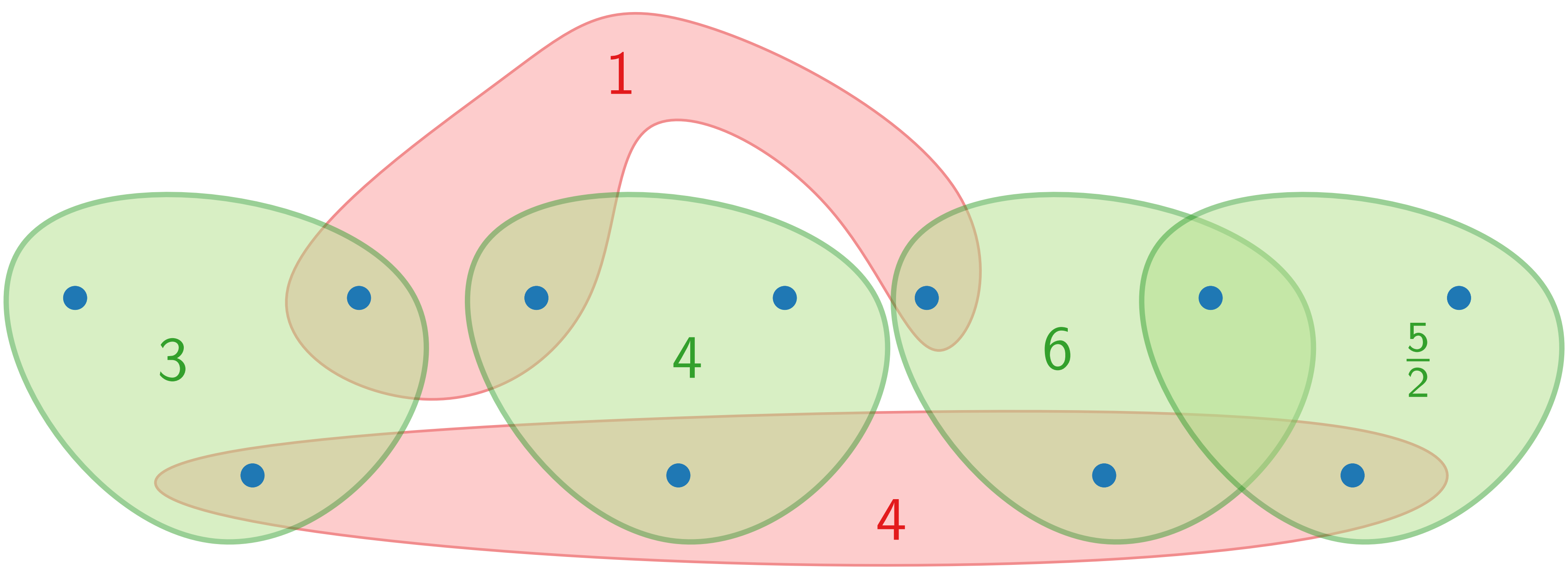

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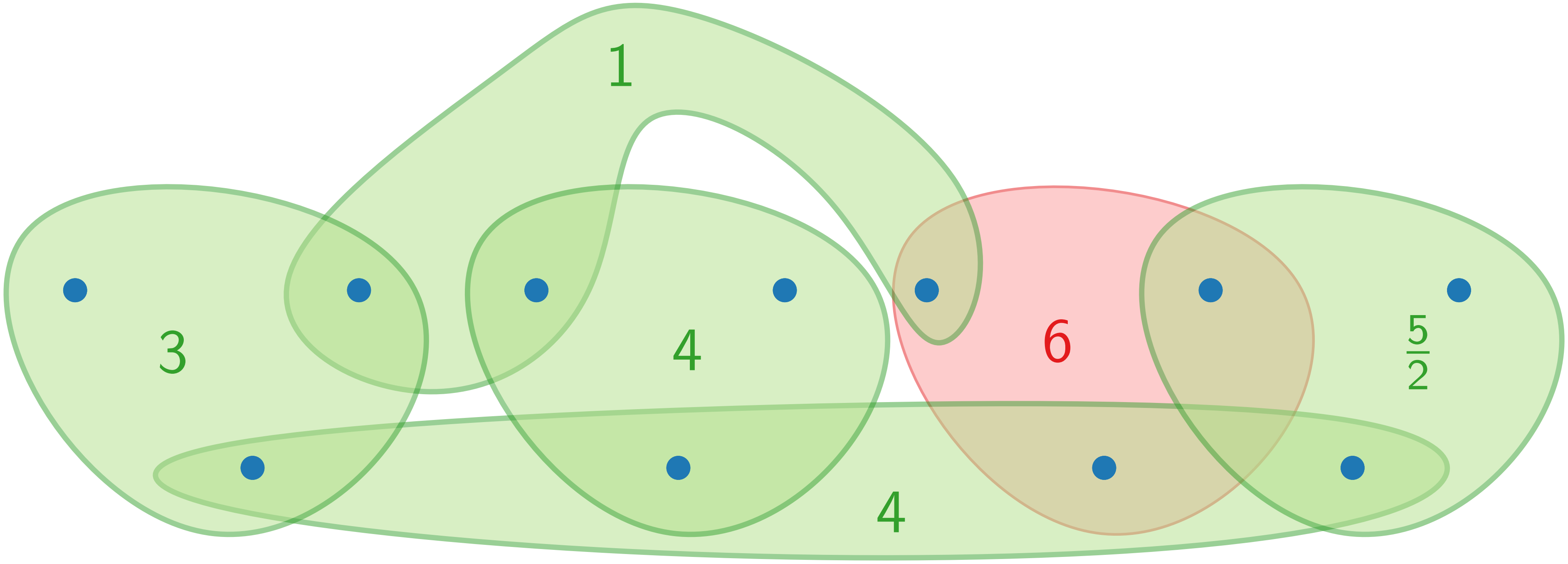

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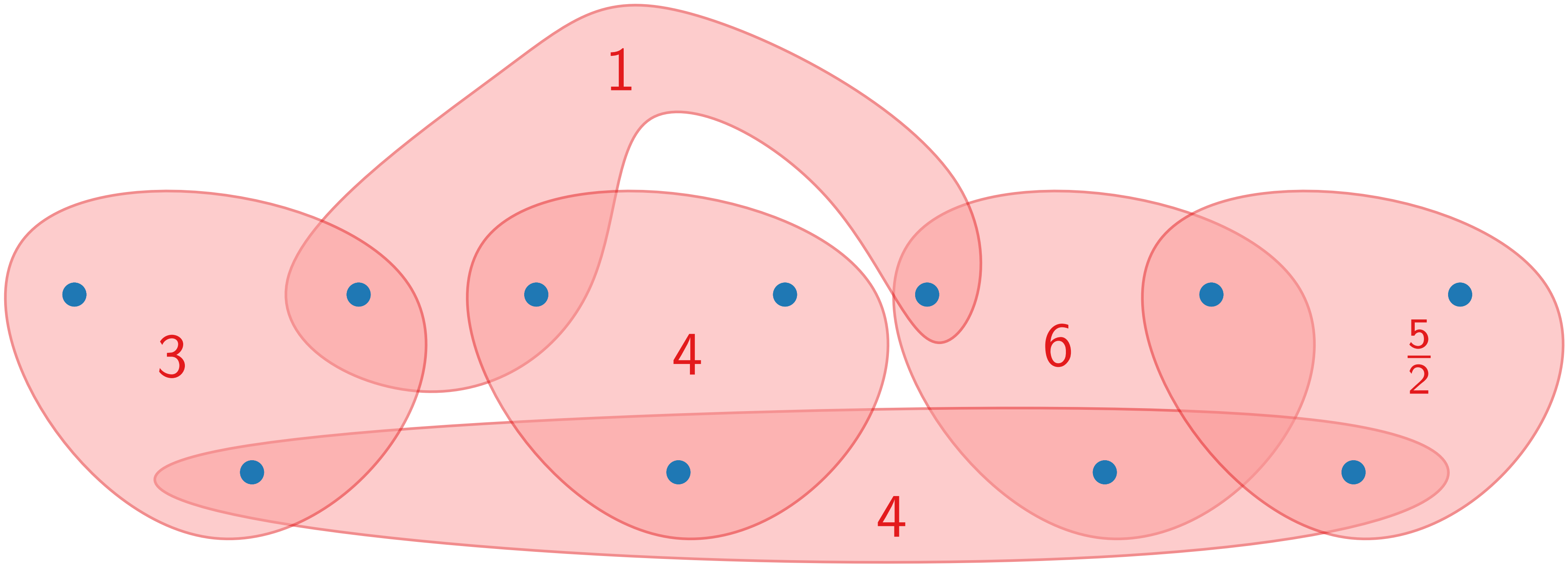
### Part II:
### Greedy for SetCover

# "Buying" Elements Iteratively

What is the real cost of picking a set?



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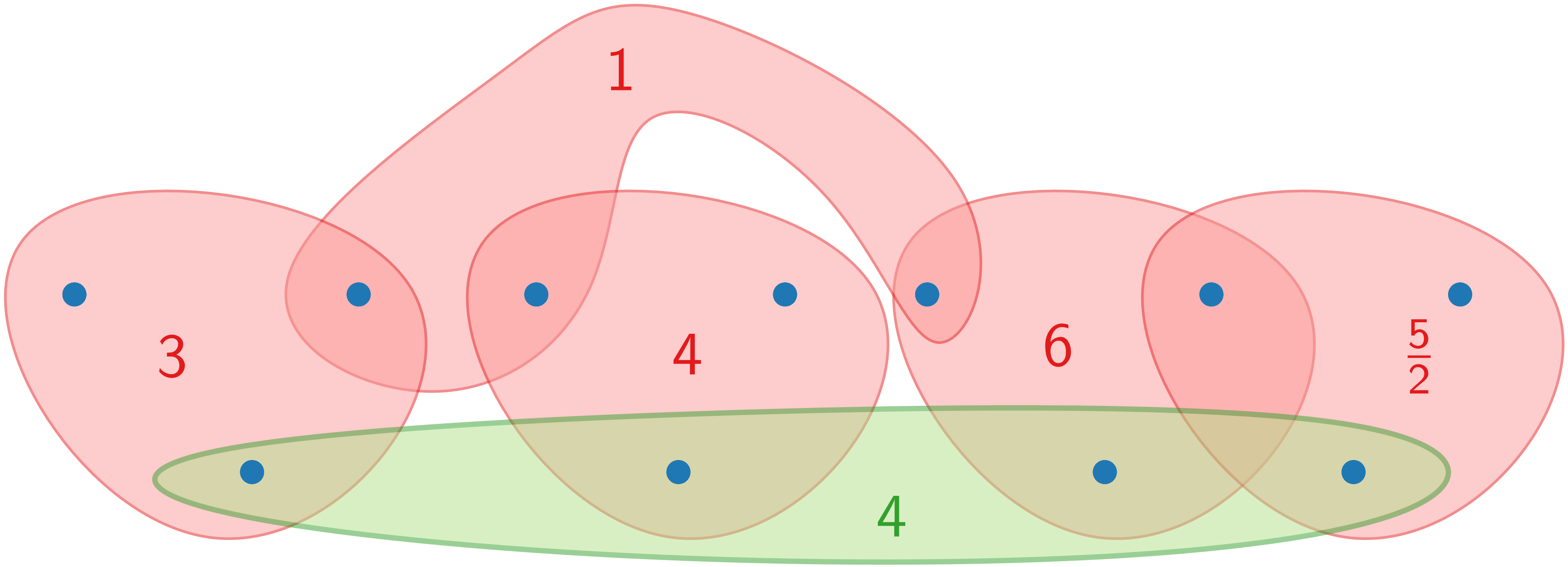

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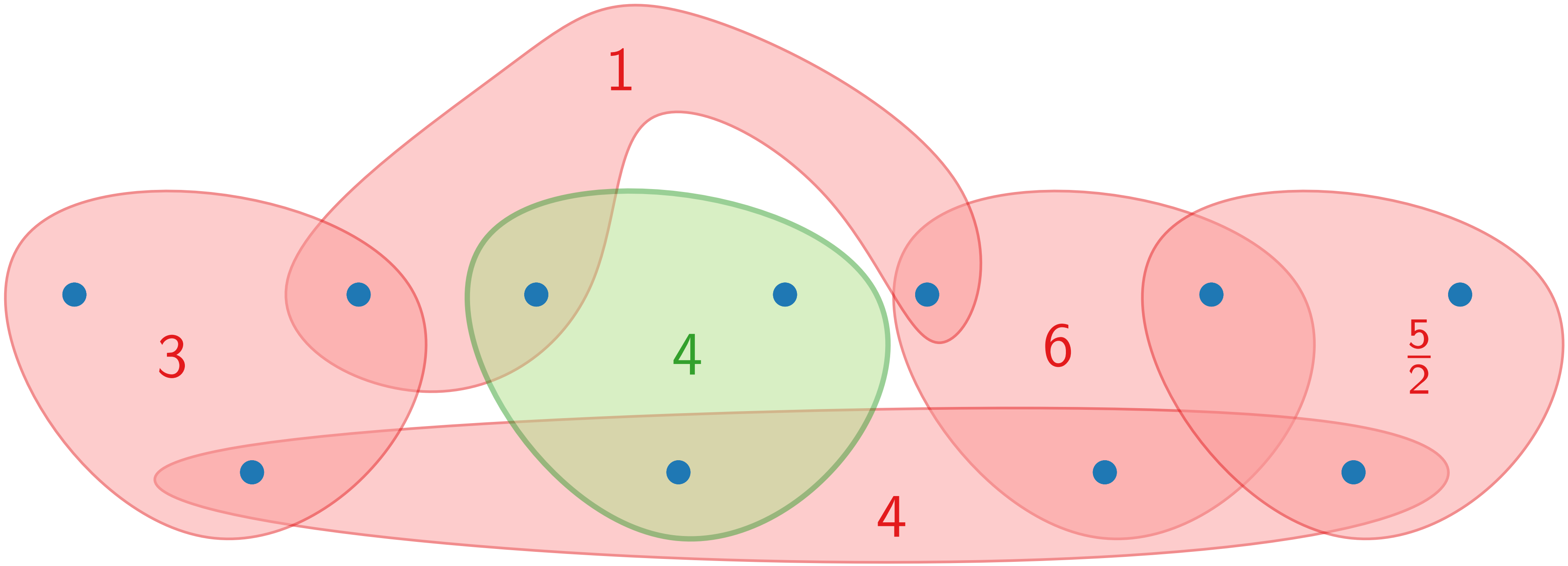

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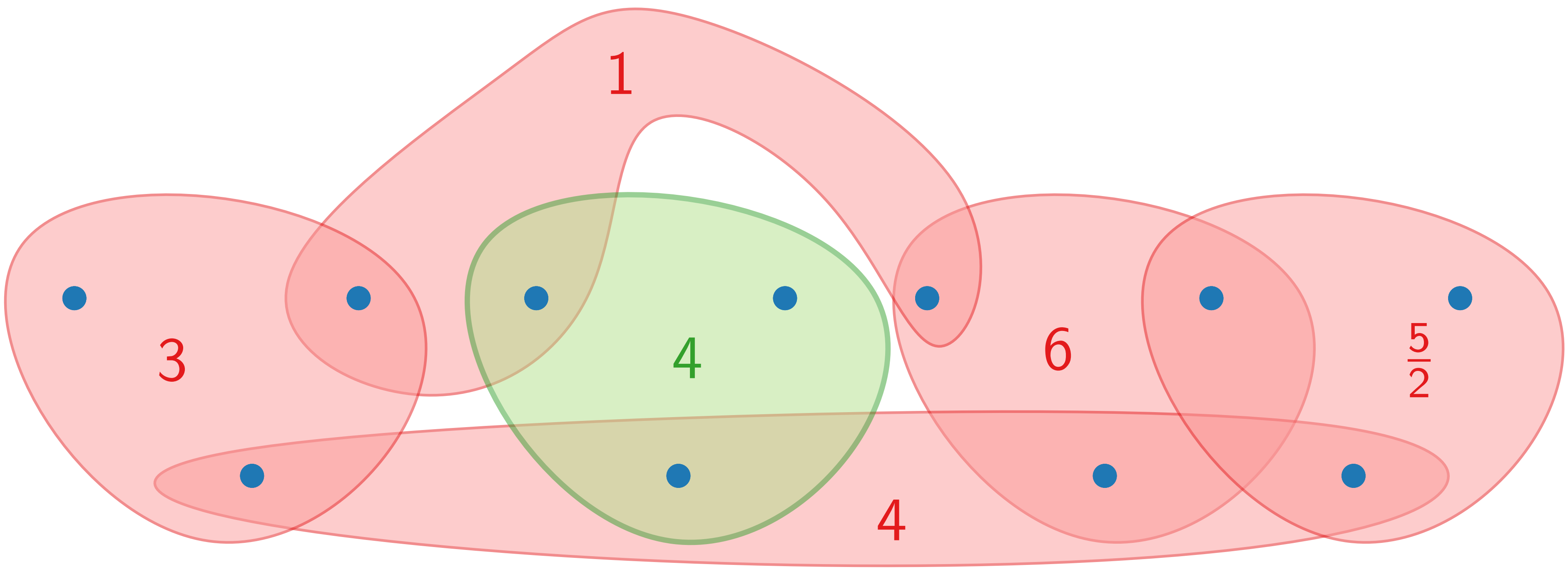

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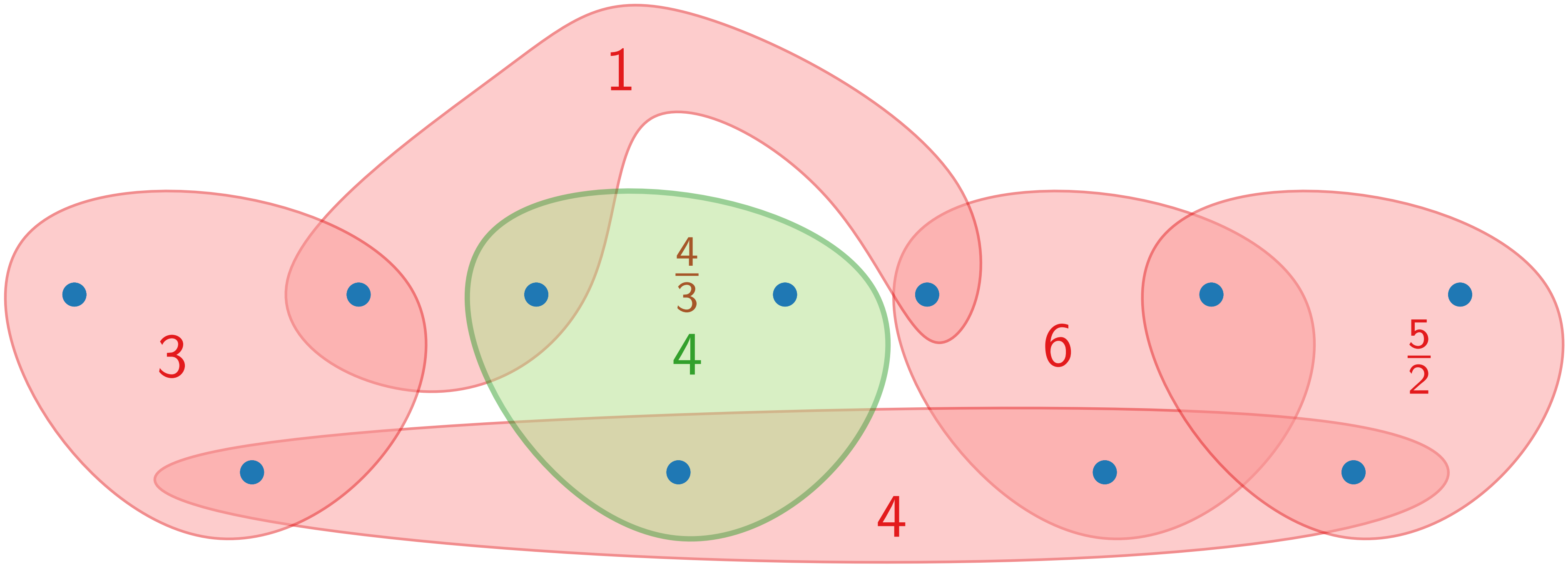

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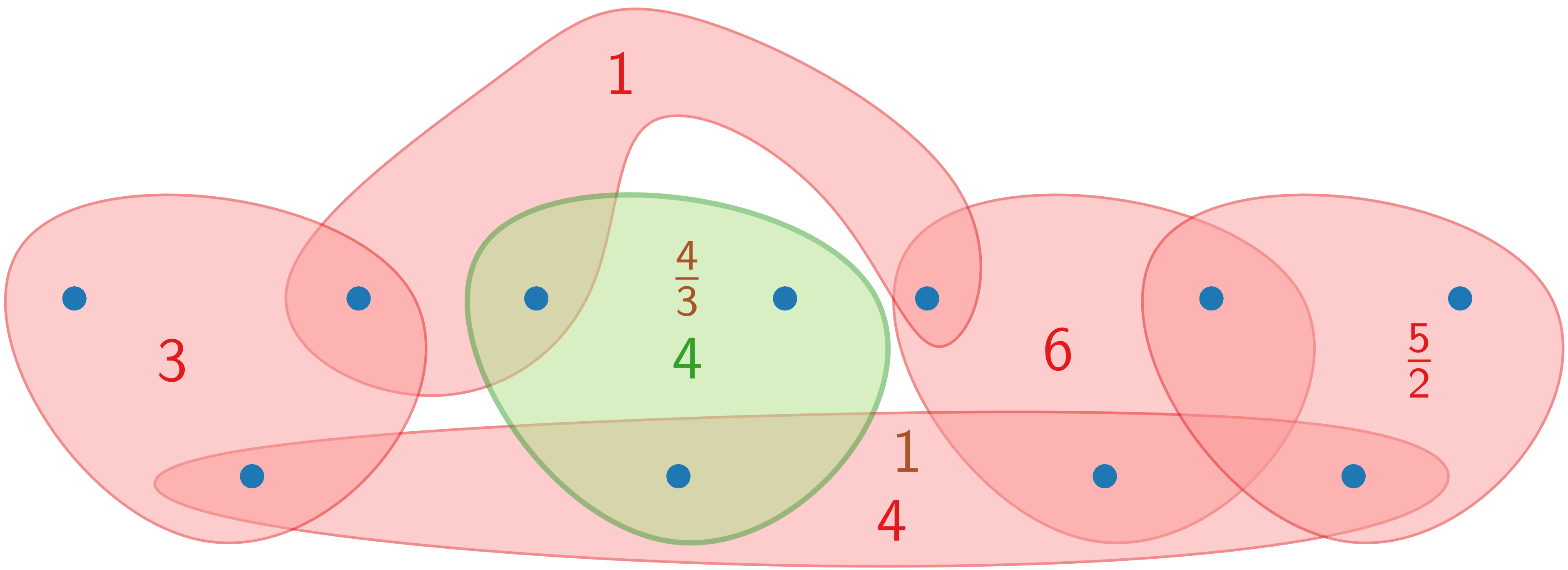

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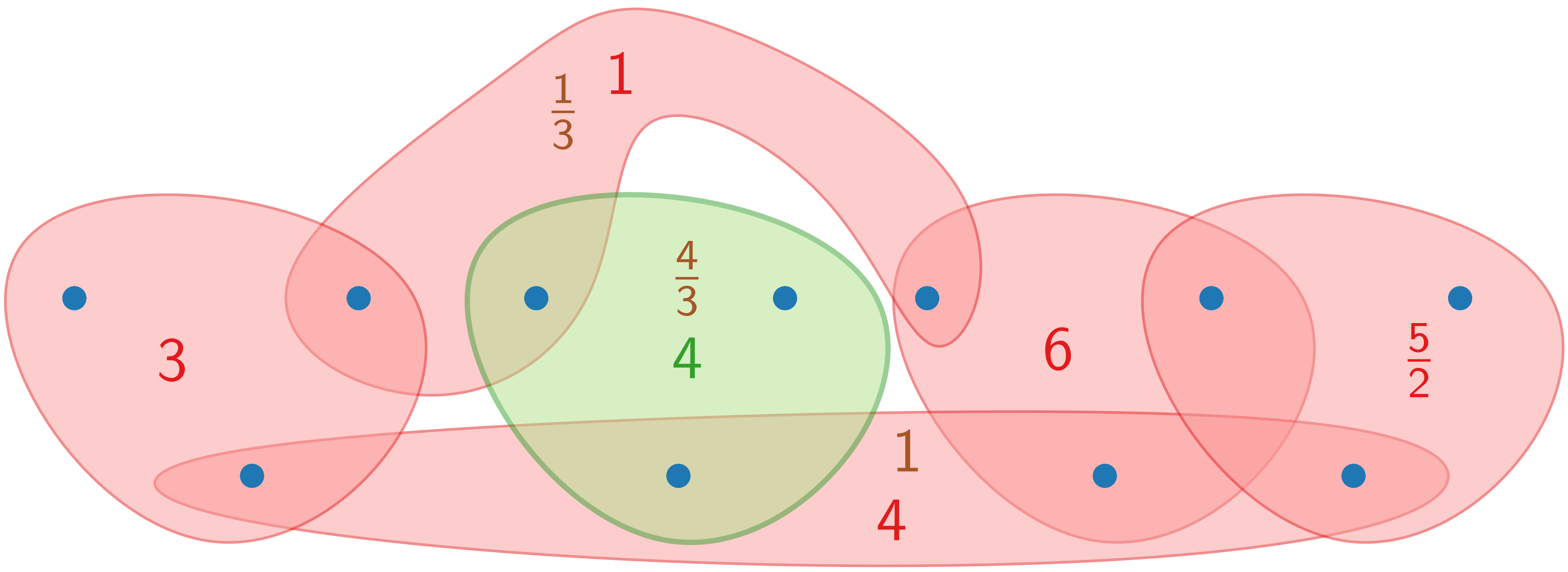

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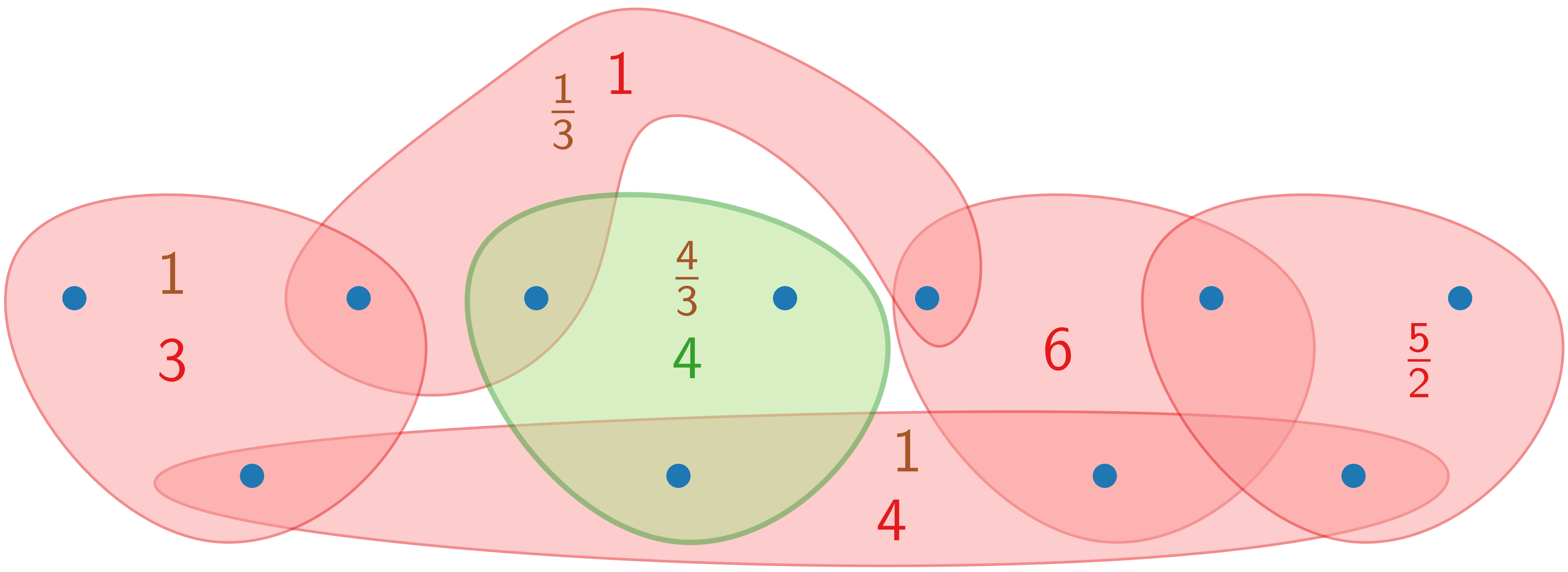

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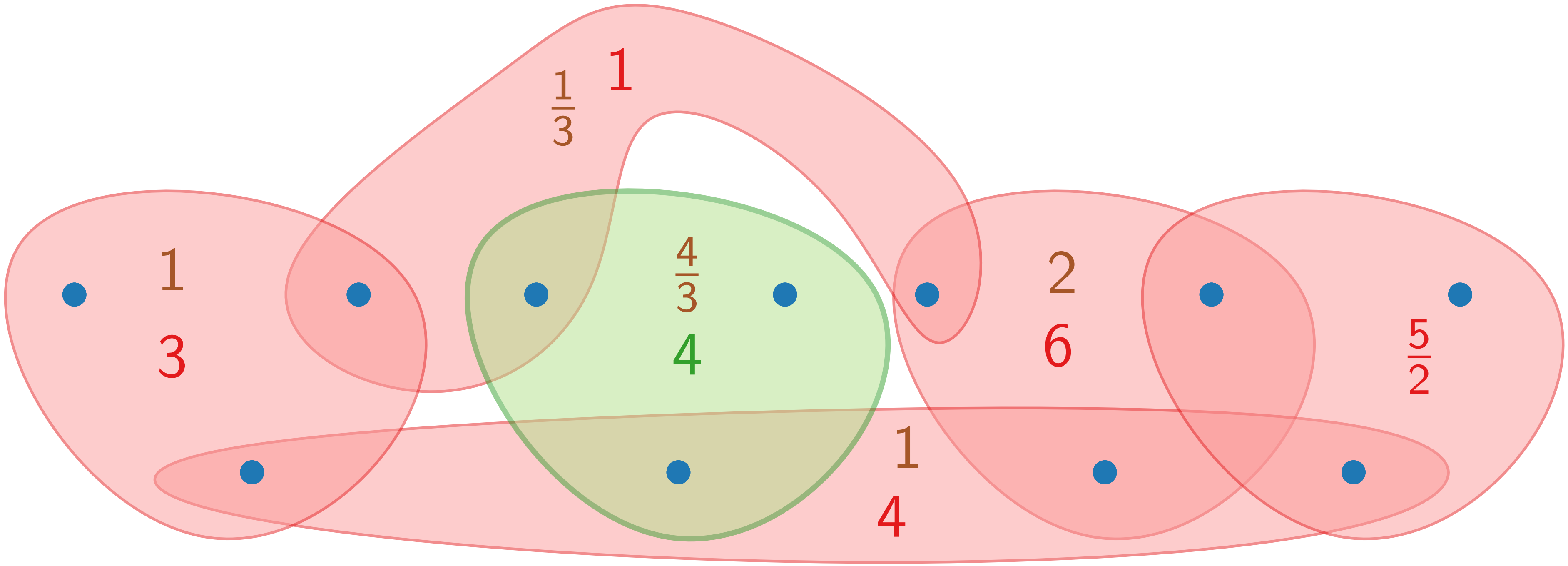

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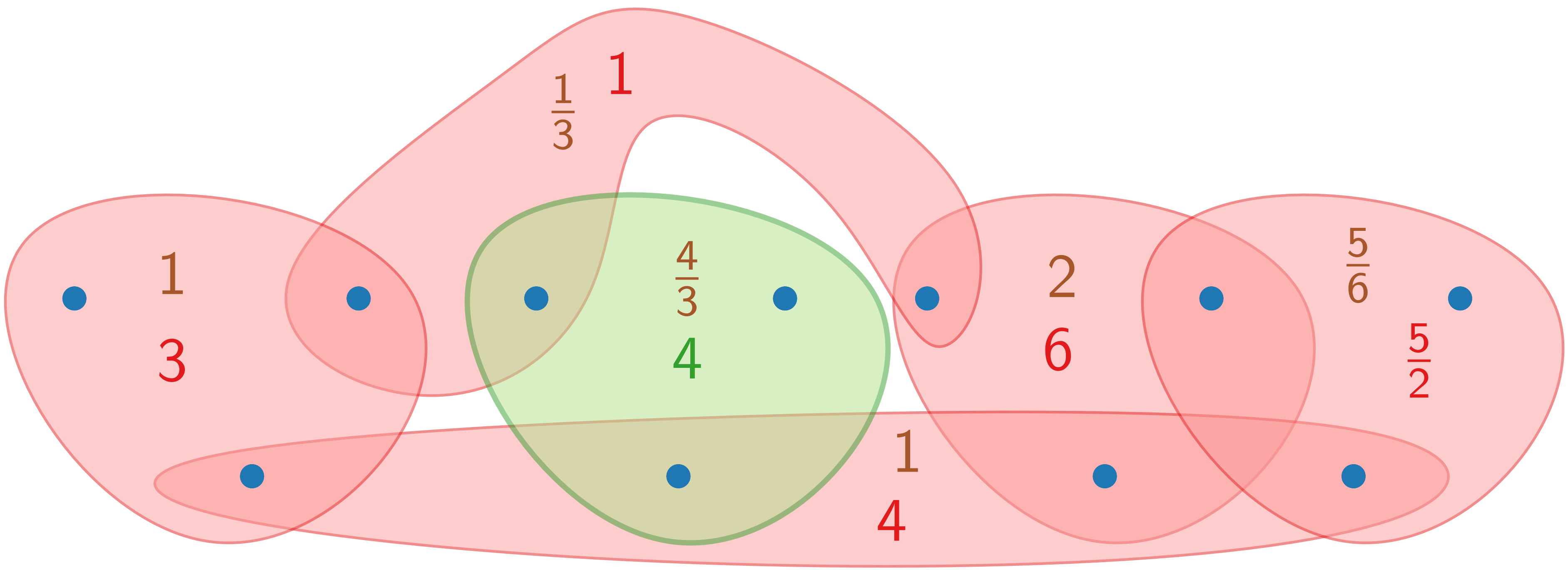

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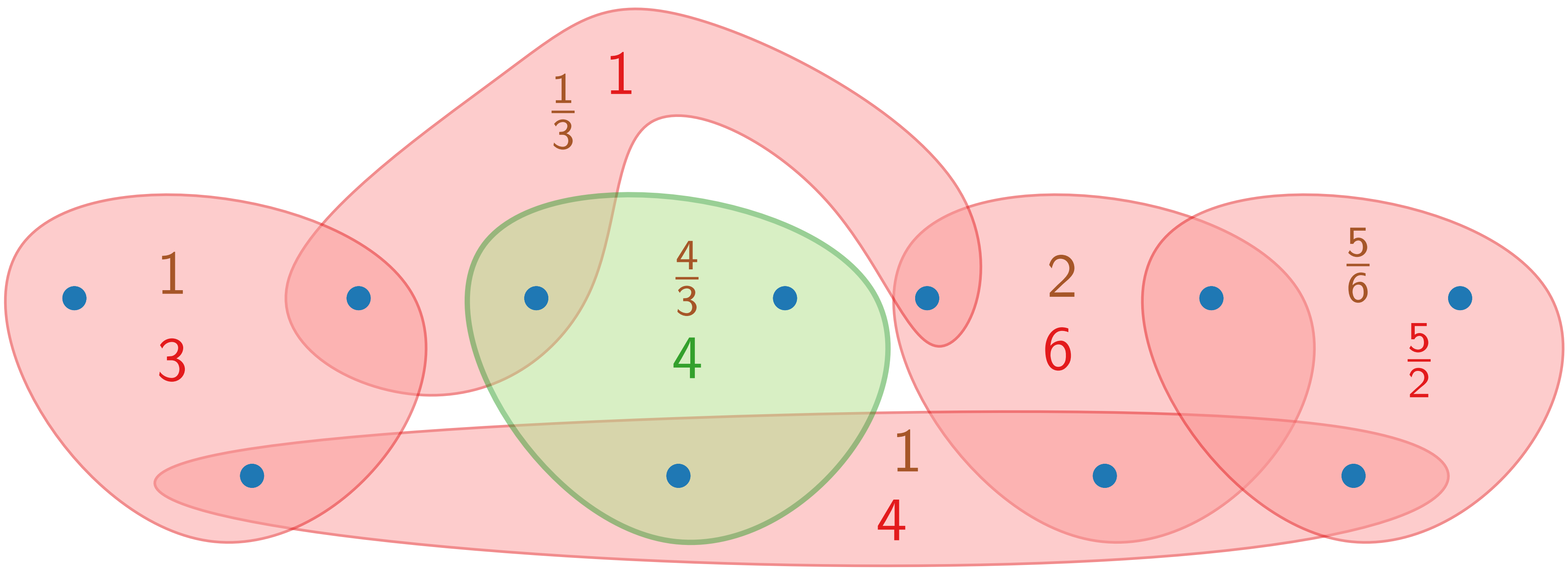

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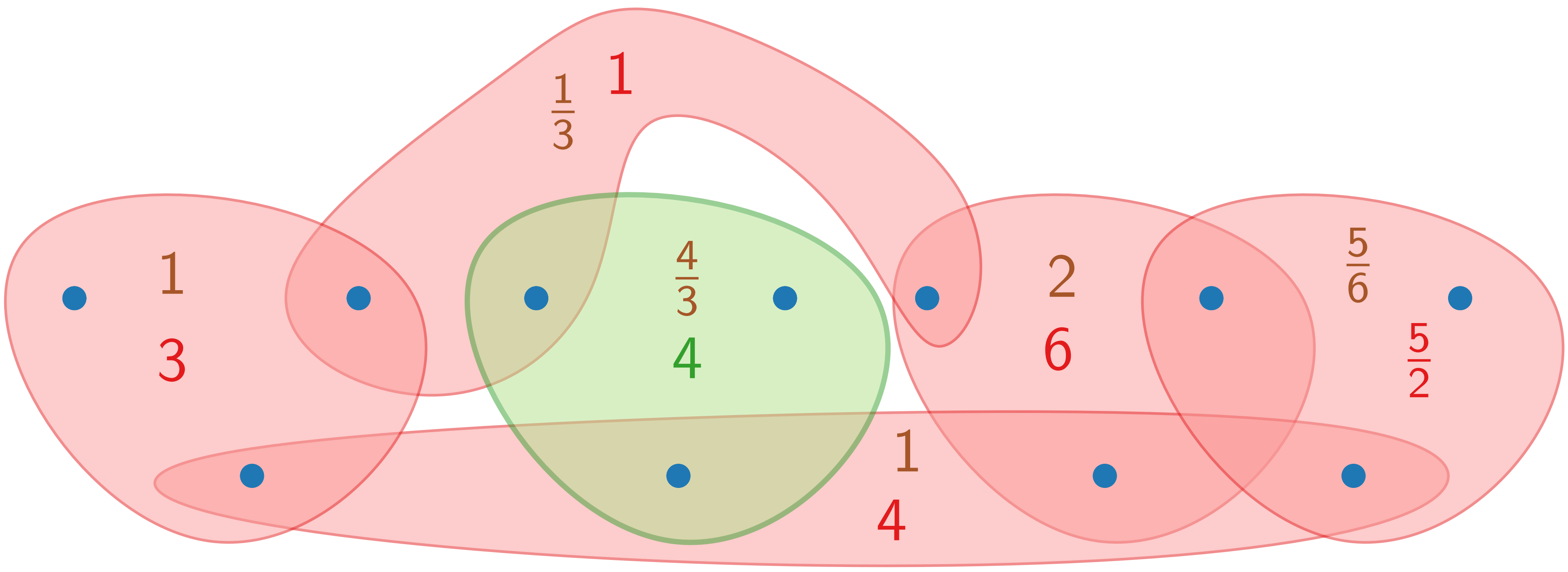

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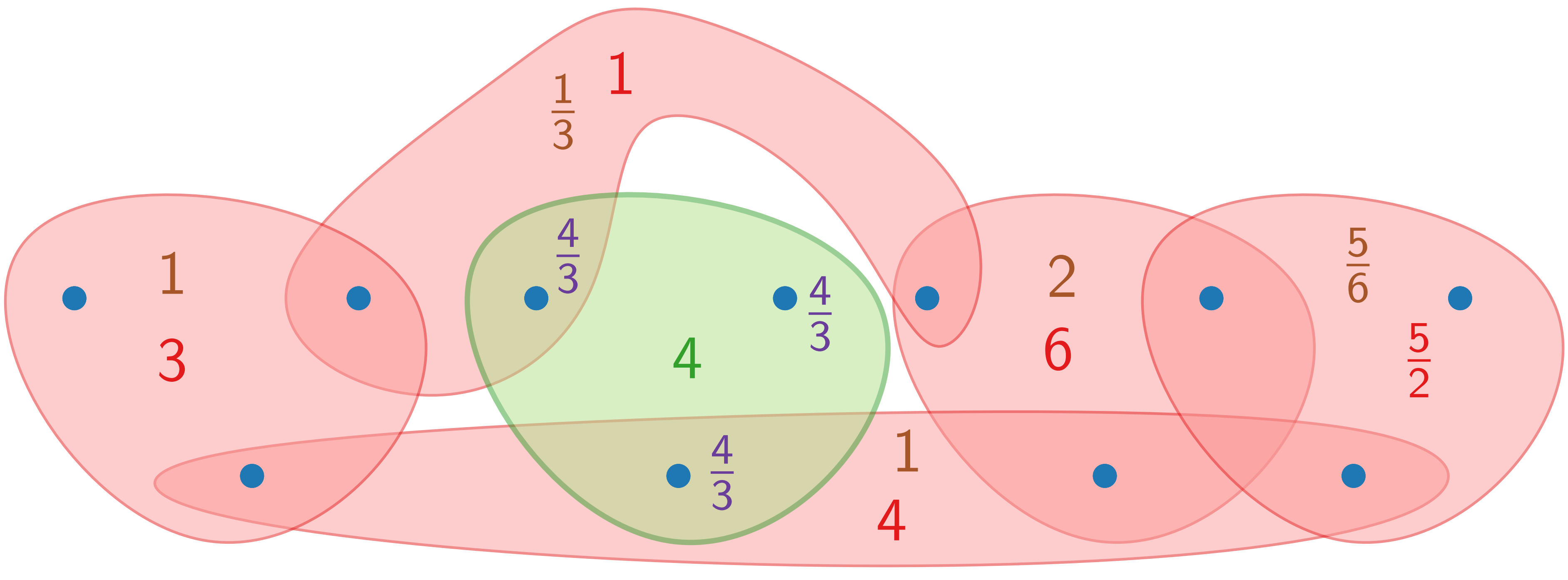

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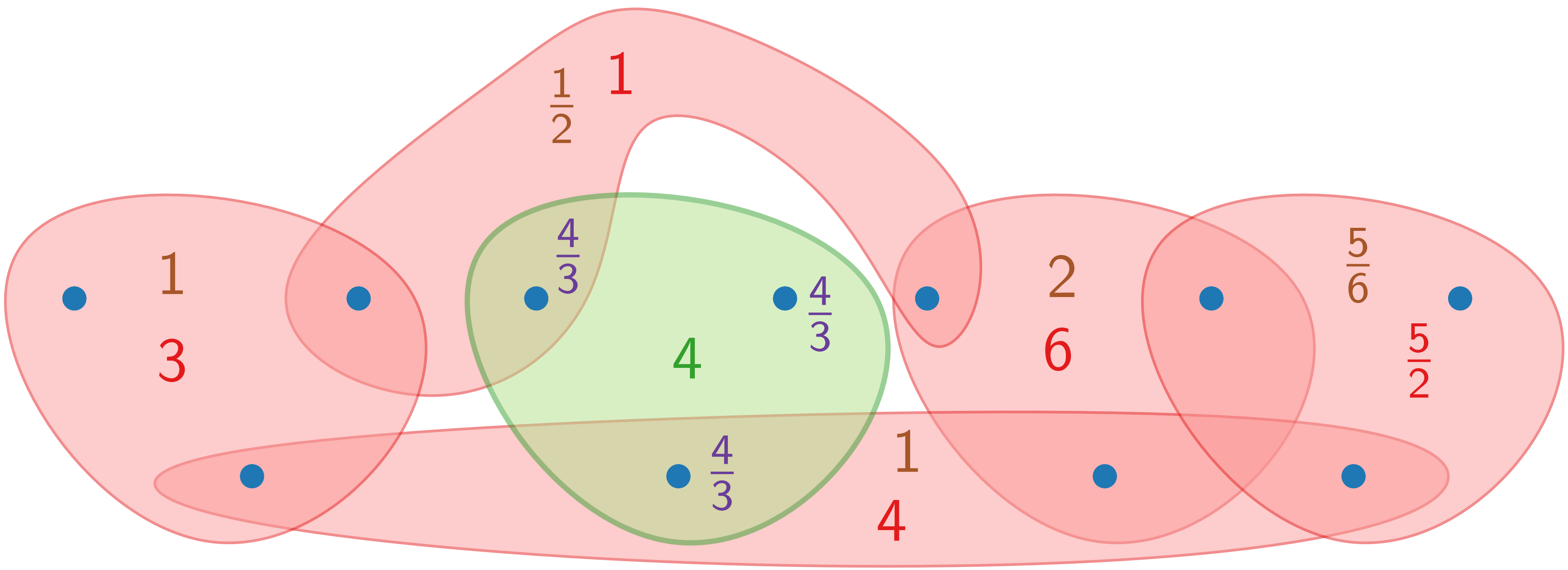

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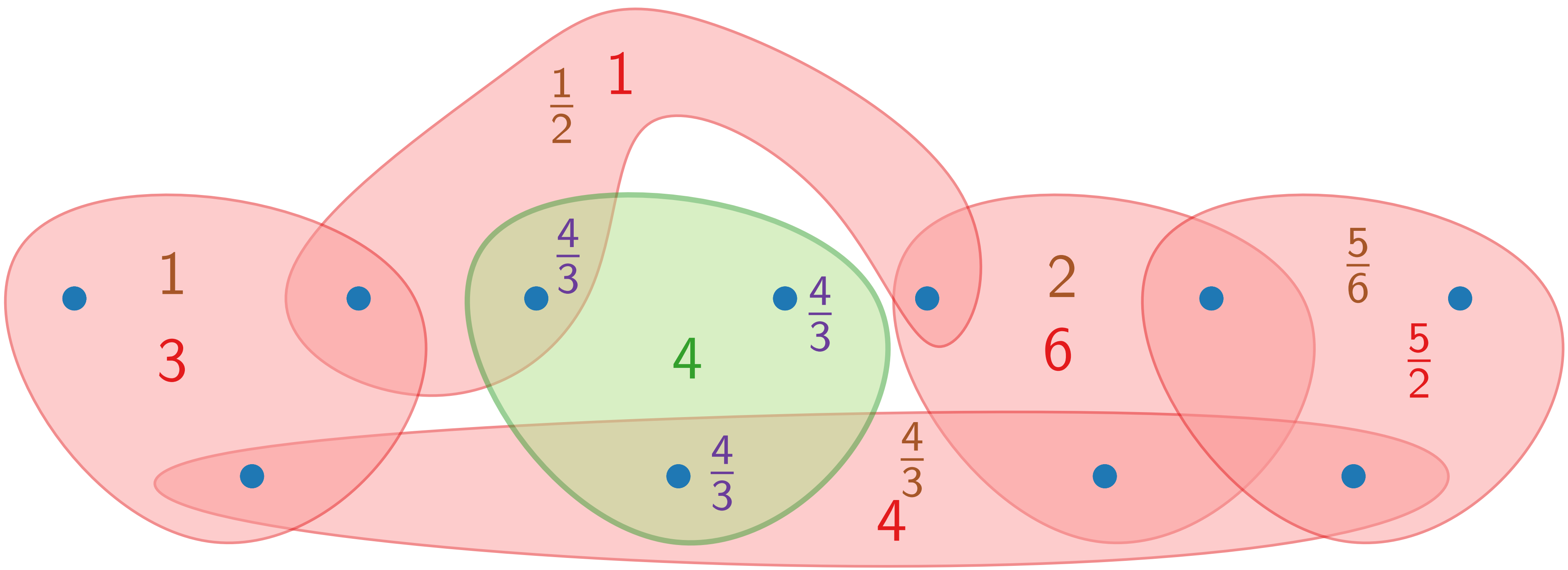

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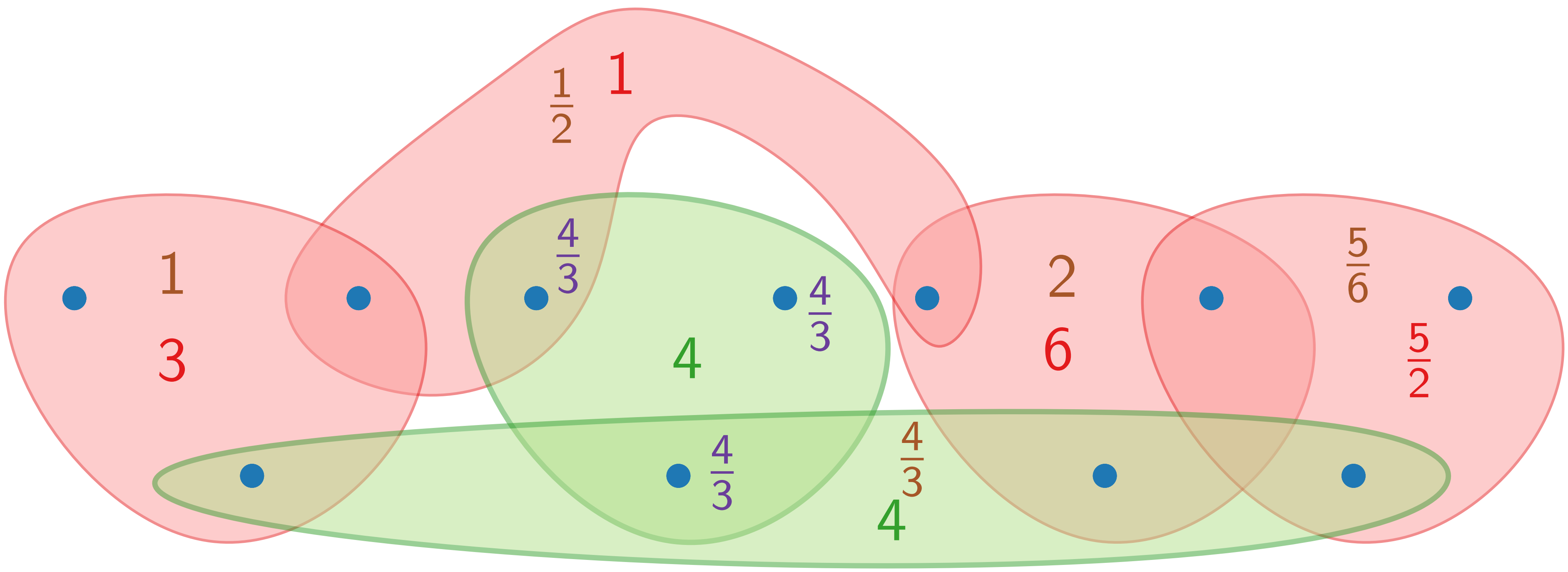

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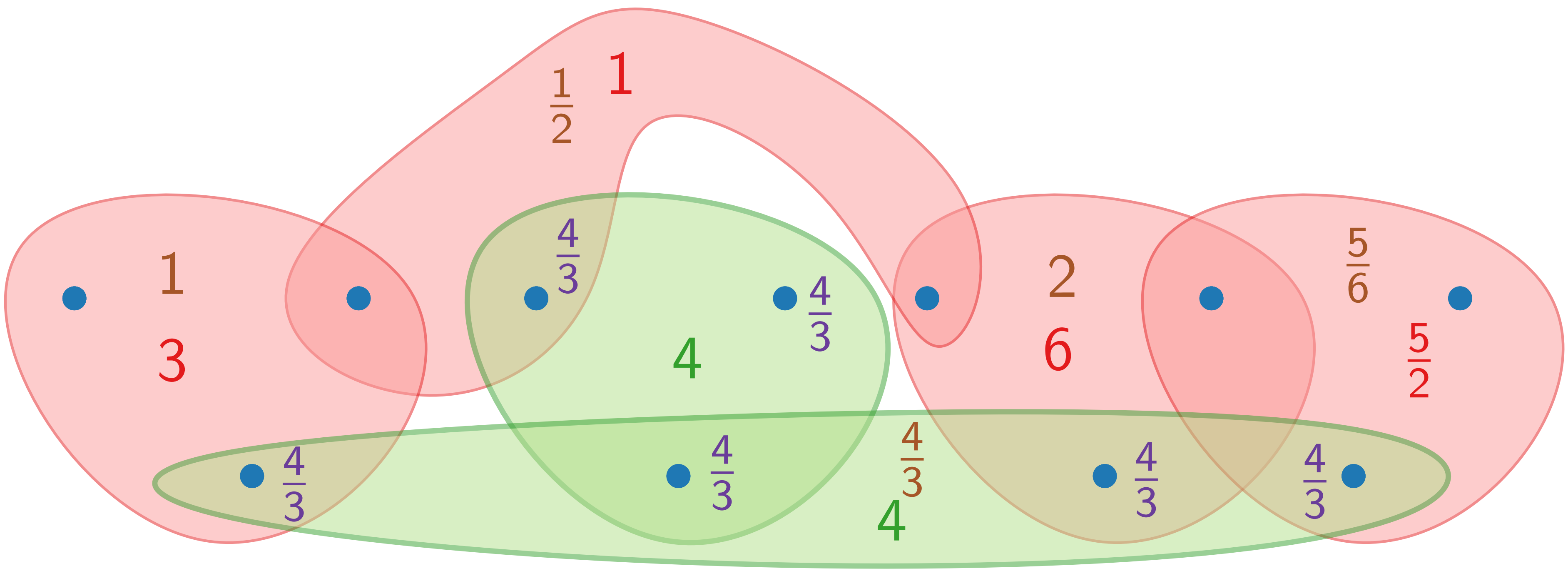

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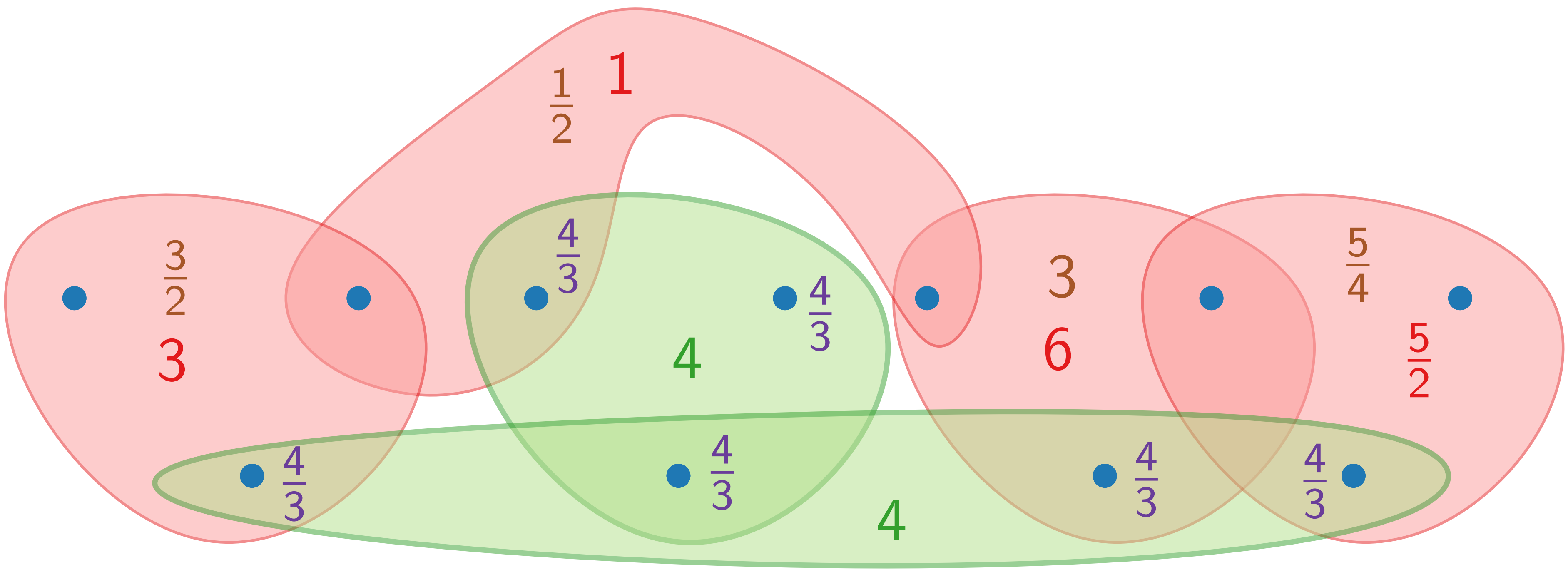

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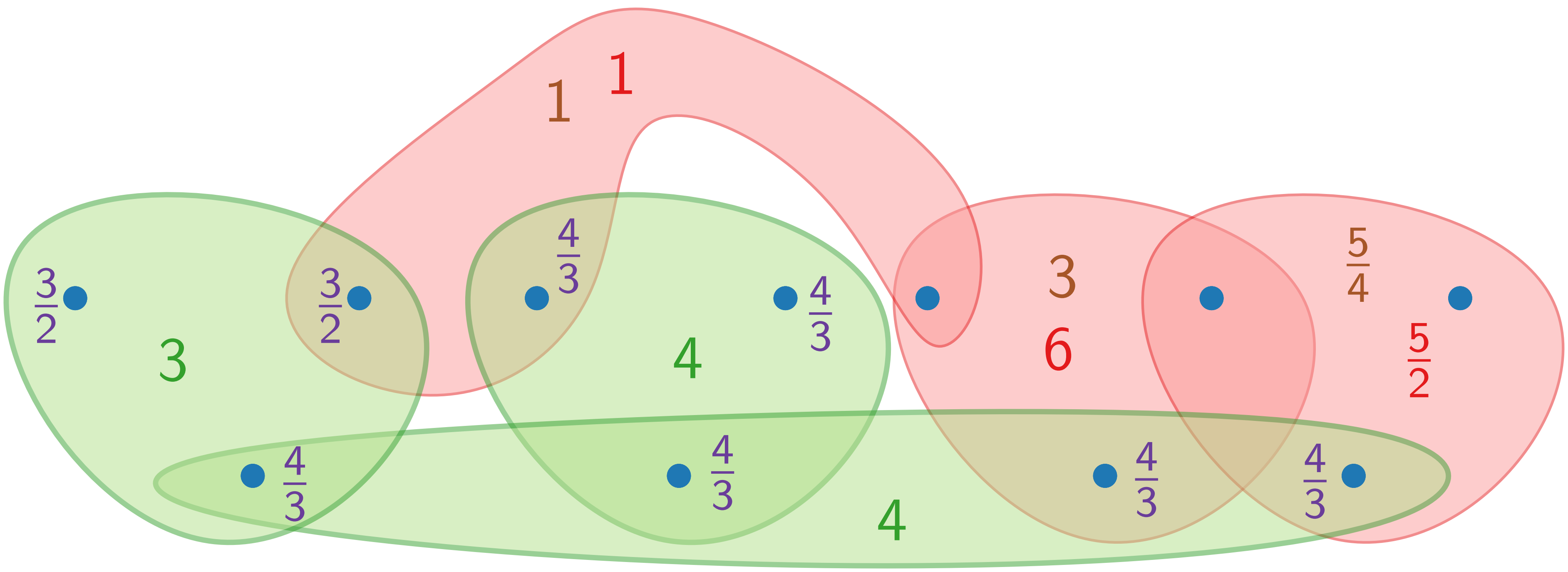

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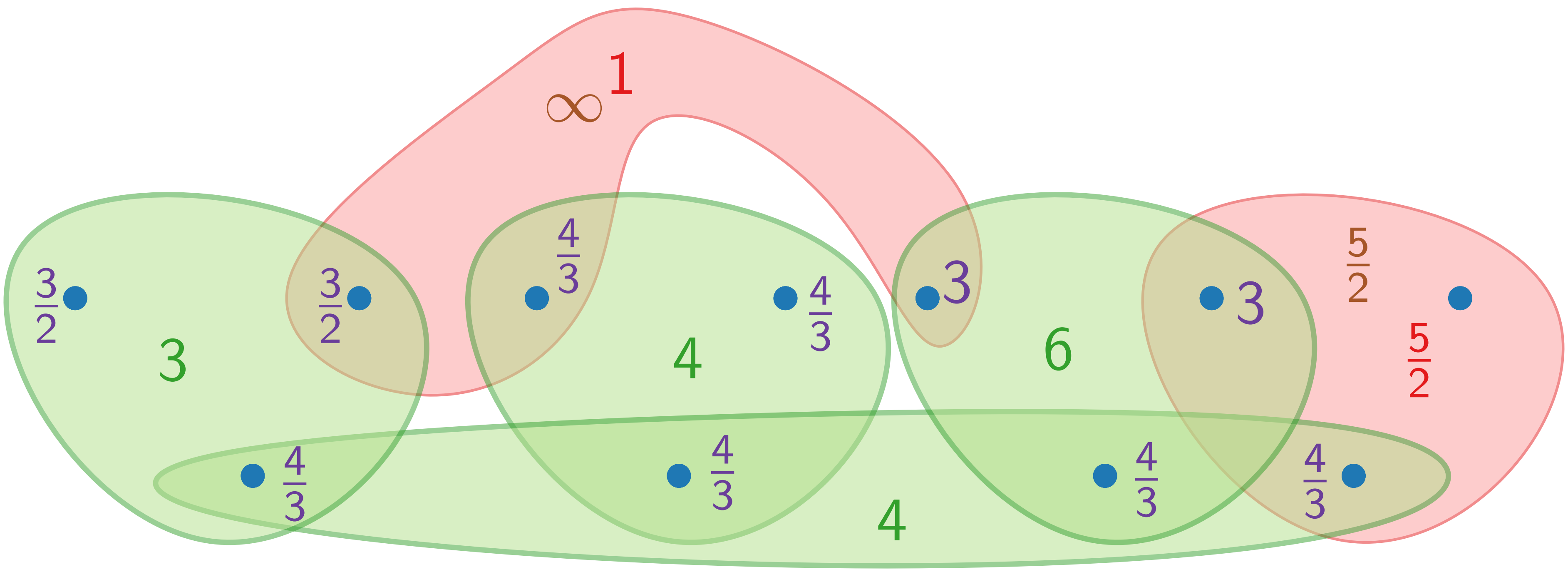

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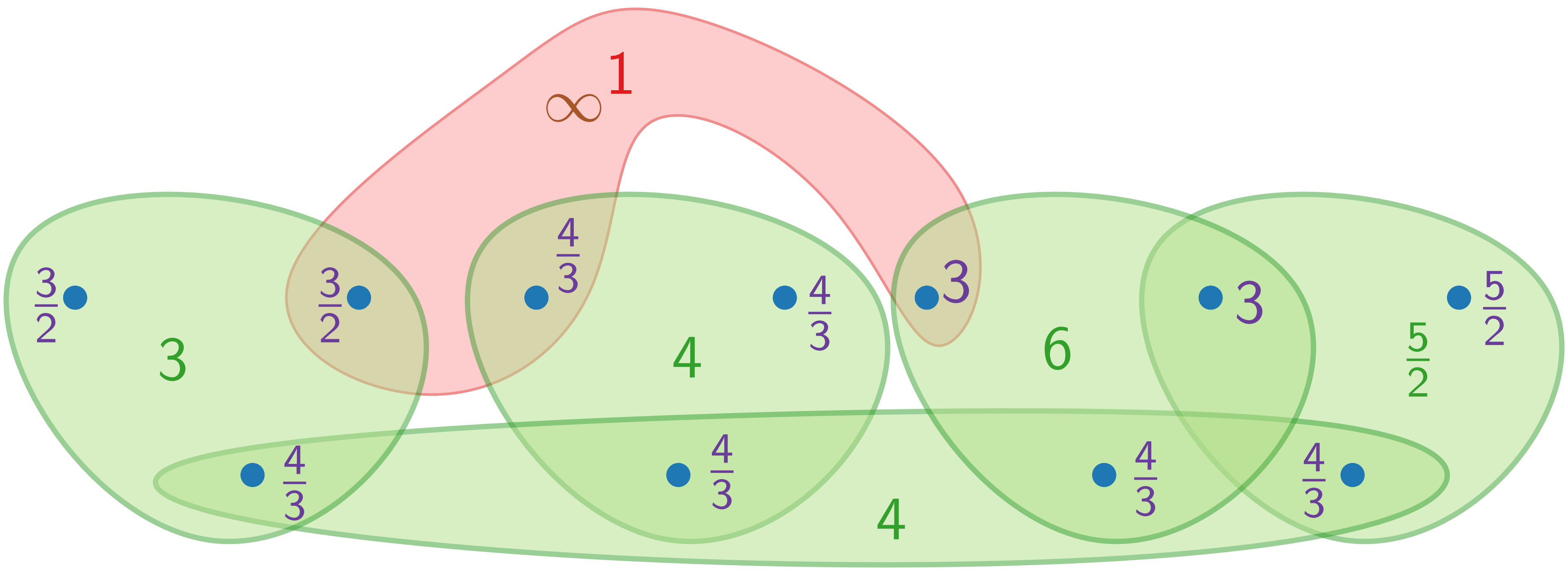

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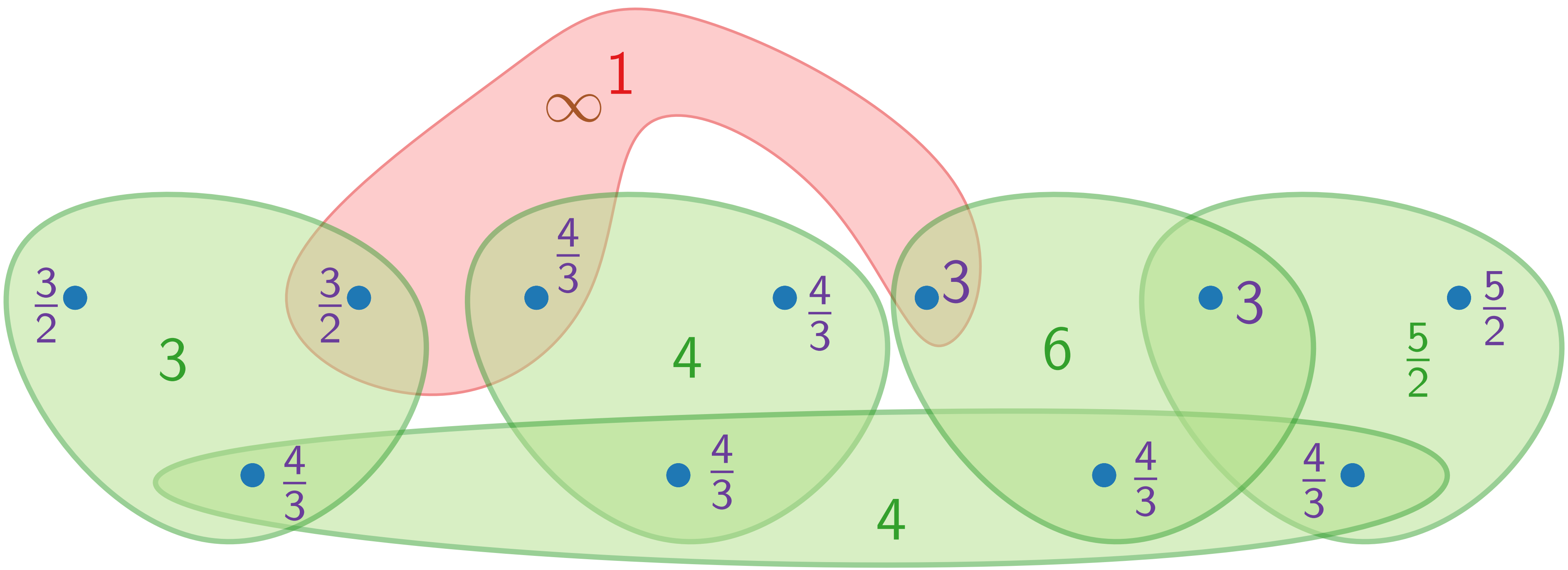

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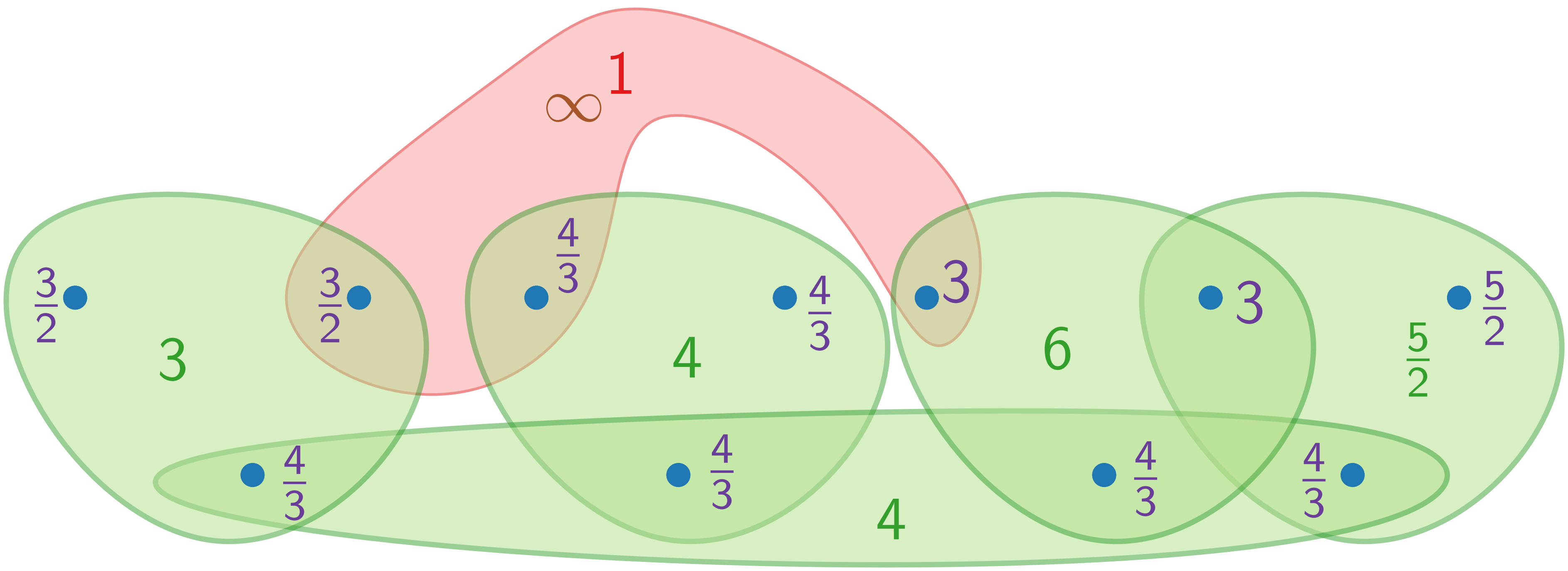
Greedy: Always choose the set with minimum per-element cost.



# Greedy for SetCover

GreedySetCover($U, \mathcal{S}, c$)

$C \leftarrow \emptyset$

$\mathcal{S}' \leftarrow \emptyset$

**return** $\mathcal{S}'$ `// Cover of` $U$

# Greedy for SetCover

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GreedySetCover(U, 𝒮, c)
  C ← ∅
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  while C ≠ U do

  return 𝒮'                                   // Cover of U
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# Approximation Algorithms

## Lecture 2:
## SetCover and ShortestSuperString

### Part III:
### Analysis

# Analysis

**Theorem.** GreedySetCover is a factor-$\mathcal{H}_k$ approximation algorithm for $\textsc{SetCover}$, where $k$ is the cardinality of the largest set in $\mathcal{S}$ and $\mathcal{H}_k := 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{k} \to 0.5 + \ln k$.

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**Lemma.** Let $S \in \mathcal{S}$, and let $u_1, \ldots, u_\ell$ be the elements of $S$ in the order in which they are covered ("bought") by GreedySetCover. Then, for every $j \in \{1, \ldots, \ell\}$: $\text{price}(u_j) \leq$

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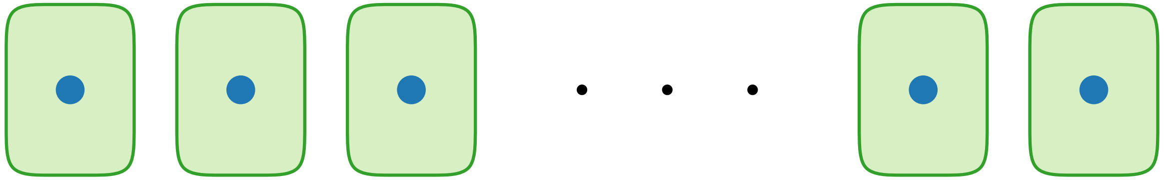

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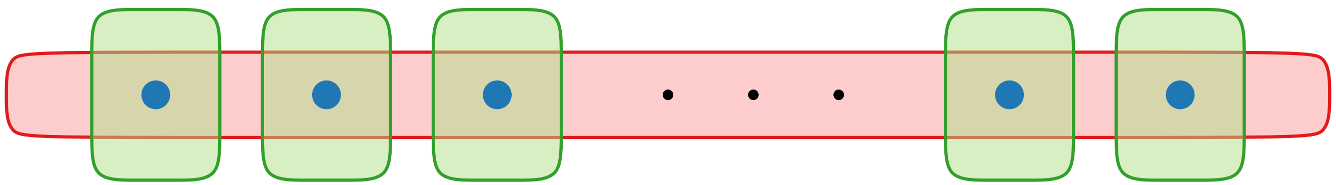

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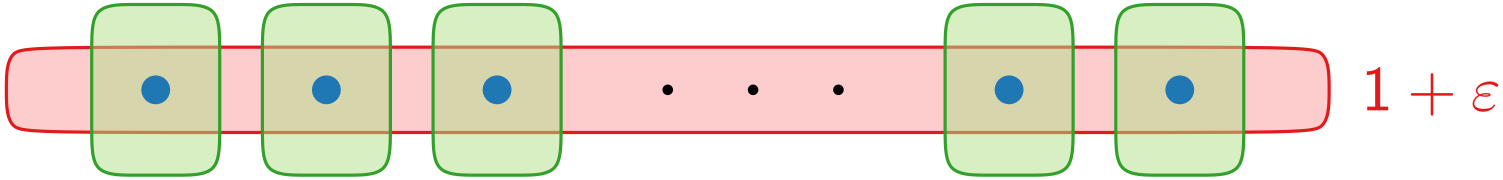

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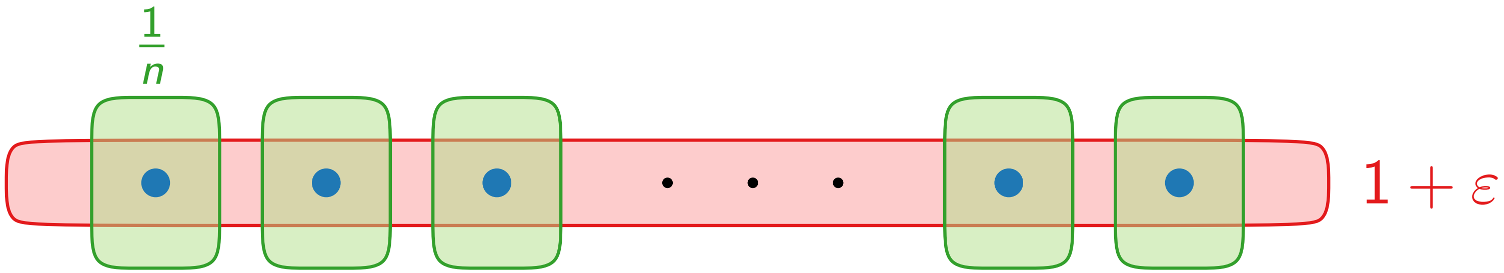

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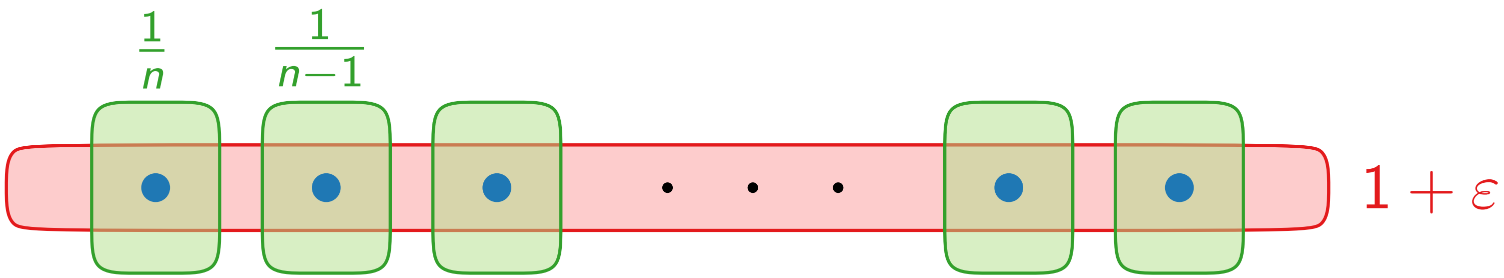

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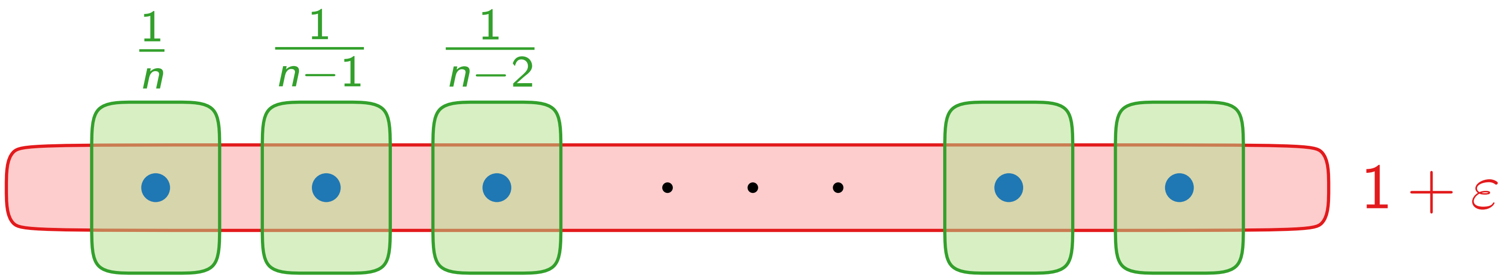

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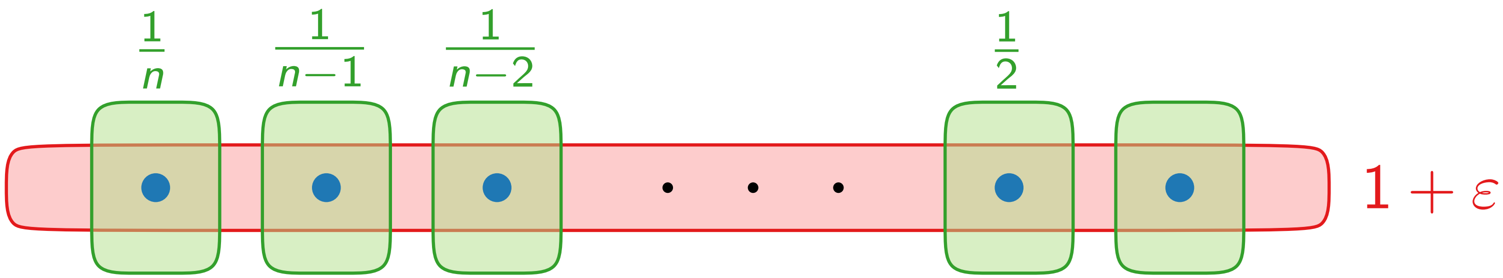

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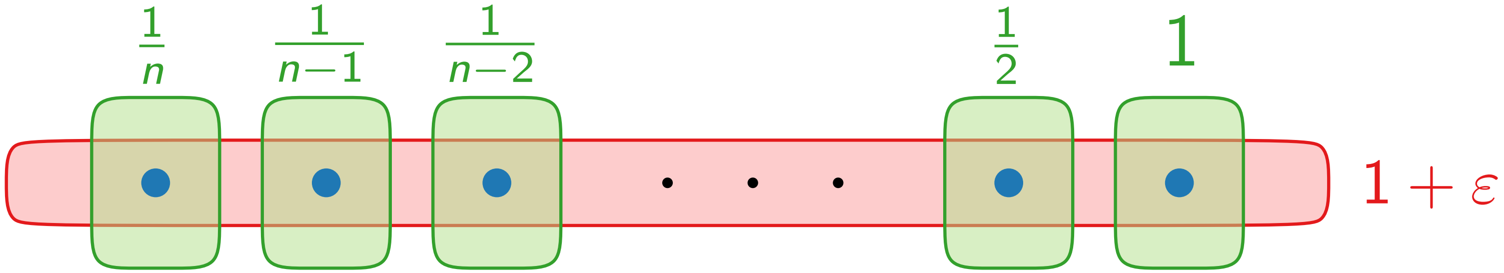

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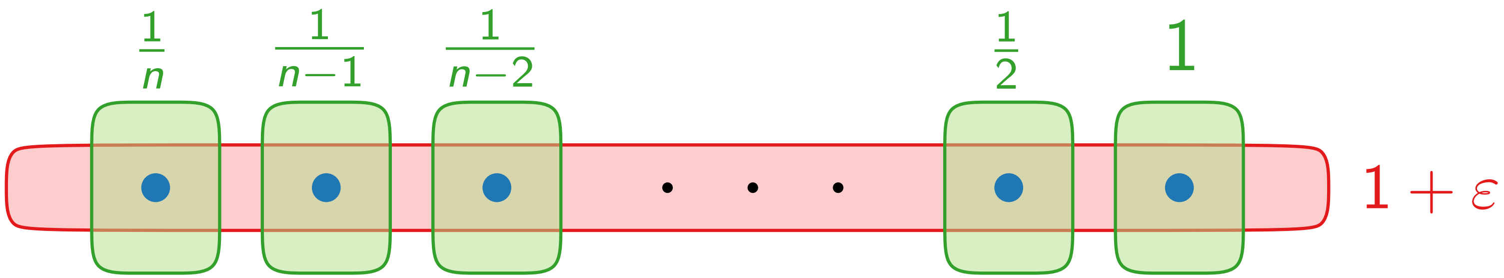


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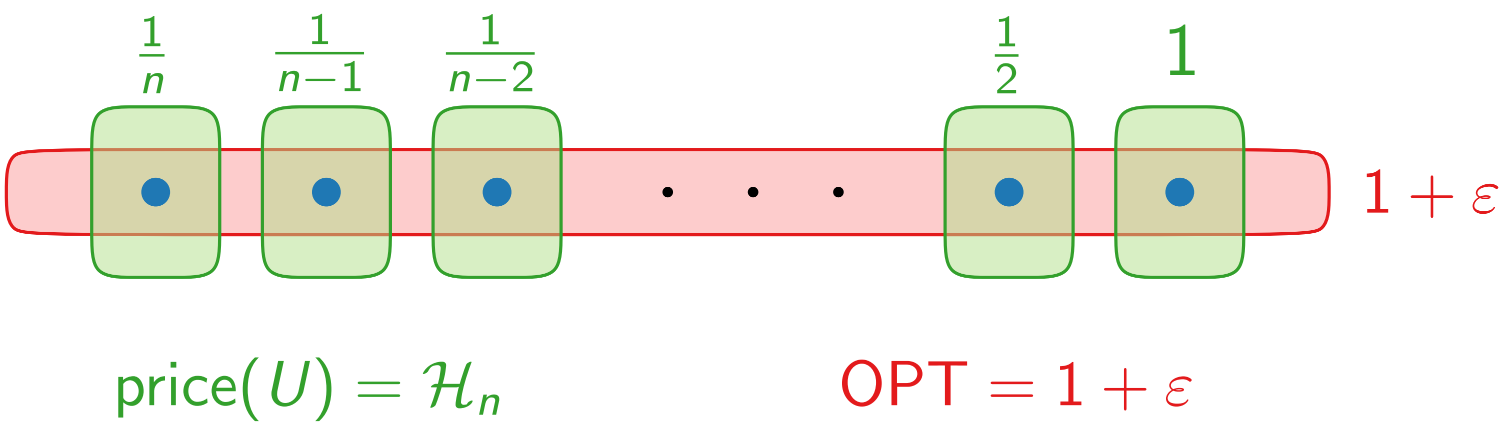


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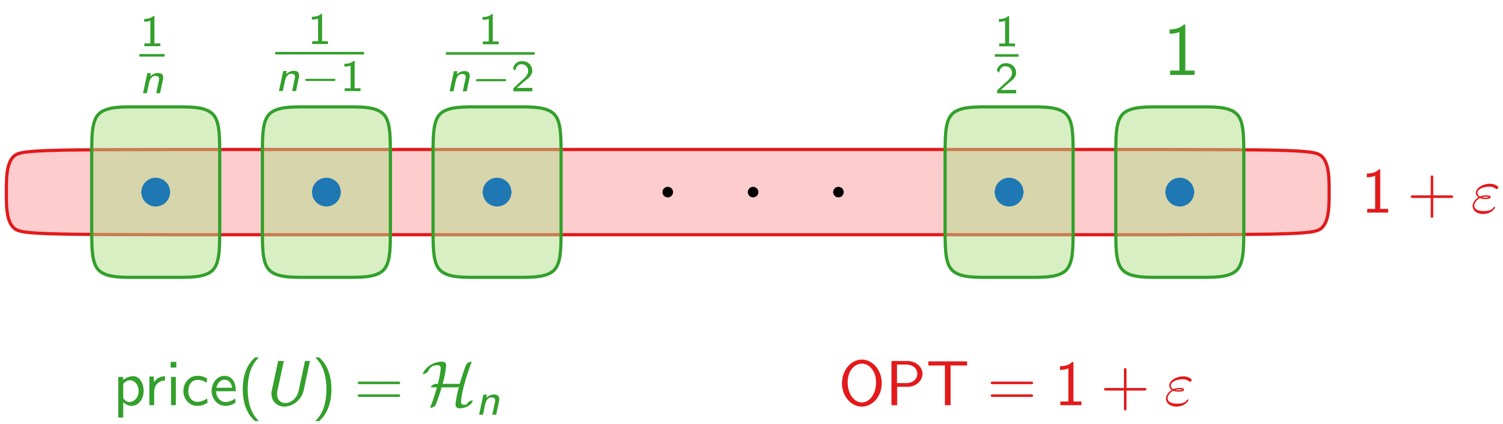


**Can we do better?**

No – for any $\varepsilon > 0$, it is NP-hard to approximate SetCover with factor $(1 - \varepsilon) \cdot \ln n$ [Feige, JACM 1998] [Dinur, Steurer, STOC 2014]

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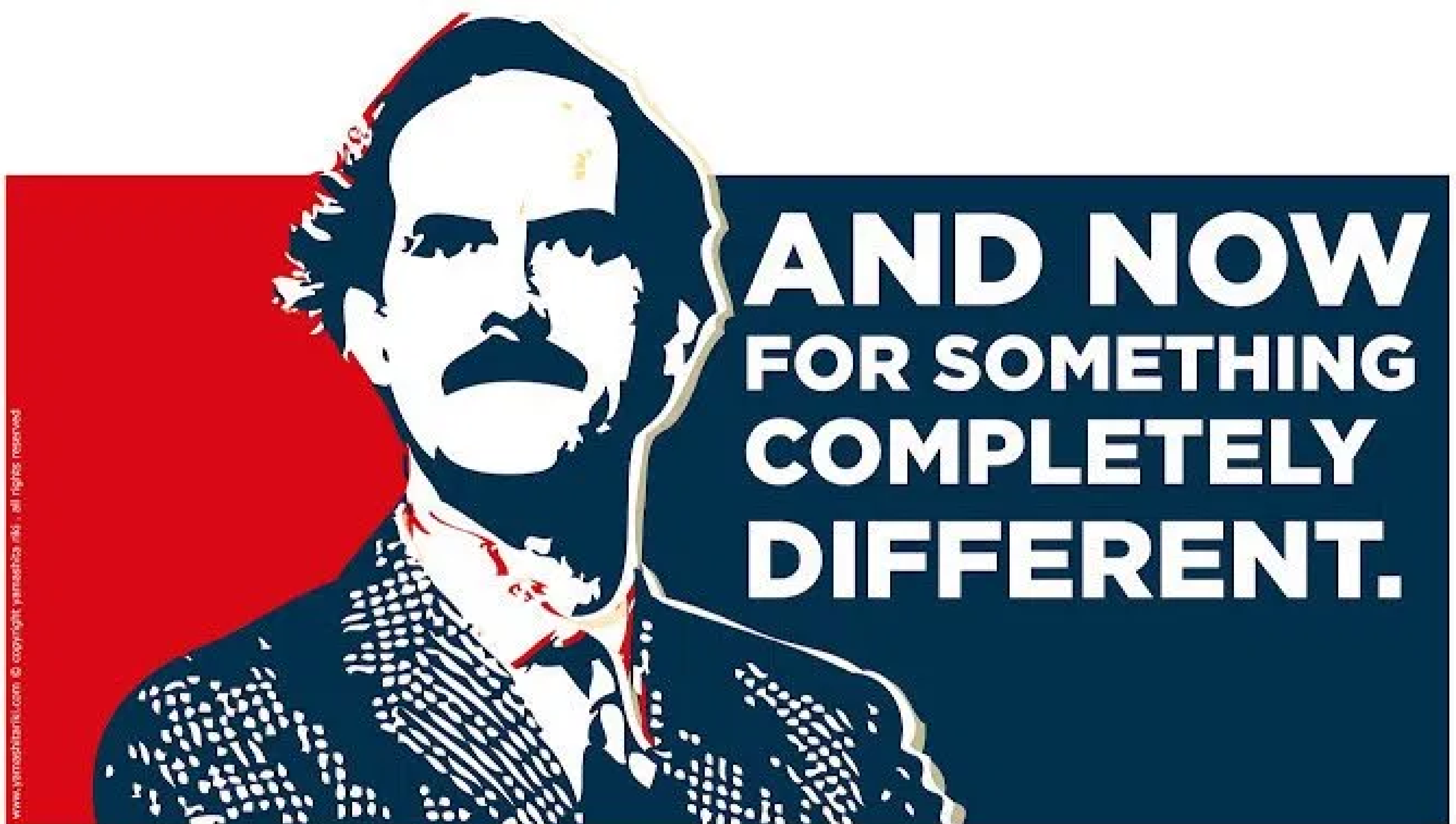


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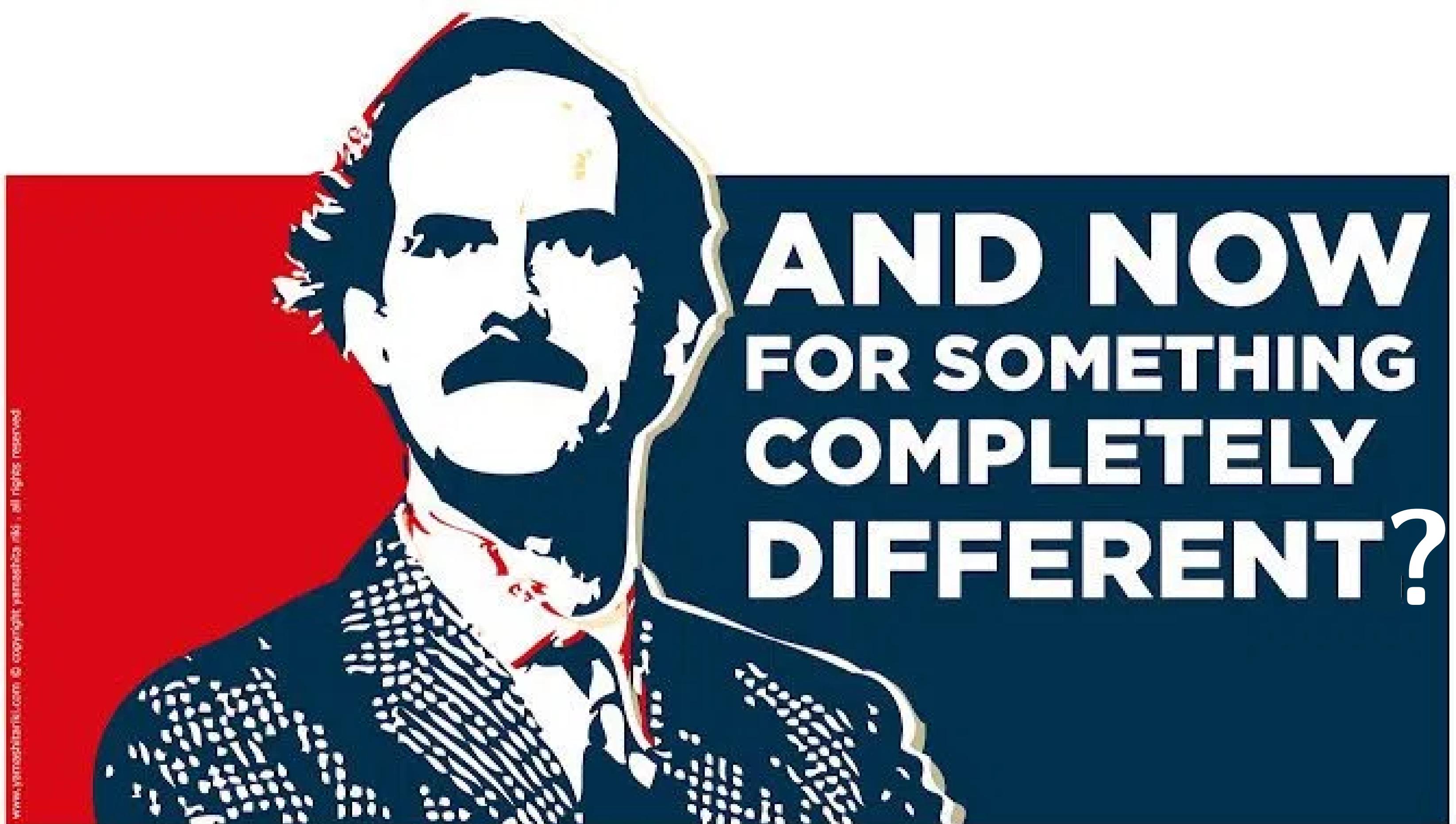


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# Approximation Algorithms

## Lecture 2:
## SetCover and ShortestSuperString

### Part IV:
### ShortestSuperString

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**Example.** $U := \{cbaa, abc, bcb\}$

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$$\begin{array}{l} abc \\ \;\;bcb \\ \;\;\;\;cbaa \end{array}$$

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$abc$
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**Example.** $U := \{cbaa, abc, bcb\}$ $\to$ $cbaabcb$ ?

W.l.o.g.: No string $s_i$ is a substring of any other string $s_j$.

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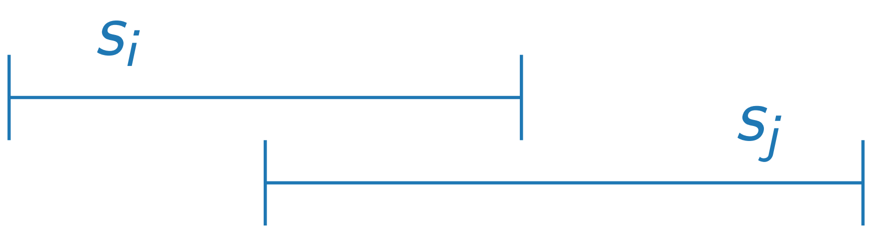

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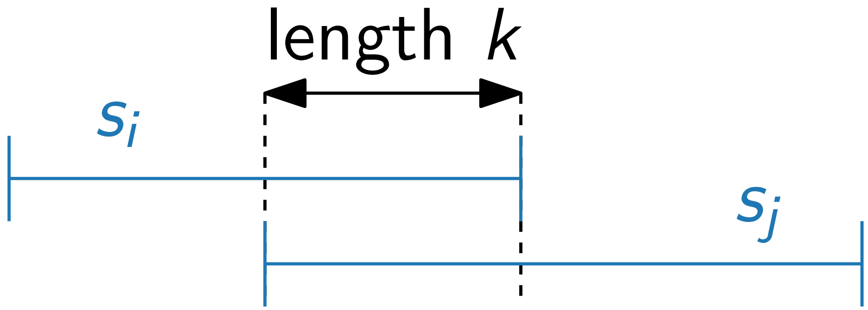

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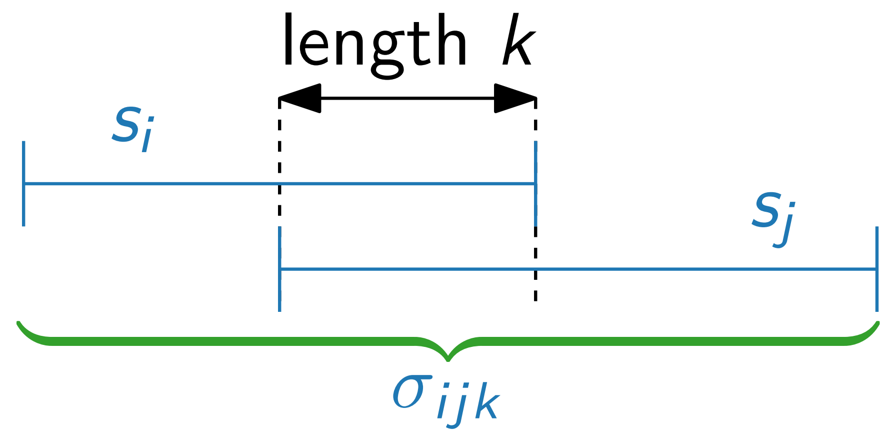

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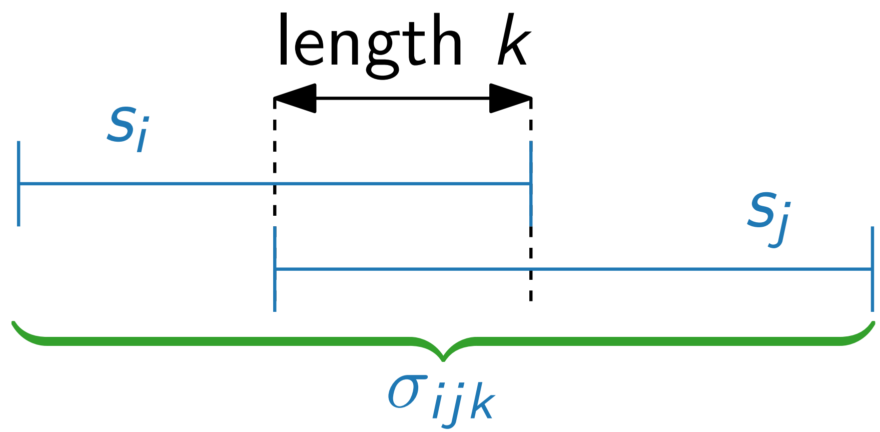

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```
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  ababc
```

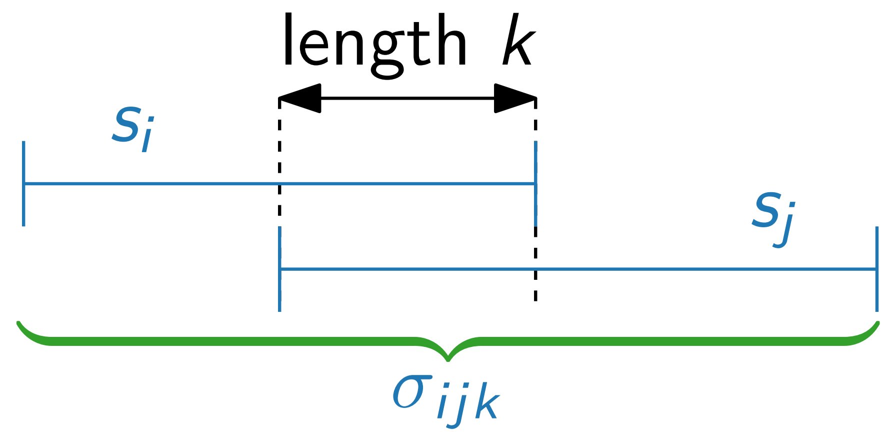

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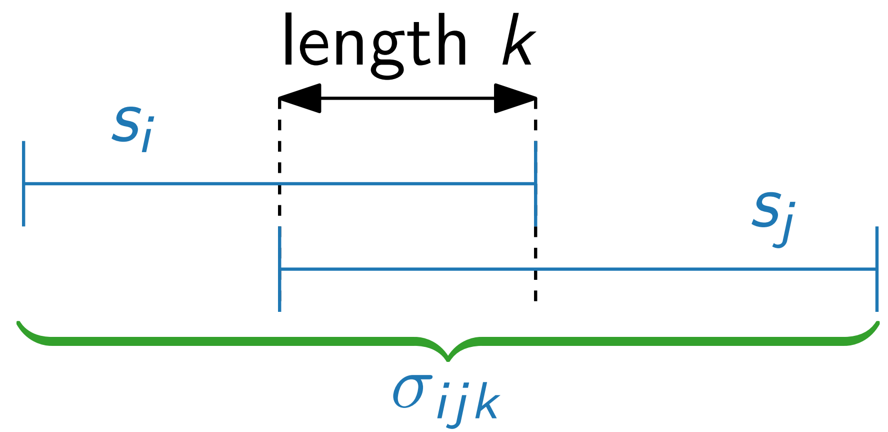

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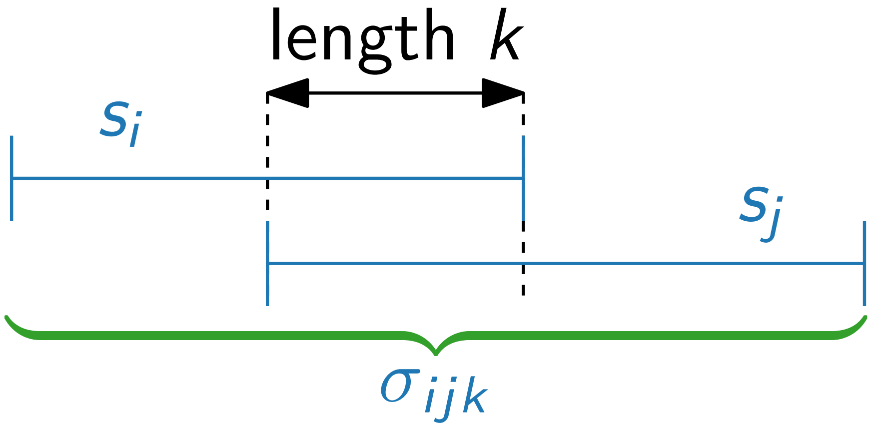

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cabab   cabab

  ababc   ababc

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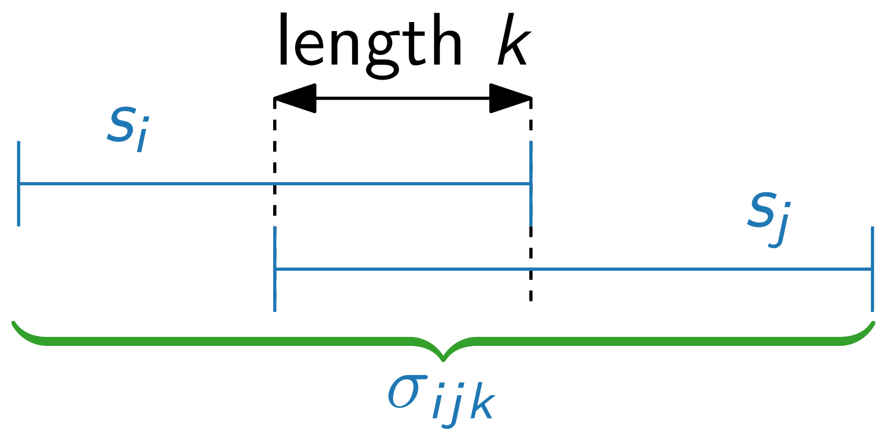

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ababc

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ababc

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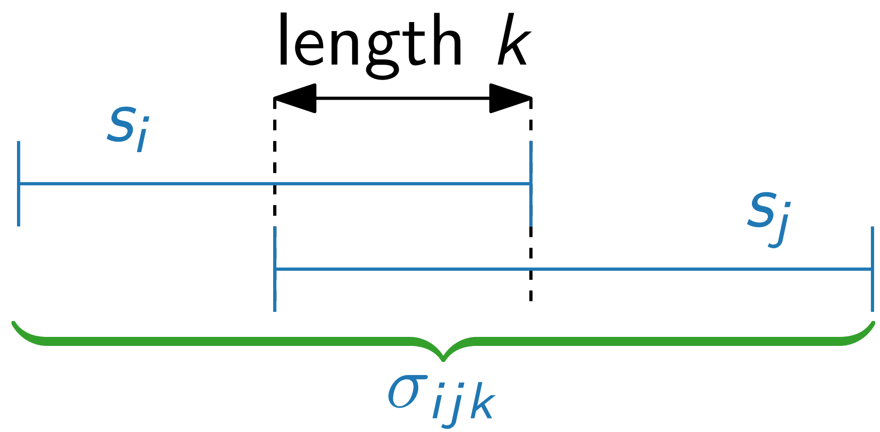

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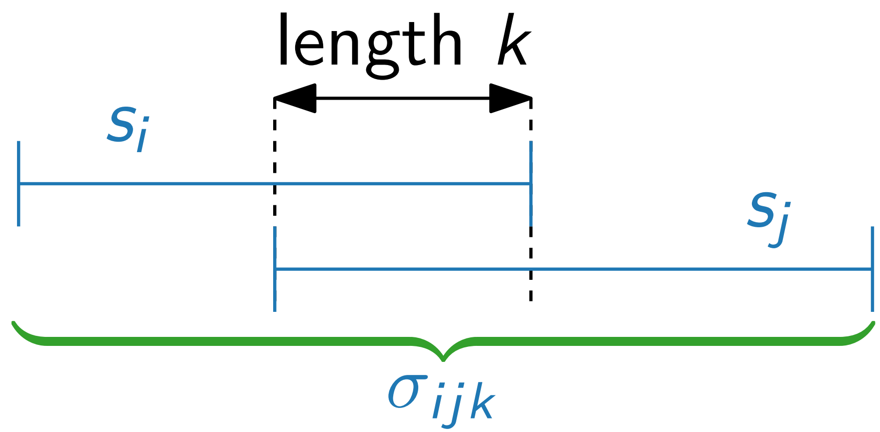


$S(\sigma_{ijk}) =$

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cabab / ababc (overlap of 2) cabab / ababc (overlap of 4)

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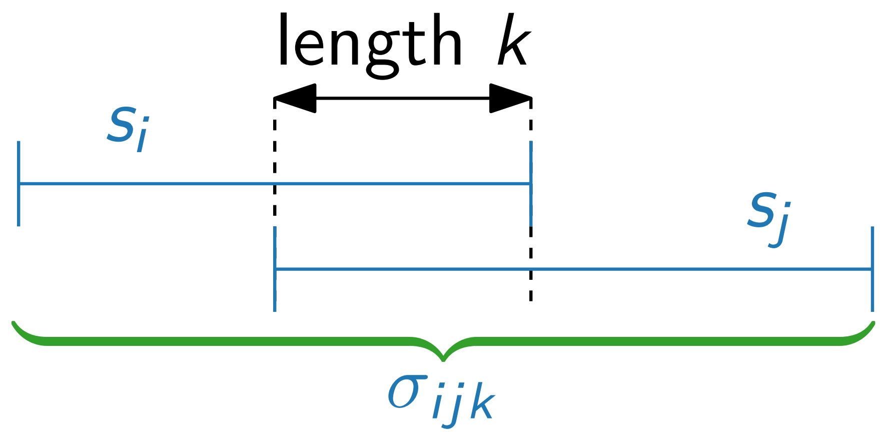


$S(\sigma_{ijk}) = \{s \in U \mid s$ substring of $\sigma_{ijk}\}$ – contains the elements of the ground set covered by $\sigma_{ijk}$.

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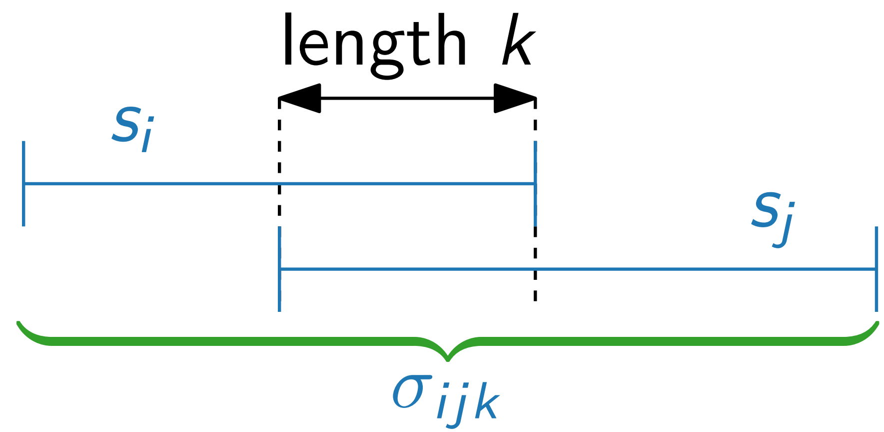


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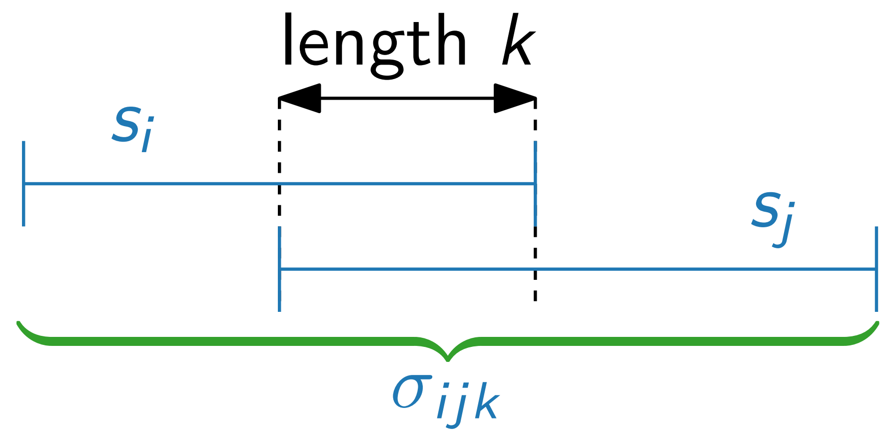


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$\mathcal{S} = \{S(\sigma_{ijk}) \mid 1 \leq i, j \leq n, \text{suitable } k \geq 0\}$

# Approximation Algorithms

## Lecture 2: SetCover and ShortestSuperString

### Part V: Solving ShortestSuperString via SetCover

# Relating SSS and SetCover

**Lemma.** Let $\mathsf{OPT}_{\mathsf{SSS}}$ be the length of a shortest superstring of $U$, and let $\mathsf{OPT}_{\mathsf{SC}}$ be the minimum cost of the corresponding SetCover instance. Then

$$\mathsf{OPT}_{\mathsf{SSS}} \leq \mathsf{OPT}_{\mathsf{SC}}.$$

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Thus, $\mathrm{OPT}_{\mathrm{SSS}} \leq |s| = \mathrm{OPT}_{\mathrm{SC}}$.

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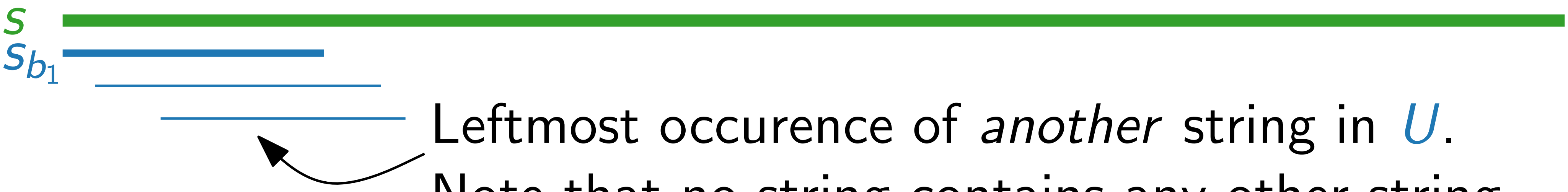


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$\Rightarrow$ Right endpoints are ordered like left endpoints.

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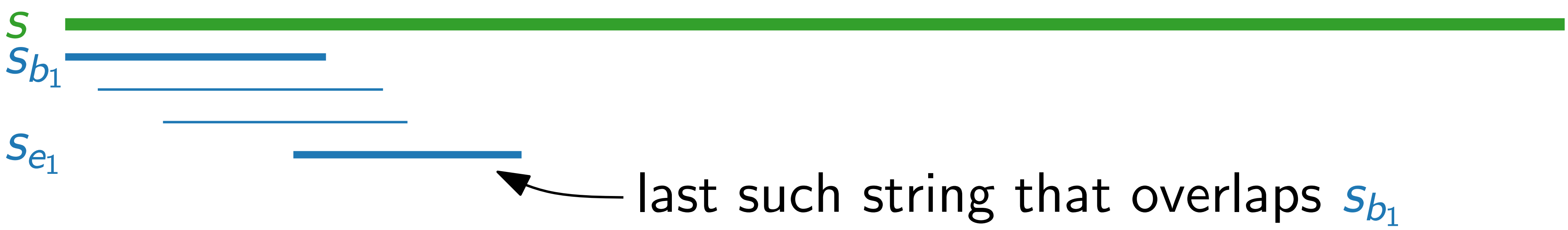

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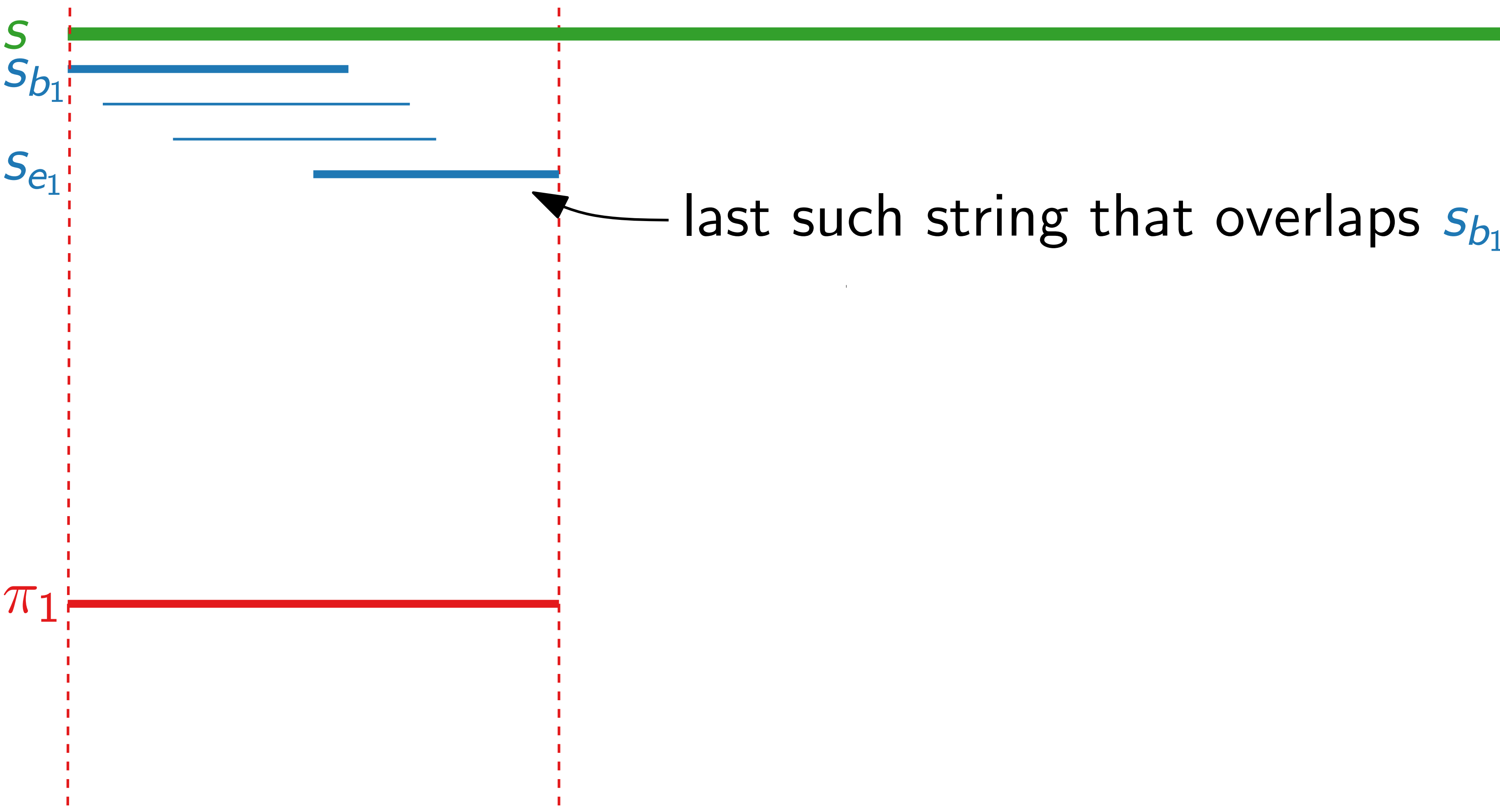

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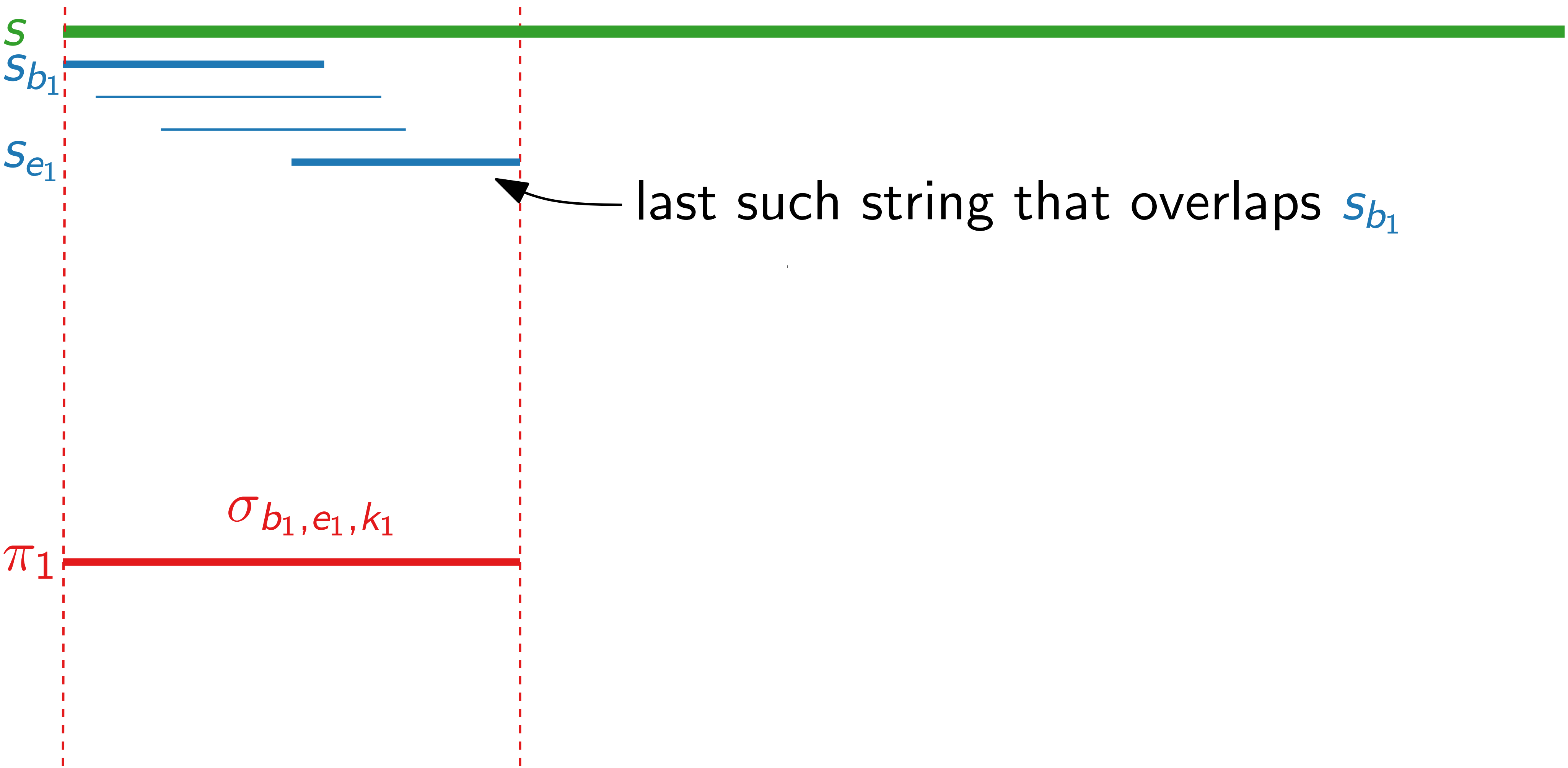

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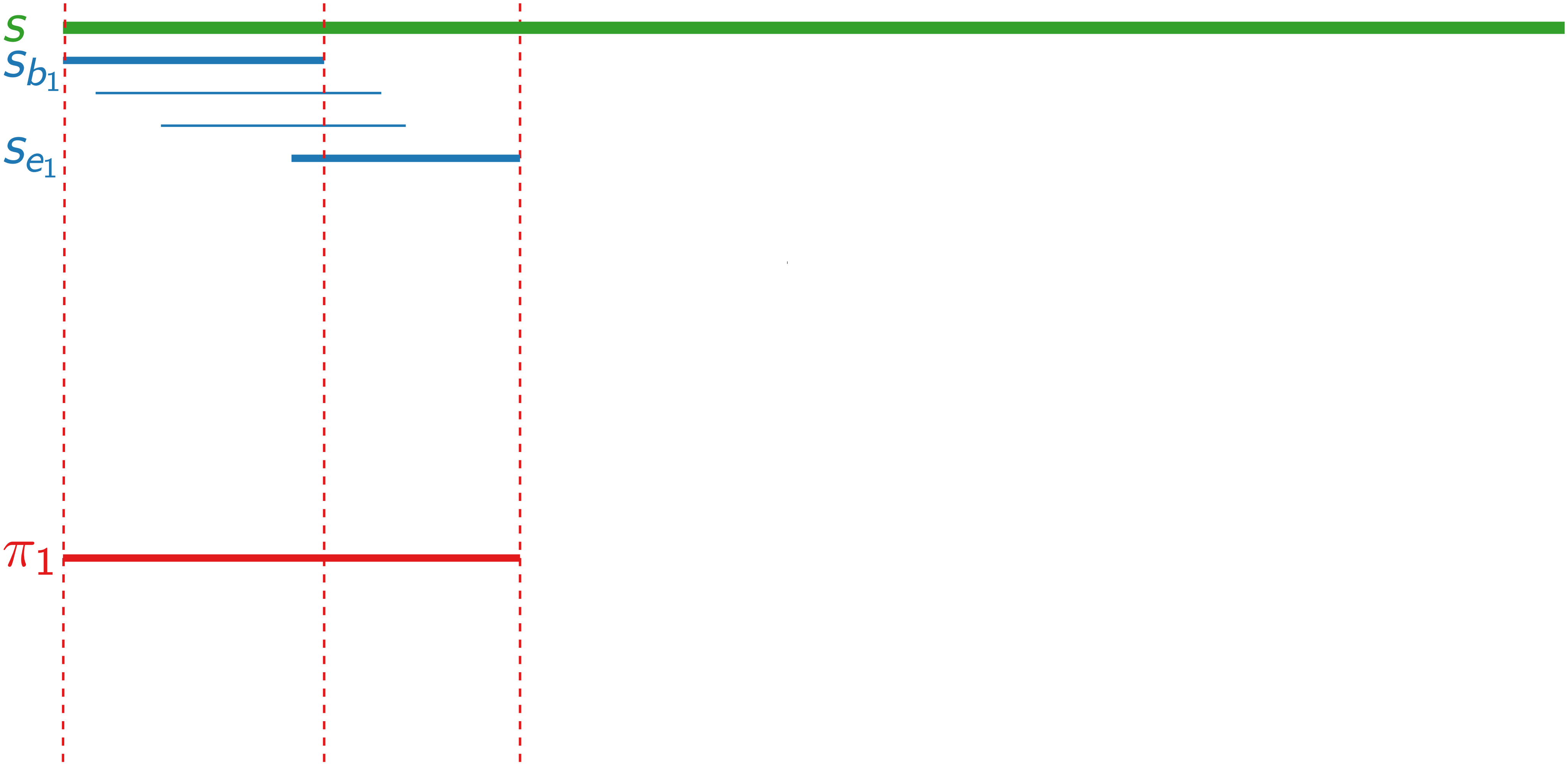

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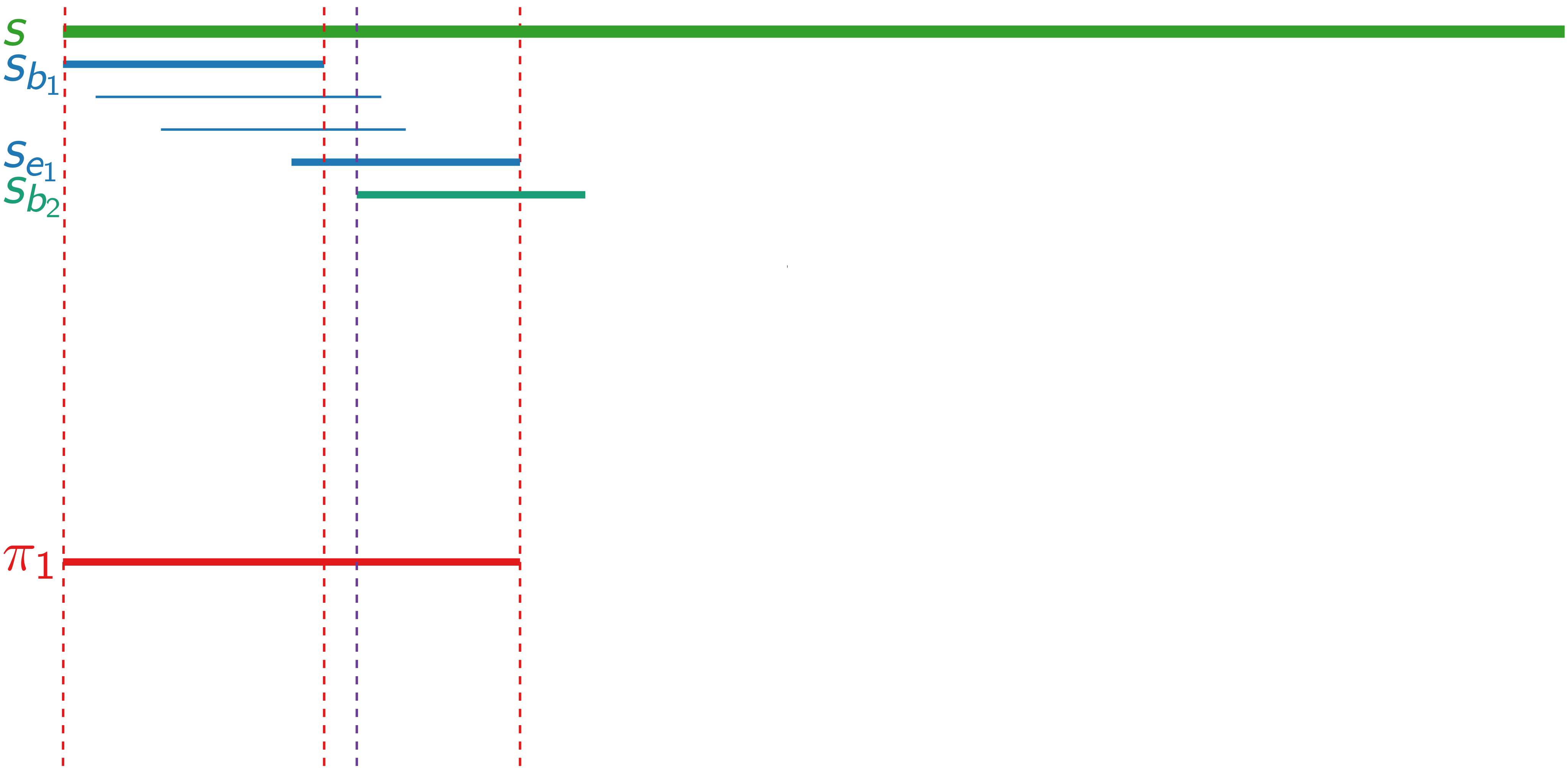

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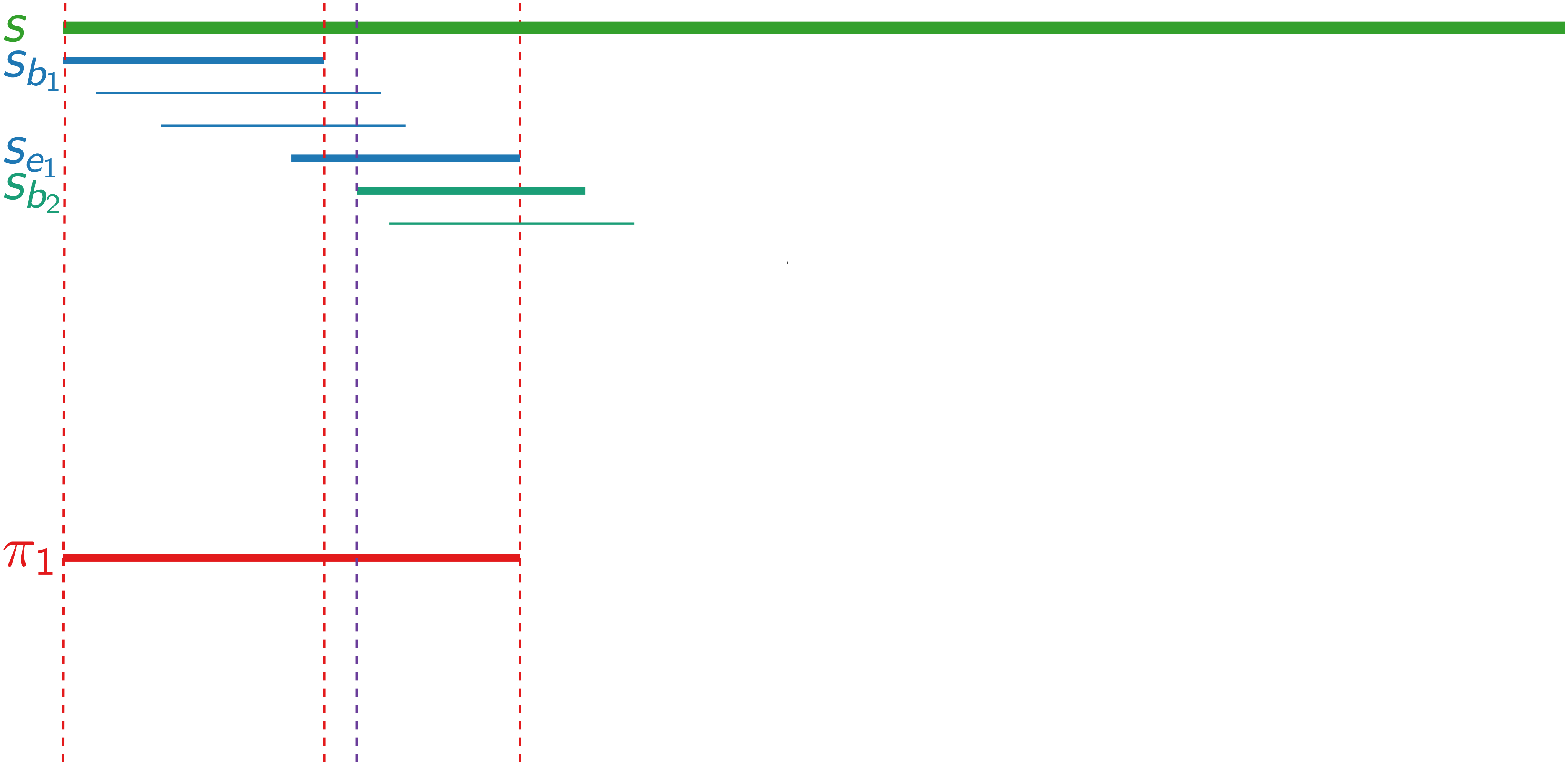

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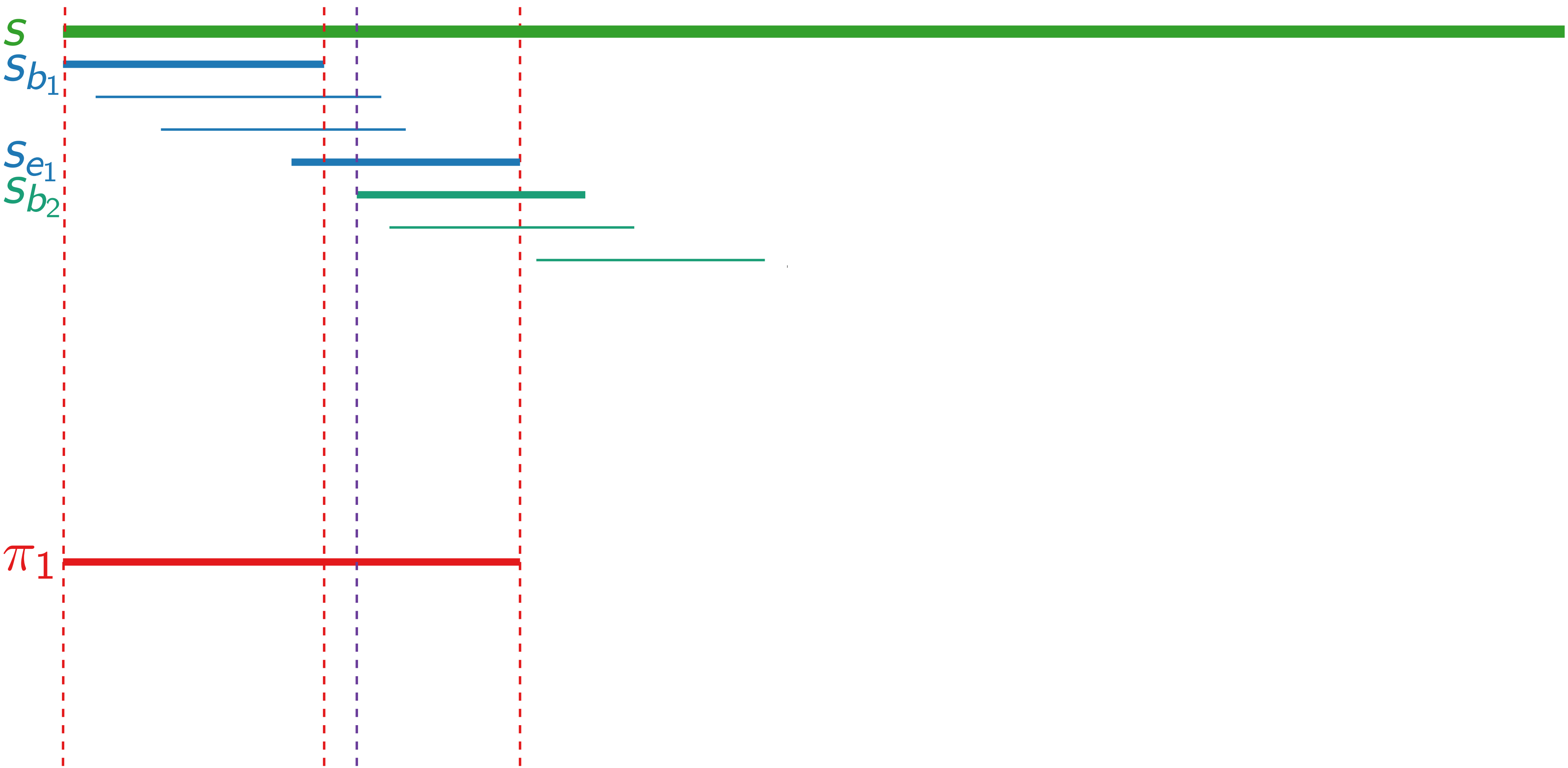

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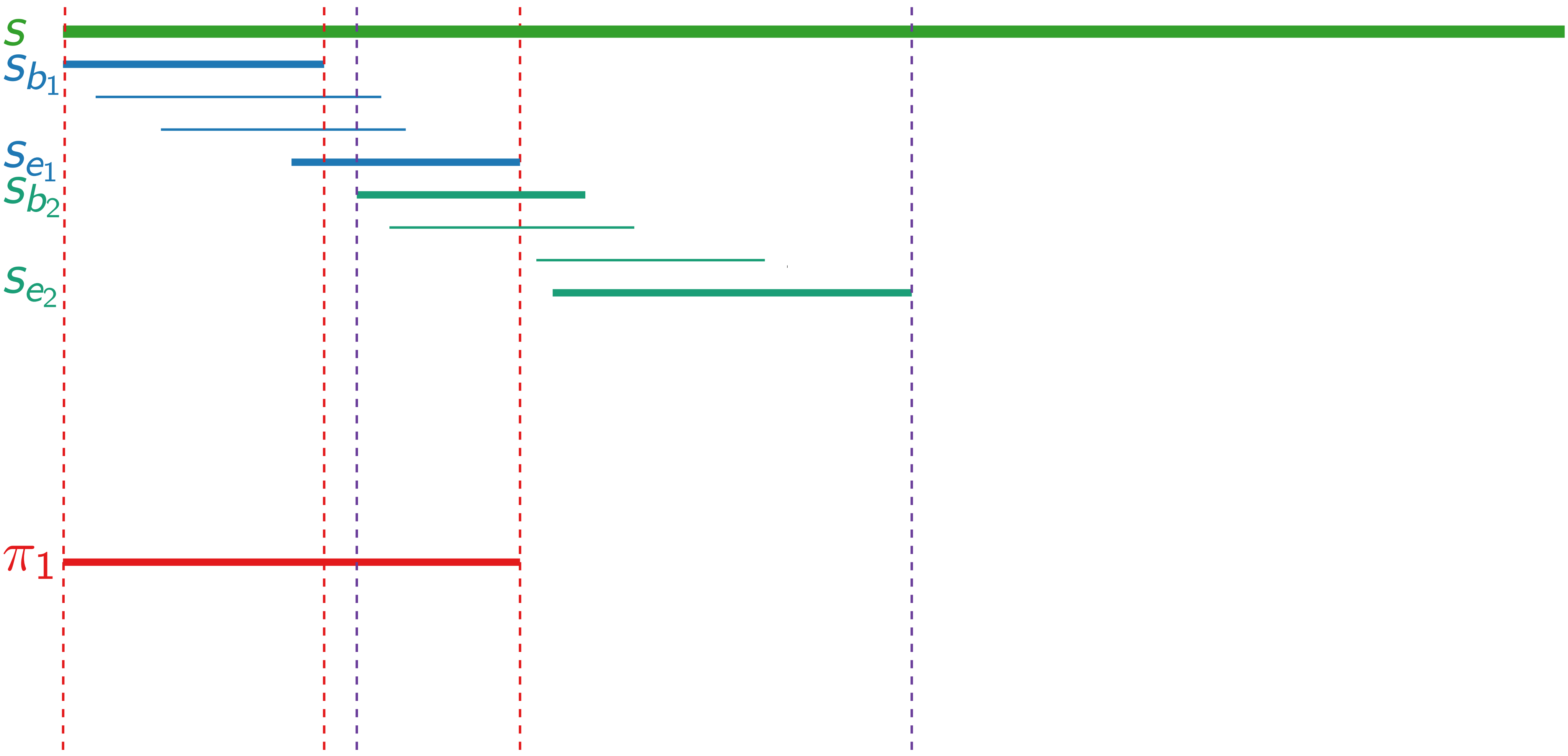

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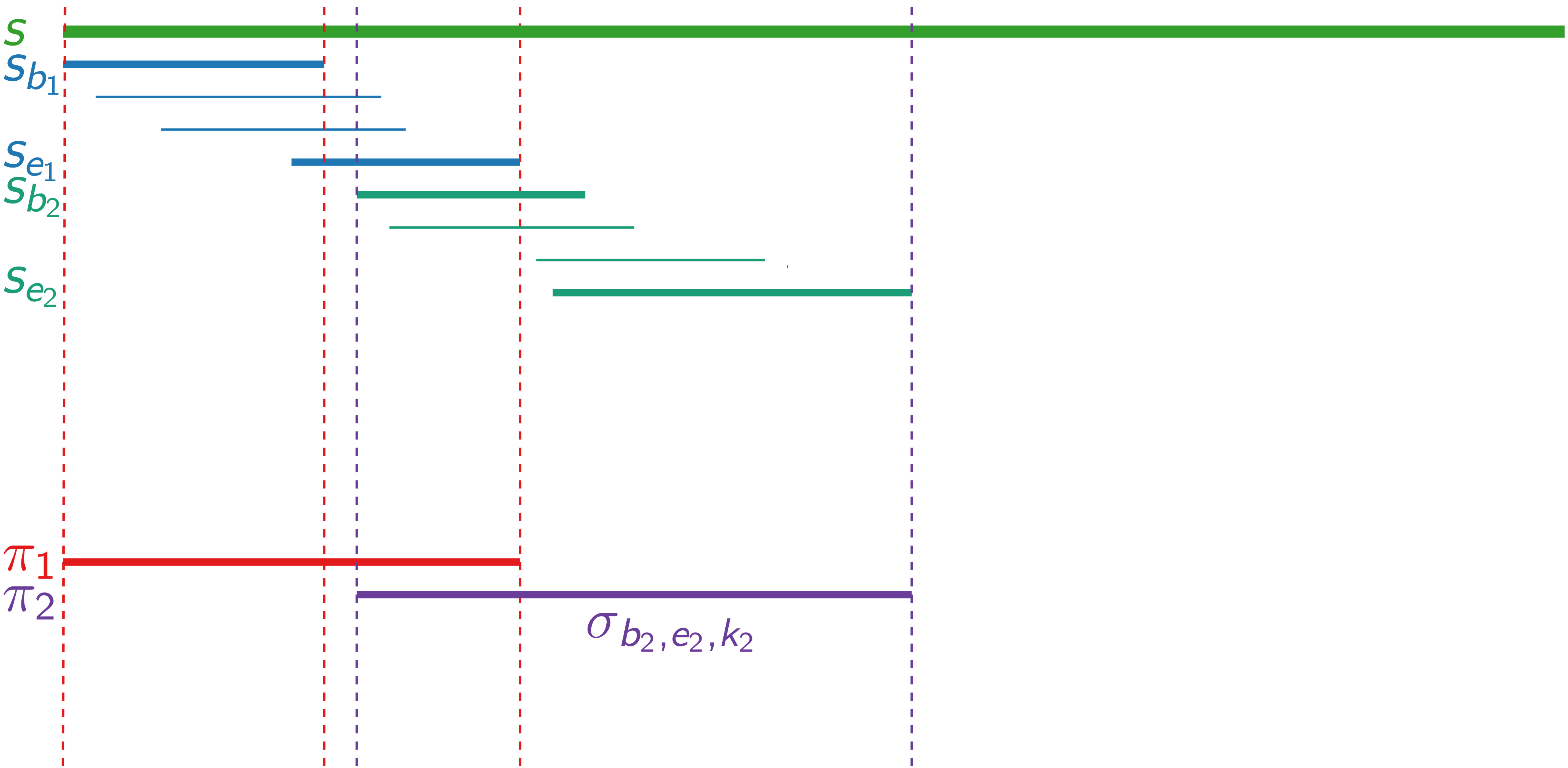

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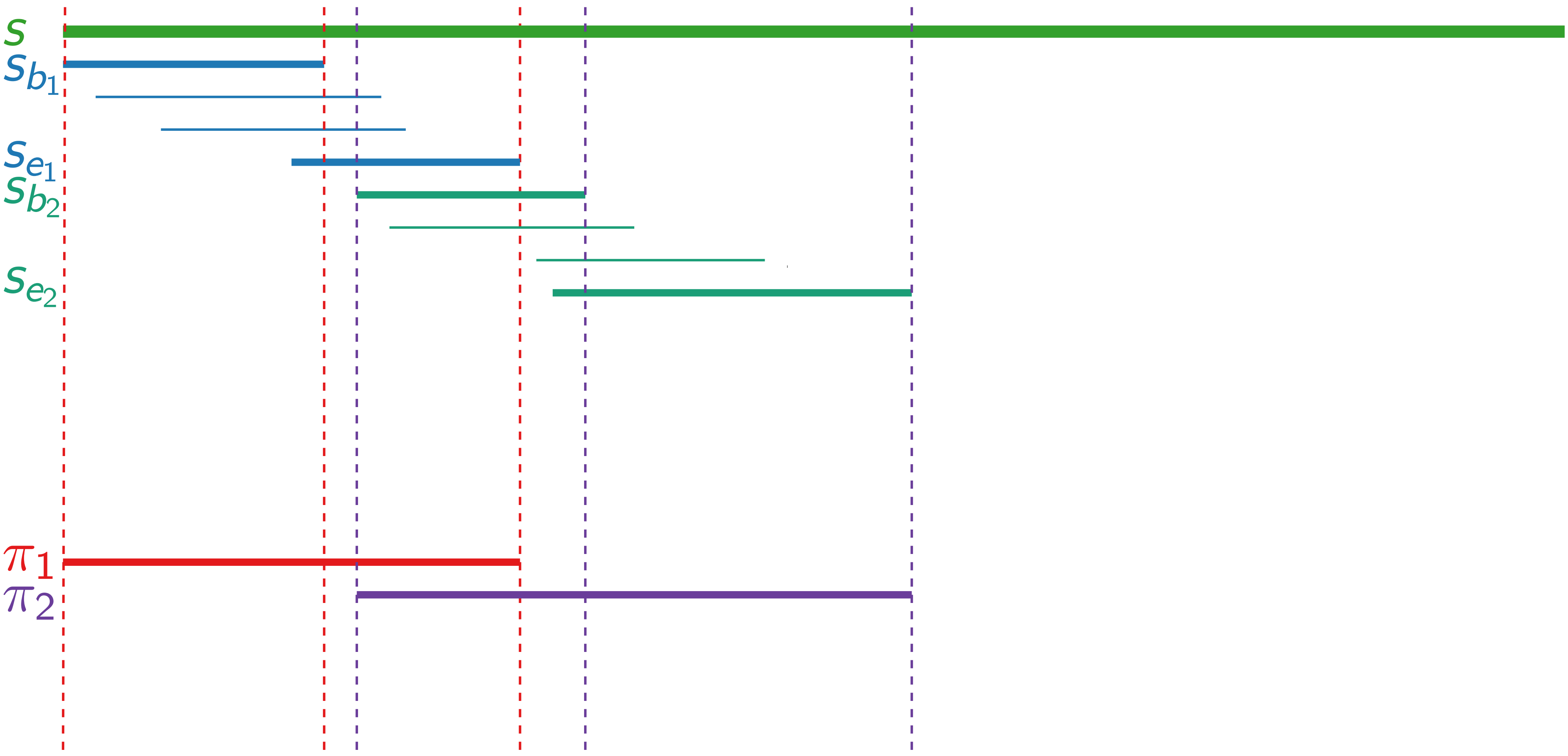

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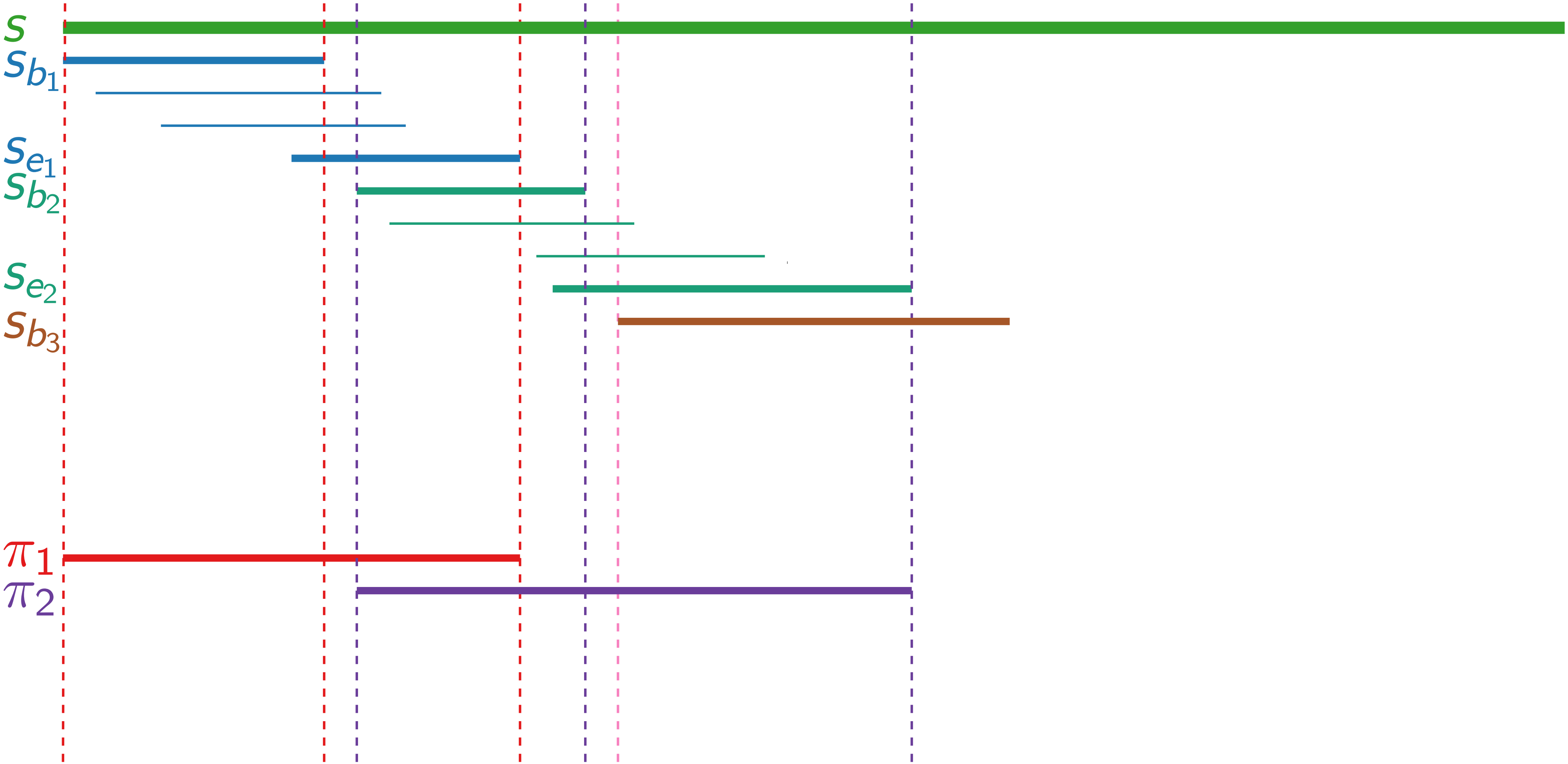

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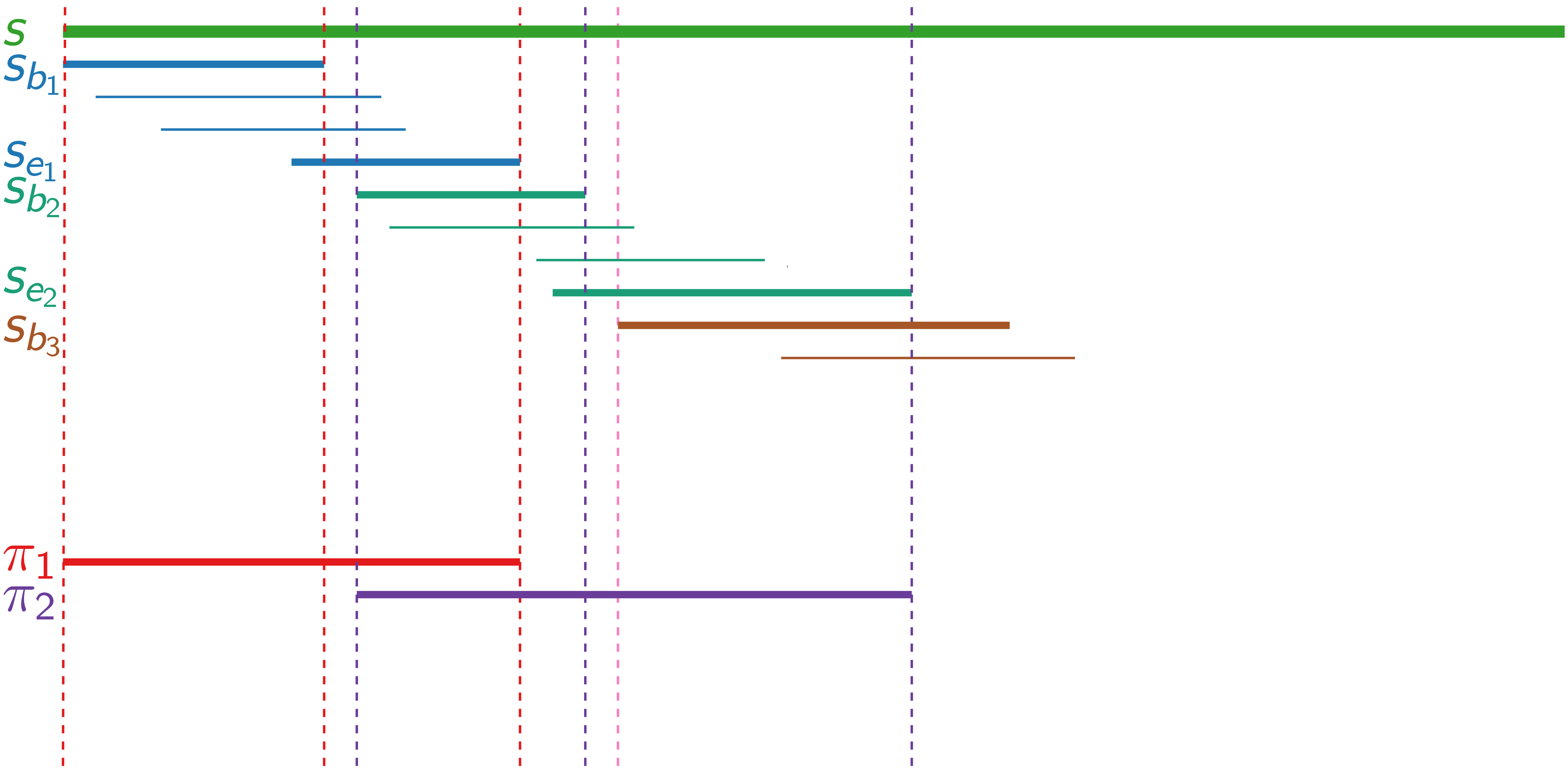

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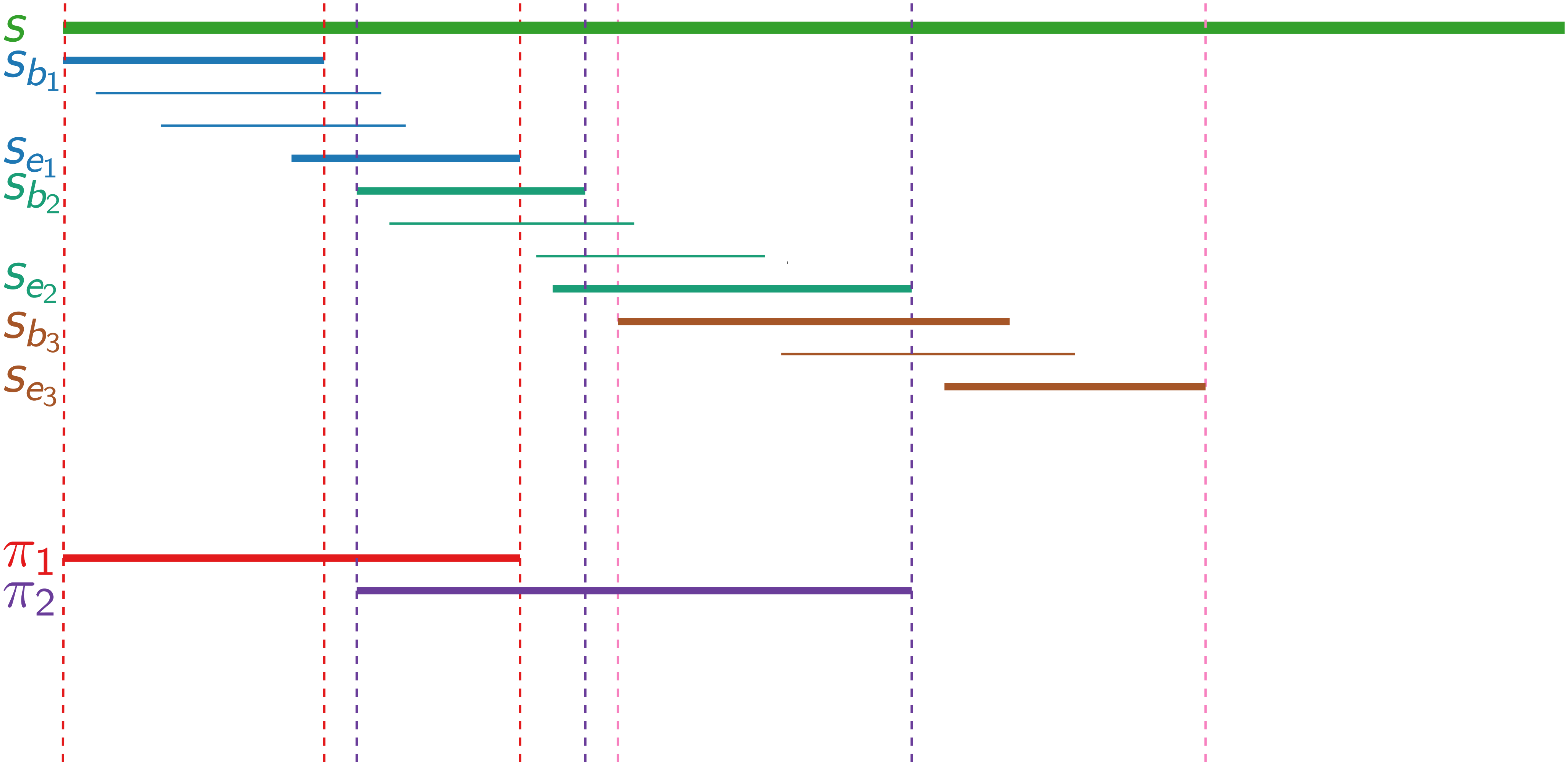

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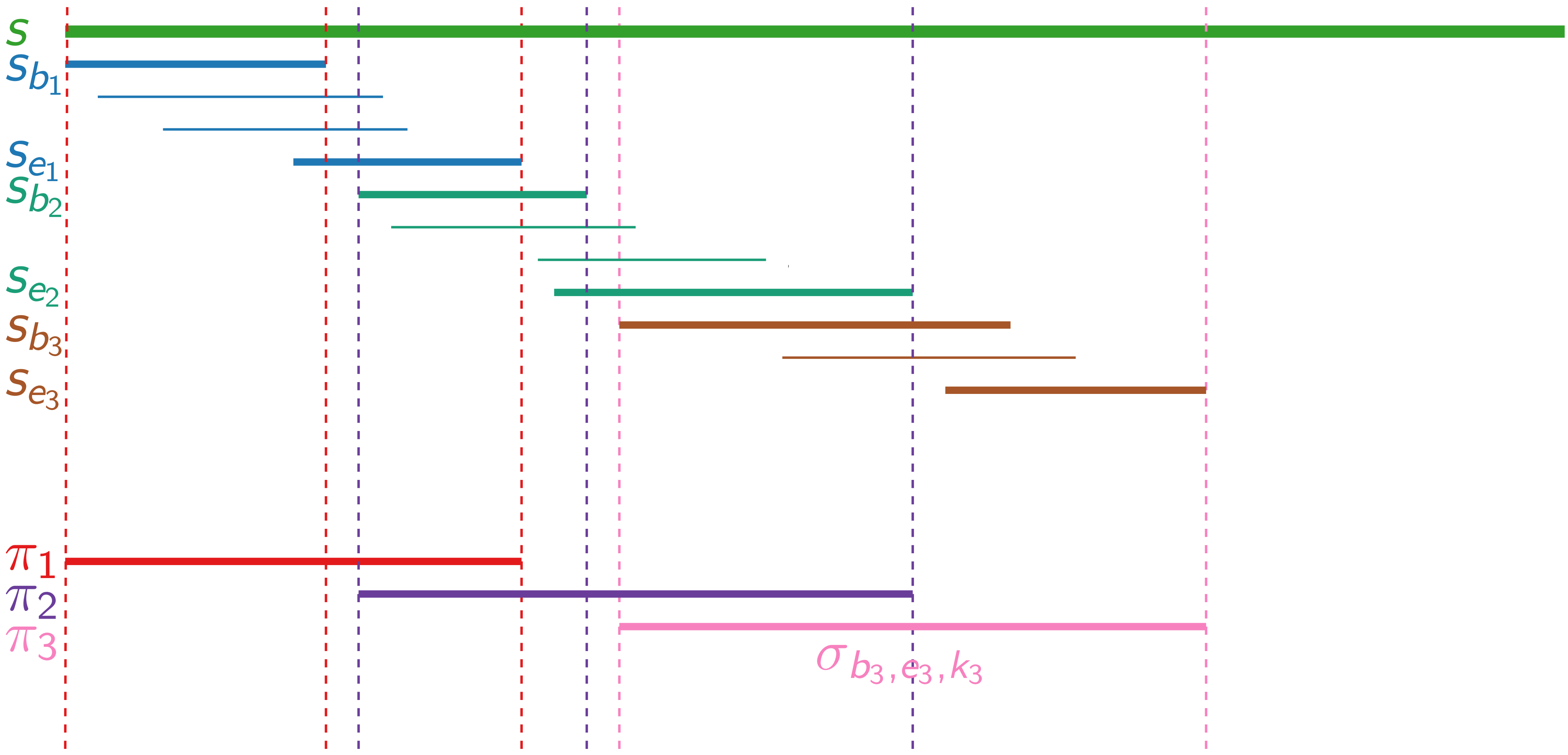

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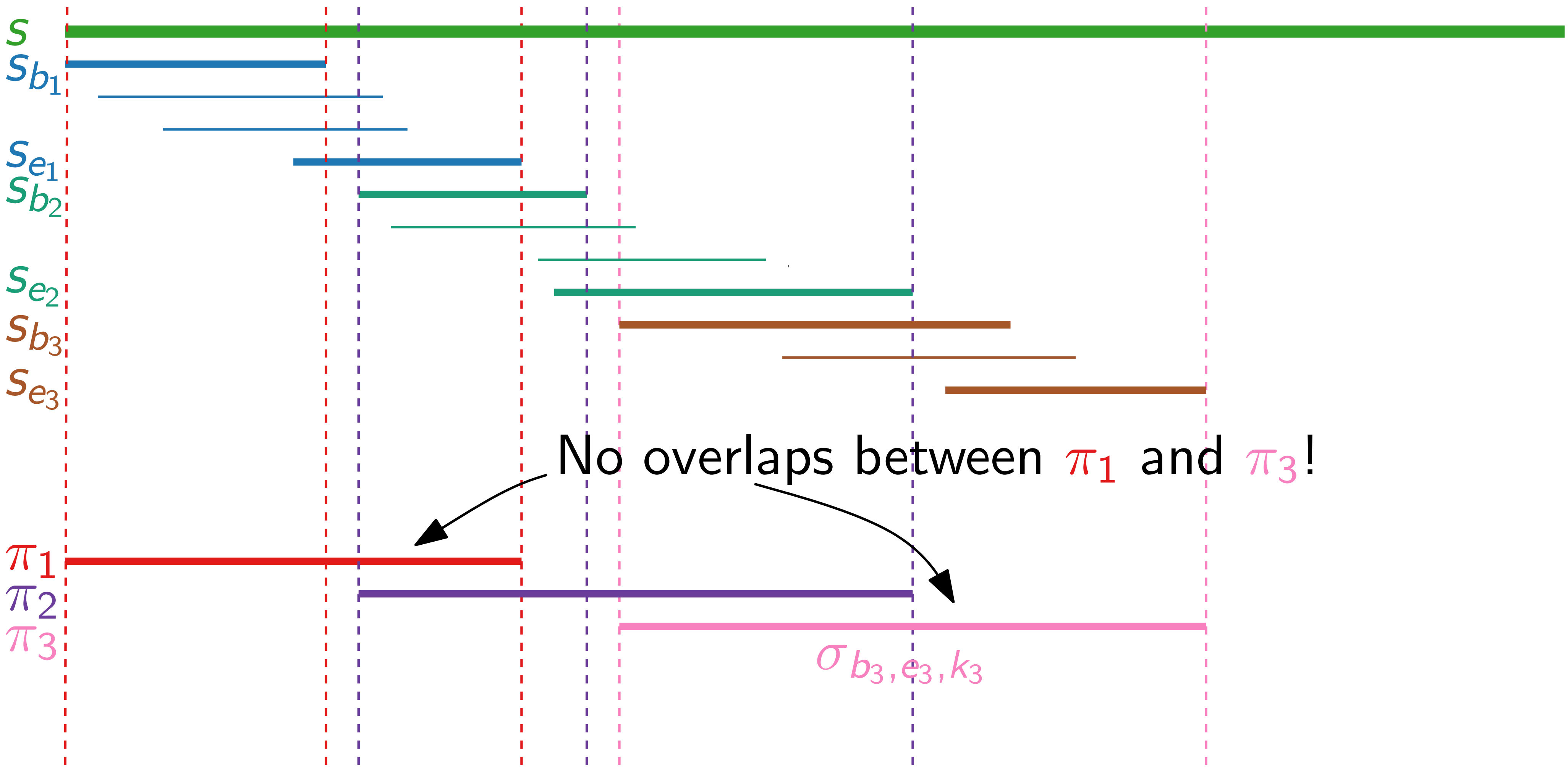

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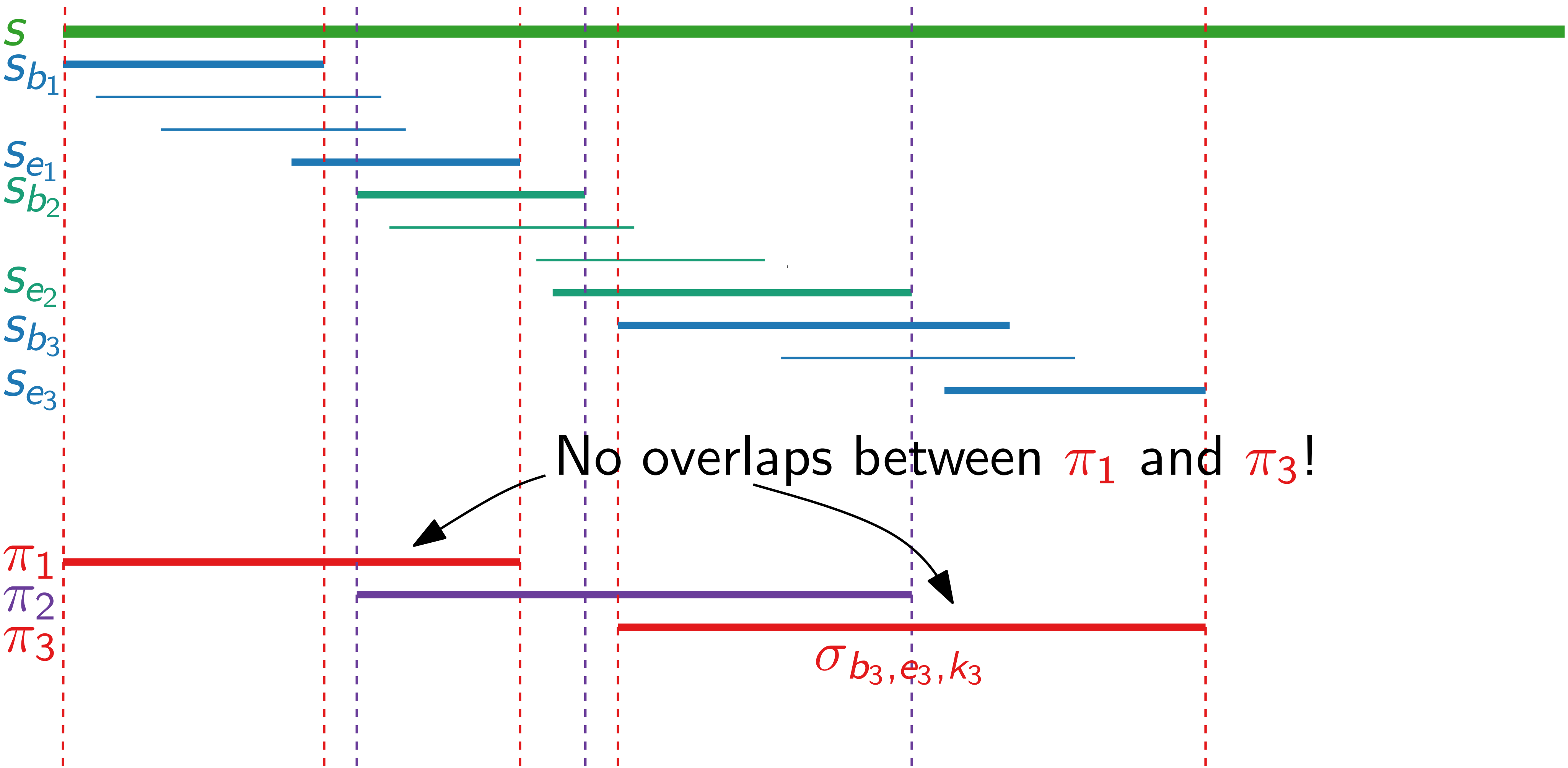

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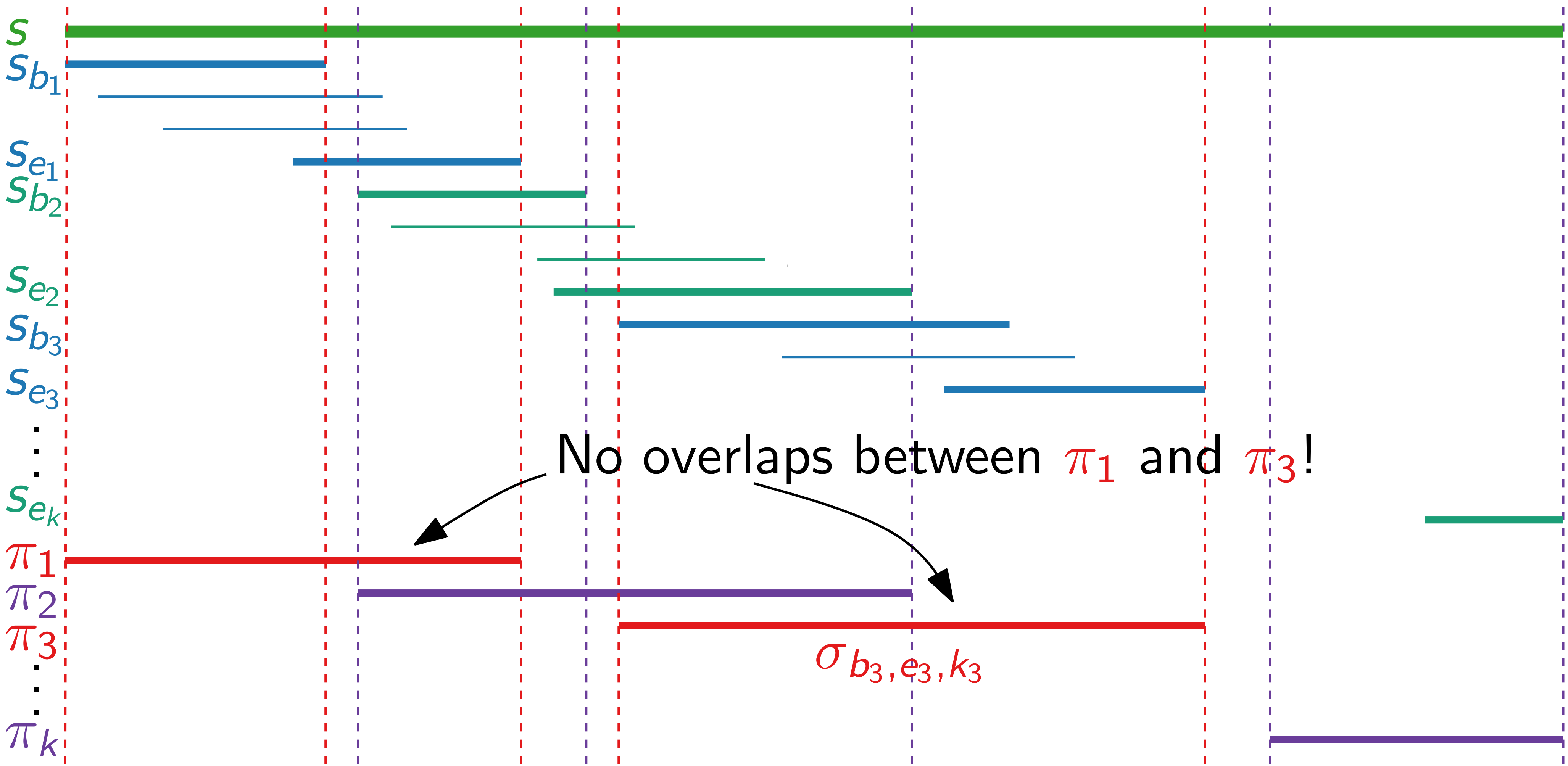

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- ShortestSuperString cannot be approximated within factor $\frac{333}{332} \approx 1.003$ (unless $\mathsf{P} = \mathsf{NP}$).
  [Karpinski & Schmied: CATS 2013]