# Approximation Algorithms

## Lecture 1:
## Introduction and Vertex Cover

### Part I:
### Organizational

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*Winter 2024/25*

# Organizational

Lectures: Fri, 10:15–11:45 (ÜR I)

English/German, depending on audience.

hands-on, with tasks/questions for audience

Tutorials: Tue, 10:15–11:45 (SE I), starting Oct. 22, 2024.

discussing old solutions and solving new tasks

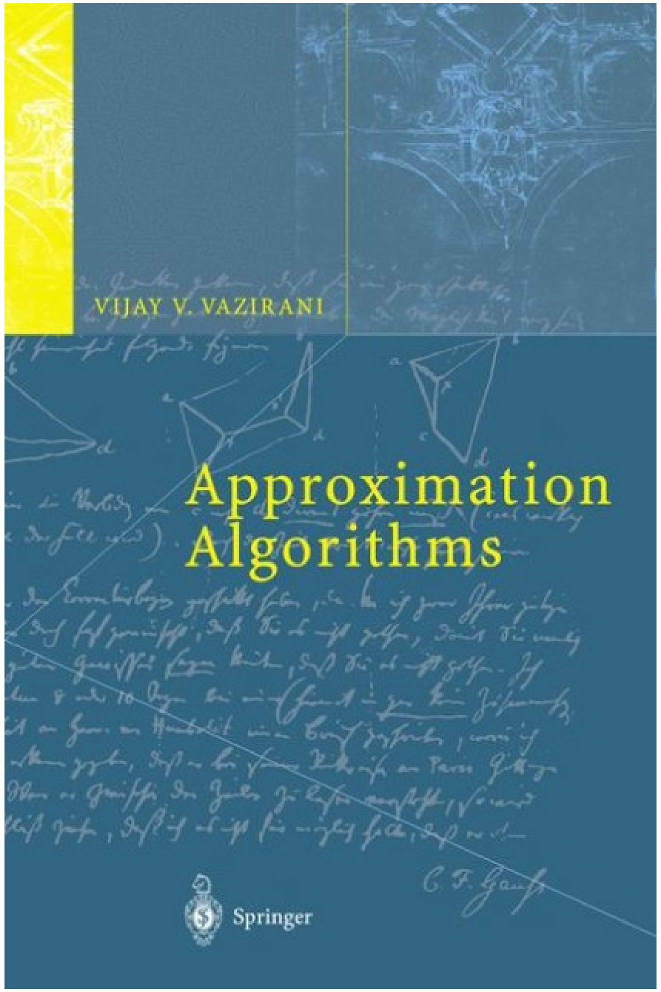
roughly one exercise sheet per lecture

bonus (+0.3 on final grade) for $\geq$ 50% points

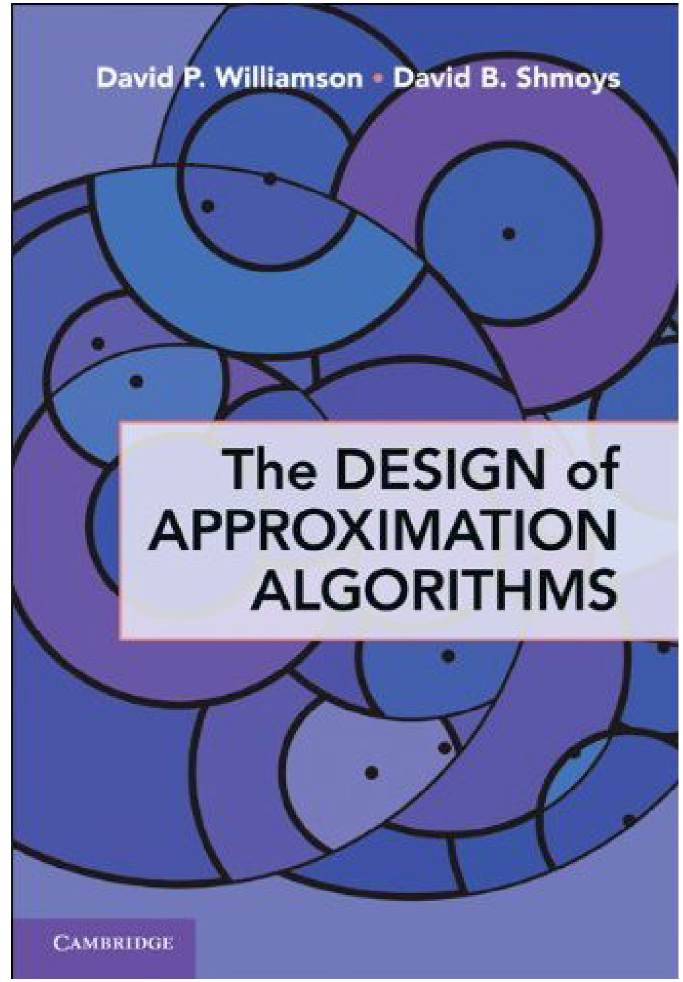
Up to two students can hand in solutions together.
Make sure to write both names!

Most slides are due to Joachim Spoerhase,
polishing & colors are due to Philipp Kindermann – thanks!

# Textbooks




Vijay V. Vazirani:
Approximation Algorithms
Springer-Verlag, 2003.




D. P. Williamson & D. B. Shmoys:
The Design of Approximation Algorithms
Cambridge-Verlag, 2011.

http://www.designofapproxalgs.com/

# Approximation Algorithms

„All exact science is dominated by the idea of approximation."

Bertrand Russell
(1872 – 1970)

# Approximation Algorithms

- Many optimization problems are NP-hard!
  (For example, the traveling salesperson problem.)
- $\leadsto$ an optimal solution cannot be efficiently computed unless P=NP.
- However, good approximate solutions can often be found efficiently!
- **Techniques** for the design and analysis of approximation algorithms arise from studying specific optimization problems.

# Overview

## Combinatorial algorithms

- Introduction (Vertex Cover)
- Set Cover via Greedy
- Shortest Superstring via reduction to SC
- Steiner Tree via MST
- Multiway Cut via Greedy
- $k$-Center via Parametrized Pruning
- Min-Degree Spanning Tree and local search
- Knapsack via DP and Scaling
- Euclidean TSP via Quadtrees

## LP-based algorithms

- introduction to LP-Duality
- Set Cover via LP Rounding
- Set Cover via Primal–Dual Schema
- Maximum Satisfiability
- Scheduling und Extreme Point Solutions
- Steiner Forest via Primal–Dual

# Approximation Algorithms

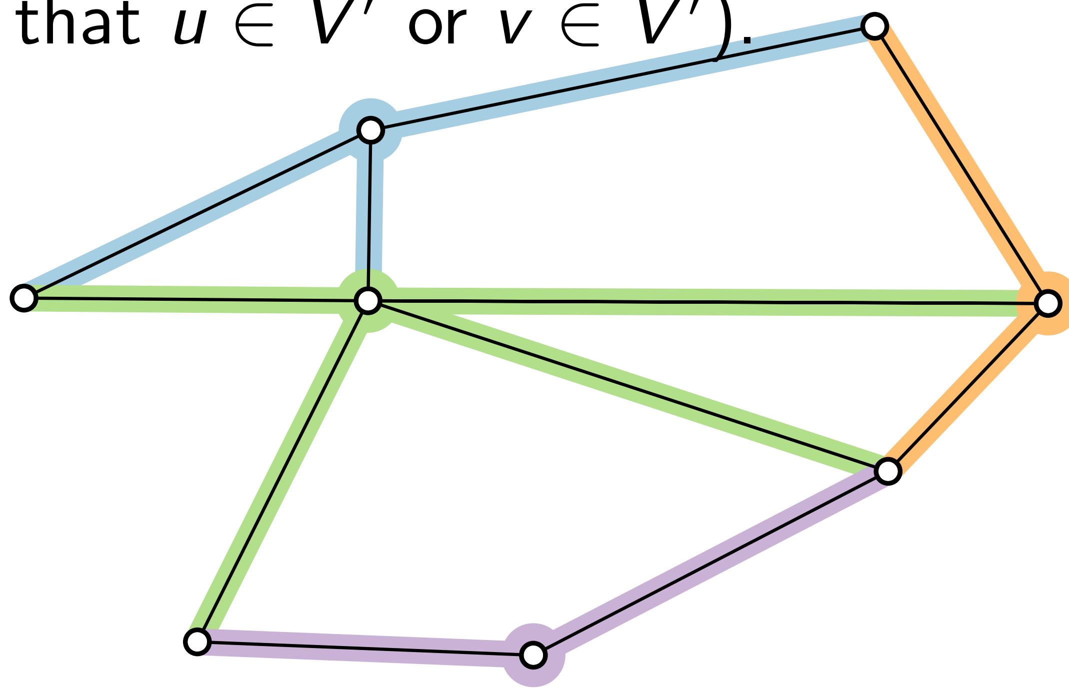
## Lecture 1:
## Introduction and Vertex Cover

### Part II:
### (Cardinality) Vertex Cover

# VertexCover (card.)

**Input:** graph $G$

**Output:** a minimum **vertex cover**, that is, a minimum-cardinality vertex set $V' \subseteq V(G)$ s. t. every edge is **covered** (i.e., for every $uv \in E(G)$, it holds that $u \in V'$ or $v \in V'$).



**Optimum** (OPT $= 4$) – but in general NP-hard to find :-(

# Approximation Algorithms

## Lecture 1:
## Introduction and Vertex Cover

### Part III:
### NP-Optimization Problem

# NP-Optimization Problem

An **NP-optimization problem** $\Pi$ is given by:

- A set $D_\Pi$ of **instances**.
  We denote the size of an instance $I \in D_\Pi$ by $|I|$.
- For each instance $I \in D_\Pi$,
  a set $S_\Pi(I) \neq \emptyset$ of **feasible solutions** for $I$ such that:
  - for each solution $s \in S_\Pi(I)$,
    its size $|s|$ is polynomially bounded in $|I|$, and
  - there is a polynomial-time algorithm that decides,
    for each pair $(s, I)$, whether $s \in S_\Pi(I)$.
- A polynomial time computable **objective function** $\mathrm{obj}_\Pi$
  which assigns a positive objective value $\mathrm{obj}_\Pi(I, s) \geq 0$ to
  any given pair $(s, I)$ with $s \in S_\Pi(I)$.
- $\Pi$ is either a minimization or maximization problem.

# VertexCover: NP-Optimization Problem

Task: Fill in the gaps for $\Pi = \textsc{Vertex Cover}$.

$D_\Pi =$ set of all graphs

For $I \in D_\Pi$: $|I| =$ number of vertices of $G$

graph $G$ $S_\Pi(I) =$ set of all vertex covers of $G$

- Why is $|s| \in \text{poly}(|I|)$ for every $s \in S_\Pi(I)$?
  $s \subseteq V \Rightarrow |s| \leq |V(G)| = |I|$
- For a given pair $(s, I)$, how can we efficiently decide whether $s \in S_\Pi(I)$? Test whether all edges are covered.

$\text{obj}_\Pi(I, s) = |s|$

$\Pi$ is a minimization problem.

# Optimum and Optimal Objective Value

maximization problem
Let $\Pi$ be a minimization problem and $I \in D_\Pi$ an instance of $\Pi$.

A feasible solution $s^* \in S_\Pi(I)$ is **optimal** if

maximal
$\mathrm{obj}_\Pi(I, s^*)$ is minimal among the objective values

attained by the feasible solutions of $I$.

The optimal value $\mathrm{obj}_\Pi(I, s^*)$ of the objective function is denoted by $\mathrm{OPT}_\Pi(I)$ or simply by $\mathrm{OPT}$ in context.

# Approximation Algorithms

maximization problem $\alpha\colon \mathbb{N} \to \mathbb{Q}$

Let $\Pi$ be a minimization problem and ~~$\alpha \in \mathbb{Q}^+$~~.

A factor-$\alpha$ approximation algorithm for $\Pi$ is an efficient algorithm that provides, for **any** instance $I \in D_\Pi$, a feasible solution $s \in S_\Pi(I)$ such that

$$\frac{\mathrm{obj}_\Pi(I, s)}{\mathrm{OPT}_\Pi(I)} \overset{\geq}{\leq} \not{\alpha}. \quad \alpha(|I|)$$

# Approximation Algorithms

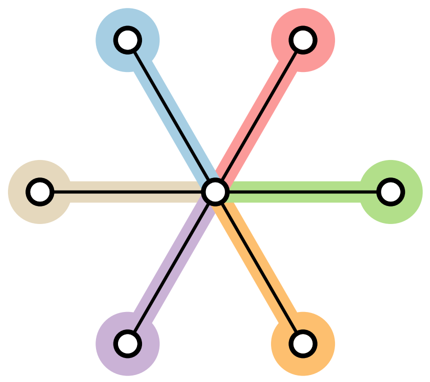
## Lecture 1:
## Introduction and Vertex Cover

### Part IV:
### Approximation Algorithm for VertexCover

# Approximation Alg. for VertexCover

Ideas?

- Edge-Greedy
- Vertex-Greedy



Quality?

**Problem:** How can we estimate $\mathrm{obj}_\Pi(I, s)/\mathrm{OPT}$ – if it is hard to compute OPT?

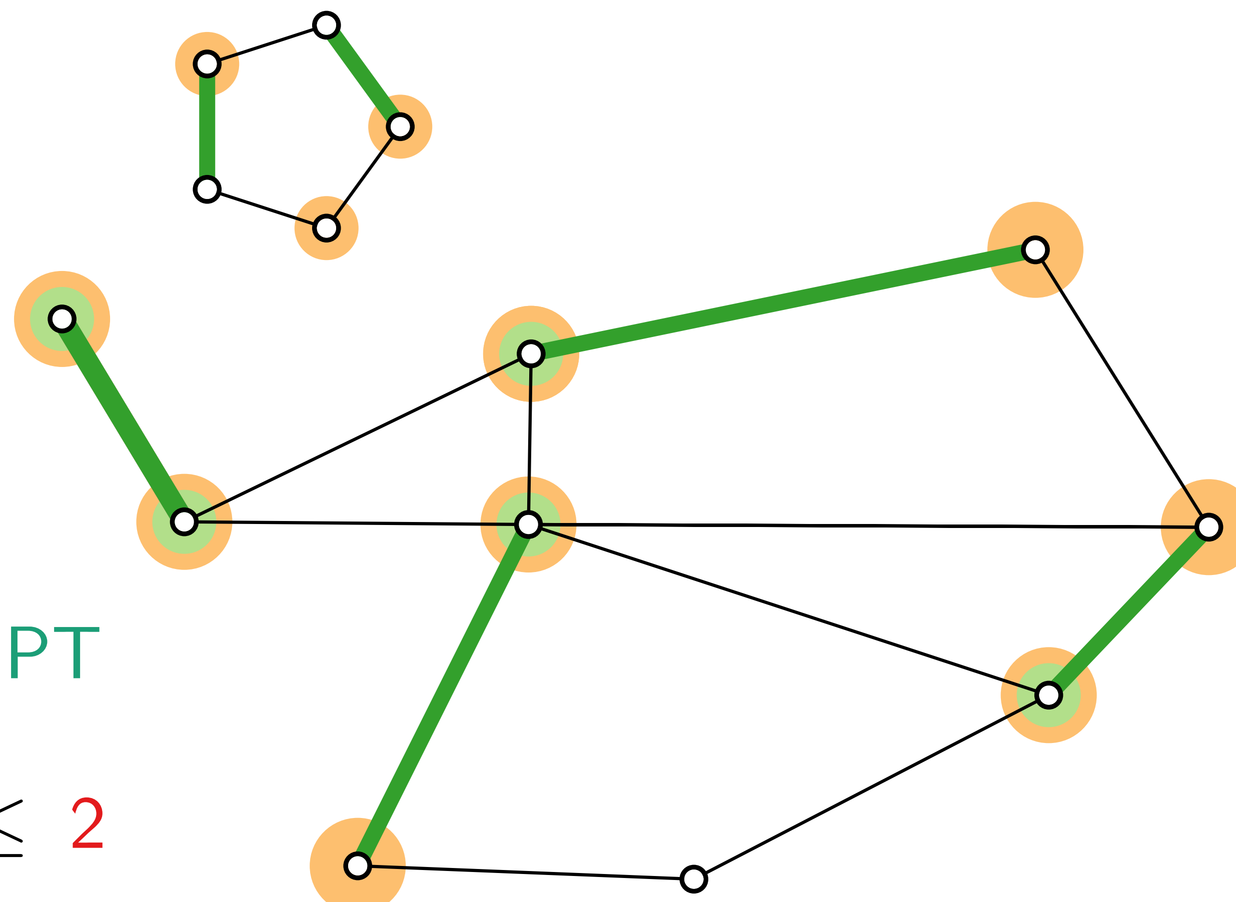
**Idea:** Find a "good" lower bound $L \leq \mathrm{OPT}$ for OPT and compare it to our approximate solution.

$$\frac{\mathrm{obj}_\Pi(I, s)}{\mathrm{OPT}} \leq \frac{\mathrm{obj}_\Pi(I, s)}{L}$$

# Lower Bound by Matchings

Given a graph $G$, a set $M$ of edges of $G$ is a **matching** if no two edges of $M$ are adjacent (i.e., share an end vertex).

$M$ is **maximal** if there is no matching $M'$ with $M' \supsetneq M$.



$\mathrm{OPT} \geq |M|$

~~$\mathrm{OPT} = |M|$~~ ?

Vertex cover of $M$

Vertex cover of $E$

$\mathrm{ALG} = 2 \cdot |M| \leq 2 \cdot \mathrm{OPT}$

$$\Rightarrow \frac{\mathrm{obj}_{\Pi}(I, s)}{\mathrm{OPT}} = \frac{\mathrm{ALG}}{\mathrm{OPT}} \leq 2$$

# Approximation Alg. for VertexCover

Algorithm VertexCover($G$)

$M \leftarrow \emptyset$
**foreach** $e \in E(G)$ **do**
  **if** $e$ is not adjacent to any edge in $M$ **then**
    $M \leftarrow M \cup \{e\}$
**return** $\{\, u, v \mid uv \in M \,\}$

**Theorem.** The above algorithm is a factor-2 approximation algorithm for VertexCover.

*Proof.* $\mathrm{ALG} = 2 \cdot |M| \leq 2 \cdot \mathrm{OPT}$ $\square$

# Approximability of Vertex Cover

The best known approximation factor for VertexCover is

$$2 - \Theta(1/\sqrt{\log n}).$$

If P $\neq$ NP, VertexCover cannot be approximated within a factor of 1.3606.

VertexCover cannot be approximated within a factor of $2 - \Theta(1)$ – if the *Unique Games Conjecture* holds.

# Approximation Algorithms

## Lecture 1:
## Introduction and Vertex Cover

### Part V:
### An LP-based Algorithm for VertexCover

# Task

Write an integer linear program (ILP) for VERTEXCOVER:

Using integer (and/or real) variables, express the problem using

- linear constraints and
- a linear objective function.

You can iterate over the vertices / edges of the given graph $G$.

Variables: for each vertex $v$ of $G$, we introduce $x_v \in \{0, 1\}$.

*v not in the solution* (0)
*v in the solution* (1)

Objective: minimize $\sum_{v \in V(G)} x_v$

Constraints: for each edge $uv$ of $G$, we require that

$$x_u + x_v \geq 1.$$

# Standard ILP Format

LP relaxation

| | | |
|---|---|---|
| minimize | $\sum_{v \in V(G)} x_v$ | |
| subject to | $x_u + x_v \geq 1$ | for each $uv \in E(G)$ |
| | $x_v \geq 0$ ~~$x_v \in \{0, 1\}$~~ | for each $v \in V(G)$ |

Problem: It's NP-hard to solve ILPs in general.

But: LPs can be solved efficiently (in $O(L \cdot n^{3.5})$ time),
where $n = \#$ variables and $L =$ total bit complexity of coefficients.

Problem′: Now we can get fractional solutions, i.e., $x_v \in (0, 1)$.

Task: Find a graph $G$ with $\mathrm{OPT}_{\mathrm{LP}} \neq \mathrm{OPT}_{\mathrm{ILP}}$!

Solution? Round the LP solution to get an integral solution!

# Rounding the LP Solution

| | | |
|---|---|---|
| minimize | $\sum_{v \in V(G)} x_v$ | |
| subject to | $x_u + x_v \geq 1$ | for each $uv \in E(G)$ |
| | $x_v \geq 0$ | for each $v \in V(G)$ |

For each $v \in V(G)$: Set $x'_v = \begin{cases} 1 & \text{if } x_v \geq 0.5, \\ 0 & \text{otherwise.} \end{cases}$

Need to check: Is $(x'_v)_{v \in V(G)}$ a feasible solution?

In other words: Is $\{v \in V(G) : x'_v = 1\}$ a vertex cover of $G$?

Need to make sure that every edge $uv$ of $G$ is covered.

Is $x'_u = 0 = x'_v$ possible? But then $x_u < 0.5$ and $x_v < 0.5$.

This contradicts $x_u + x_v \geq 1$. $\Rightarrow$ $x'_u = 1$ or $x'_v = 1$ $\Rightarrow$ $(x'_v)$ feasible!

# Cost of the Solution

| | | |
|---|---|---|
| minimize | $\sum_{v \in V(G)} x_v$ | |
| subject to | $x_u + x_v \geq 1$ | for each $uv \in E(G)$ |
| | $x_v \geq 0$ | for each $v \in V(G)$ |

For each $v \in V(G)$: Set $x'_v = \begin{cases} 1 & \text{if } x_v \geq 0.5, \\ 0 & \text{otherwise.} \end{cases}$

$$\text{ALG} = \sum_{v \in V(G)} x'_v \leq 2 \cdot \sum_{v \in V(G)} x_v = 2 \cdot \text{OPT}_{\text{LP}} \leq 2 \cdot \text{OPT}_{\text{ILP}}$$

**Theorem.** The LP rounding algorithm is a factor-2 approximation algorithm for VertexCover.

# Cost of the Solution

$$\begin{array}{lll} \text{minimize} & \sum_{v \in V(G)} x_v \cdot w(v) & \\ \text{subject to} & x_u + x_v \geq 1 & \text{for each } uv \in E(G) \\ & x_v \geq 0 & \text{for each } v \in V(G) \end{array}$$

For each $v \in V(G)$: Set $x'_v = \begin{cases} 1 & \text{if } x_v \geq 0.5, \\ 0 & \text{otherwise.} \end{cases}$

$$\text{ALG} = \sum_{v \in V(G)} x'_v \cdot w(v) \leq 2 \cdot \sum_{v \in V(G)} x_v \cdot w(v) = 2 \cdot \text{OPT}_{\text{LP}} \leq 2 \cdot \text{OPT}_{\text{ILP}}$$

**Theorem.** The LP rounding algorithm is a factor-2 approximation algorithm for WeightedVertexCover.