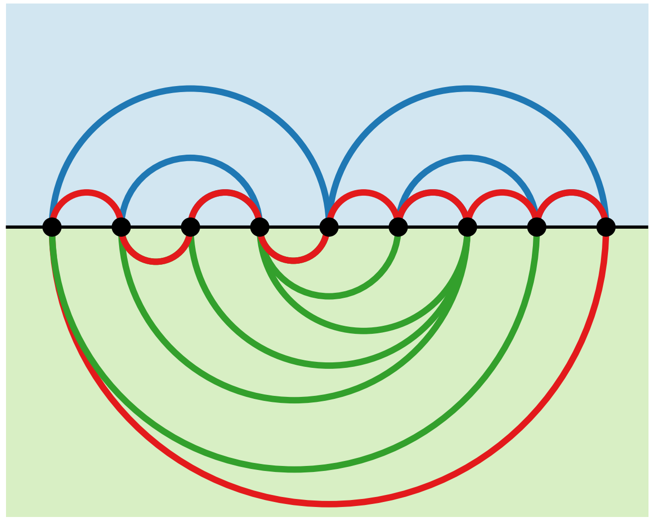
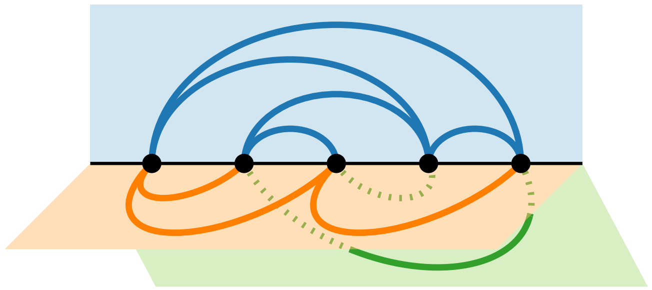
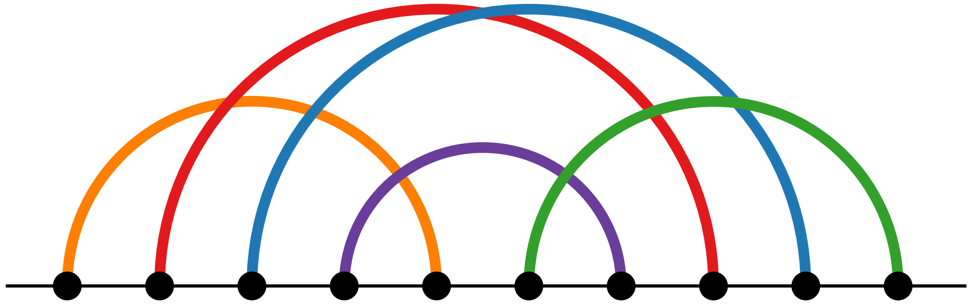
Julius-Maximilians-
UNIVERSITÄT
WÜRZBURG

Visualization of Graphs



Lecture 12:
Linear Layouts
(Book Embeddings)

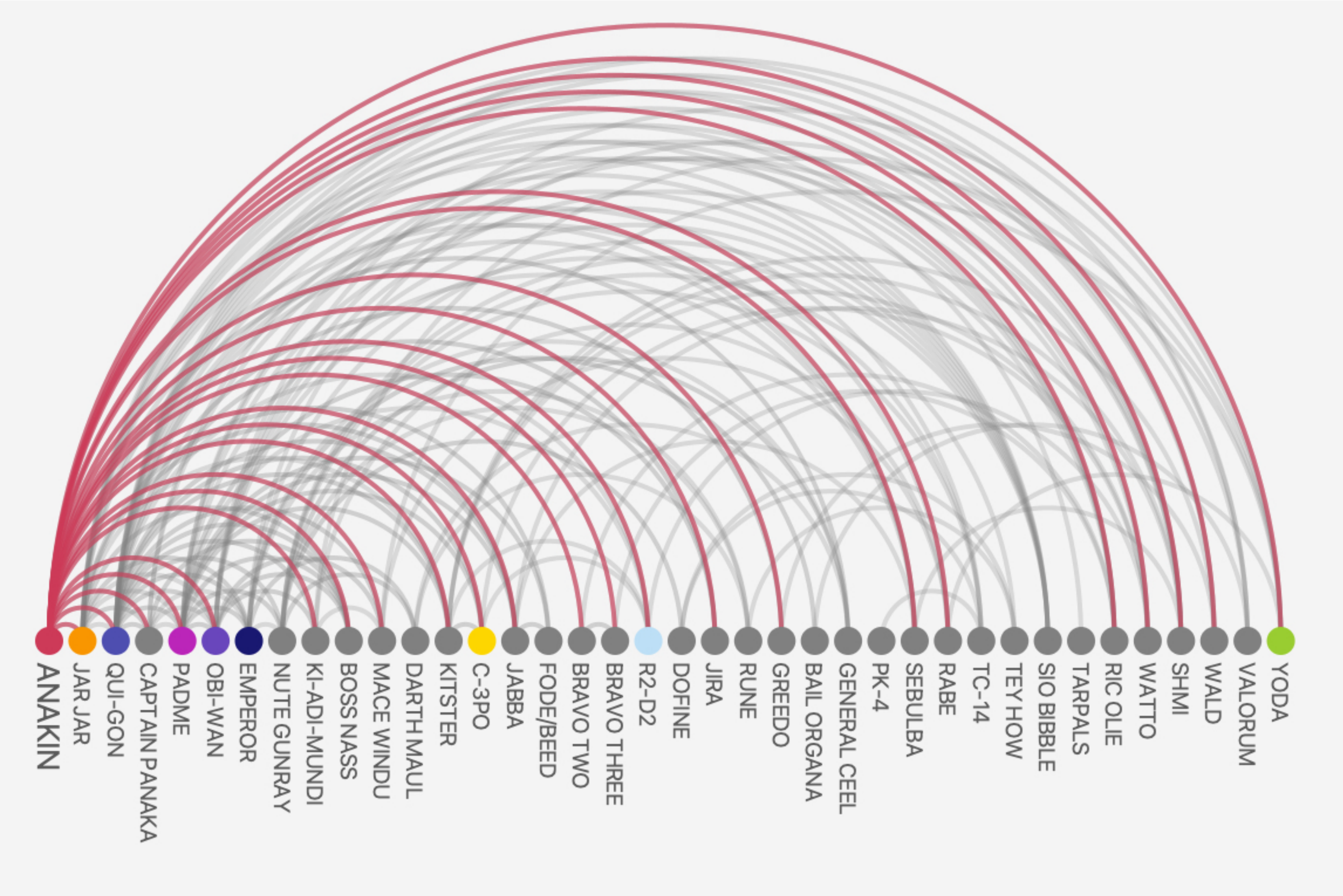




Johannes Zink

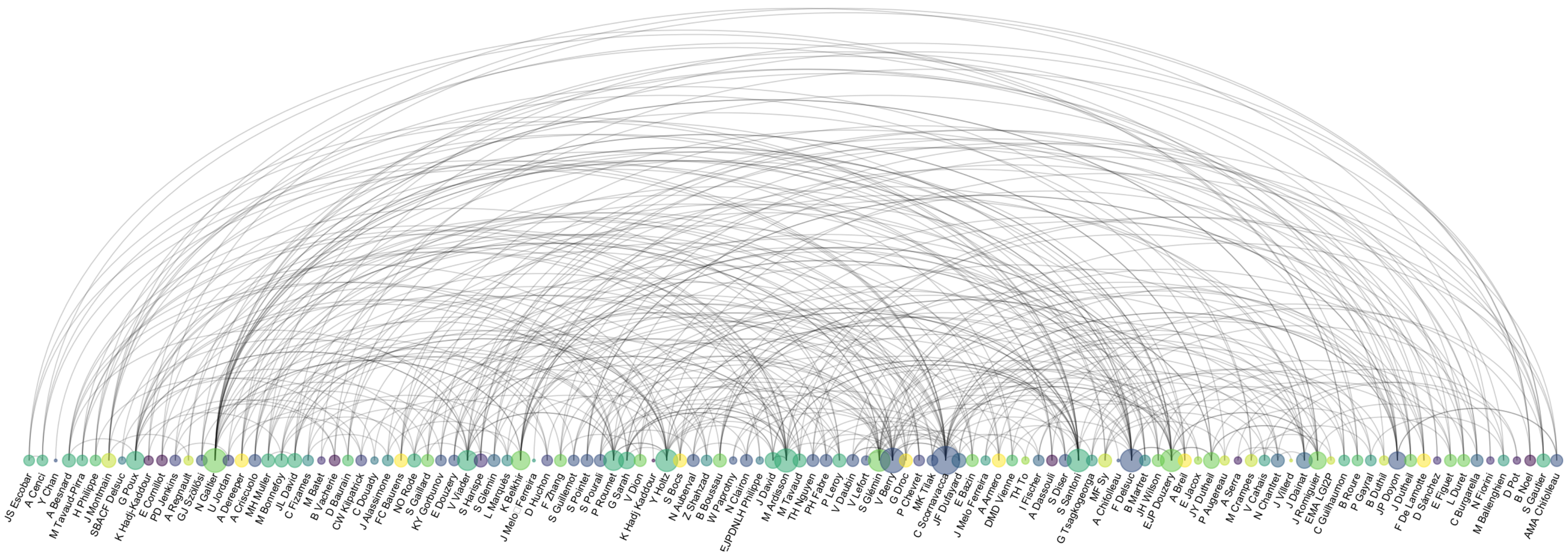
Summer semester 2024

Drawing Style: Arc Diagrams



interactions in Star Wars Episode I
[https://harmoniccode.blogspot.com/2020/11/arc-charts.html]

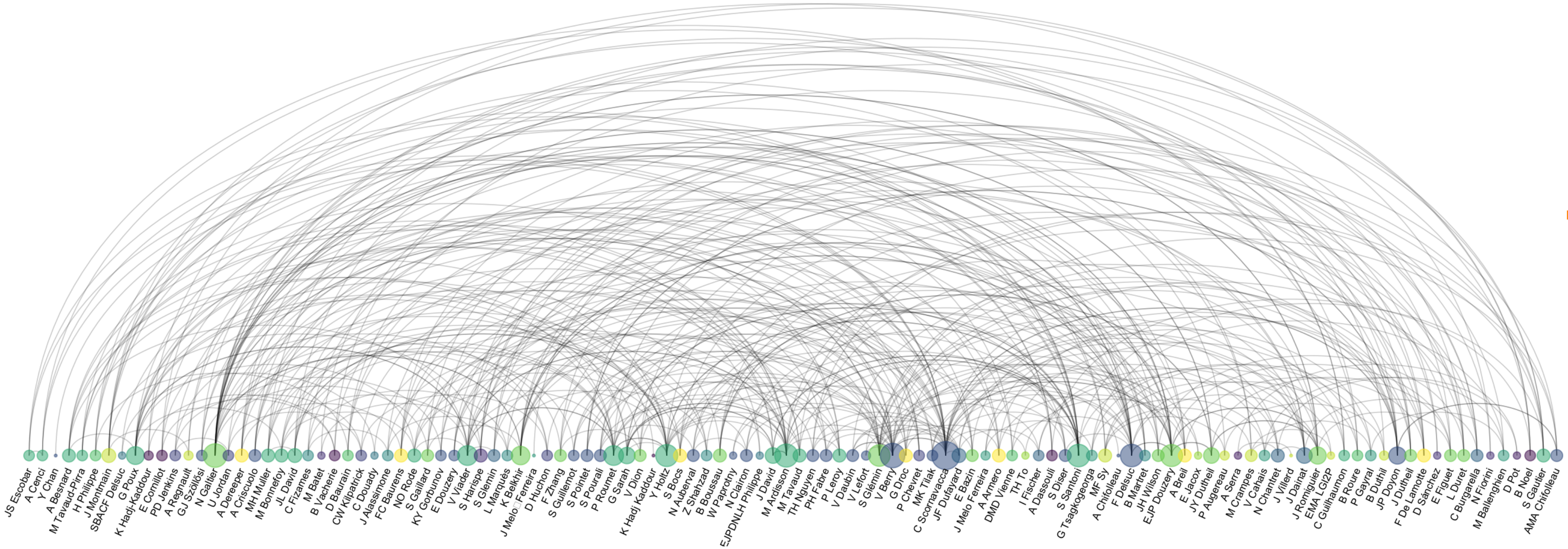
Drawing Style: Arc Diagrams



network of co-authors of Vincent Ranwez (edge ⇔ co-authors)

[https://www.data-to-viz.com/graph/arc.html]

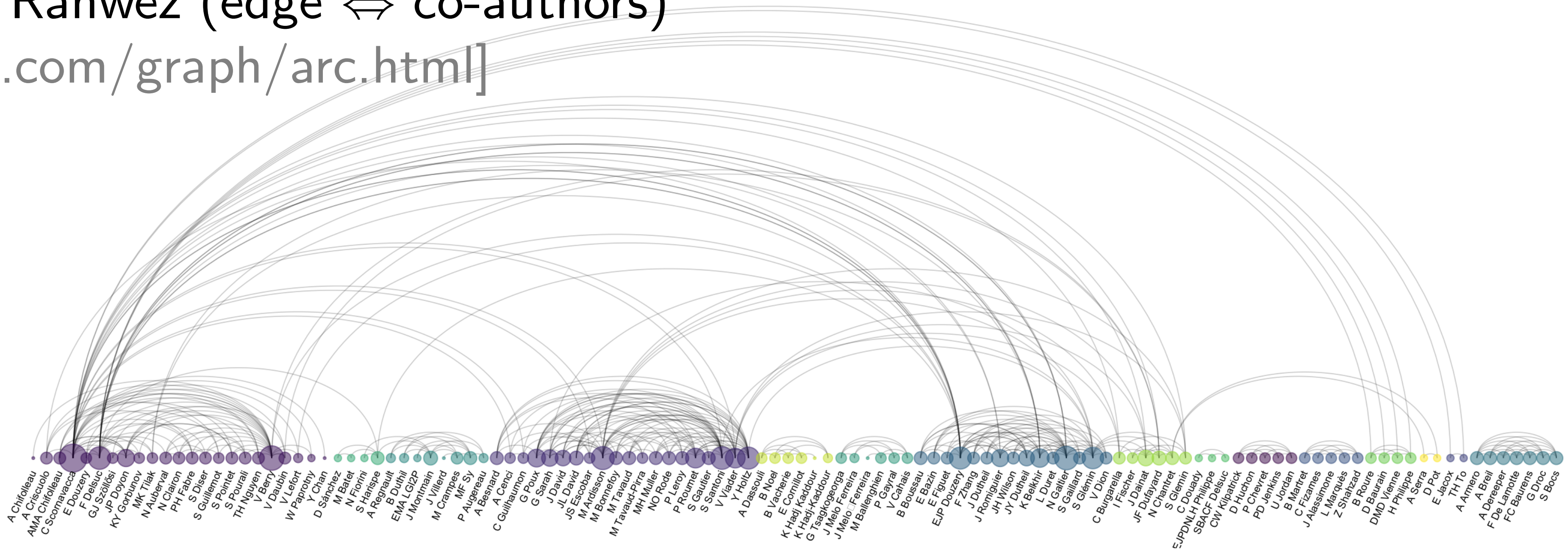
Drawing Style: Arc Diagrams



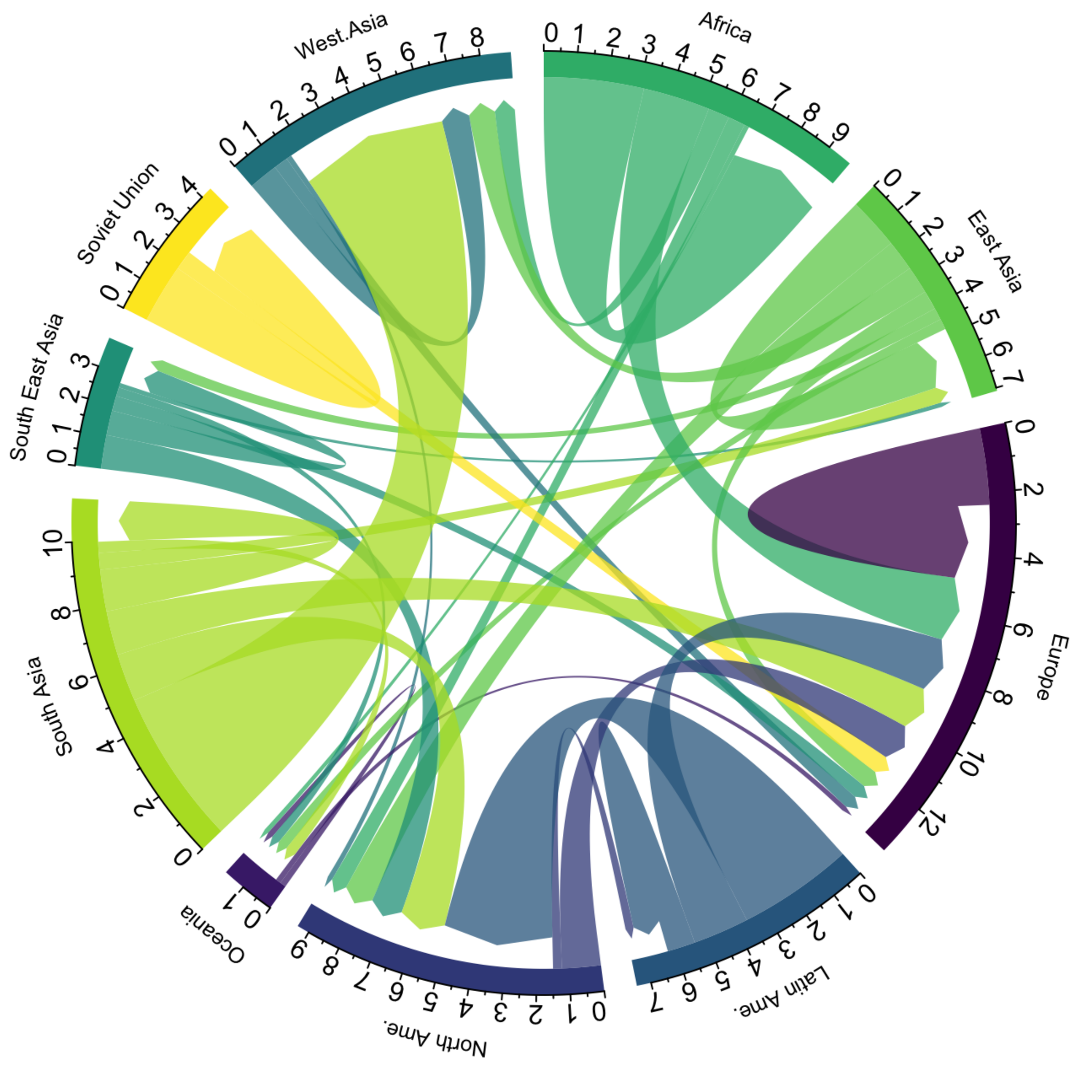
re-order vertices

network of co-authors of Vincent Ranwez (edge ⇔ co-authors)

[https://www.data-to-viz.com/graph/arc.html]



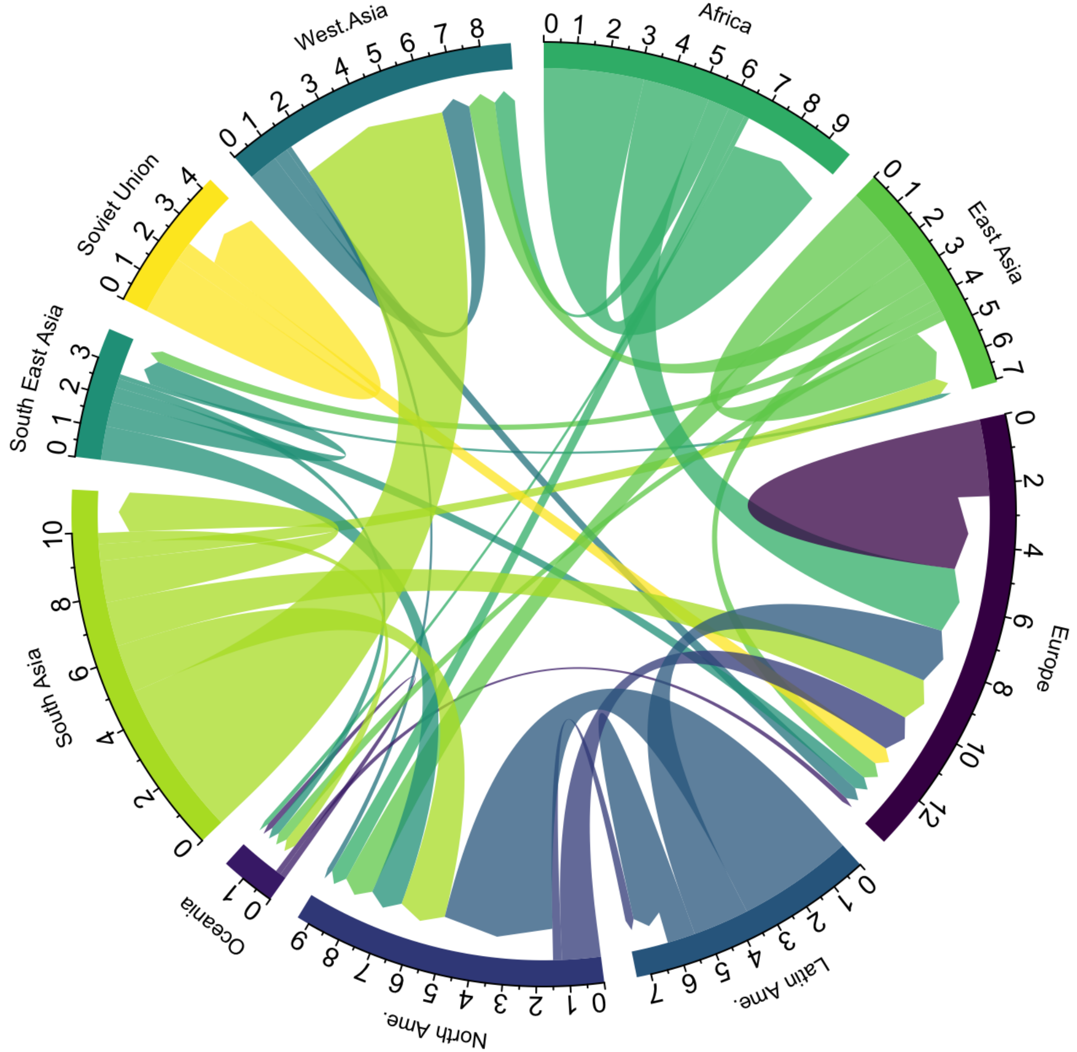
Drawing Style: Chord Diagrams



migration between continents

[https://www.data-to-viz.com/story/AdjacencyMatrix.html]

Drawing Style: Chord Diagrams



migration between continents

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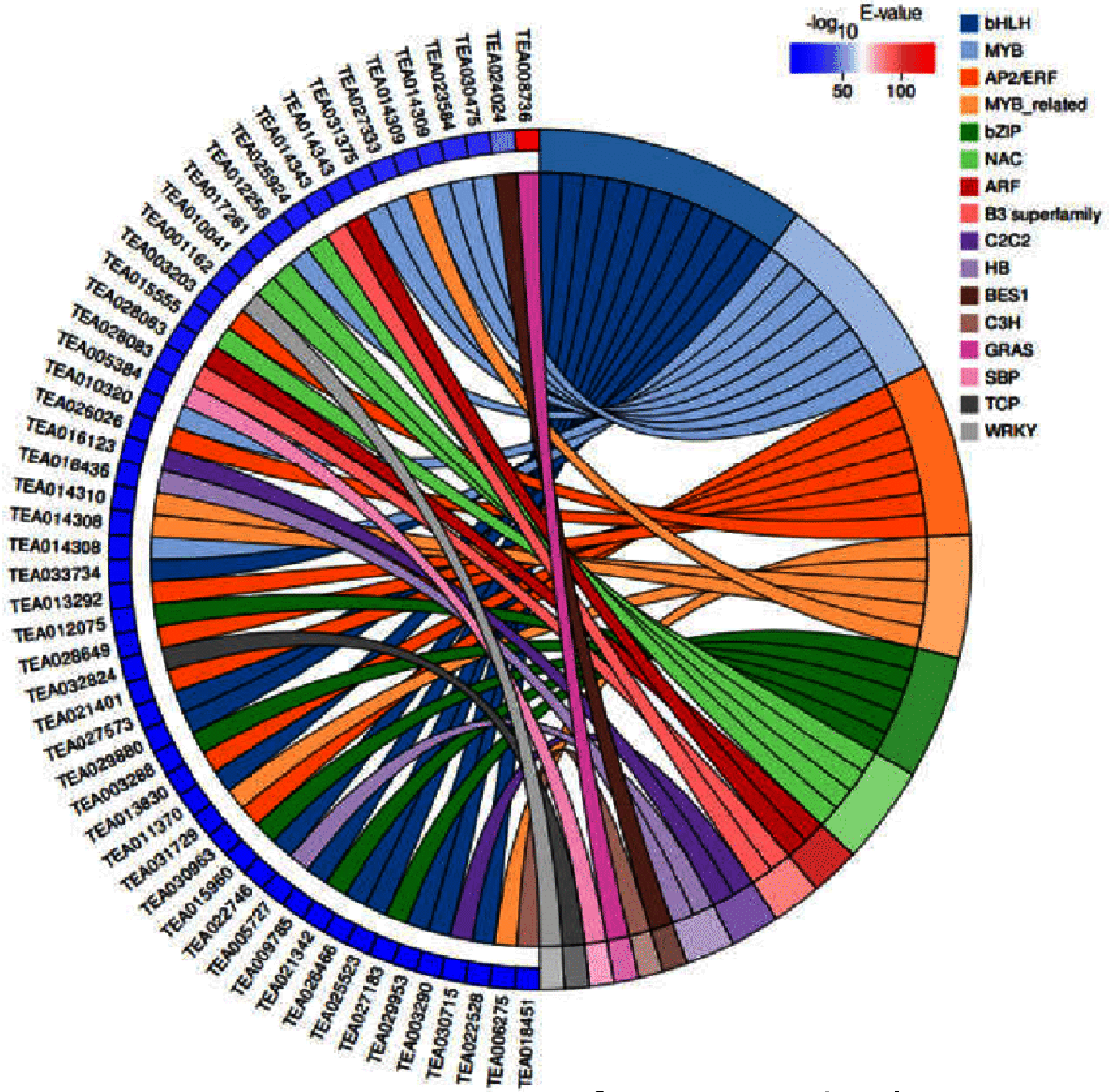
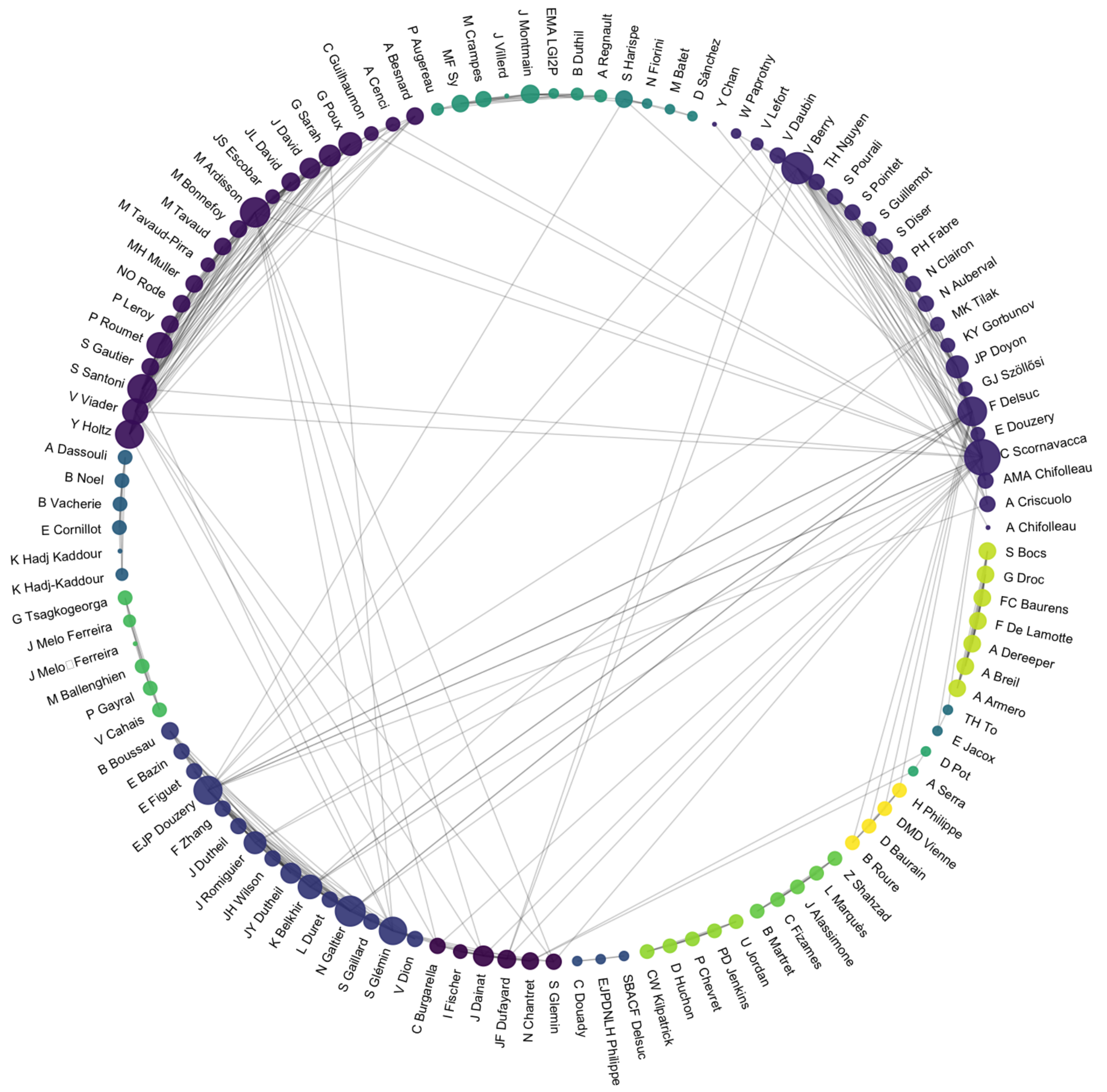


transcription factors in biology

[Wang, Xuejin, Zhao 2020: *Exploration of the Effects of Different Blue LED Light Intensities on Flavonoid and Lipid Metabolism in Tea Plants via Transcriptomics and Metabolomics*]

Drawing Style: Chord Diagrams



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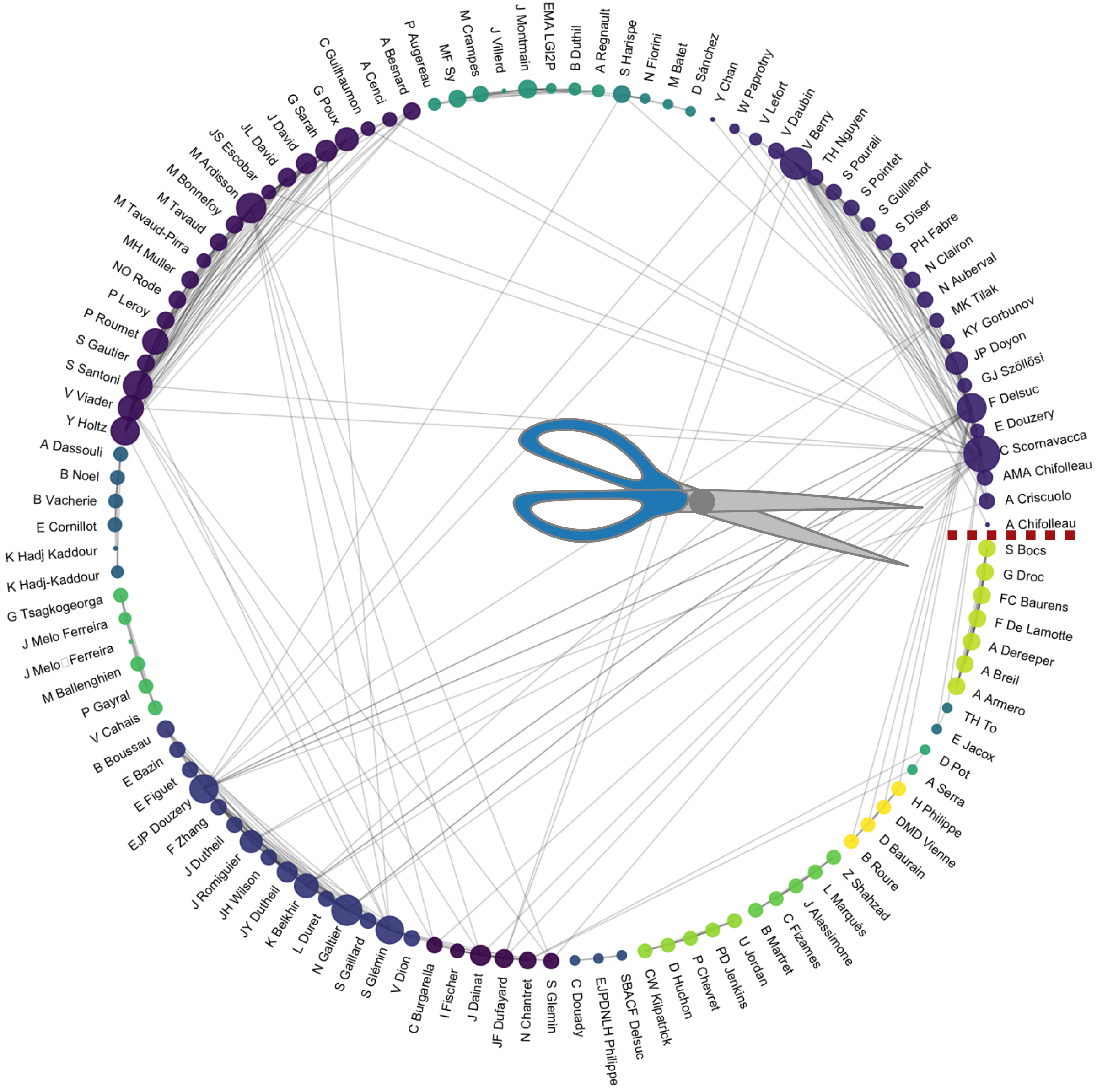

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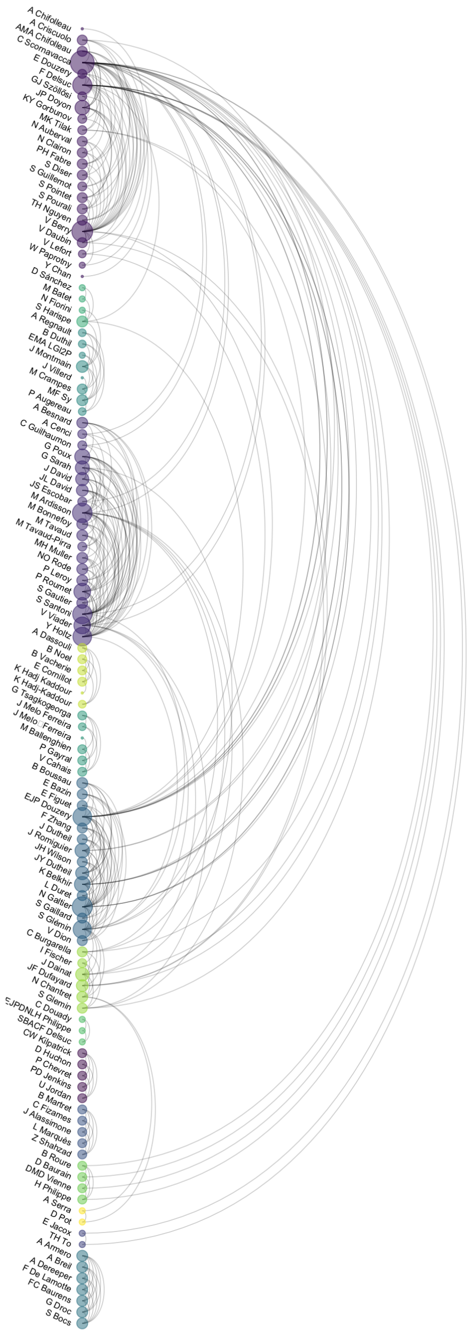
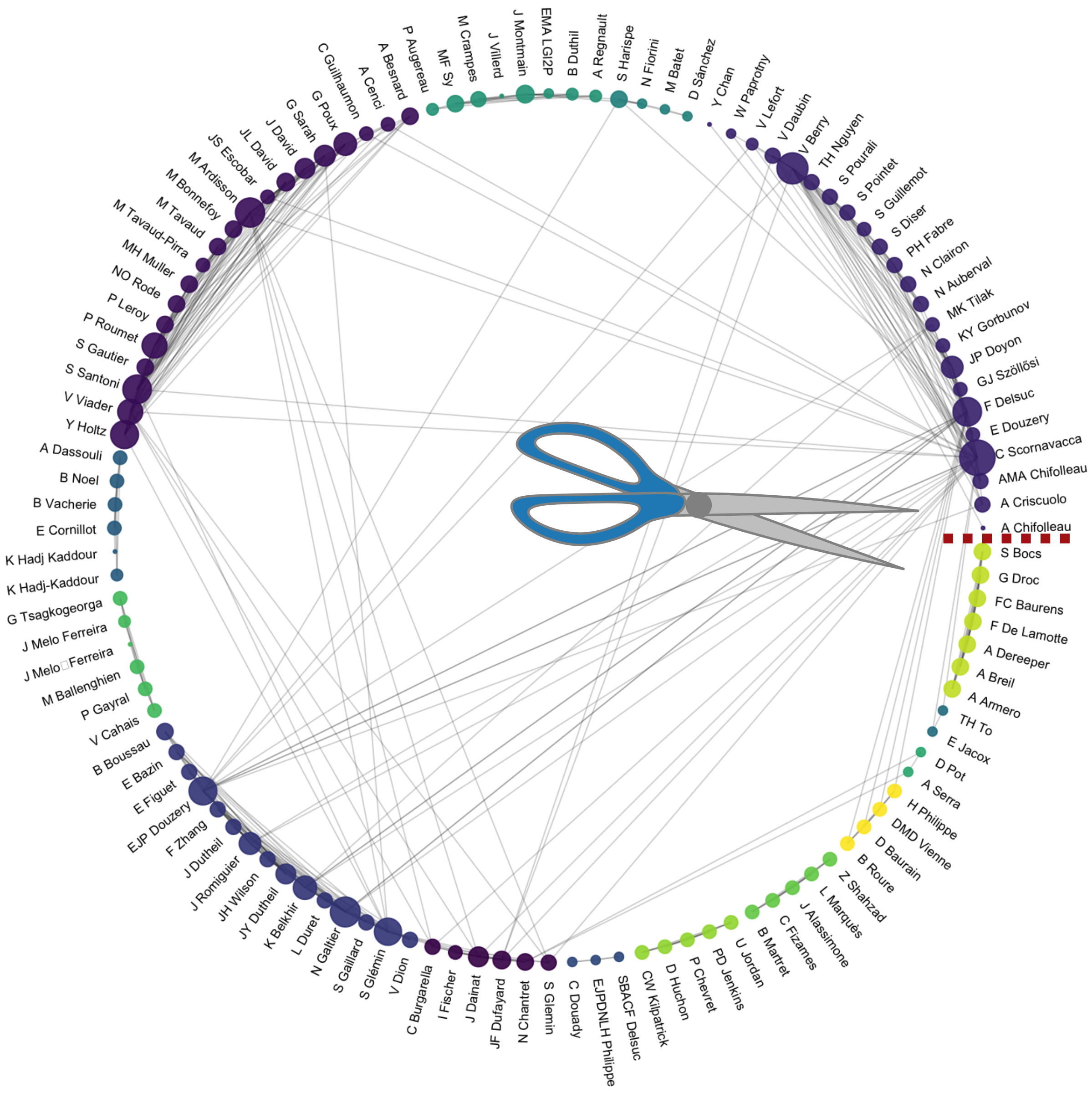


"cut" circle



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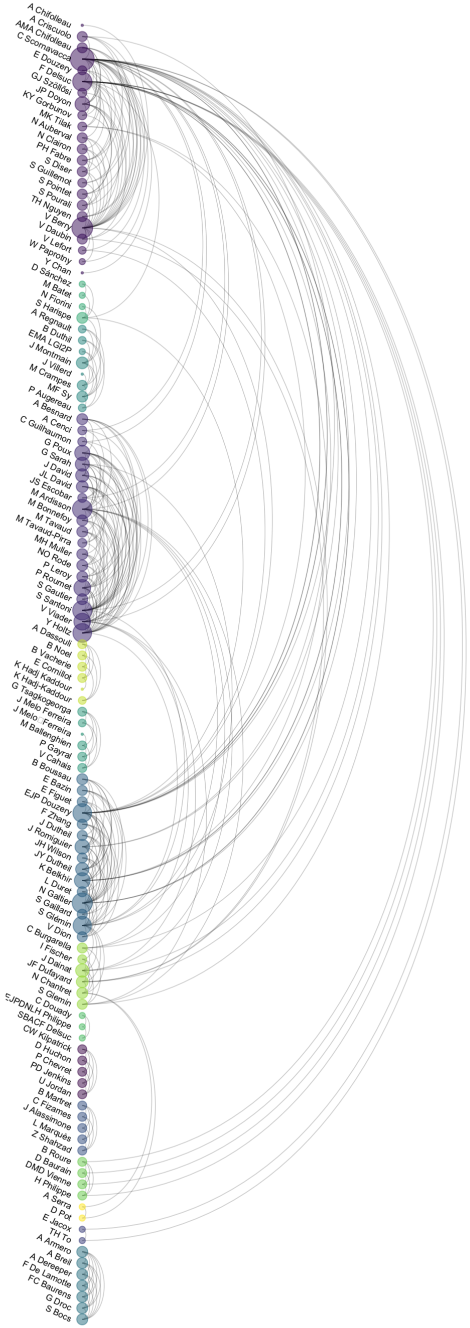
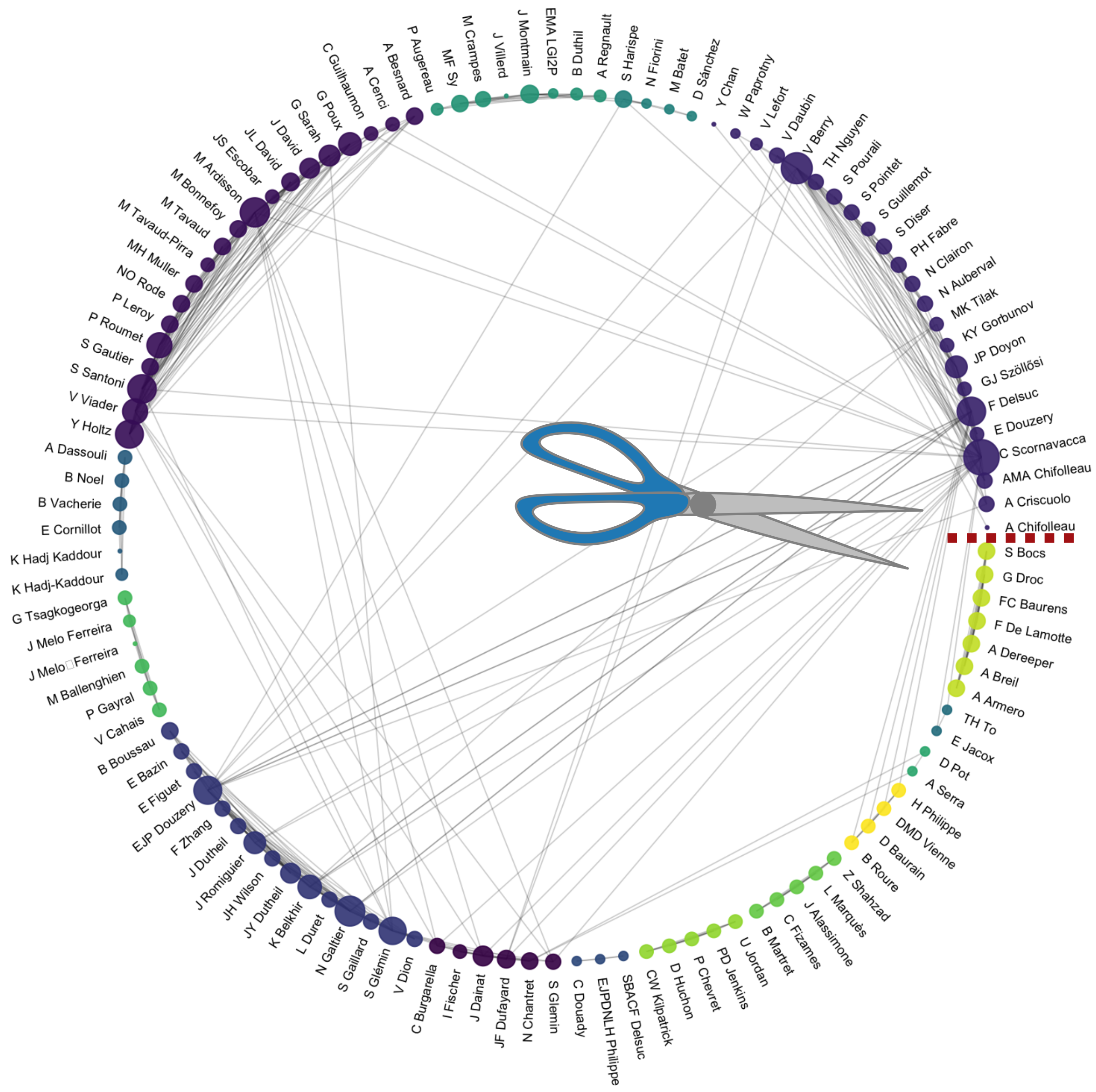


"cut" circle

"glue" sequence



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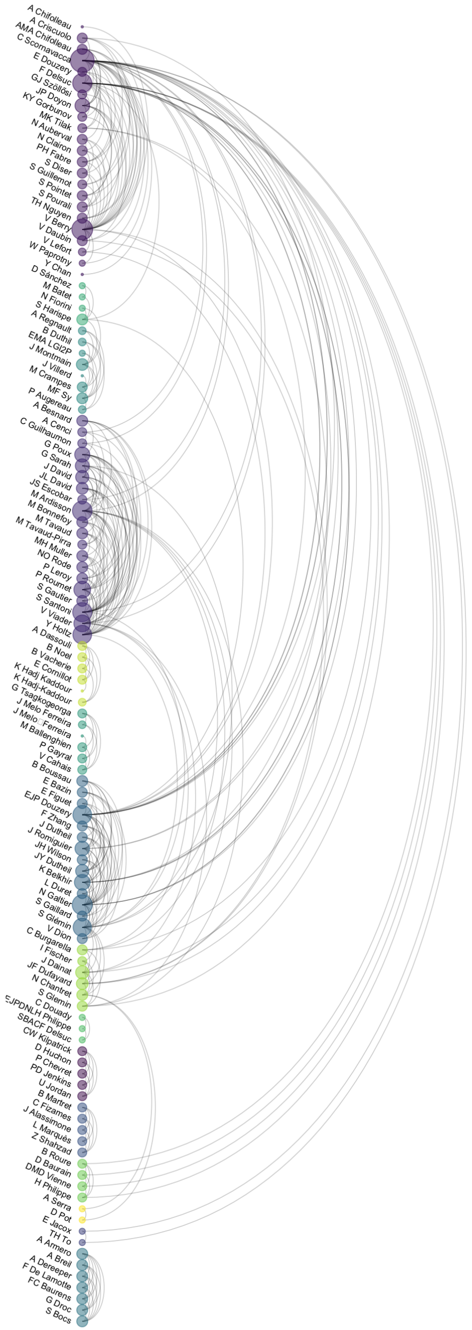

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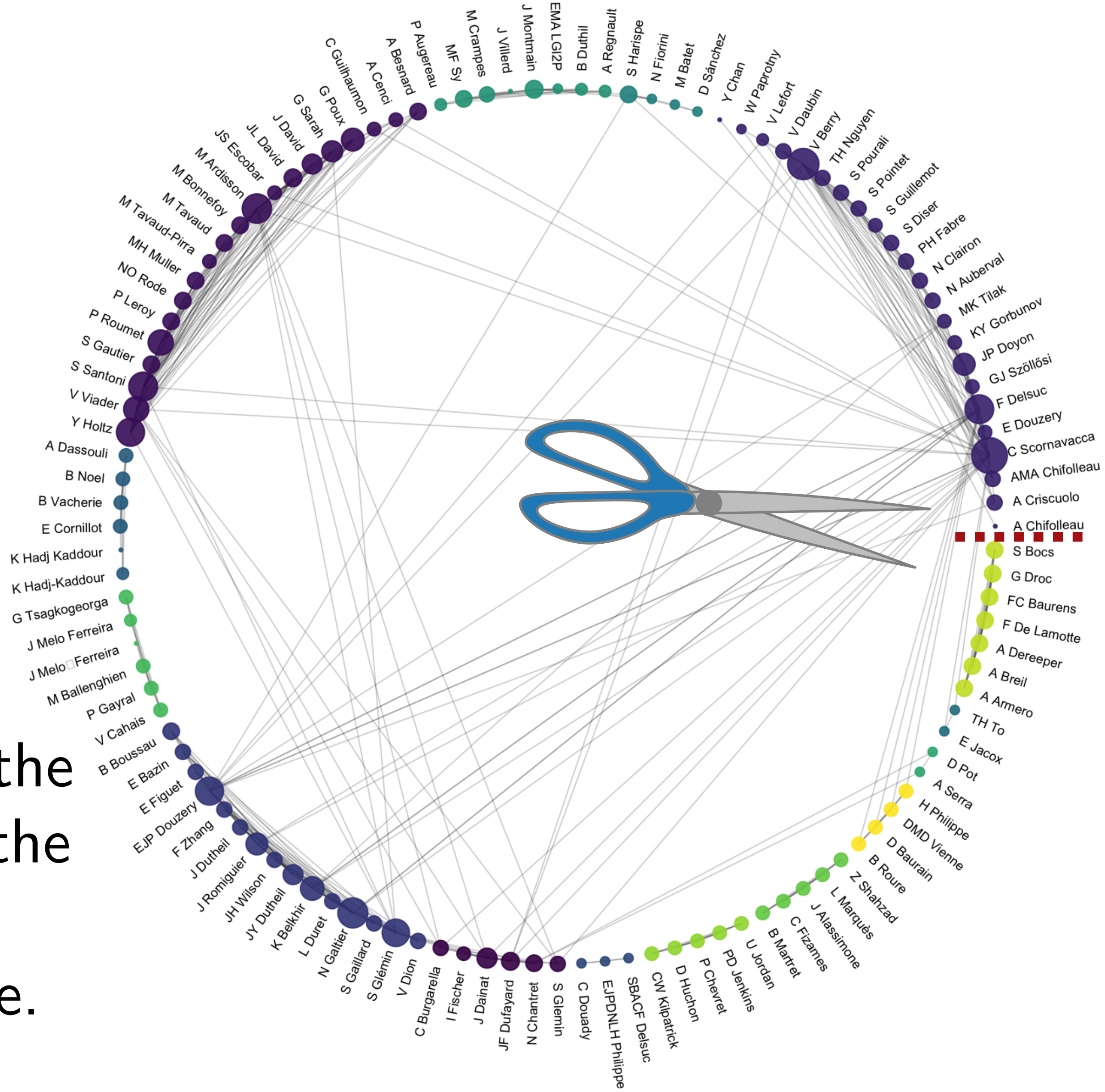


Note that, if we keep the order of the vertices, the crossings between the edges remain the same.

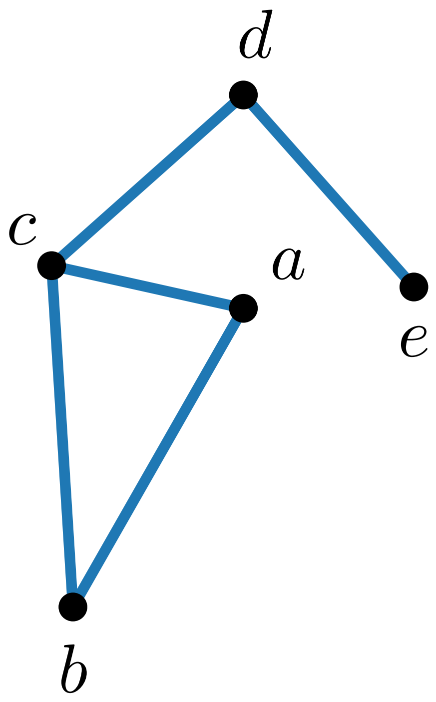
network of co-authors of Vincent Ranwez (edge ⇔ co-authors)

[https://www.data-to-viz.com/story/AdjacencyMatrix.html]

Planarity + Arc/Chord Diagrams?

Planarity + Arc/Chord Diagrams?

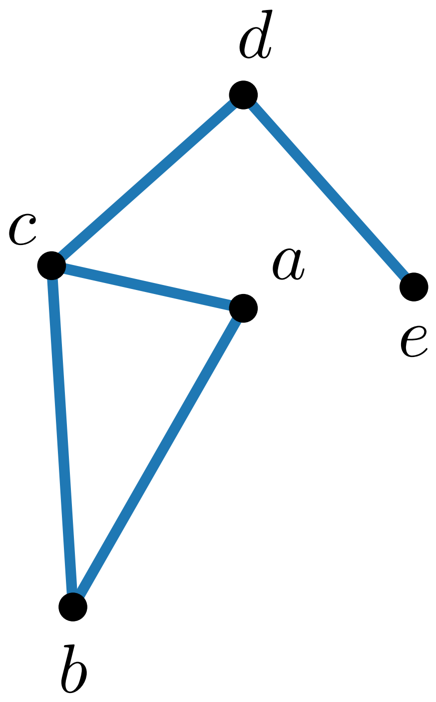
Given: ■ graph G



Planarity + Arc/Chord Diagrams?

Given:
- graph G

Task: Find a linear order $\prec$ of $V(G)$



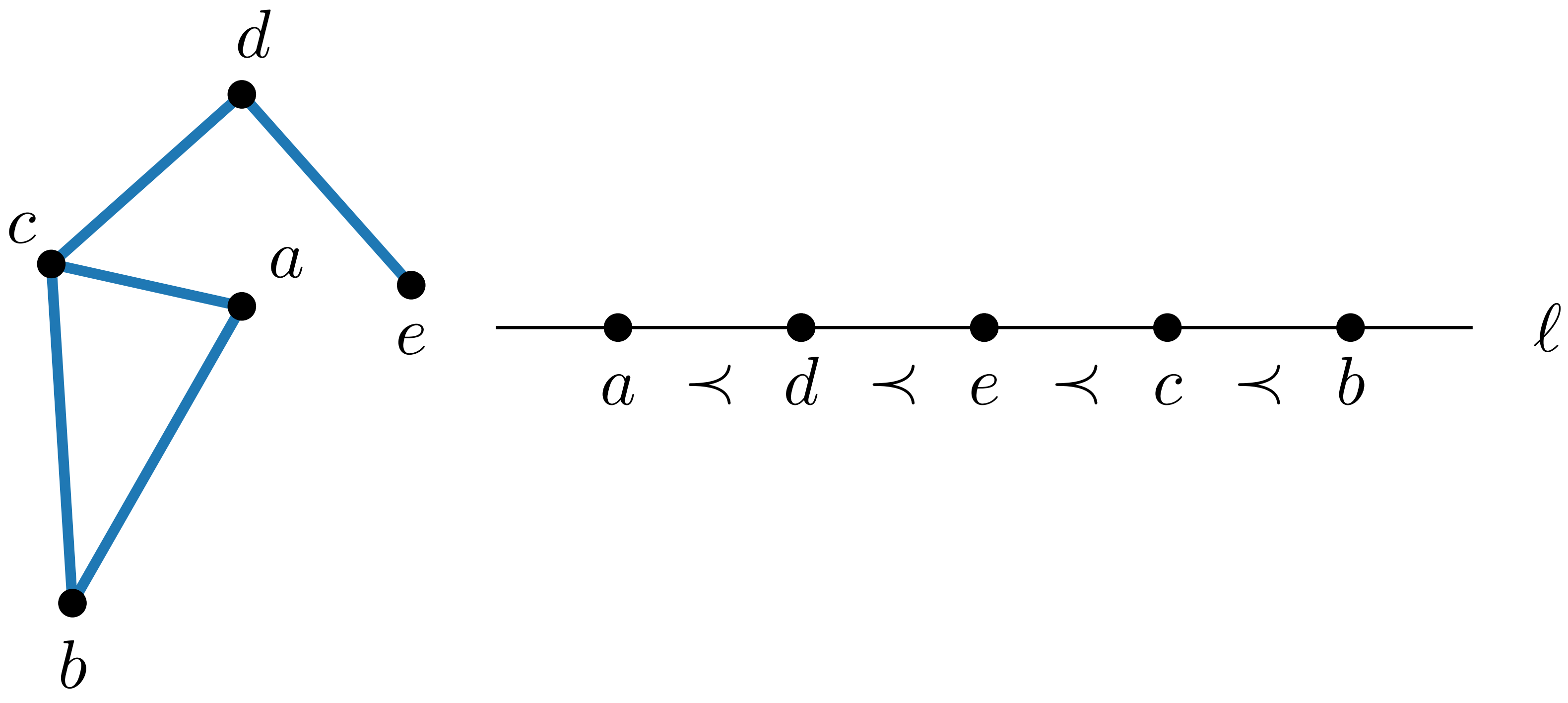
$$a \prec d \prec e \prec c \prec b$$

Planarity + Arc/Chord Diagrams?

Given: ■ graph G

Task: Find a linear order $\prec$ of $V(G)$ such that there is a planar drawing where

- the vertices $V(G)$ in order $\prec$ are arranged along a horizontal line ℓ

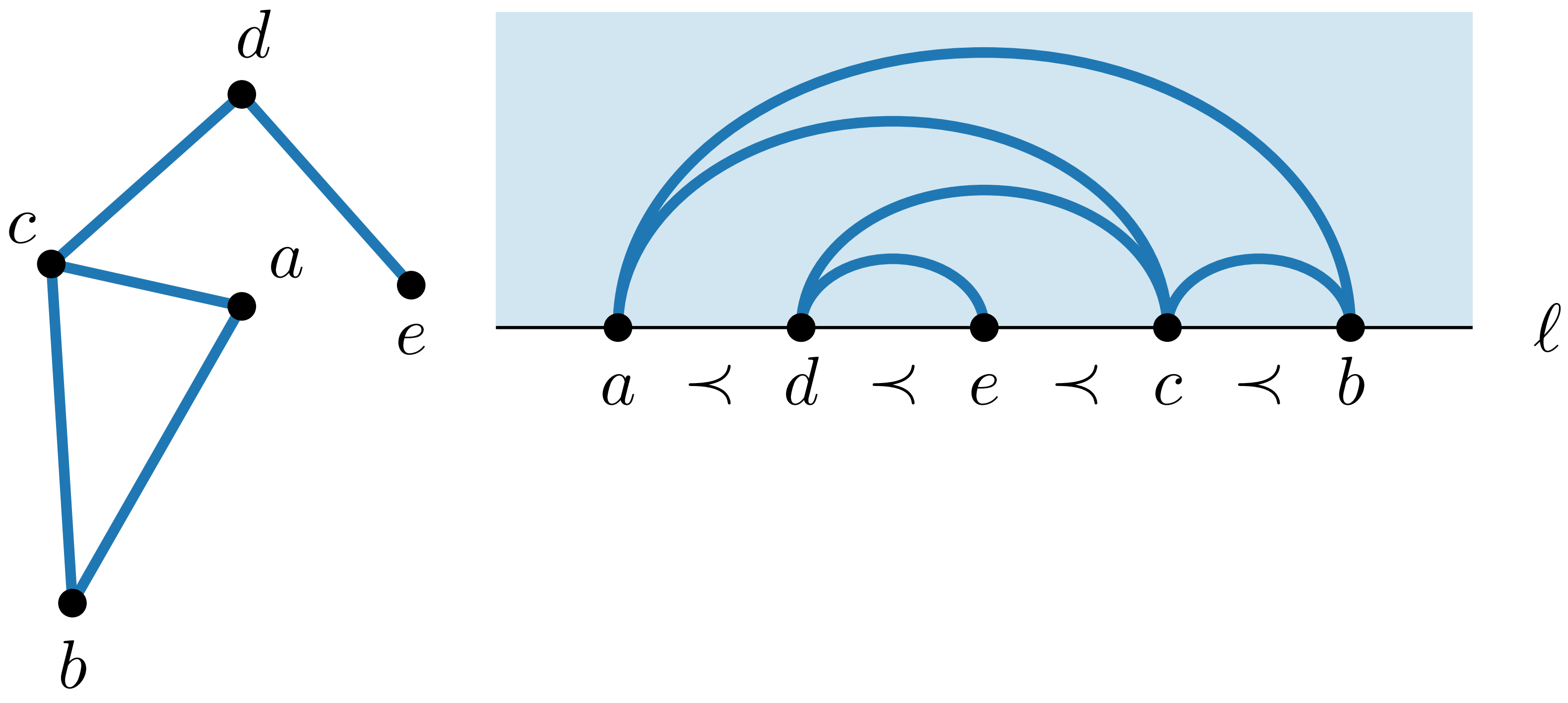


Planarity + Arc/Chord Diagrams?

Given: graph G

Task: Find a linear order $\prec$ of $V(G)$ such that there is a planar drawing where

- the vertices $V(G)$ in order $\prec$ are arranged along a horizontal line ℓ and
- the edges $E(G)$ are drawn as x-monotone arcs in the half plane above ℓ.



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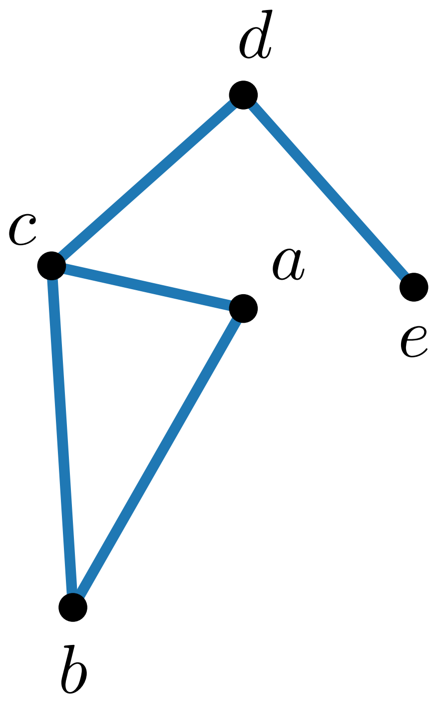

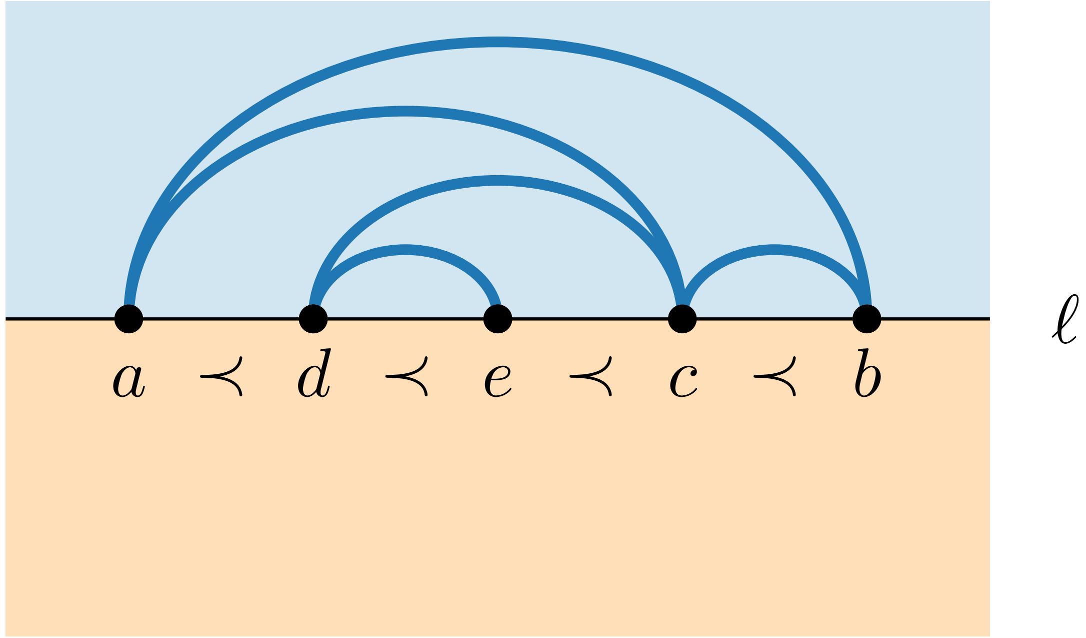

Why not using the half plane below ℓ?

Planarity + Arc/Chord Diagrams?

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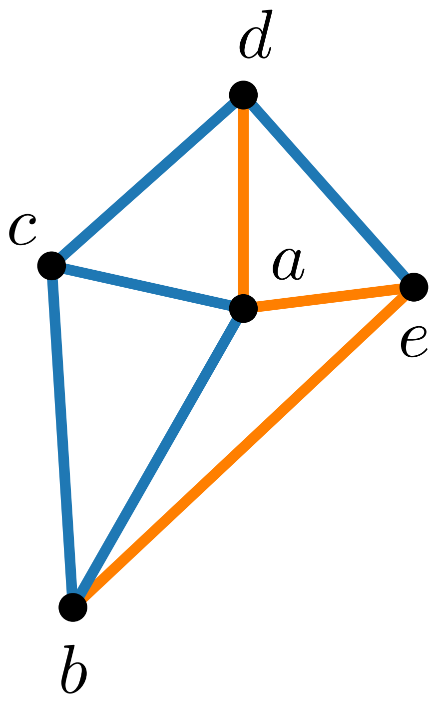

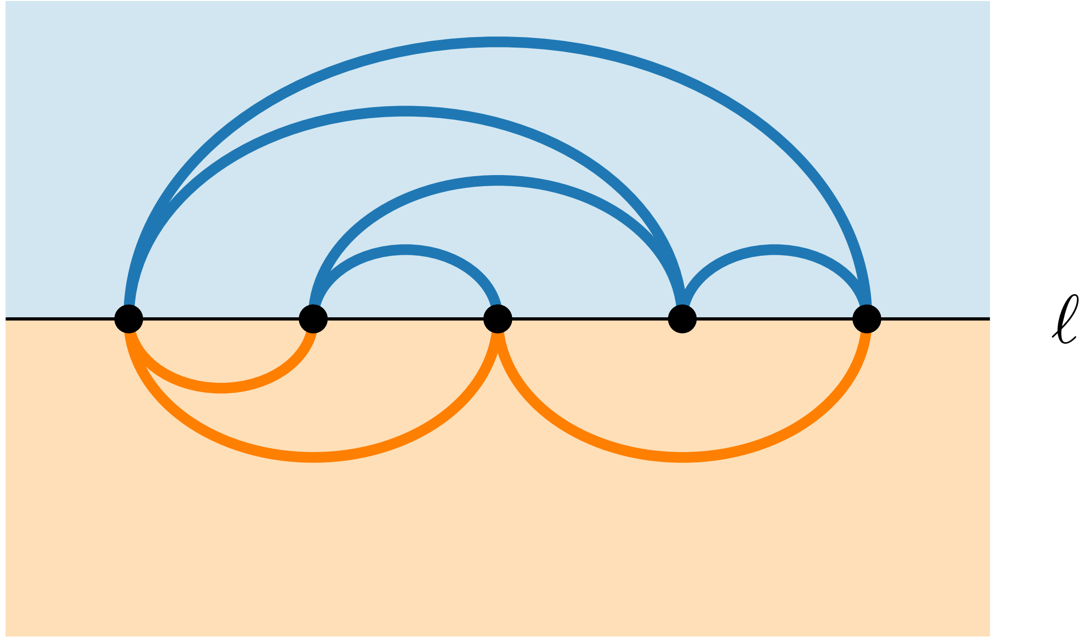

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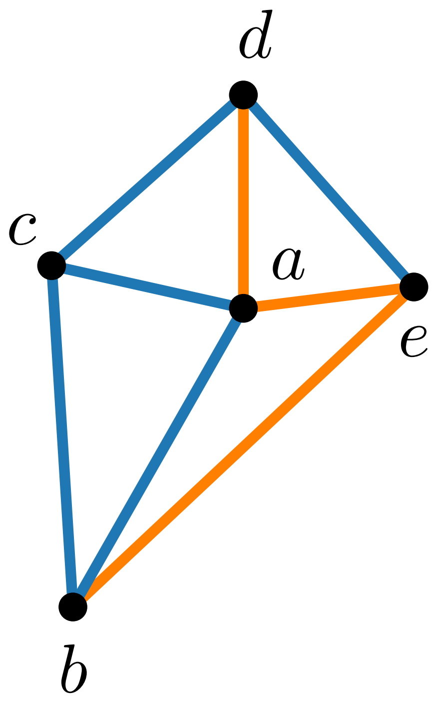
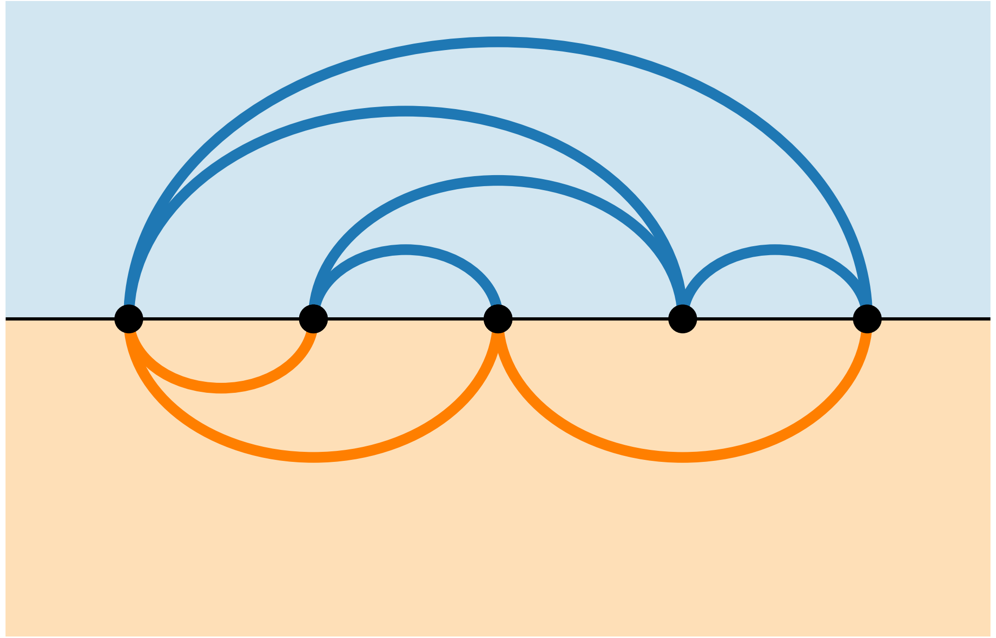
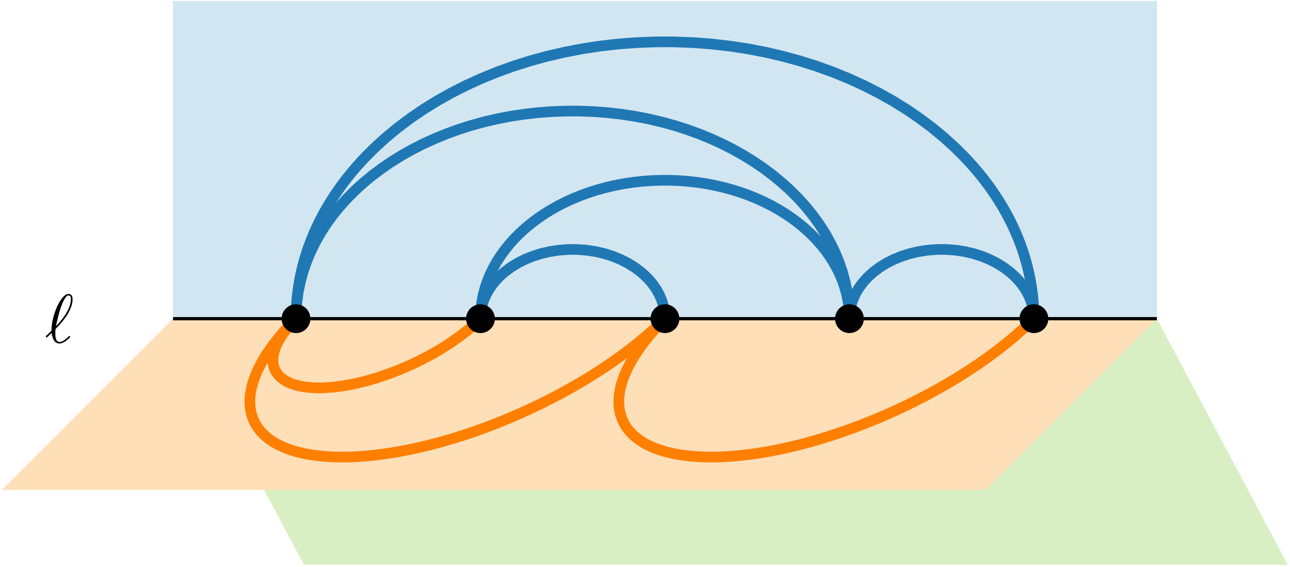




ℓ



Why not using the half plane below ℓ?

Or even more half planes?

Planarity + Arc/Chord Diagrams?

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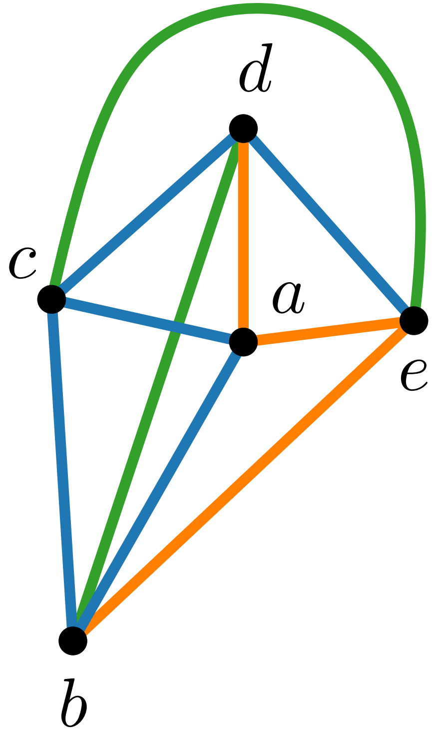

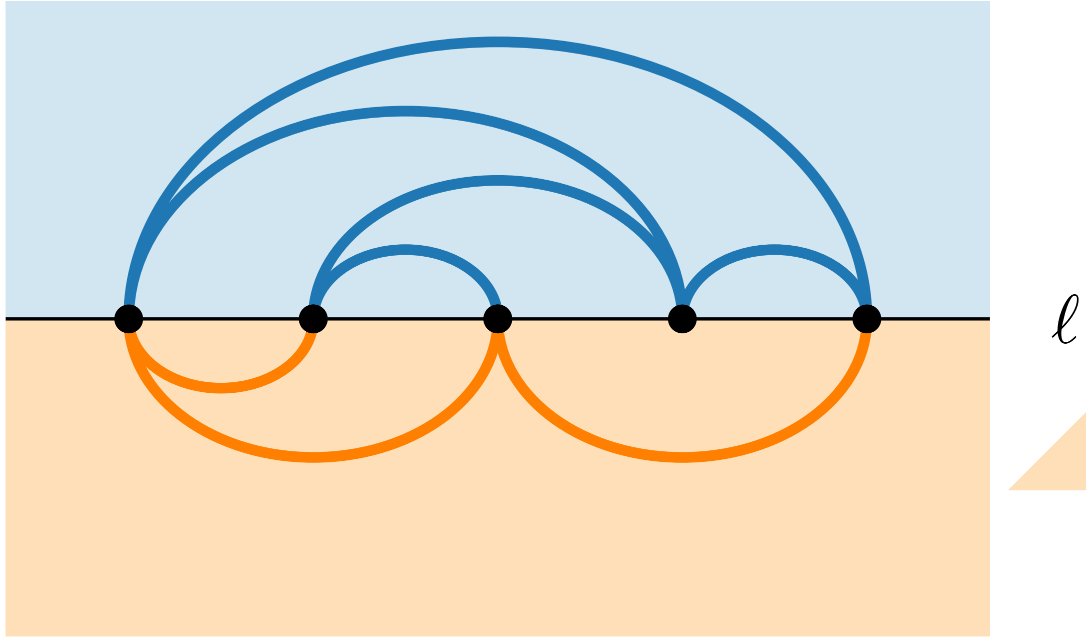

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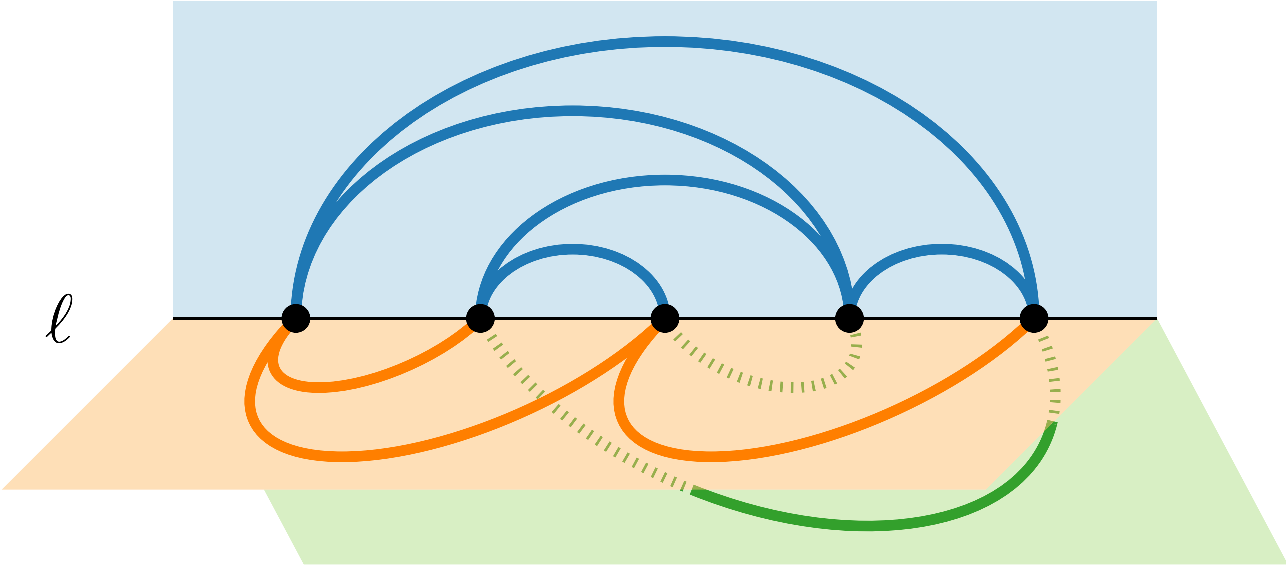

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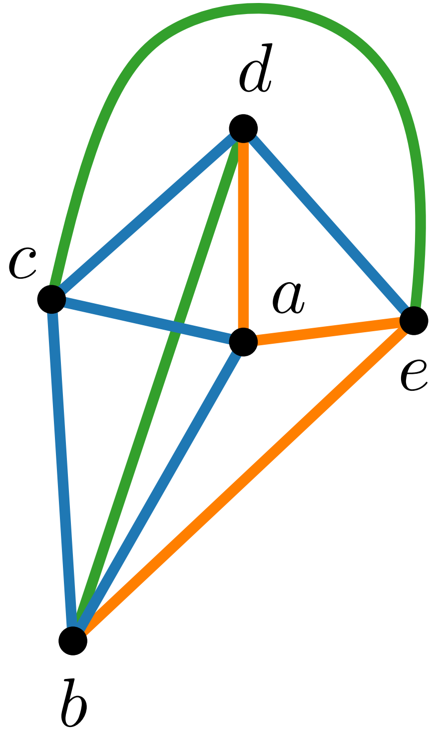

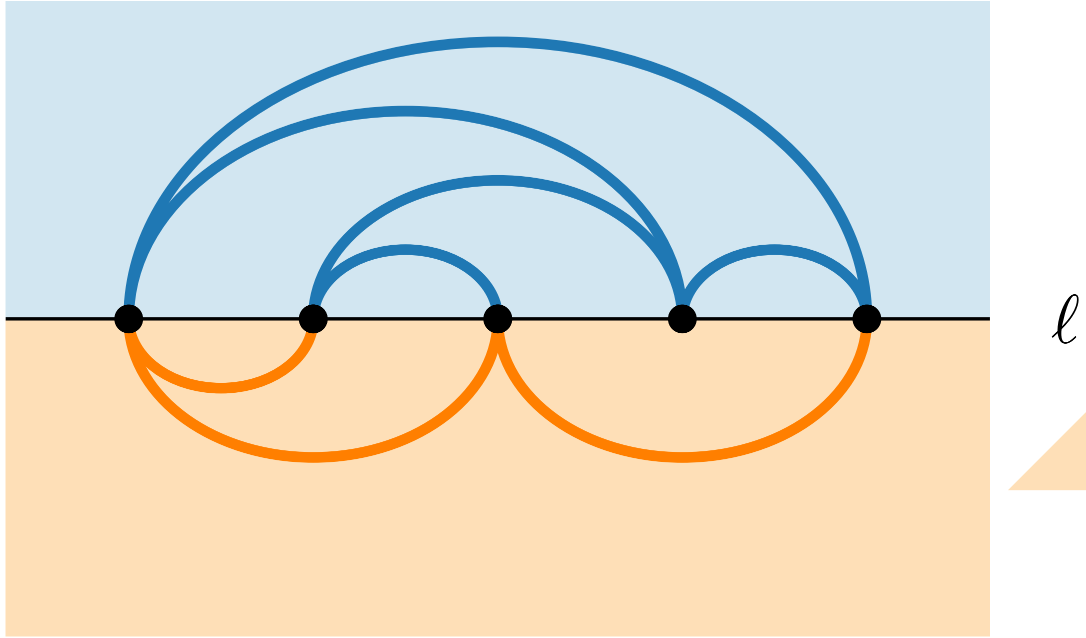

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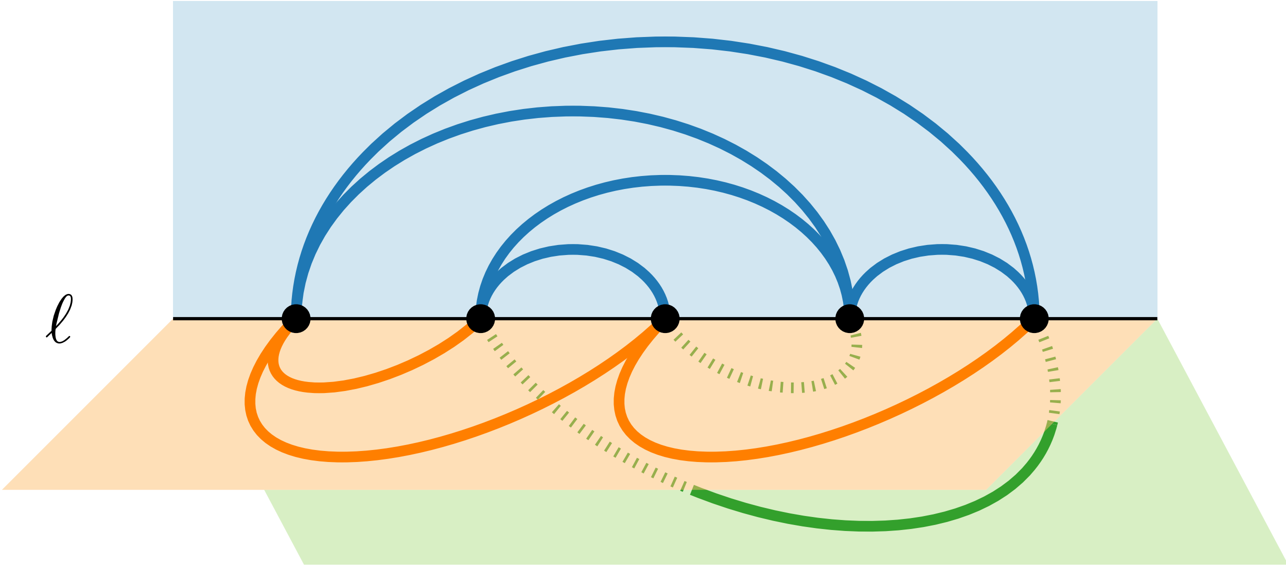
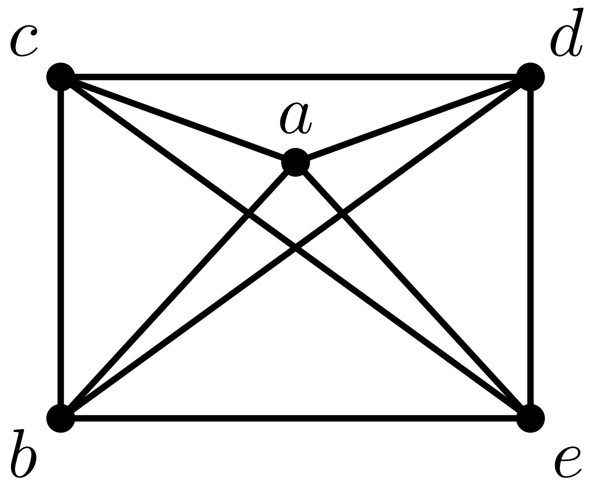


Or even more half planes?

→ **book embeddings**

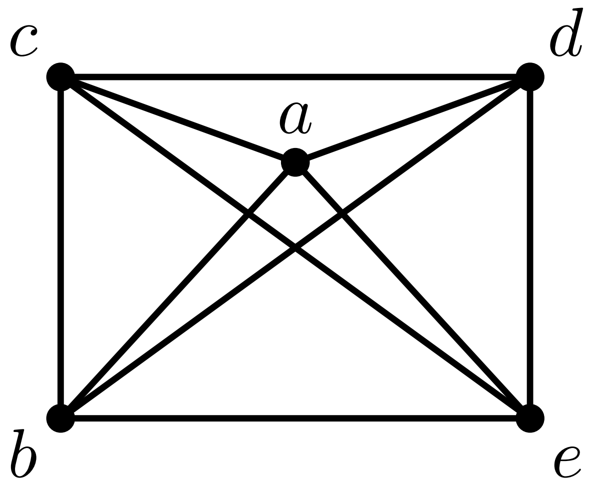
Book Embeddings

Given: graph G



Book Embeddings

Given:
- graph G
- integer k



$\boxed{k = 3}$

Book Embeddings

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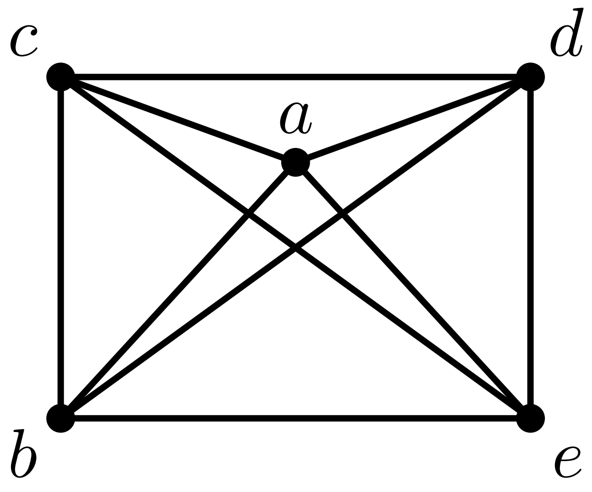
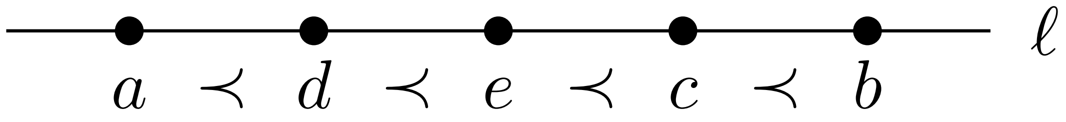


$k = 3$



Book Embeddings

Given:
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Task: Find (i) a linear order $\prec$ of $V(G)$ and (ii) an assigment $p\colon E(G) \to \{1, \ldots, k\}$ such that ...

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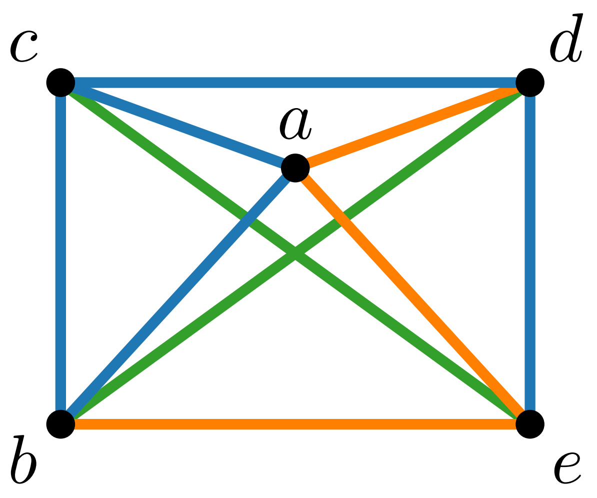

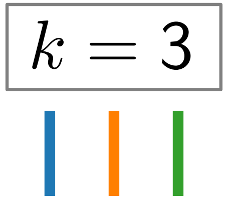

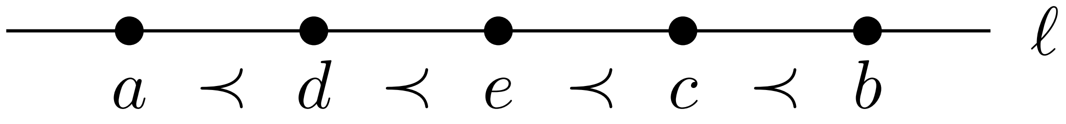

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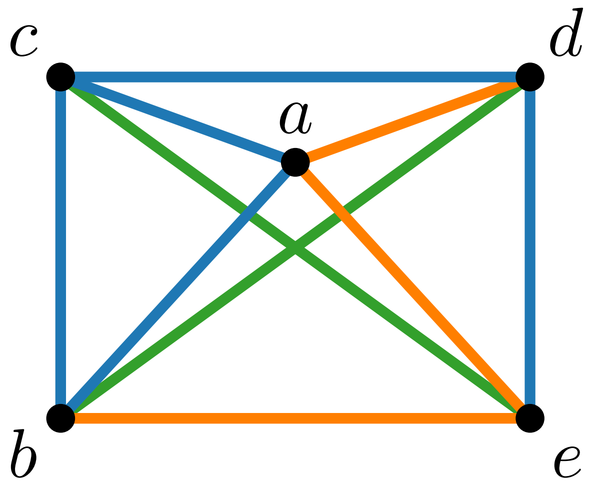

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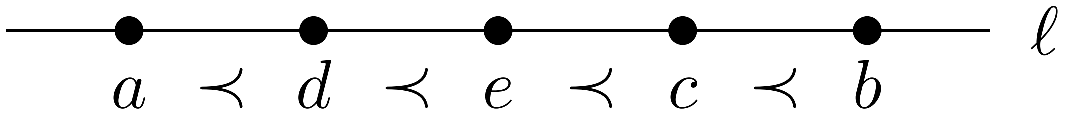

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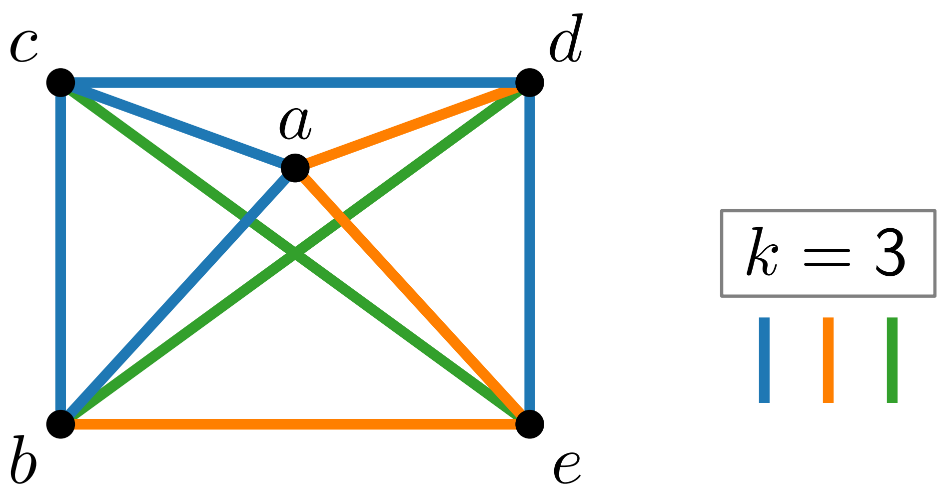

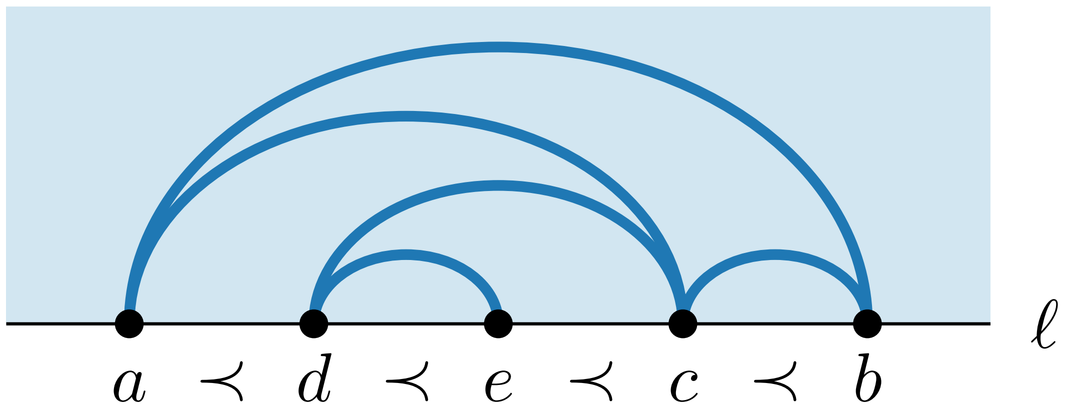

Book Embeddings

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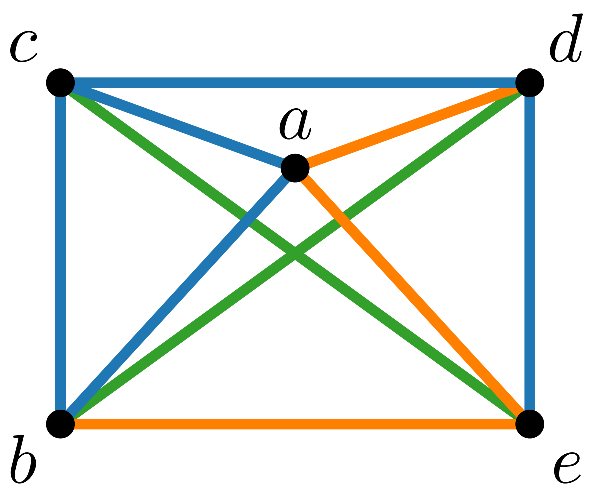

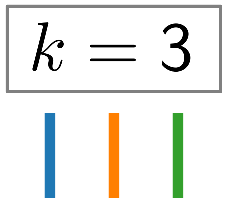

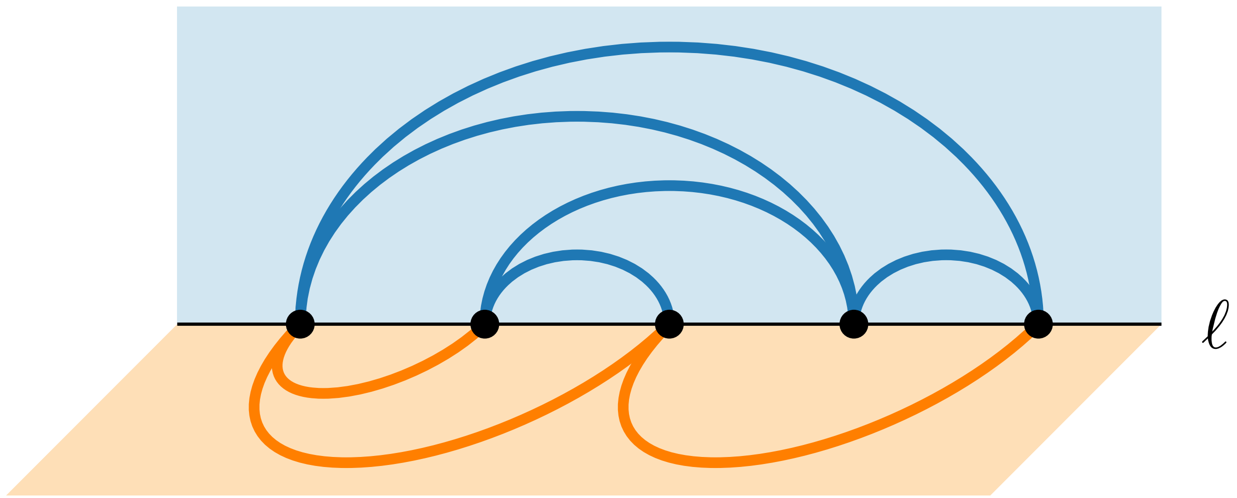

Book Embeddings

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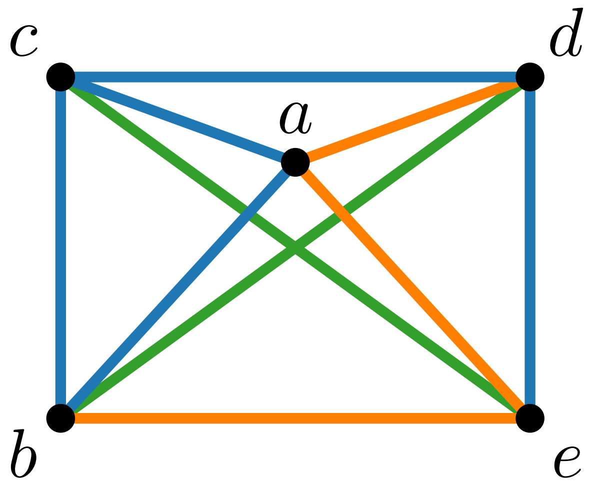
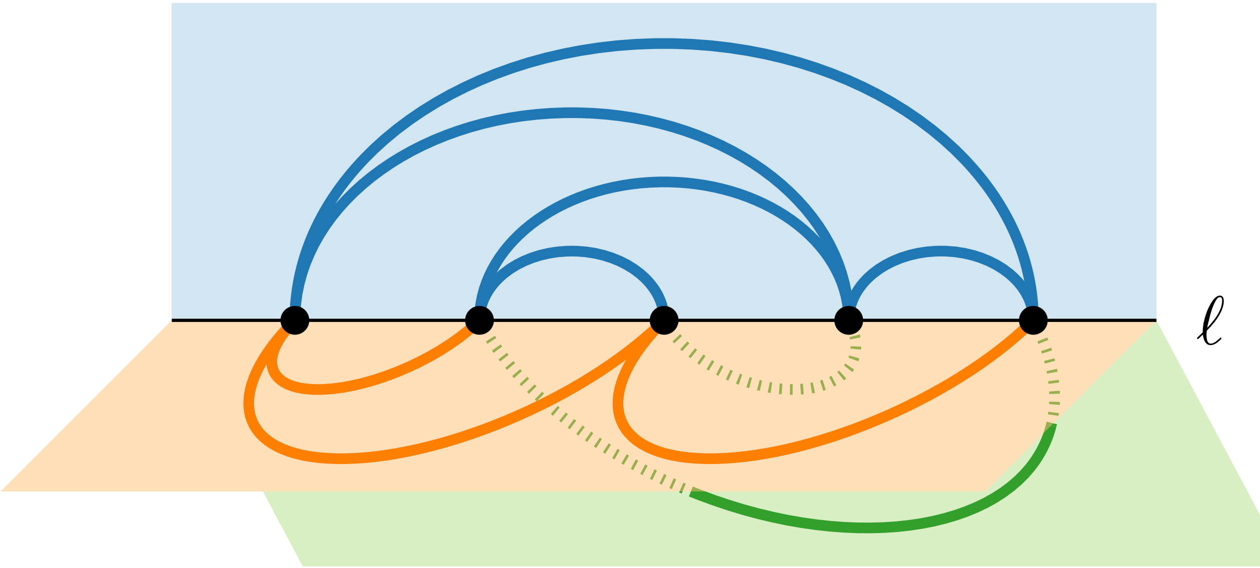


$k = 3$



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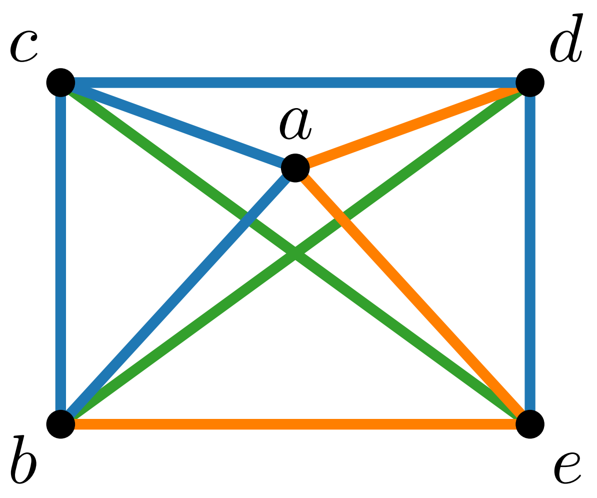
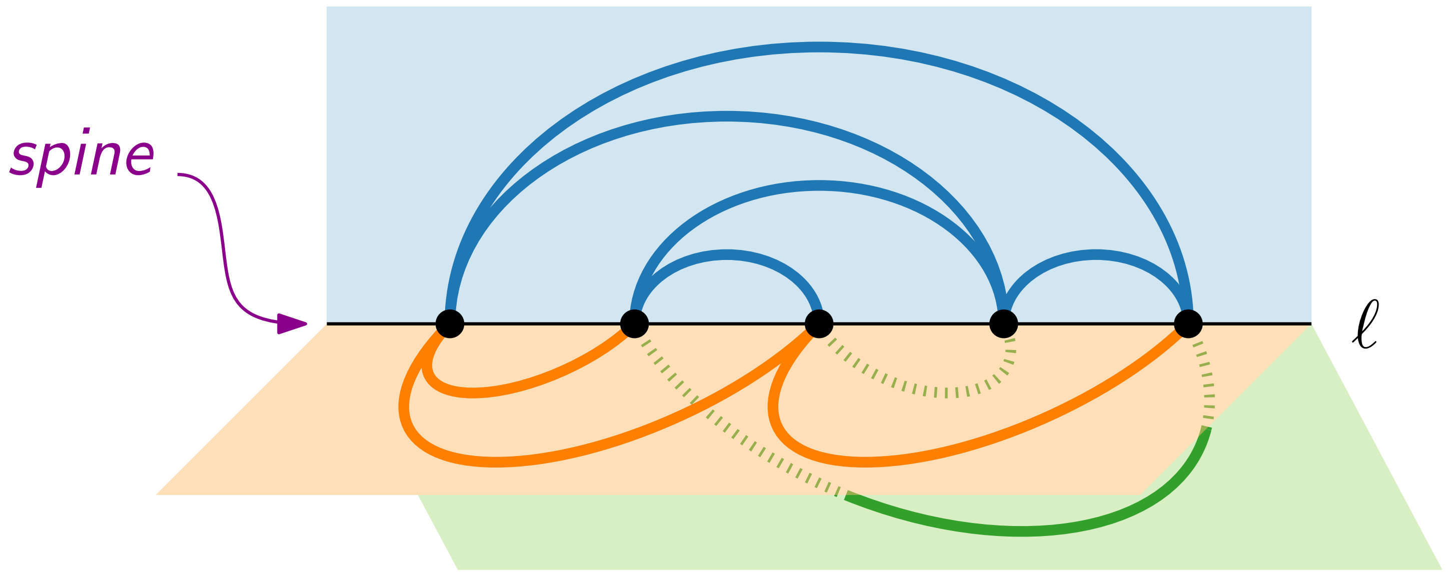


$\boxed{k = 3}$



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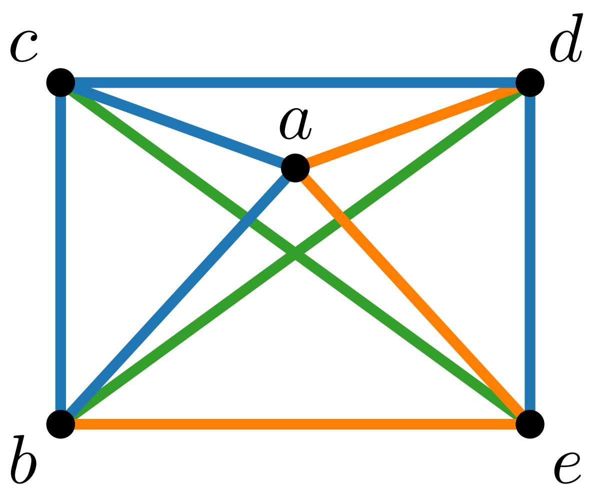

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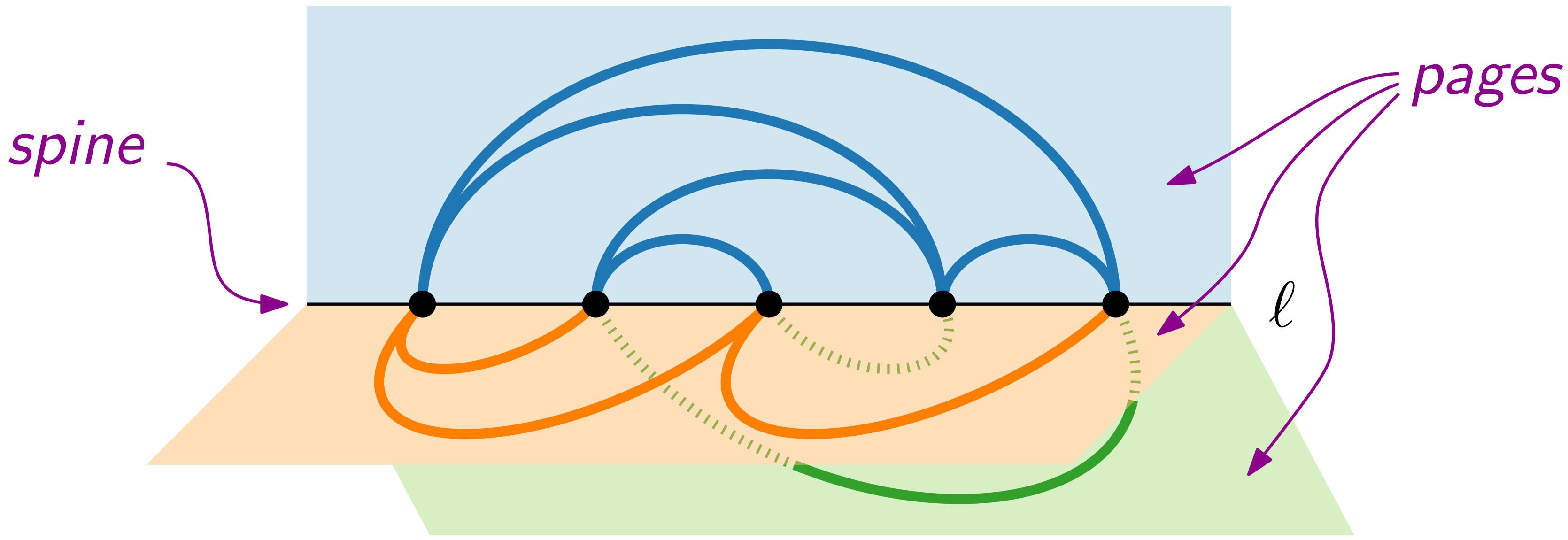

Book Embeddings (Stack Layouts)

Given:
- graph G
- integer k

Task: Find (i) a linear order $\prec$ of $V(G)$ and (ii) an assigment $p\colon E(G) \to \{1, \dots, k\}$ such that ...

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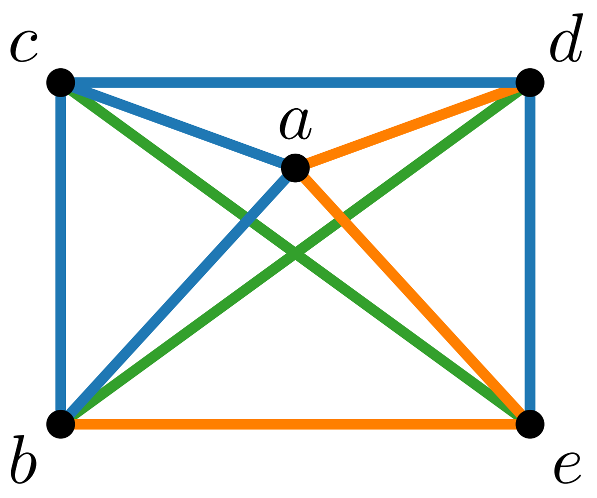

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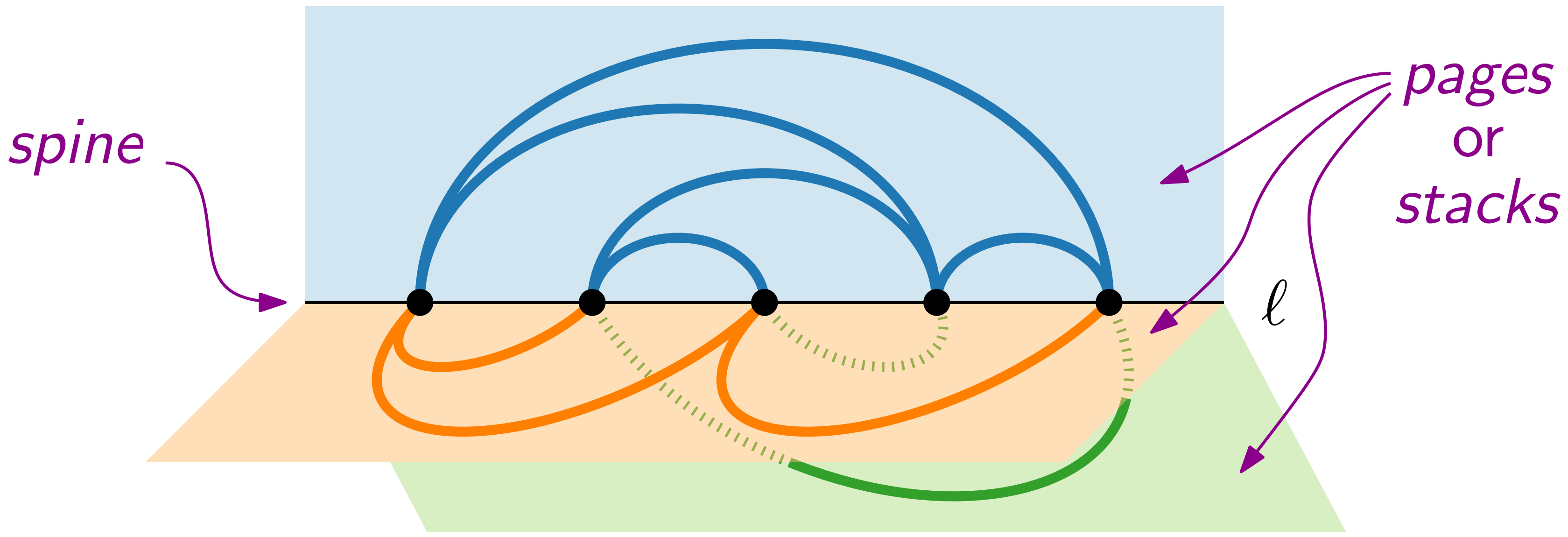

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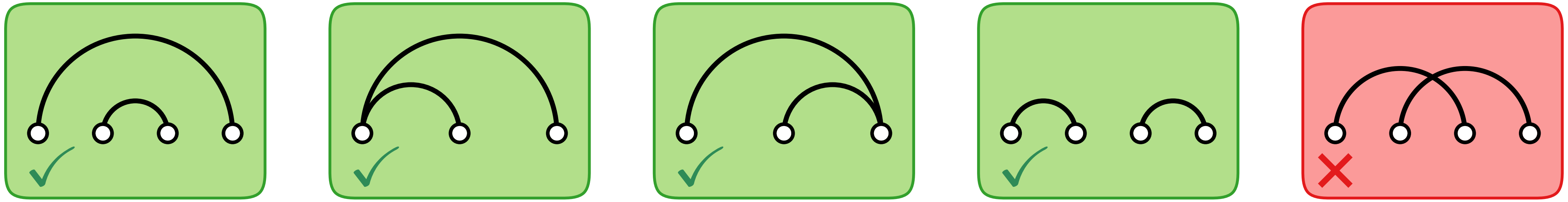

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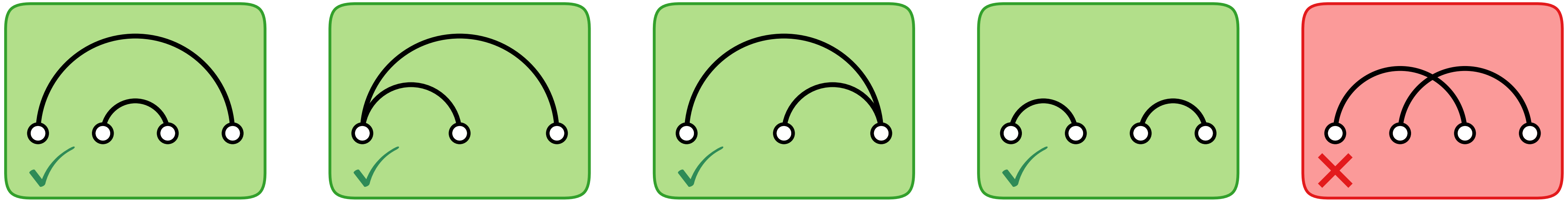

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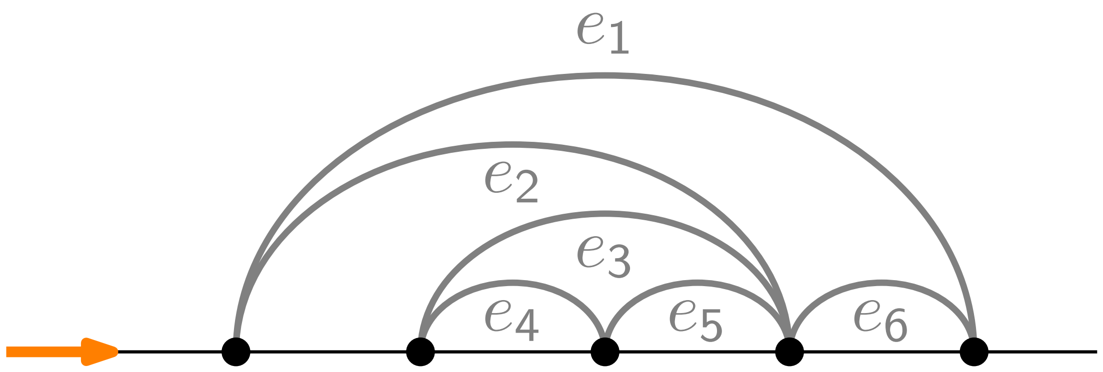

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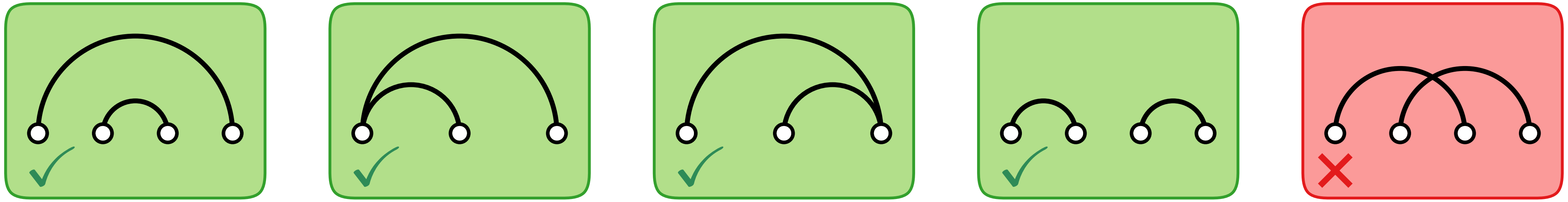

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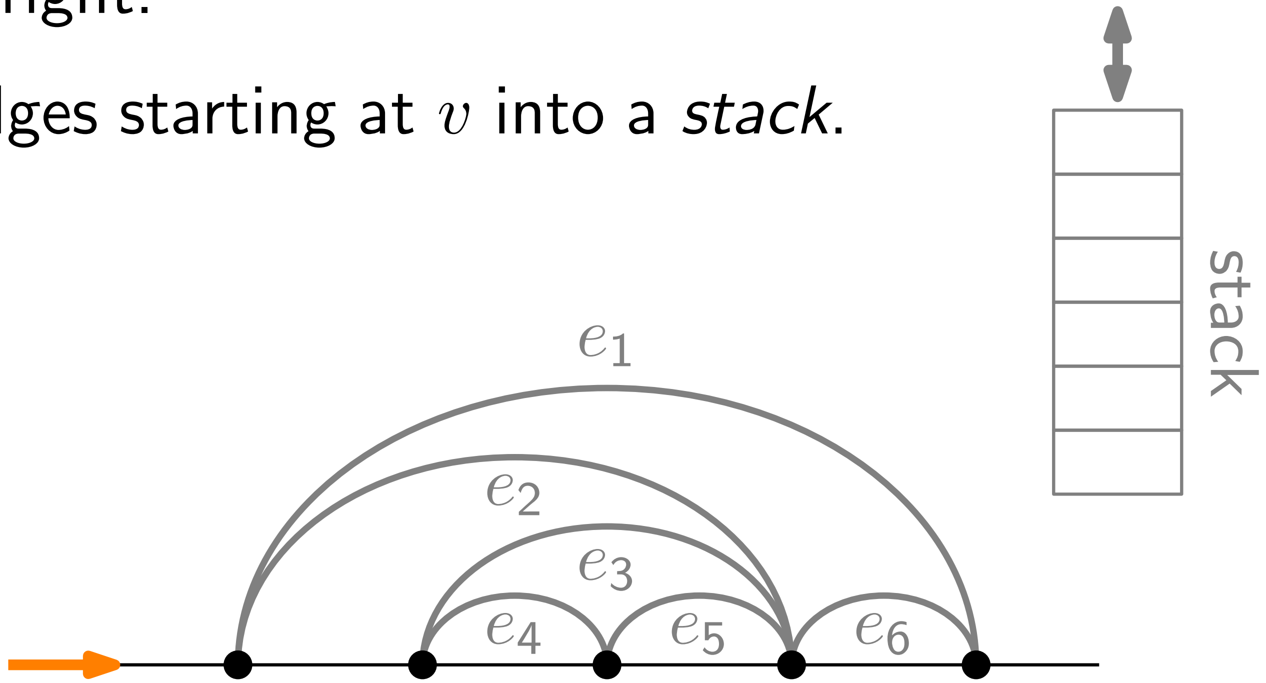

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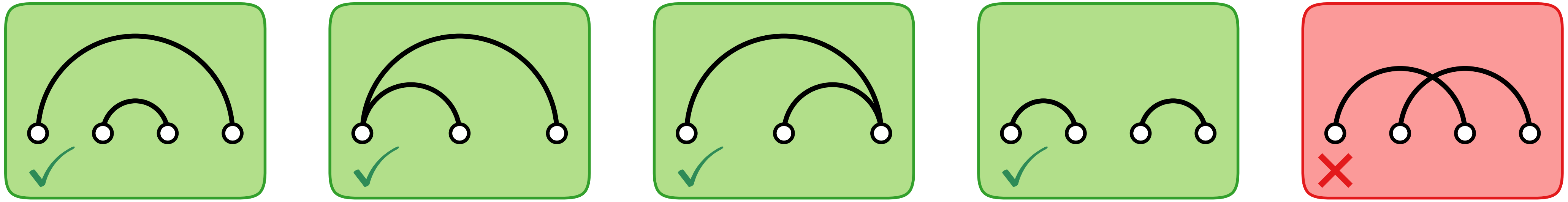

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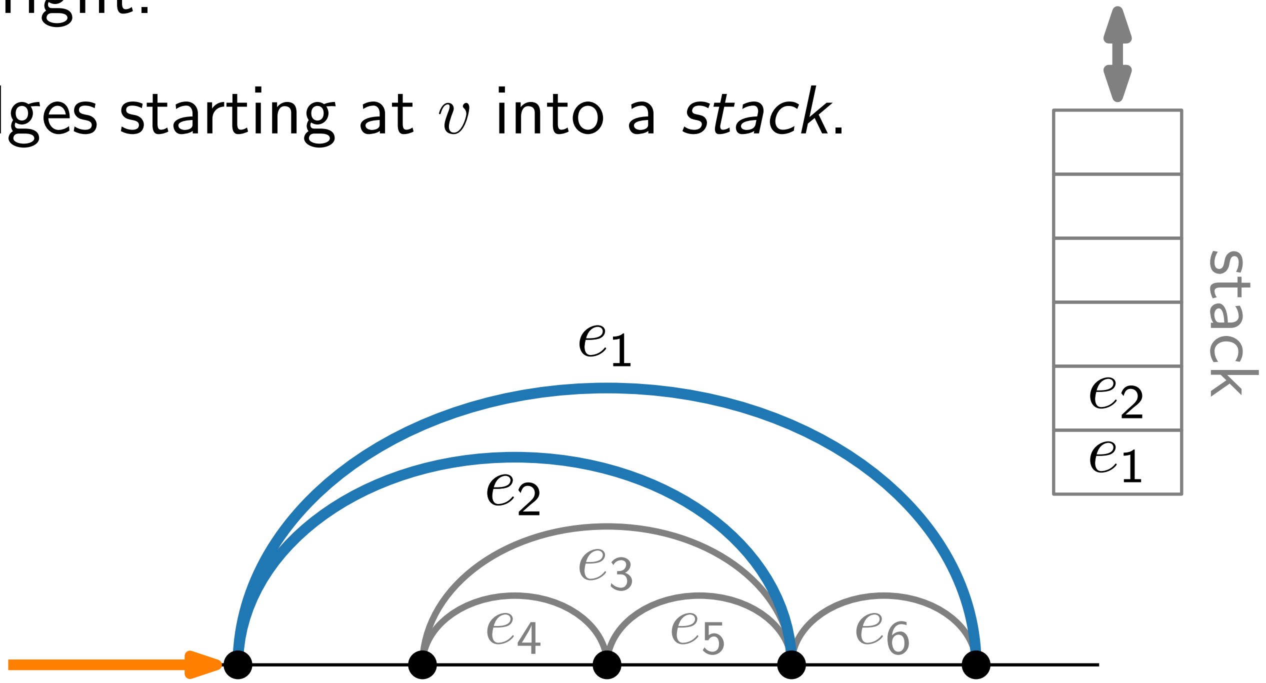

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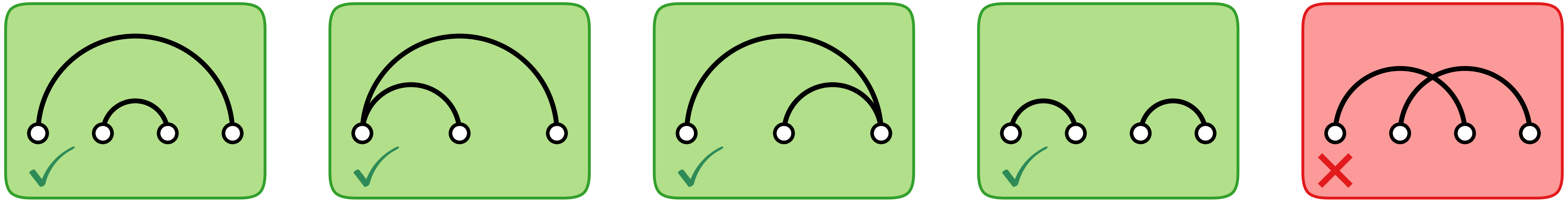

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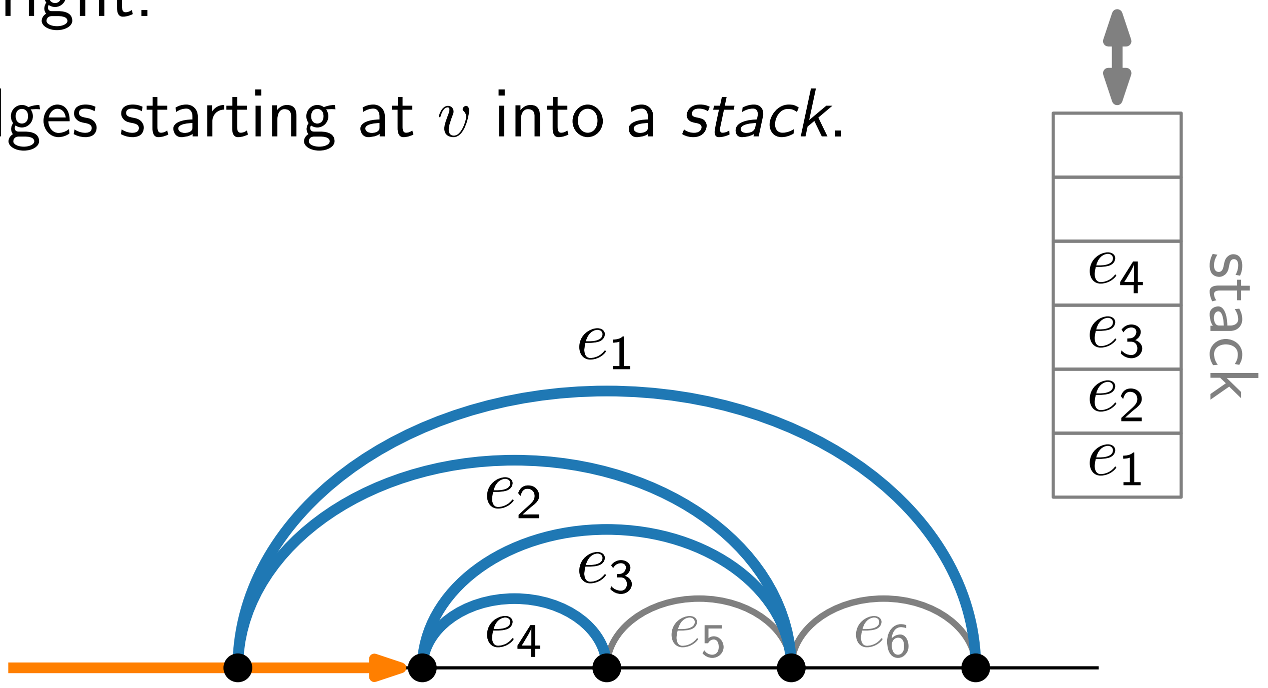

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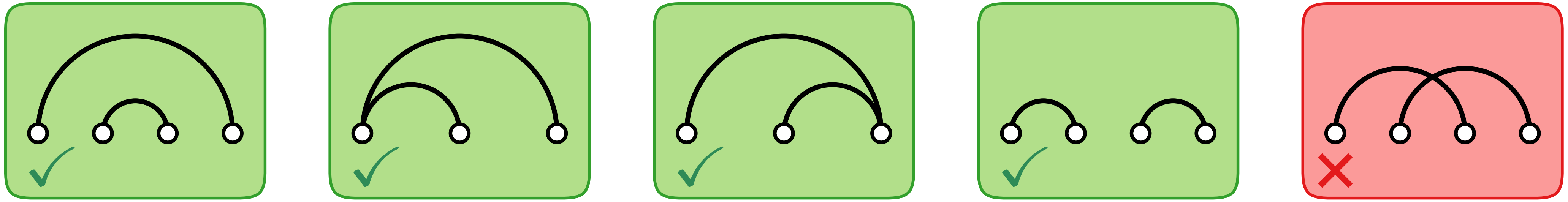

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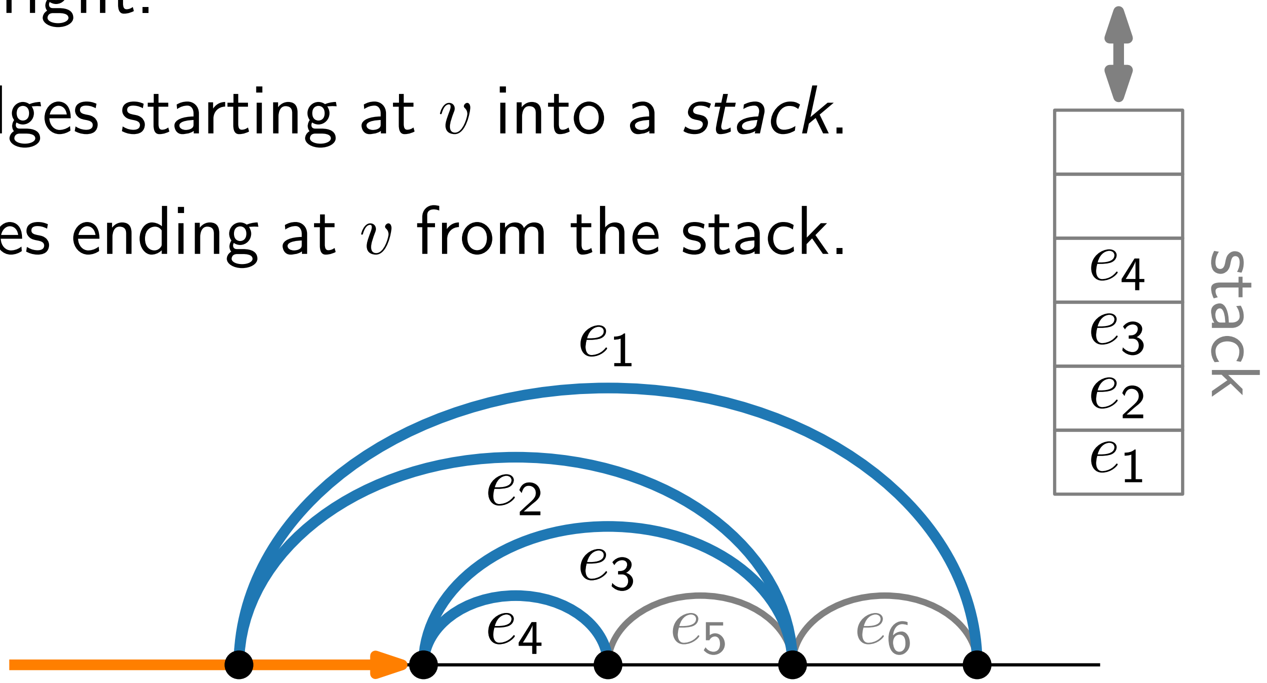

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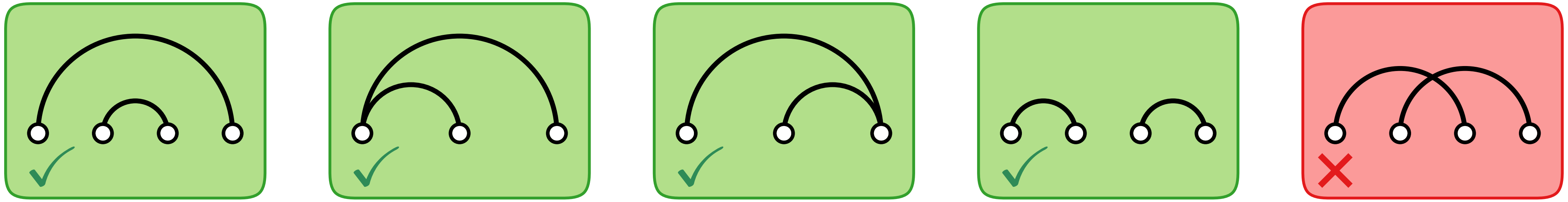

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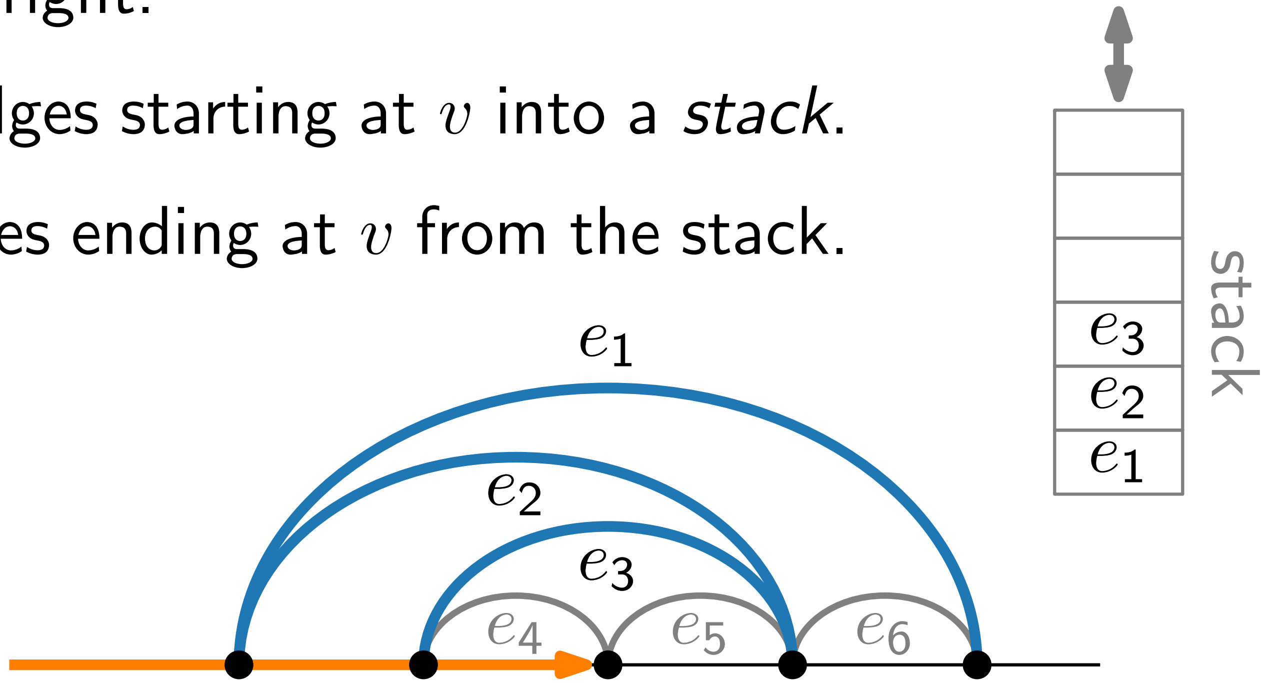

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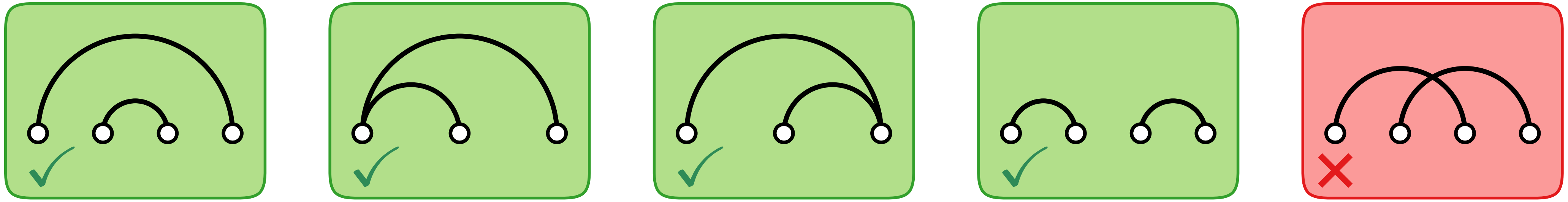

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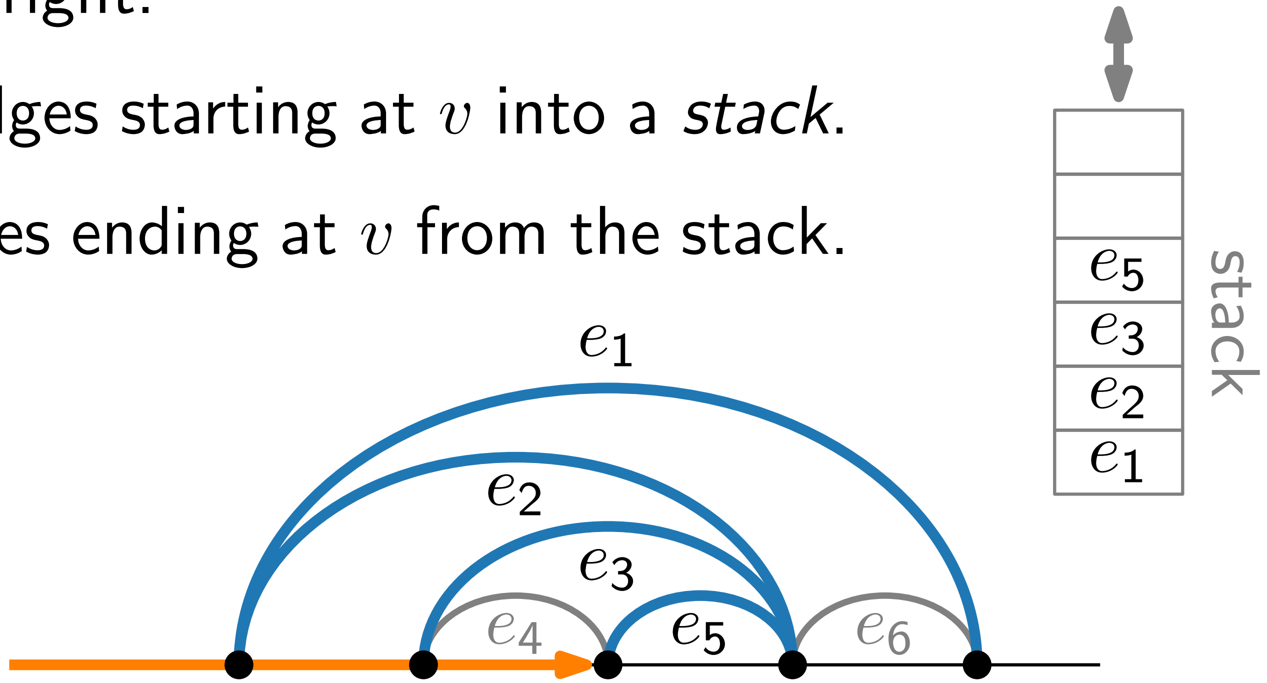

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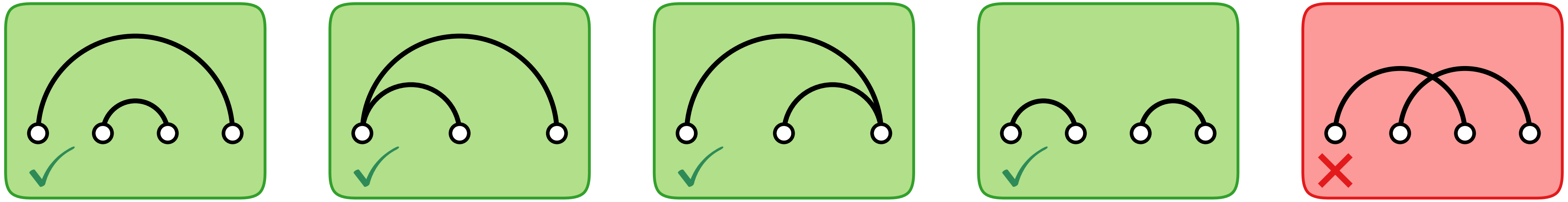

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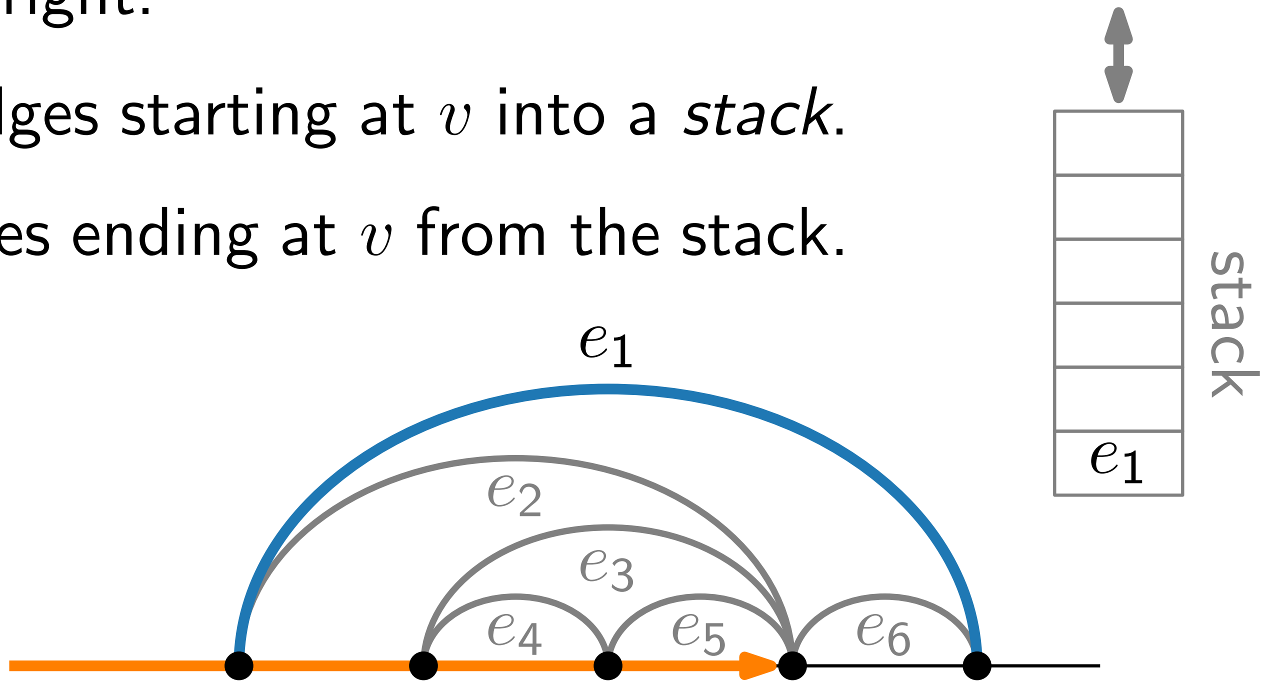

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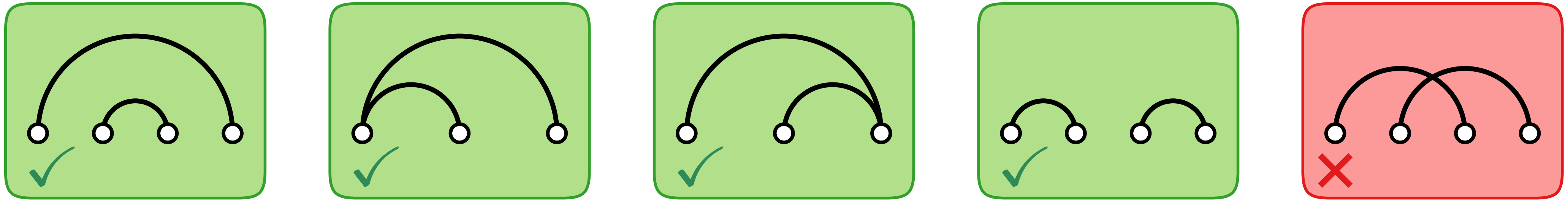

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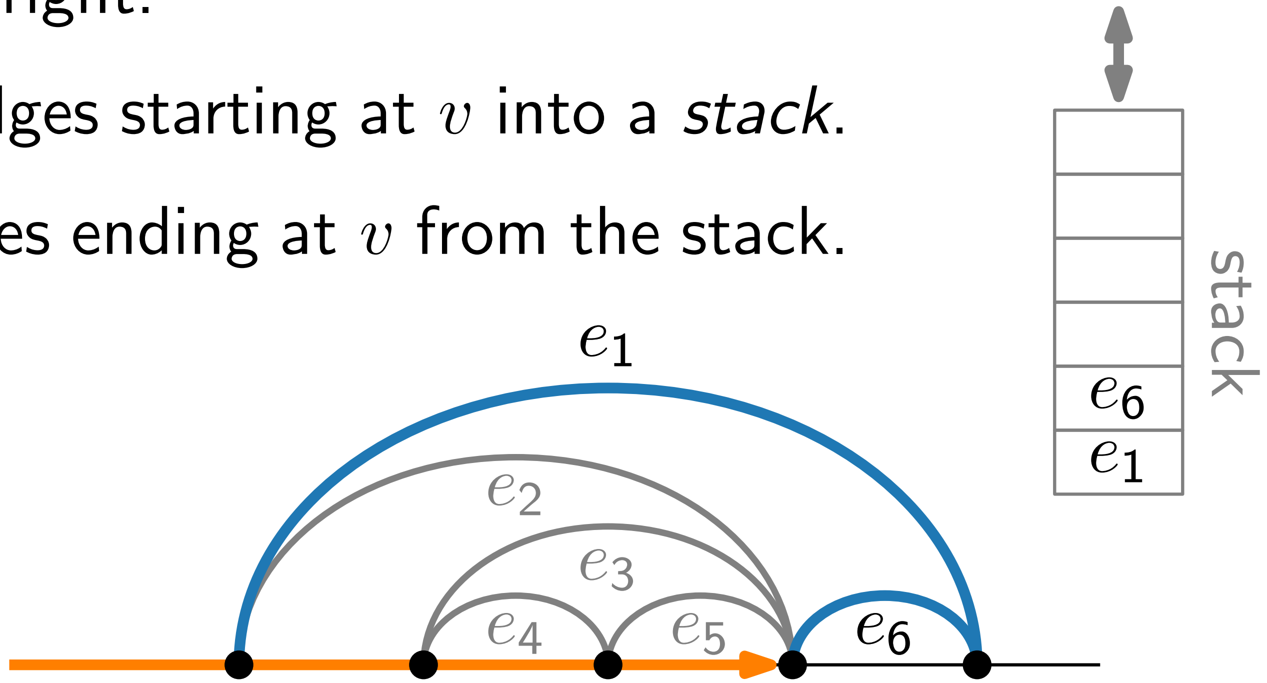

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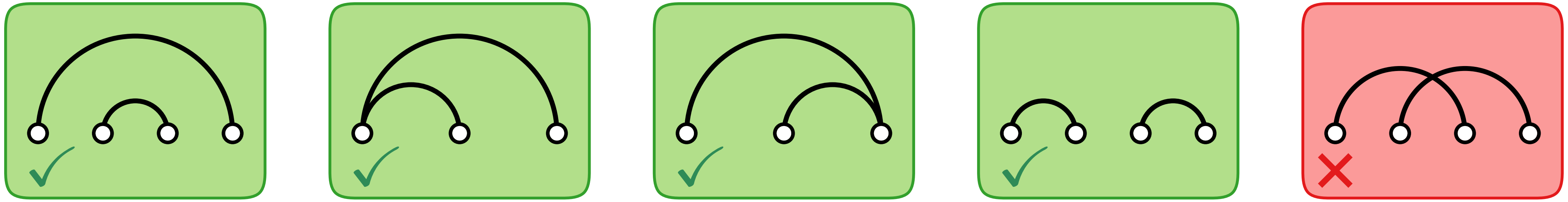

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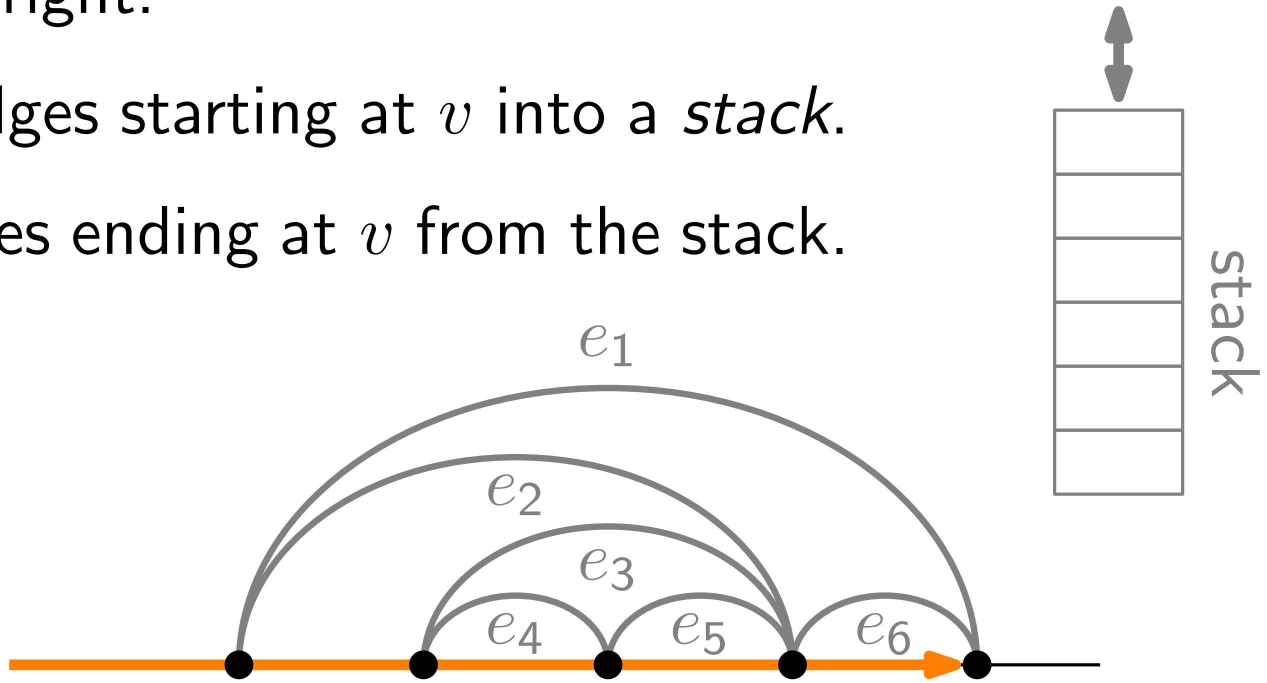

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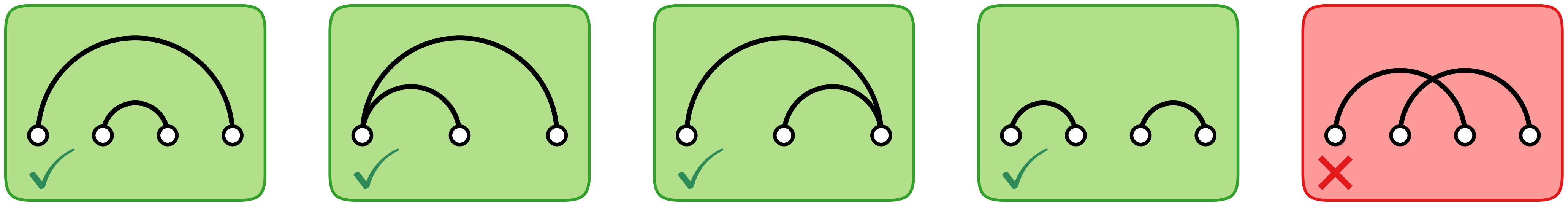

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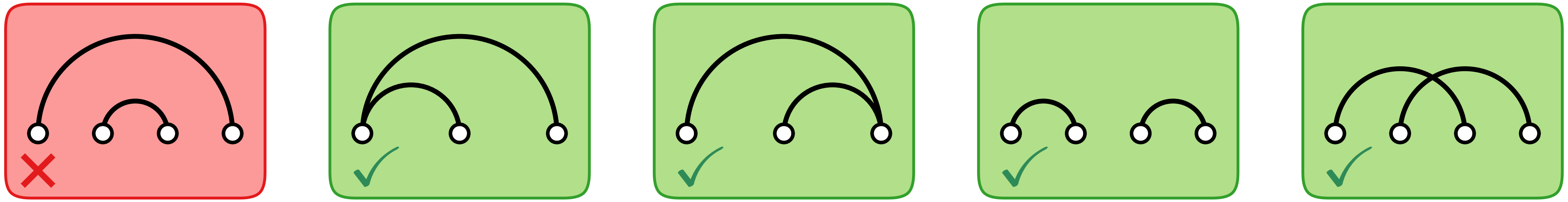

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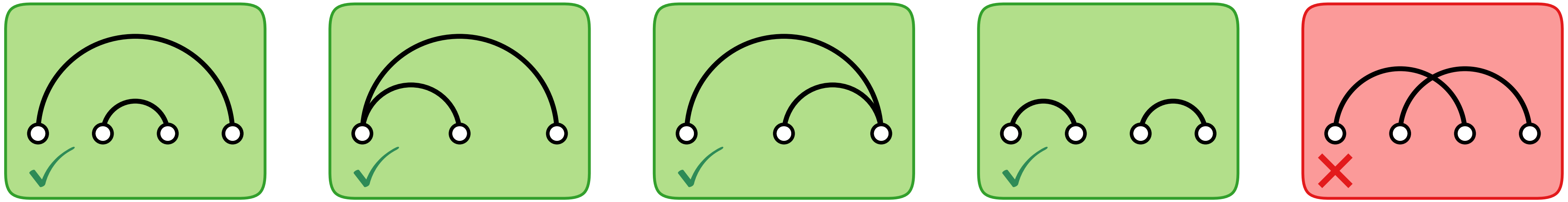

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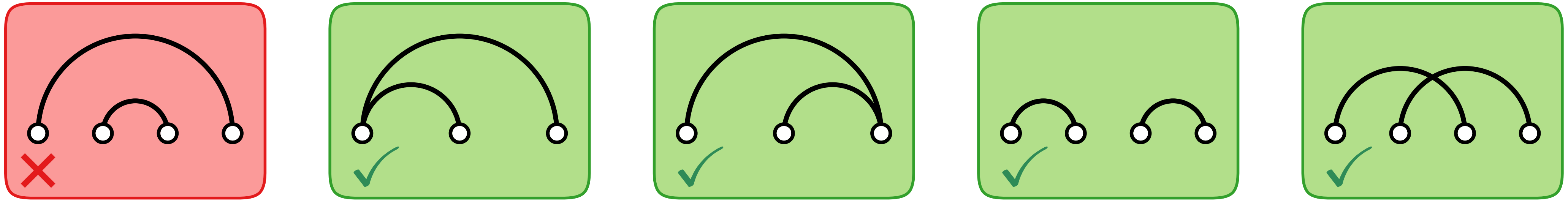

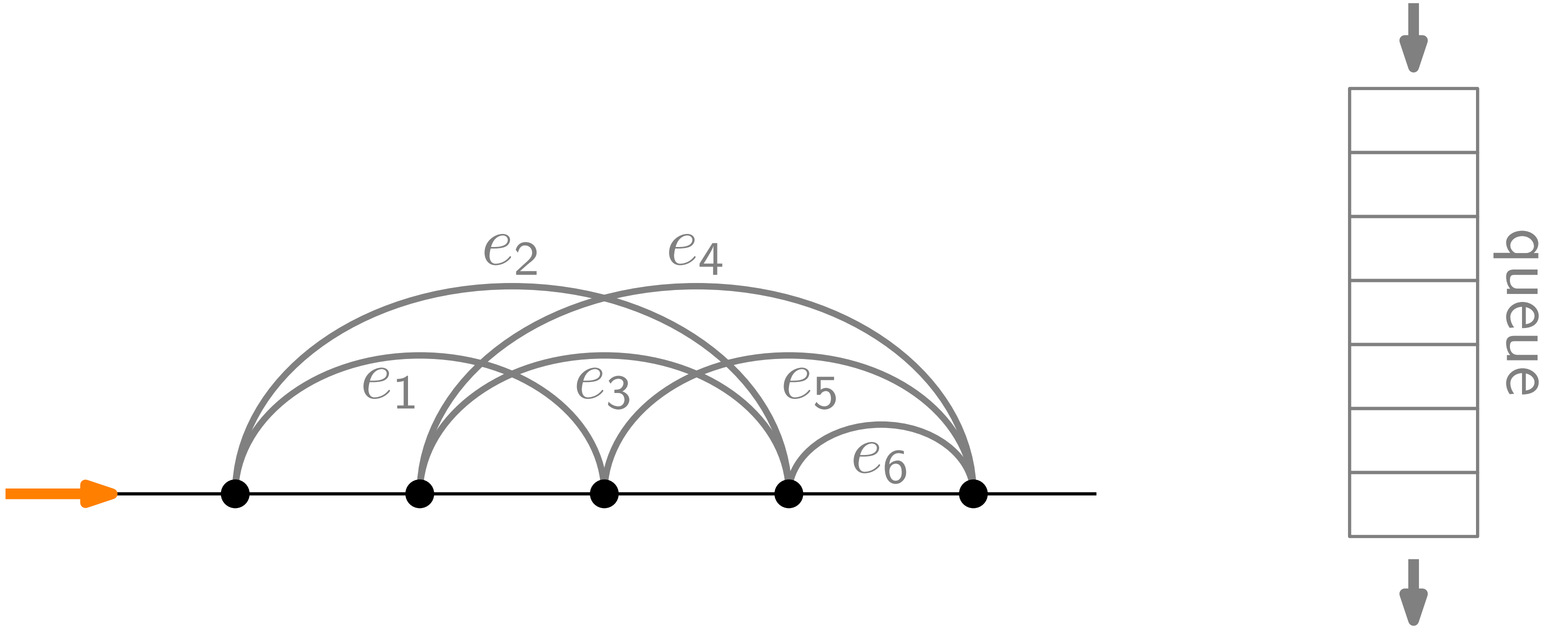

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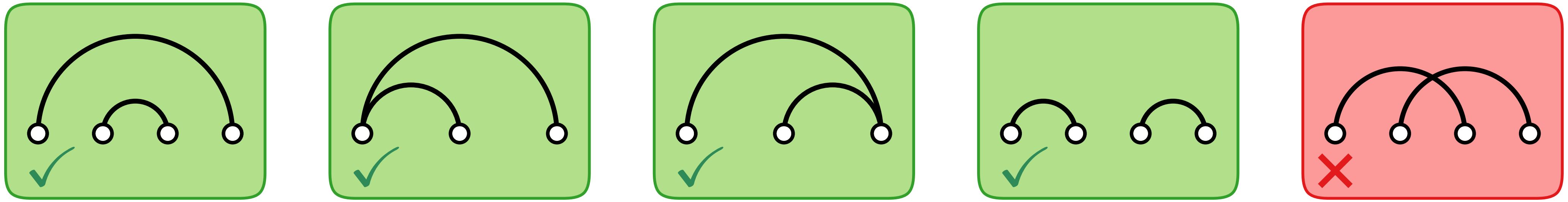

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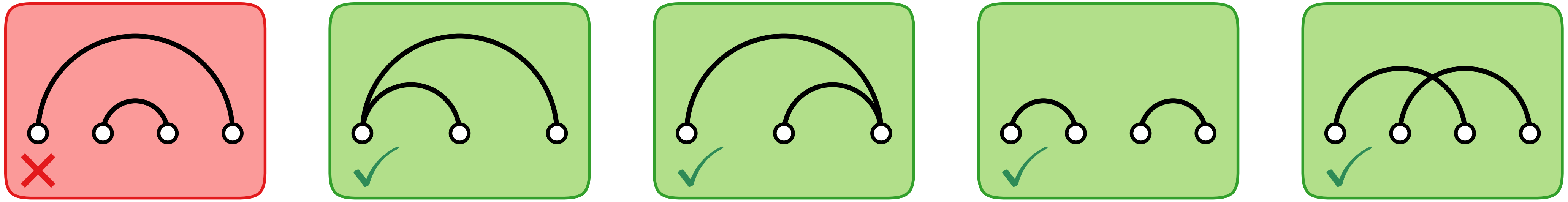

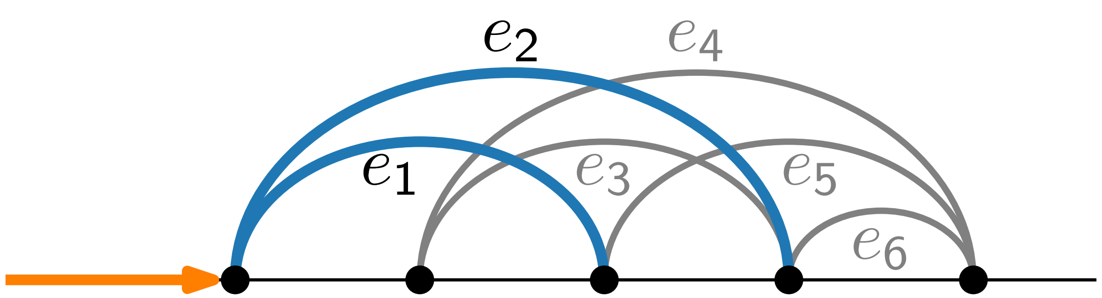

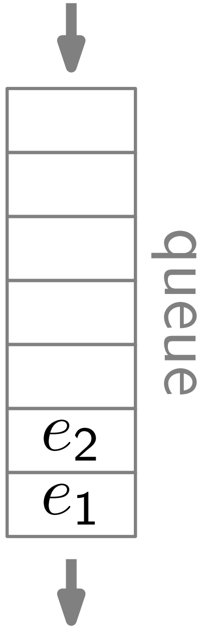

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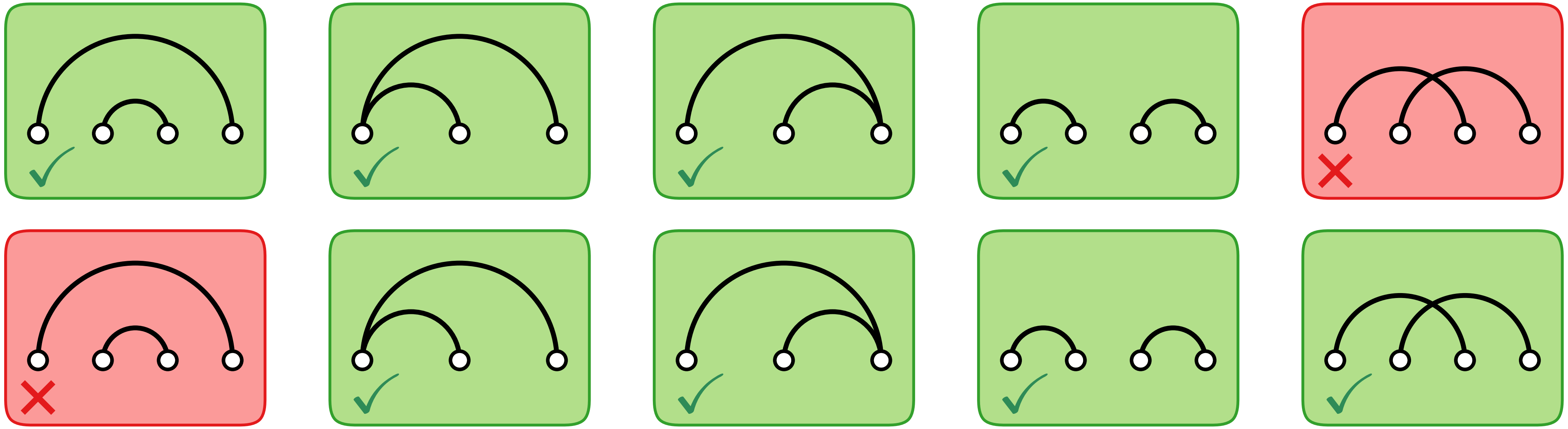

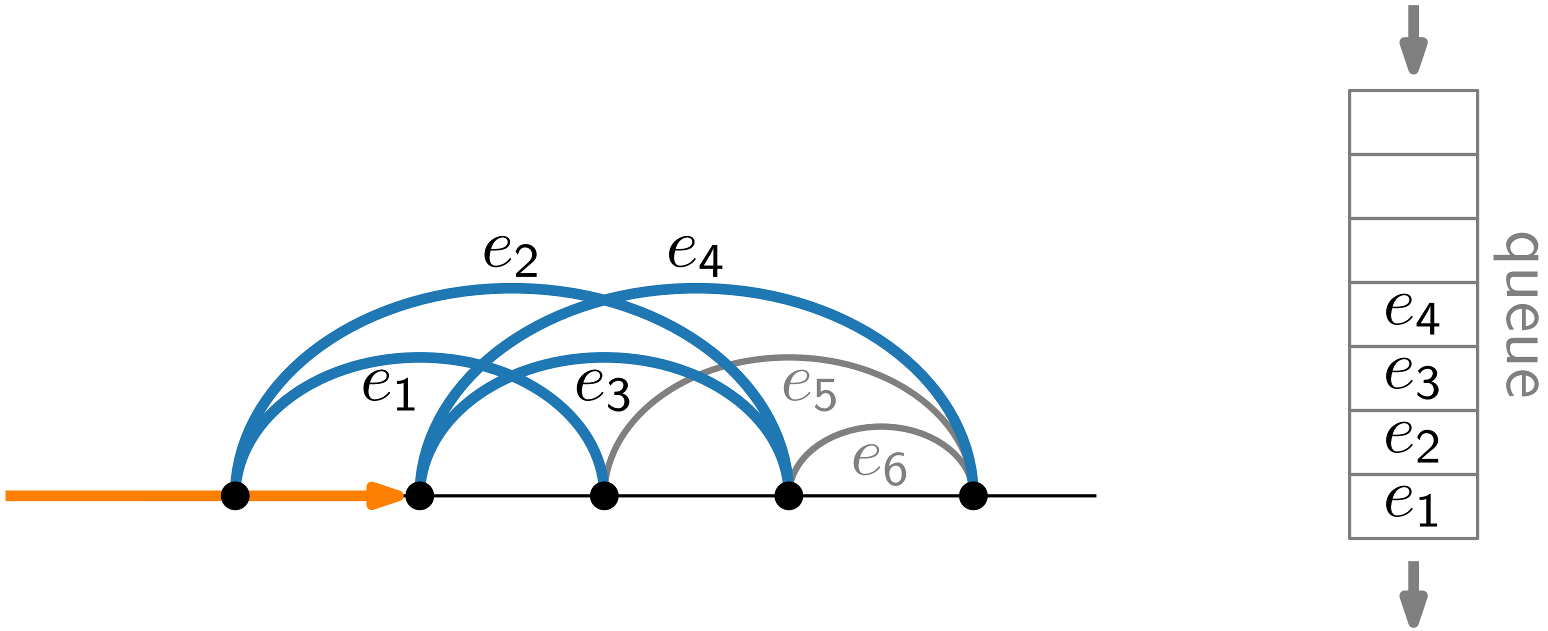

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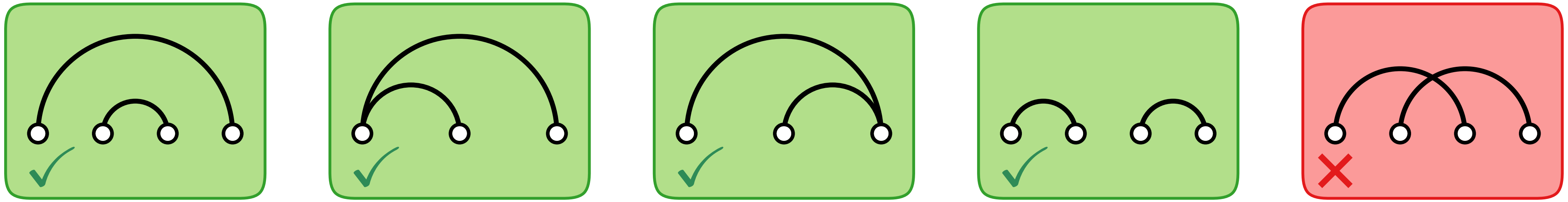

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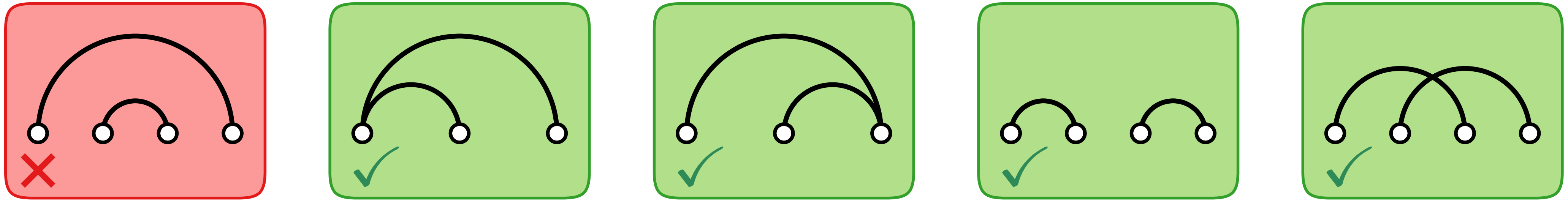

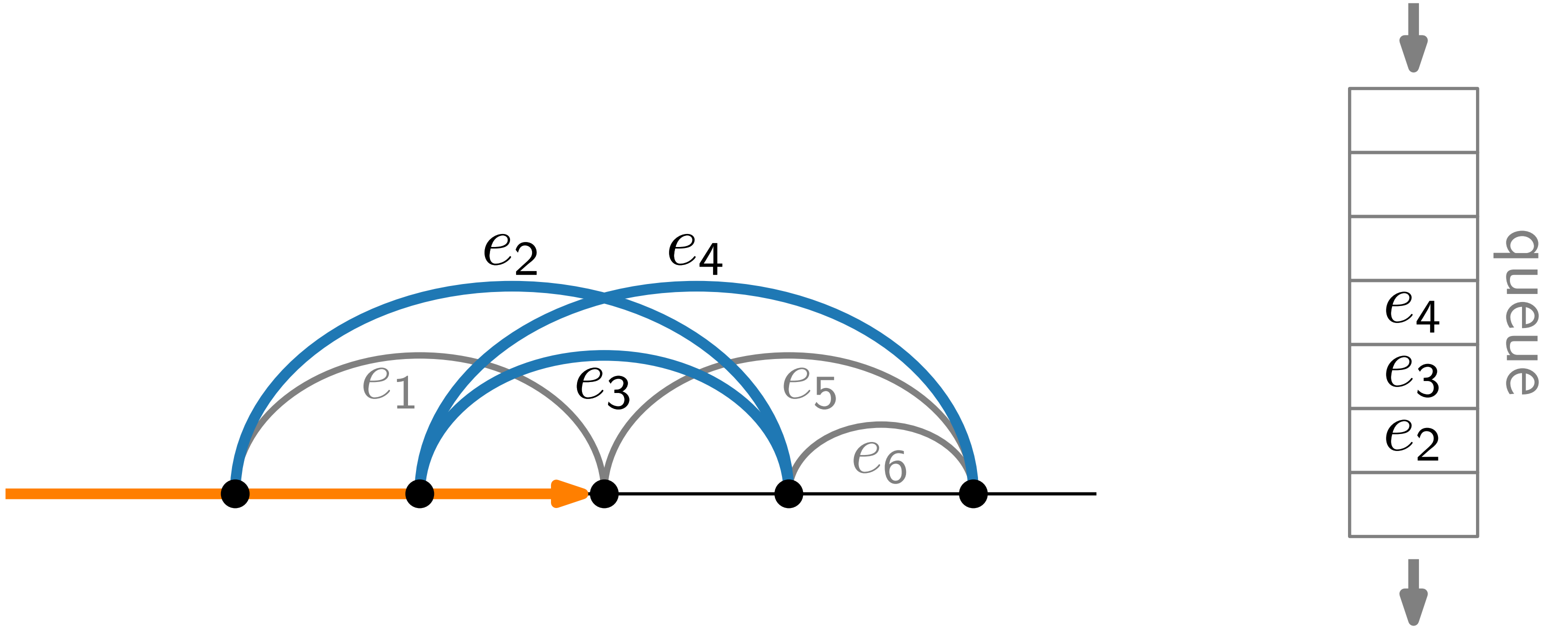

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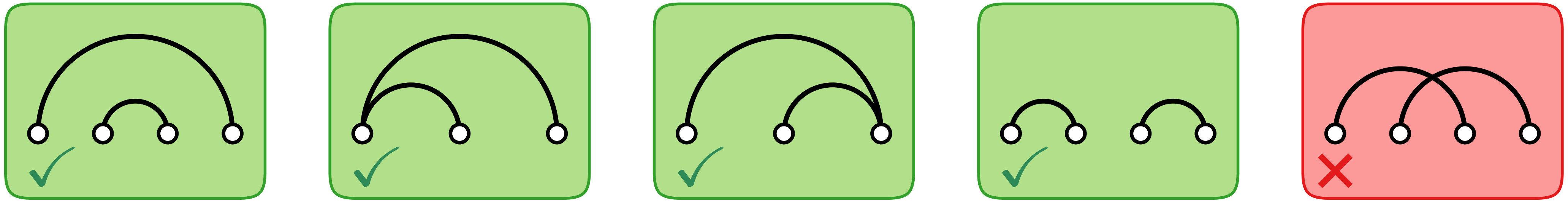

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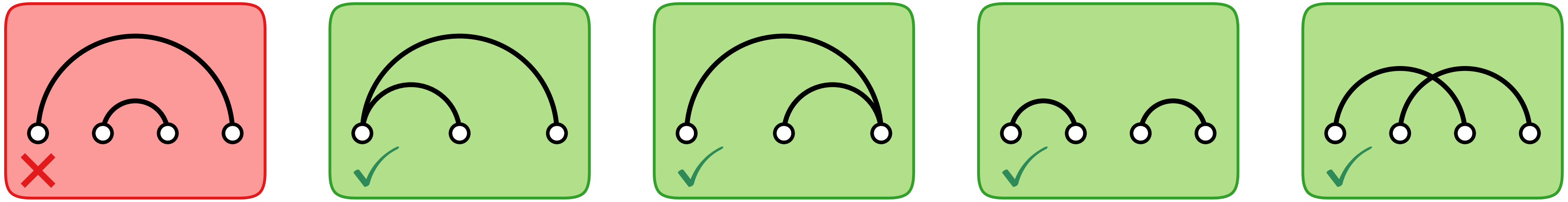

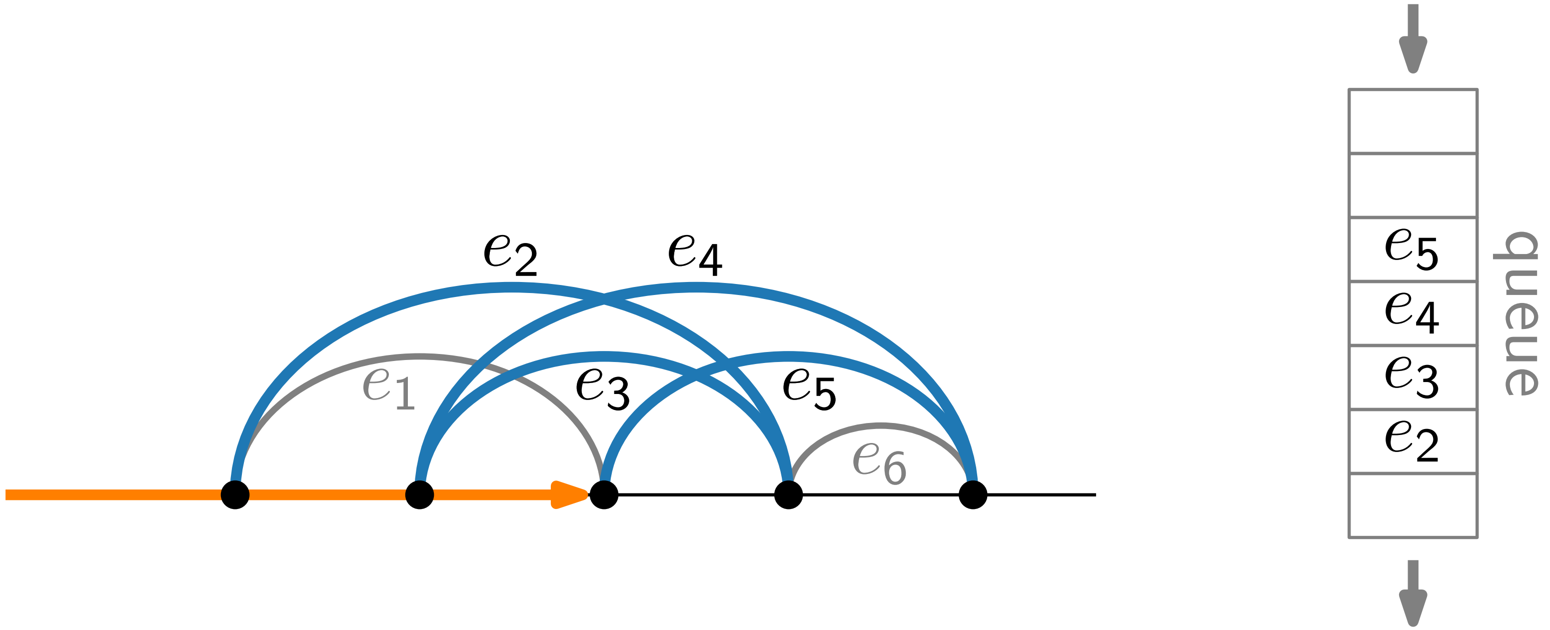

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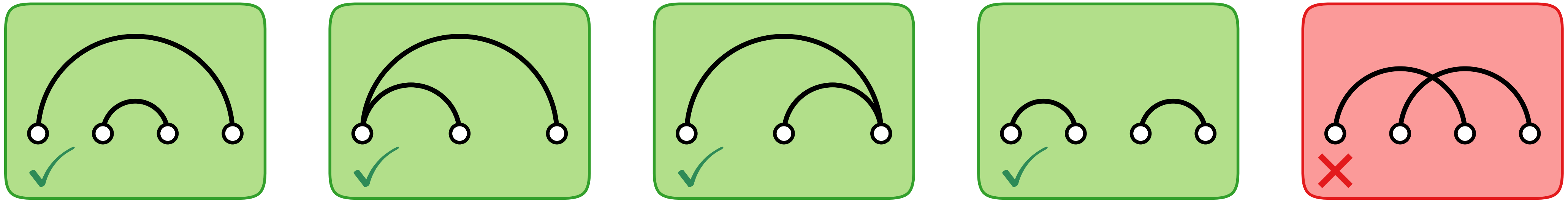

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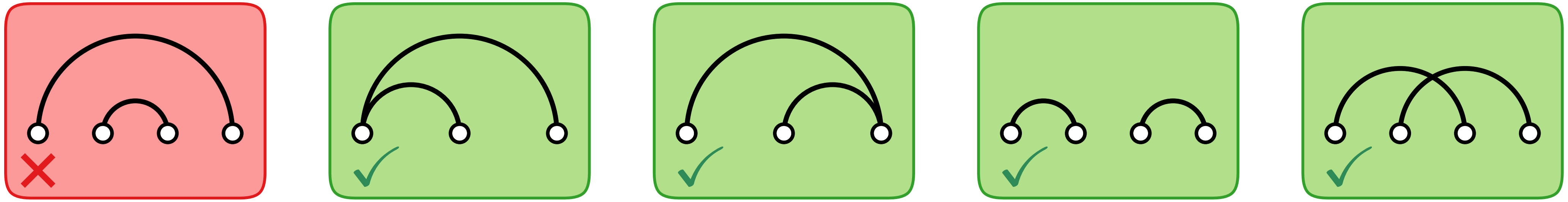

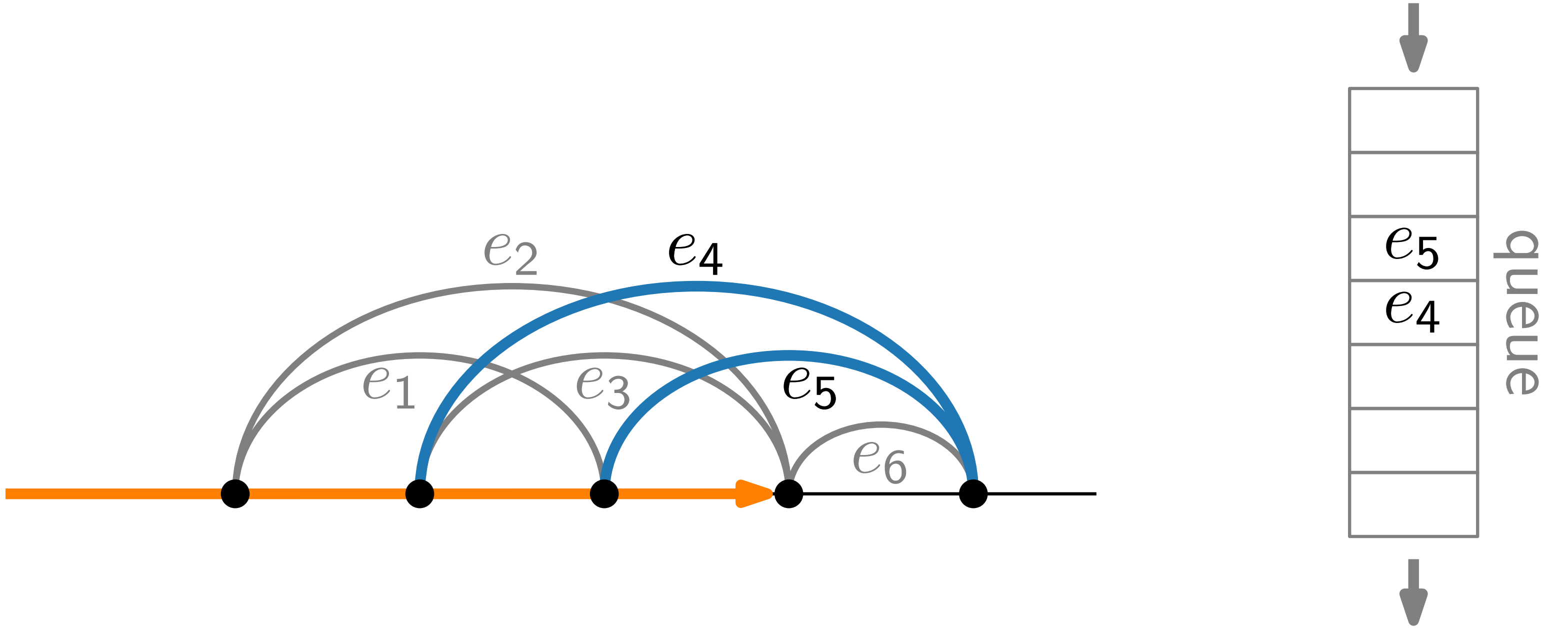

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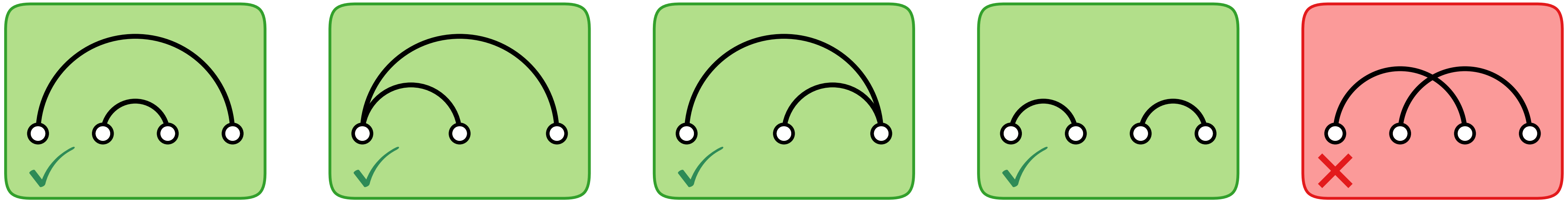

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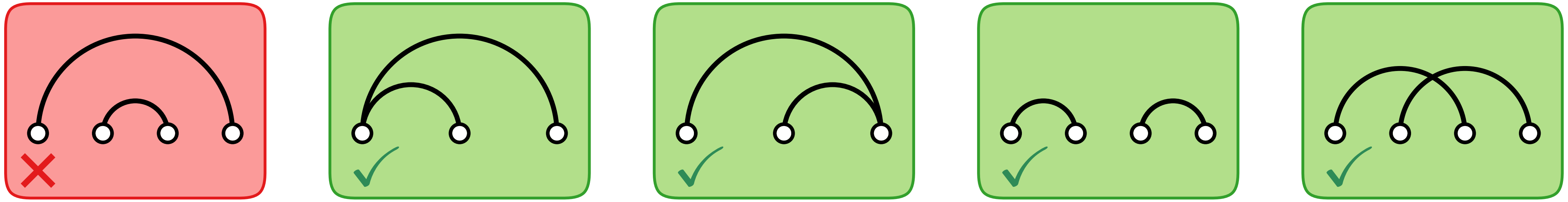

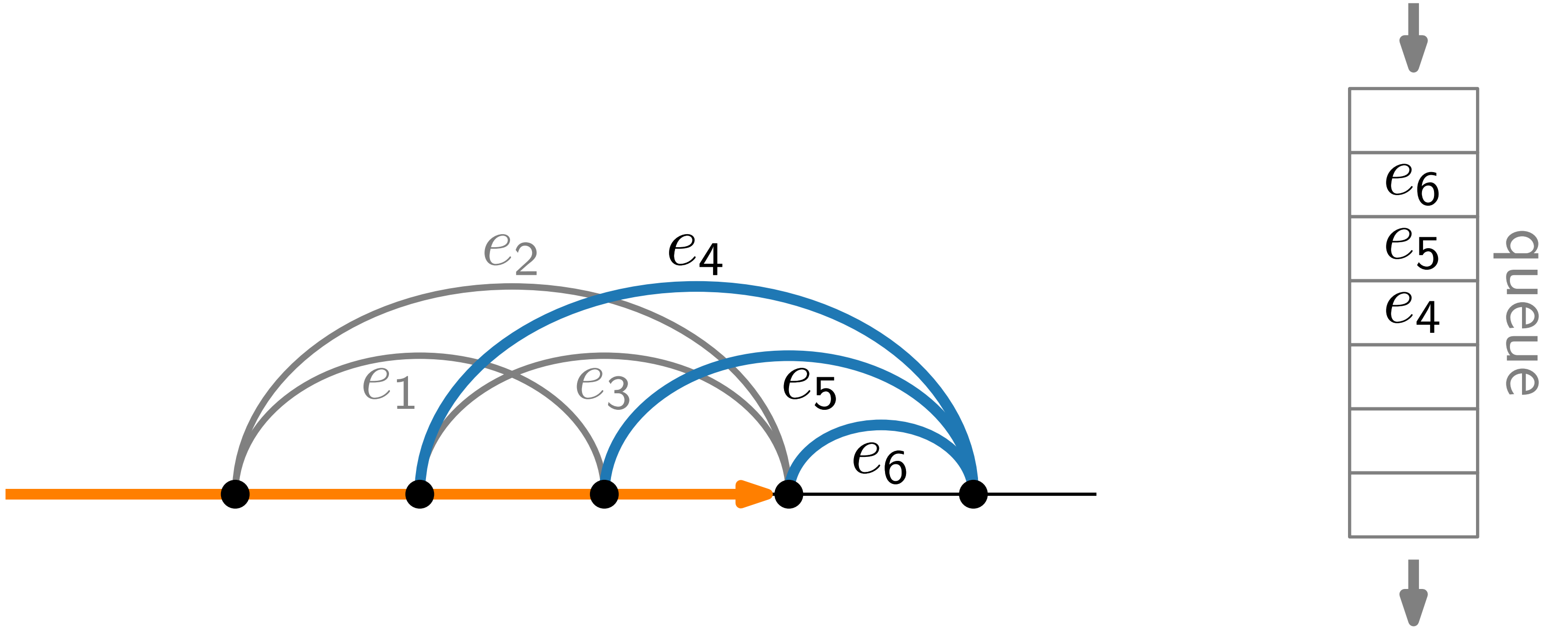

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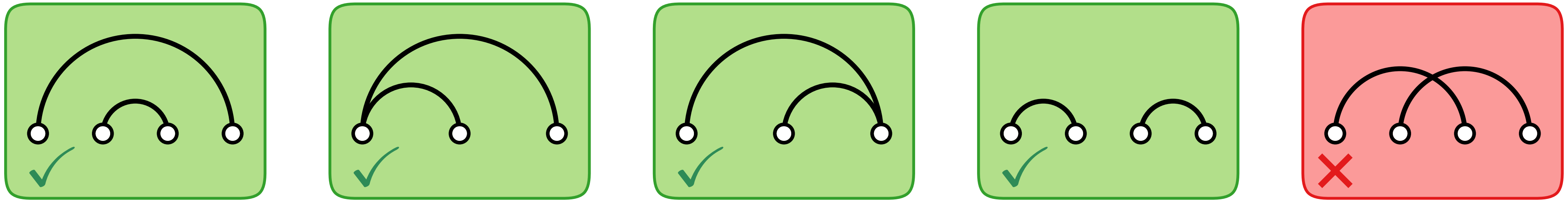

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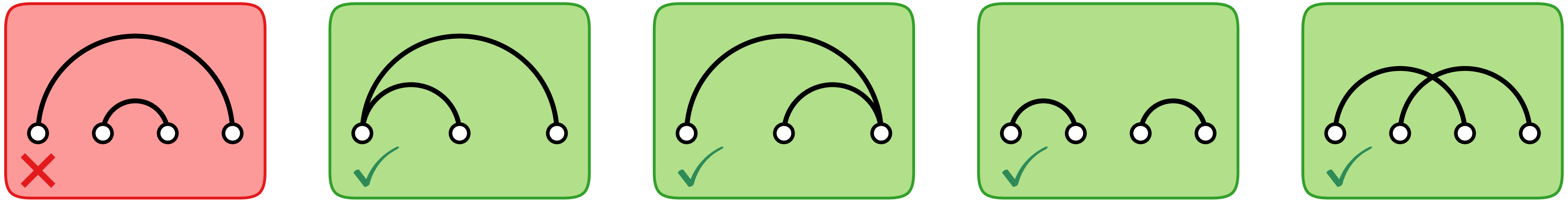

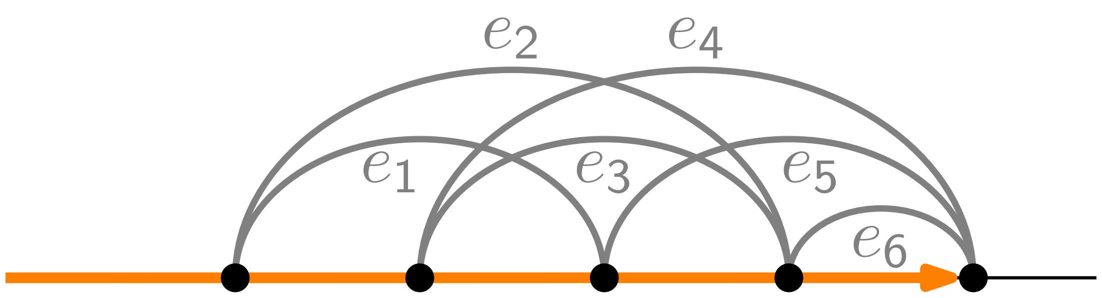

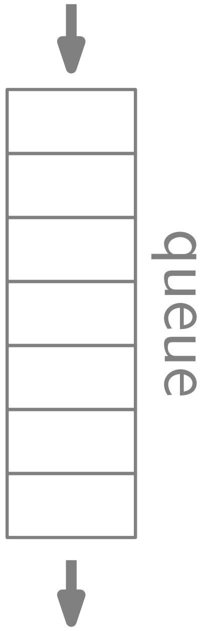

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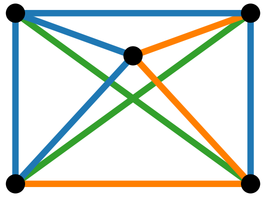

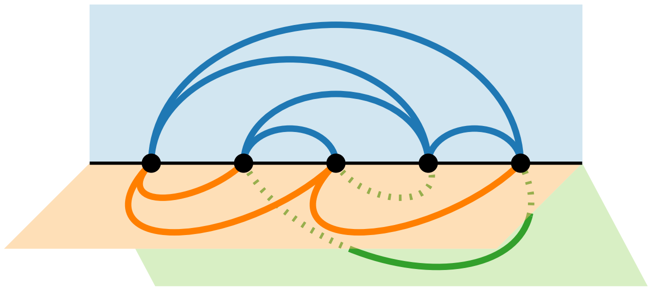

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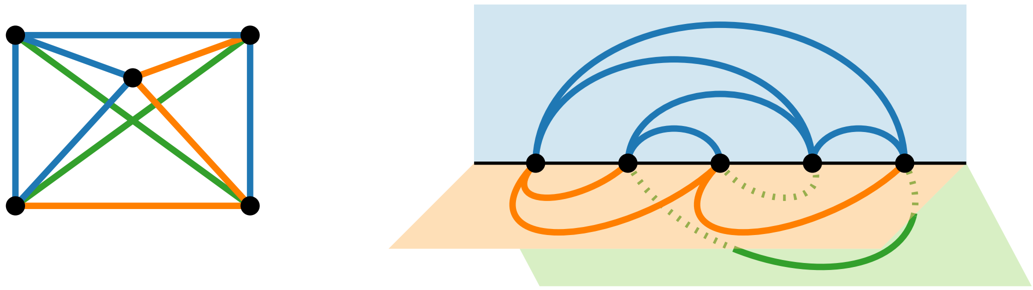

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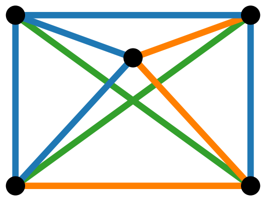

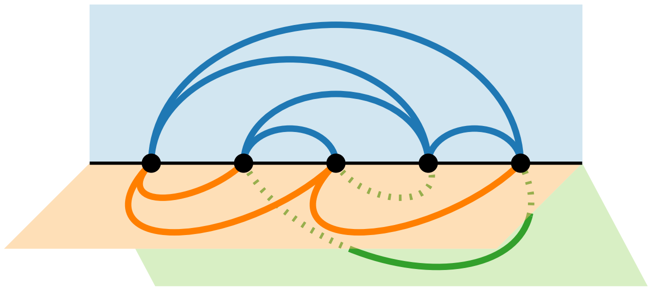

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- Does K_5 have a 2-page stack layout?

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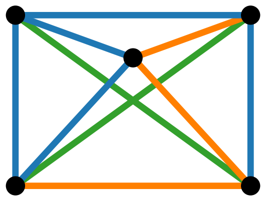

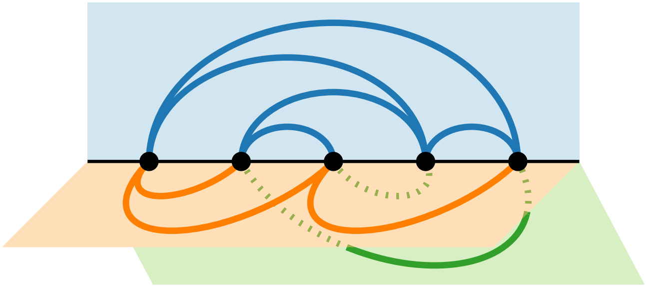

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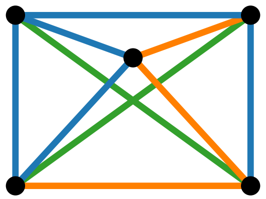

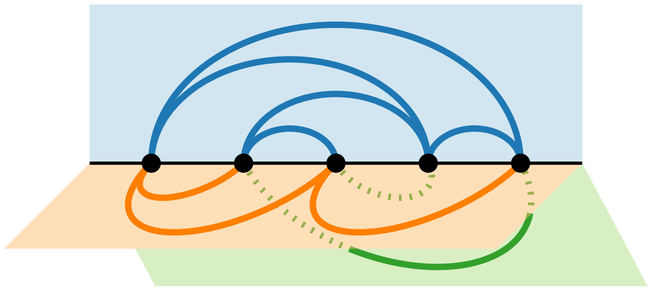

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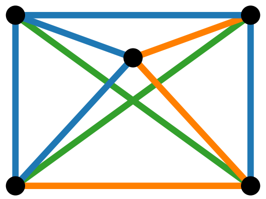

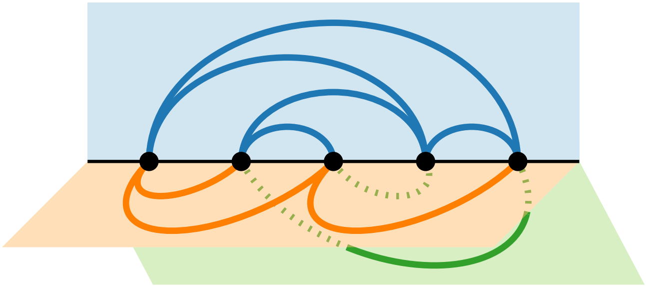

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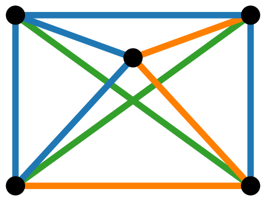

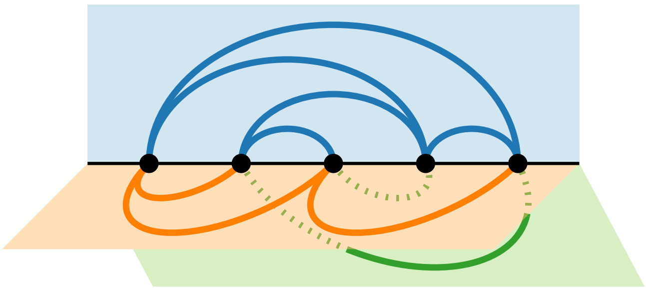

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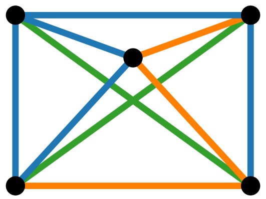

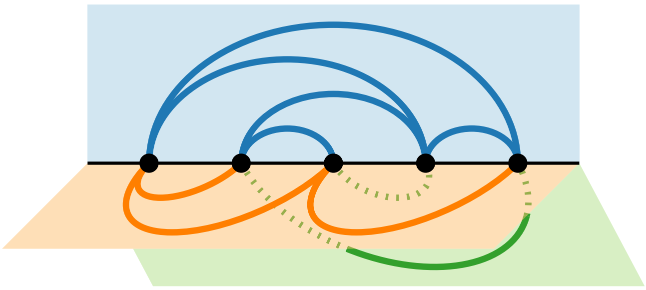

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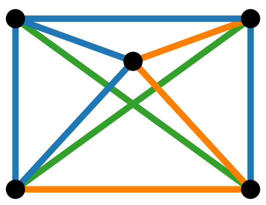

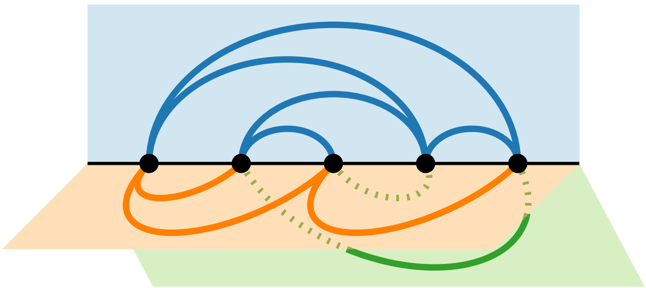

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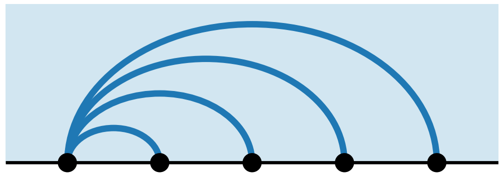

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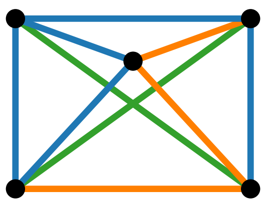

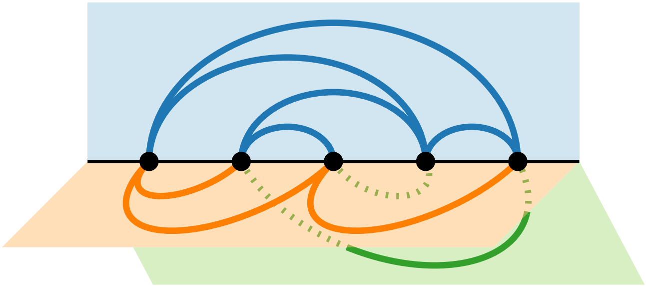

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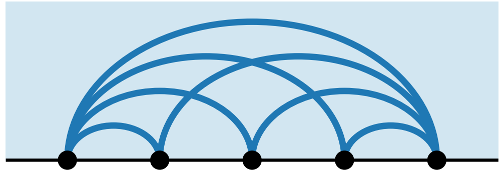

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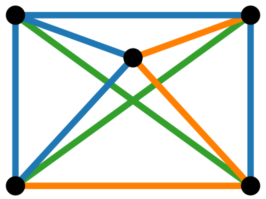

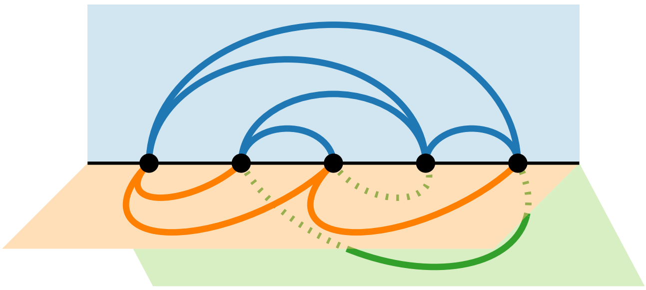

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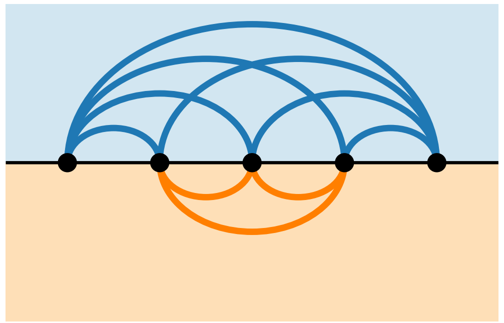

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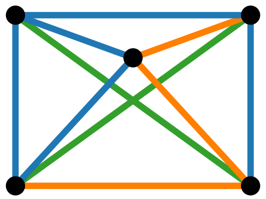

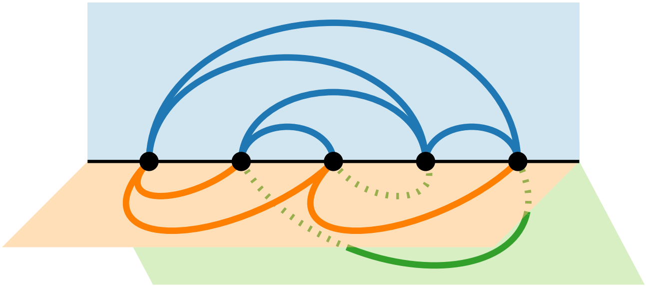

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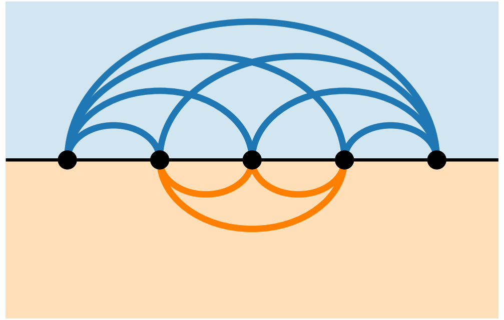

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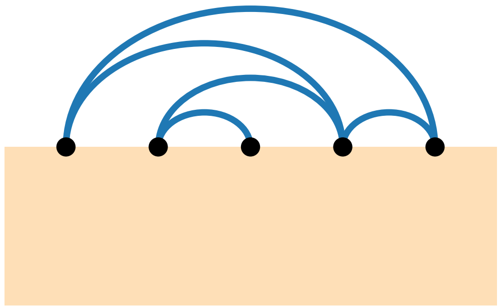

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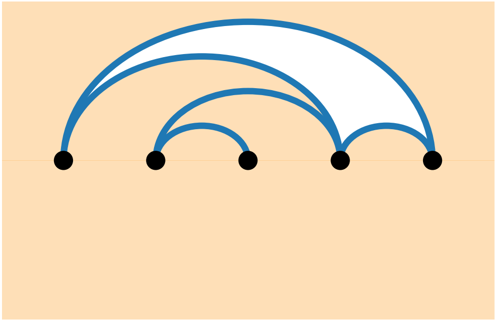

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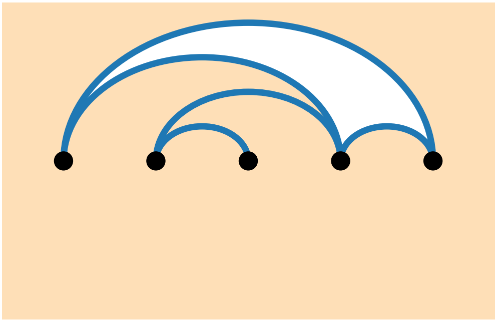

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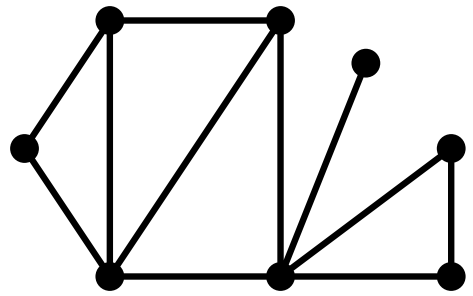

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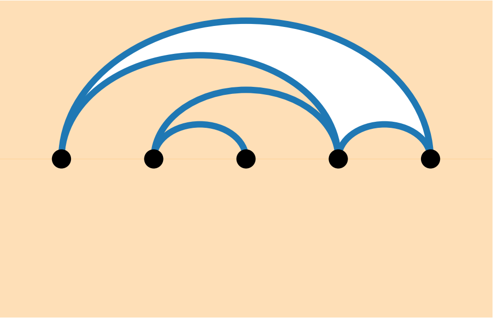

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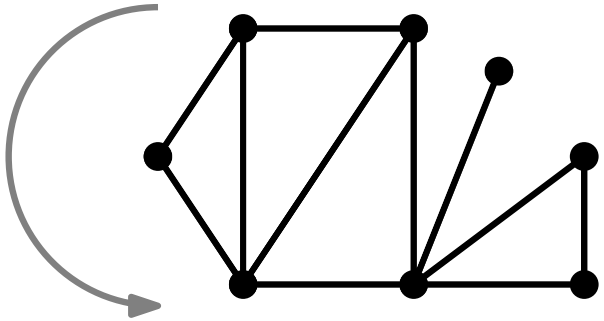

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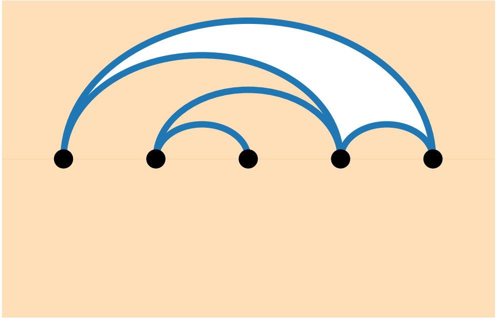

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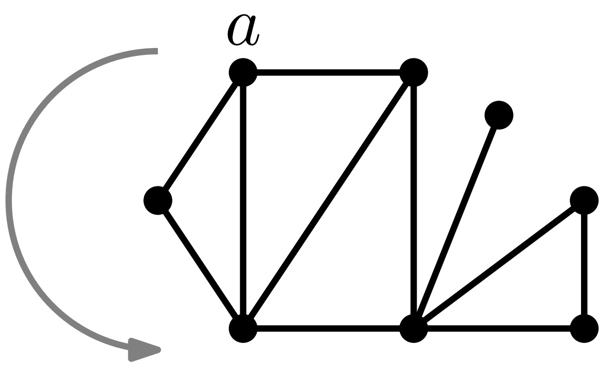

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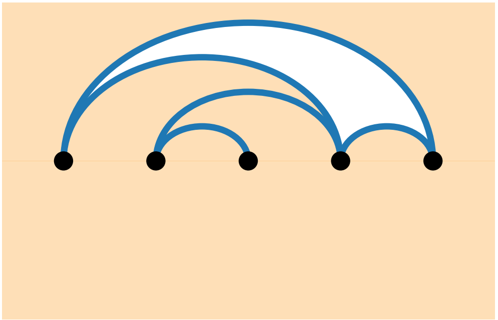

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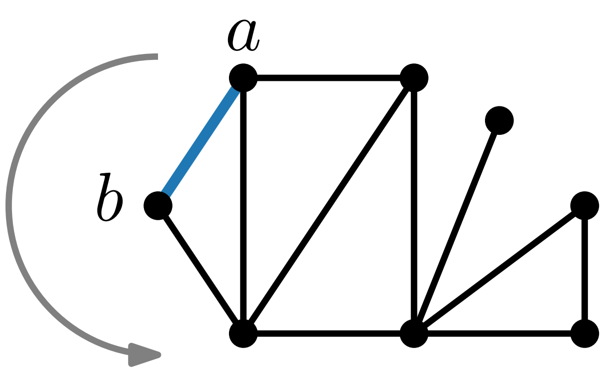

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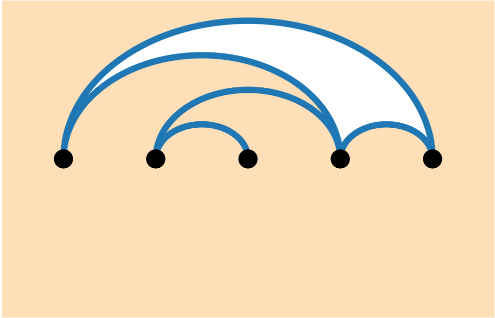

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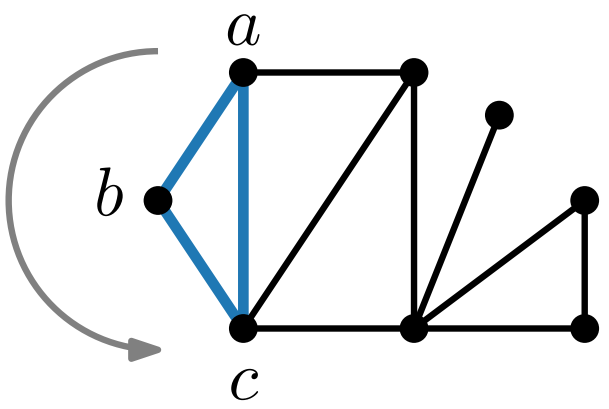

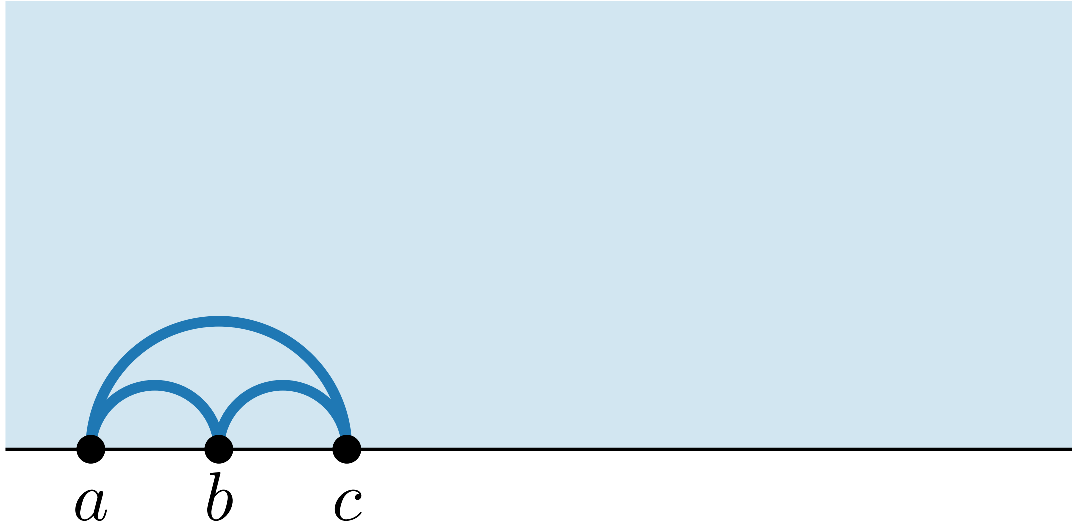

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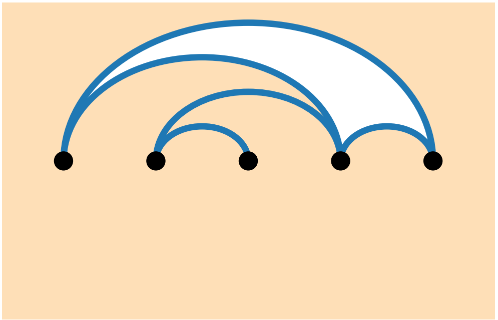

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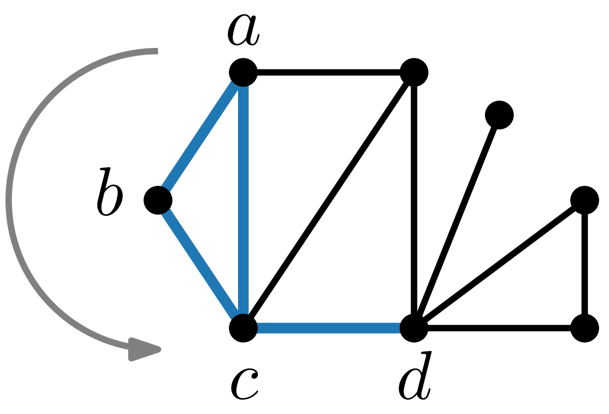

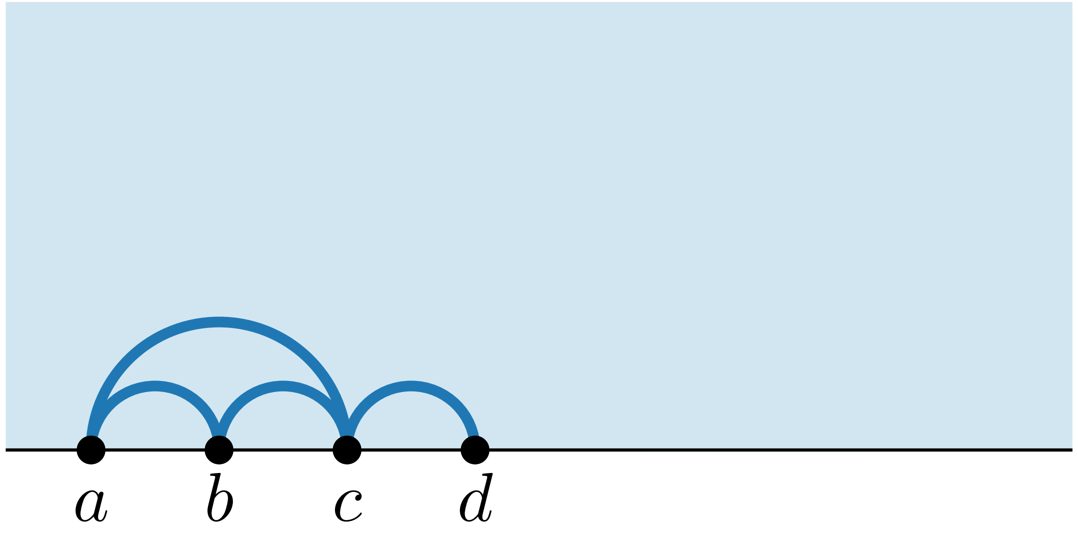

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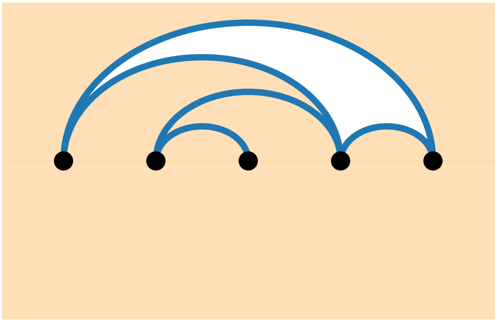

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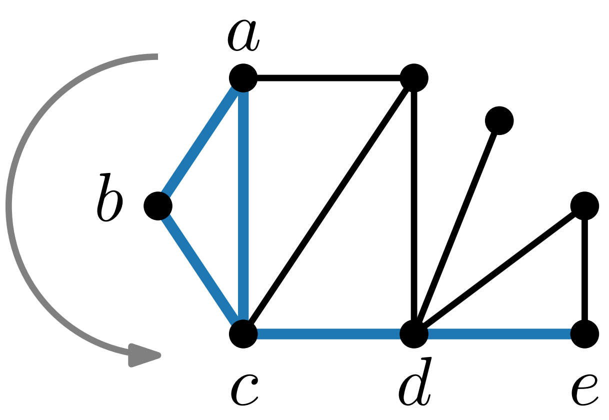

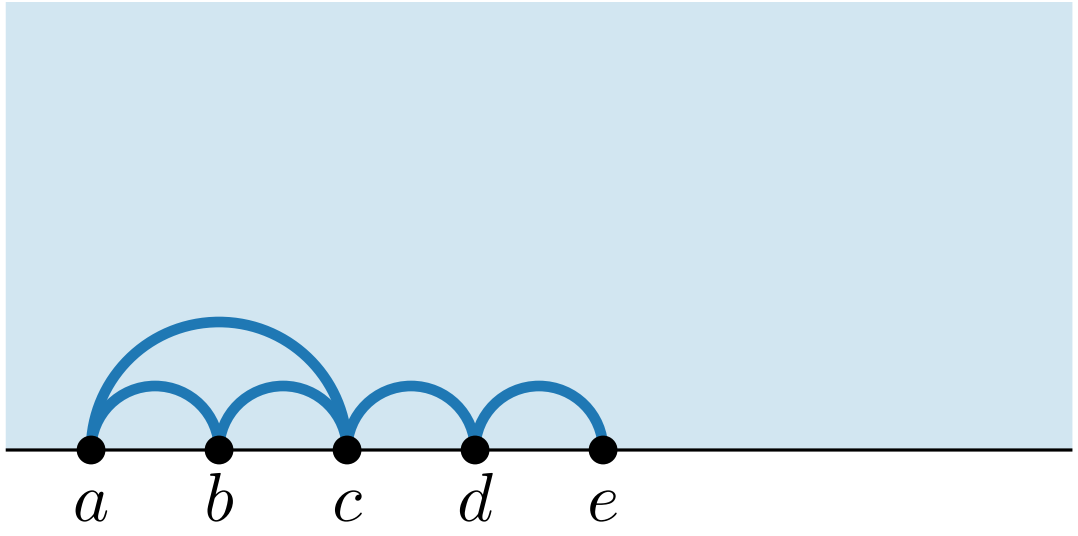

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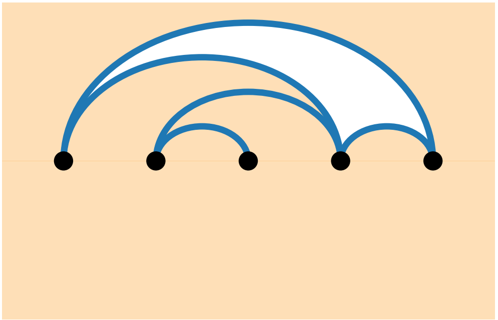

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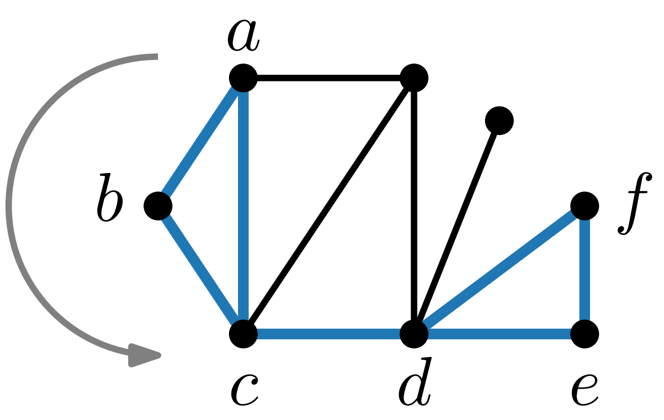

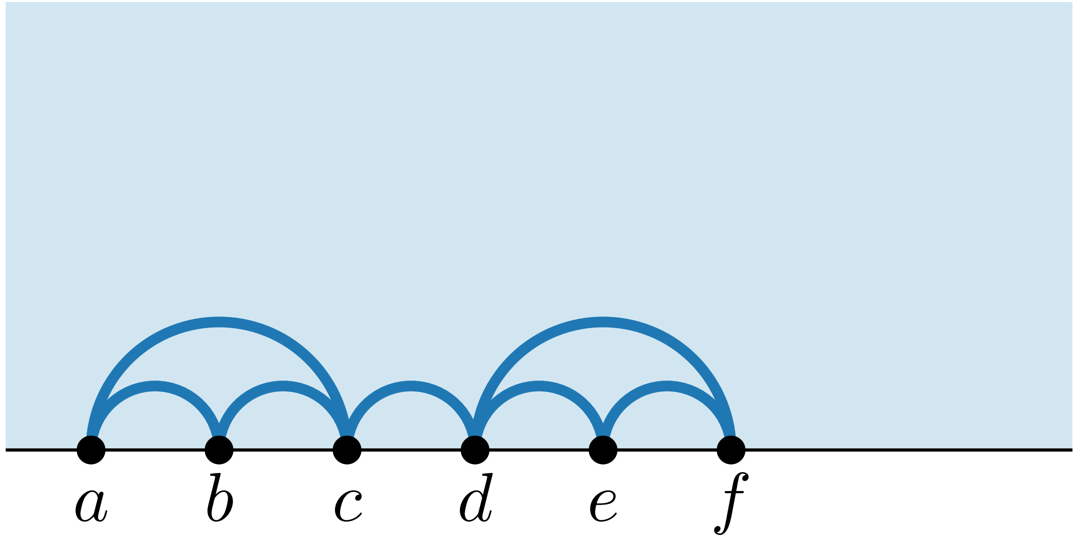

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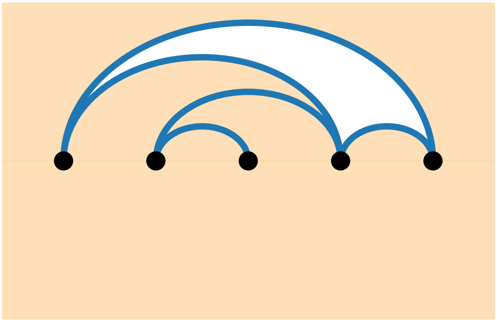
For a graph G holds: sn$(G) = 1 \Leftrightarrow G$ is outerplanar

Recall: a graph is outerplanar if it has a planar drawing where every vertex is incident to the outer face.



Proof Idea.

"$\Rightarrow$": Clearly, a 1-page stack layout can be perceived as a planar drawing where the vertices lie at the outer face.

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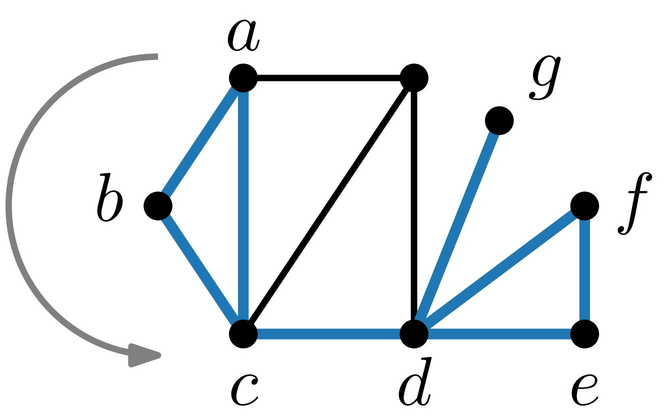

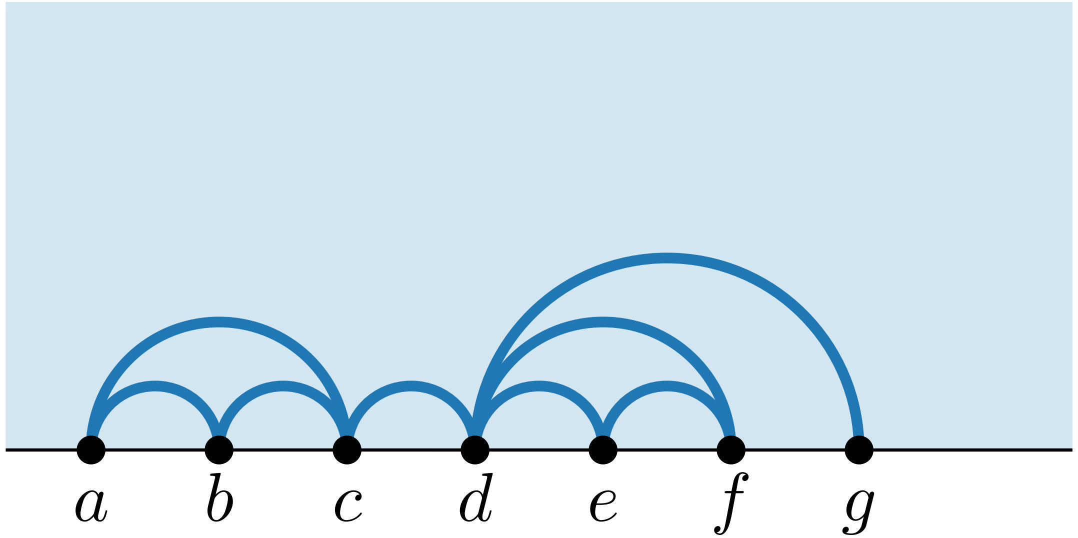

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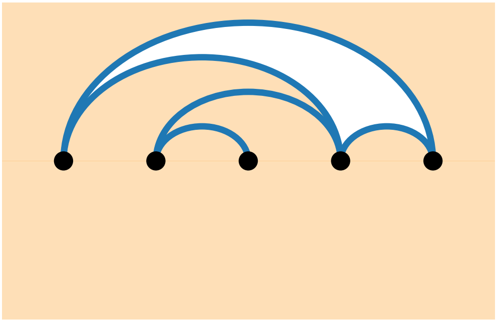

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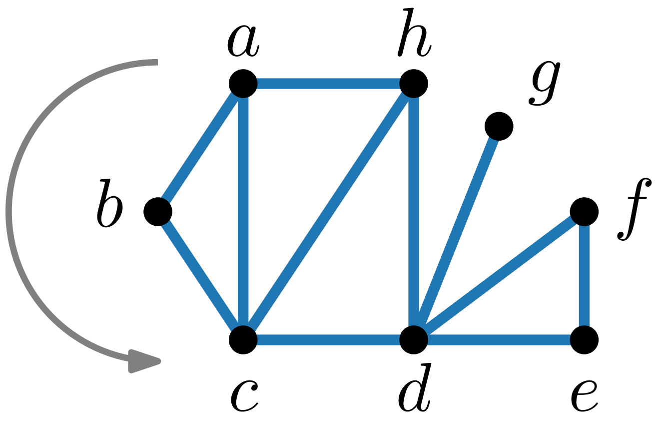

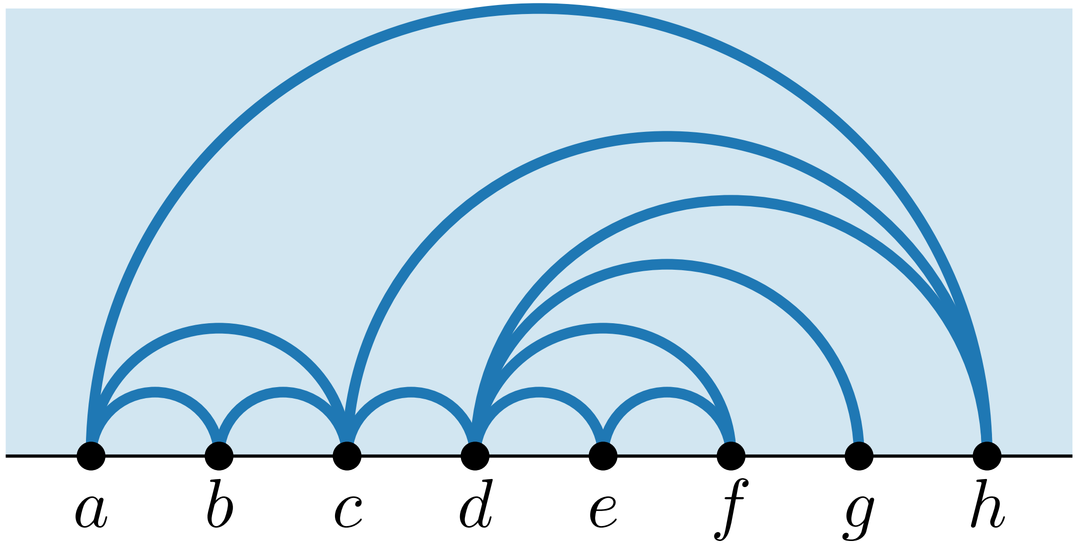

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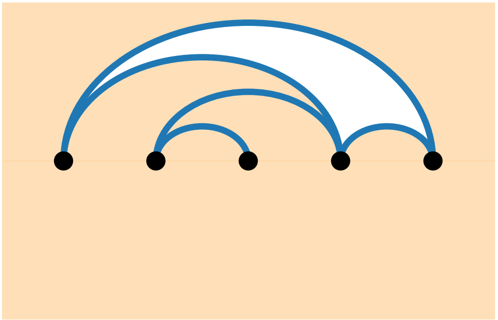

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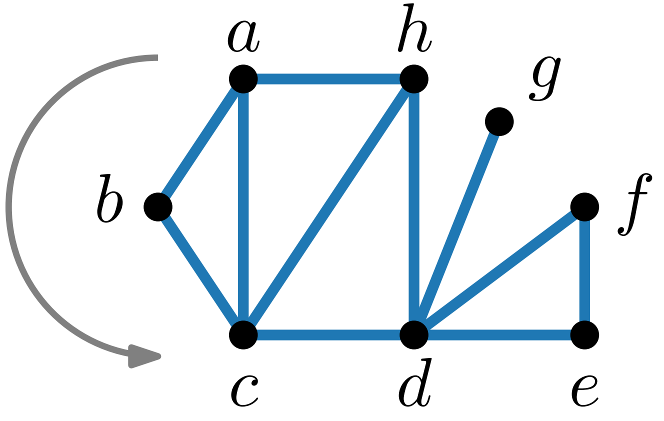

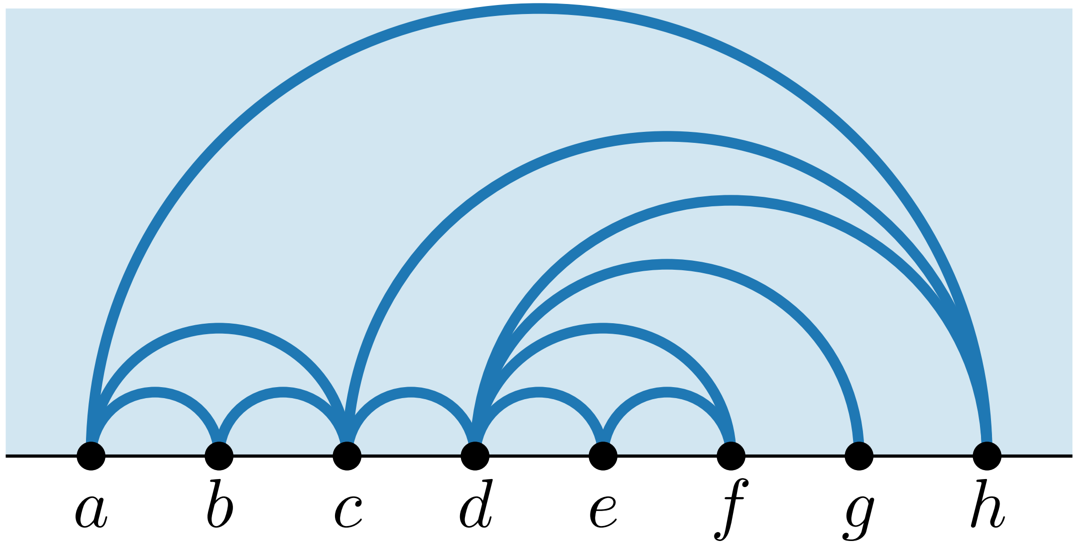

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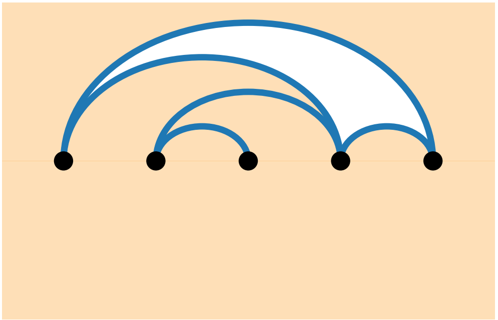

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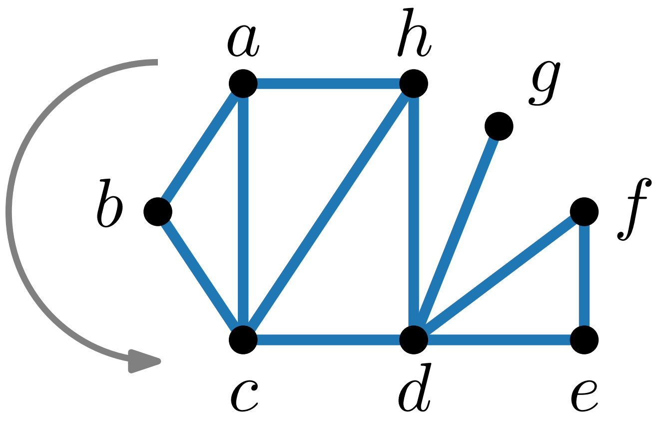

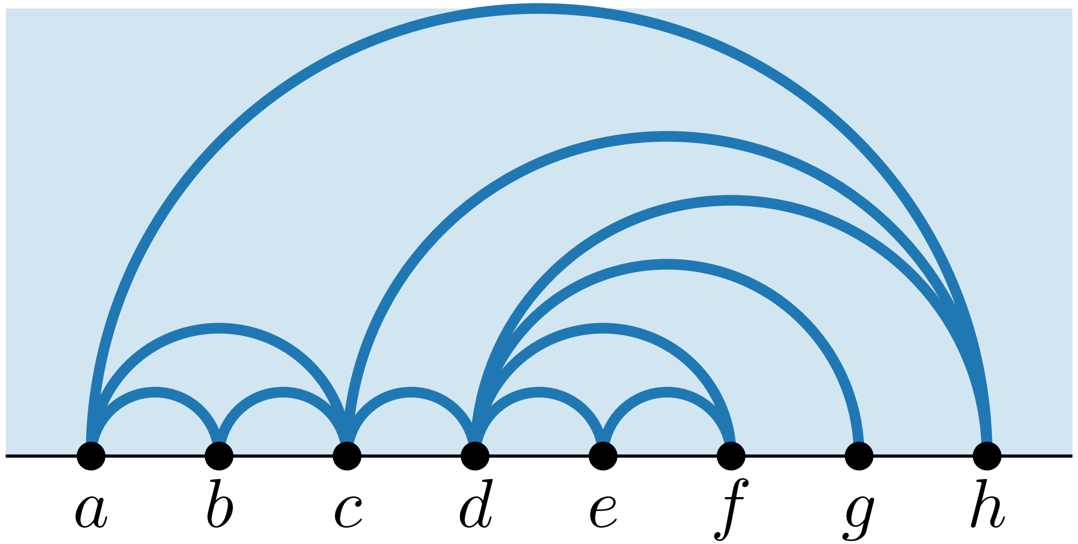

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We can think of "morphing" the one drawing into the other. $\square$

2-Page Stack Layouts

Theorem. [Bernhart & Kainen 1979]
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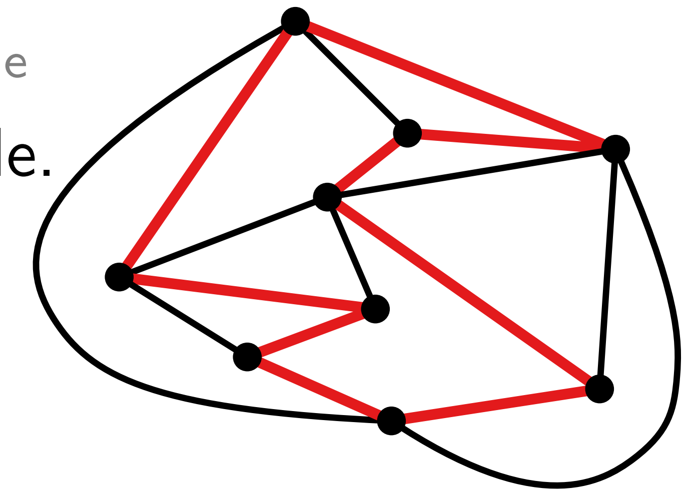

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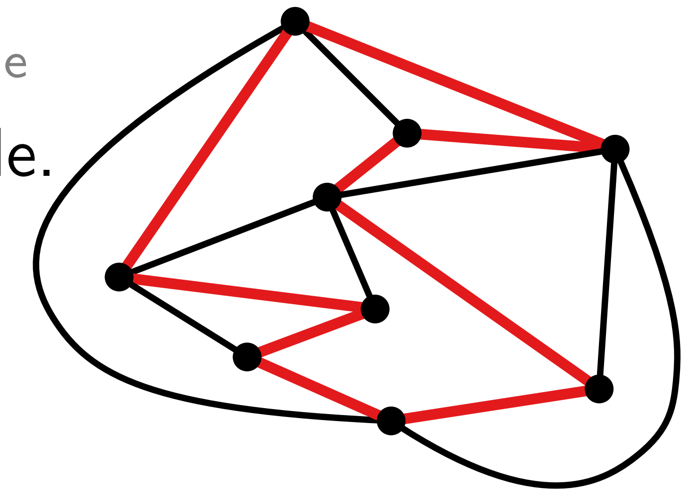

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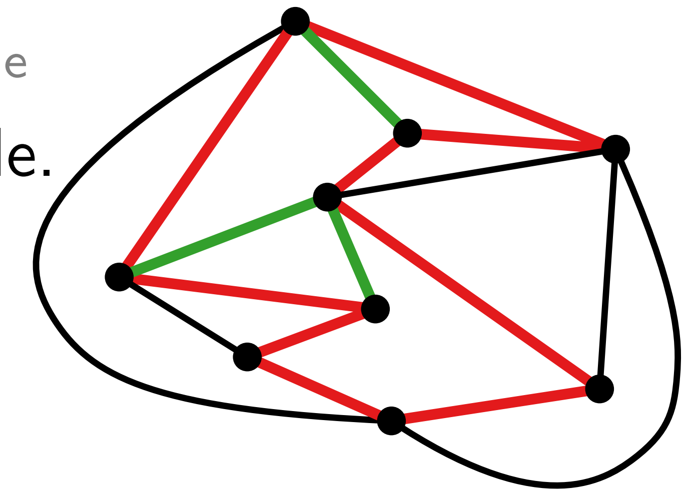

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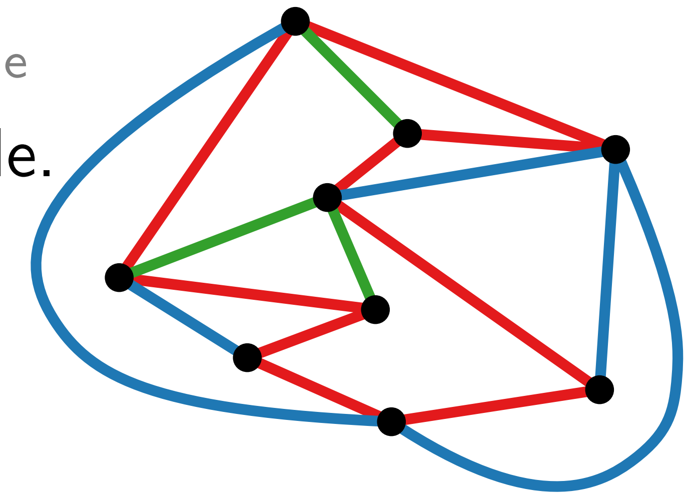

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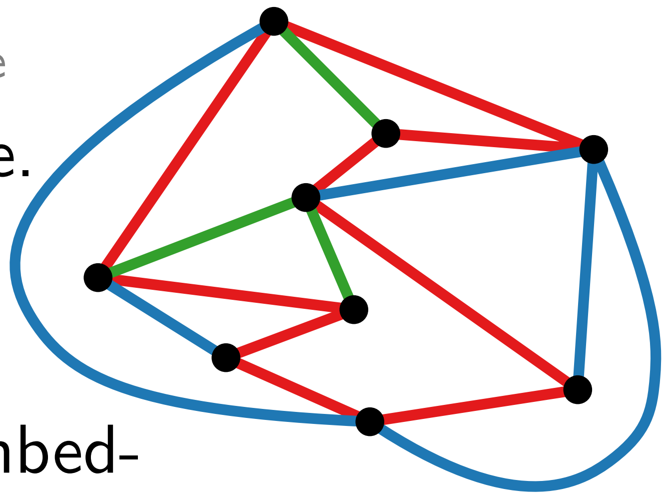

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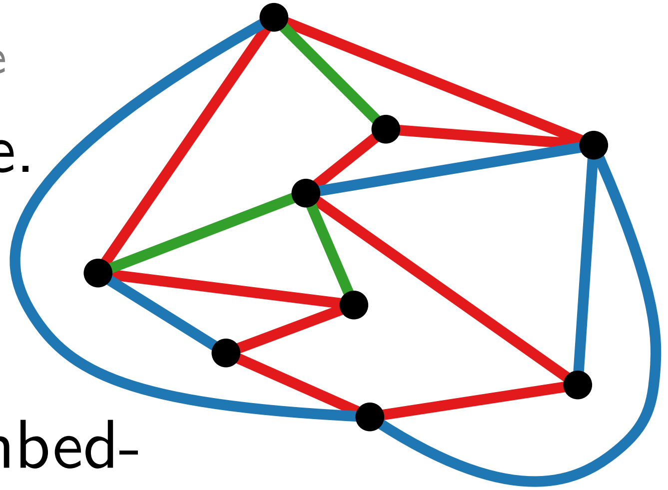

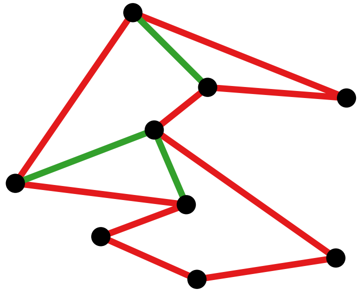

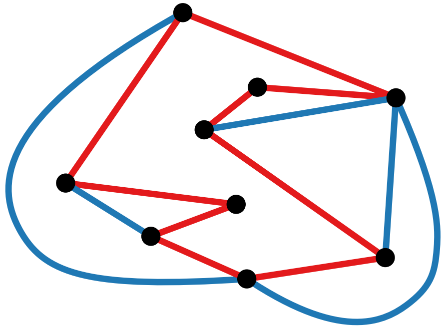

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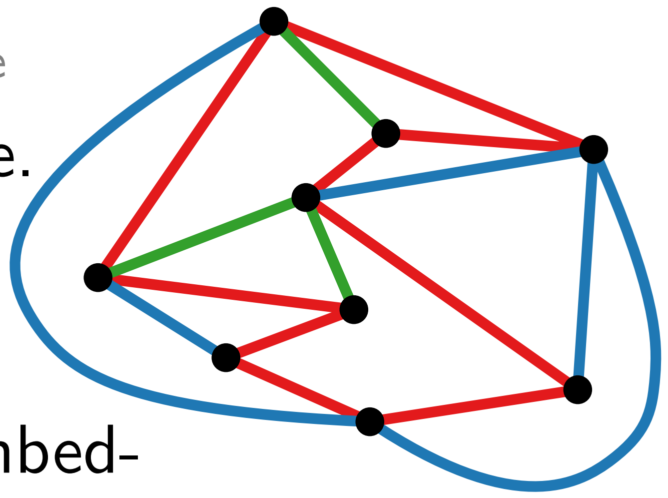

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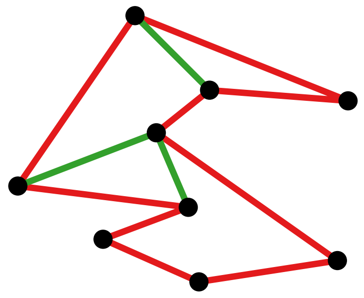

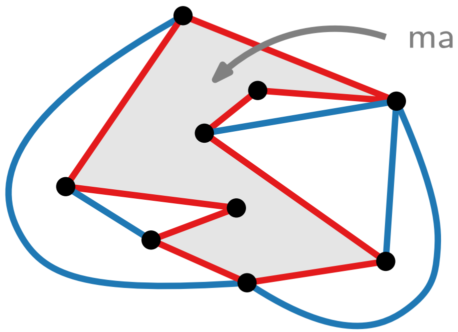

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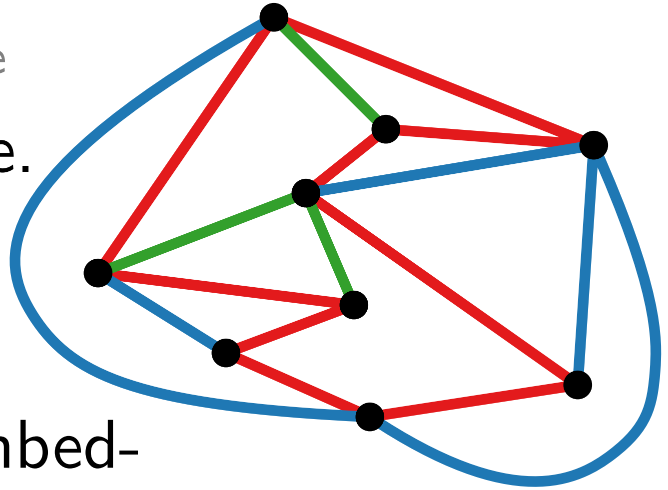

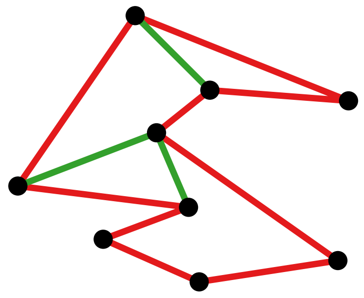

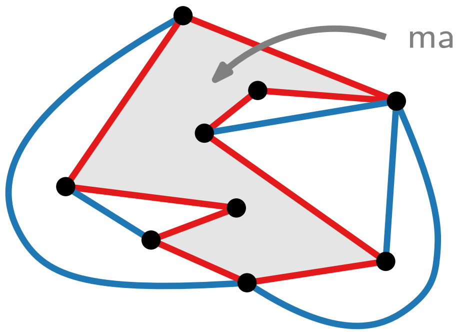

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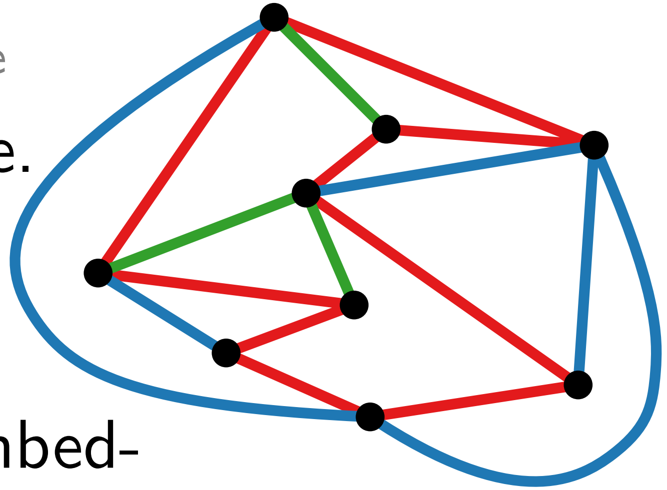

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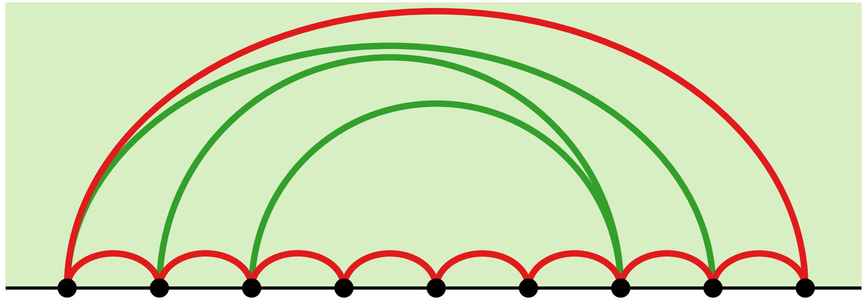

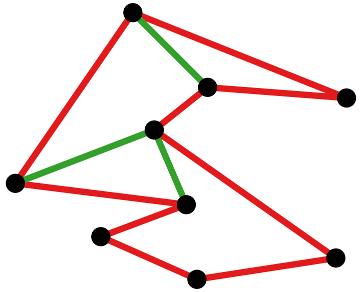

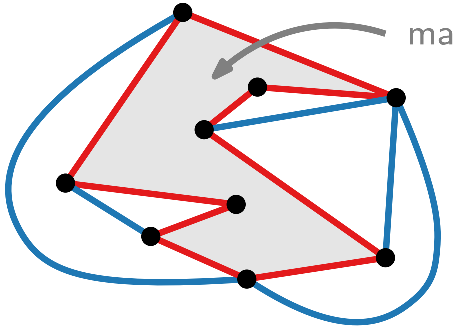

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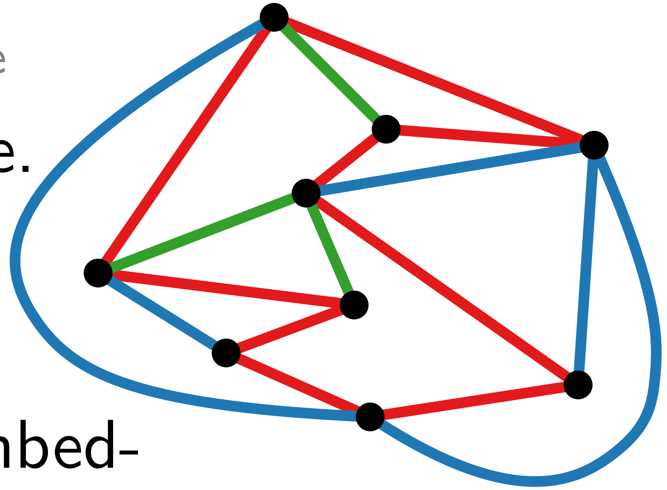

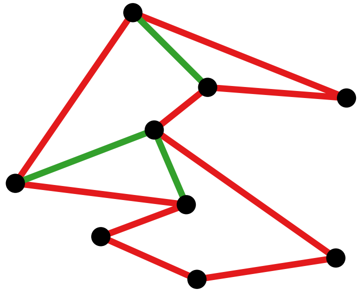

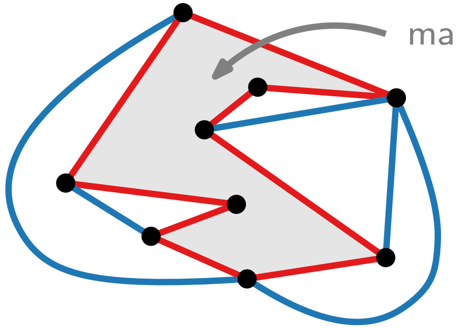

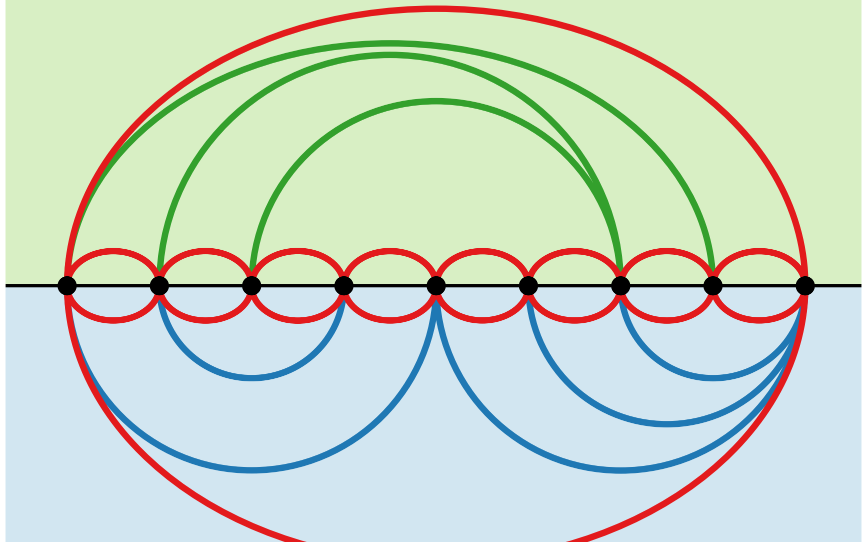

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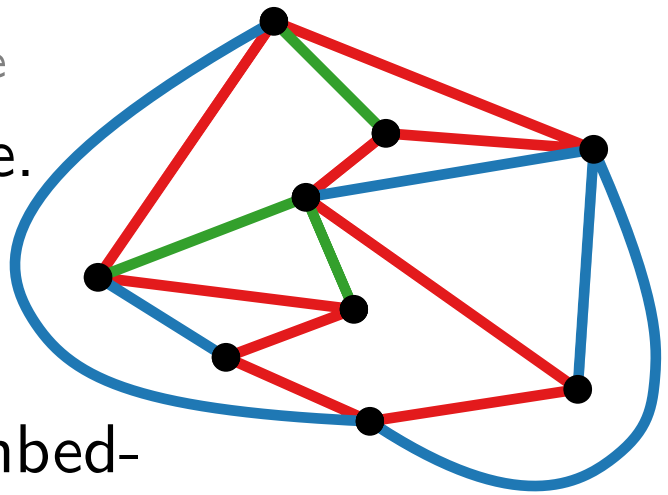

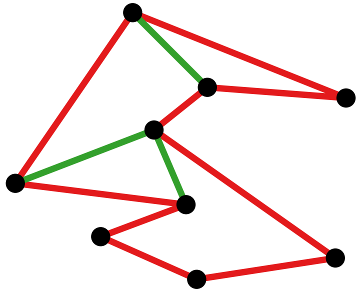

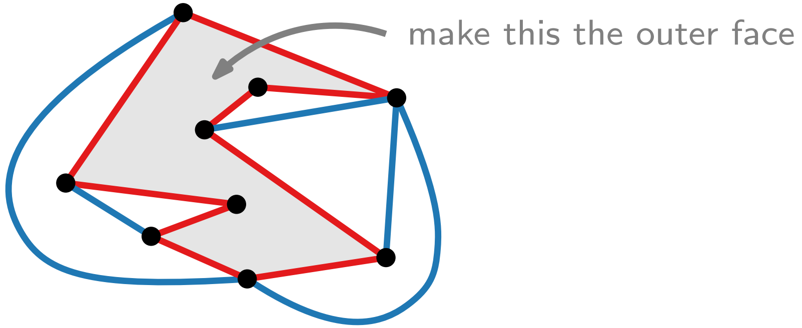

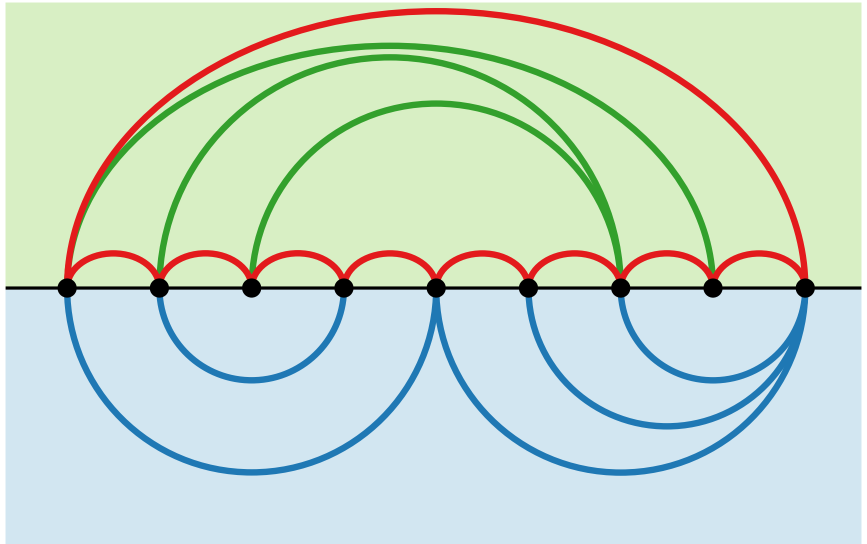

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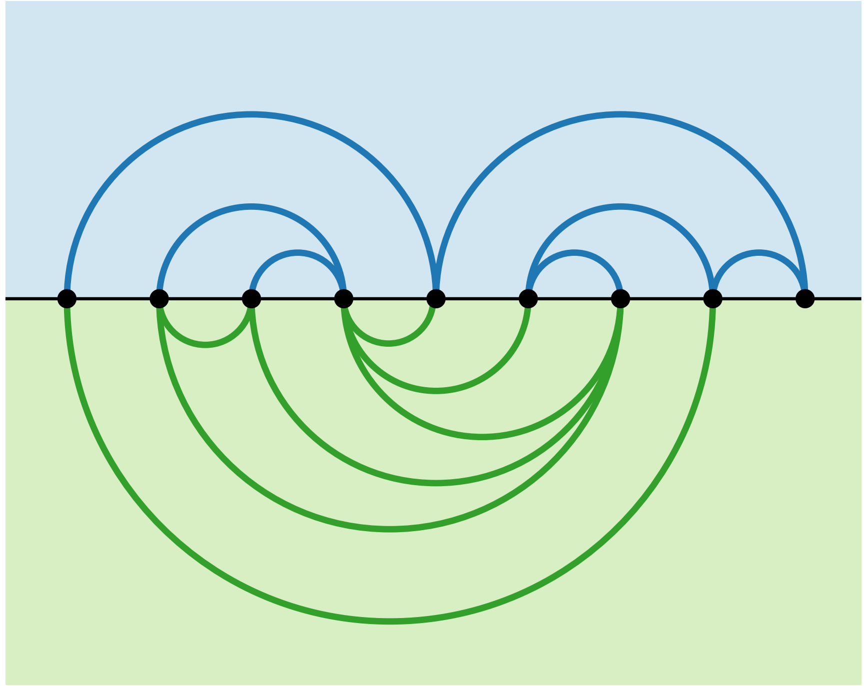

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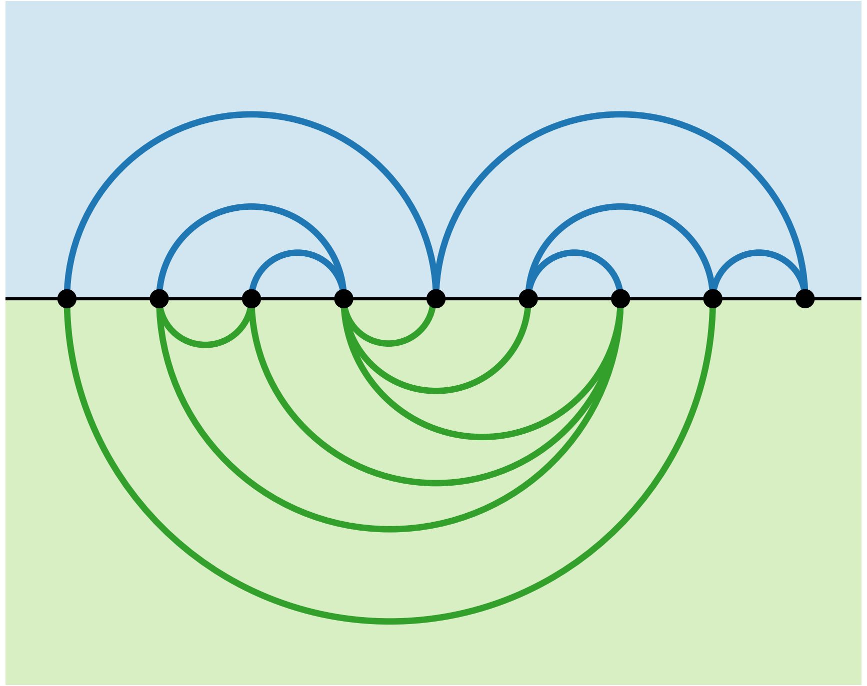

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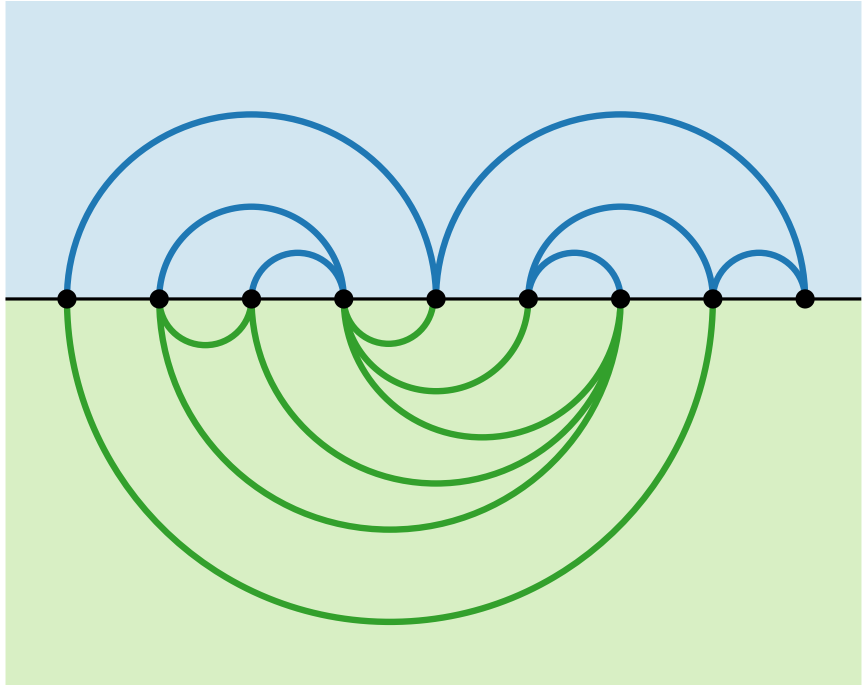

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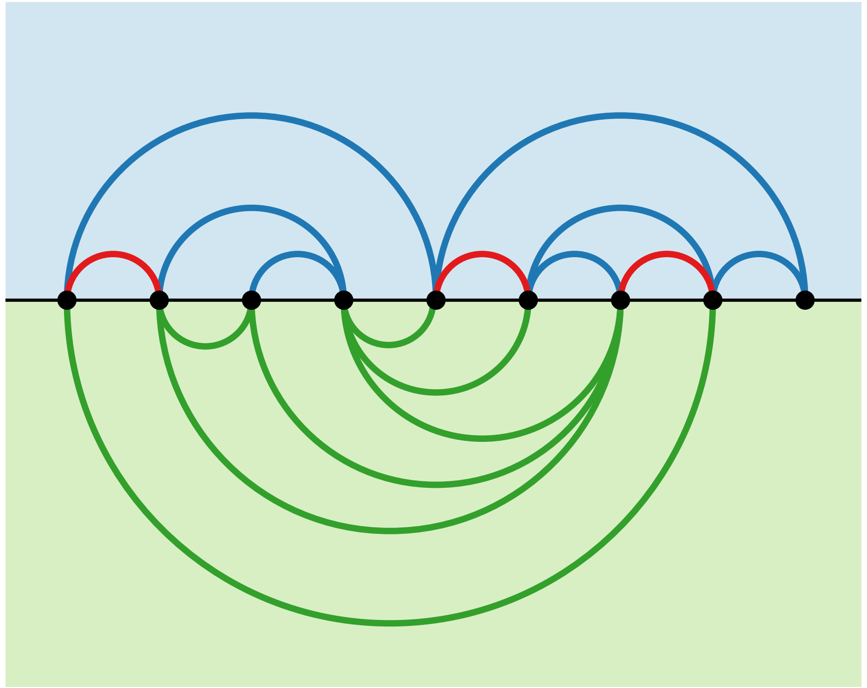

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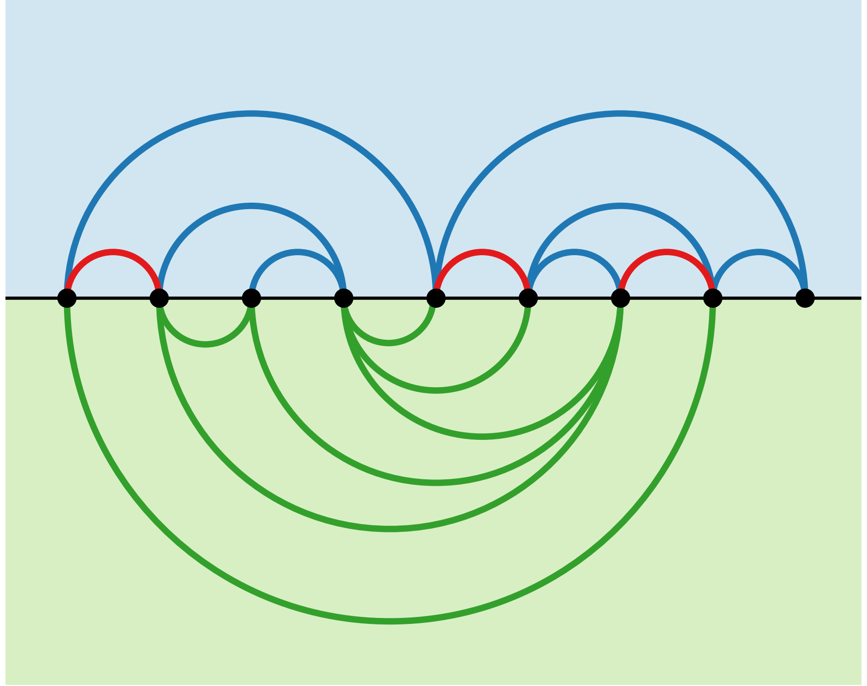

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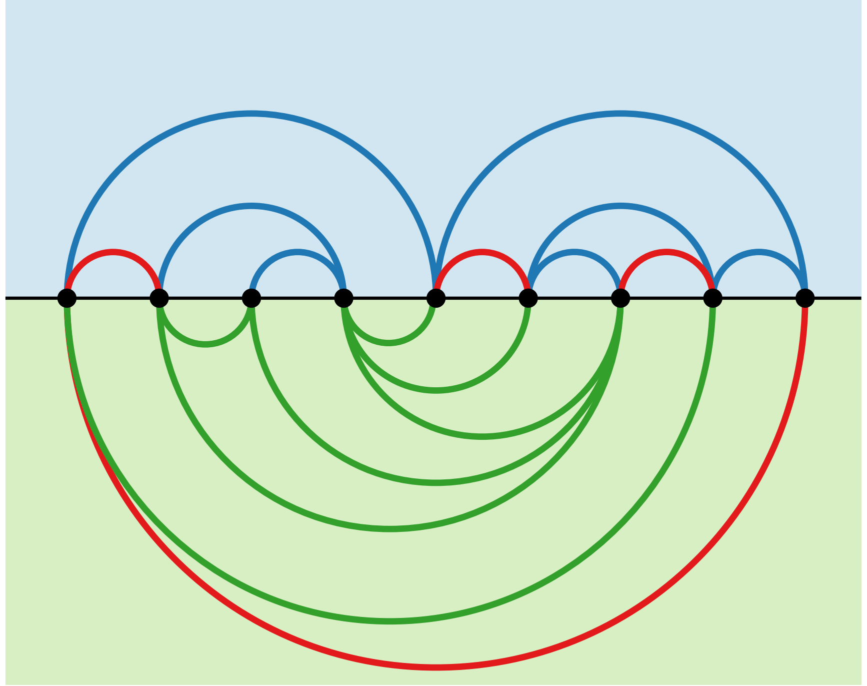

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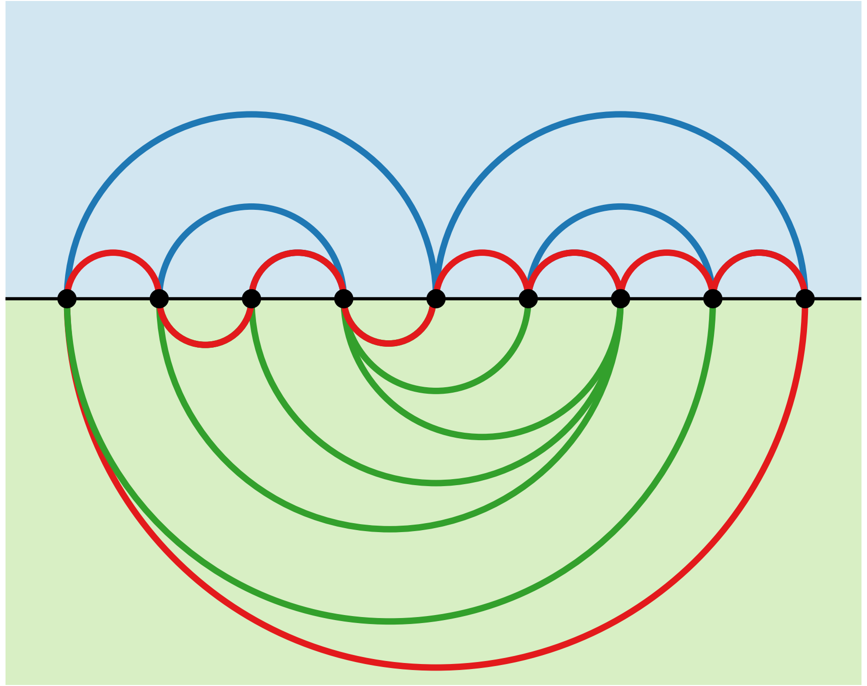
$\square$



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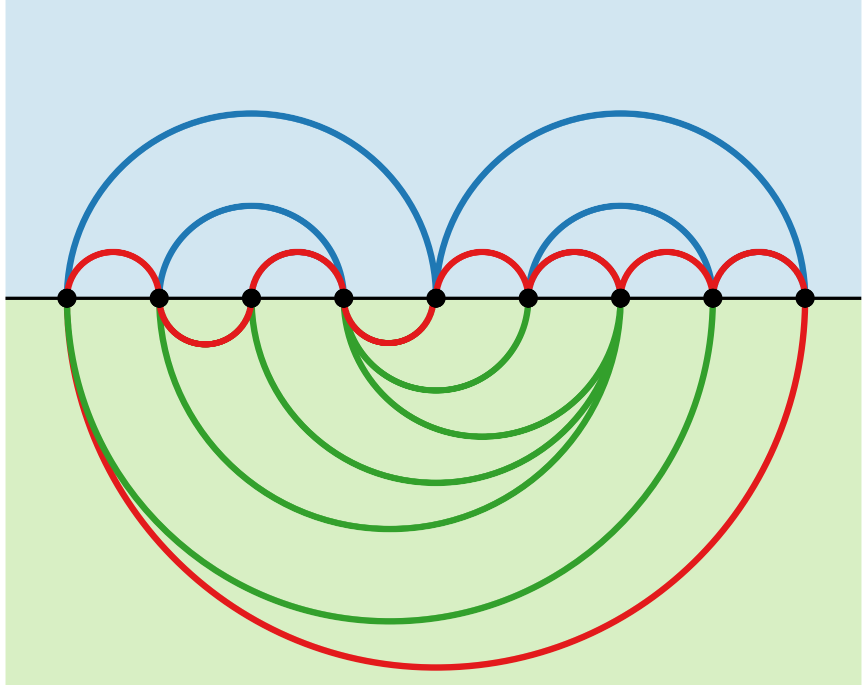
- Consider a 2-page stack layout as a drawing Γ.
- Clearly, Γ is planar.
- Add missing edges such that all pairs of neighboring vertices on the spine are connected (always possible).
- If absent, add edge from the first to the last vertex (always possible).
- The Hamiltonian cycle traverses all vertices on the spine in order.

$\square$



This result includes planar bipartite and series-parallel graphs.

Stack Layouts of Planar Graphs

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(not all planar graphs can be extended to have a Hamiltonian cycle)

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Theorem. [Yannakakis 2020, Bekos, Kaufmann, Klute, Pupyrev, Raftopoulou & Ueckerdt 2020]
There is a planar graph G with $\mathrm{sn}(G) \ge 4$.

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Yes! (The planar graph presented by Bekos et al. has 275 vertices and 819 edges.)

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Do all graphs have constant stack number?

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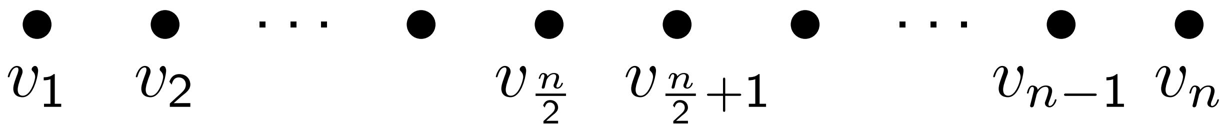

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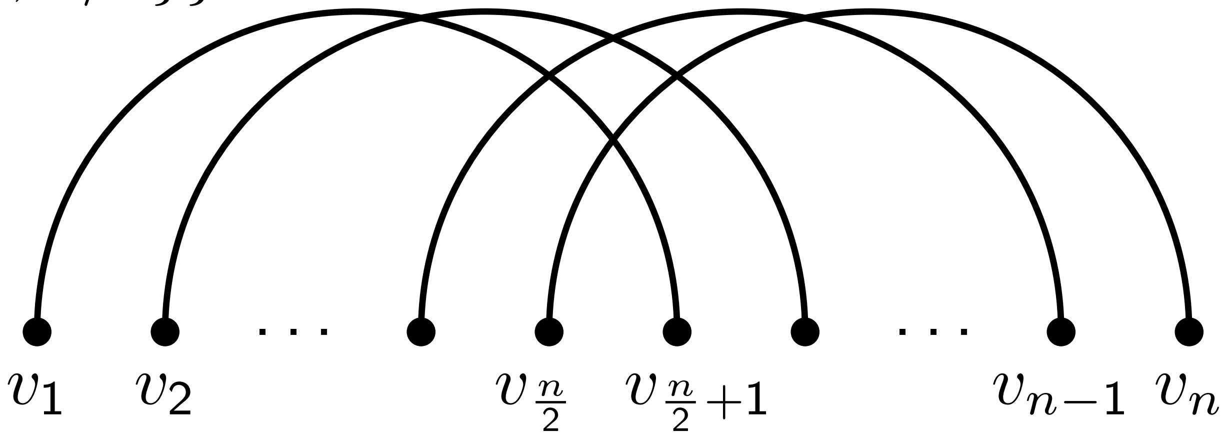
- Consider any order $\prec$ of the vertices on the spine and name them $v_1, \ldots, v_n$ accordingly.
- Consider the set of edges $\{v_i v_{n/2+i} \mid i \in \{1, \ldots, n/2\}\}$.



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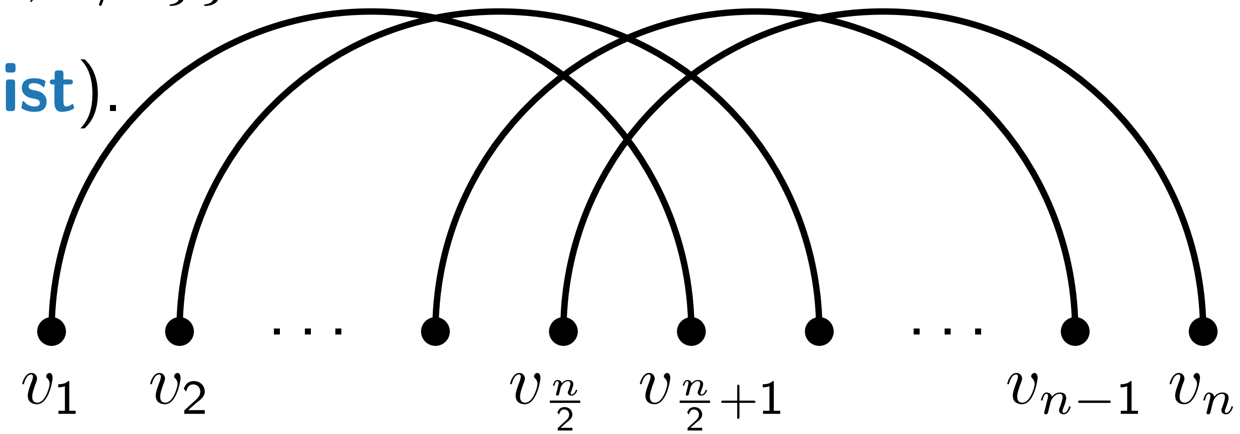

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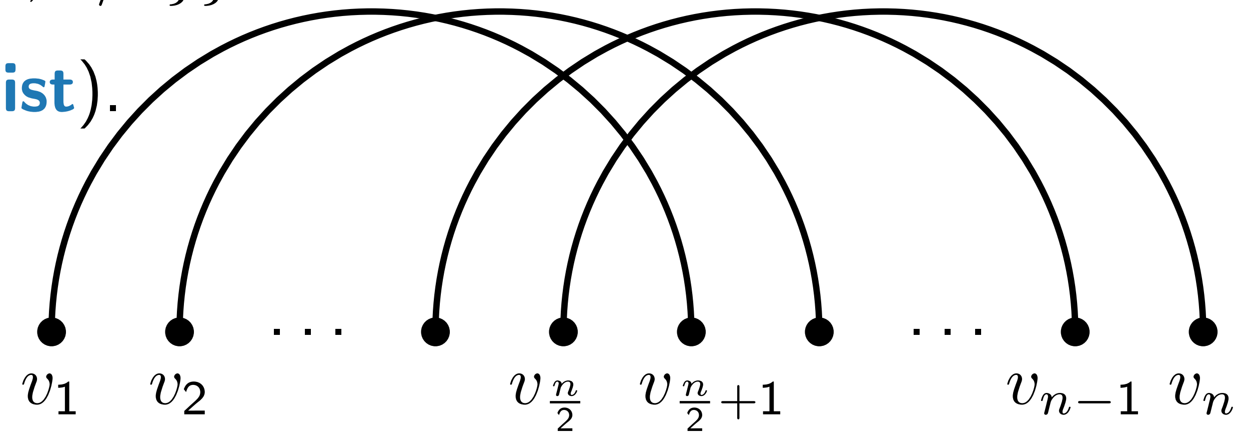
- Each of these edges needs a separate stack.



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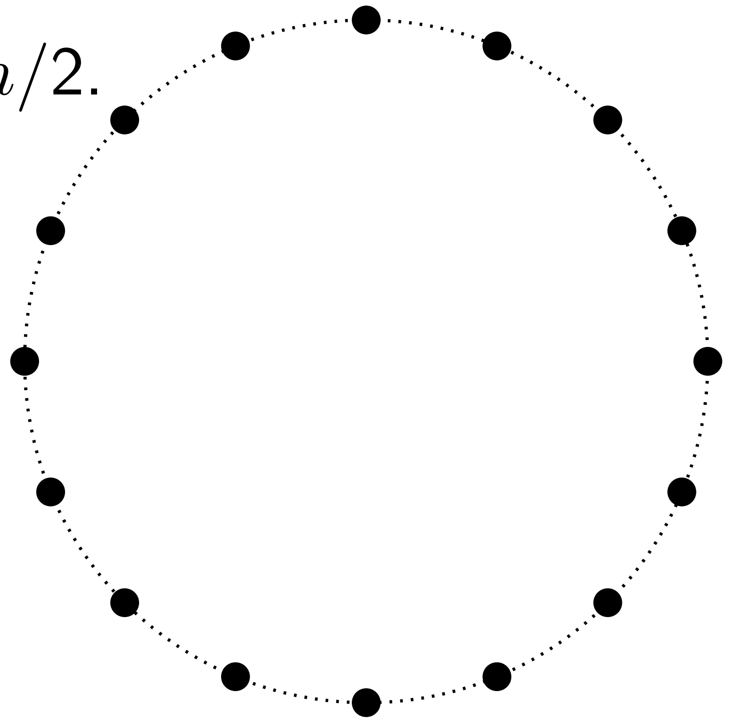
- Arrange the vertices of K_n on a circle.



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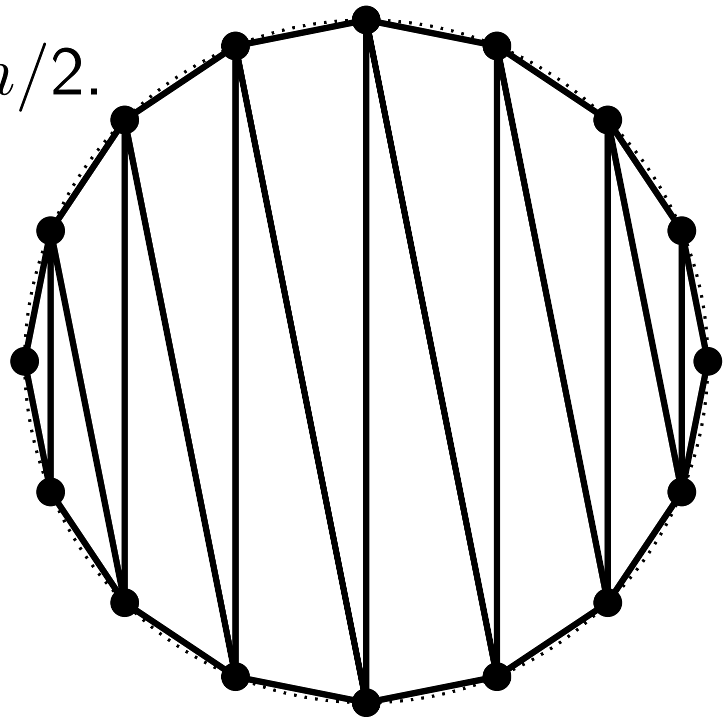

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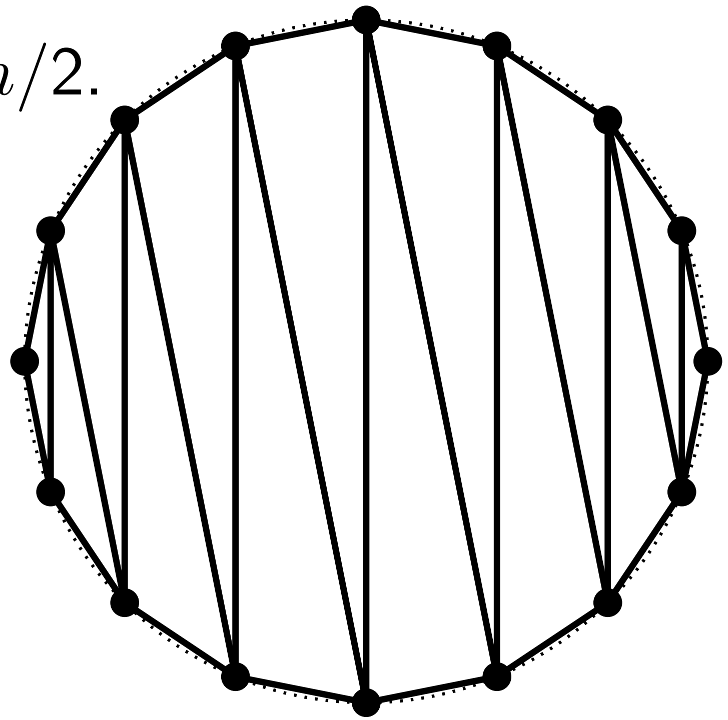

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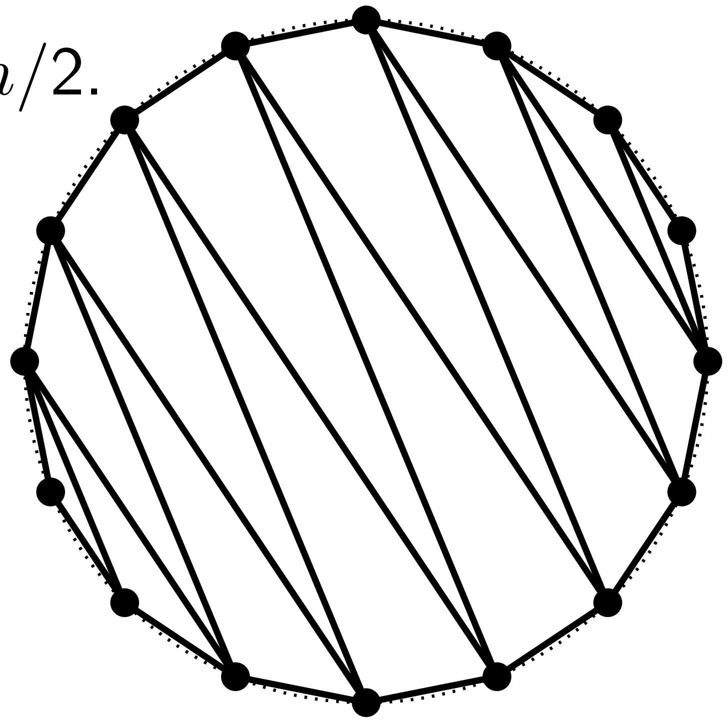

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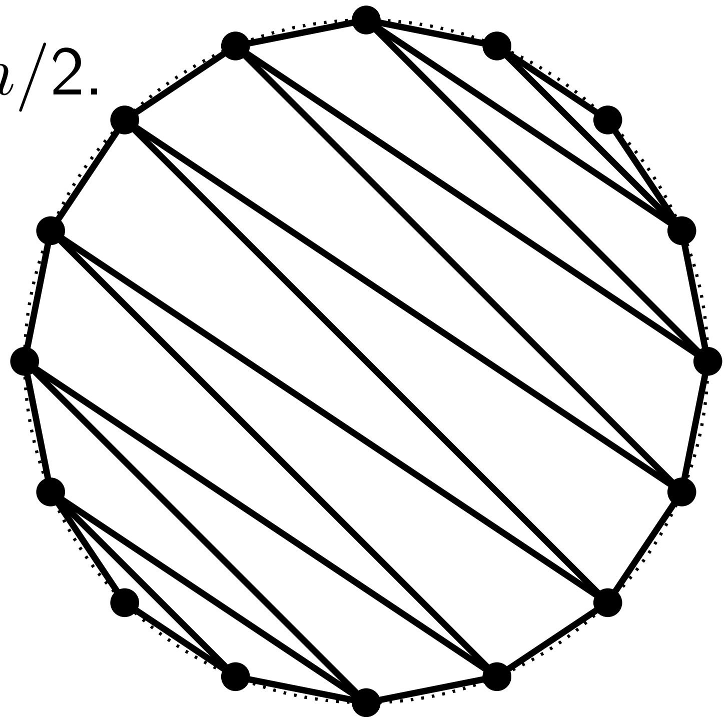

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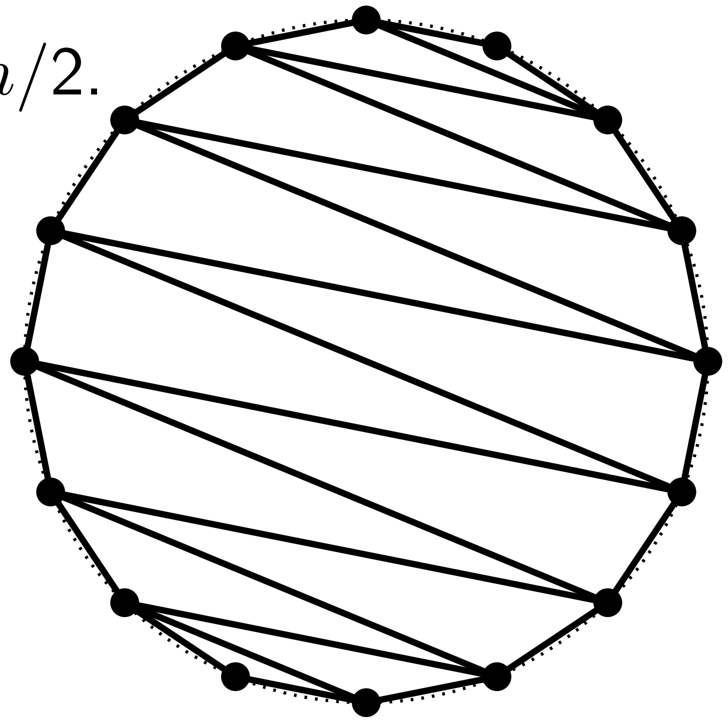

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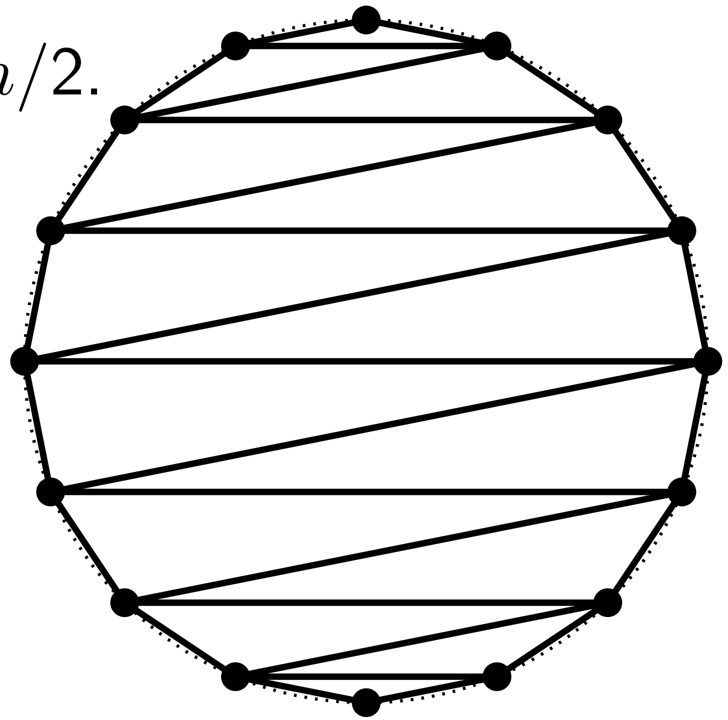

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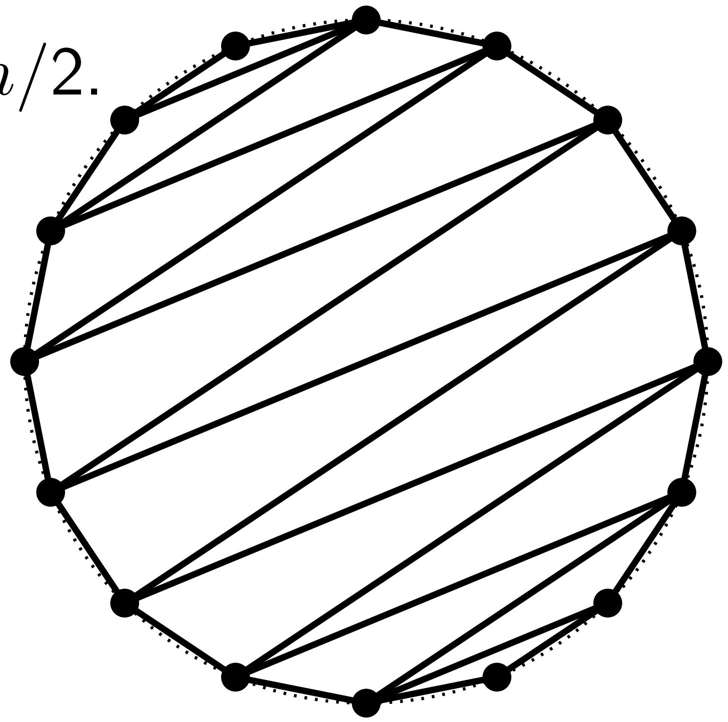

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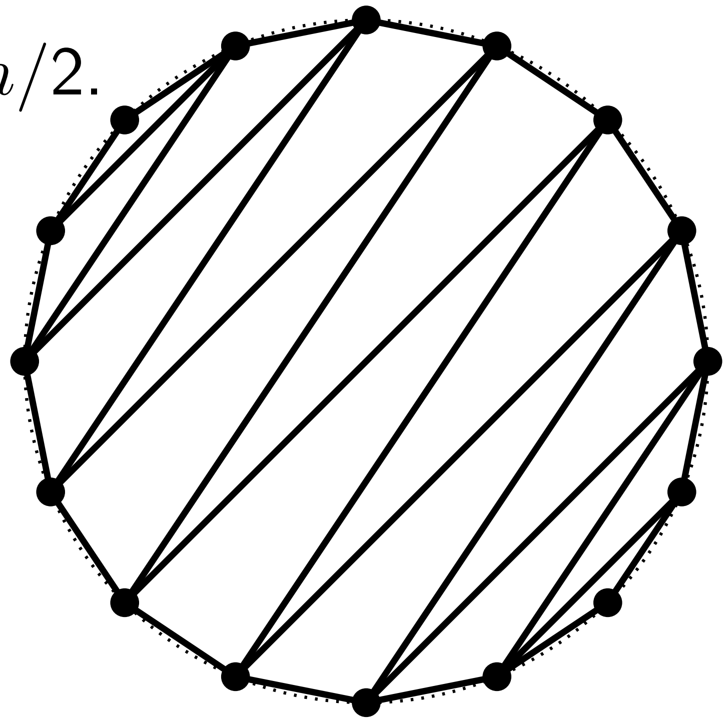

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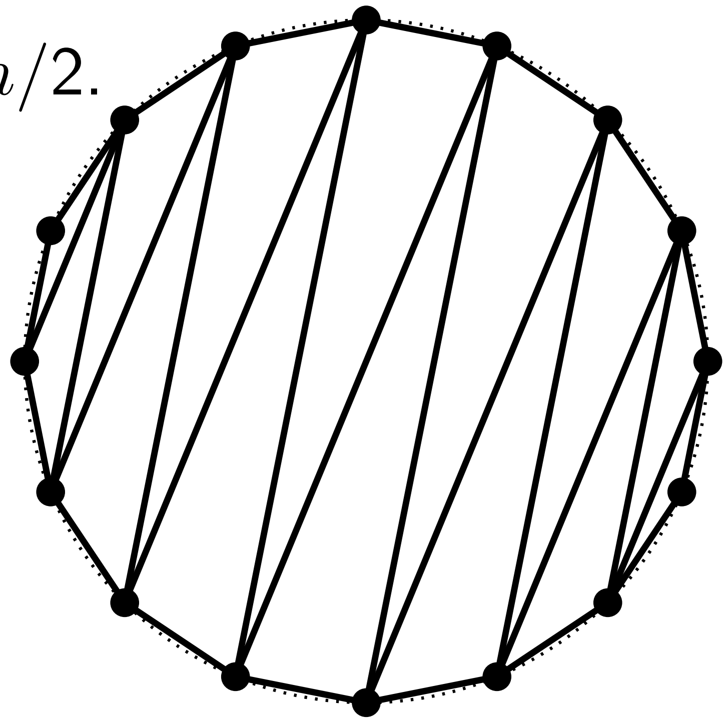

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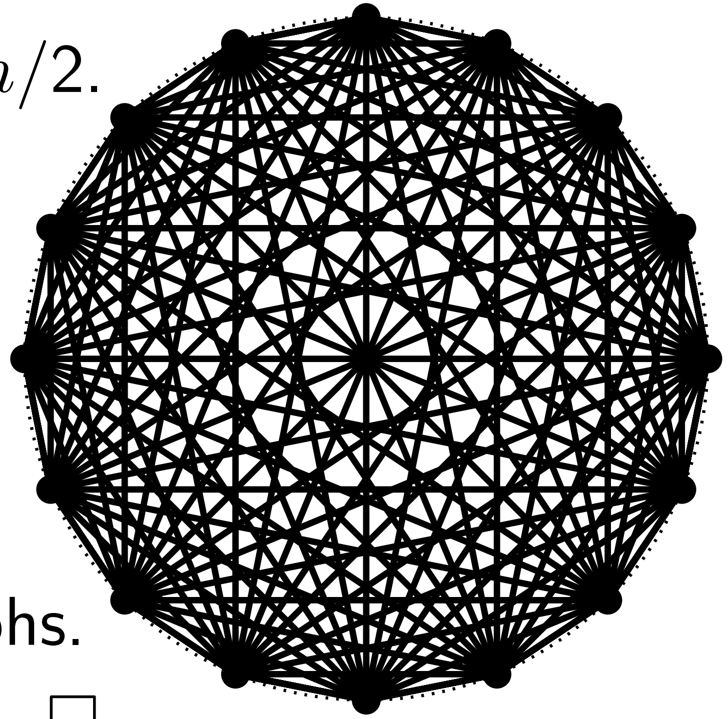

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- Clearly, every edge appears in some of these $n/2$ outerplanar graphs.

$\square$

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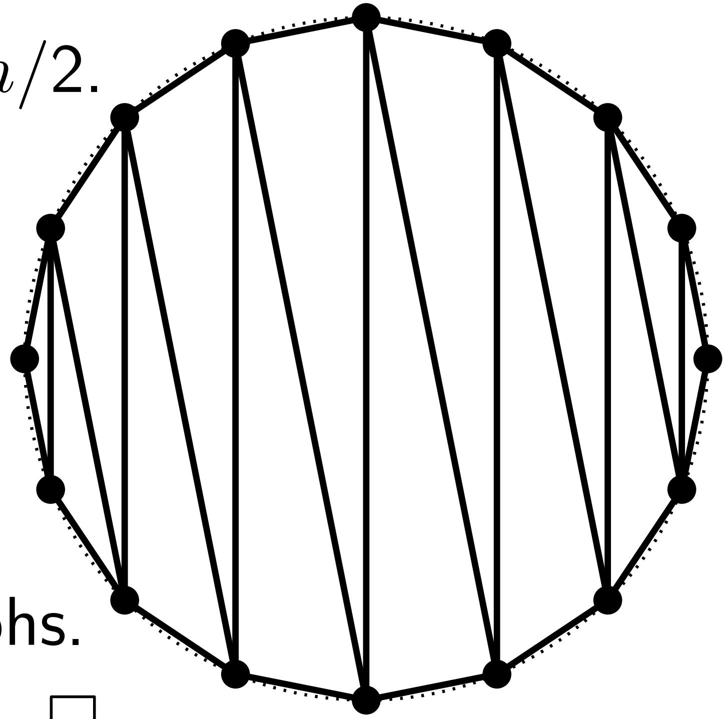

- Arrange the vertices of K_n on a circle.
- Add boundary edges and inner diagonals as follows:
- This is an outerplanar graph drawing and can go to one stack.
- "Rotate" the inner diagonals by 1, 2, . . . , $n/2 - 1$ position(s).
- Clearly, every edge appears in some of these $n/2$ outerplanar graphs.

$\square$

1-Page Queue Layouts

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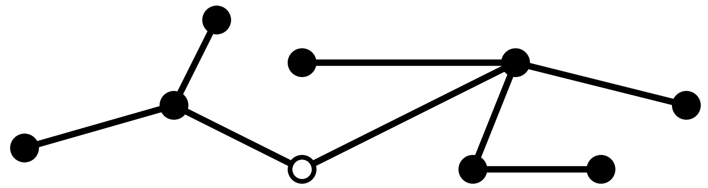

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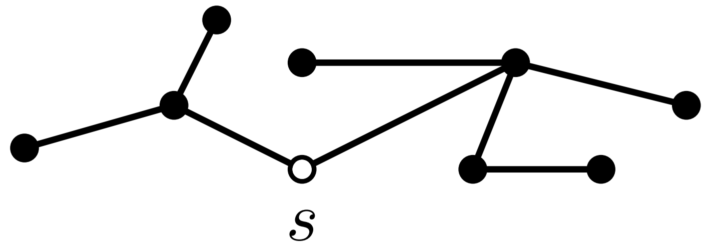

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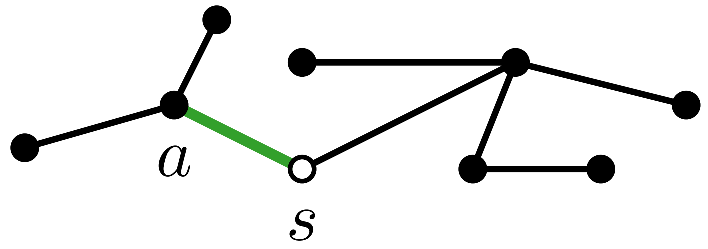

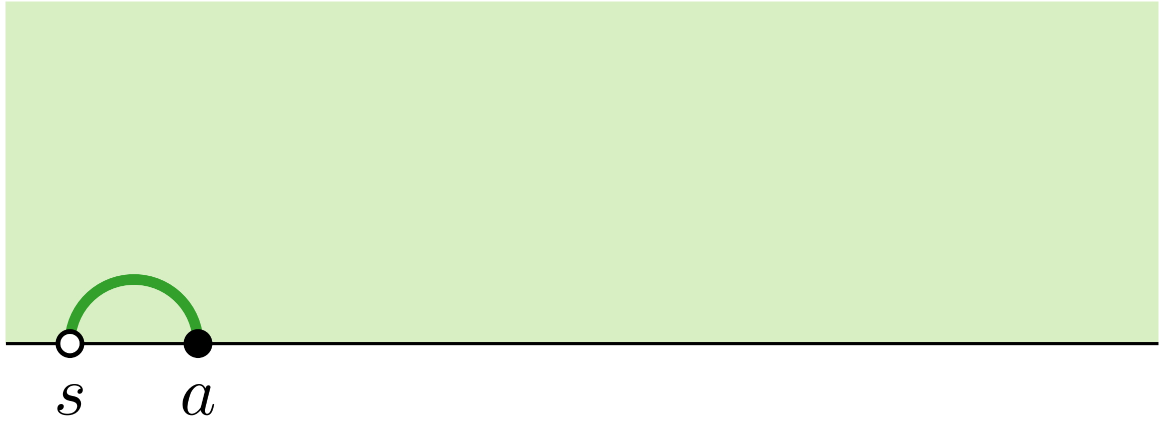

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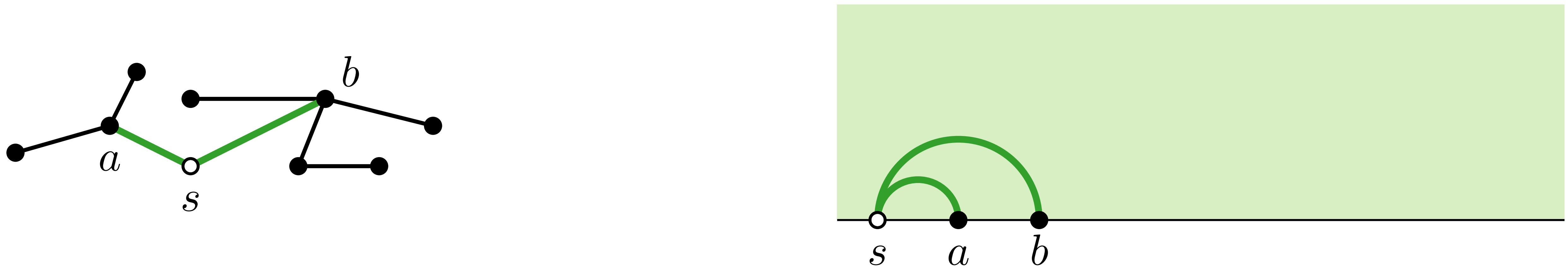

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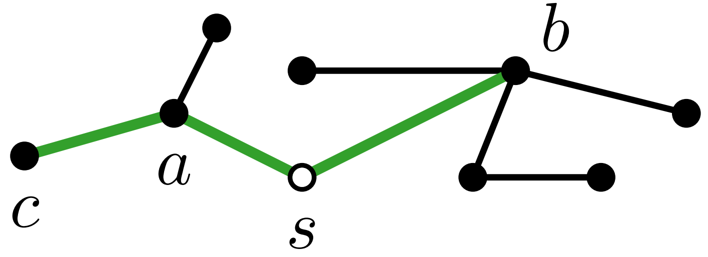

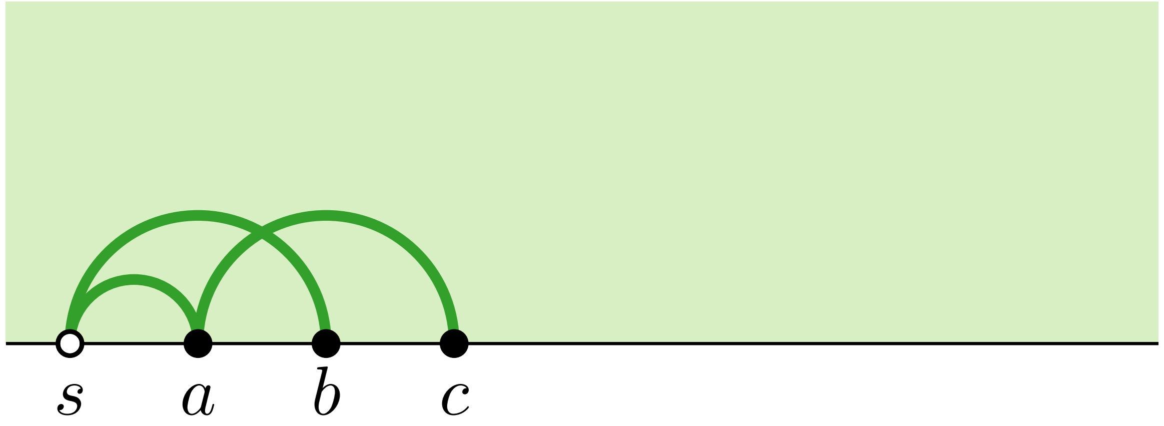

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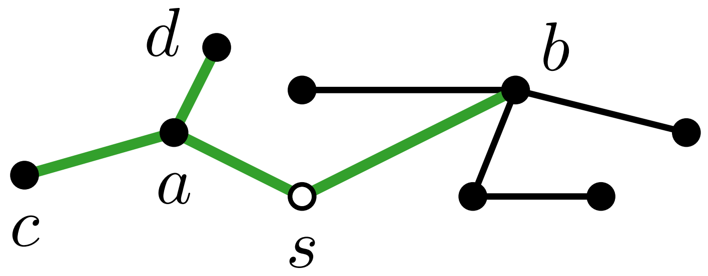

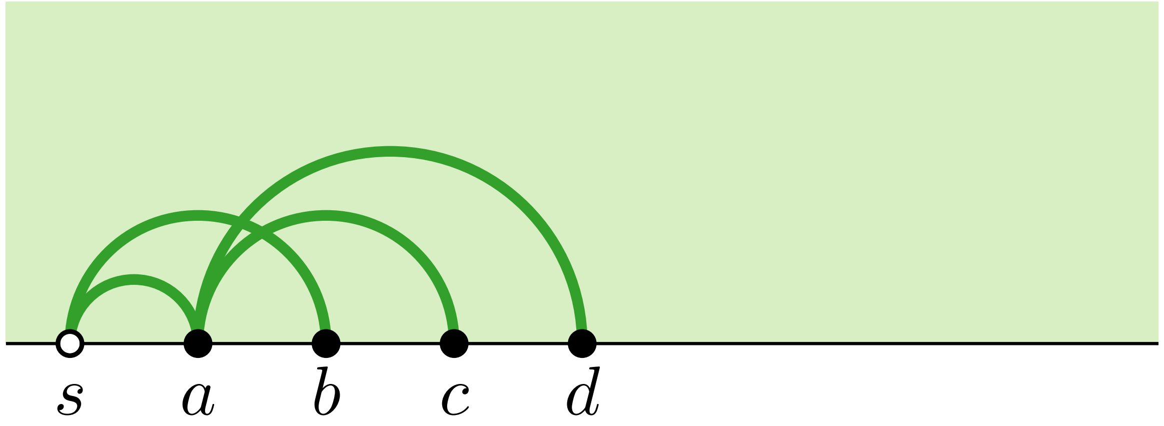

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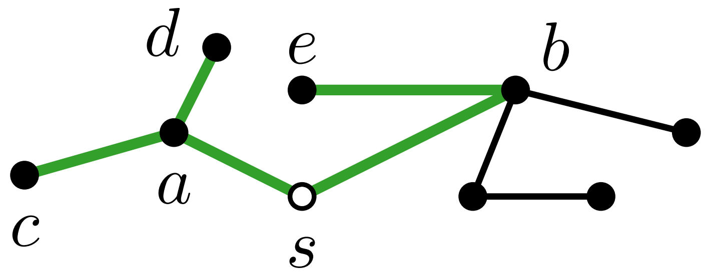

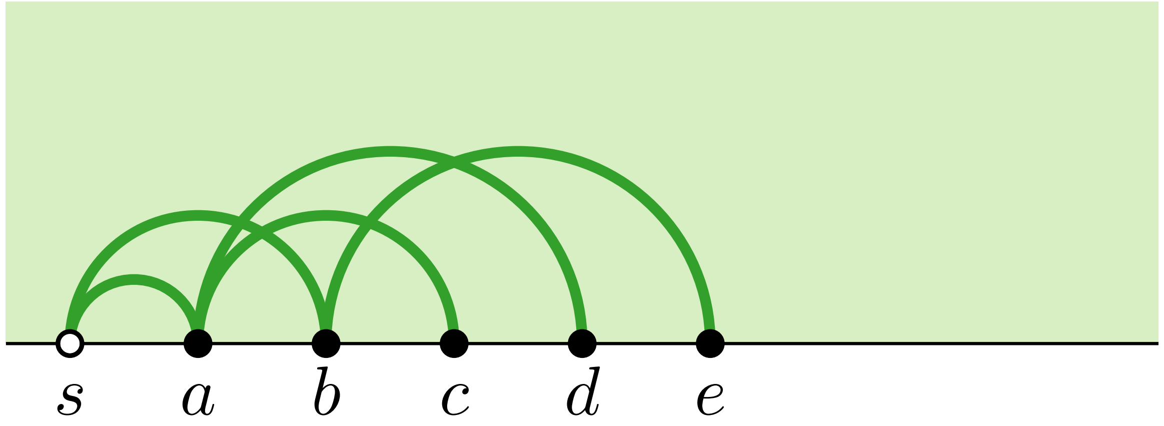

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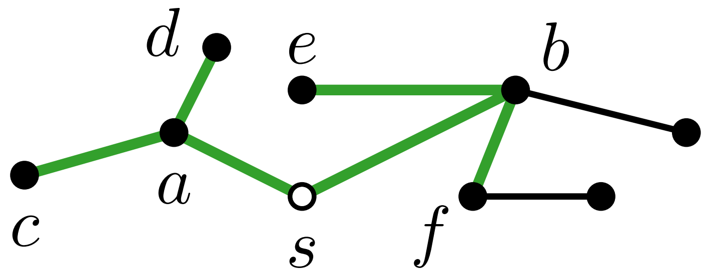

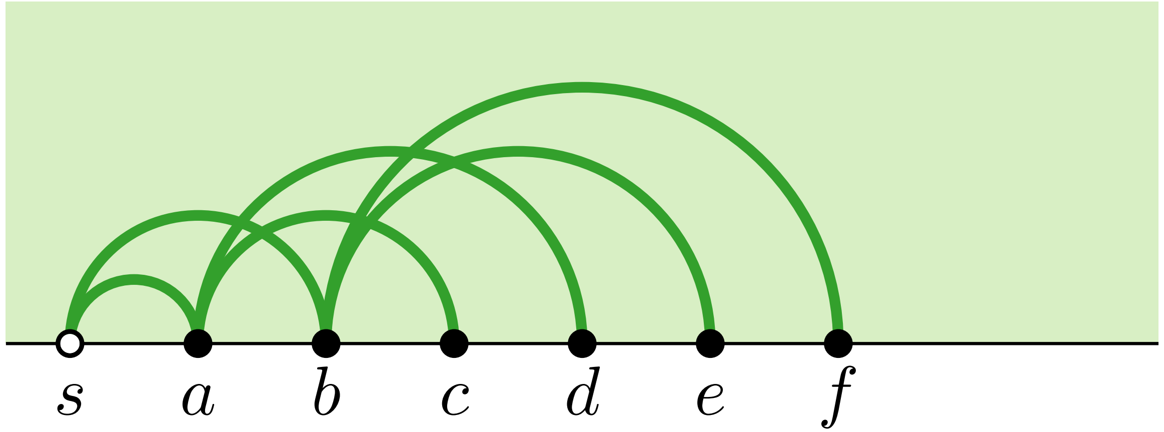

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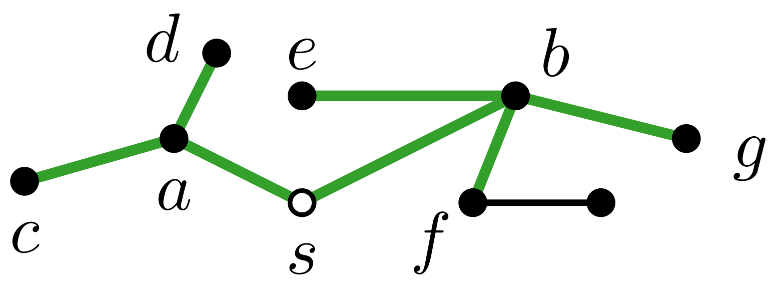

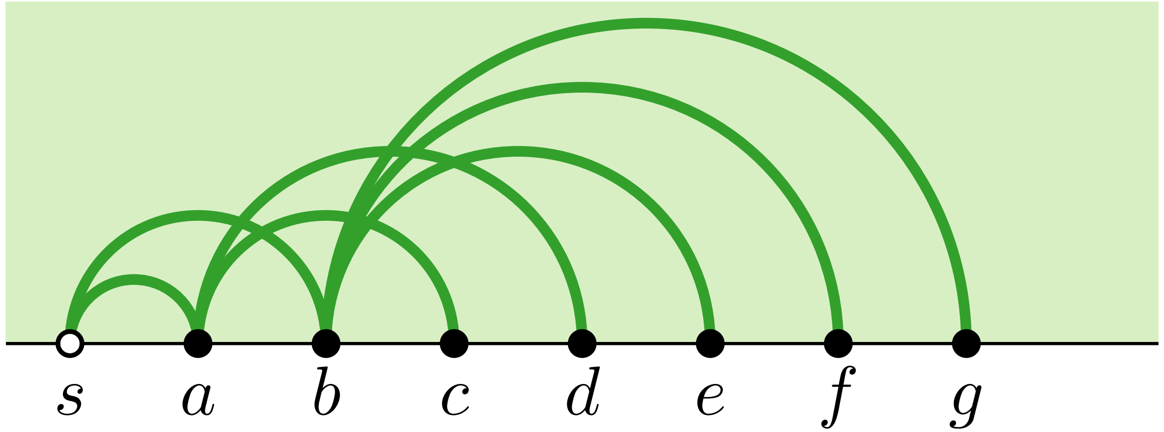

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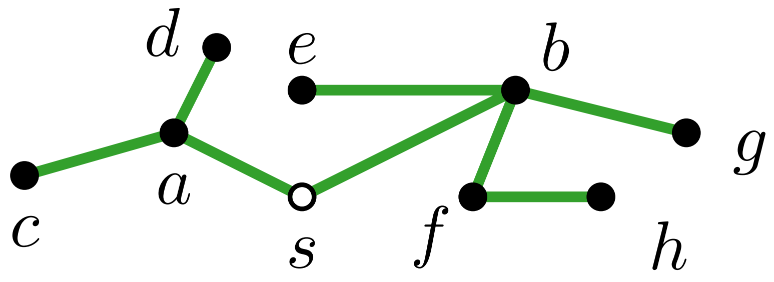

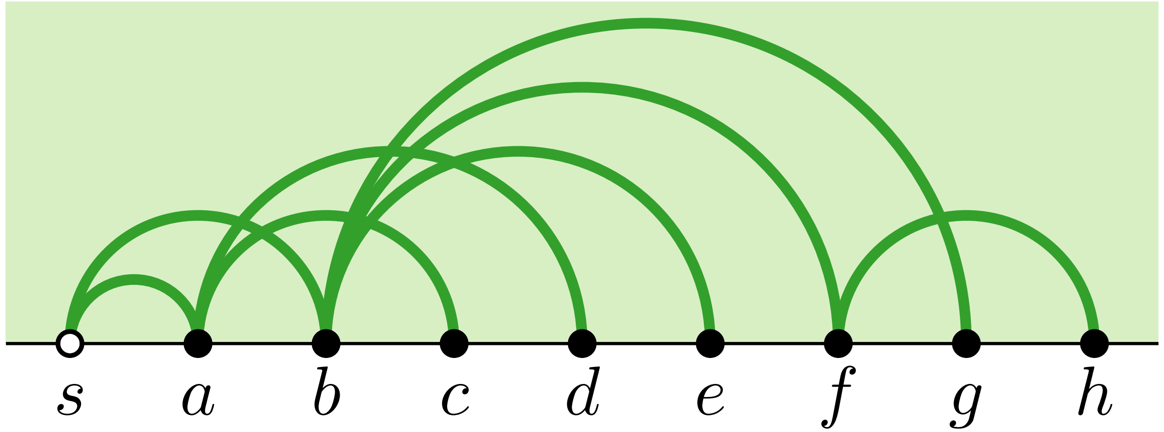

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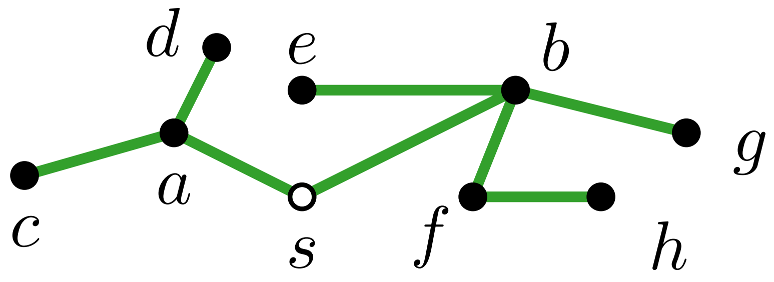

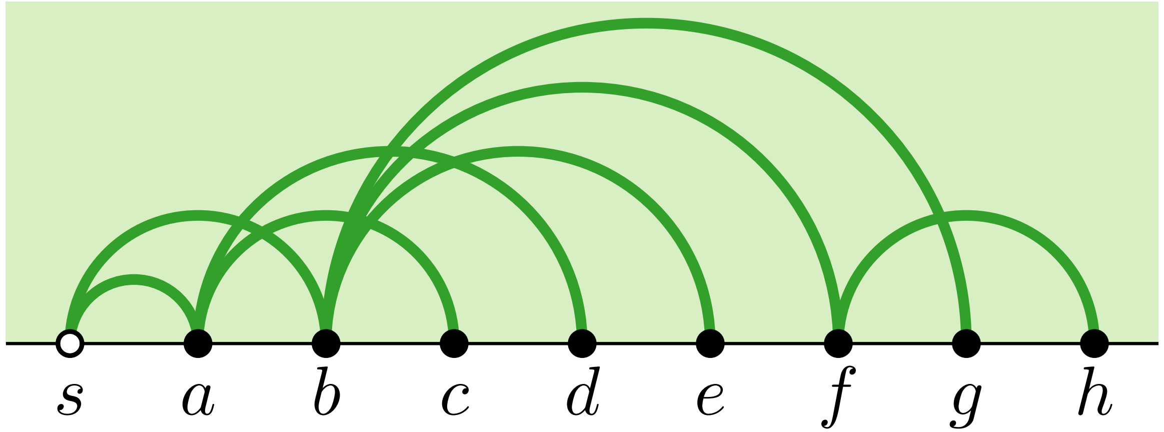

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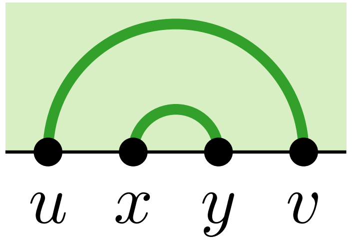

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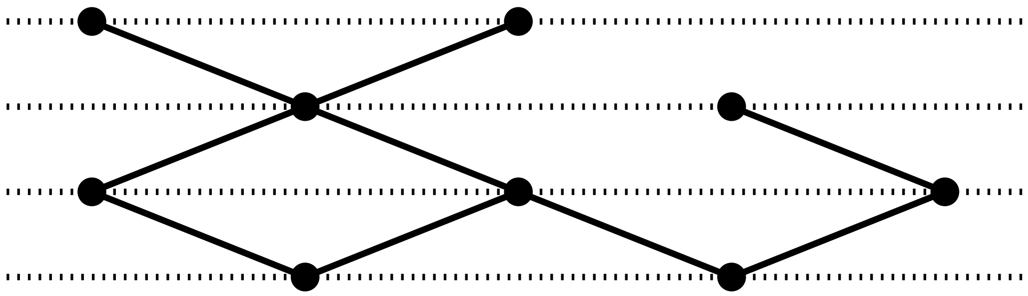

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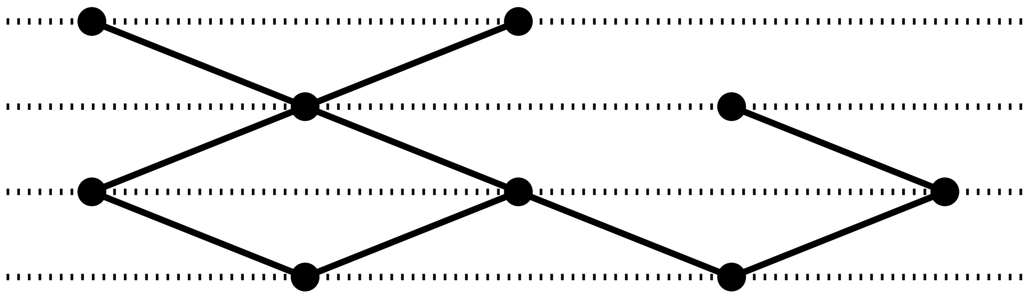

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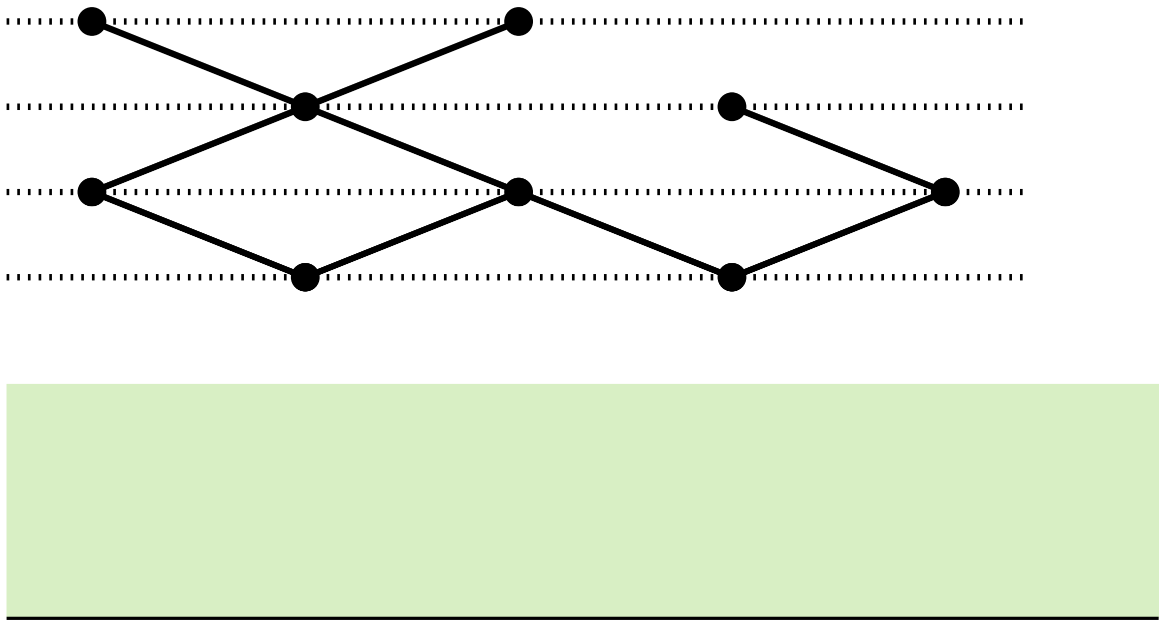

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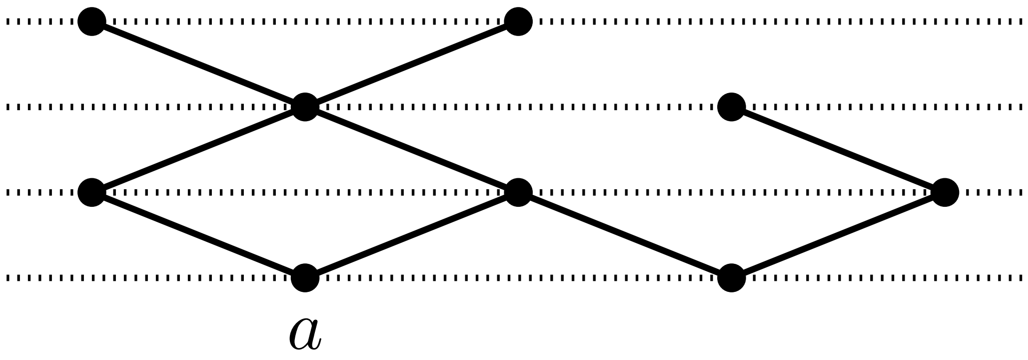

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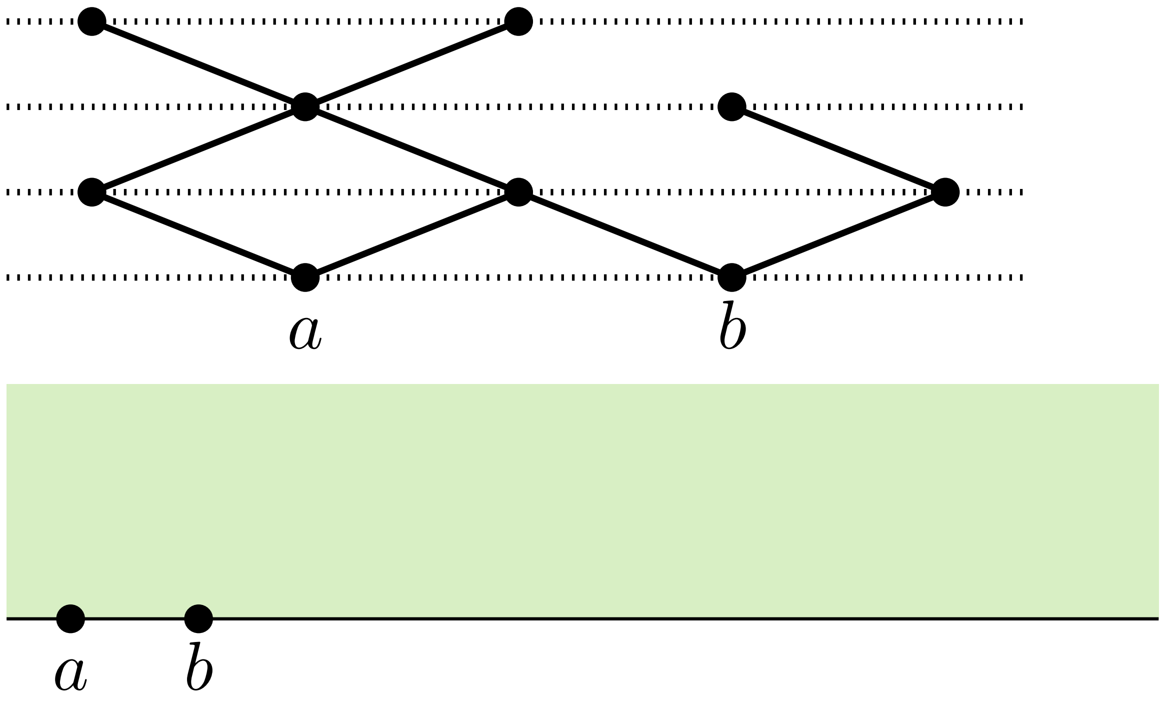

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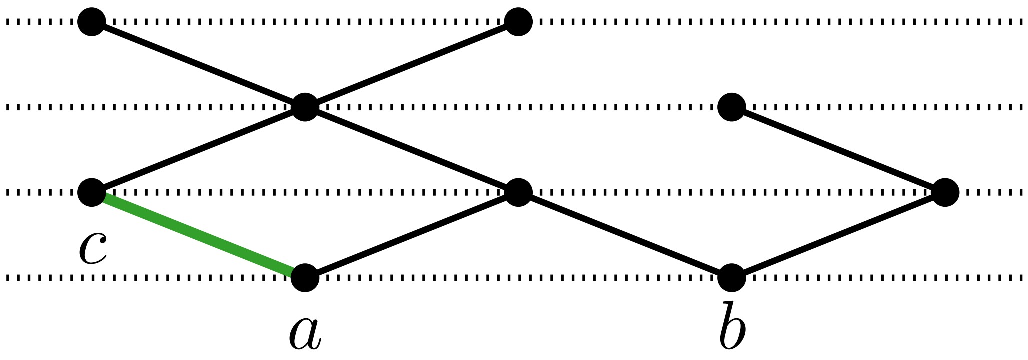

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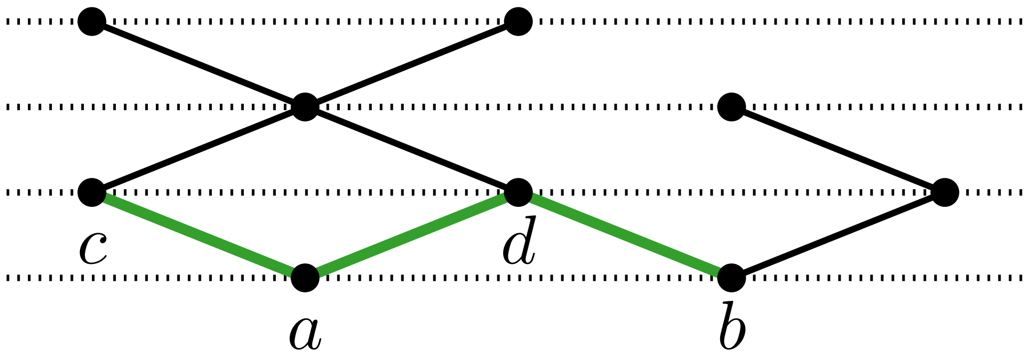

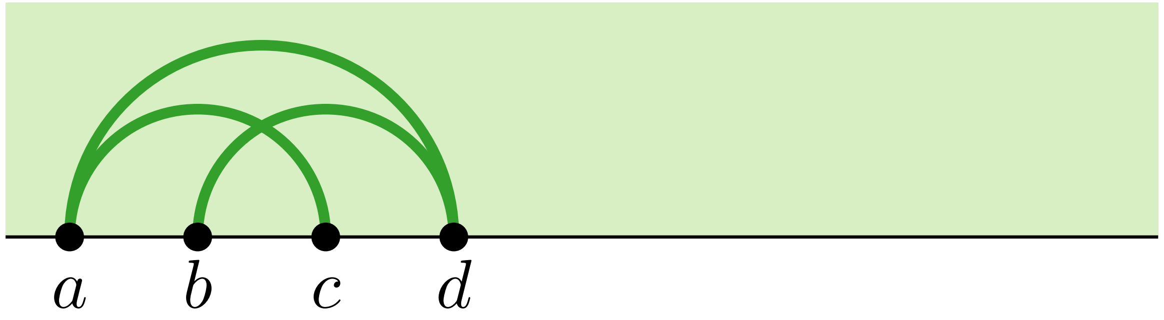

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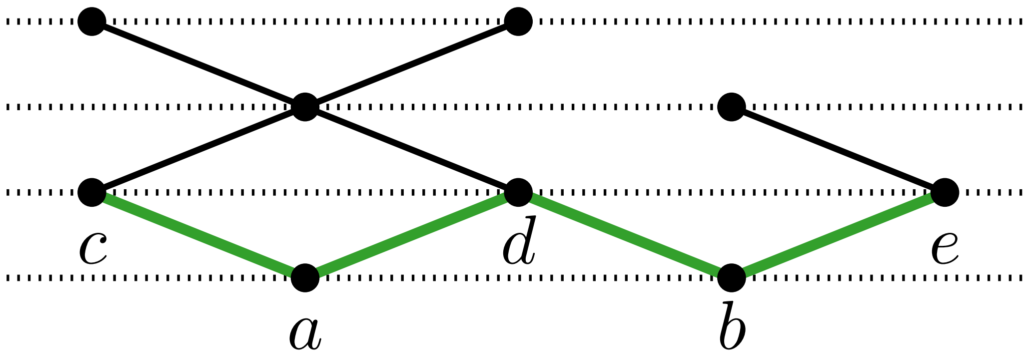

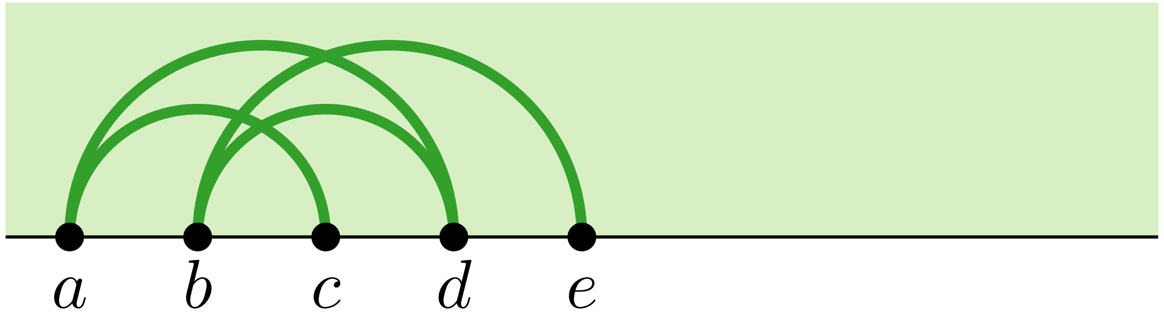

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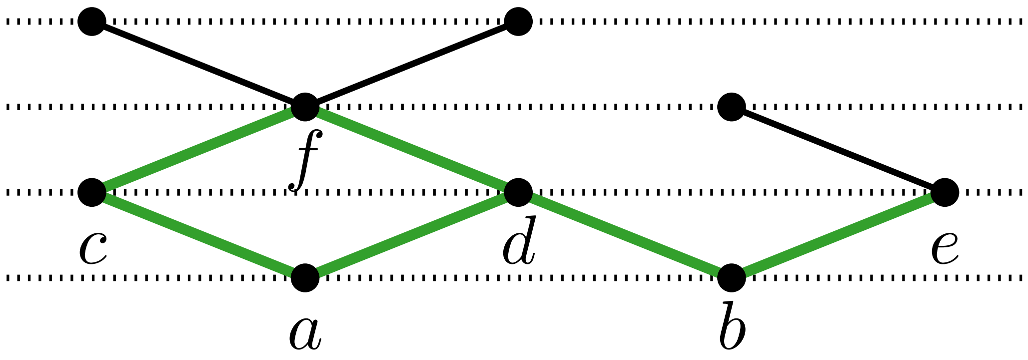

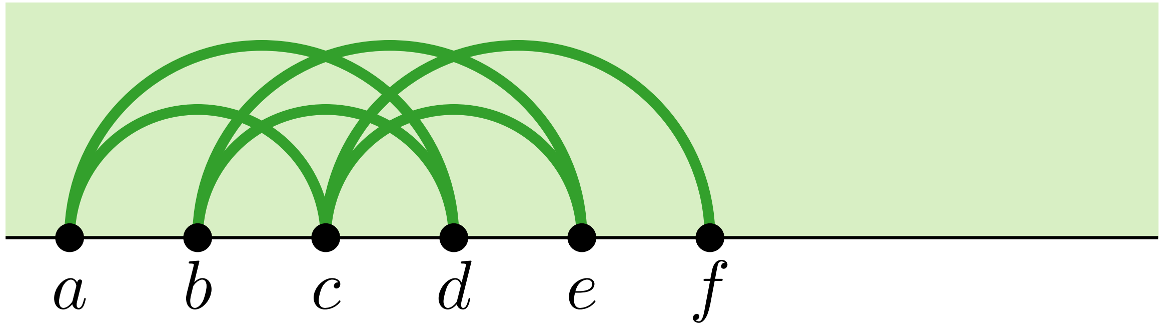

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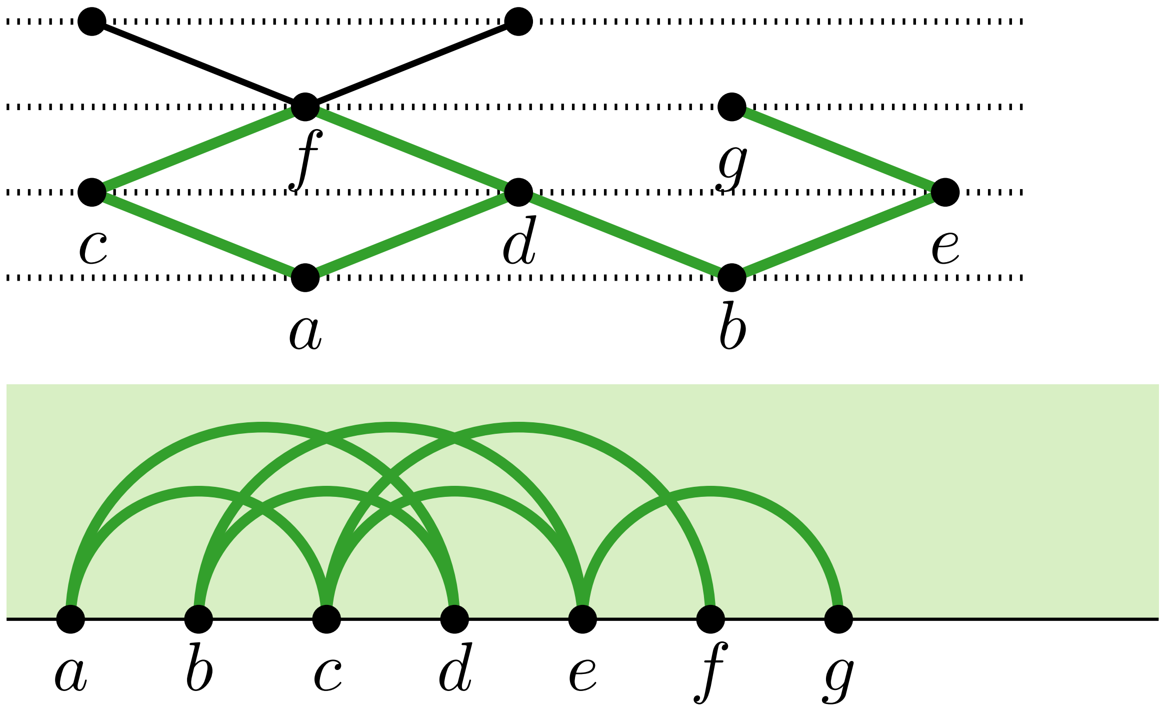

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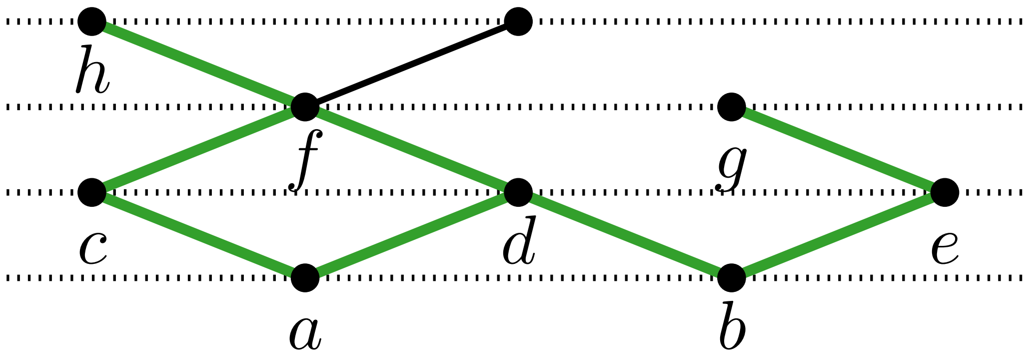

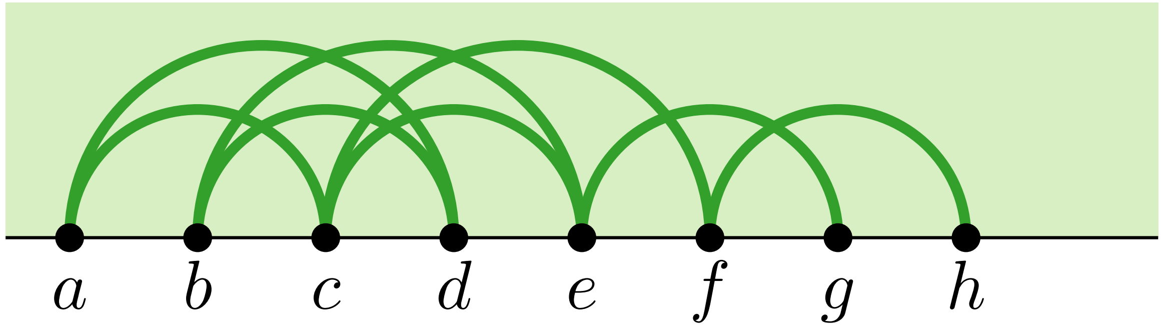

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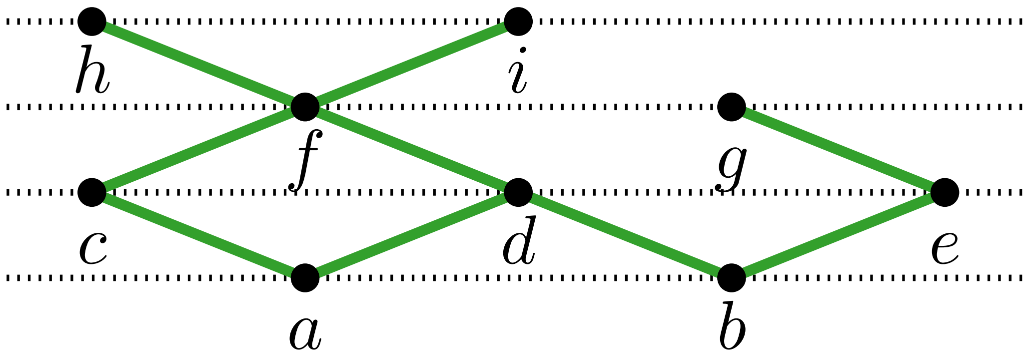

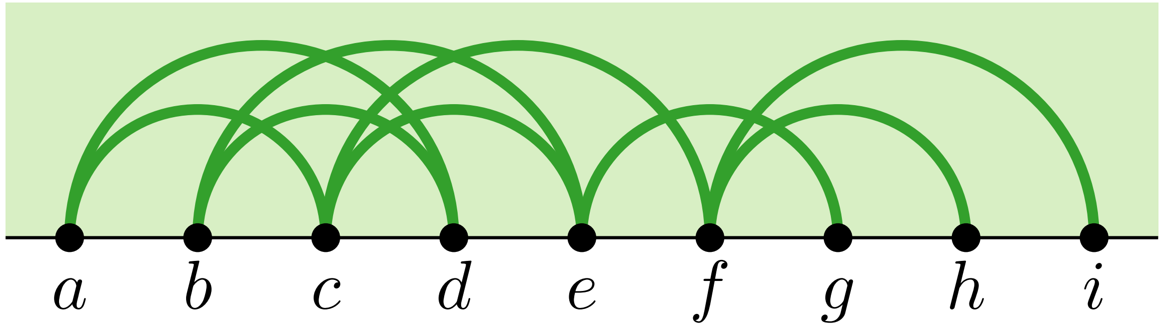

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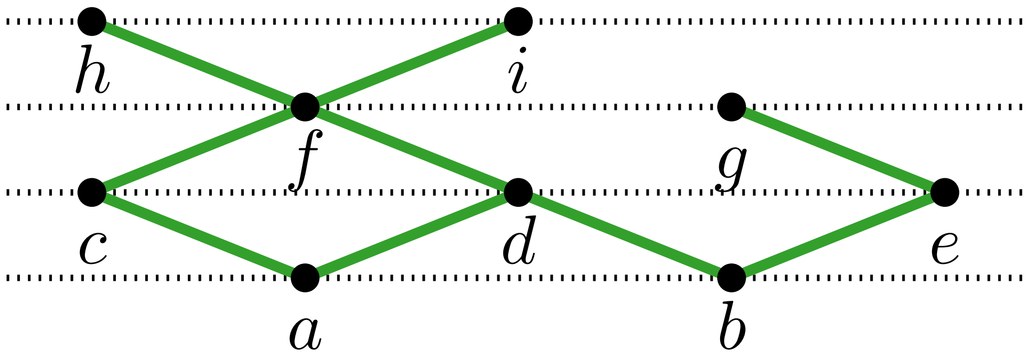

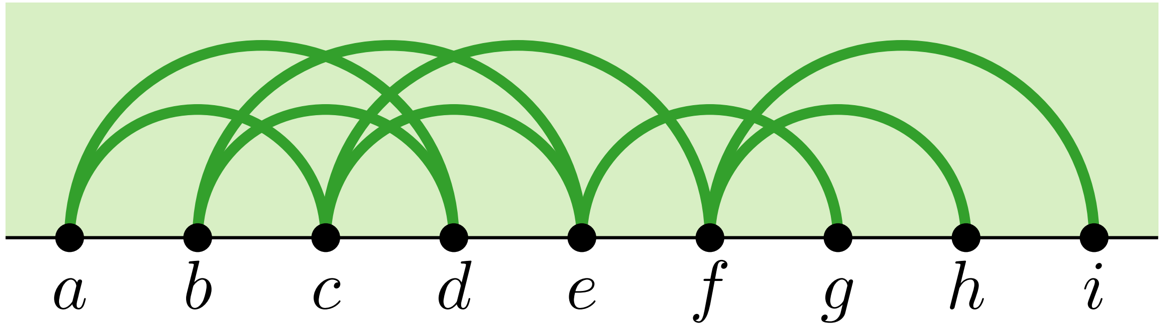

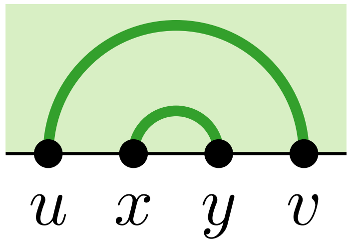

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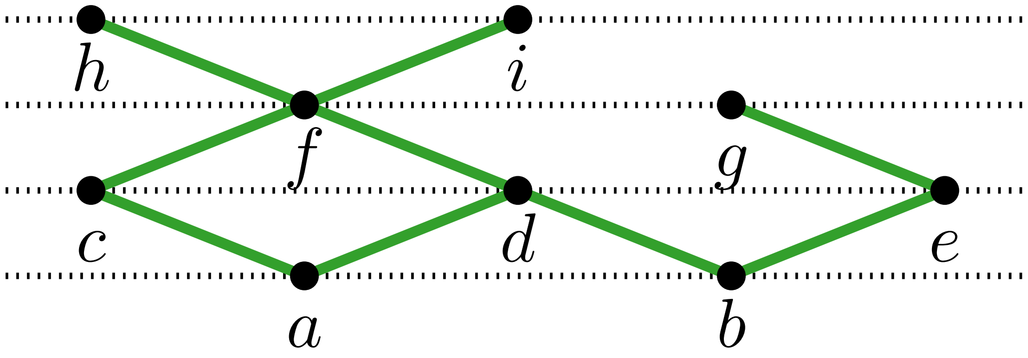

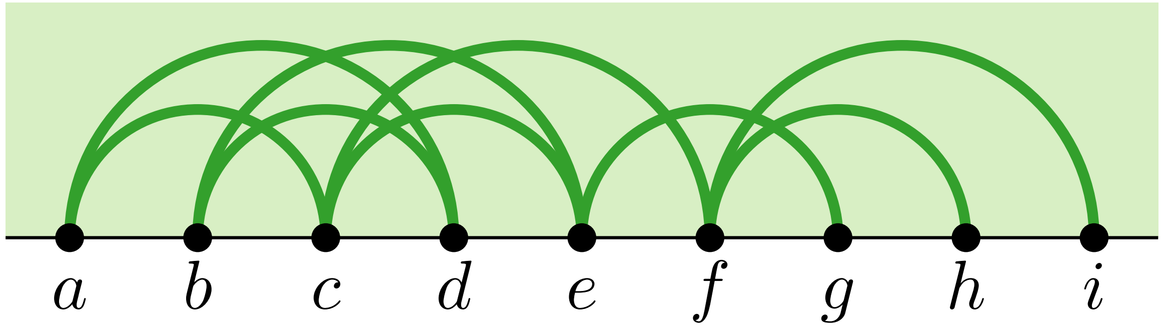

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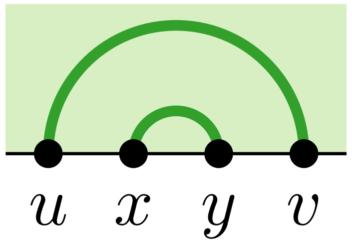
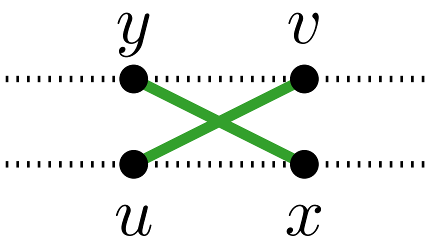
- If there was a nesting uv above xy, u would be to the left of x on one level, and y would be to the left of v on the level above; this would not be planar. □





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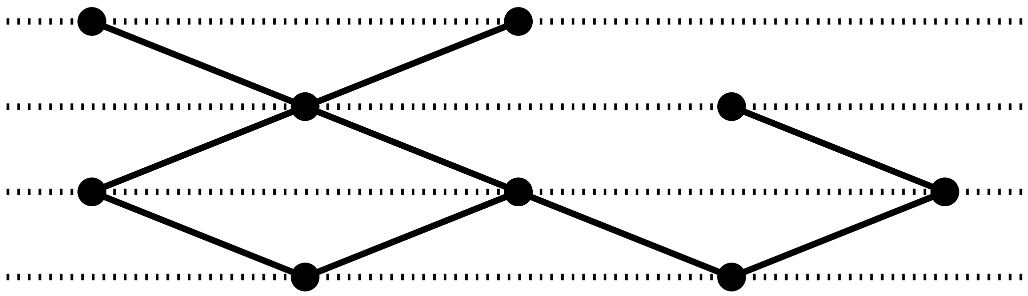

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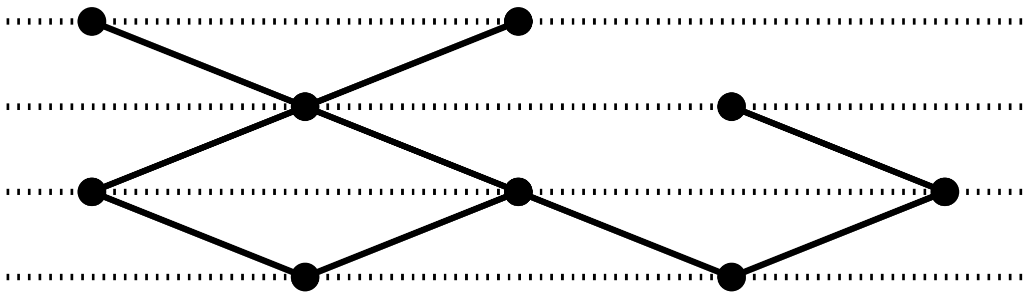

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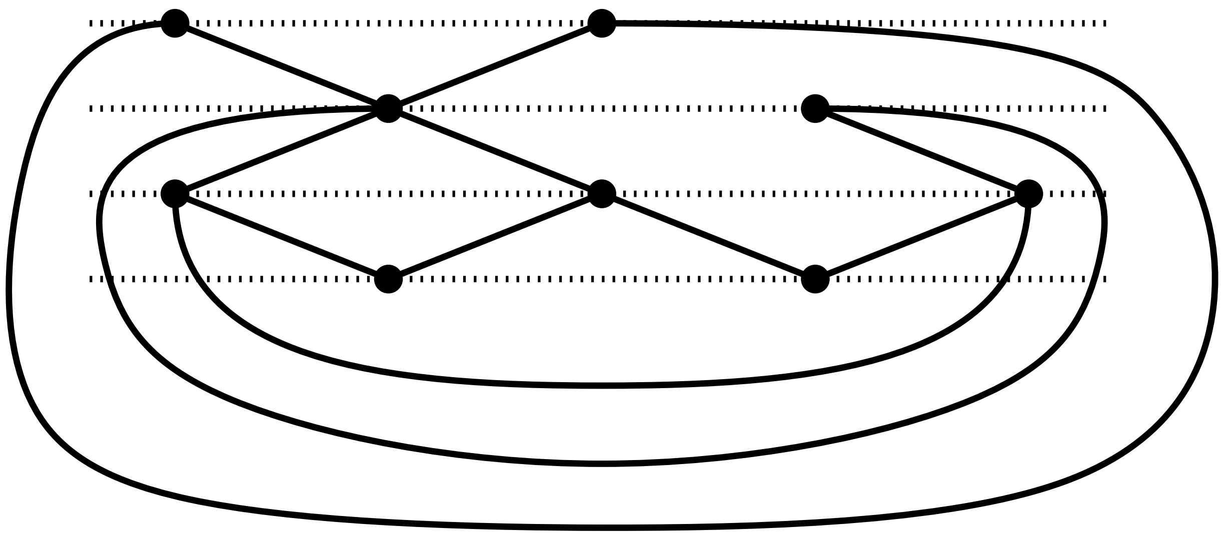
A graph is **arched leveled-planar** if it has a leveled-planar drawing where additionally vertices on the same level may be connected by edges that enclose all lower levels.



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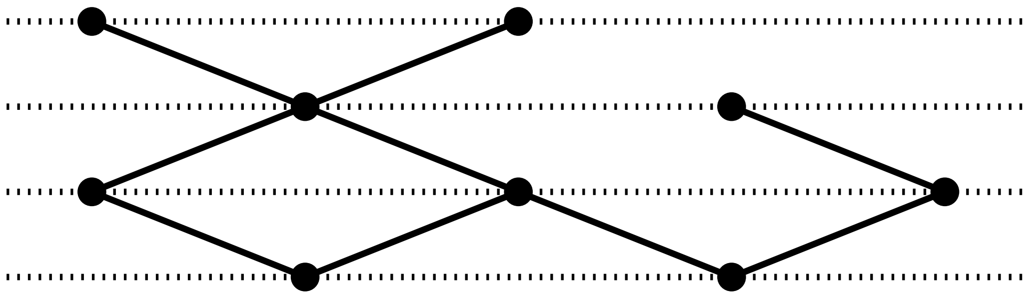

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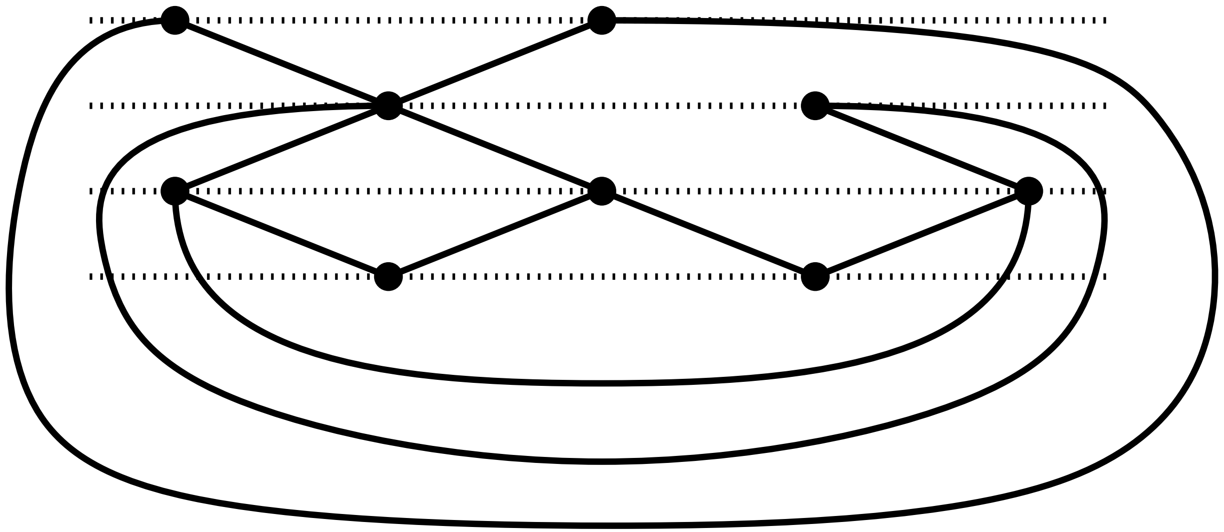
Proof. $\rightarrow$ *Exercise!*

A graph is **arched leveled-planar** if it has a leveled-planar drawing where additionally vertices on the same level may be connected by edges that enclose all lower levels.



2-Page and 3-Page Queue Layouts

Theorem. [Heath & Rosenberg 1992, Rengarajan & Veni Madhavan 1995.]
For every outerplanar graph G, $\mathrm{qn}(G) \leq 2$.

Theorem. [Rengarajan & Veni Madhavan 1996.]
For every series-parallel graph G, $\mathrm{qn}(G) \leq 3$.

Queue Layouts of Planar Graphs

We have seen planar graphs have **stack** number at most 4. What is the max. **queue** number?

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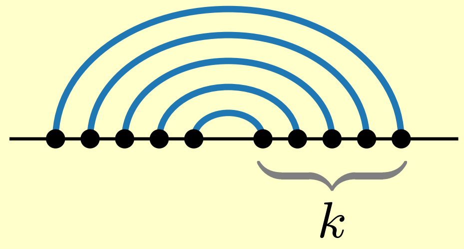

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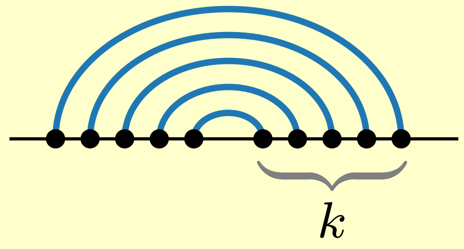

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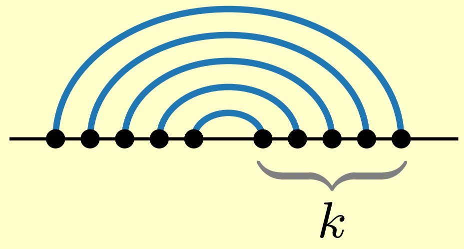

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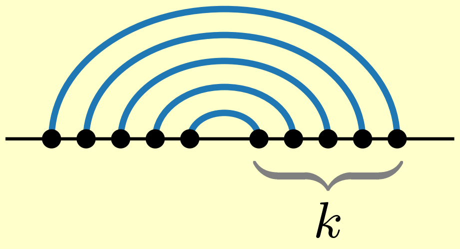

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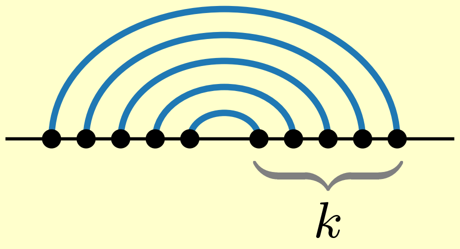

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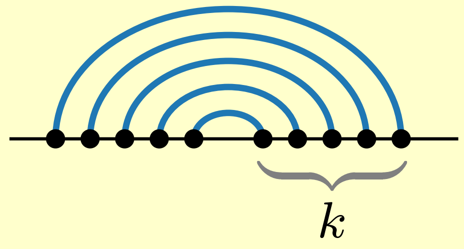

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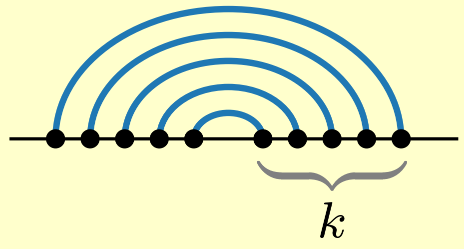

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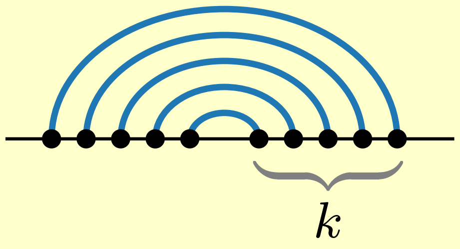

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- For the running time, see the implementation described by [Heath & Rosenberg 1992]. $\square$

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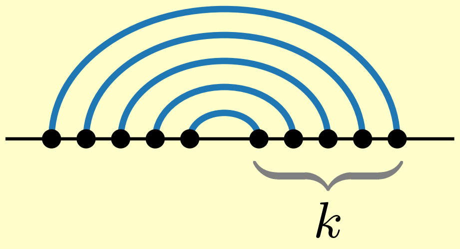

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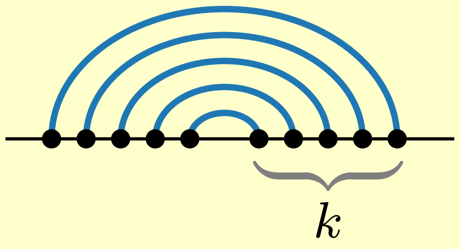

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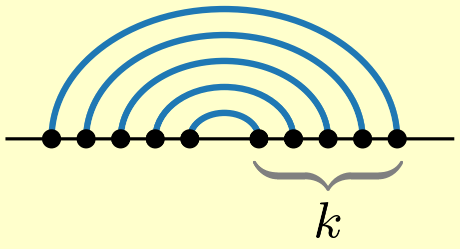

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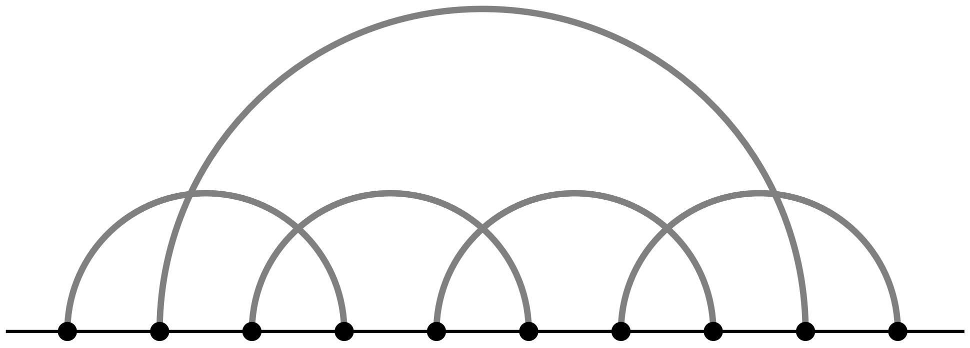

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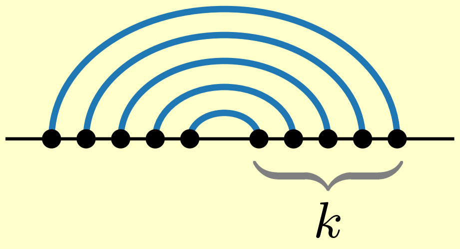

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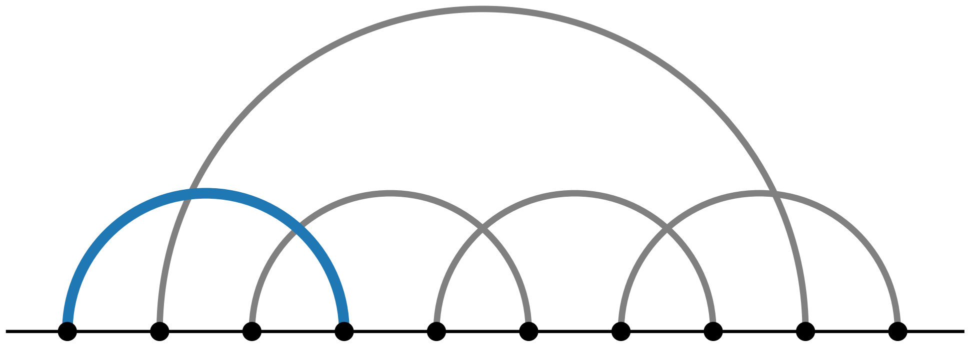

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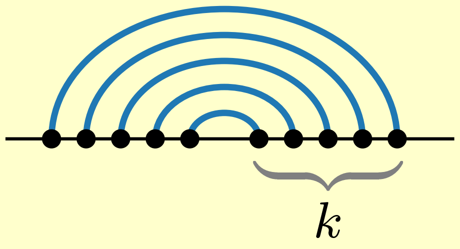
Lemma 1. [Heath & Rosenberg 1992]
For a graph G, let an order $\prec$ of $V(G)$ be given. If k is the size of a largest rainbow in G under vertex order $\prec$, then there is a k-page queue layout of G with vertex order $\prec$ on the spine.
Such a layout can be found in $\mathcal{O}(|E(G)| \log \log n)$ time.

A **rainbow** of size k (or k-**rainbow**) is a set of k pairwise nesting edges.



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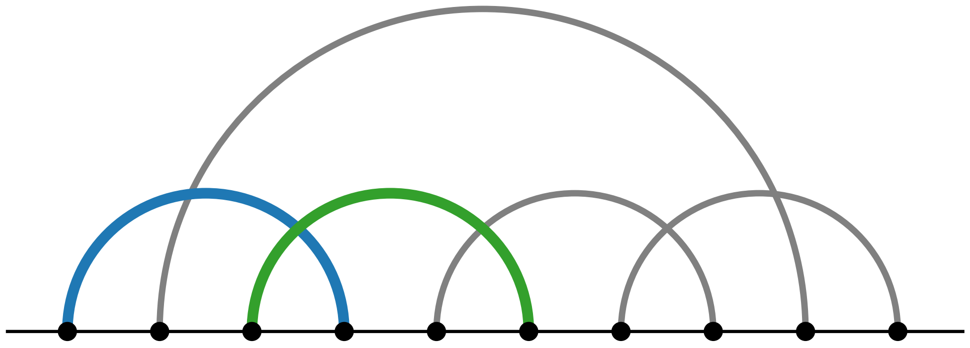

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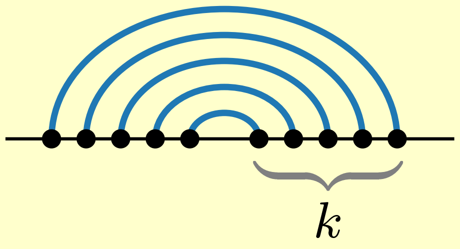

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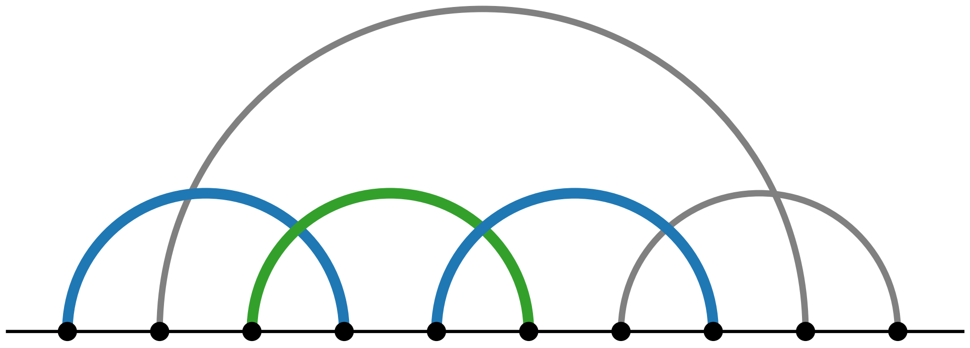

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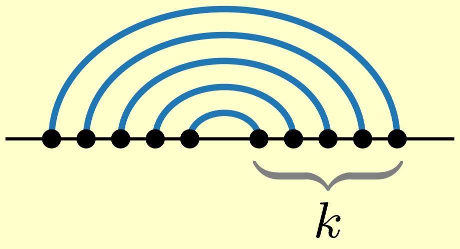

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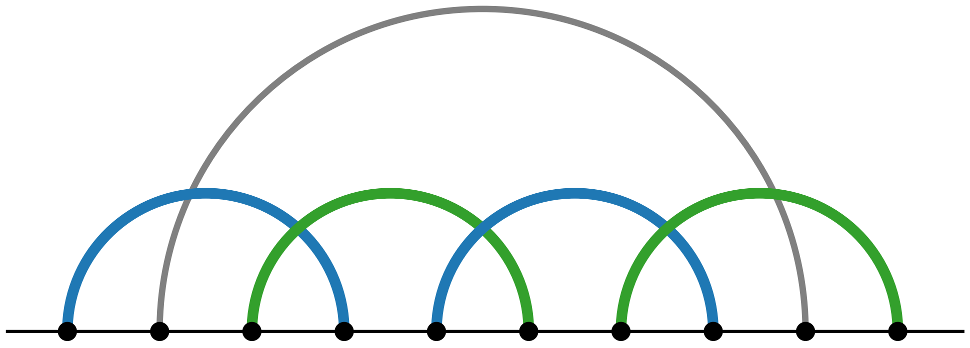

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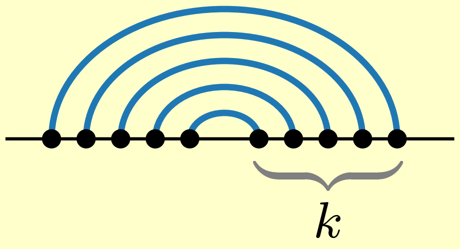

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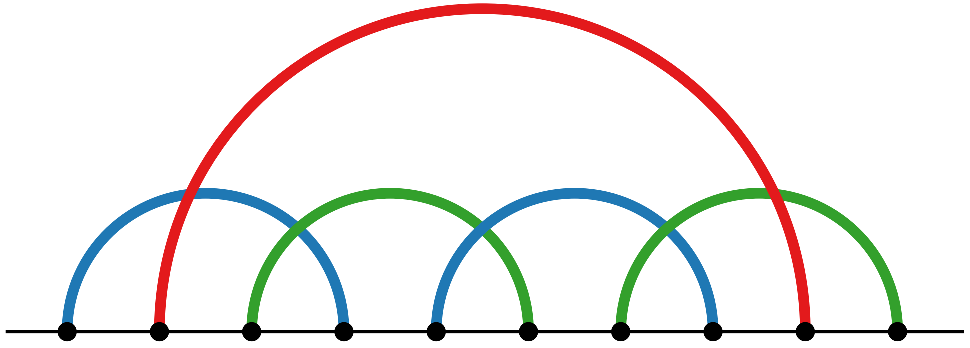

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Queue Layouts of Complete Graphs

The stack number can be linear in n. What about the the queue number of K_n?

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- These are $n/2$ pairwise nesting edges ($n/2$-**rainbow**).
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- With n vertices, there cannot be any rainbow having a size larger than $n/2$.
- Then, $\mathrm{qn}(K_n) \leq n/2$, follows directly from Lemma 1. □

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So, is determining the stack number easier if the order of the vertices on the spine is given?

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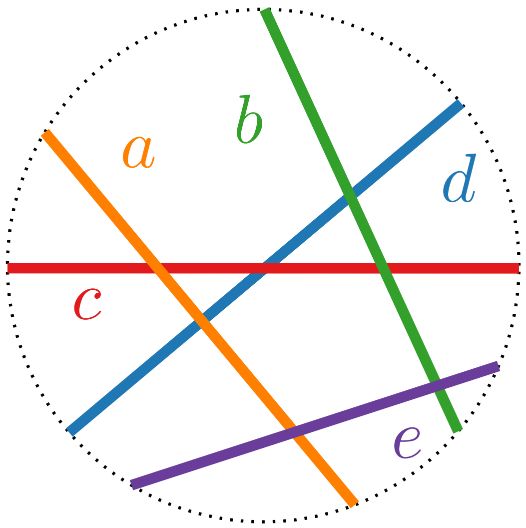

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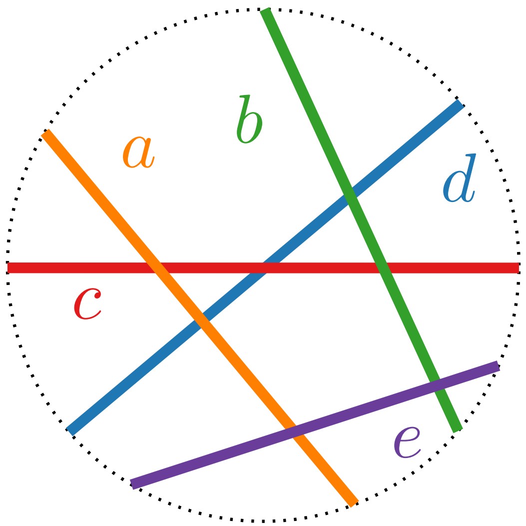

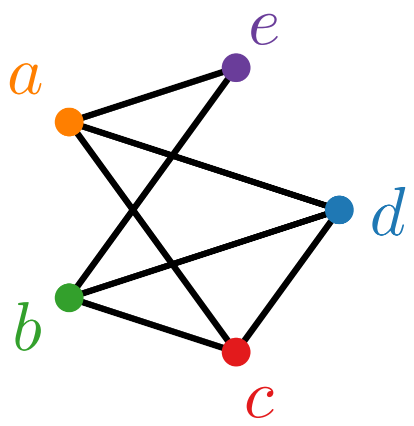

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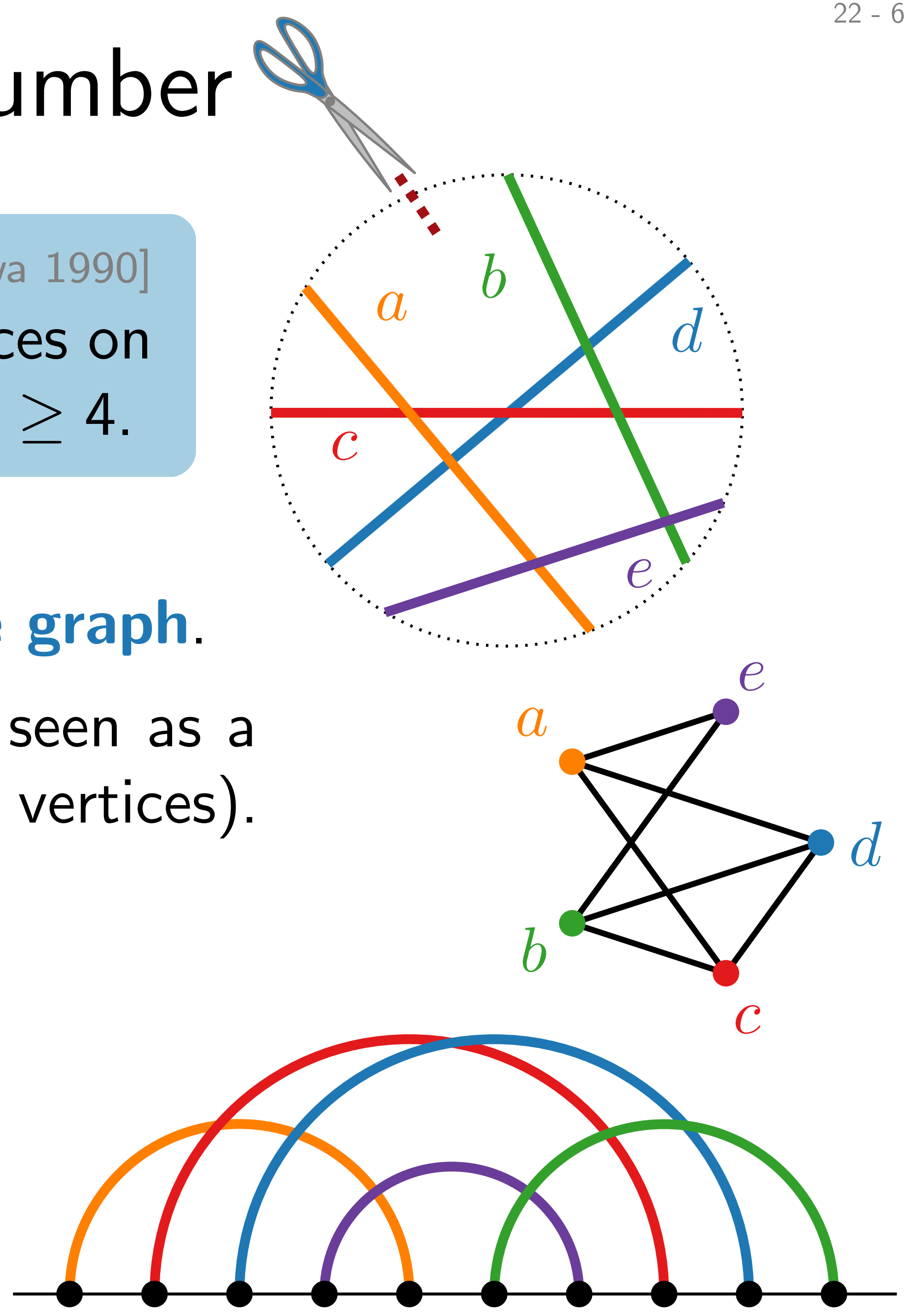

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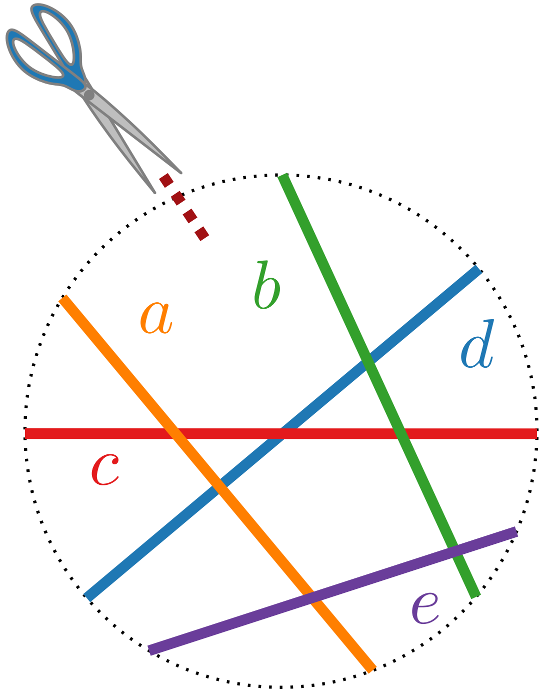

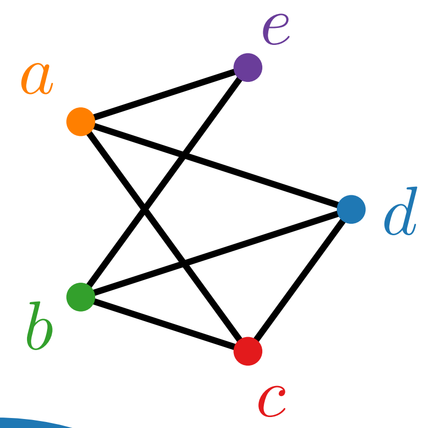

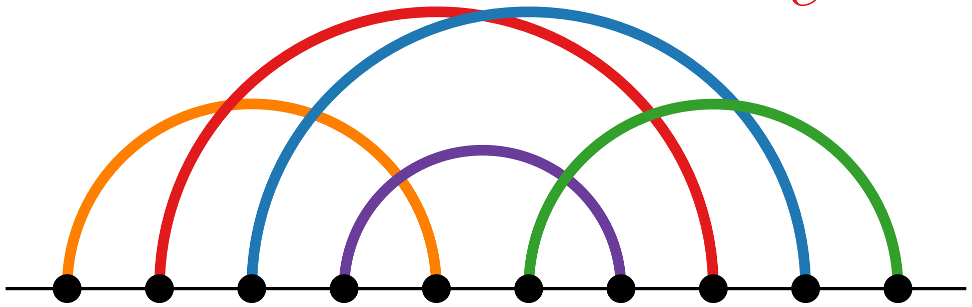

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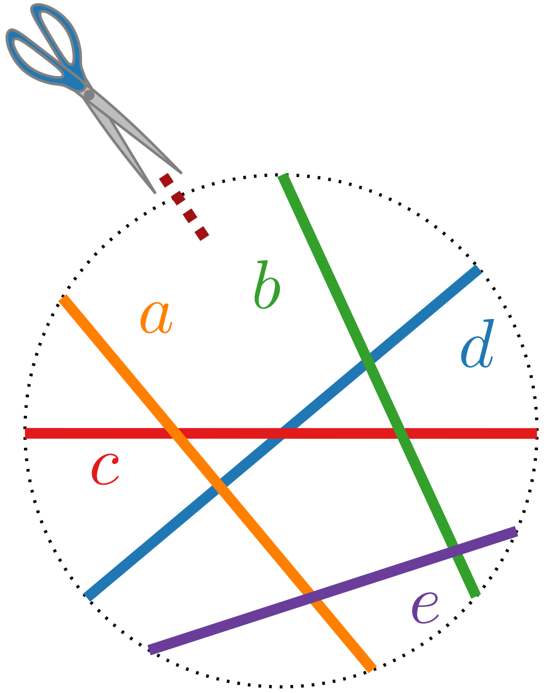

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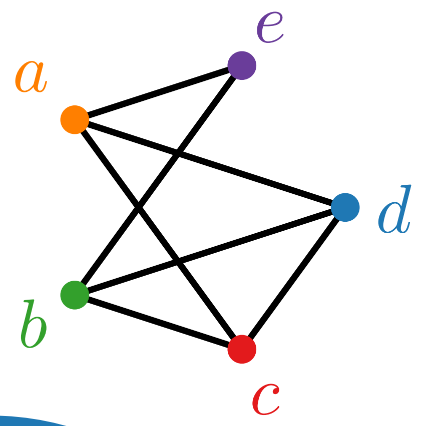

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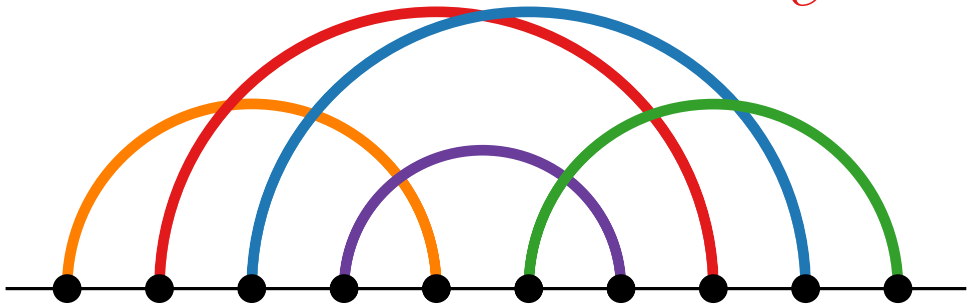

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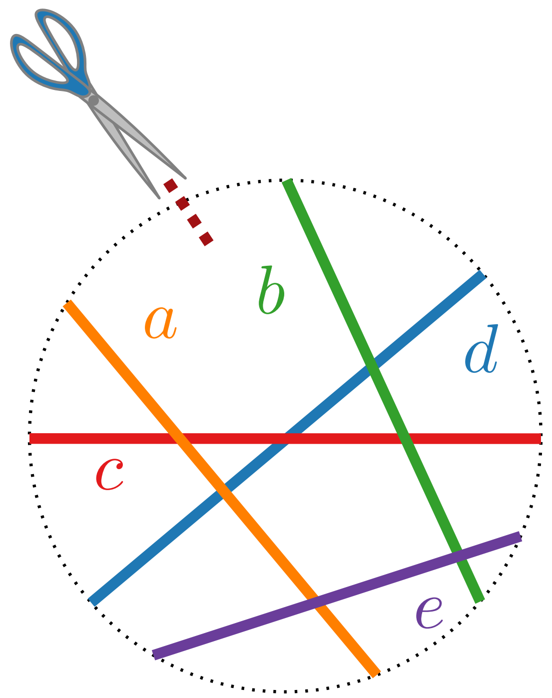

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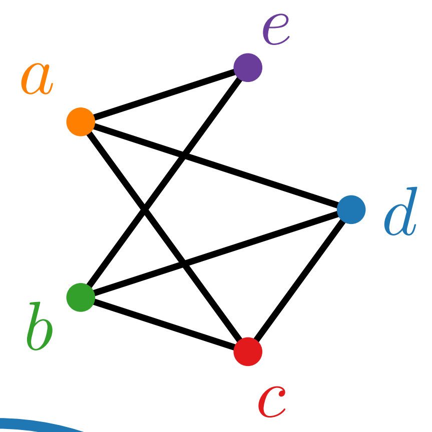

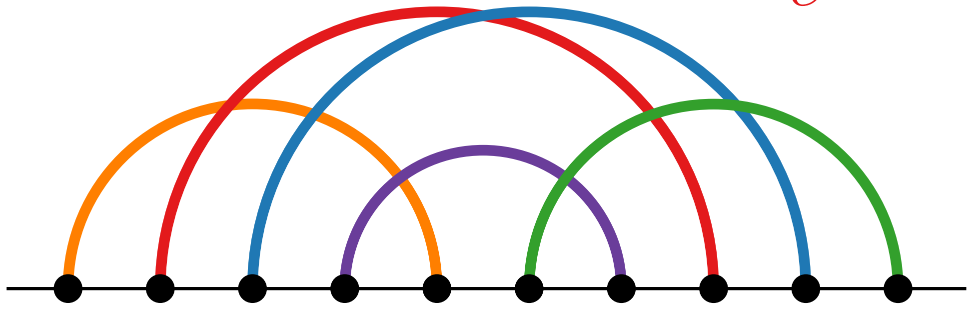

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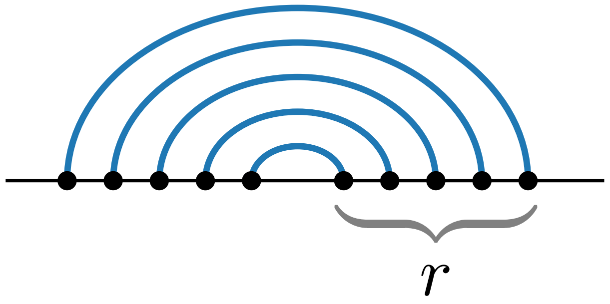

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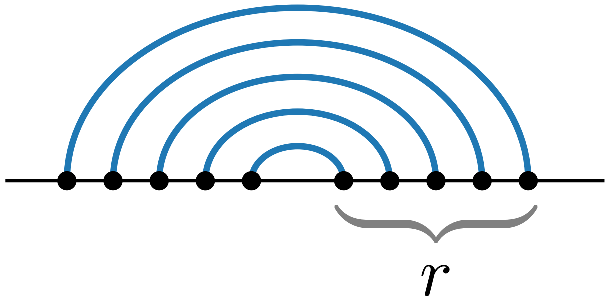
Details in exercise!

- Determine the size r of the largest rainbow in polynomial time.



Complexity of Determining the Queue Number

Theorem. [Heath & Rosenberg 1992]
Deciding whether a graph G has queue number $\mathrm{qn}(G) \leq k$ is NP-complete for $k \geq 1$.

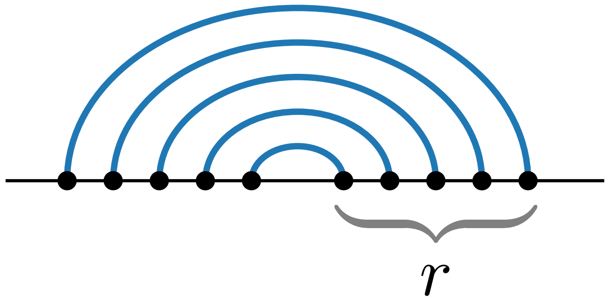
Proof Sketch.

- Deciding whether a given graph is arched leveled-planar is NP-complete.
  Hence, deciding whether $\mathrm{qn}(G) = 1$ is NP-hard. $\square$

Theorem. [Heath & Rosenberg 1992]
Deciding whether a graph G given with an order of the vertices on the spine has queue number $\mathrm{qn}(G) \leq k$ is polynomial-time solvable.

Proof Sketch.

Details in exercise!



- Determine the size r of the largest rainbow in polynomial time.
- If $r \leq k$, then there is k-page queue layout due to Lemma 1. $\square$

Discussion

- There are surprisingly many applications of stack and queue layouts, e.g., in computational biology (RNA folding), VLSI design, traffic control, . . .

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- There are surprisingly many applications of stack and queue layouts, e.g., in computational biology (RNA folding), VLSI design, traffic control, . . .
- By the book-embedding paradigm, **page number** and **book thickness** are alternative terms for *stack number*.
- There are many more variants, e.g., for fixed vertex order, directed graphs, using other data structures, . . .

Literature

Sources for the overview:

- [Ueckerdt 2022] Invited Talk on WG 2022: *Stack and queue layouts of planar graphs.*
- [Pupyrev 2024] Website on Linear Layouts:
  `https://spupyrev.github.io/linearlayouts.html`

Some of the referenced papers:

- [Bernhart & Kainen 1979] *The book thickness of a graph.*
- [Yannakakis 1986] *Embedding planar graphs in four pages.*
- [Heath & Rosenberg 1992] *Laying out graphs using queues.*
- [Bekos, Kaufmann, Klute, Pupyrev, Raftopoulou & Ueckerdt 2020]
  Four pages are indeed necessary for planar graphs.
- [Dujmović, Joret, Micek, Morin, Ueckerdt & Wood 2020]
  Planar graphs have bounded queue-number.
- [Bekos, Gronemann & Raftopoulou 2021]
  An improved upper bound on the queue number of planar graphs.