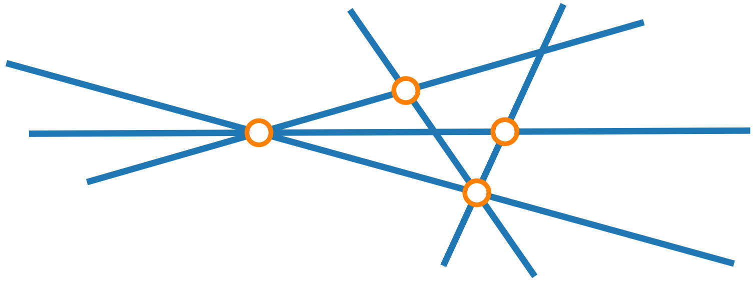
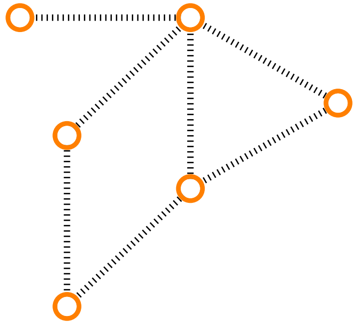
Visualization of Graphs

Lecture 9:
The Crossing Lemma
and Its Applications





Johannes Zink

Summer semester 2024

Crossing Number and Topological Graphs

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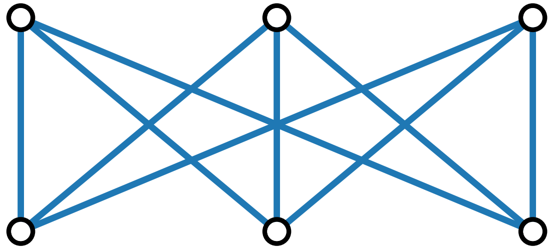

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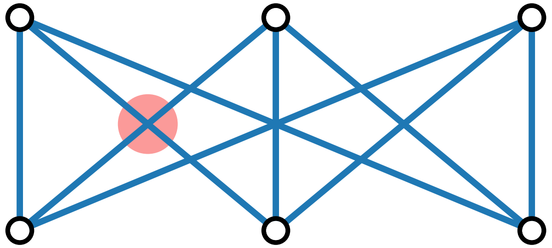

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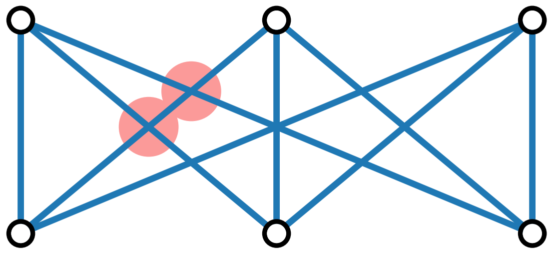

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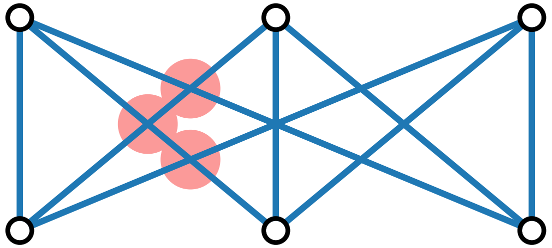

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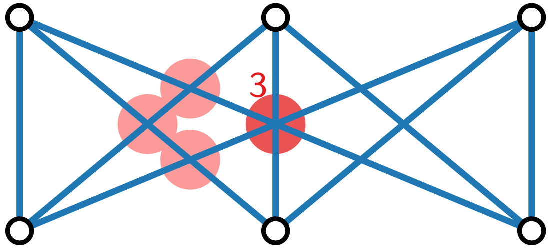

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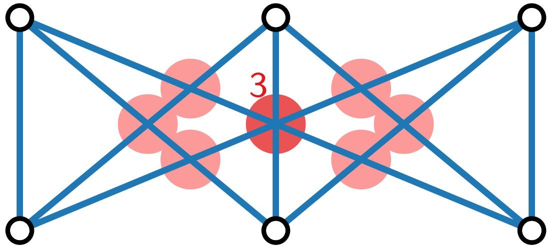

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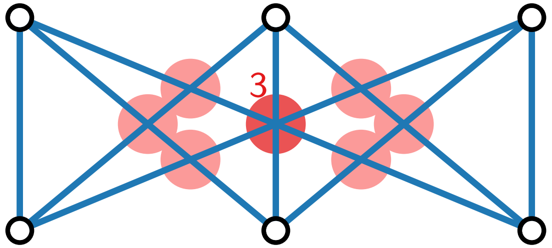

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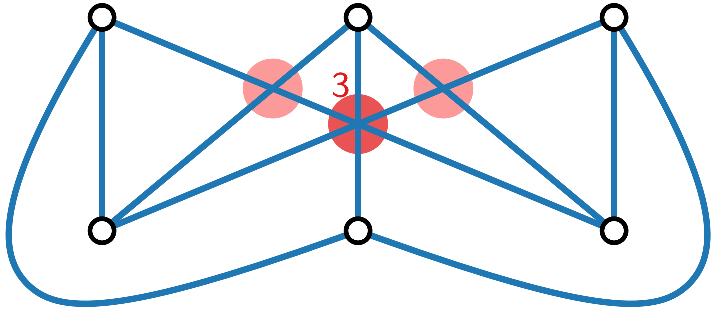

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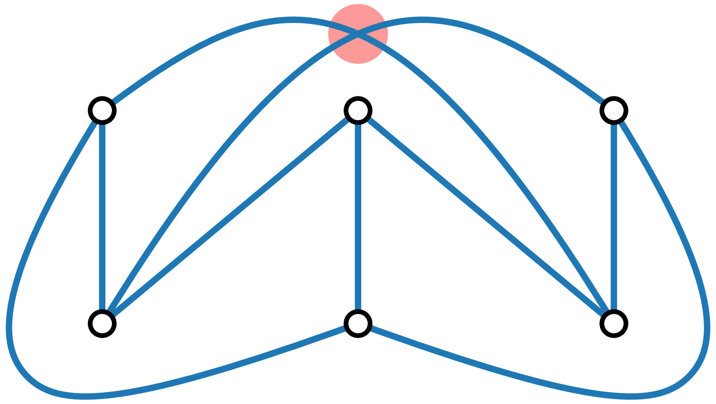

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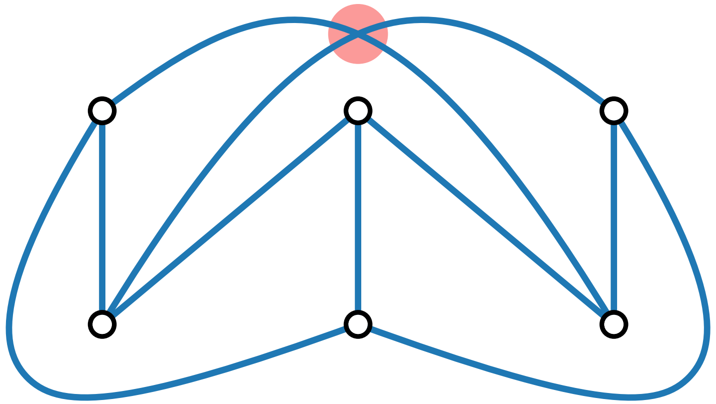

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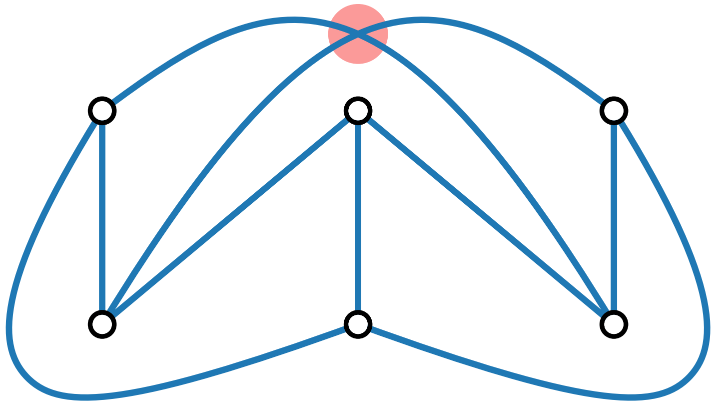

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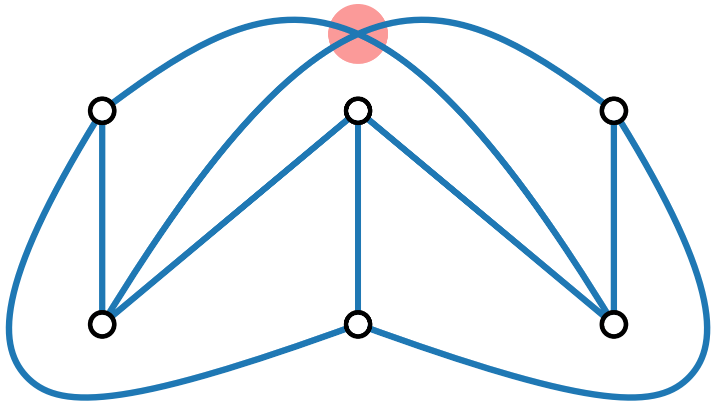

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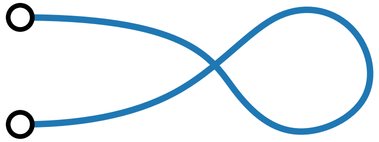

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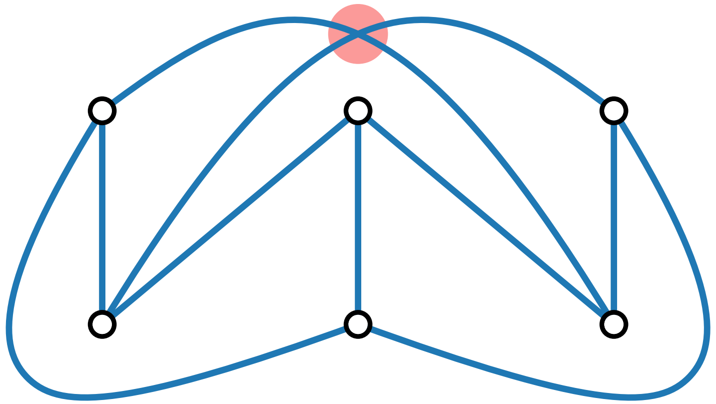

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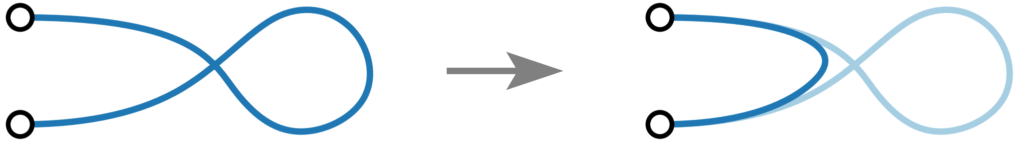

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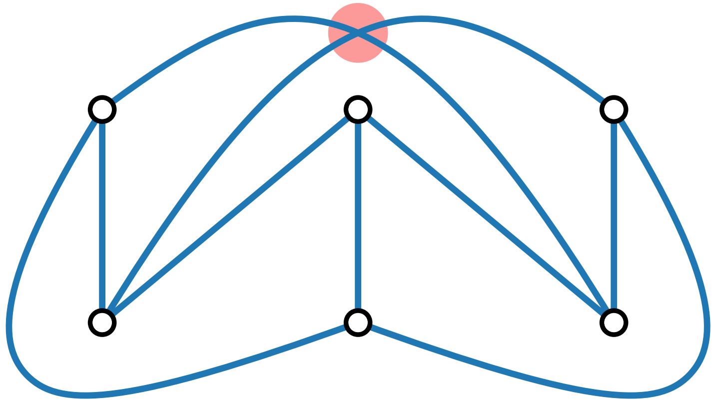

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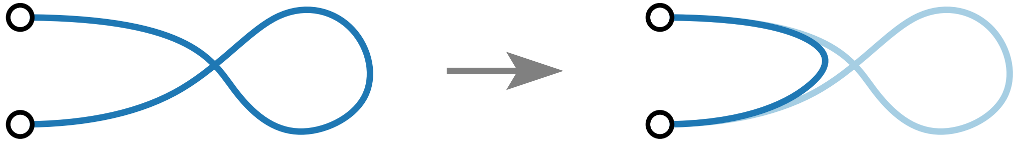

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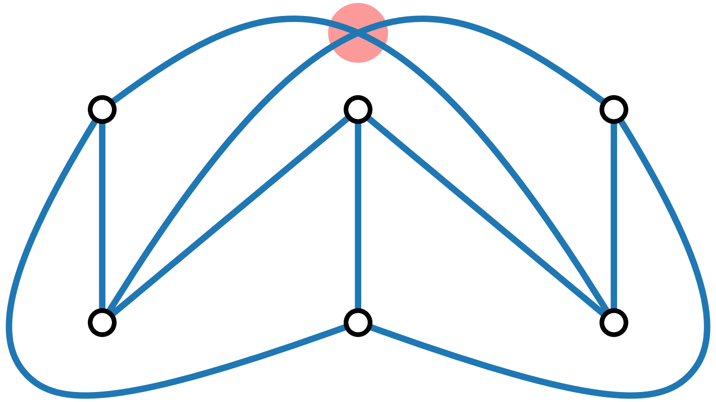

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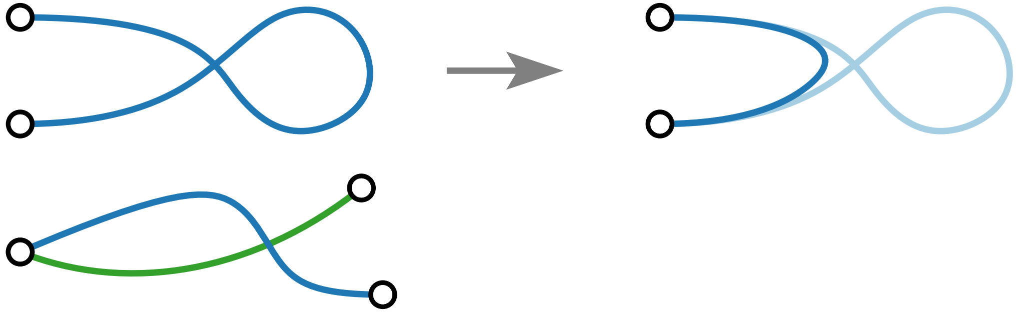

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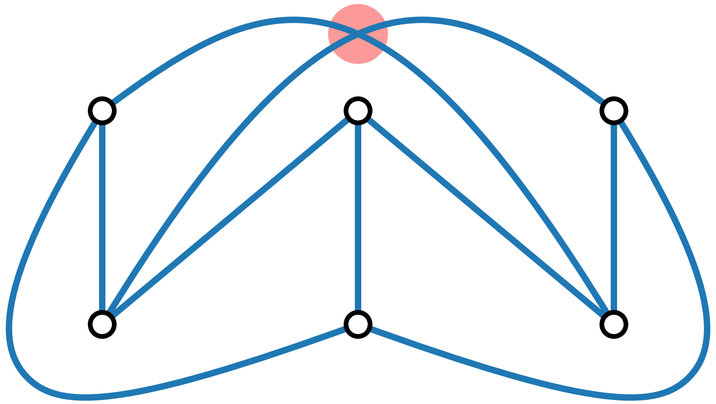

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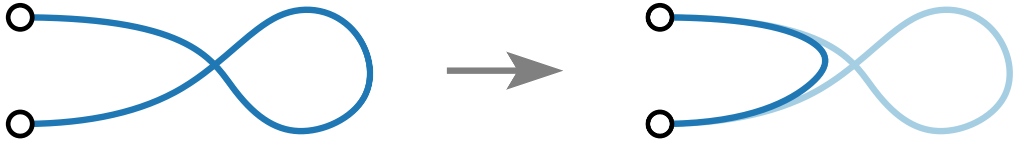

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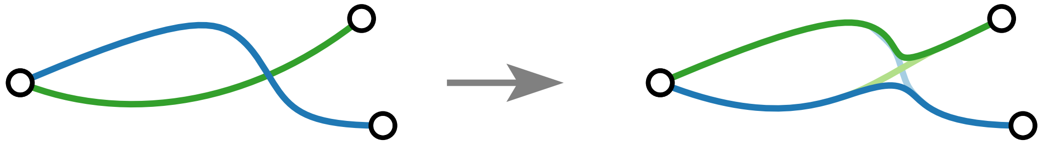

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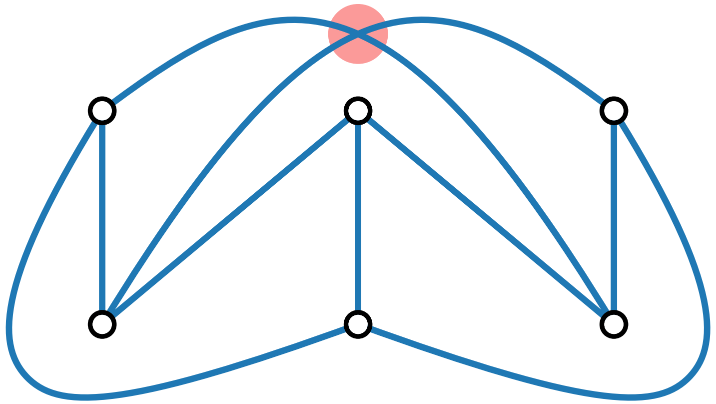

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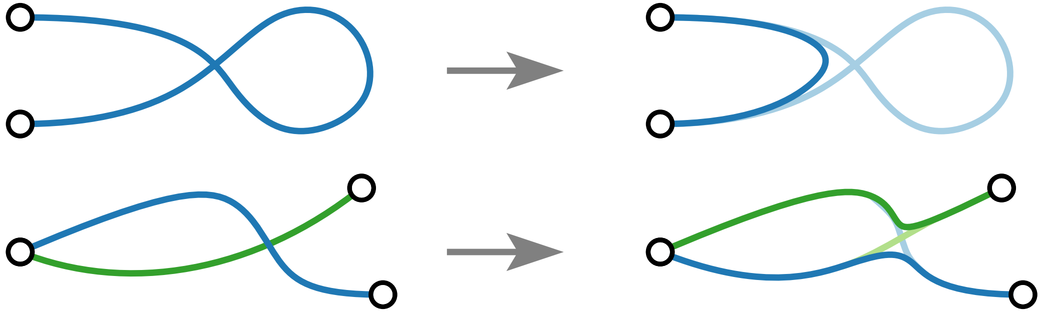

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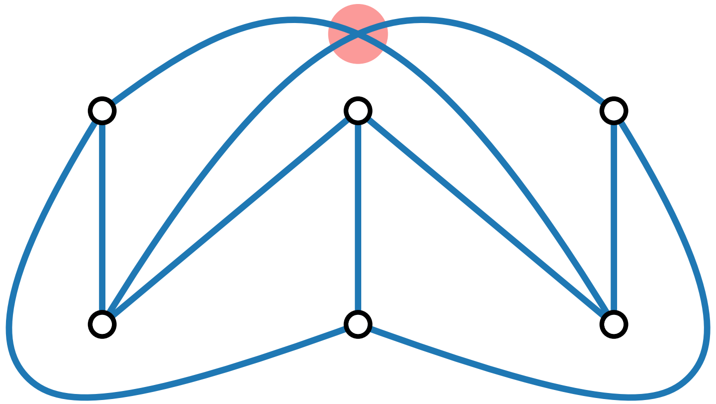

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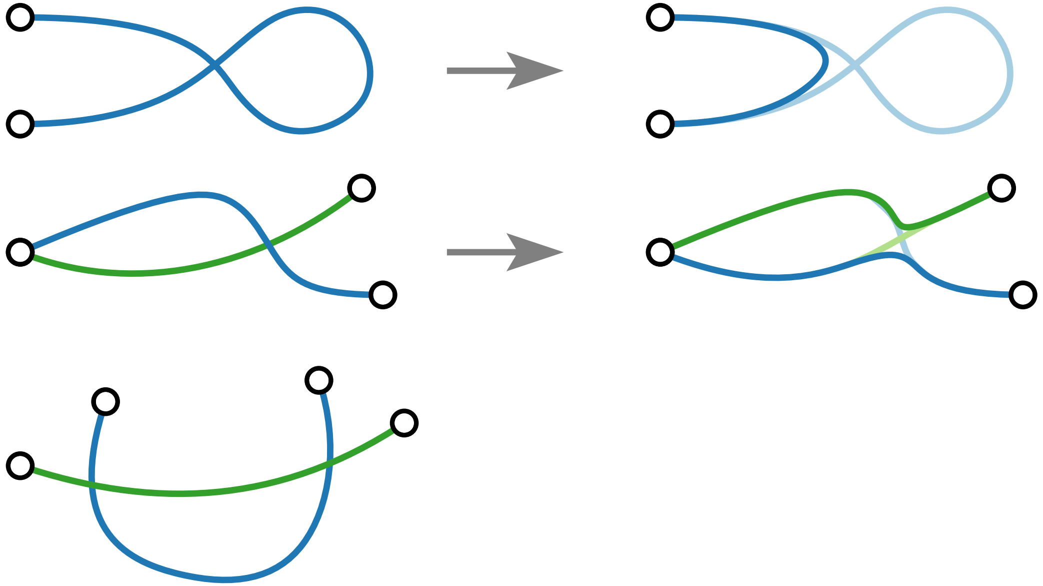

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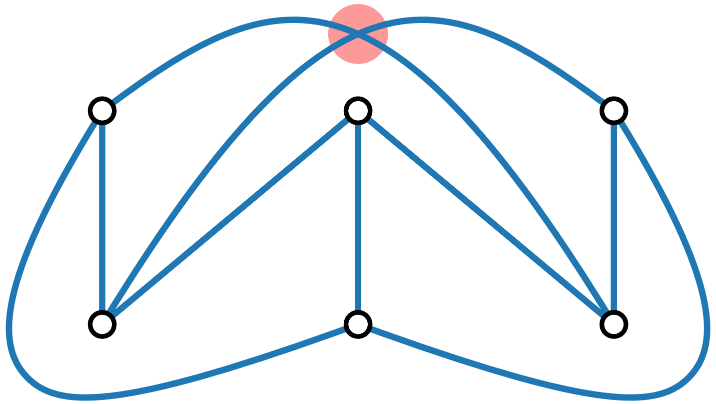

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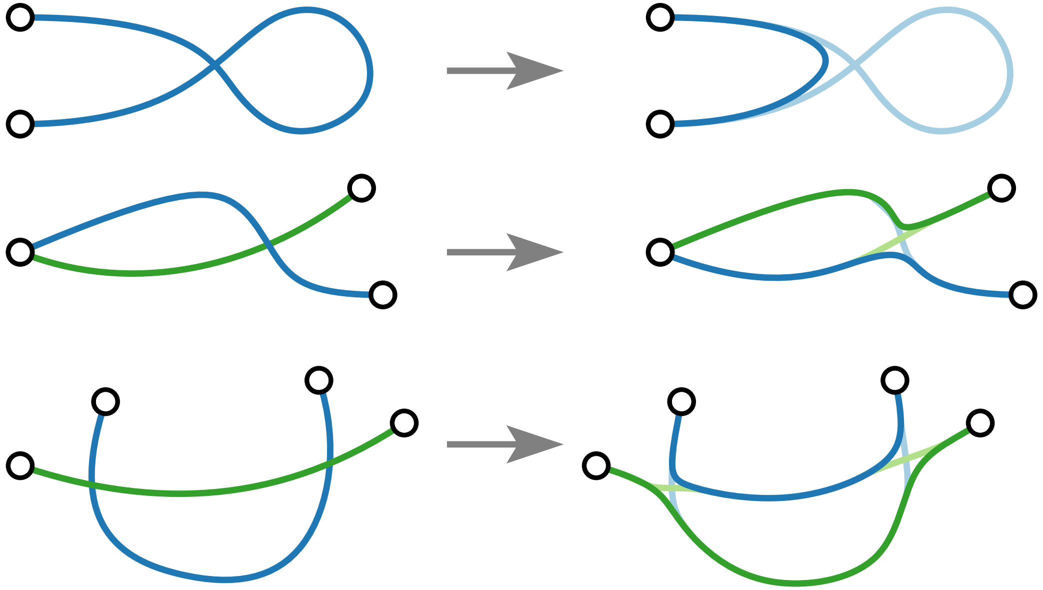

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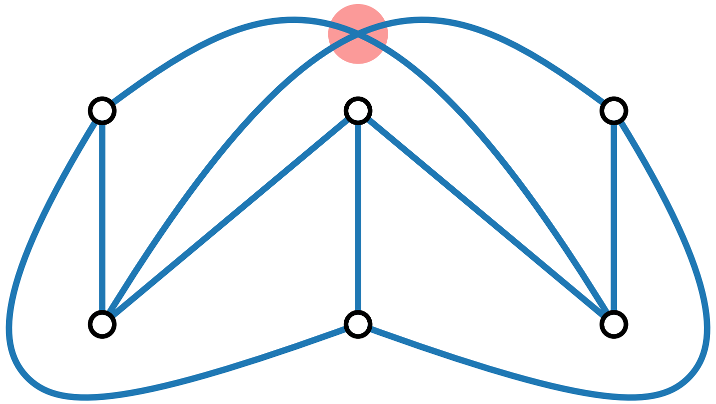

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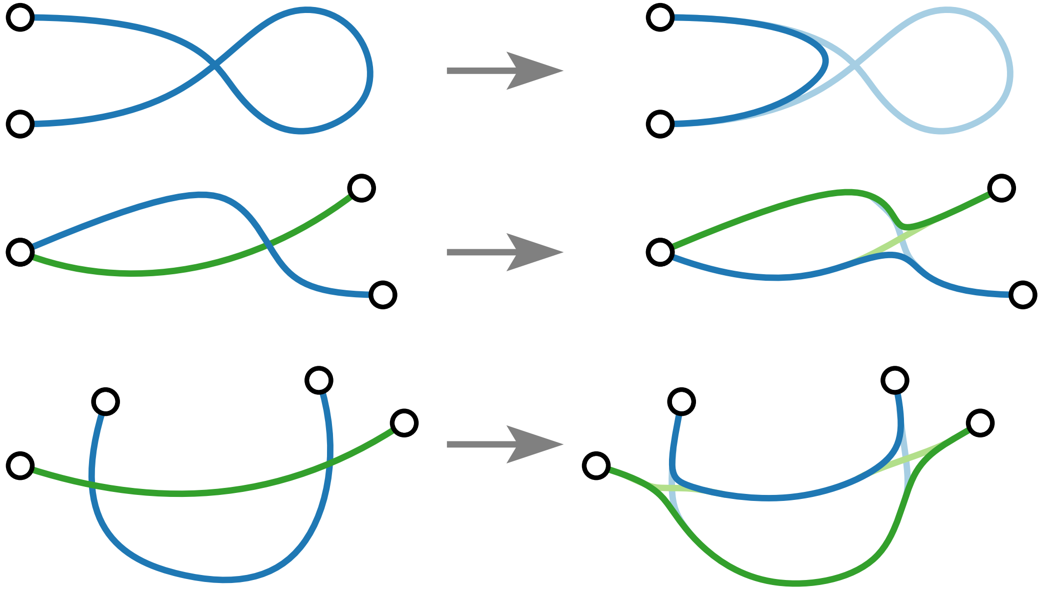

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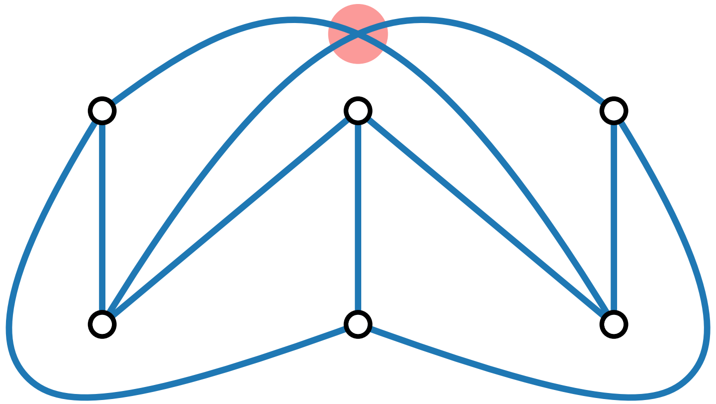

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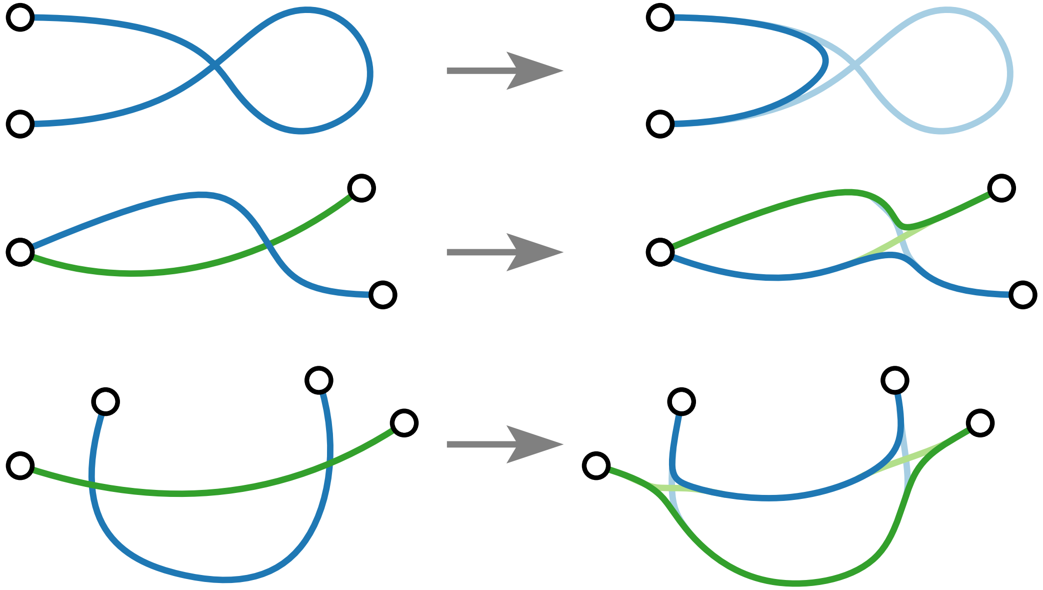

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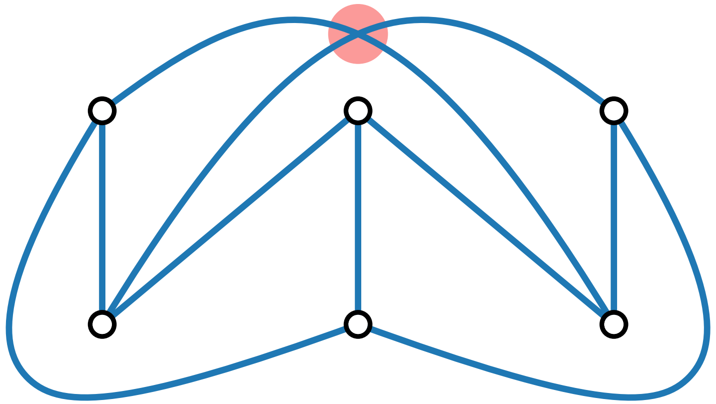

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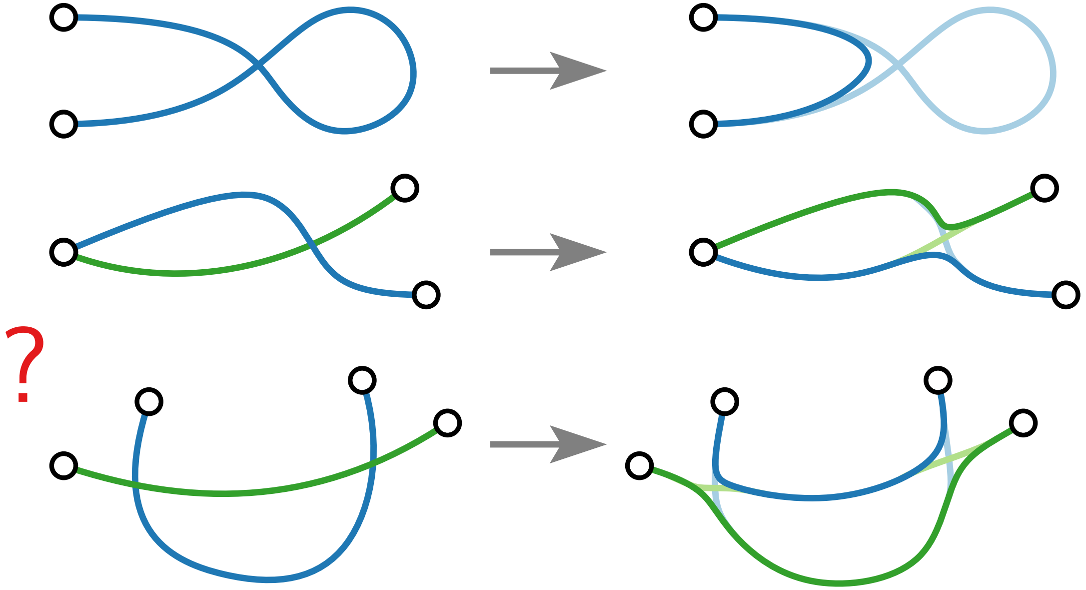

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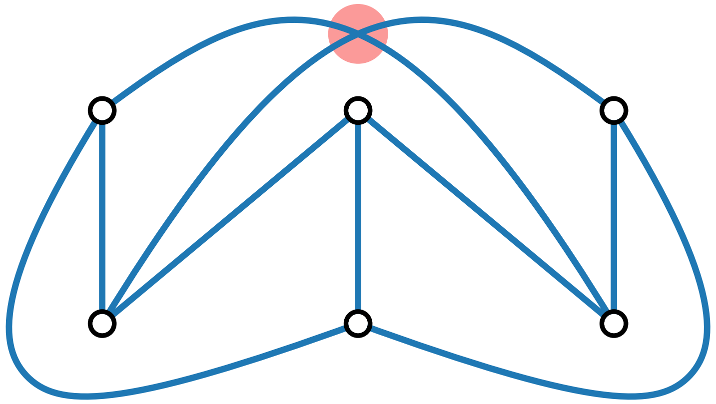

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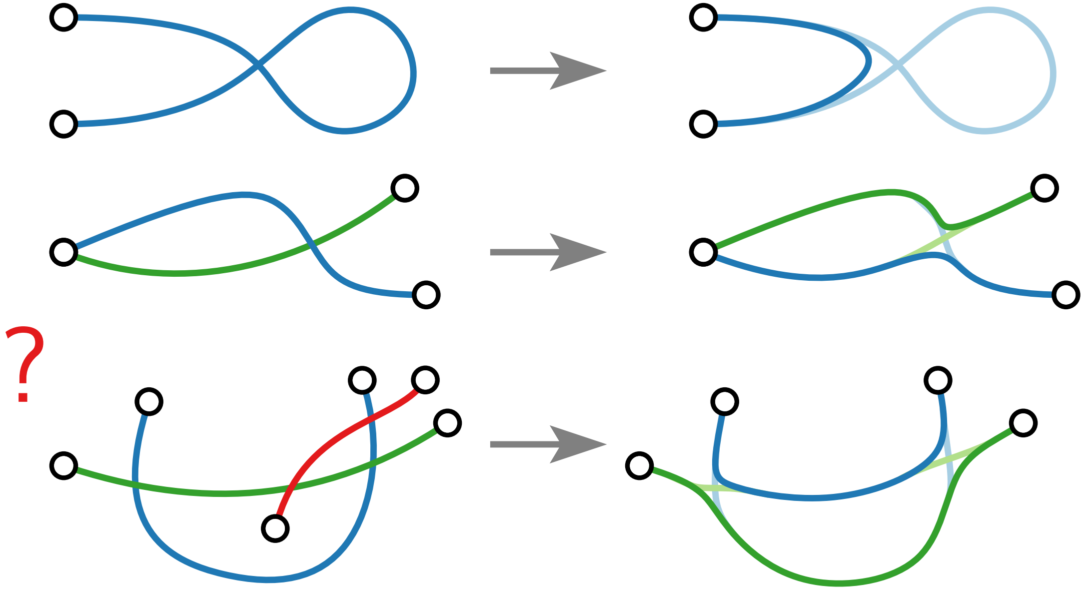

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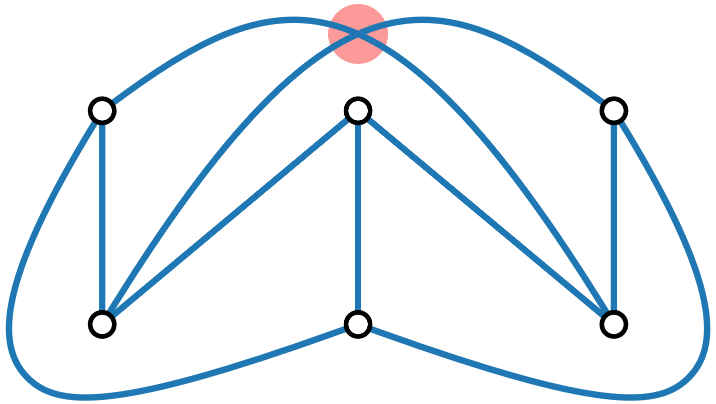

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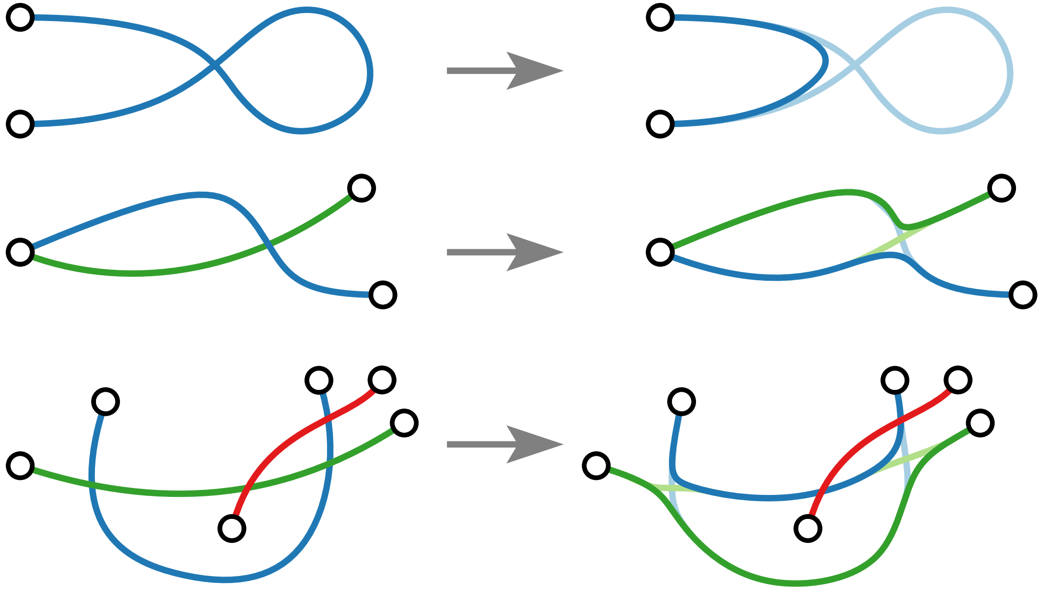

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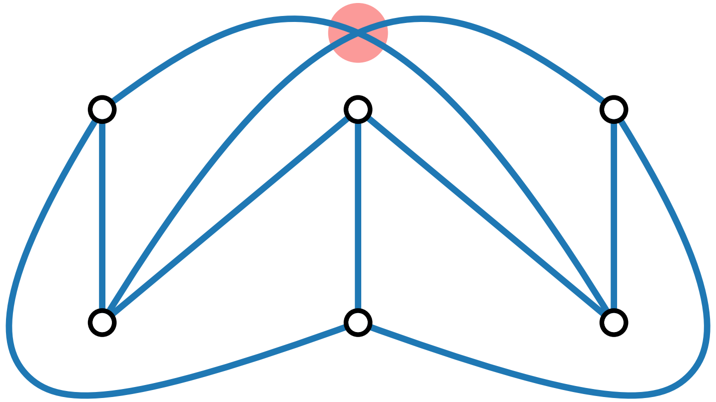

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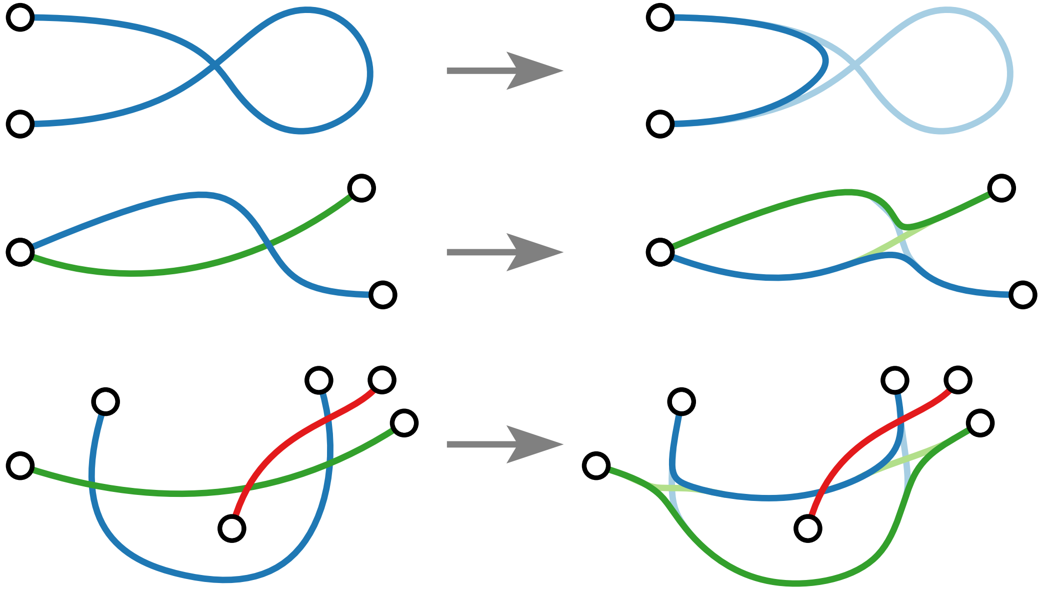

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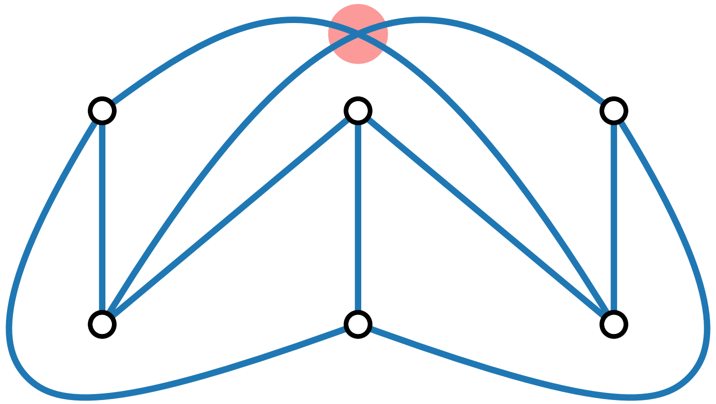

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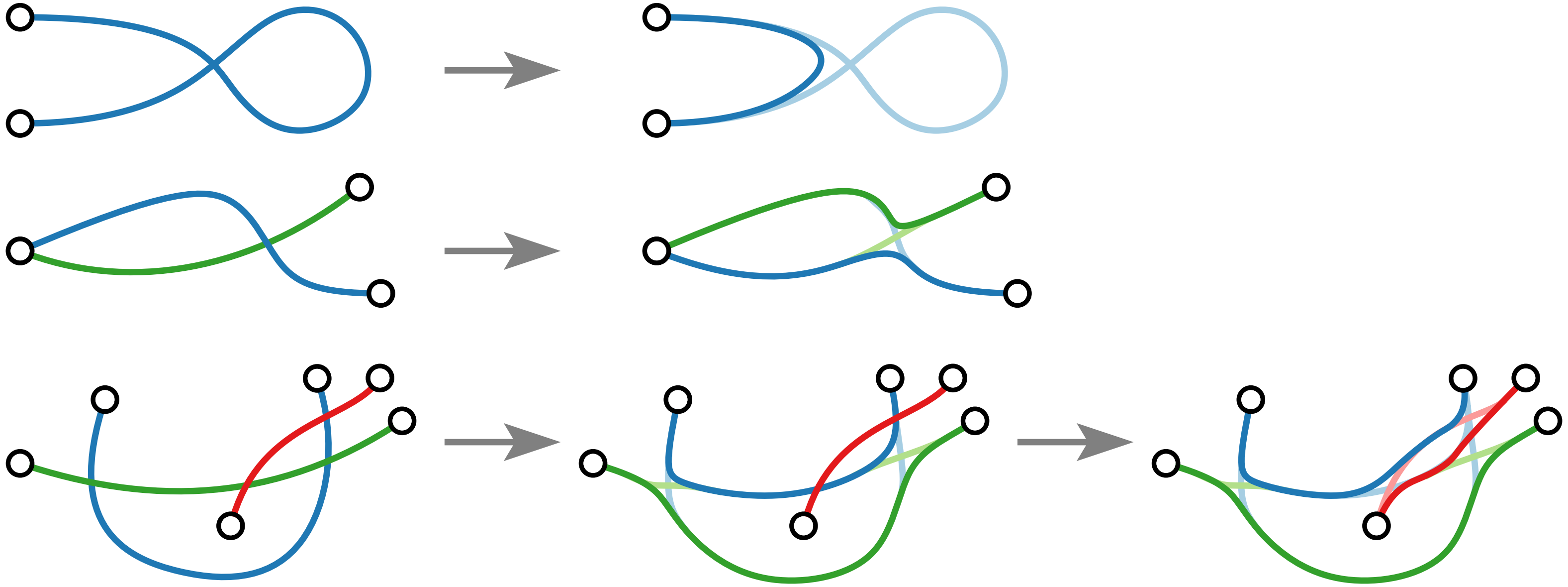

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Theorem. [Hanani '43, Tutte '70]

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Hence, there must be two edges on these paths that cross an odd number of times. $\square$

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Is $\text{ocr}(G) = \text{cr}(G)$?

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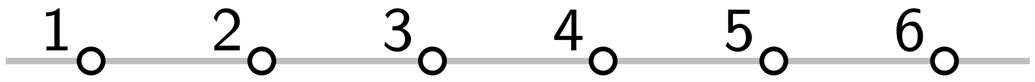

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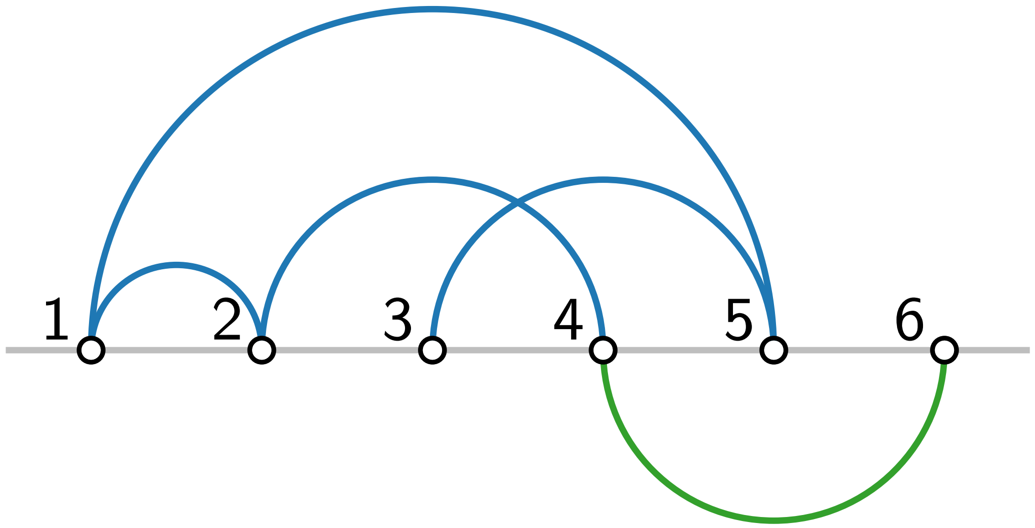

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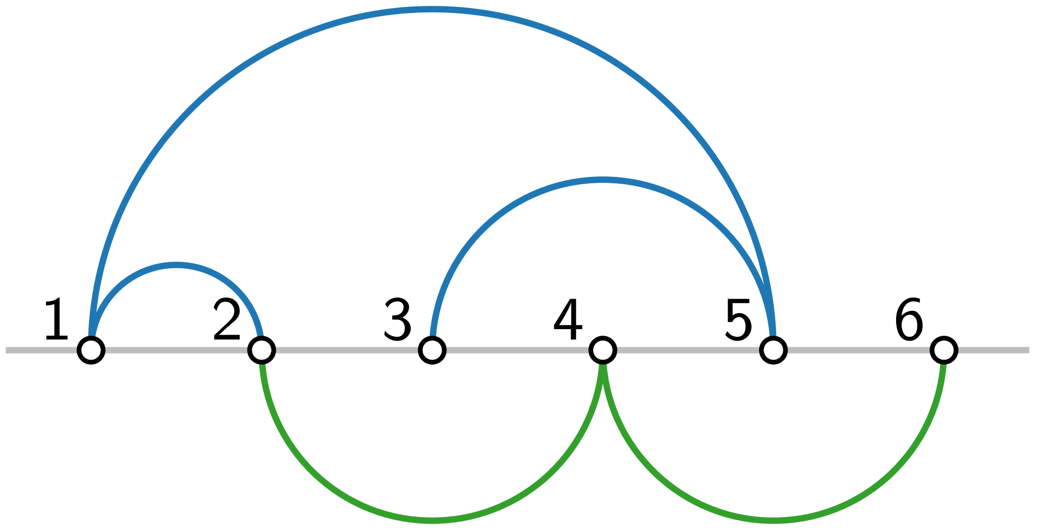

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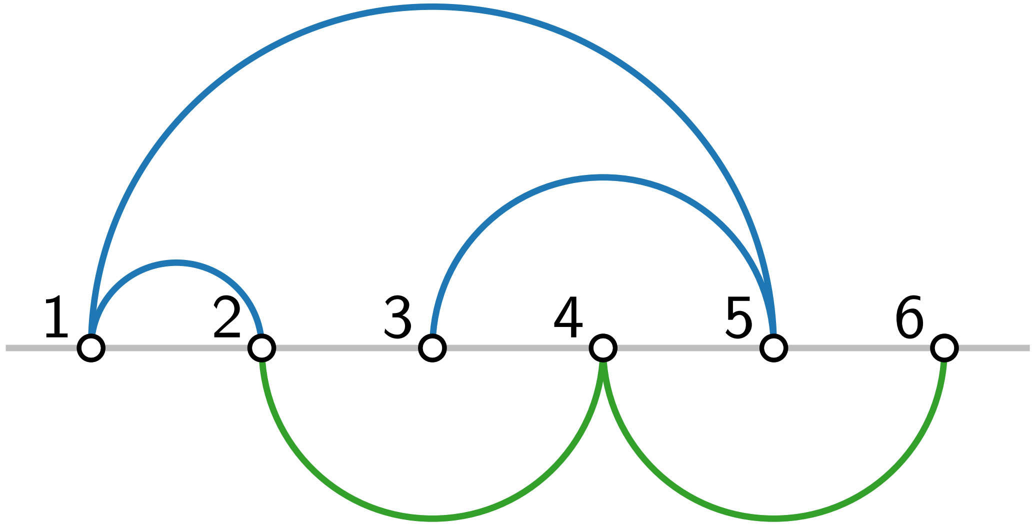

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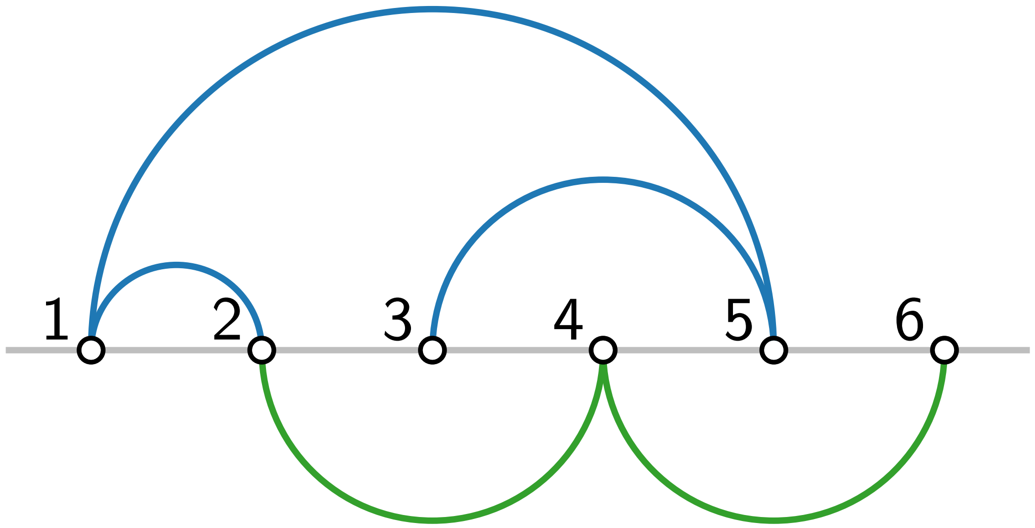

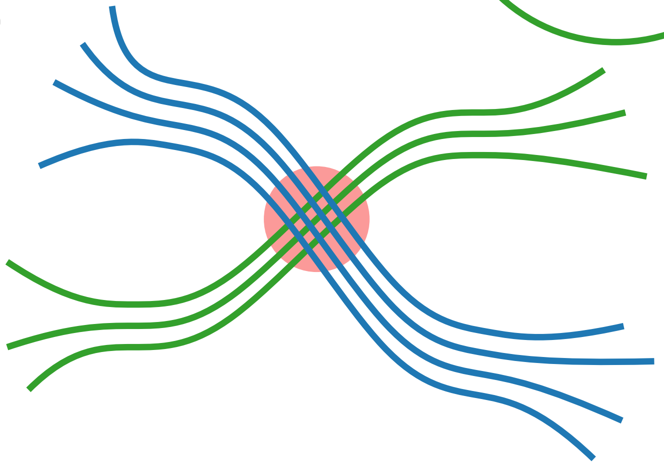

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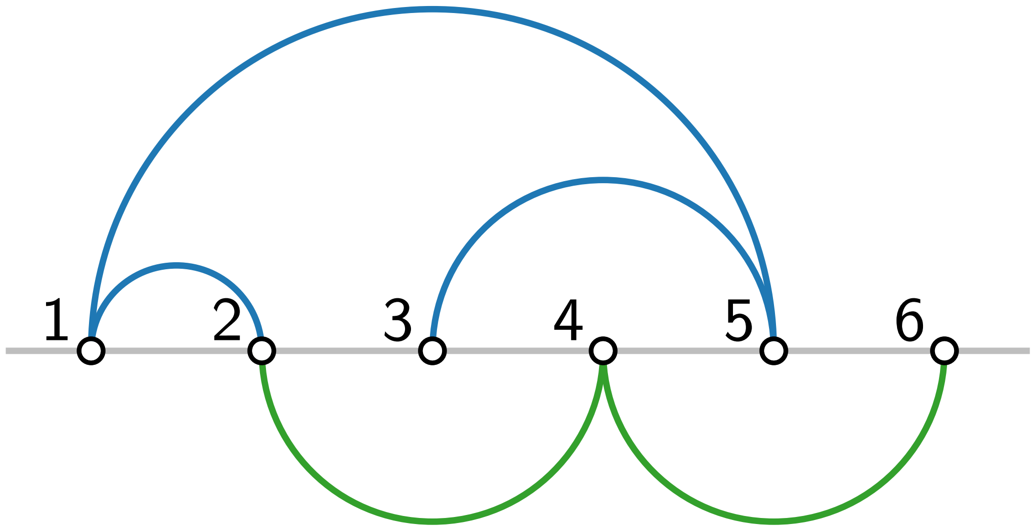

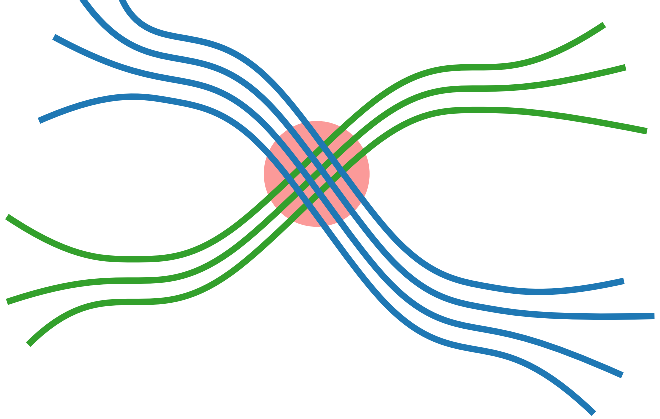

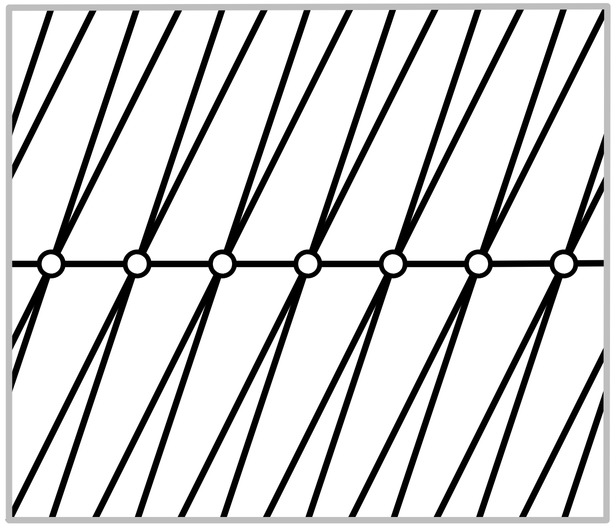

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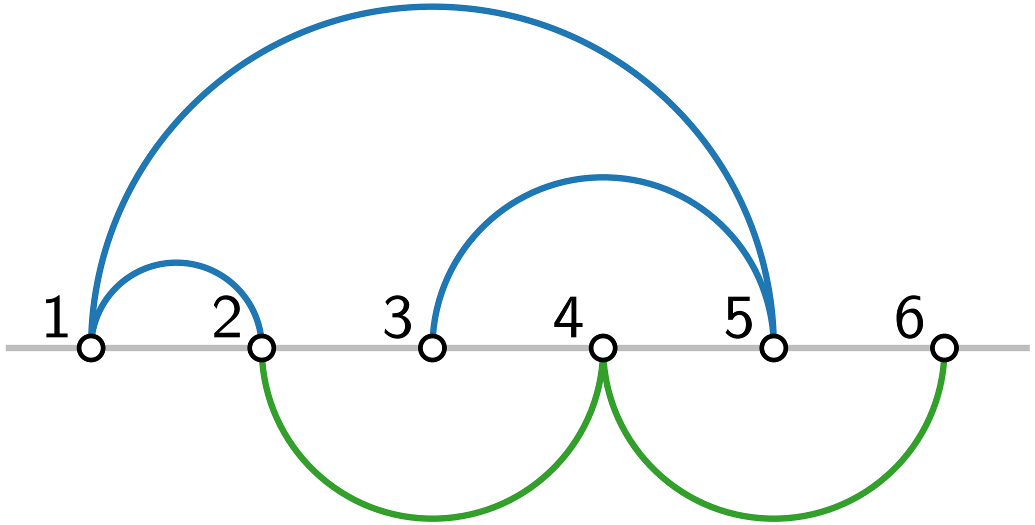

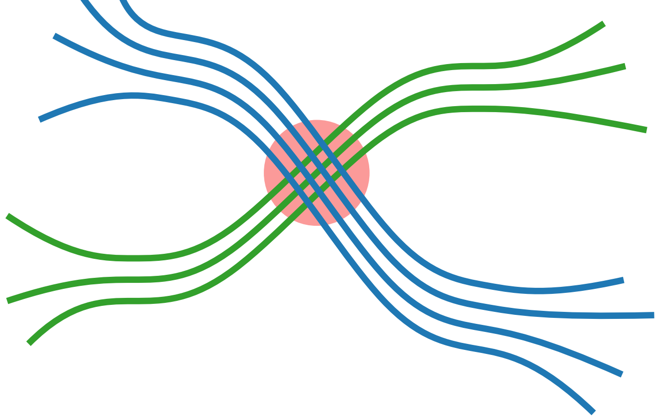

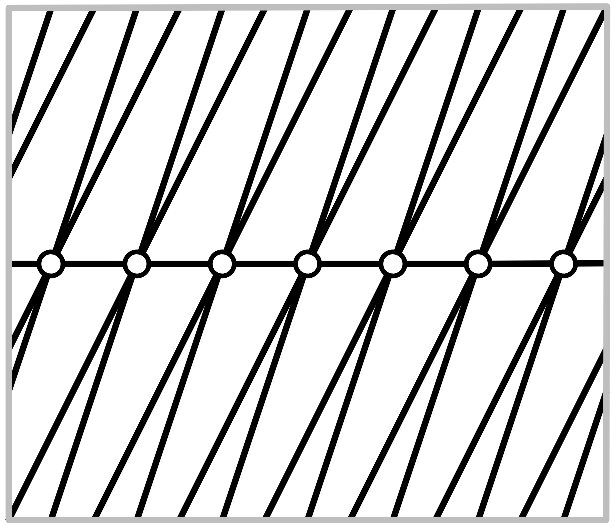

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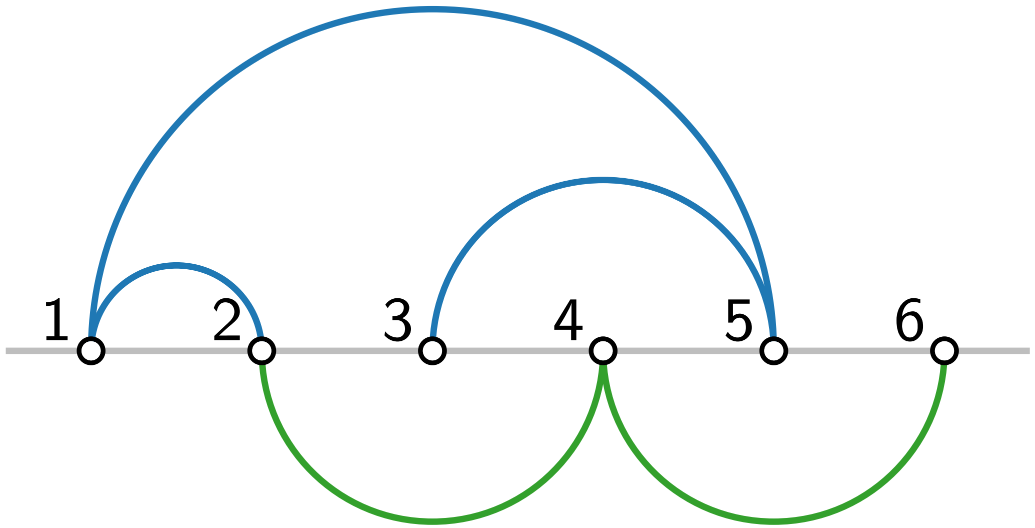

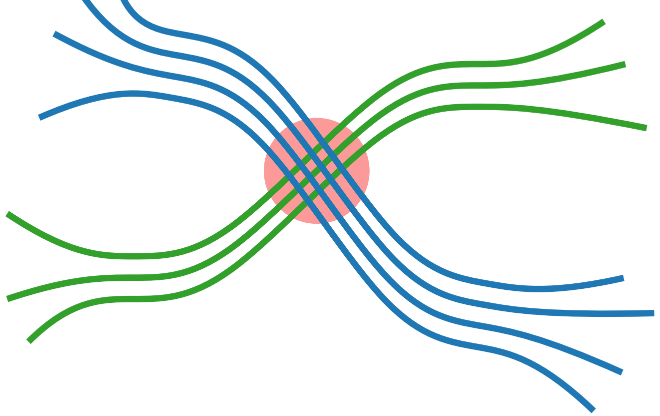

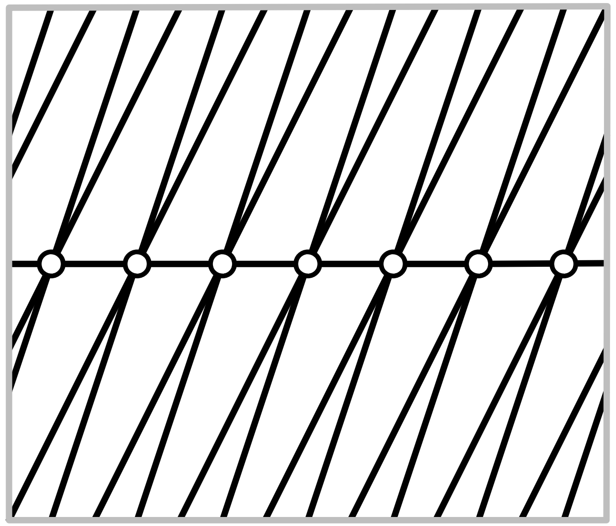

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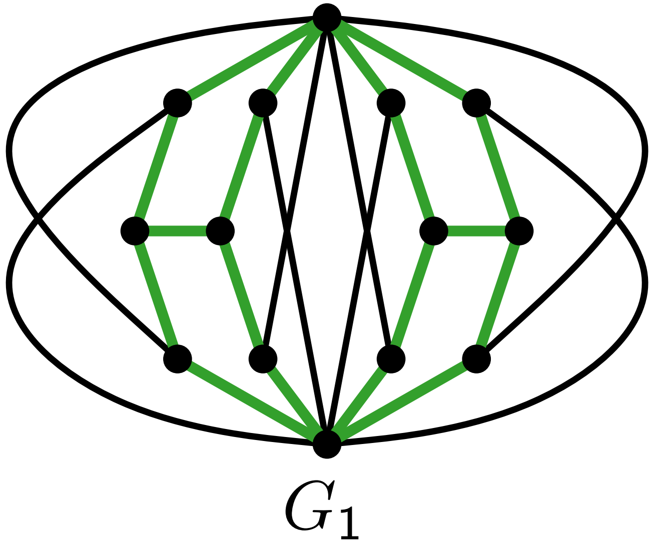

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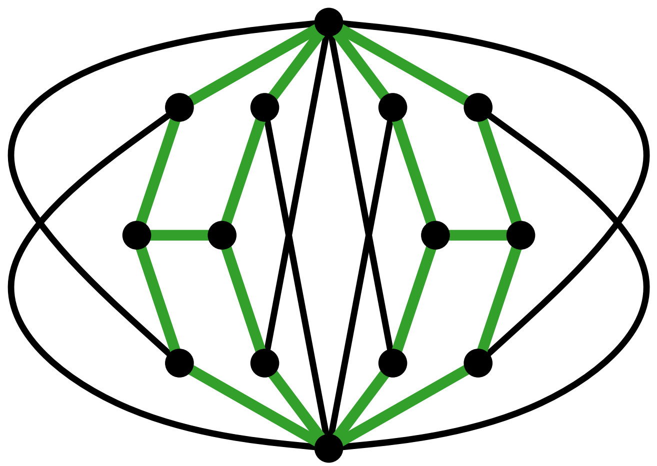

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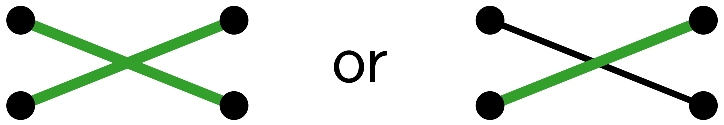

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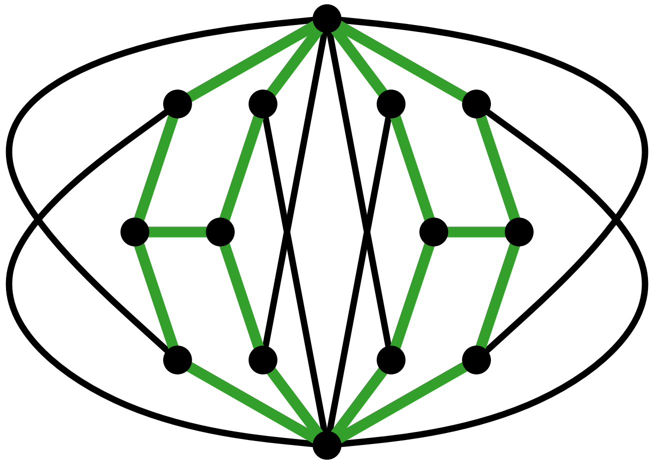

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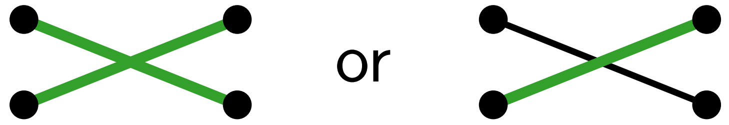

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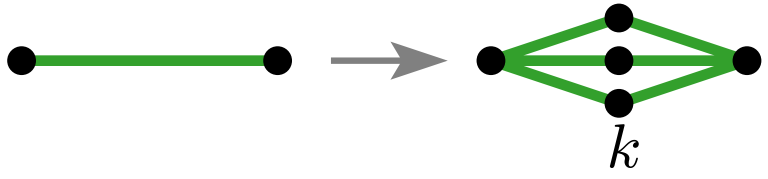

Bounds for Complete Graphs

Theorem. [Guy '60]

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Turán's brick factory problem (1944)

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Check out `http://www.ist.tugraz.at/staff/aichholzer/crossings.html`

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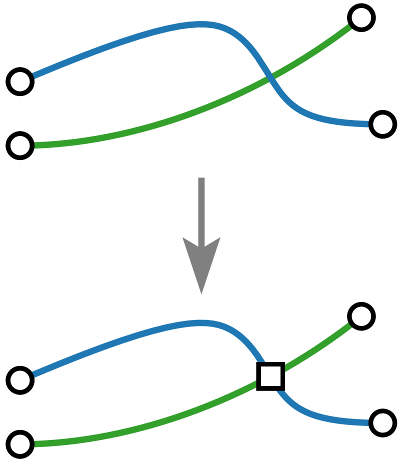

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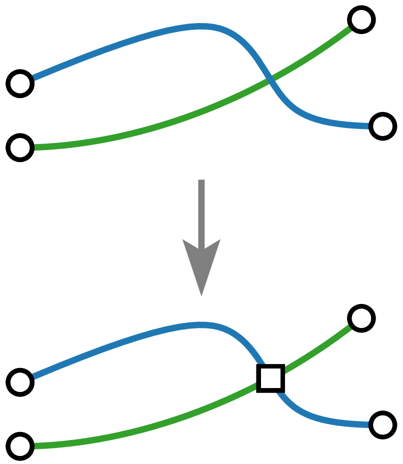

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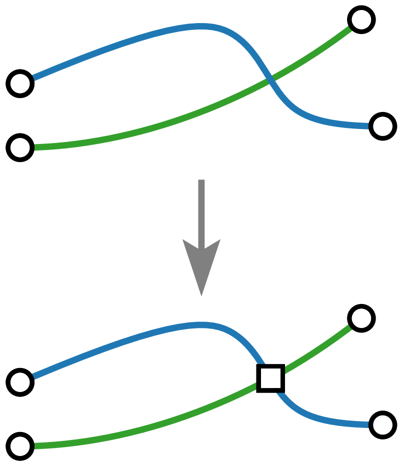

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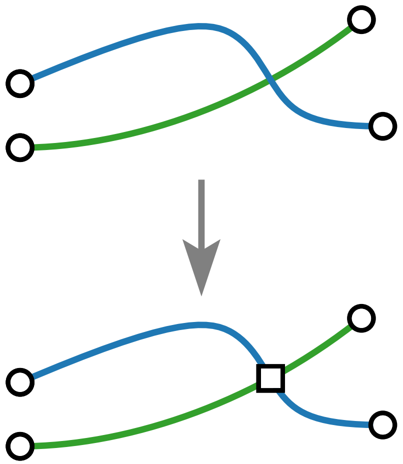

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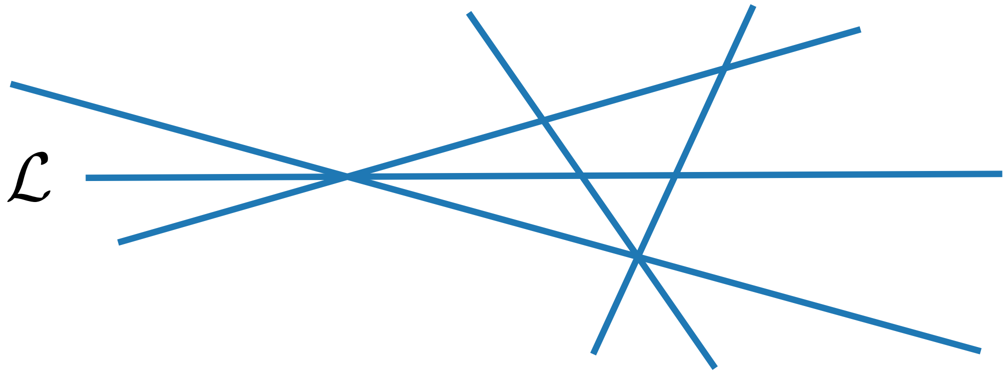

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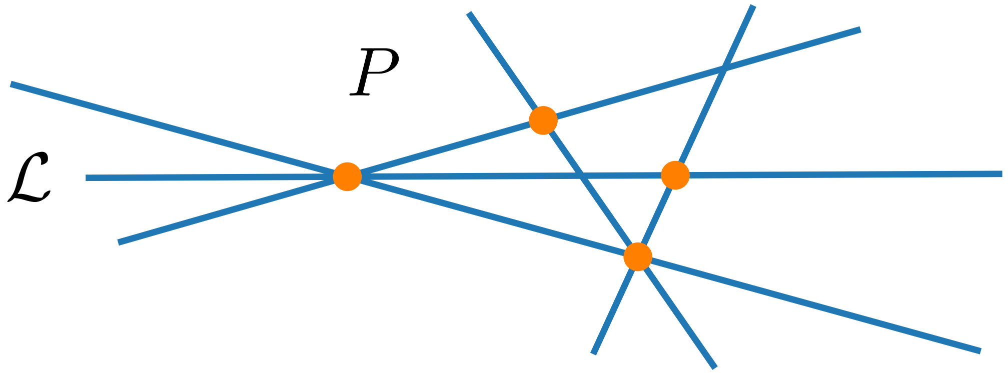

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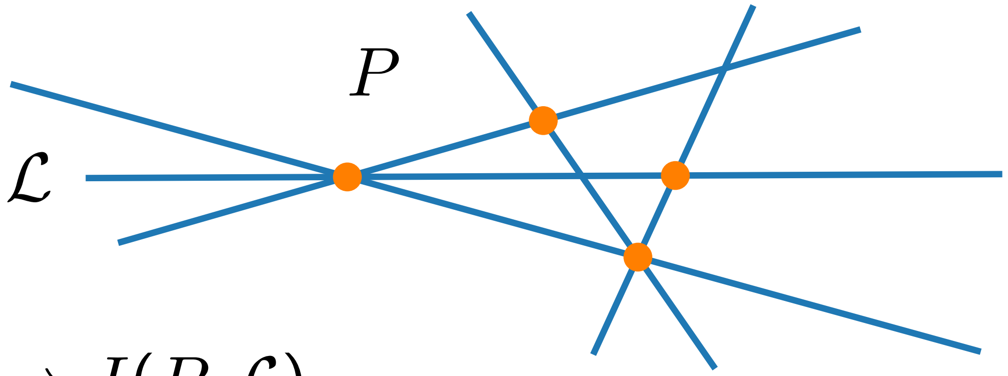

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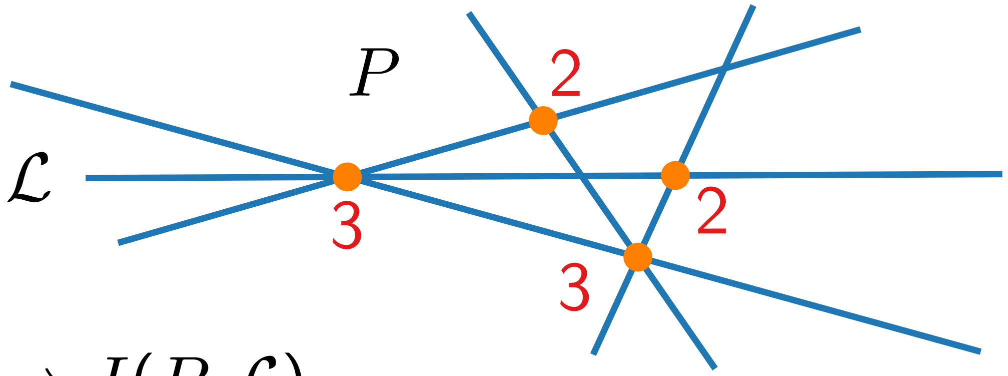

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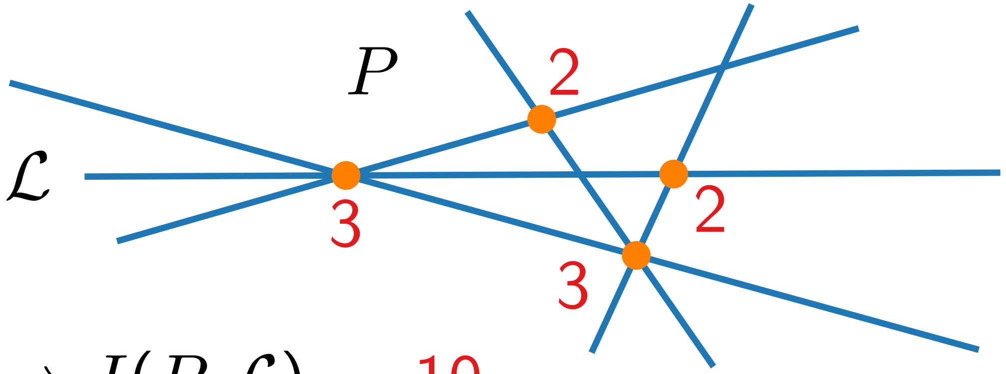

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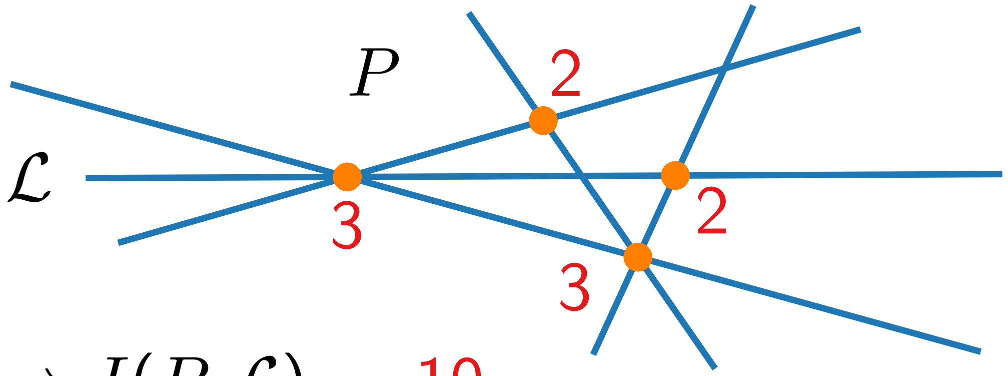

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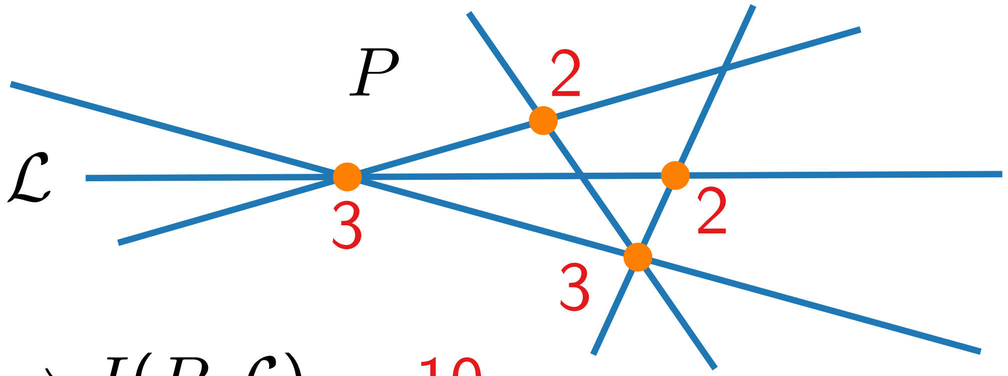

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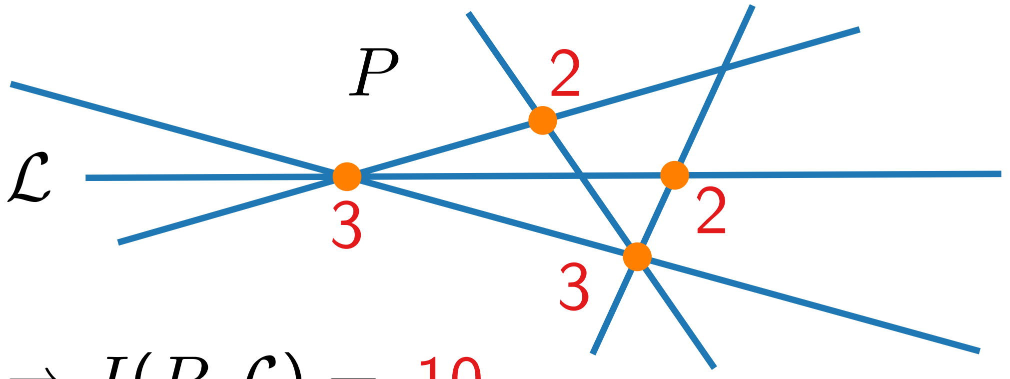

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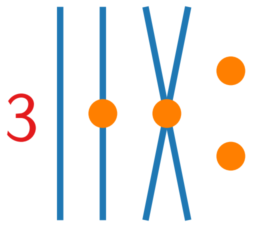

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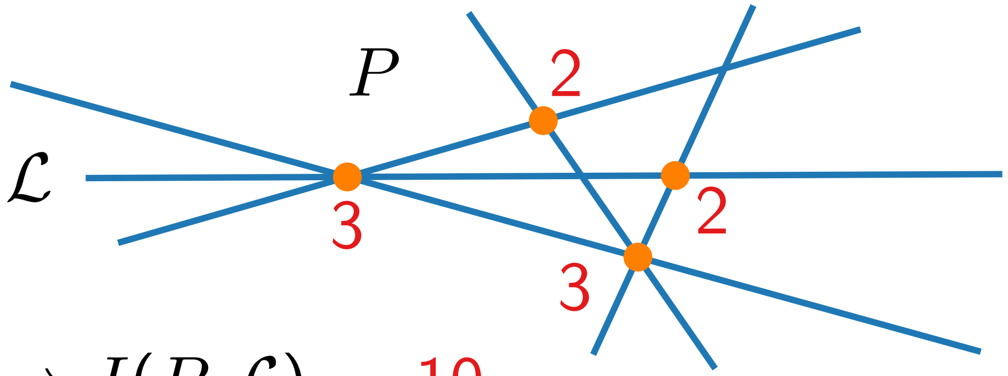

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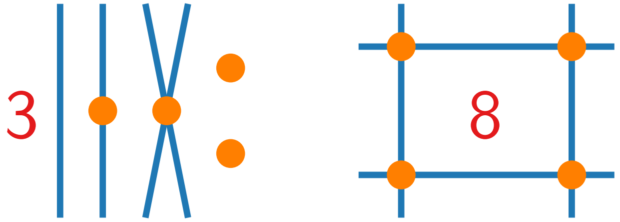

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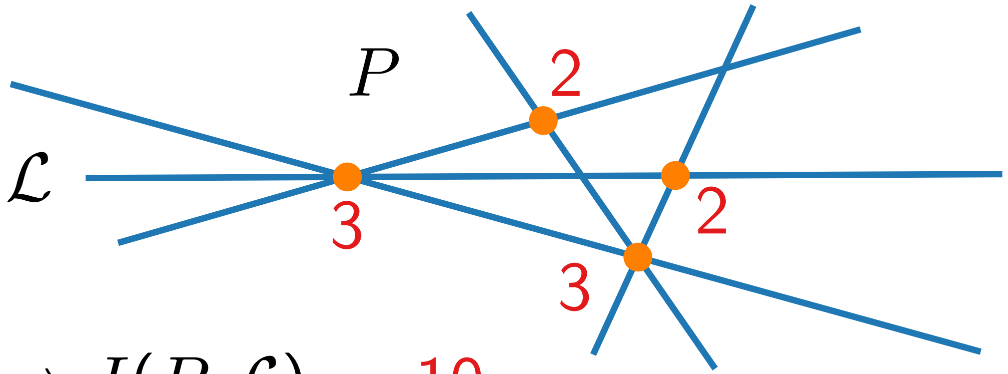

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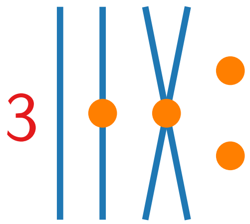

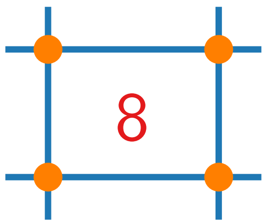

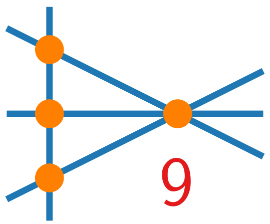

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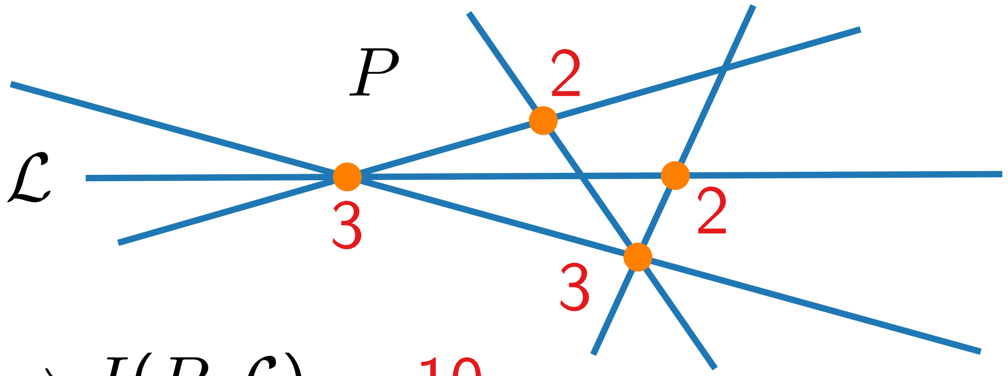

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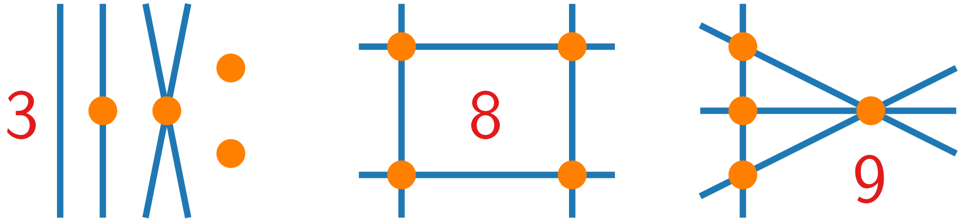

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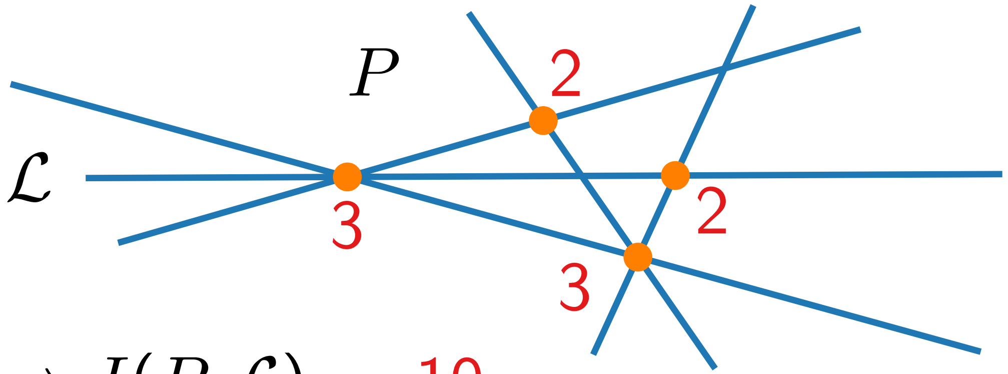

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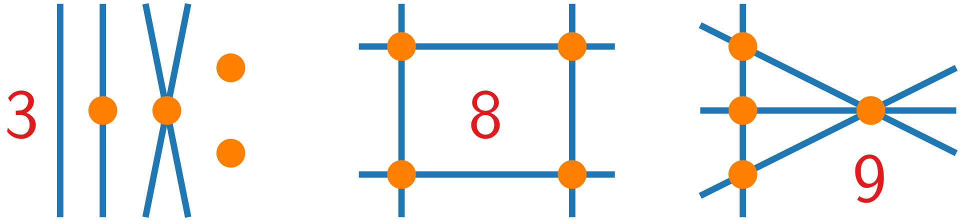

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[Szemerédi, Trotter '83, Székely '97]
$I(n, k) \le 2.7 n^{2/3} k^{2/3} + 6n + 2k$.

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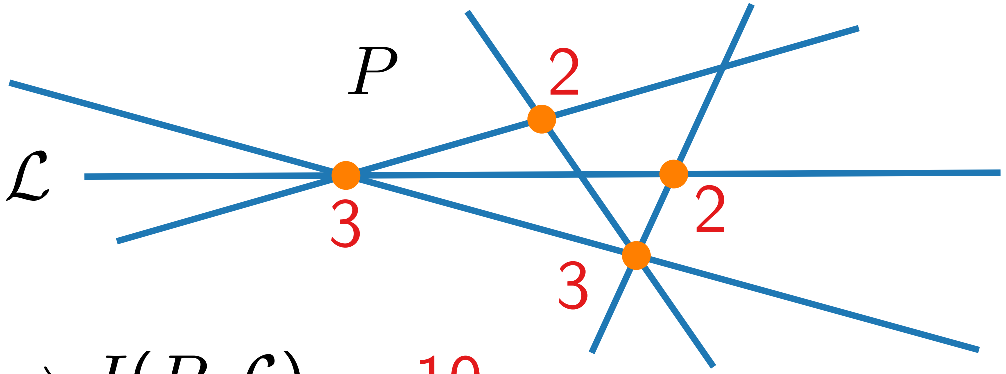

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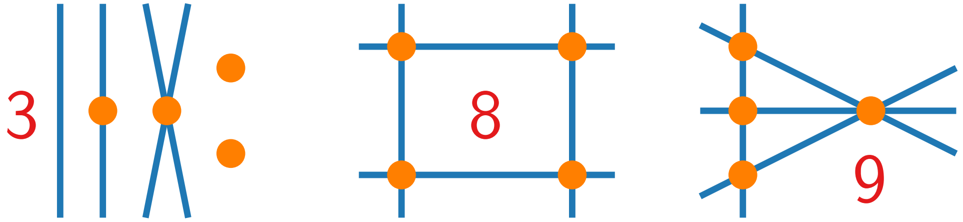

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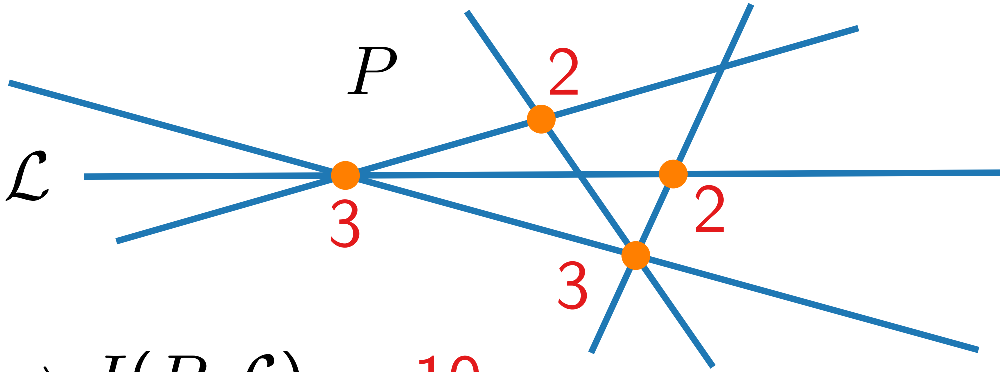

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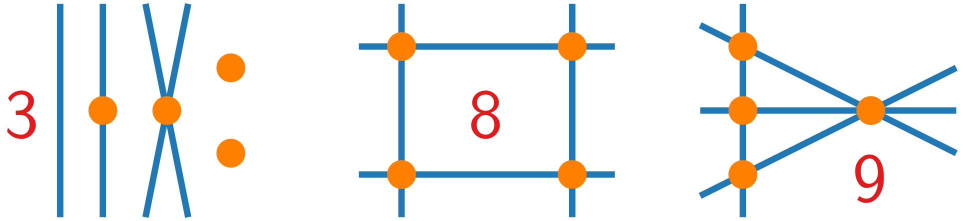

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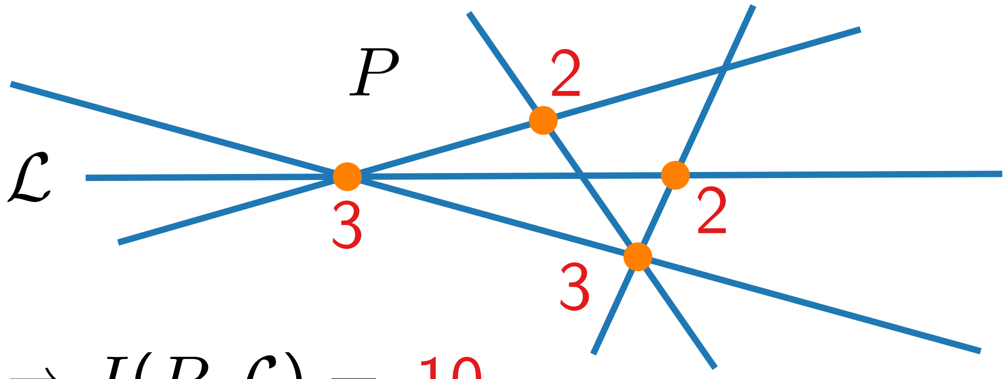

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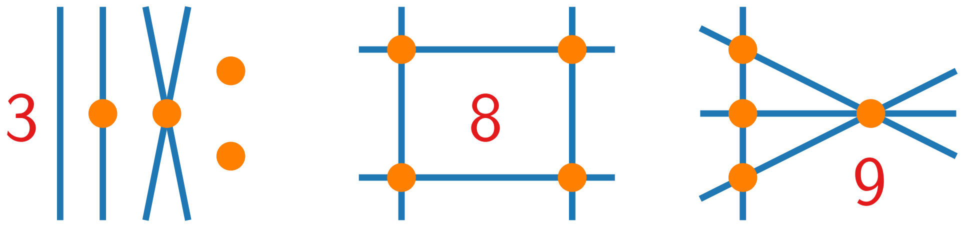

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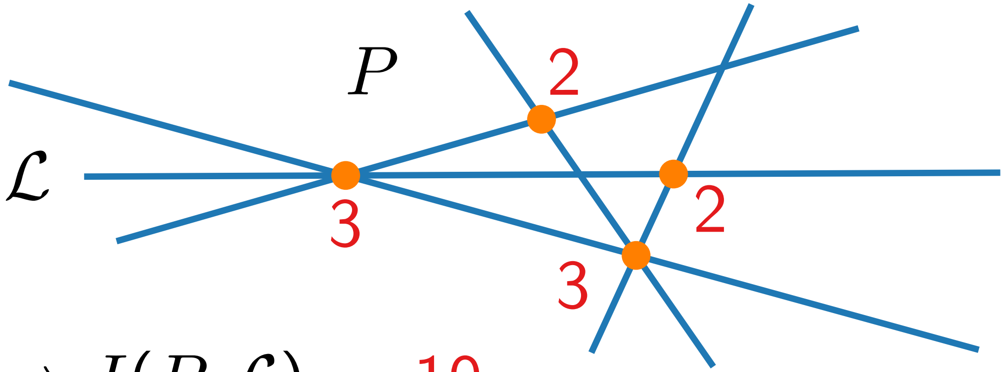

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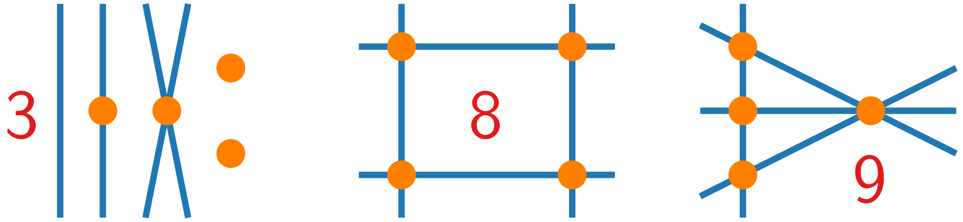

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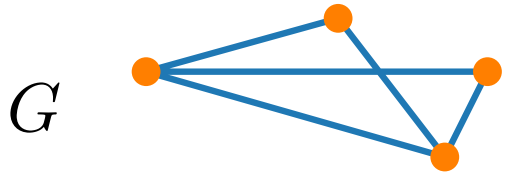

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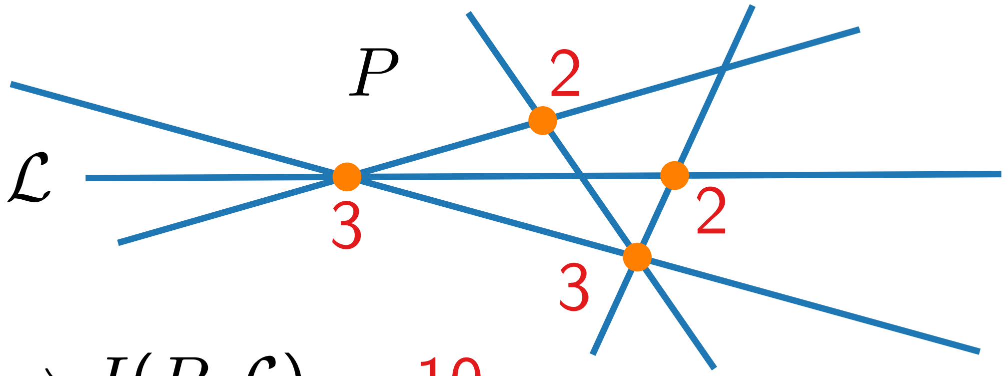

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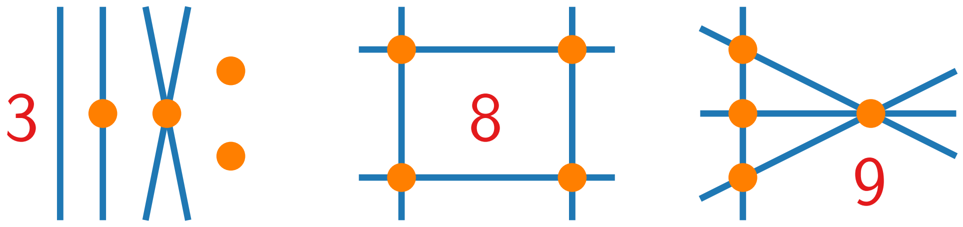

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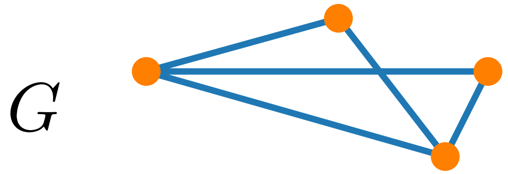

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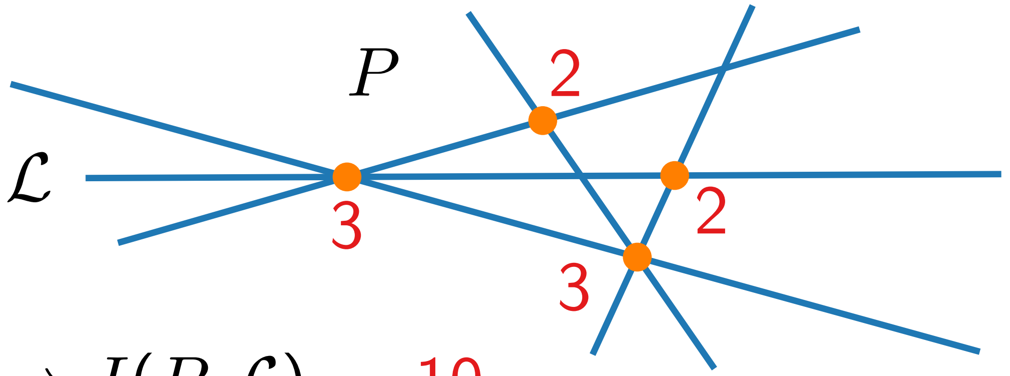

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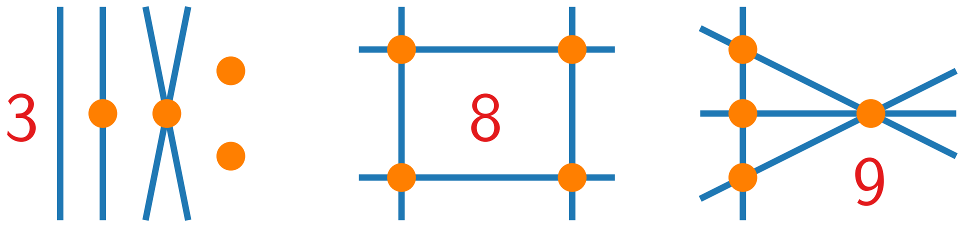

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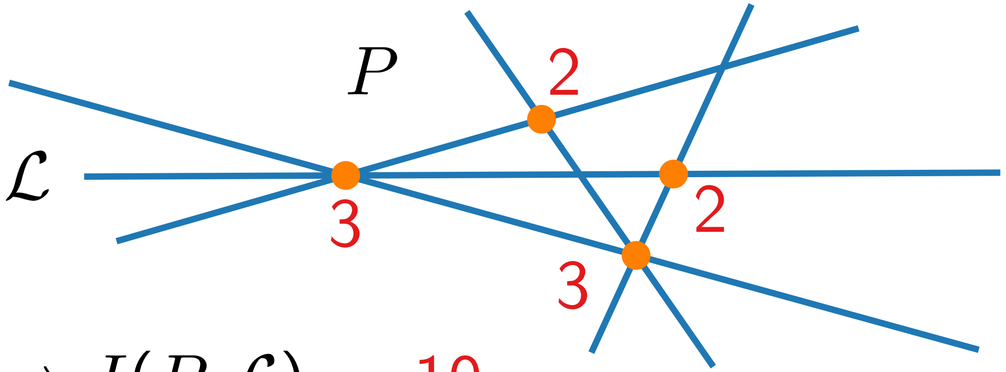

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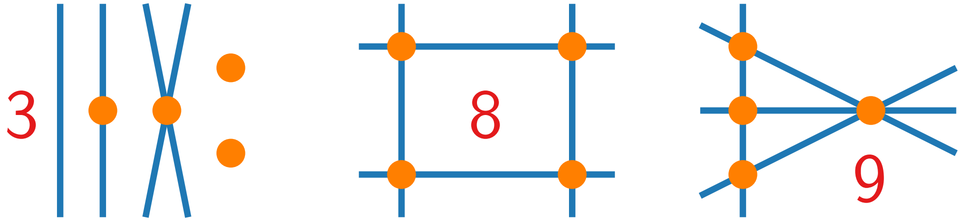

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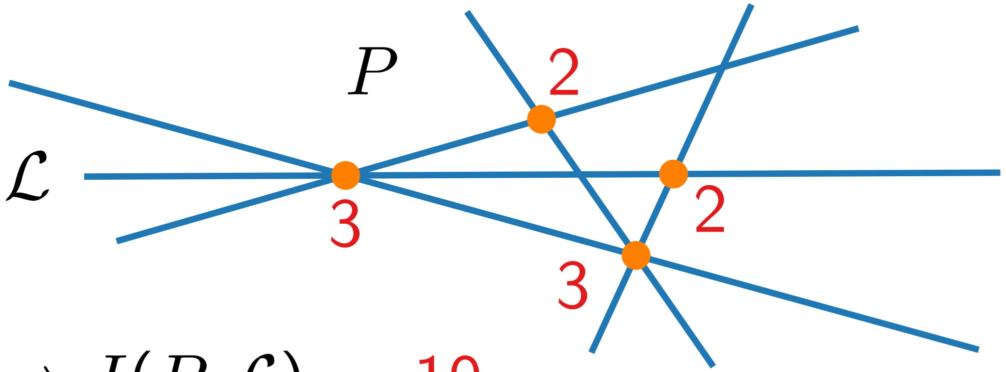

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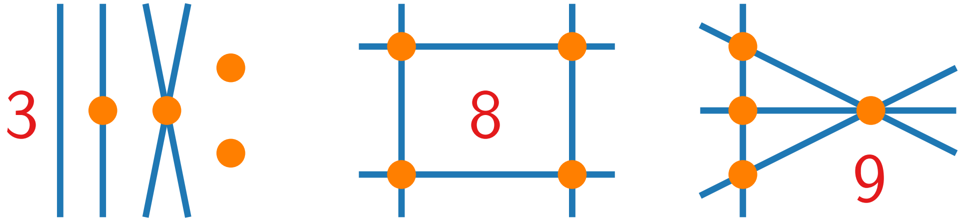

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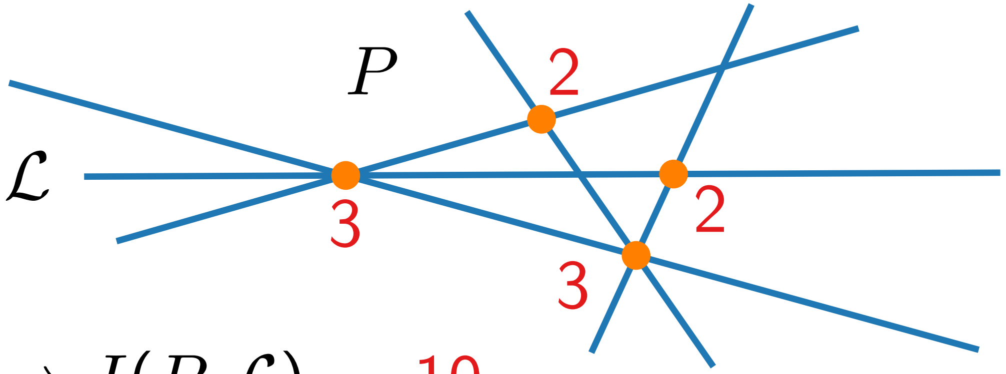

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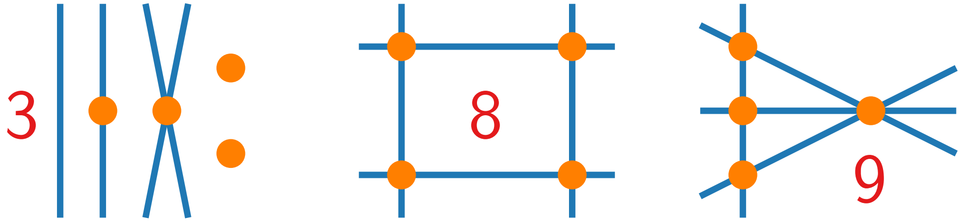

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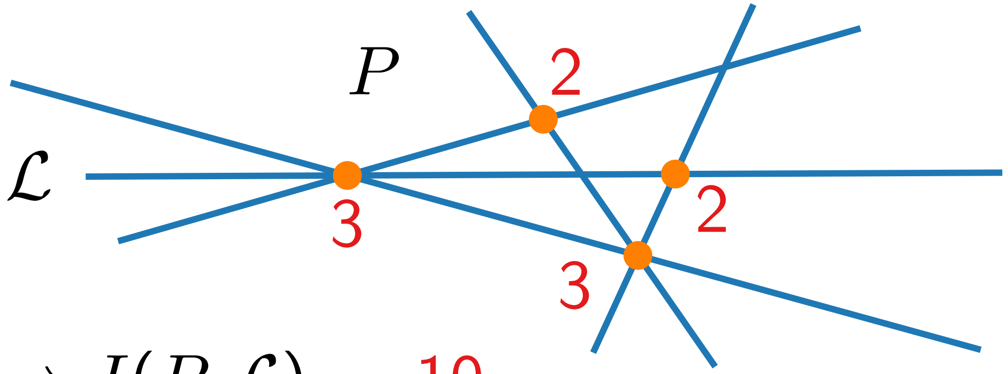

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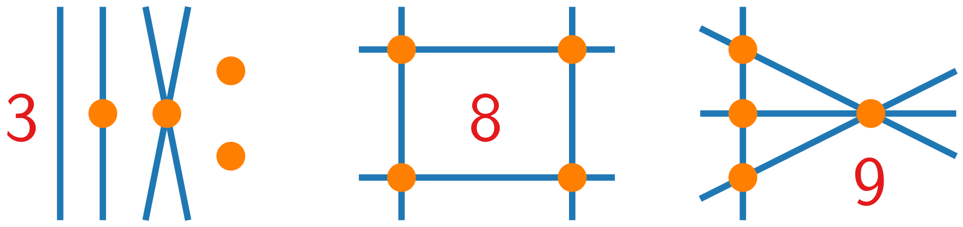

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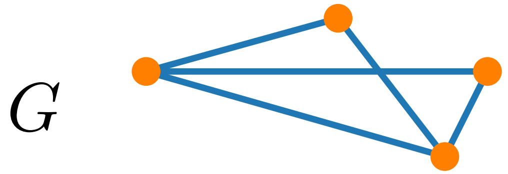

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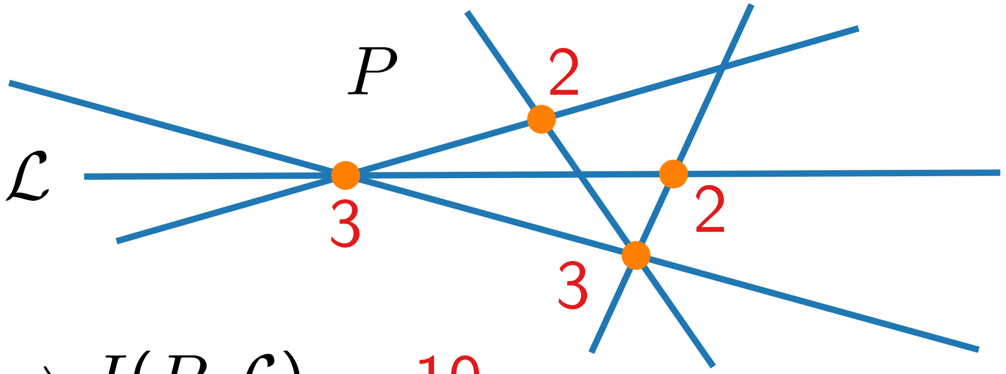

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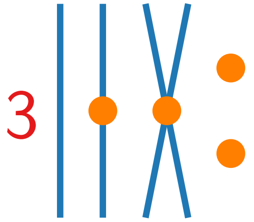

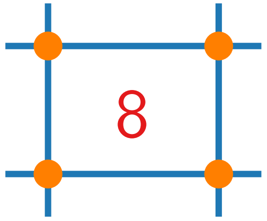

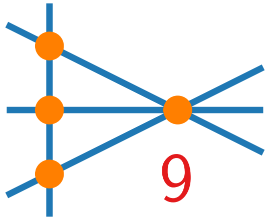

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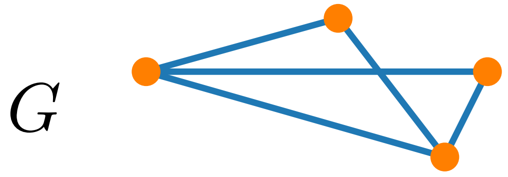

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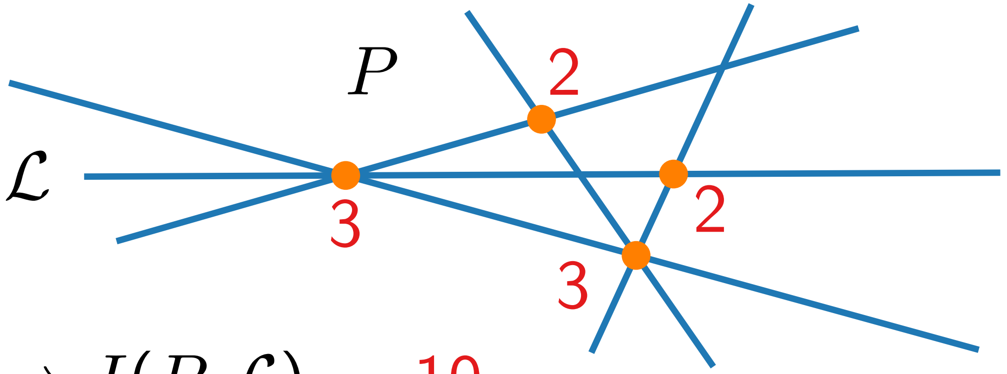

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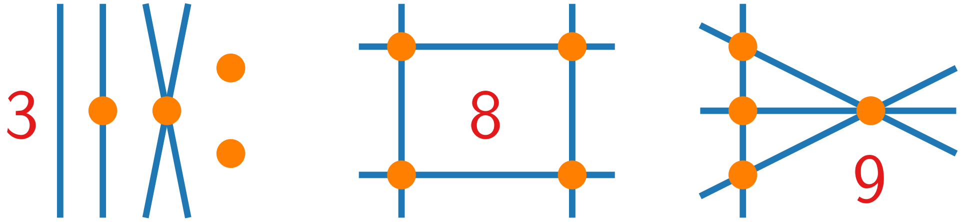

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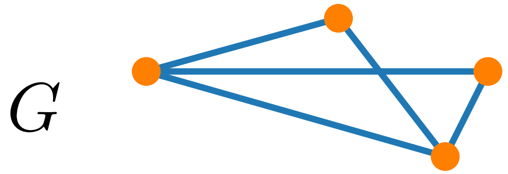

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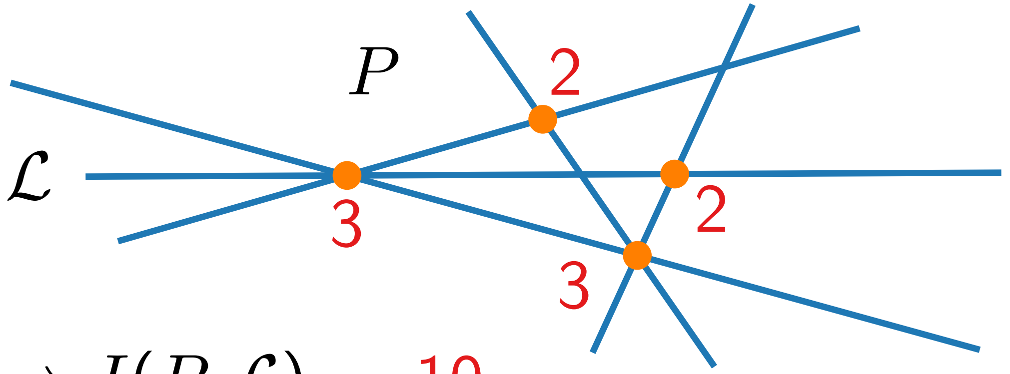

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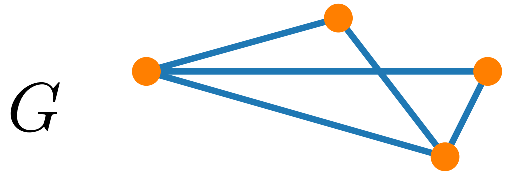

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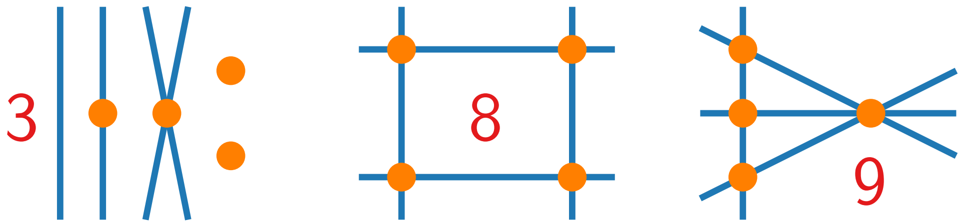

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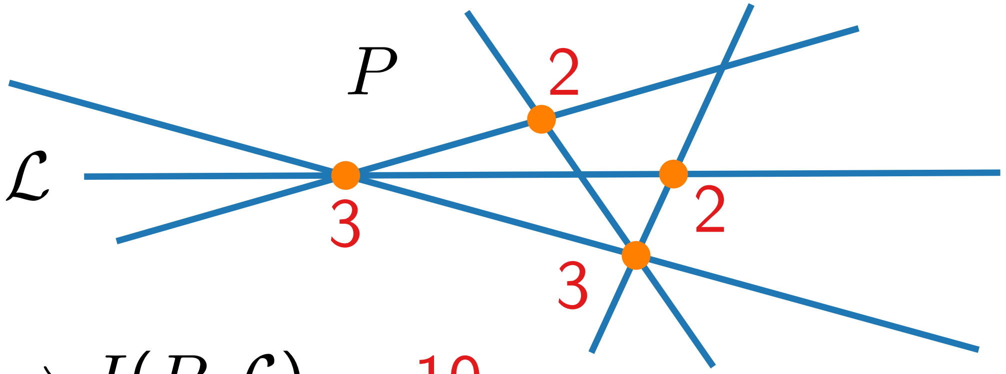

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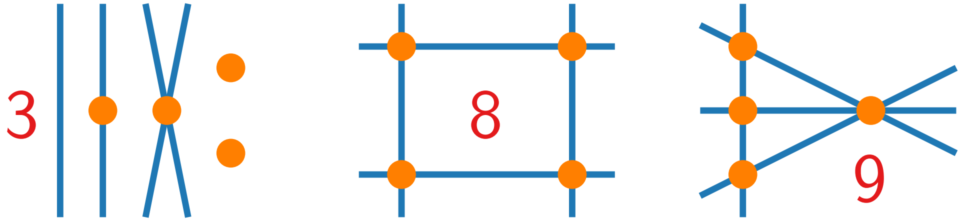

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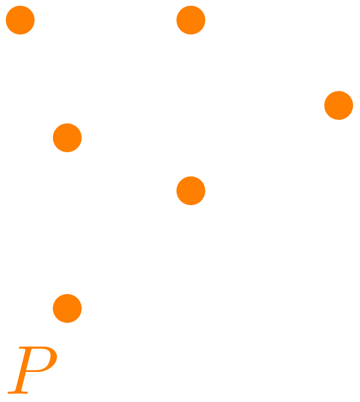

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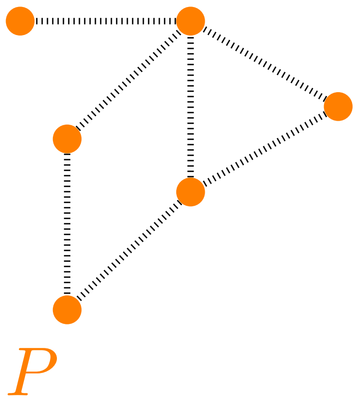

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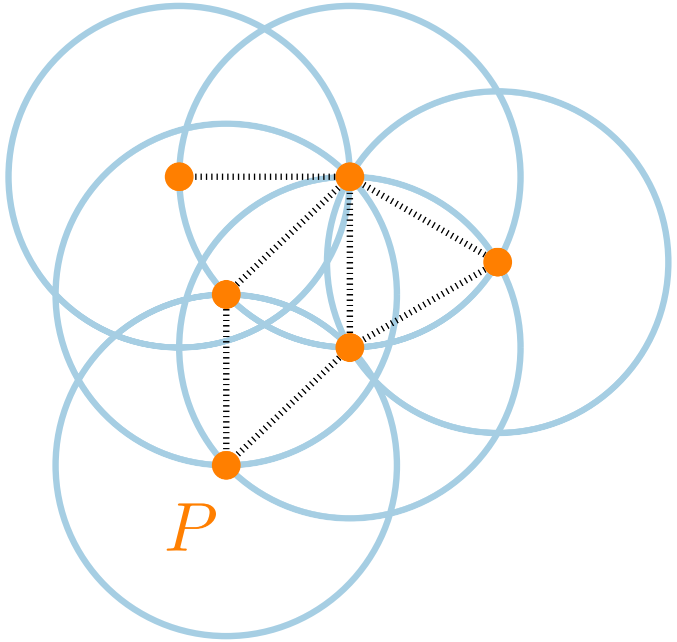

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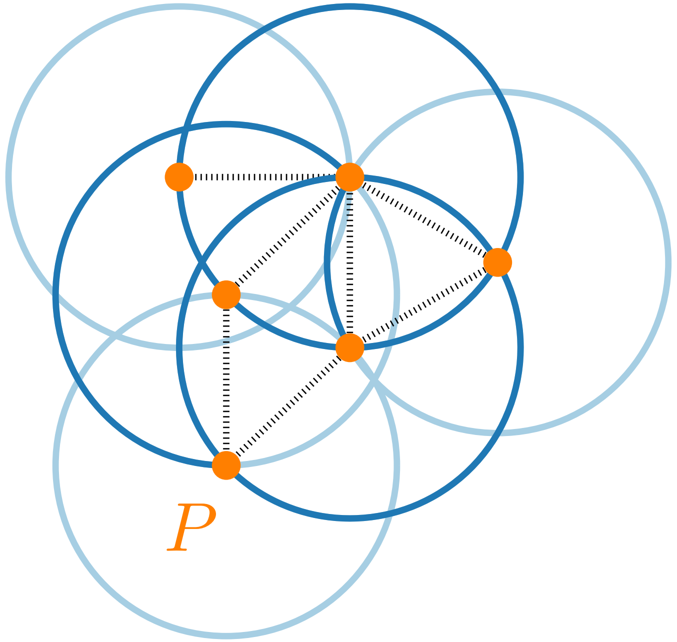

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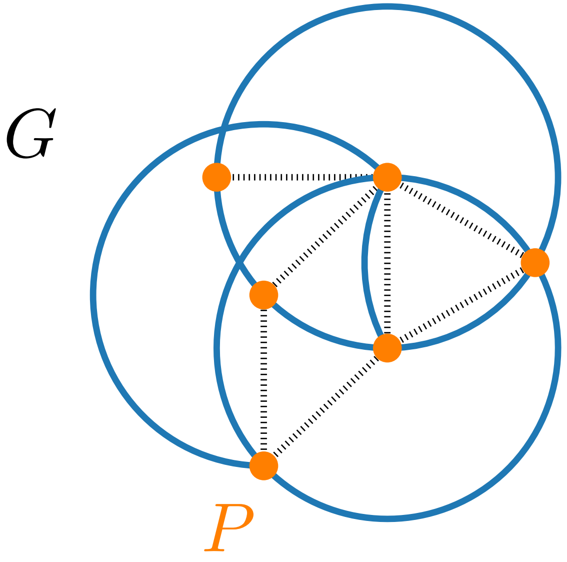

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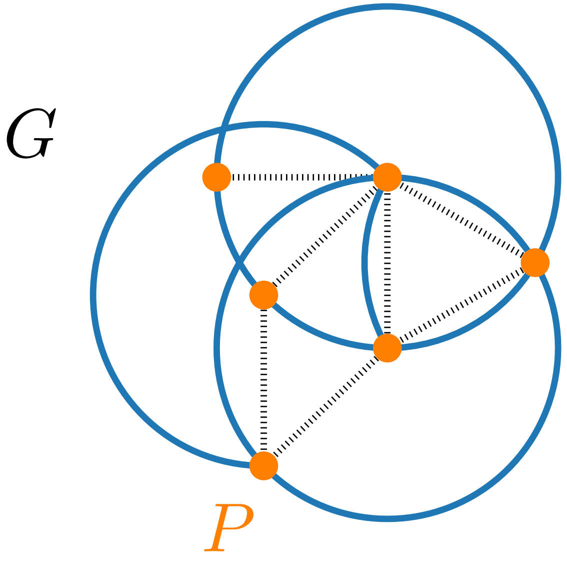

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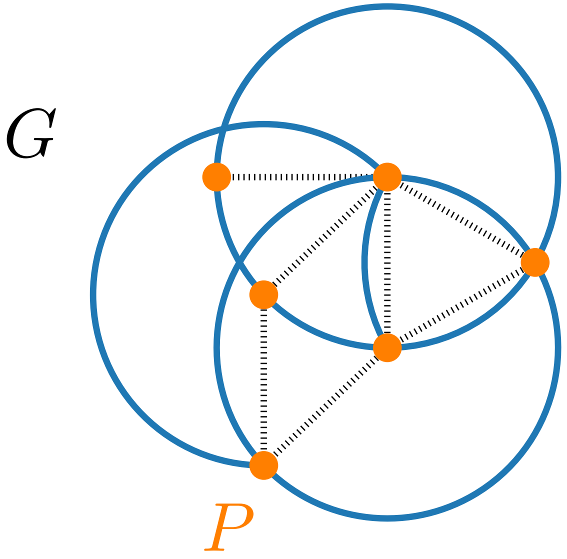

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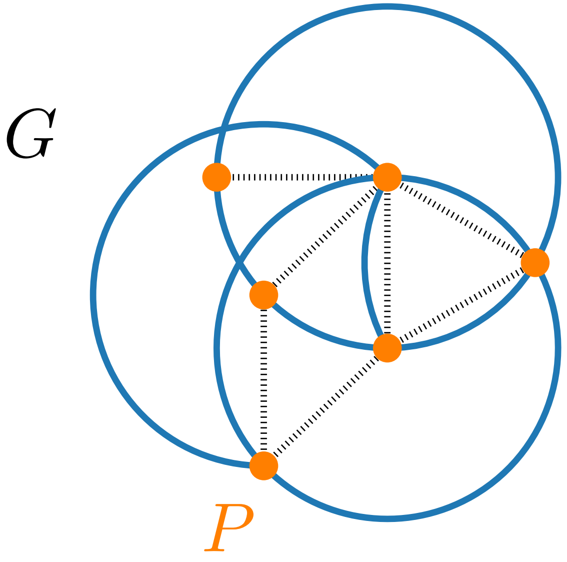

- $U(P) \le \underbrace{c''}_{\text{some constant}}\, \underbrace{m}_{\text{number of edges in } G}$
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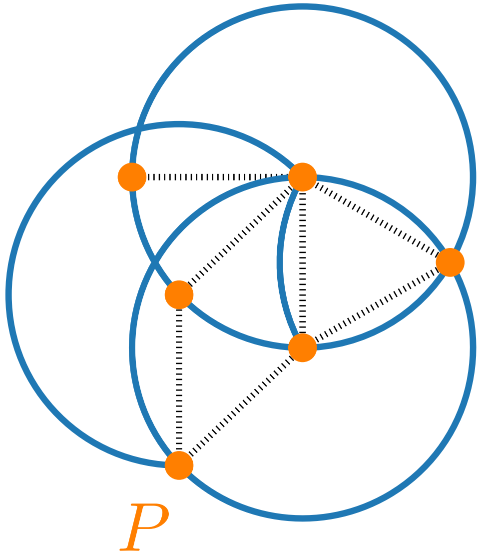

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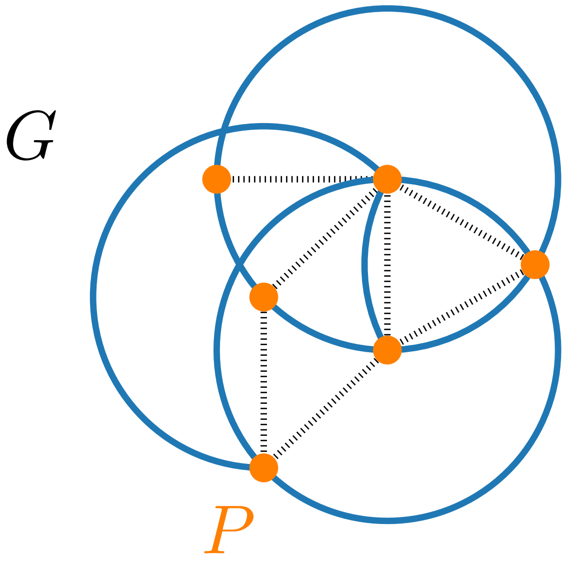

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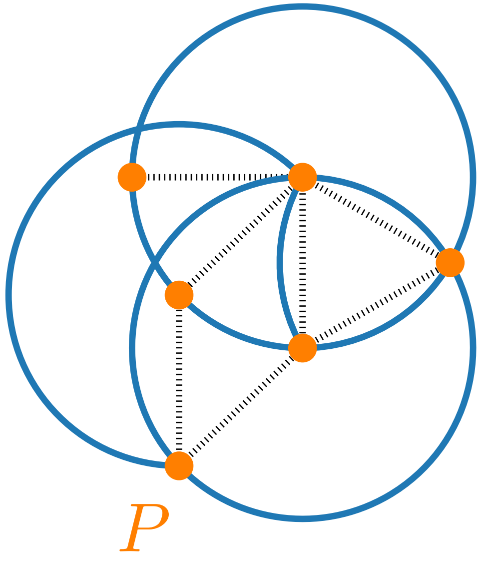

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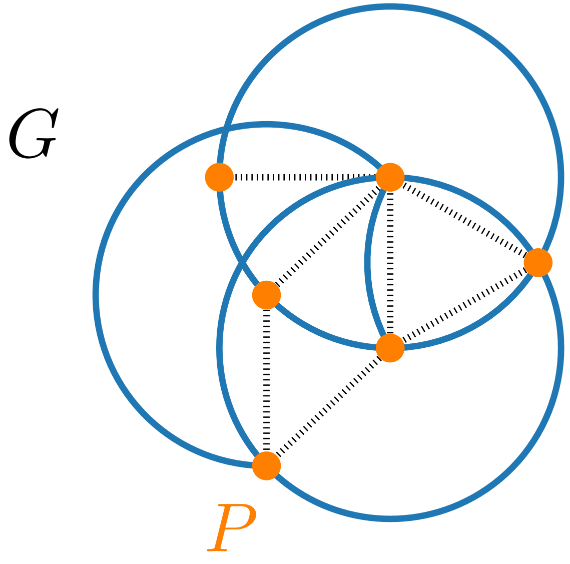

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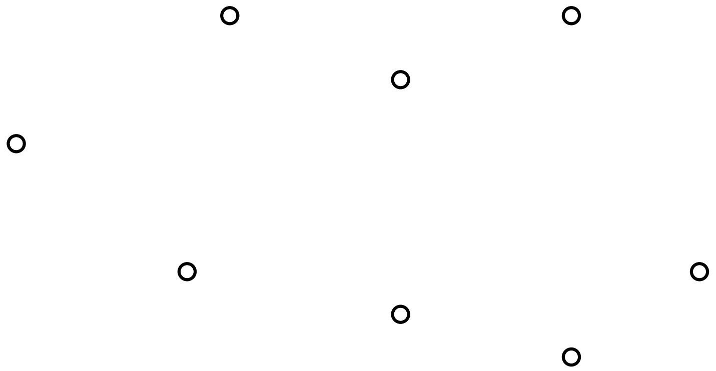

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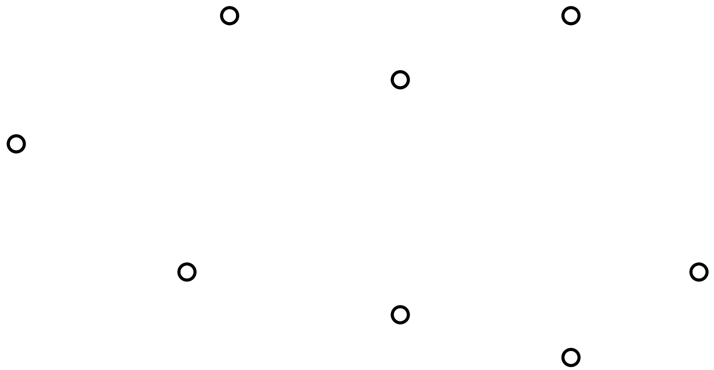

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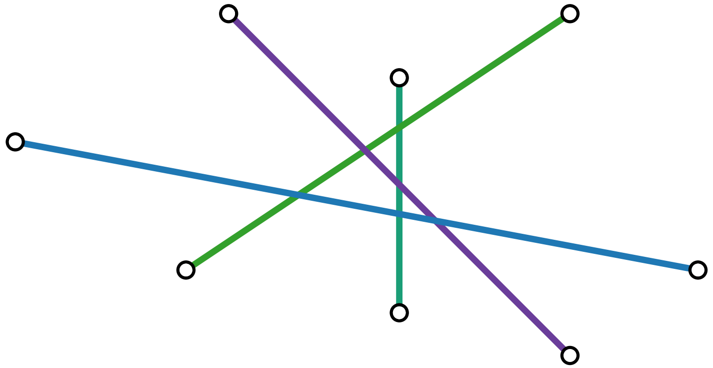

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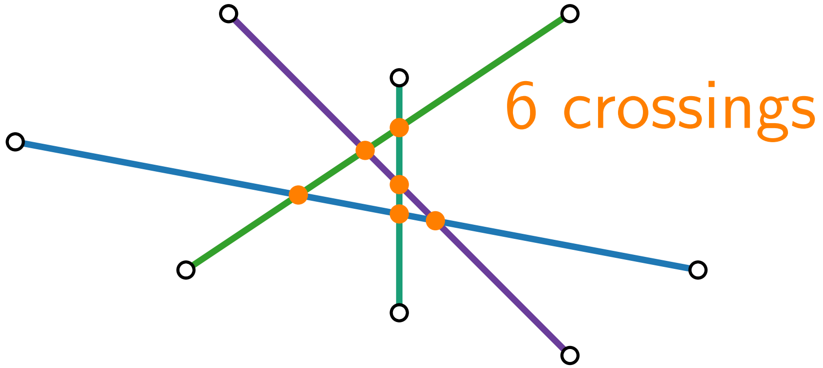

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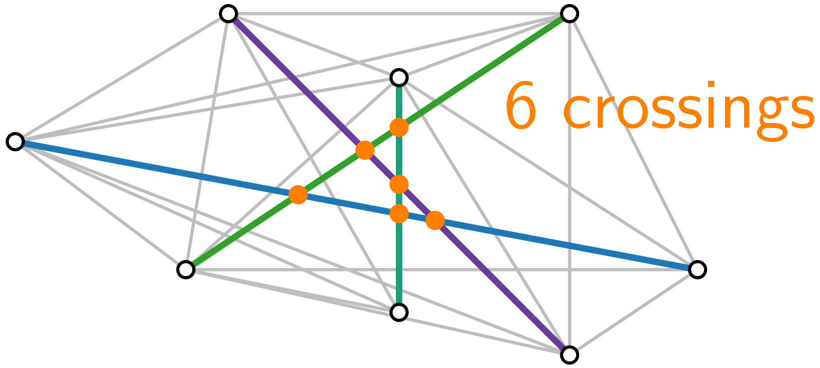

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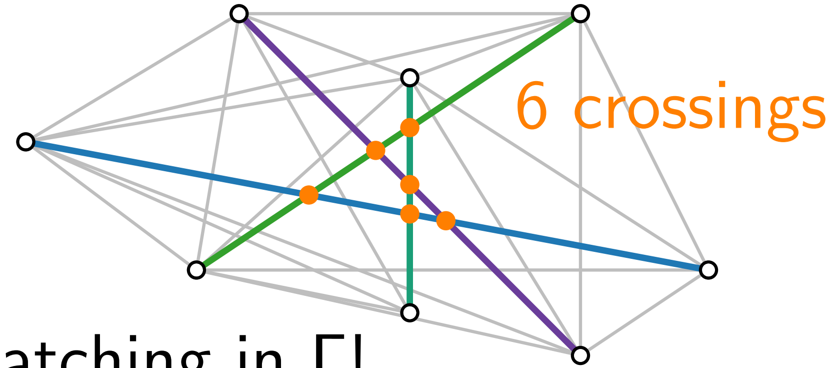

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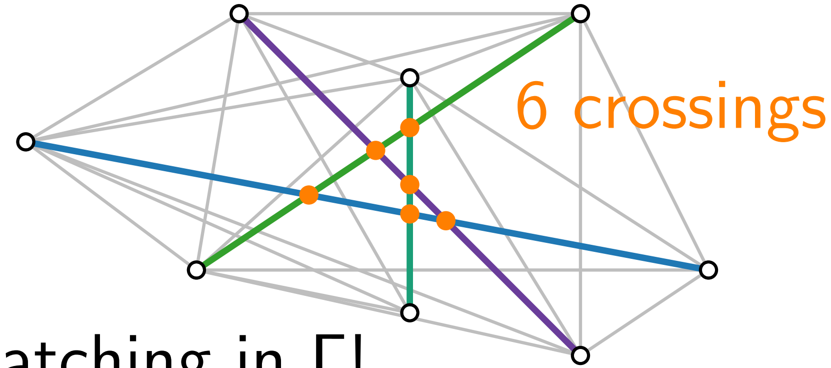

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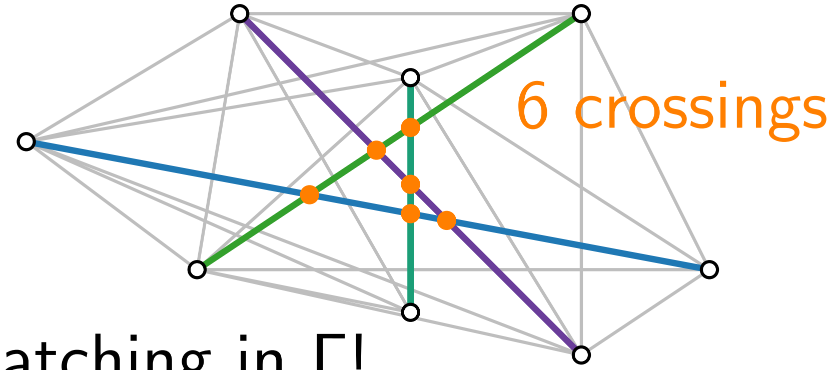

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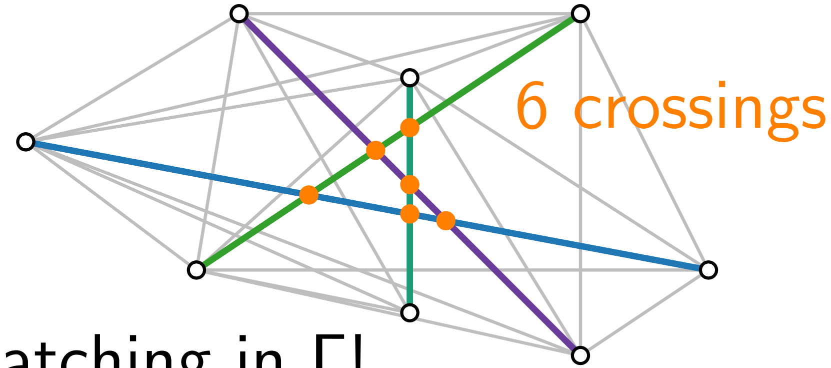

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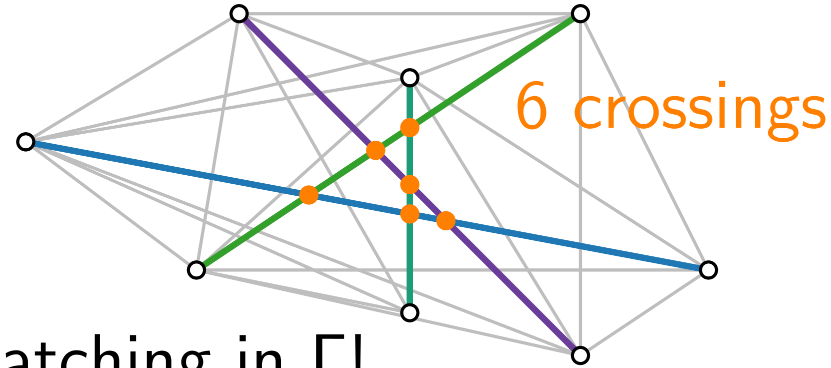

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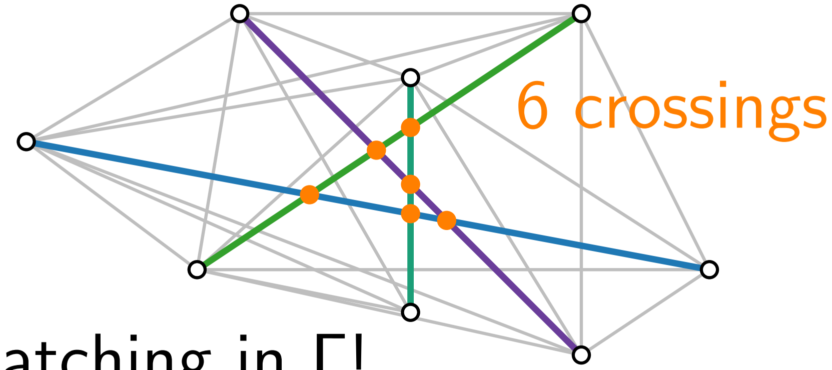

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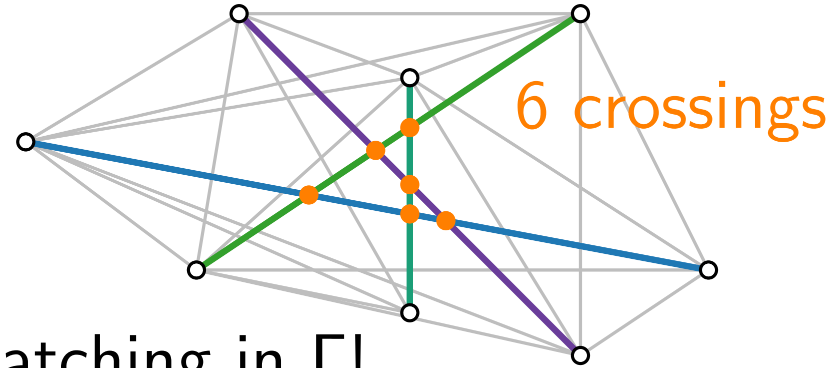

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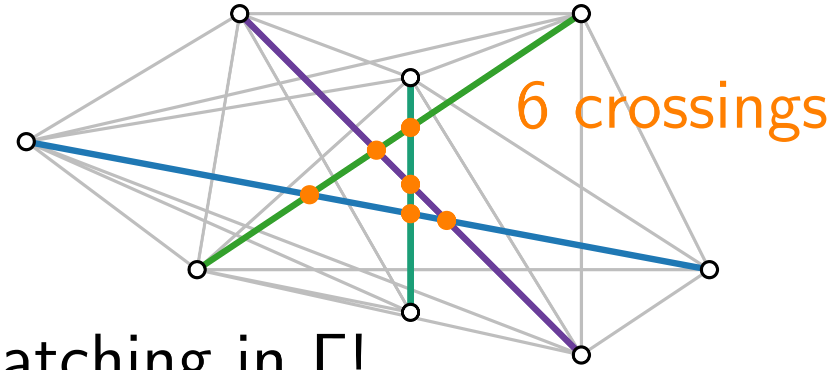

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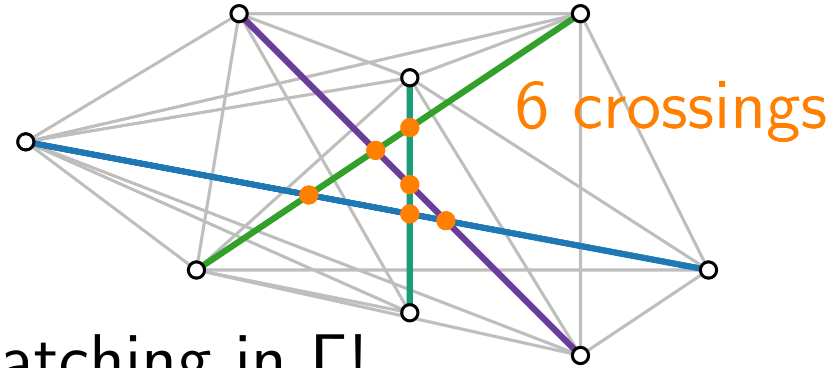

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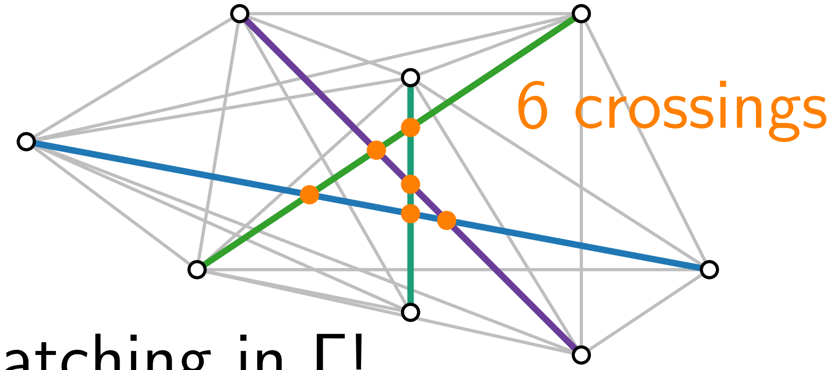

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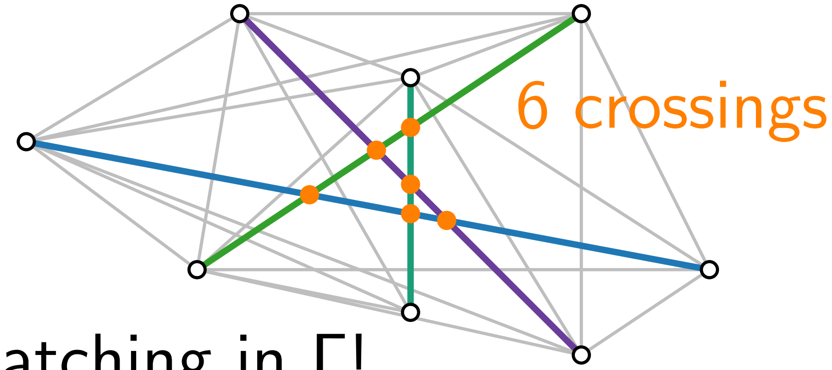

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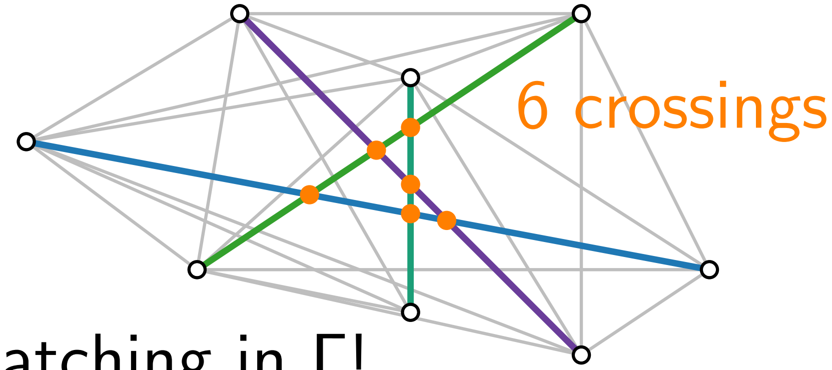

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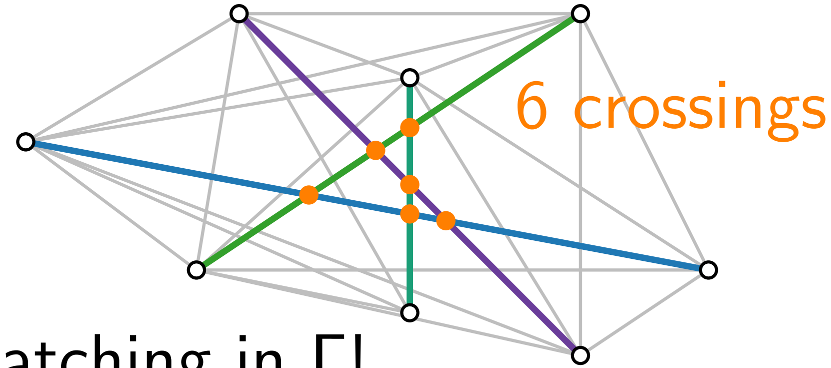

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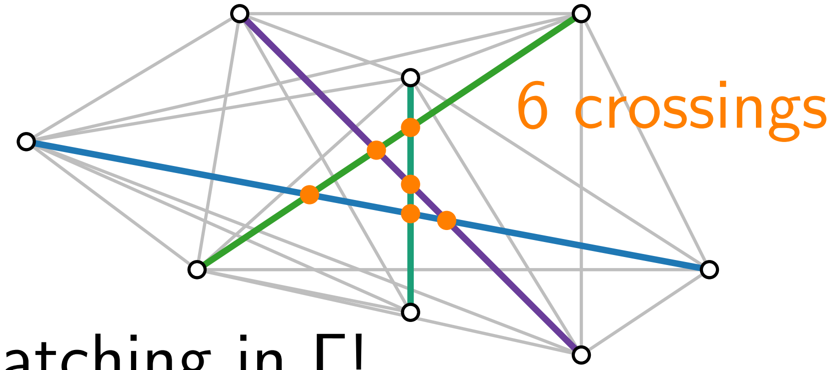

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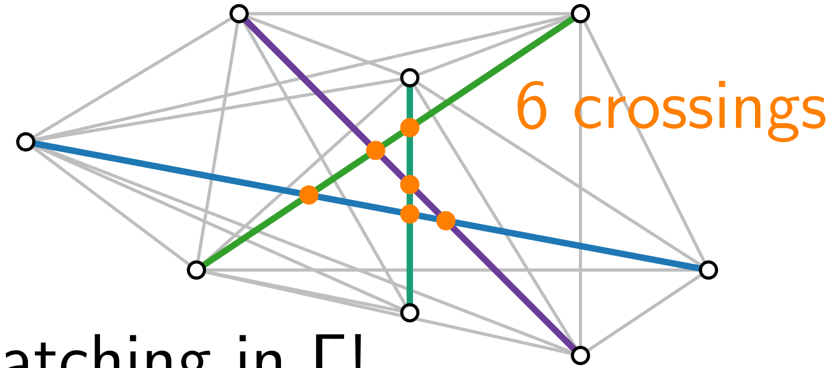

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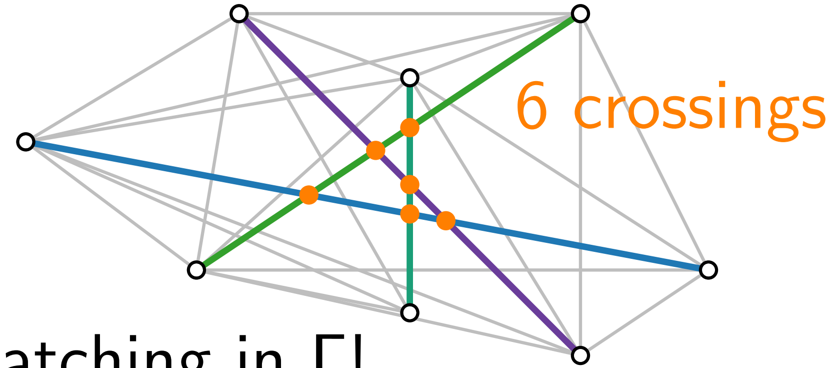
Application 3: Expected Number of Crossings in a Matching

Given set of n points (in general position, n even) –
what is the average number of crossings in a perfect matching?



Point set spans drawing Γ of K_n.
We will analyze the number of crossings in a **random** perfect matching in Γ!

Number of crossings in Γ $\geq \overline{\text{cr}}(K_n) > \frac{3}{8}\binom{n}{4}$ [Lovász et al. '04, Aichholzer et al. '06]
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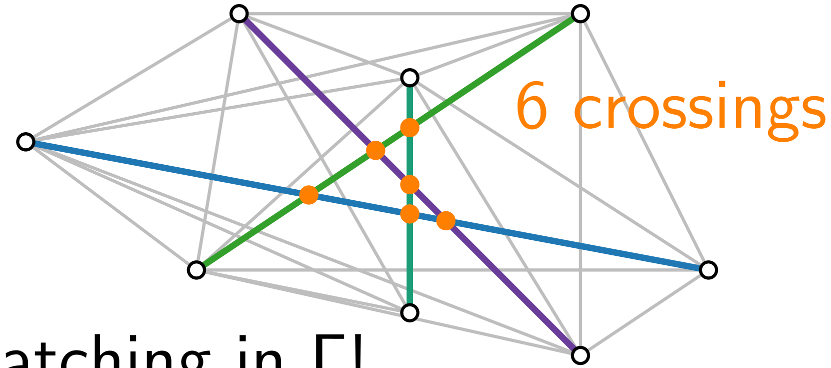

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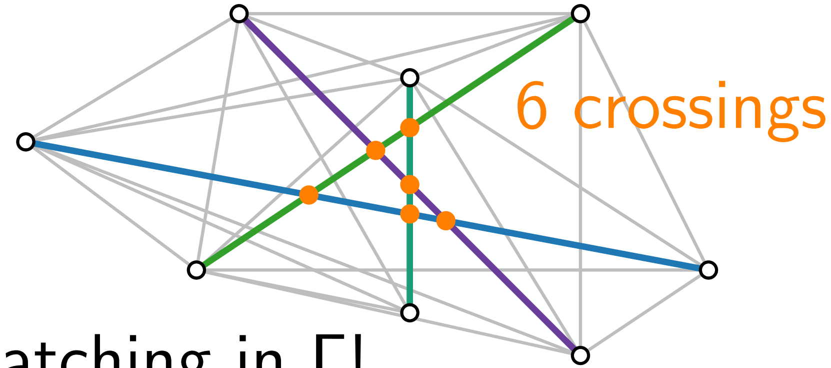

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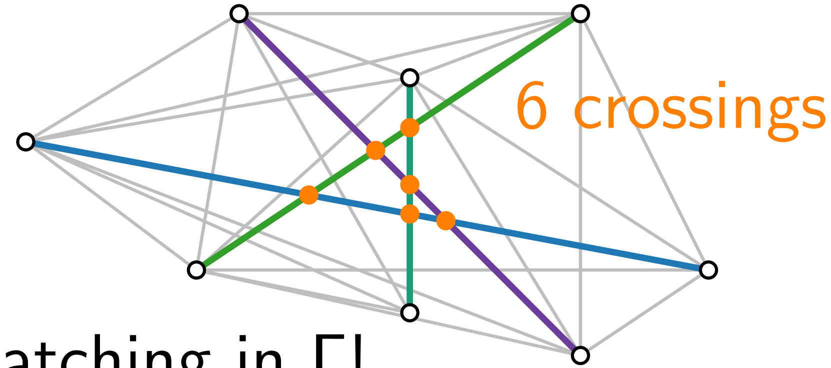

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Literature

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- [Tutte '70] Toward a theory of crossing numbers
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- Documentary/Biography "N Is a Number: A Portrait of Paul Erdős"
- Exact computations of crossing numbers: `http://crossings.uos.de`