## Visualization of Graphs

Lecture 9:<br>The Crossing Lemma

and Its Applications

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## Crossing Number and Topological Graphs

For a graph $G$, the crossing number $\operatorname{cr}(G)$ is the smallest number of pairwise edge crossings in a drawing of $G$ (in the plane).

## Hanani-Tutte Theorem

## Theorem.

[Hanani '43, Tutte '70]
A graph is planar if and only if it has a drawing in which all pairs of vertex-disjoint edges cross an even number of times.

## Proof sketch.

Hanani showed that every drawing of $K_{5}$ and $K_{3,3}$ must have a pair of edges that crosses an odd number of times.

Every non-planar graph has $K_{5}$ or $K_{3,3}$ as a minor, so there are two paths that cross an odd number of times.

Hence, there must be two edges on these paths that cross an odd number of times.

## Hanani-Tutte Theorem

## Theorem.

[Hanani '43, Tutte '70]
A graph is planar if and only if it has a drawing in which all pairs of vertex-disjoint edges cross an even number of times.

Corollary. $\quad \operatorname{ocr}(G)=0 \Rightarrow \operatorname{pcr}(G)=0 \Rightarrow \operatorname{cr}(G)=0$
Theorem. [Pelsmajer, Schaefer \& Štefankovič '08, Tóth '08]
There is a graph $G$ with $\operatorname{ocr}(G)<\operatorname{cr}(G) \leq 10$

The odd crossing number $\operatorname{ocr}(G)$ of $G$ is the smallest number of pairs of edges that cross oddly in a drawing of $G$.

$$
\begin{array}{ll}
\text { Is ocr }(G)=\operatorname{cr}(G) ? & \text { No! } \\
\text { Is ocr }(G) & =\operatorname{pcr}(G) ? \\
& \text { No! } \\
\text { Is } \operatorname{pcr}(G) & =\operatorname{cr}(G) ?
\end{array} \quad \text { Open! }
$$

## Theorem. [Pelsmajer, Schaefer \& Štefankovič '07] [Pach \& Tóth '00]

If $\Gamma$ is a drawing of $G$ and $E_{0}$ is the set of edges that cross any other edge an even number of times in $\Gamma$, then $G$ can be drawn such that no edge in $E_{0}$ is involved in any crossings and no new pairs of edges cross.

The pairwise crossing number $\operatorname{pcr}(G)$ of $G$ is the smallest number of pairs of edges that cross in a drawing of $G$. By definition ocr $(G) \leq \operatorname{pcr}(G) \leq \operatorname{cr}(G)$

Note that, in the resulting drawing of $G$, an edge might cross some edges an odd number of times and some other edges an even number of times. So, no implications on $\operatorname{ocr}(G)=\operatorname{pcr}(G)$.

## Computing the Crossing Number

- Computing $\operatorname{cr}(G)$ is NP-hard. ... even if $G$ is a planar graph plus one edge!
[Garey \& Johnson '83] [Cabello \& Mohar '08]

■ $\operatorname{cr}(G)$ often unknown, only conjectures exist (for $K_{n}$ it is only known for up to $\approx 12$ vertices)

- In practice, $\operatorname{cr}(G)$ is often not computed directly but rather drawings of $G$ are optimized with
■ force-based methods,
- multidimensional scaling,

■ heuristics, ...

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For exact computations,
check out http://crossings.uos.de!
```

- $\operatorname{cr}(G)$ is a measure of how far $G$ is from being planar.
- For planarization, where we replace crossings with dummy vertices, also only heuristic approaches are known.


## Other Crossing Numbers

- Schaefer [Sch20] wrote a survey on many variants of crossing numbers (including precise definitions).
- One-sided crossing minimization (see lecture 8)
- Fixed linear crossing number

■ Book embeddings (vertices on a line, edges assigned to few "pages" where edges do not cross)

- Crossings of edge bundles

■ On other surfaces, such as donuts

- Weighted crossings

■ Crossing minimization is NP-hard for most variants.


## Rectilinear Crossing Number

## Definition.

For a graph $G$, the rectilinear (straight-line) crossing number $\overline{\operatorname{cr}}(G)$ is the smallest number of crossings in a straight-line drawing of $G$.

## Even more ...

## Lemma 1. [Bienstock, Dean '93]

For every $k \geq 4$, there exists a graph $G_{k}$ with $\operatorname{cr}\left(G_{k}\right)=4$ and $\overline{\operatorname{cr}}\left(G_{k}\right) \geq k$.

■ Each straight-line drawing of $G_{1}$ has at least one crossing of the following types:

- From $G_{1}$ to $G_{k}$ do

$$
\begin{aligned}
& \text { Separation. } \\
& \operatorname{cr}\left(K_{8}\right)=18 \text {, but } \overline{\operatorname{cr}}\left(K_{8}\right)=19 .
\end{aligned}
$$



## Bounds for Complete Graphs

Theorem. Conjecture. [Guy '60] $\operatorname{cr}\left(K_{n}\right) \leq \frac{1}{4}\left\lceil\frac{n}{2}\right\rceil\left\lceil\frac{n-1}{2}\right\rceil\left\lceil\frac{n-2}{2}\right\rceil\left\lceil\frac{n-3}{2}\right\rceil=\frac{3}{8}\binom{n}{4}+O\left(n^{3}\right)$
Bound is tight for $n \leq 12$. complete bipartite graph with $m \times n$ edges
Theorem. Conjecture. [Zarankiewicz '54, Urbaník '55]
$\operatorname{cr}\left(K_{m, n}\right) \nsubseteq \frac{1}{4}\left\lceil\frac{n}{2}\right\rceil\left\lceil\frac{n-1}{2}\right\rceil\left\lceil\frac{m}{2}\right\rceil\left\lceil\frac{m-1}{2}\right\rceil$
Turán's brick
$\begin{aligned} & \text { Pál Turán } \\ & \text { Budapest, Hungary }\end{aligned}$


## Bounds for Complete Graphs

Theorem. Conjecture.
[Guy '60]

$$
\operatorname{cr}\left(K_{n}\right)<\frac{1}{4}\left\lceil\frac{n}{2}\right\rceil\left\lceil\frac{n-1}{2}\right\rceil\left\lceil\frac{n-2}{2}\right\rceil\left\lceil\frac{n-3}{2}\right\rceil=\frac{3}{8}\binom{n}{4}+O\left(n^{3}\right)
$$

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Theorem. [Lovász et al. '04, Aichholzer et al. '06]
$\left(\frac{3}{8}+\varepsilon\right)\binom{n}{4}+O\left(n^{3}\right)<\overline{\operatorname{cr}}\left(K_{n}\right)<0.3807\binom{n}{4}+O\left(n^{3}\right)$
Exact numbers are known for $n \leq 27$.

## First Lower Bounds on $\operatorname{cr}(G)$

## Lemma 2.

For a graph $G$ with $n$ vertices and $m$ edges,

$$
\operatorname{cr}(G) \geq m-3 n+6
$$

## Proof.

- Consider a drawing of $G$ with $\operatorname{cr}(G)$ crossings.

■ Obtain a graph $H$ by turning crossings into dummy vertices.

- $H$ has $n+\operatorname{cr}(G)$ vertices and $m+2 \operatorname{cr}(G)$ edges.

■ $H$ is planar, so

$$
m+2 \operatorname{cr}(G) \leq 3(n+\operatorname{cr}(G))-6 .
$$

## First Lower Bounds on $\operatorname{cr}(G)$

## Lemma 3.

For a non-planar graph $G$ with $n$ vertices and $m$ edges,

$$
\operatorname{cr}(G) \geq r \cdot\binom{\lfloor m / r\rfloor}{ 2} \in \Omega\left(\frac{m^{2}}{n}\right)
$$

where $r \leq 3 n-6$ is the maximum number of edges in a planar subgraph of $G$.

## Proof sketch.

■ Take $\lfloor m / r\rfloor$ edge-disjoint subgraphs $G_{1}, G_{2}, \ldots, G_{\lfloor m / r\rfloor}$ of $G$ with (at least) $r$ edges.
■ In the best case, they are all planar.
■ For every $i<j$, any edge of $G_{j}$ induces at least one crossings with $G_{i}$. (Otherwise, we could add an edge to $G_{i}$ and obtain a planar subgraph of $G$ with $r+1$ edges.)

- So, for each of the $\binom{\lfloor m / r\rfloor}{ 2}$ pairs of subgraphs, there are at least $r$ crossings.


## The Crossing Lemma

■ In 1973 Erdős and Guy conjectured that $\operatorname{cr}(G) \in \Omega\left(m^{3} / n^{2}\right)$.

■ In 1982 Leighton and, independently, Ajtai, Chávtal, Newborn, and Szemerédi showed that

$$
\begin{array}{ll}
\operatorname{cr}(G) \geq \frac{1}{64} \cdot \frac{m^{3}}{n^{2}} . & \begin{array}{l}
\text { Consider this bound for } \\
\text { graphs with } \Theta(n) \text { and } \\
\Theta\left(n^{2}\right) \text { many edges. }
\end{array}
\end{array}
$$

- Bound is asymptotically tight.

■ Result stayed hardly known until Székely demonstrated its usefulness (in 1997).

- We go through the proof of Chazelle, Sharir, and Welzl (see "THE BOOK").
- Factor $\frac{1}{64}$ was later (with intermediate steps) improved to $\frac{1}{29}$ by Ackerman in 2013.


## The Crossing Lemma

## Crossing Lemma.

For a graph $G$ with $n$ vertices and $m$ edges, $m \geq 4 n$,

$$
\operatorname{cr}(G) \geq \frac{1}{64} \cdot \frac{m^{3}}{n^{2}}
$$

## Proof.

- Consider a crossing-minimal drawing of $G$.
- Let $p$ be a number in $(0,1]$.

■ Keep every vertex of $G$ independently with probability $p$.
■ $G_{p}=$ remaining graph (with drawing $\Gamma_{p}$ ).

- Let $n_{p}, m_{p}, X_{p}$ be the random variables counting the numbers of vertices / edges / crossings of $\Gamma_{p}$, resp.
■ By Lemma 2, $\operatorname{cr}\left(G_{p}\right)-m_{p}+3 n_{p} \geq 6$.
$\square \mathrm{E}\left[n_{p}\right]=p n$ and $\mathrm{E}\left[m_{p}\right]=p^{2} m$
■ $\mathrm{E}\left[X_{p}\right]=p^{4} \operatorname{cr}(G)$
$\square 0 \leq \mathrm{E}\left[X_{p}\right]-\mathrm{E}\left[m_{p}\right]+3 \mathrm{E}\left[n_{p}\right]$

$$
=p^{4} \mathrm{cr}(G)-p^{2} m+3 p n
$$

$■ \operatorname{cr}(G) \geq \frac{p^{2} m-3 p n}{p^{4}}=\frac{m}{p^{2}}-\frac{3 n}{p^{3}}$
$\square$ Set $p=\frac{4 n}{m}$.
$■ \operatorname{cr}(G) \geq \frac{m^{3}}{16 n^{2}}-\frac{3 m^{3}}{64 n^{2}}=\frac{1}{64} \frac{m^{3}}{n^{2}}$
$\operatorname{cr}(G) \geq m-3 n+6 \quad \Rightarrow \mathrm{E}\left[X_{p}-m_{p}+3 n_{p}\right] \geq 0$.

## Application 1: Point-Line Incidences

For a set $P \subset \mathbb{R}^{2}$ of points and a set $\mathcal{L}$ of lines, let $I(P, \mathcal{L})=\#$ point-line incidences in $(P, \mathcal{L})$.

$$
\Rightarrow I(P, \mathcal{L})=10
$$

■ Define $I(n, k)=\max _{|P|=n,|\mathcal{L}|=k} I(P, \mathcal{L})$.
■ For example: $I(4,4)=9$


Theorem 1.
[Szemerédi, Trotter '83, Székely '97]

$$
I(n, k) \leq 2.7 n^{2 / 3} k^{2 / 3}+6 n+2 k
$$

## Application 1: Point-Line Incidences

For a set $P \subset \mathbb{R}^{2}$ of points and a set $\mathcal{L}$ of lines, let $I(P, \mathcal{L})=\#$ point-line incidences in $(P, \mathcal{L})$.

## Theorem 1.

[Szemerédi, Trotter '83, Székely '97]

$$
I(n, k) \leq c\left(n^{2 / 3} k^{2 / 3}+n+k\right) .
$$

$$
\Rightarrow I(P, \mathcal{L})=10
$$

## Proof.

- $\#($ points on a line $\ell)-1=\#($ edges on $\ell)$

$$
\Rightarrow I(n, k)-k=m \quad(\text { sum up over } \mathcal{L} \text { in an }
$$

"optimal" instance)

■ Define $I(n, k)=\max _{|P|=n,|\mathcal{L}|=k} I(P, \mathcal{L})$.
■ For example: $I(4,4)=9$

- If $m \leq 4 n$, then $I(n, k)-k=m \leq 4 n$.

$$
\Rightarrow I(n, k) \leq 4 n+k \leq c\left(n+k+n^{2 / 3} k^{2 / 3}\right)
$$

■ Otherwise, employ the Crossing Lemma:

$$
\begin{aligned}
& \frac{m^{3}}{64 n^{2}} \leq \operatorname{cr}(G) \leq k^{2} / 2 \Rightarrow \frac{(I(n, k)-k)^{3}}{64 n^{2}} \leq k^{2} / 2 \\
& \Leftrightarrow I(n, k) \leq c\left(n^{2 / 3} k^{2 / 3}+k\right) \\
& \quad \leq c\left(n^{2 / 3} k^{2 / 3}+k+n\right) .
\end{aligned}
$$

## Application 2: Unit Distances

For a set $P \subset \mathbb{R}^{2}$ of points, define
■ $U(P)=$ number of pairs in $P$ at unit distance and
$\square U(n)=\max _{|P|=n} U(P)$.

## Theorem 2. <br> [Spencer, Szemerédi, Trotter '84, Székely '97] $U(n)<6.7 n^{4 / 3}$

## Proof sketch.



## Application 2: Unit Distances

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Proof sketch.

$\square U(P) \leq c^{\prime \prime} m$ number of edges in $G$
$\square \operatorname{cr}(G) \leq 2\binom{n}{2} \leq n^{2}$ (Circles intersect each other at most twice.)

- $n^{2} \geq \operatorname{cr}(G) \geq \frac{m^{3}}{64 n^{2}} \geq$ by the Crossing Lemma.


## Application 3: Expected Number of Crossings in a Matching

Given set of $n$ points (in general position, $n$ even) what is the average number of crossings in a perfect matching?

Point set spans drawing $\Gamma$ of $K_{n}$.
We will analyze the number of crossings in a random perfect matching in $\Gamma$ !
Number of crossings in $\Gamma \geq \overline{\operatorname{cr}}\left(K_{n}\right)>\frac{3}{8}\binom{n}{4}$
[Lovász et al. '04, Aichholzer et al. '06]
Number of edges in $K_{n}:\binom{n}{2}$
Number of potential crossings (all pairs of edges): $\operatorname{pot}\left(K_{n}\right)=\left(\begin{array}{c}n \\ 2 \\ 2\end{array}\right) \approx 3\binom{n}{4}$
Pick two random edges $e_{1}$ and $e_{2}$.

$$
\operatorname{Pr}\left[e_{1} \text { and } e_{2} \operatorname{cross}\right] \geq \overline{\operatorname{cr}}\left(K_{n}\right) / \operatorname{pot}\left(K_{n}\right)>\frac{1}{8} .
$$

Pick random perfect matching $M$; it has $n / 2$ edges, so $\binom{n / 2}{2}=\frac{1}{8} n(n-2)$ pairs of edges. By linearity of expectation, the expected number of crossings in $M$ is $>\frac{1}{8}\binom{n / 2}{2}=\frac{1}{64} n(n-2) \in \Omega\left(n^{2}\right)$.

## Literature

■ [Aigner, Ziegler] Proofs from THE BOOK [https://doi.org/10.1007/978-3-662-57265-8]
■ [Schaefer '20] The Graph Crossing Number and its Variants: A Survey

- Terrence Tao's blog post "The crossing number inequality" from 2007
- [Hanani '43] Über wesentlich unplättbare Kurven im dreidimensionalen Raume
- [Tutte '70] Toward a theory of crossing numbers
- [Pach \& Tóth '00] Which crossing number is it anyway?
- [Pelsmajer, Schaefer \& Štefankovič '07] Removing even crossings
- [Pelsmajer, Schaefer \& Štefankovič '08] Odd Crossing Number and Crossing Number Are Not the Same
- [Tóth '08] Note on the Pair-crossing Number and the Odd-crossing Number

■ [Garey, Johnson '83] Crossing number is NP-complete

- [Bienstock, Dean '93] Bounds for rectilinear crossing numbers

■ [Lovász et al. '04] Towards a theory of geometric graphs

- [Aichholzer et al. '06] On the Crossing Number of Complete Graphs

■ [Székely '97] Crossing Numbers and Hard Erdős Problems in Discrete Geometry

- Documentary/Biography " $N$ Is a Number: A Portrait of Paul Erdős"

■ Exact computations of crossing numbers: http://crossings.uos.de

