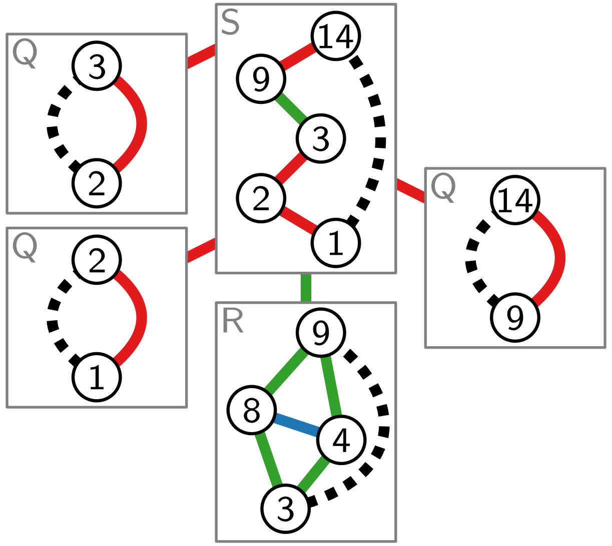
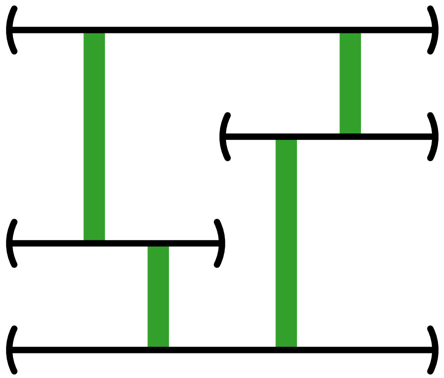
Julius-Maximilians-
UNIVERSITÄT
WÜRZBURG

Visualization of Graphs

Lecture 10:
Partial Visibility Representation Extension



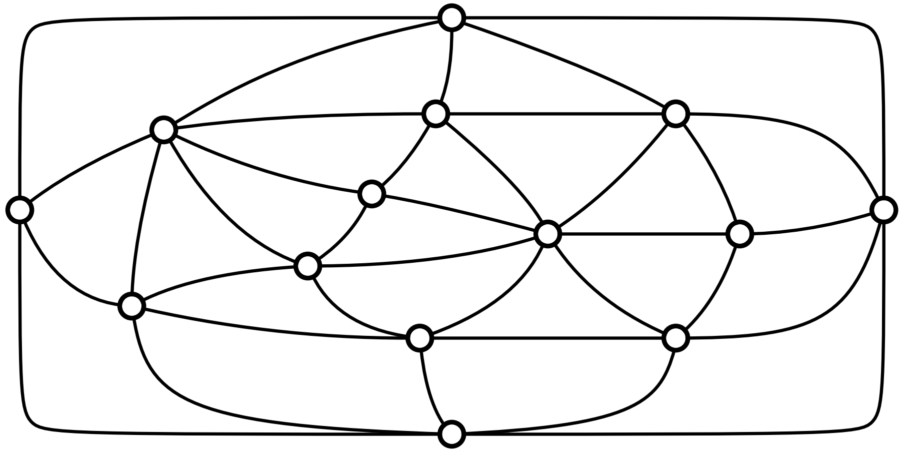
Johannes Zink



Summer semester 2024

Partial Representation Extension Problem

Let G be a graph.



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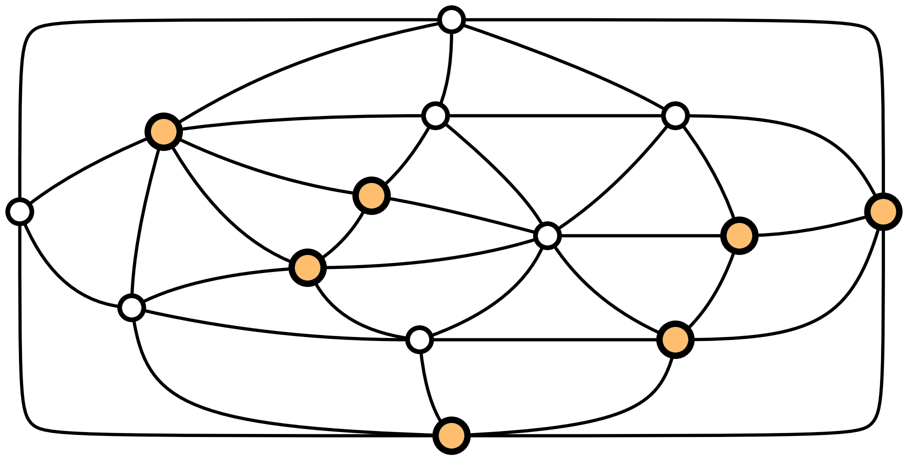
Let $V' \subseteq V(G)$



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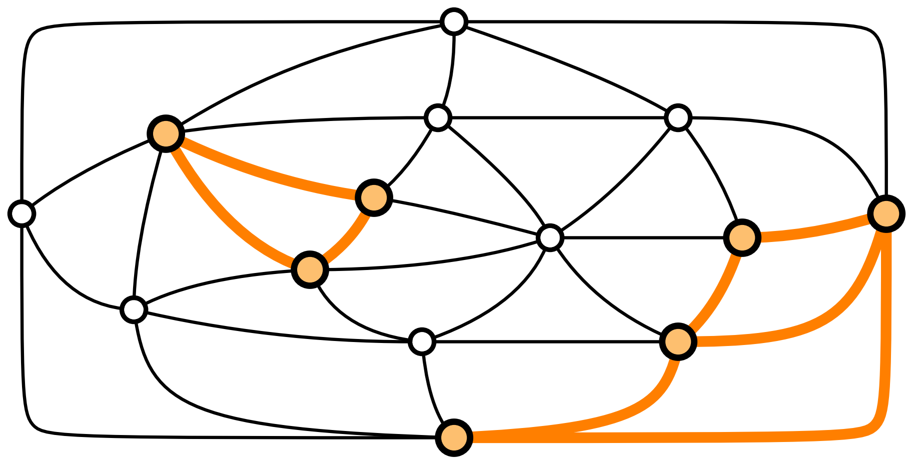
Let $V' \subseteq V(G)$ and $H = G[V']$



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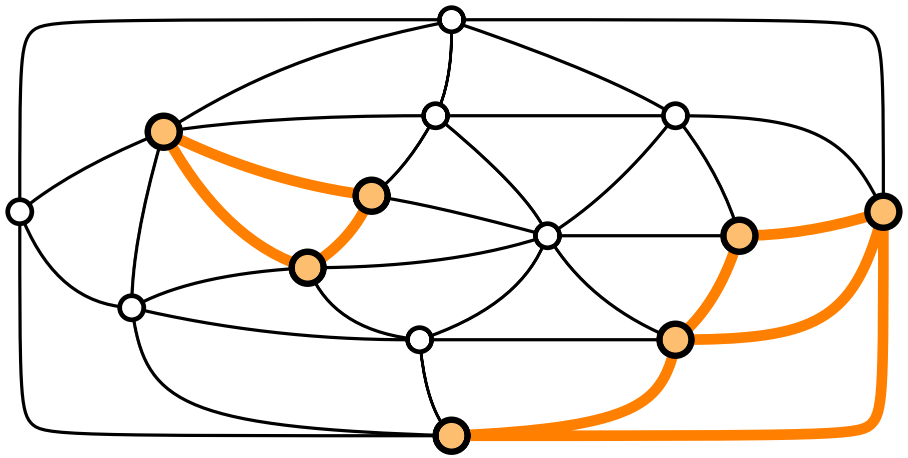

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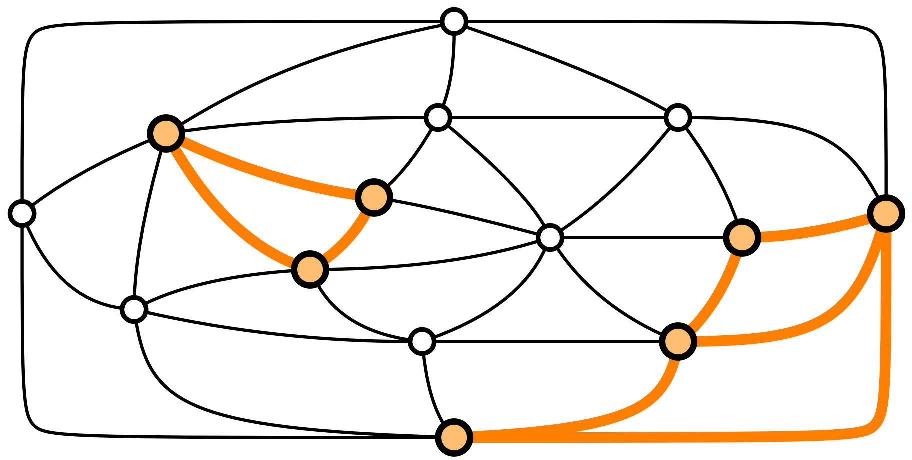

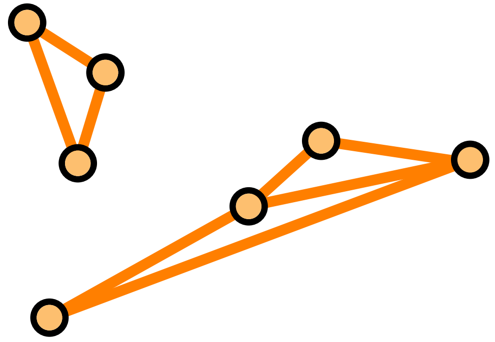

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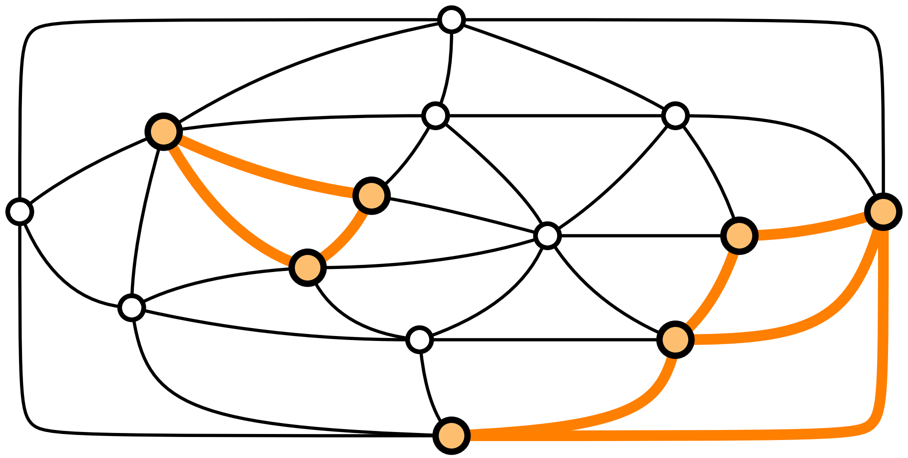

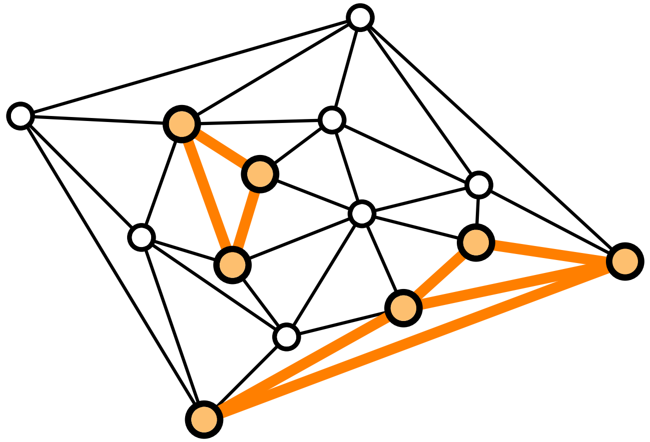

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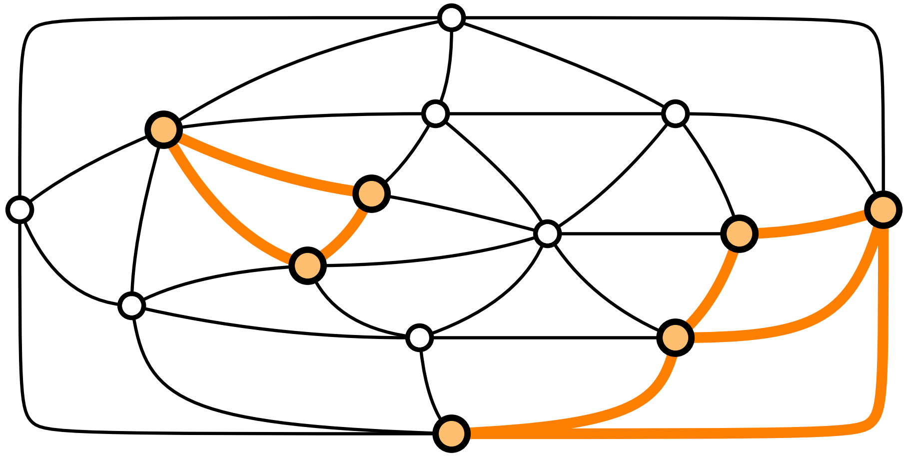

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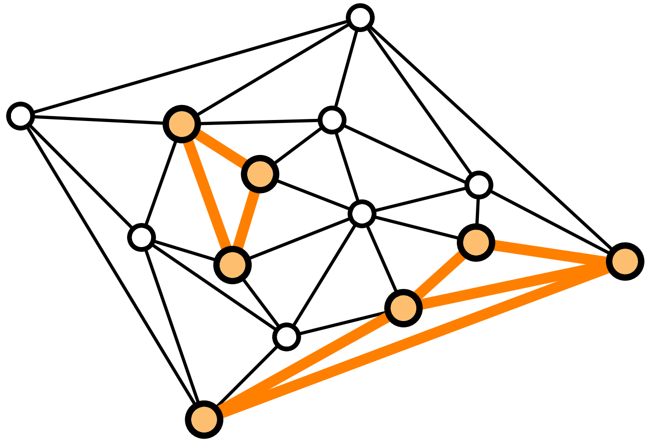

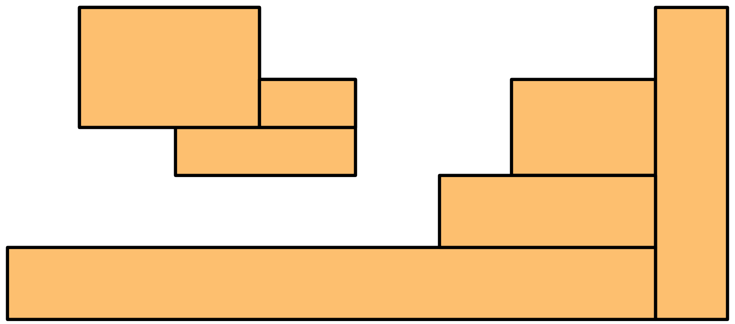

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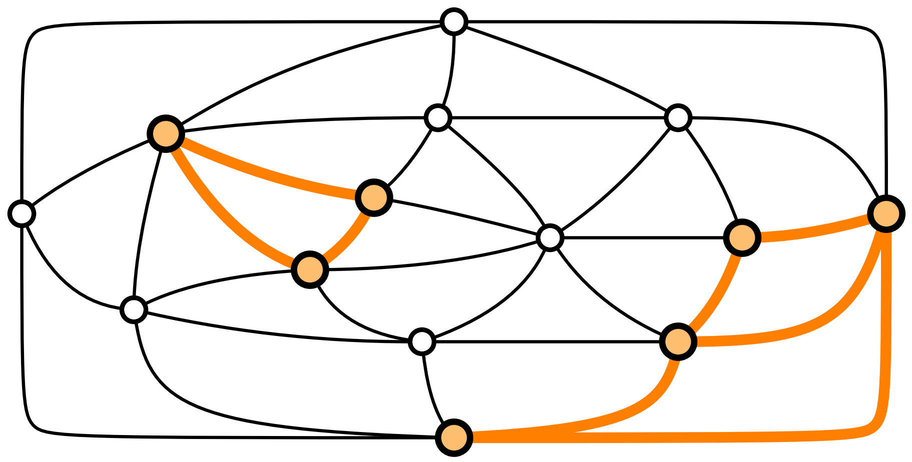

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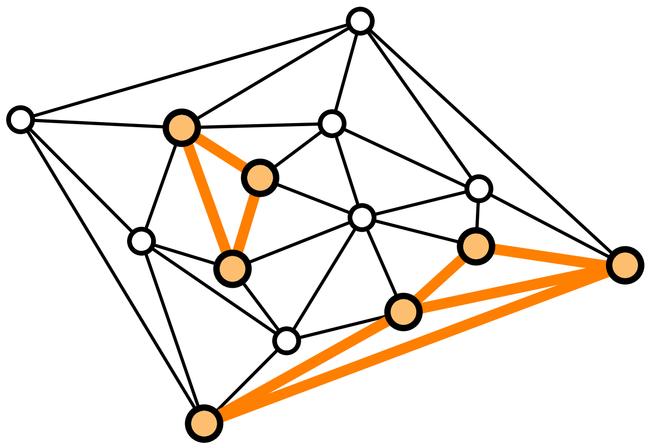

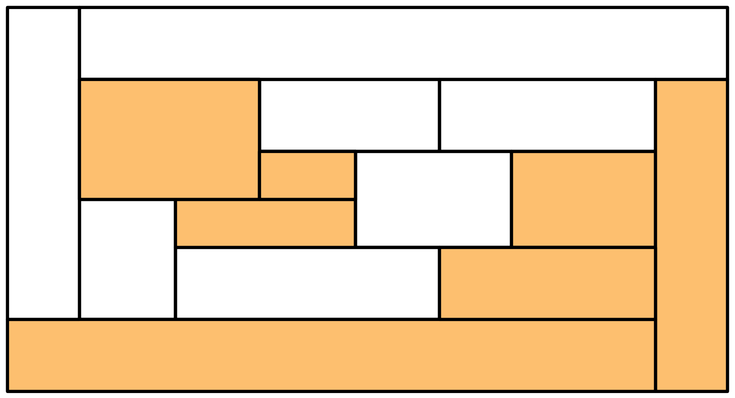

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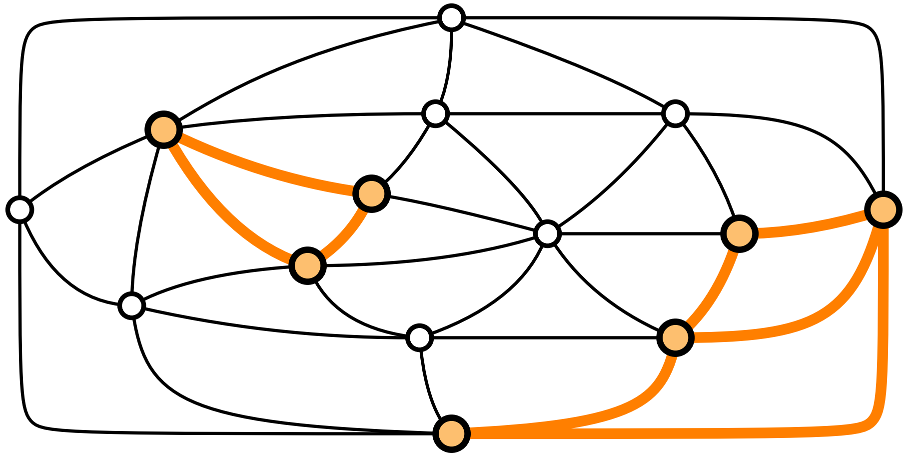

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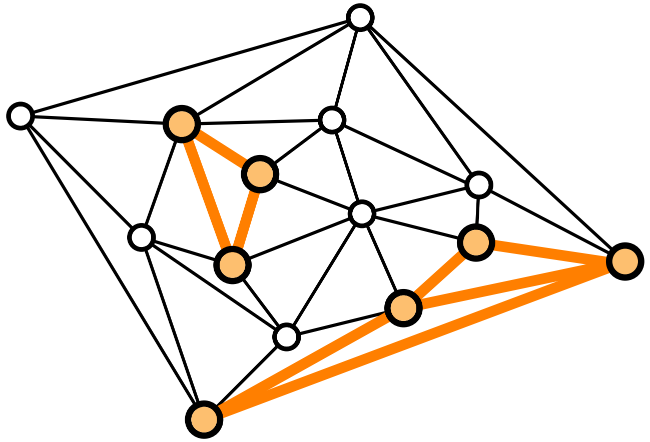

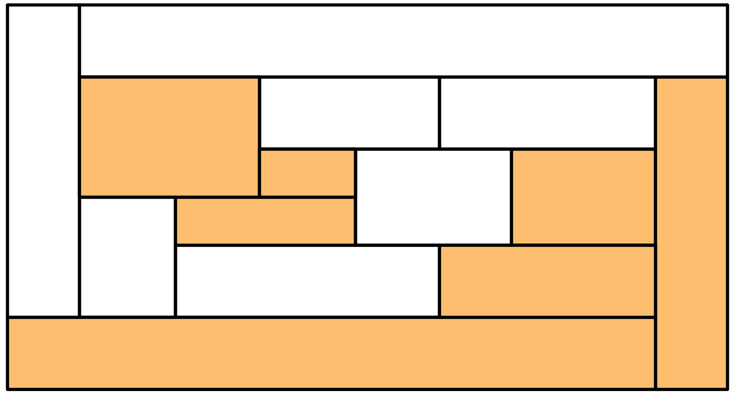

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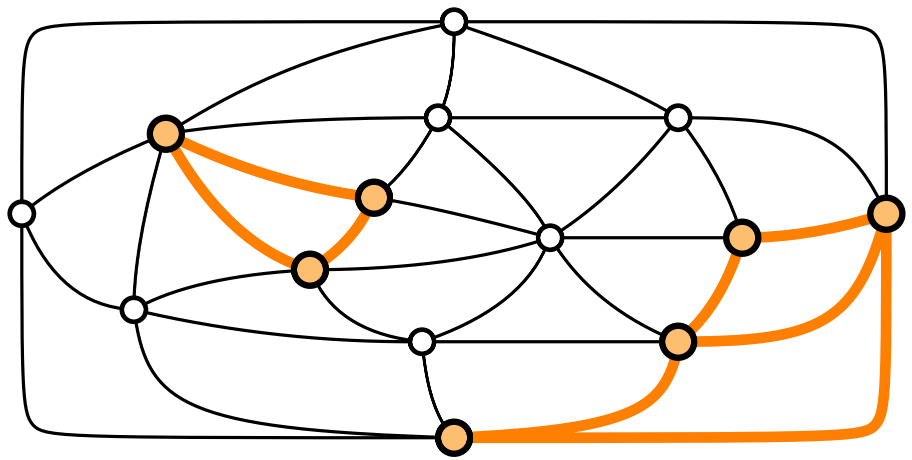
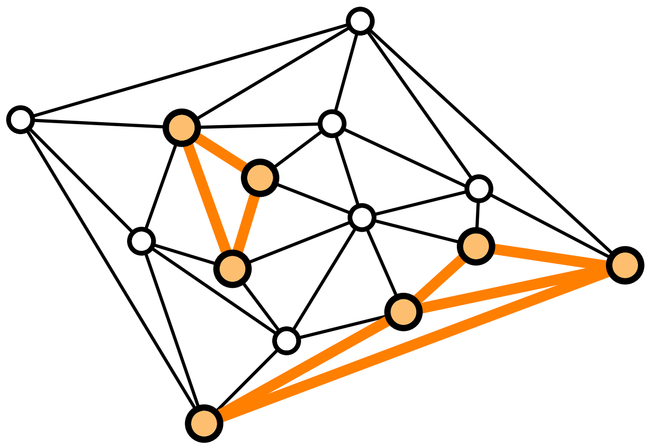
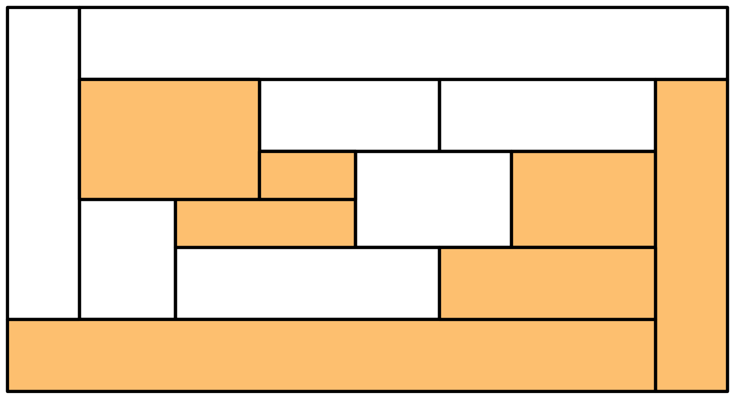
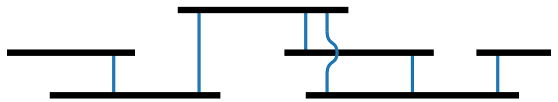






Polytime for:

- (unit) interval graphs



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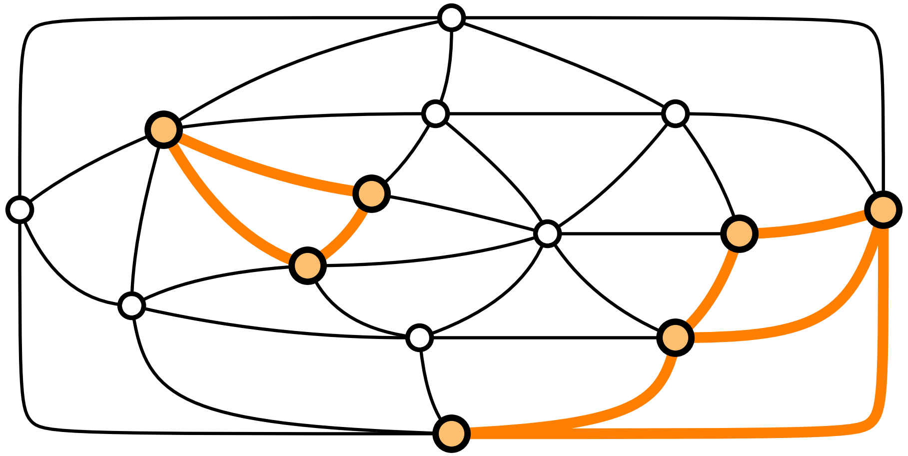

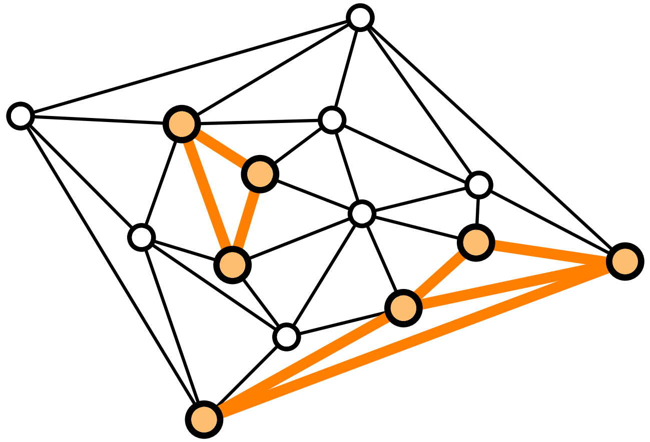

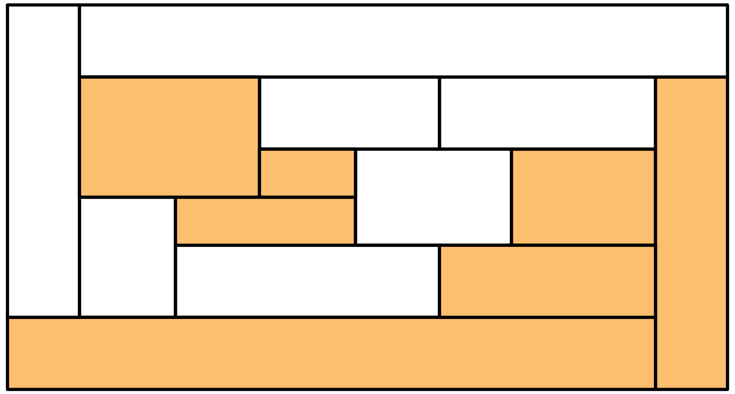

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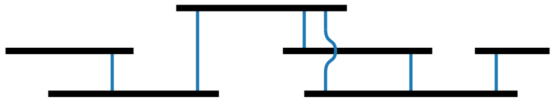

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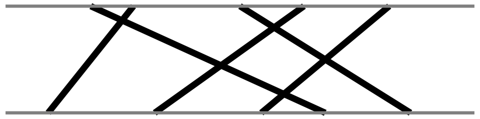

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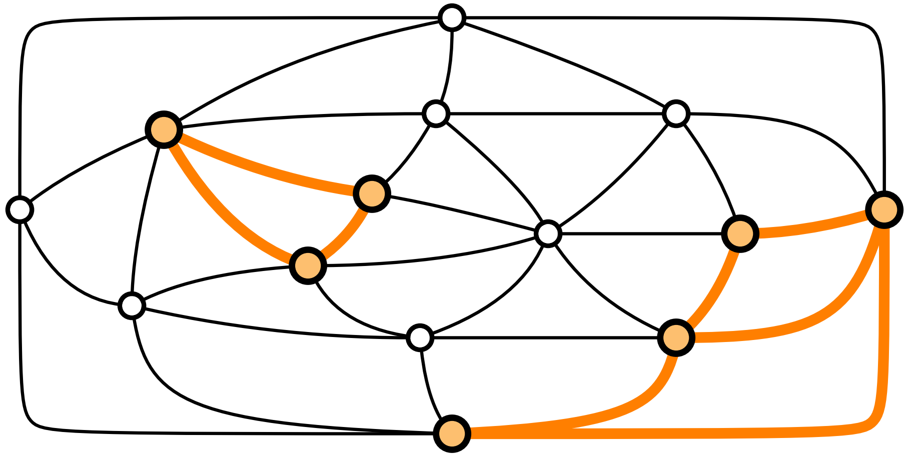

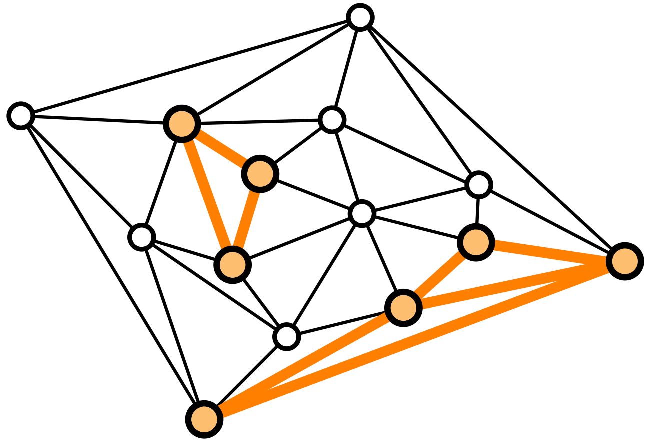

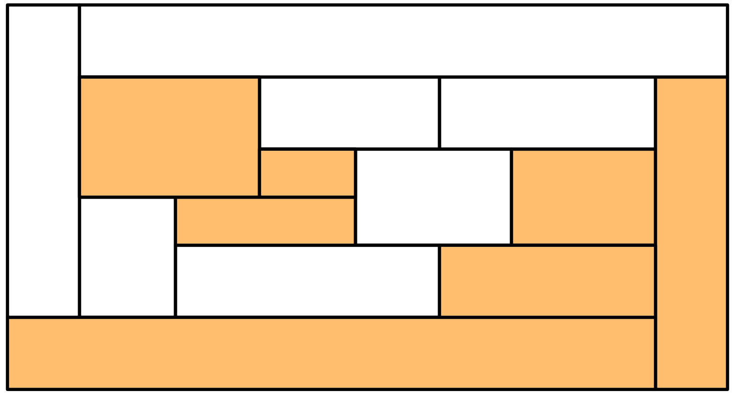

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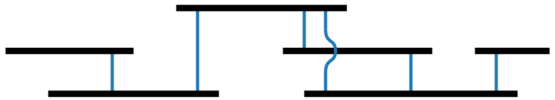

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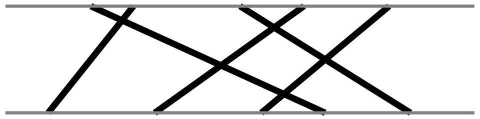

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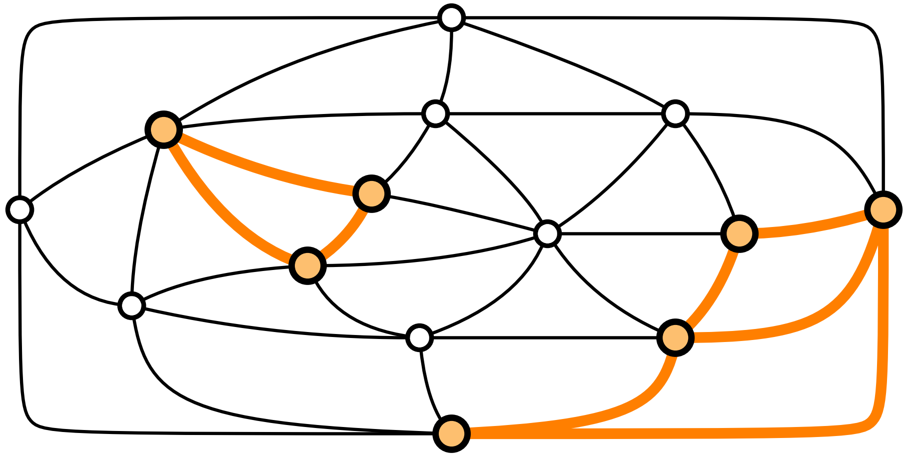

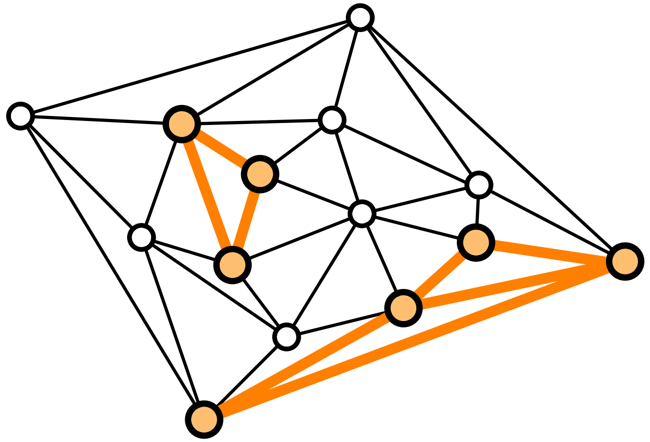

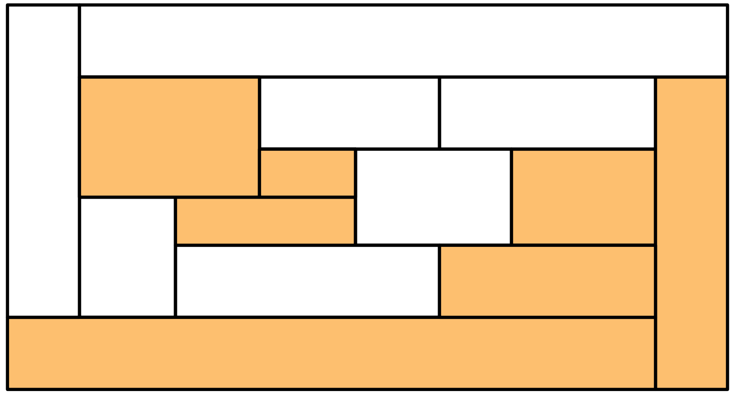

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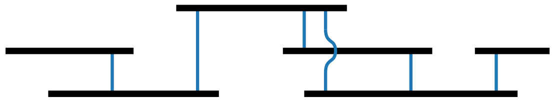

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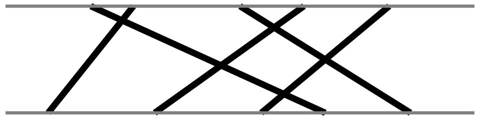

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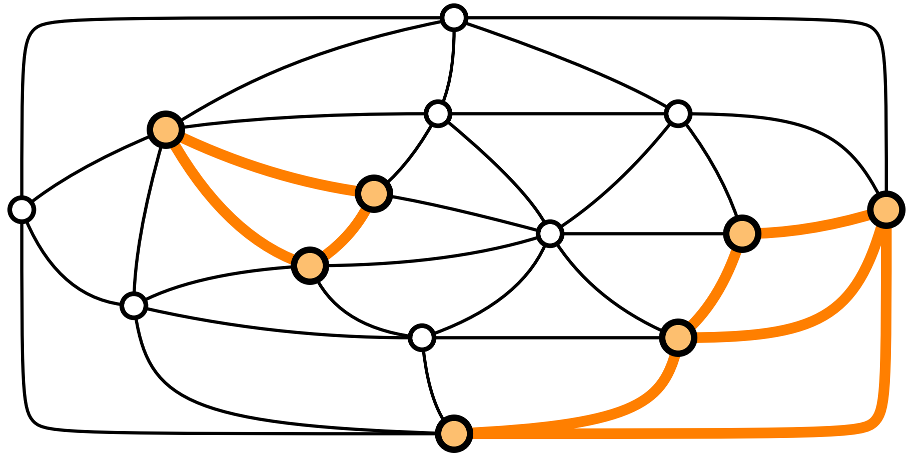

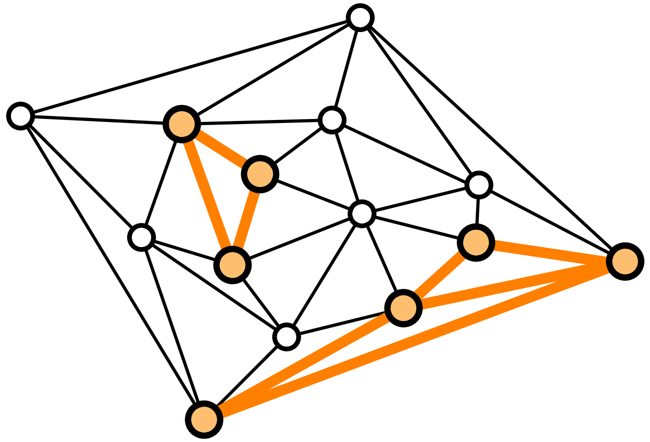

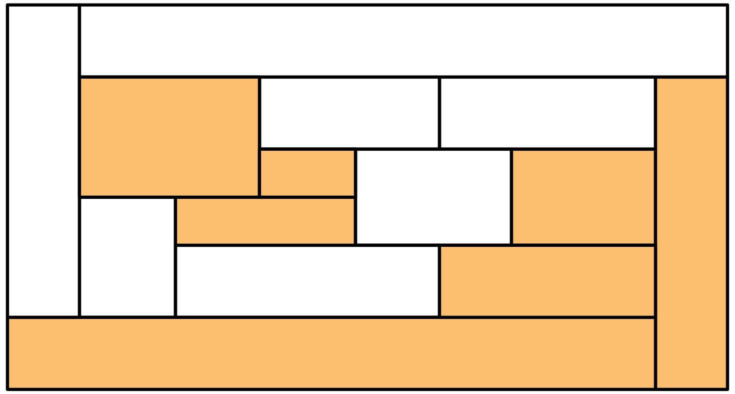

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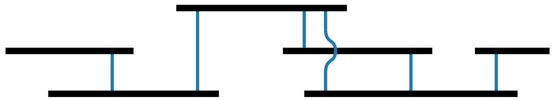

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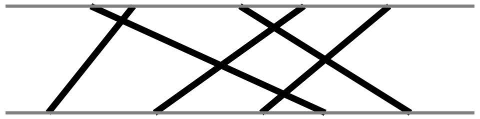

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NP-hard for:

- planar straight-line drawings

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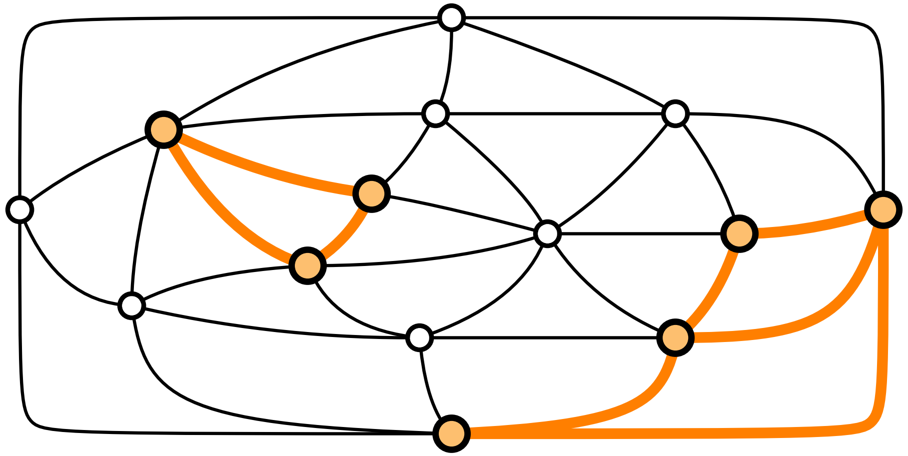

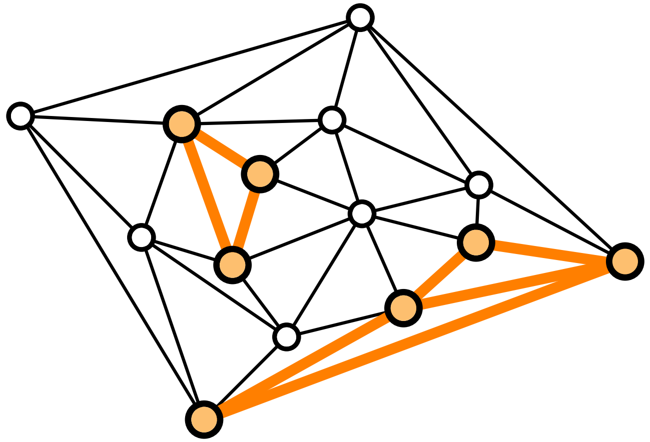

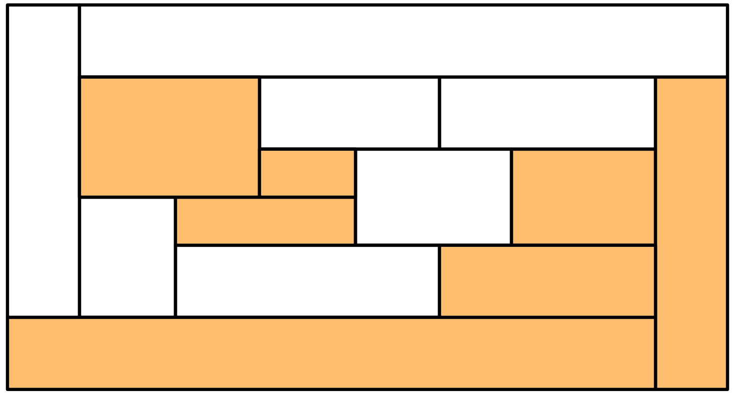

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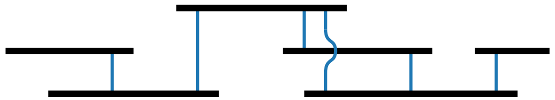

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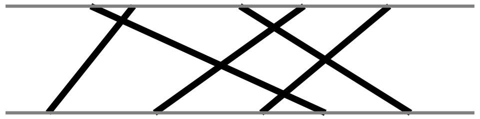

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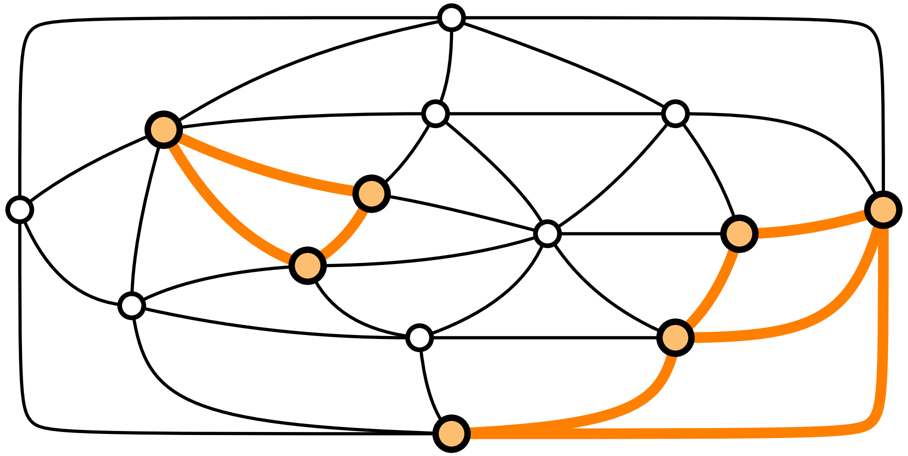

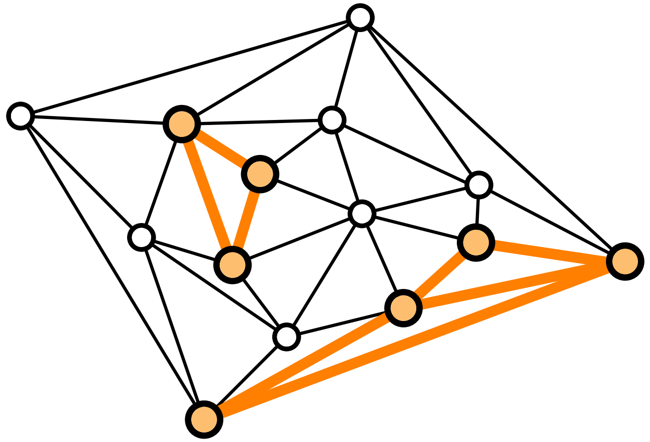

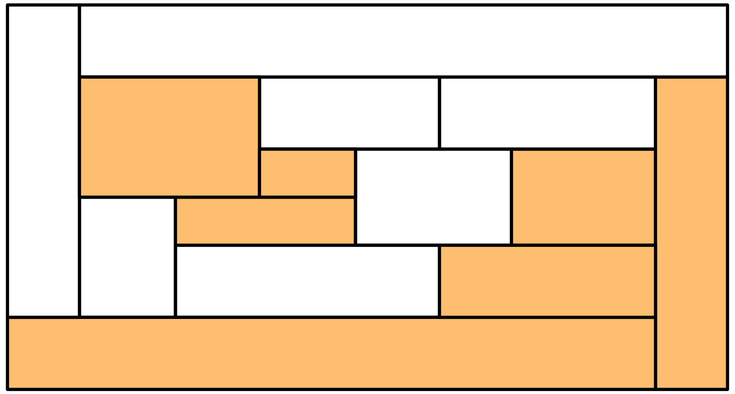

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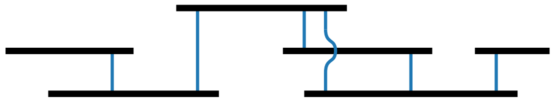

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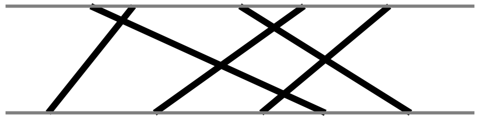

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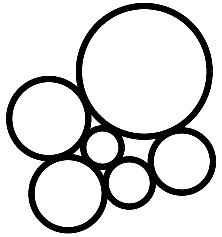

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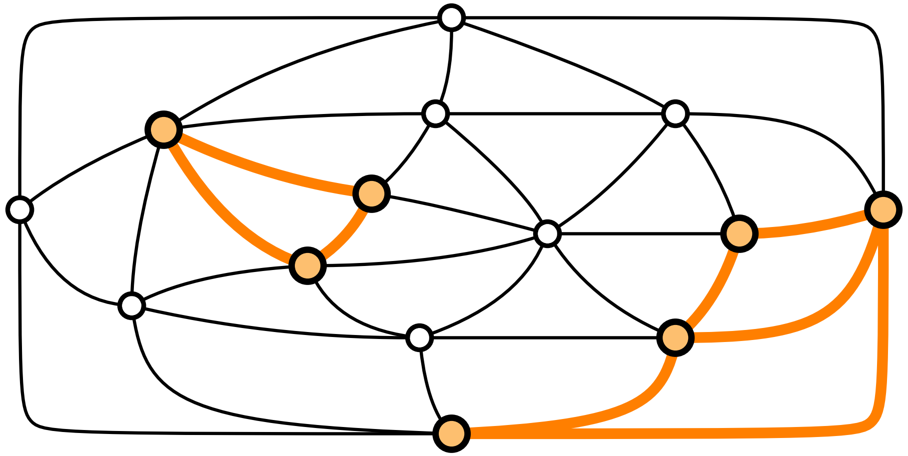

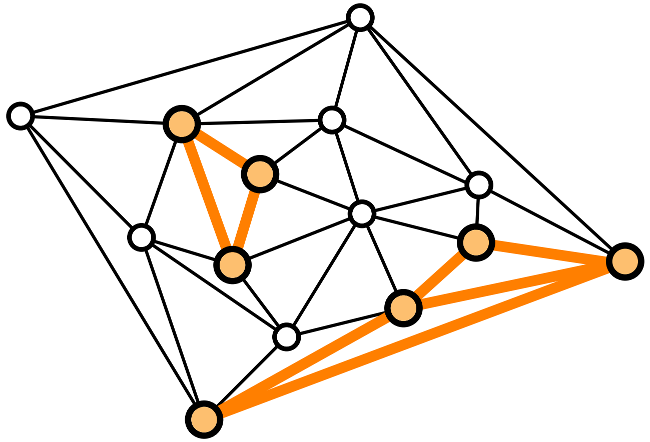

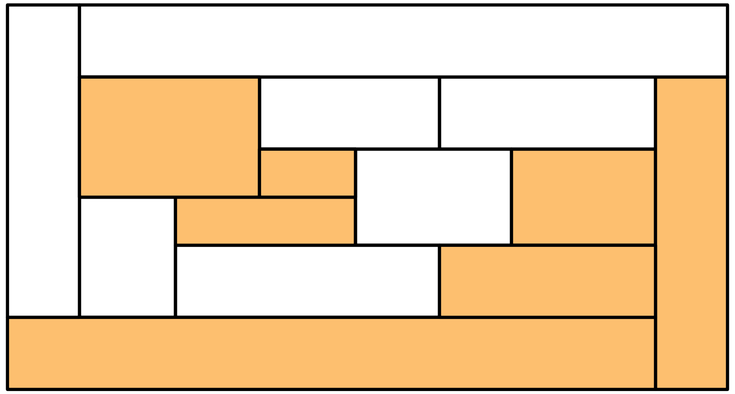

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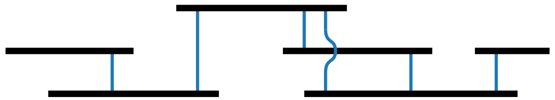

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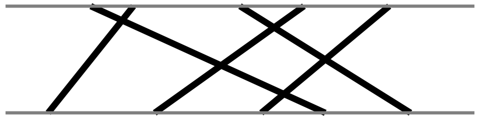

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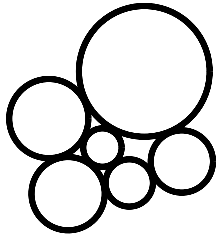

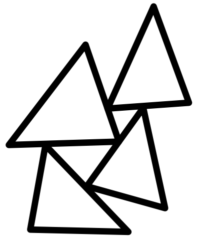

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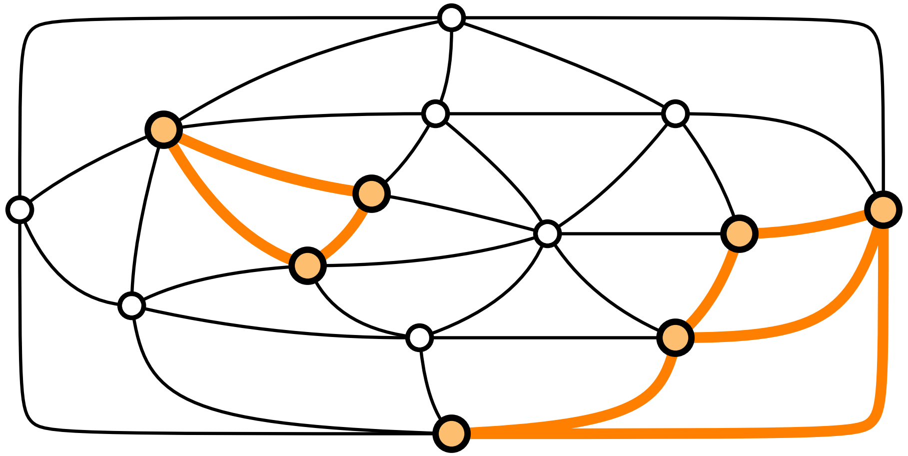

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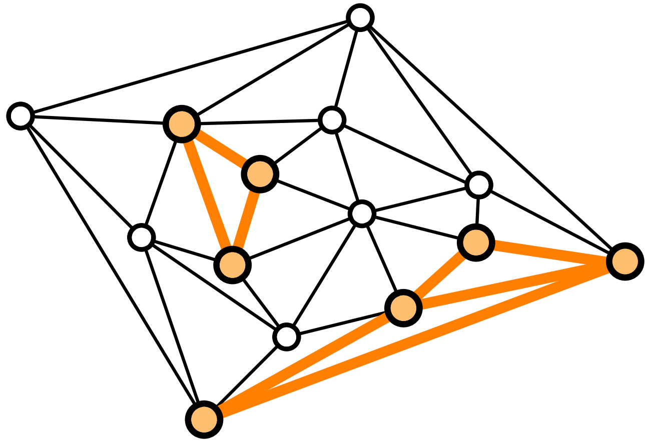
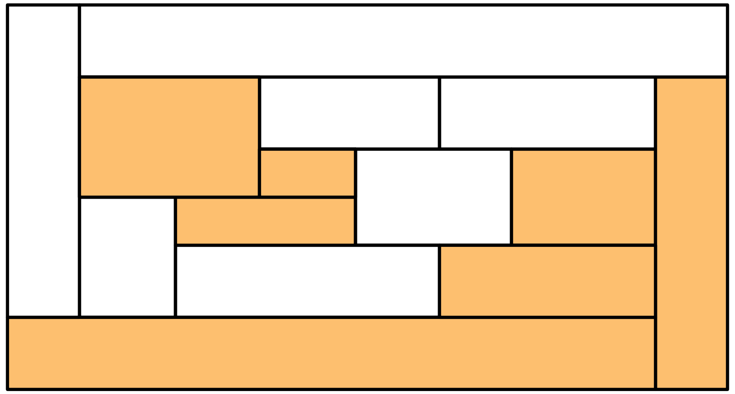
Let $V' \subseteq V(G)$ and $H = G[V']$

Let Γ_H be a representation of H.

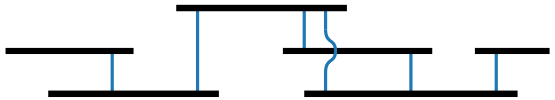
Find a representation Γ_G of G that *extends* Γ_H.



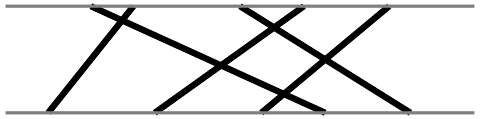


Polytime for:

- (unit) interval graphs



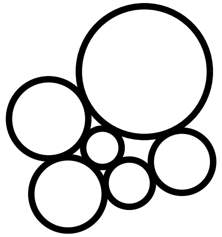
- permutation graphs



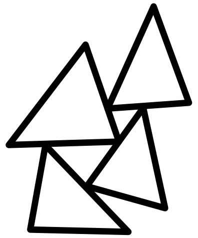
- circle graphs

NP-hard for:

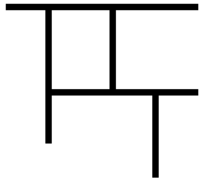
- planar straight-line drawings
- contacts of
  - disks



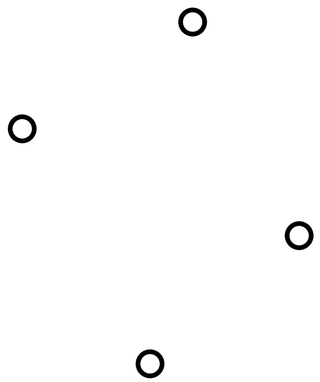
  - triangles



  - orthogonal segments

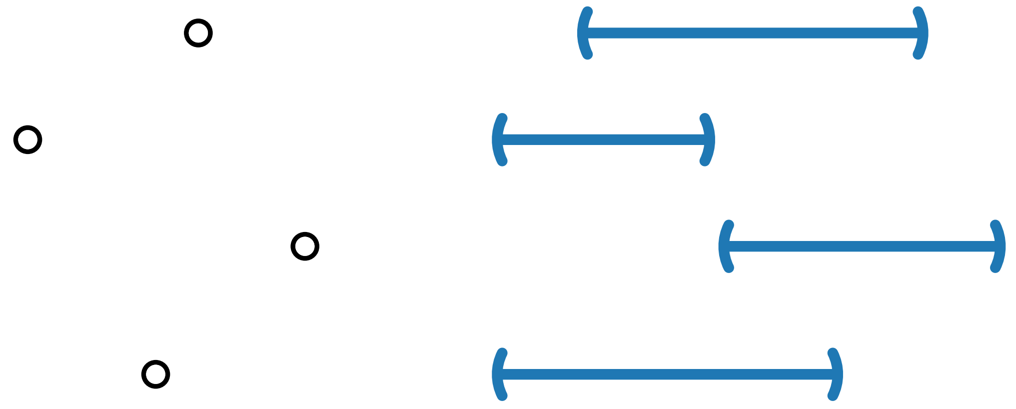


Bar Visibility Representation



Bar Visibility Representation

- Vertices correspond to horizontal (open) line segments called **bars**.



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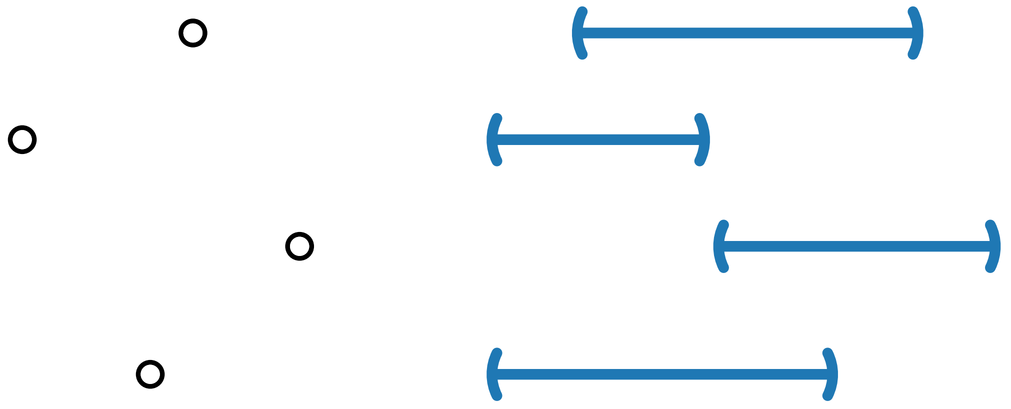

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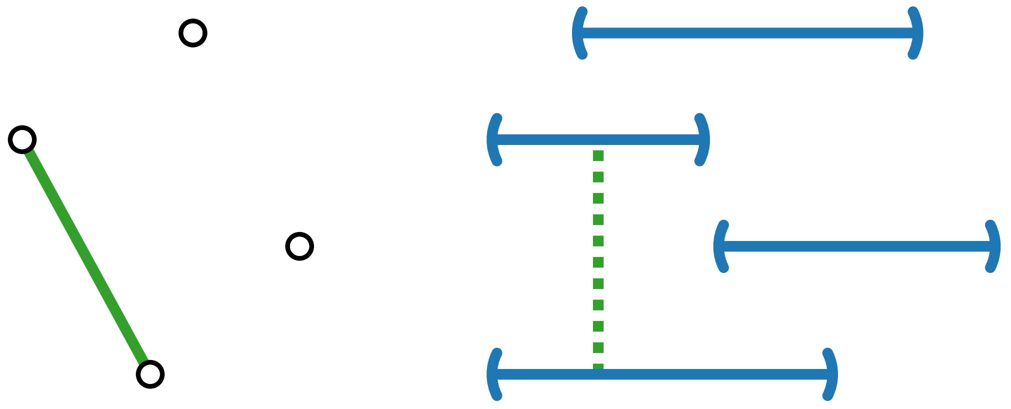

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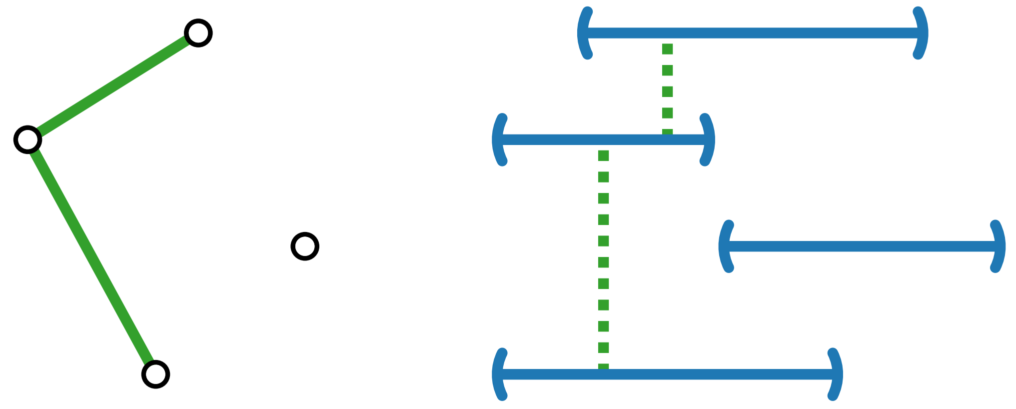

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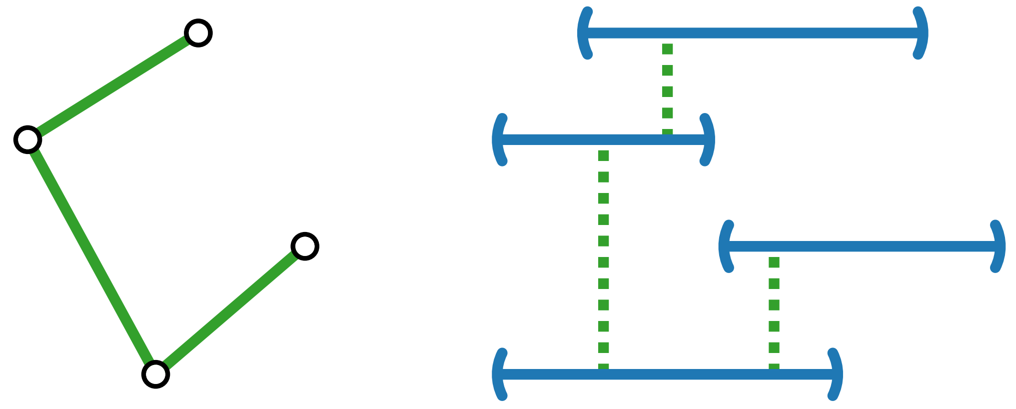

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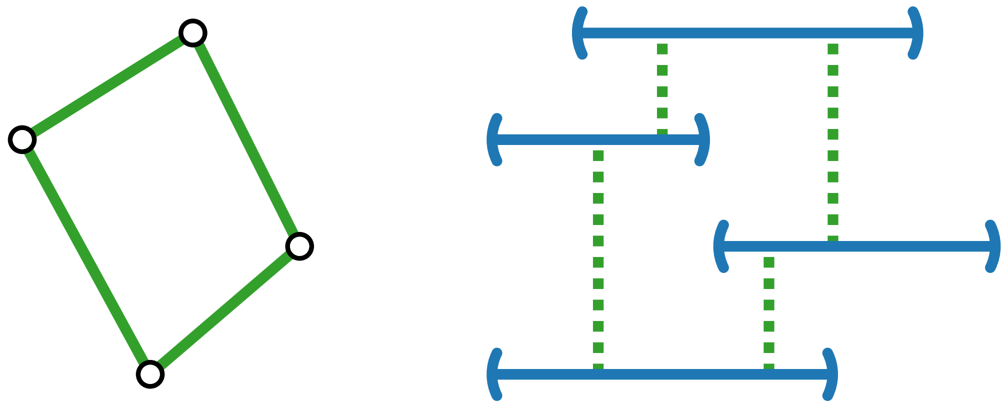

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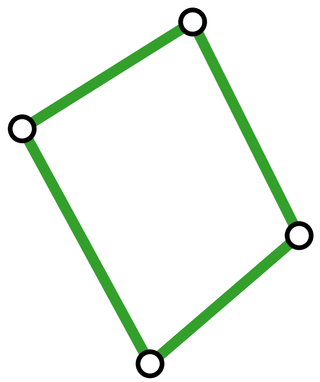

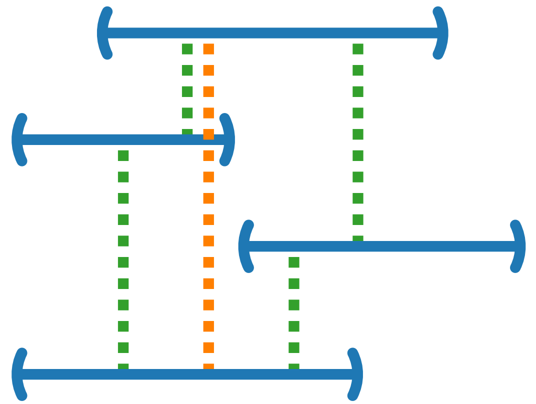

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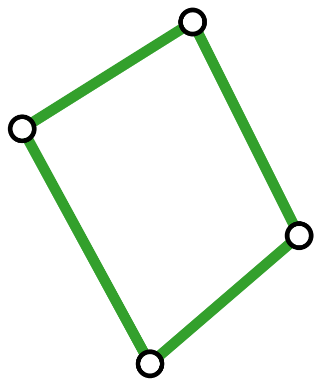

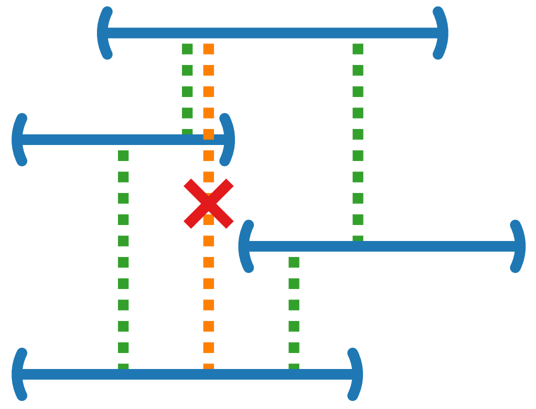

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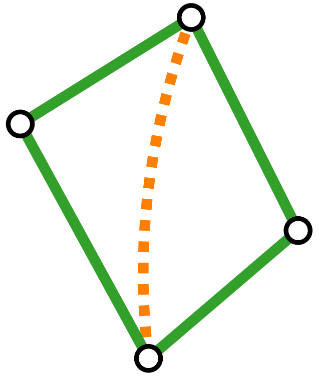

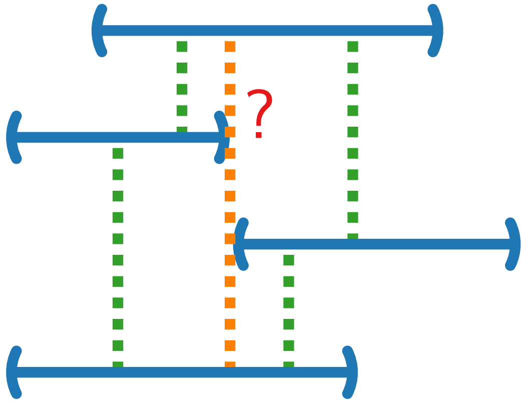

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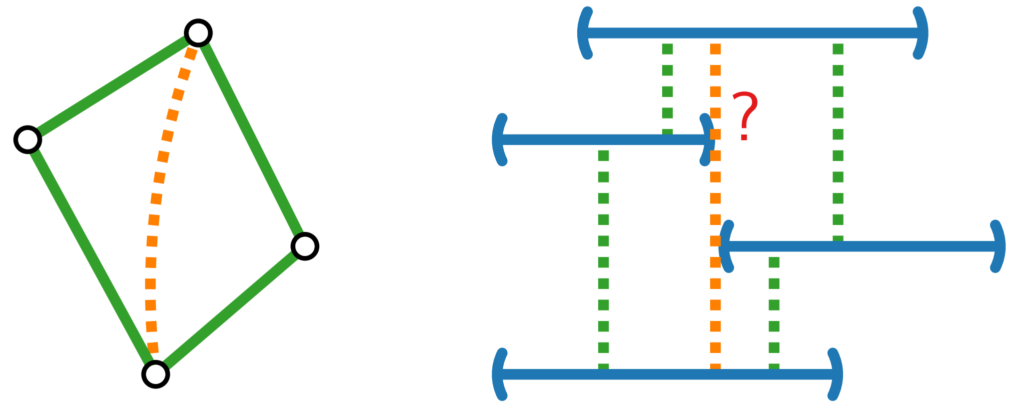

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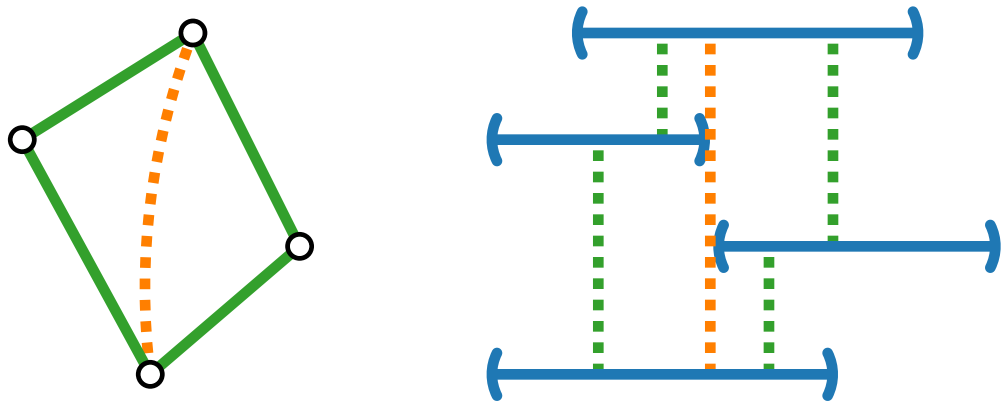

Models.

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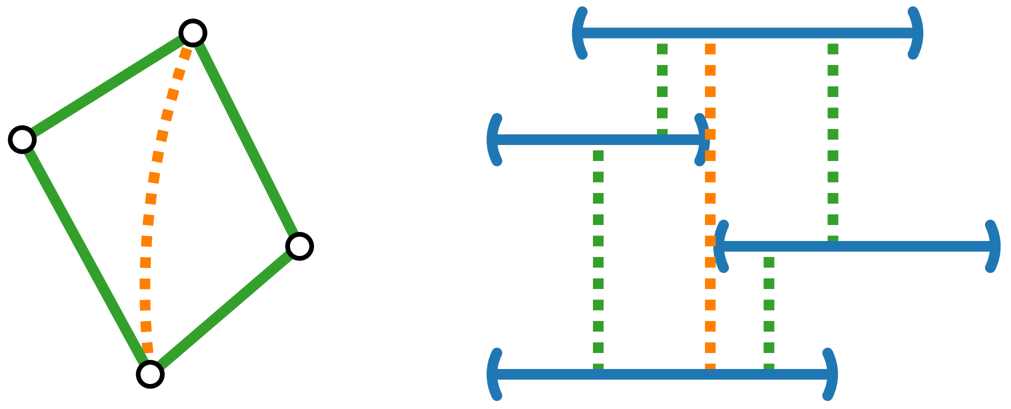

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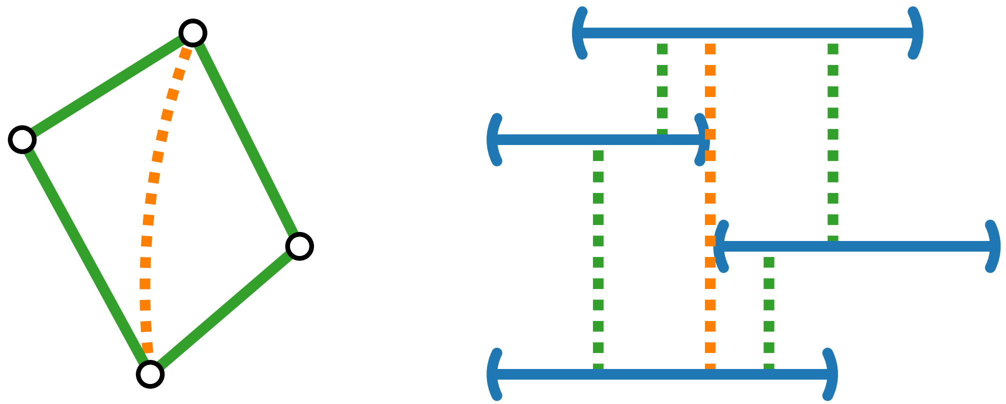

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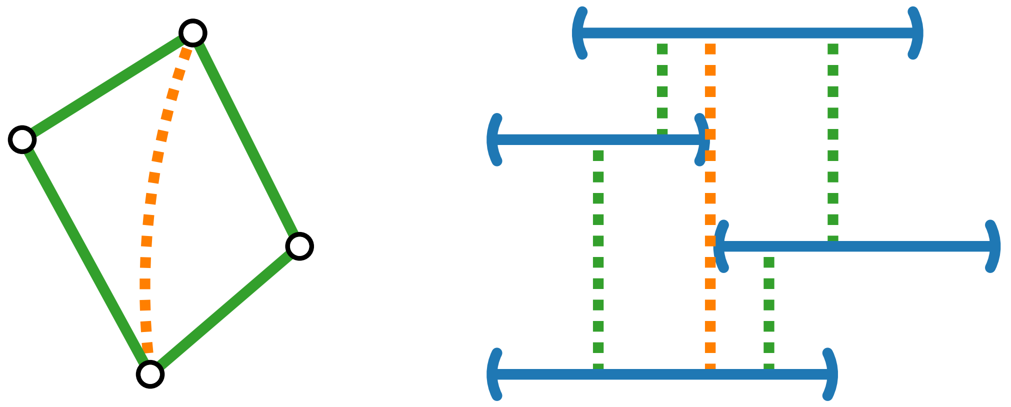

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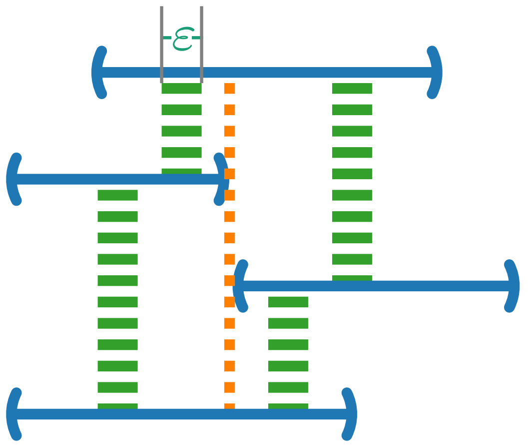

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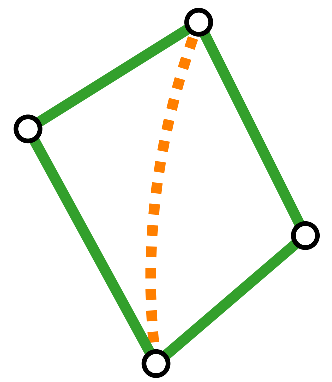

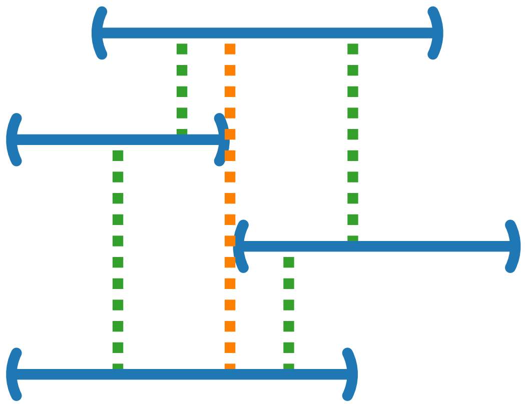

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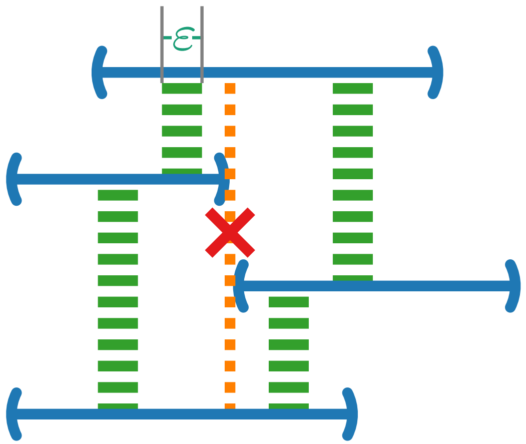

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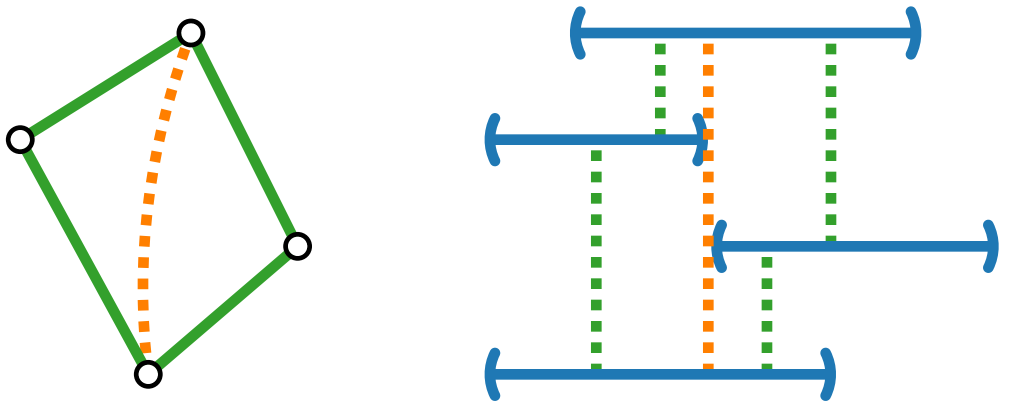

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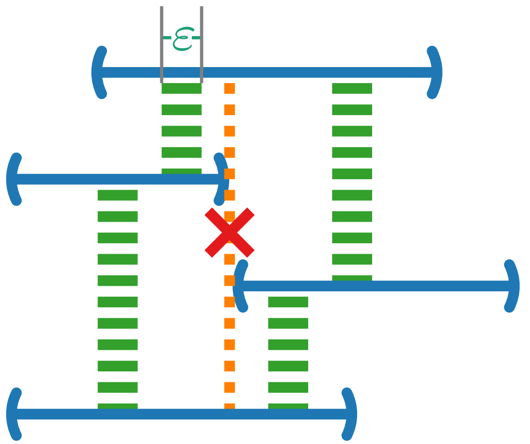

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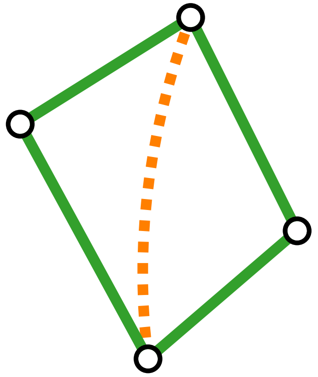

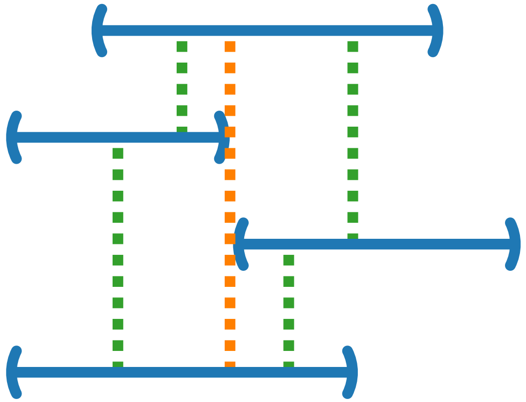

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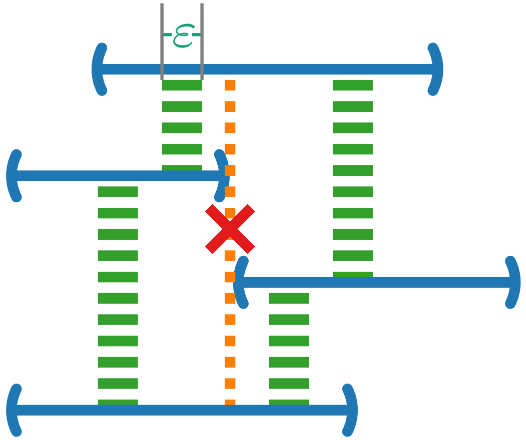
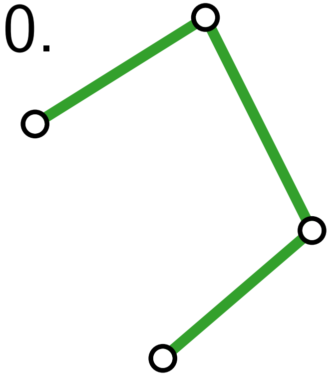
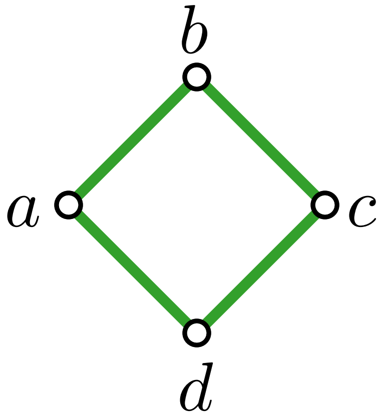
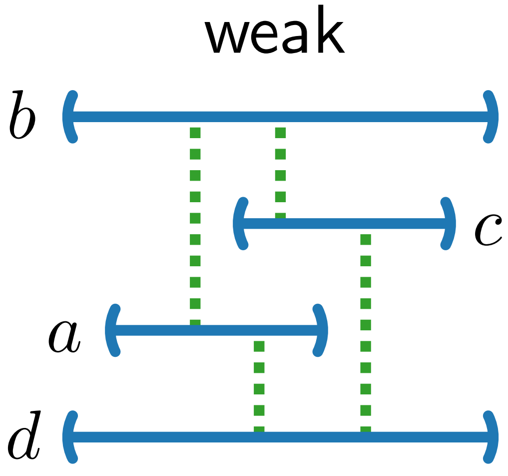




Problems





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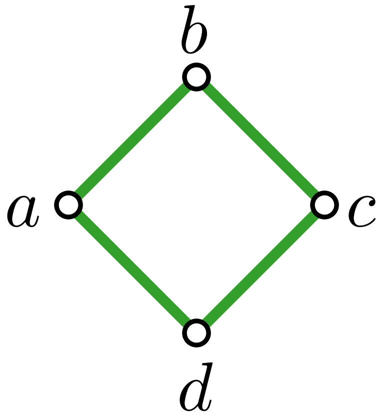
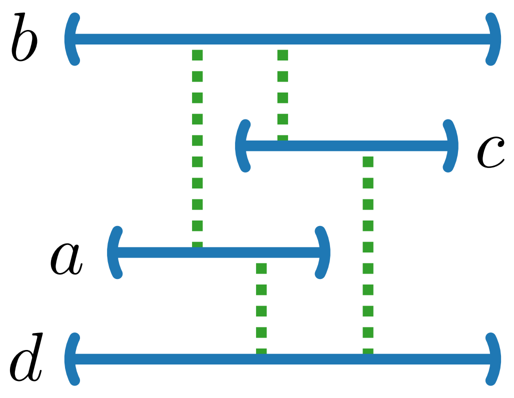
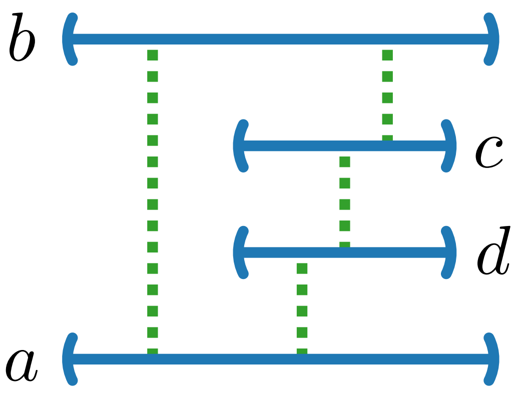


weak



strong



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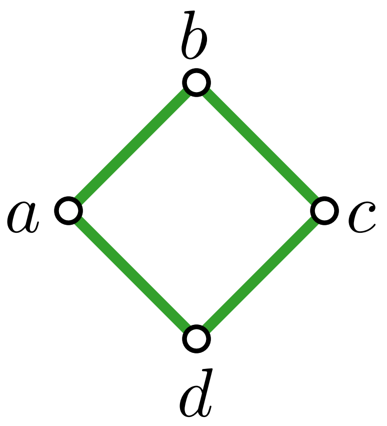

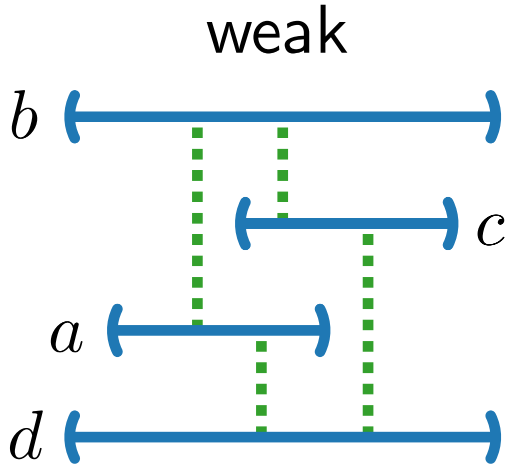

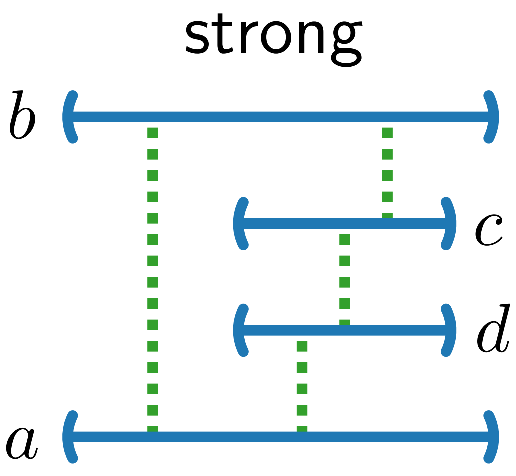

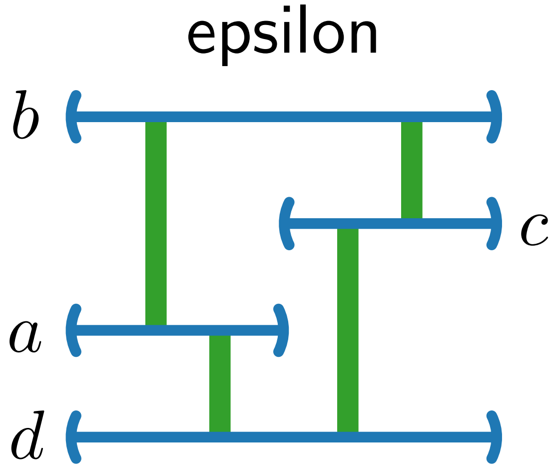

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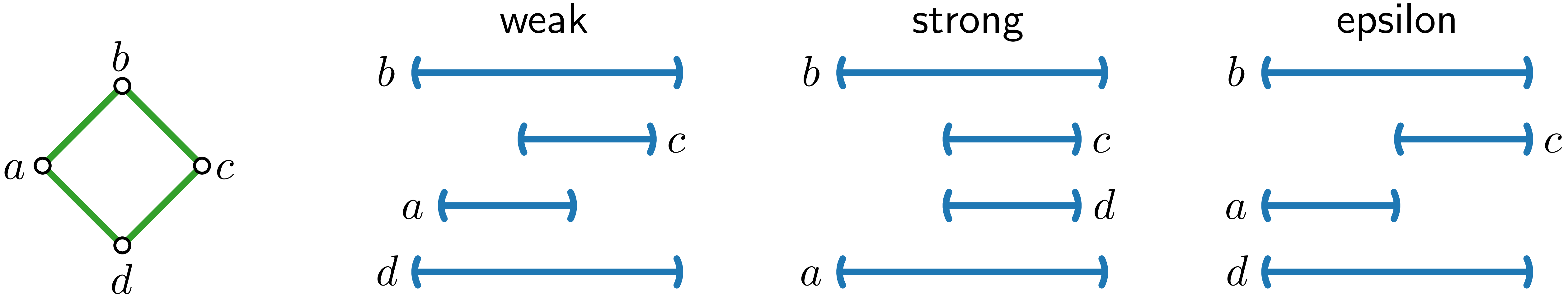

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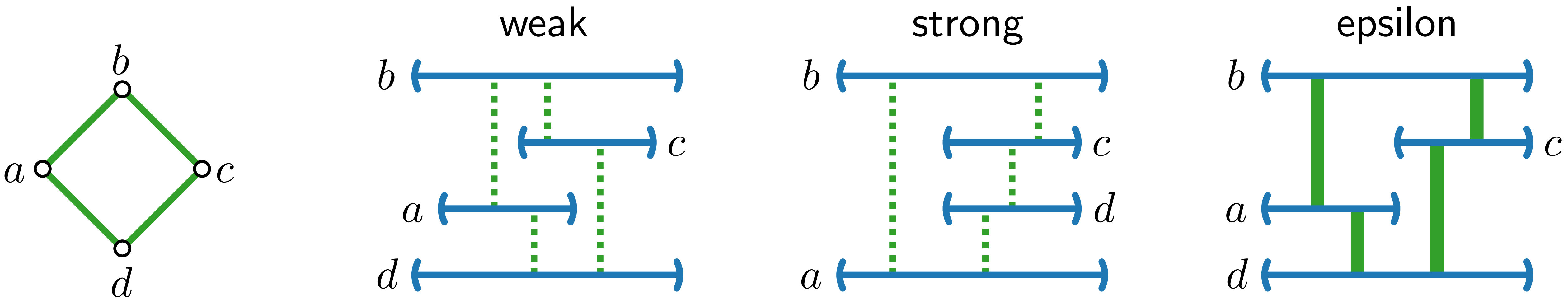

Recognition Problem.
Given a graph G, **decide** whether there exists a weak/strong/ε-bar visibility representation ψ of G.

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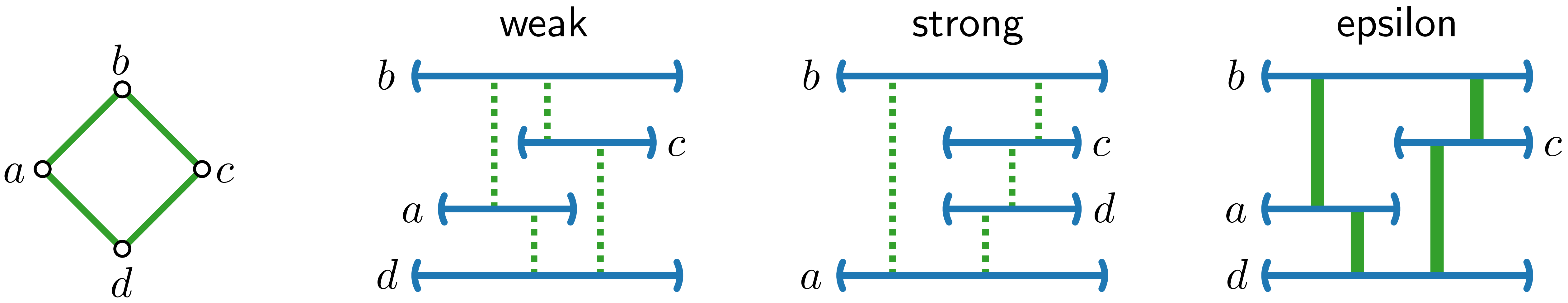

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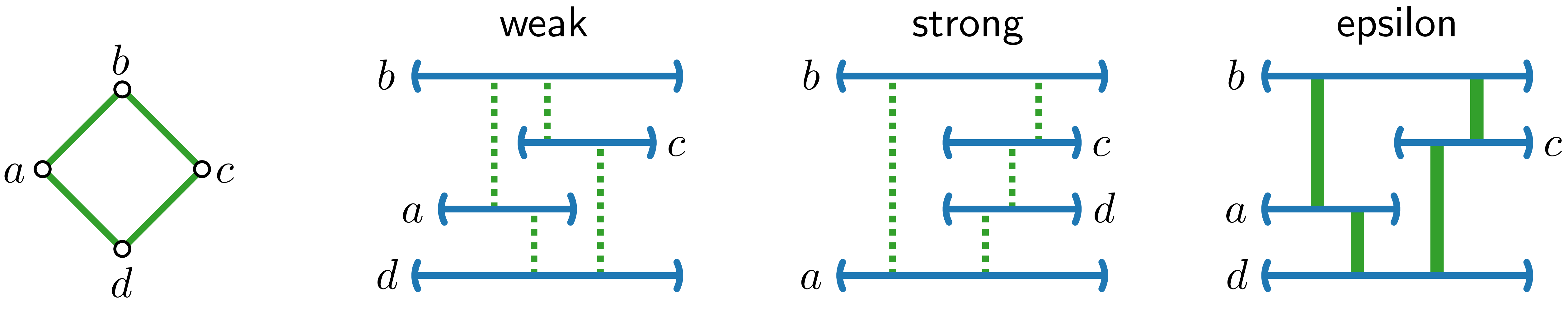

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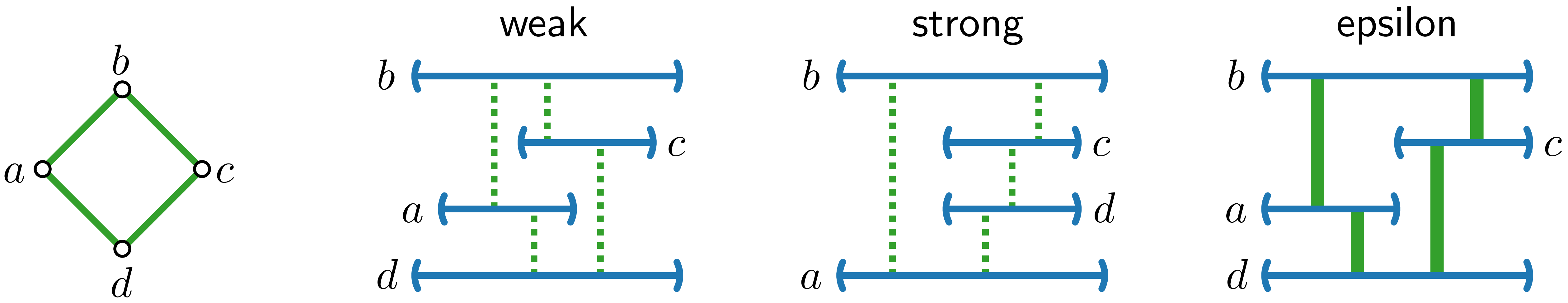
Partial Representation Extension Problem.
Given a graph G and a **set of bars ψ' of $V' \subseteq V(G)$**, **decide** whether there exists a weak/strong/ε-bar visibility representation ψ of G **where $\psi|_{V'} = \psi'$** (and **construct** ψ if a representation exists).

Background



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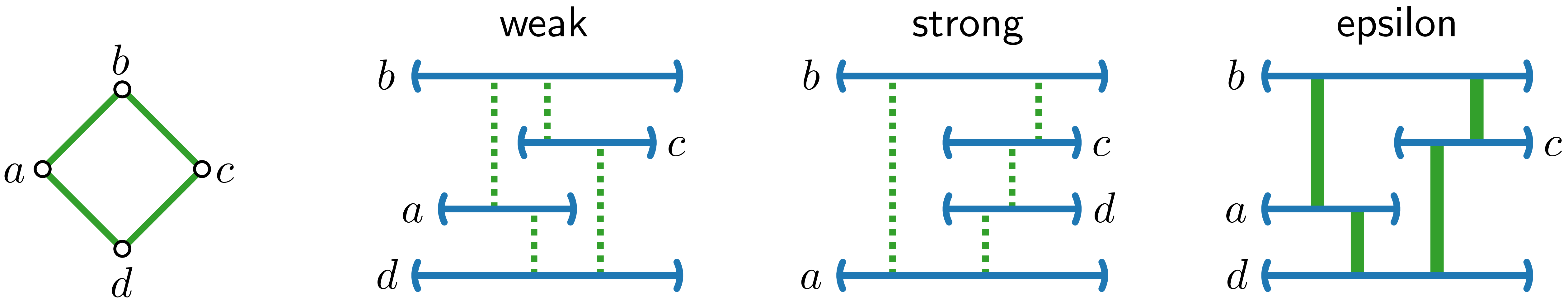

Weak Bar Visibility.

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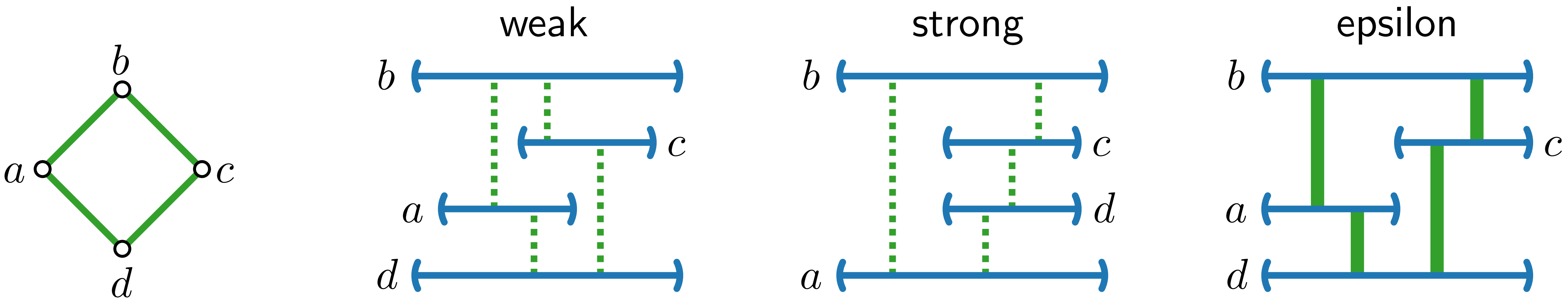

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- Exactly all planar graphs [Tamassia & Tollis '86; Wismath '85]

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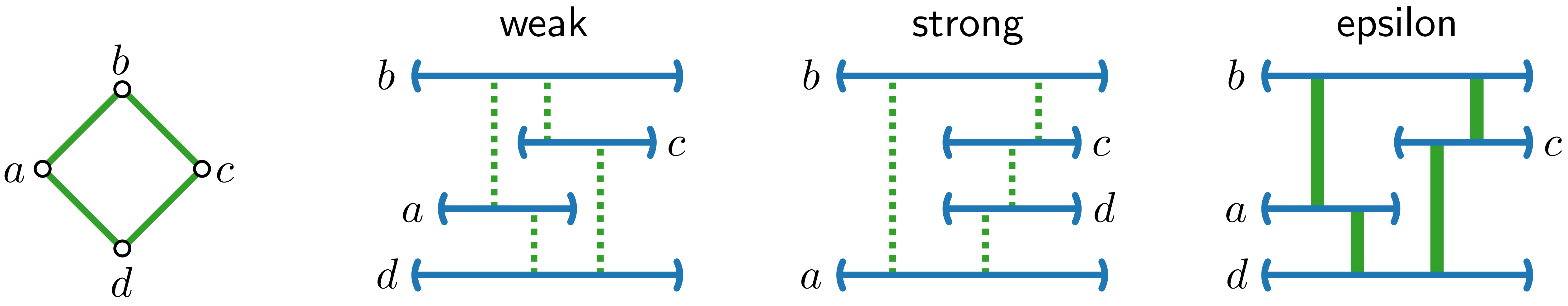

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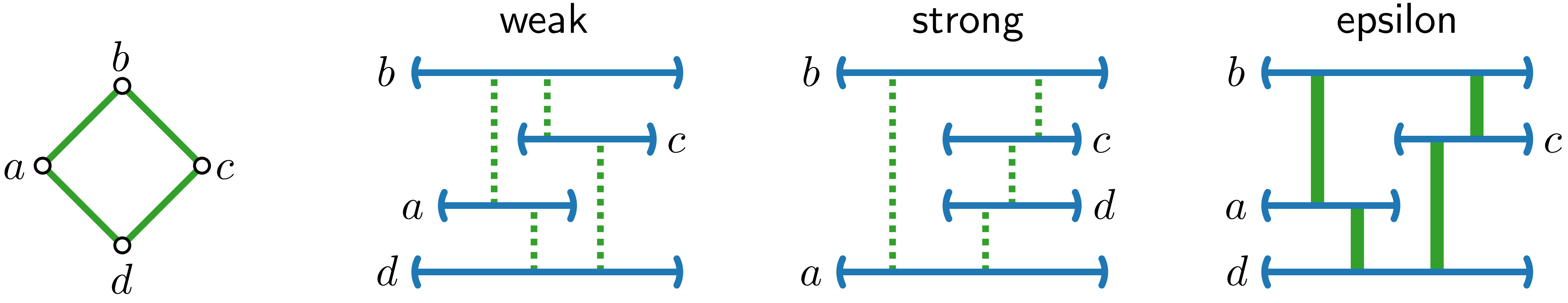

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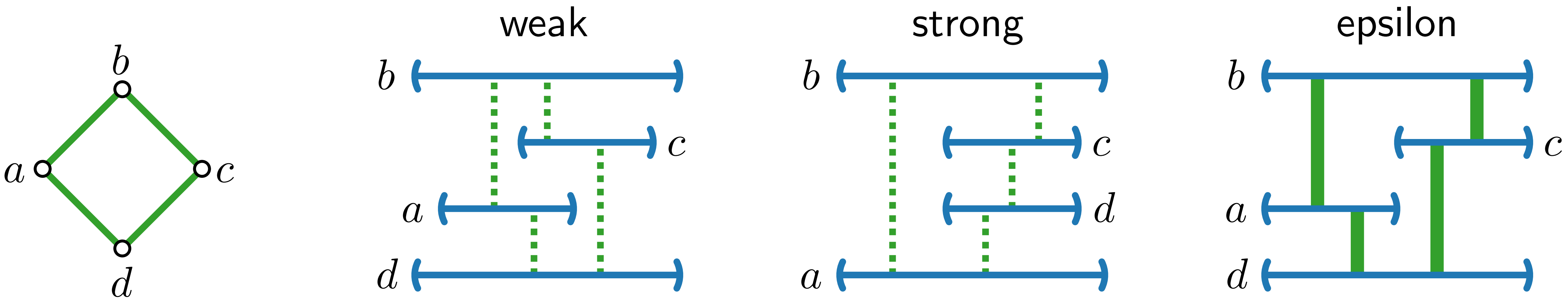

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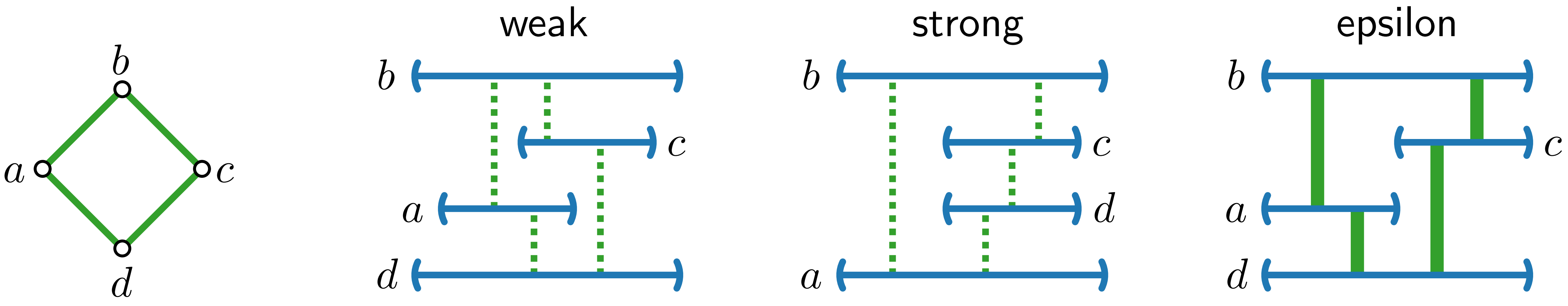

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Strong Bar Visibility.

- NP-complete to recognize [Andreae '92]

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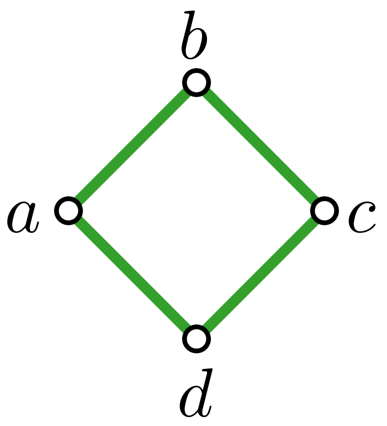

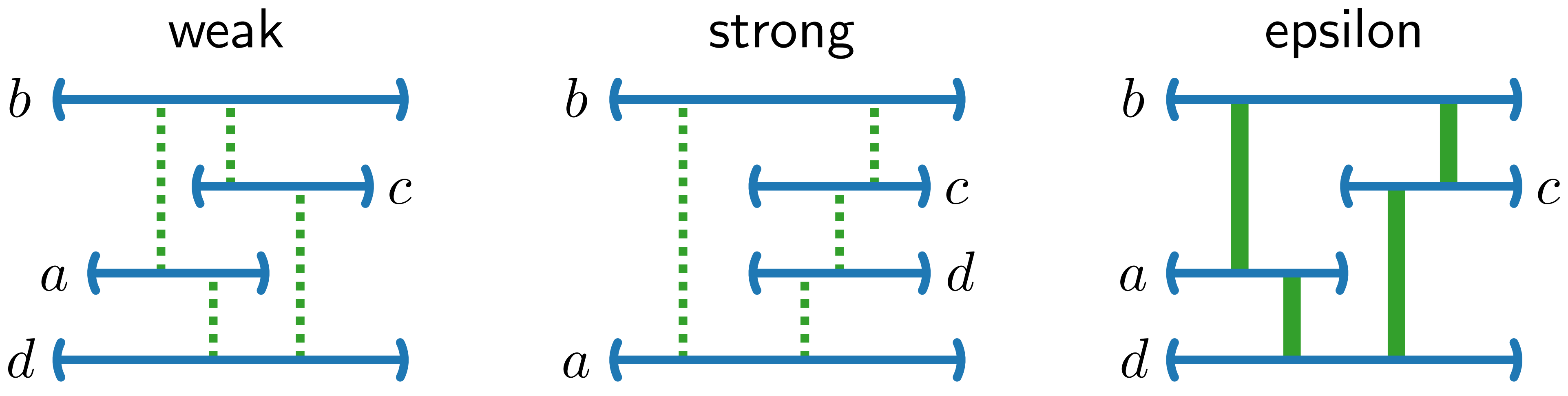

ε-Bar Visibility.

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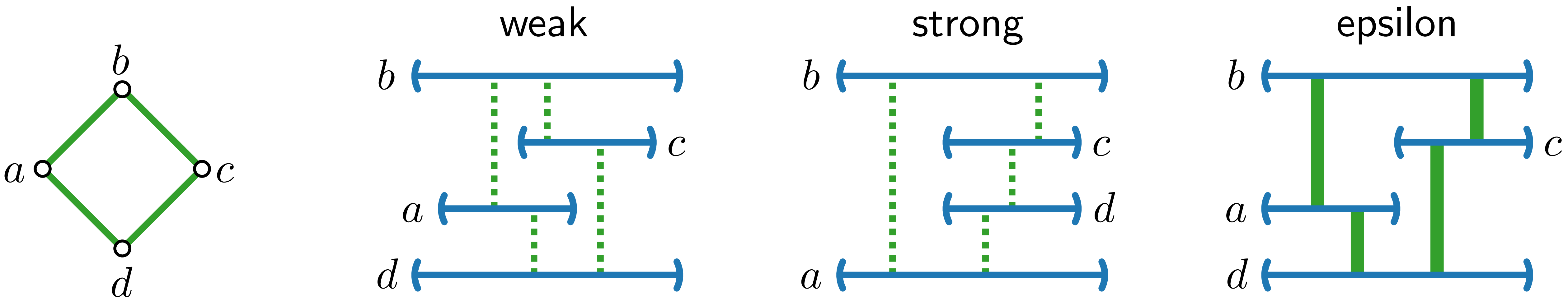
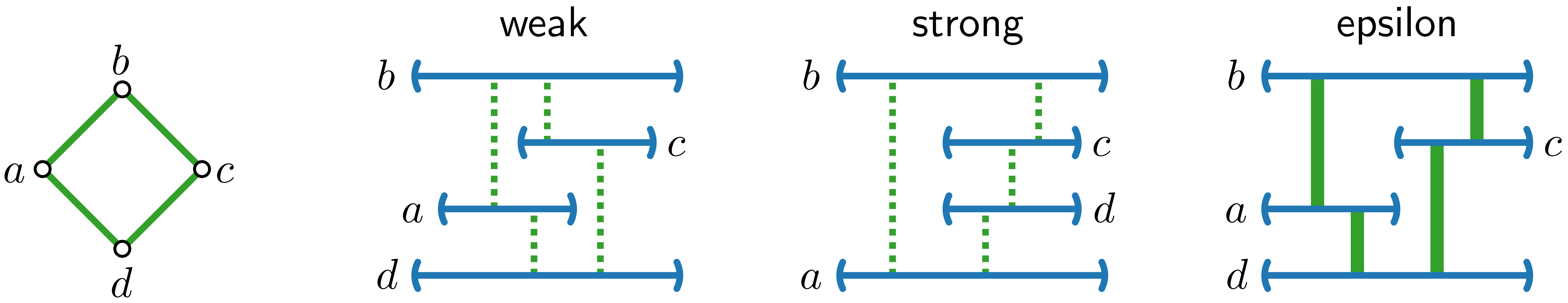


ε-Bar Visibility.

- Exactly all planar graphs that can be embedded with all **cut vertices** on the outerface [T&T '86, Wismath '85]

Background



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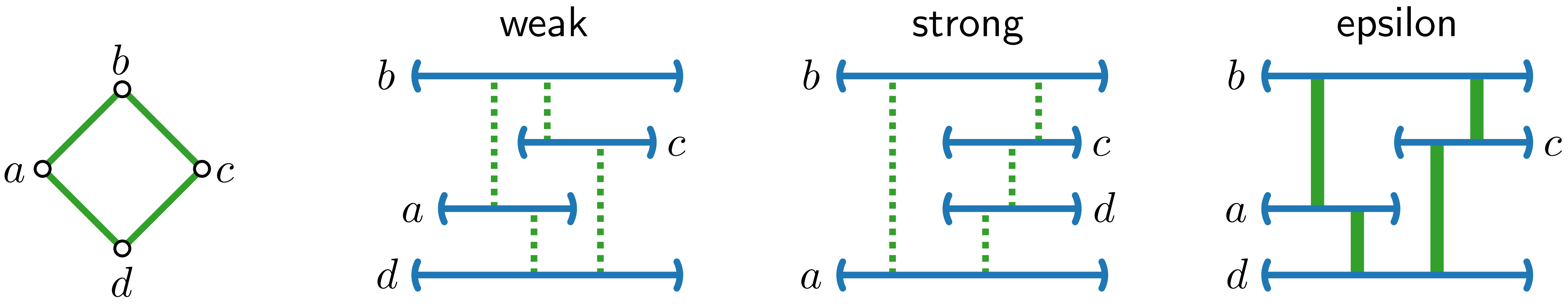

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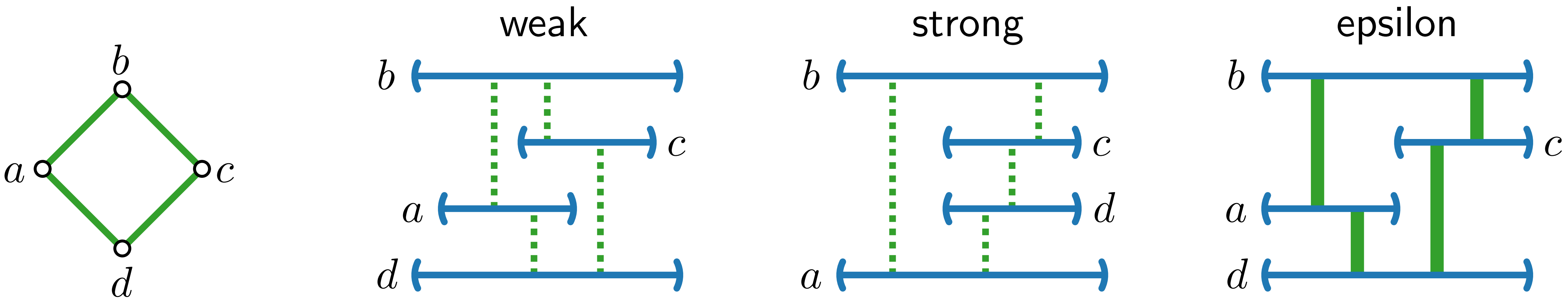

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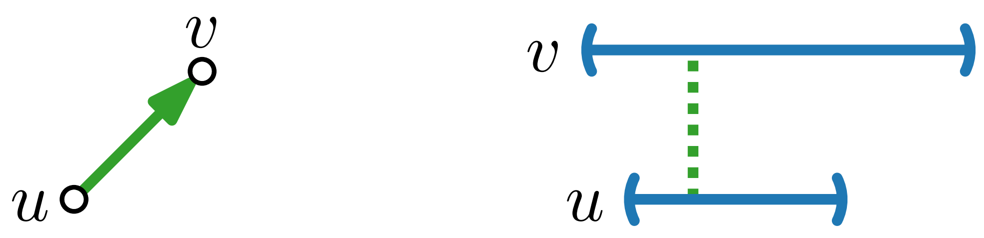

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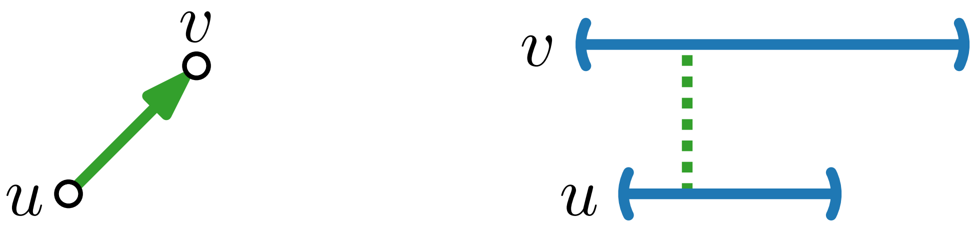

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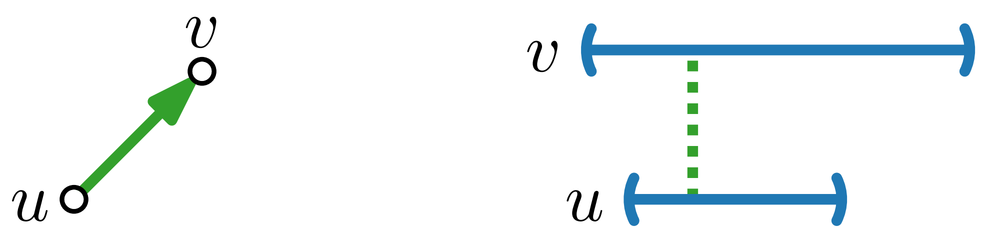

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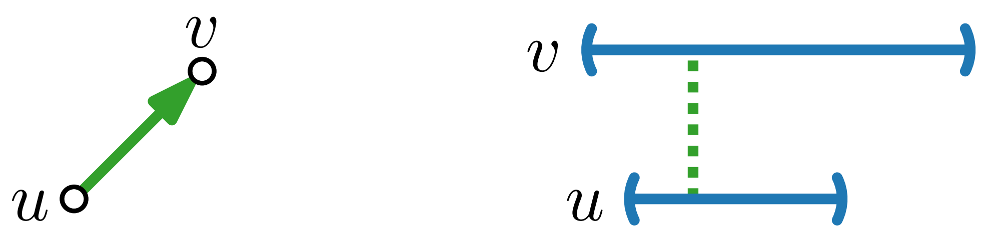

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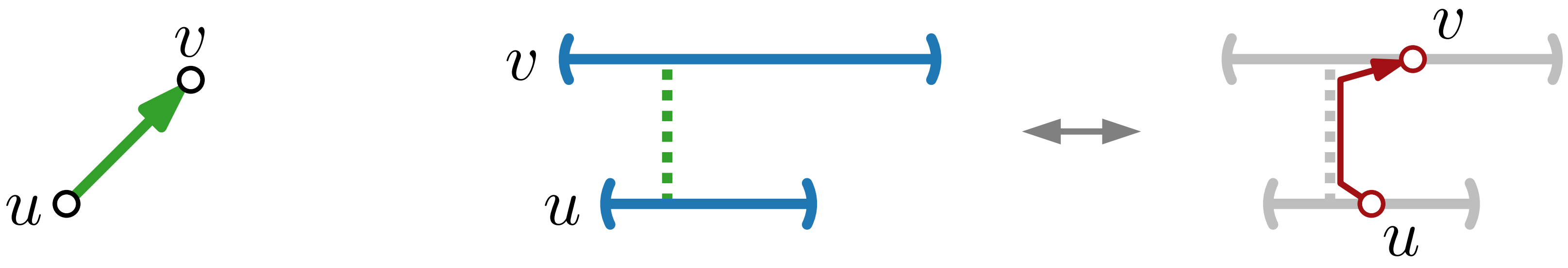

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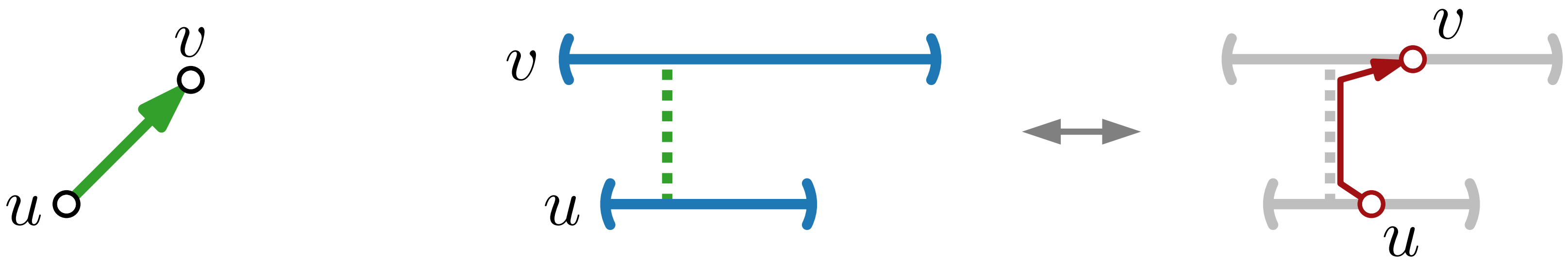

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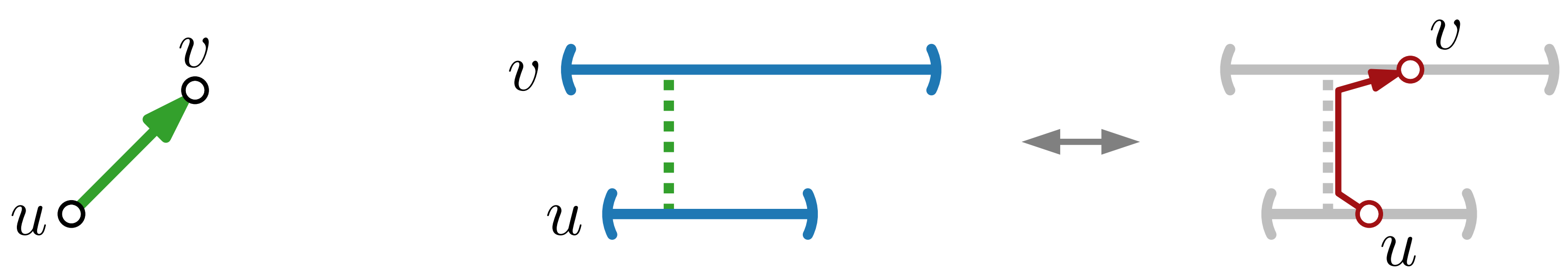

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Next, we consider ε-bar visibility representations of specific directed graphs ($\rightarrow$ st-graphs)

ε-Bar Visibility and st-Graphs

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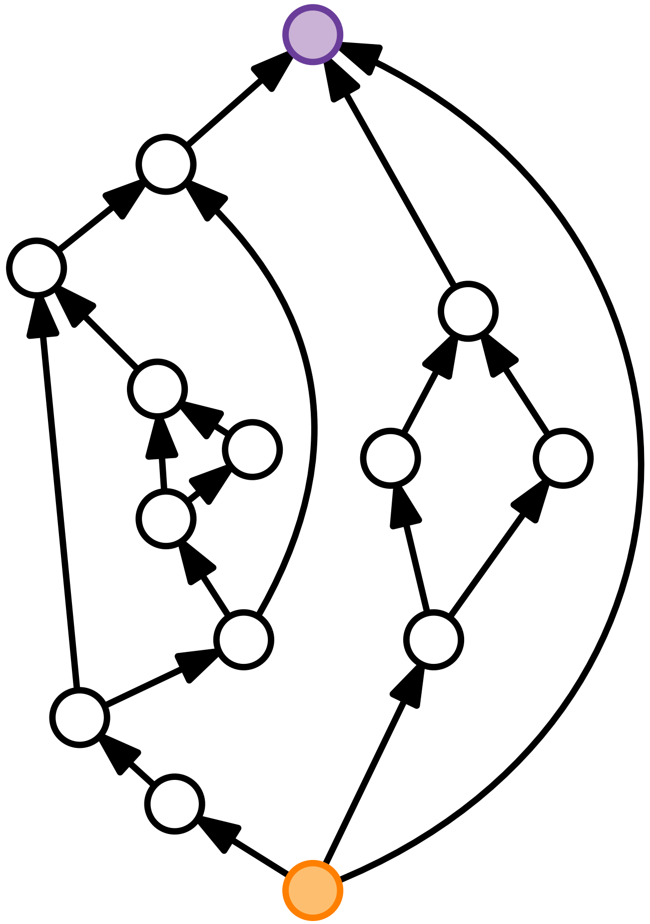

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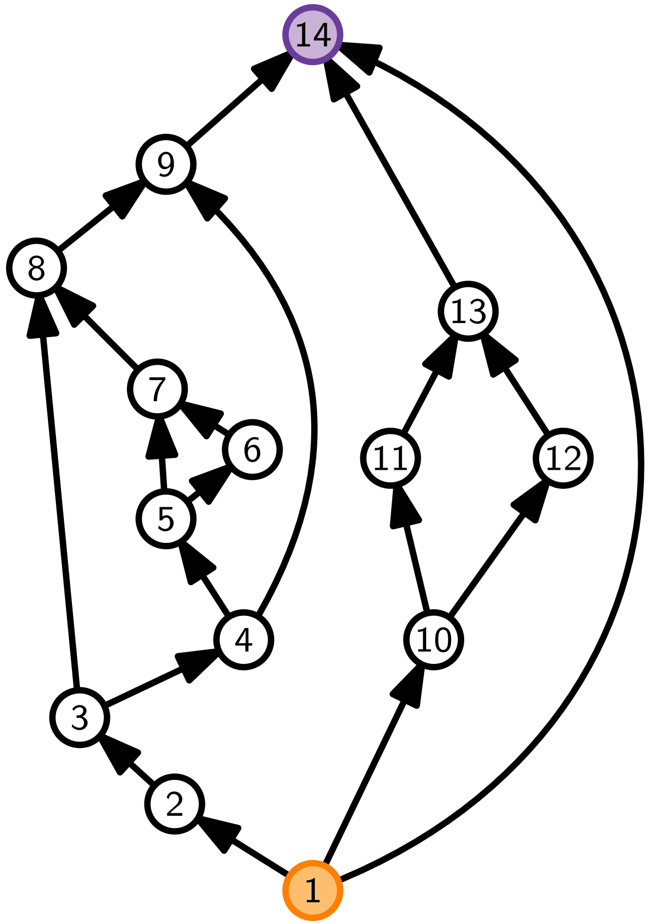

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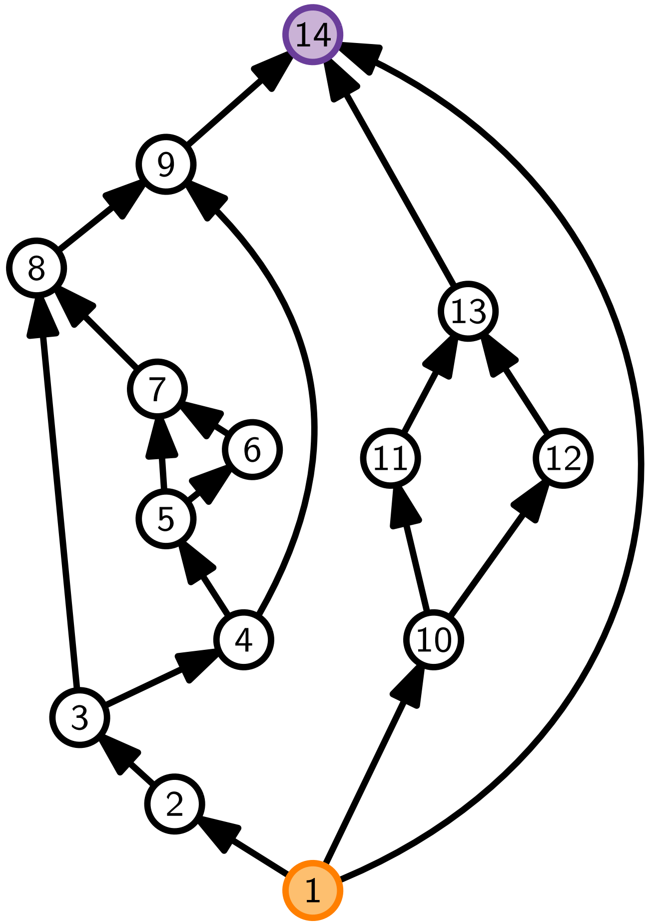

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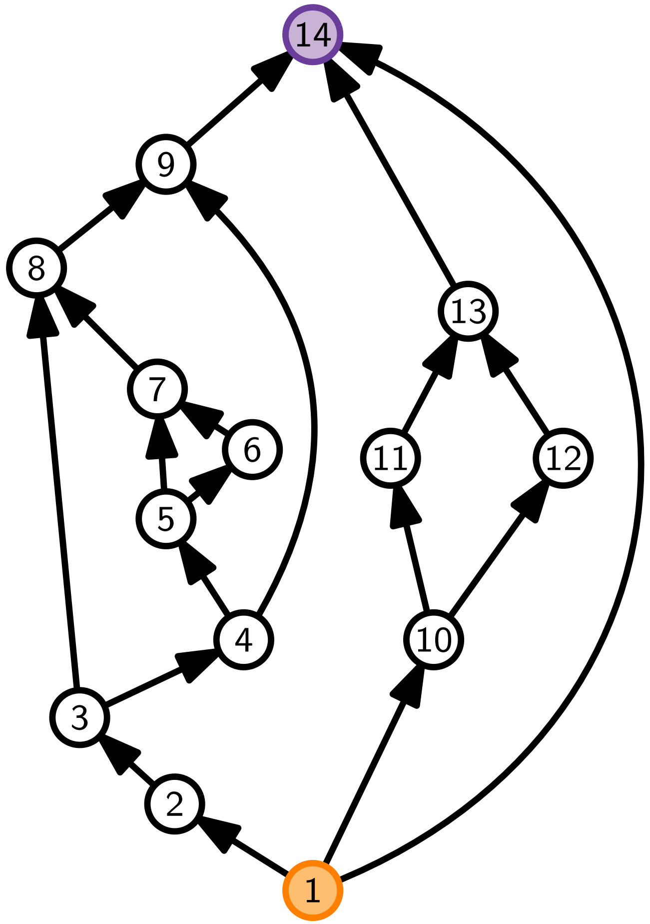

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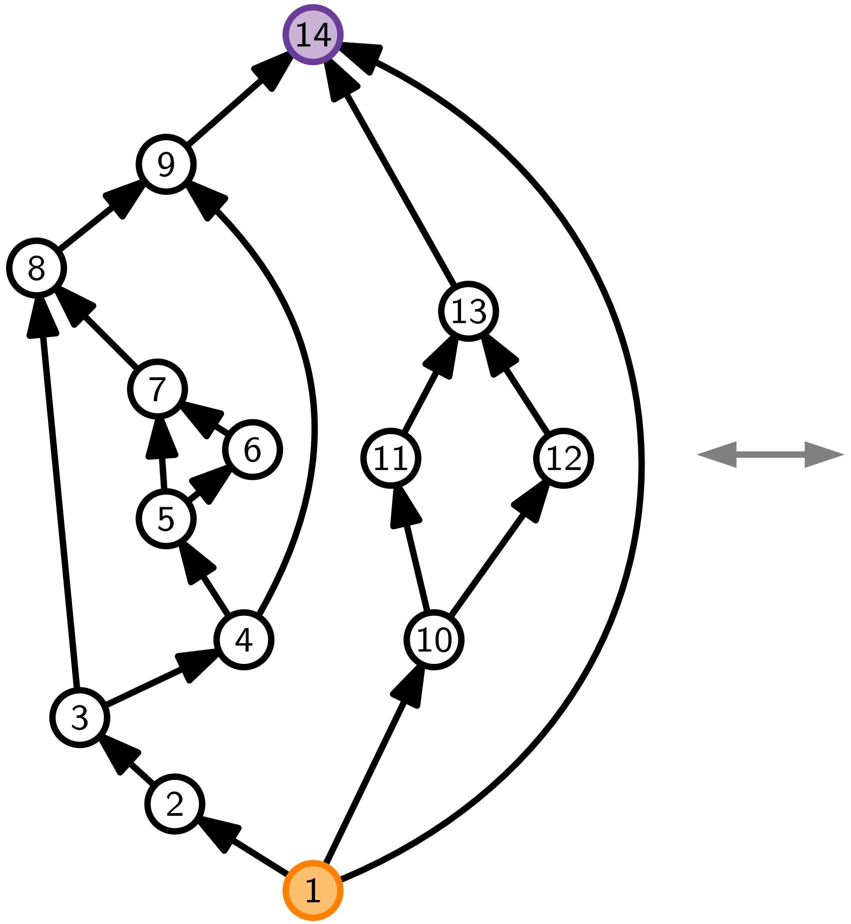

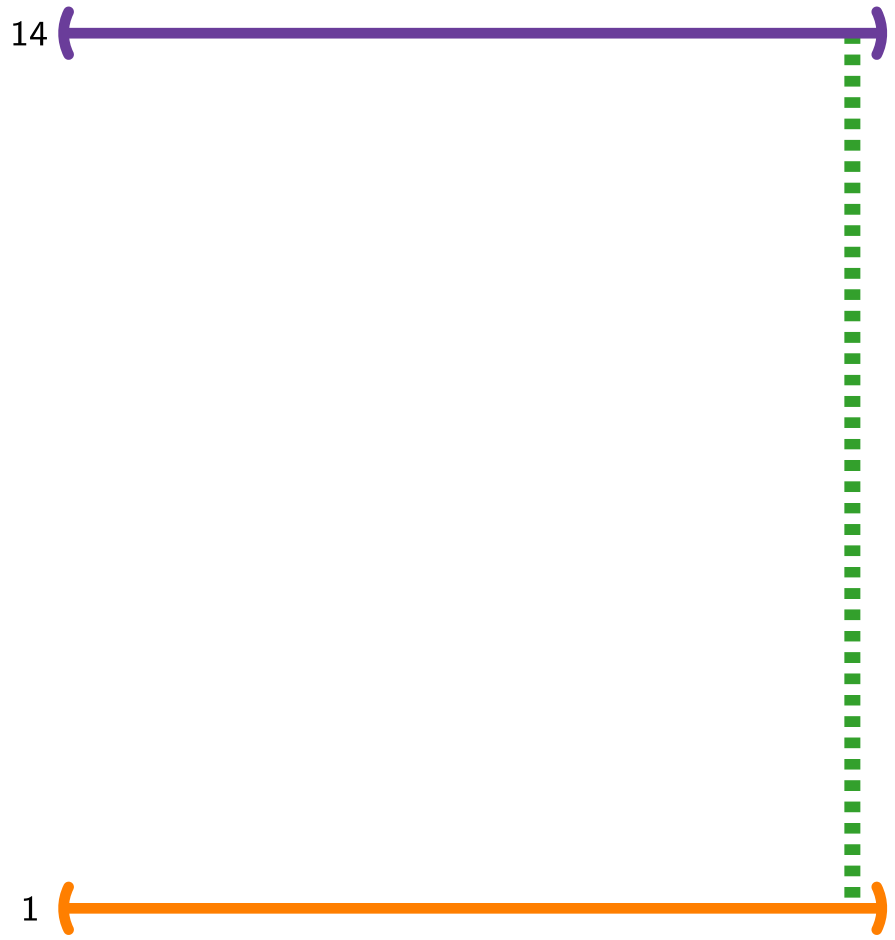

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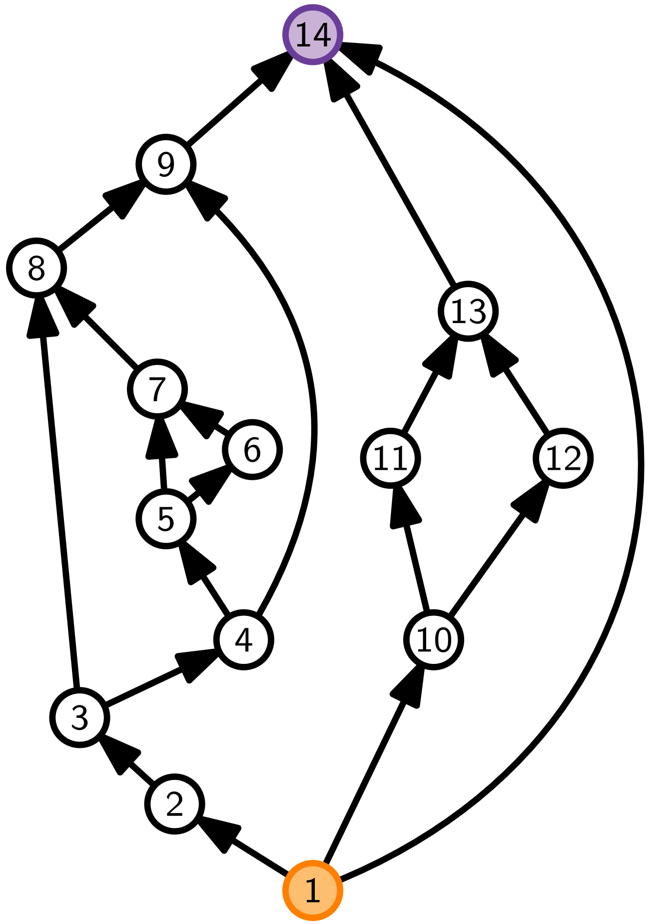

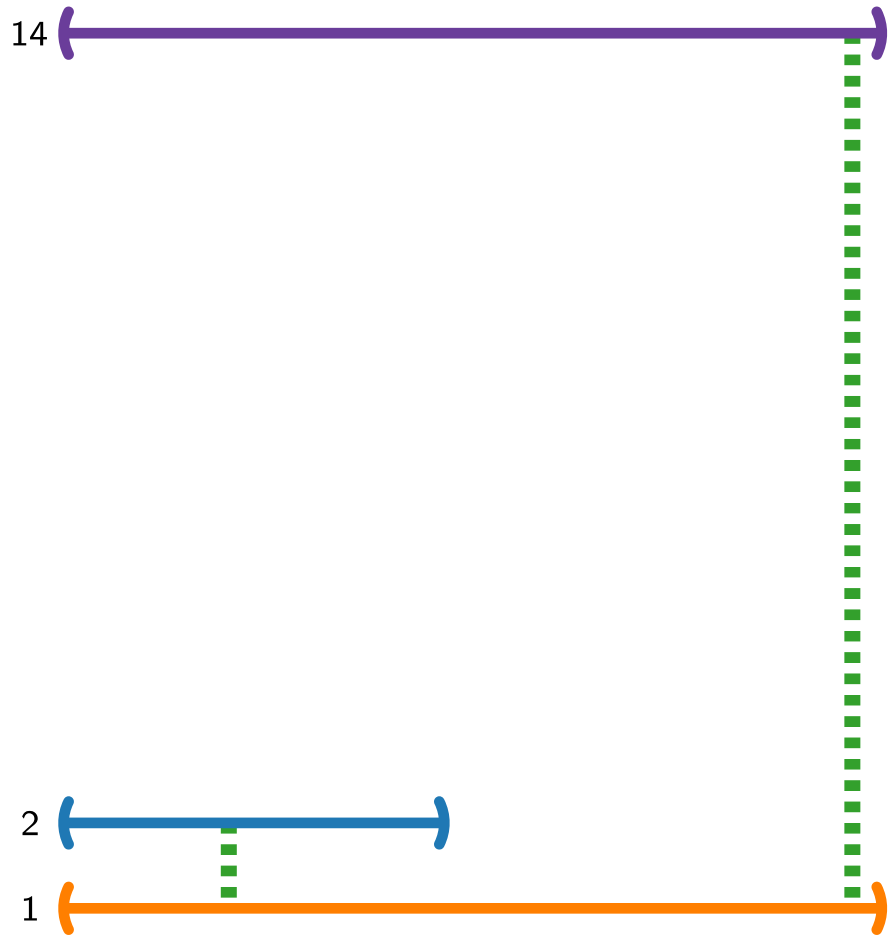

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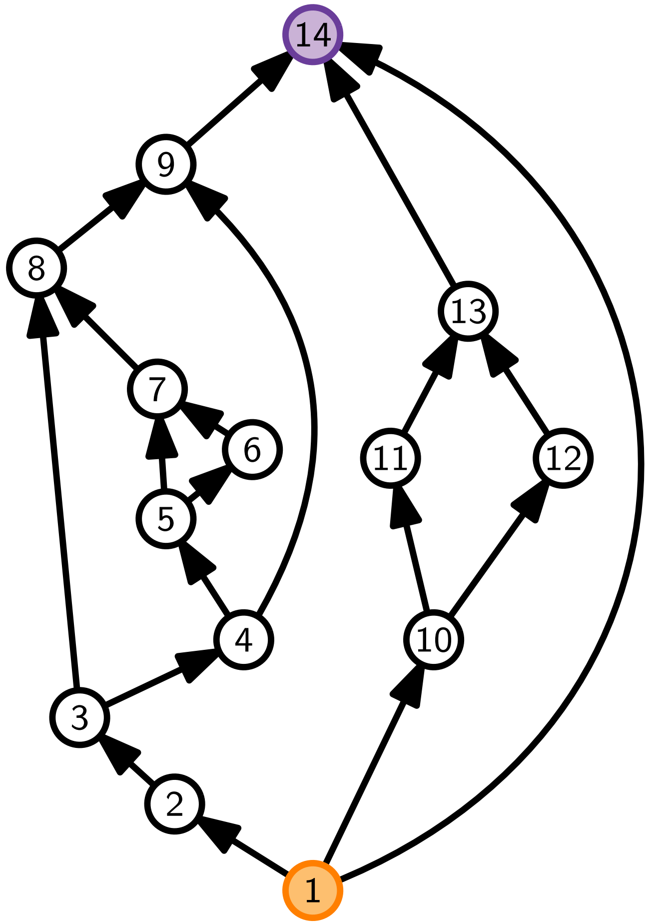

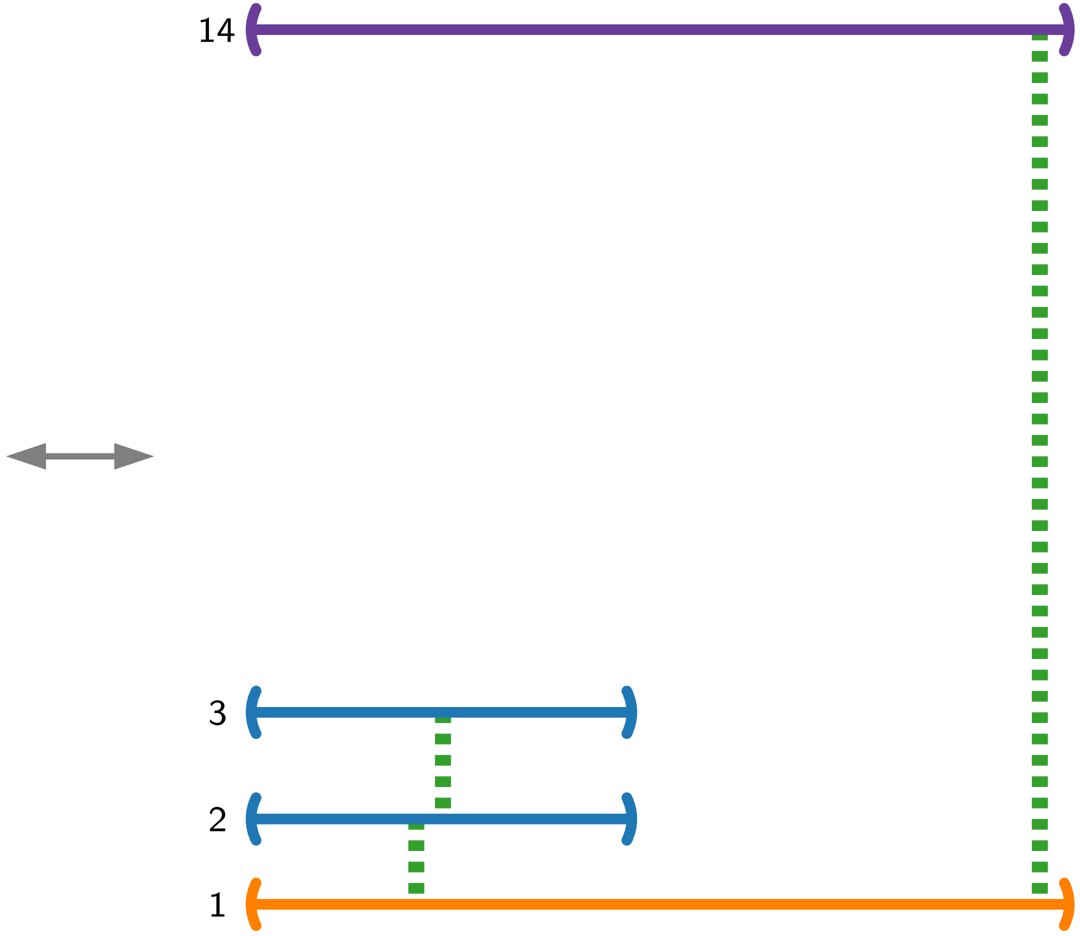

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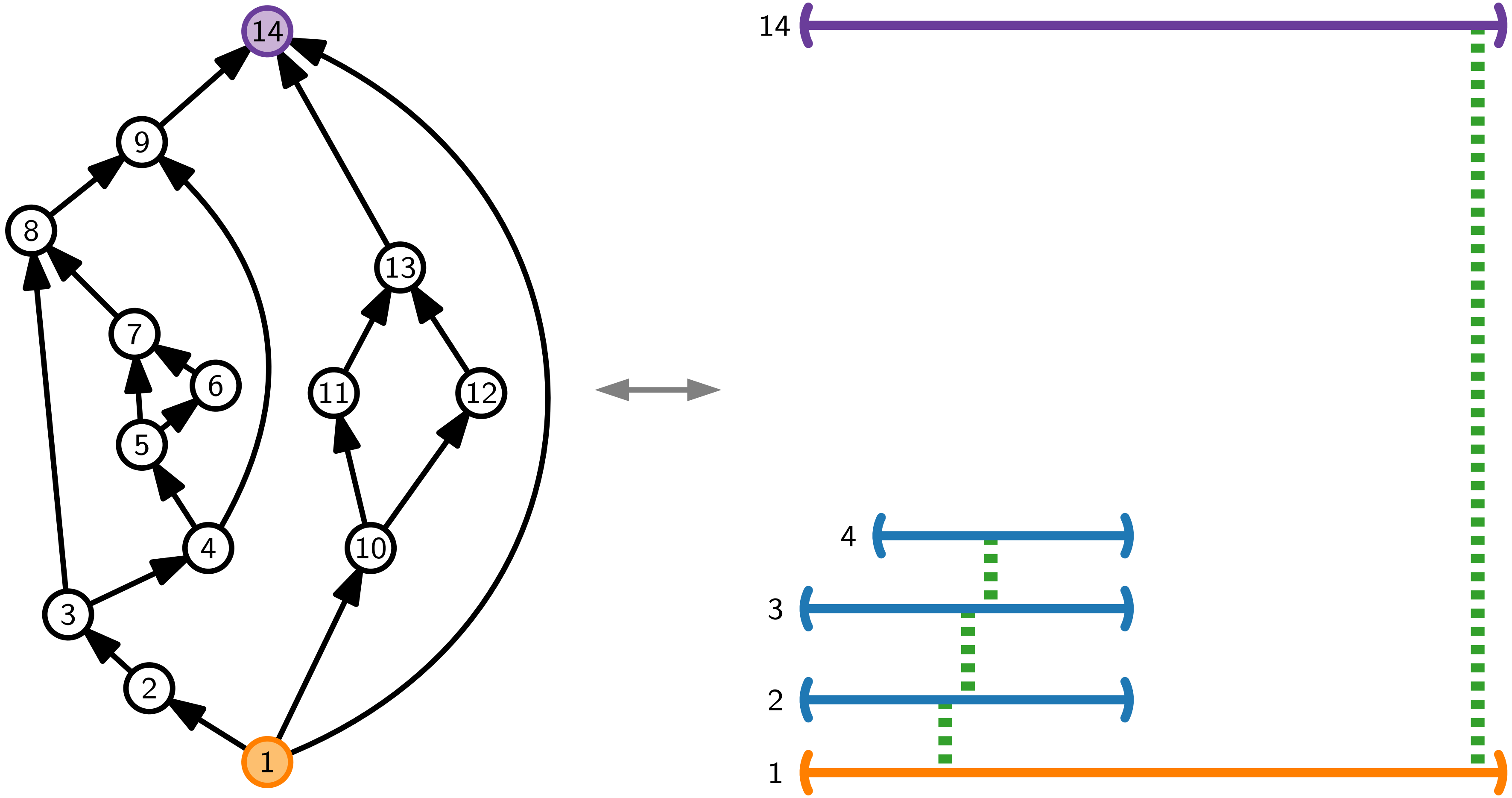

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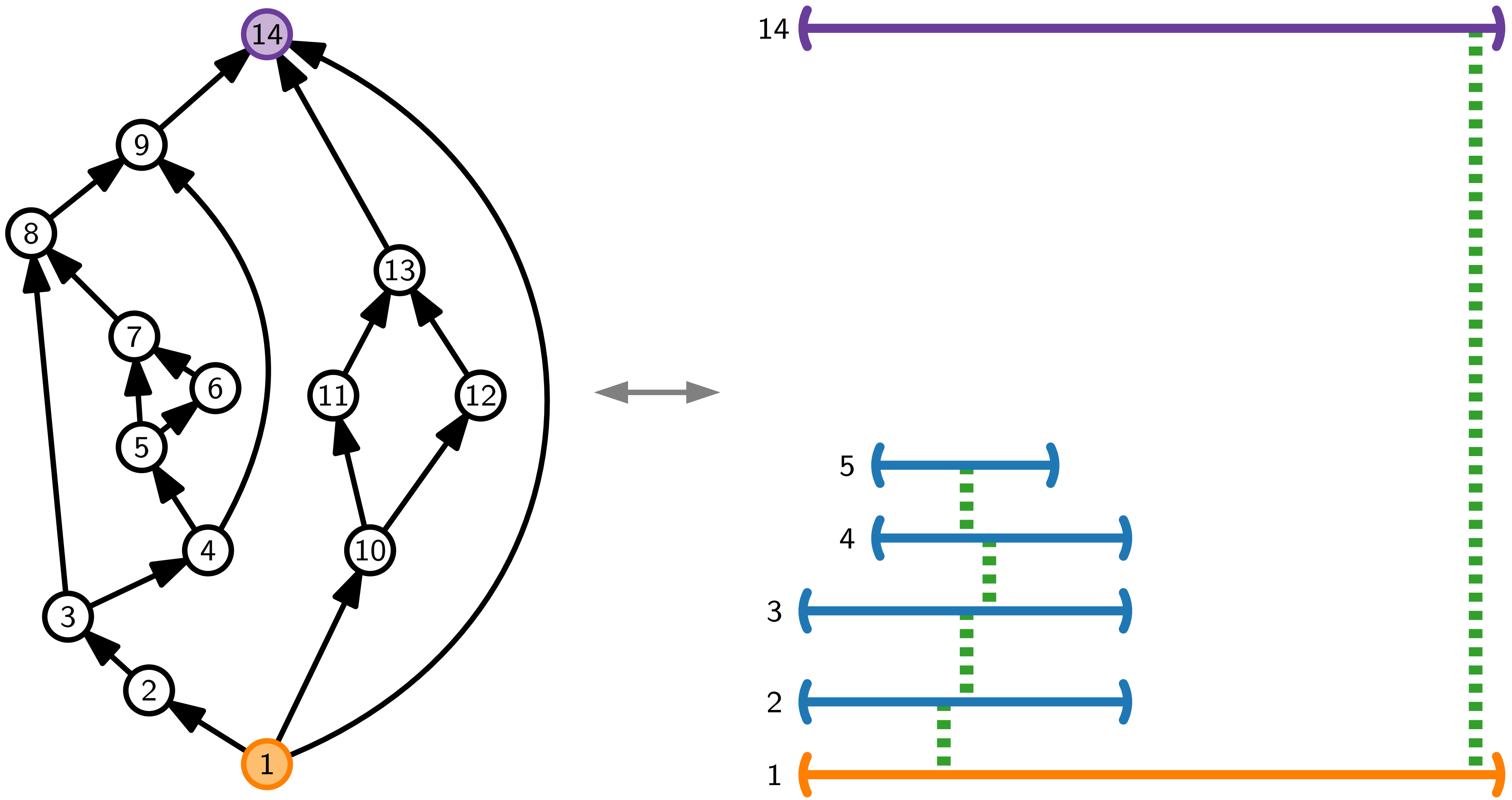

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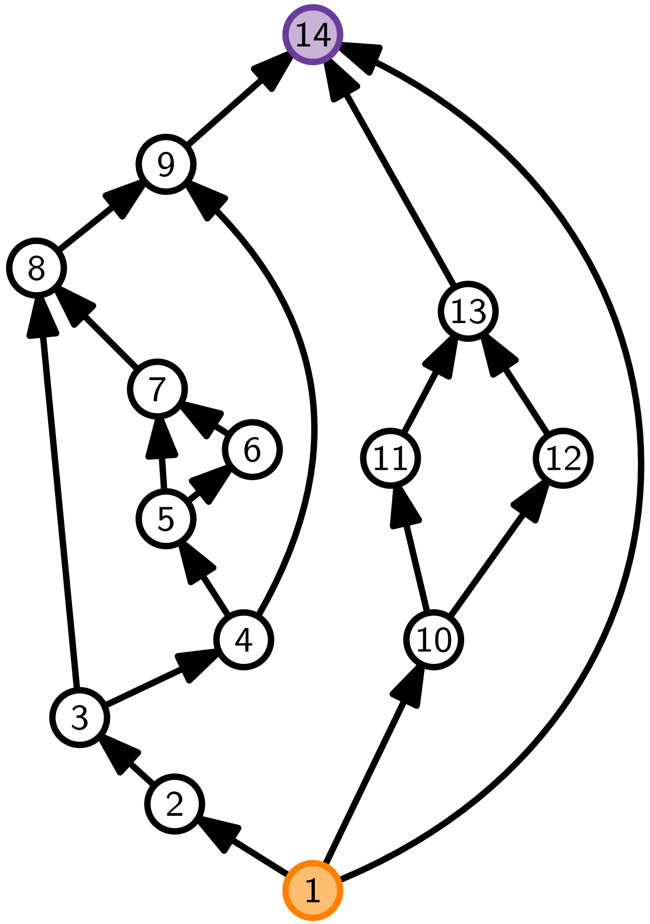

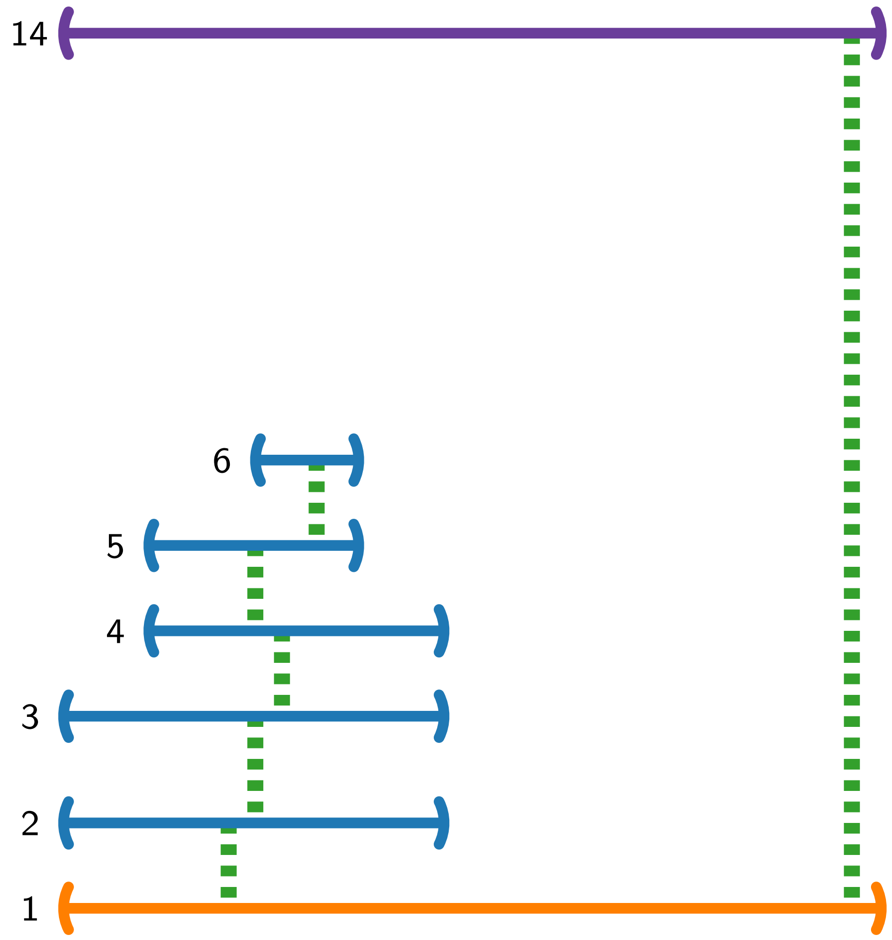

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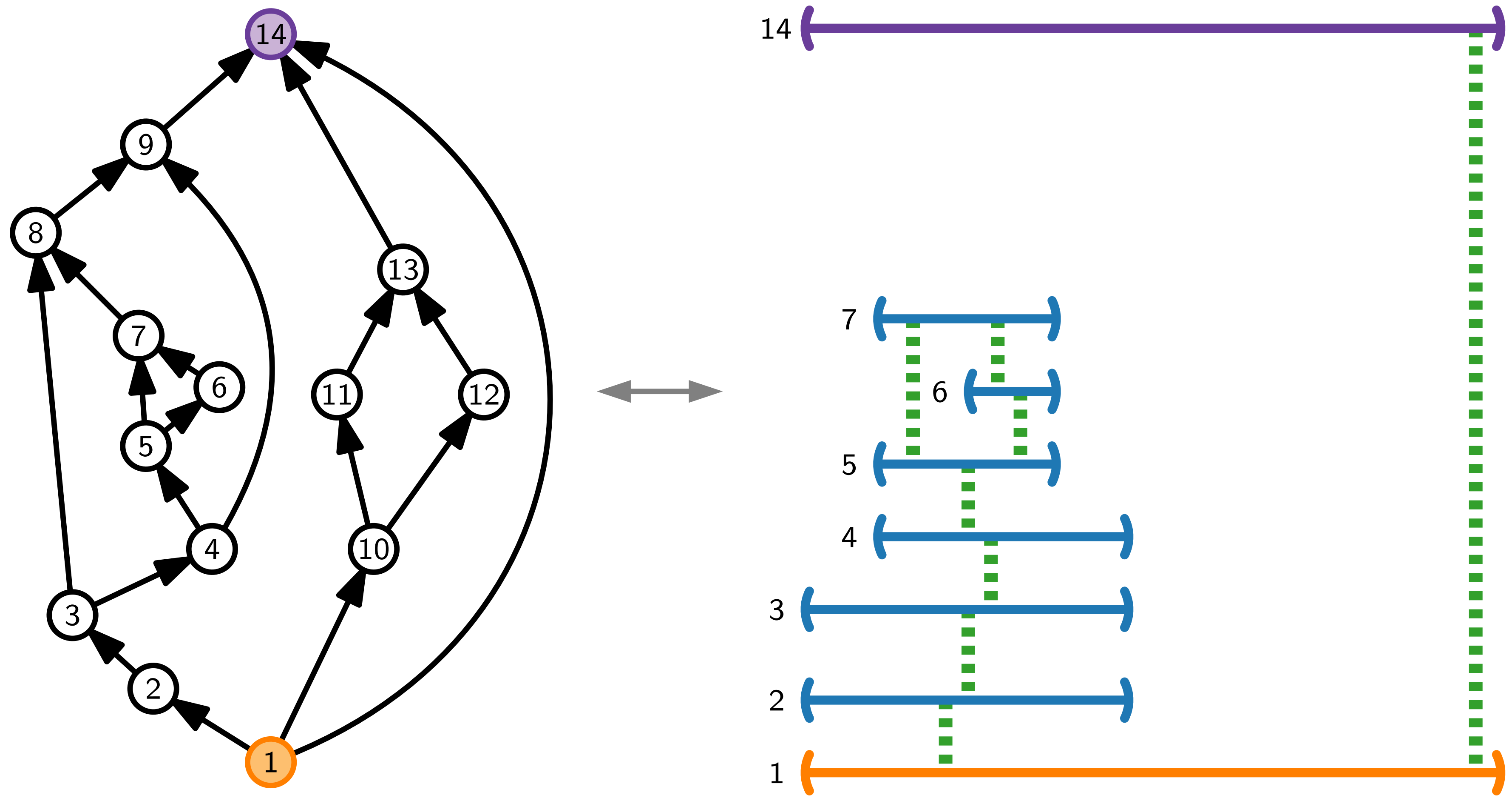

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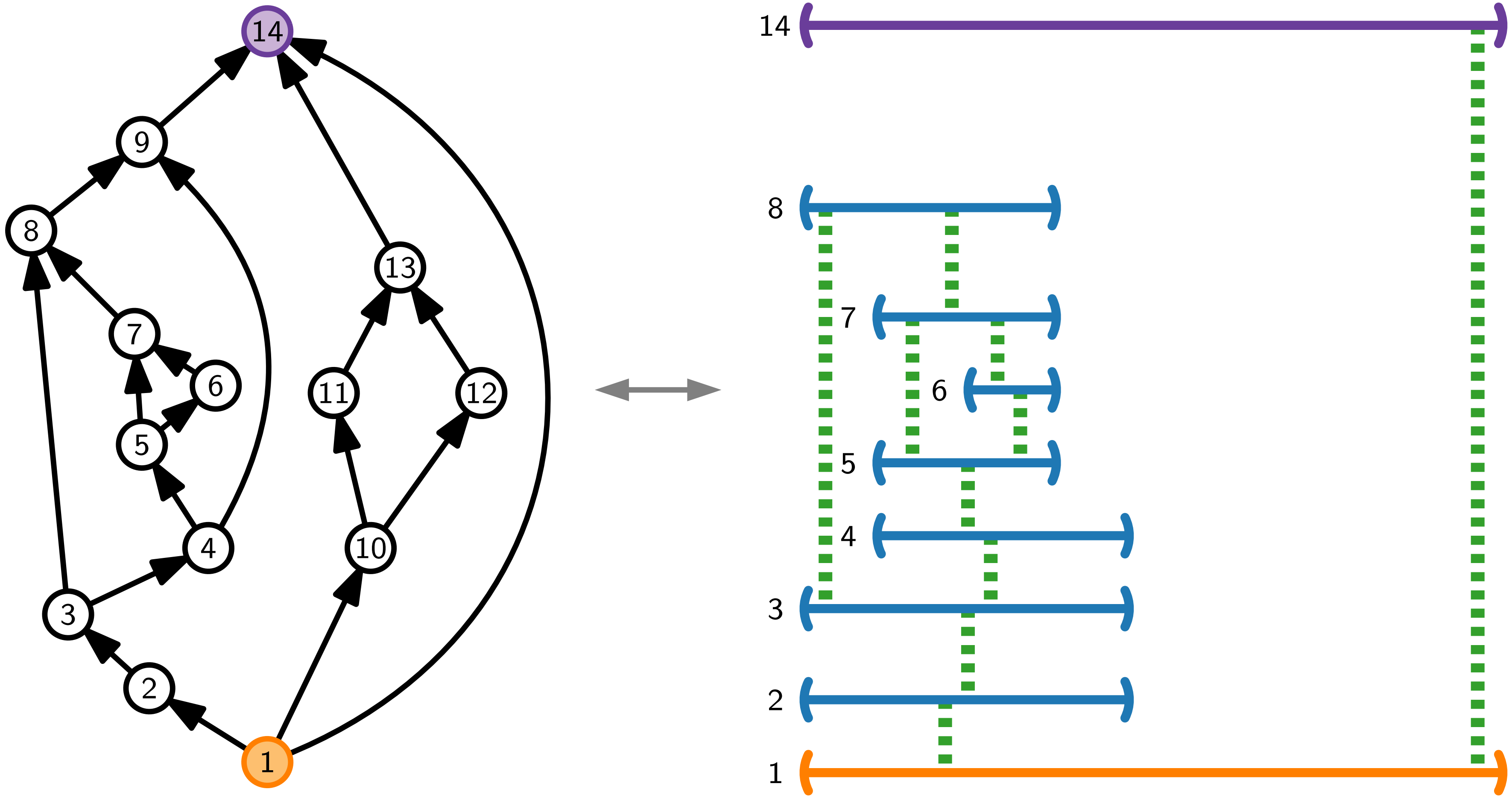

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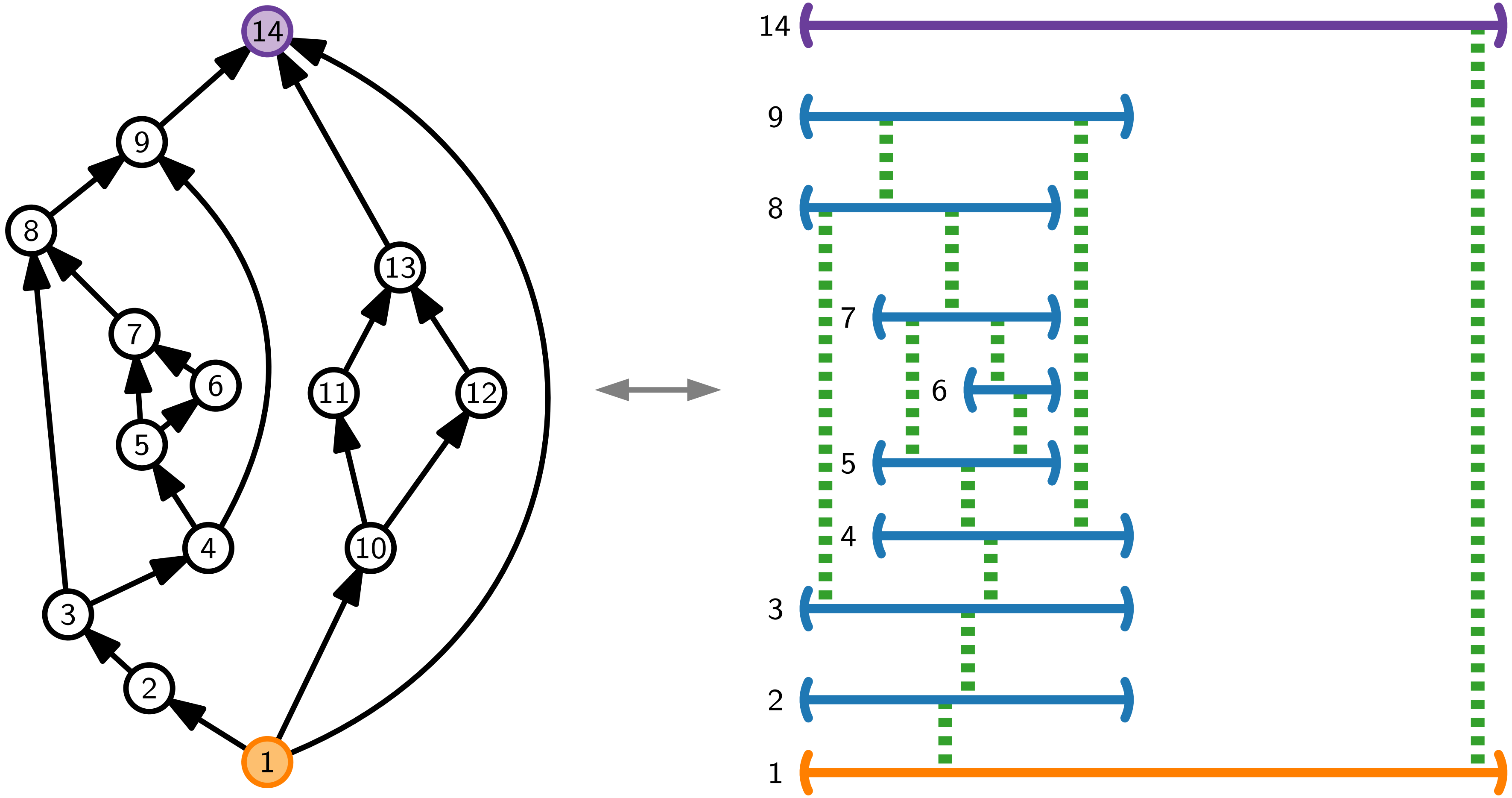

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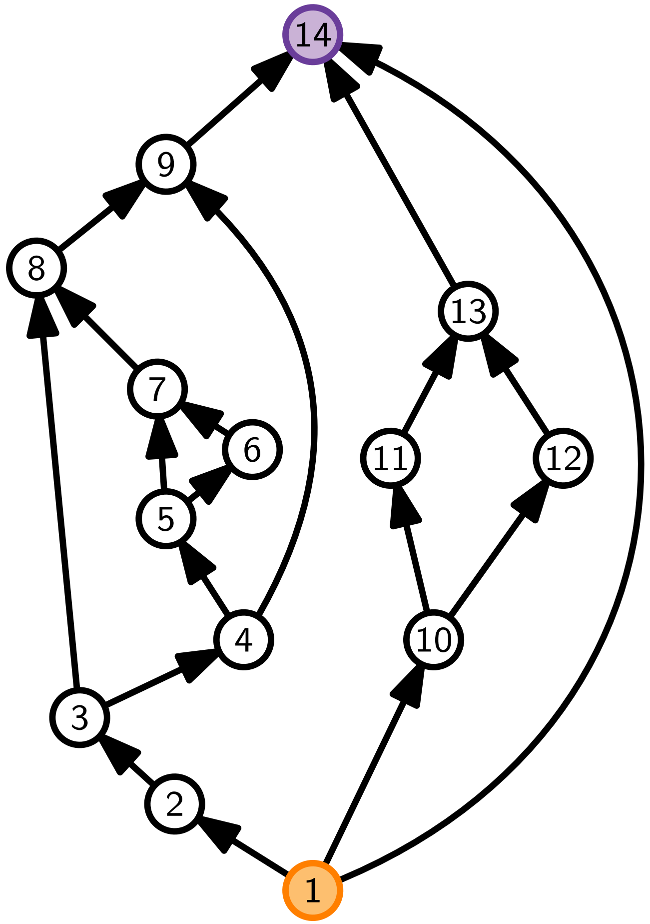

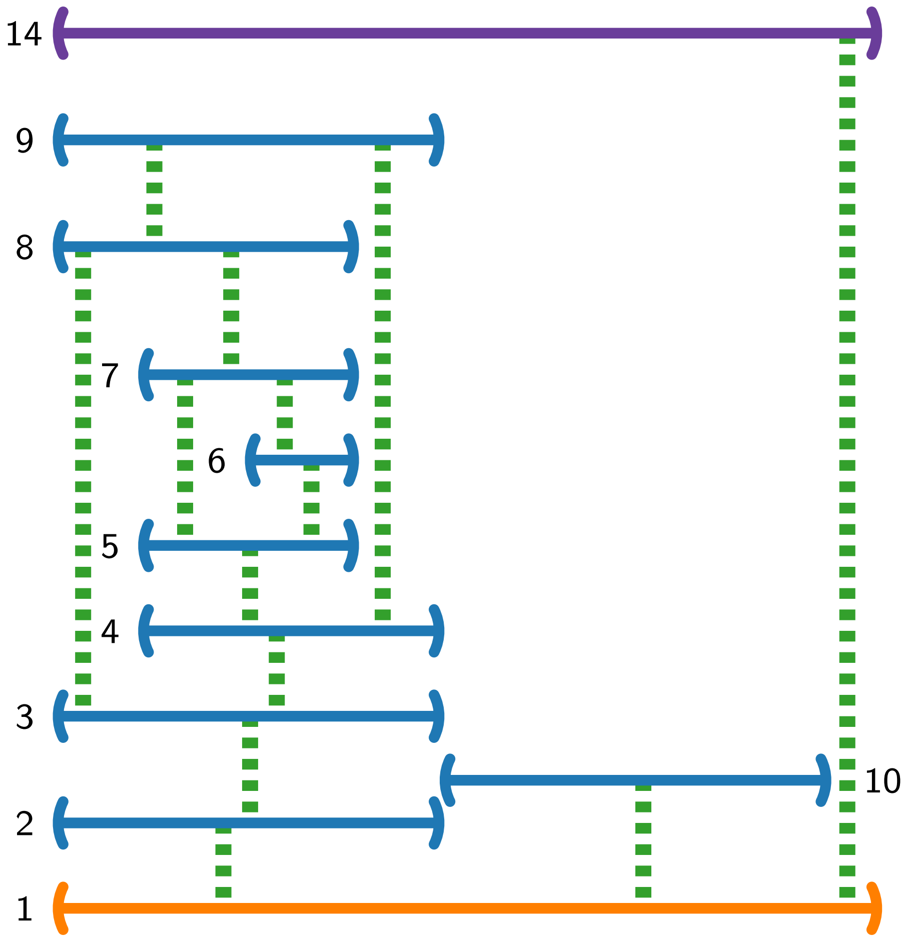

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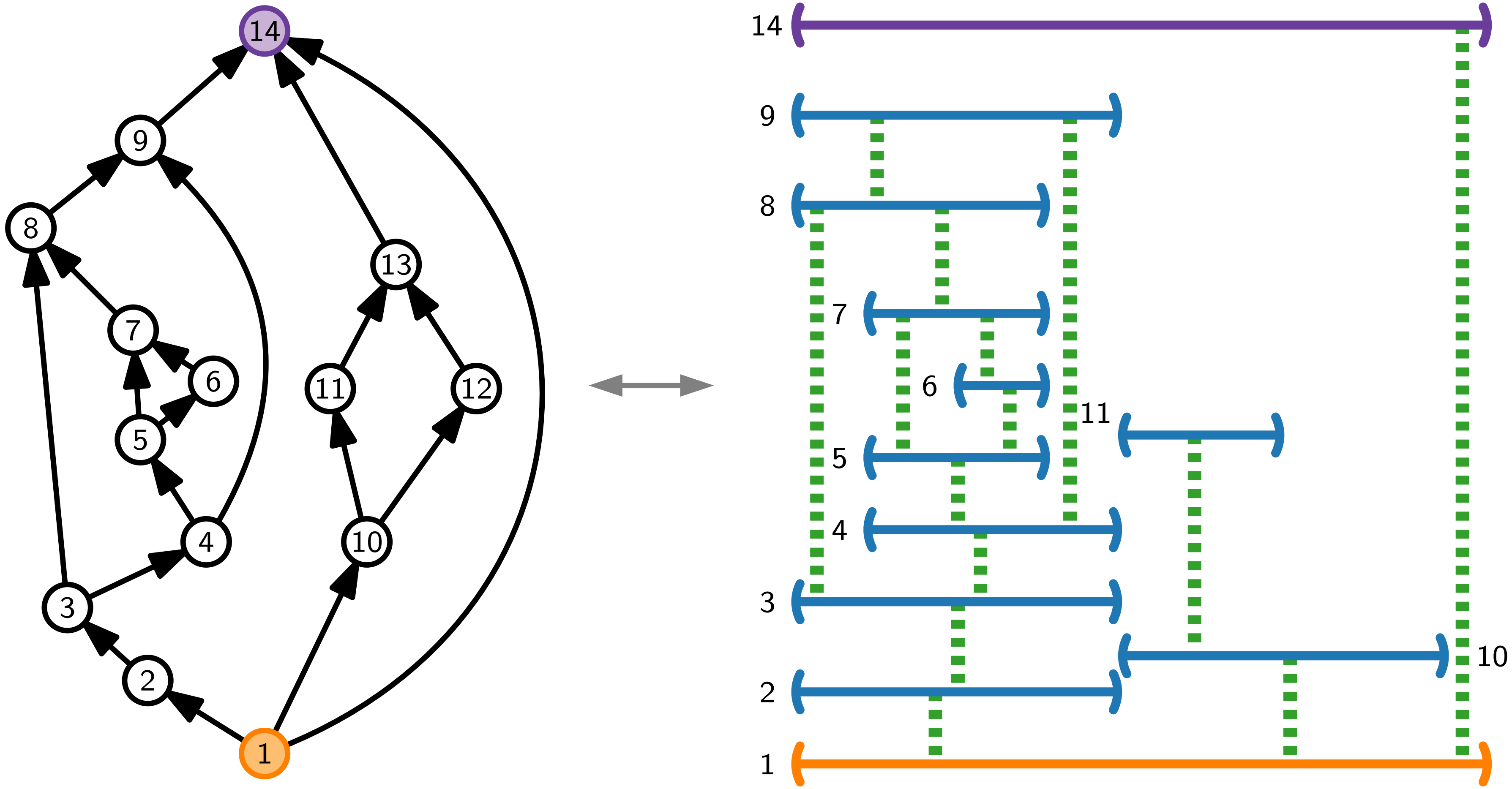

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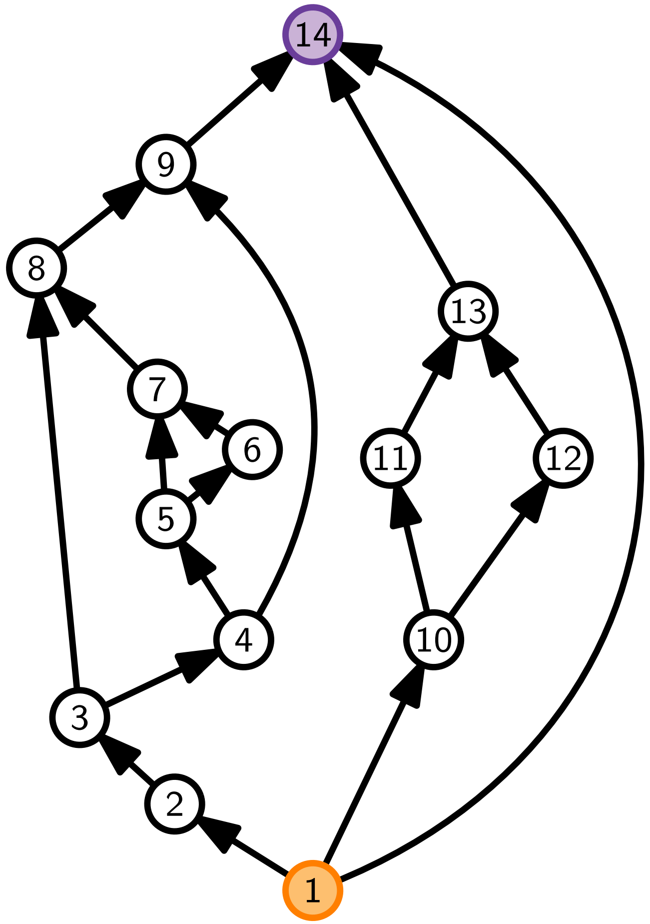

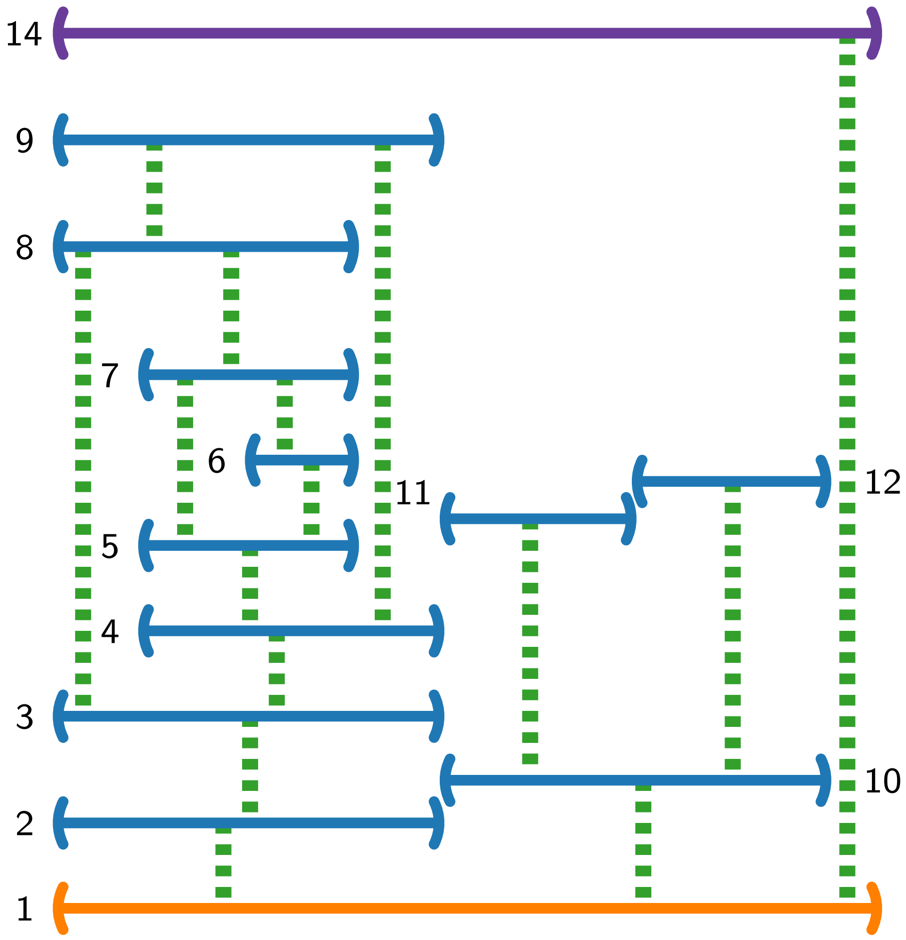

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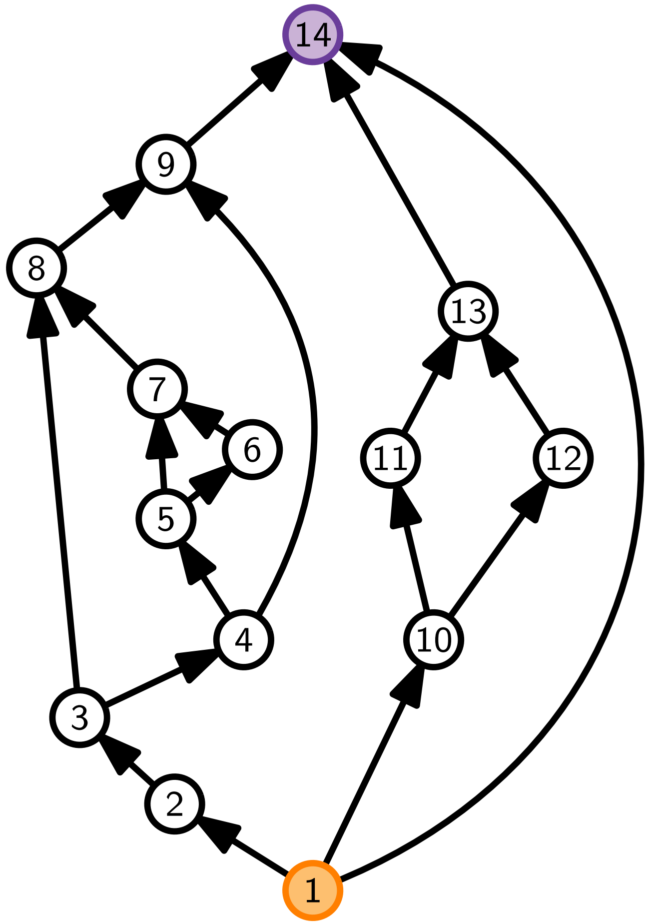

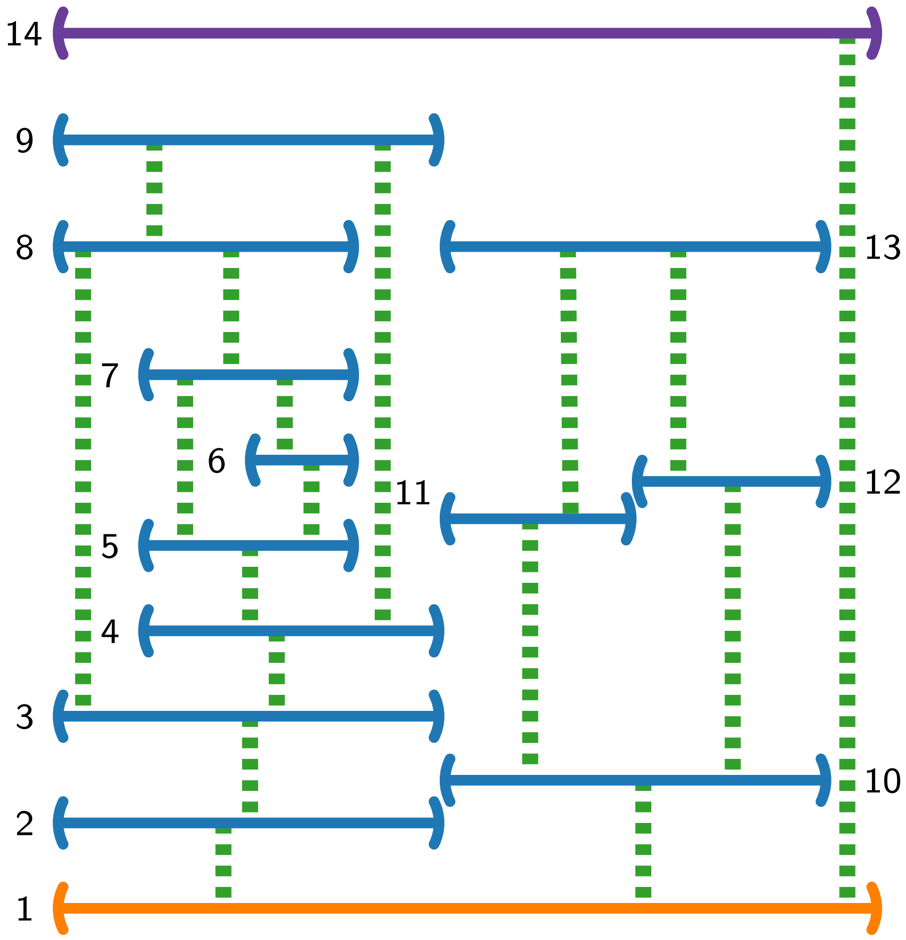

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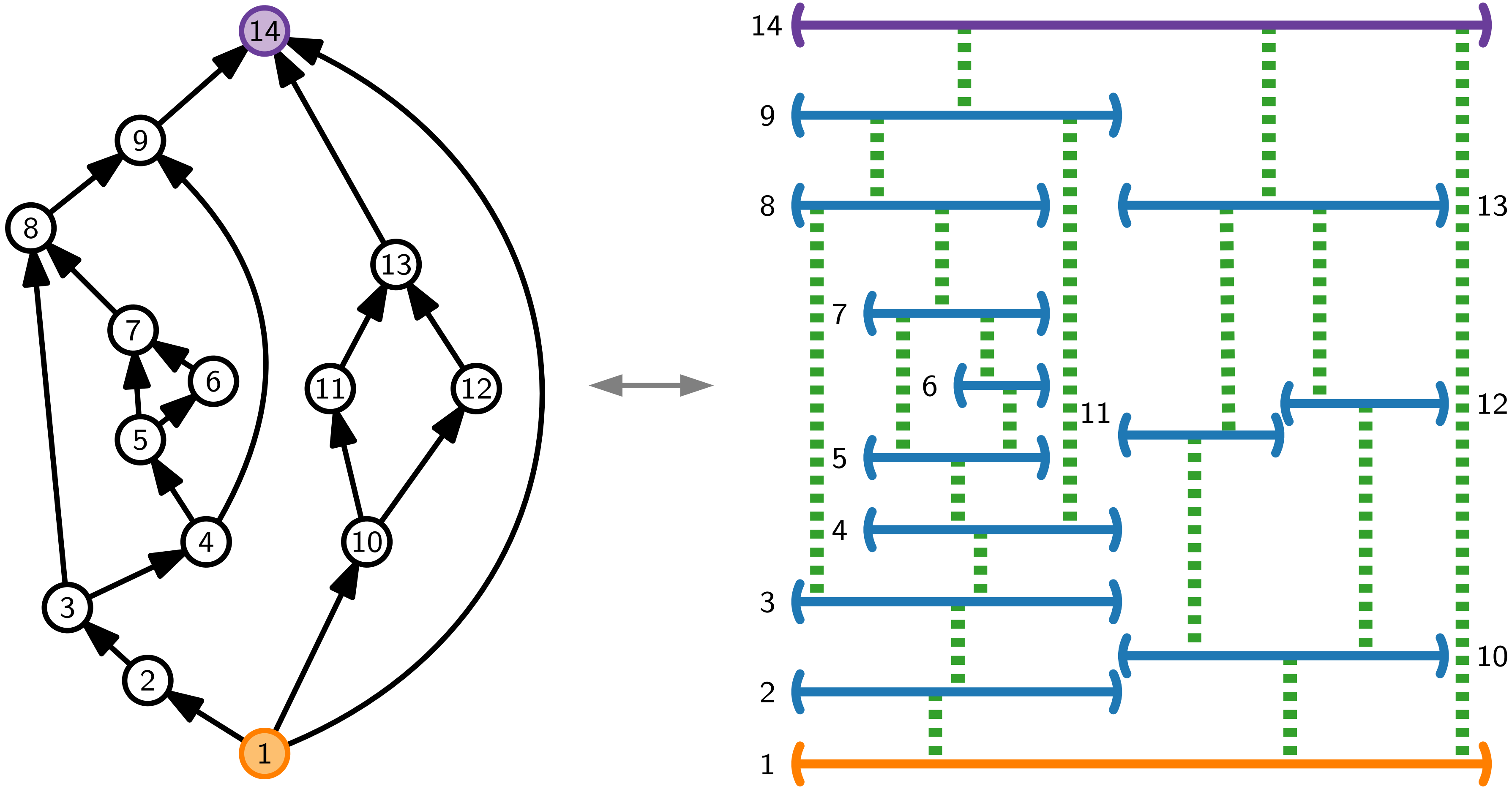

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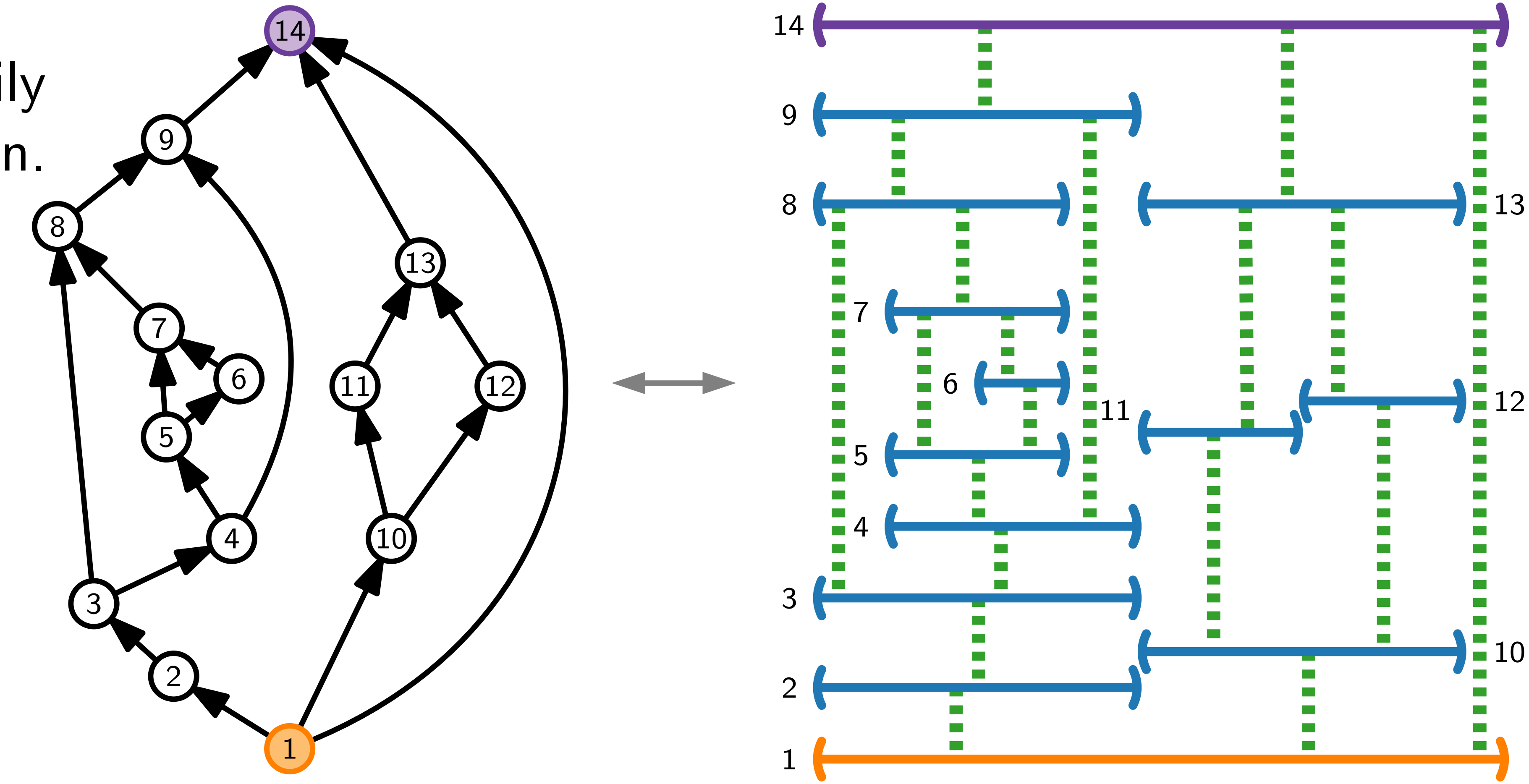

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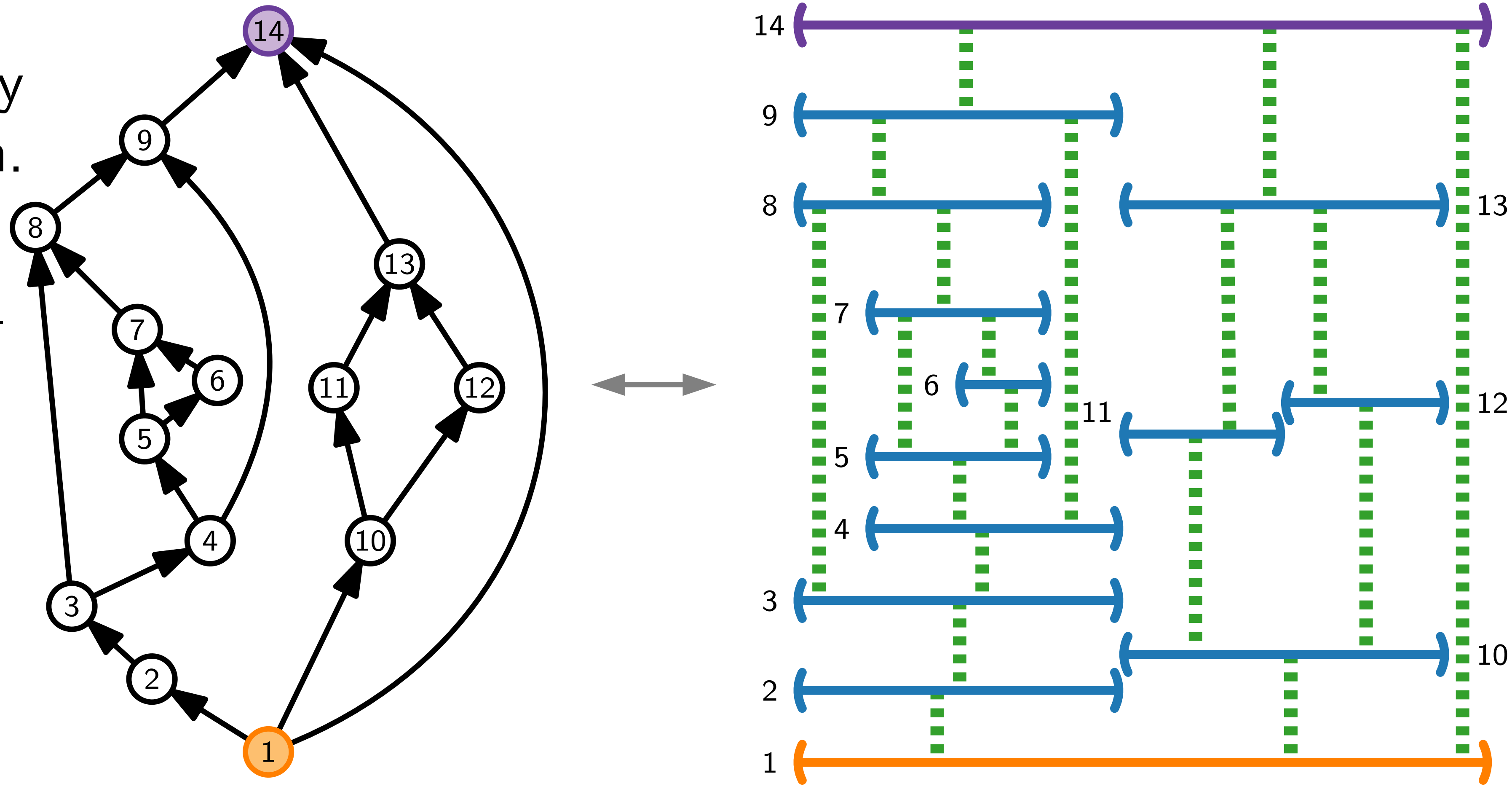

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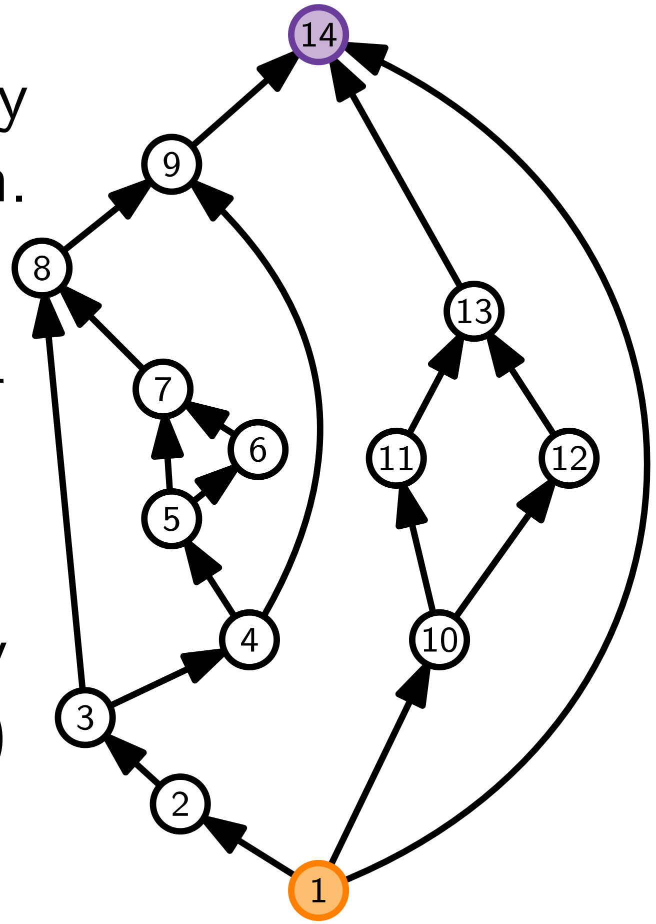

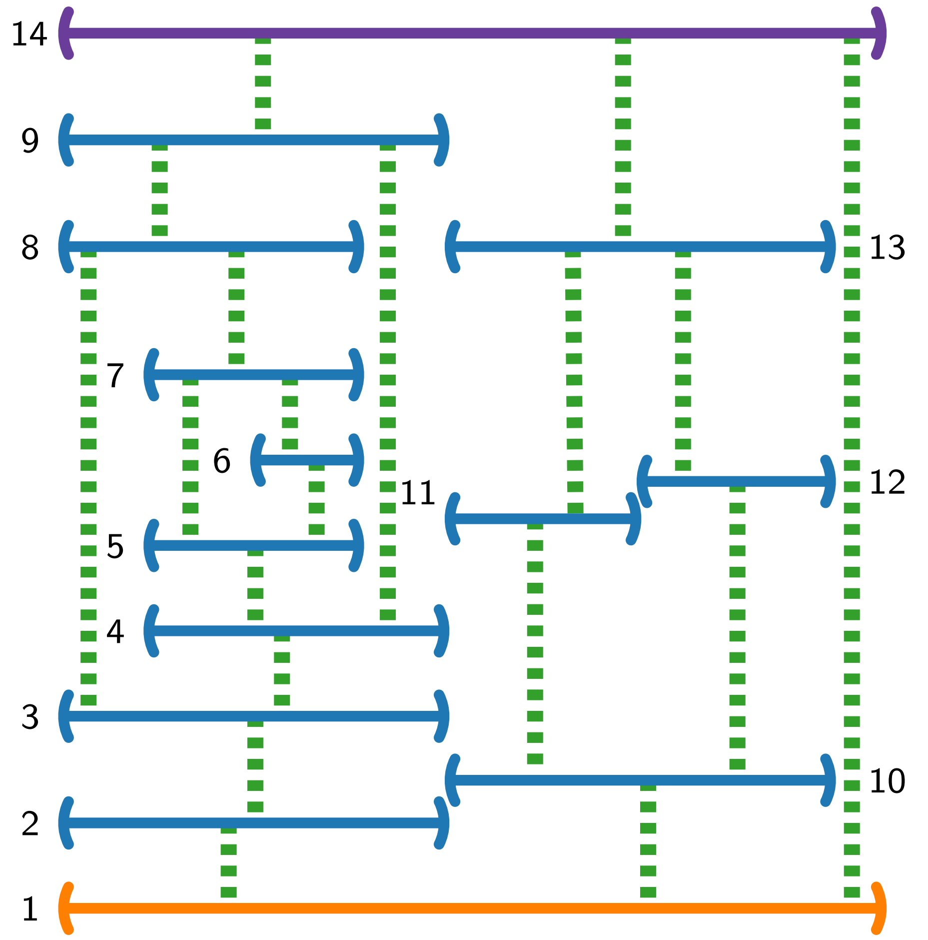

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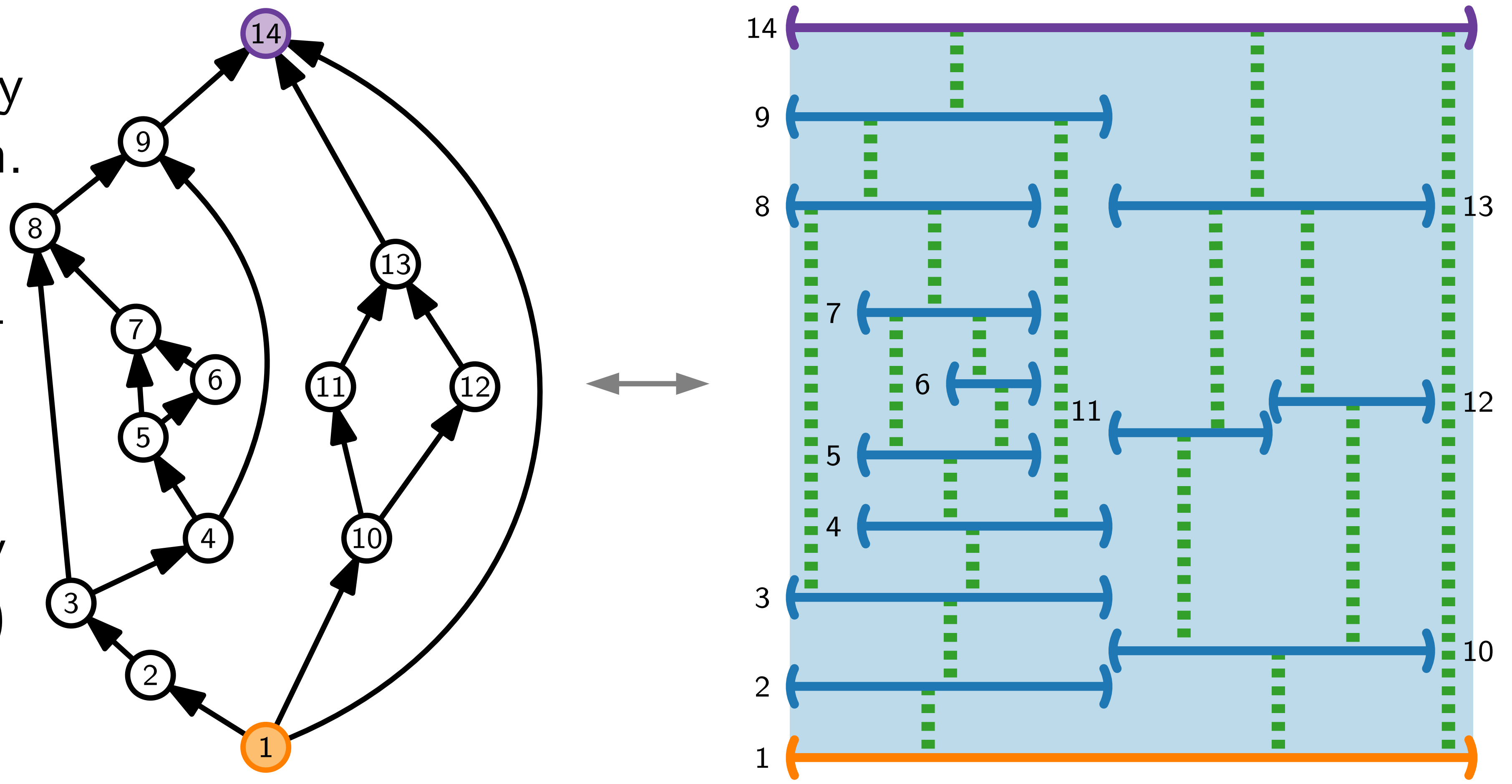

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[Chaplick, Guśpiel, Gutowski, Krawczyk, Liotta '18]

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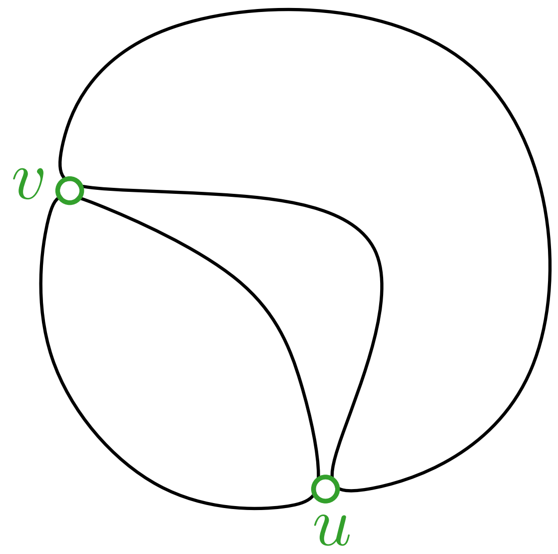

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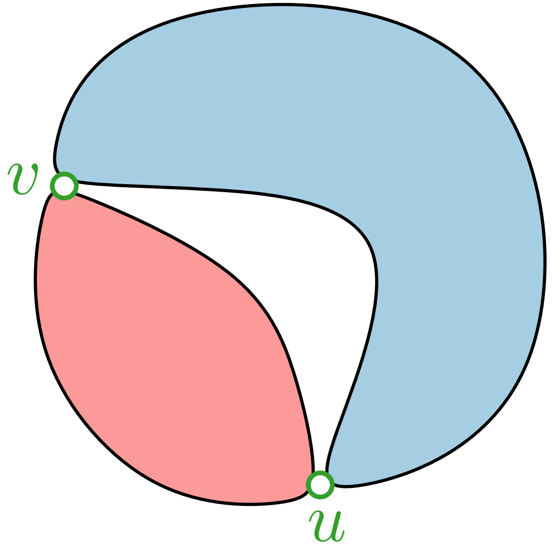

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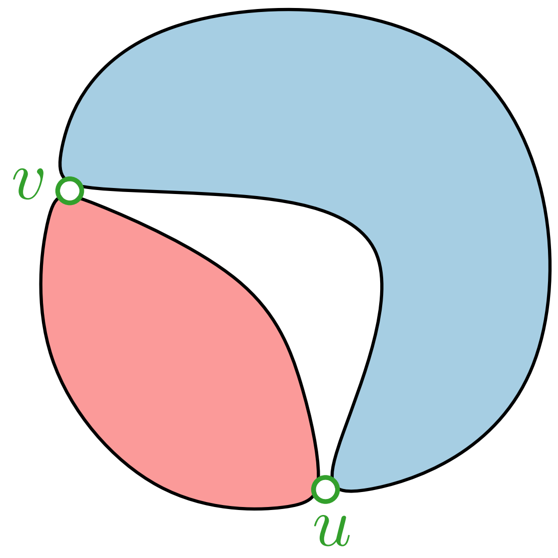

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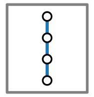

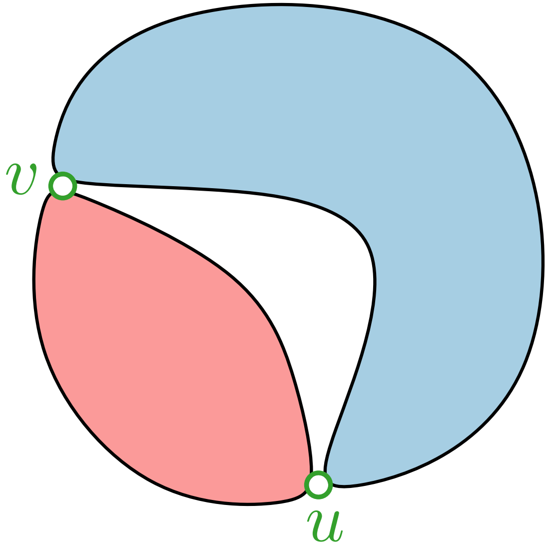

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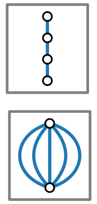

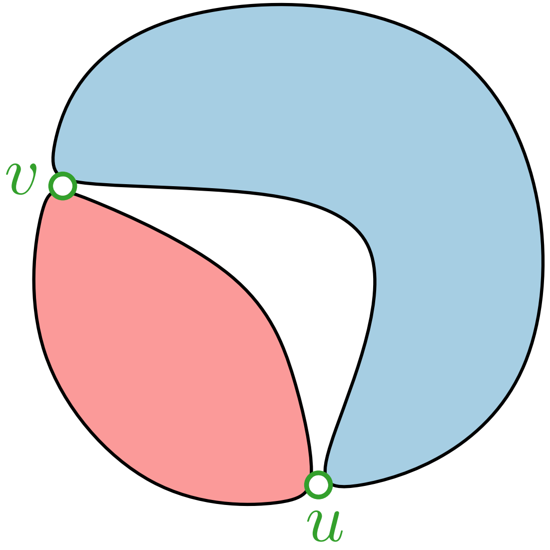

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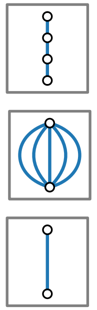

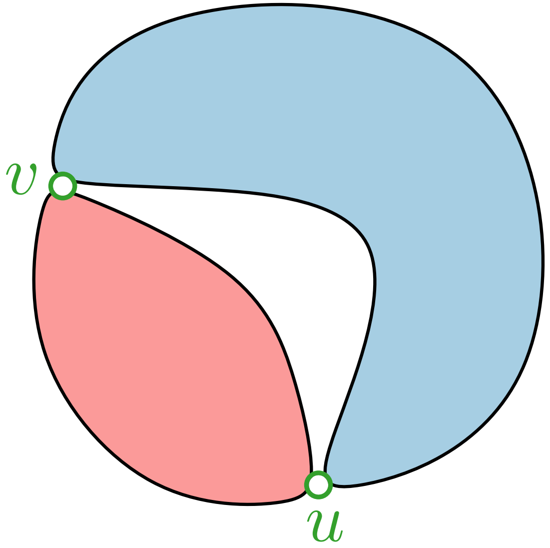

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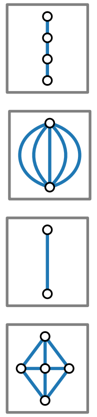

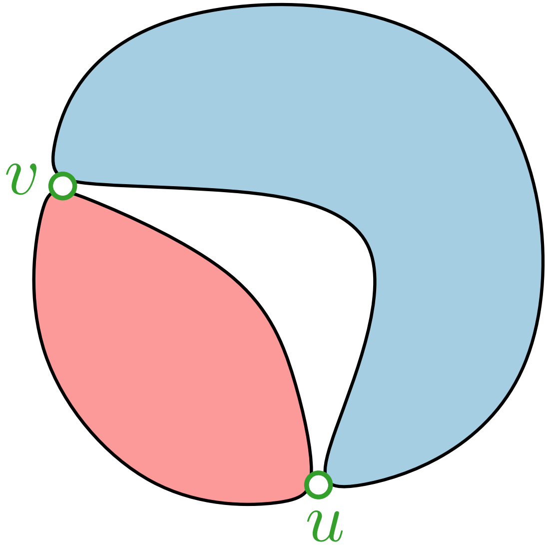

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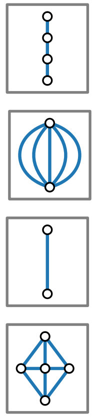

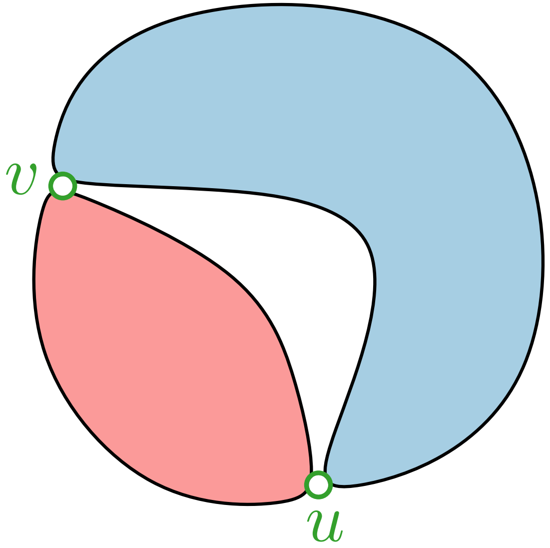

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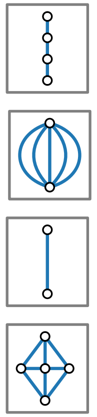

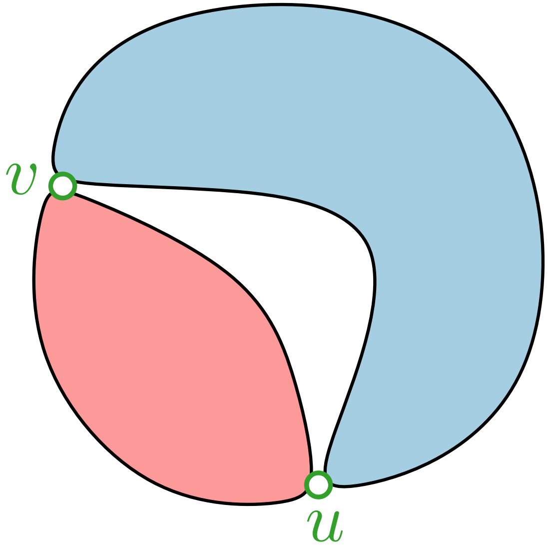

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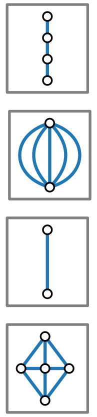

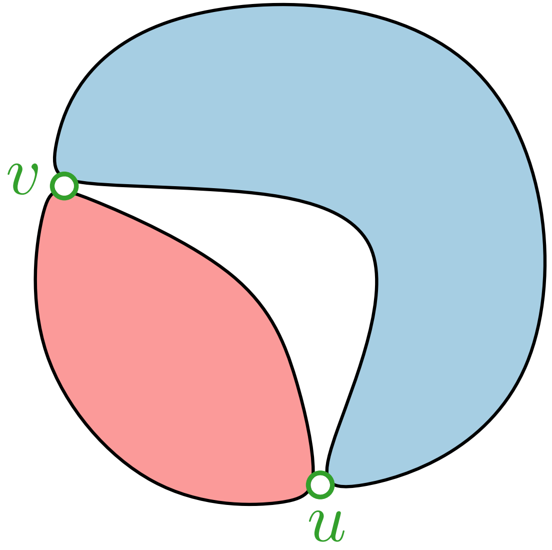

- A decomposition tree of a series-parallel graph is an SPQR-tree without **R**-nodes.
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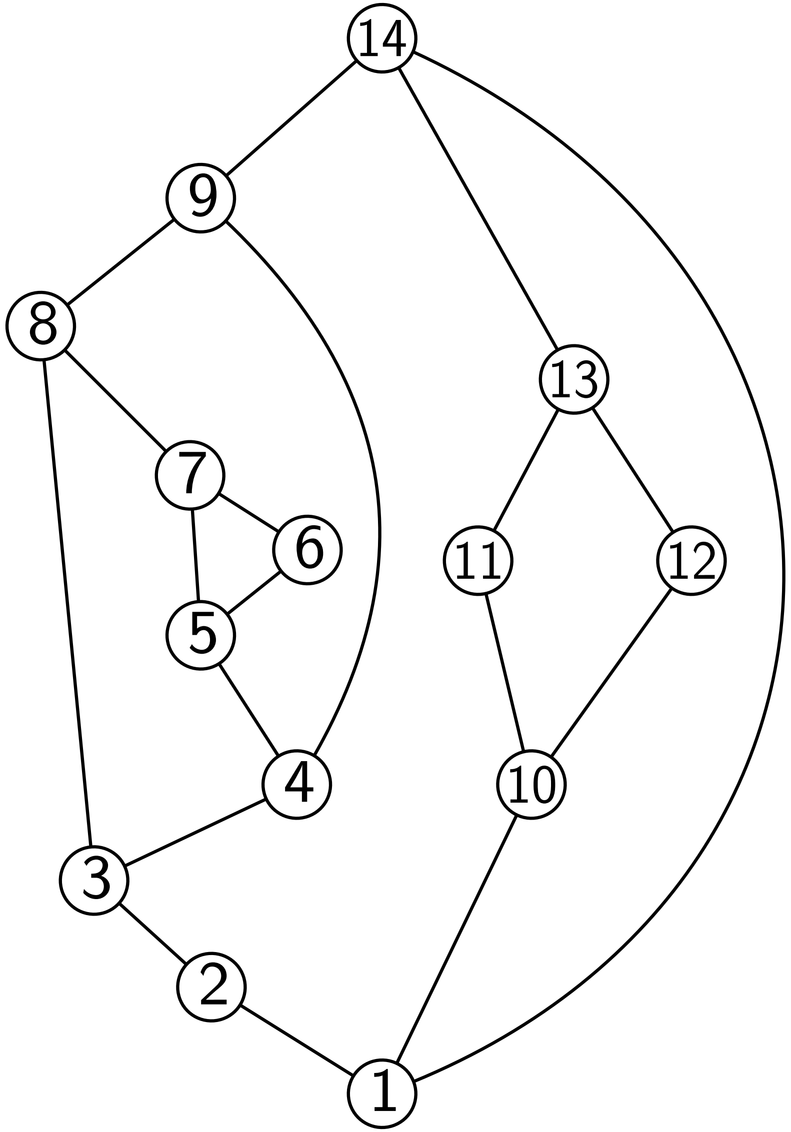
- T can be computed in time linear in the size of G. [Gutwenger, Mutzel '01]

SPQR-Tree – Example

G



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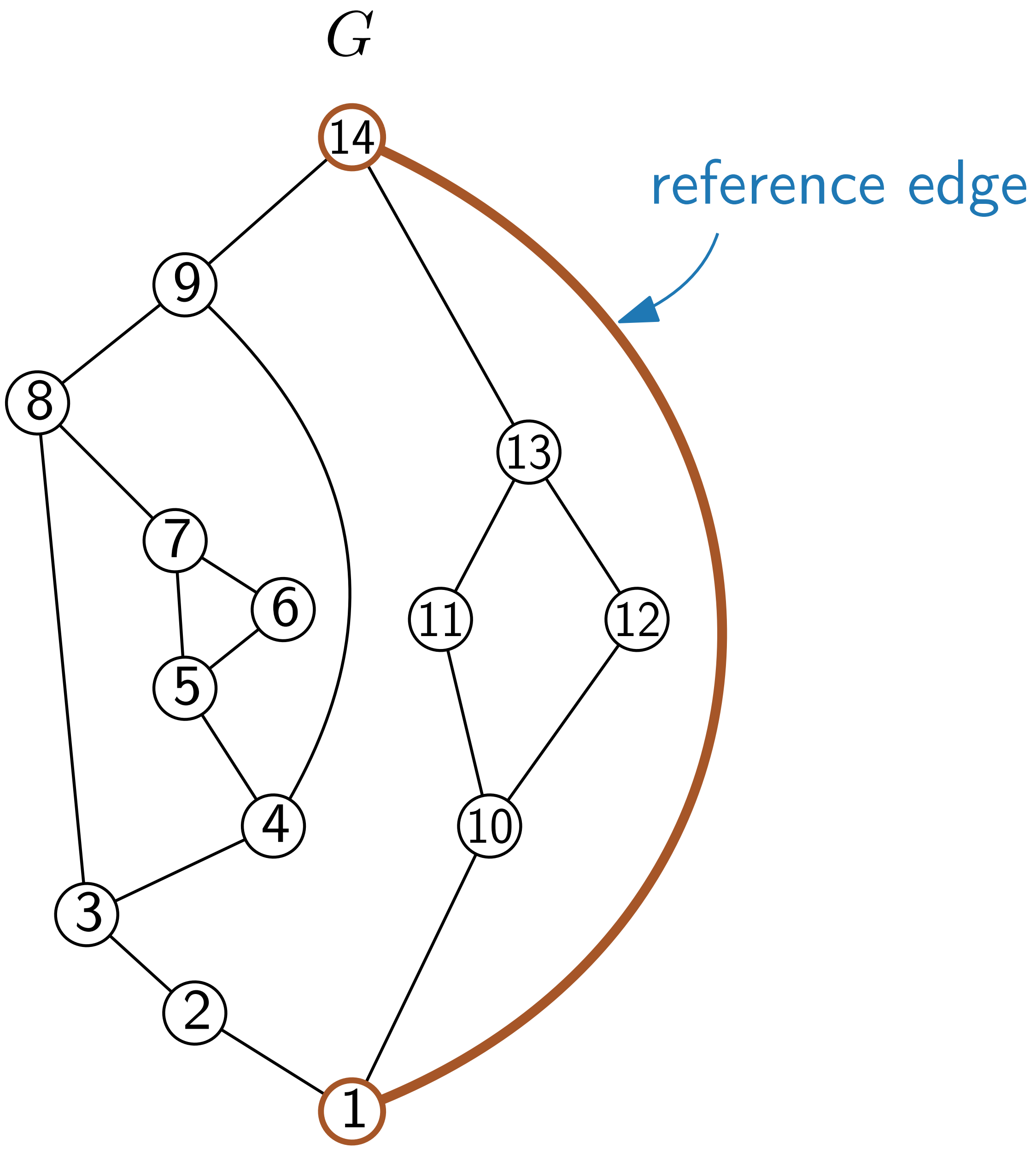

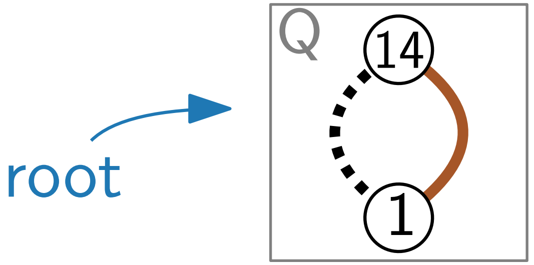

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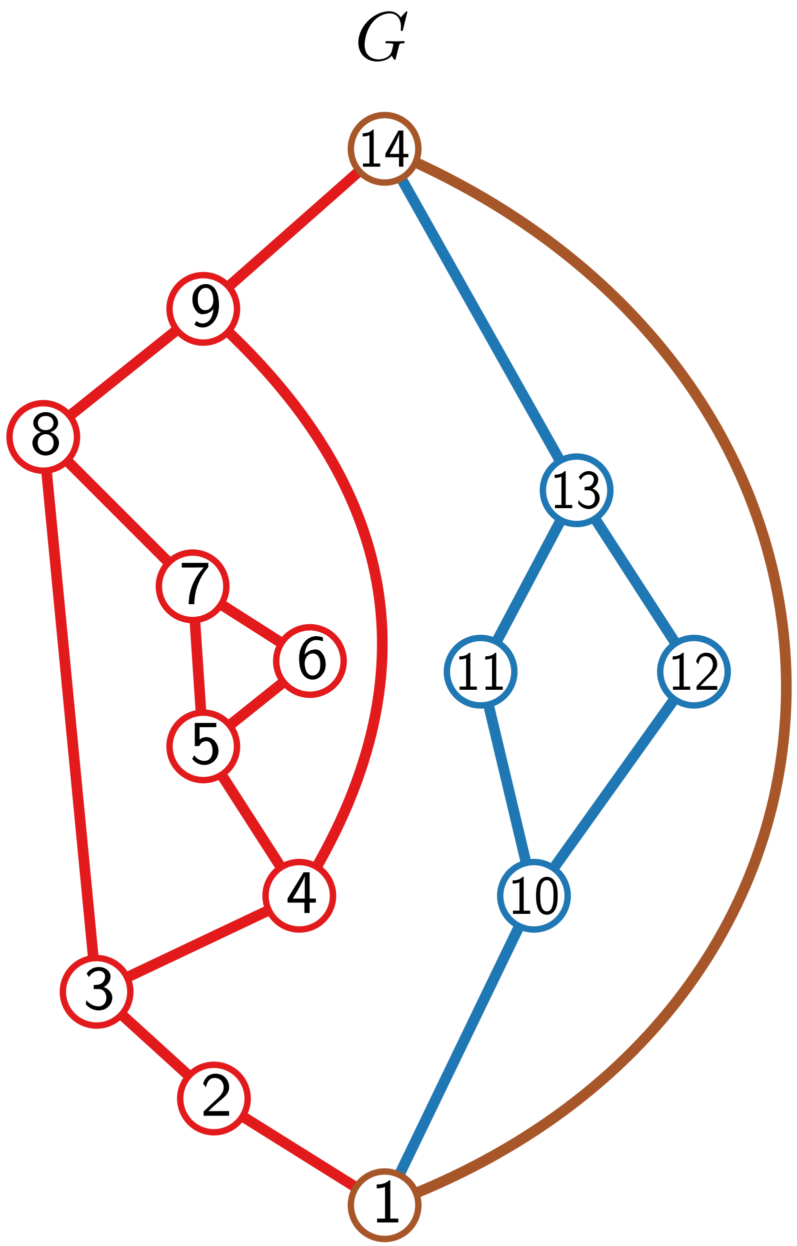

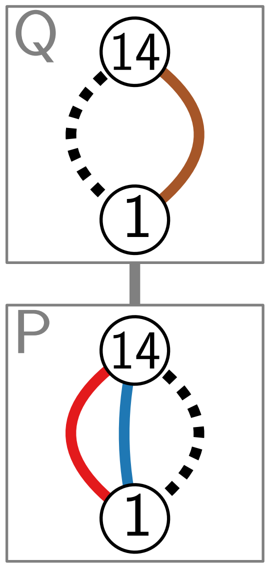

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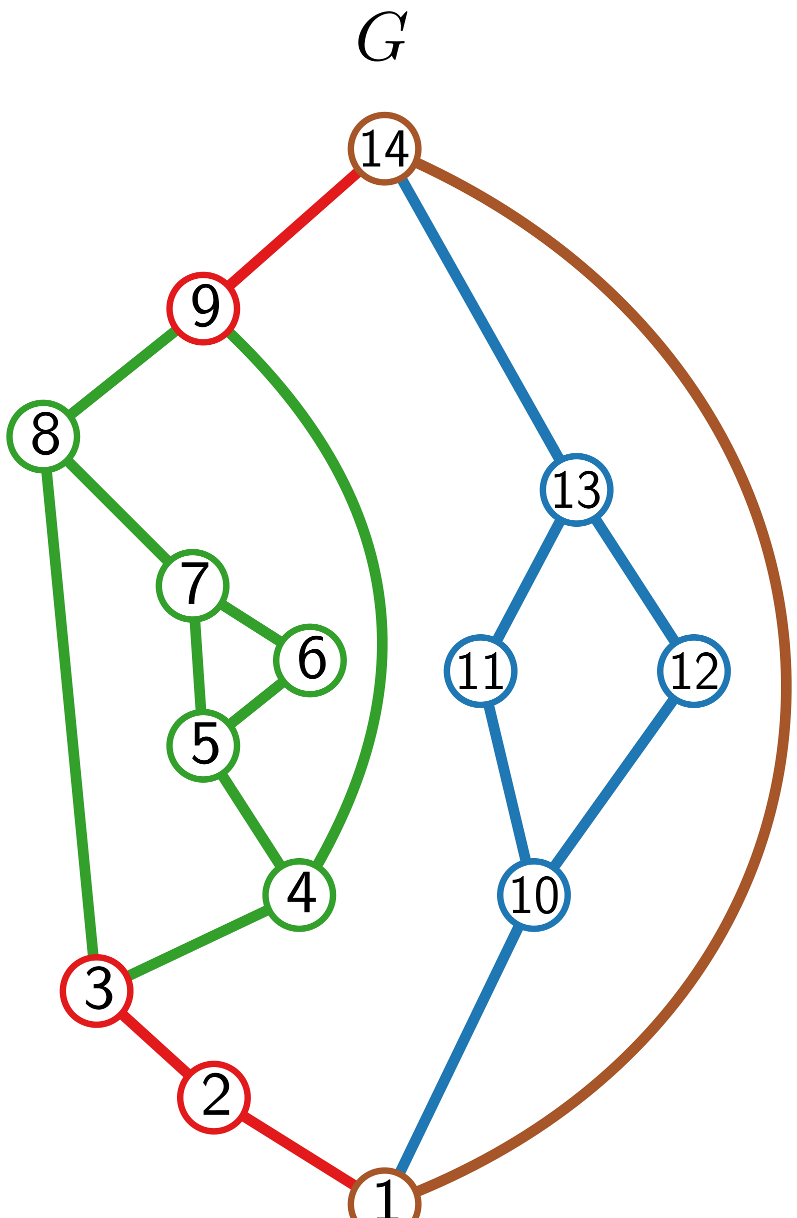

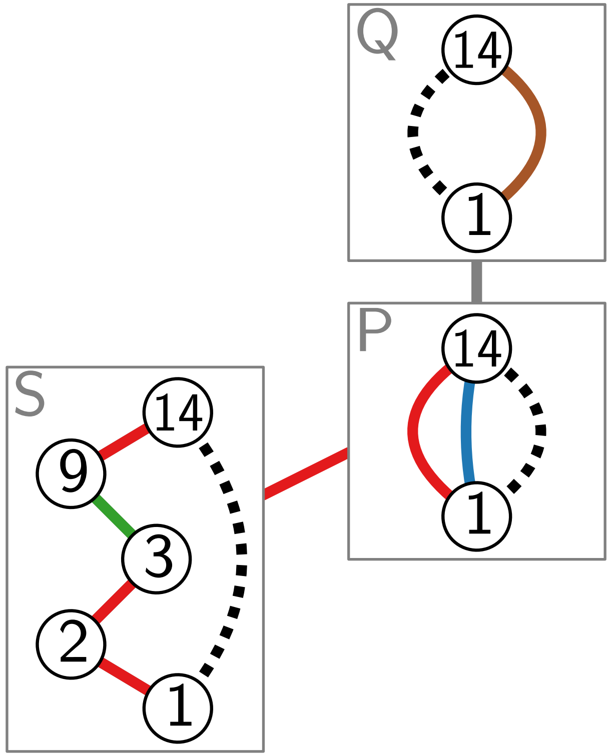

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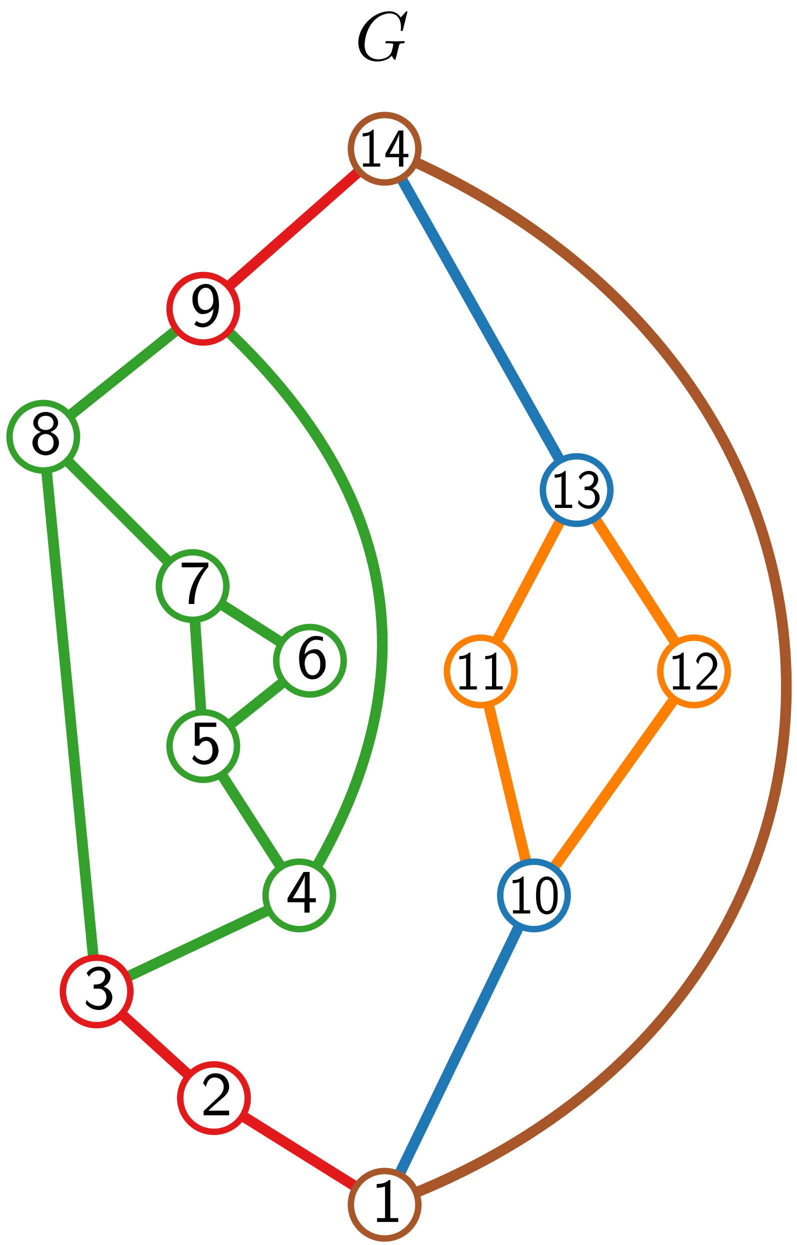

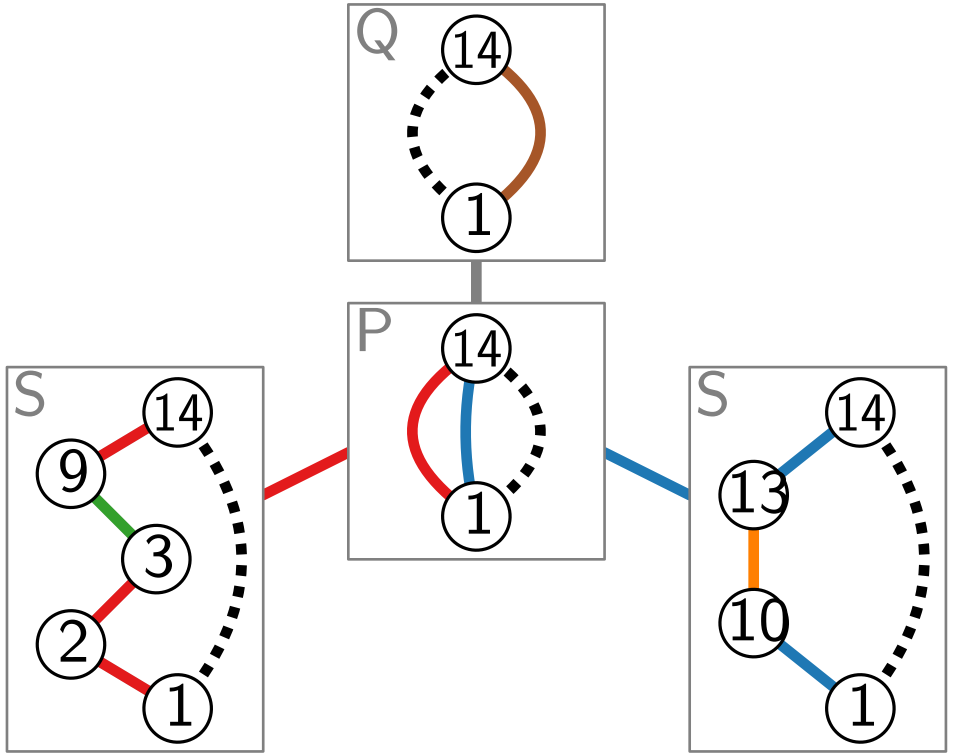

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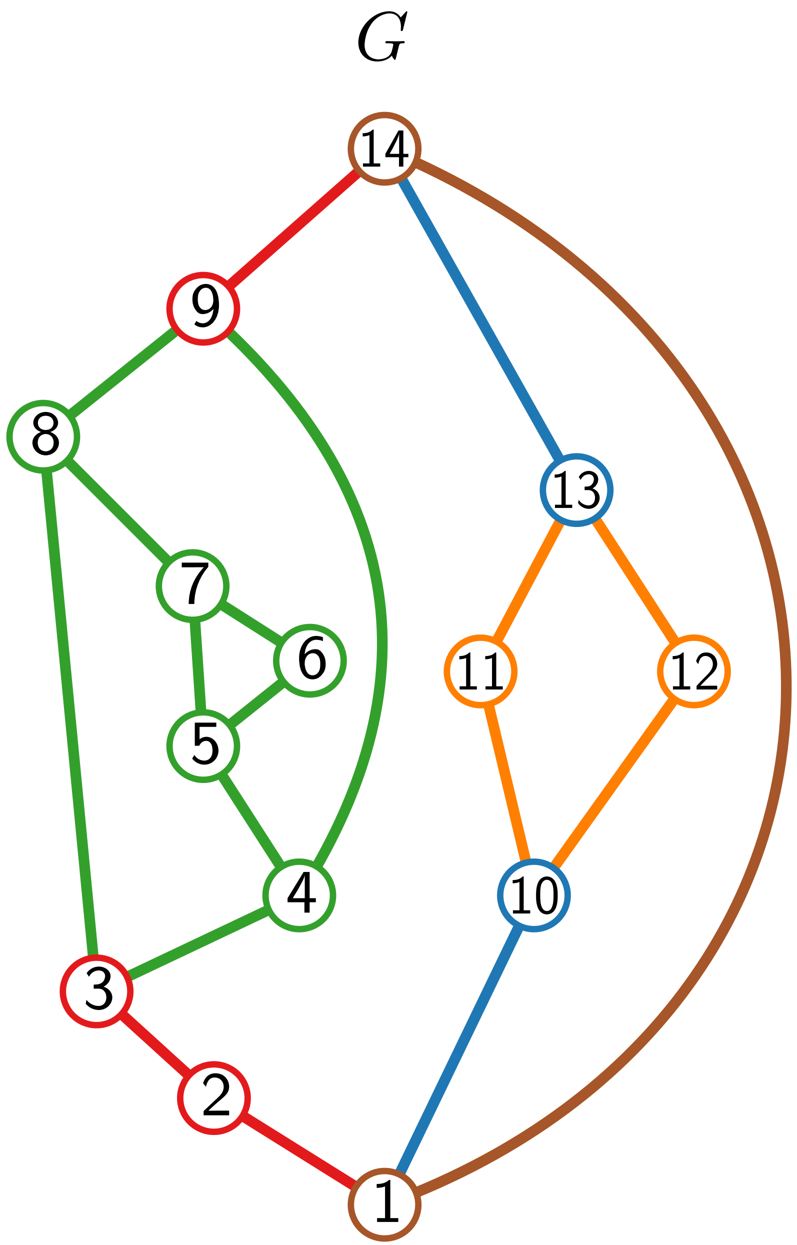

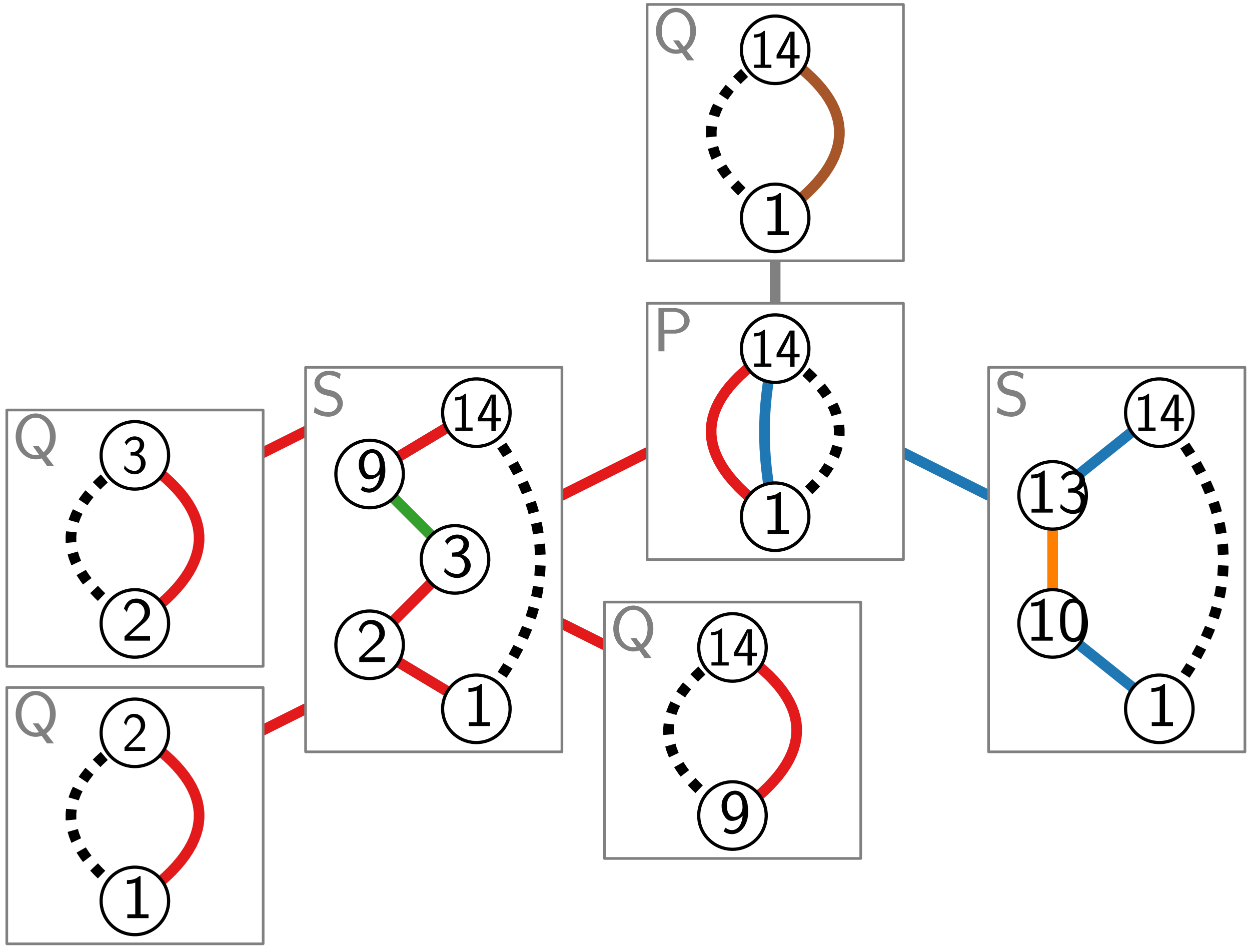

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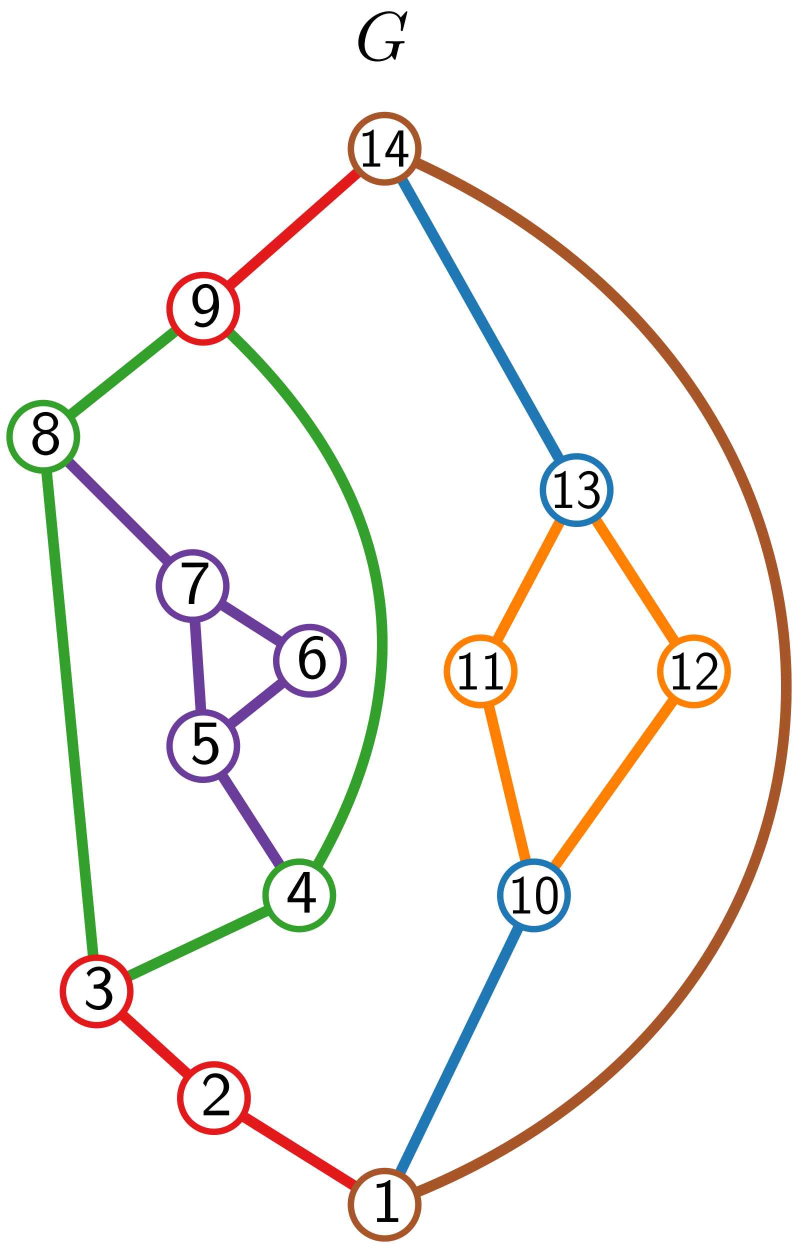

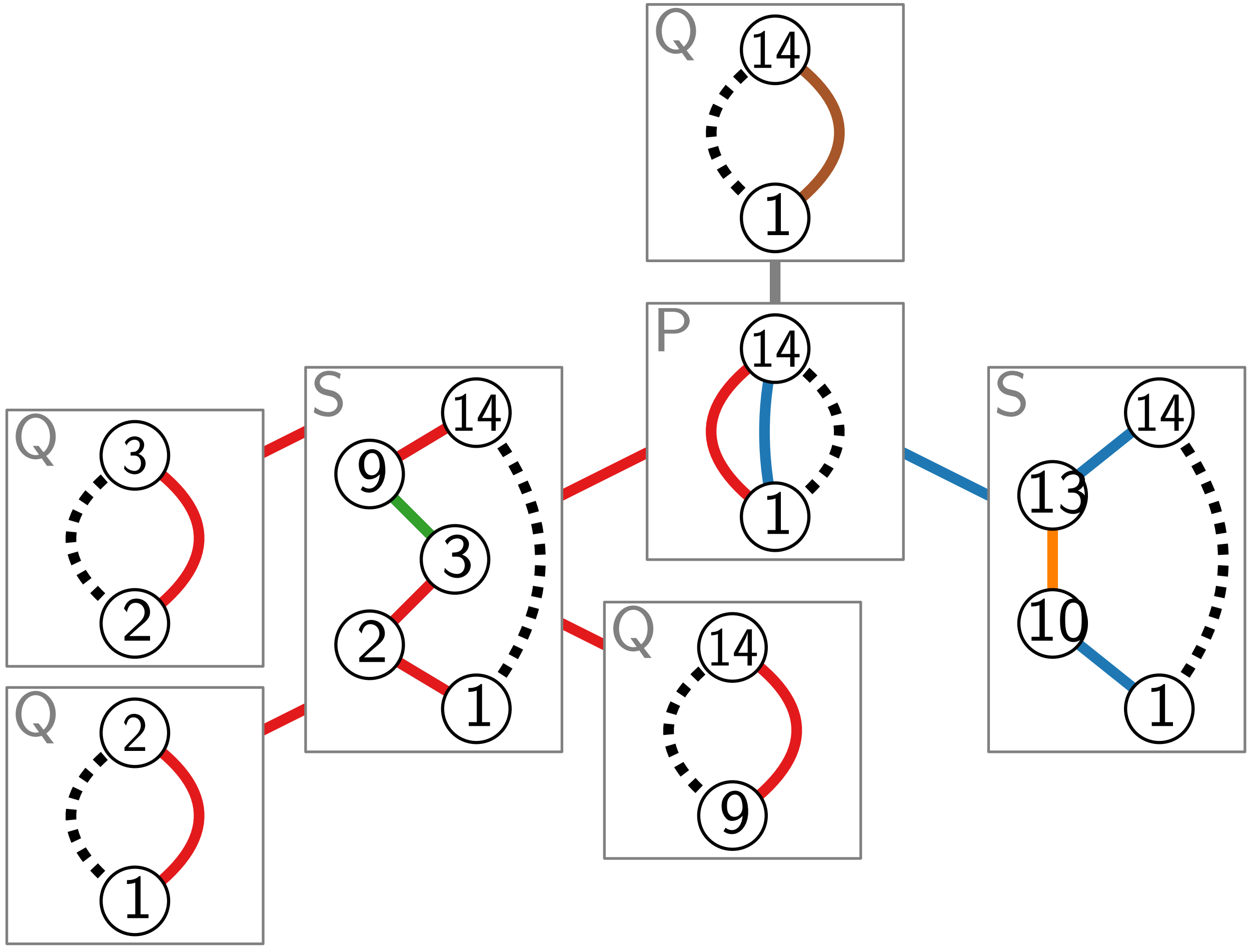

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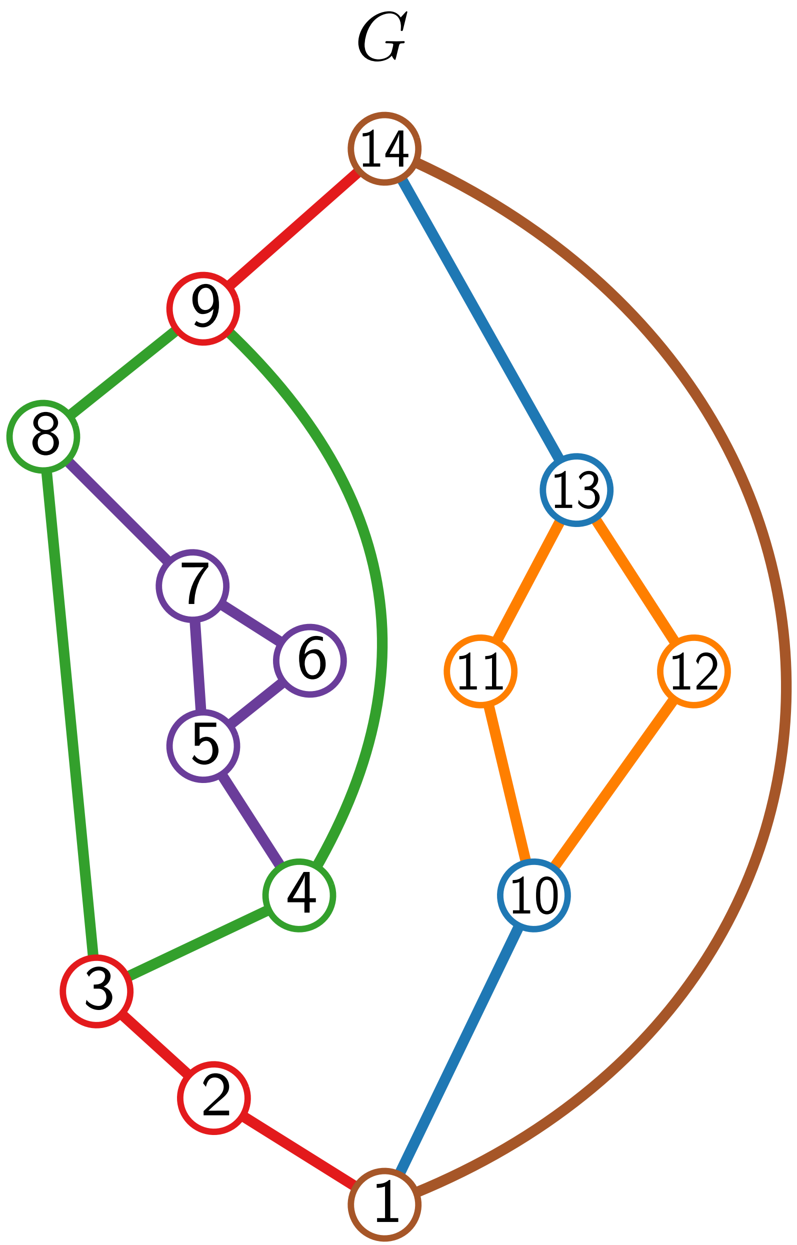

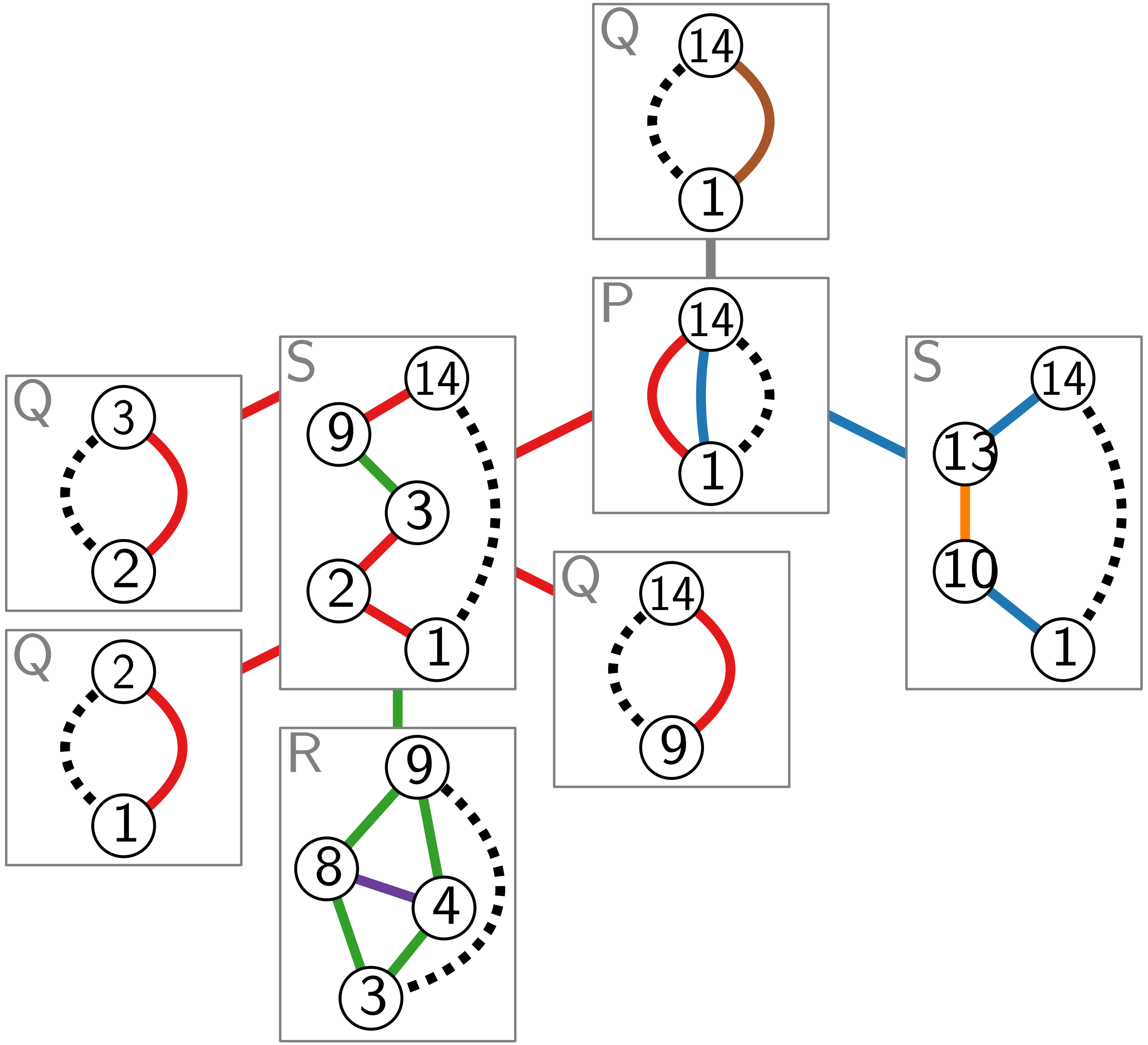

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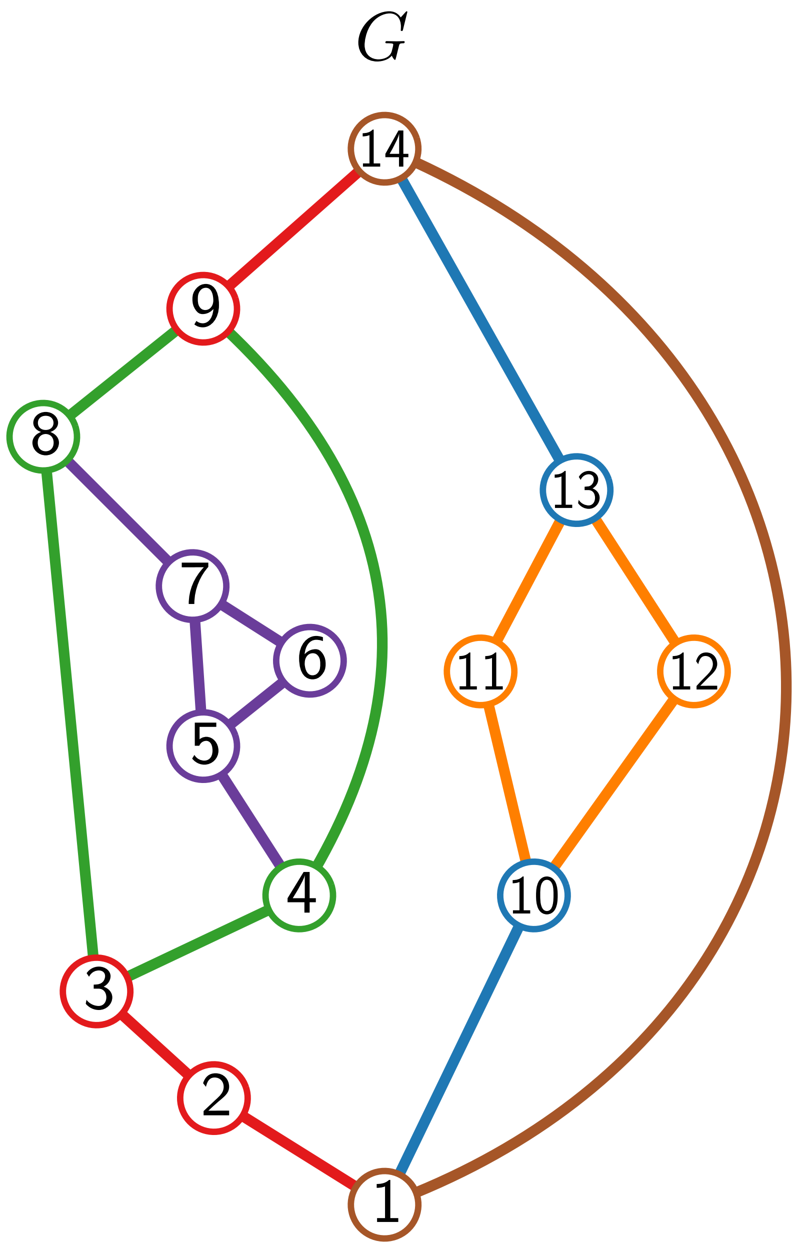

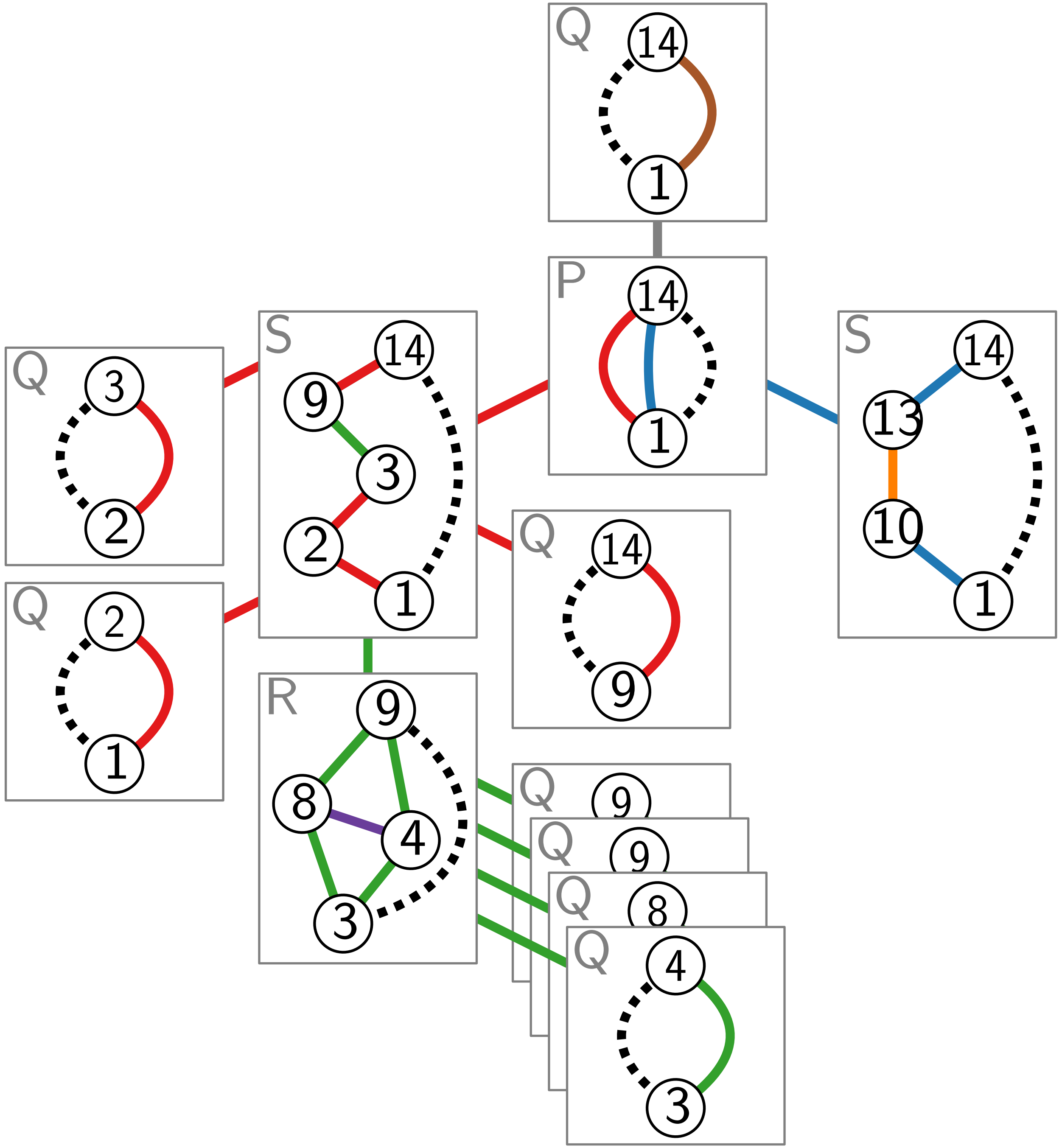

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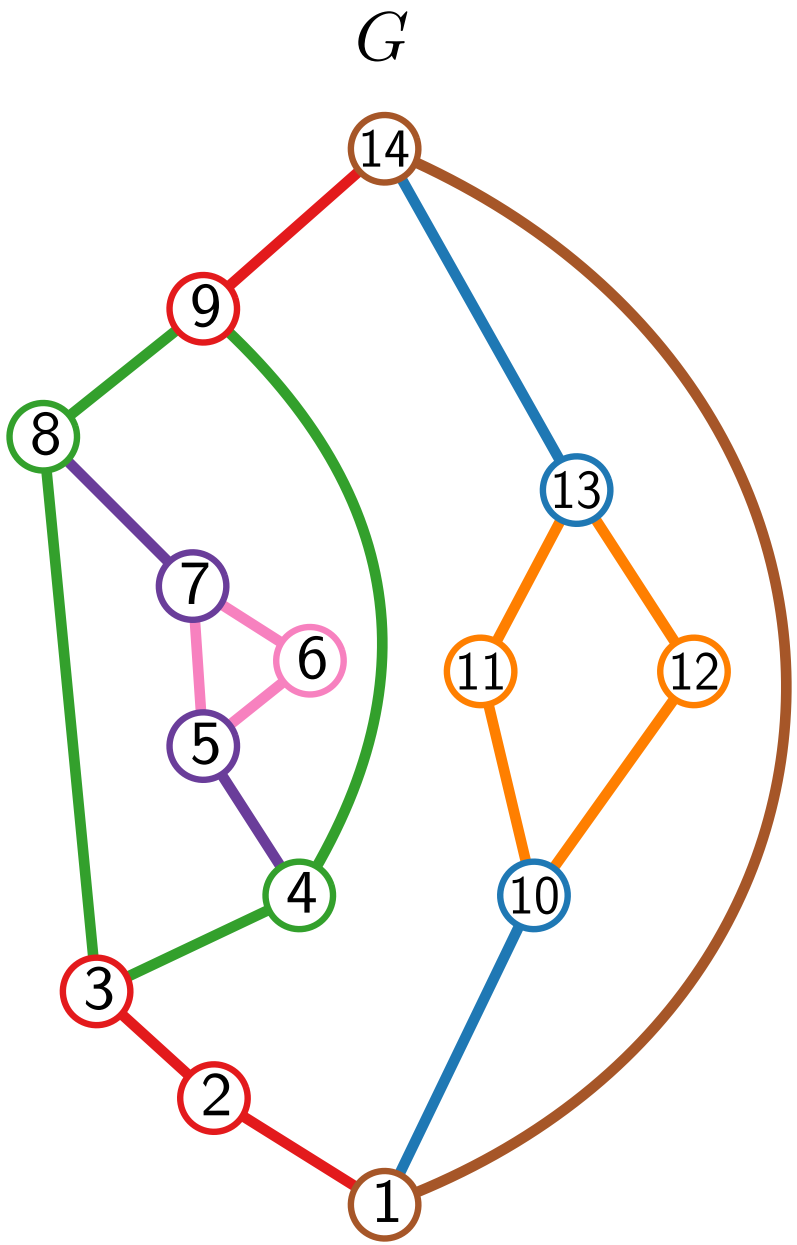

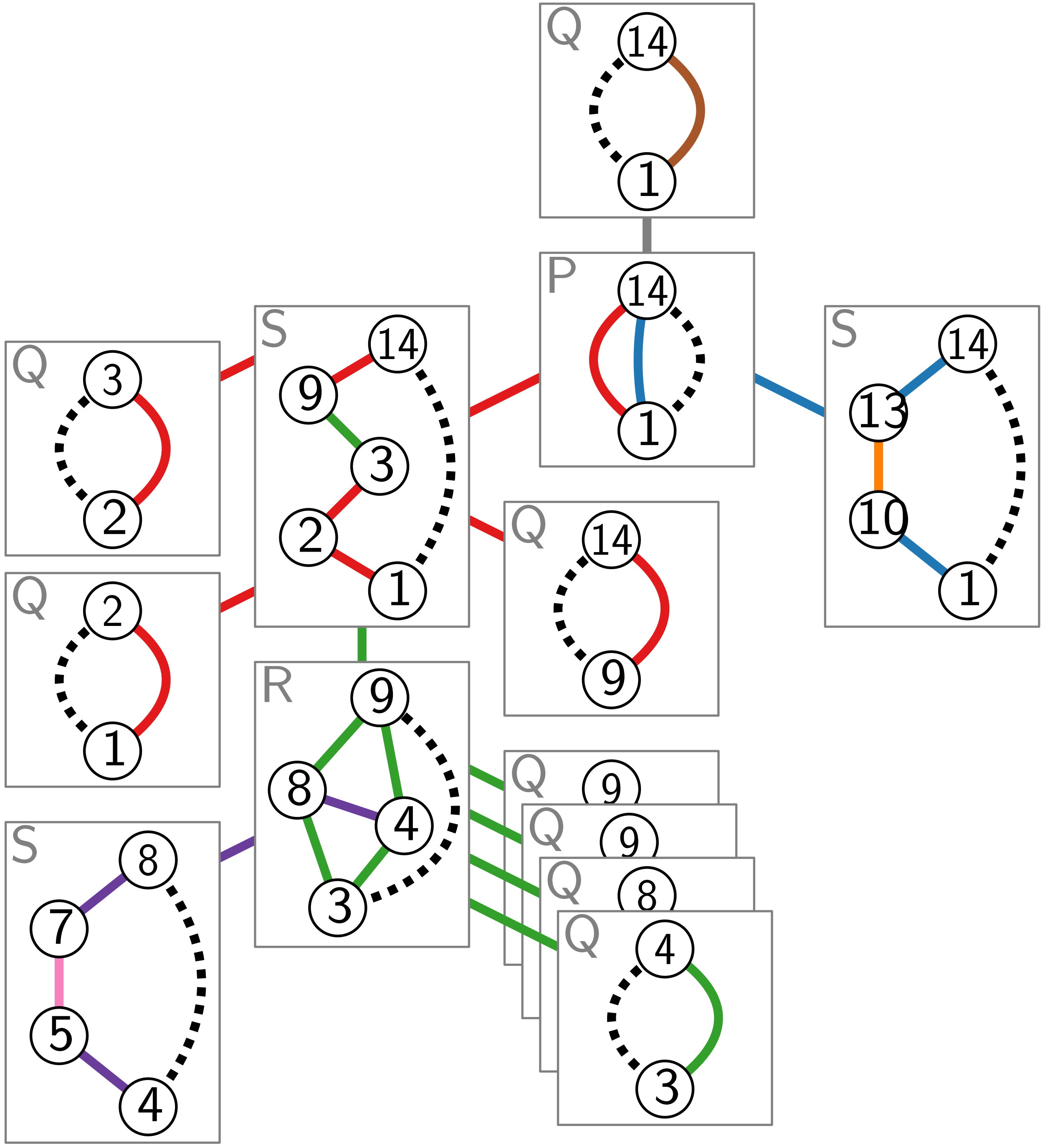

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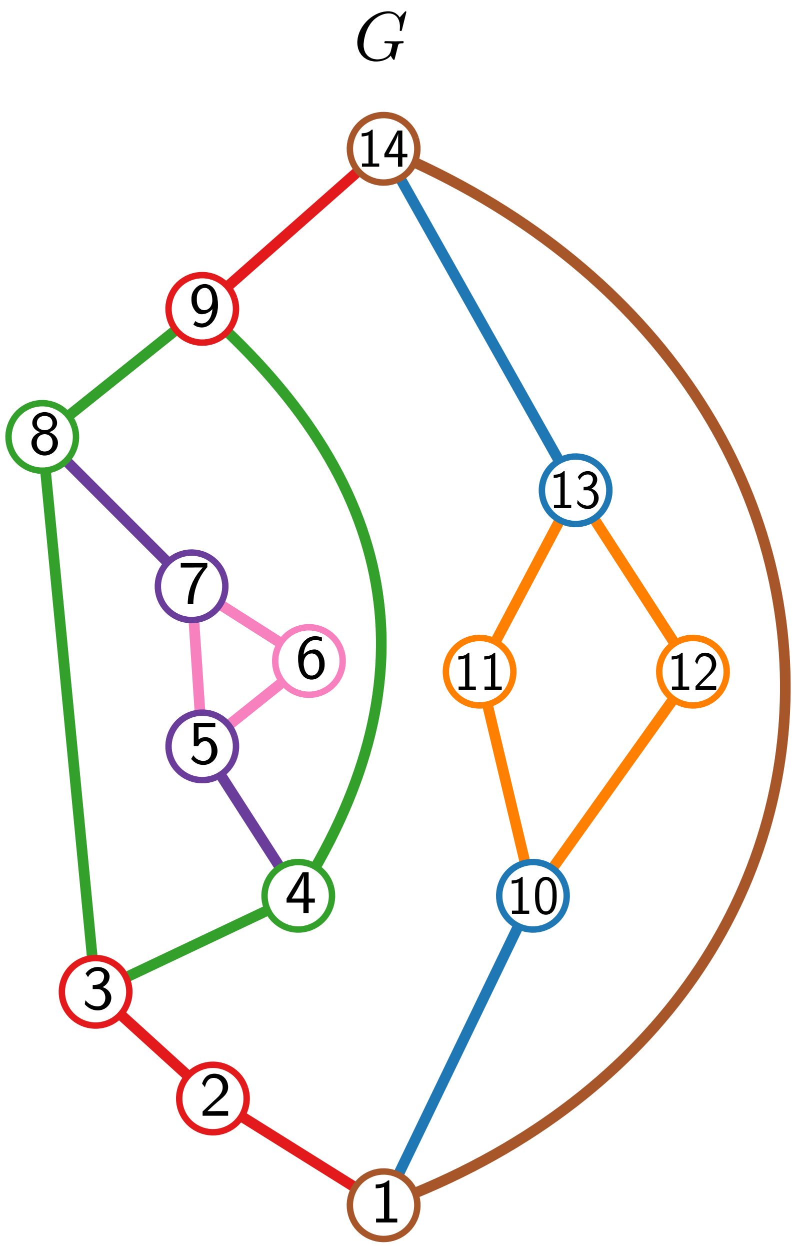

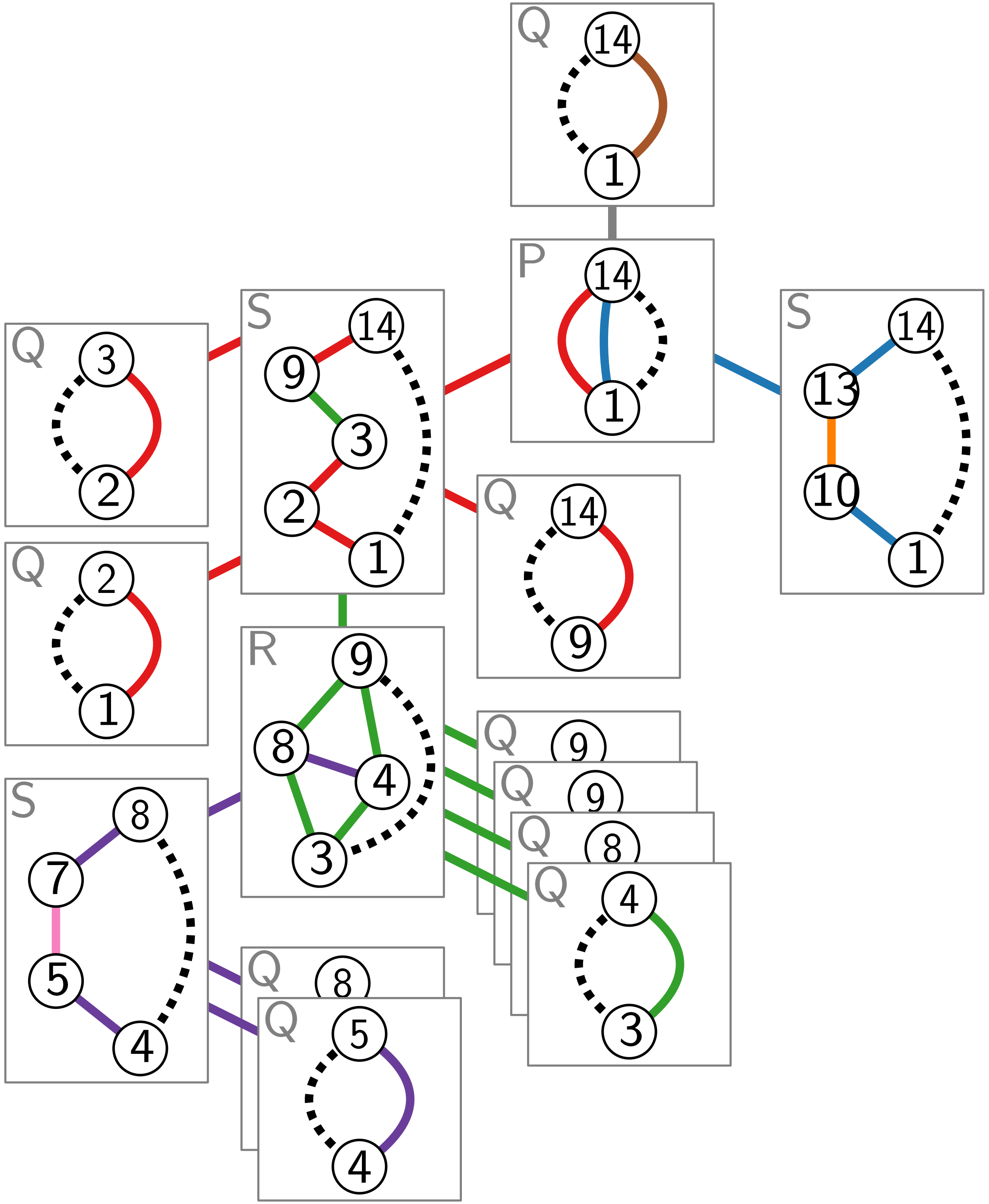

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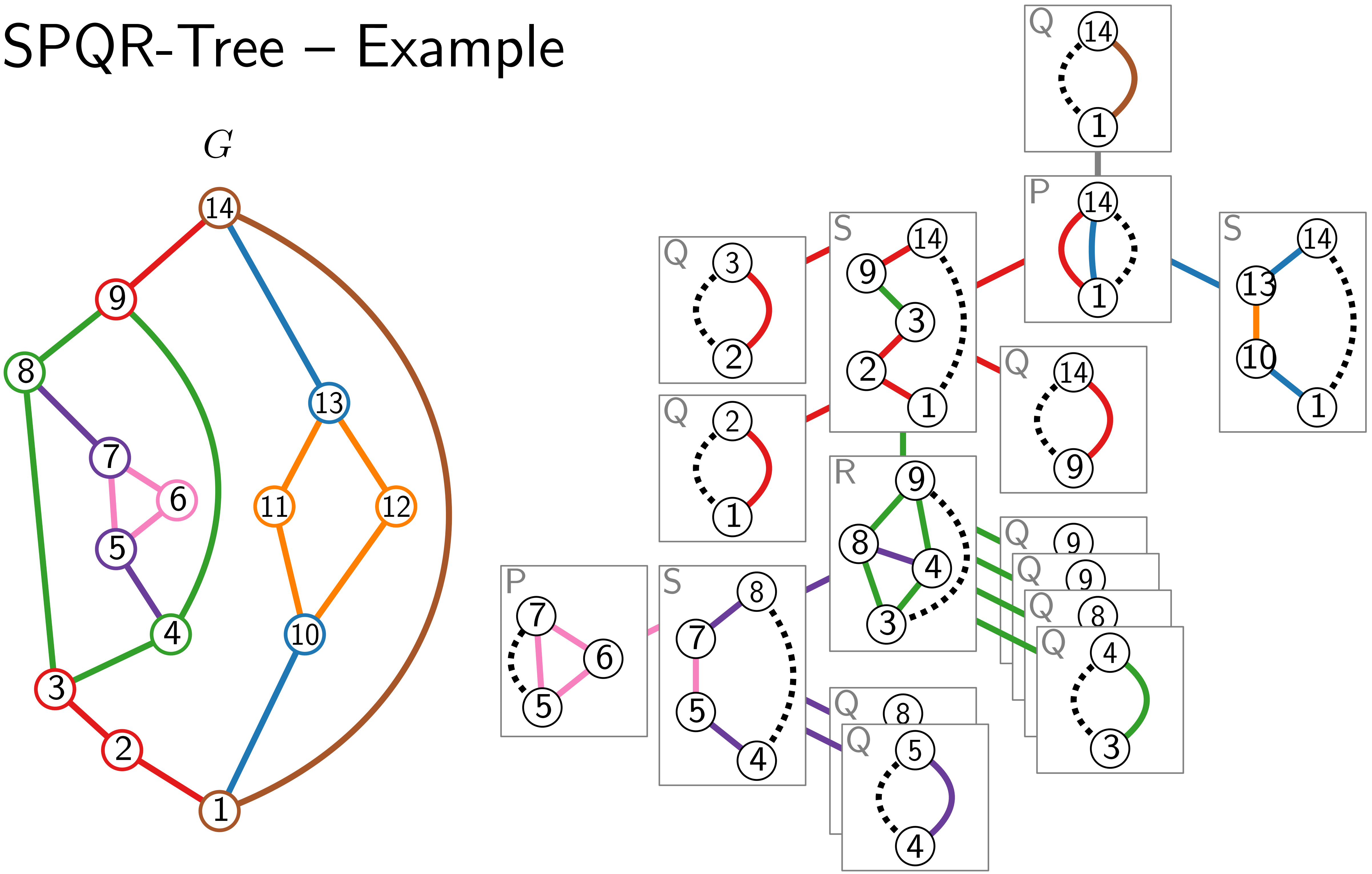

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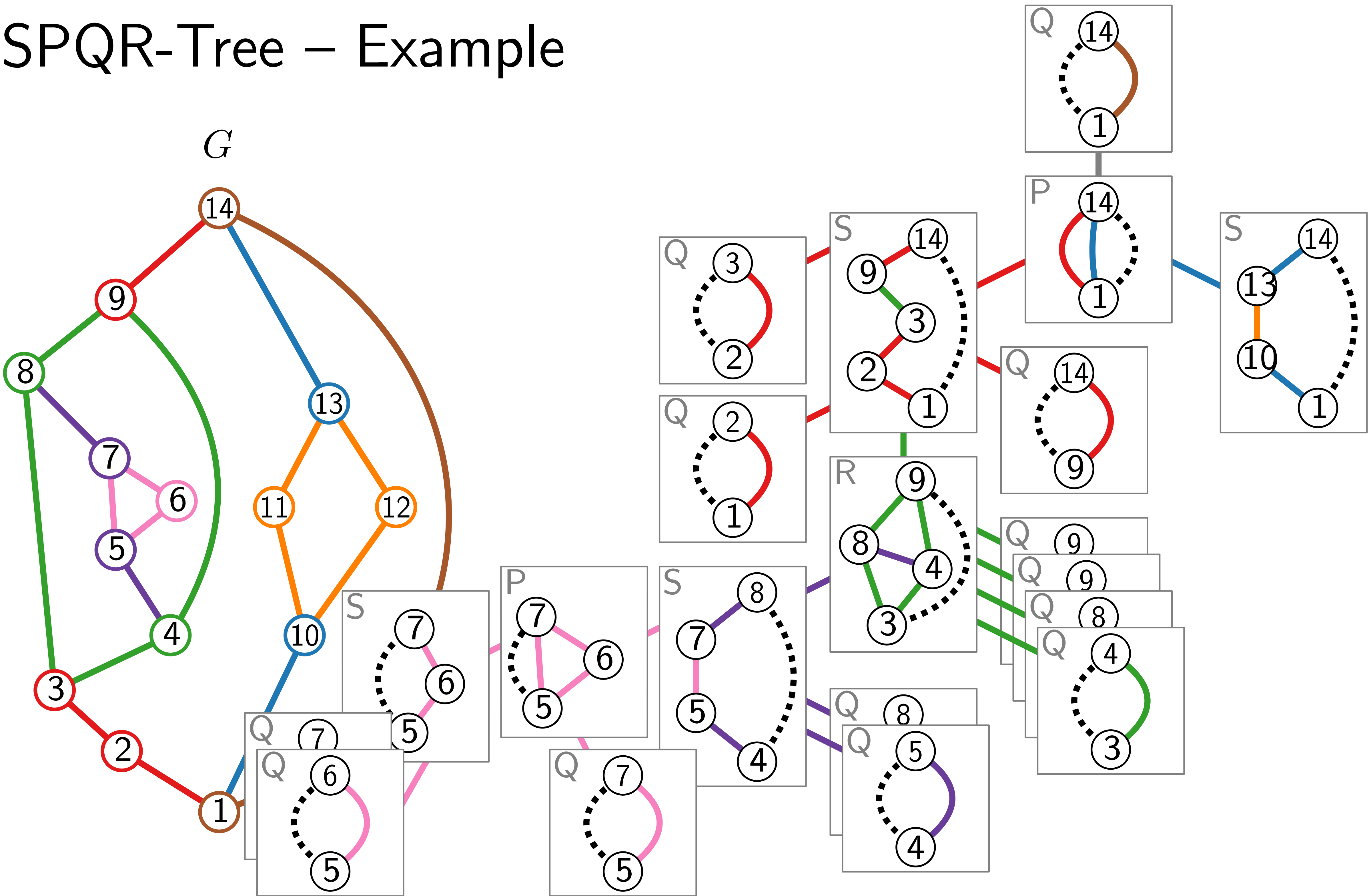

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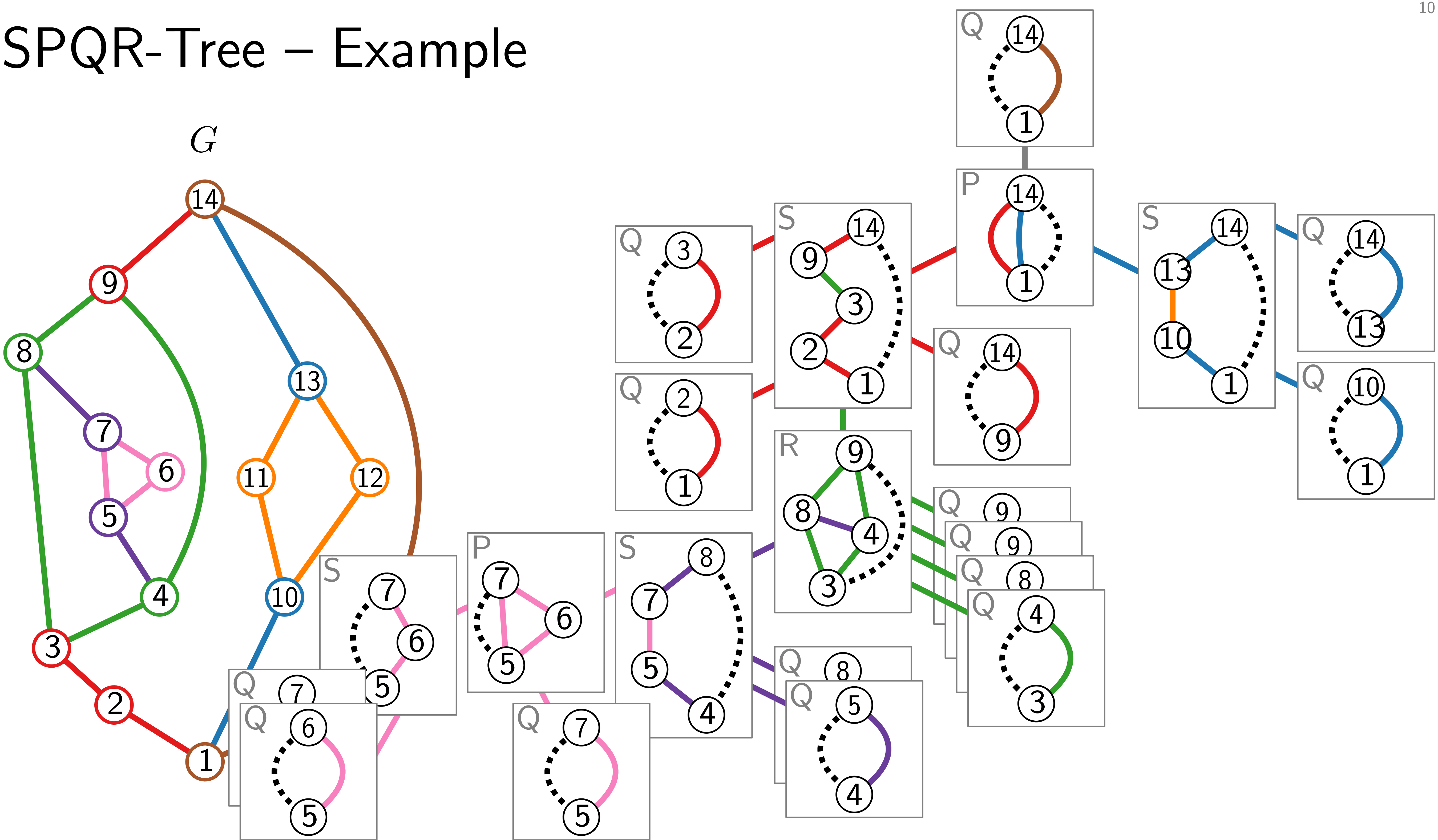

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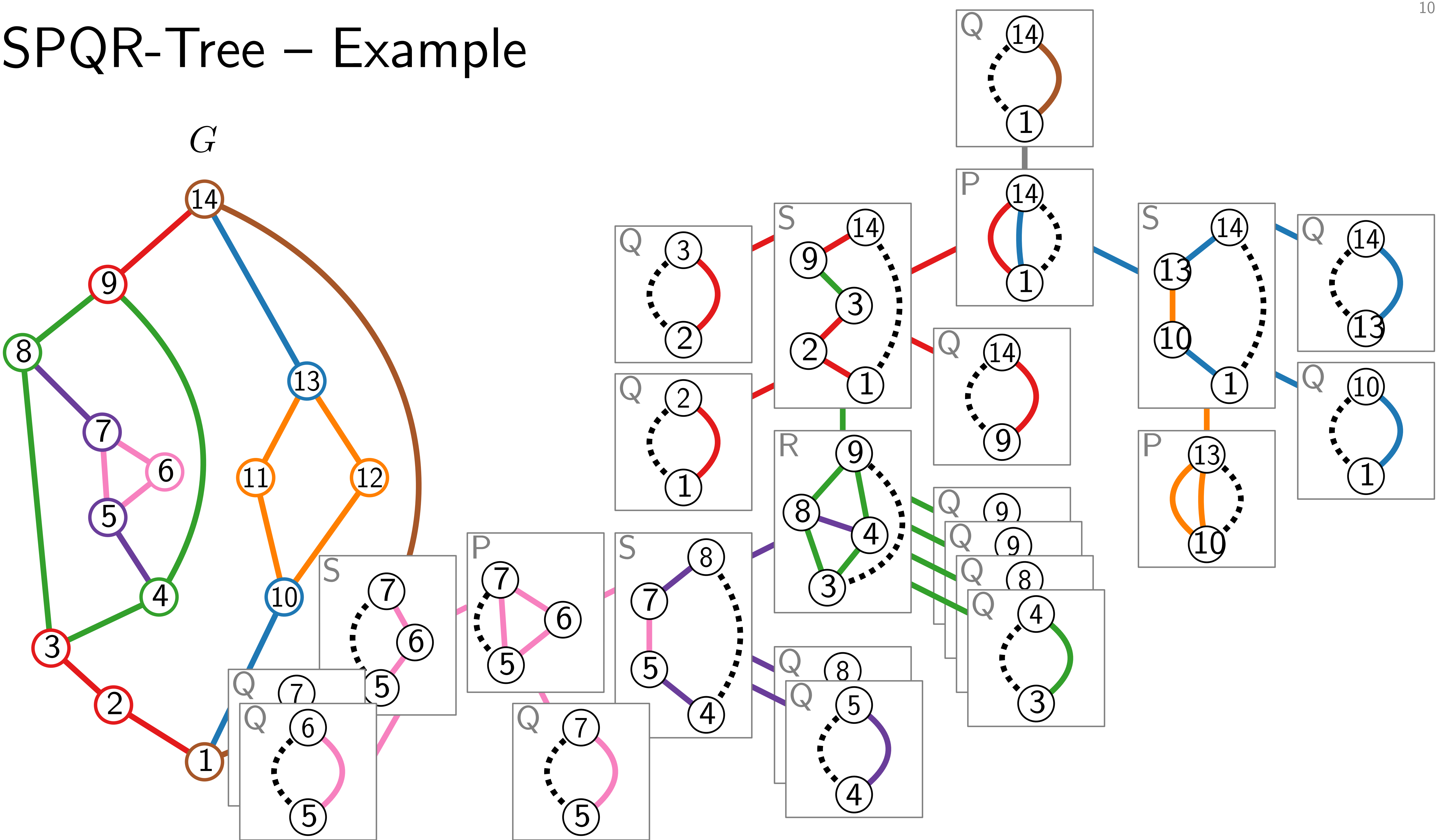

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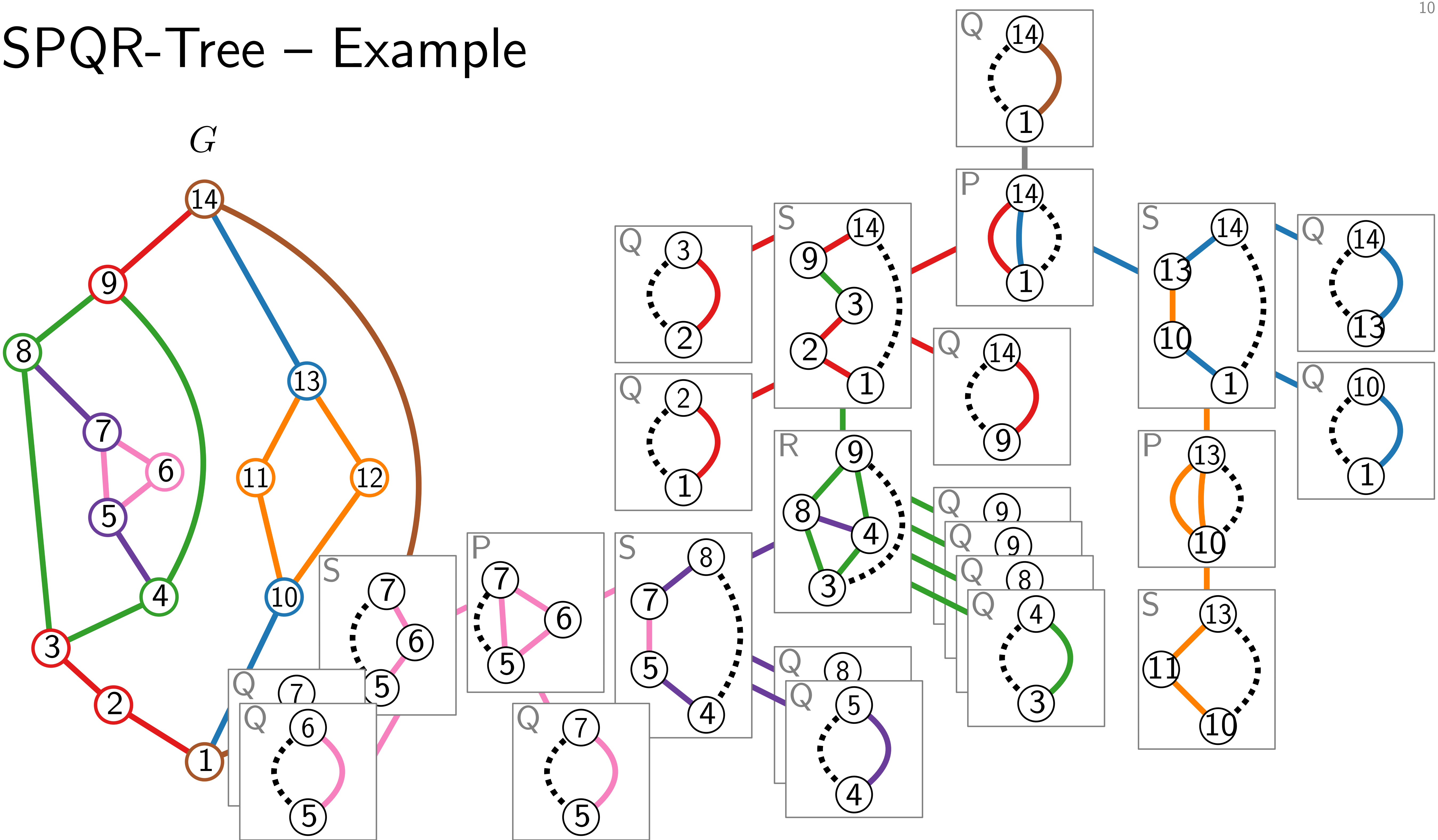

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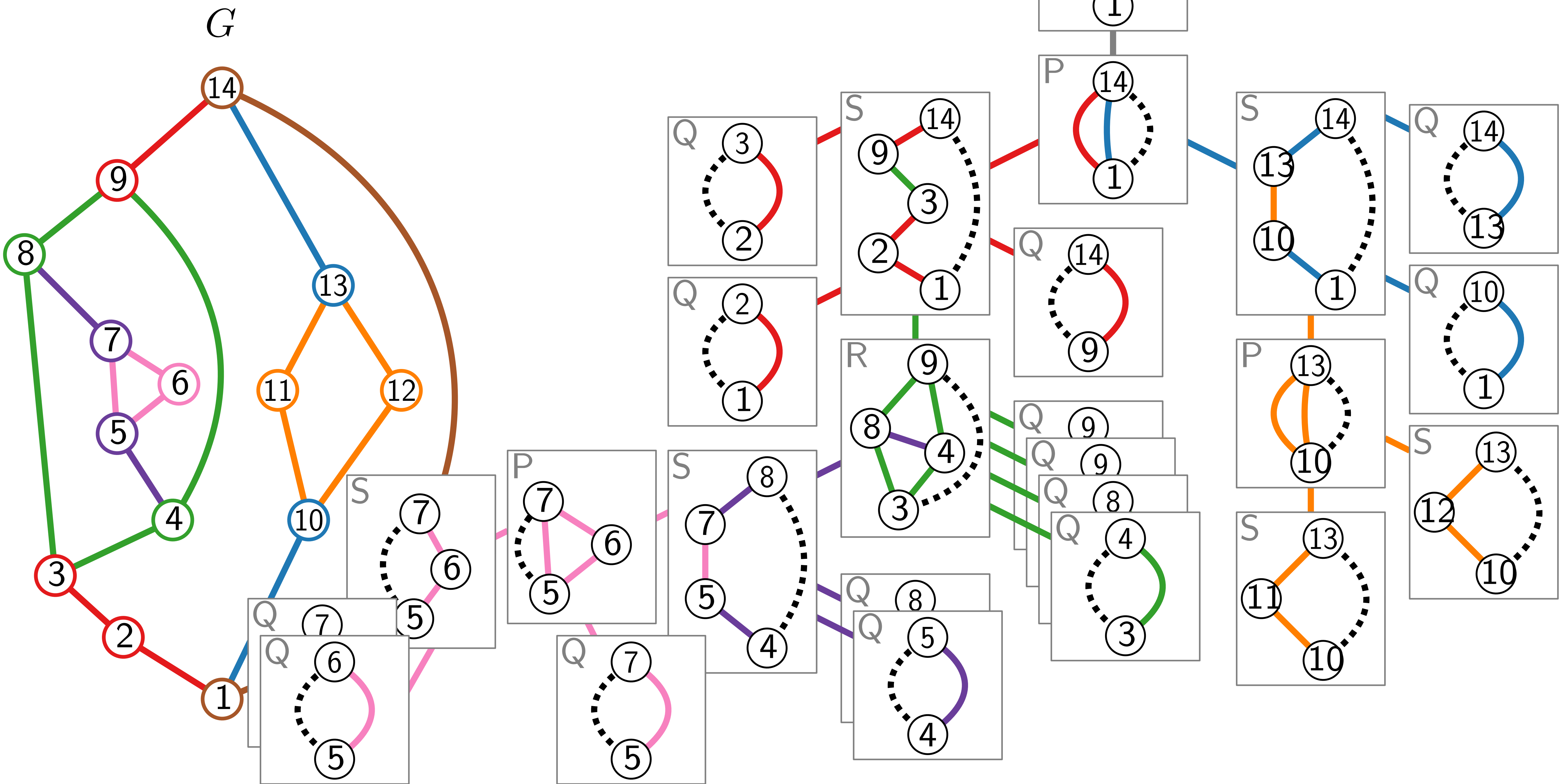

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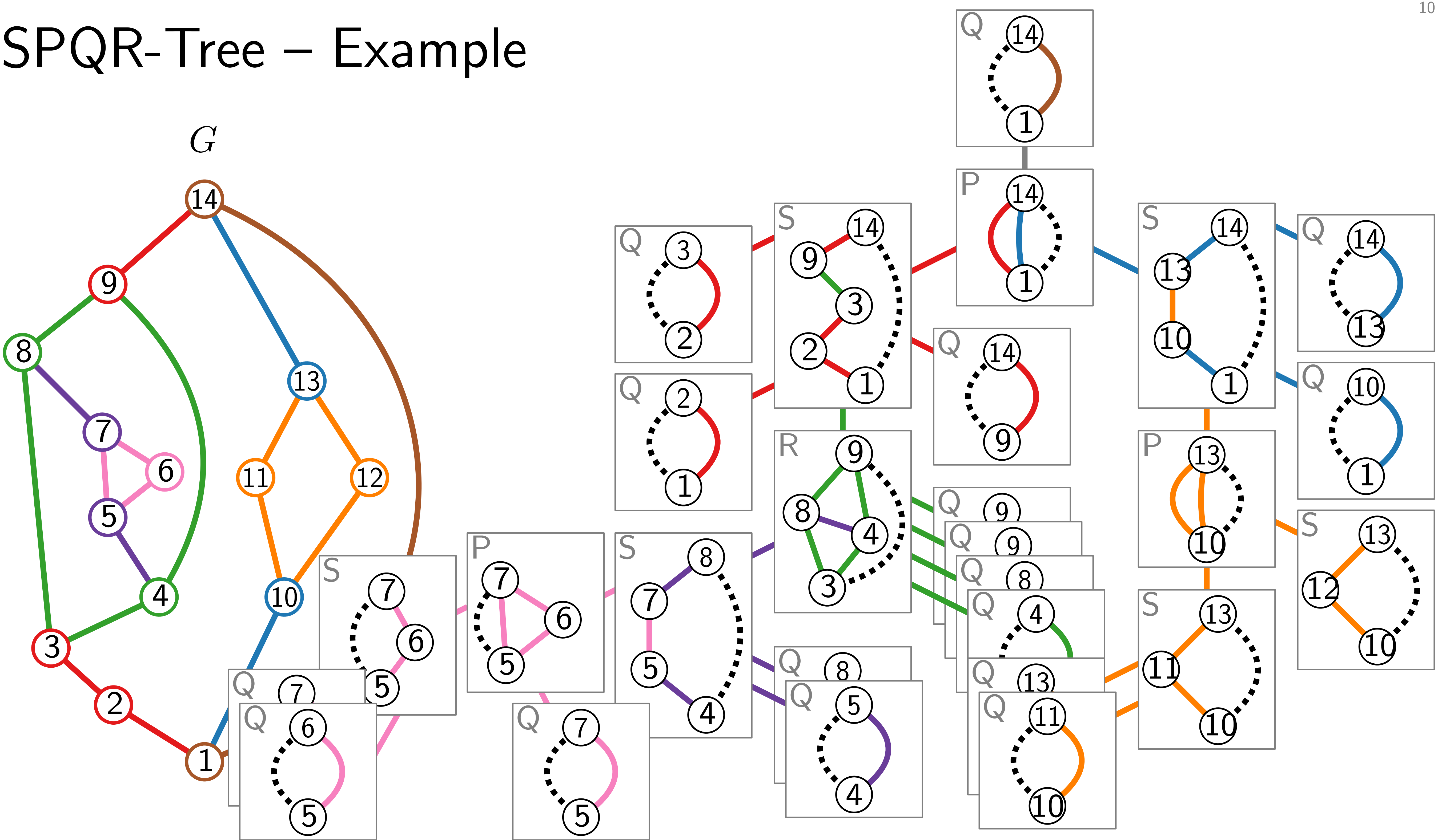

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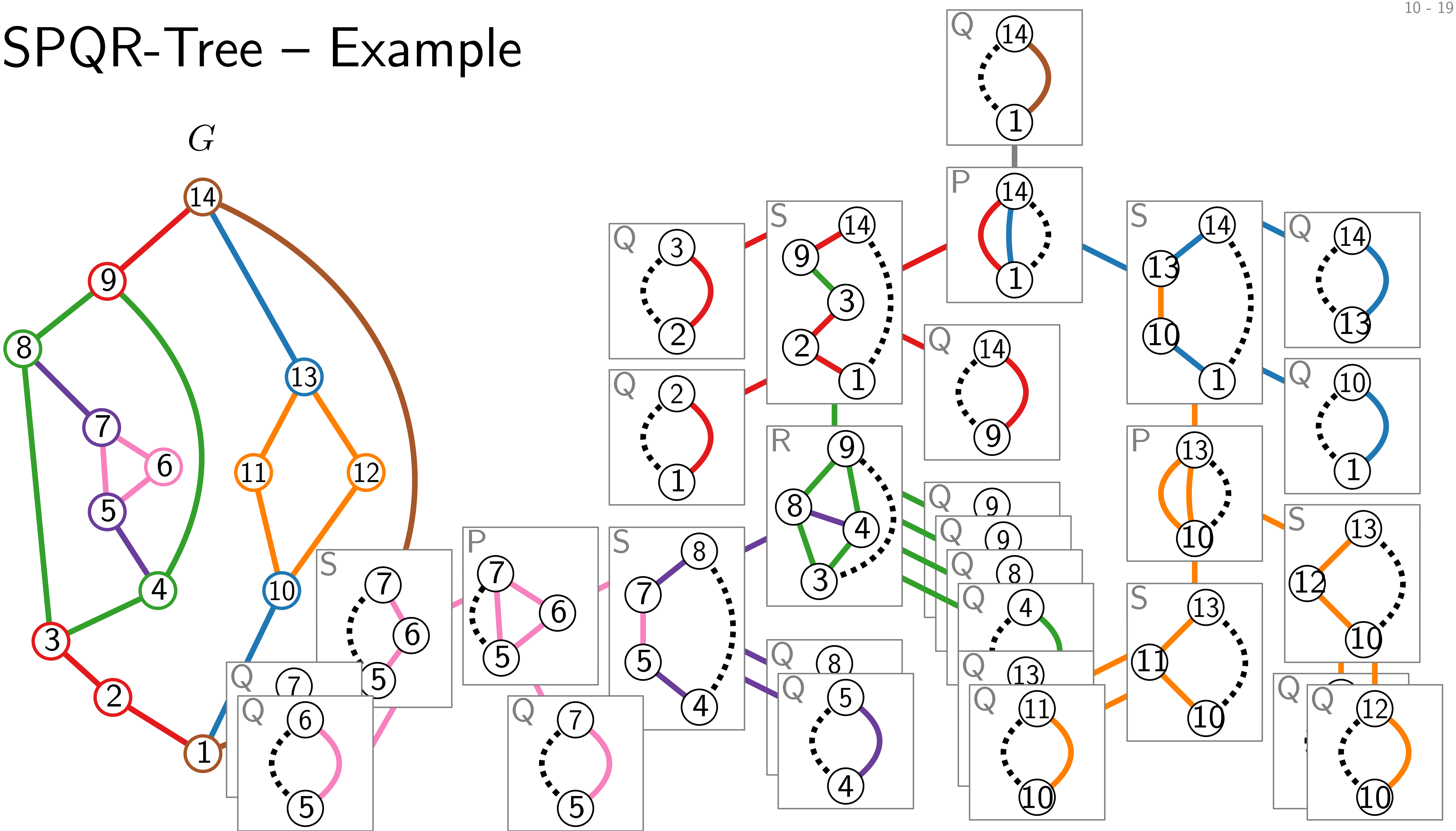

Representation Extension for st-Graphs

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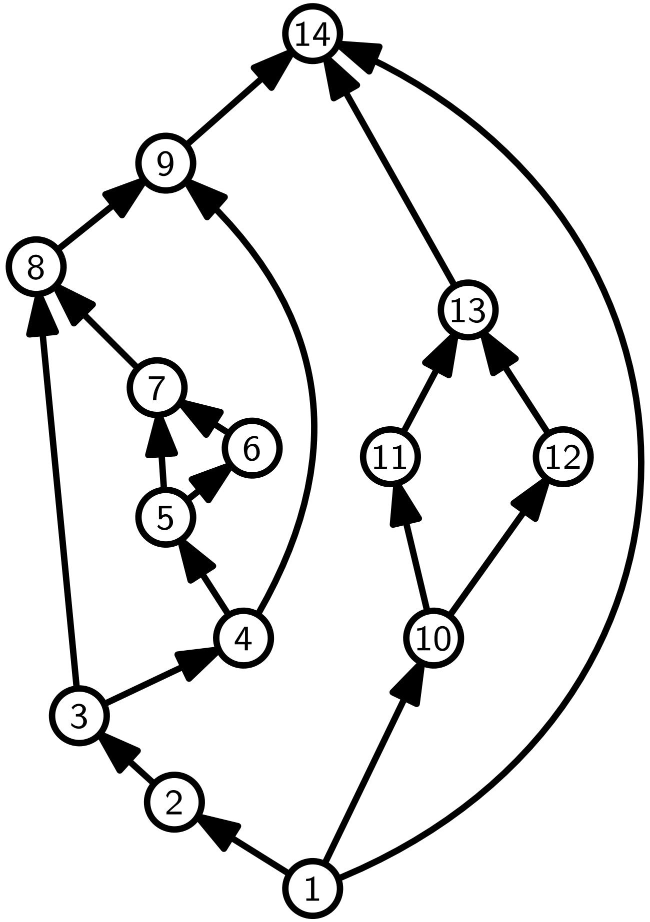
Rectangular ε-bar visibility representation extension can be solved in $\mathcal{O}(n^2)$ time for st-graphs.



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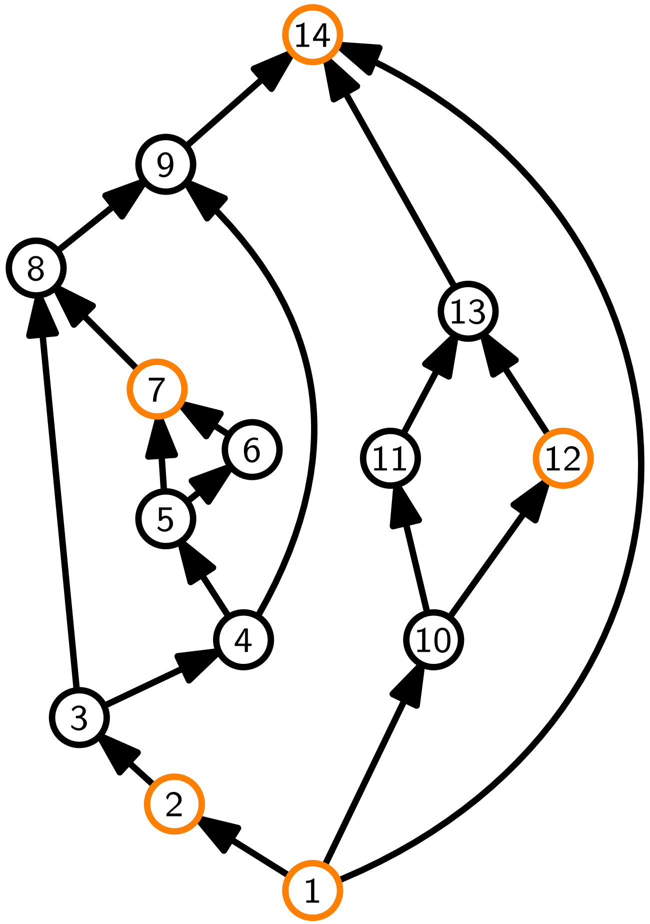

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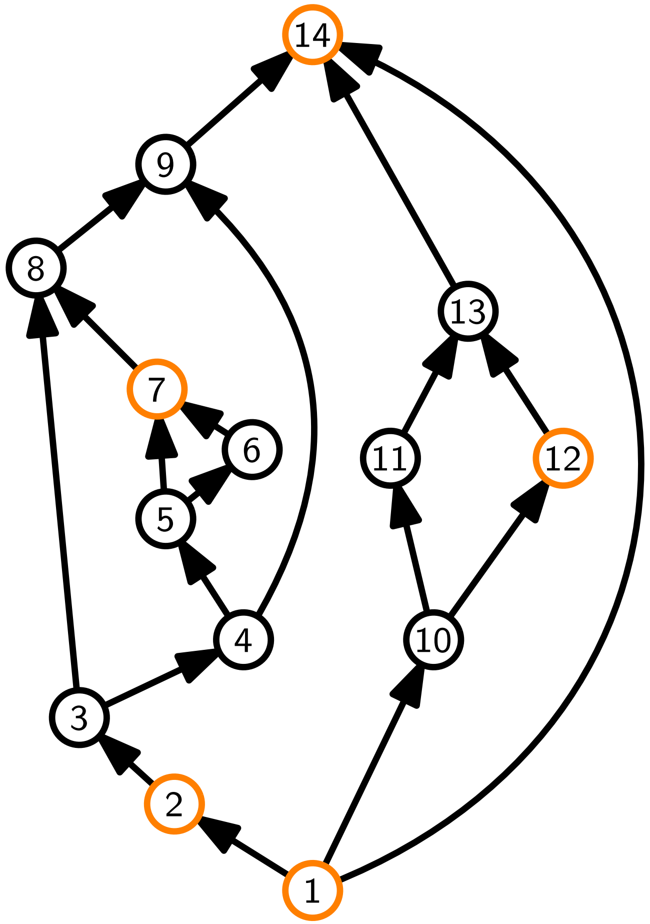

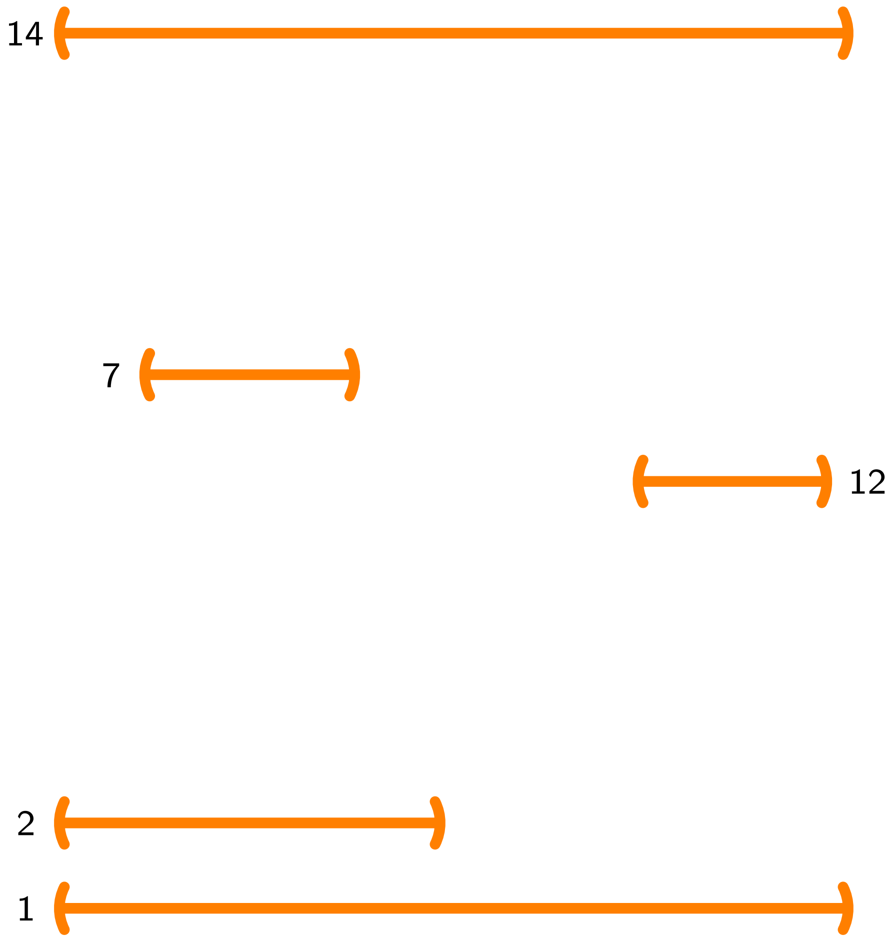

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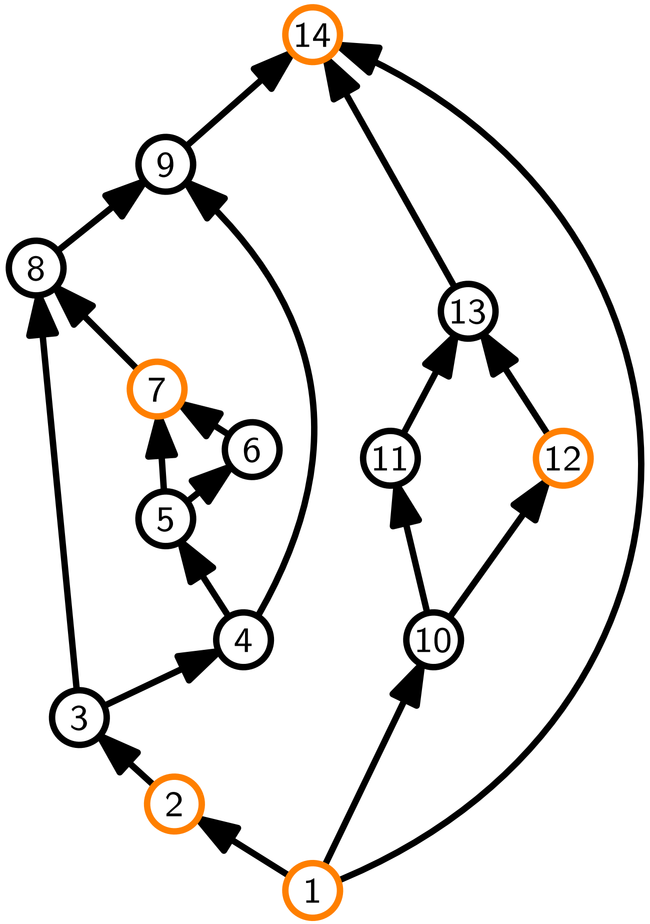

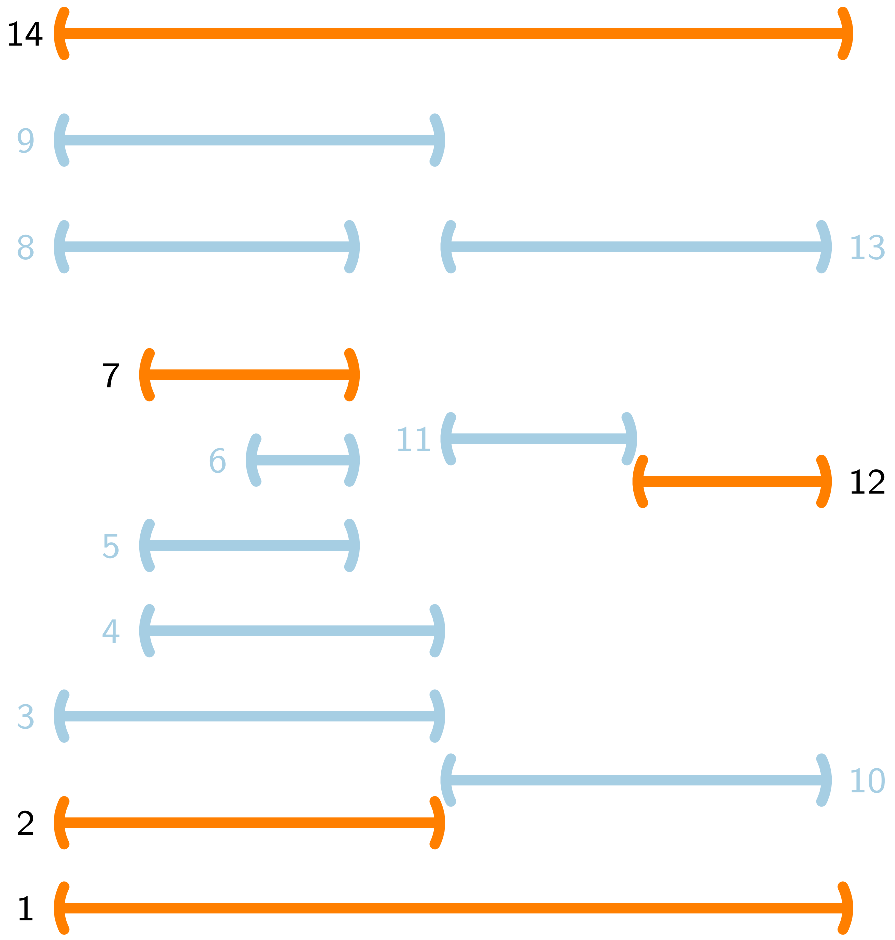

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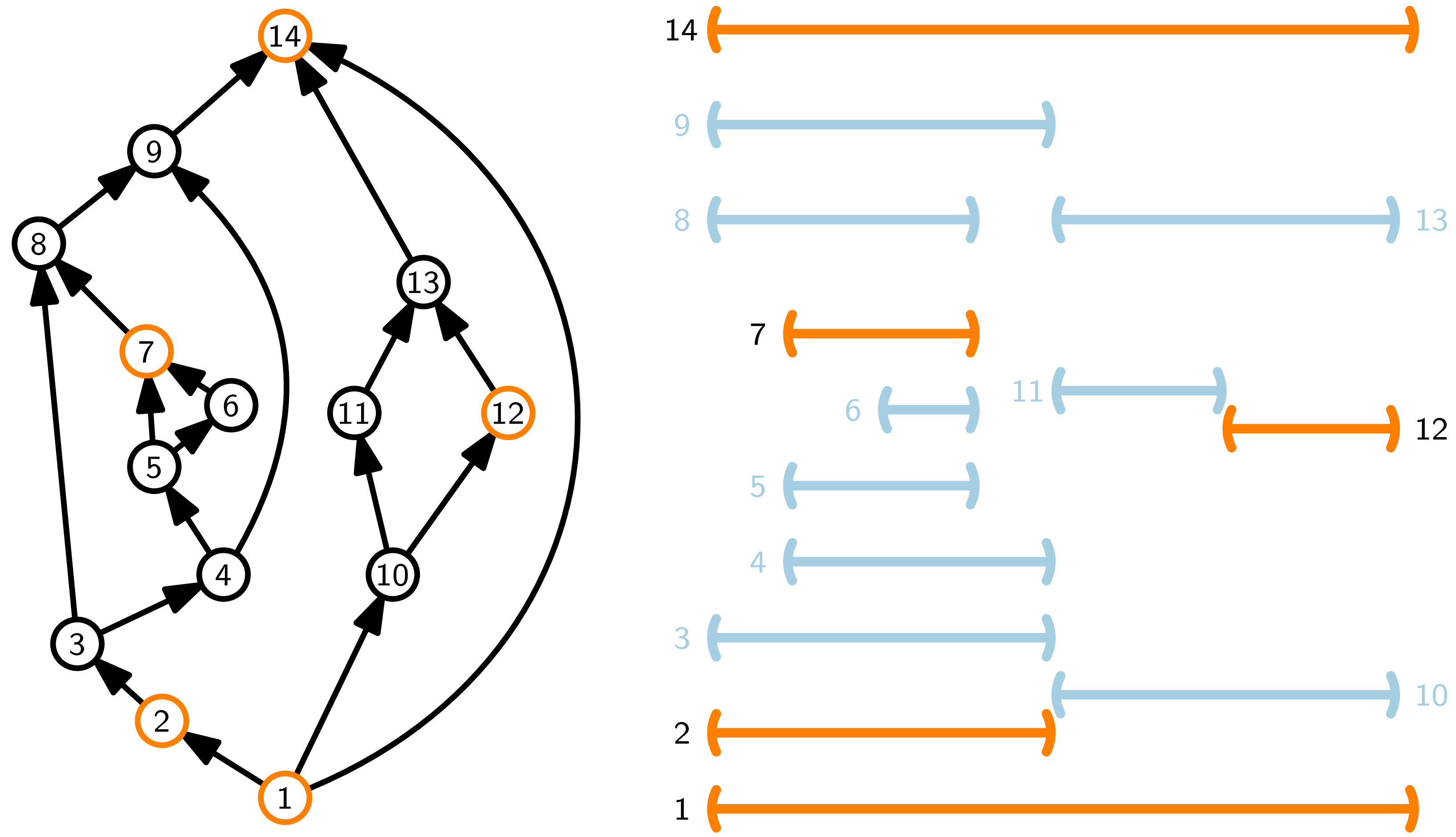

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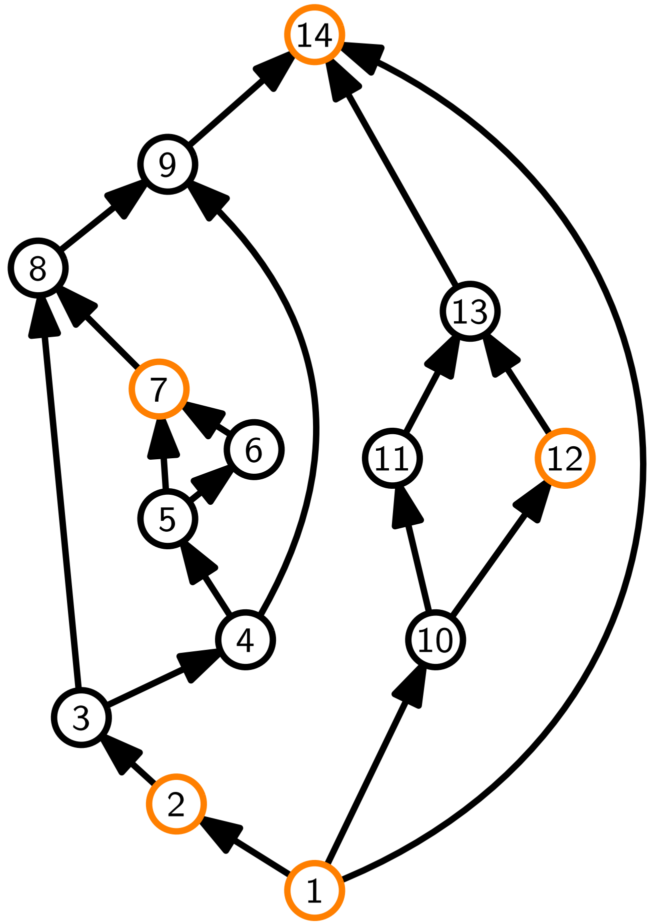

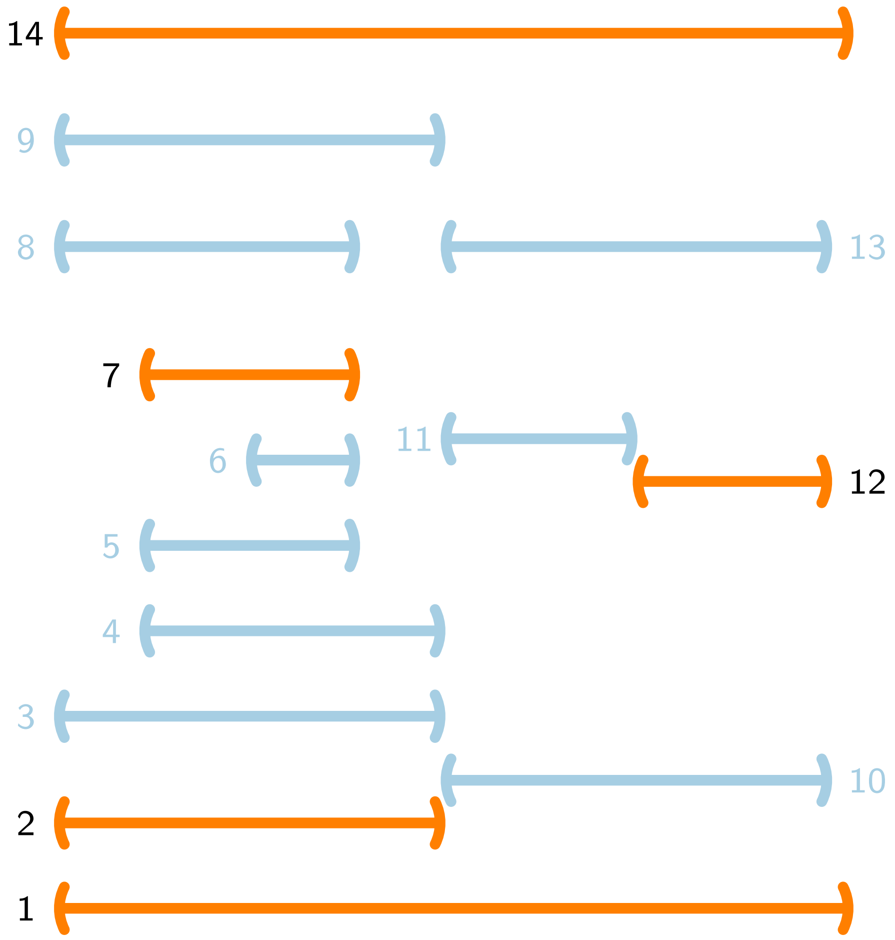

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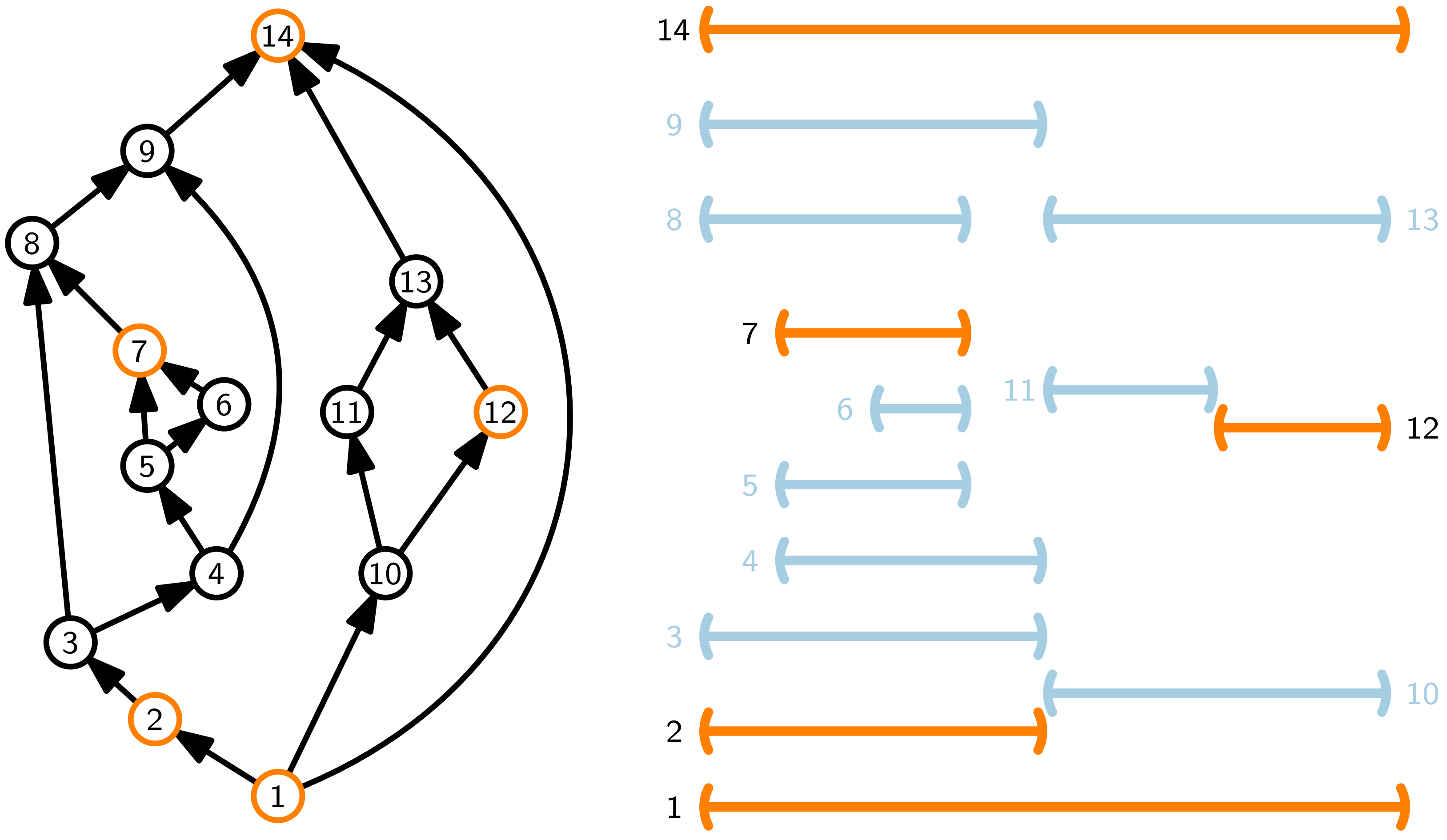

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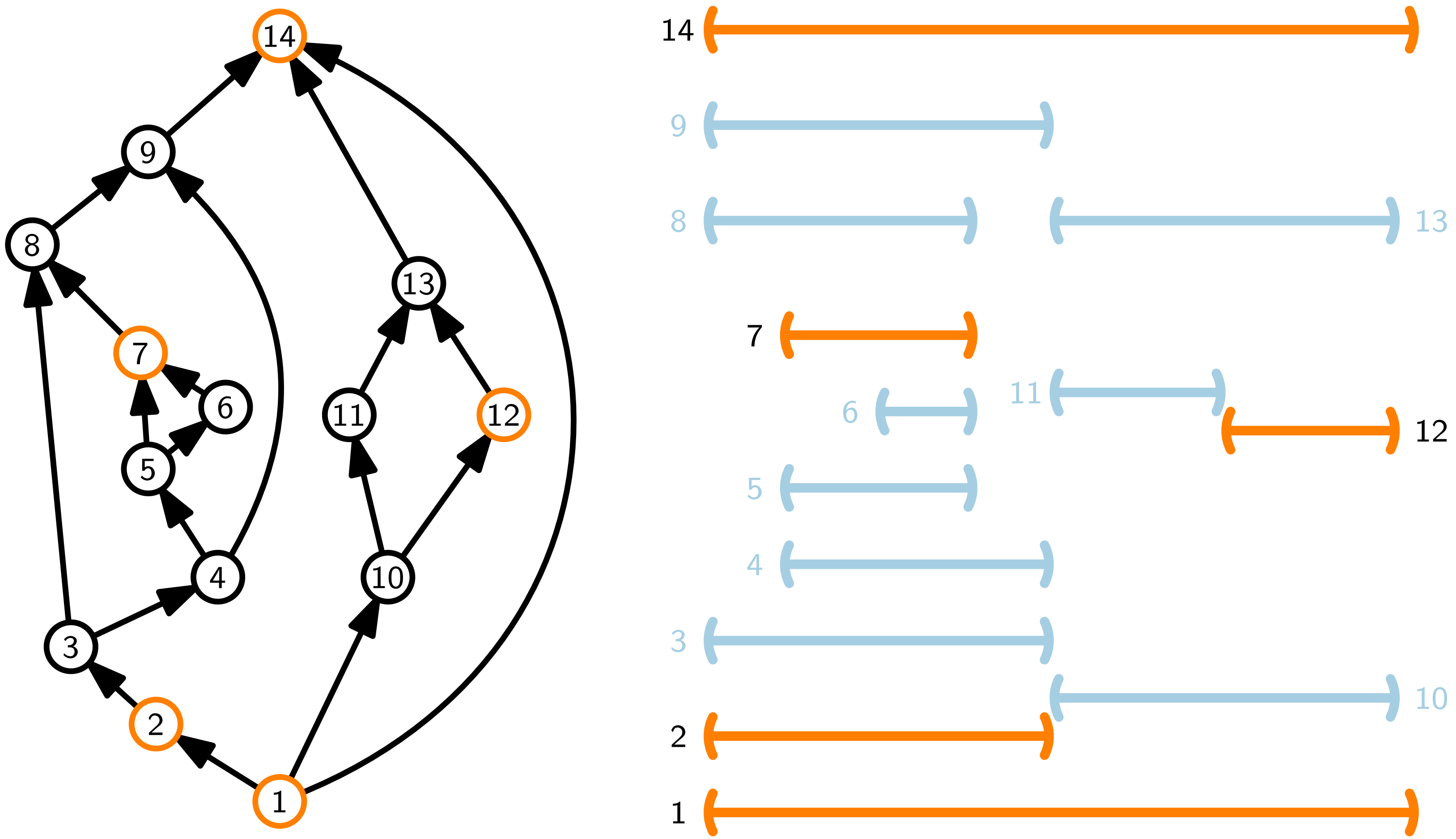

- Simplify problem via assumption regarding y-coordinates
- Exploit connection between SPQR-trees and rectangle tiling
- Solve problems for **S**-, **P**-, and **R**-nodes
- Dynamic program via structure of SPQR-tree

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Proof idea. The relative positions of **adjacent** bars must match the order given by y.
So, we can adjust the y-coordinates of any solution to be as in y by sweeping from bottom to top.

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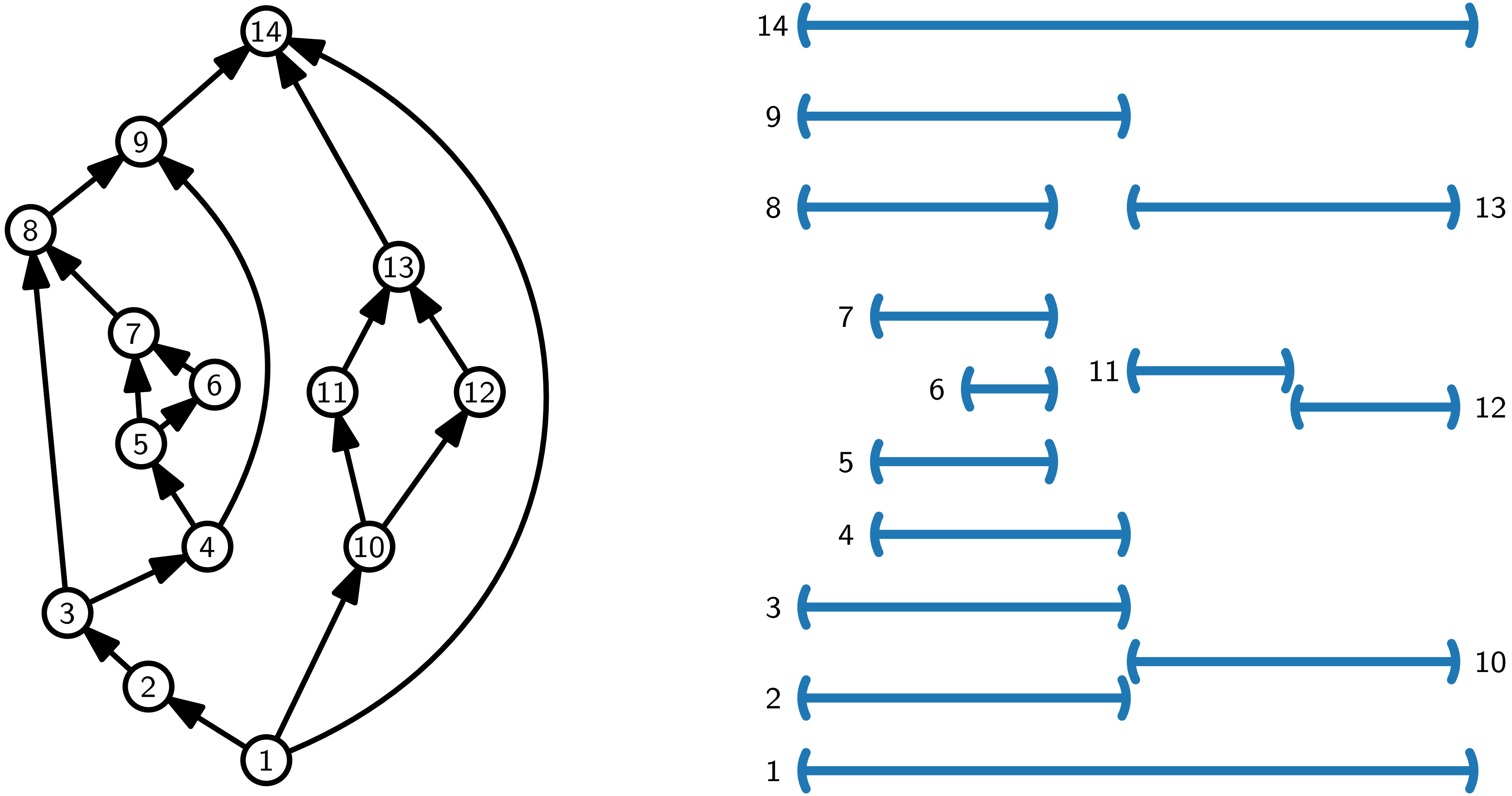
where the y-coordinates of the bars are as in y.

We can now assume that all y-coordinates are given!

Proof idea. The relative positions of **adjacent** bars must match the order given by y.
So, we can adjust the y-coordinates of any solution to be as in y by sweeping from bottom to top.

But Why Do SPQR-Trees Help?



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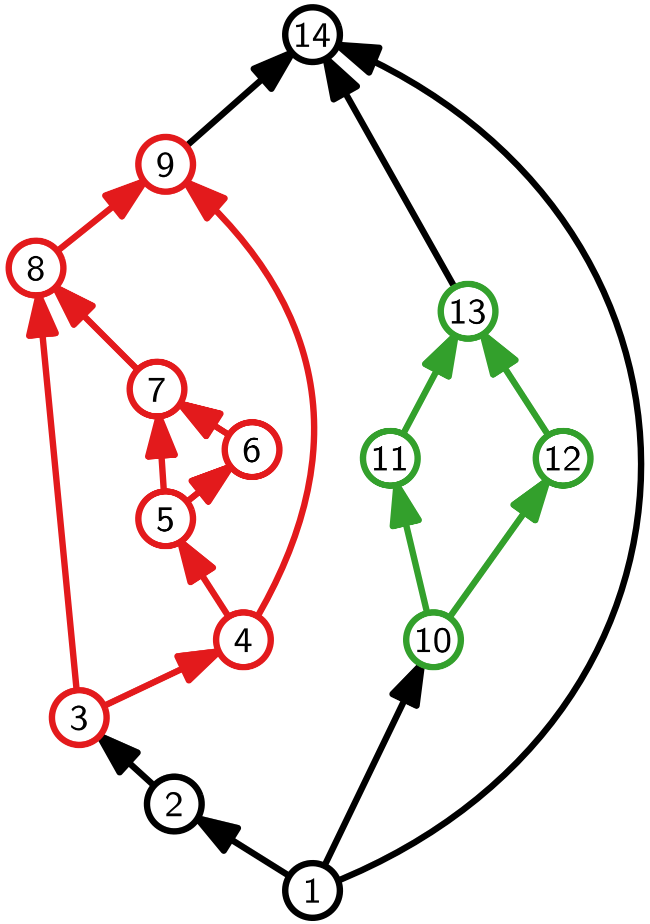

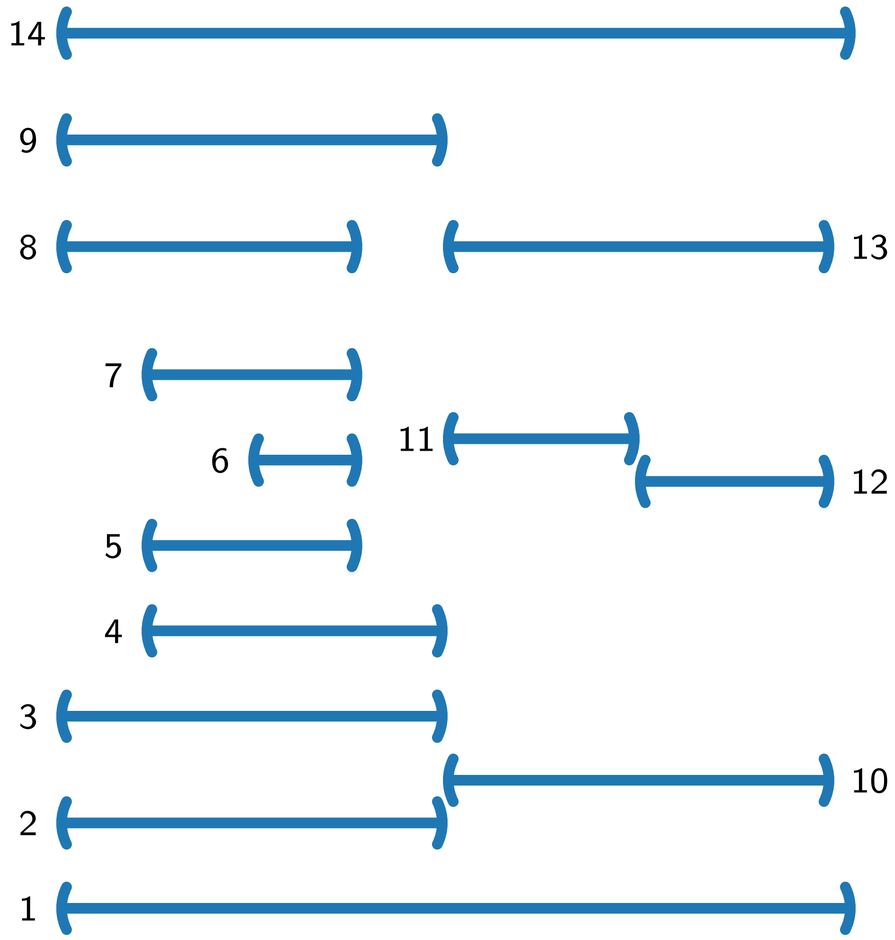

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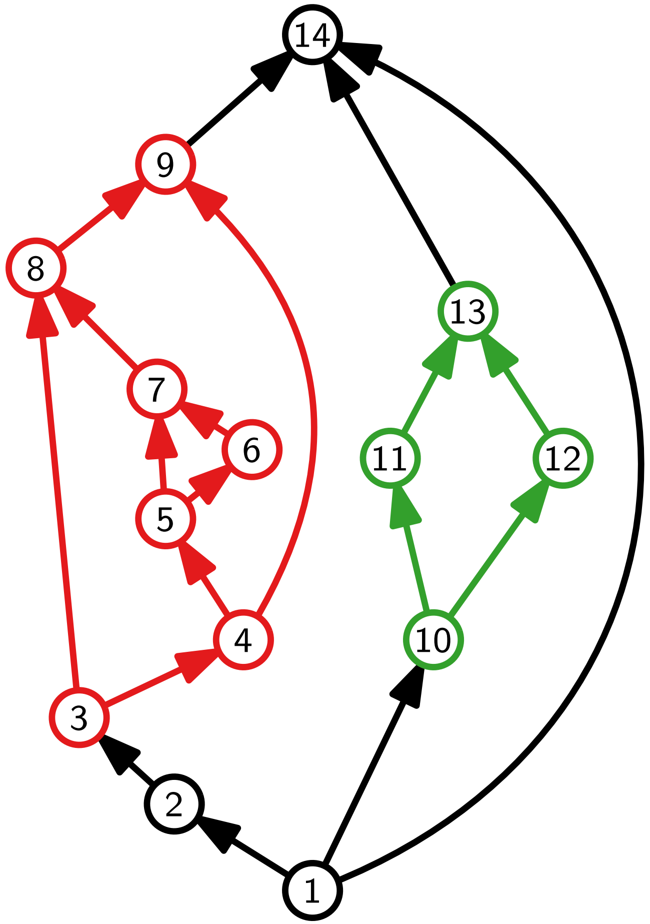

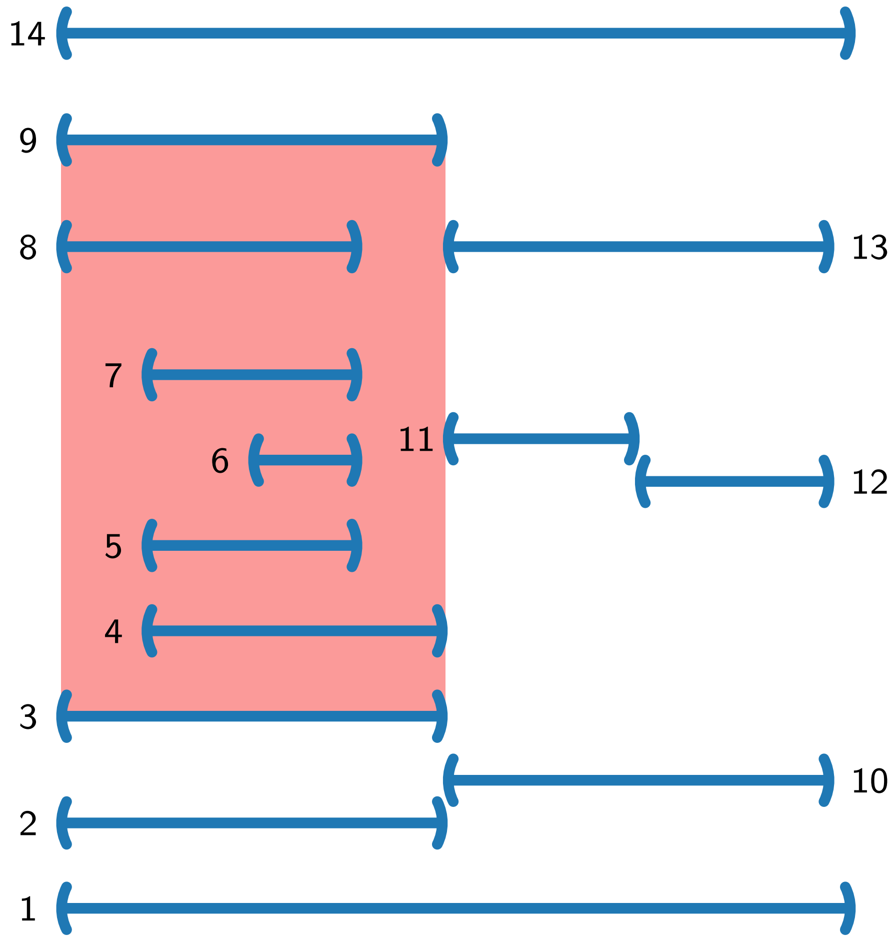

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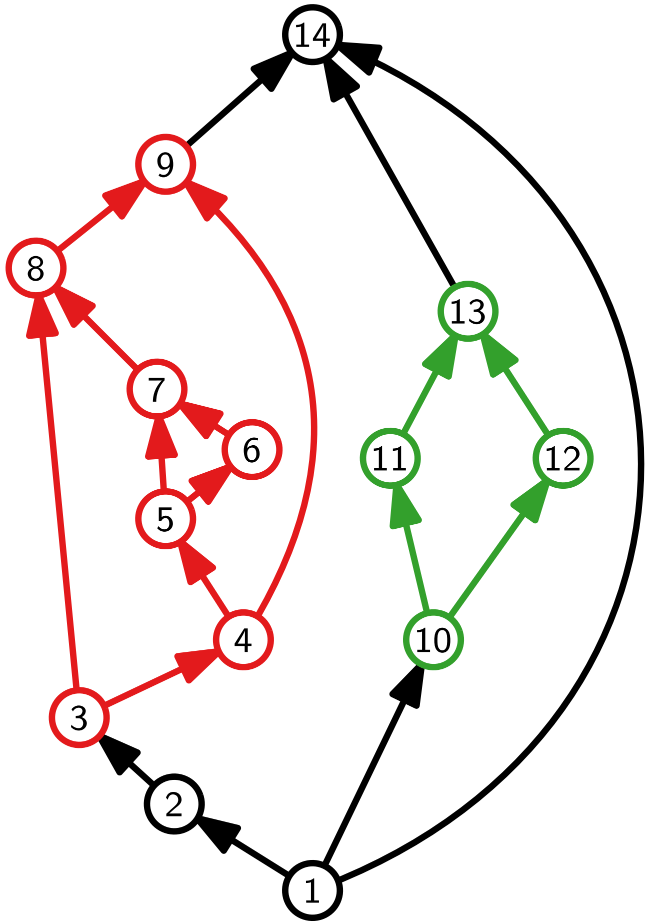

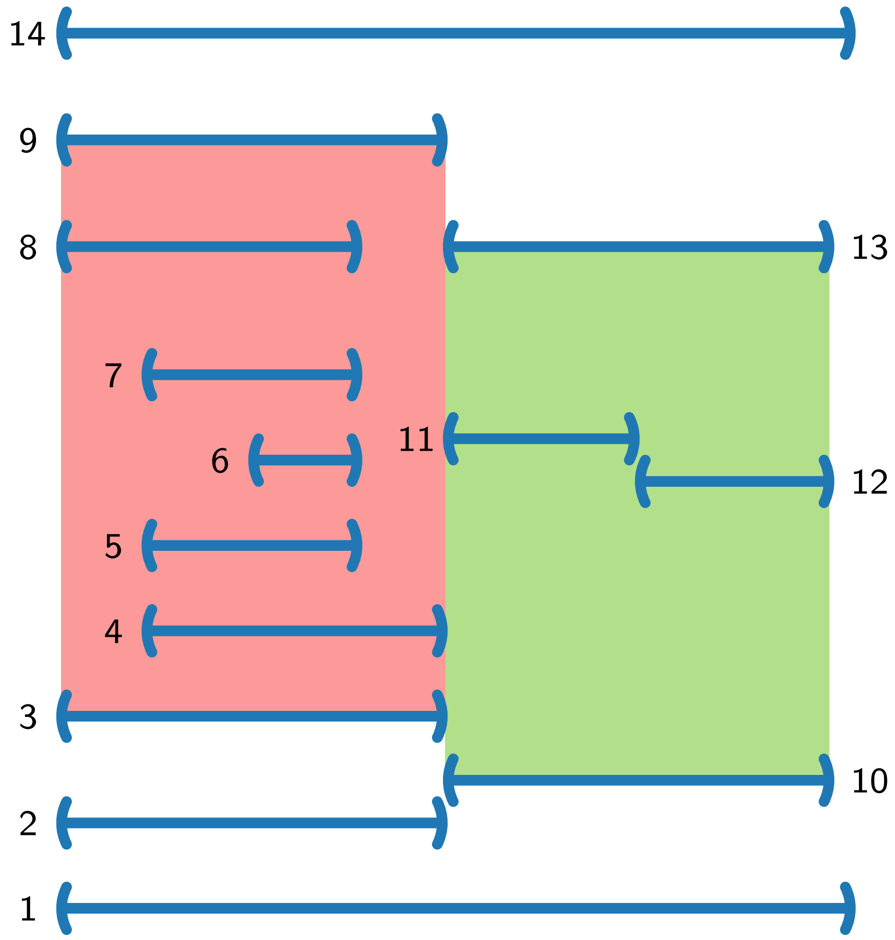

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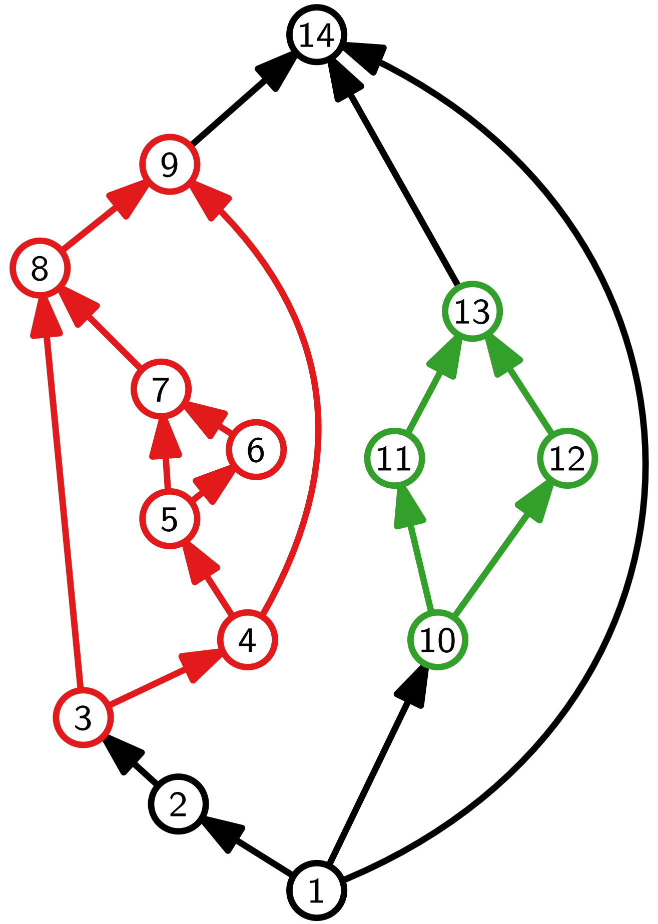
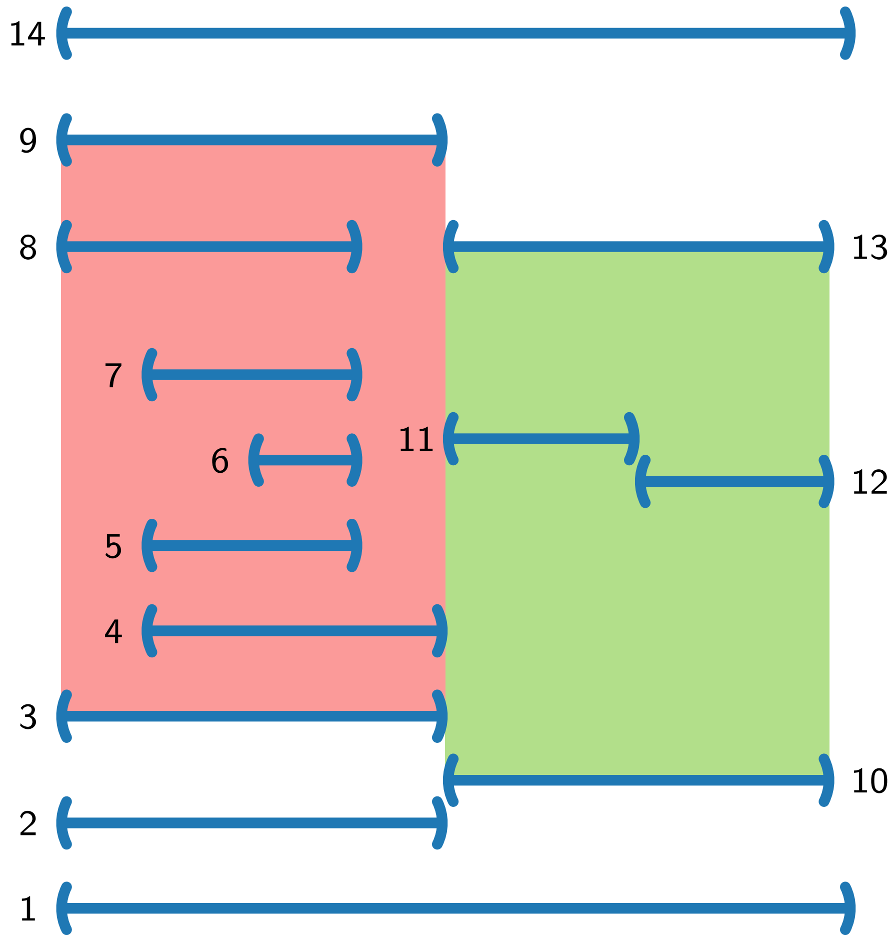
Lemma 2.
The SPQR-tree of an st-graph G induces a recursive **tiling** of any ε-bar visibility representation of G.





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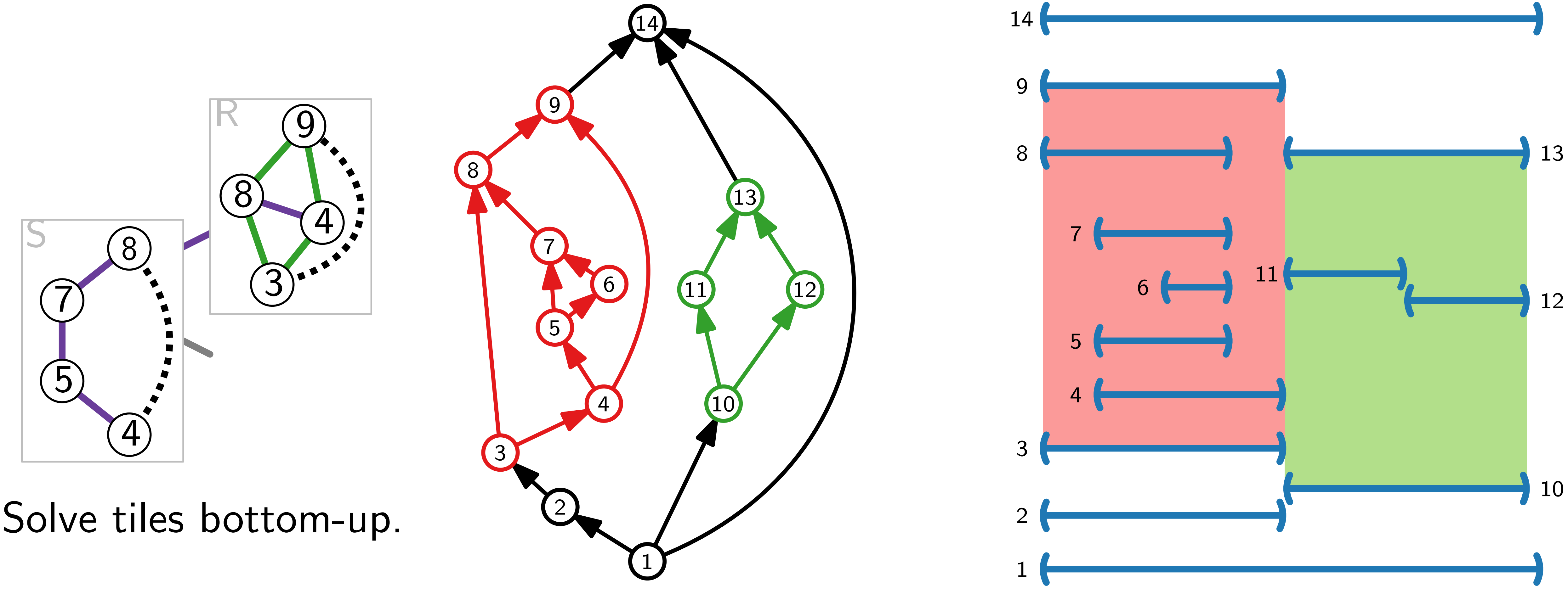
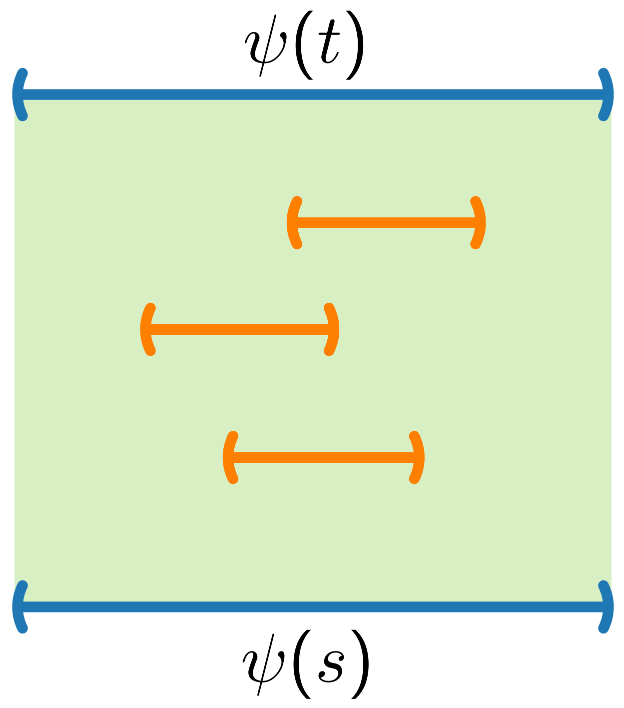


Solve tiles bottom-up.

Tiles

Convention. **Orange** bars are from the given partial representation.



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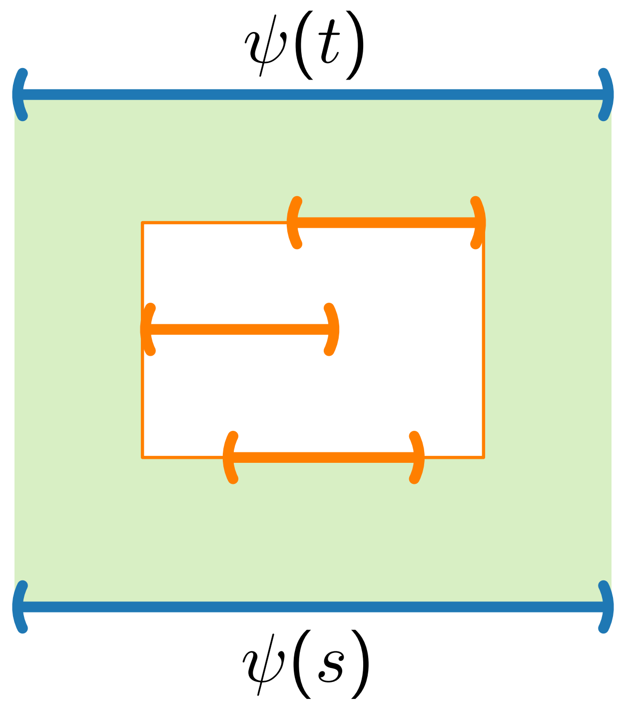

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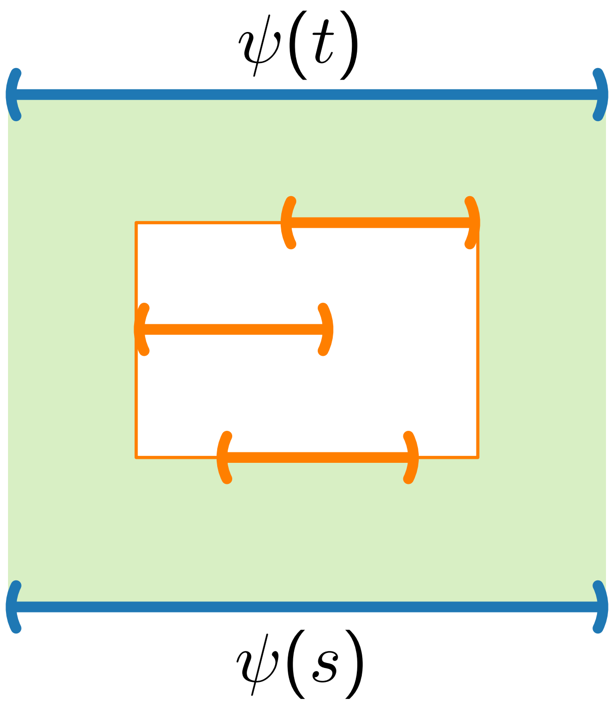
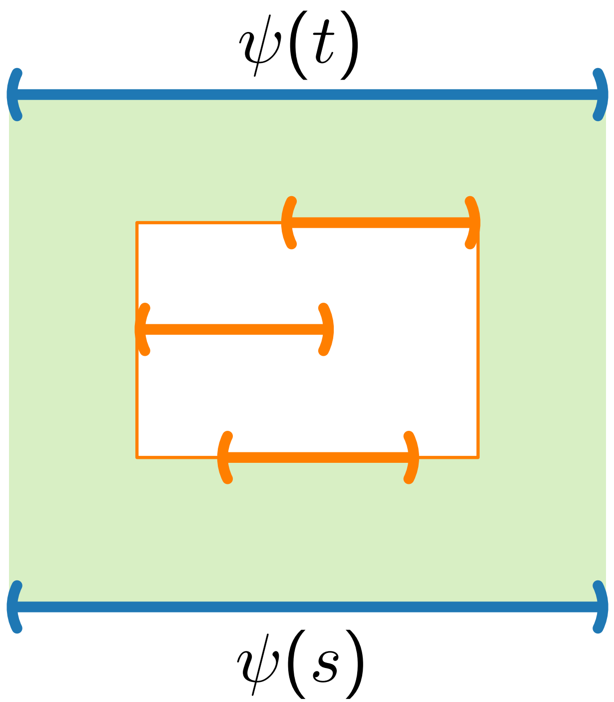


Observation.
The bounding box (tile) of any solution ψ **contains** the bounding box of the partial representation.

Tiles

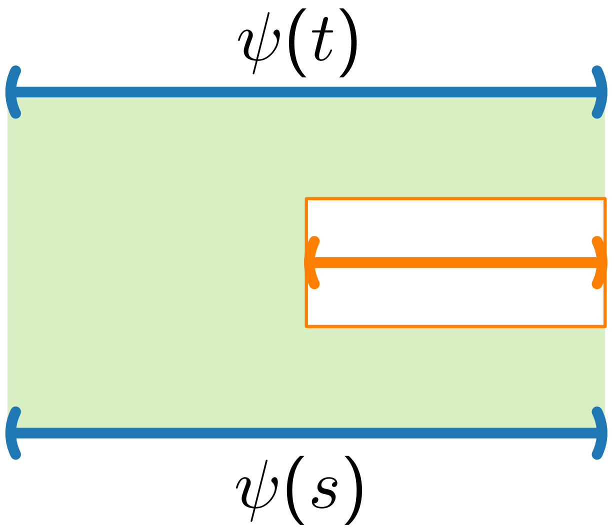
Convention. **Orange** bars are from the given partial representation.



Observation.
The bounding box (tile) of any solution ψ **contains** the bounding box of the partial representation.

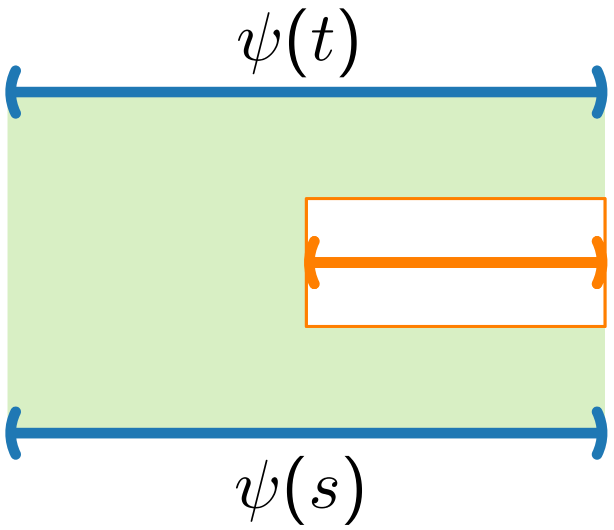
How many different **types** of tiles are there?

Types of Tiles



- Right **F**ixed
- Left **L**oose

Types of Tiles



- Right **F**ixed
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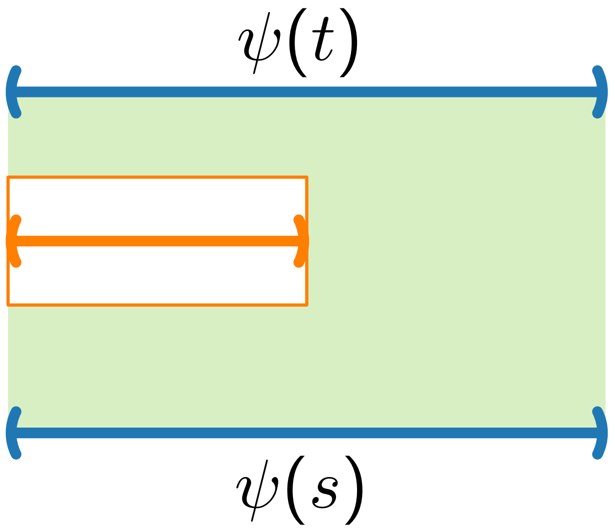
- Left **F**ixed
- Right **L**oose



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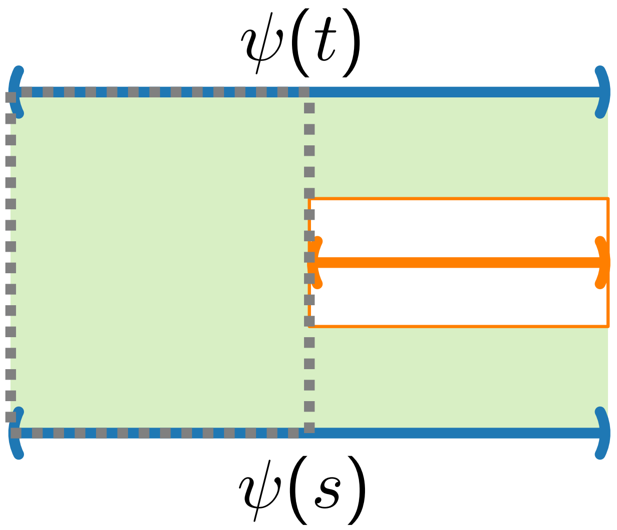

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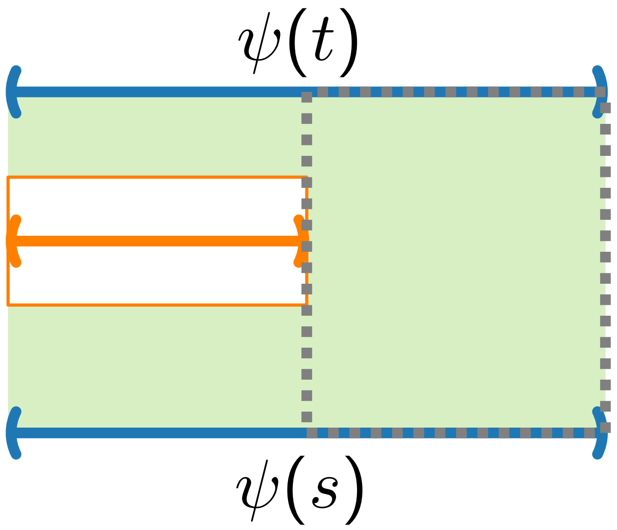

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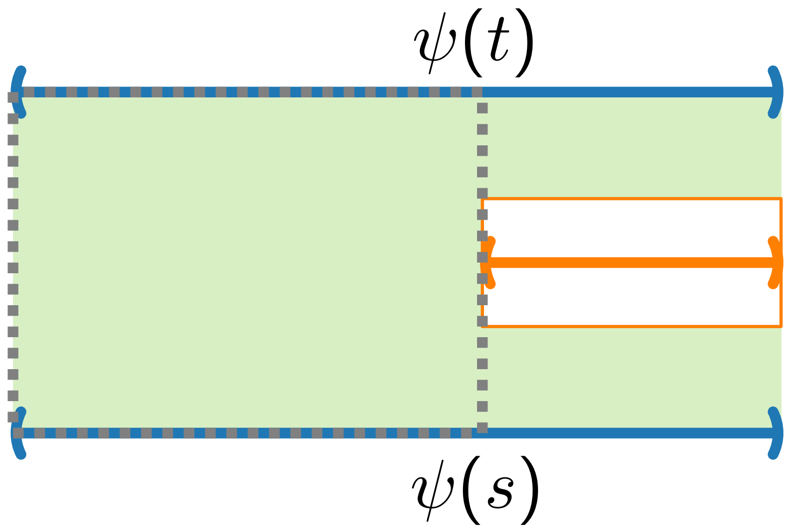

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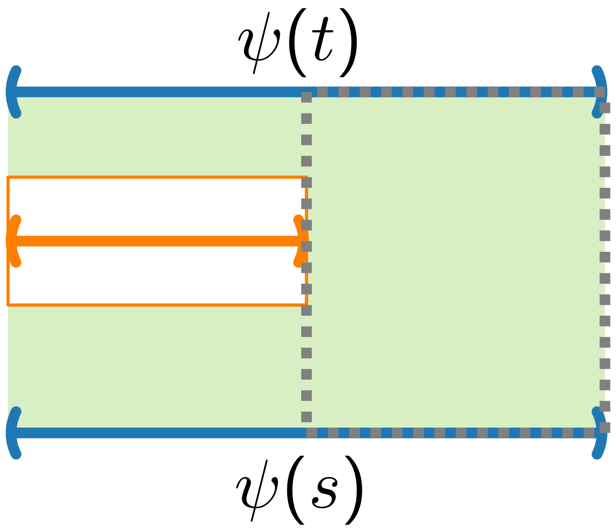

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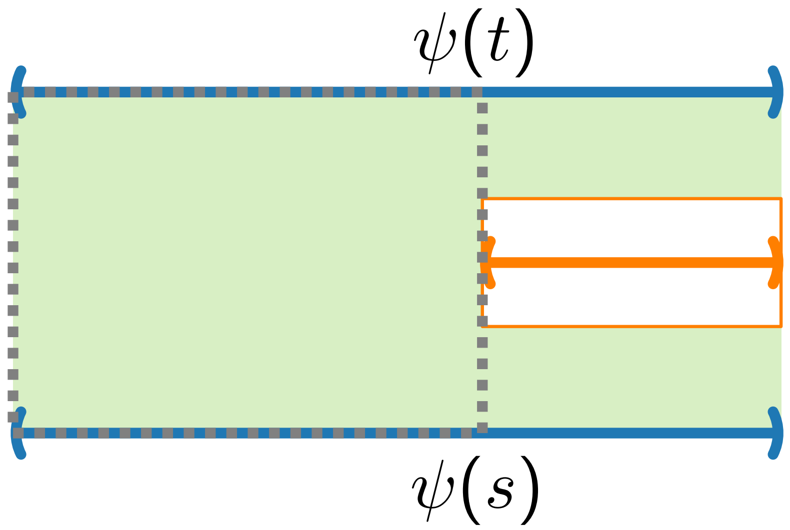

- Right **F**ixed
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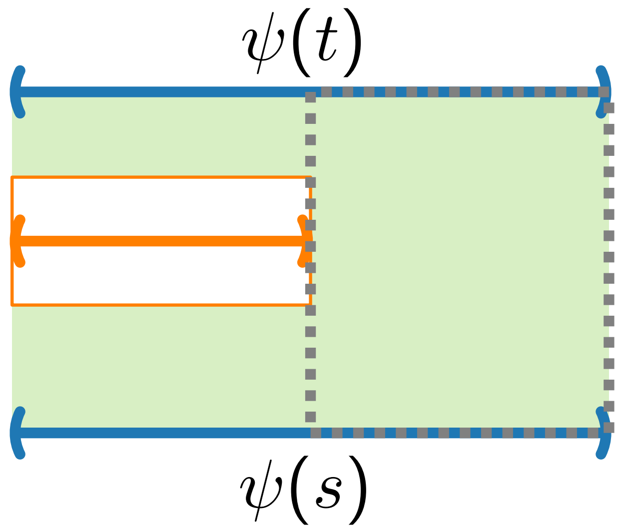
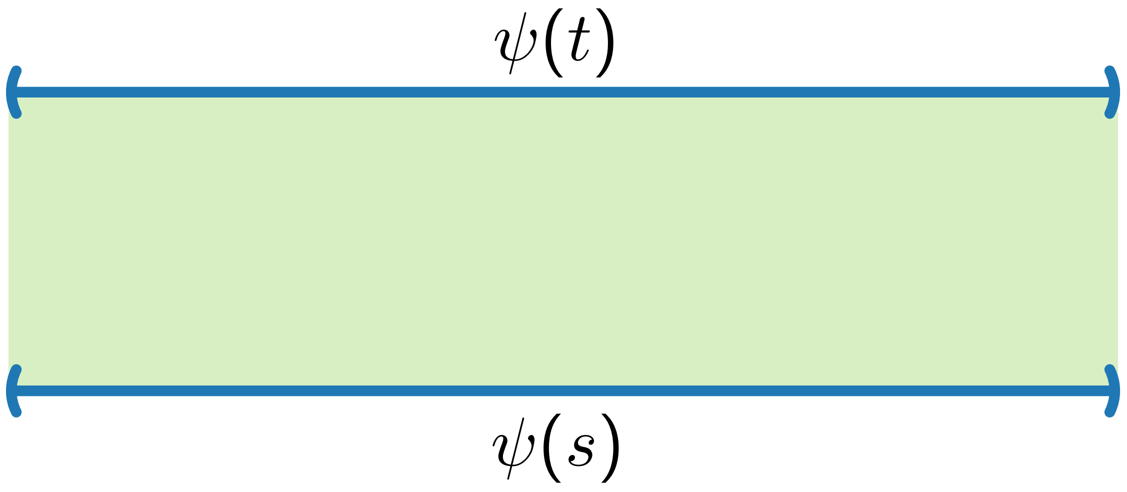
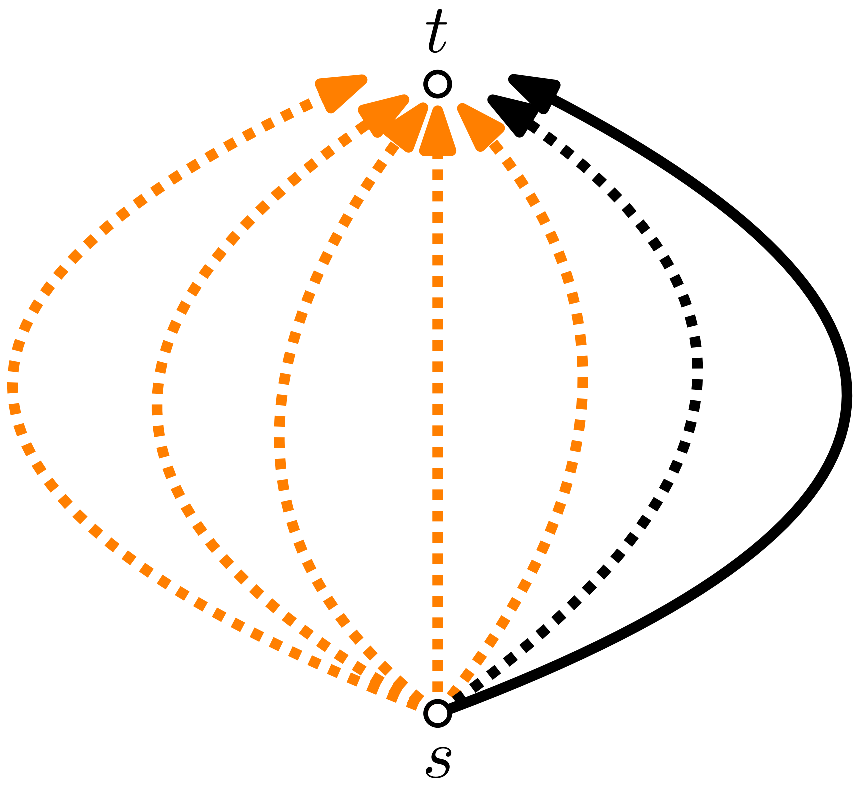


- Left **F**ixed
- Right **L**oose

Four different types: **FF, FL, LF, LL**

P-Nodes





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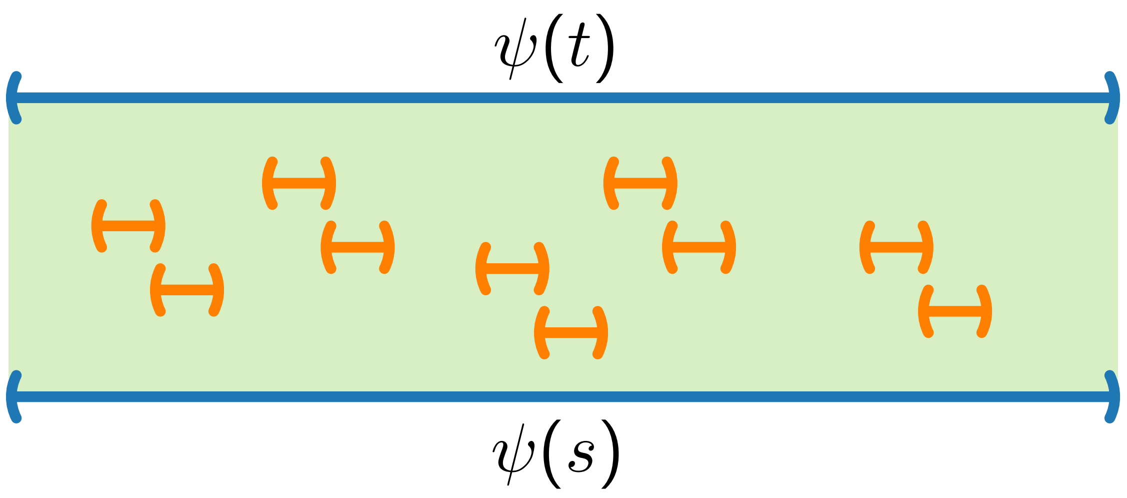

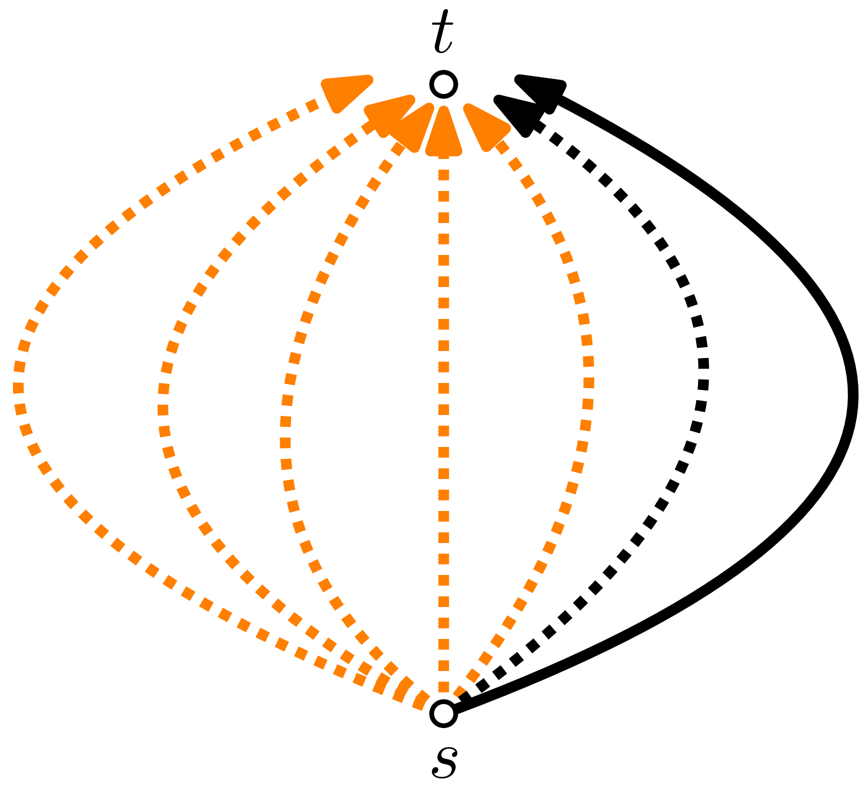

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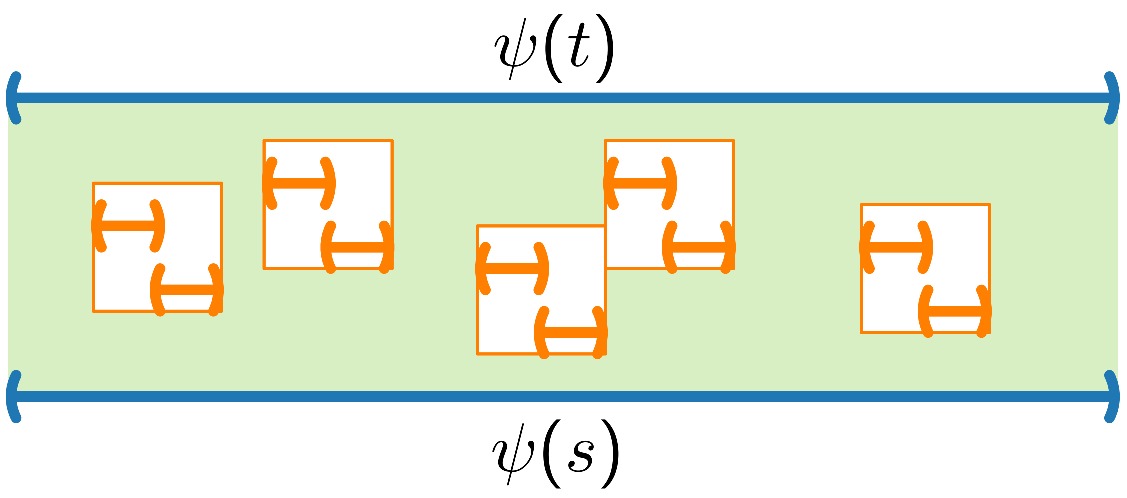

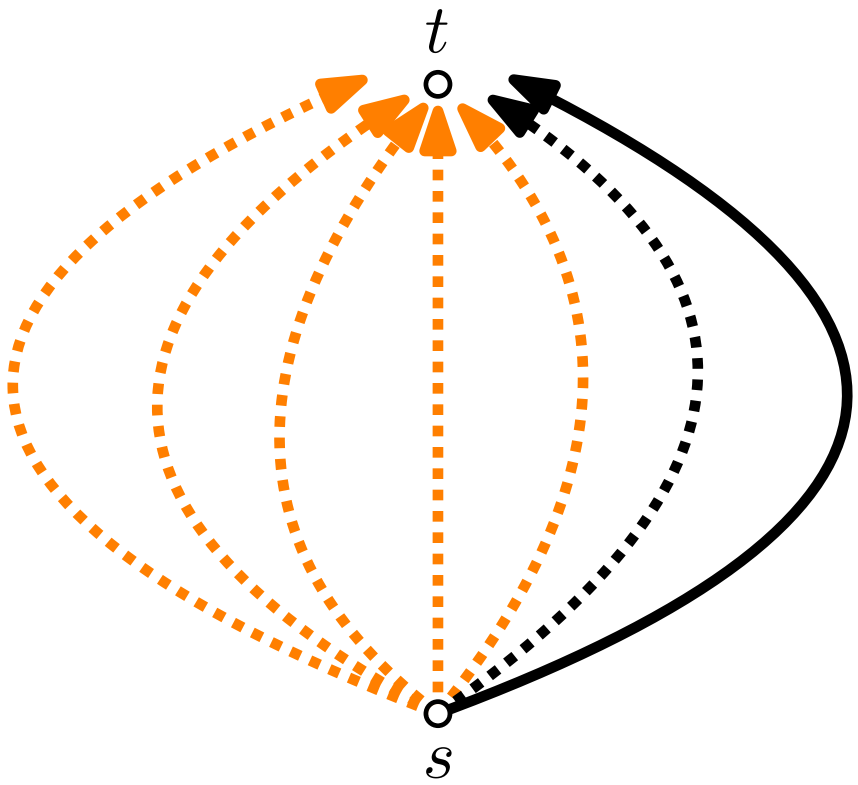

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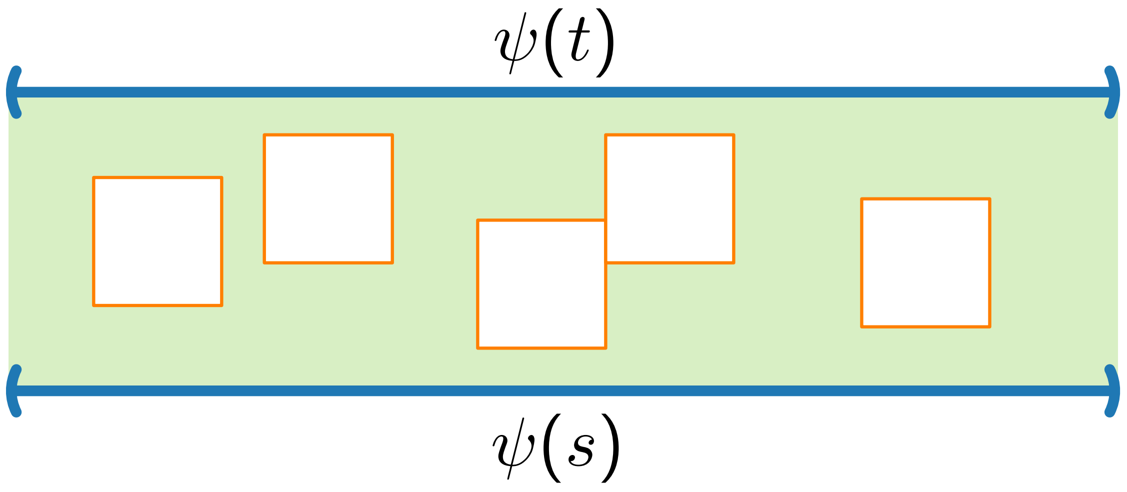

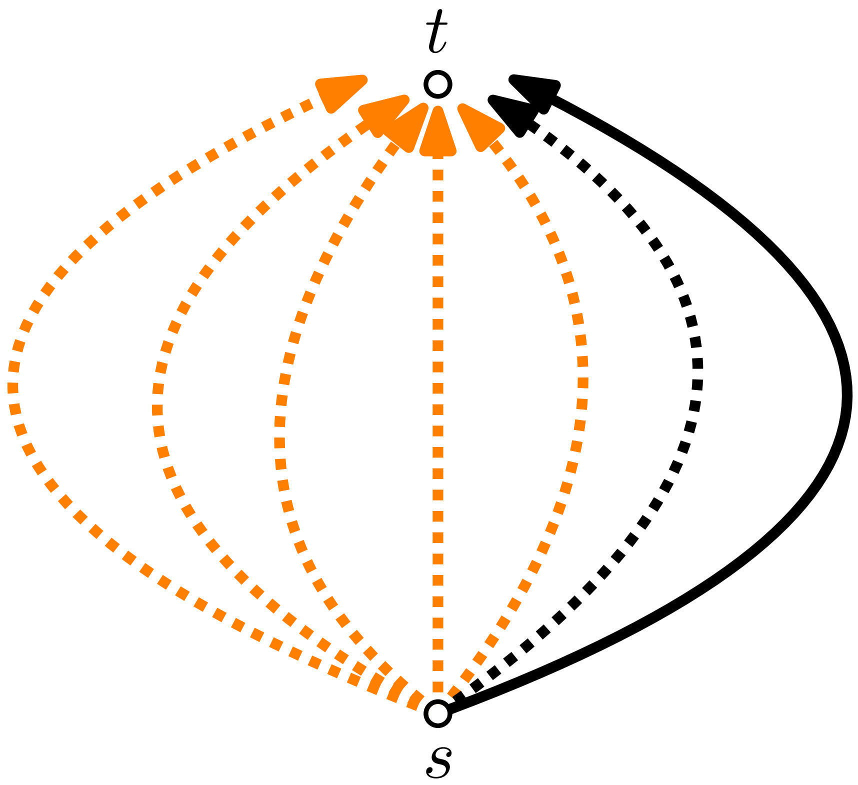

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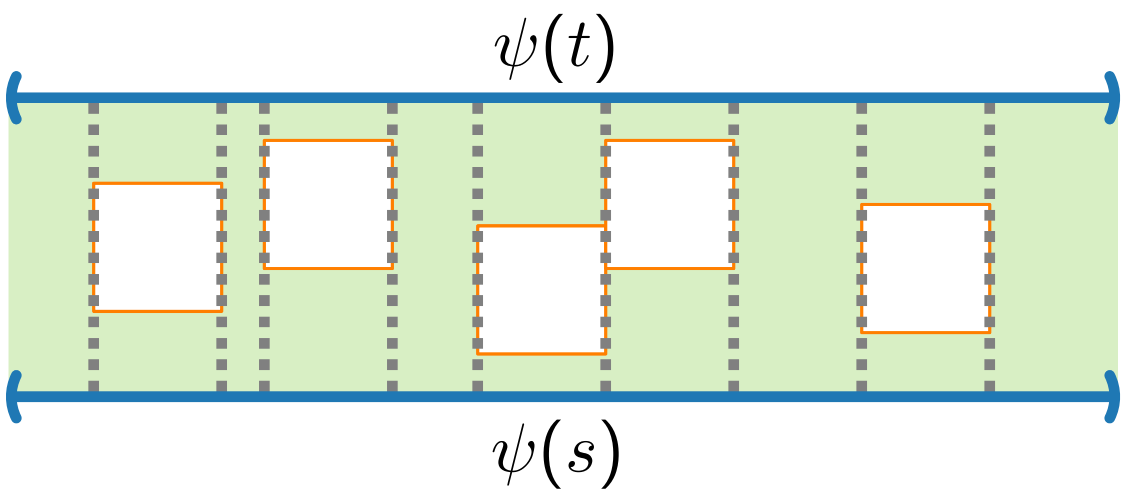

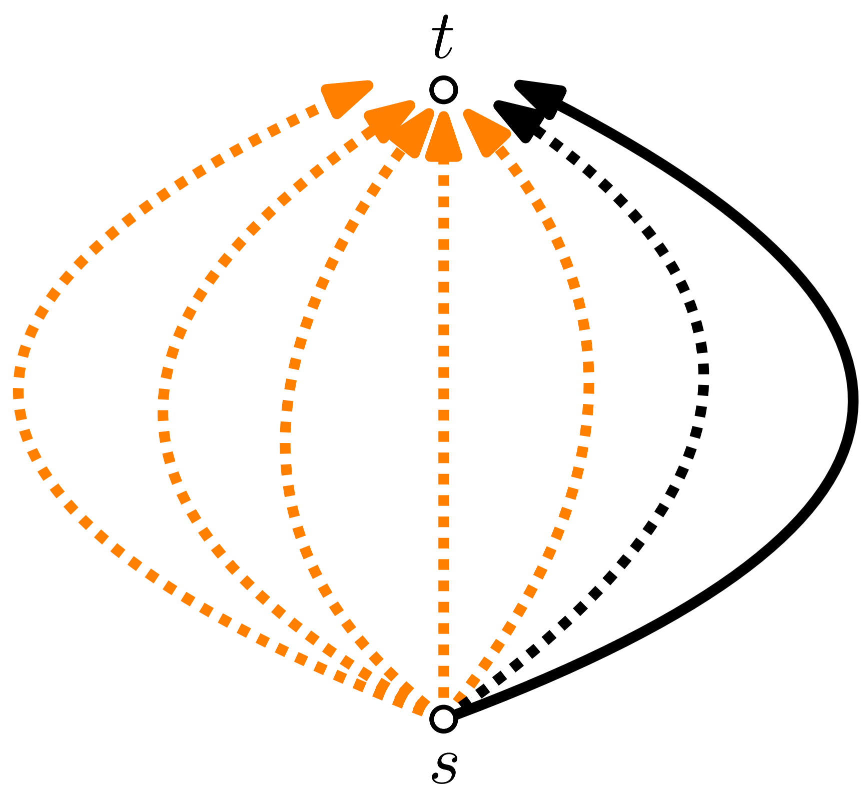

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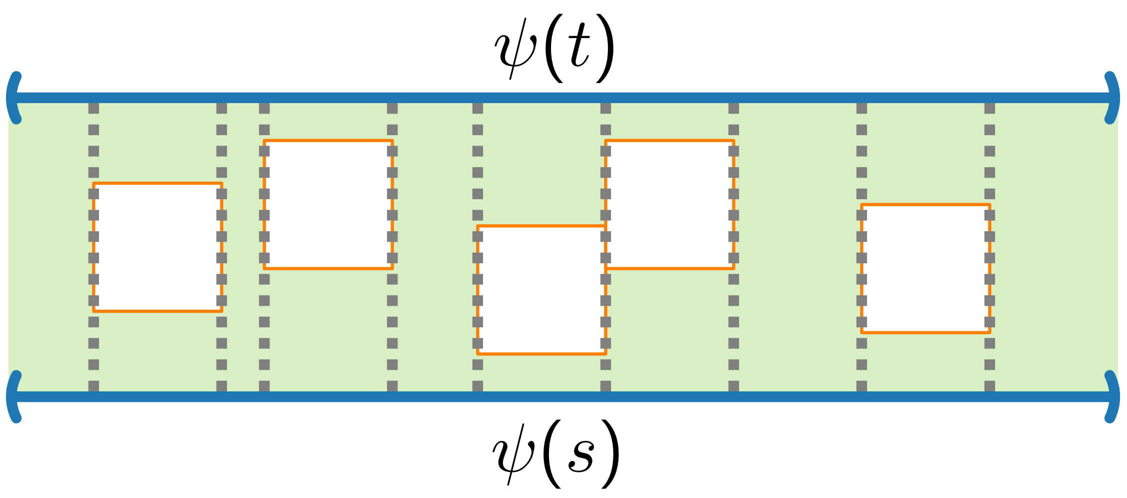
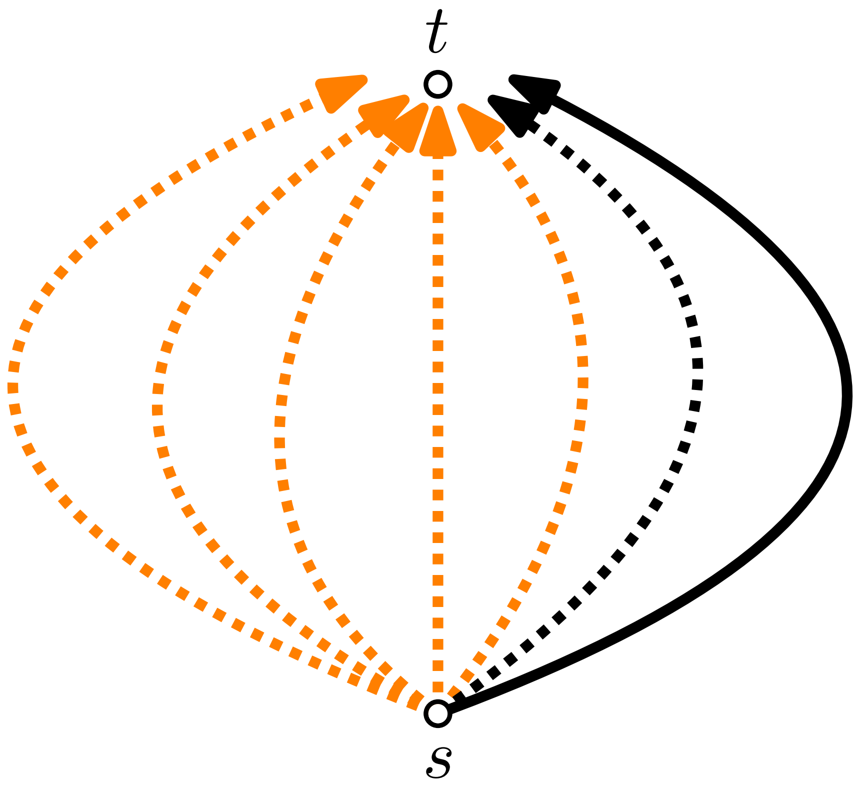


- Children of **P**-node with prescribed bars occur in given left-to-right order



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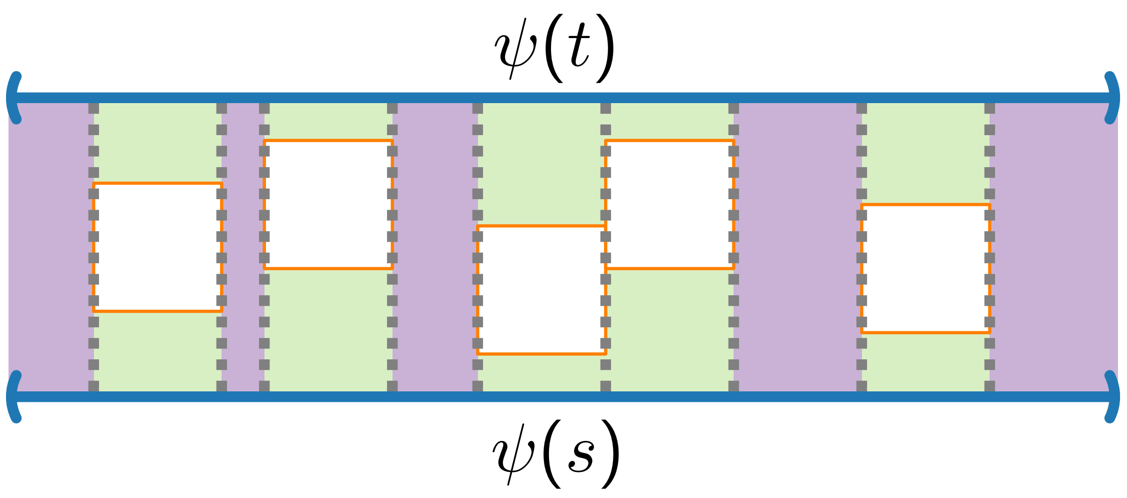

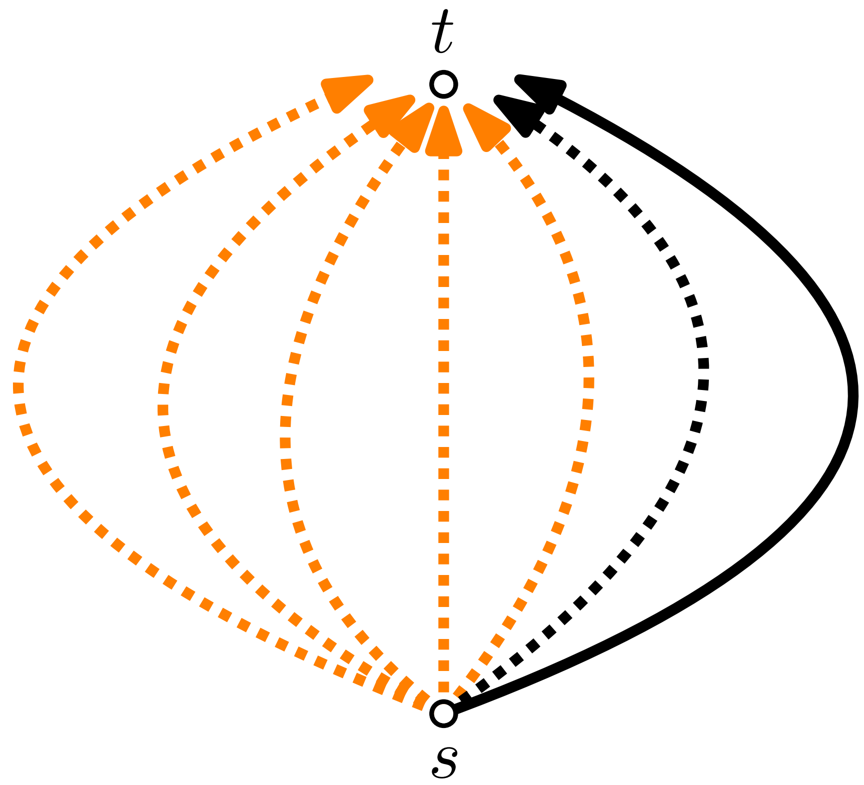

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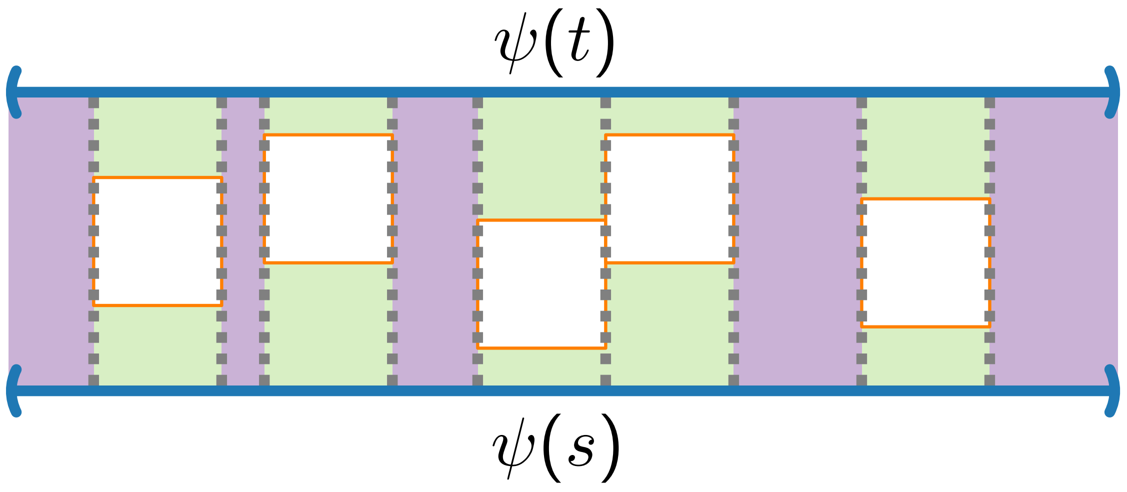
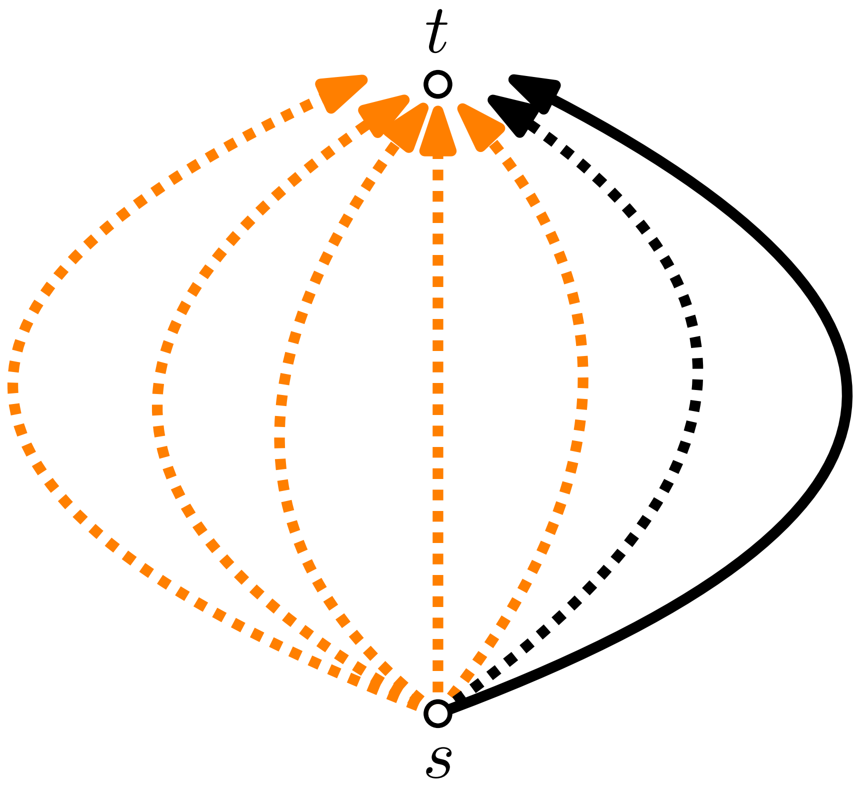
- But there might be some gaps. . .

P-Nodes





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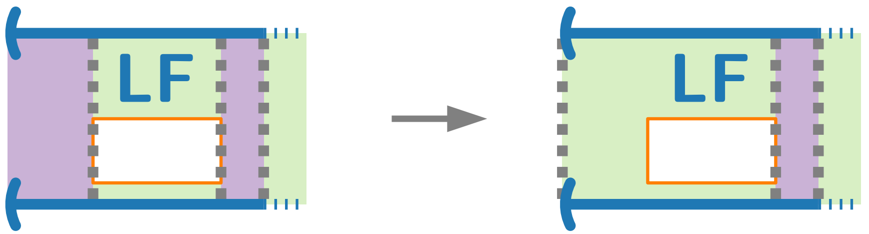
Idea.
Greedily *fill* the gaps by preferring to "stretch" the children with prescribed bars.



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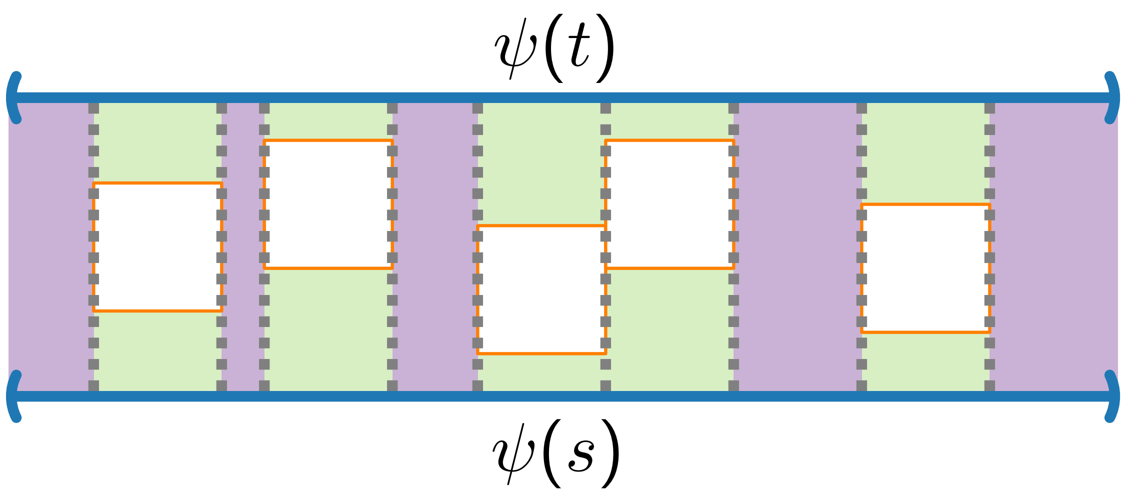

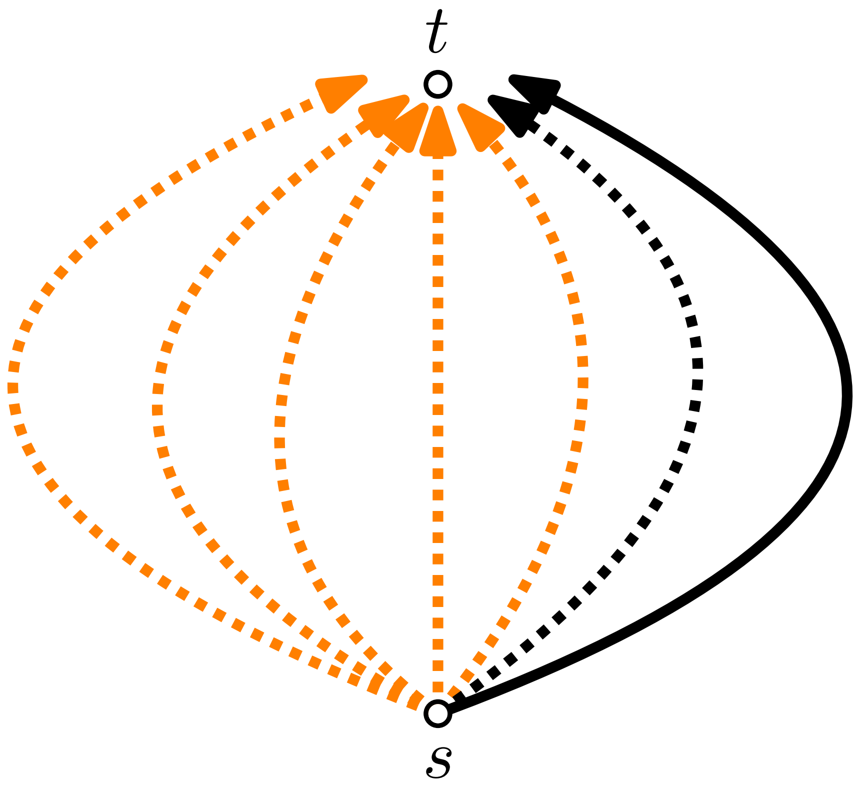

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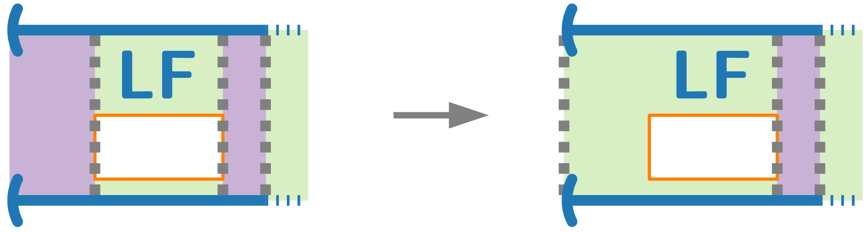
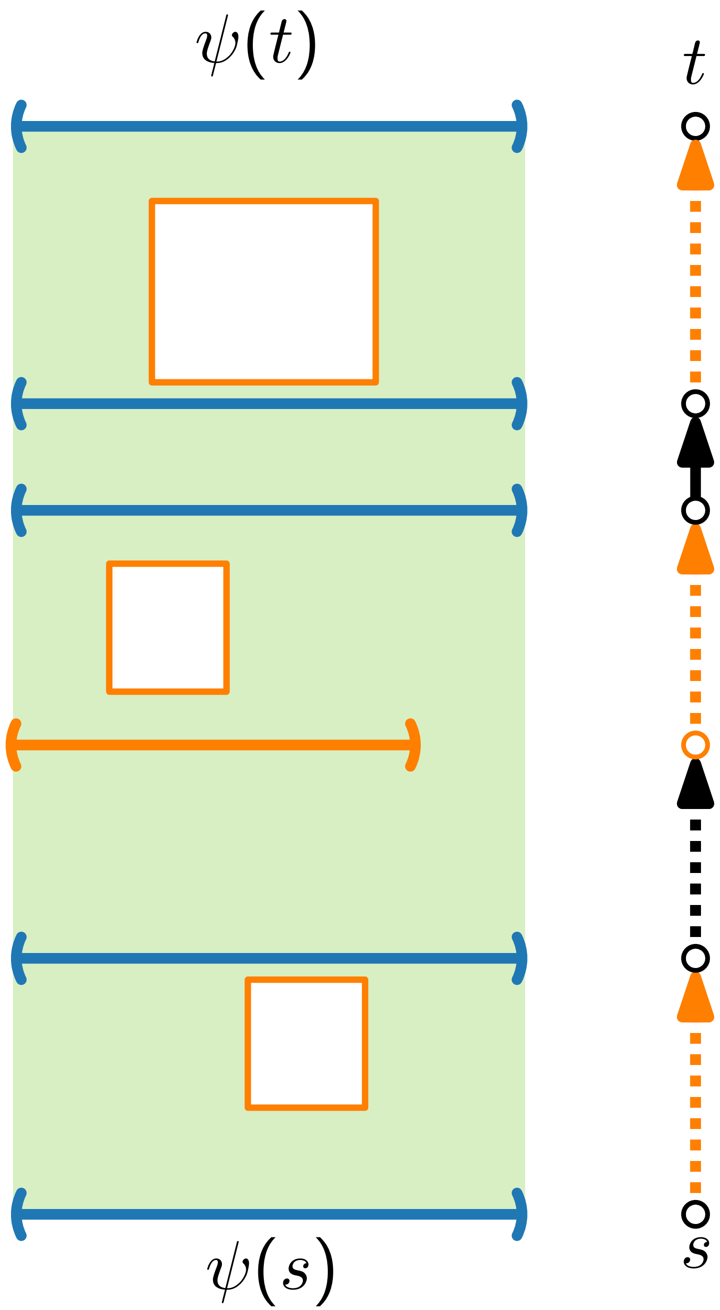


Outcome.

After processing, we must know the valid types for the corresponding subgraphs.

S-Nodes



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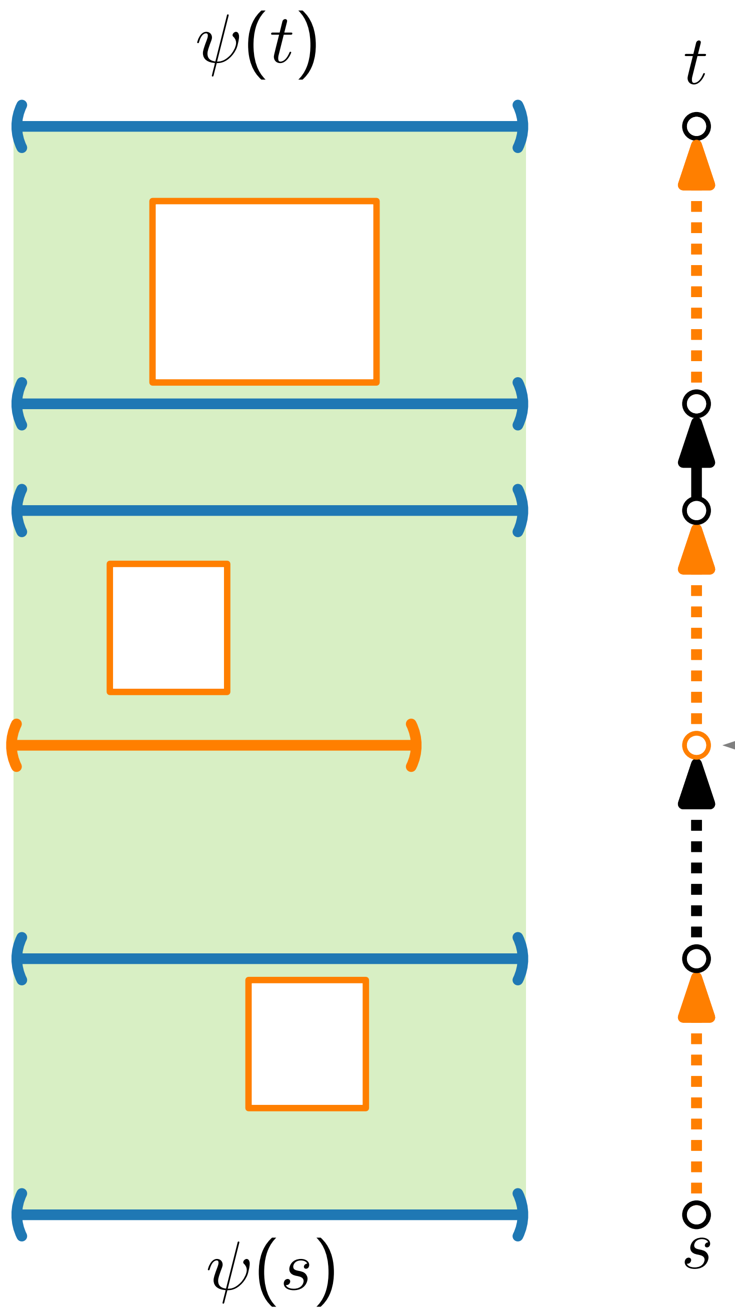
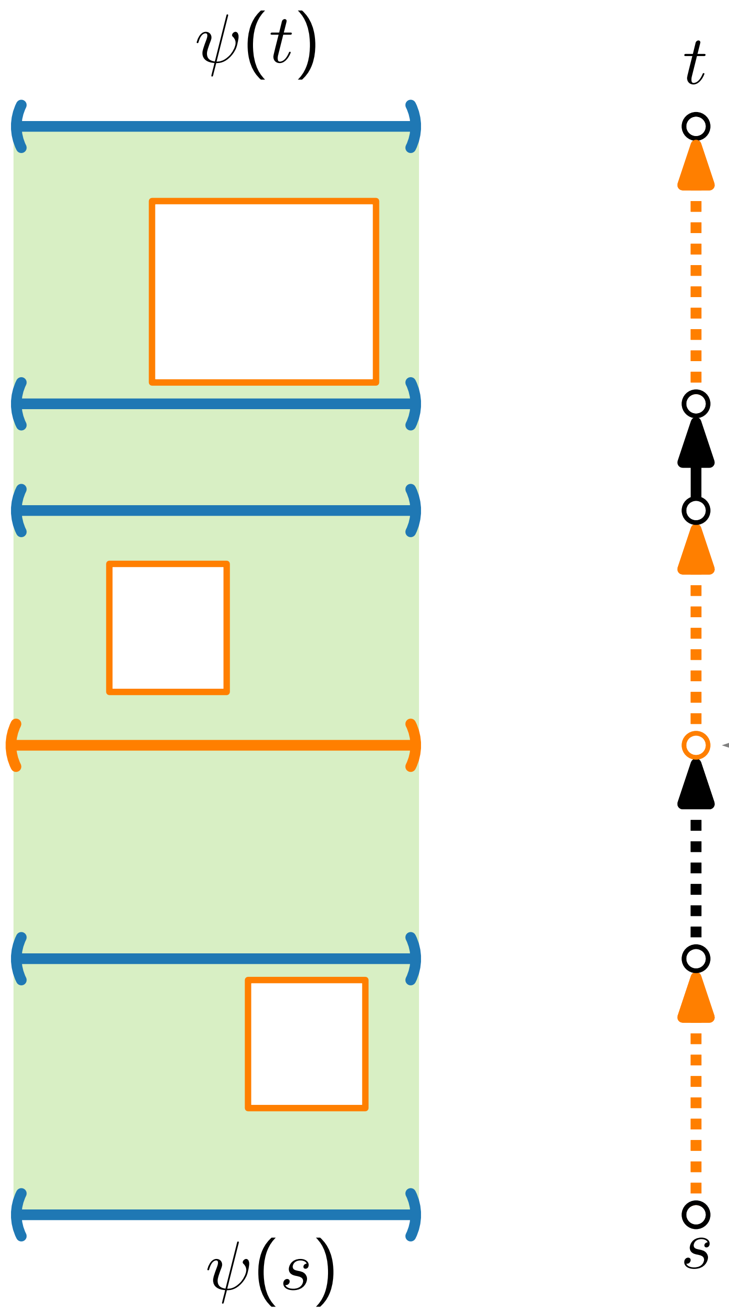


This fixed vertex means we can only make a **F**ixed-**F**ixed representation!

S-Nodes



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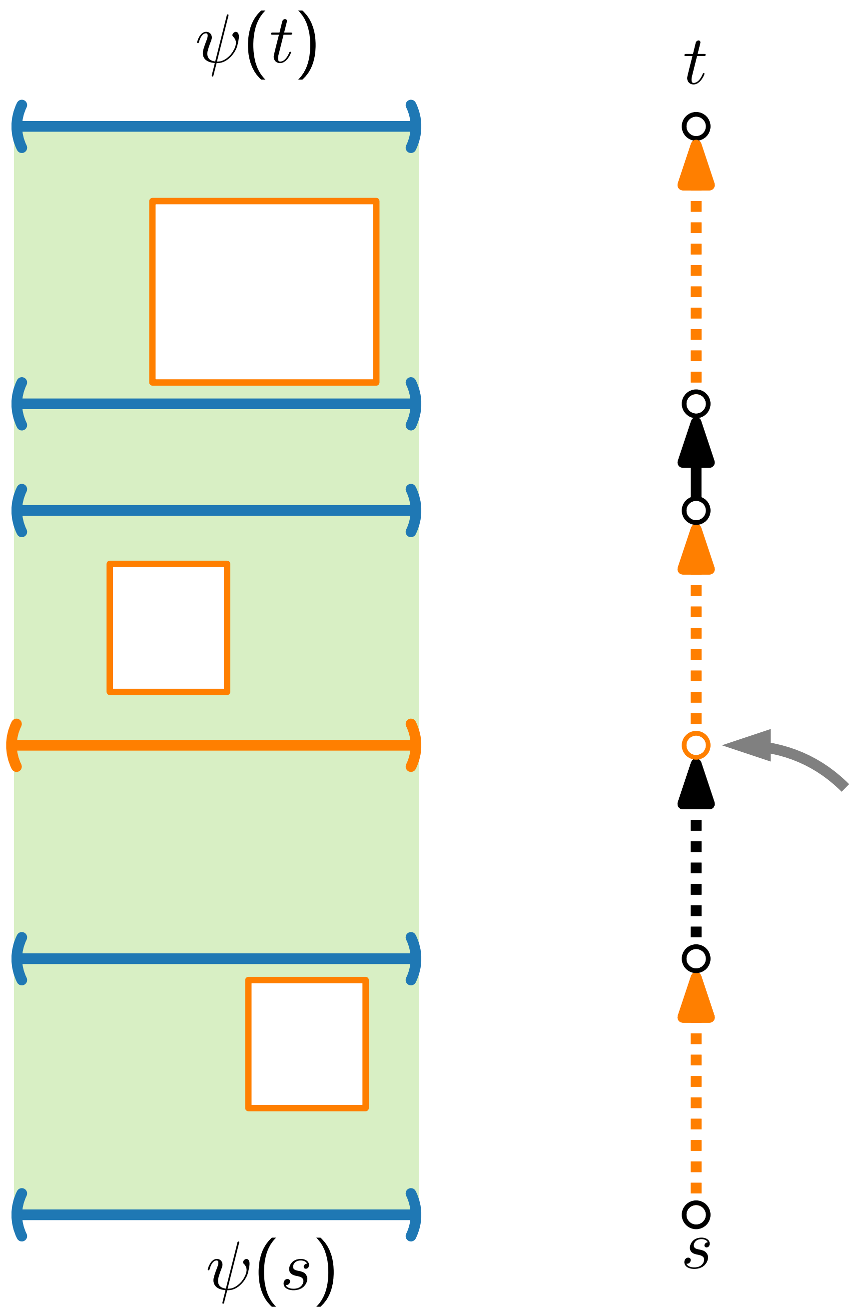

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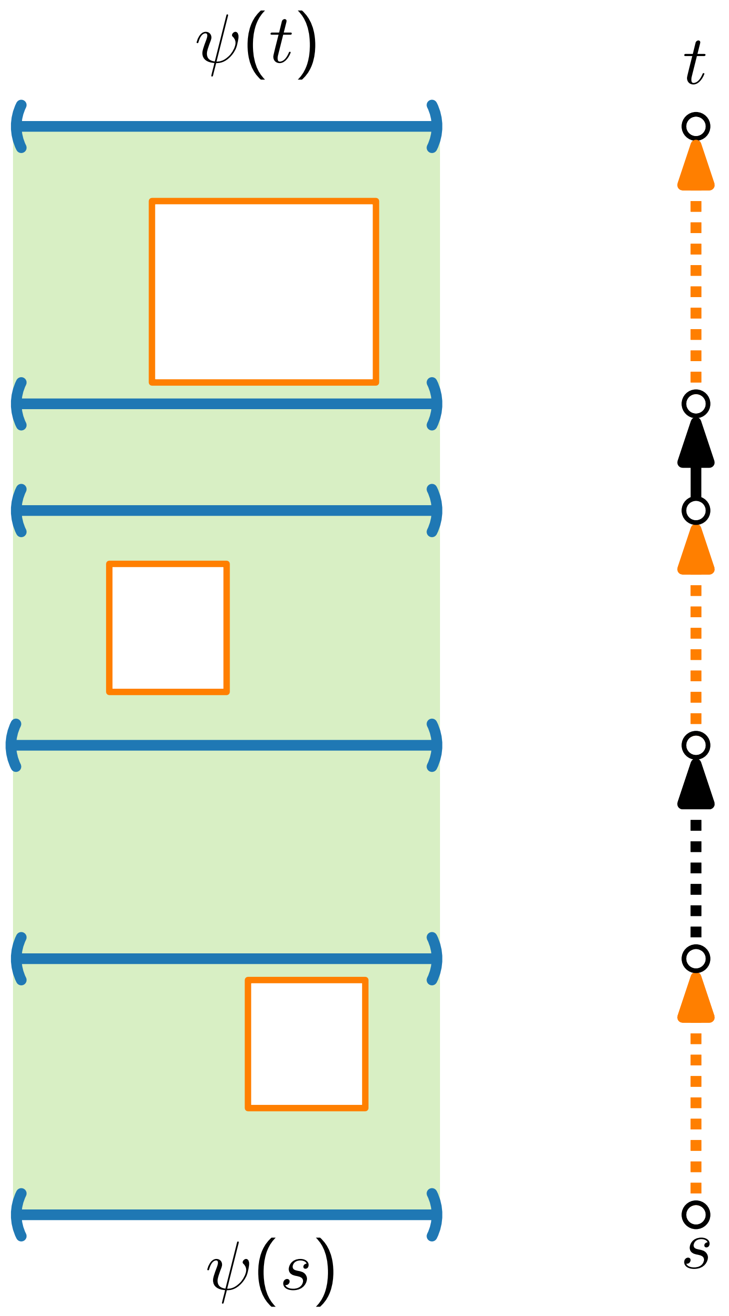

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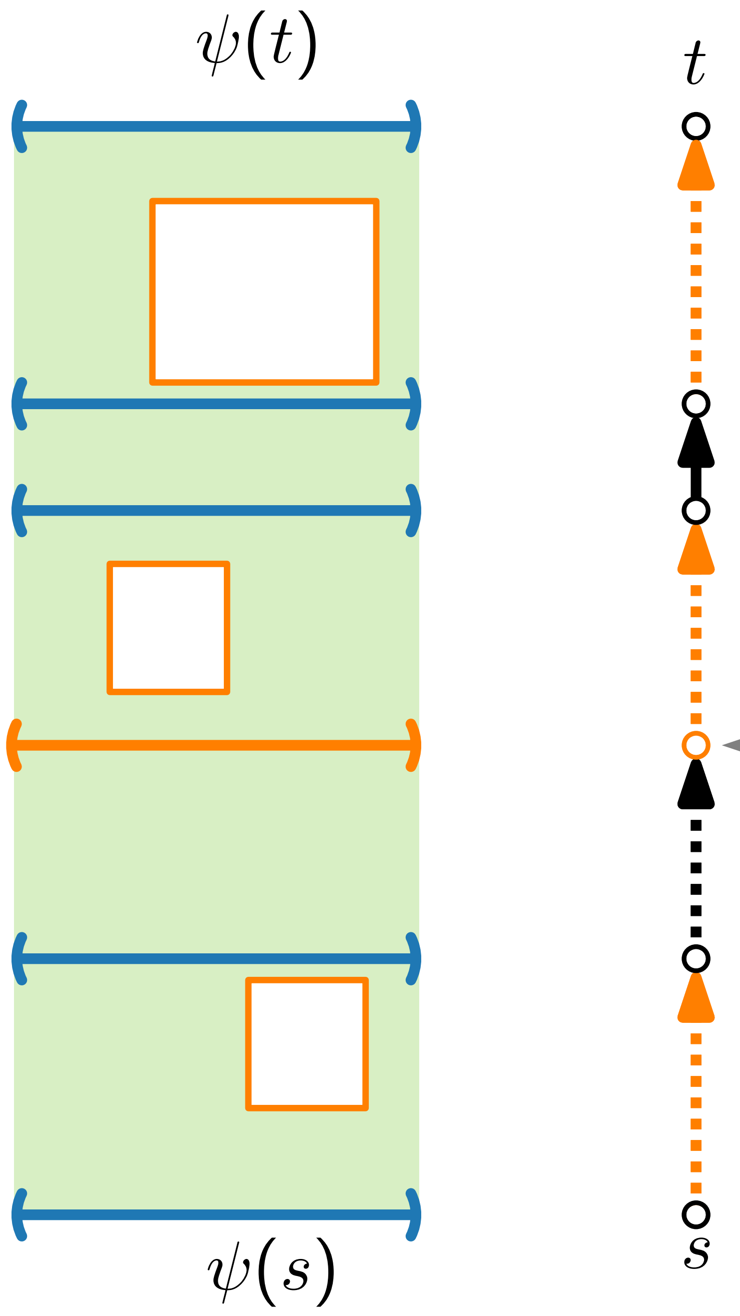
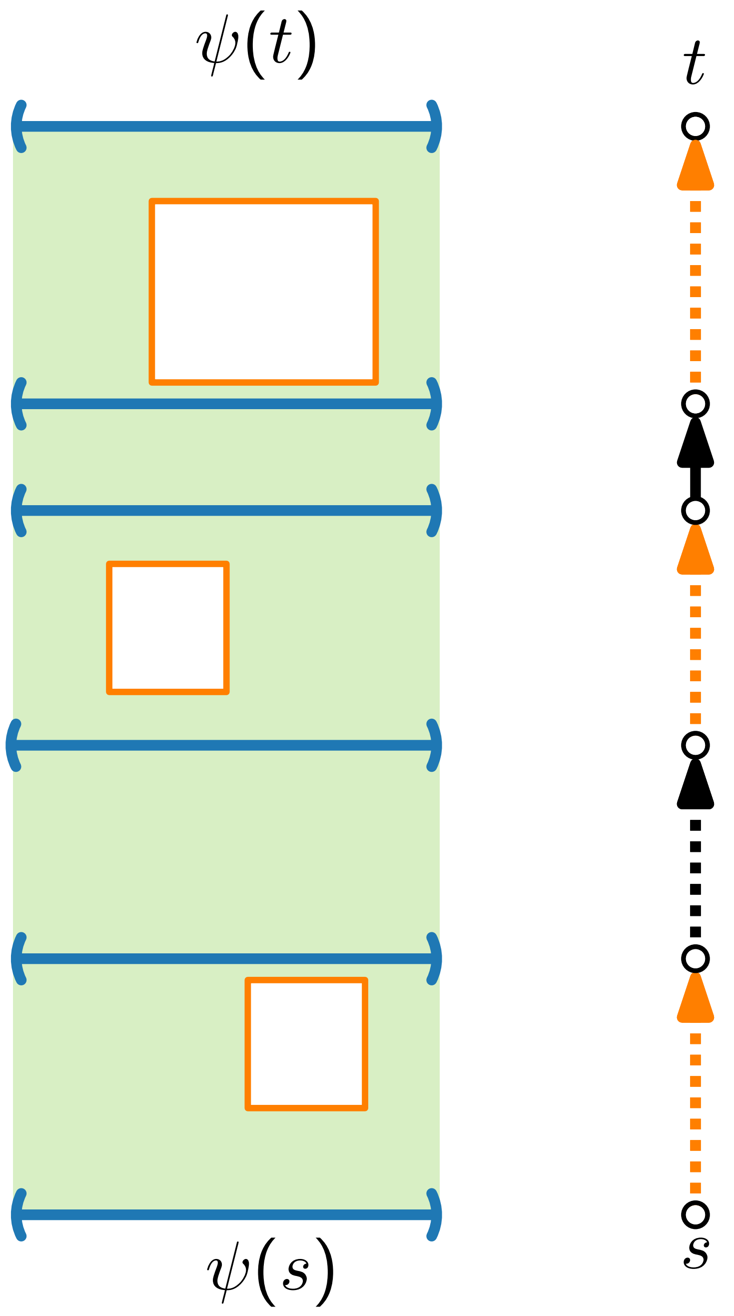
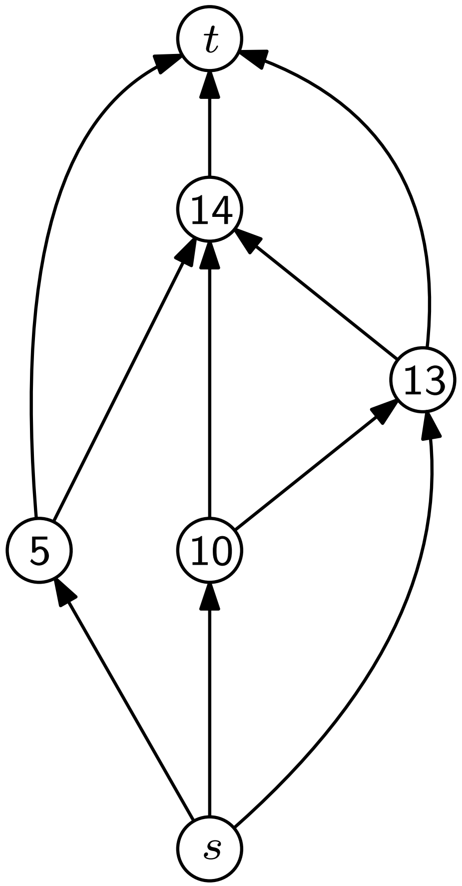


Here we have a chance to make all (**LL, FL, LF, FF**) types.

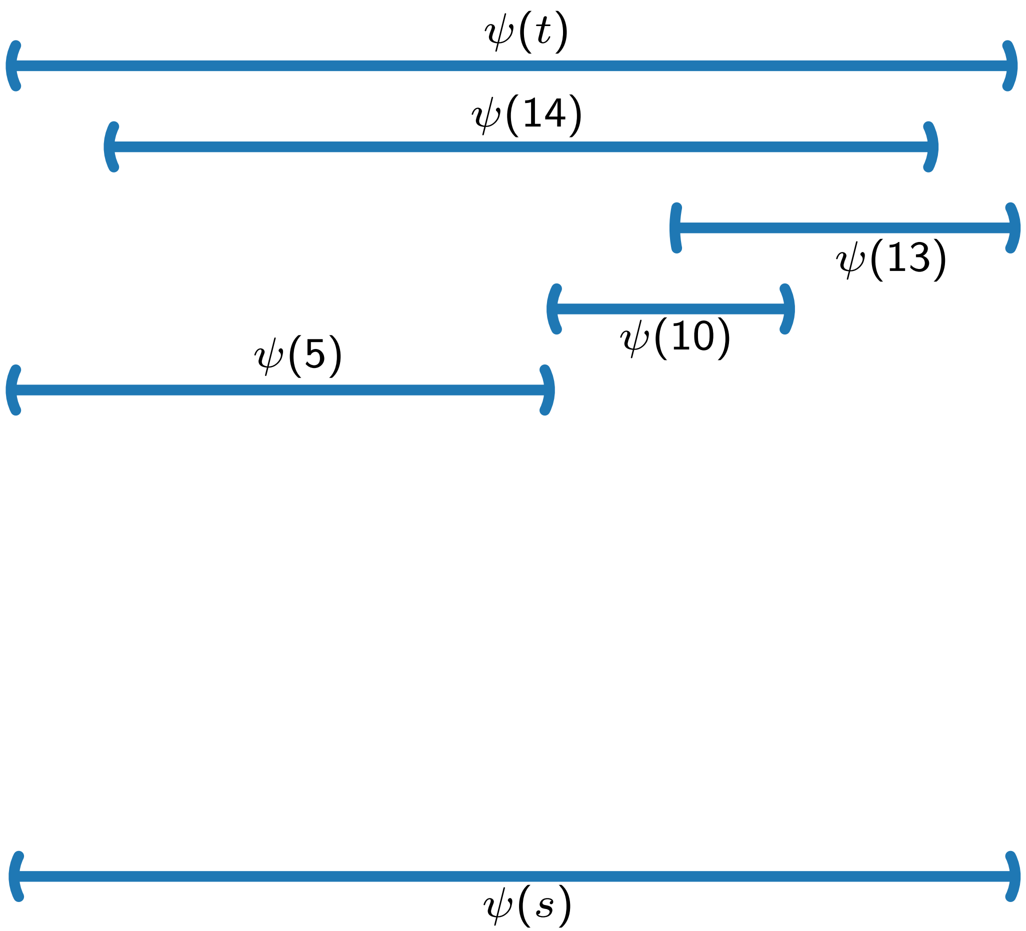
This fixed vertex means we can only make a **F**ixed-**F**ixed representation!

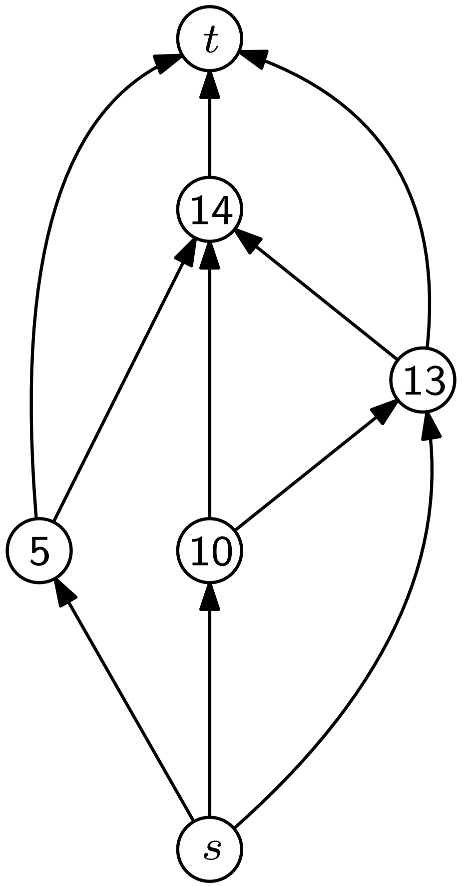


R-Nodes



R-Nodes





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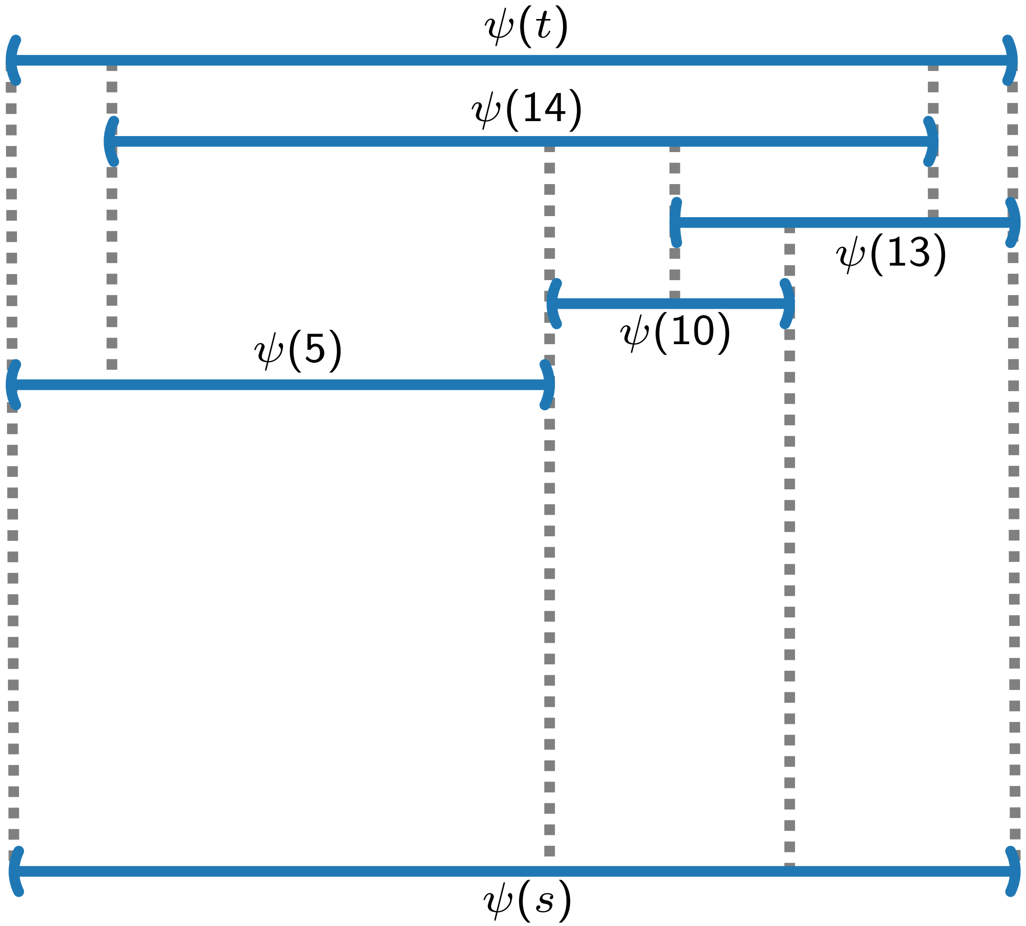

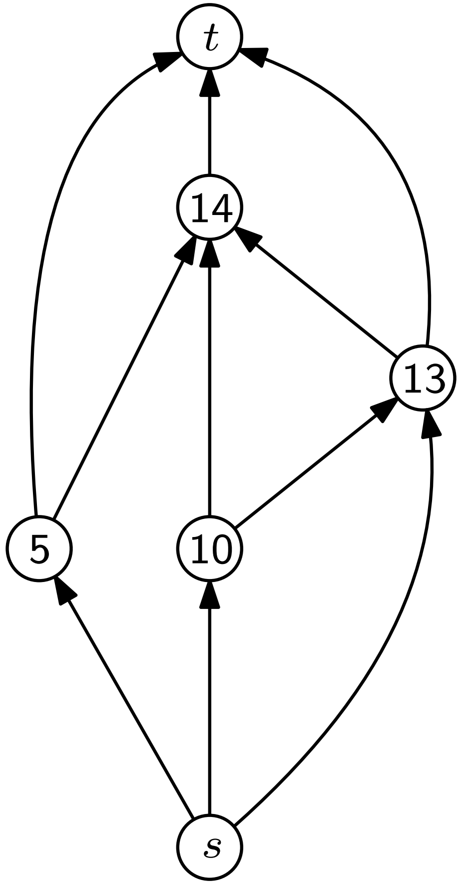

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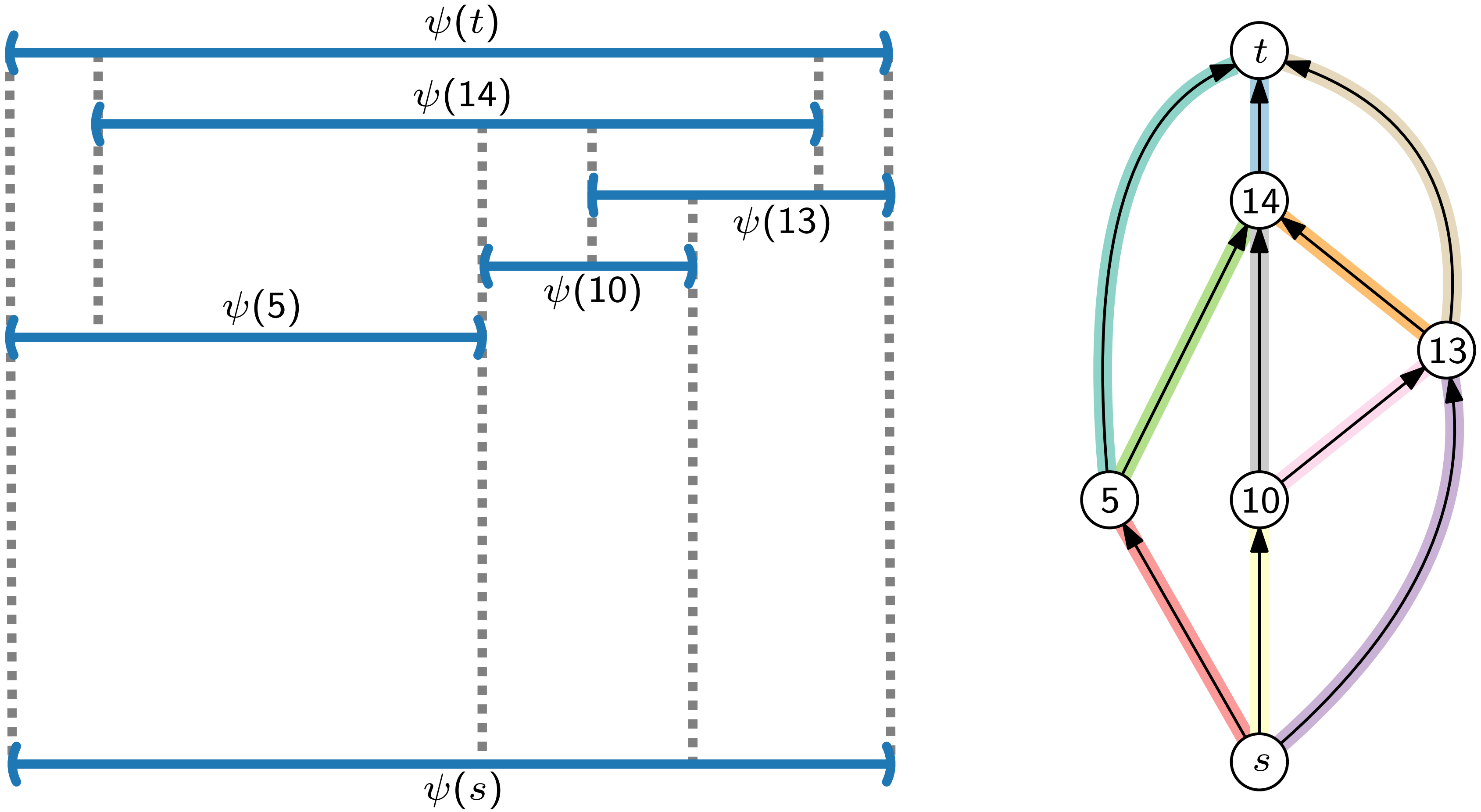

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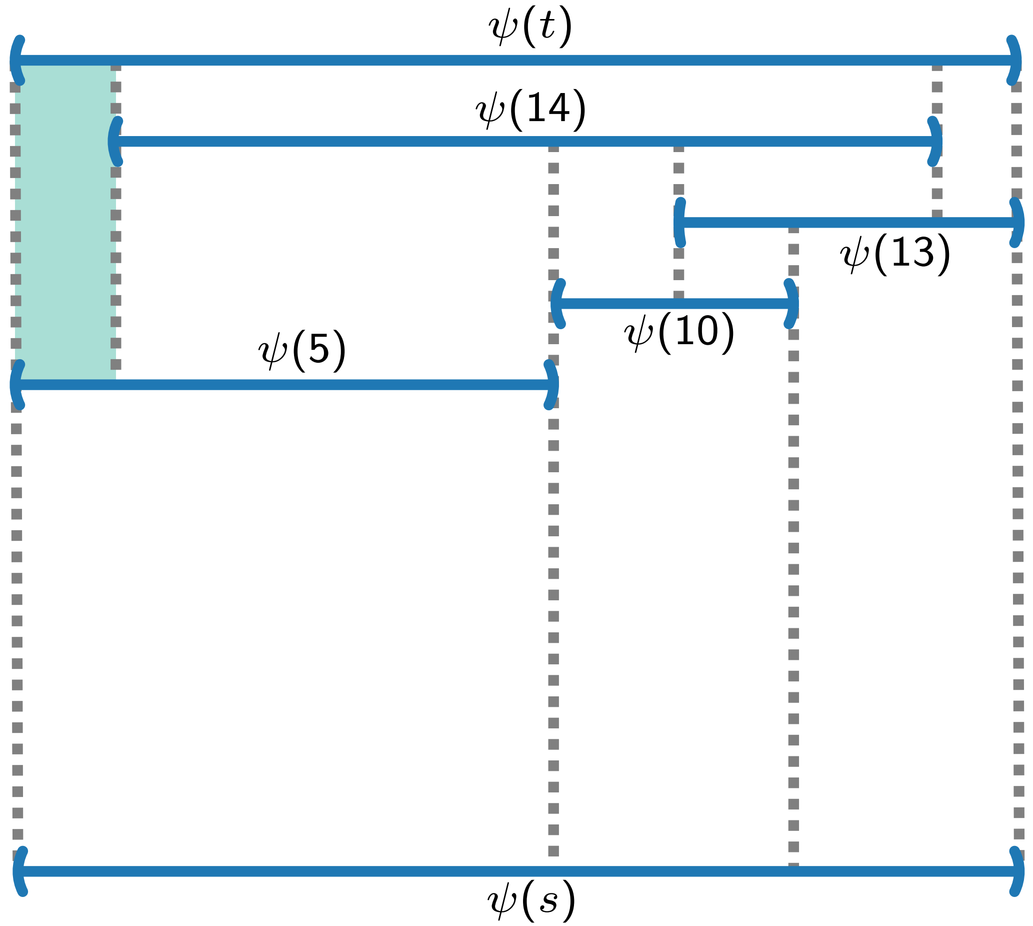

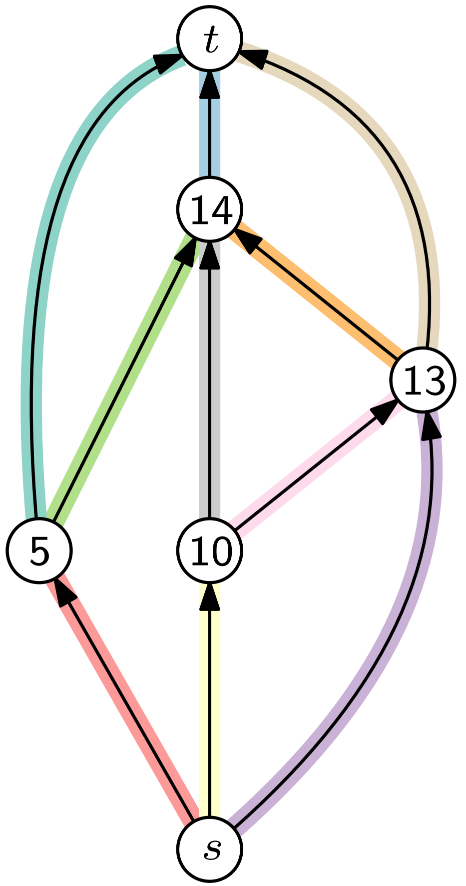

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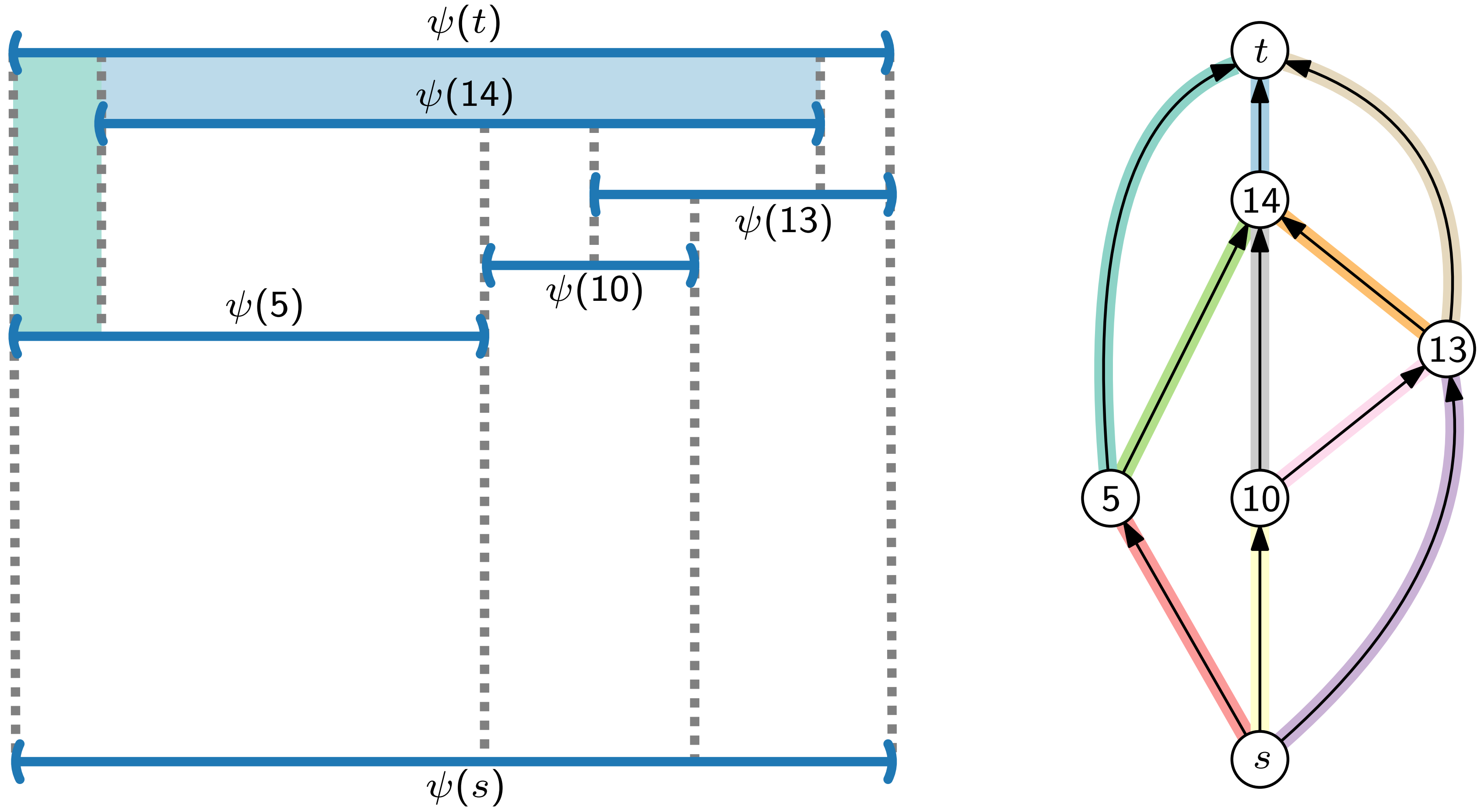

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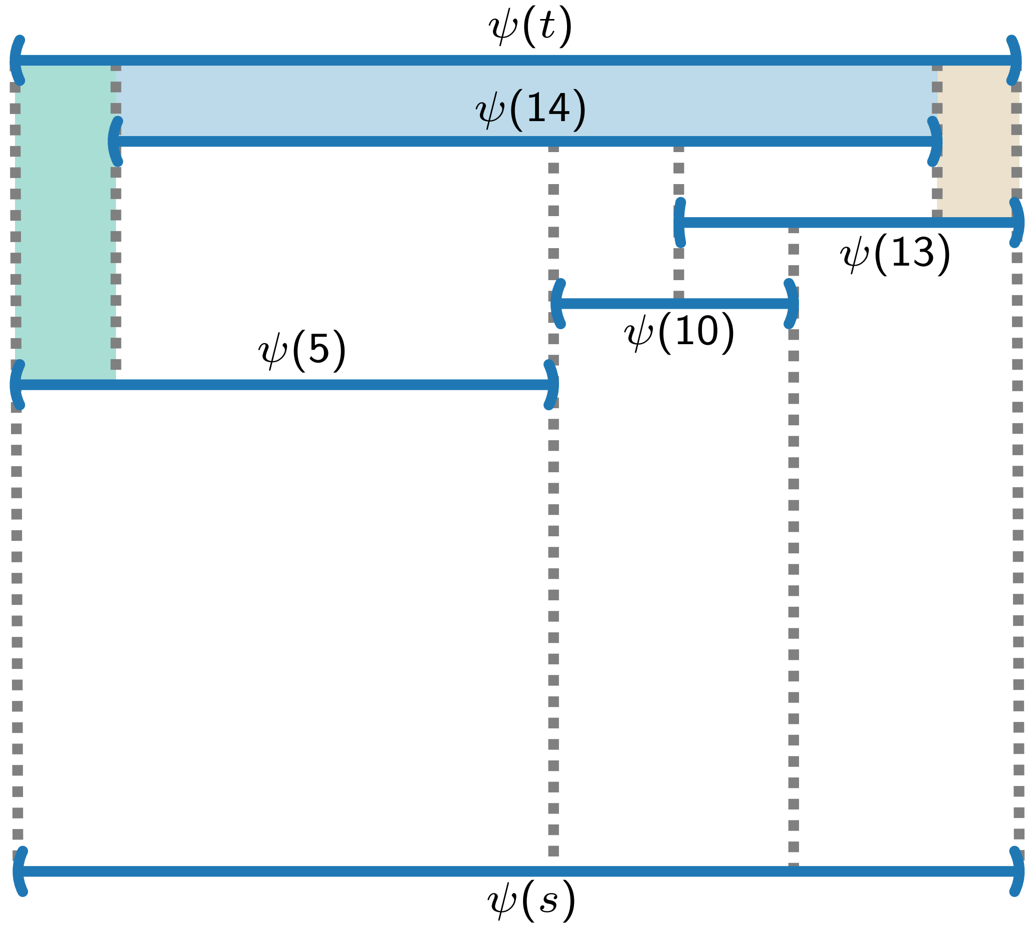

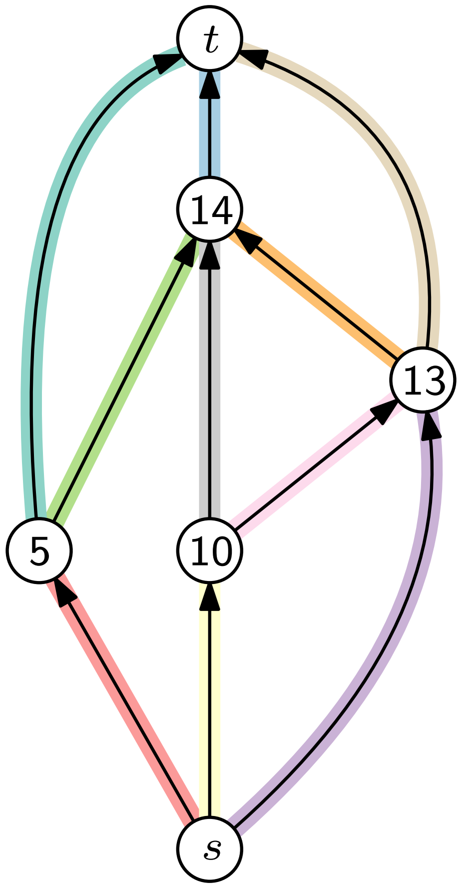

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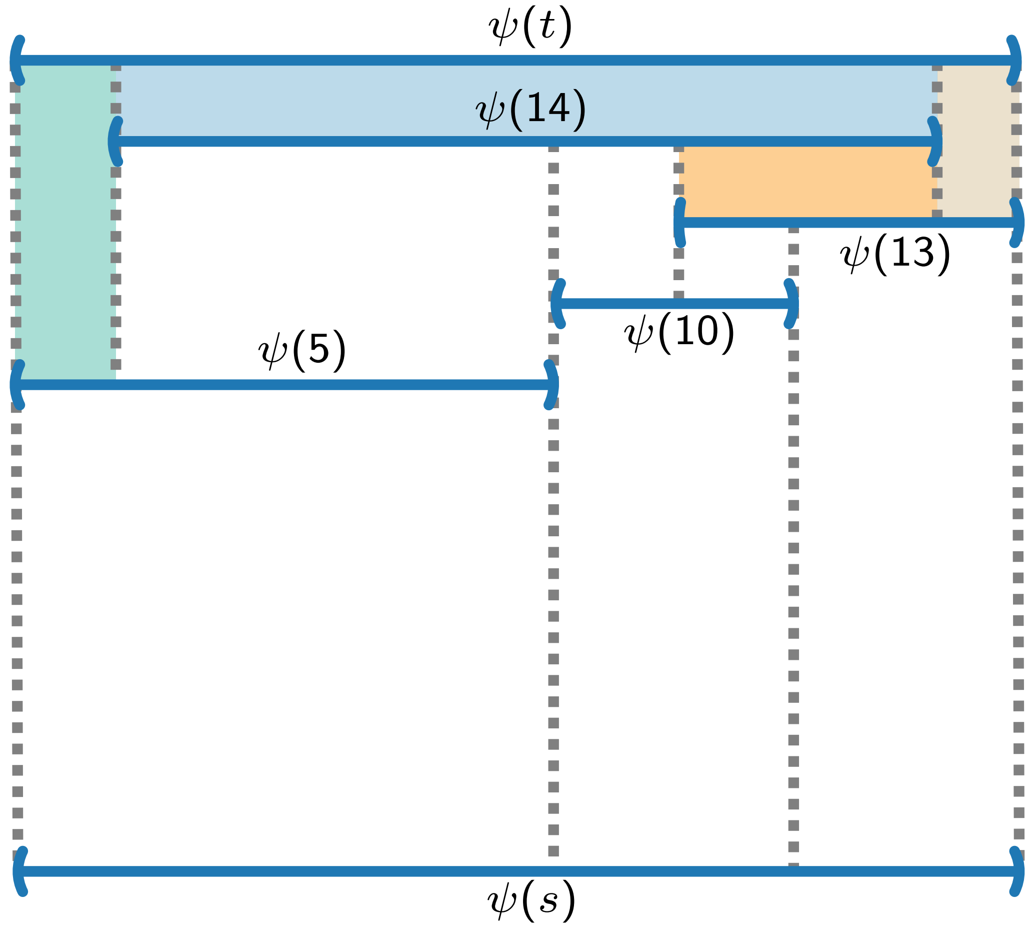

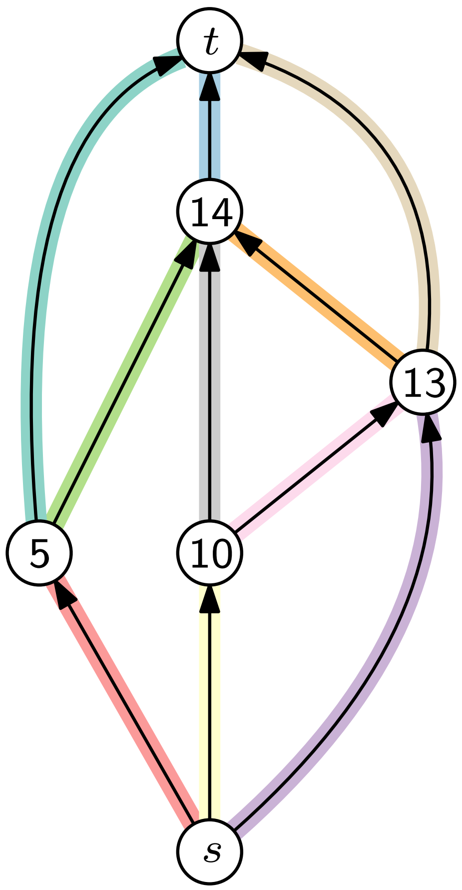

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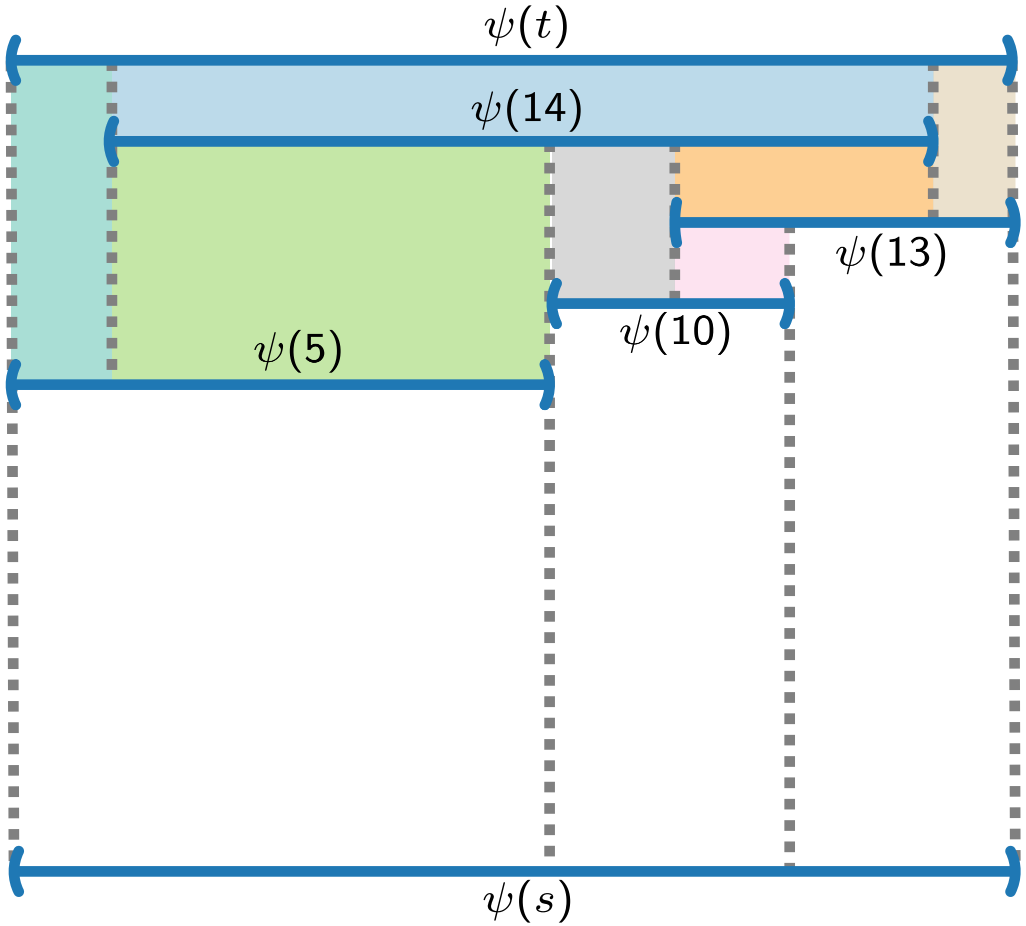

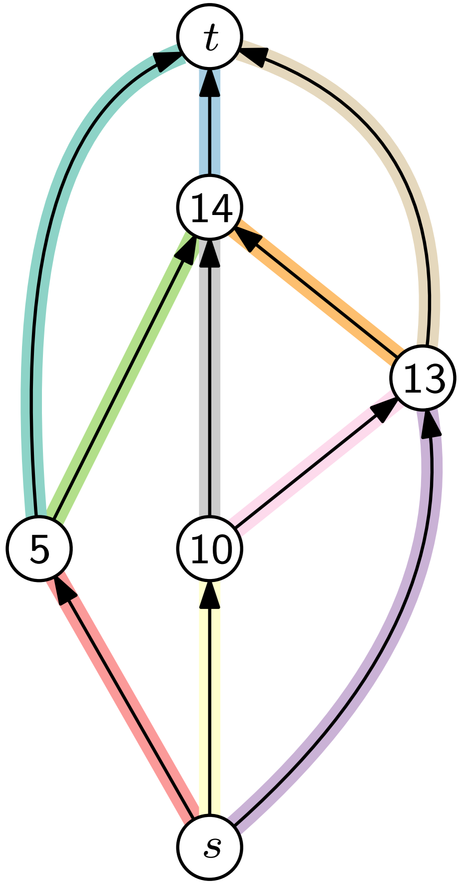

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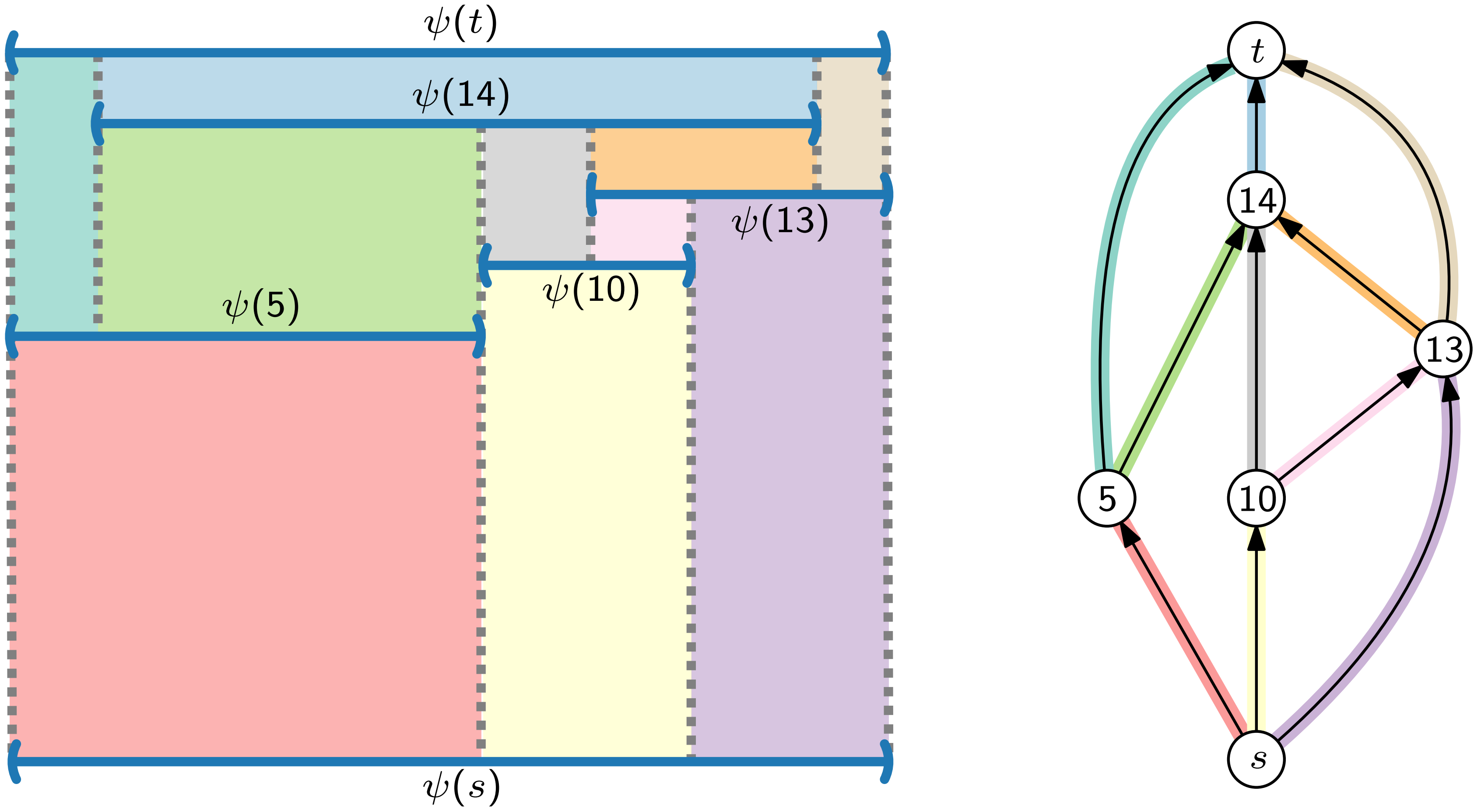

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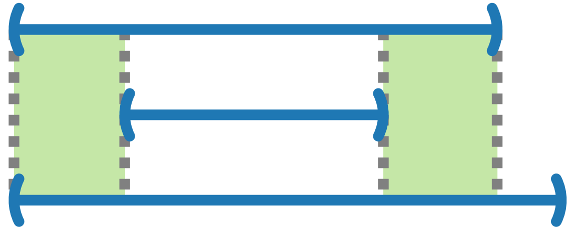
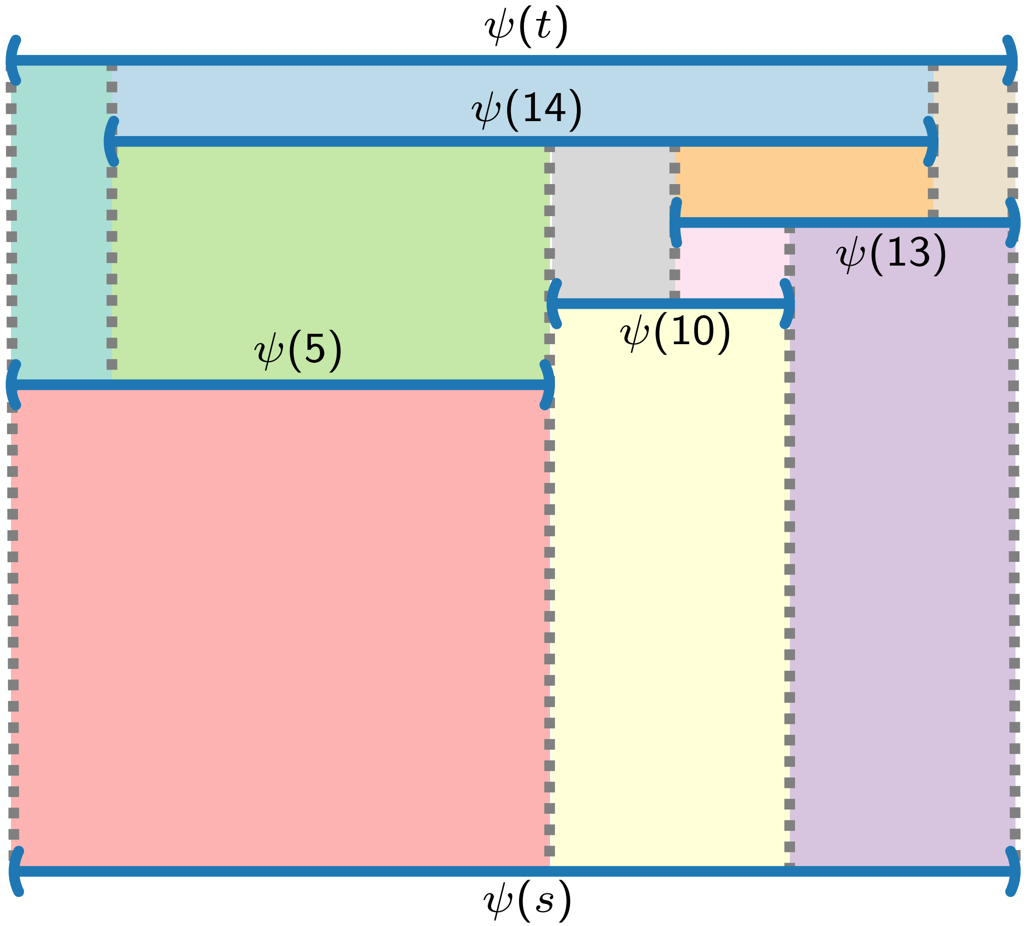
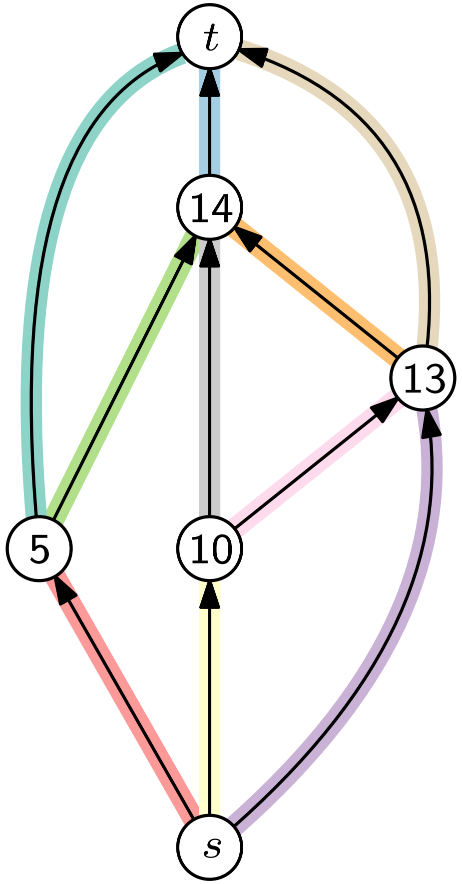
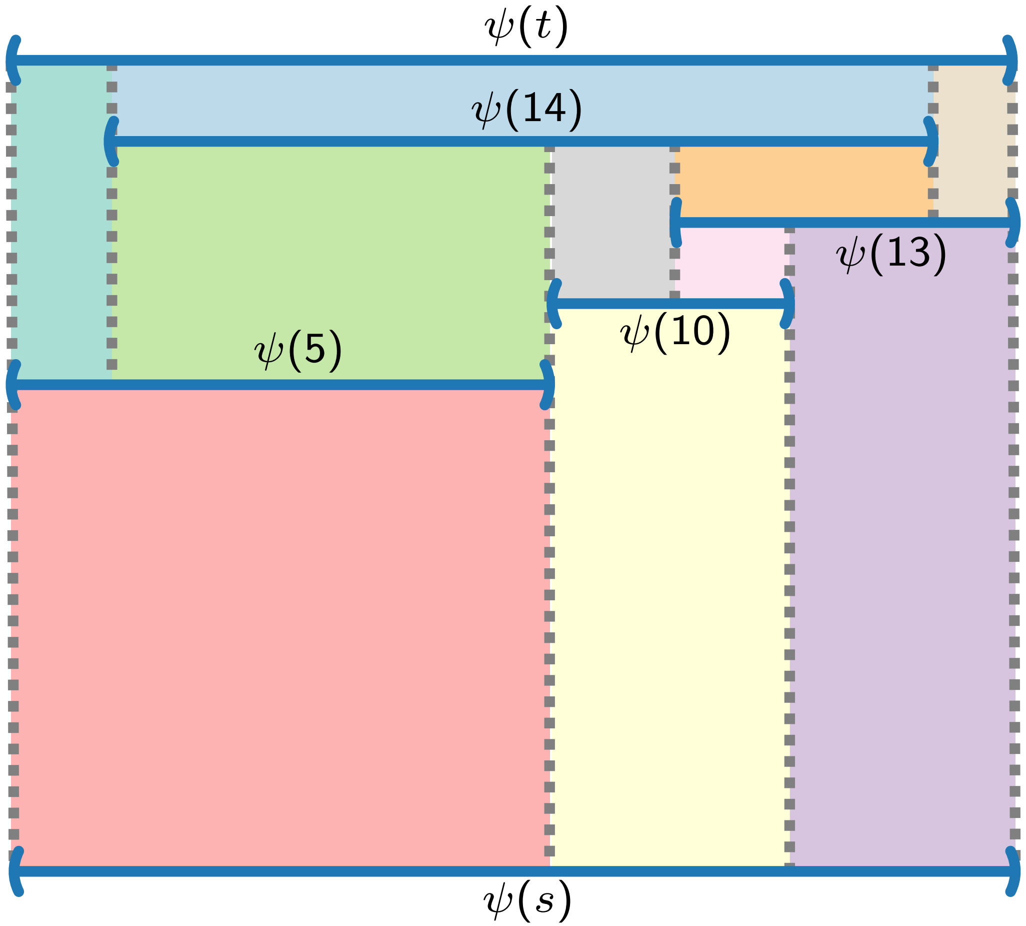
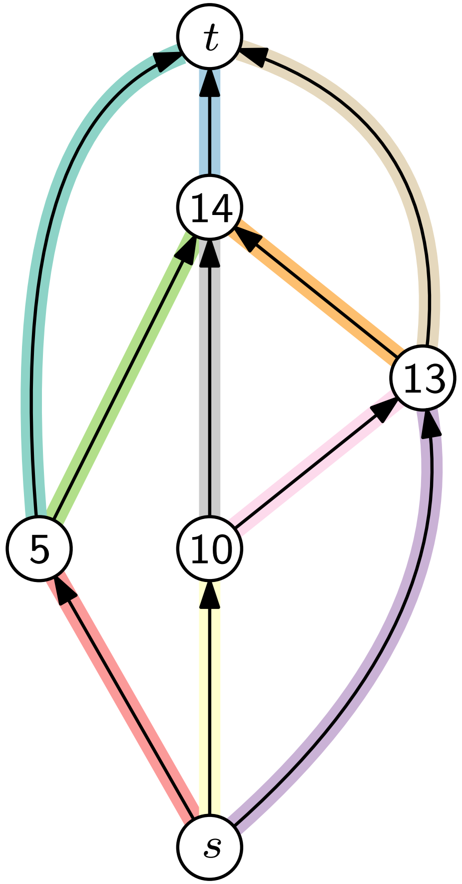


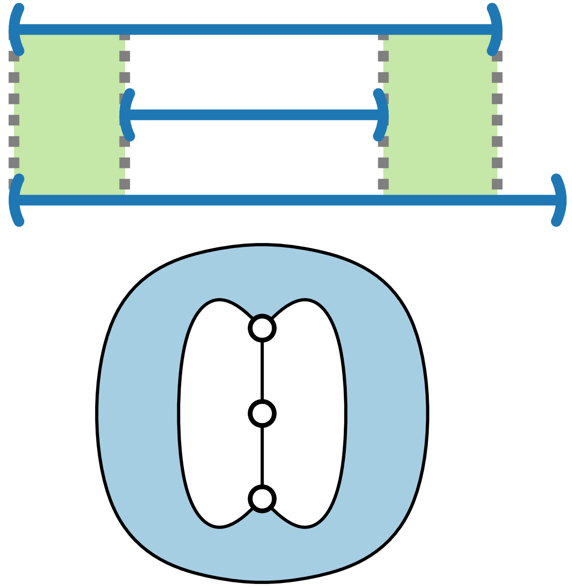




R-Nodes







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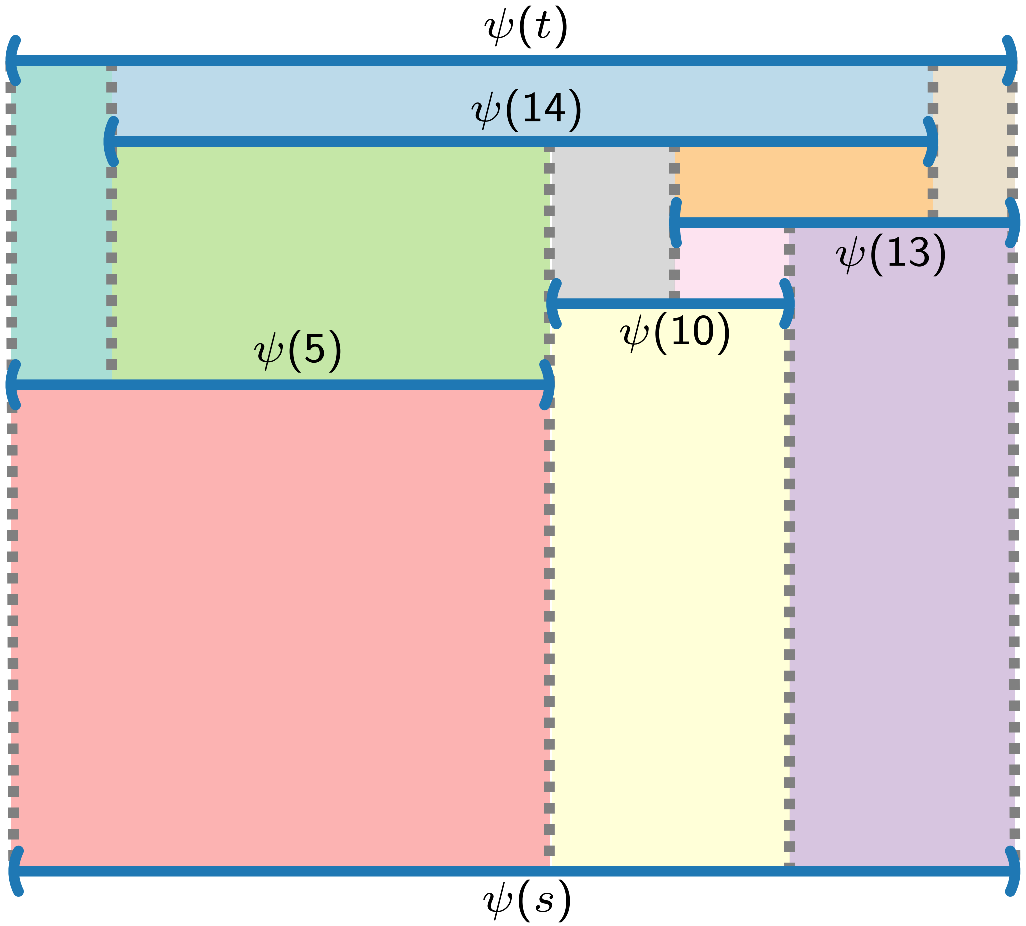
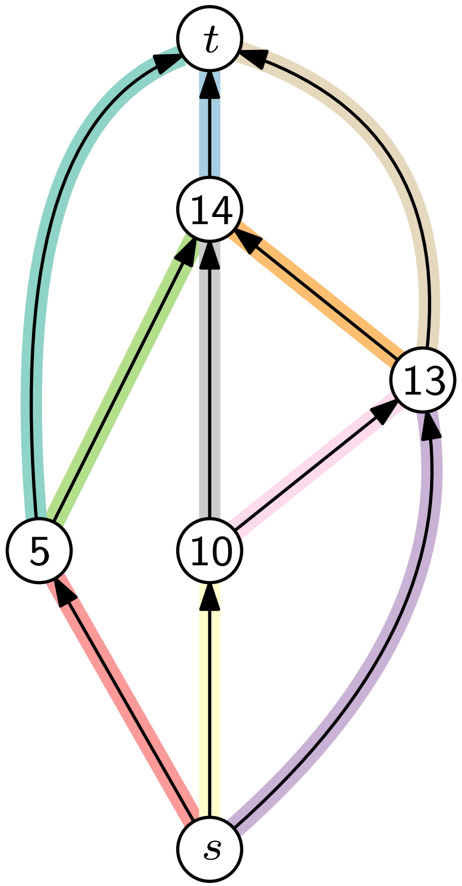
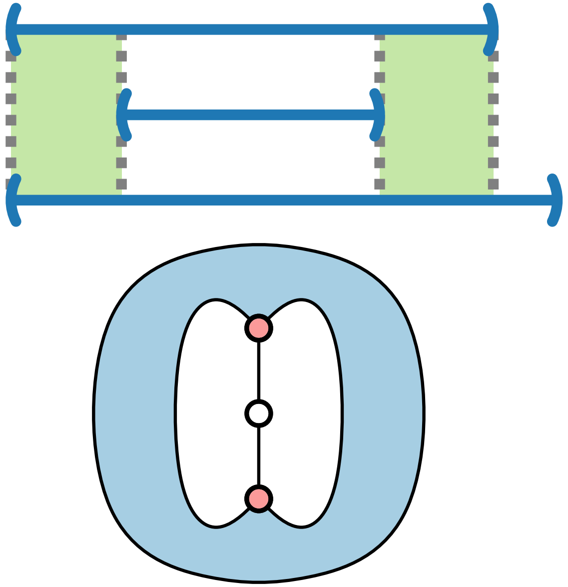
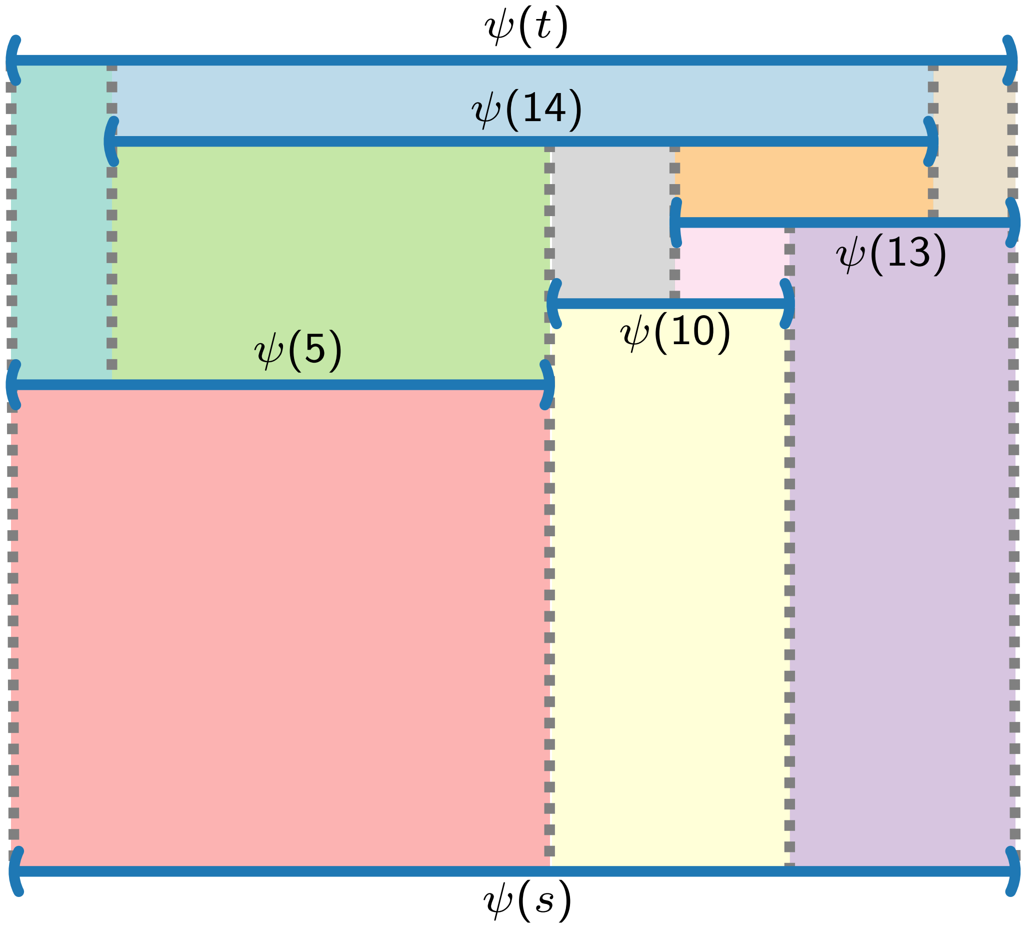


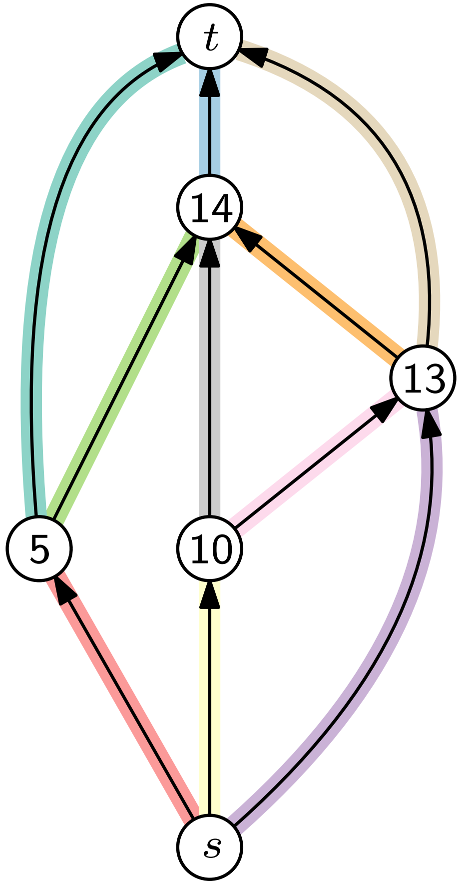


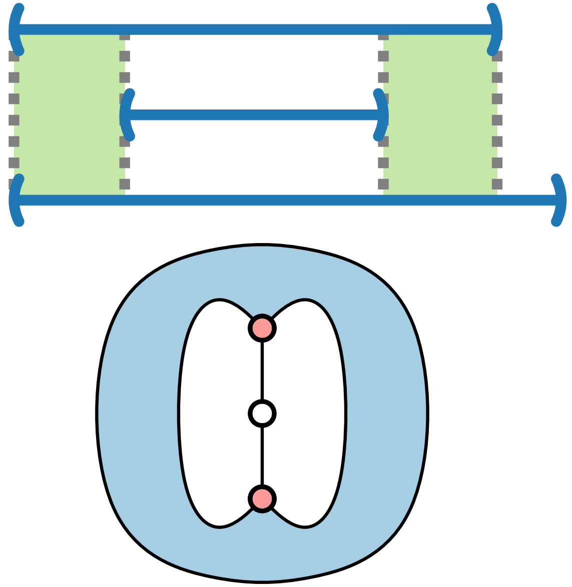


Separation pair!

R-Nodes



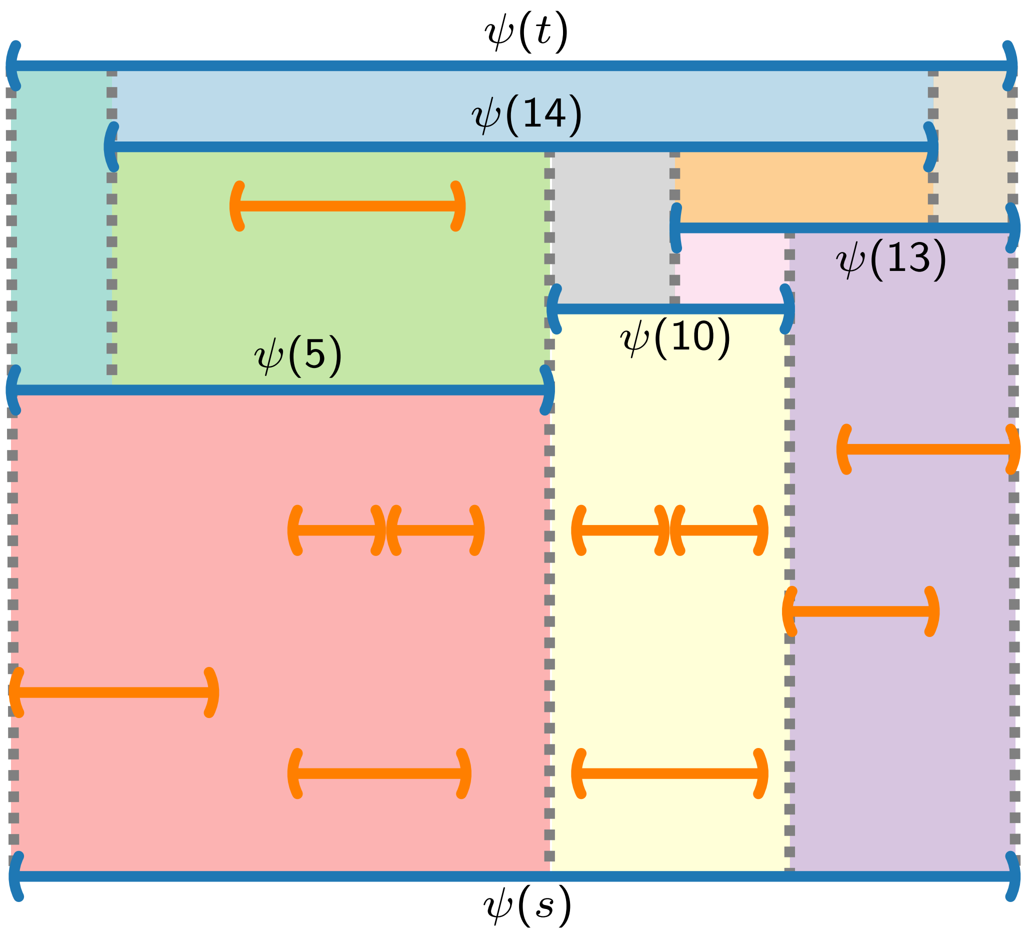


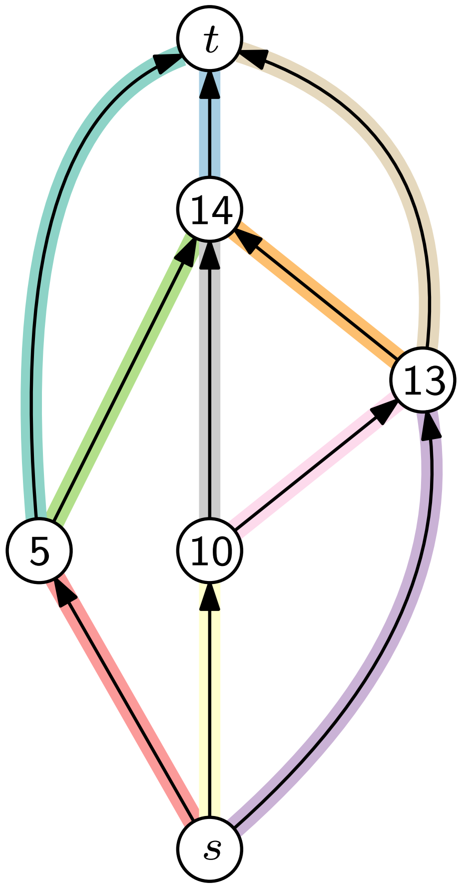


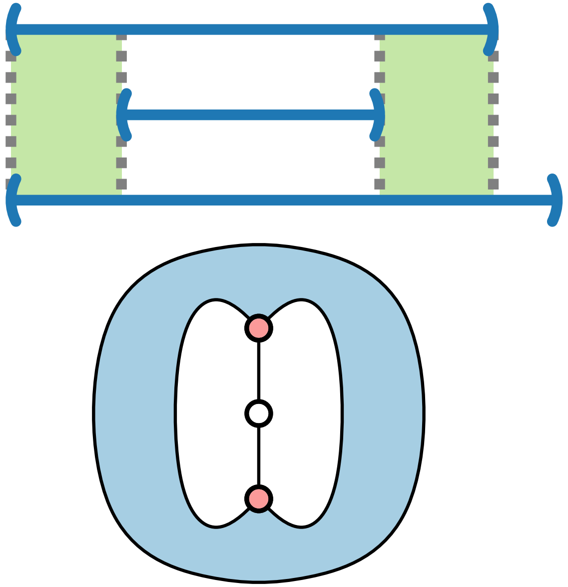
Separation pair!

($\nexists$ in **R**-component.)

R-Nodes





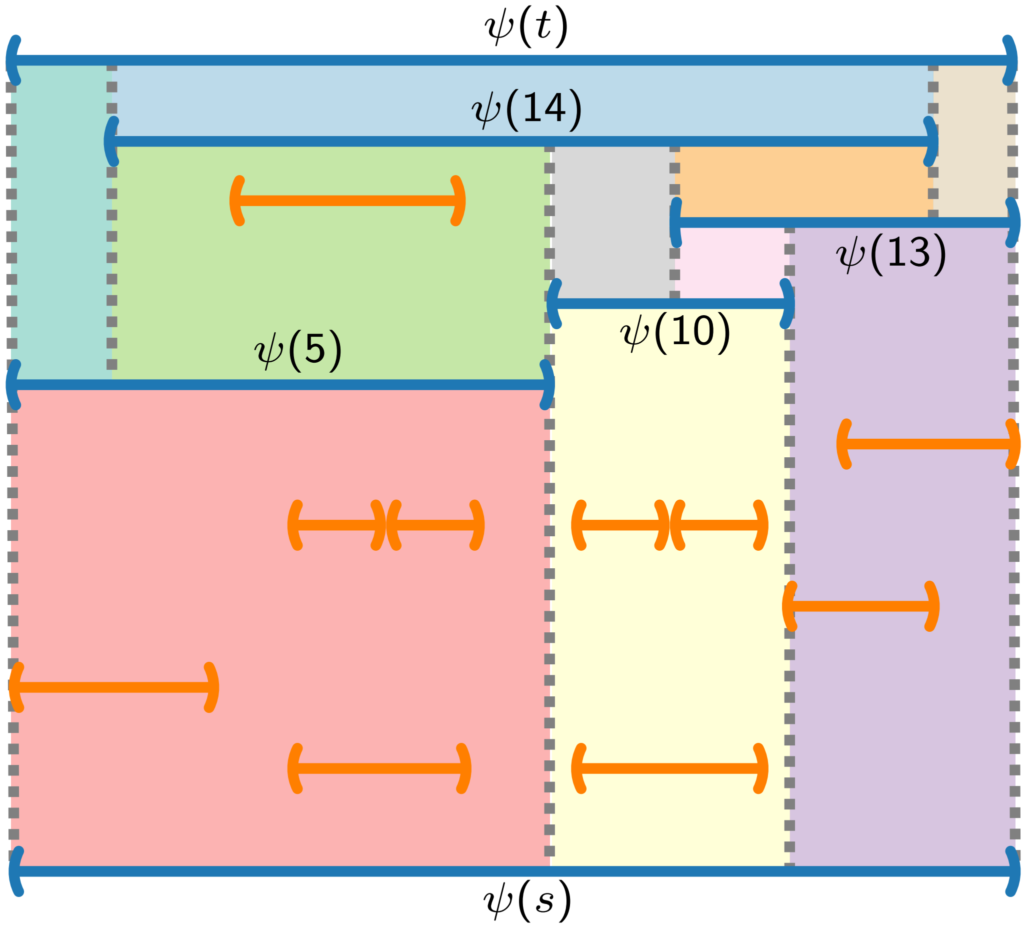


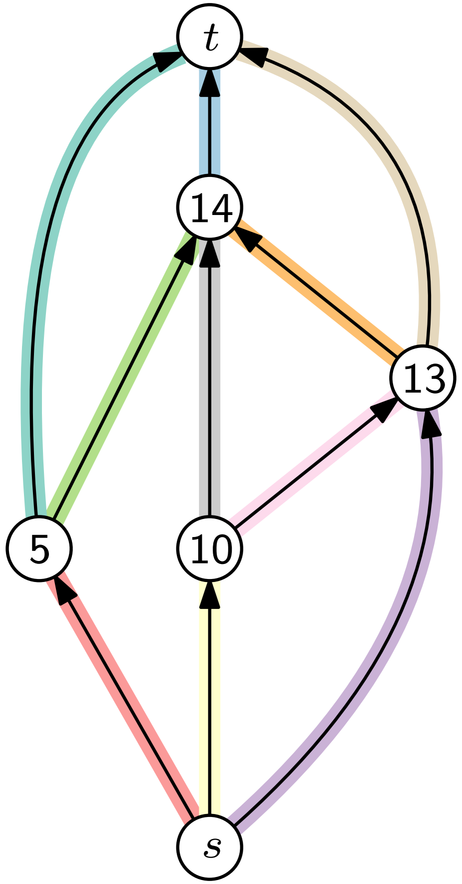
Separation pair!

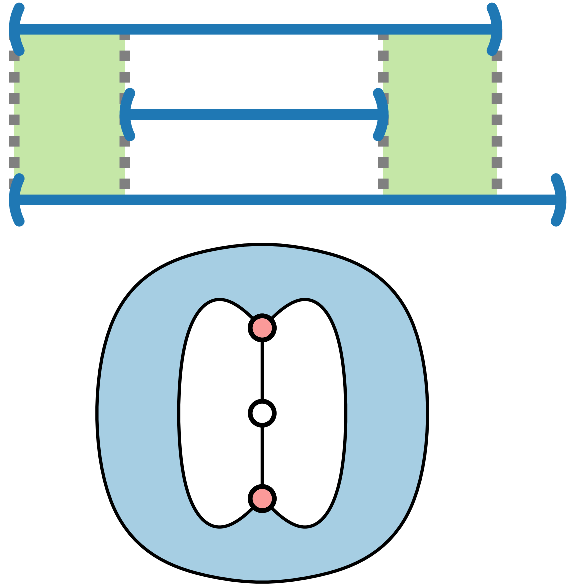
(∄ in **R**-component.)

R-Nodes

- For each child (edge) e:







Separation pair!

($\nexists$ in **R**-component.)

R-Nodes

- For each child (edge) e:
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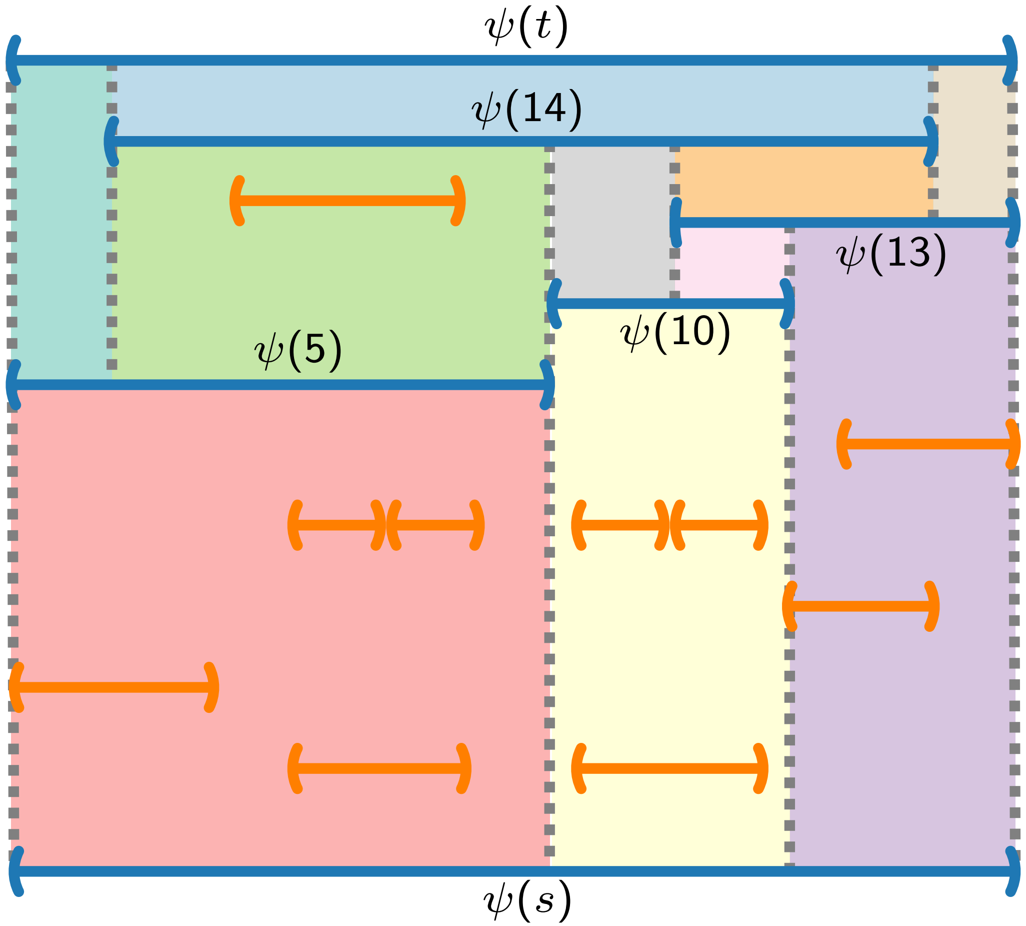

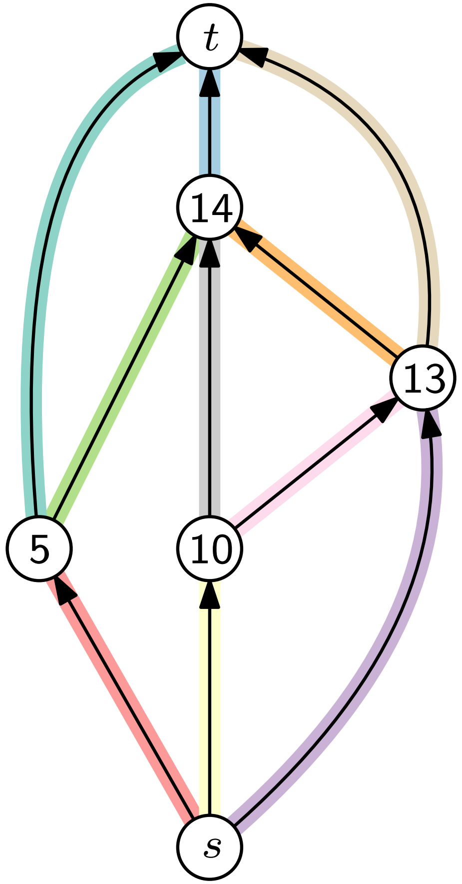

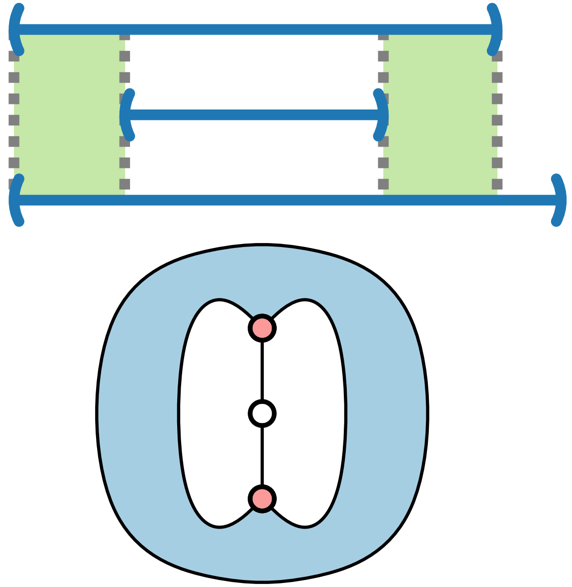

Separation pair!

($\nexists$ in **R**-component.)

R-Nodes with 2-SAT Formulation

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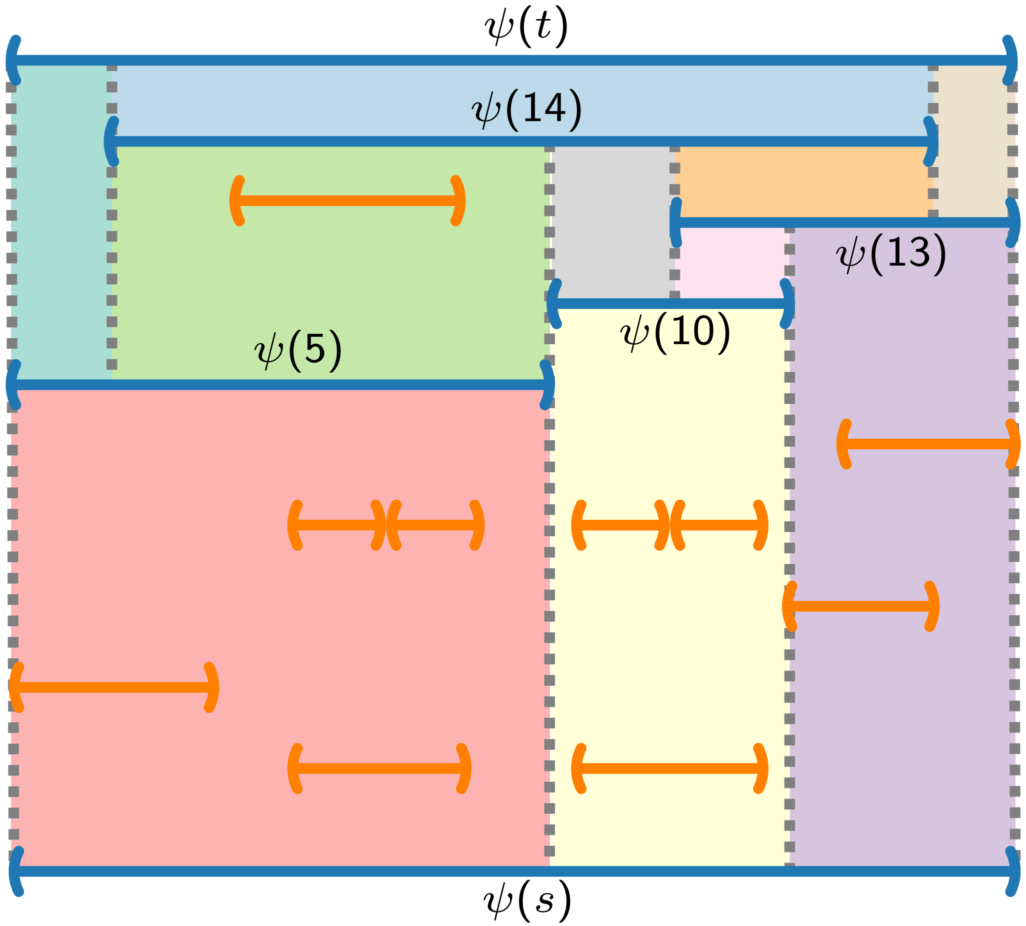

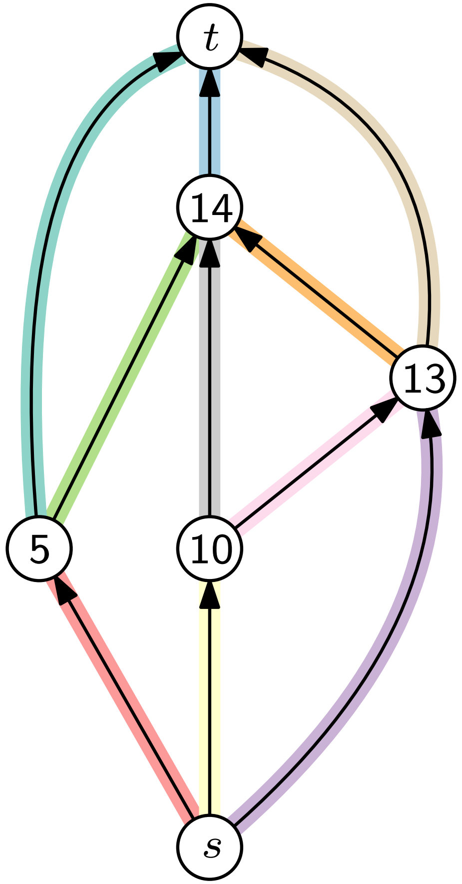

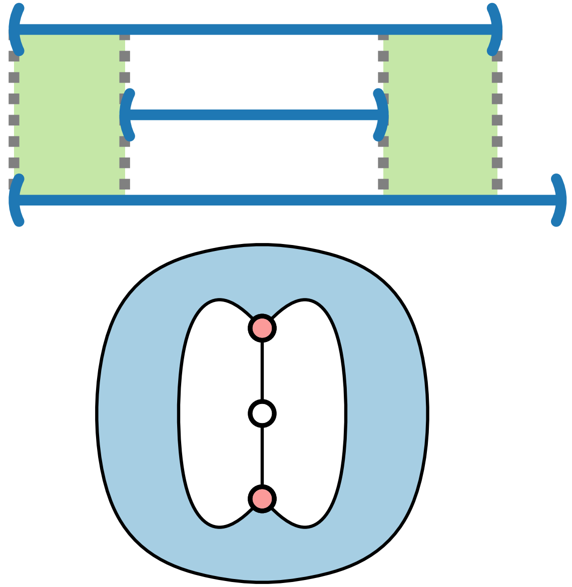

Separation pair!
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R-Nodes with 2-SAT Formulation

- For each child (edge) e:
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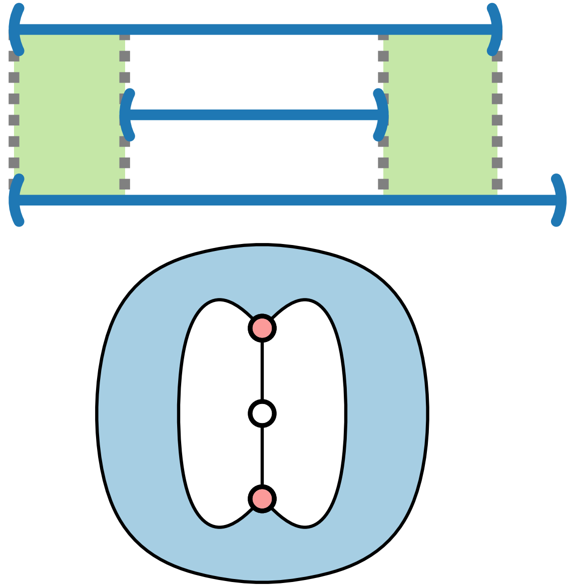

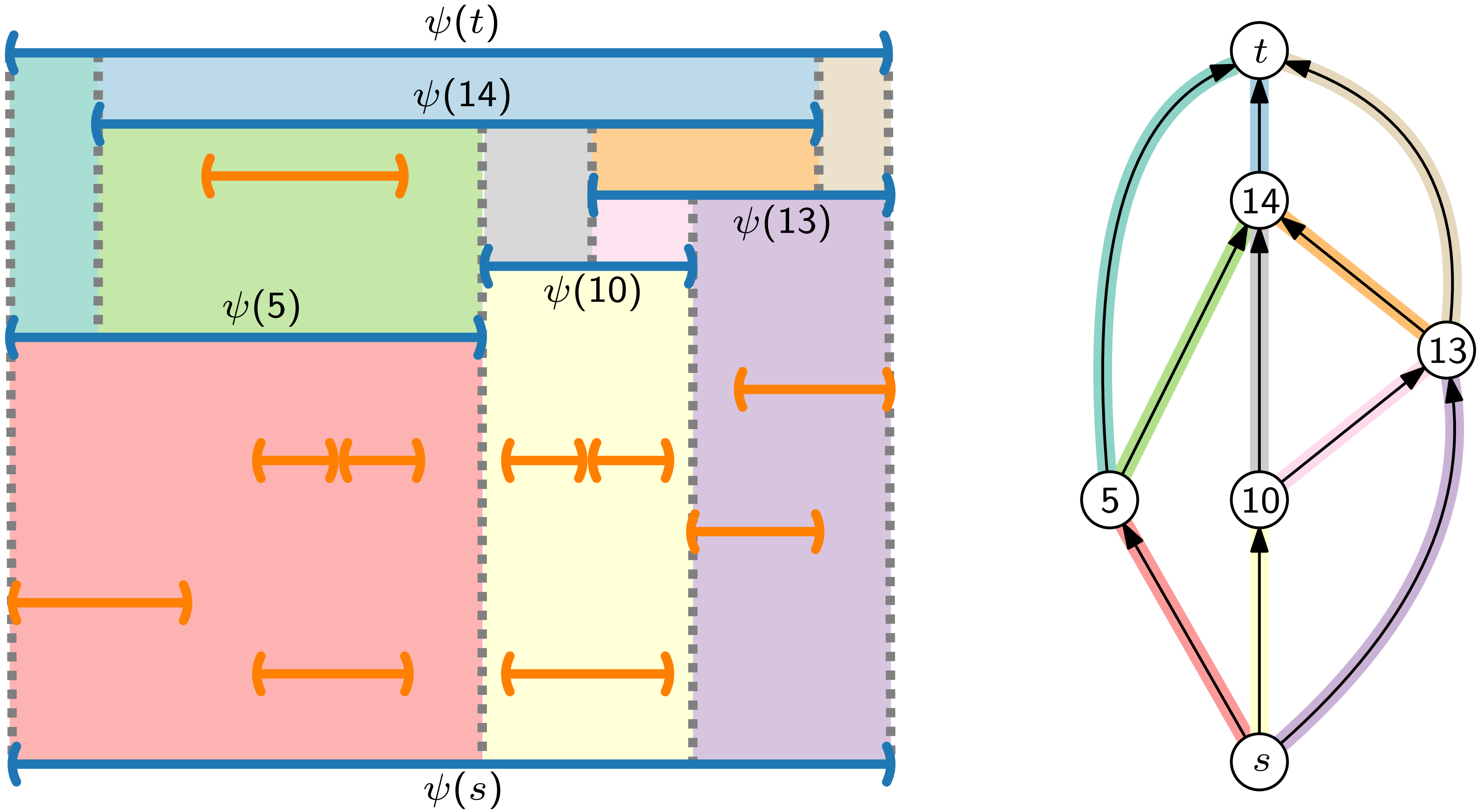

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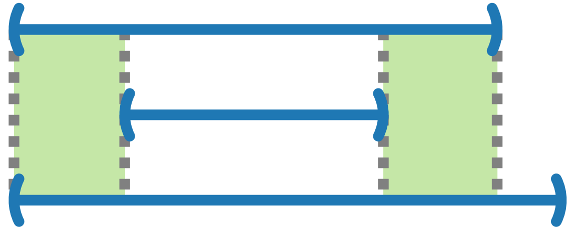

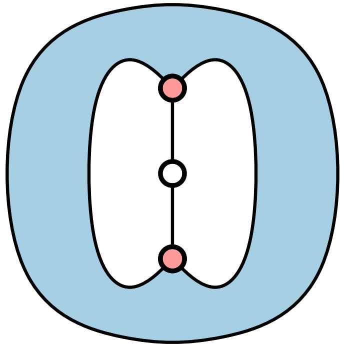

Separation pair!

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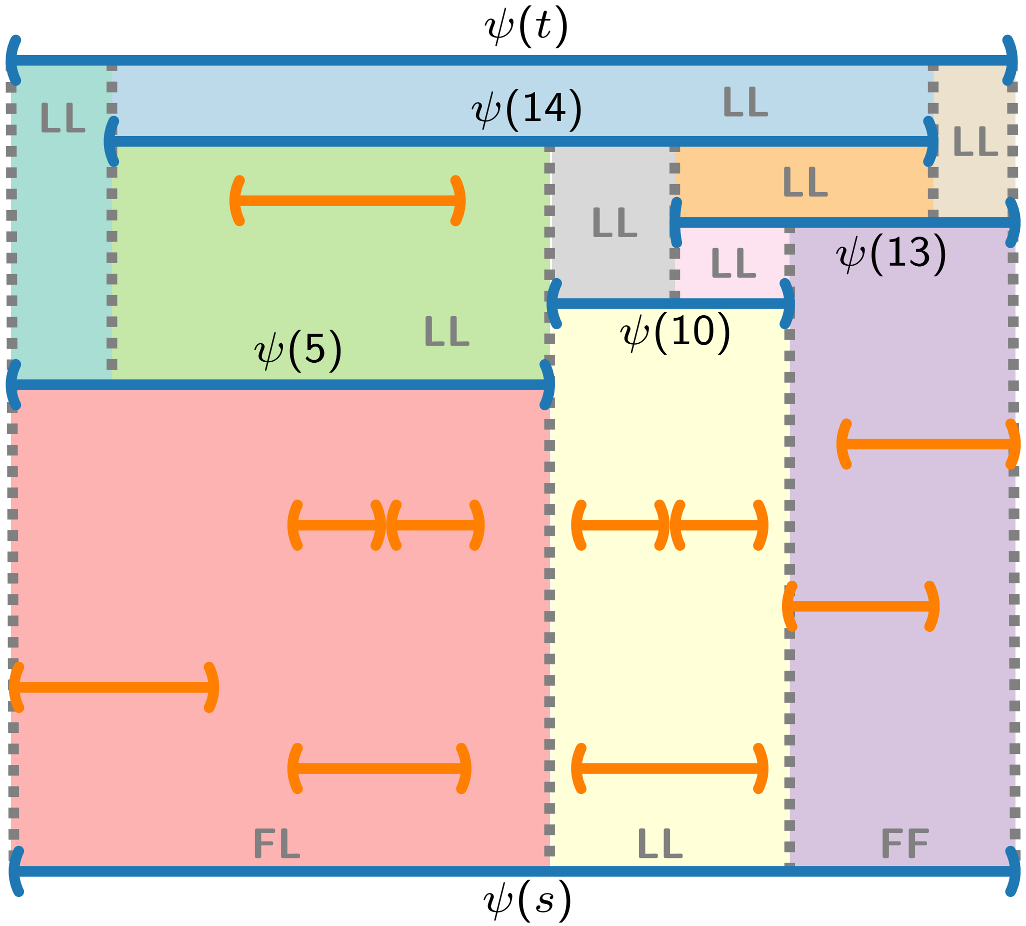

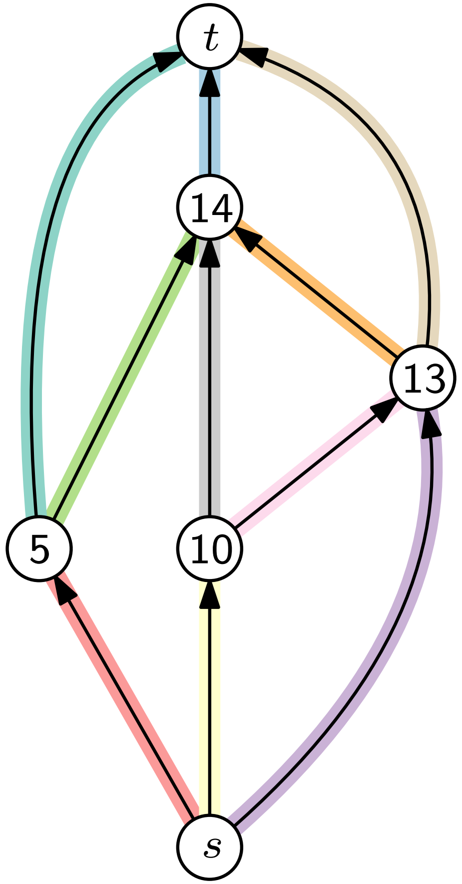

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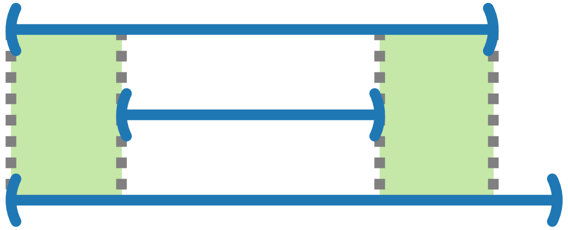

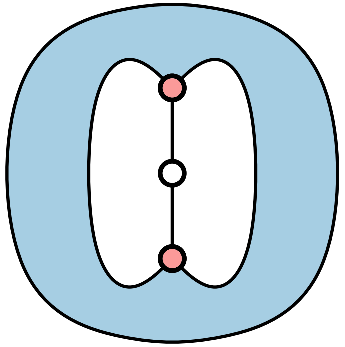
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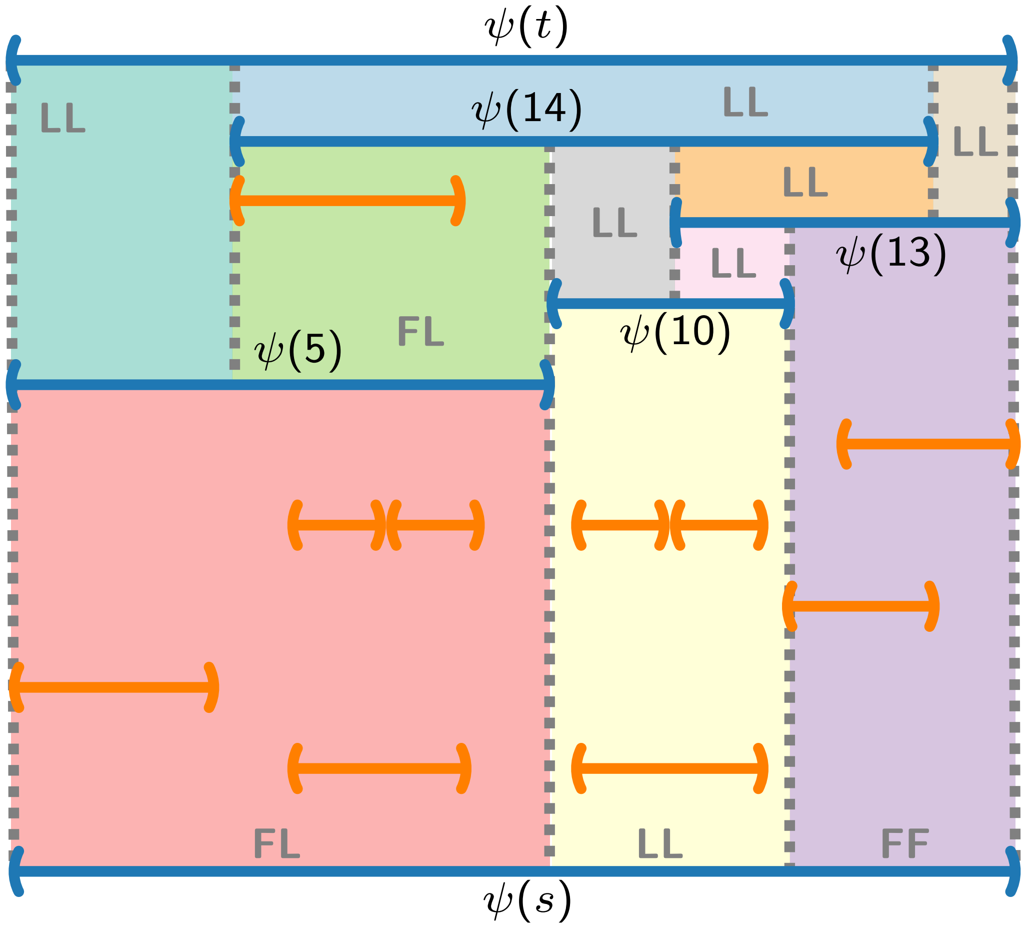

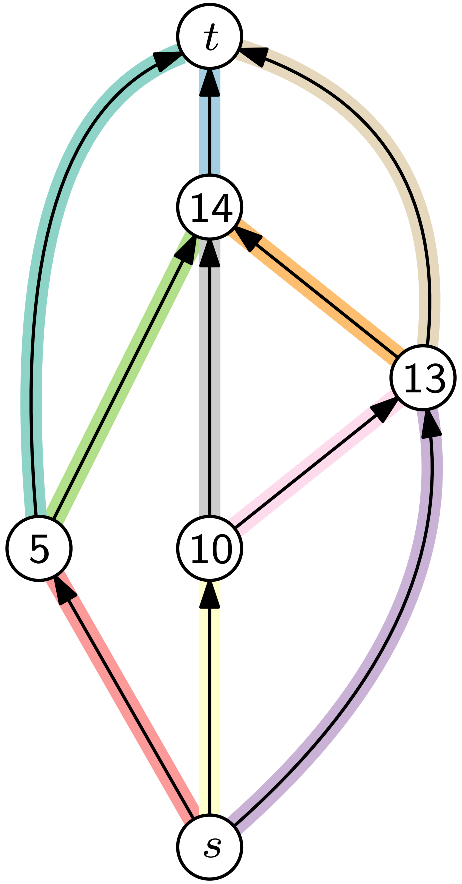

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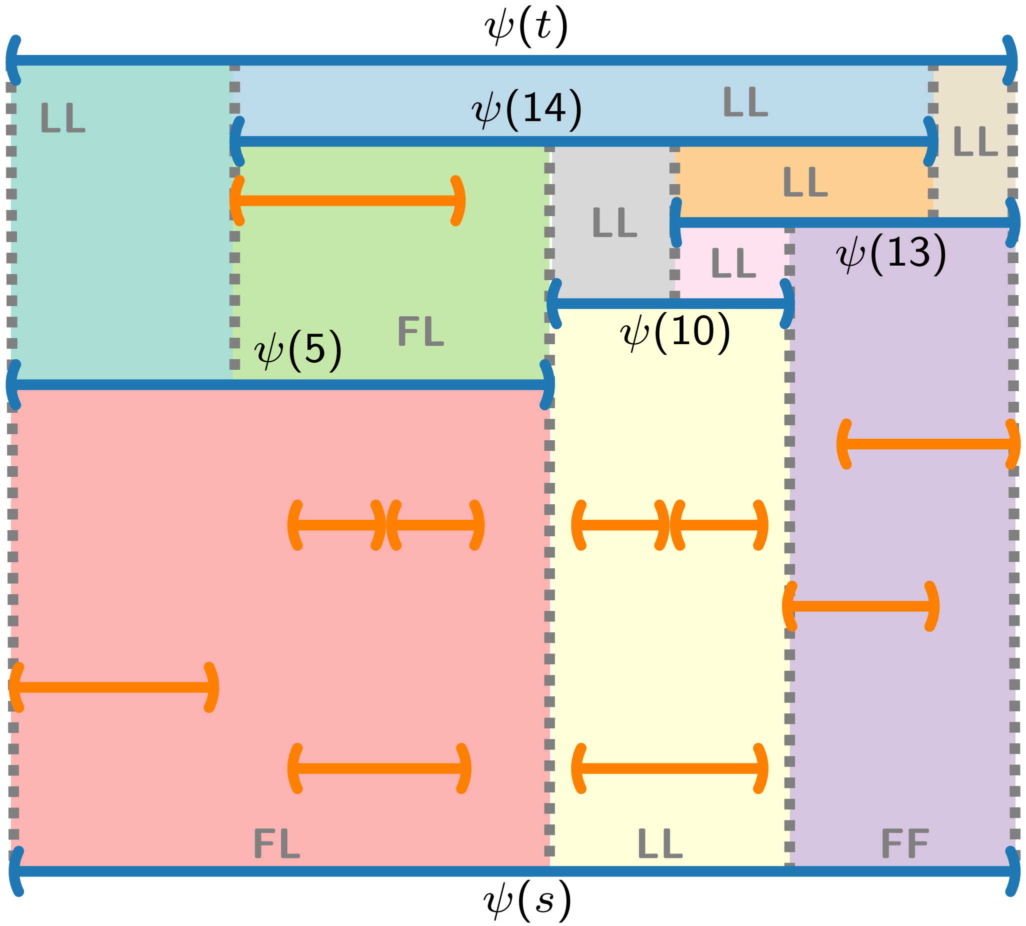

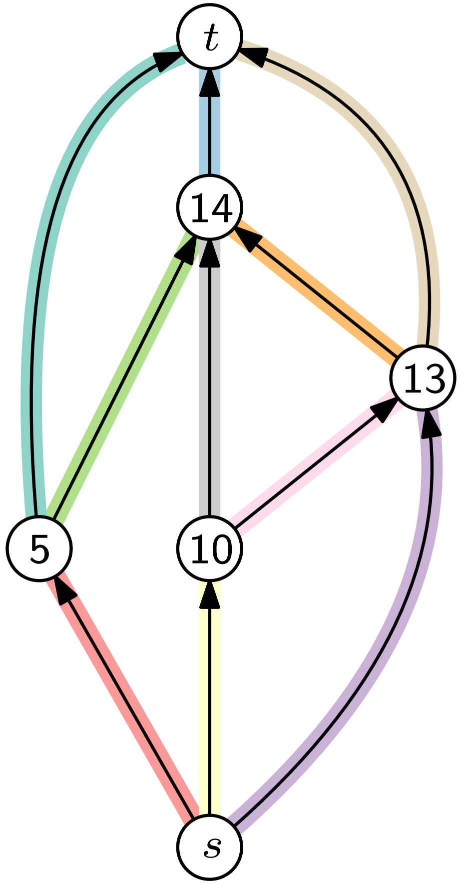

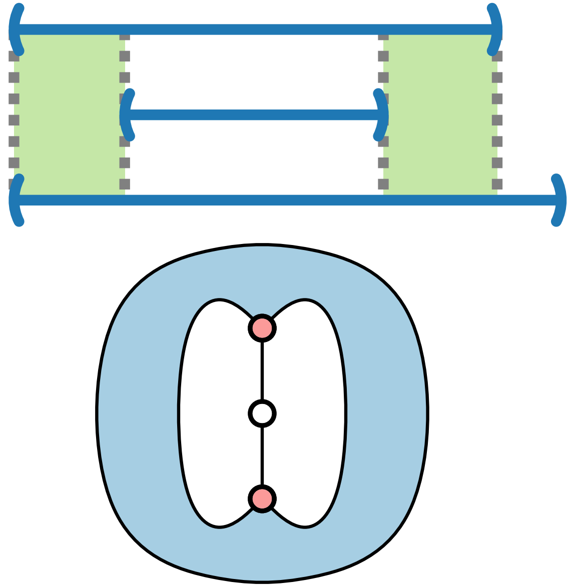

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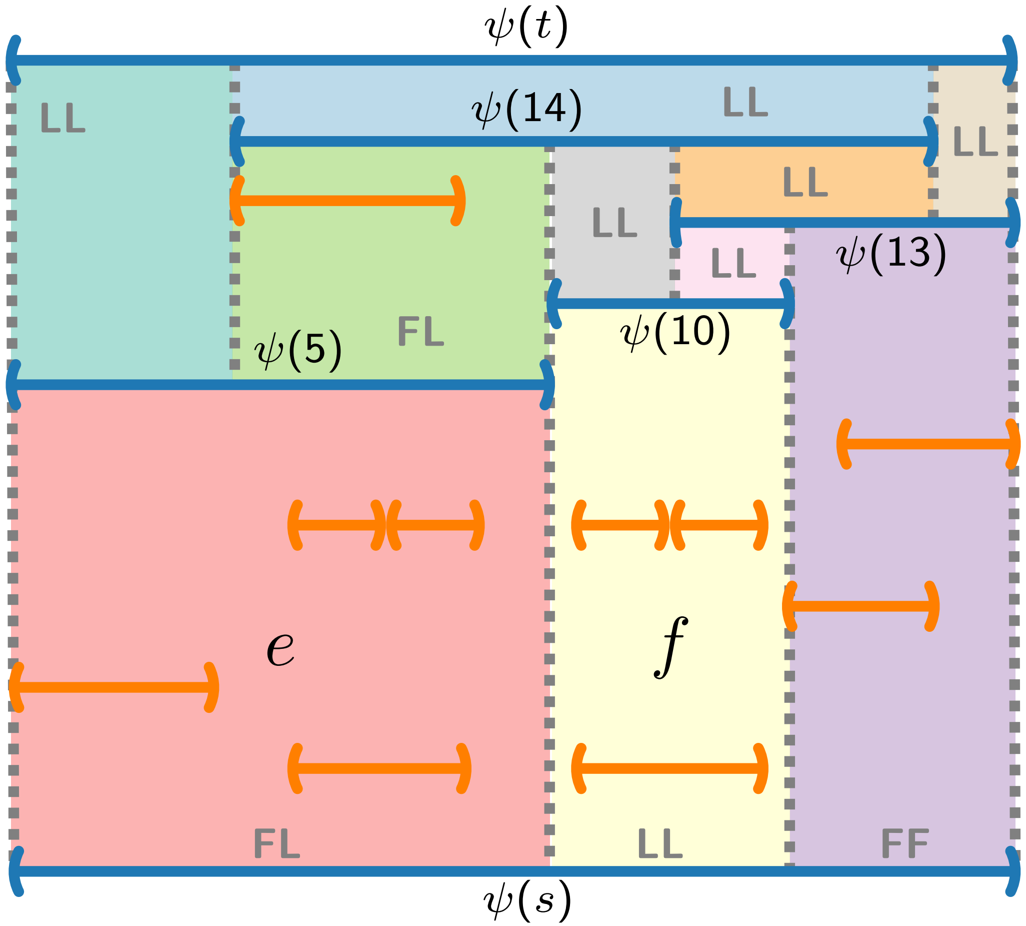

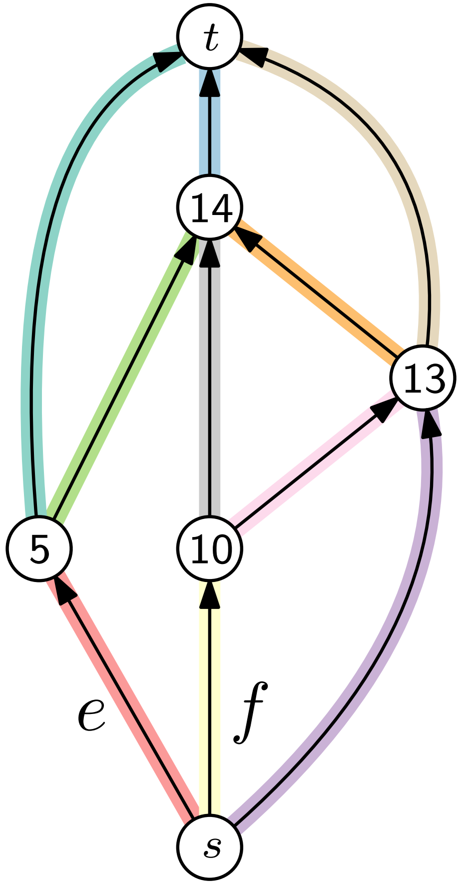

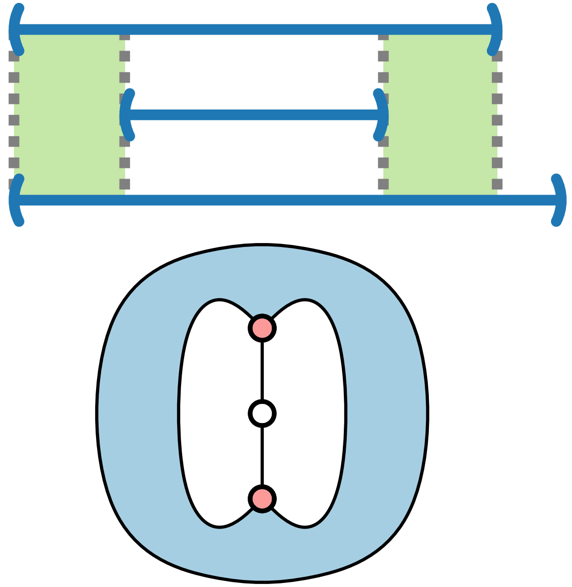

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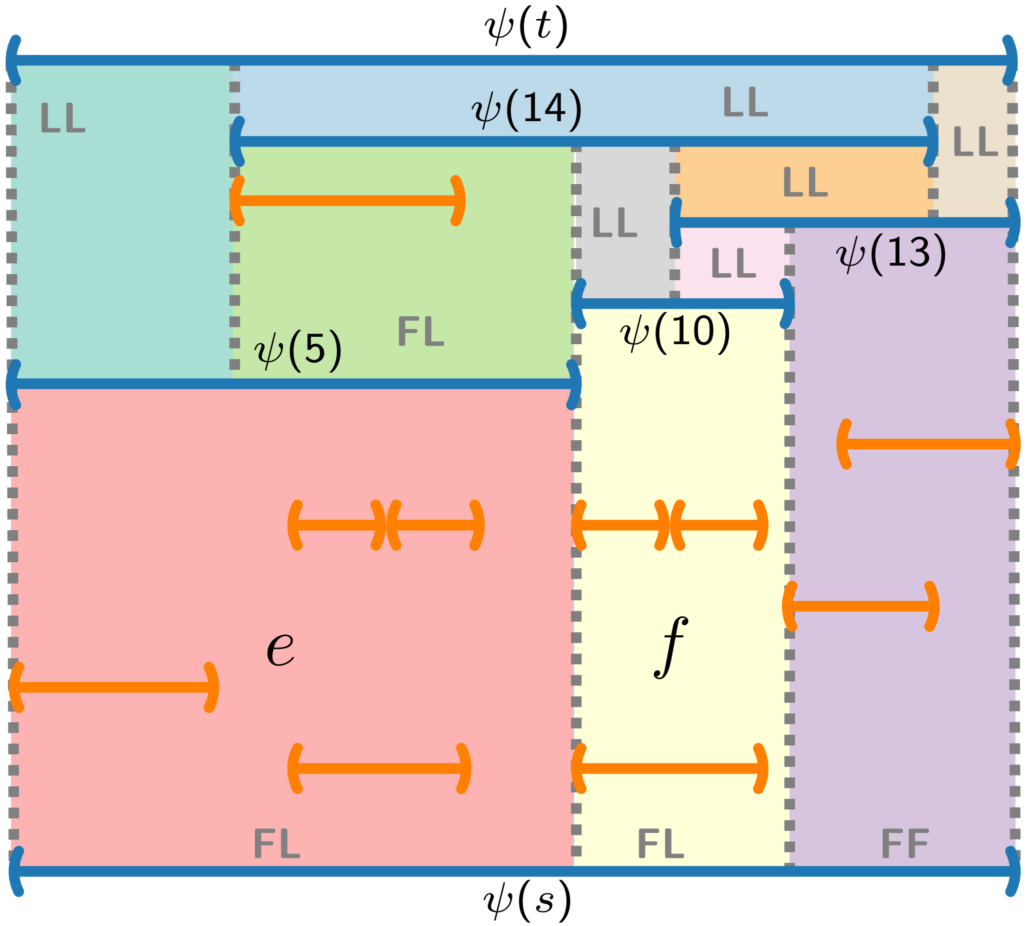

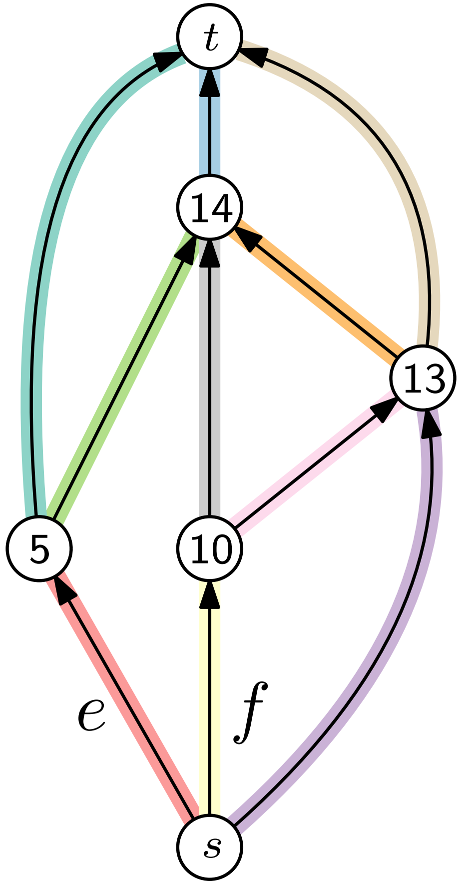

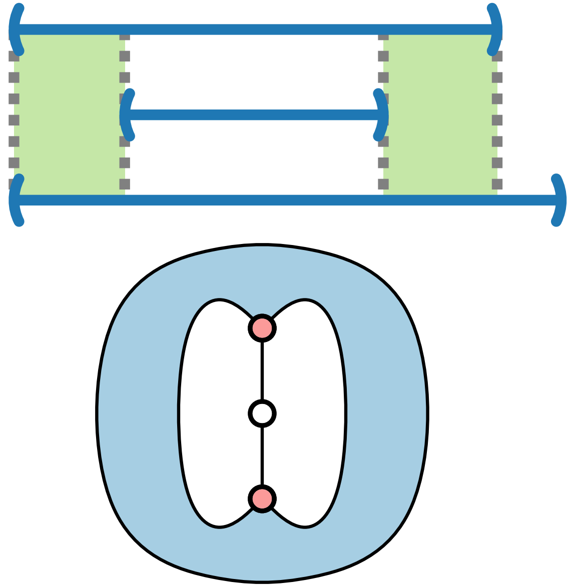

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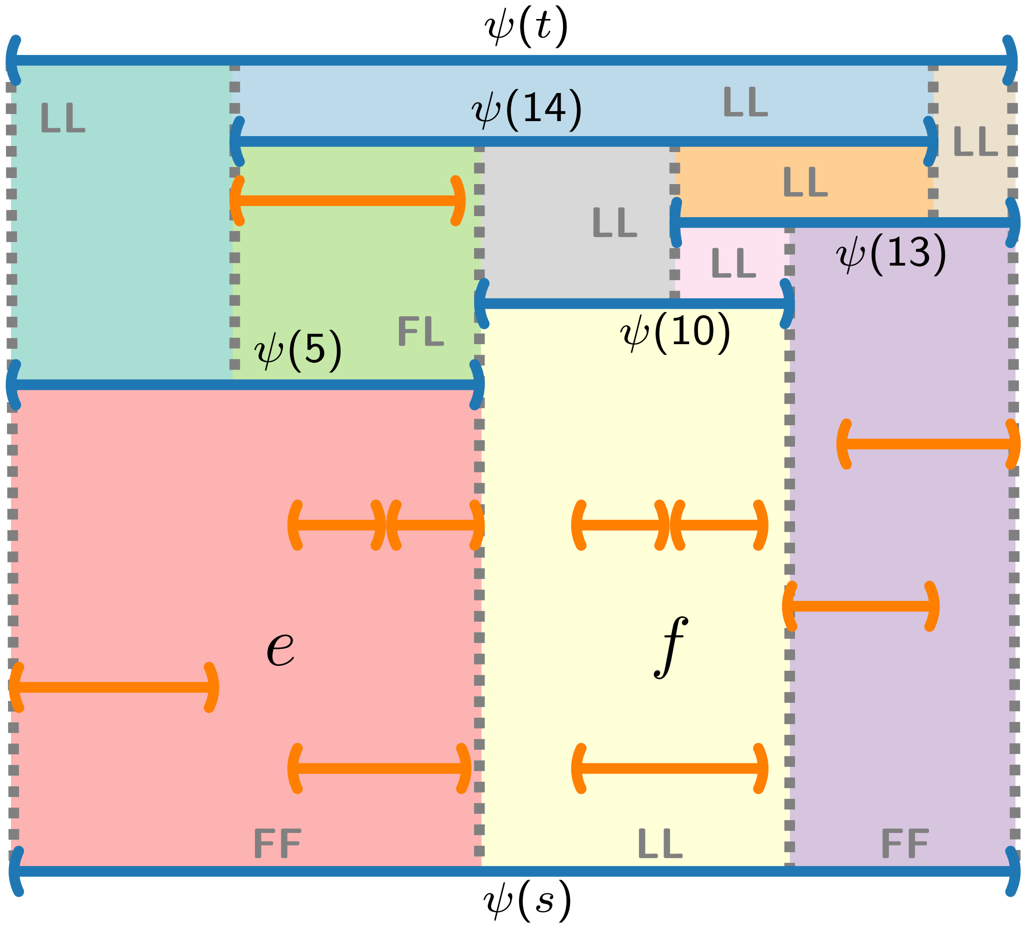

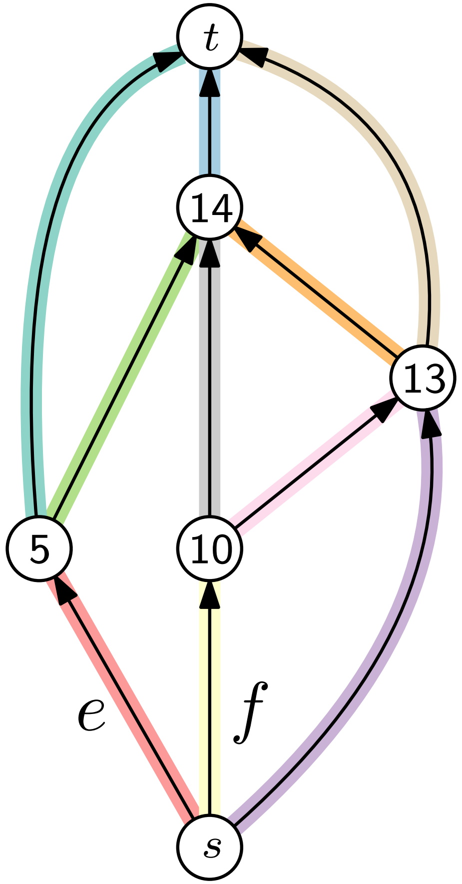

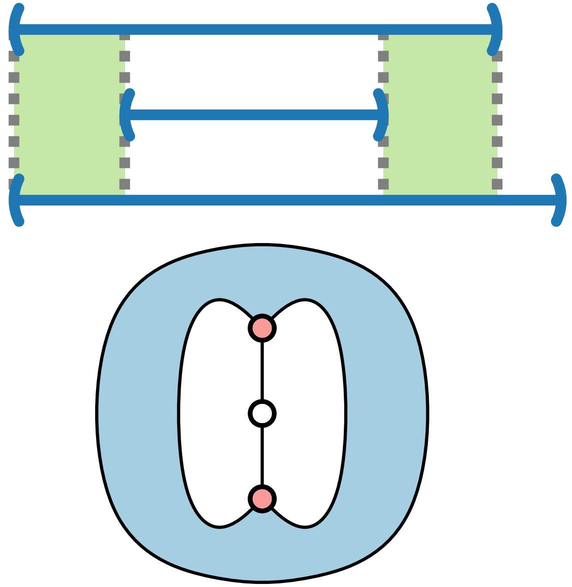

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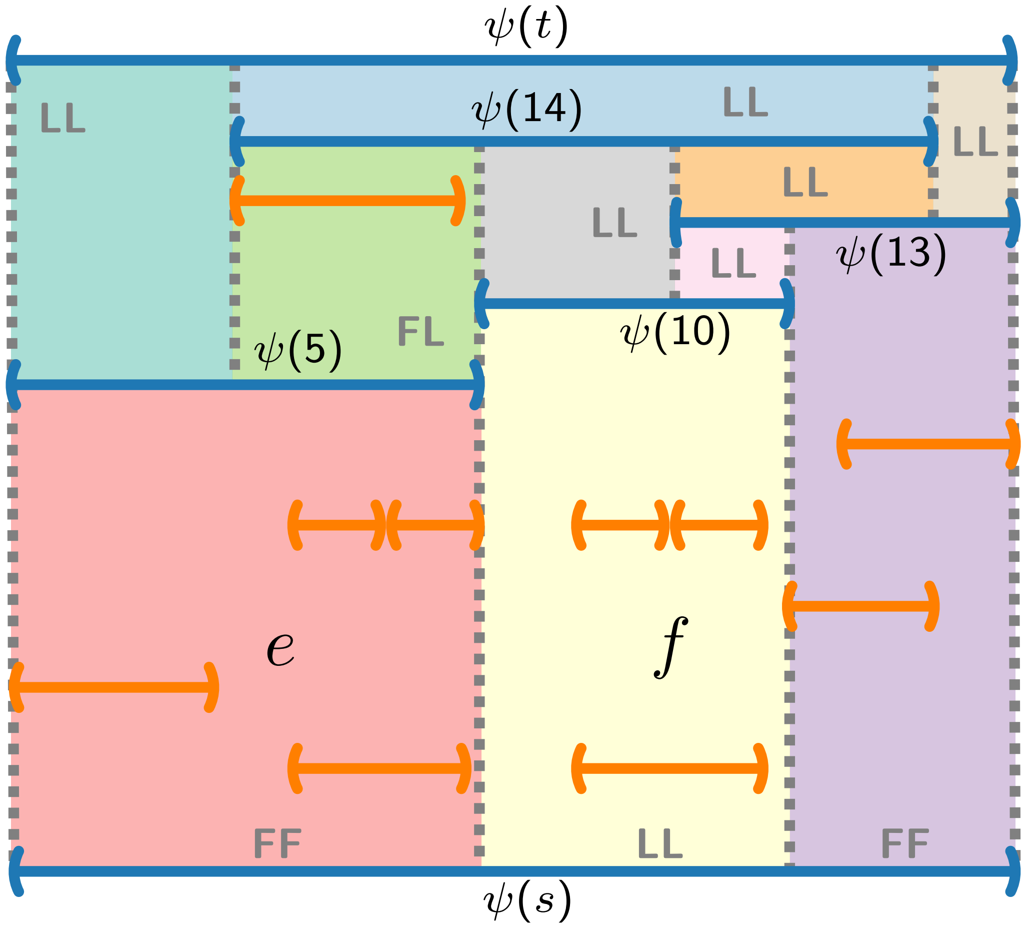

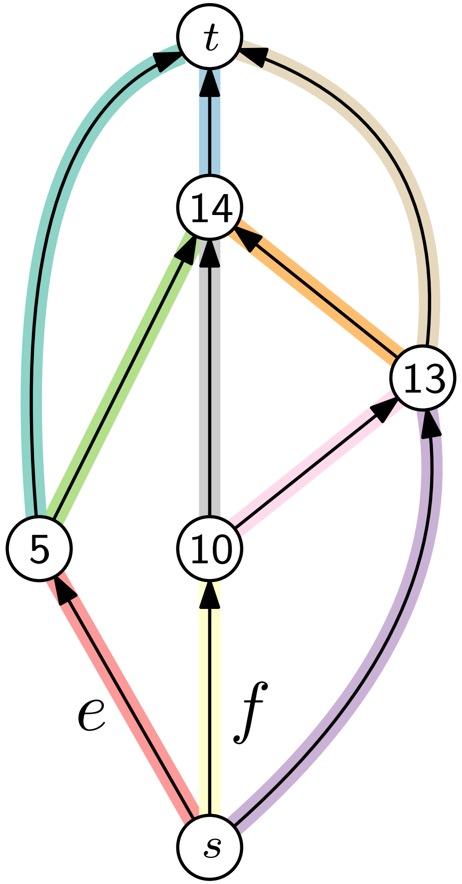

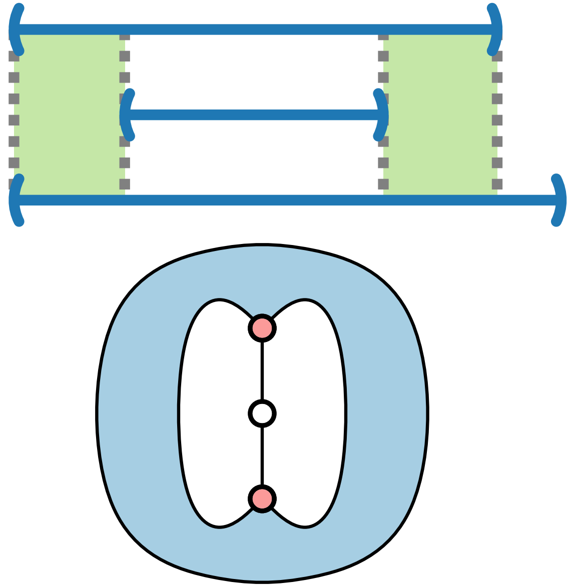

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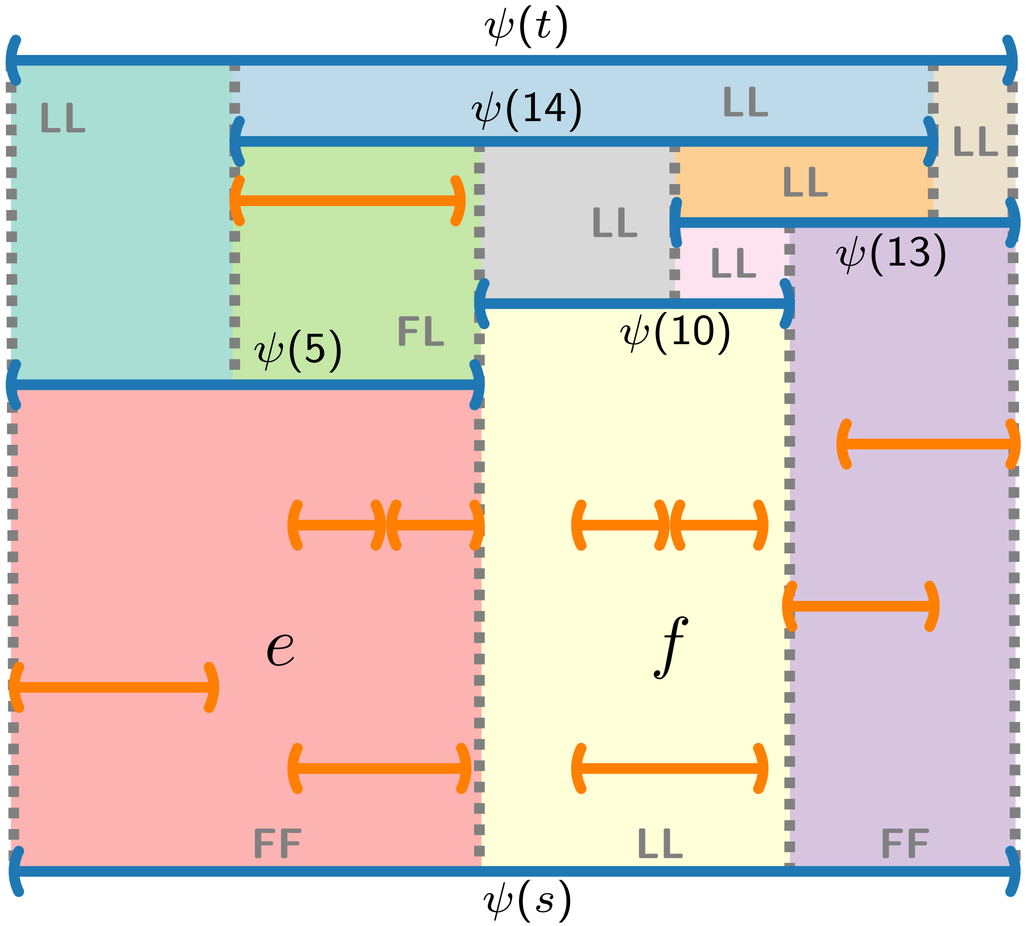
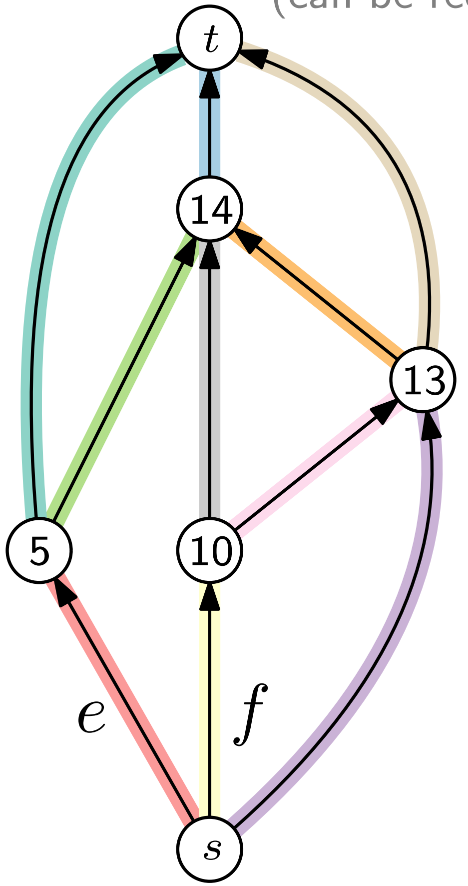
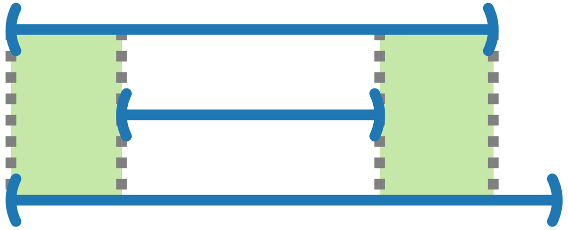
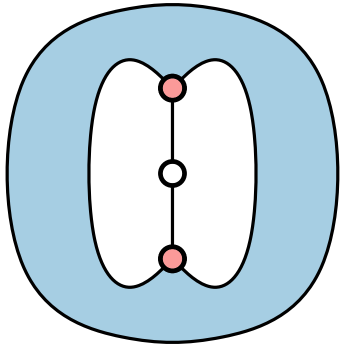
(can be reduced to $O(n \log^2 n)$)









Separation pair!

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R-Nodes with 2-SAT Formulation

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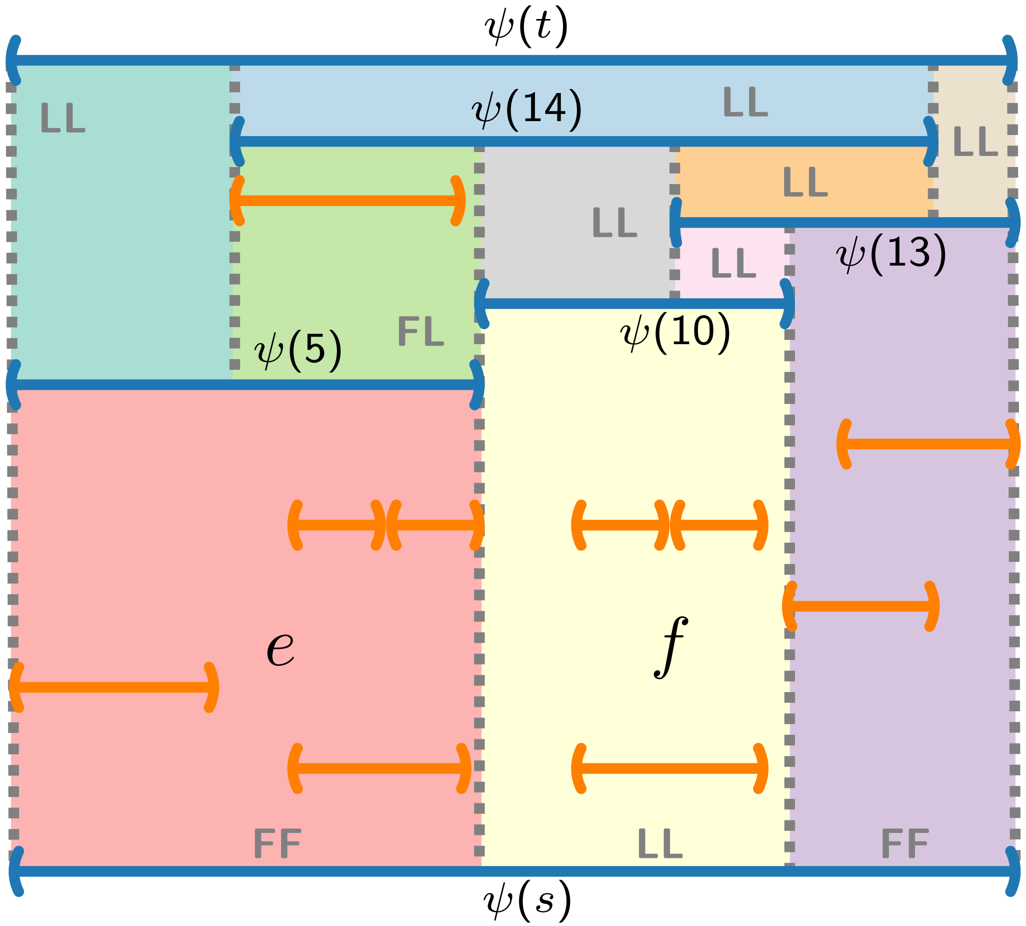

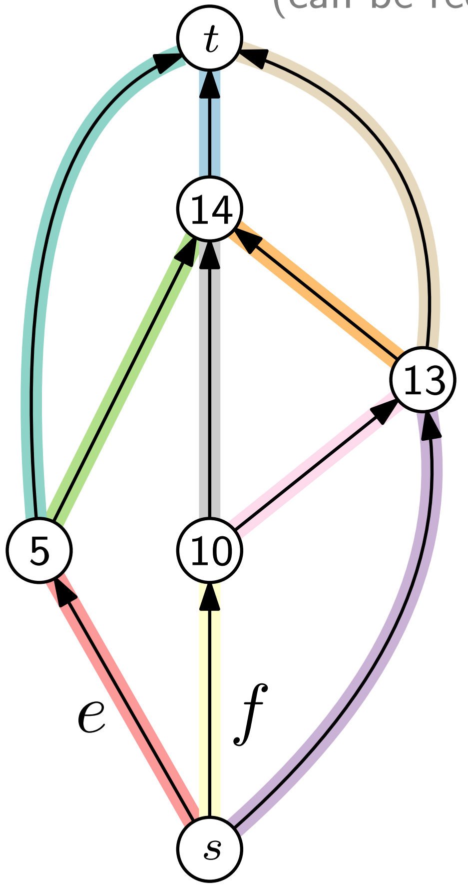

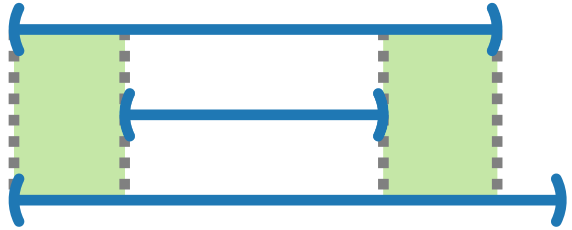

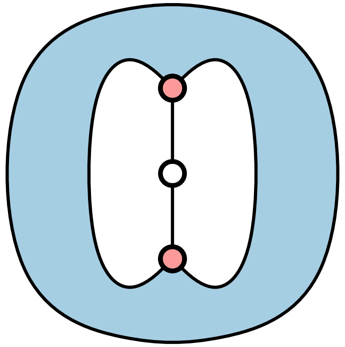

Separation pair!

($\nexists$ in **R**-component.)

- Finding a satisfying assignment of a 2-SAT formula can be done in linear time!

R-Nodes with 2-SAT Formulation

- For each child (edge) e:
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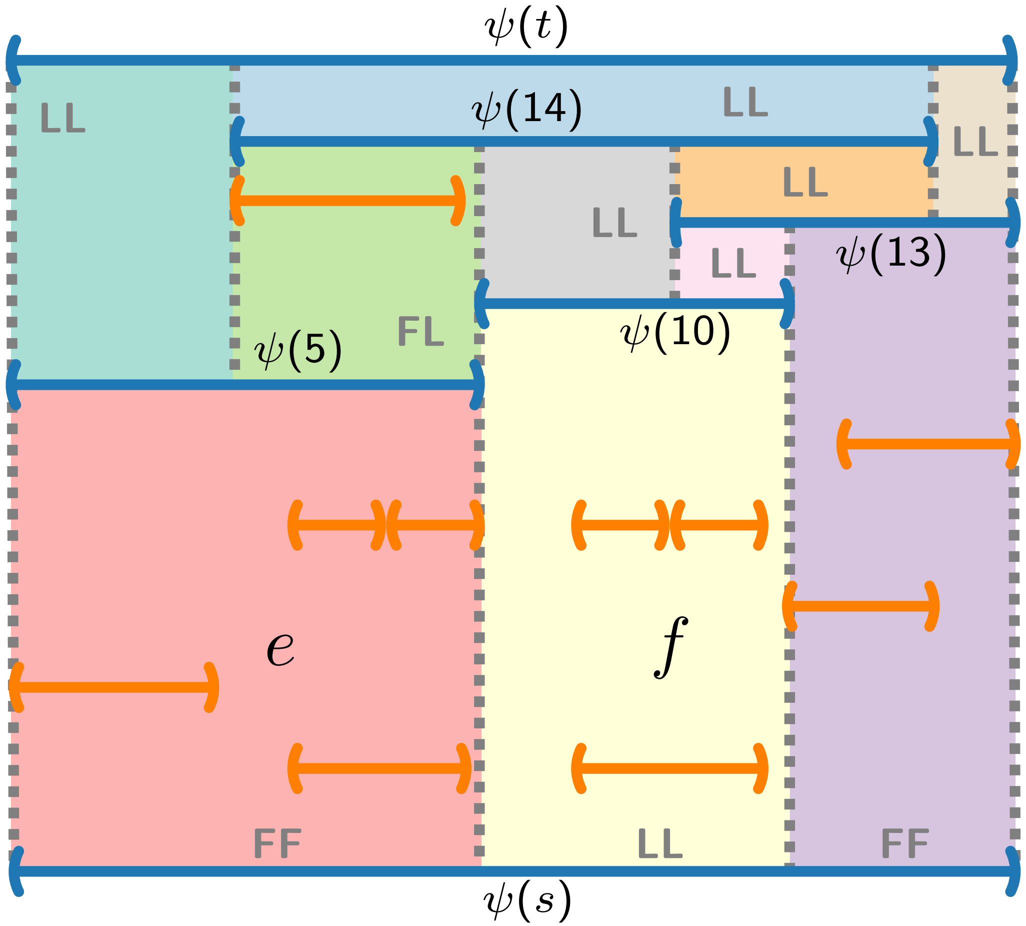

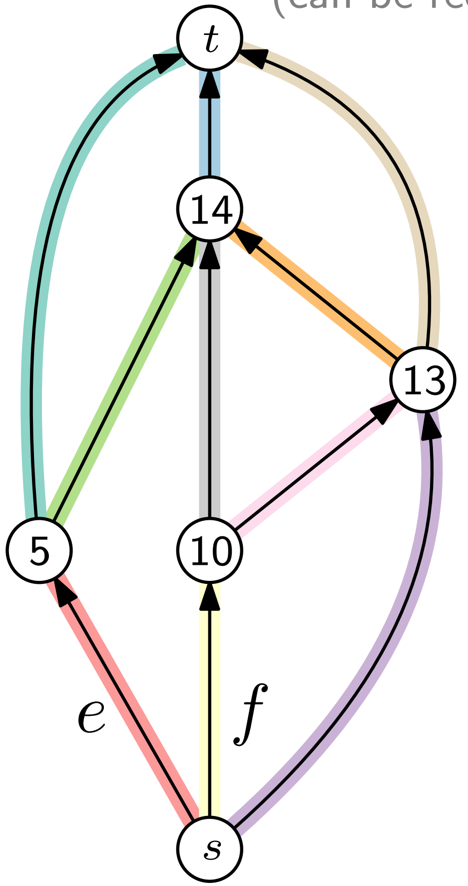

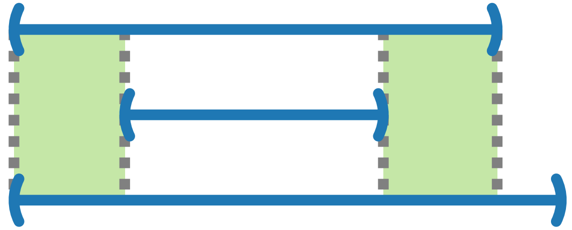

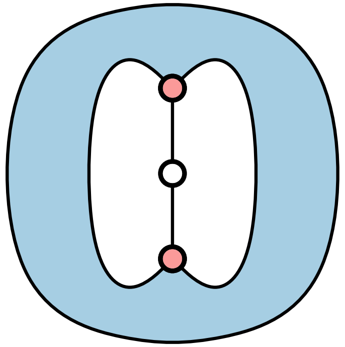

Separation pair!
($\nexists$ in **R**-component.)

- Finding a satisfying assignment of a 2-SAT formula can be done in linear time!

$\Rightarrow O(n^2)$ time in total

R-Nodes with 2-SAT Formulation

- For each child (edge) e:
  - Find all types of {**FF, FL, LF, LL**} that admit a drawing.
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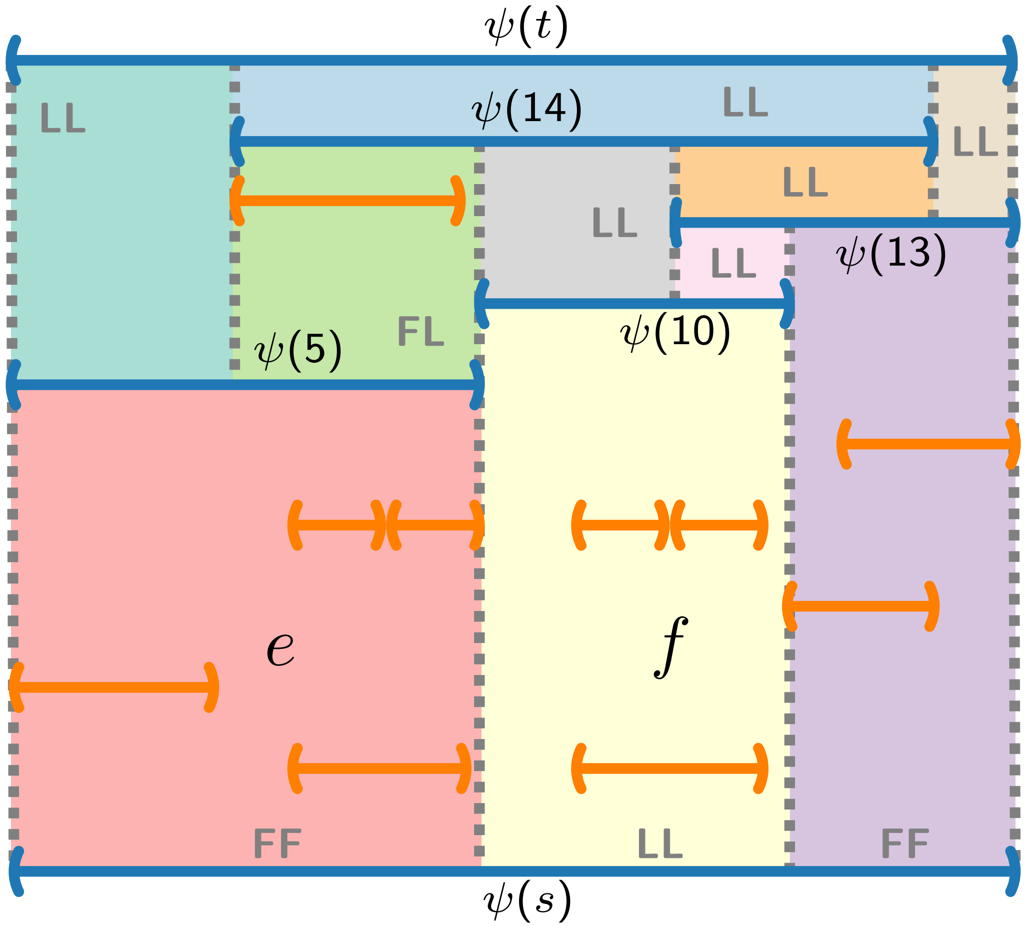
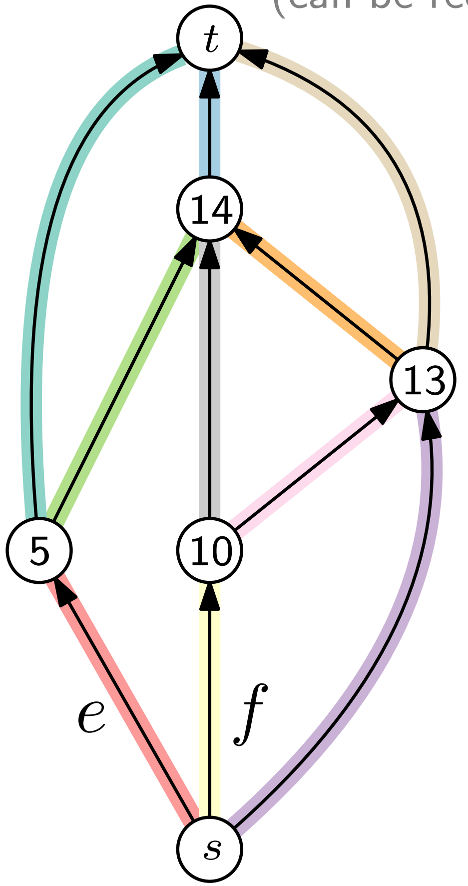
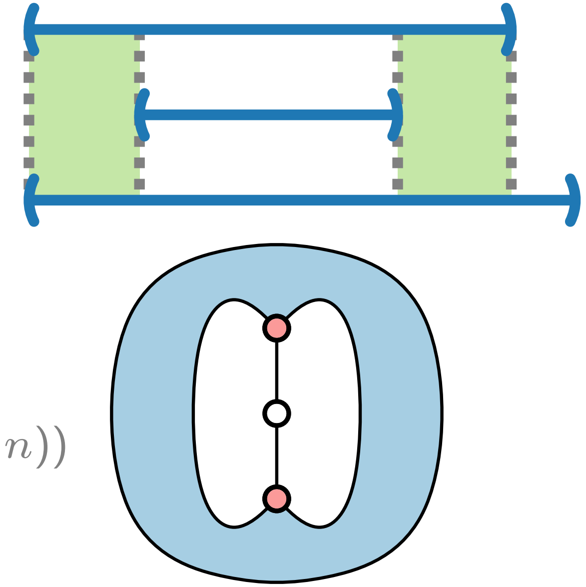
(can be reduced to $O(n \log^2 n)$)







Separation pair!

($\nexists$ in **R**-component.)

- Finding a satisfying assignment of a 2-SAT formula can be done in linear time!

$\Rightarrow O(n^2)$ time in total or $O(n \log^2 n)$

Results and Outline

[Chaplick, Guśpiel, Gutowski, Krawczyk, Liotta '18]

Theorem 1.
Rectangular ε-bar visibility representation extension can be solved in $\mathcal{O}(n \log^2 n)$ time for st-graphs.

- Dynamic program via SPQR-trees
- Easier version: $\mathcal{O}(n^2)$

Theorem 2.
ε-bar visibility representation extension is NP-complete.

- Reduction from Planar Monotone 3-SAT

Theorem 3.
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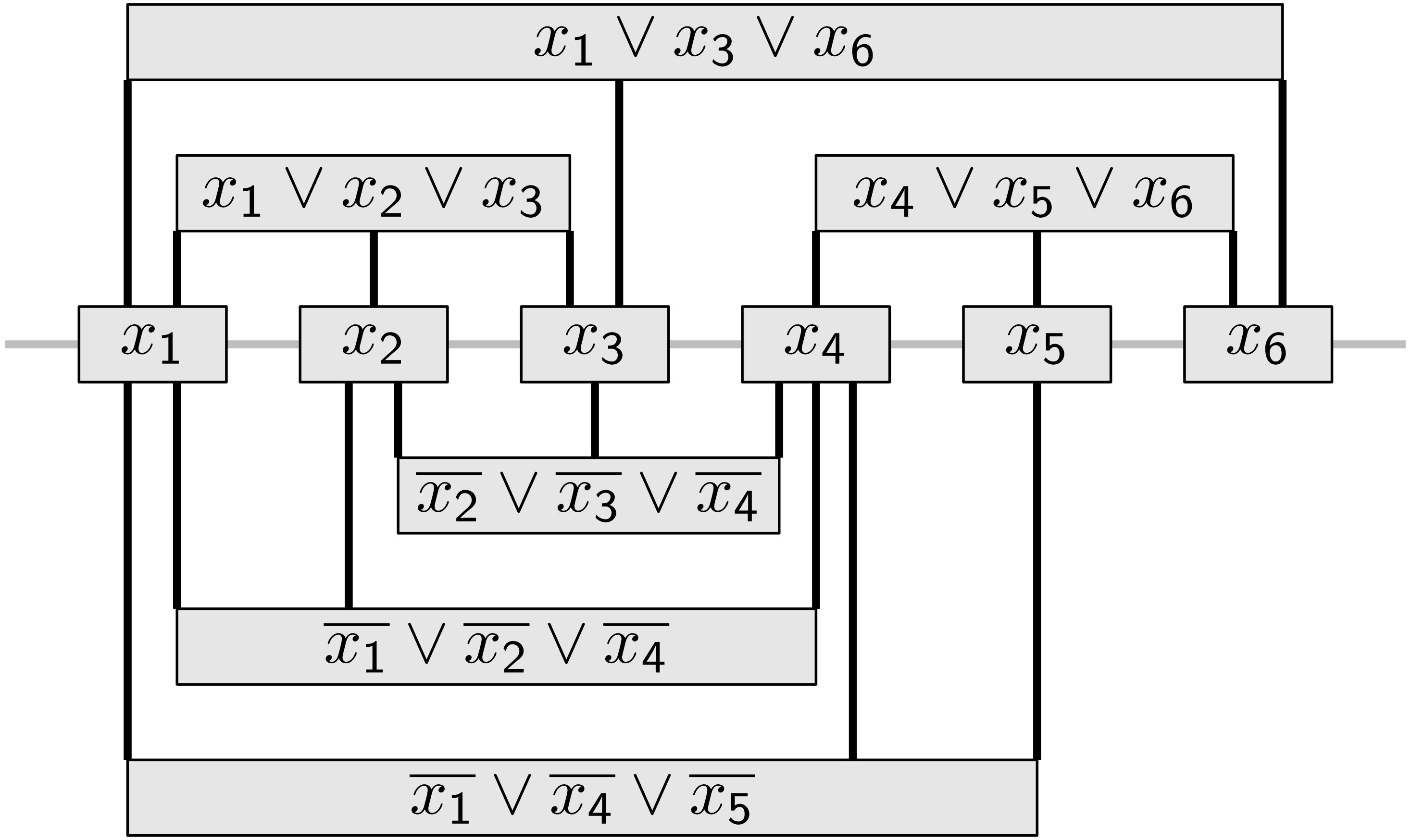

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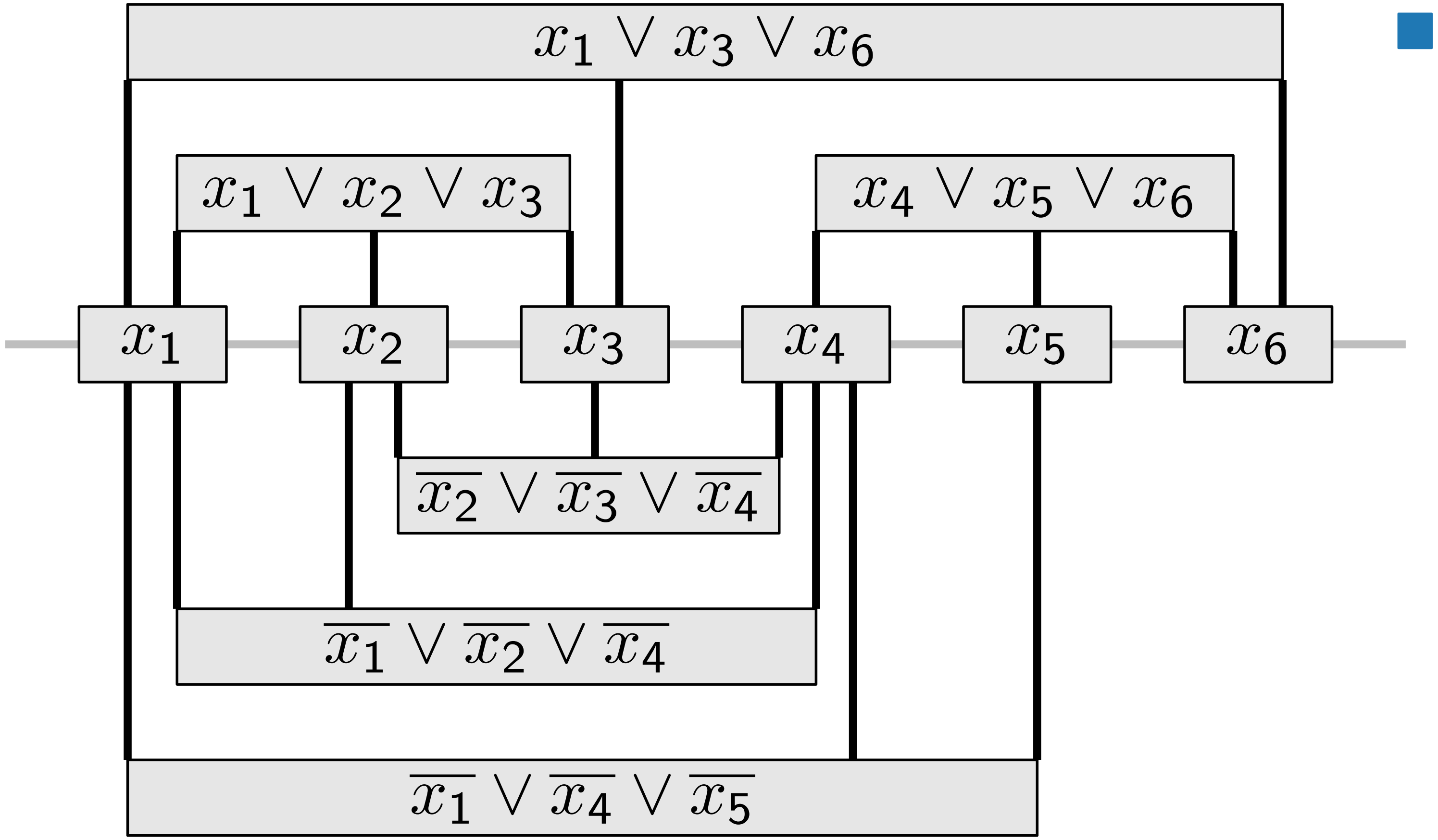

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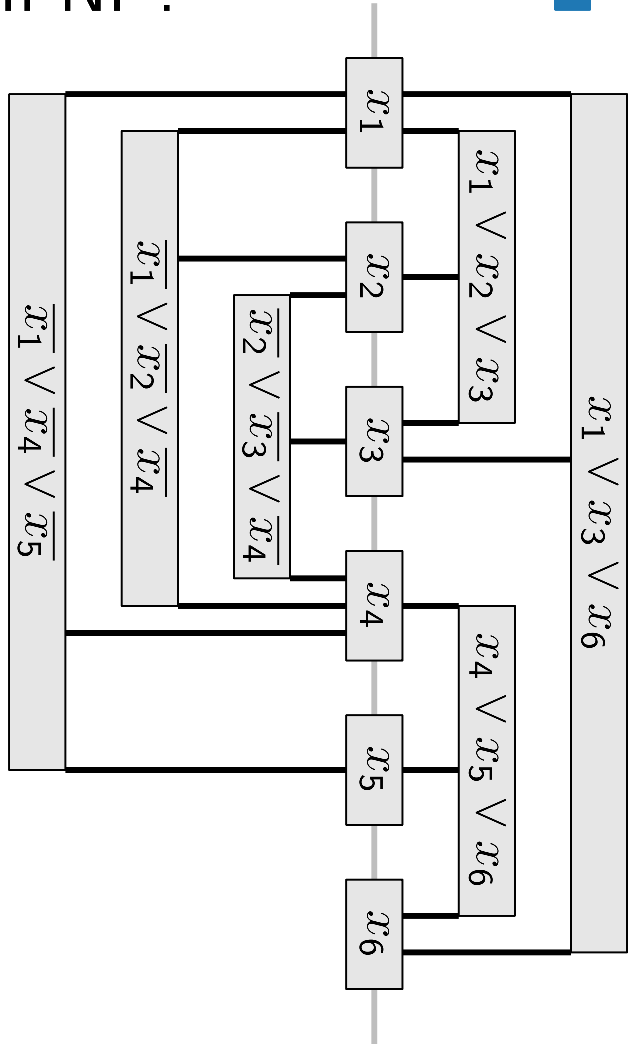

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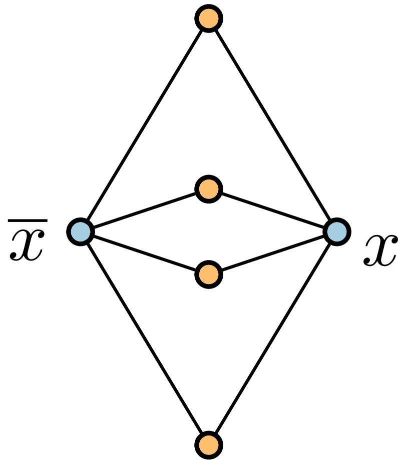

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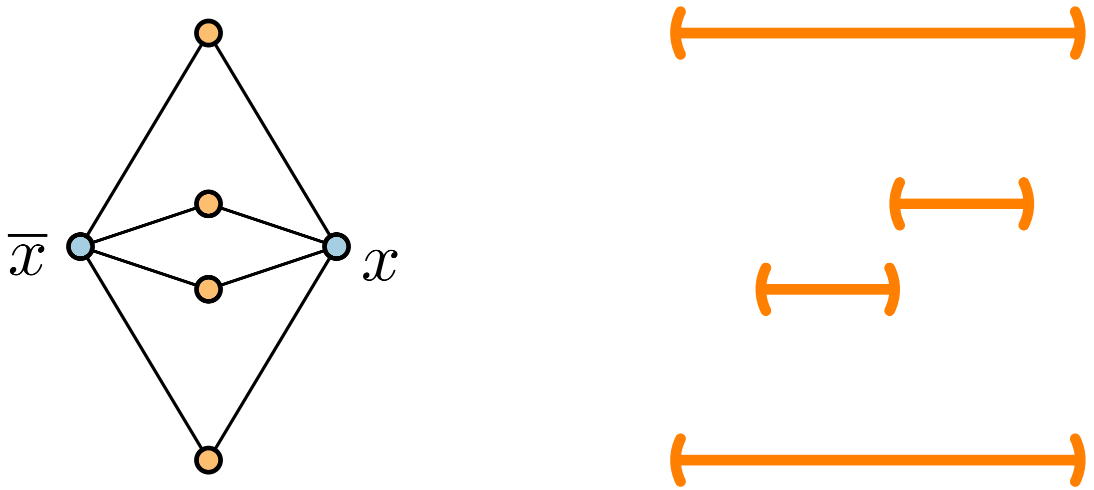

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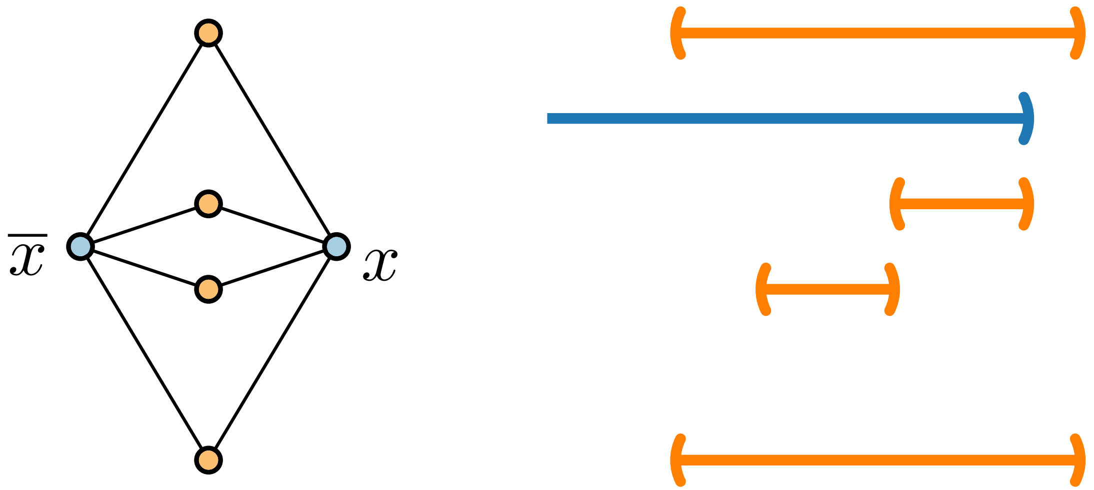

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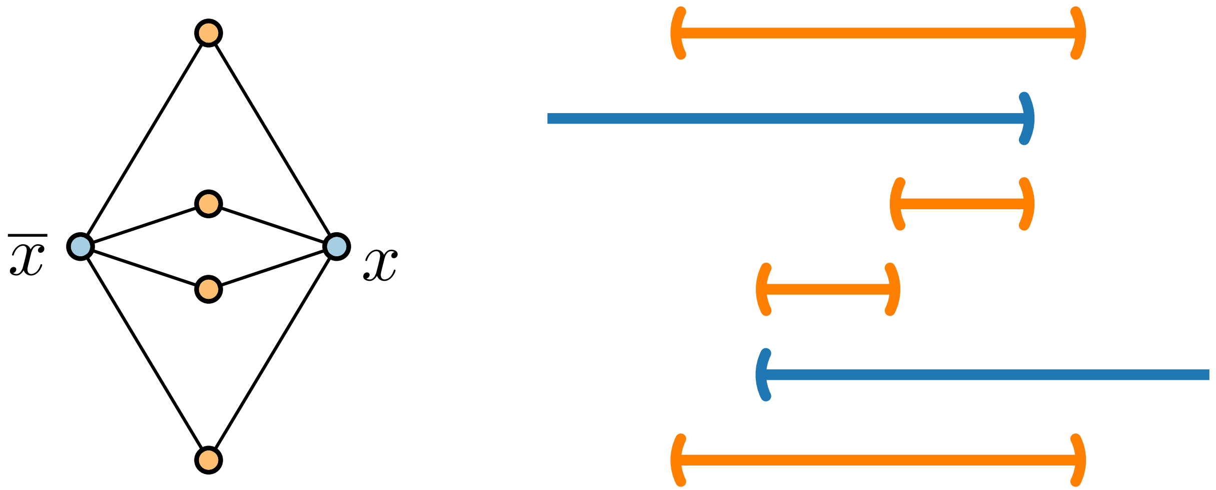

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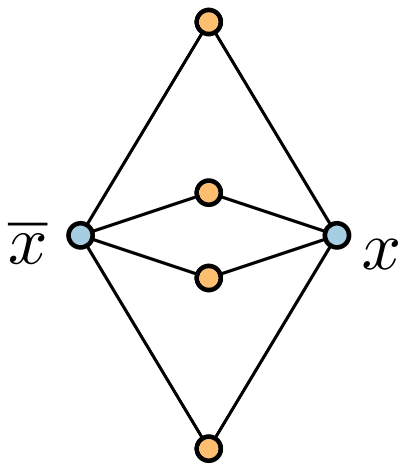

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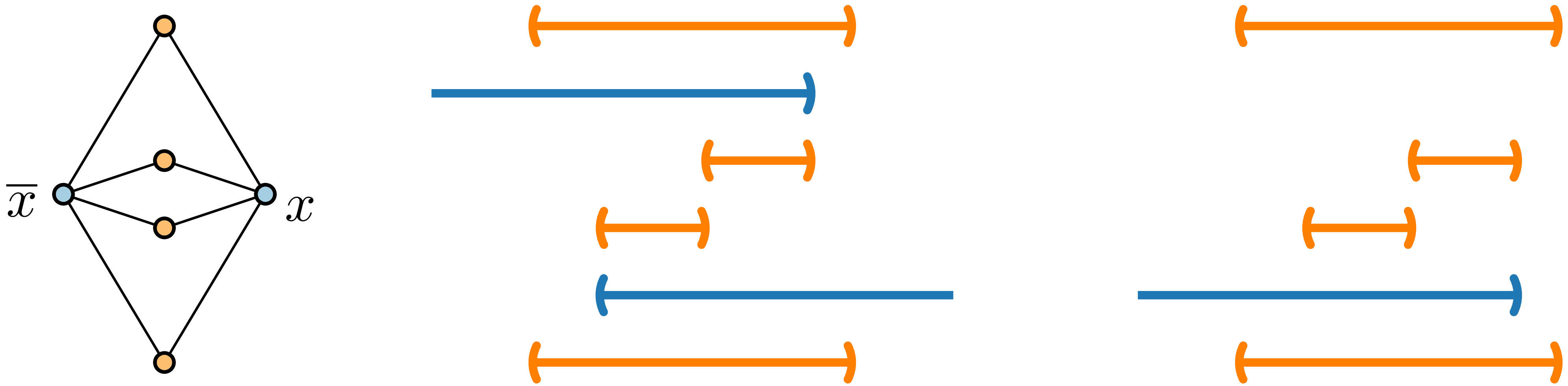

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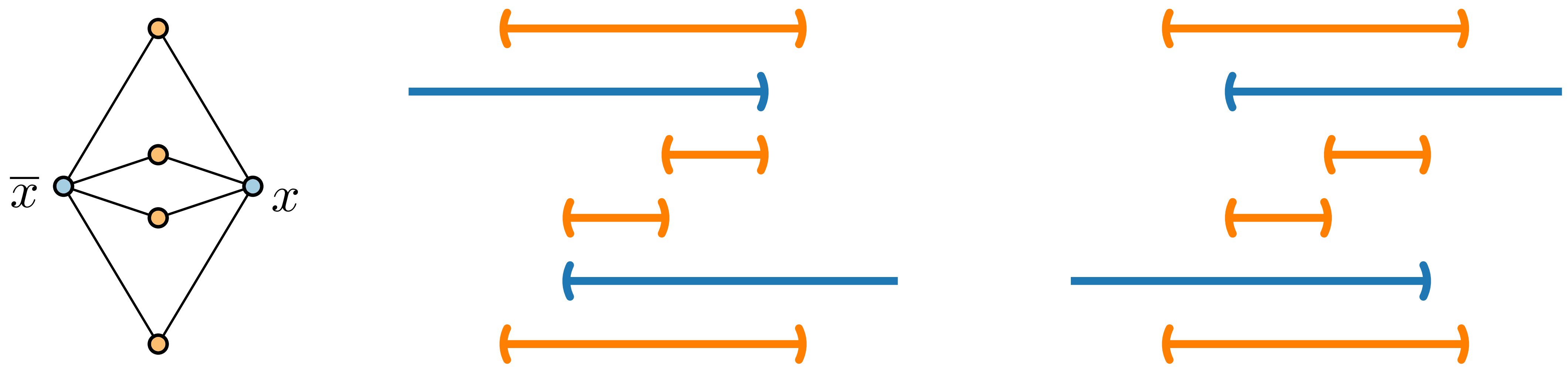

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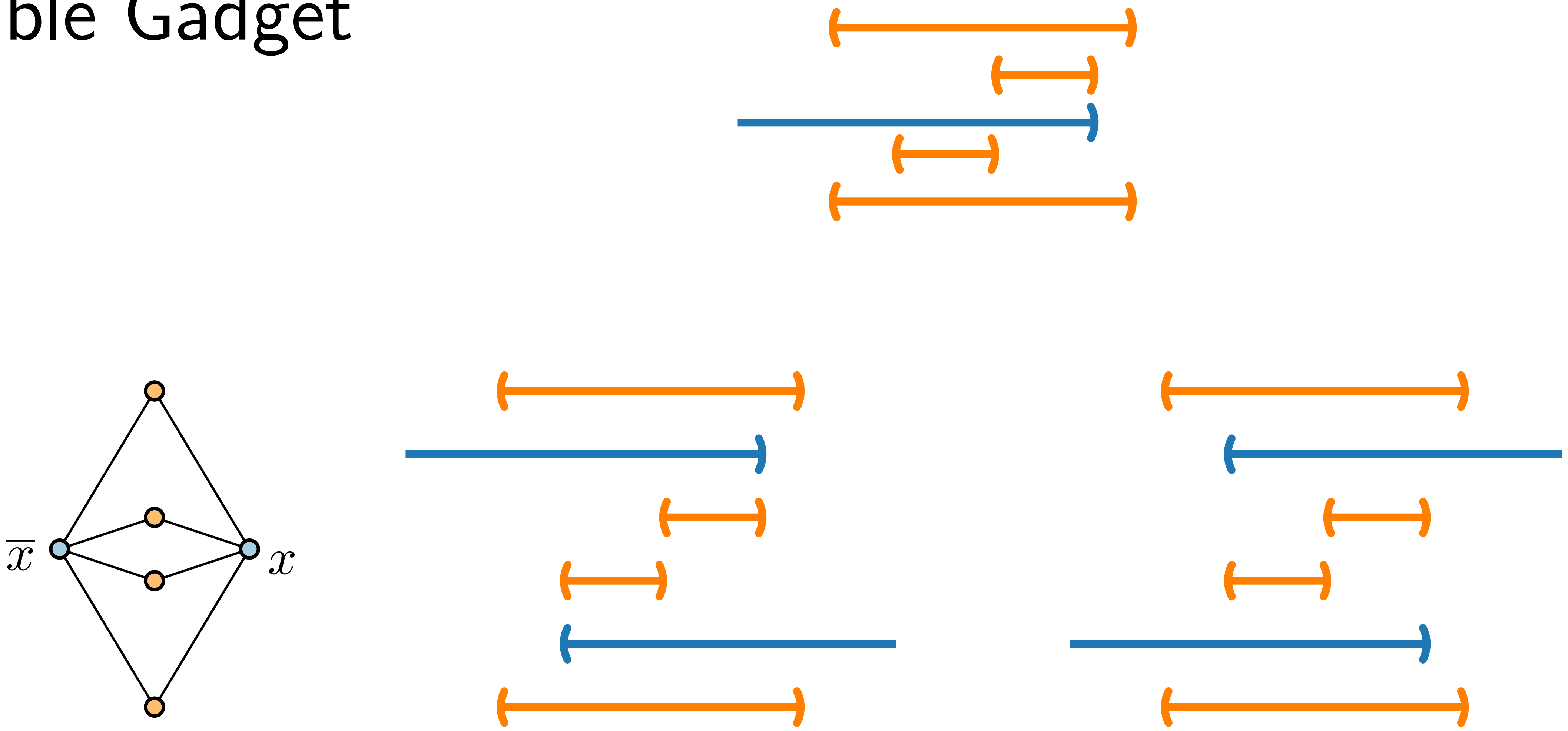

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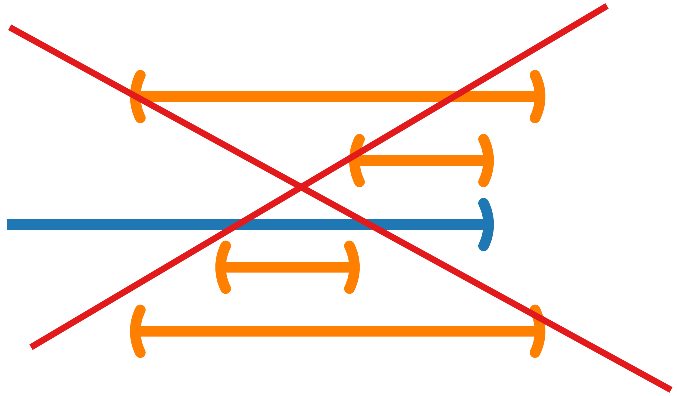

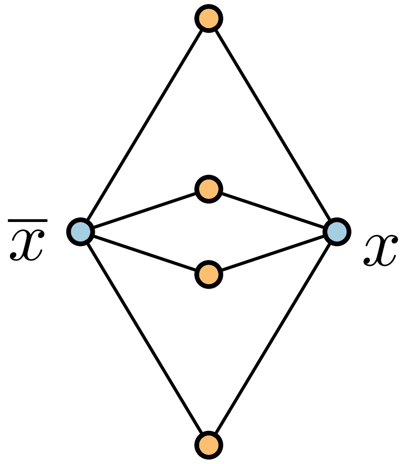

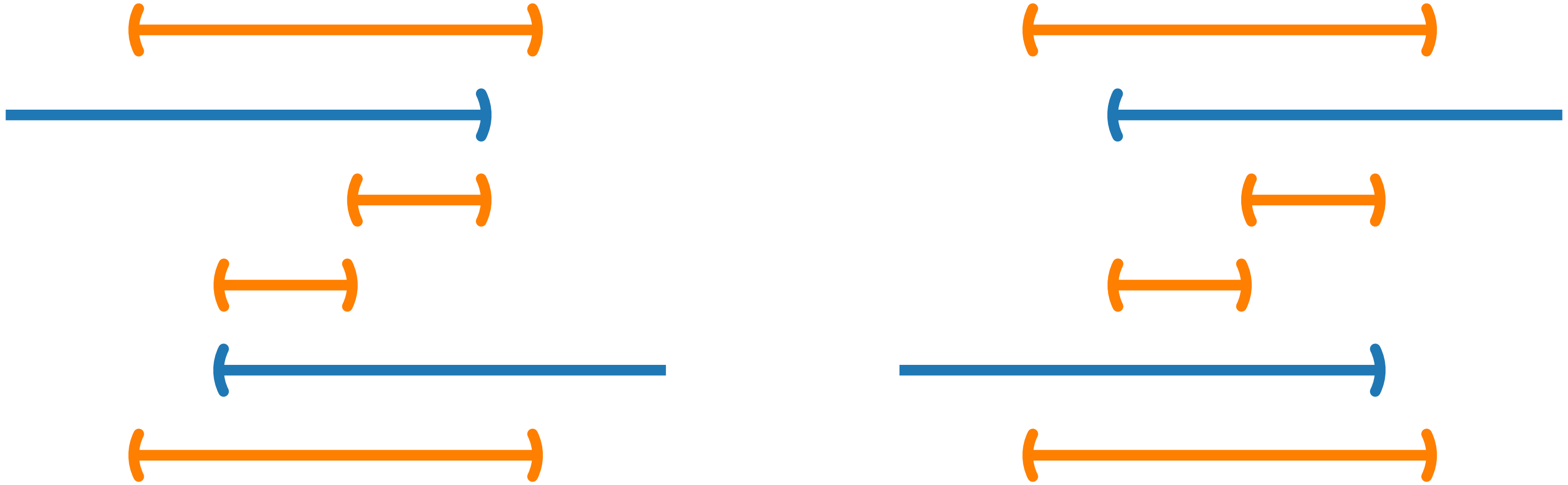

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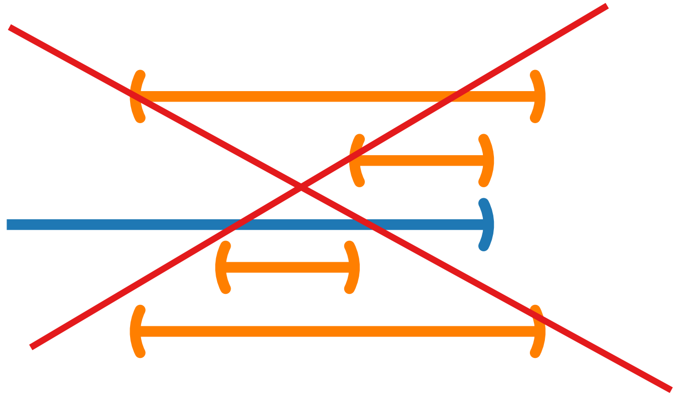
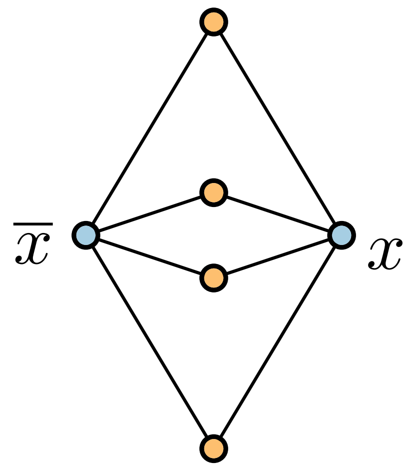
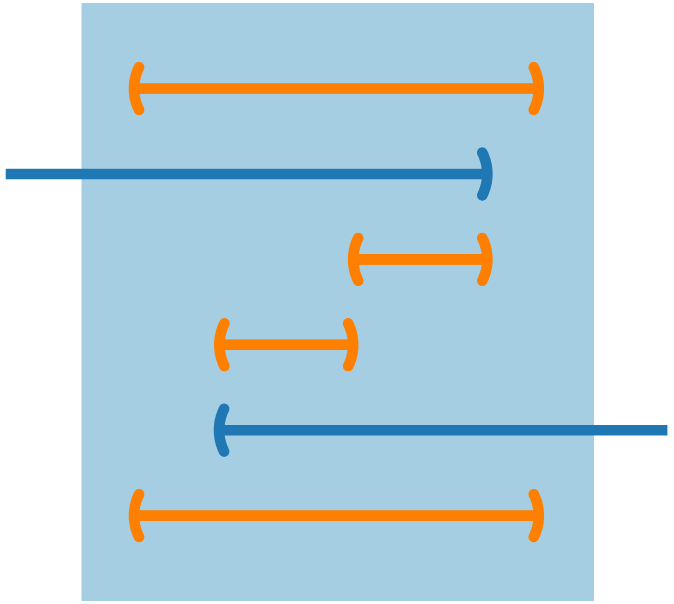




$x = \textsc{False}$



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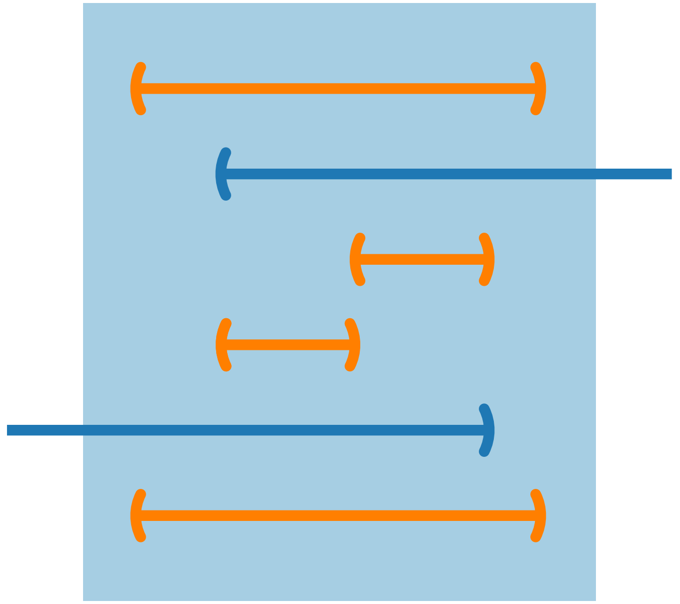

Clause Gadget

$$x \vee y \vee z$$

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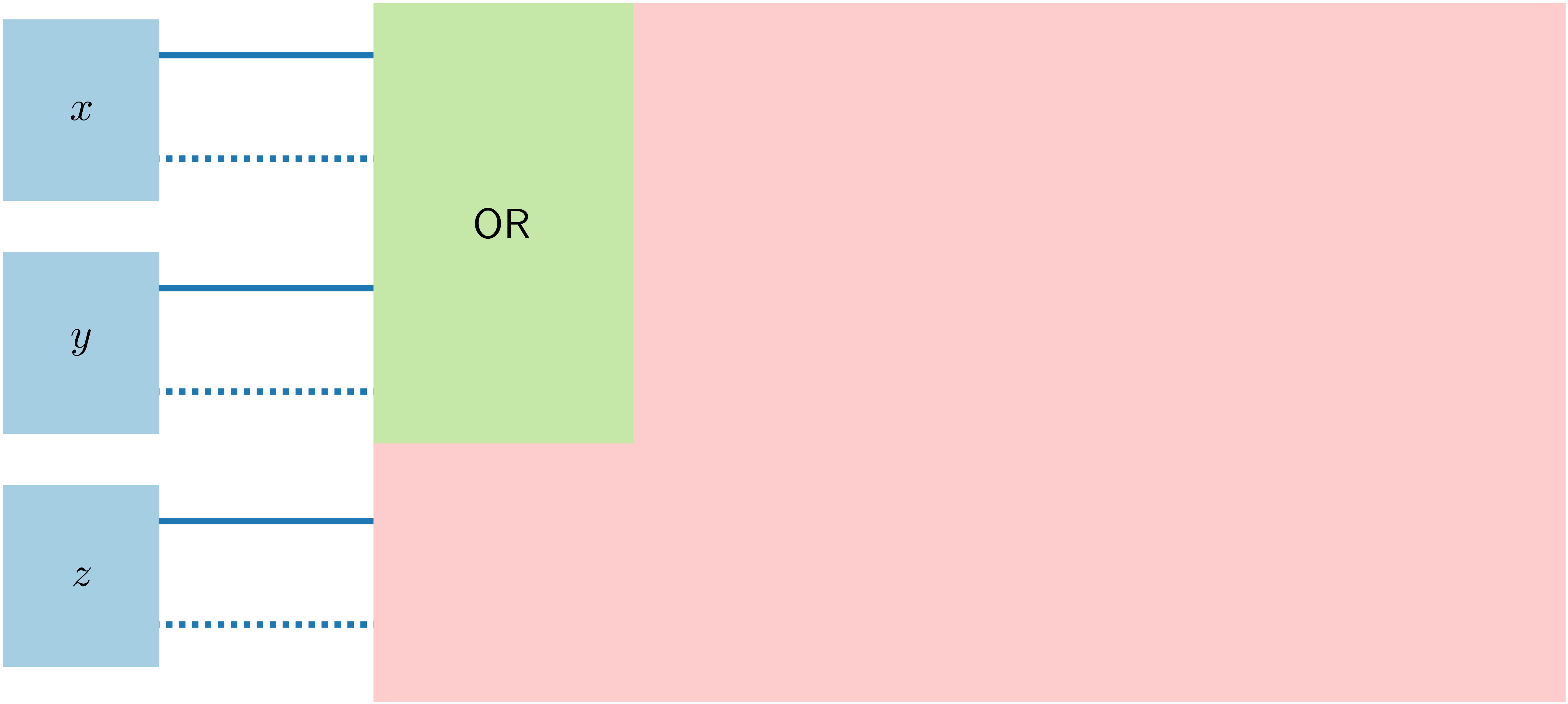

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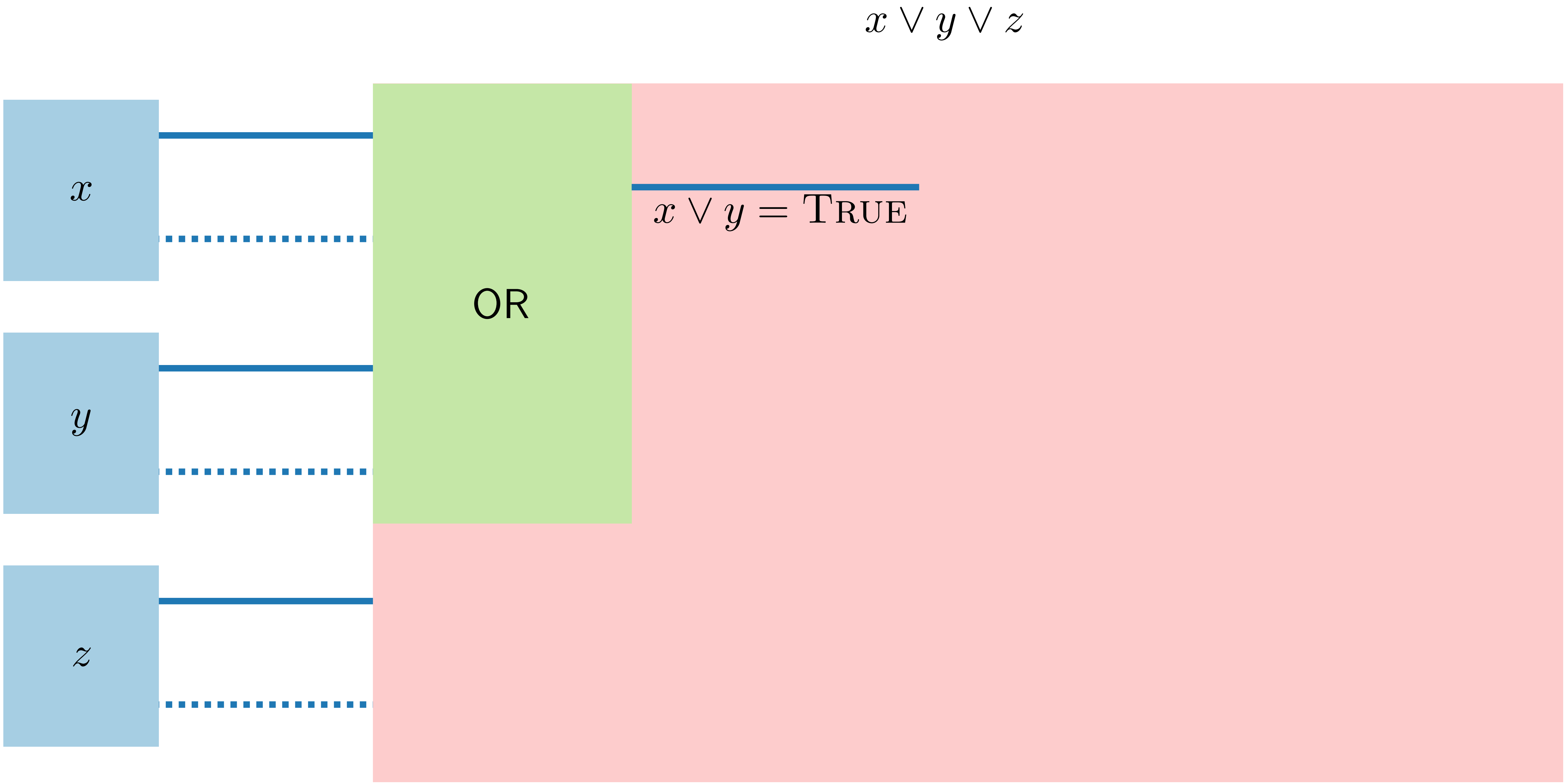

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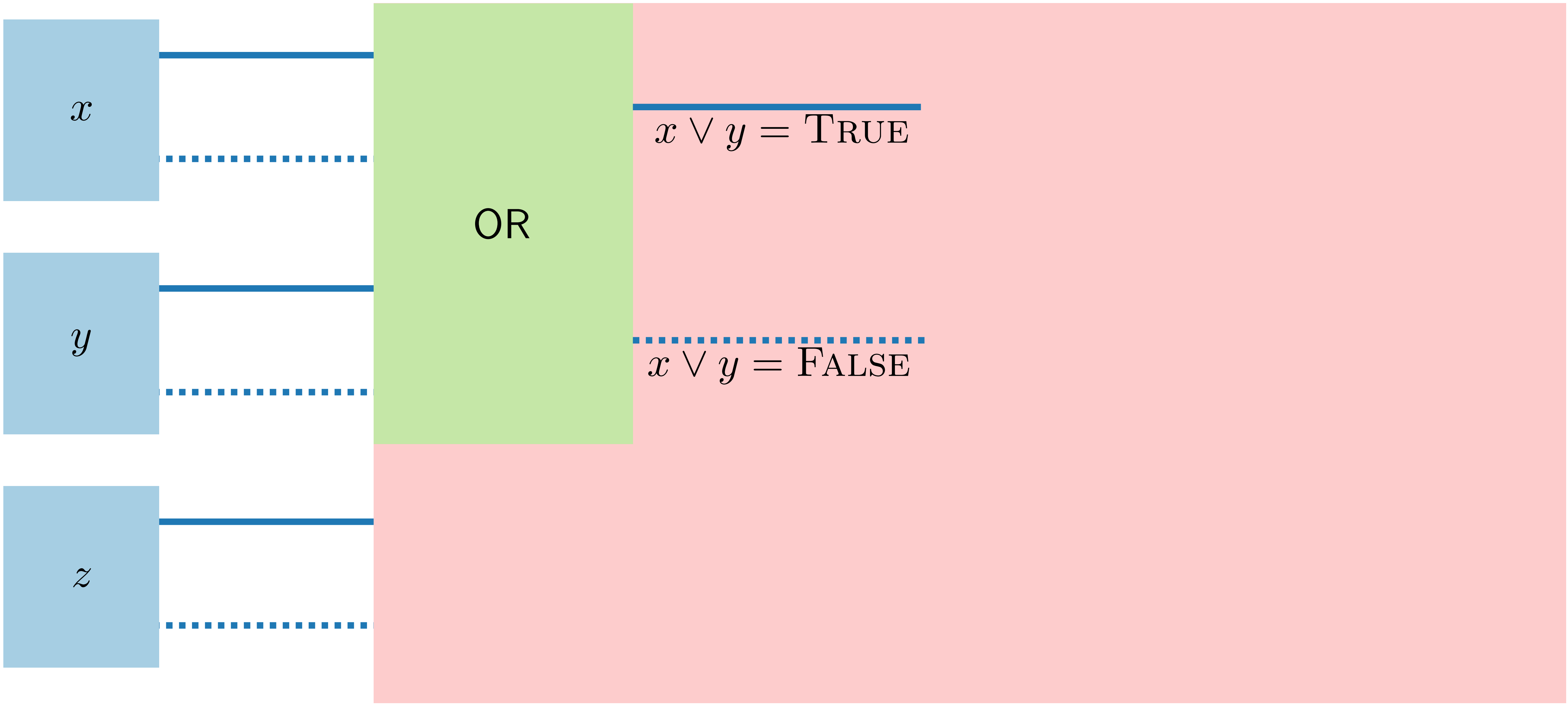

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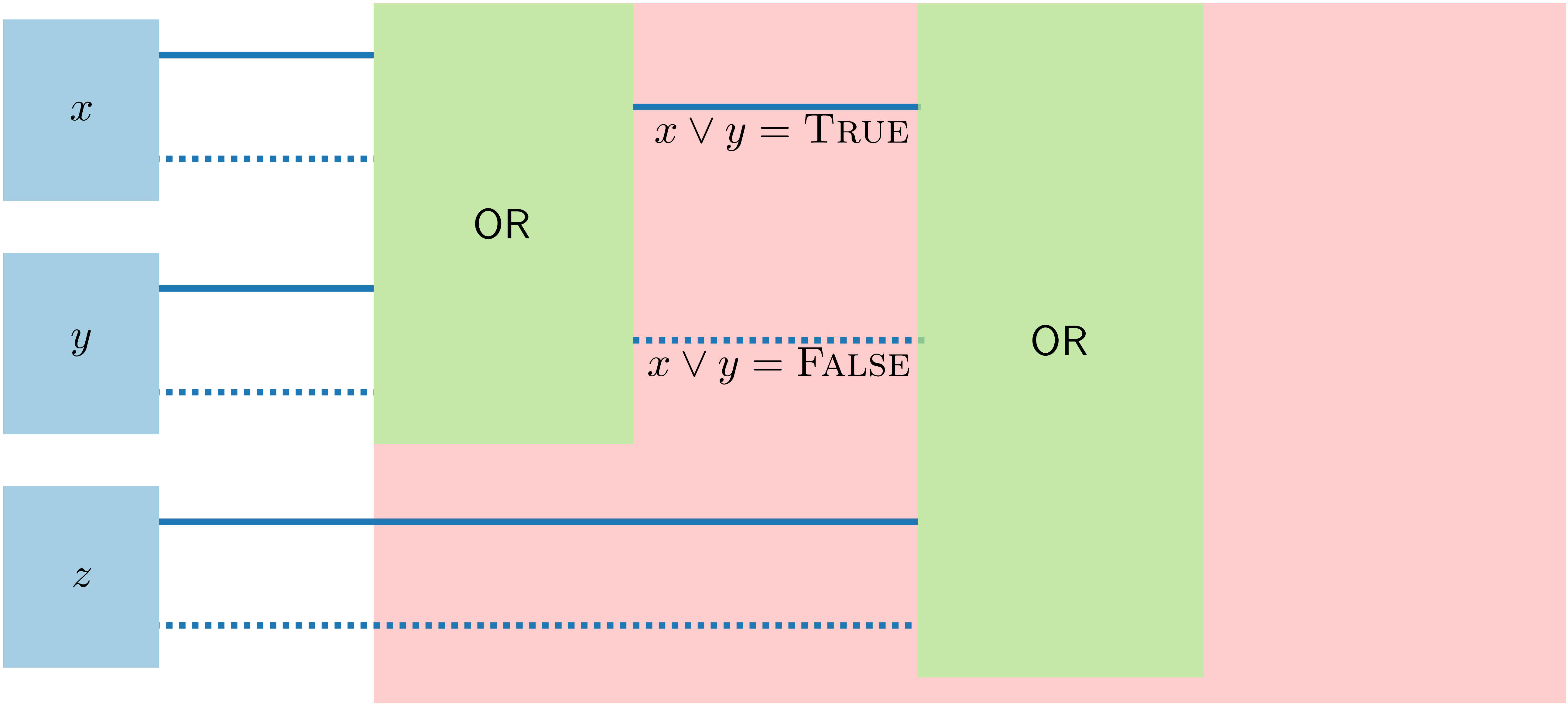

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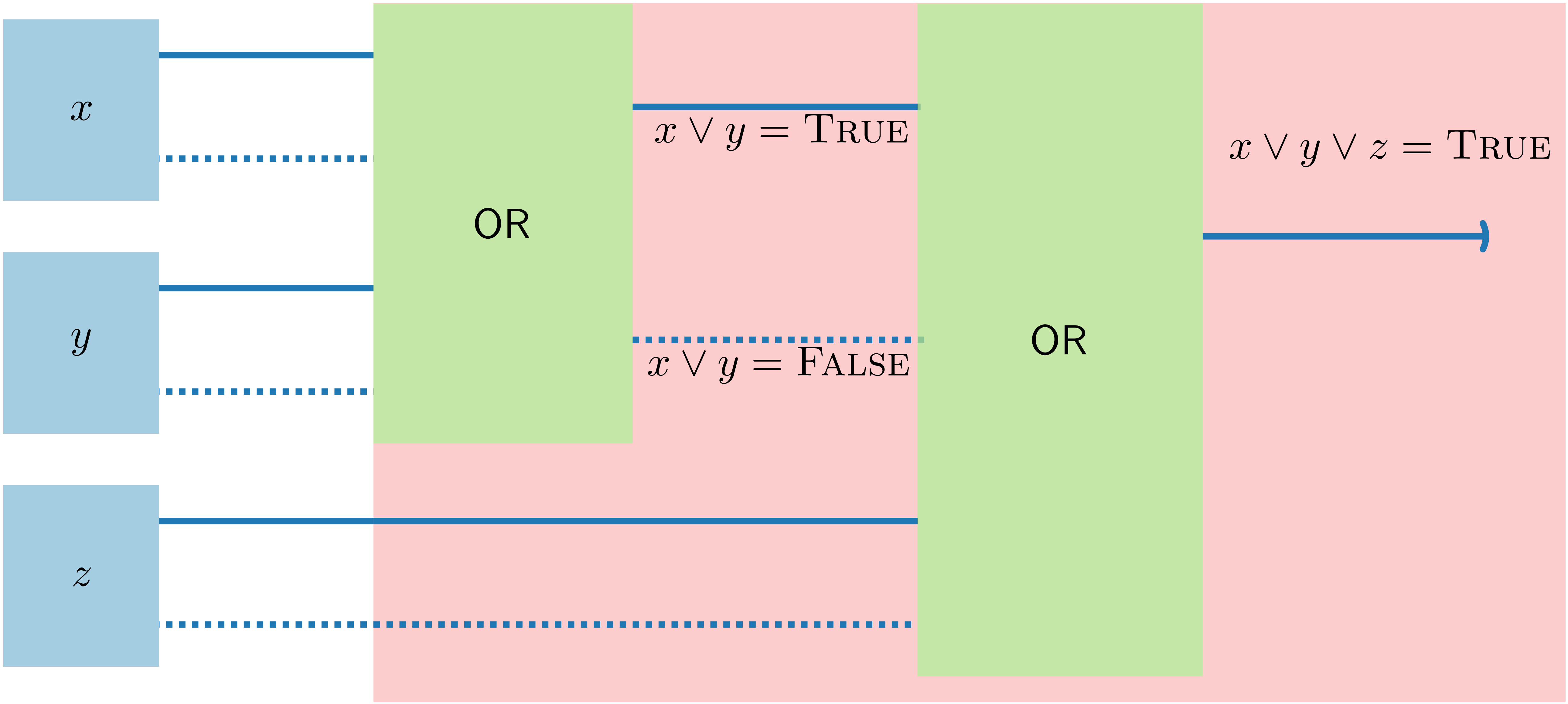

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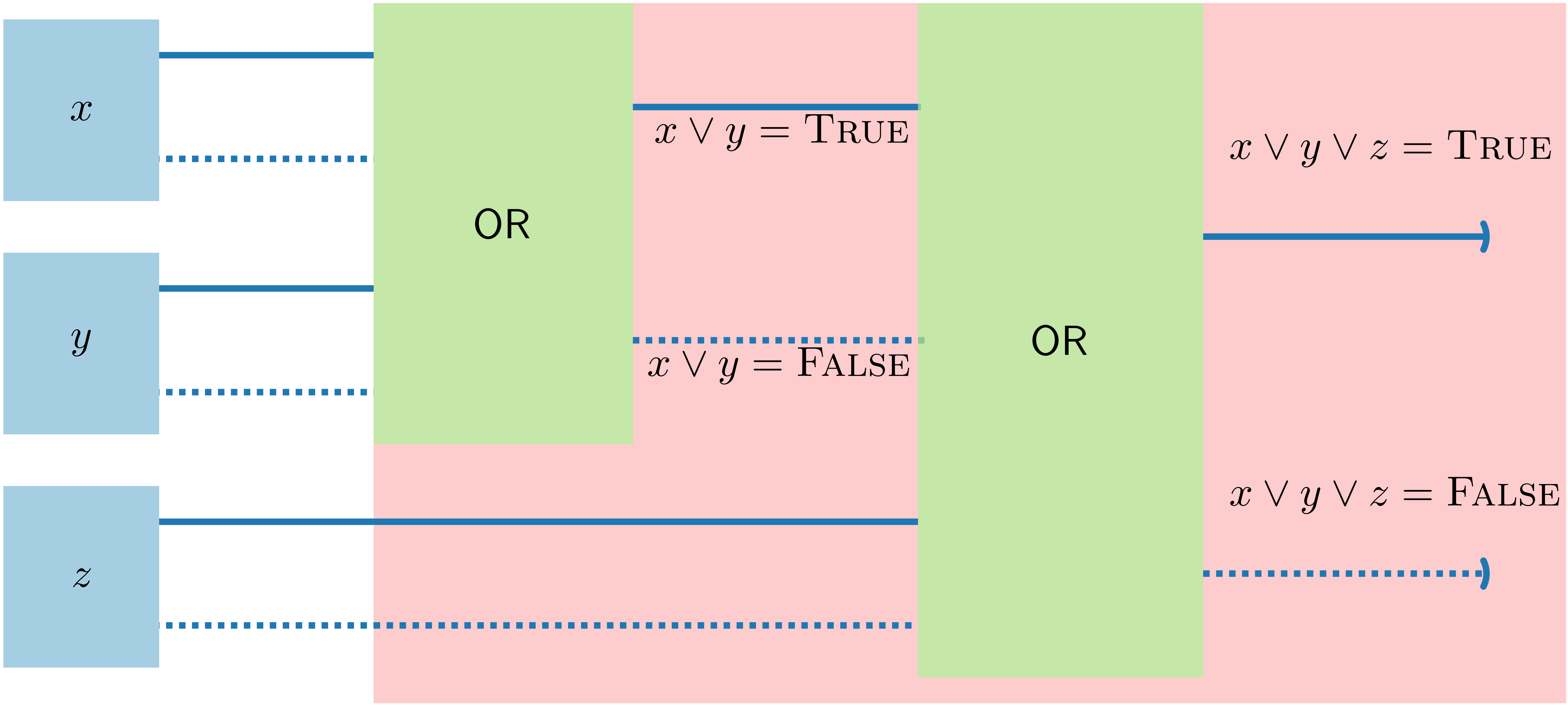

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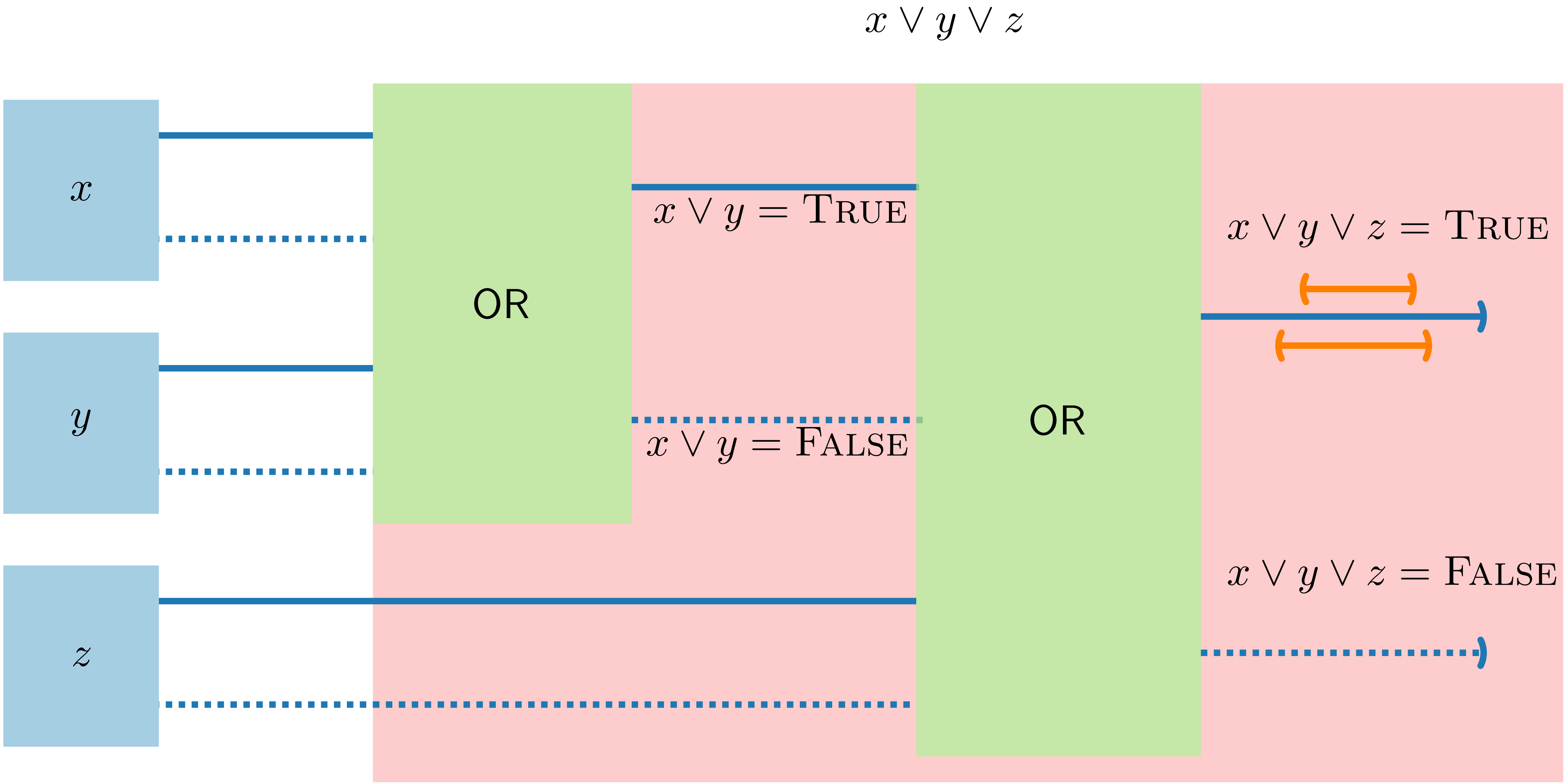

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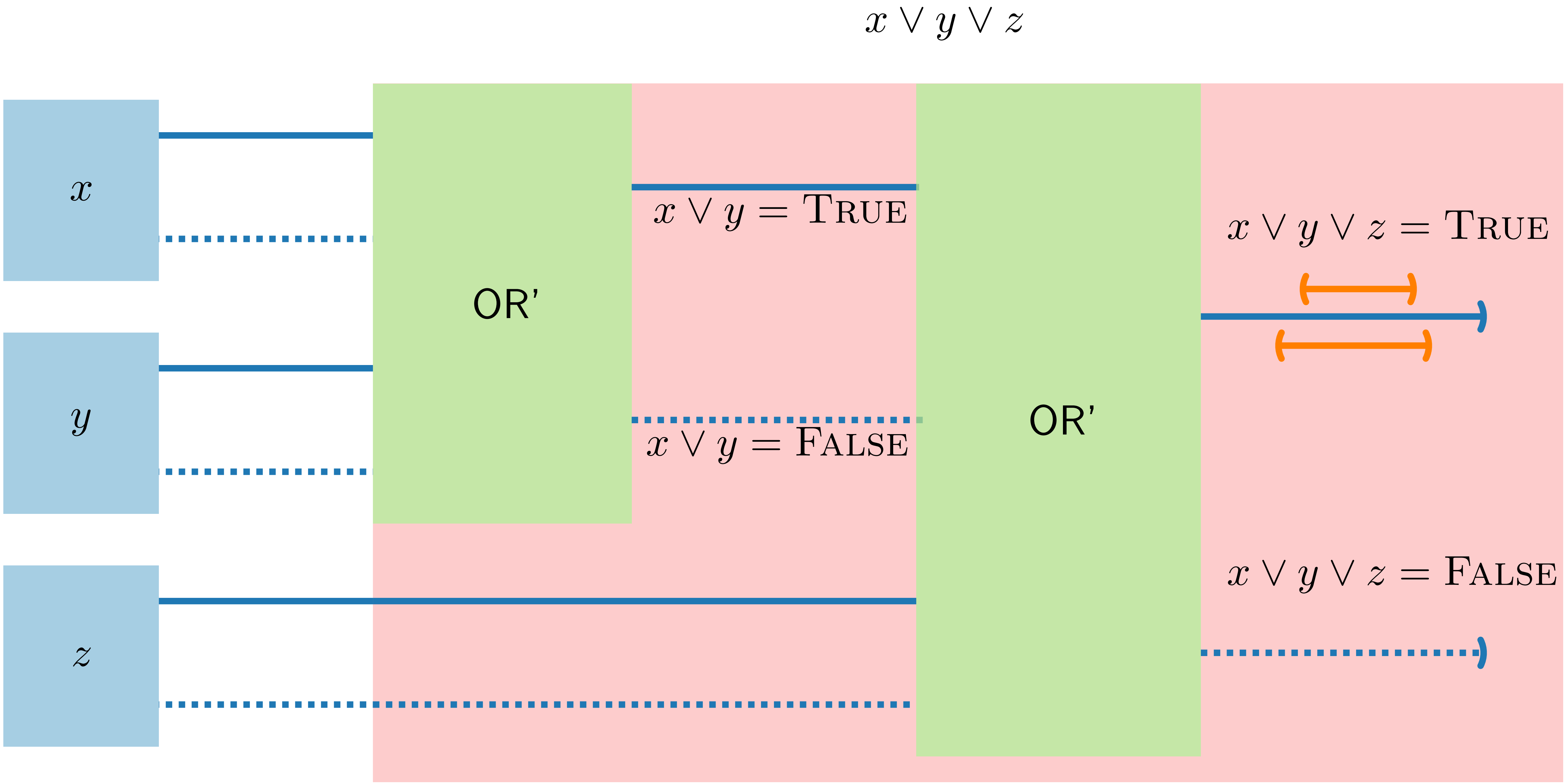

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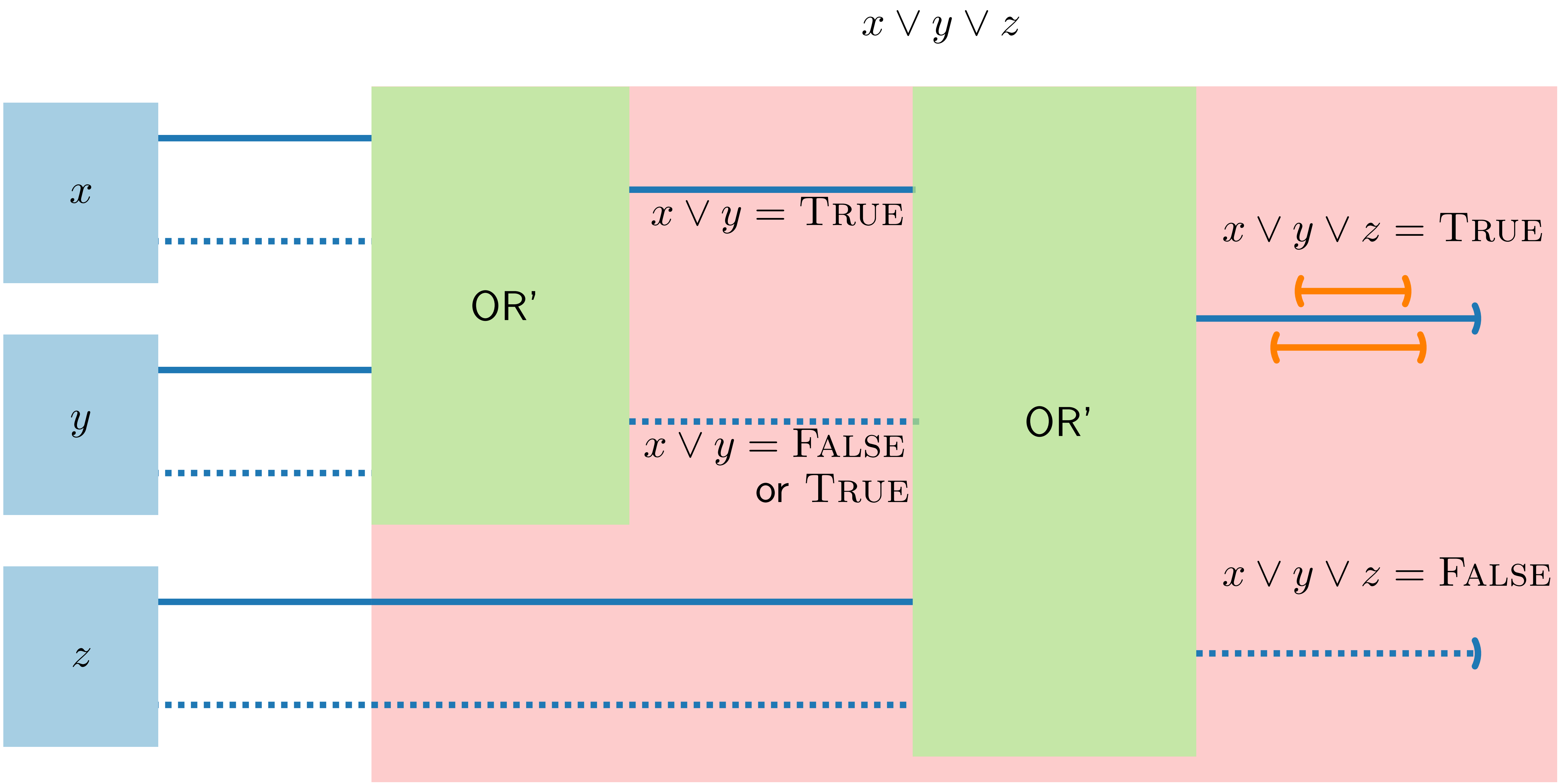

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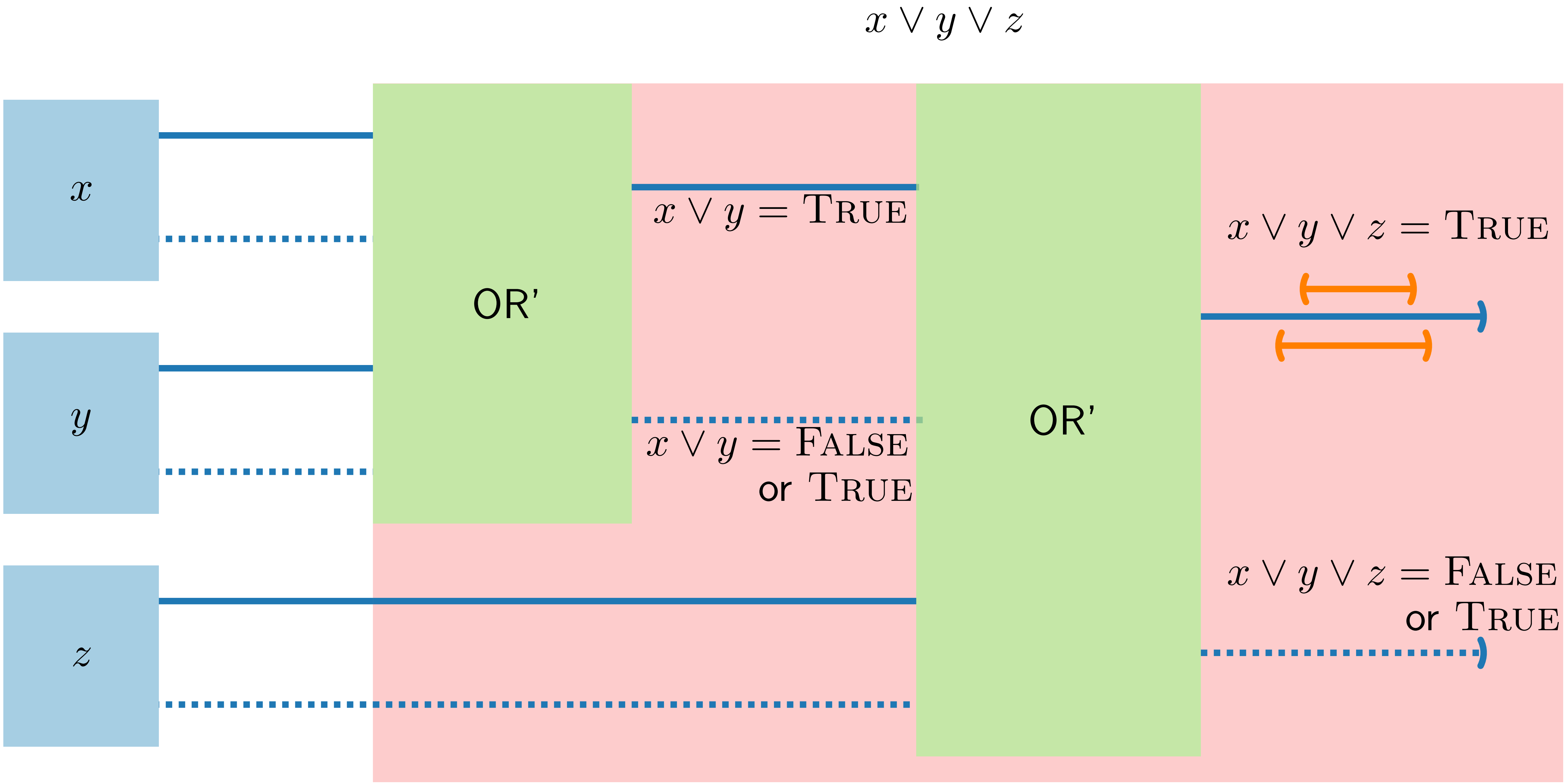

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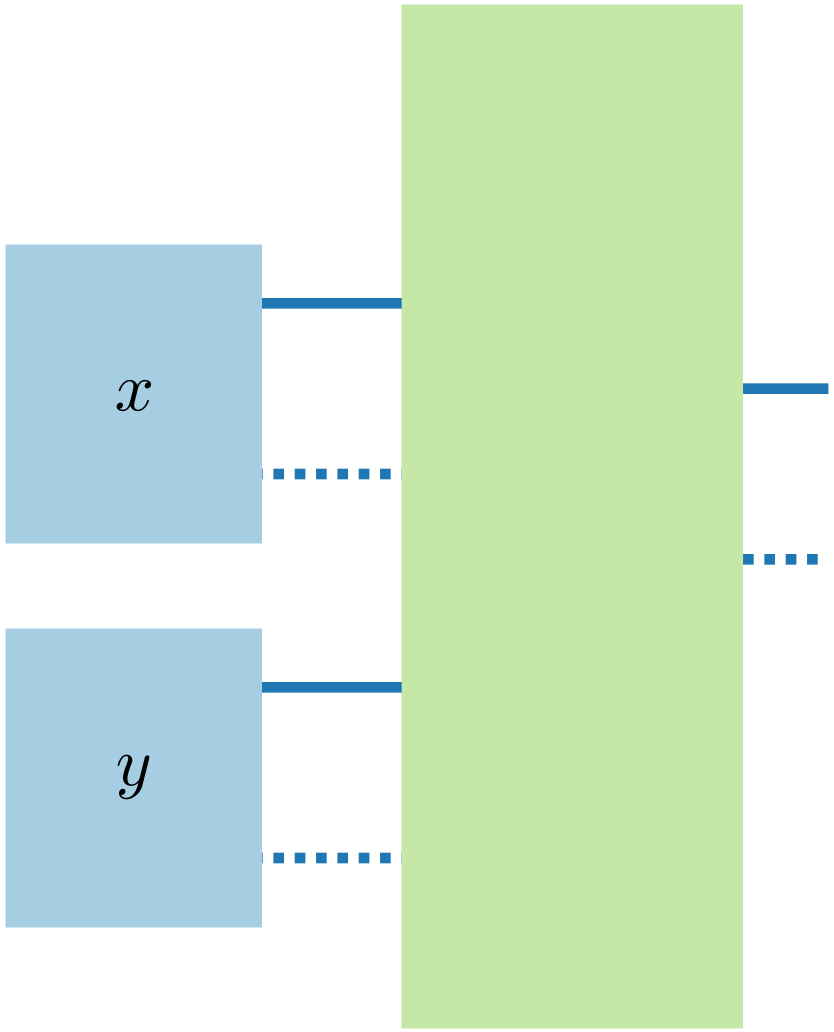

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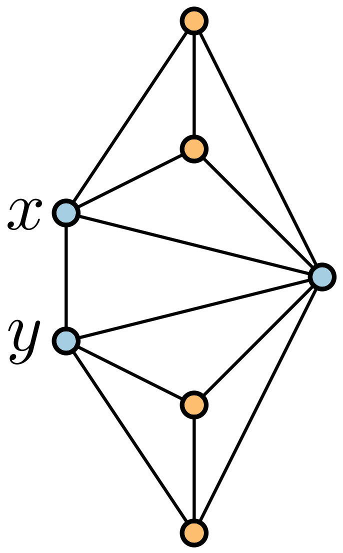

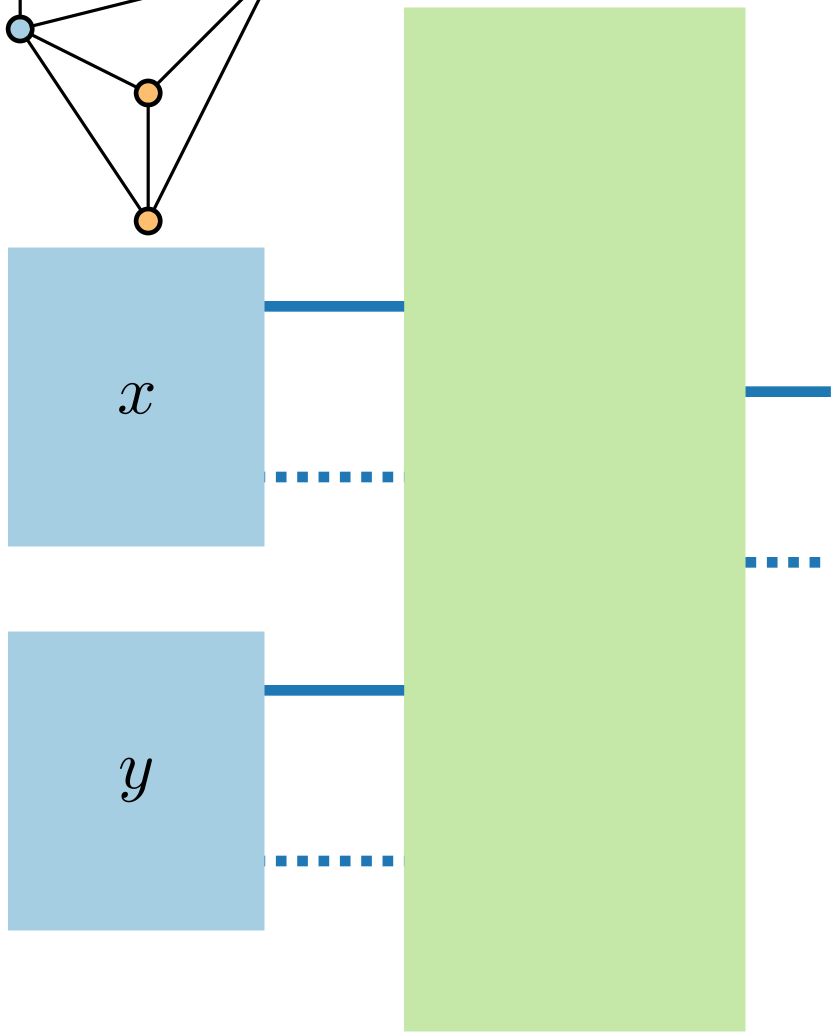

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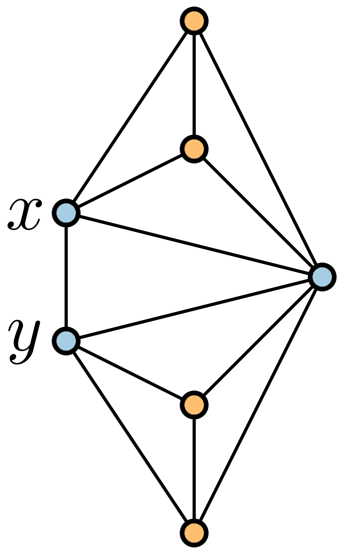

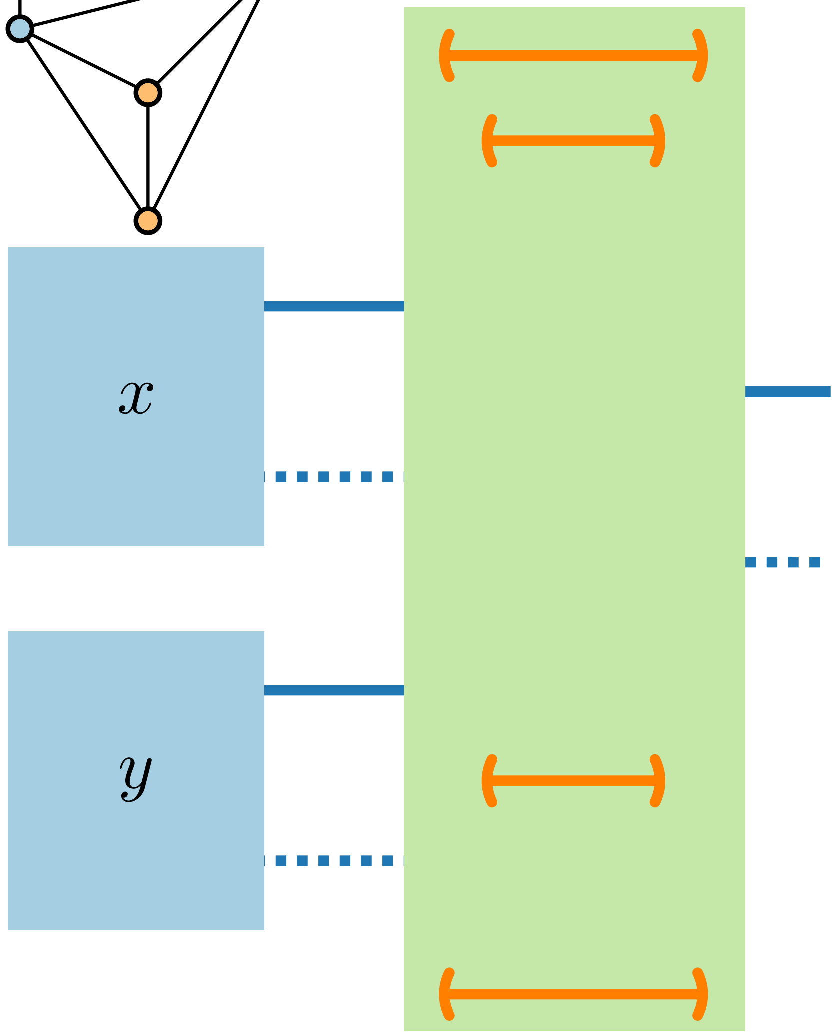

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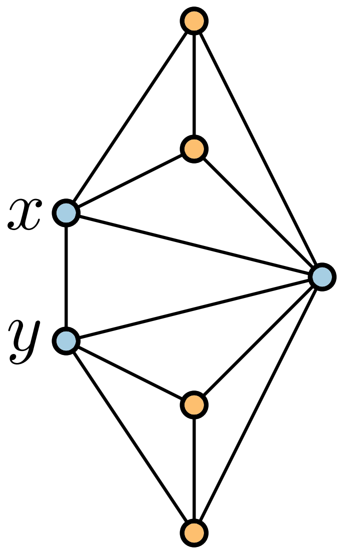

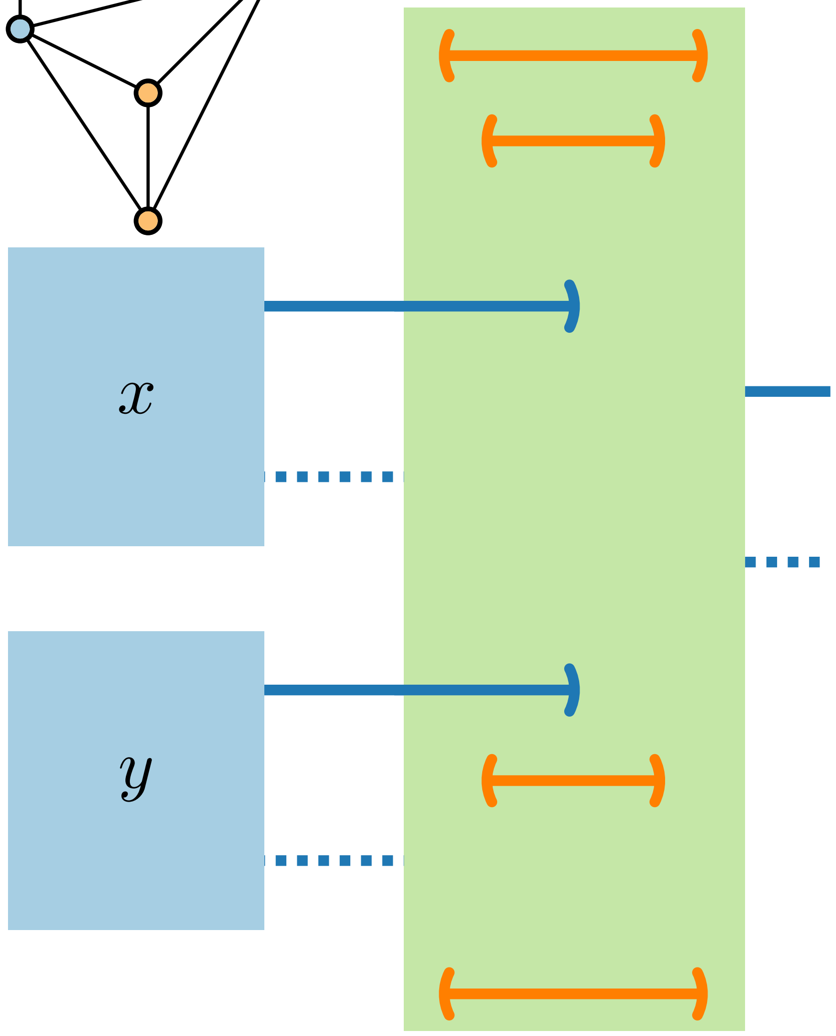

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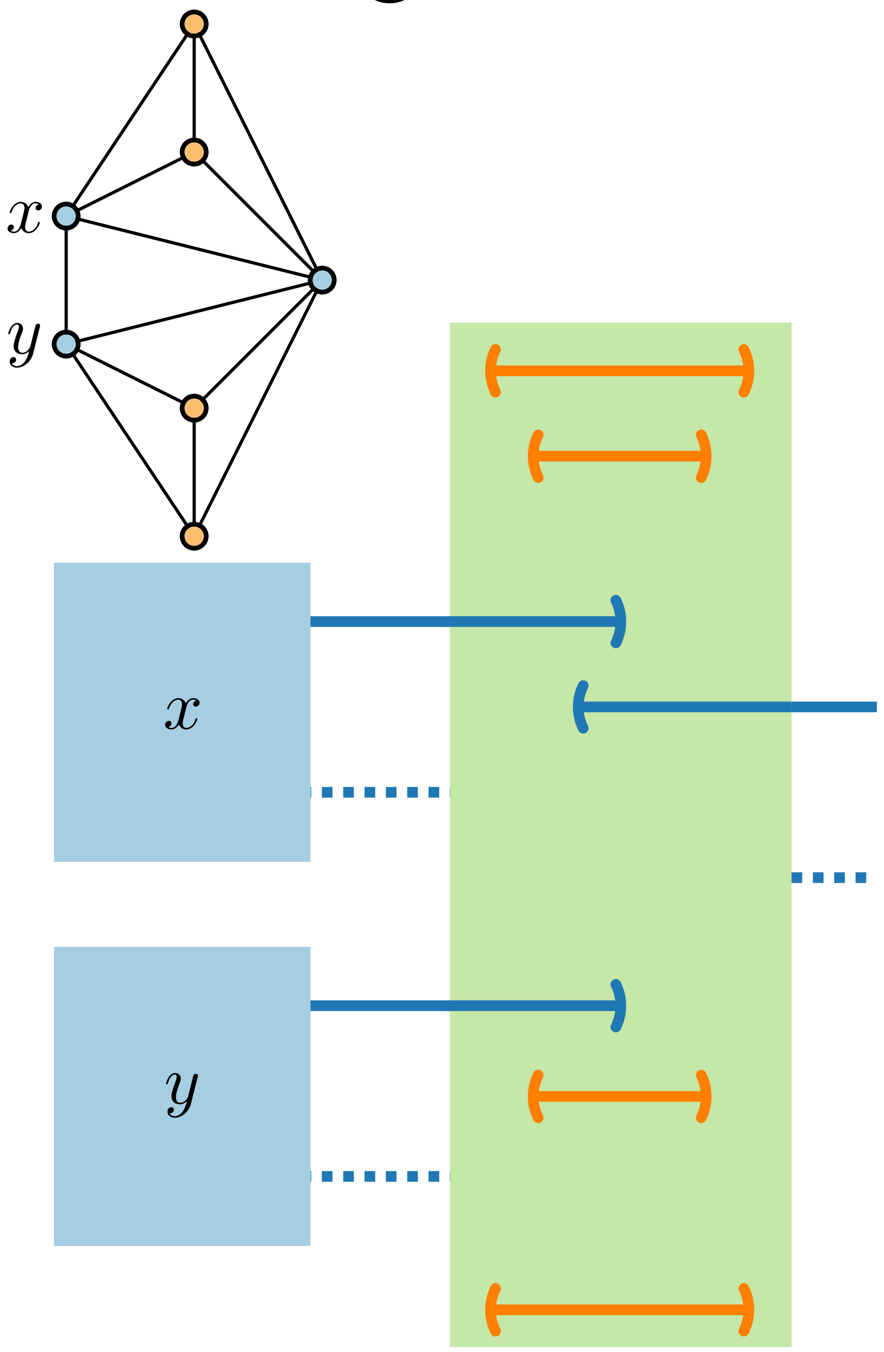

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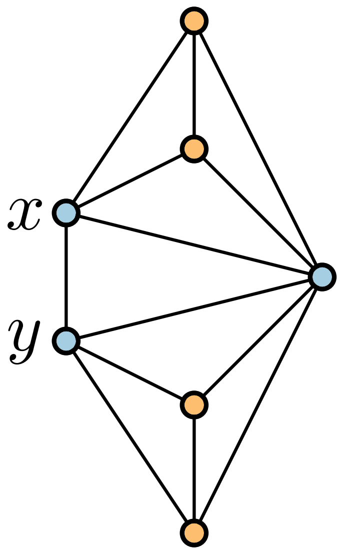

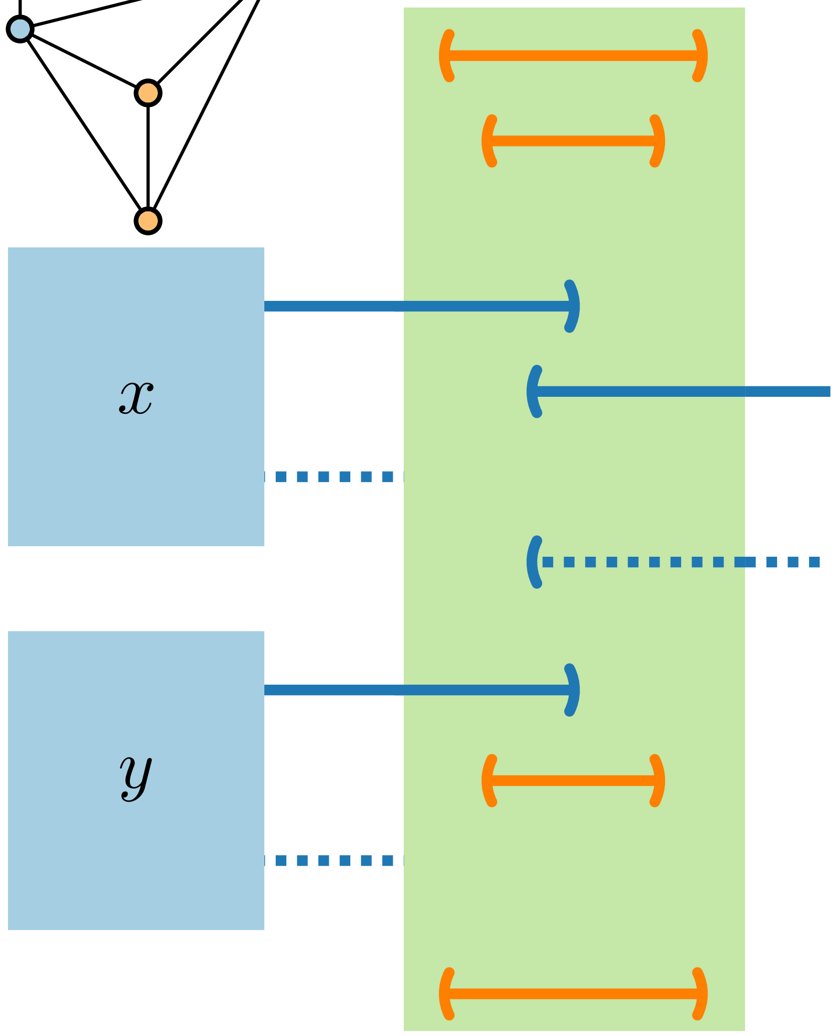

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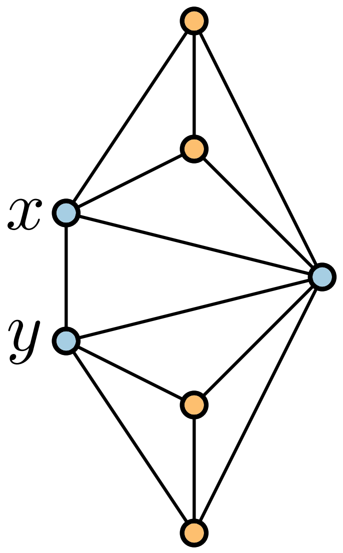

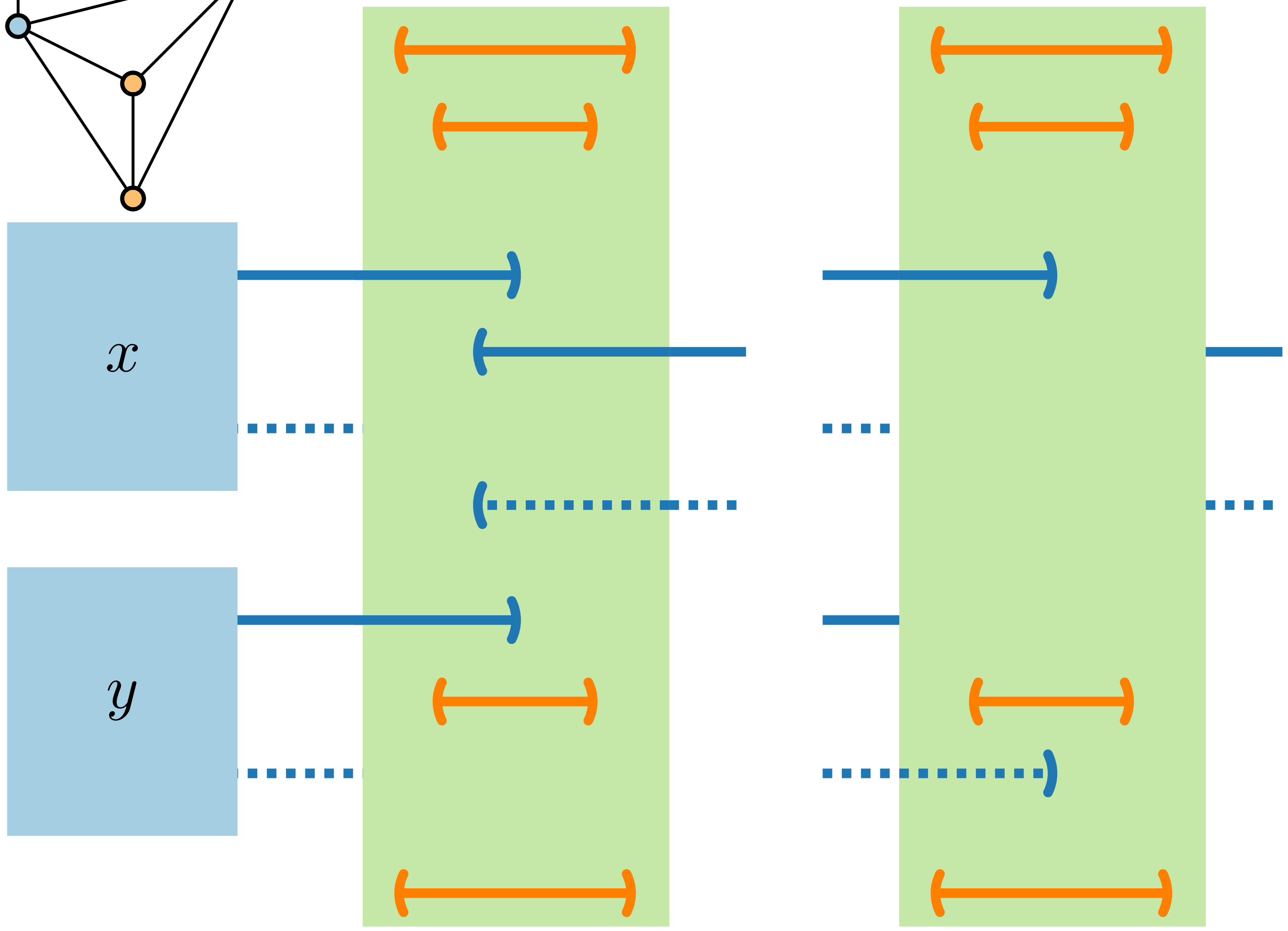

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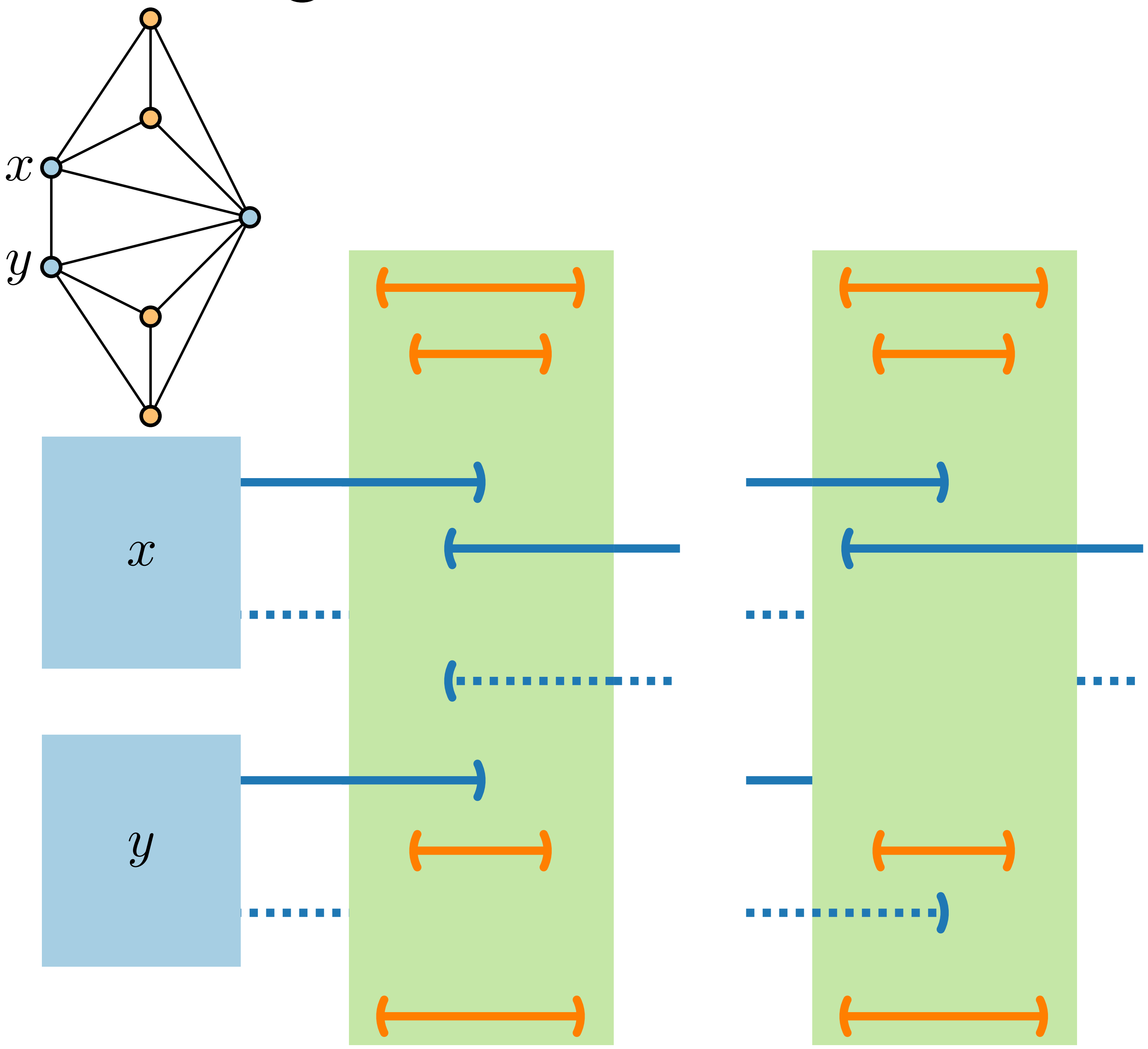

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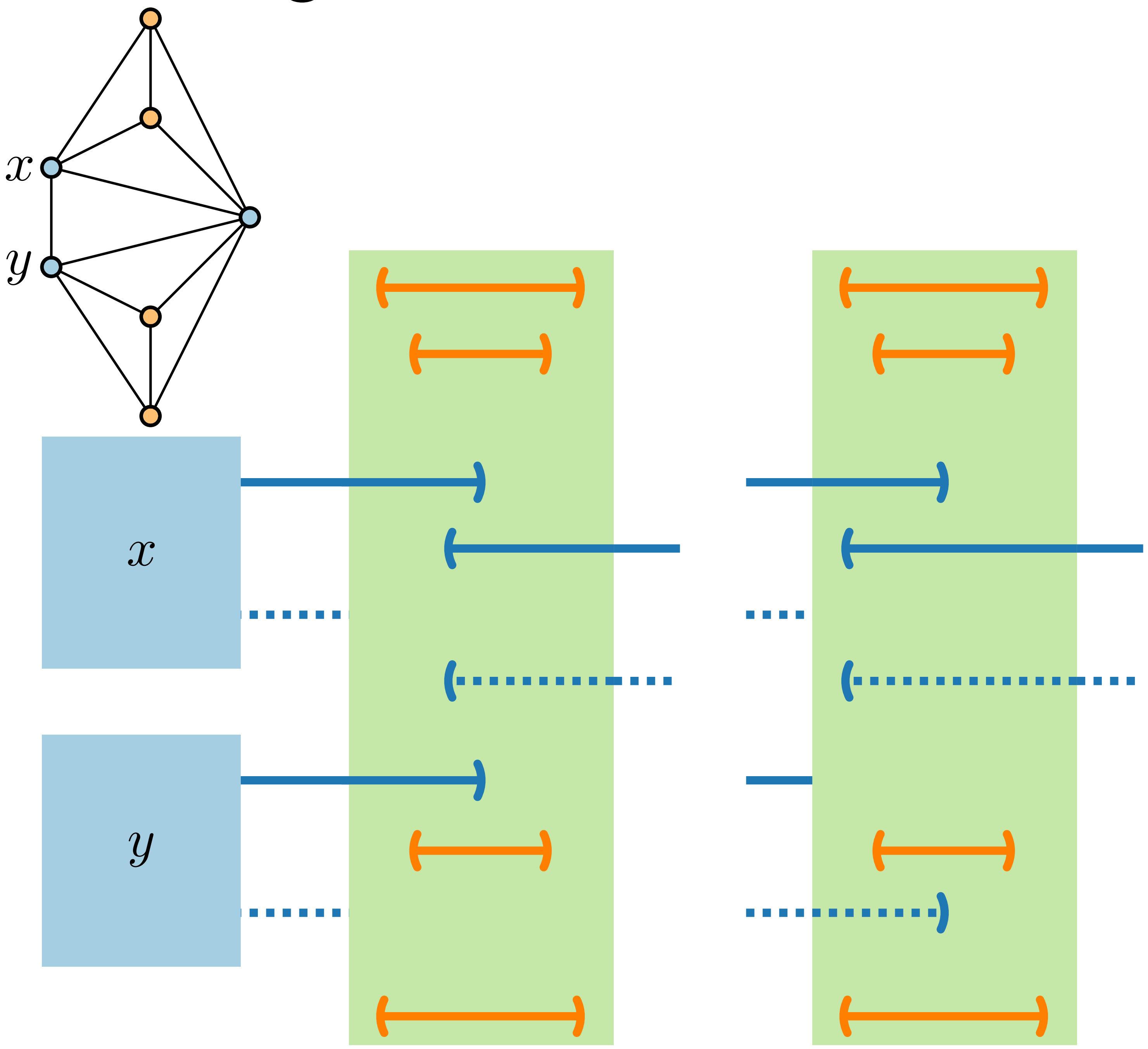

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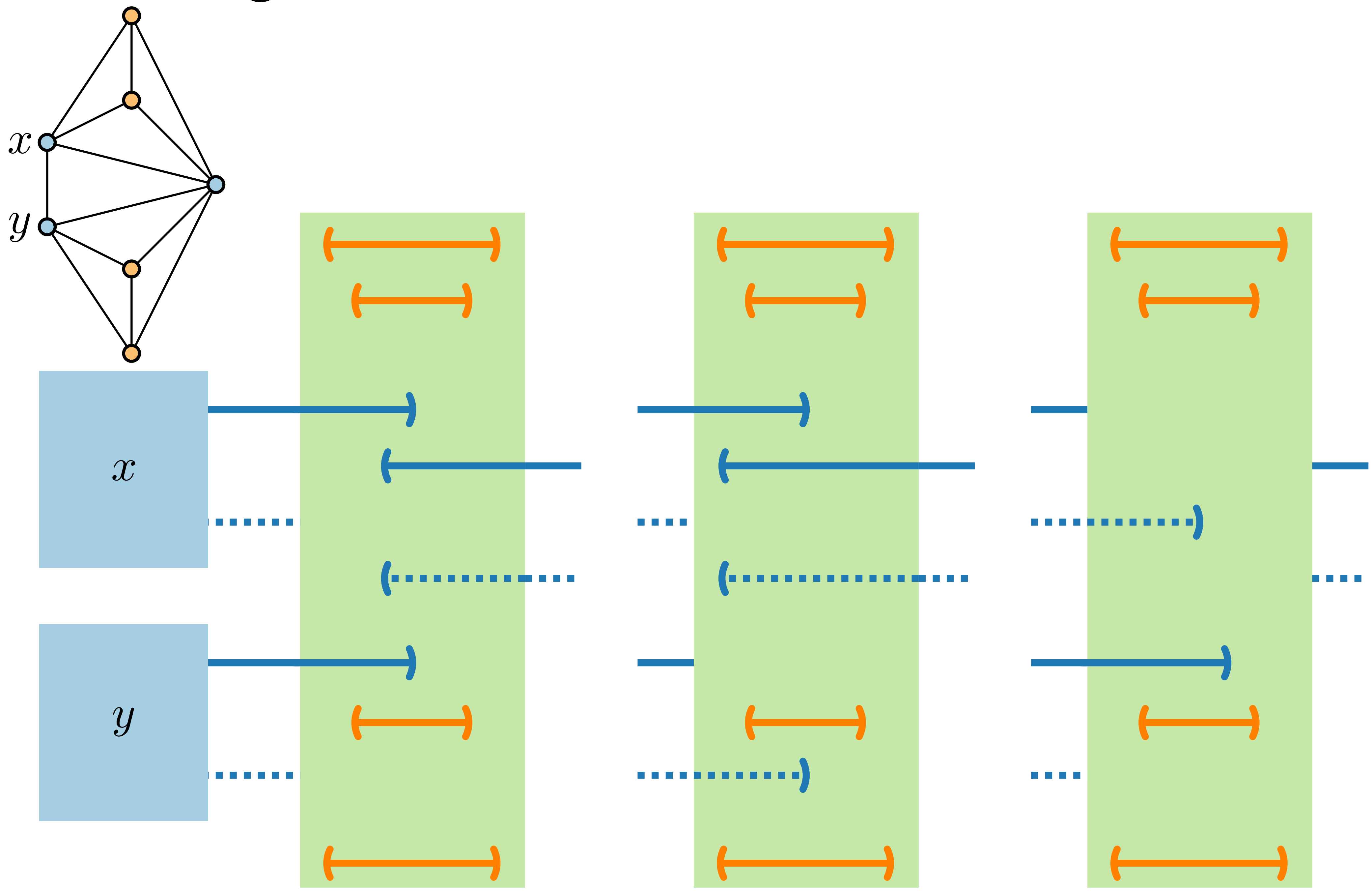

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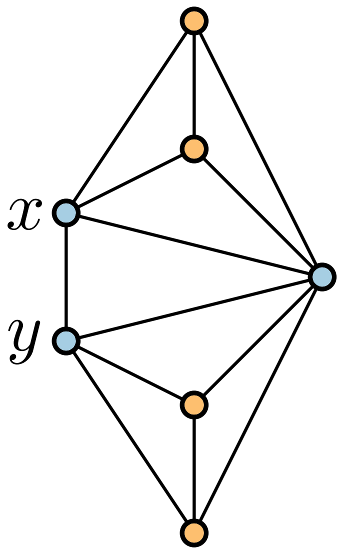

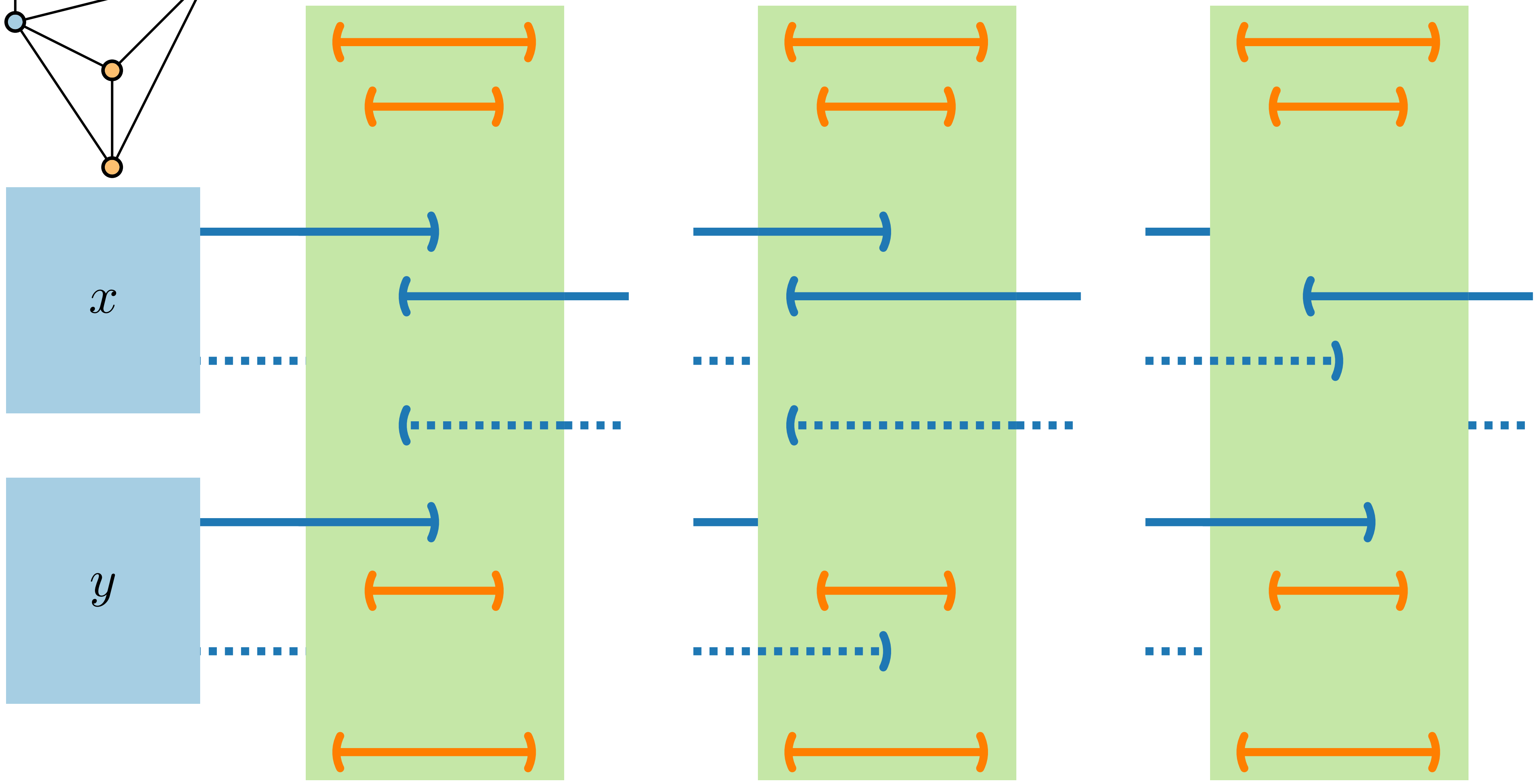

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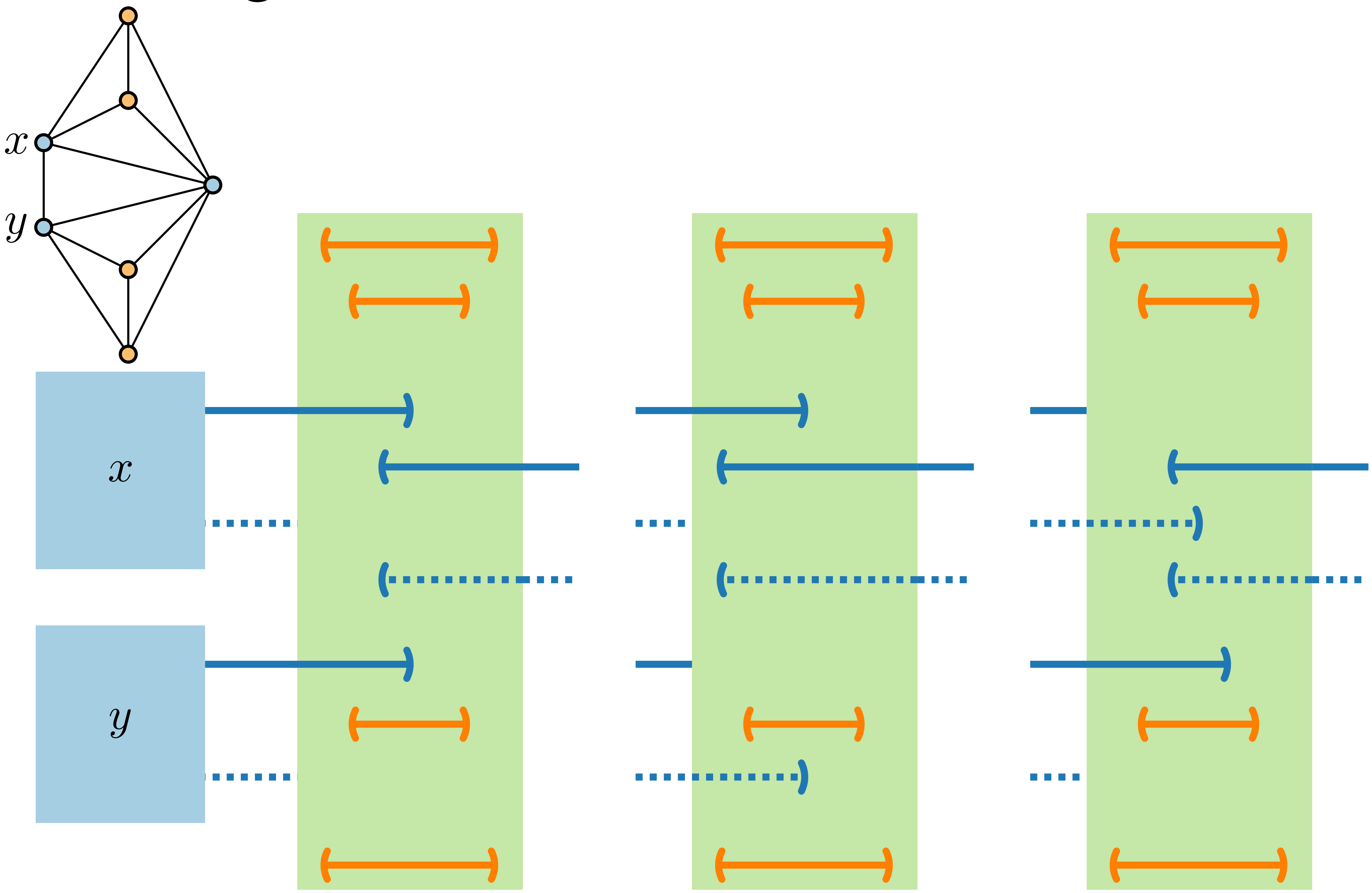

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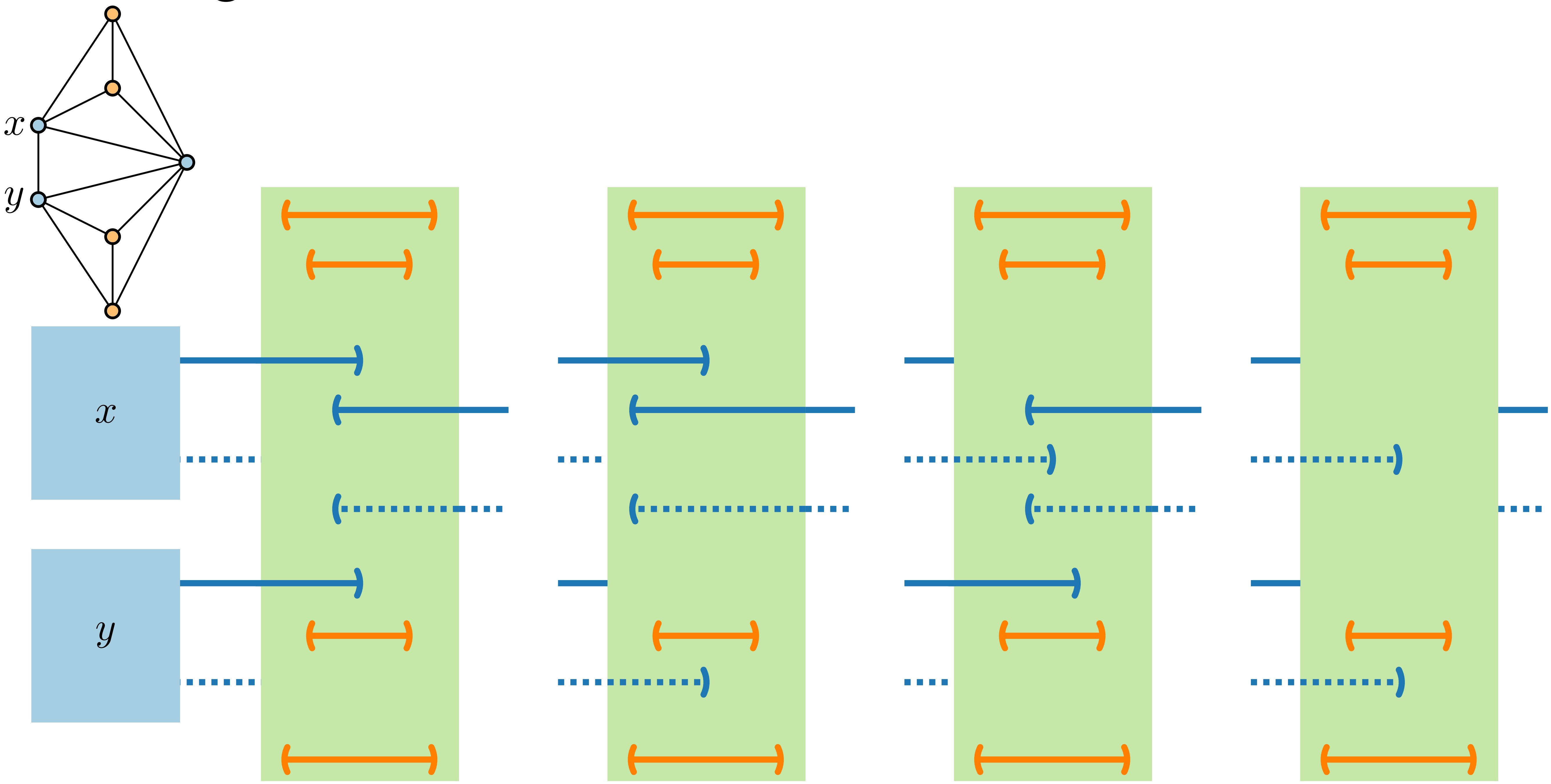

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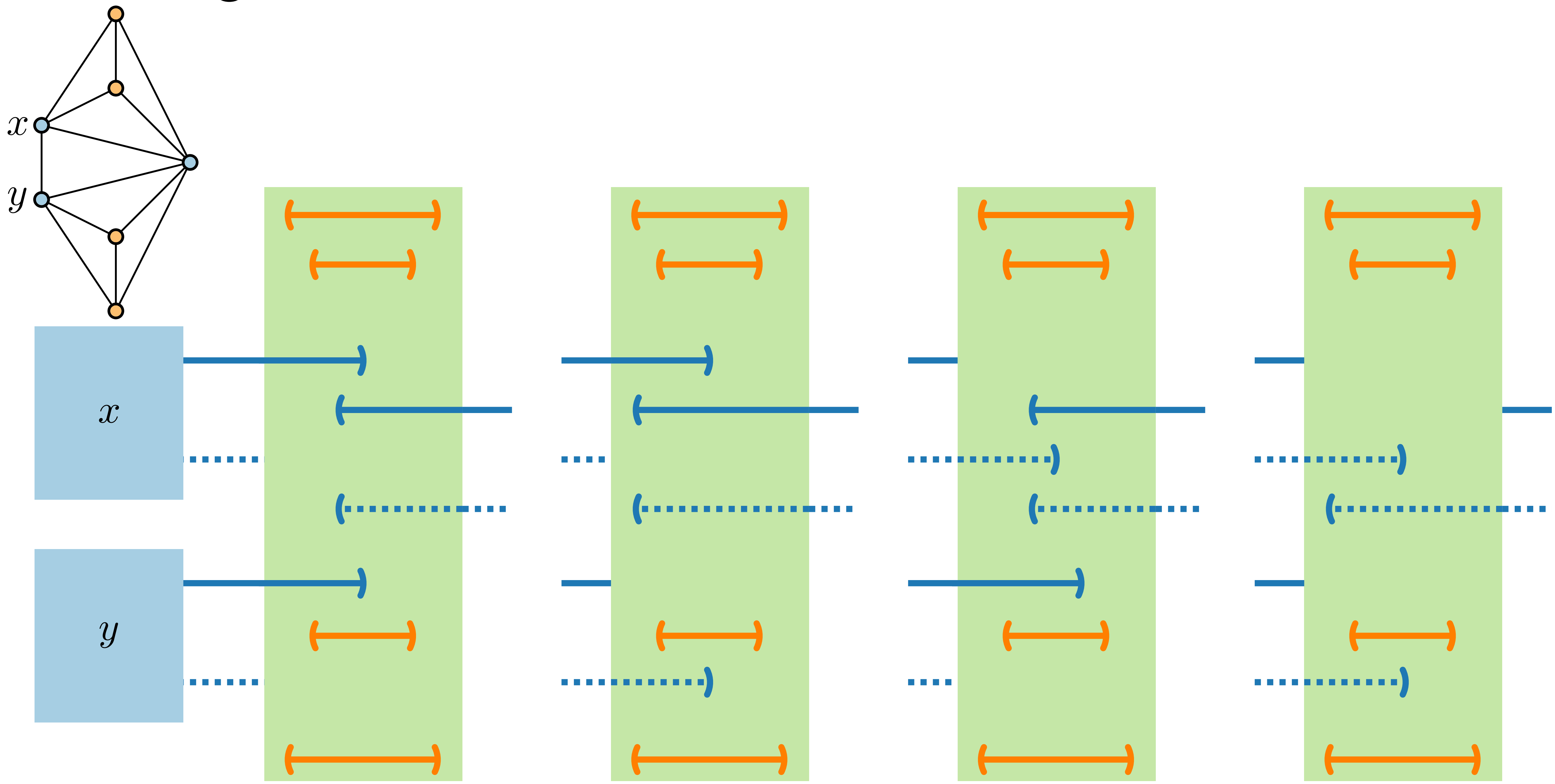

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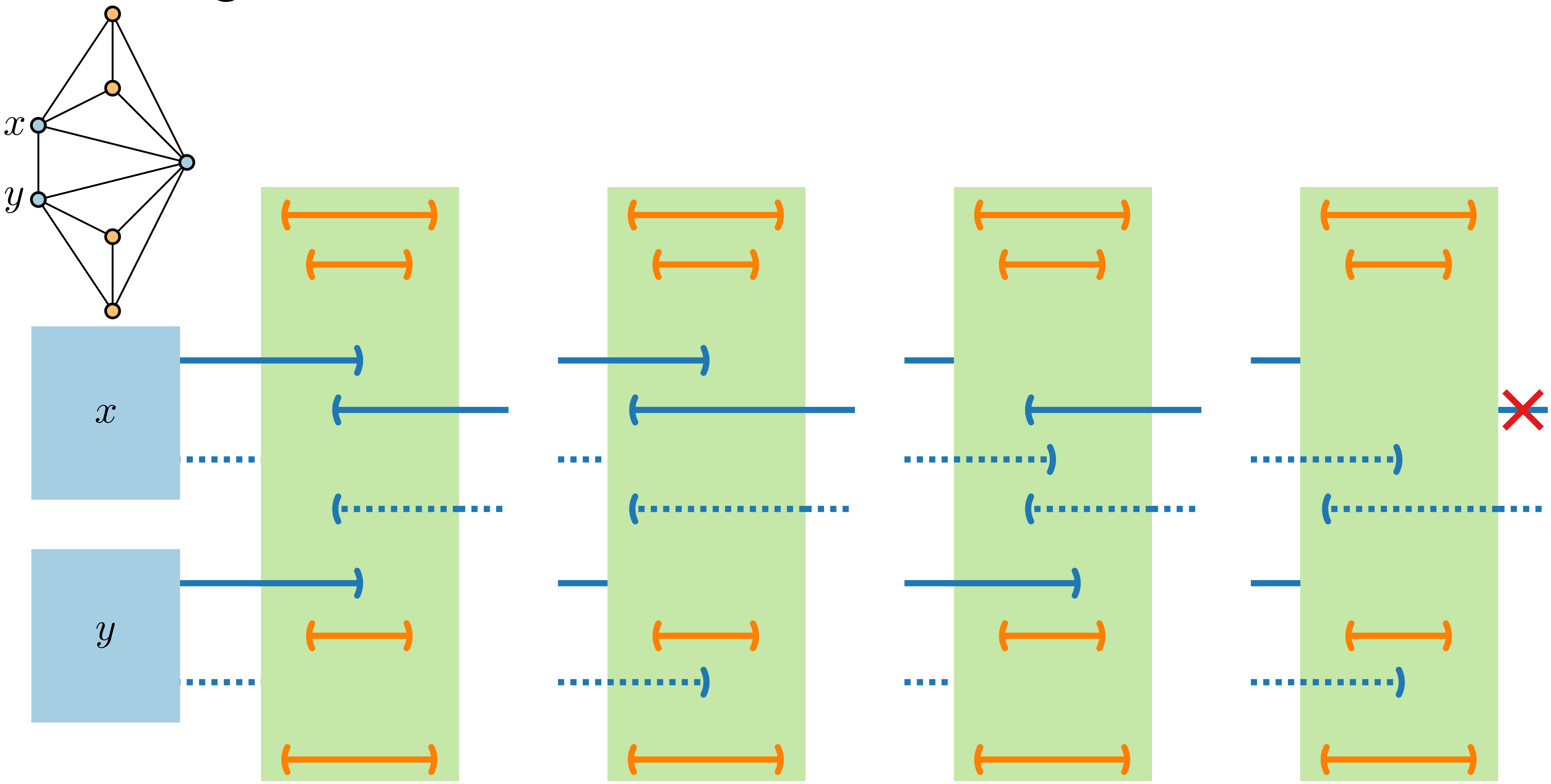

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- Can *strong* bar visibility recognition / representation extension be solved in polynomial time for st-graphs?

Literature

Main source:

- [Chaplick, Guśpiel, Gutowski, Krawczyk, Liotta '18]
  The Partial Visibility Representation Extension Problem

Referenced papers:

- [Tamassia, Tollis '86] Algorithms for visibility representations of planar graphs
- [Wismath '85] Characterizing bar line-of-sight graphs
- [Chaplick, Dorbec, Kratochvíl, Montassier, Stacho '14]
  Contact representations of planar graphs: Extending a partial representation is hard
- [Andreae '92] Some results on visibility graphs
- [Garg, Tamassia '01]
  On the Computational Complexity of Upward and Rectilinear Planarity Testing
- [Gutwenger, Mutzel '01] A Linear Time Implementation of SPQR-Trees
- [de Berg, Khosravi '10] Optimal Binary Space Partitions in the Plane