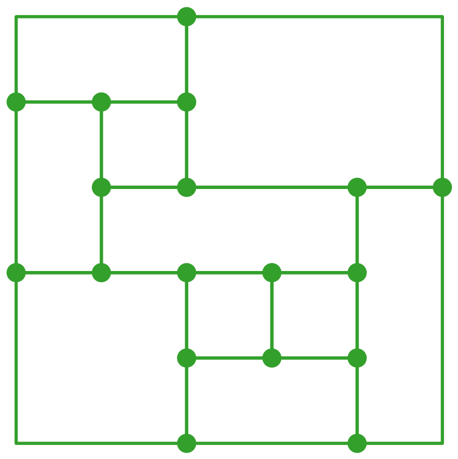
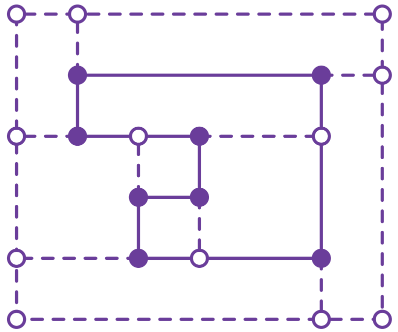
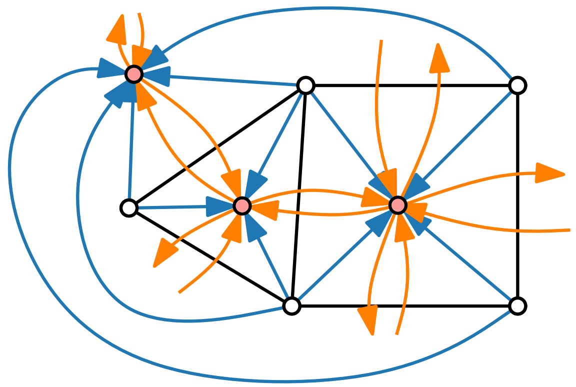
Visualization of Graphs



Lecture 6: Orthogonal Layouts

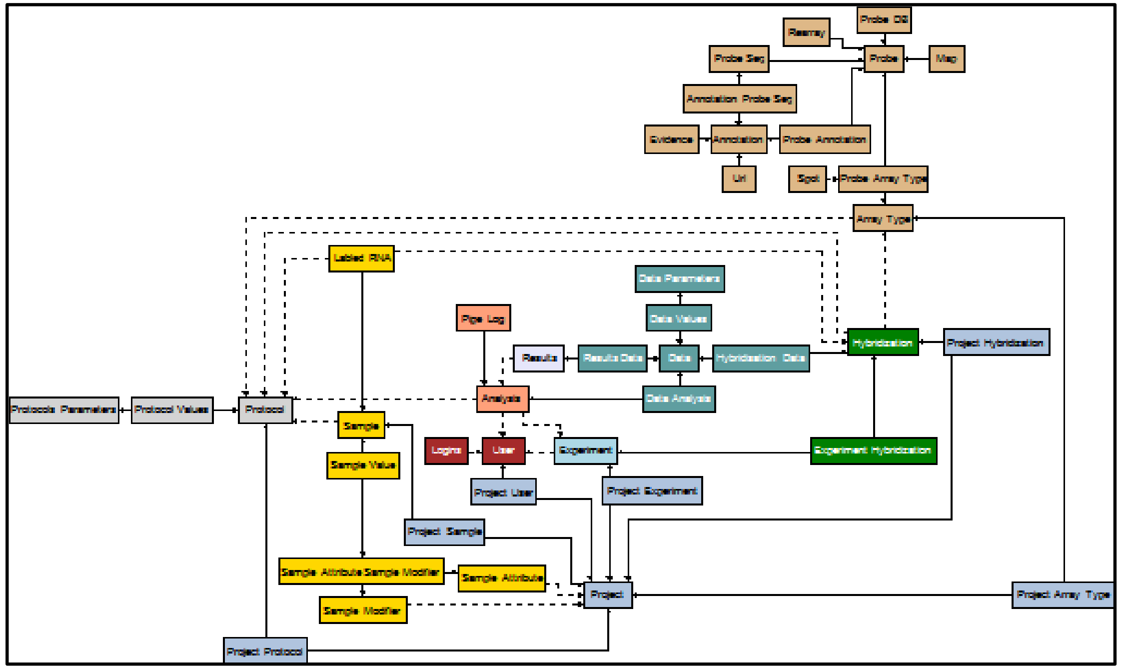




Johannes Zink

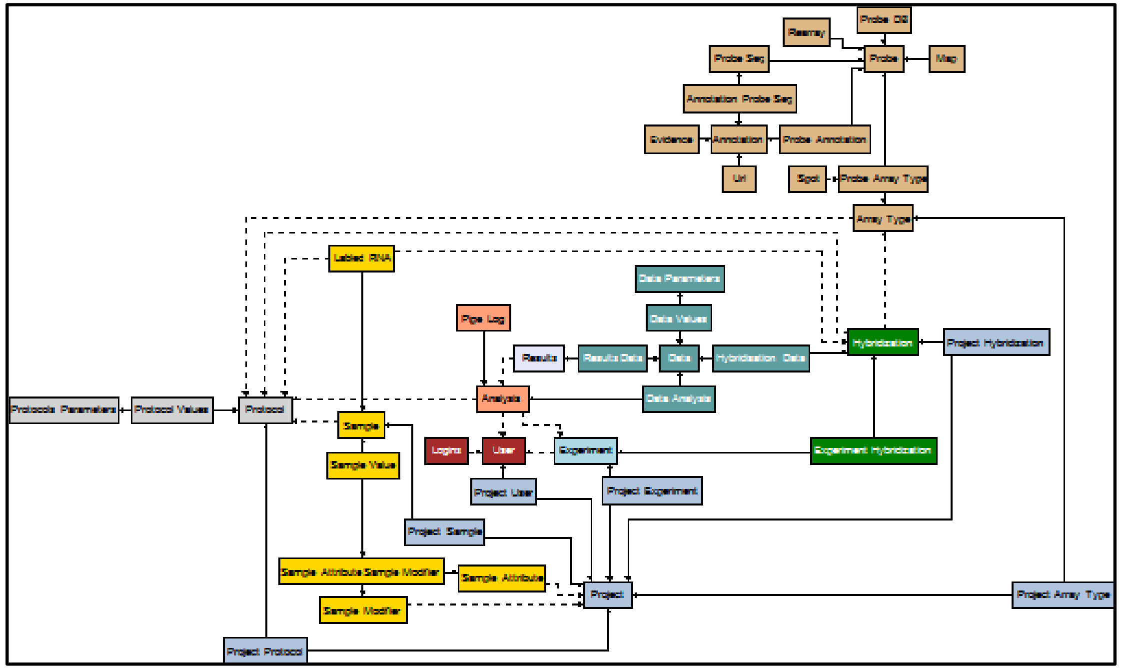
Summer semester 2024

Orthogonal Layout – Applications



ER diagram in OGDF

Orthogonal Layout – Applications



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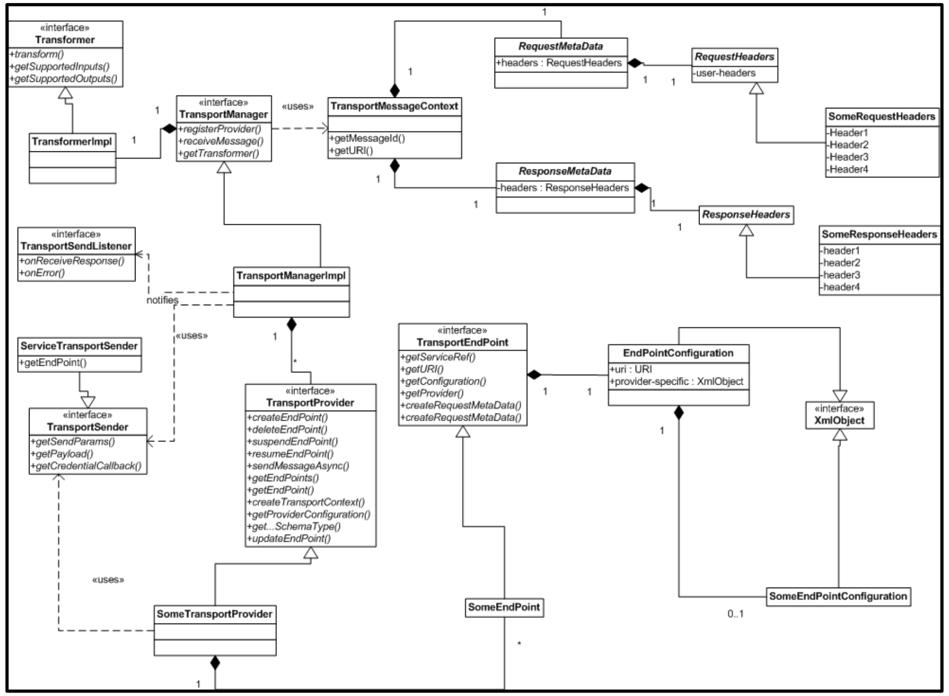
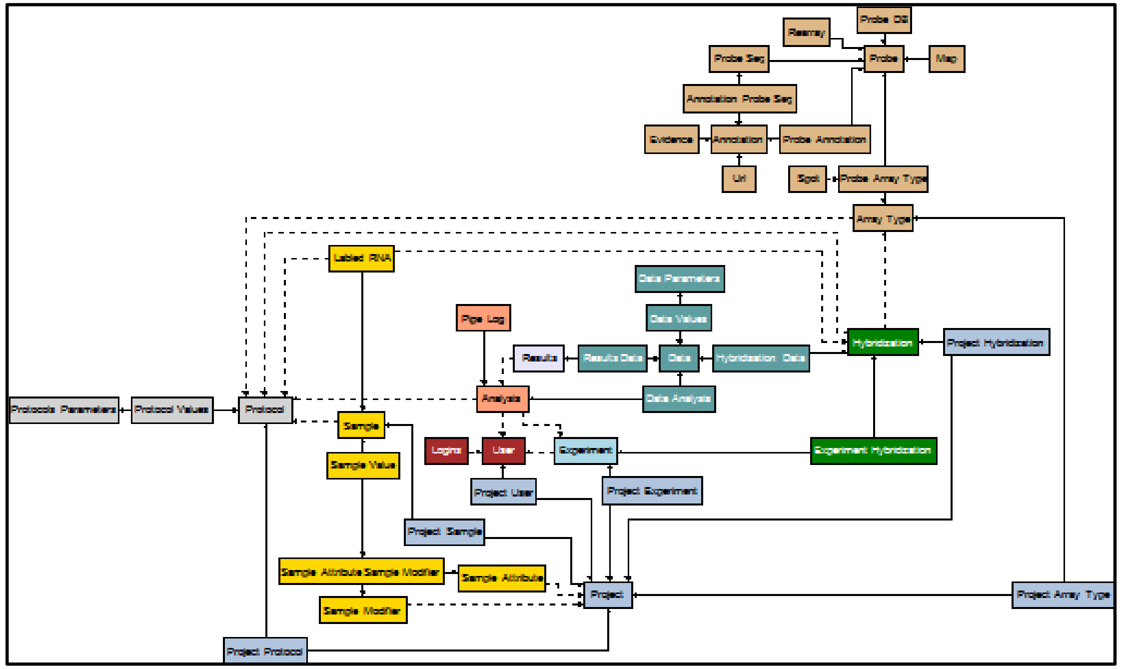


UML diagram by Oracle

Orthogonal Layout – Applications



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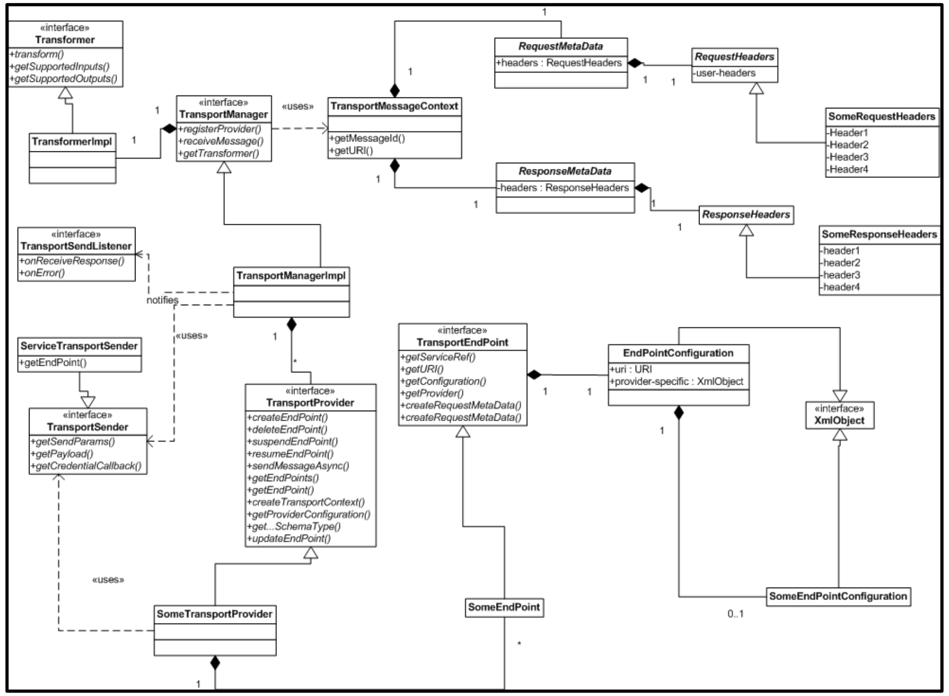

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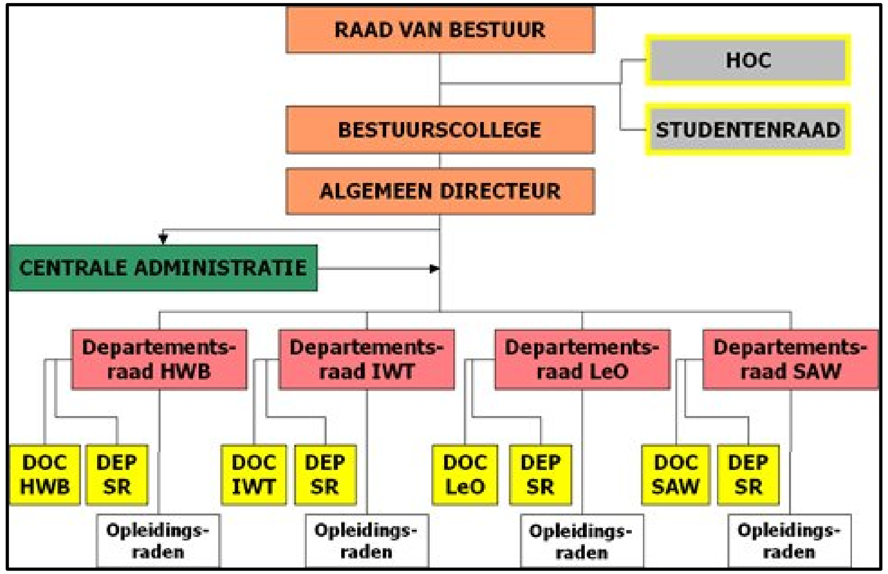
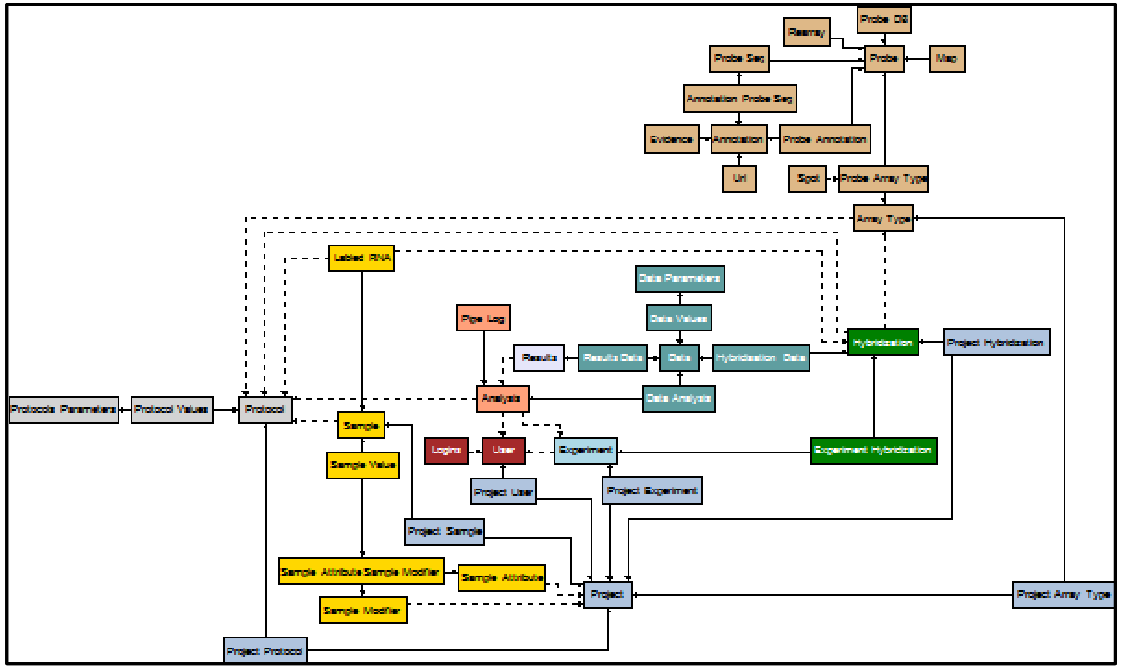


Organigram of HS Limburg

Orthogonal Layout – Applications



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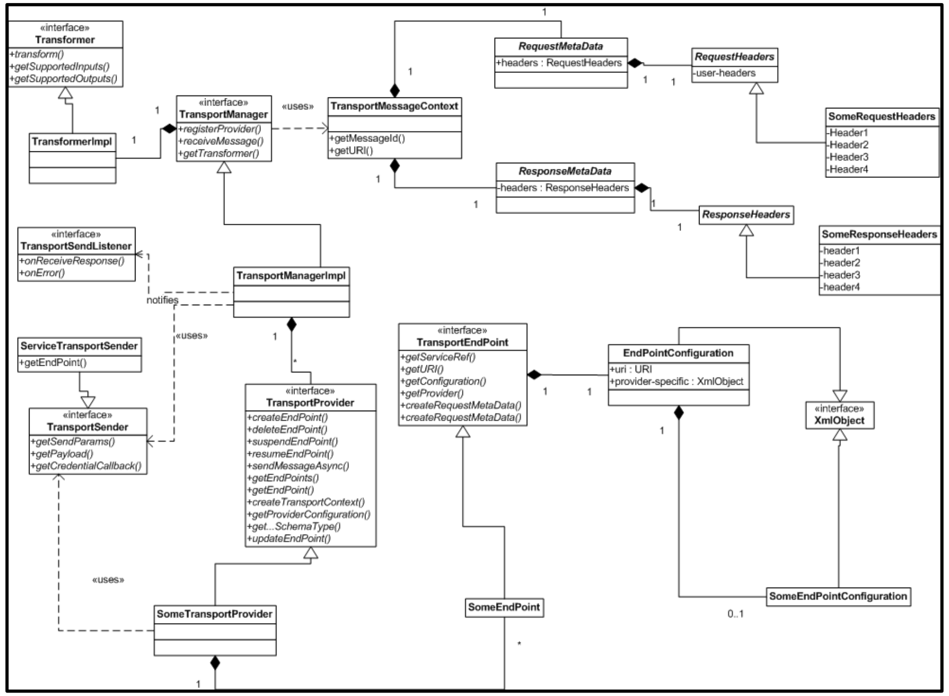

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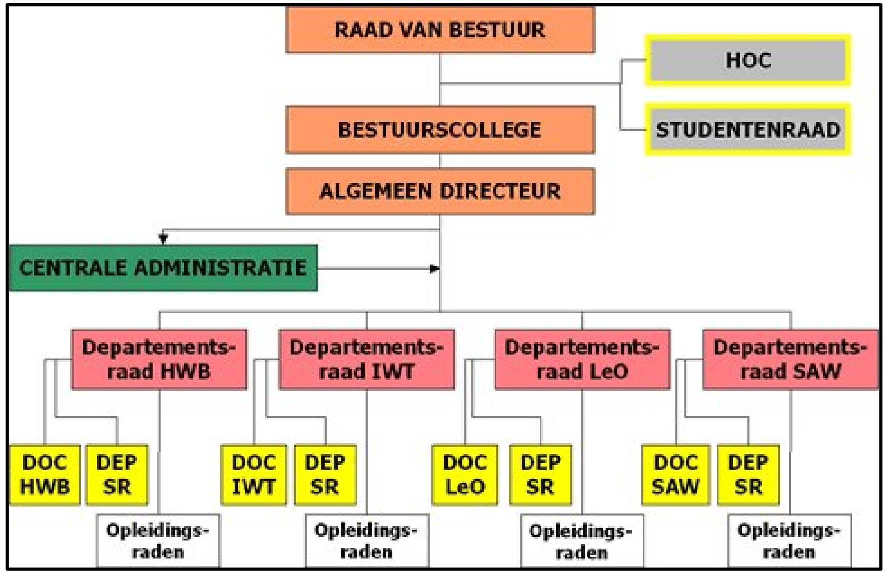

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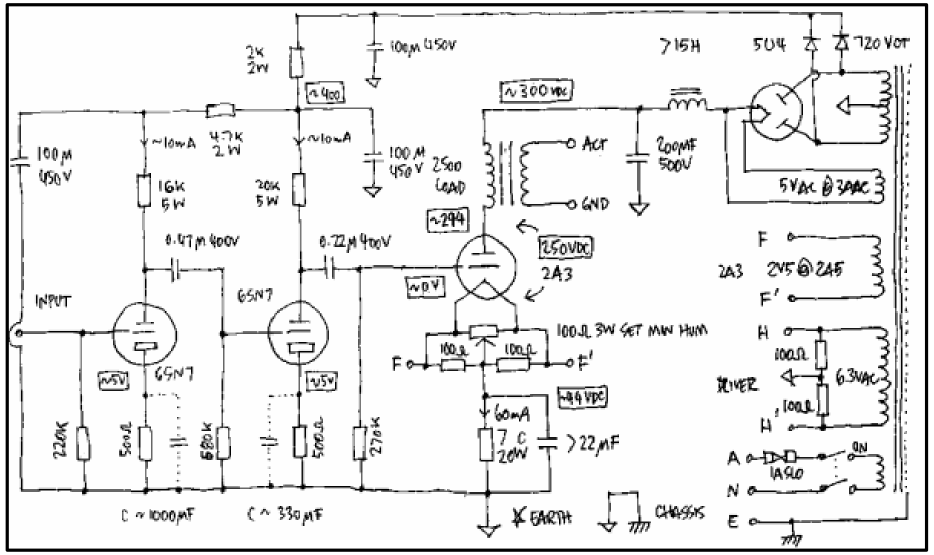

Circuit diagram by Jeff Atwood

Orthogonal Layout – Definition

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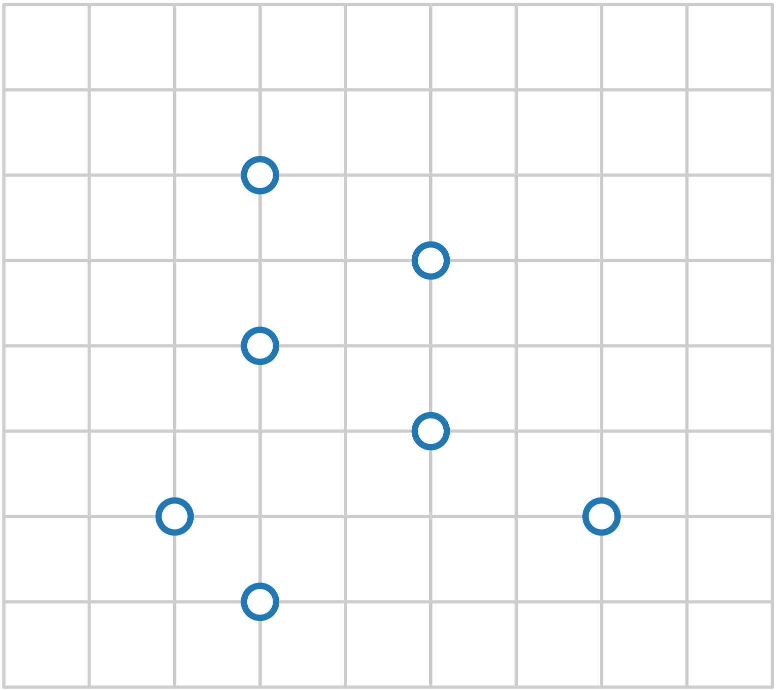
Orthogonal Layout – Definition



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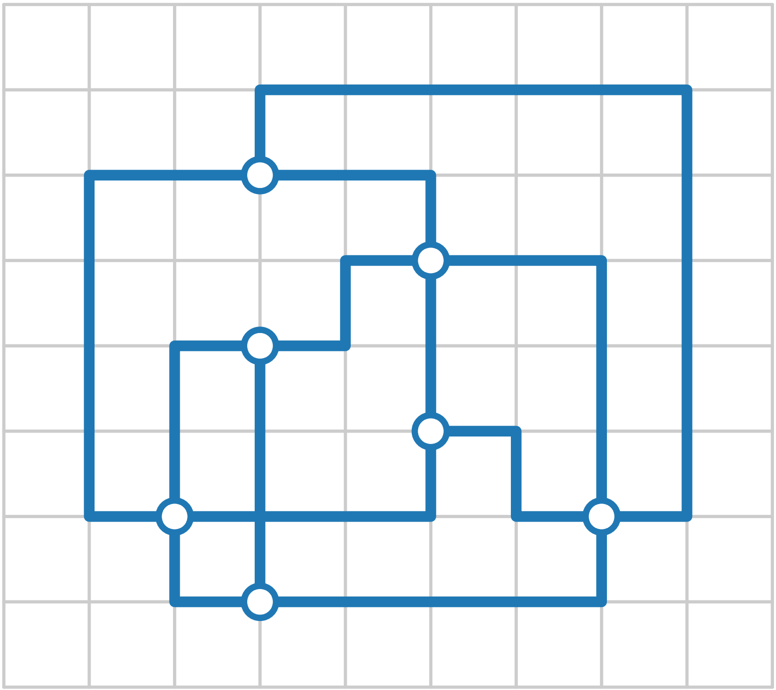

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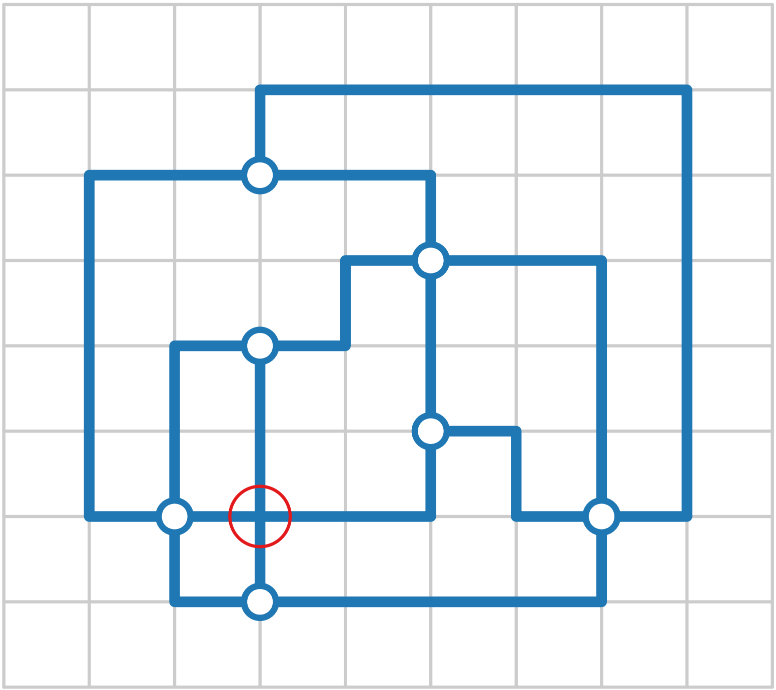

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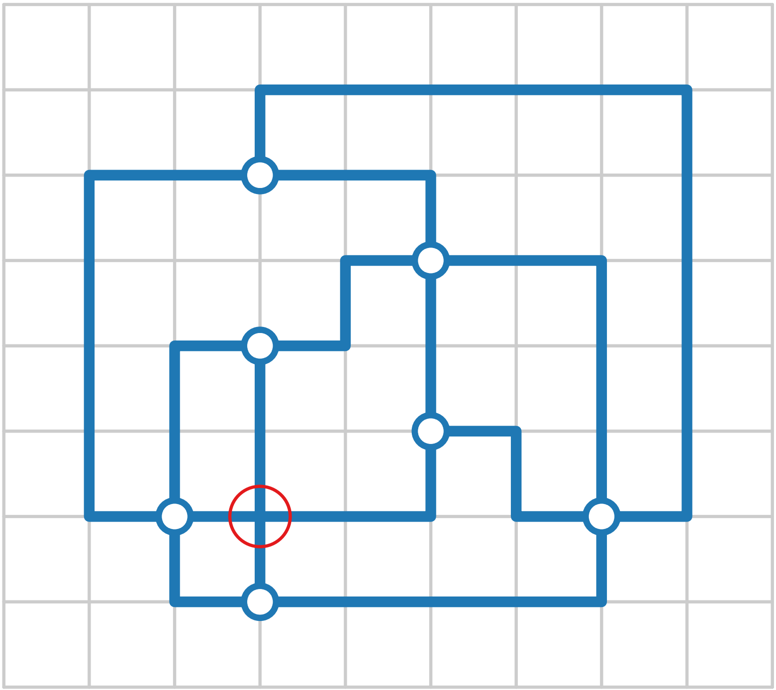

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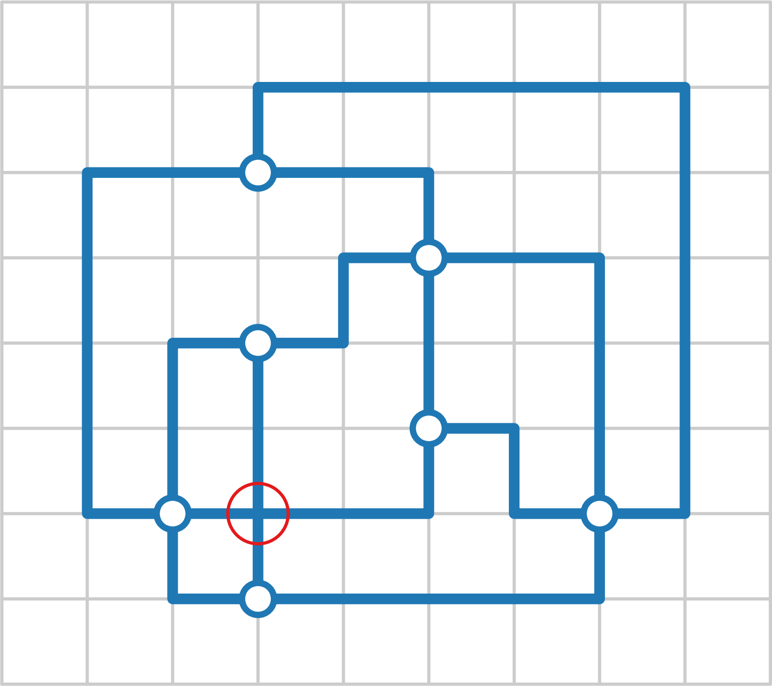

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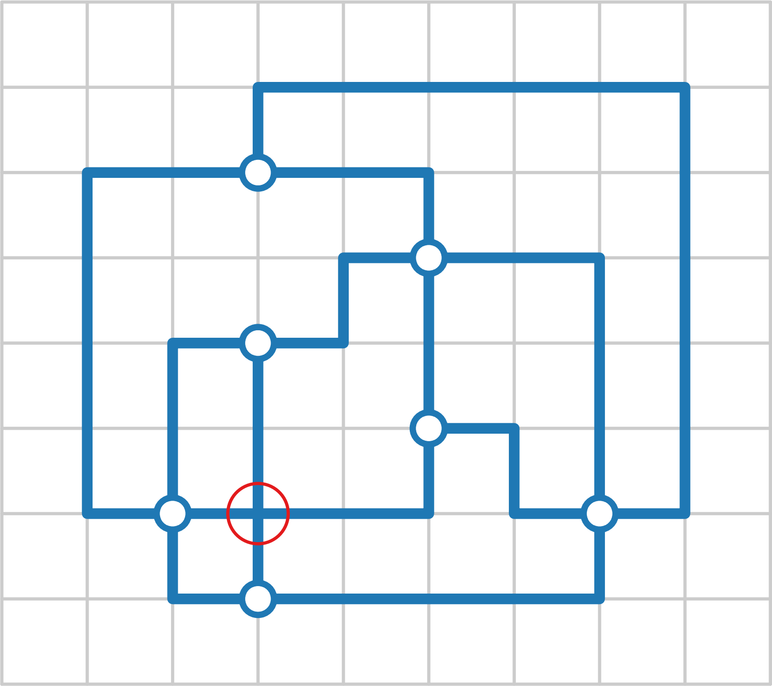

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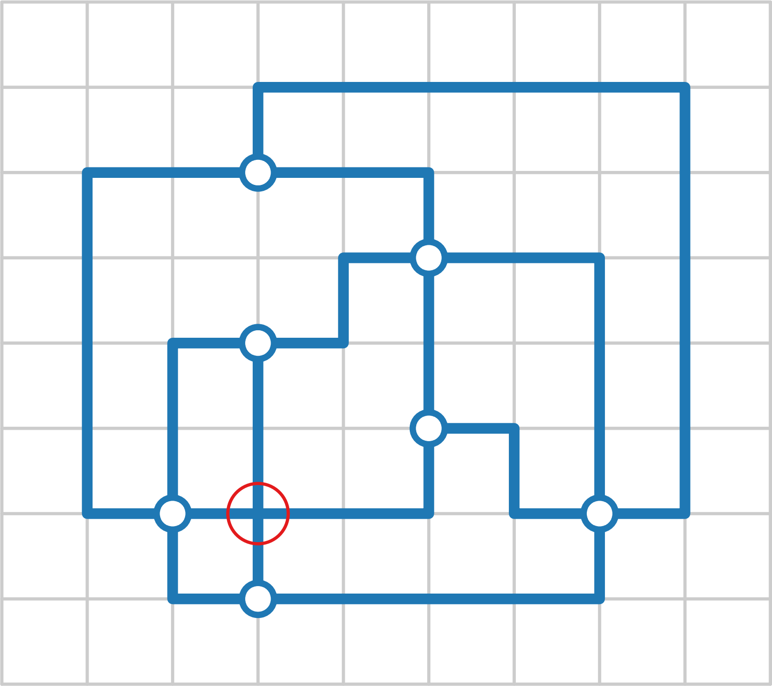

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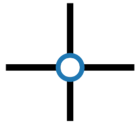
- Otherwise 

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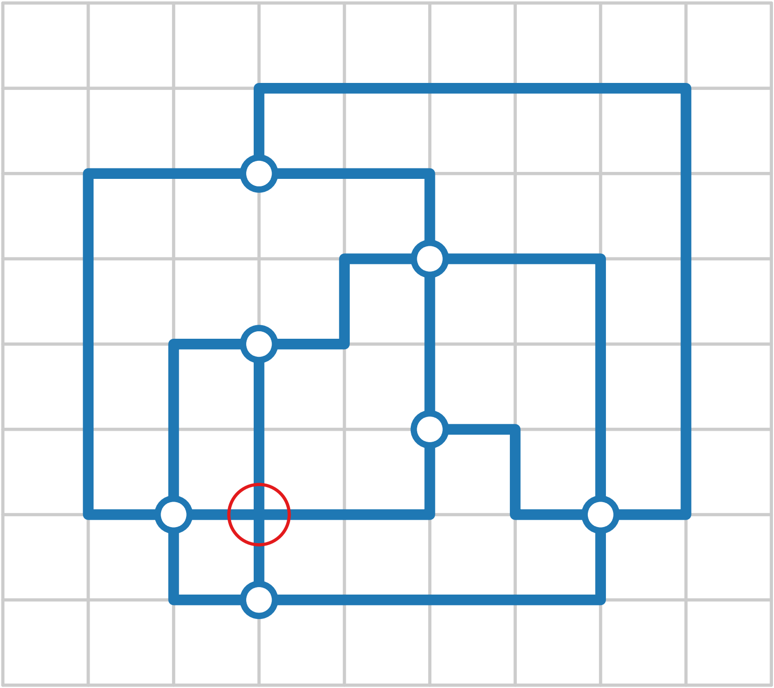

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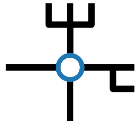
- Otherwise 

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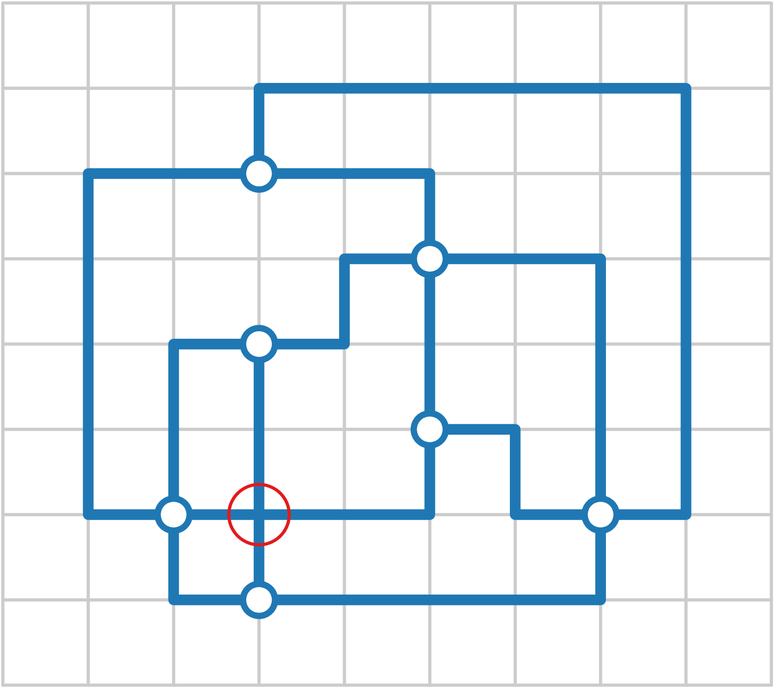

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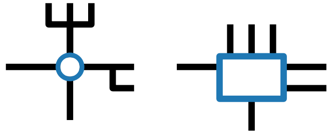
- Otherwise 

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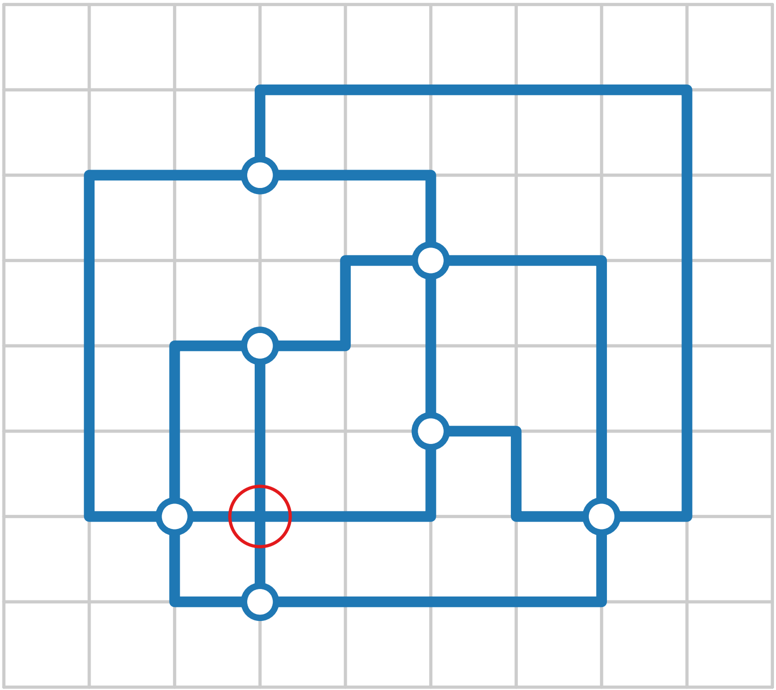

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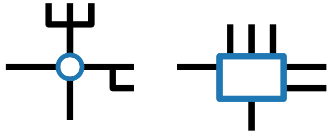
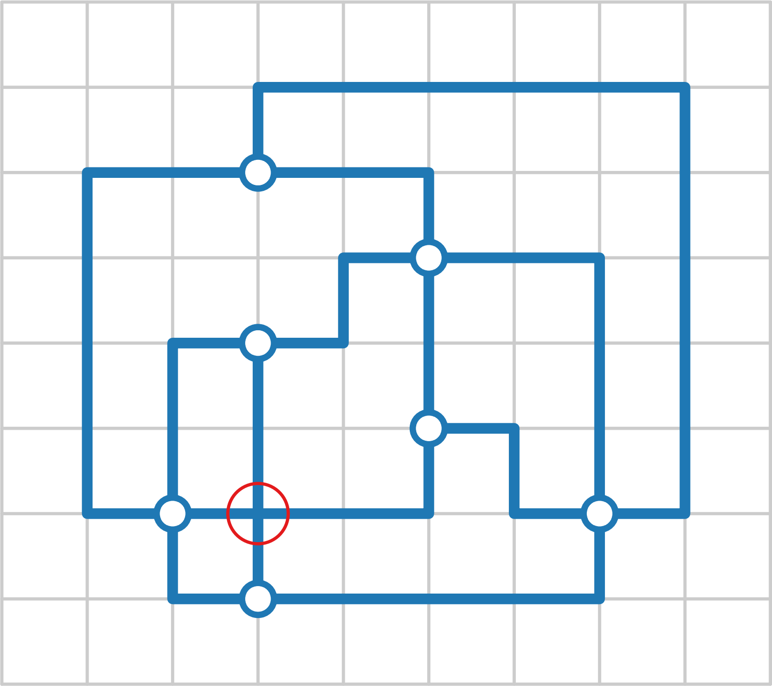


Planarization.

Orthogonal Layout – Definition



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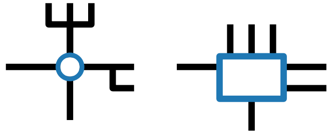
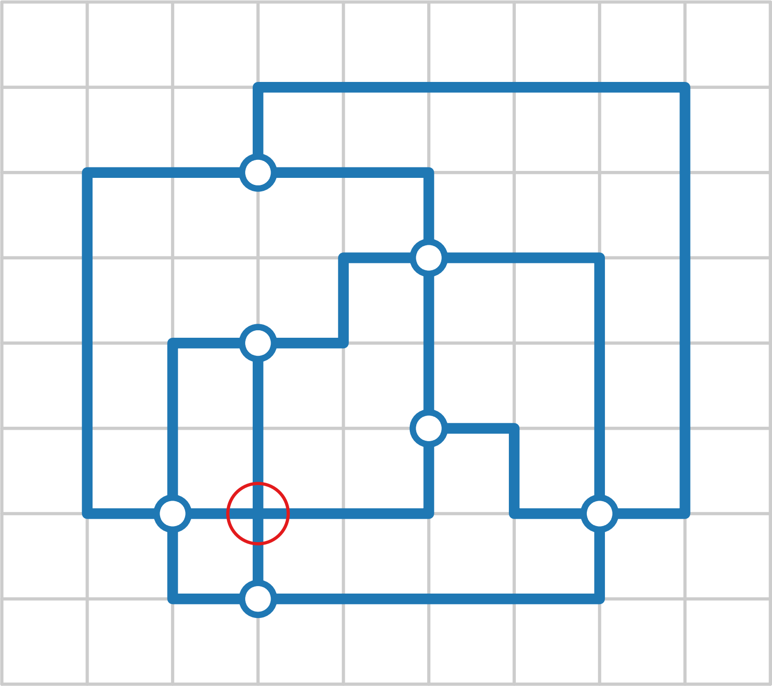
- Otherwise 

Planarization.
- Fix embedding

Orthogonal Layout – Definition



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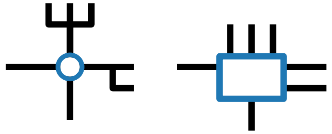
- Otherwise 

Planarization.
- Fix embedding
- Crossings become vertices

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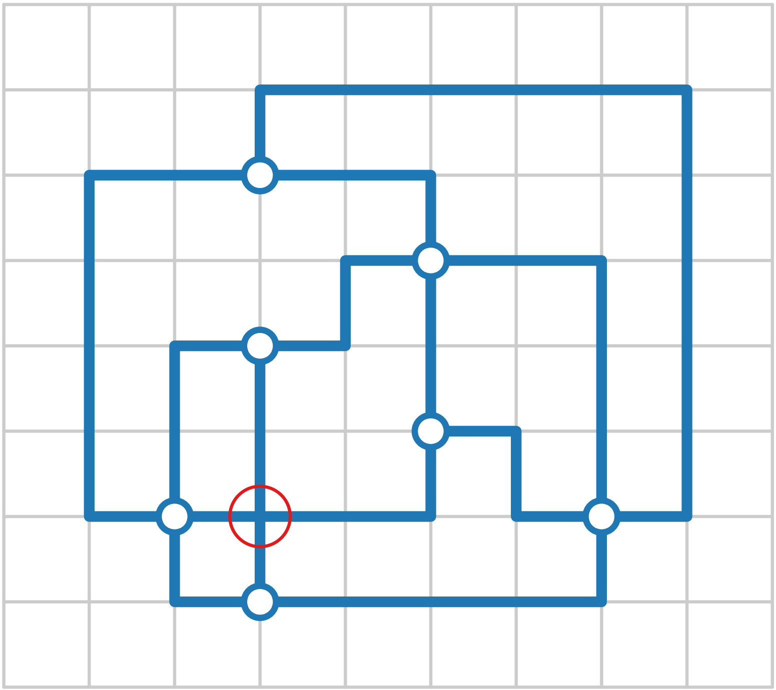

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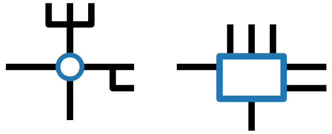

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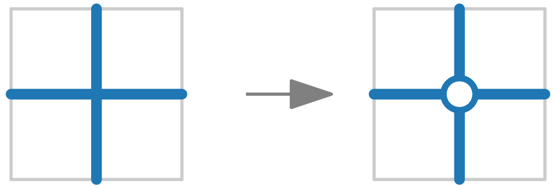

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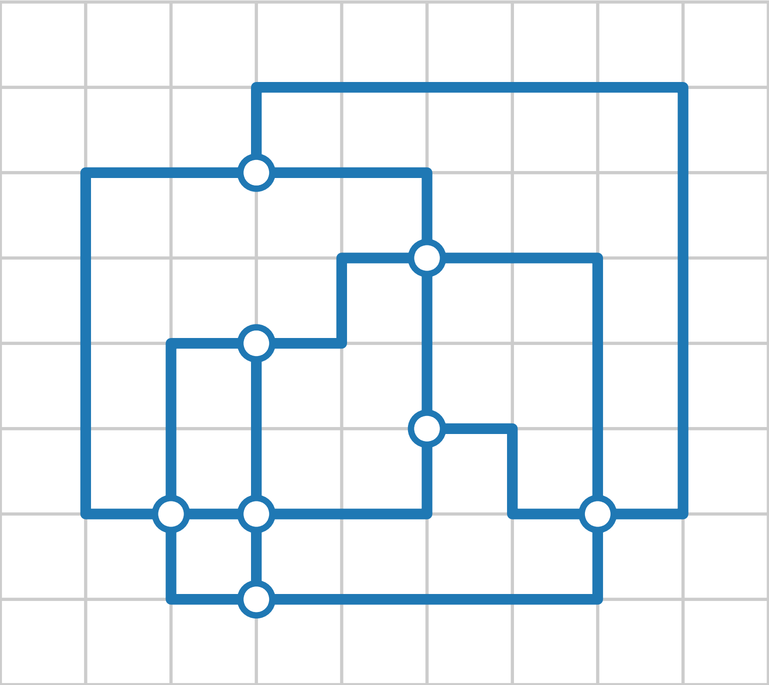

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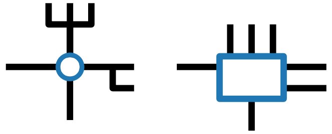

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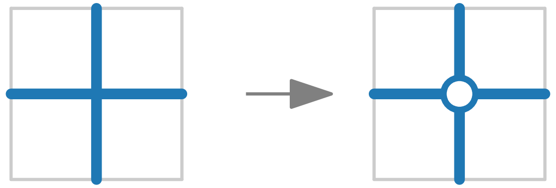

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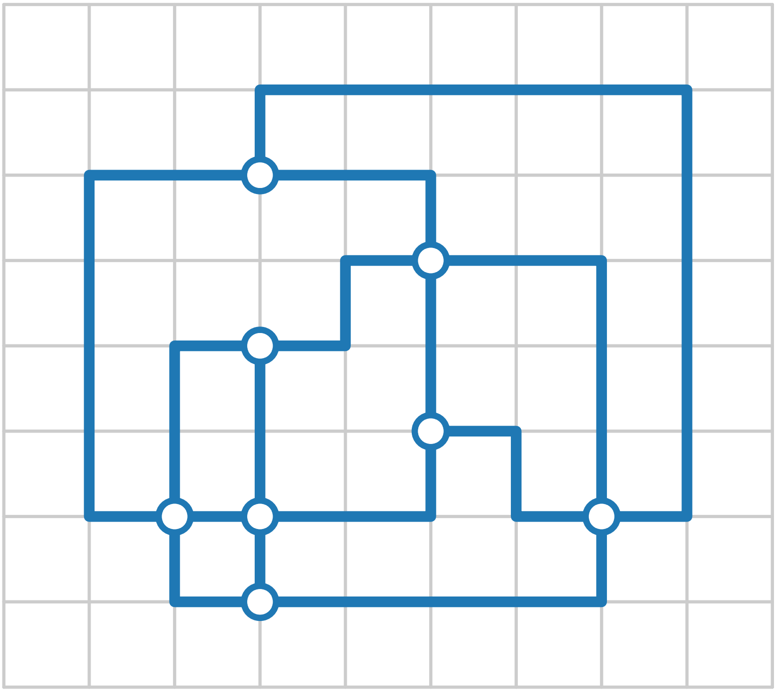

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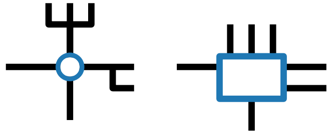

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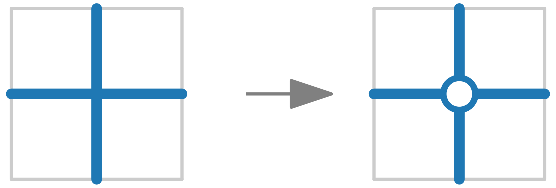
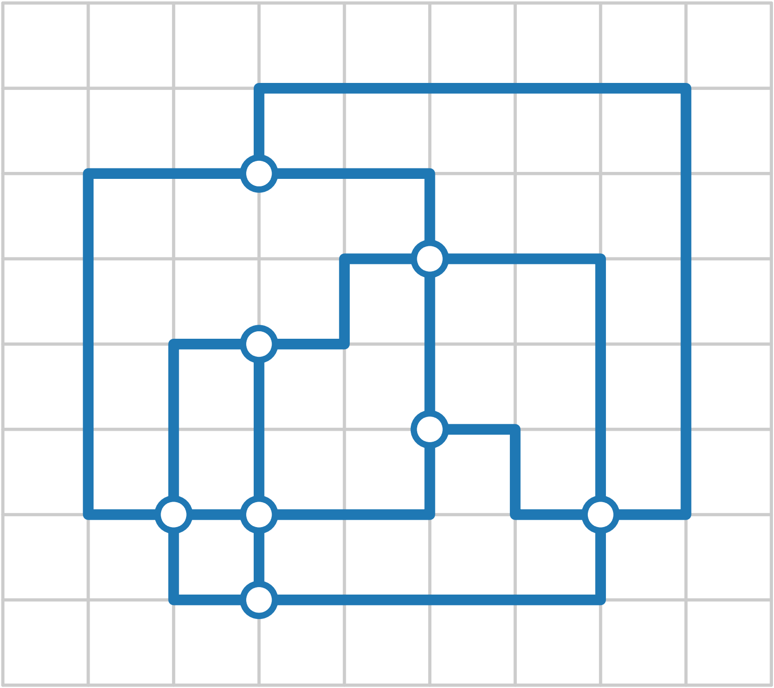


Aesthetic criteria to optimize.

Orthogonal Layout – Definition



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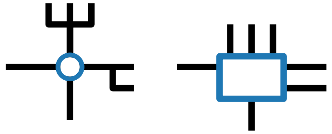

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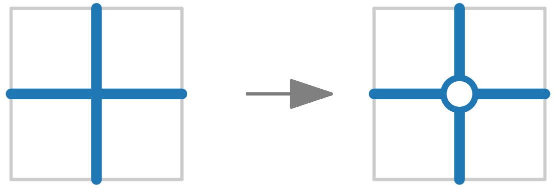
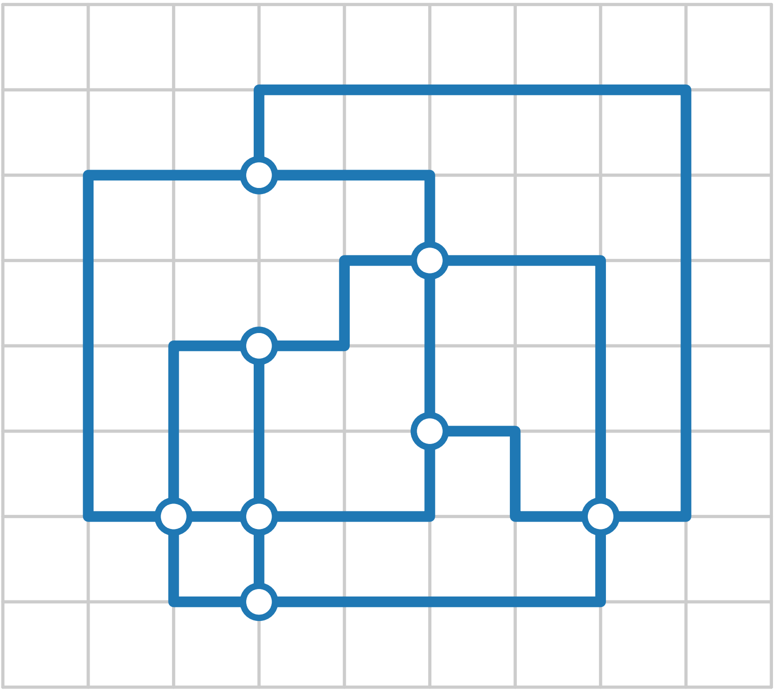


Aesthetic criteria to optimize.
- Number of bends

Orthogonal Layout – Definition



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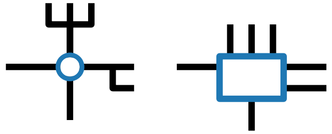

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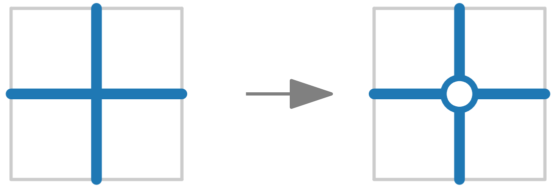
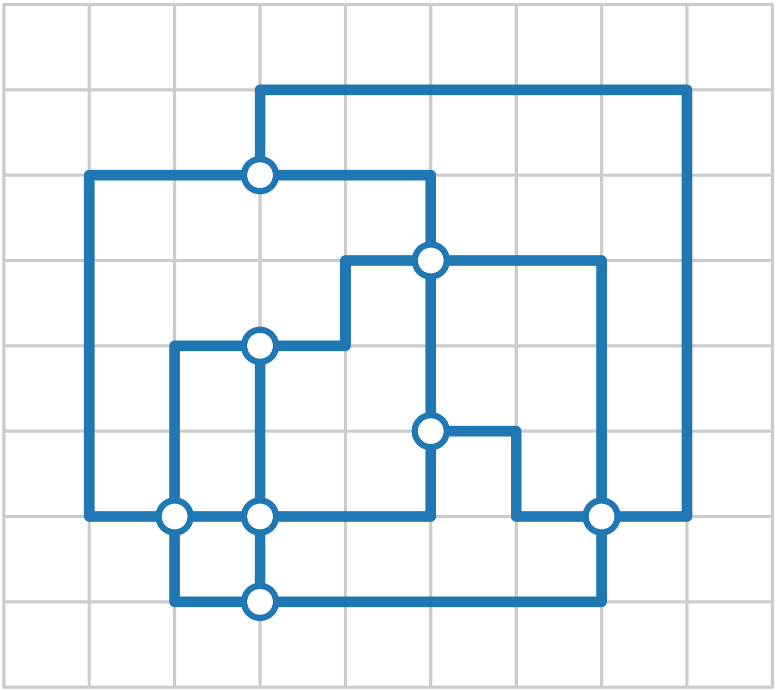


Aesthetic criteria to optimize.
- Number of bends
- Length of edges

Orthogonal Layout – Definition



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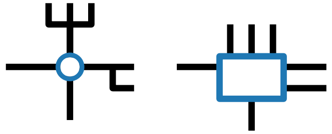

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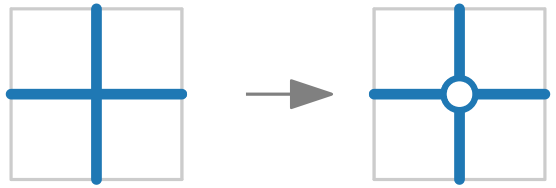
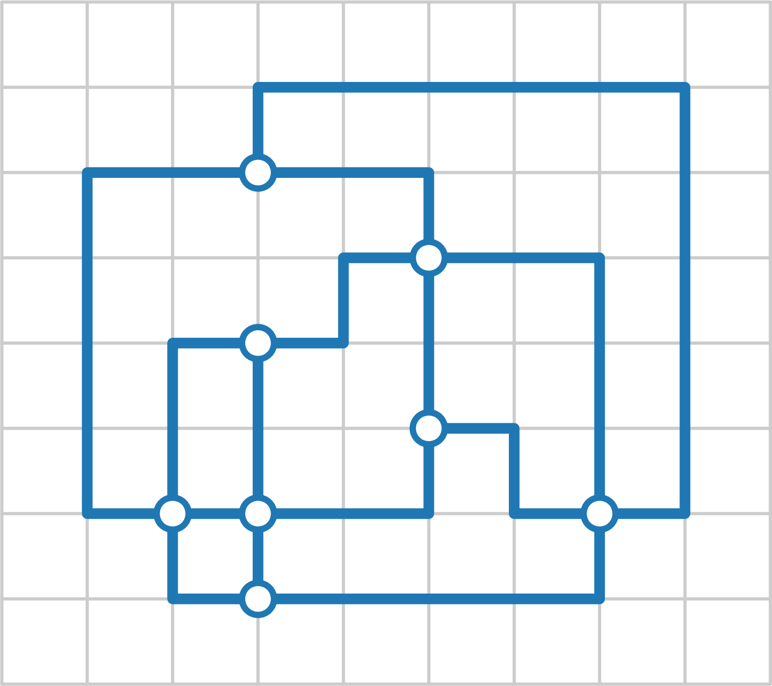


Aesthetic criteria to optimize.
- Number of bends
- Length of edges
- Width, height, area

Orthogonal Layout – Definition



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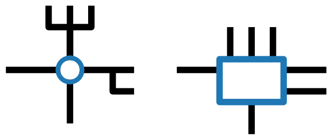

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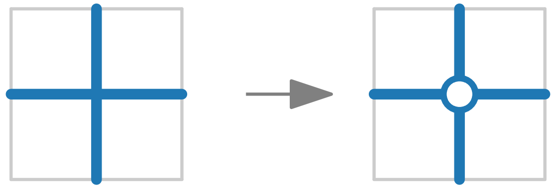

Aesthetic criteria to optimize.
- Number of bends
- Length of edges
- Width, height, area
- Monotonicity of edges
- ...

Topology – Shape – Metrics

Three-step approach: [Tamassia 1987]

TOPOLOGY – SHAPE – METRICS

Topology – Shape – Metrics

Three-step approach: [Tamassia 1987]

$V(G) = \{v_1, v_2, v_3, v_4\}$
$E(G) = \{v_1v_2, v_1v_3, v_1v_4, v_2v_3, v_2v_4\}$

TOPOLOGY — SHAPE — METRICS

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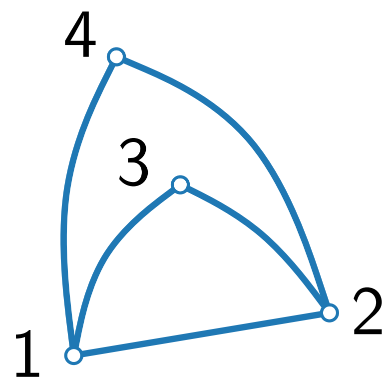
combinatorial embedding/ planarization



TOPOLOGY — SHAPE — METRICS

Topology – Shape – Metrics

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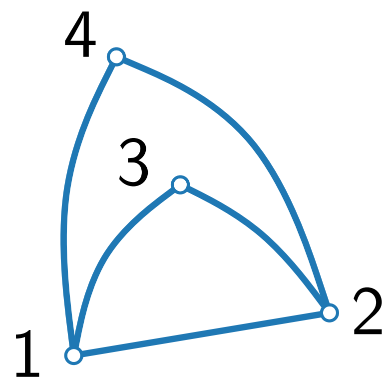

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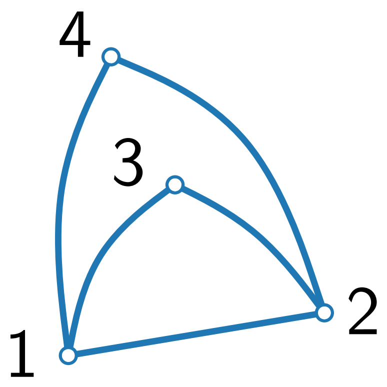
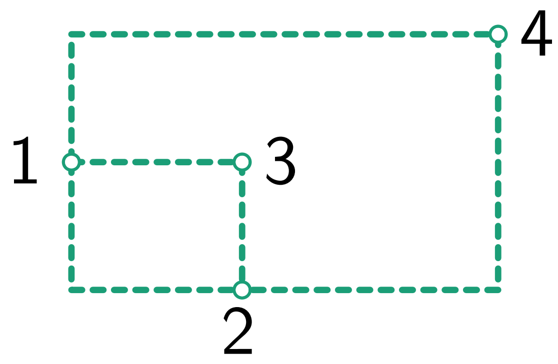


orthogonal representation



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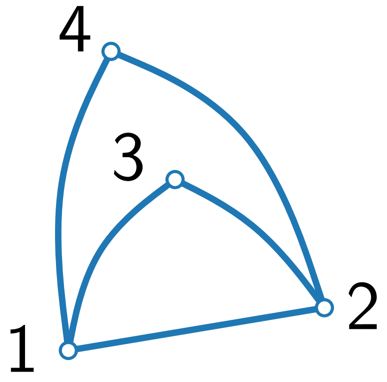

bend minimization

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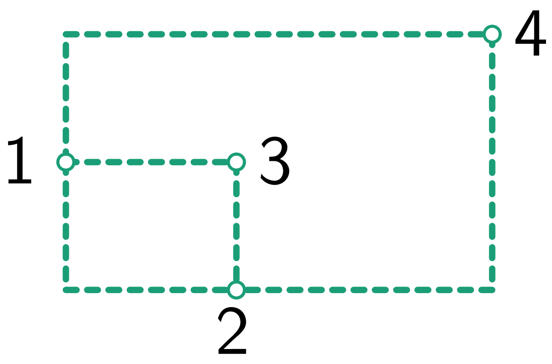

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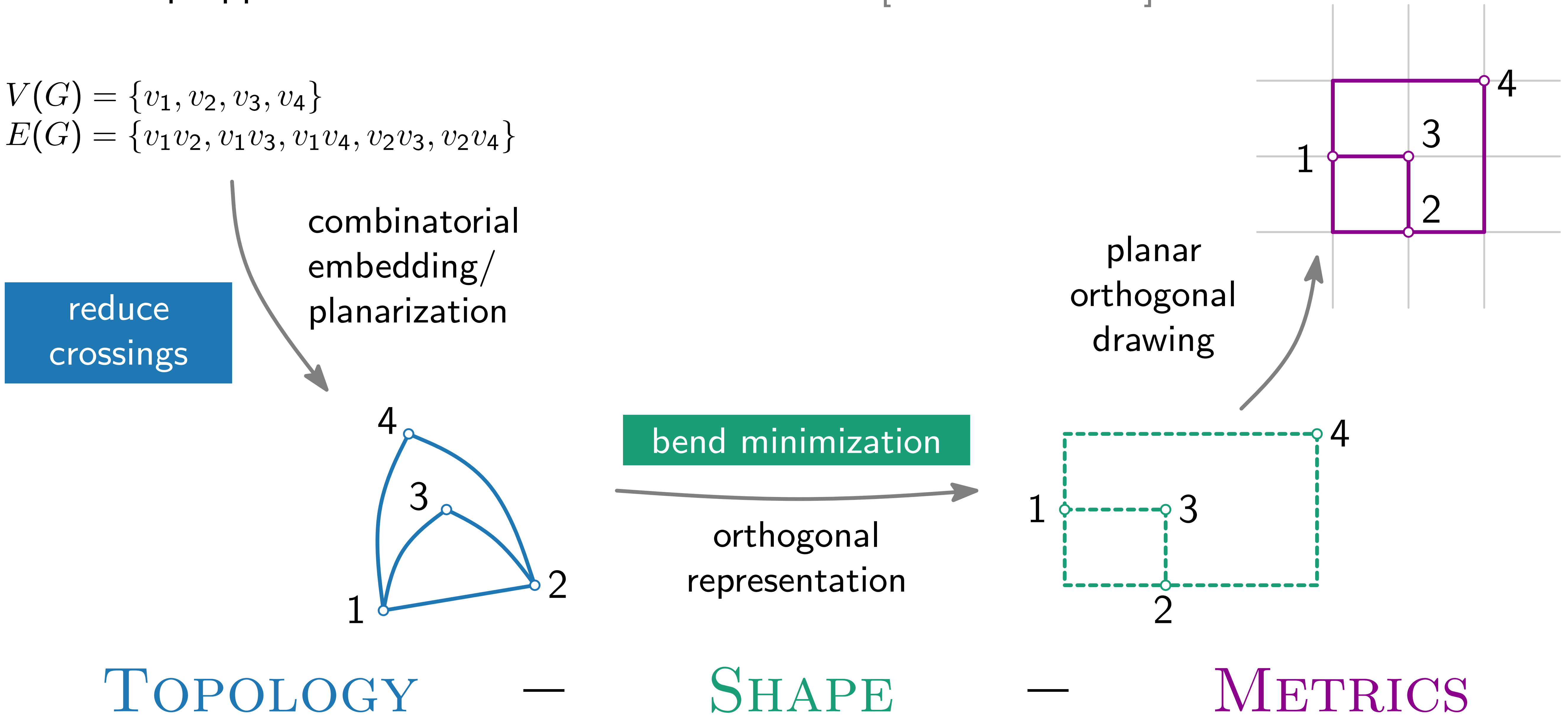

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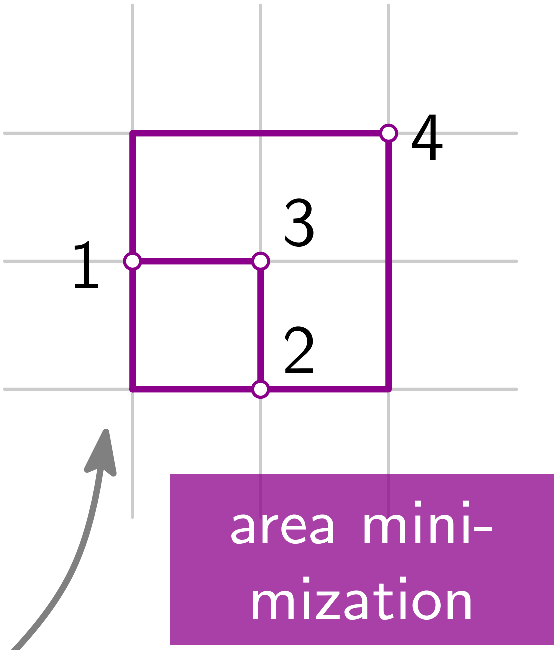

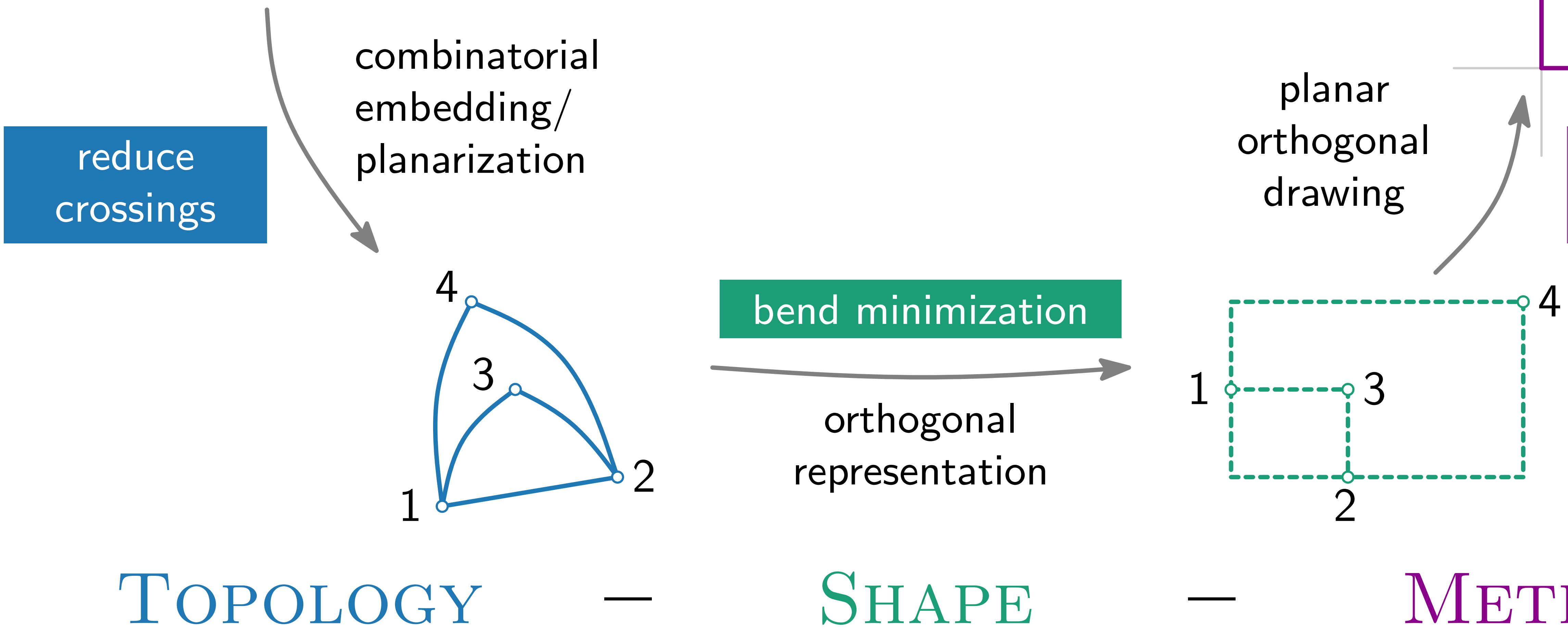

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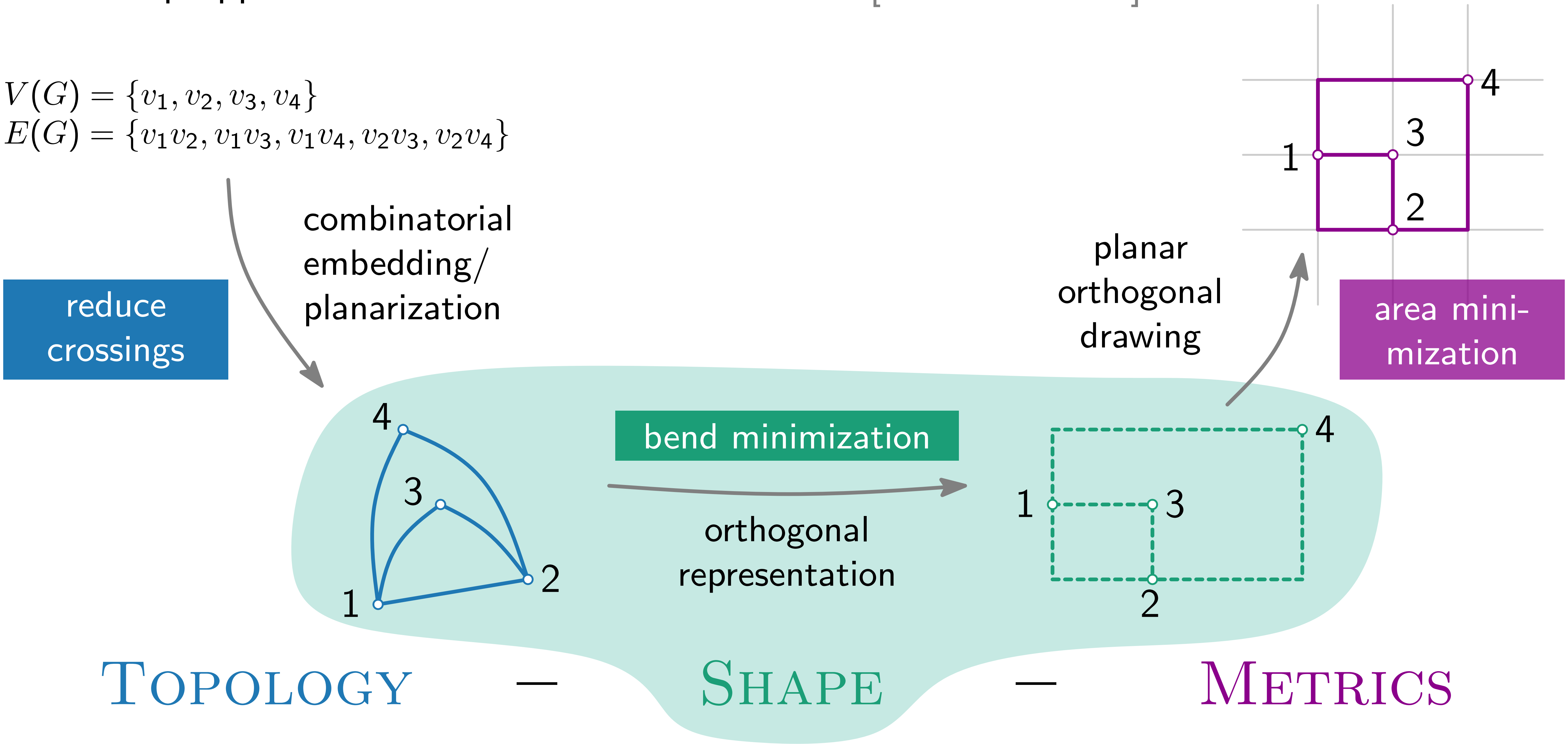

Orthogonal Representation

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Describe orthogonal drawing combinatorially.

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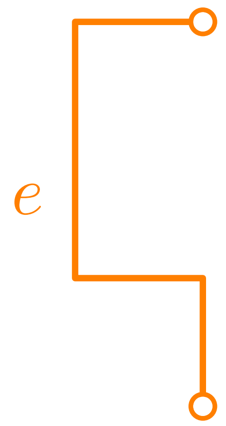

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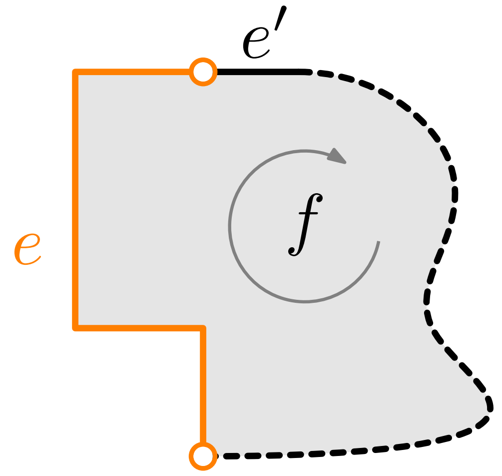

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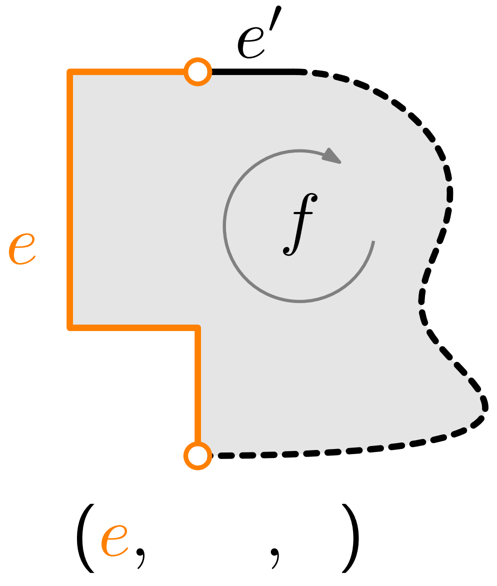

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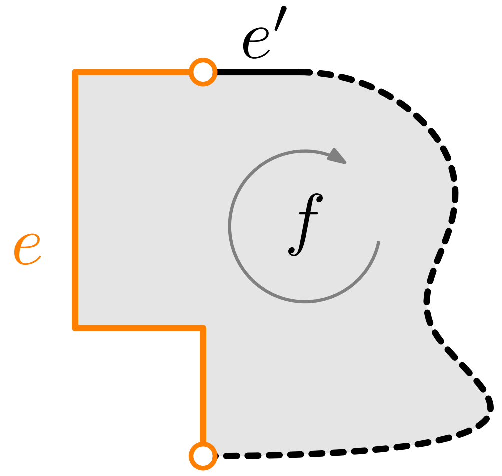

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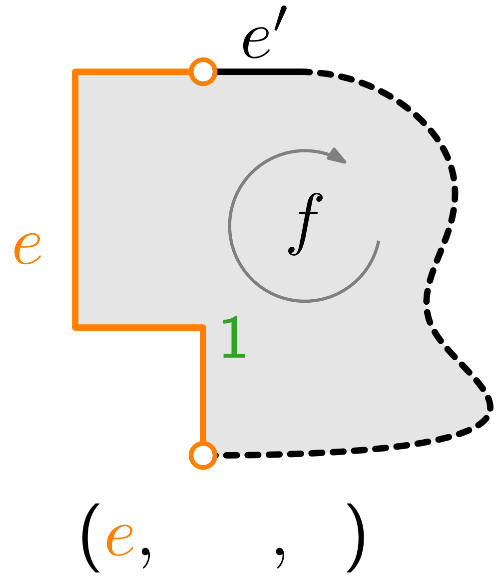

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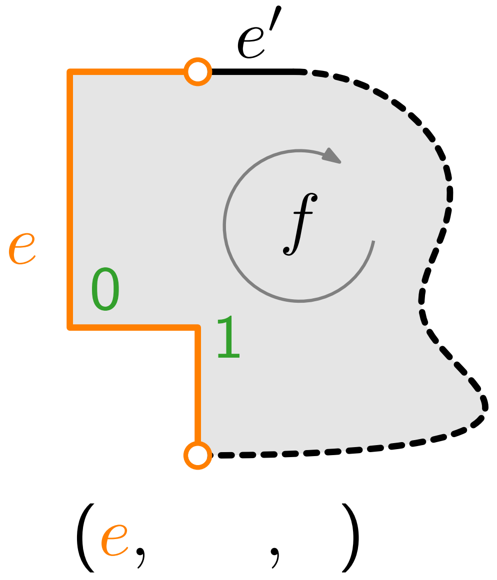

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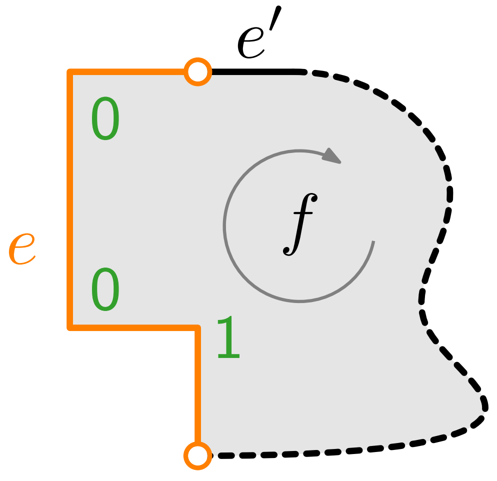

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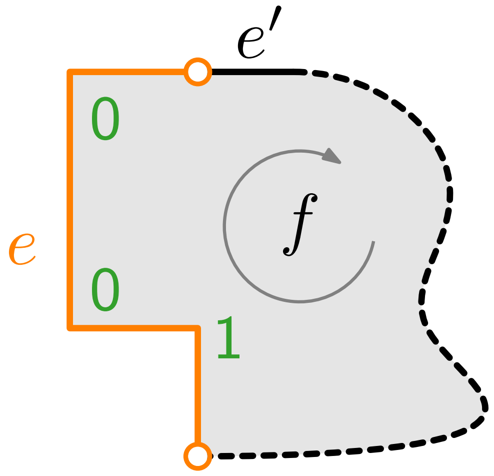

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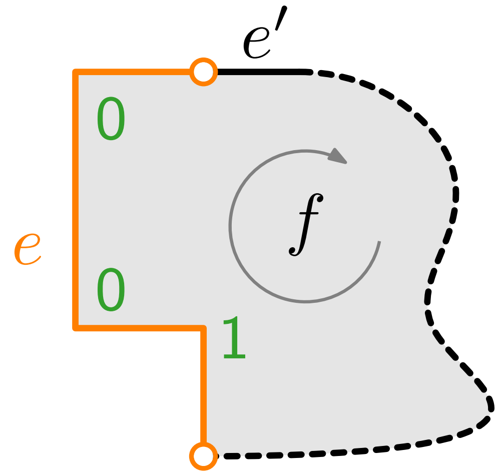

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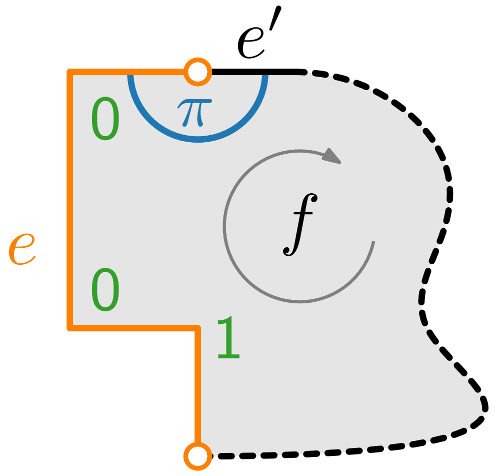

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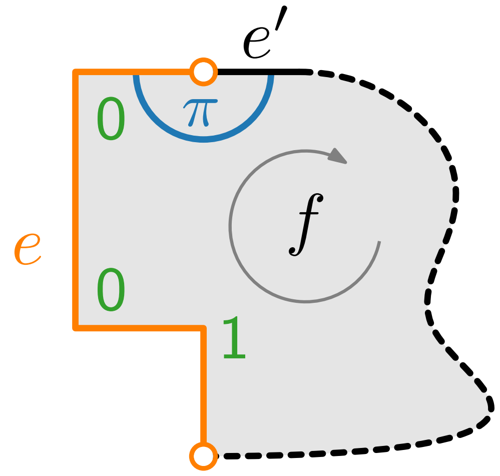

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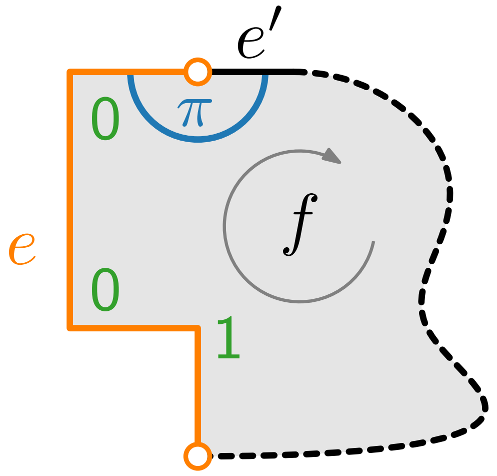

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- A **face representation** $H(f)$ of a face f is a clockwise ordered sequence $(e_1, \delta_1, \alpha_1), (e_2, \delta_2, \alpha_2), \dots, (e_{\deg(f)}, \delta_{\deg(f)}, \alpha_{\deg(f)})$ of edge descriptions w.r.t. f.

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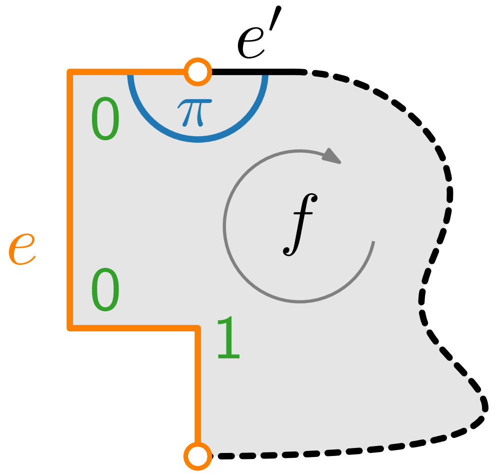

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- An **orthogonal representation** $H(G)$ of G is defined as

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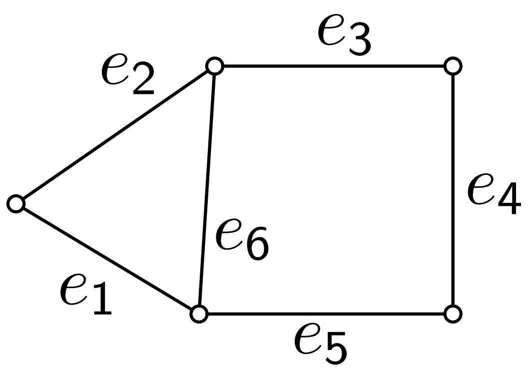
Orthogonal Representation – Example



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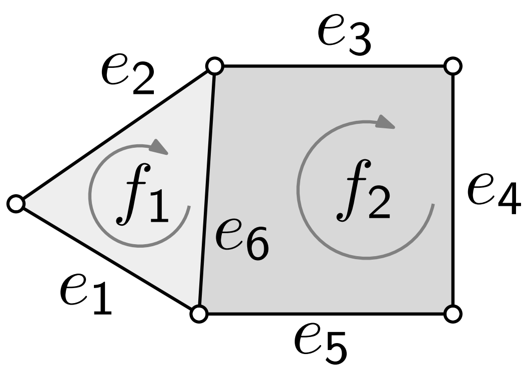

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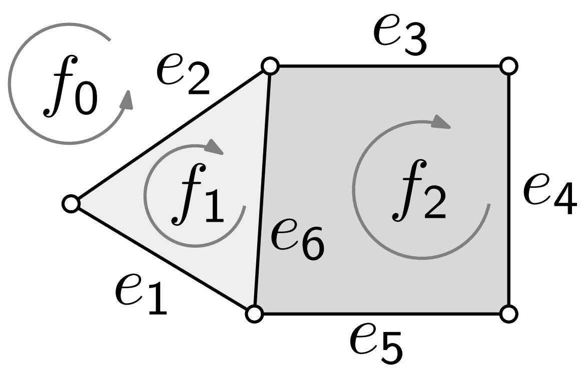

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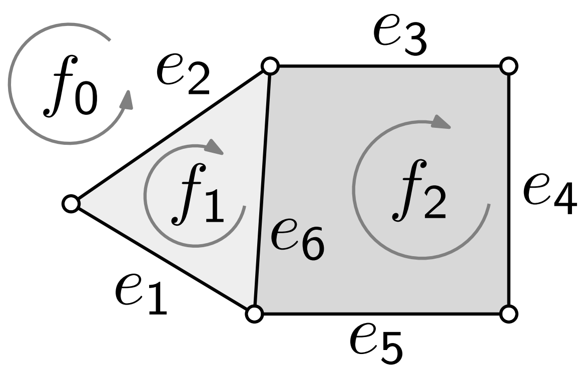

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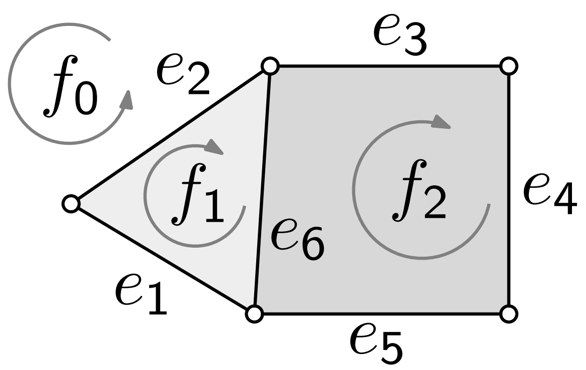

Combinatorial "drawing" of $H(G)$?

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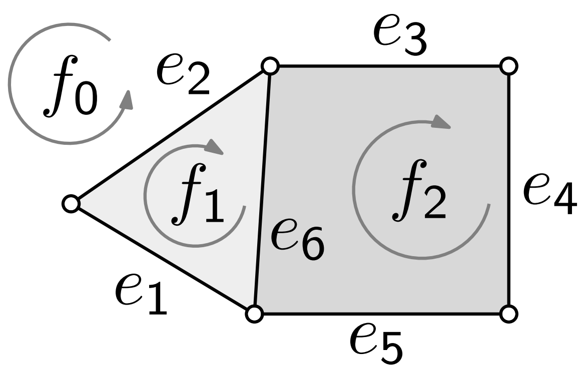

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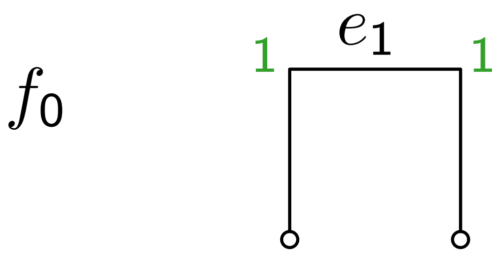

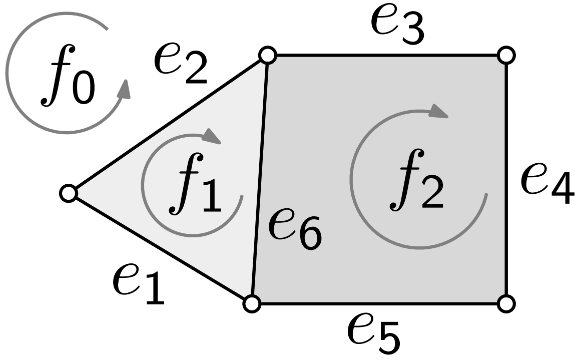

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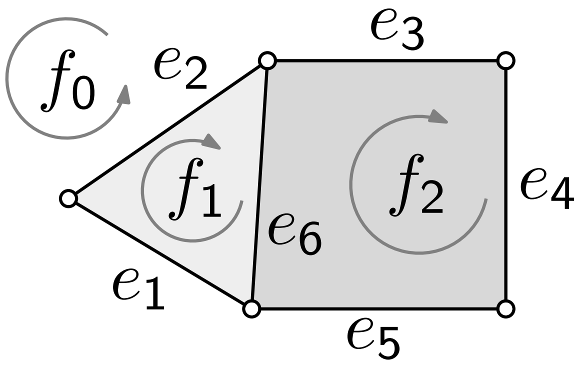

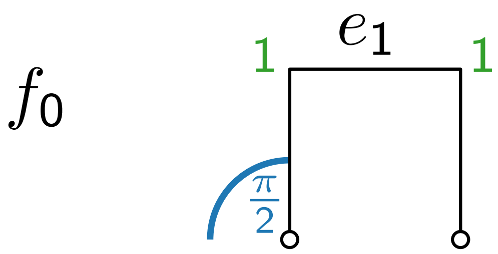

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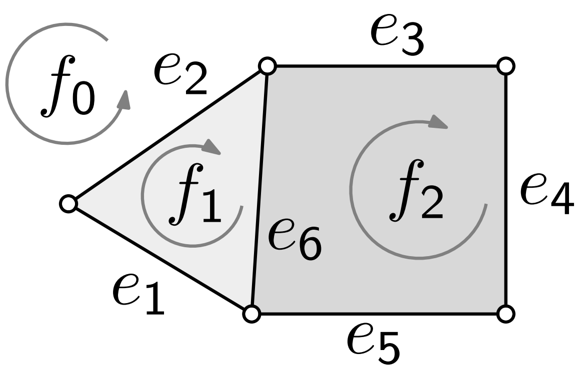

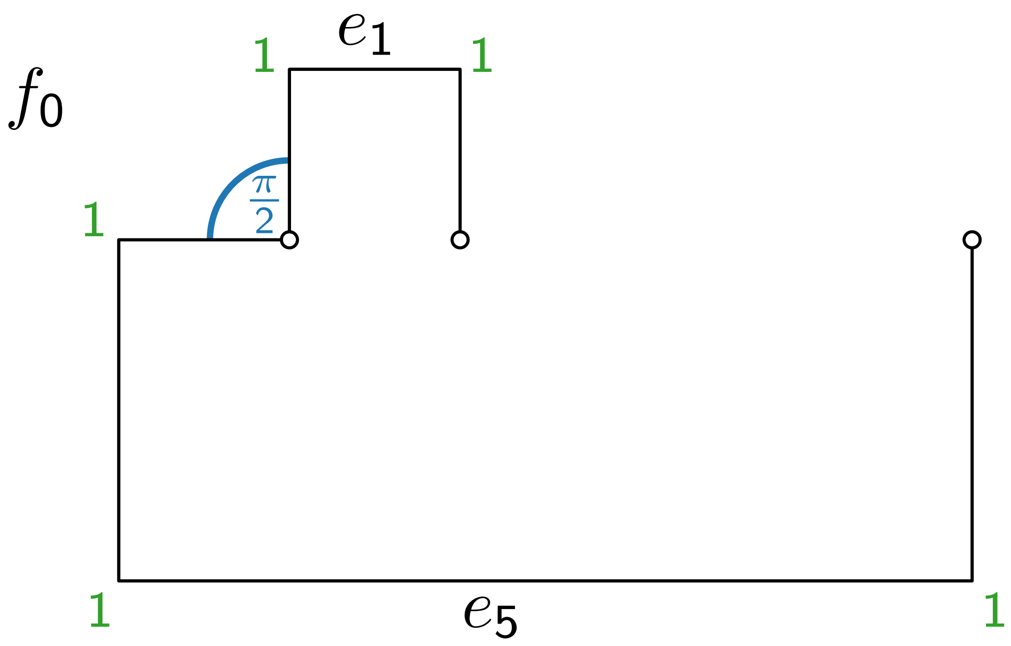

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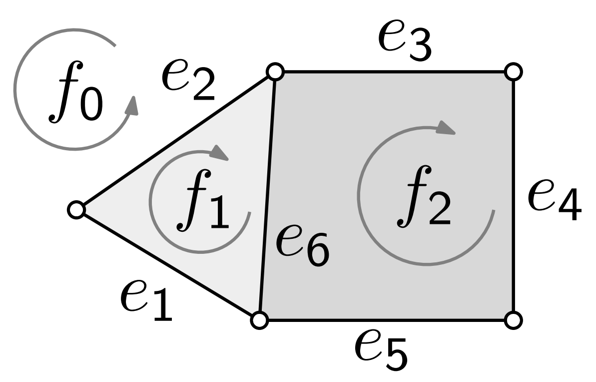

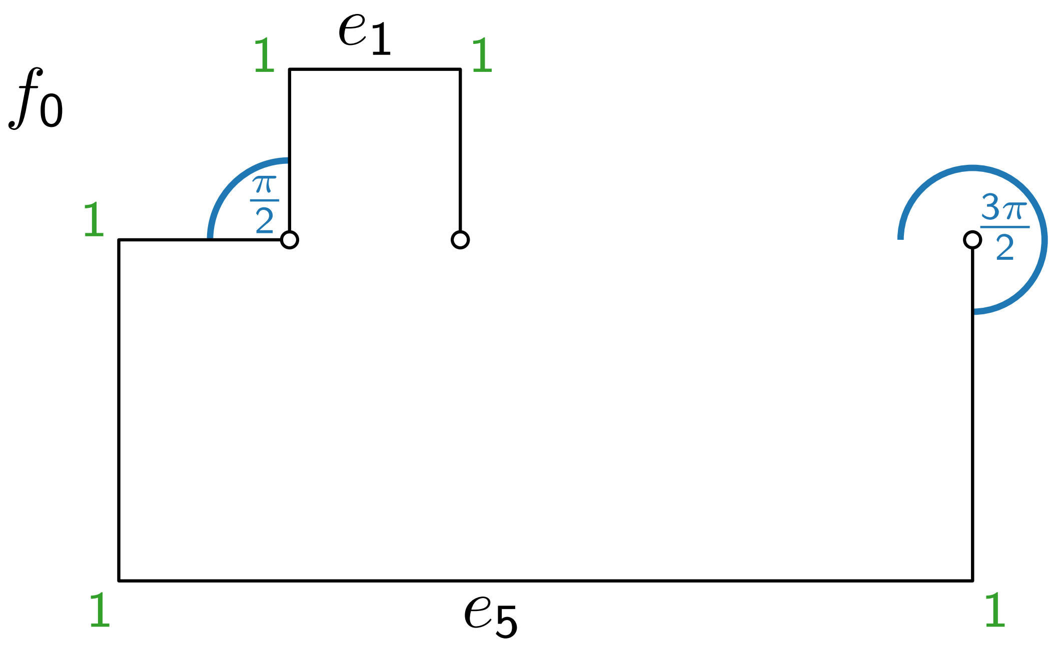

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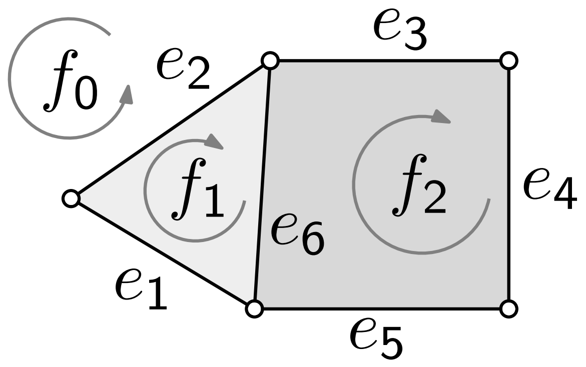
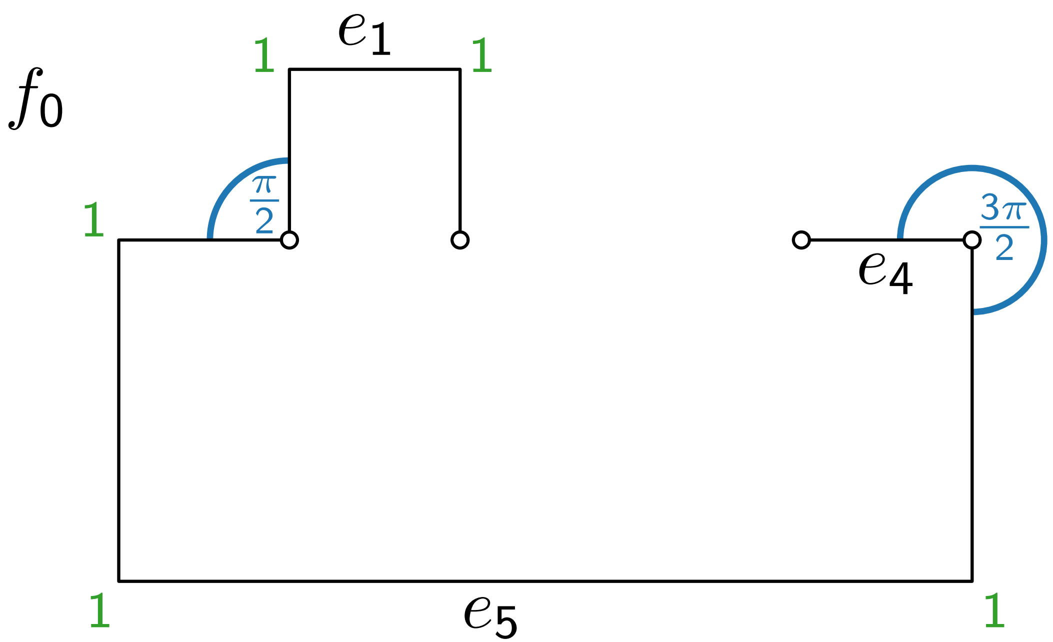
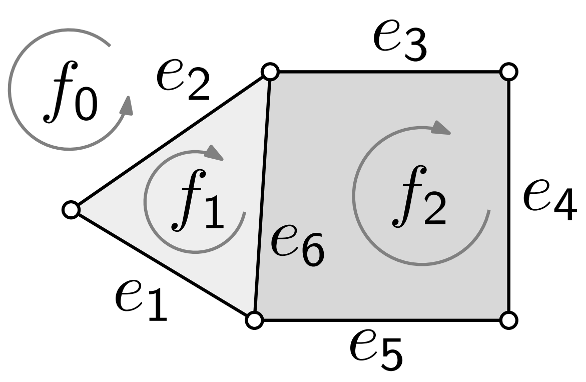
$$H(f_2) = ((e_5, 000, \tfrac{\pi}{2}), (e_6, 11, \tfrac{\pi}{2}), (e_3, \emptyset, \pi), (e_4, \emptyset, \tfrac{\pi}{2}))$$

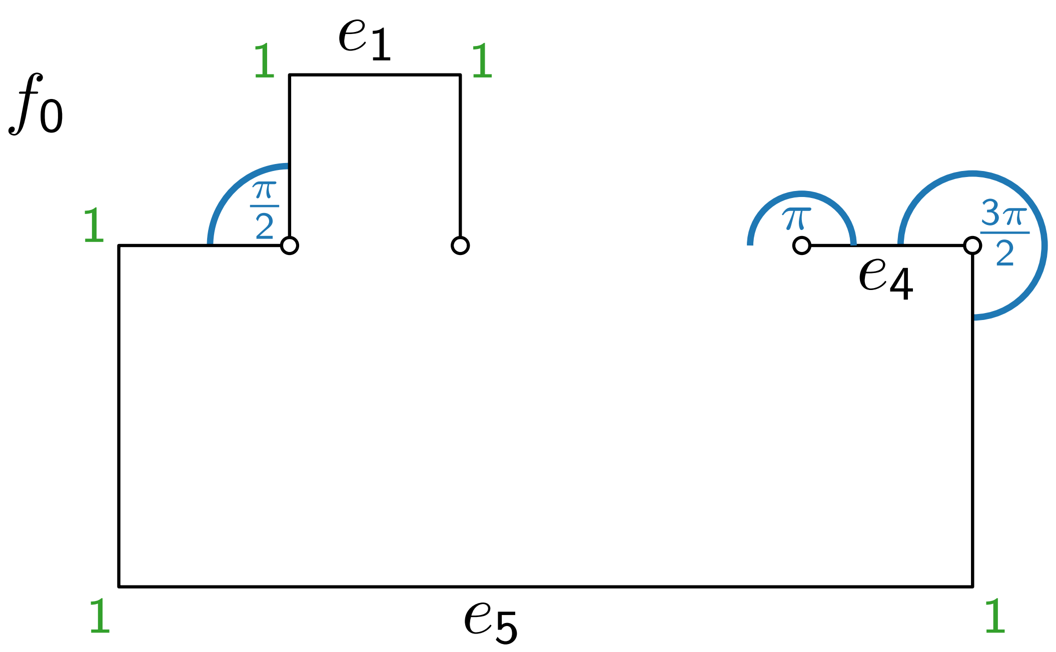




Orthogonal Representation – Example

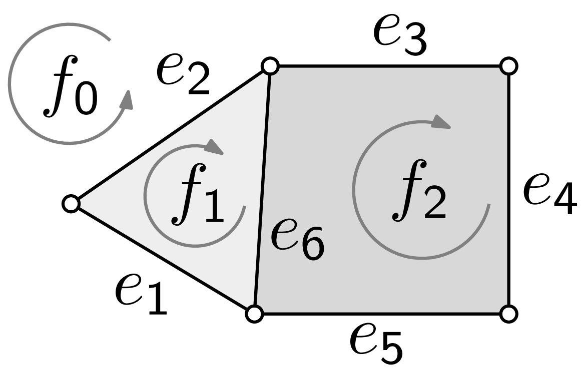
$$H(f_0) = ((e_1, 11, \tfrac{\pi}{2}), (e_5, 111, \tfrac{3\pi}{2}), (e_4, \emptyset, \pi), (e_3, \emptyset, \pi), (e_2, \emptyset, \tfrac{\pi}{2}))$$
$$H(f_1) = ((e_1, 00, \tfrac{3\pi}{2}), (e_2, \emptyset, \tfrac{\pi}{2}), (e_6, 00, \pi))$$
$$H(f_2) = ((e_5, 000, \tfrac{\pi}{2}), (e_6, 11, \tfrac{\pi}{2}), (e_3, \emptyset, \pi), (e_4, \emptyset, \tfrac{\pi}{2}))$$

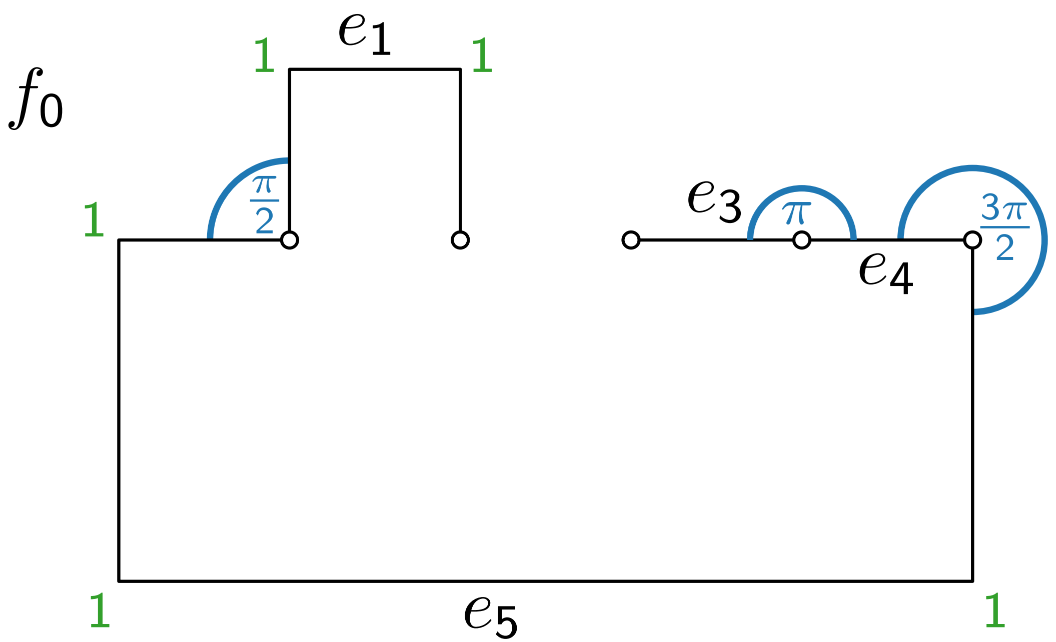




Orthogonal Representation – Example

$$H(f_0) = ((e_1, 11, \tfrac{\pi}{2}), (e_5, 111, \tfrac{3\pi}{2}), (e_4, \emptyset, \pi), (e_3, \emptyset, \pi), (e_2, \emptyset, \tfrac{\pi}{2}))$$
$$H(f_1) = ((e_1, 00, \tfrac{3\pi}{2}), (e_2, \emptyset, \tfrac{\pi}{2}), (e_6, 00, \pi))$$
$$H(f_2) = ((e_5, 000, \tfrac{\pi}{2}), (e_6, 11, \tfrac{\pi}{2}), (e_3, \emptyset, \pi), (e_4, \emptyset, \tfrac{\pi}{2}))$$





Orthogonal Representation – Example

$$H(f_0) = ((e_1, 11, \tfrac{\pi}{2}), (e_5, 111, \tfrac{3\pi}{2}), (e_4, \emptyset, \pi), (e_3, \emptyset, \pi), (e_2, \emptyset, \tfrac{\pi}{2}))$$
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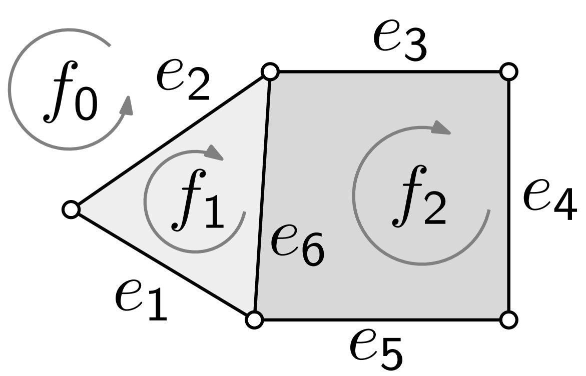
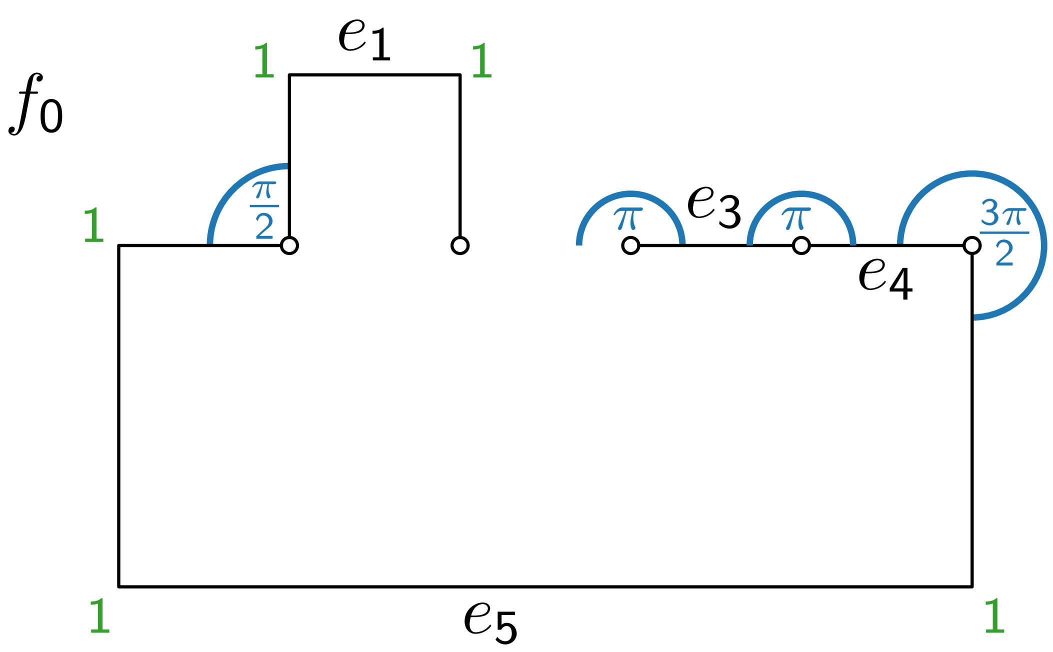
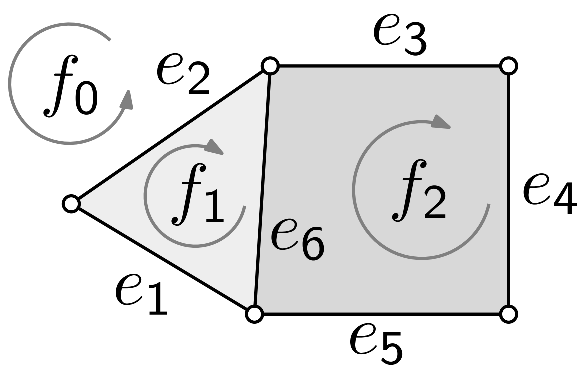
$$H(f_2) = ((e_5, 000, \tfrac{\pi}{2}), (e_6, 11, \tfrac{\pi}{2}), (e_3, \emptyset, \pi), (e_4, \emptyset, \tfrac{\pi}{2}))$$

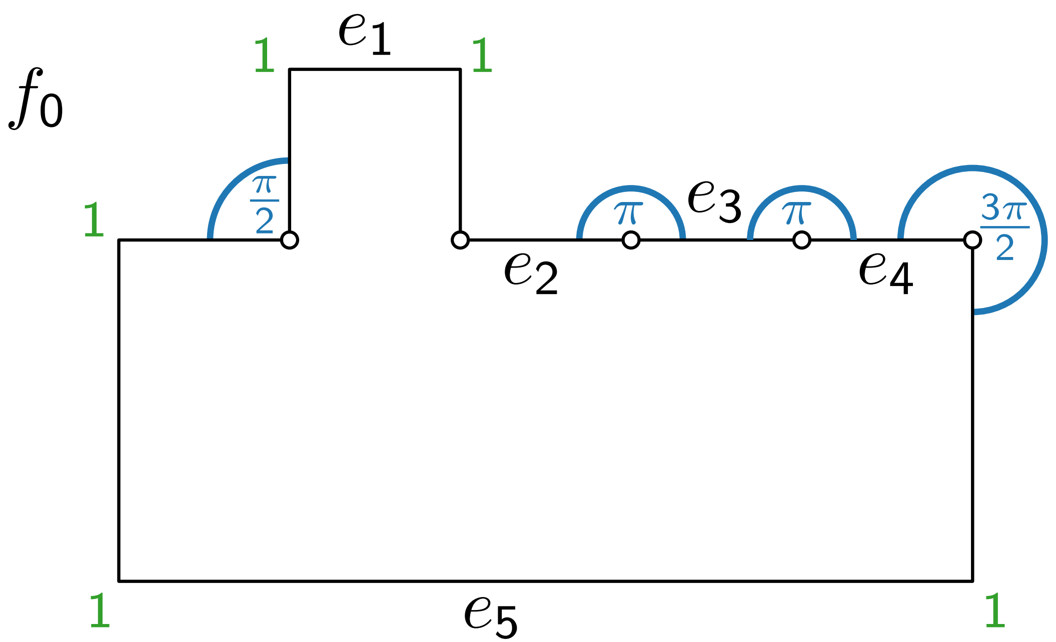




Orthogonal Representation – Example

$$H(f_0) = ((e_1, 11, \tfrac{\pi}{2}), (e_5, 111, \tfrac{3\pi}{2}), (e_4, \emptyset, \pi), (e_3, \emptyset, \pi), (e_2, \emptyset, \tfrac{\pi}{2}))$$
$$H(f_1) = ((e_1, 00, \tfrac{3\pi}{2}), (e_2, \emptyset, \tfrac{\pi}{2}), (e_6, 00, \pi))$$
$$H(f_2) = ((e_5, 000, \tfrac{\pi}{2}), (e_6, 11, \tfrac{\pi}{2}), (e_3, \emptyset, \pi), (e_4, \emptyset, \tfrac{\pi}{2}))$$





Orthogonal Representation – Example

$$H(f_0) = ((e_1, 11, \tfrac{\pi}{2}), (e_5, 111, \tfrac{3\pi}{2}), (e_4, \emptyset, \pi), (e_3, \emptyset, \pi), (e_2, \emptyset, \tfrac{\pi}{2}))$$
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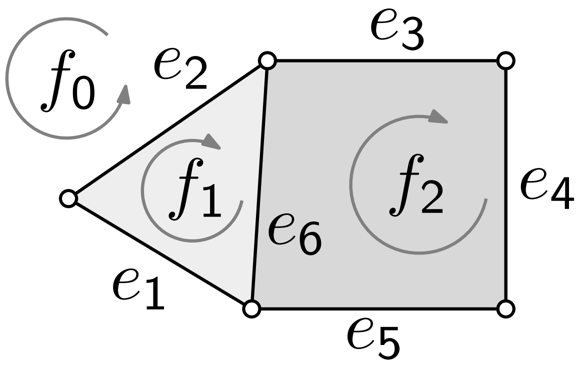
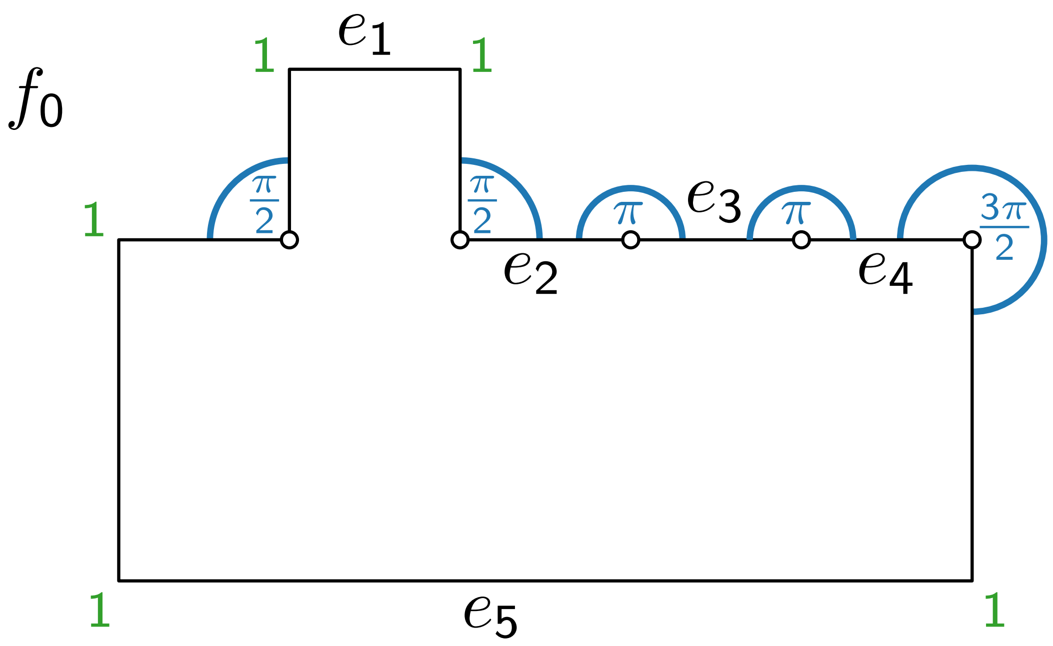


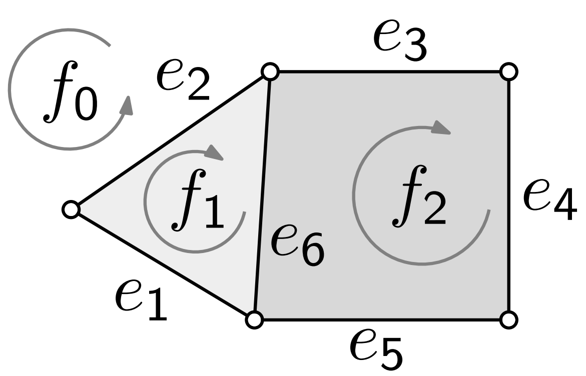


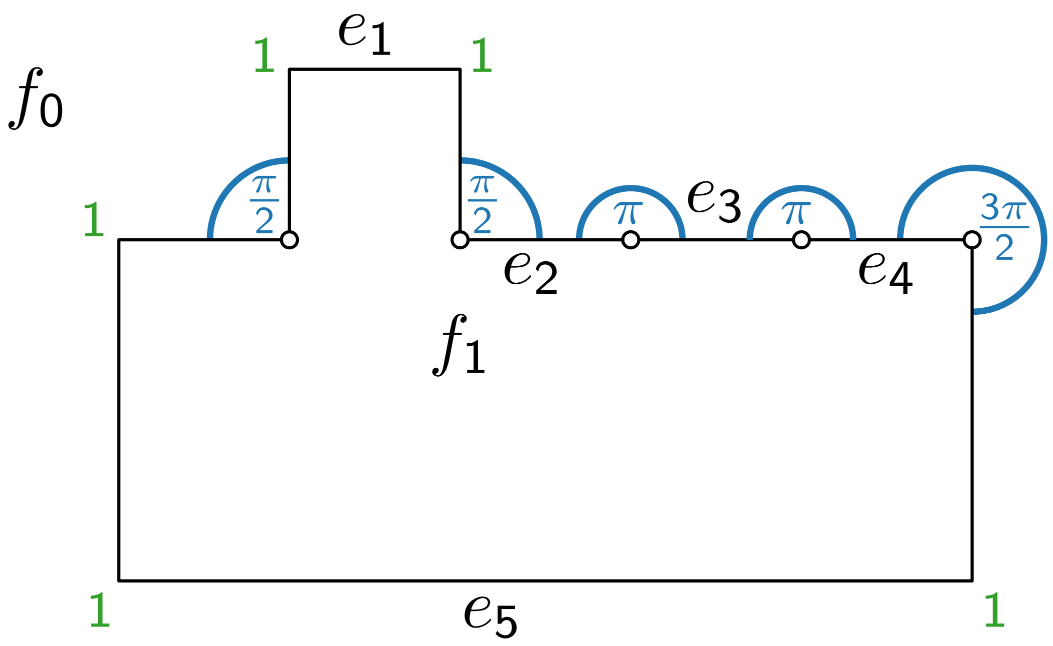
Orthogonal Representation – Example

$$H(f_0) = ((e_1, 11, \tfrac{\pi}{2}), (e_5, 111, \tfrac{3\pi}{2}), (e_4, \emptyset, \pi), (e_3, \emptyset, \pi), (e_2, \emptyset, \tfrac{\pi}{2}))$$

$$H(f_1) = ((e_1, 00, \tfrac{3\pi}{2}), (e_2, \emptyset, \tfrac{\pi}{2}), (e_6, 00, \pi))$$

$$H(f_2) = ((e_5, 000, \tfrac{\pi}{2}), (e_6, 11, \tfrac{\pi}{2}), (e_3, \emptyset, \pi), (e_4, \emptyset, \tfrac{\pi}{2}))$$





Orthogonal Representation – Example

$$H(f_0) = ((e_1, 11, \tfrac{\pi}{2}), (e_5, 111, \tfrac{3\pi}{2}), (e_4, \emptyset, \pi), (e_3, \emptyset, \pi), (e_2, \emptyset, \tfrac{\pi}{2}))$$
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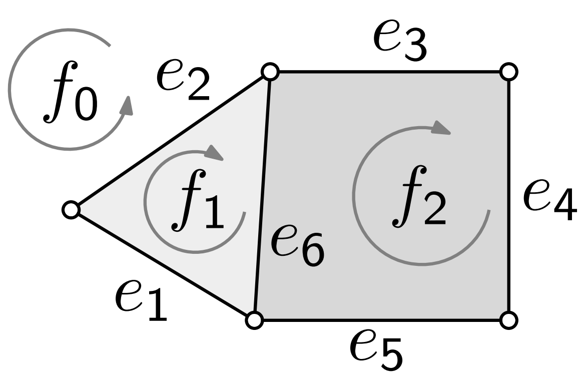
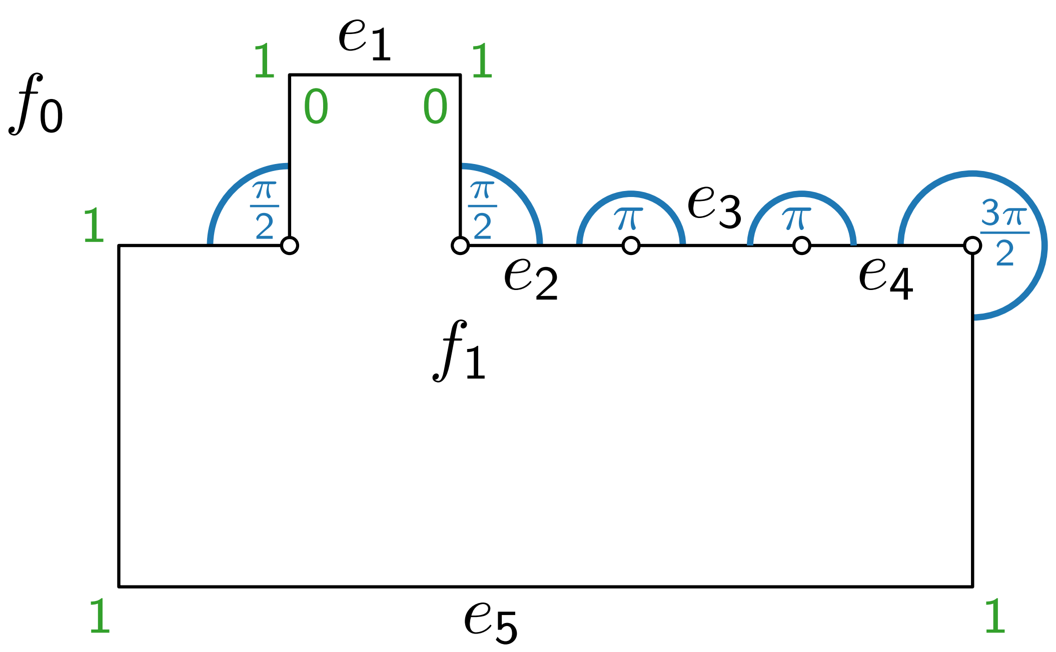
$$H(f_2) = ((e_5, 000, \tfrac{\pi}{2}), (e_6, 11, \tfrac{\pi}{2}), (e_3, \emptyset, \pi), (e_4, \emptyset, \tfrac{\pi}{2}))$$





Orthogonal Representation – Example

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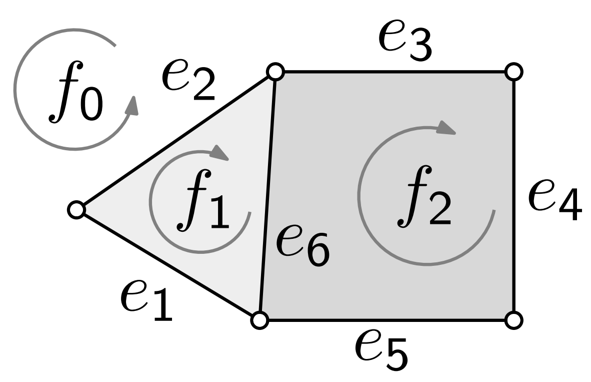

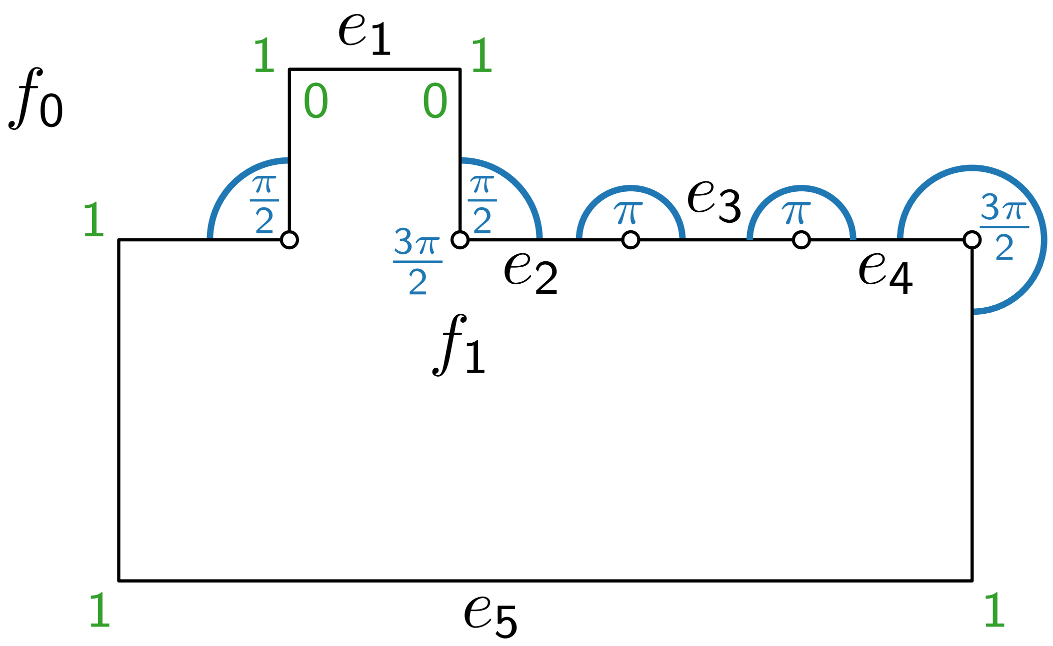

Orthogonal Representation – Example

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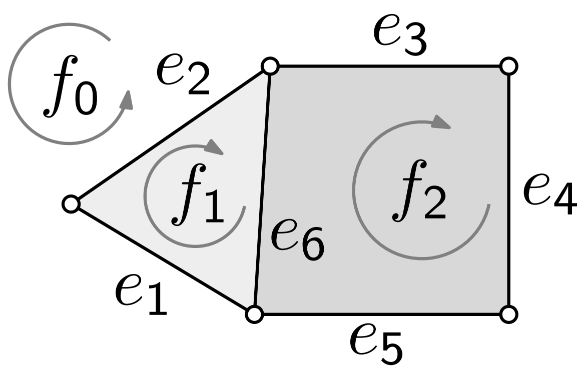
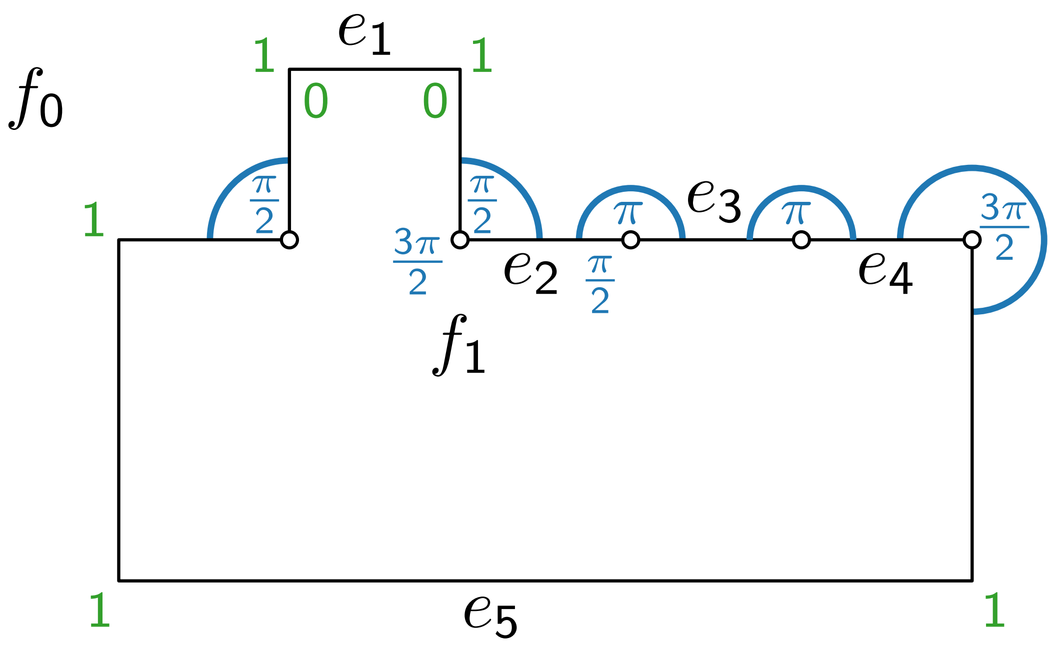
$$H(f_1) = ((e_1, 00, \tfrac{3\pi}{2}), (e_2, \emptyset, \tfrac{\pi}{2}), (e_6, 00, \pi))$$
$$H(f_2) = ((e_5, 000, \tfrac{\pi}{2}), (e_6, 11, \tfrac{\pi}{2}), (e_3, \emptyset, \pi), (e_4, \emptyset, \tfrac{\pi}{2}))$$





Orthogonal Representation – Example

$$H(f_0) = ((e_1, 11, \tfrac{\pi}{2}), (e_5, 111, \tfrac{3\pi}{2}), (e_4, \emptyset, \pi), (e_3, \emptyset, \pi), (e_2, \emptyset, \tfrac{\pi}{2}))$$
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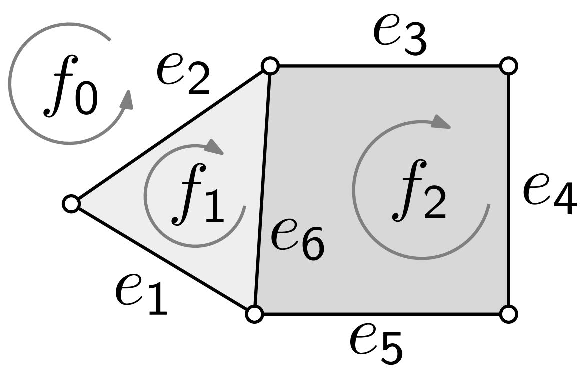
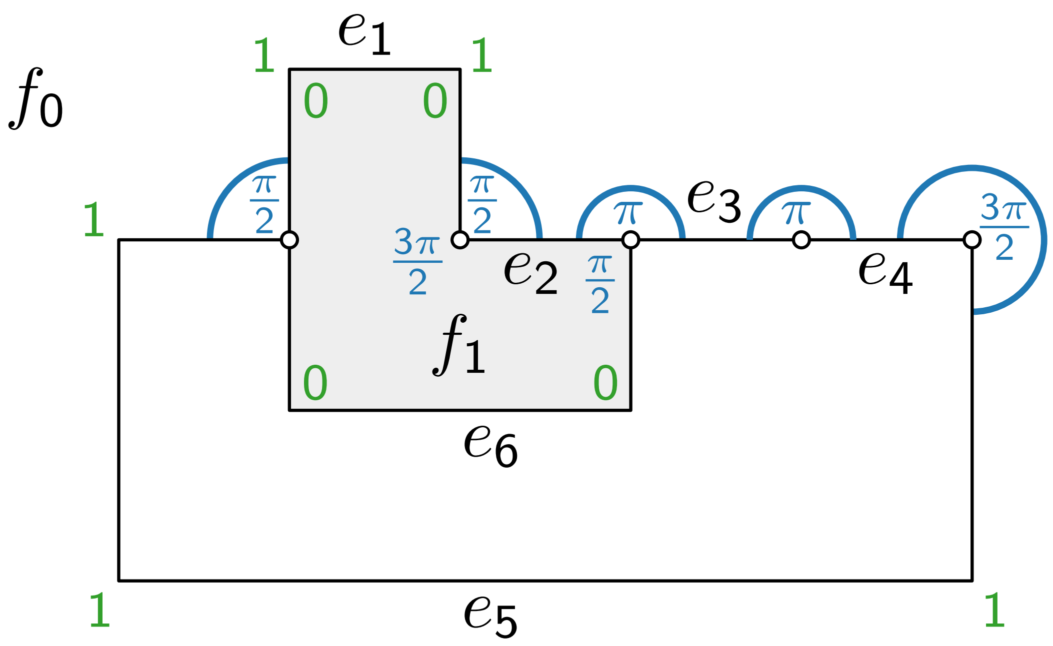
$$H(f_2) = ((e_5, 000, \tfrac{\pi}{2}), (e_6, 11, \tfrac{\pi}{2}), (e_3, \emptyset, \pi), (e_4, \emptyset, \tfrac{\pi}{2}))$$





Orthogonal Representation – Example

$$H(f_0) = ((e_1, 11, \tfrac{\pi}{2}), (e_5, 111, \tfrac{3\pi}{2}), (e_4, \emptyset, \pi), (e_3, \emptyset, \pi), (e_2, \emptyset, \tfrac{\pi}{2}))$$

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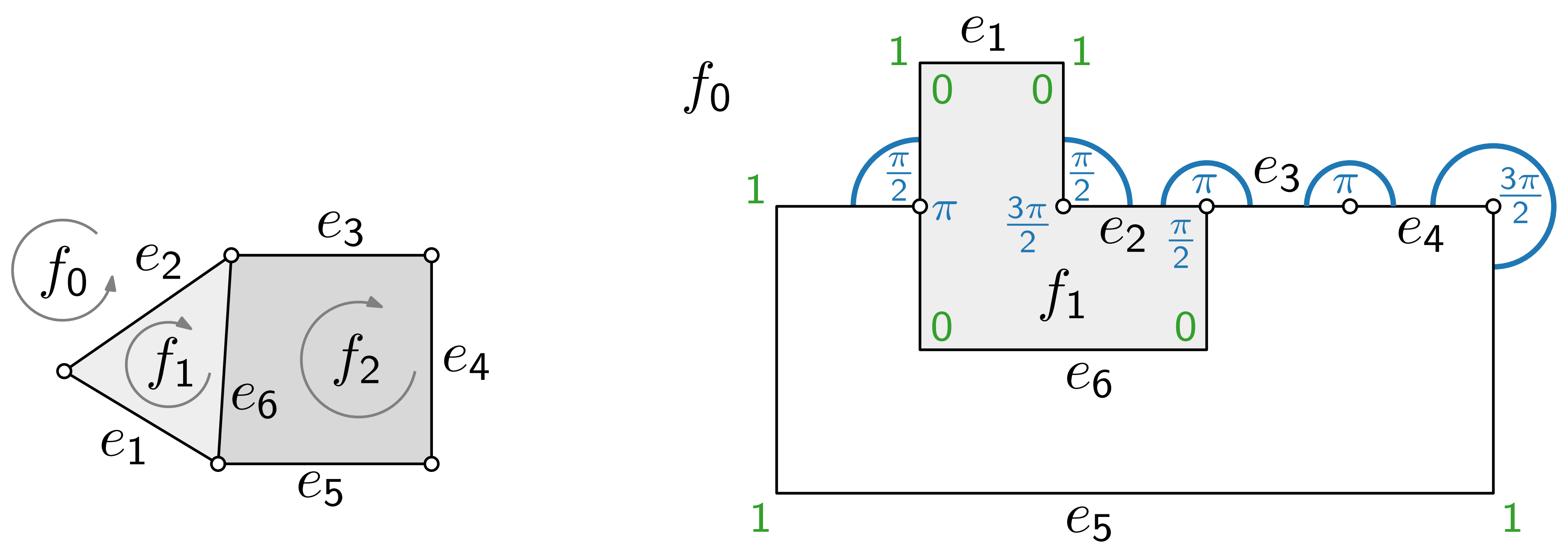

Orthogonal Representation – Example

$$H(f_0) = ((e_1, 11, \tfrac{\pi}{2}), (e_5, 111, \tfrac{3\pi}{2}), (e_4, \emptyset, \pi), (e_3, \emptyset, \pi), (e_2, \emptyset, \tfrac{\pi}{2}))$$
$$H(f_1) = ((e_1, 00, \tfrac{3\pi}{2}), (e_2, \emptyset, \tfrac{\pi}{2}), (e_6, 00, \pi))$$
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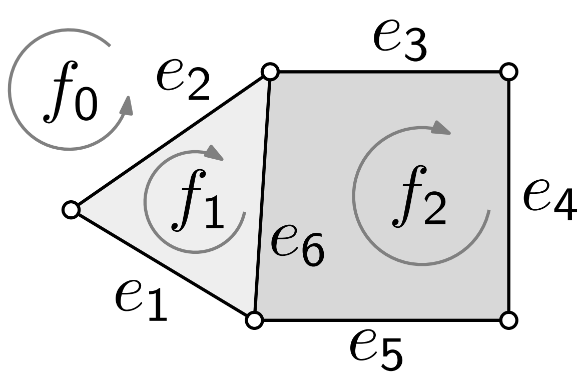

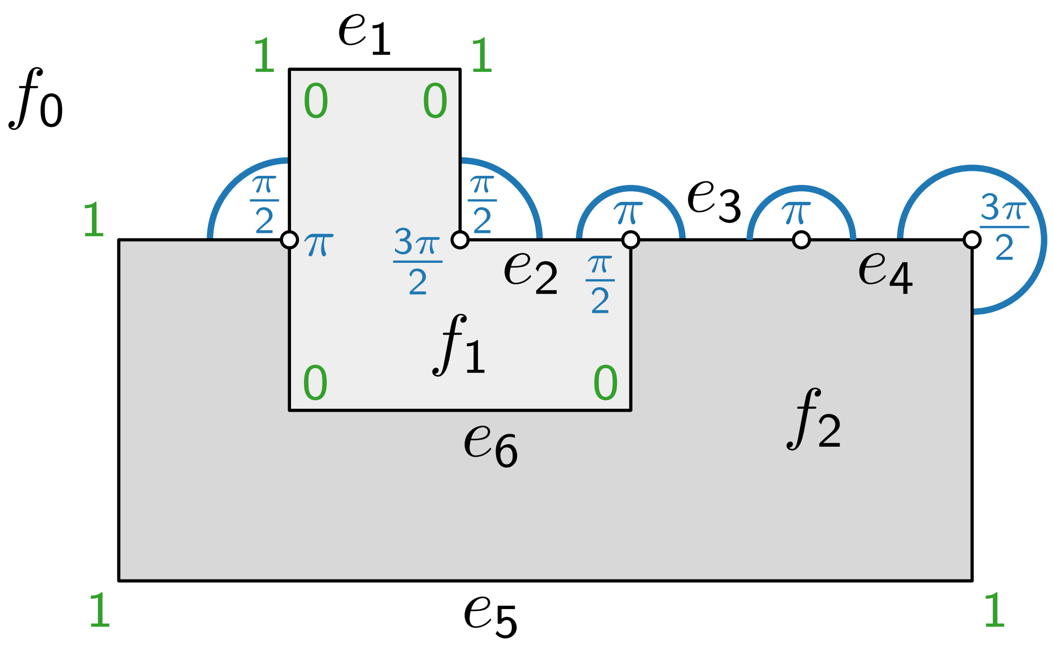

Orthogonal Representation – Example

$$H(f_0) = ((e_1, 11, \tfrac{\pi}{2}), (e_5, 111, \tfrac{3\pi}{2}), (e_4, \emptyset, \pi), (e_3, \emptyset, \pi), (e_2, \emptyset, \tfrac{\pi}{2}))$$
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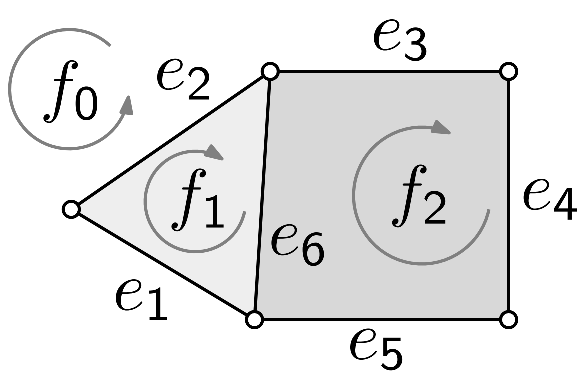

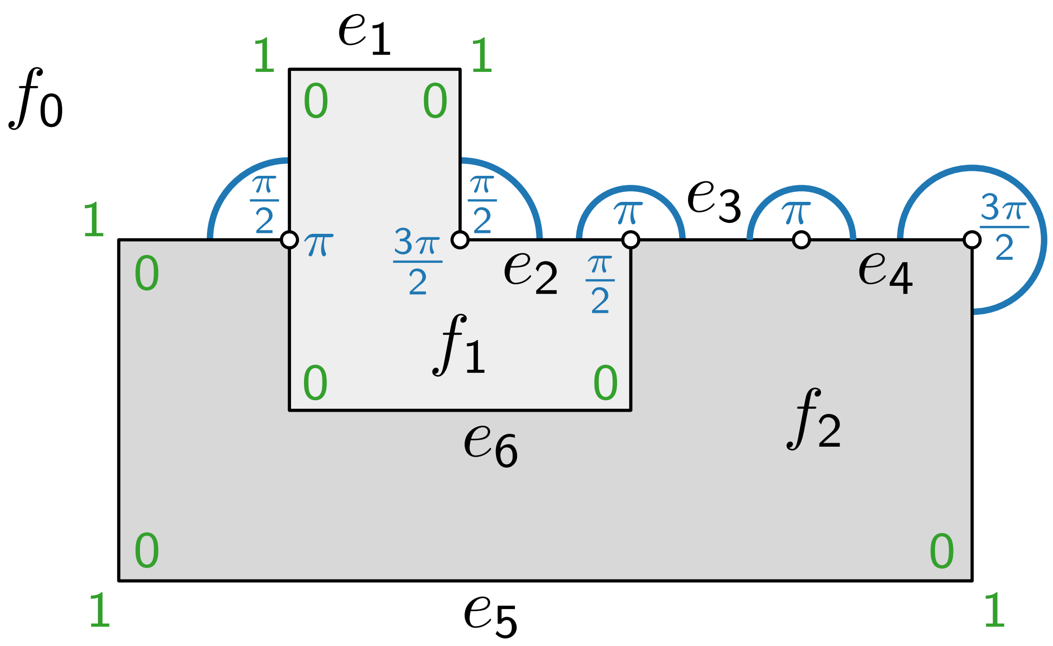

Orthogonal Representation – Example

$$H(f_0) = ((e_1, 11, \tfrac{\pi}{2}), (e_5, 111, \tfrac{3\pi}{2}), (e_4, \emptyset, \pi), (e_3, \emptyset, \pi), (e_2, \emptyset, \tfrac{\pi}{2}))$$
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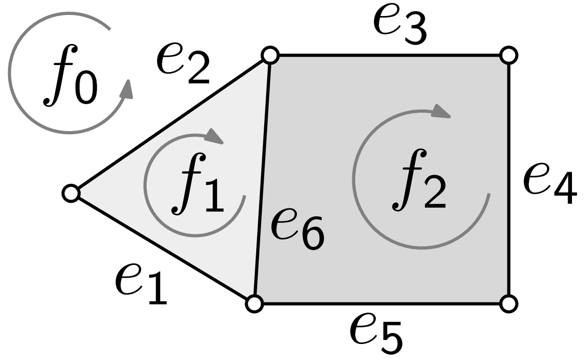

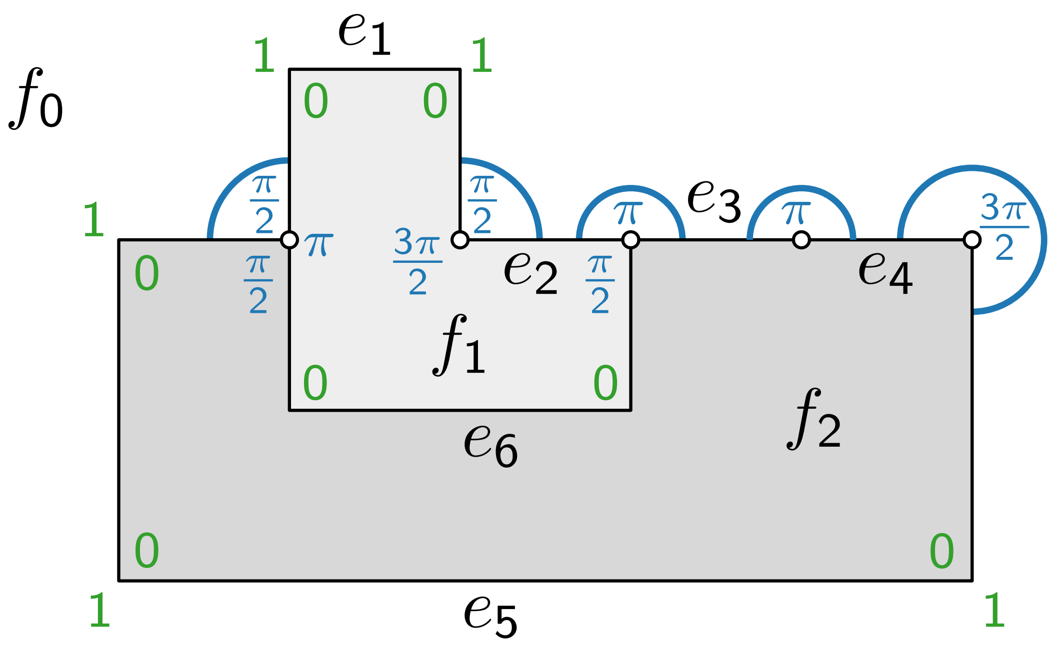

Orthogonal Representation – Example

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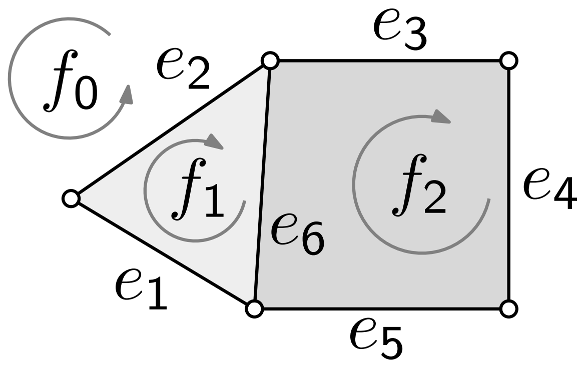

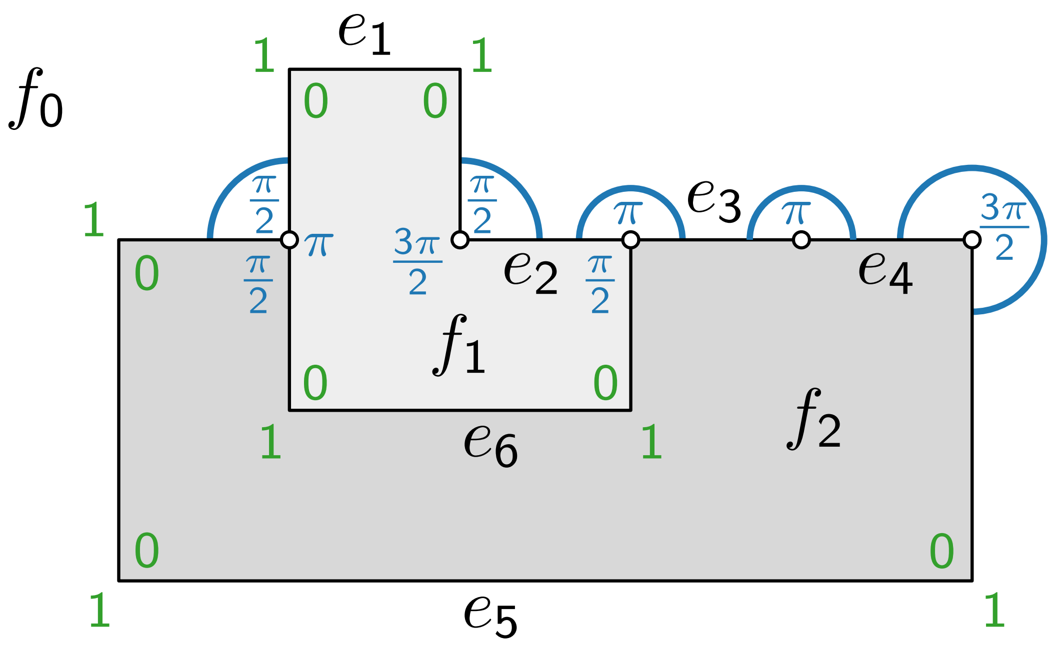

Orthogonal Representation – Example

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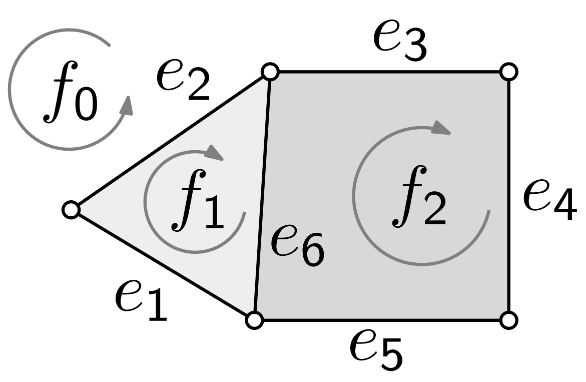

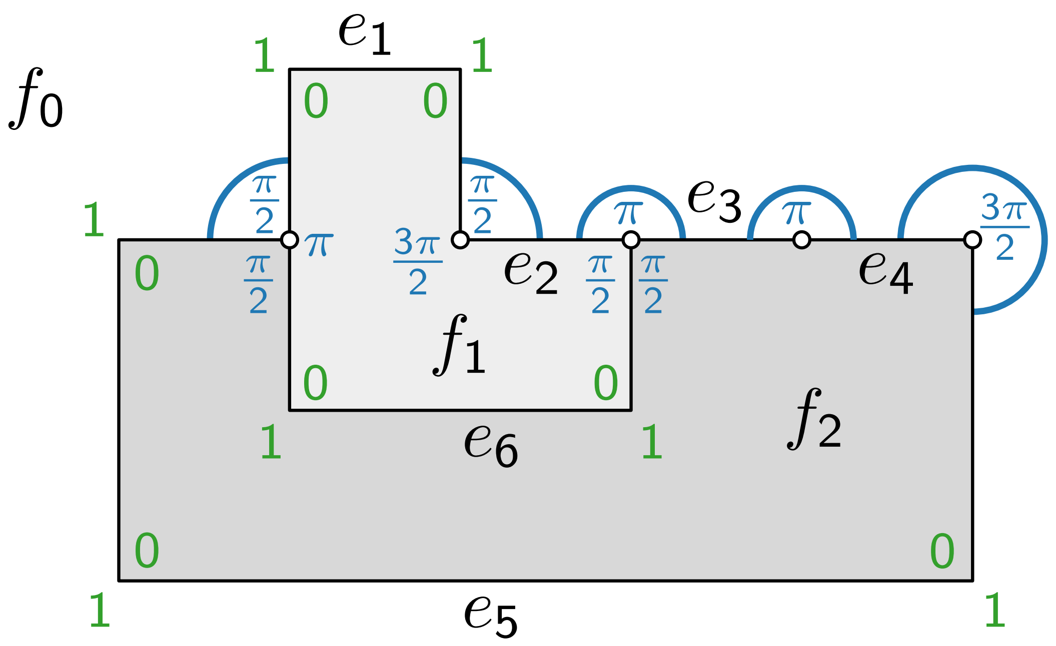

Orthogonal Representation – Example

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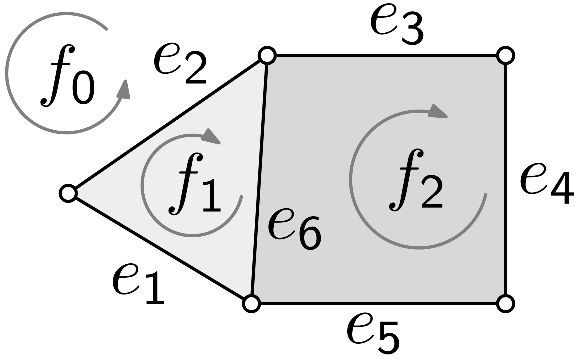

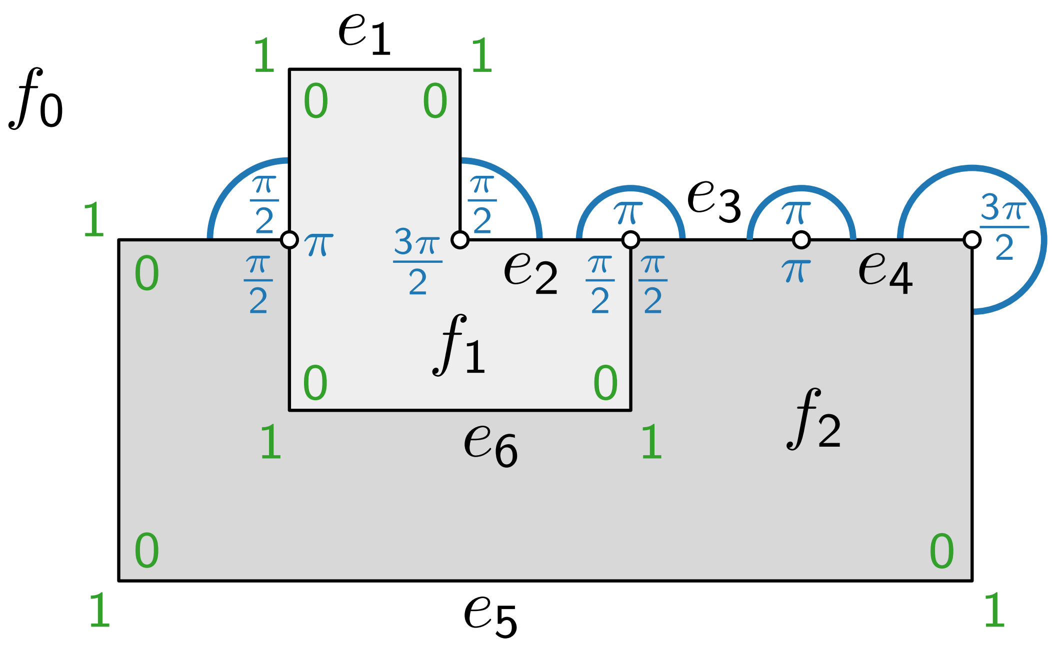

Orthogonal Representation – Example

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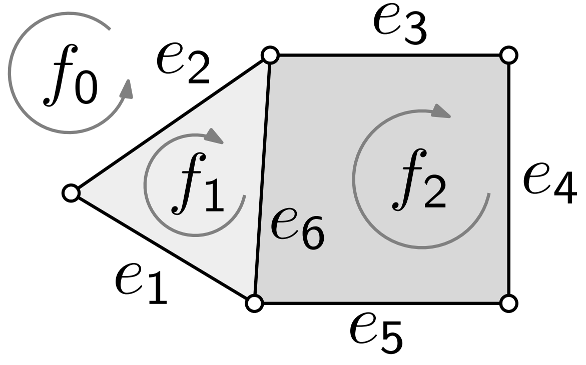

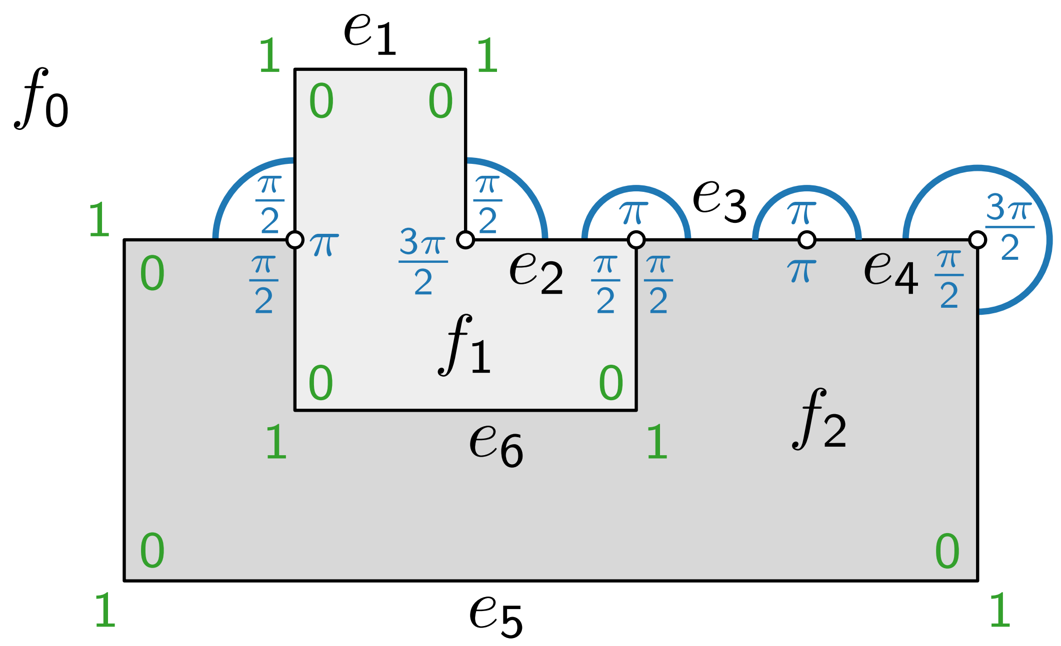

Orthogonal Representation – Example

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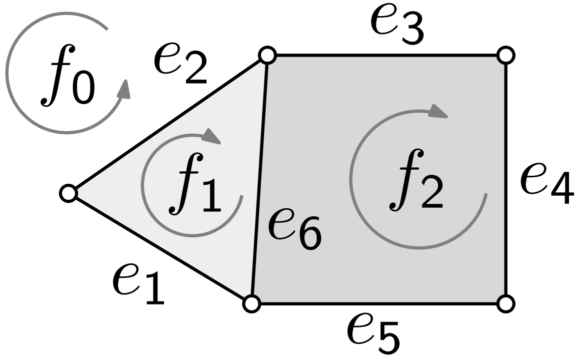
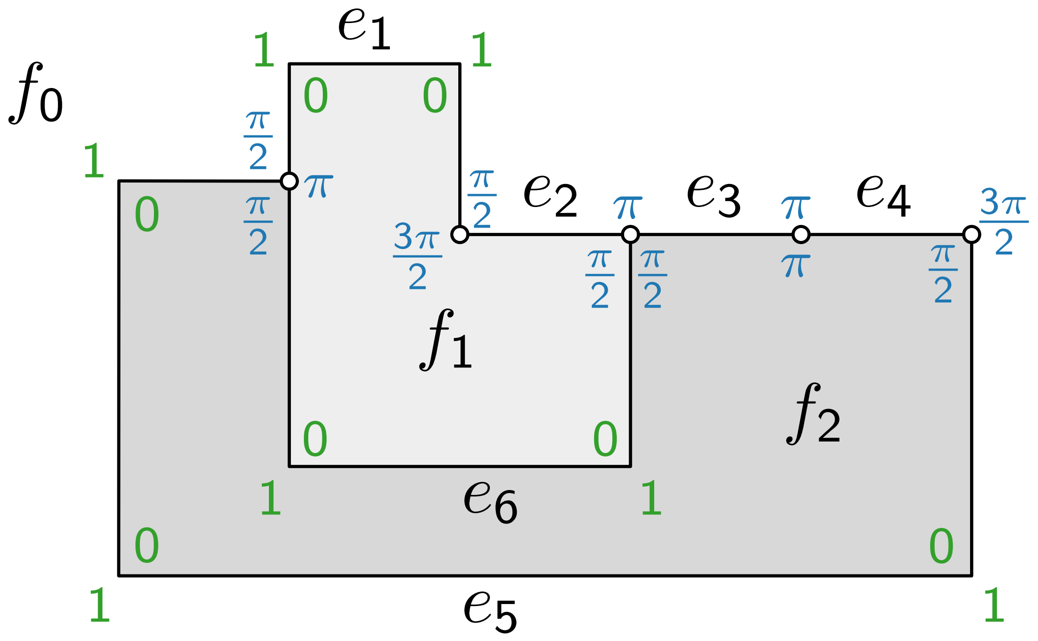
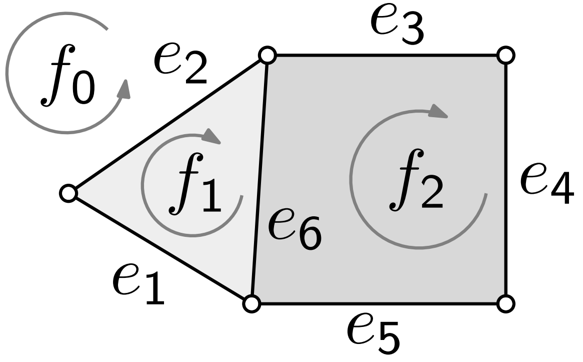
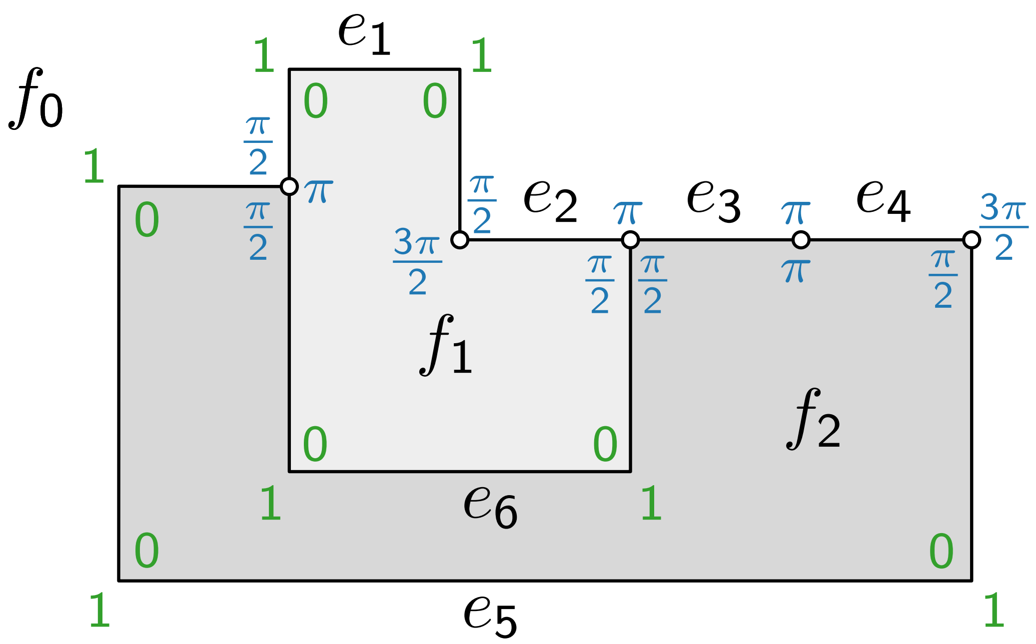




Orthogonal Representation – Example

$$H(f_0) = ((e_1, 11, \tfrac{\pi}{2}), (e_5, 111, \tfrac{3\pi}{2}), (e_4, \emptyset, \pi), (e_3, \emptyset, \pi), (e_2, \emptyset, \tfrac{\pi}{2}))$$
$$H(f_1) = ((e_1, 00, \tfrac{3\pi}{2}), (e_2, \emptyset, \tfrac{\pi}{2}), (e_6, 00, \pi))$$
$$H(f_2) = ((e_5, 000, \tfrac{\pi}{2}), (e_6, 11, \tfrac{\pi}{2}), (e_3, \emptyset, \pi), (e_4, \emptyset, \tfrac{\pi}{2}))$$

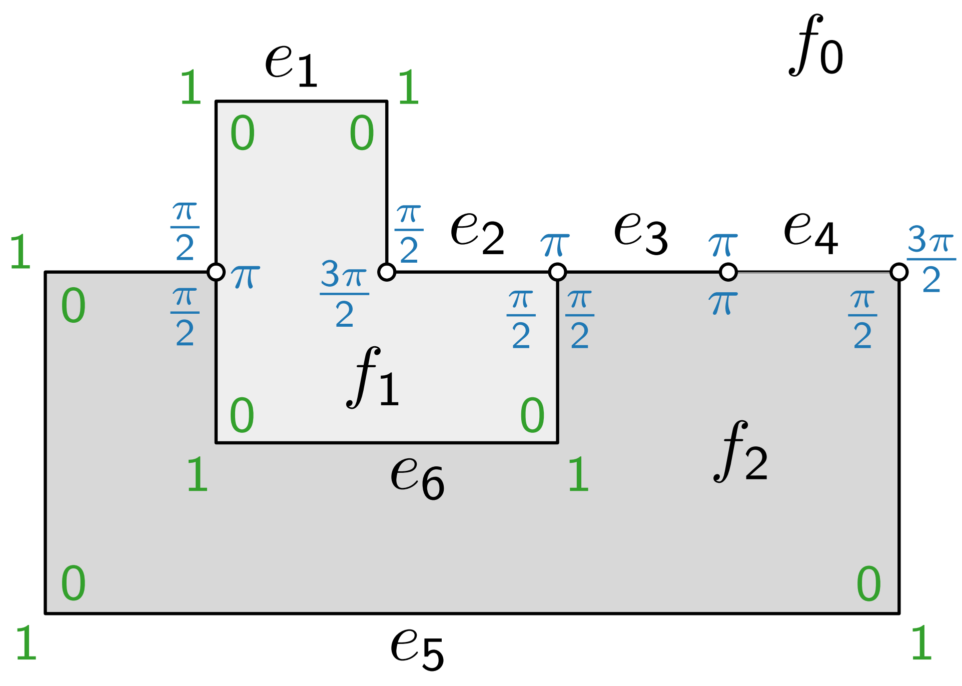




Coordinates are not fixed yet!

Correctness of an Orthogonal Representation

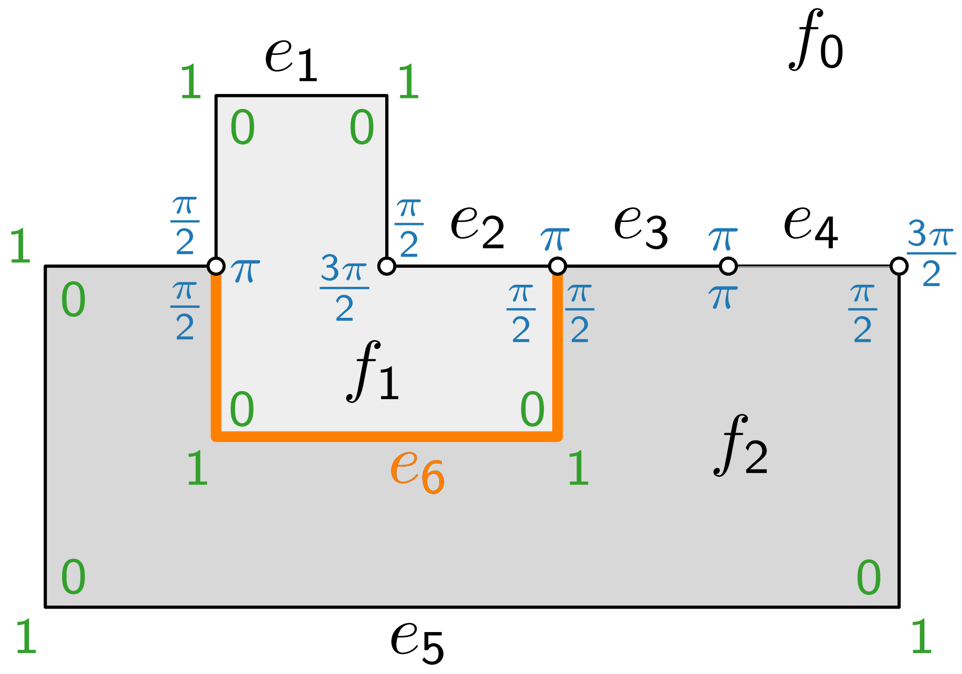
(H1) $H(G)$ corresponds to F, f_0.



Correctness of an Orthogonal Representation

(H1) $H(G)$ corresponds to F, f_0.

(H2) For each **edge** $\{u, v\}$ shared by faces f and g with $((u, v), \delta_1, \alpha_1) \in H(f)$ and $((v, u), \delta_2, \alpha_2) \in H(g)$



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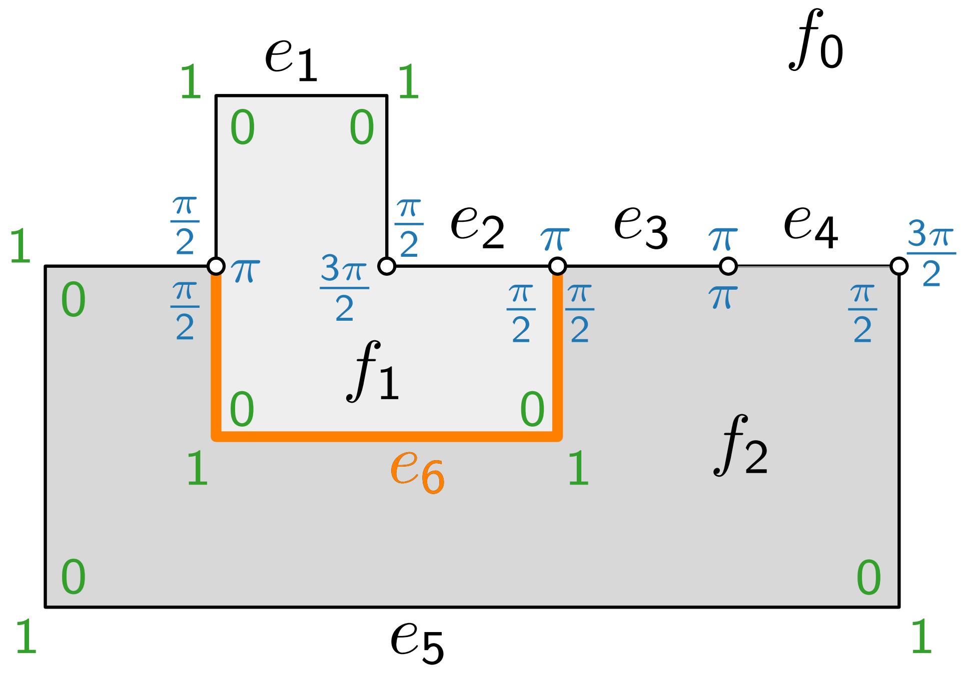

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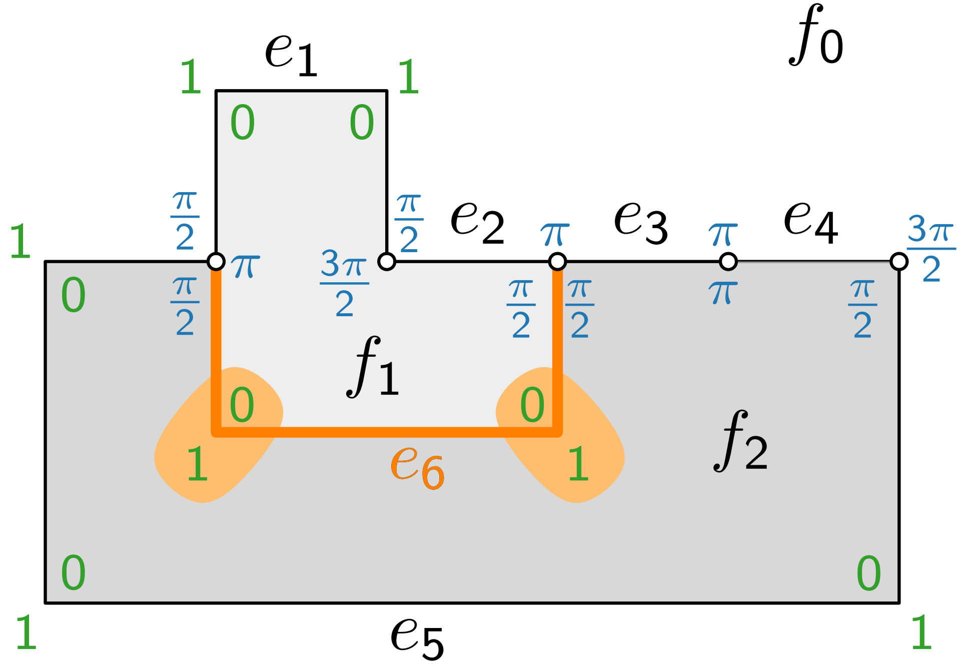

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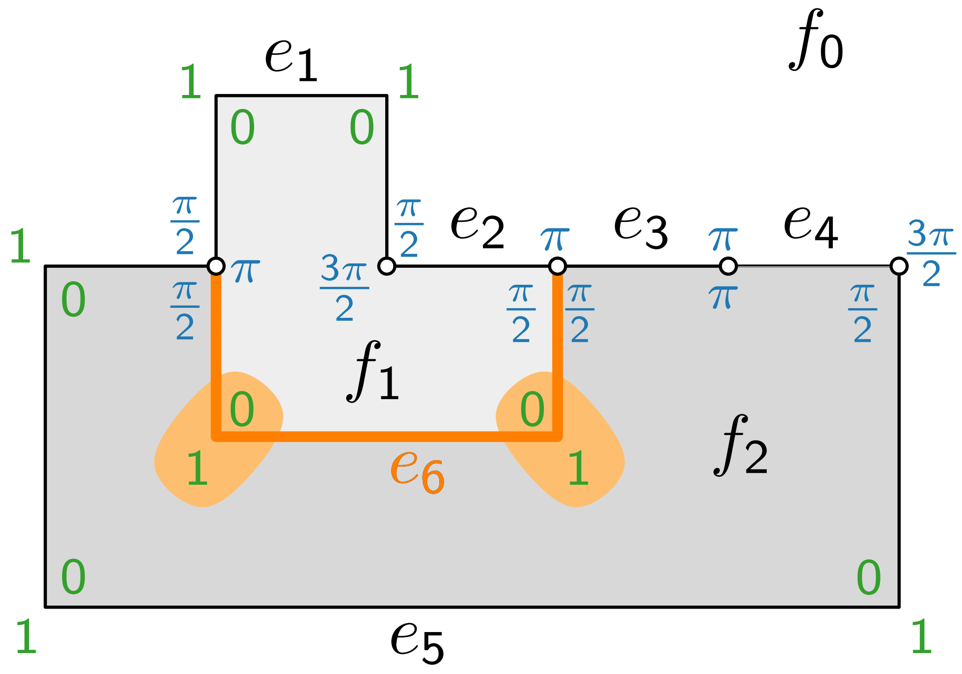

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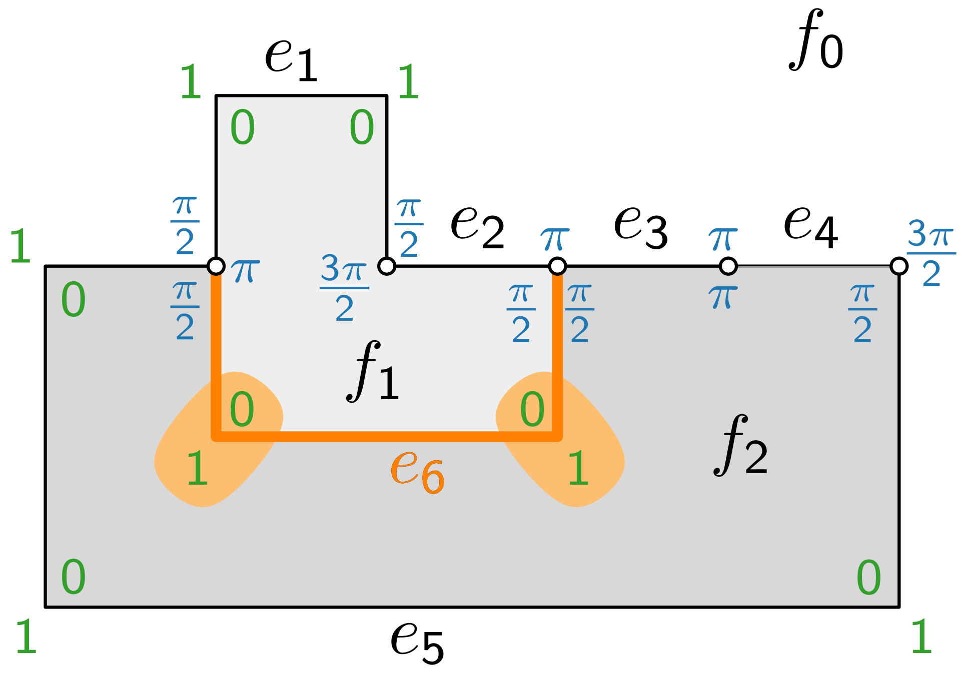

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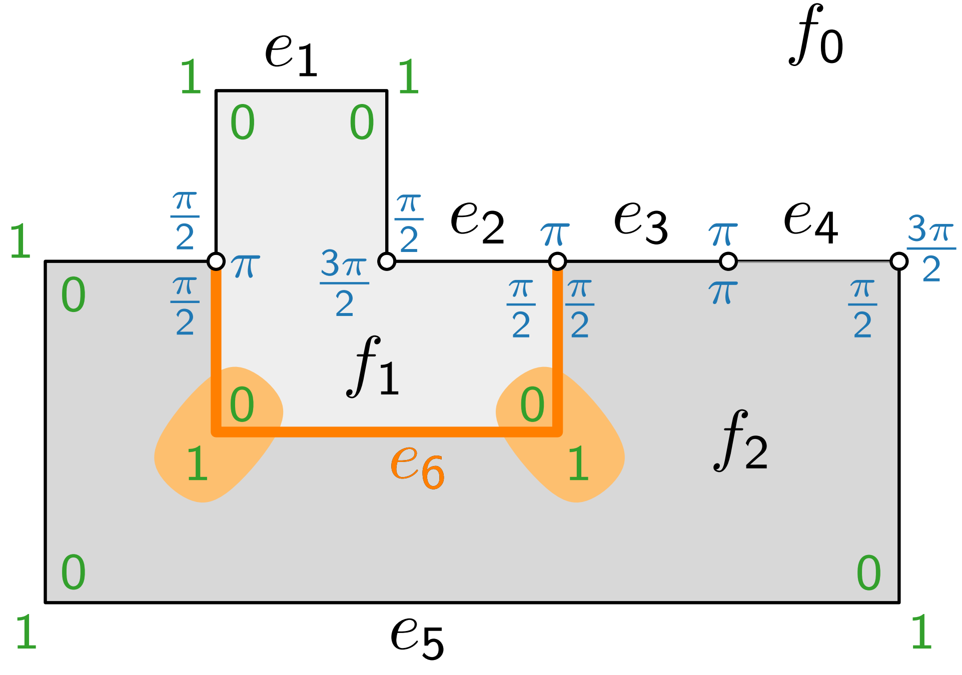

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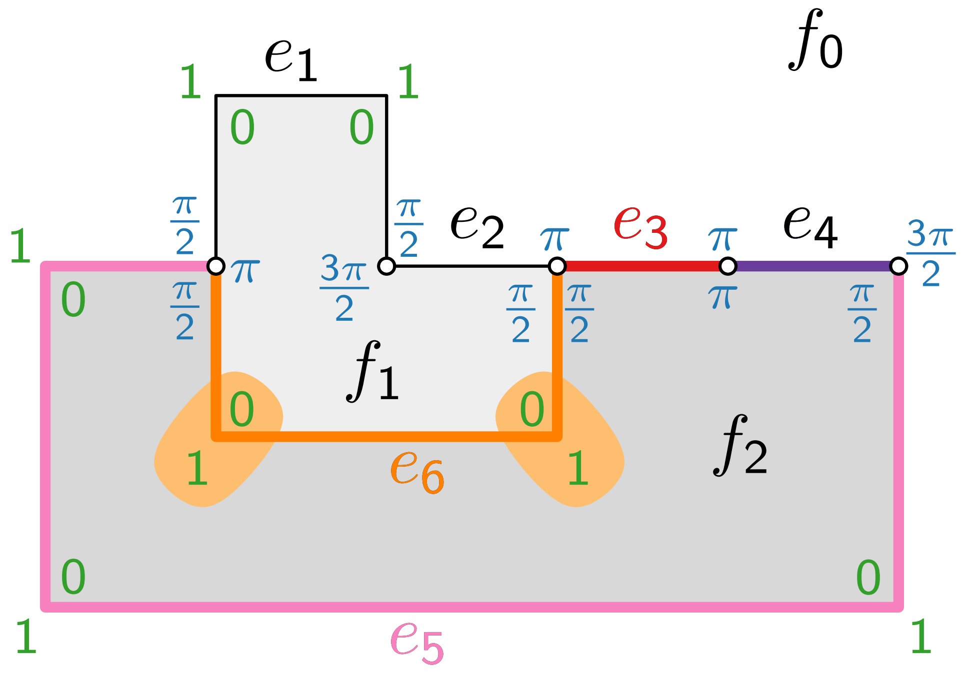

$$C((e_3, \emptyset, \pi)) = \quad - \quad - \quad + 2 =$$

$$C((e_4, \emptyset, \tfrac{\pi}{2})) = \quad - \quad - \quad + 2 =$$

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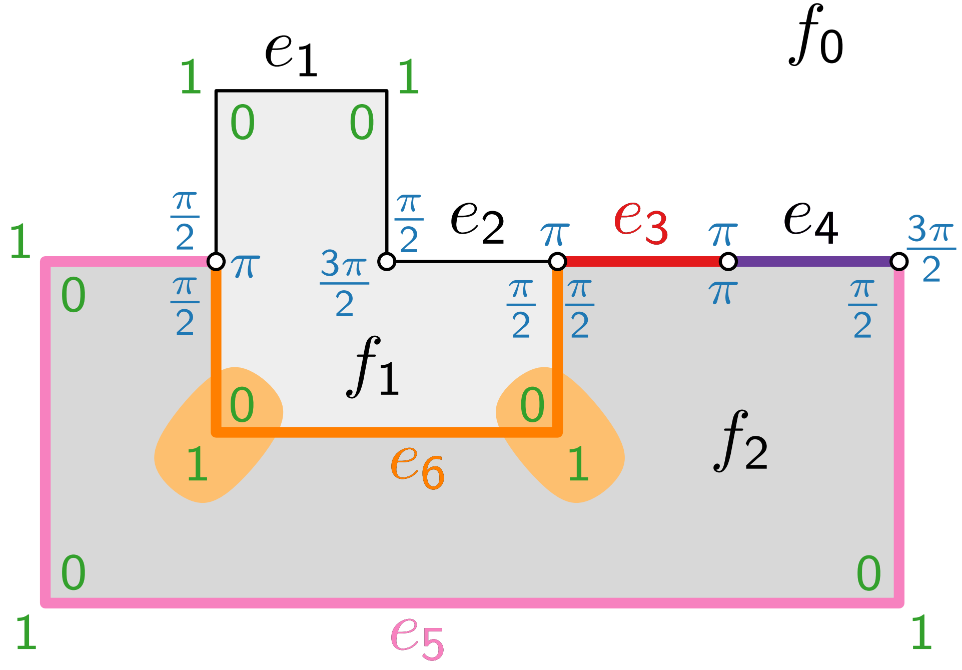

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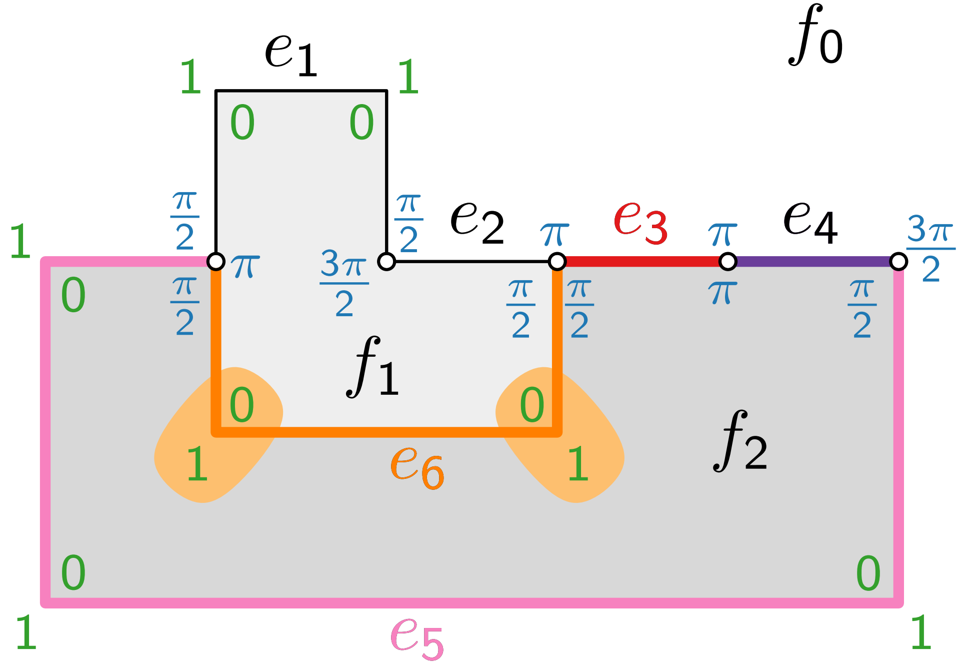

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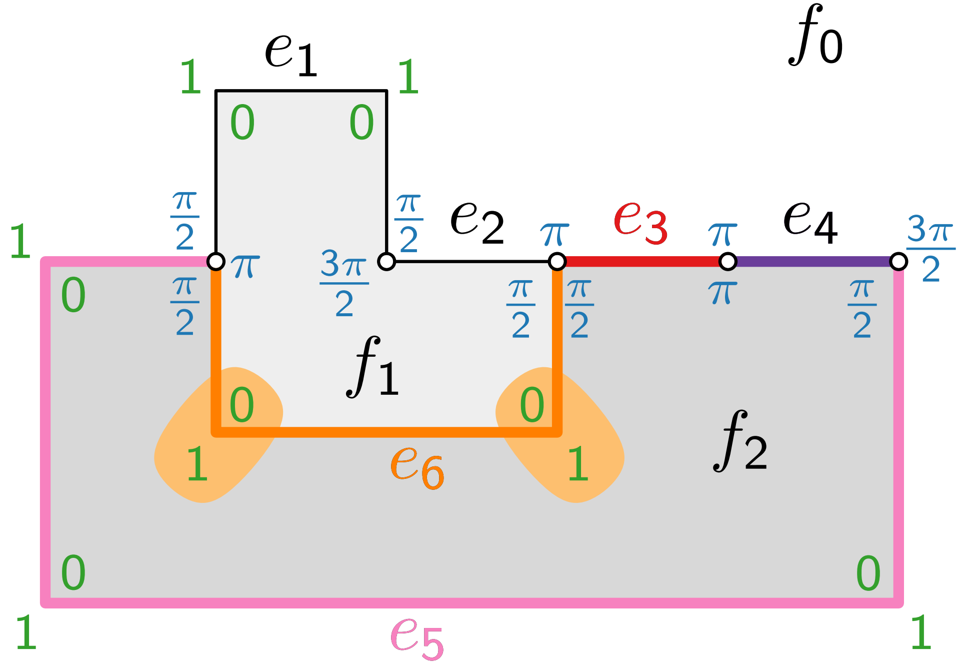

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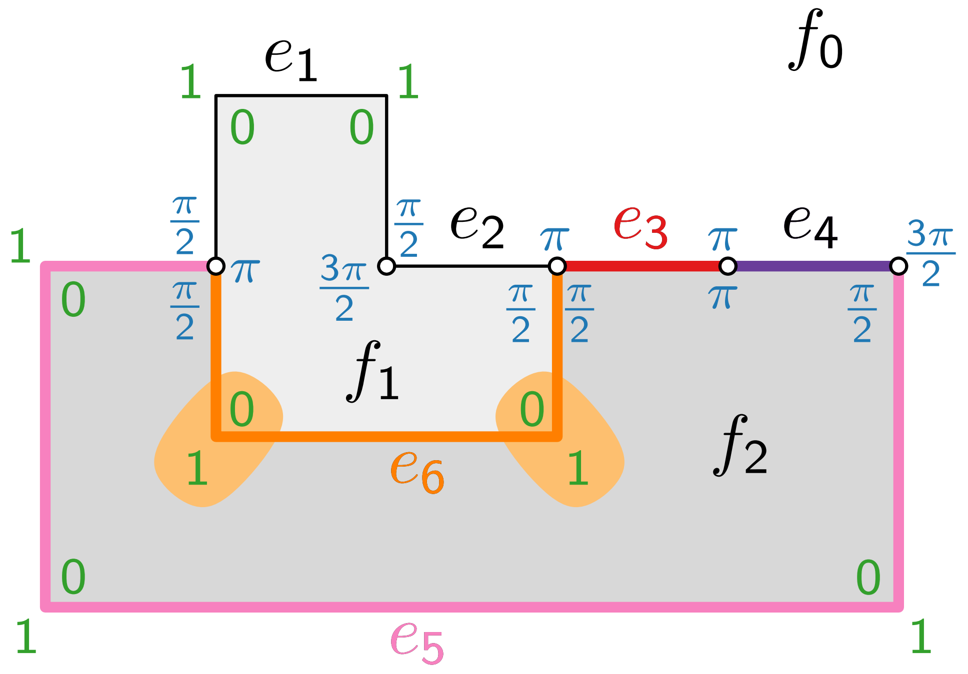

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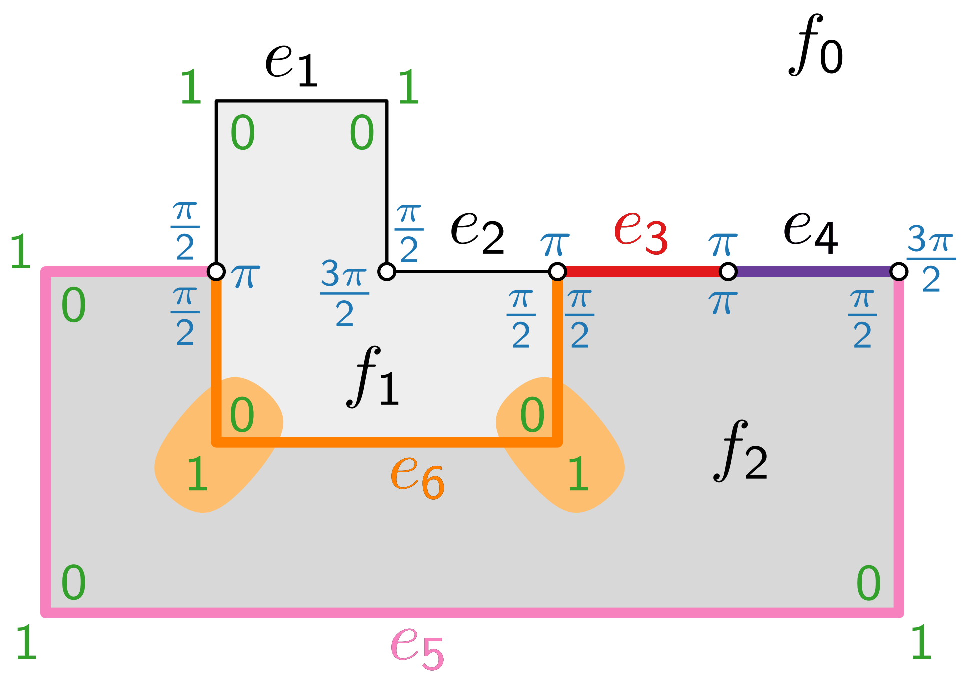

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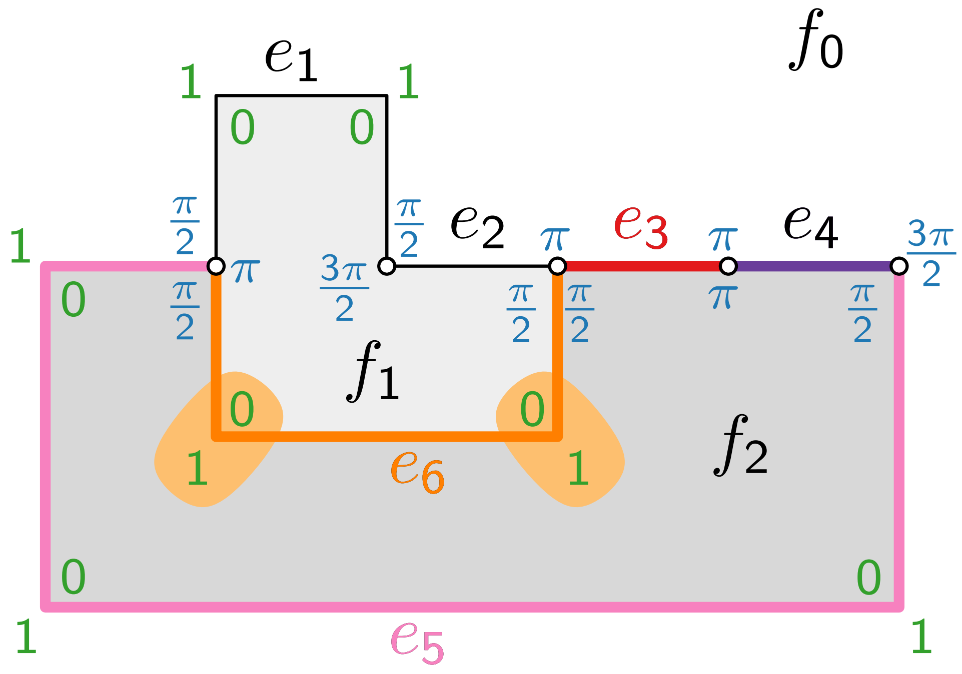

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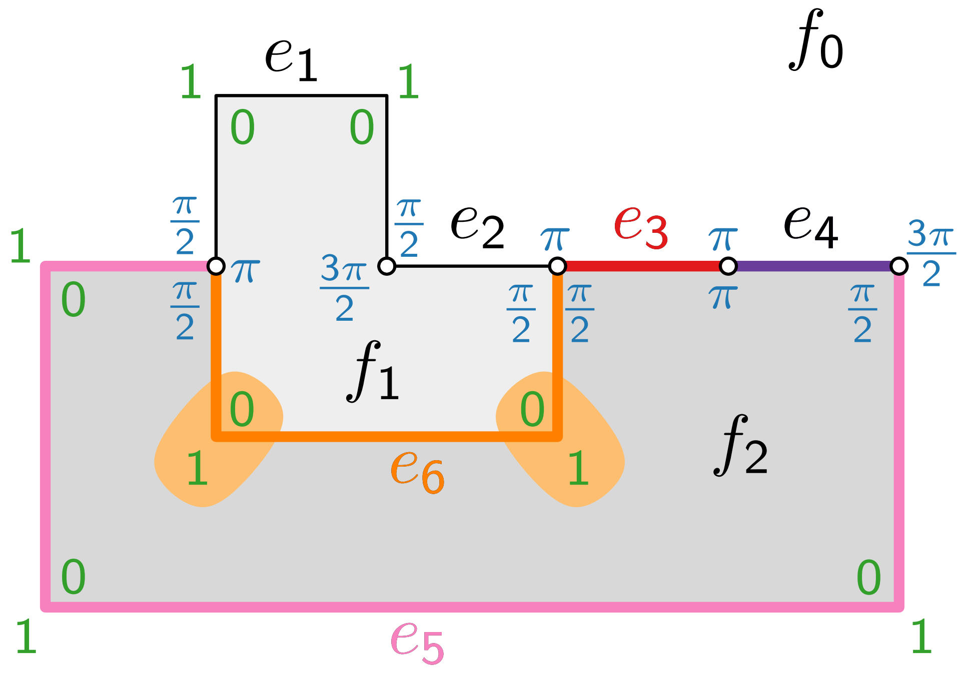

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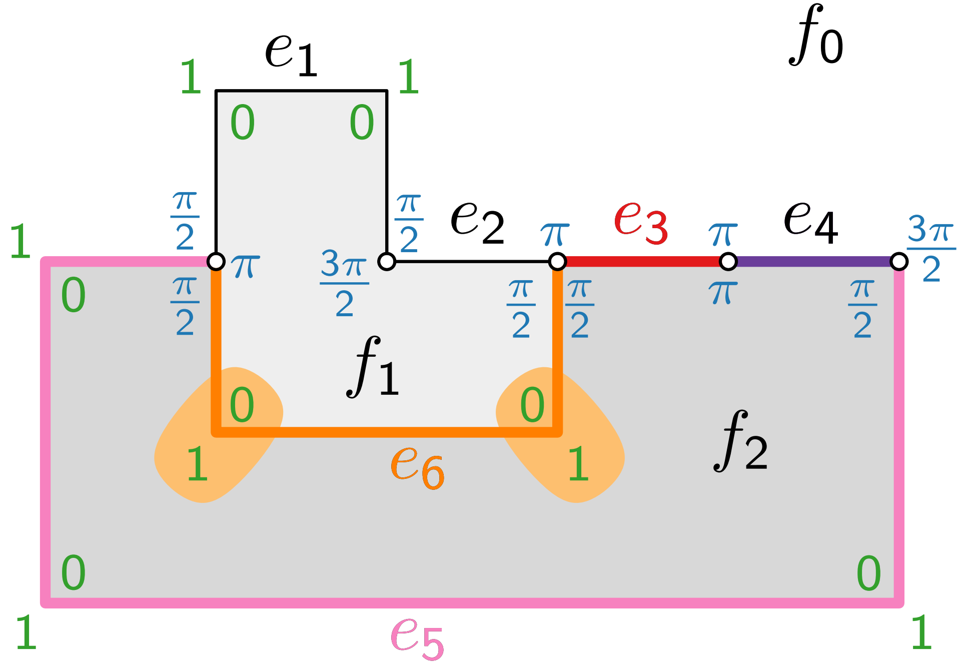

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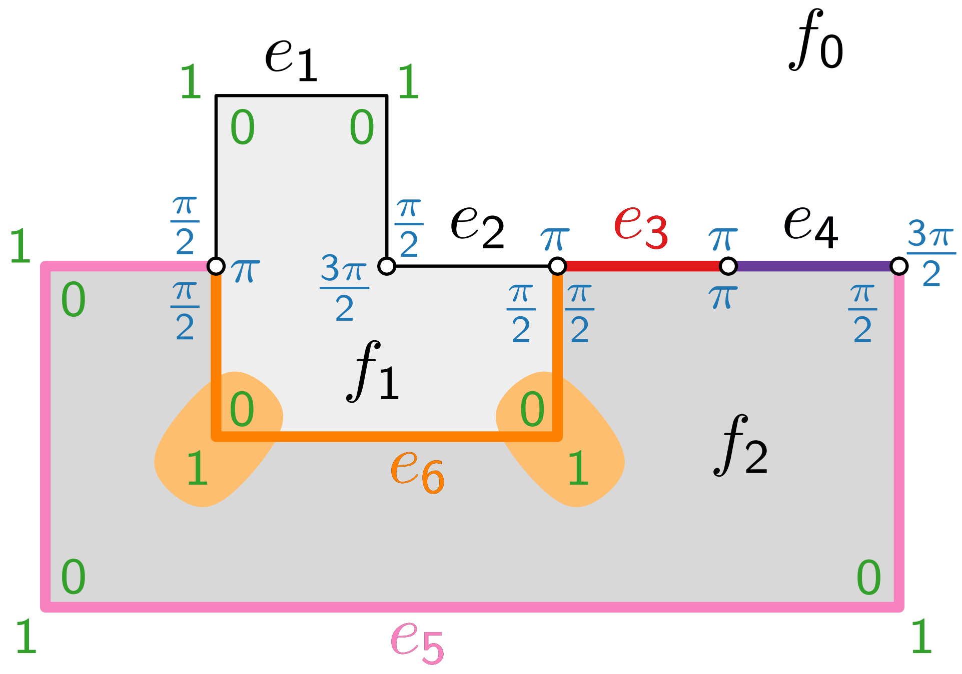

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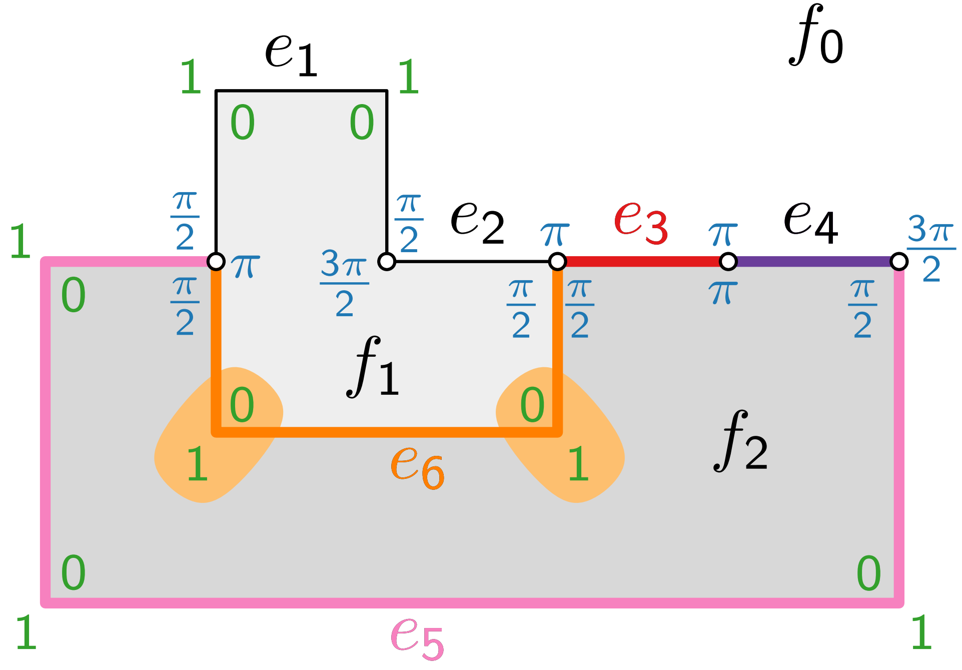

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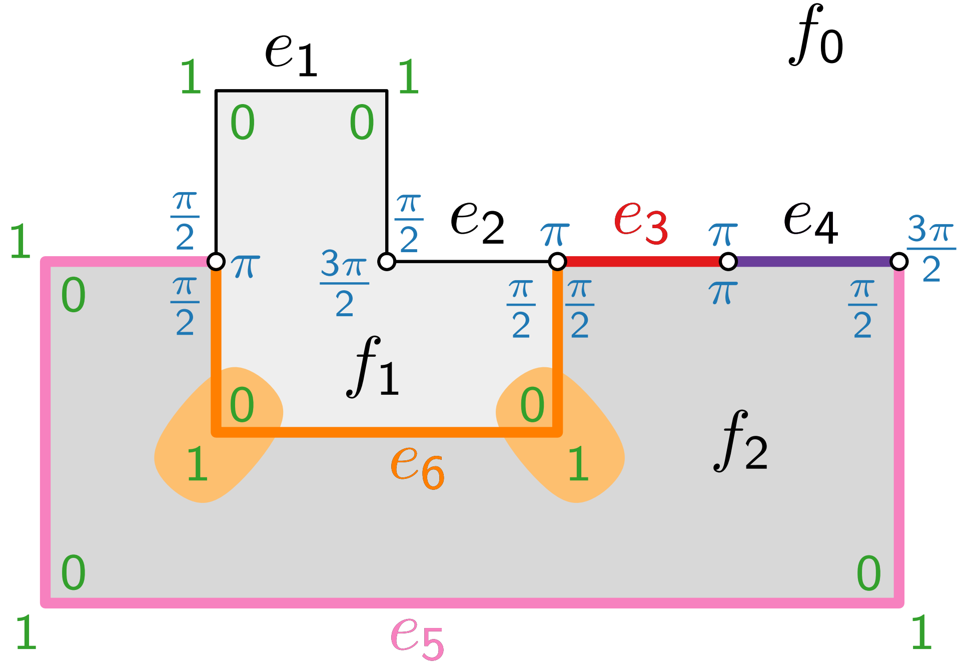

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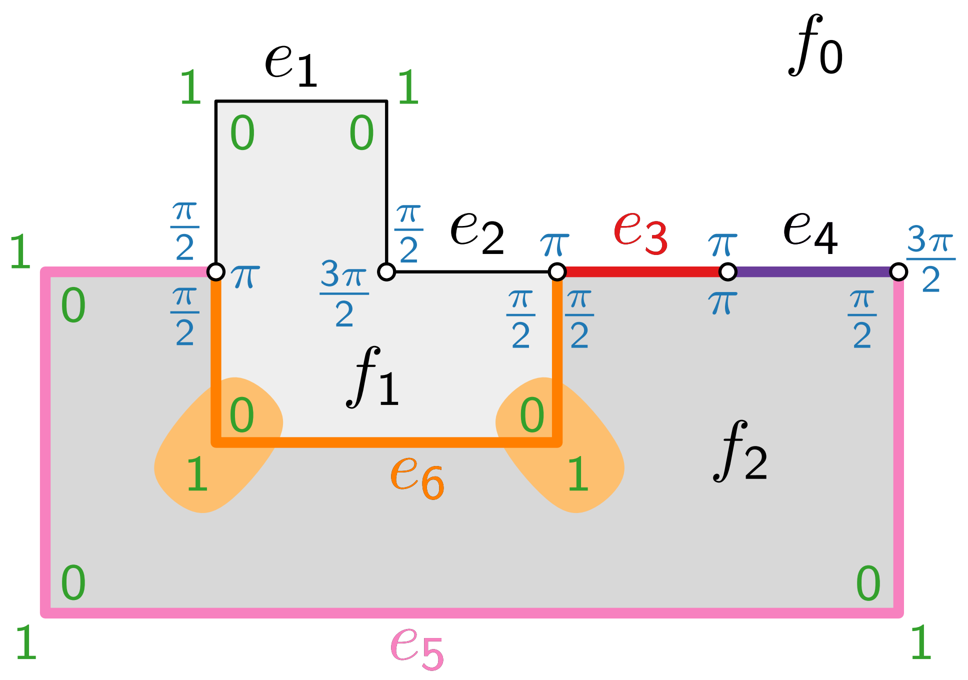
$$\sum_{r \in H(f)} C(r) = \begin{cases} -4 & \text{if } f = f_0 \\ +4 & \text{otherwise.} \end{cases}$$



$$C((e_3, \emptyset, \pi)) = 0 - 0 - 2 + 2 = 0$$

$$C((e_4, \emptyset, \tfrac{\pi}{2})) = 0 - 0 - 1 + 2 = 1$$

$$C((e_5, 000, \tfrac{\pi}{2})) = 3 - 0 - 1 + 2 = 4$$

$$C((e_6, 11, \tfrac{\pi}{2})) = 0 - 2 - 1 + 2 = -1$$

$$\sum_{r \in H(f_2)} C(r) =$$

Correctness of an Orthogonal Representation

(H1) $H(G)$ corresponds to F, f_0.

(H2) For each **edge** $\{u,v\}$ shared by faces f and g with $((u,v),\delta_1,\alpha_1) \in H(f)$ and $((v,u),\delta_2,\alpha_2) \in H(g)$, the sequence δ_1 is like δ_2, but reversed and inverted.

(H3) Let $|\delta|_0$ (resp. $|\delta|_1$) be the number of zeros (resp. ones) in δ, and let $r = (e,\delta,\alpha)$.
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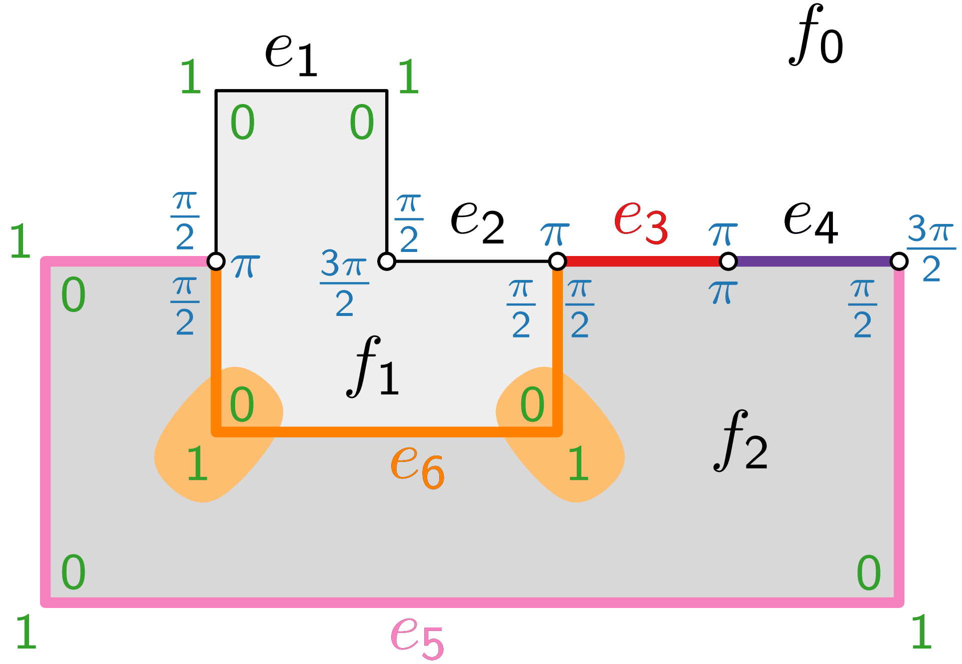

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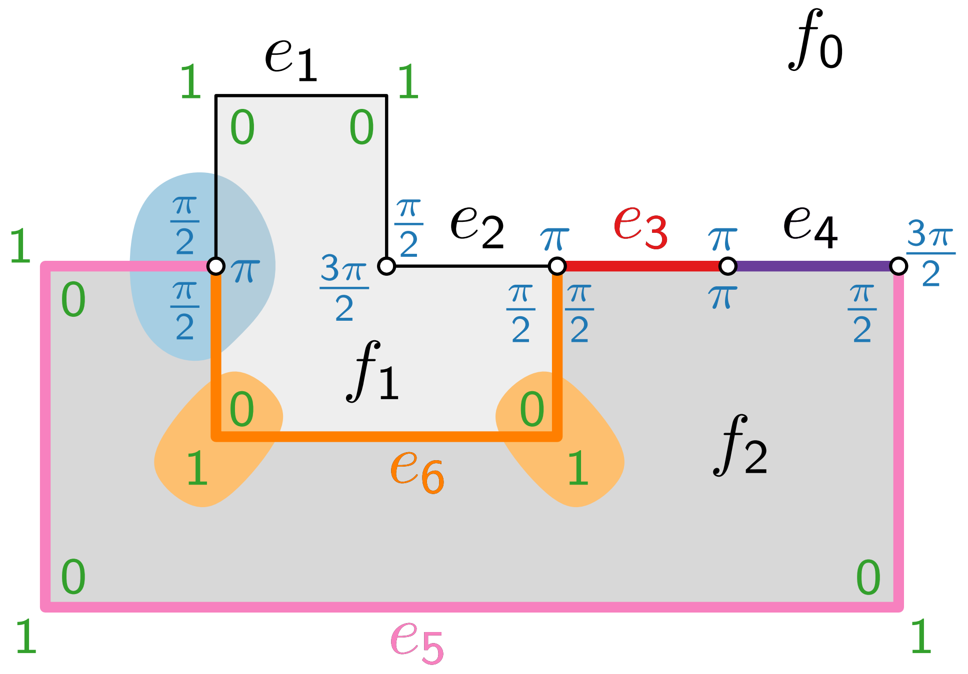
(H4) For each **vertex** v, the sum of incident angles is 2π.



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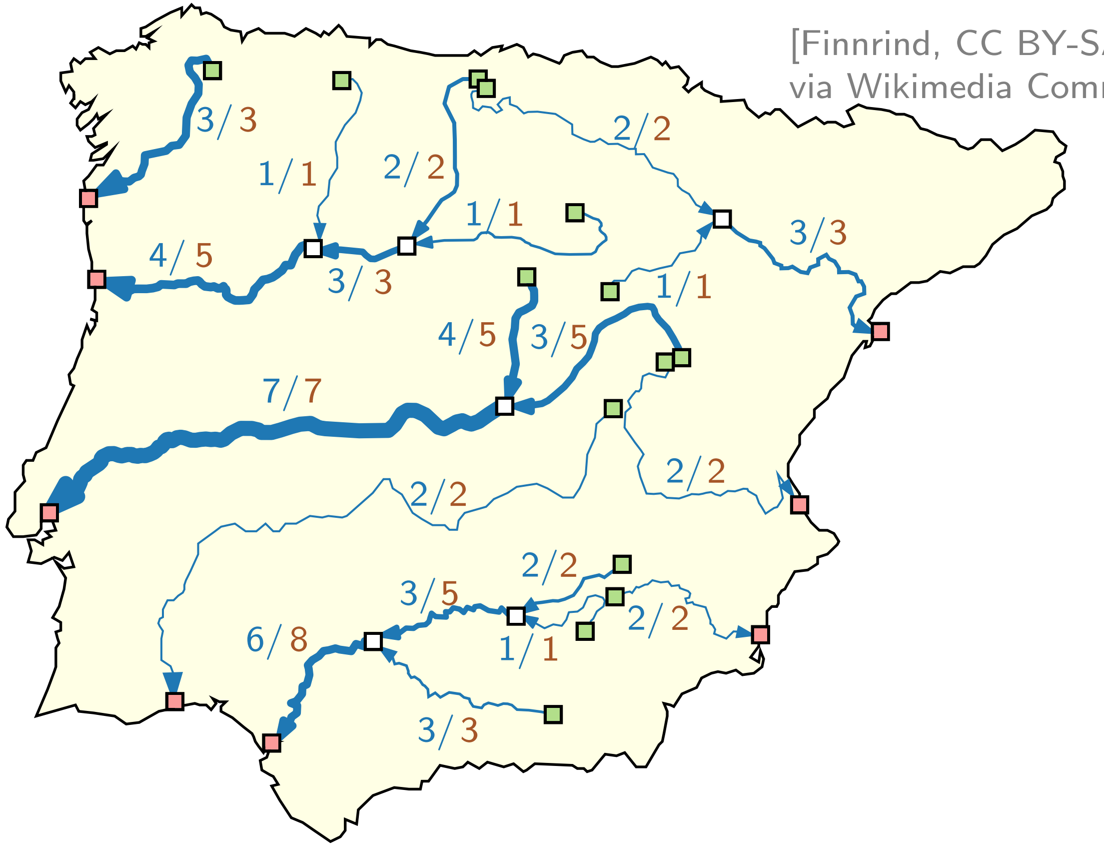
Reminder: s–t Flow Networks



Flow network $(G; S, T; u)$ with

- directed graph G
- *sources* $S \subseteq V(G)$, *sinks* $T \subseteq V(G)$
- edge *capacity* $u\colon E(G) \to \mathbb{R}_0^+ \cup \{\infty\}$

A function $X\colon E(G) \to \mathbb{R}_0^+$ is called S–T **flow** if:

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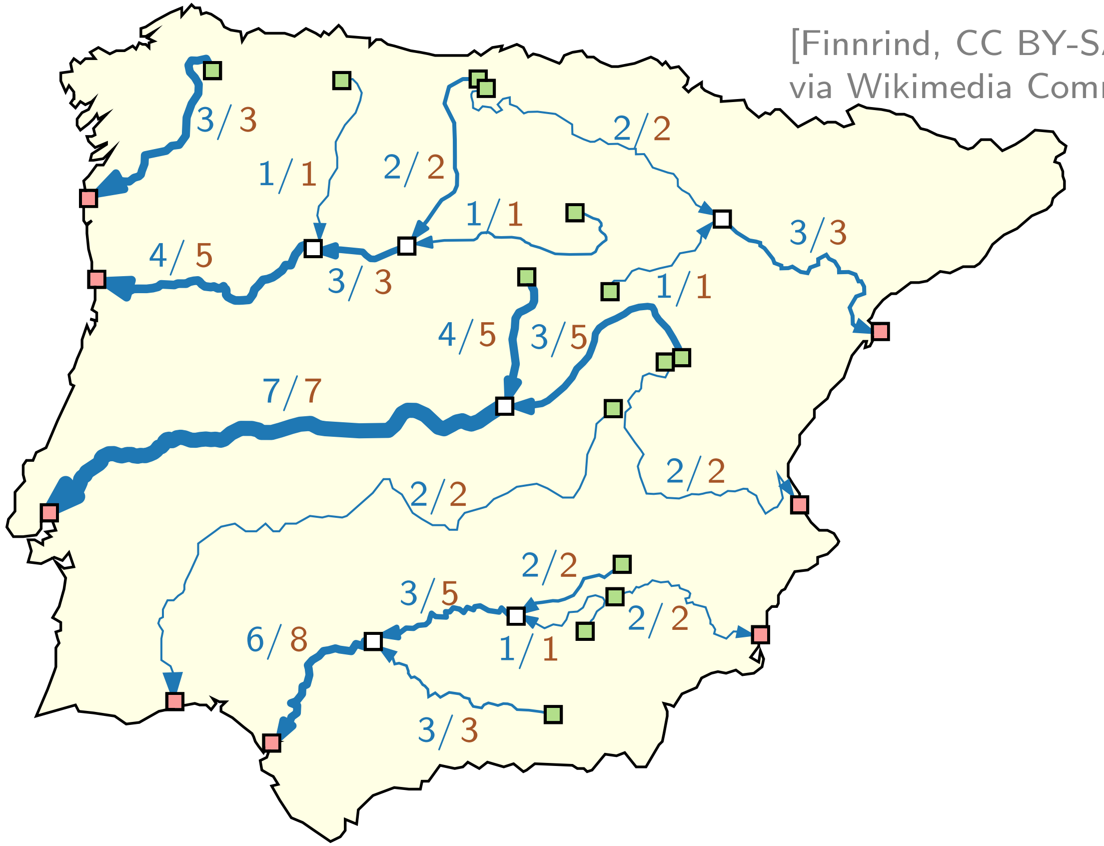

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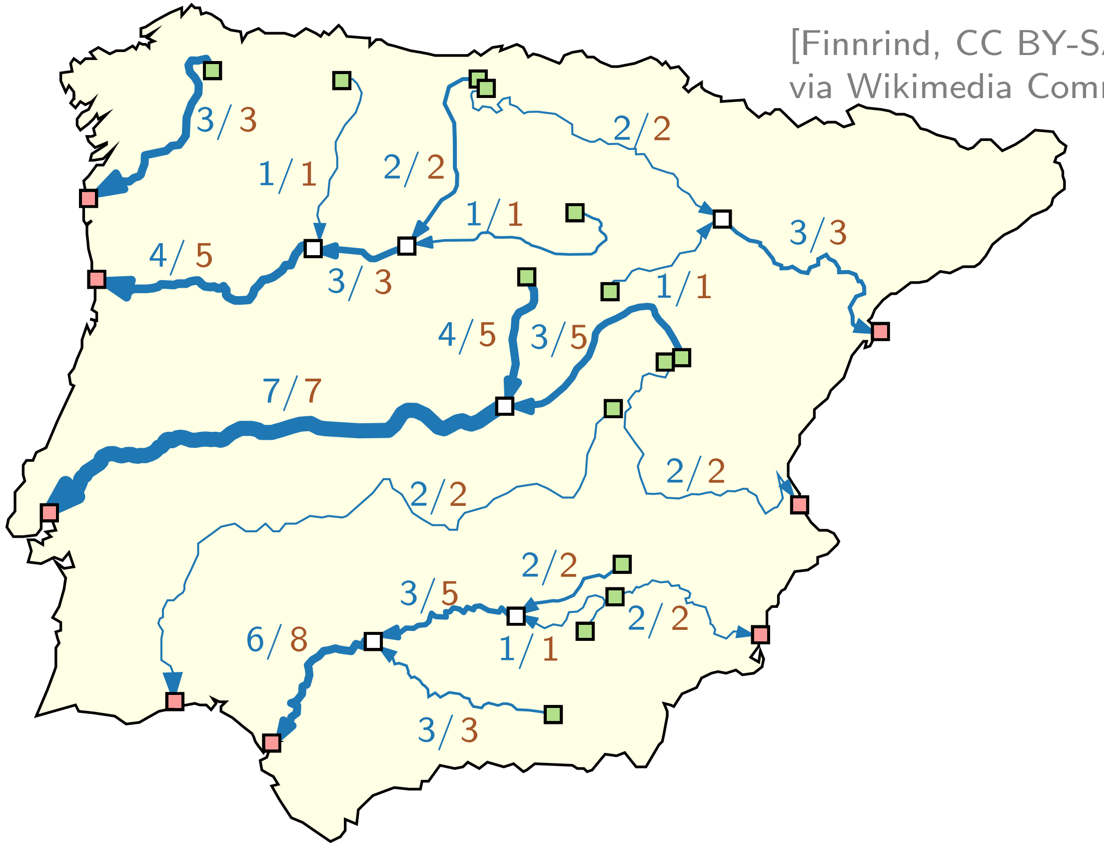

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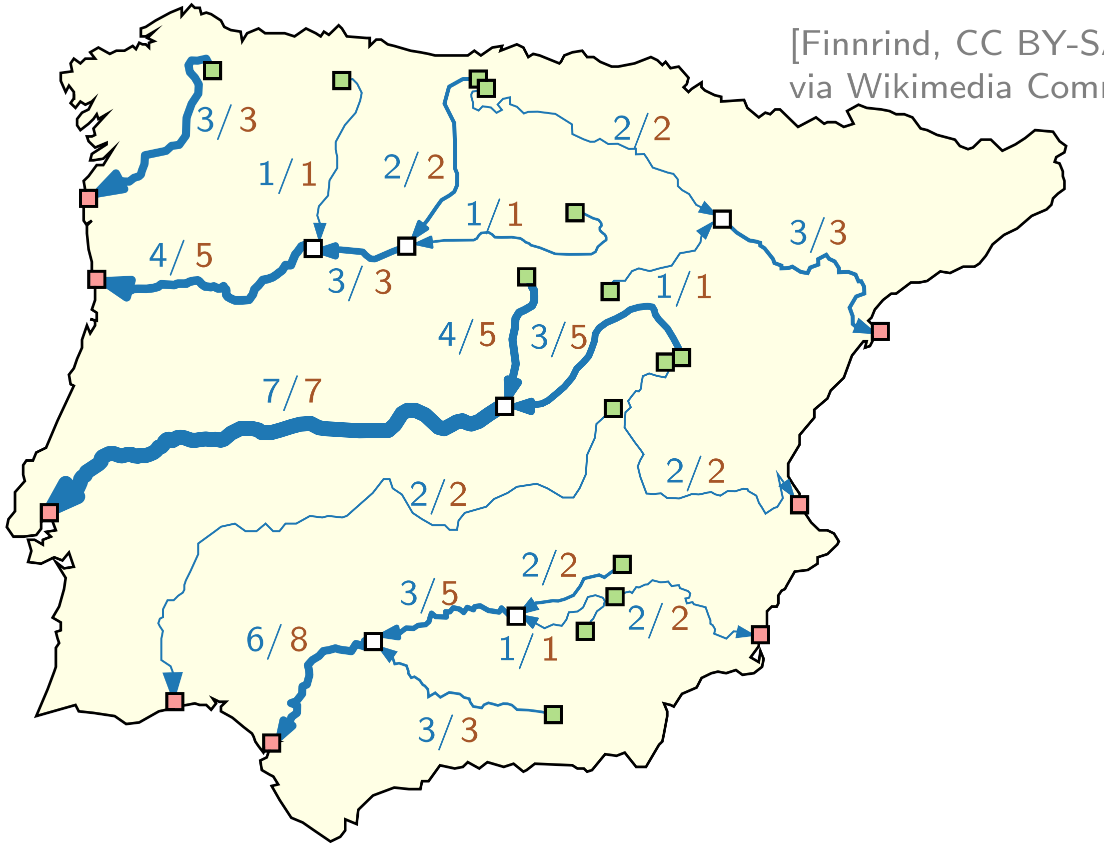

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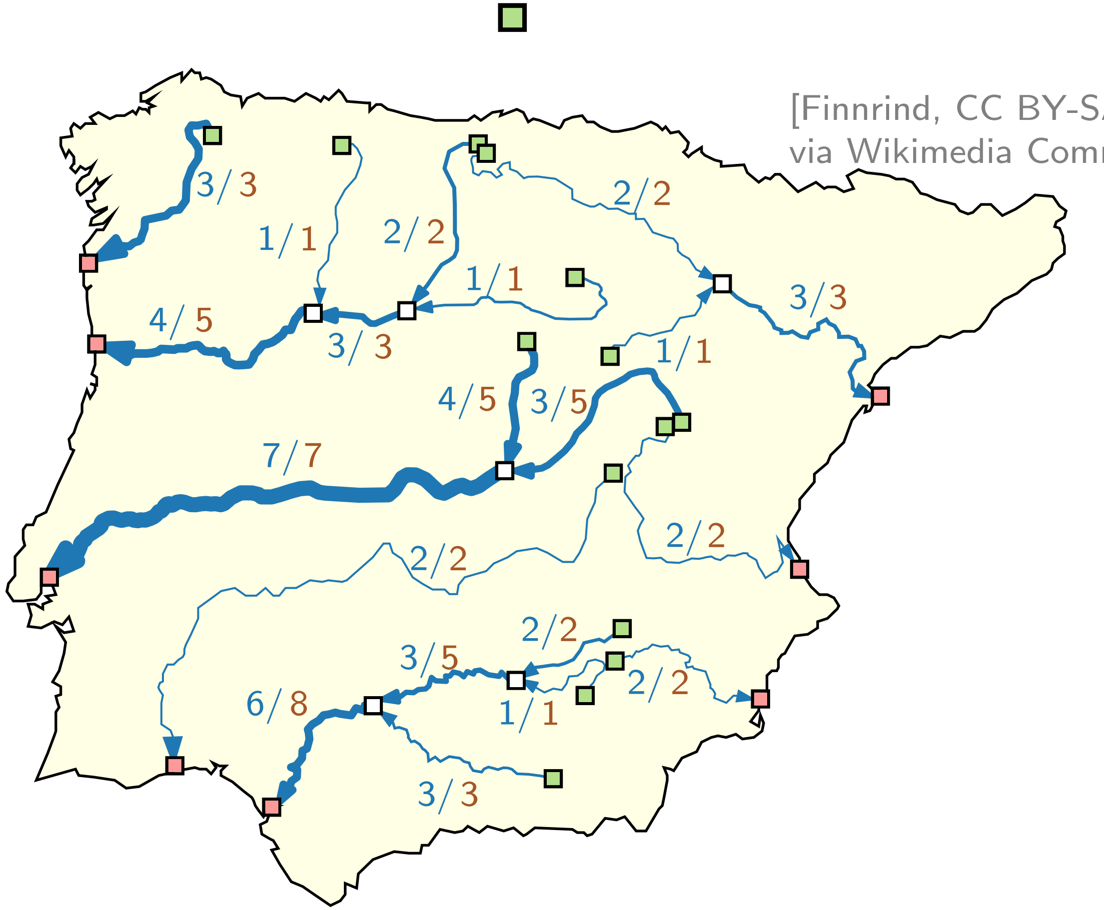

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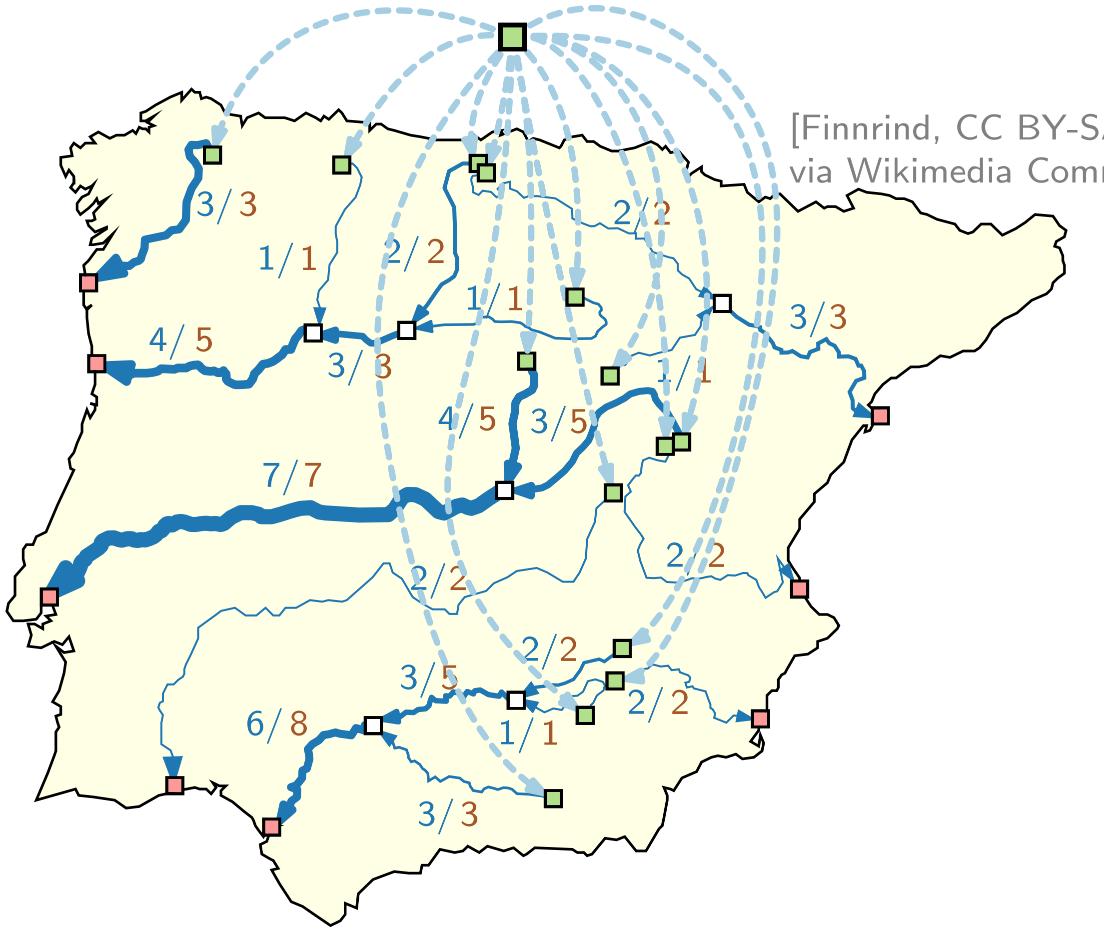

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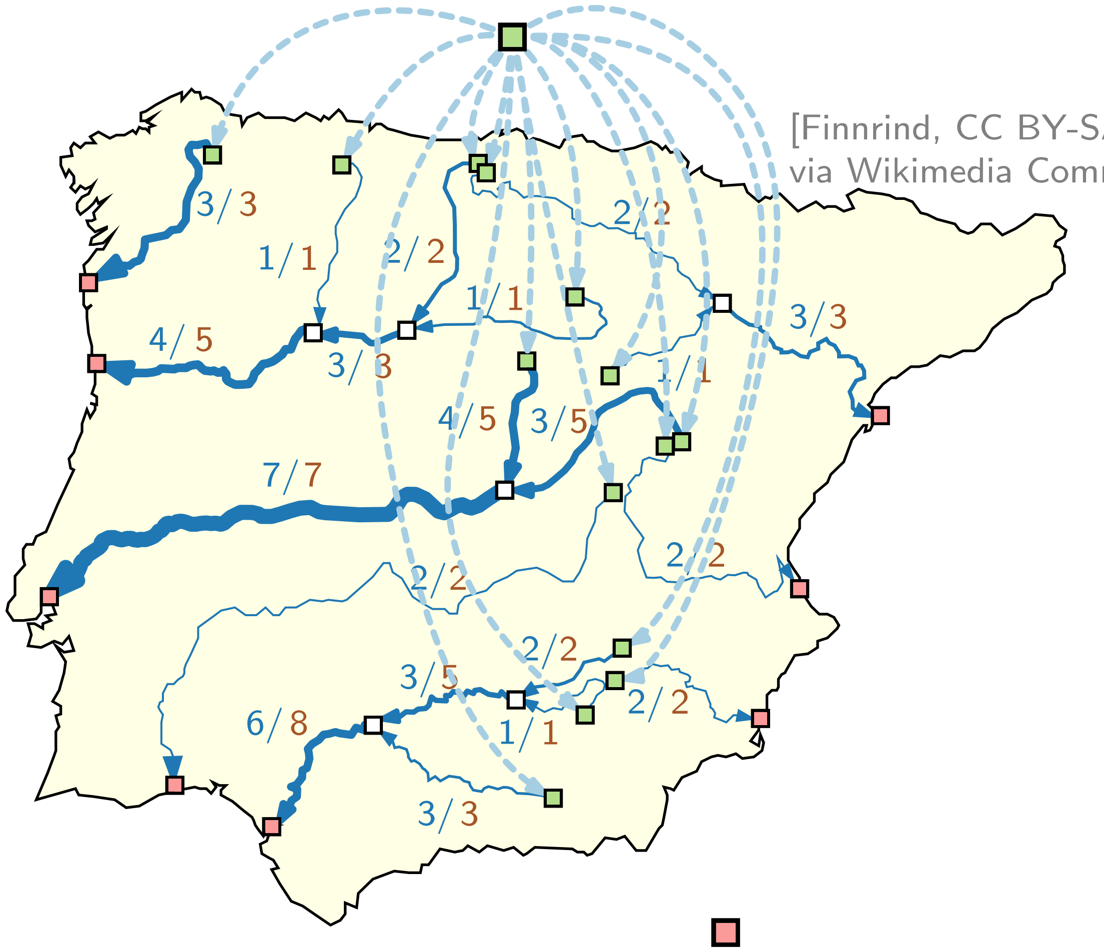

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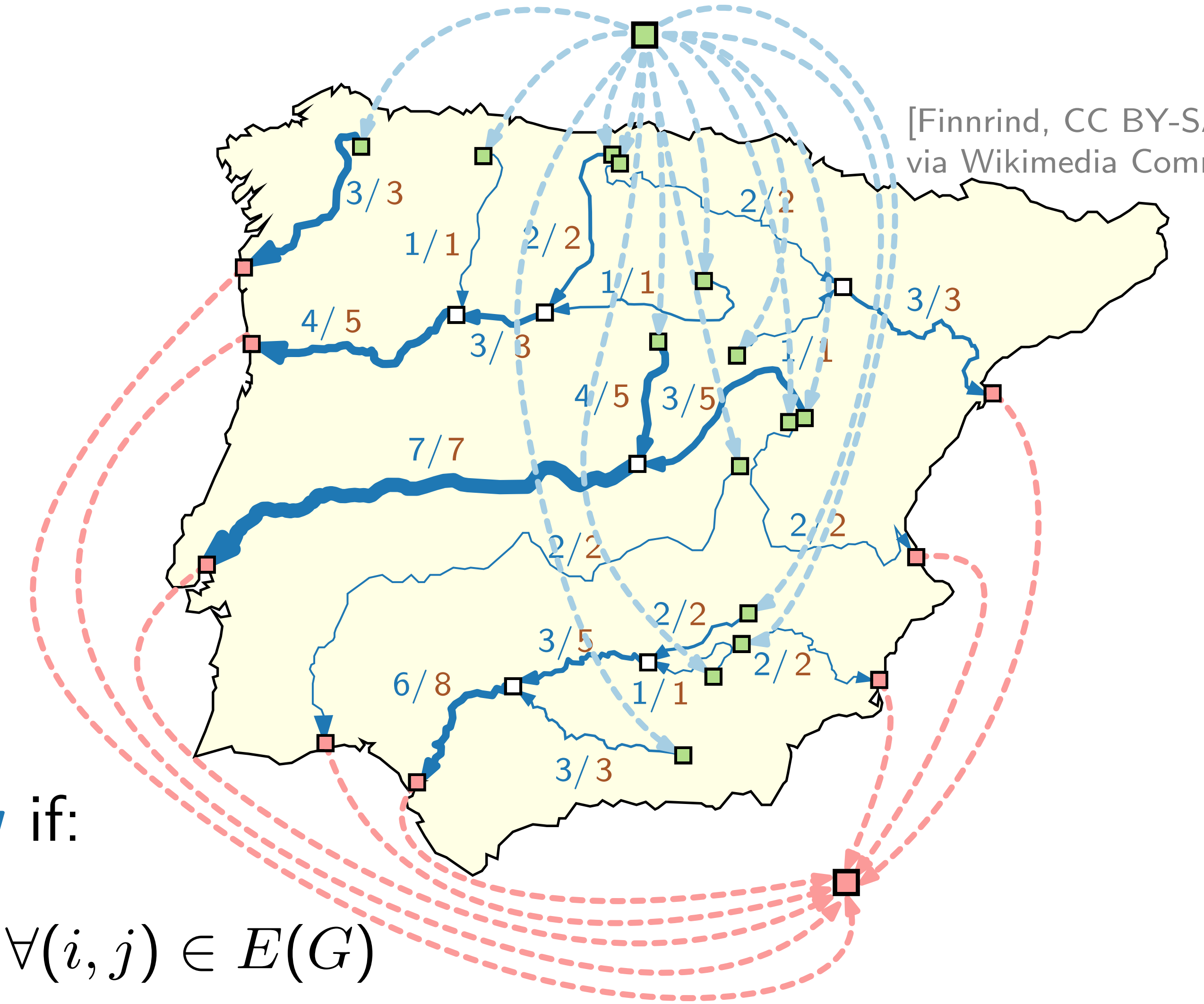

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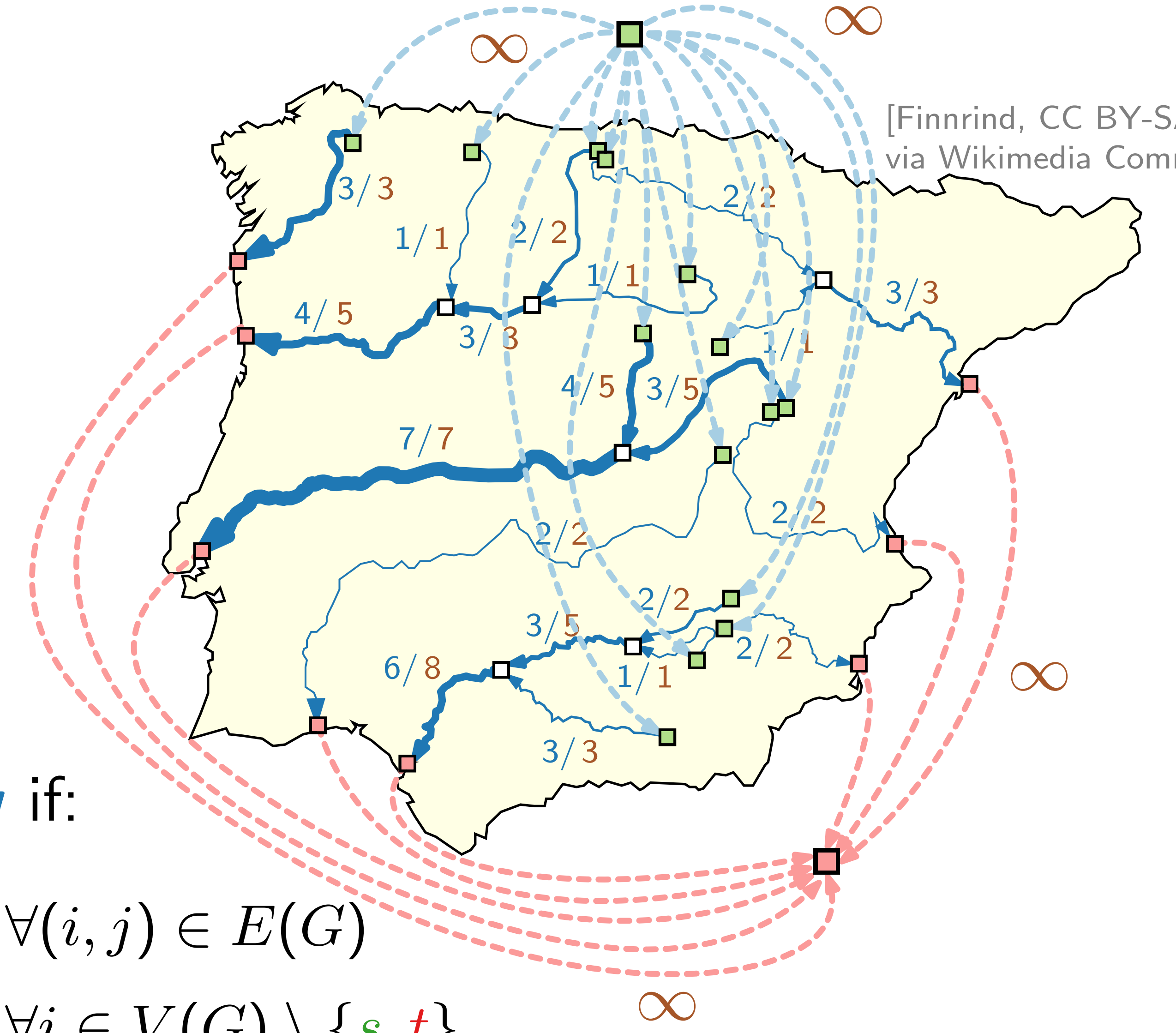

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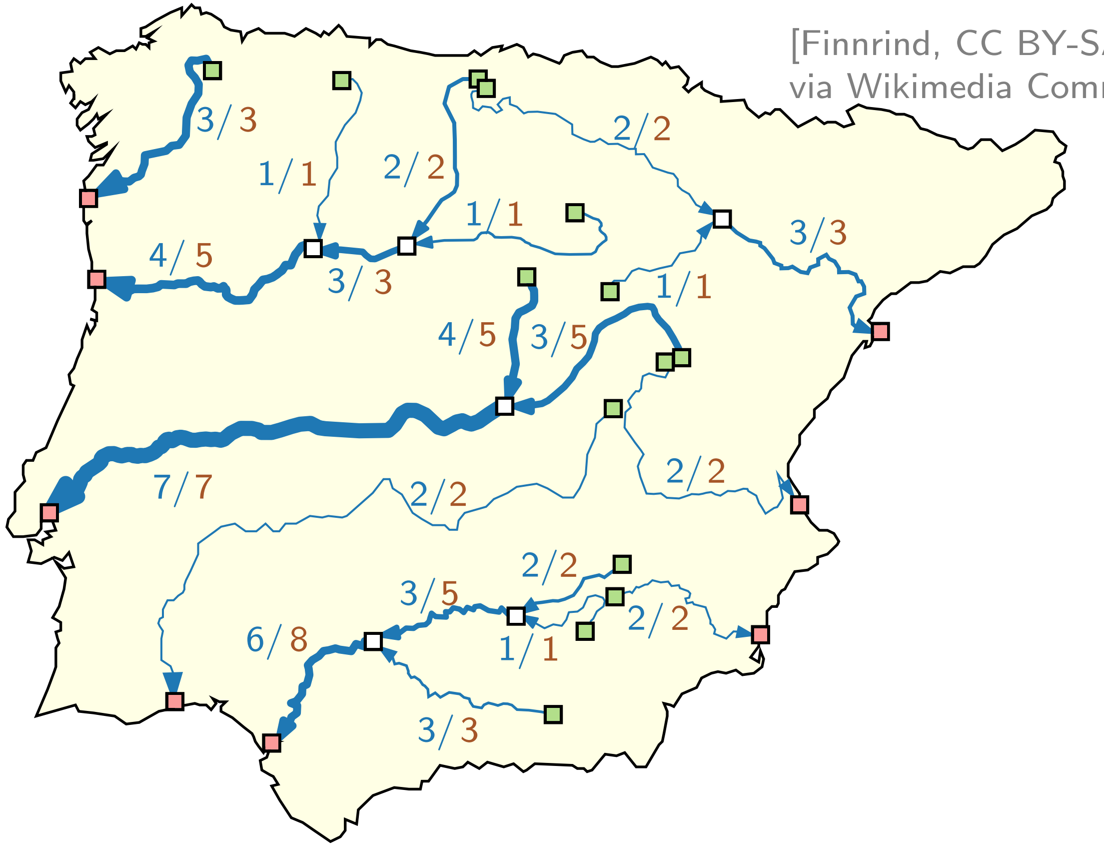
General Flow Network



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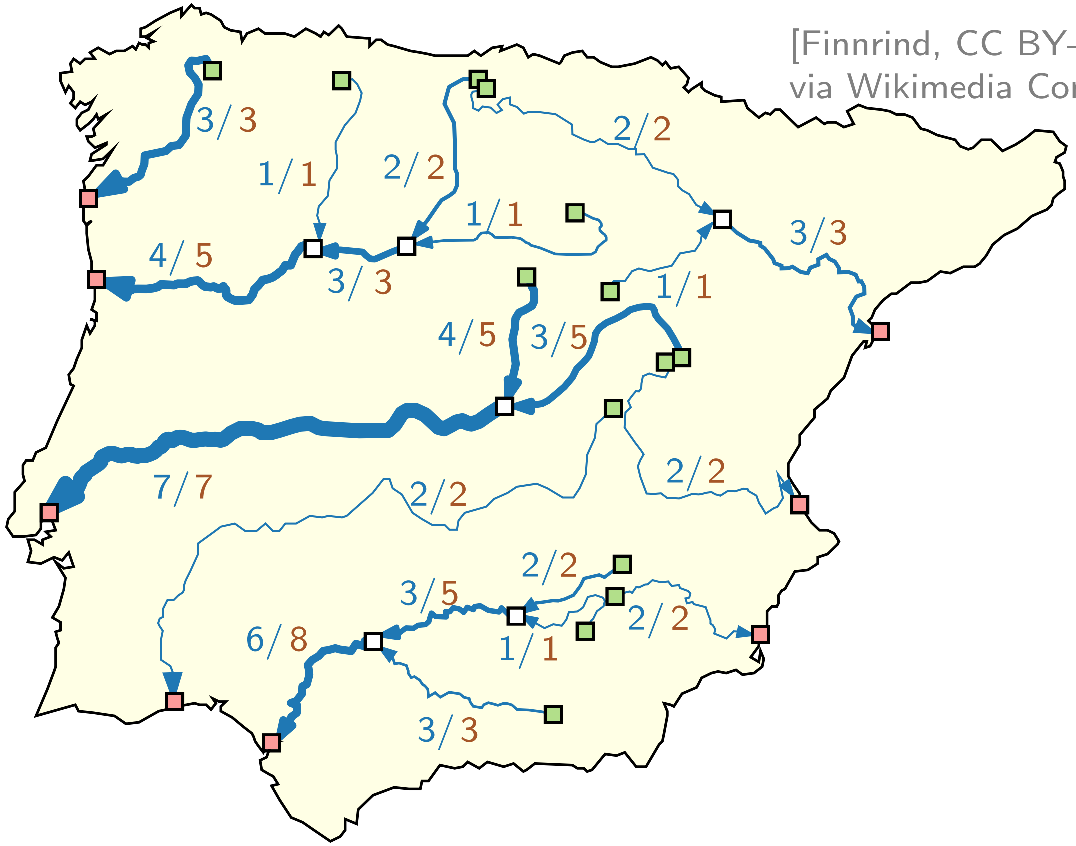

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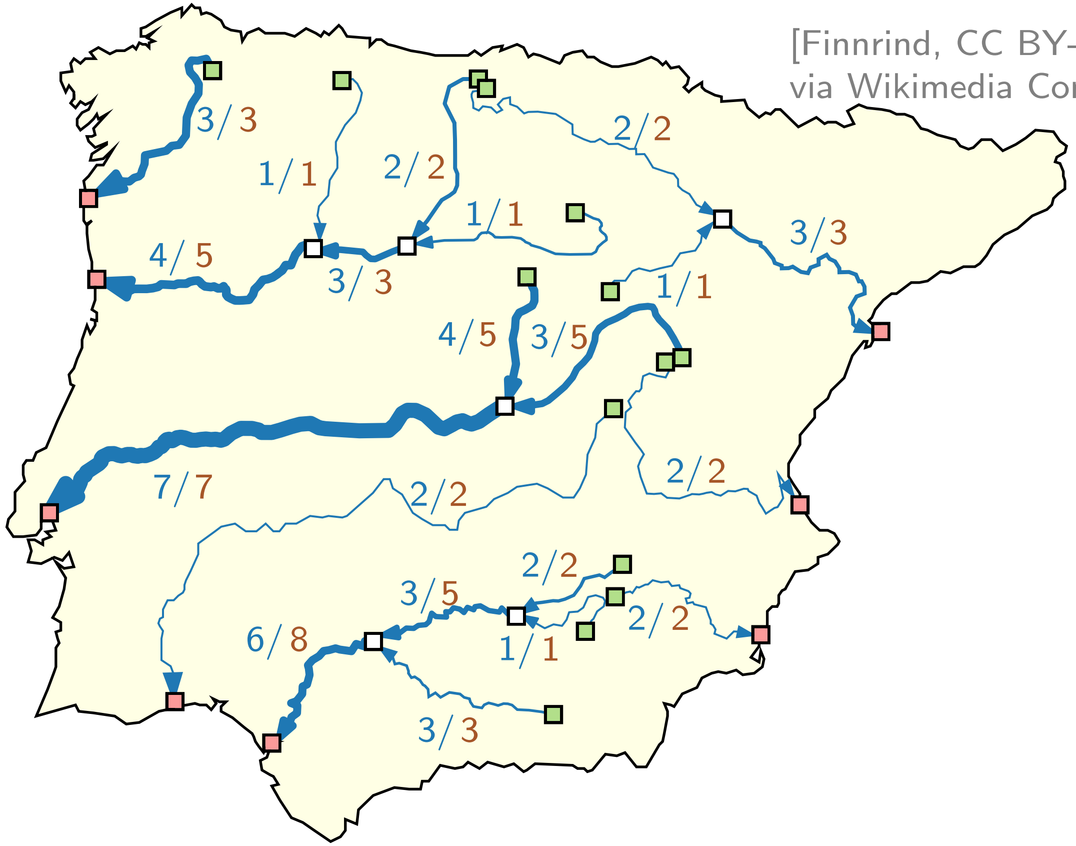

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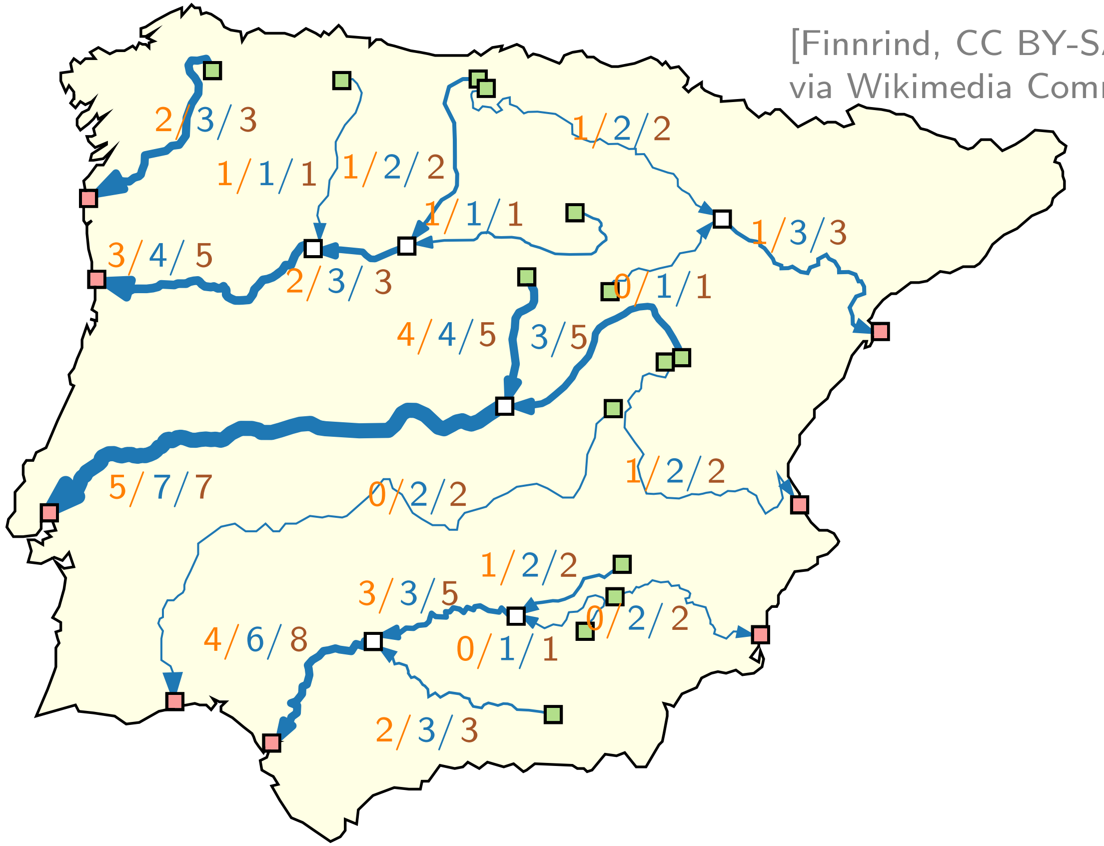

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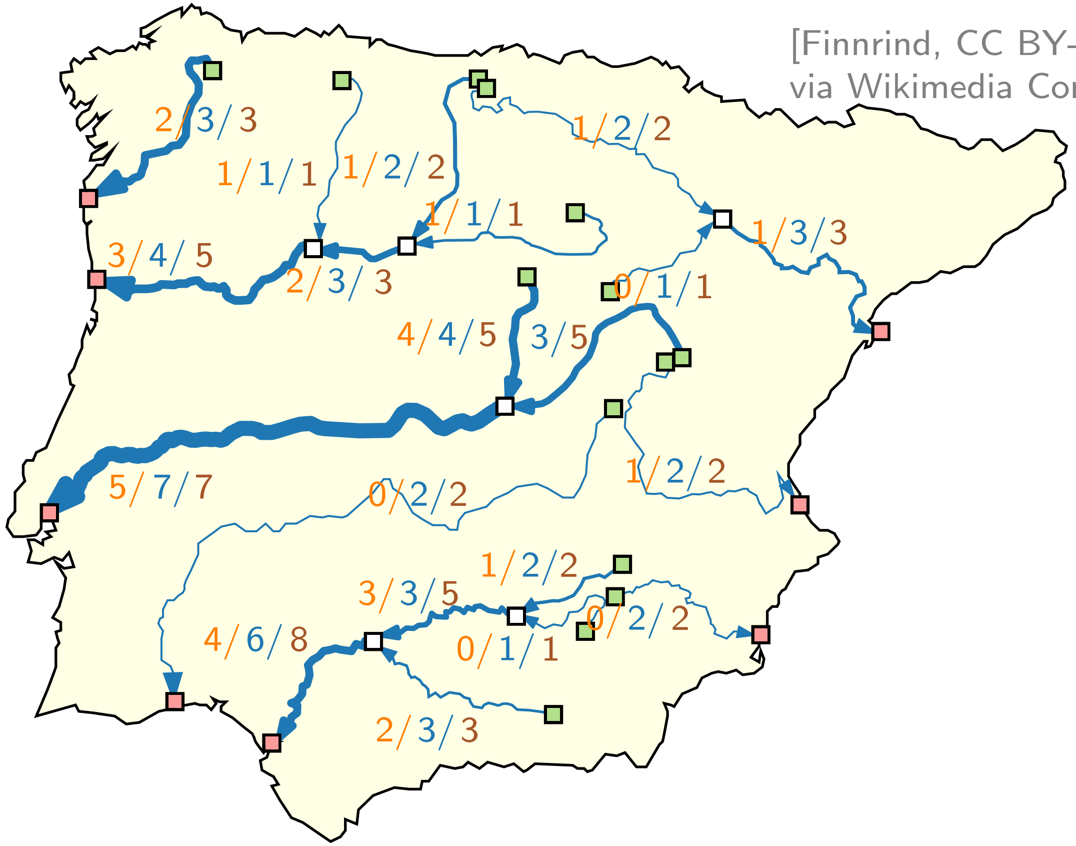

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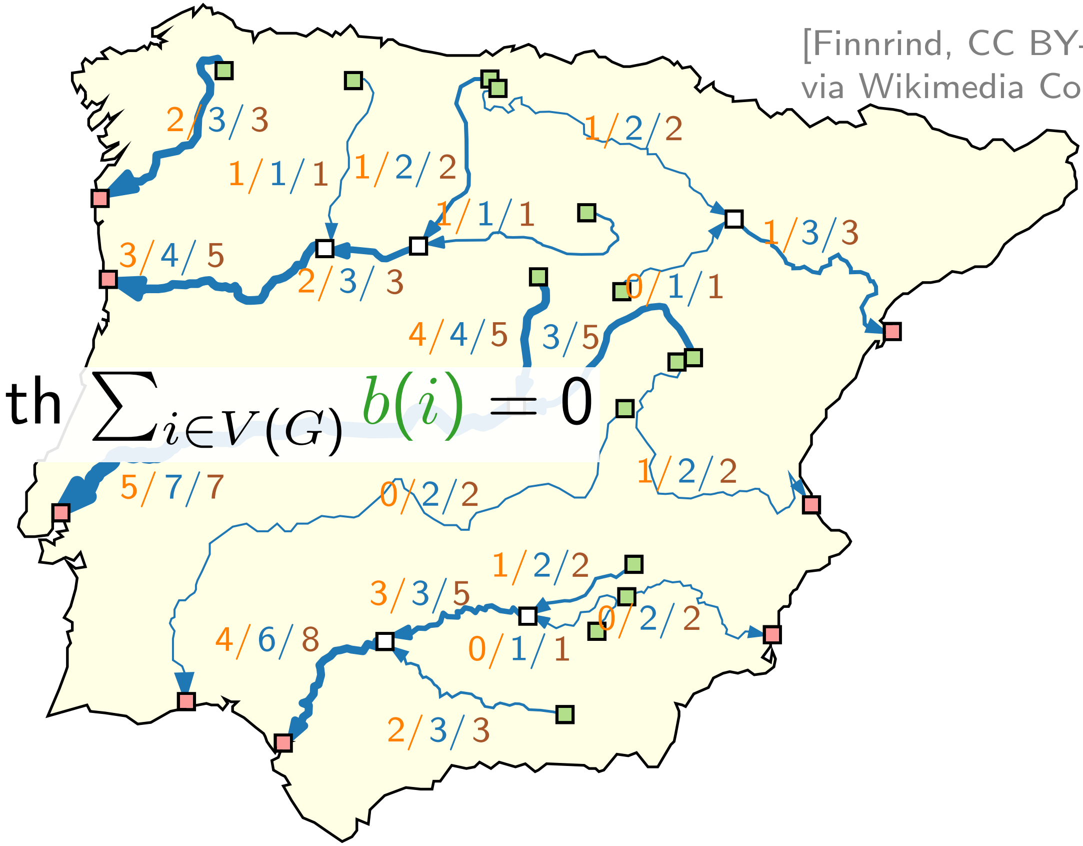

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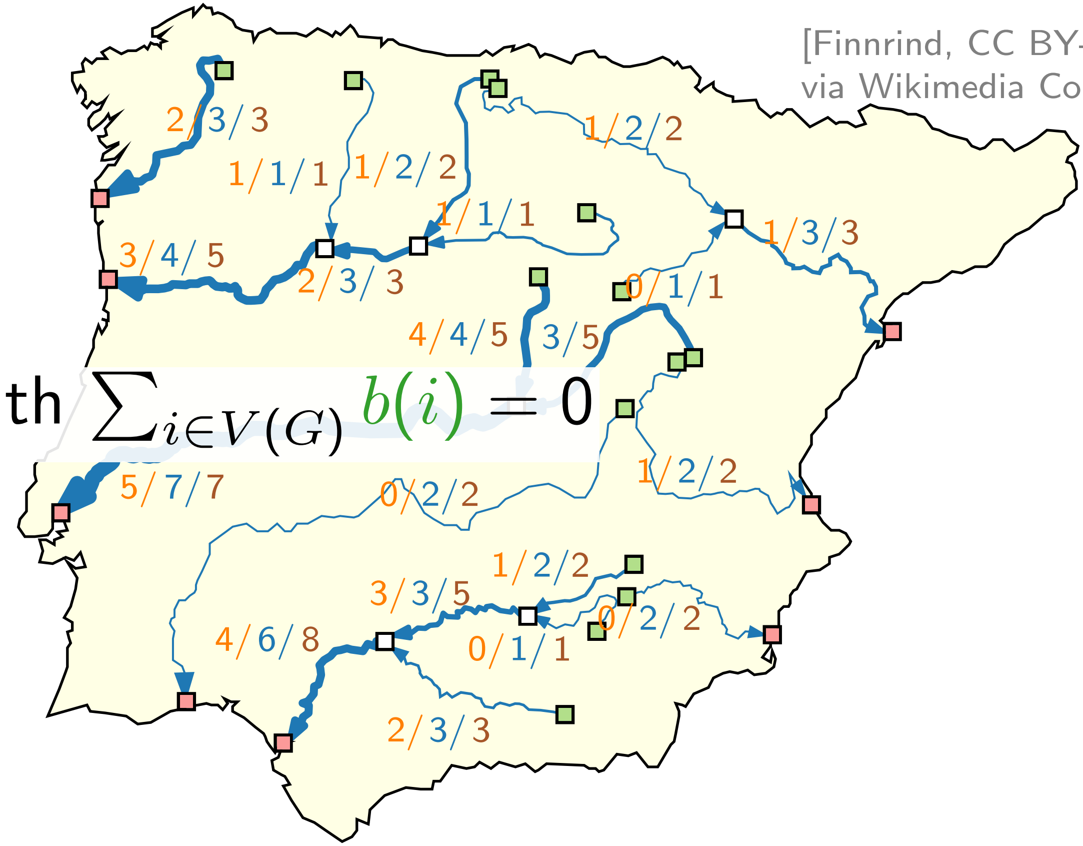

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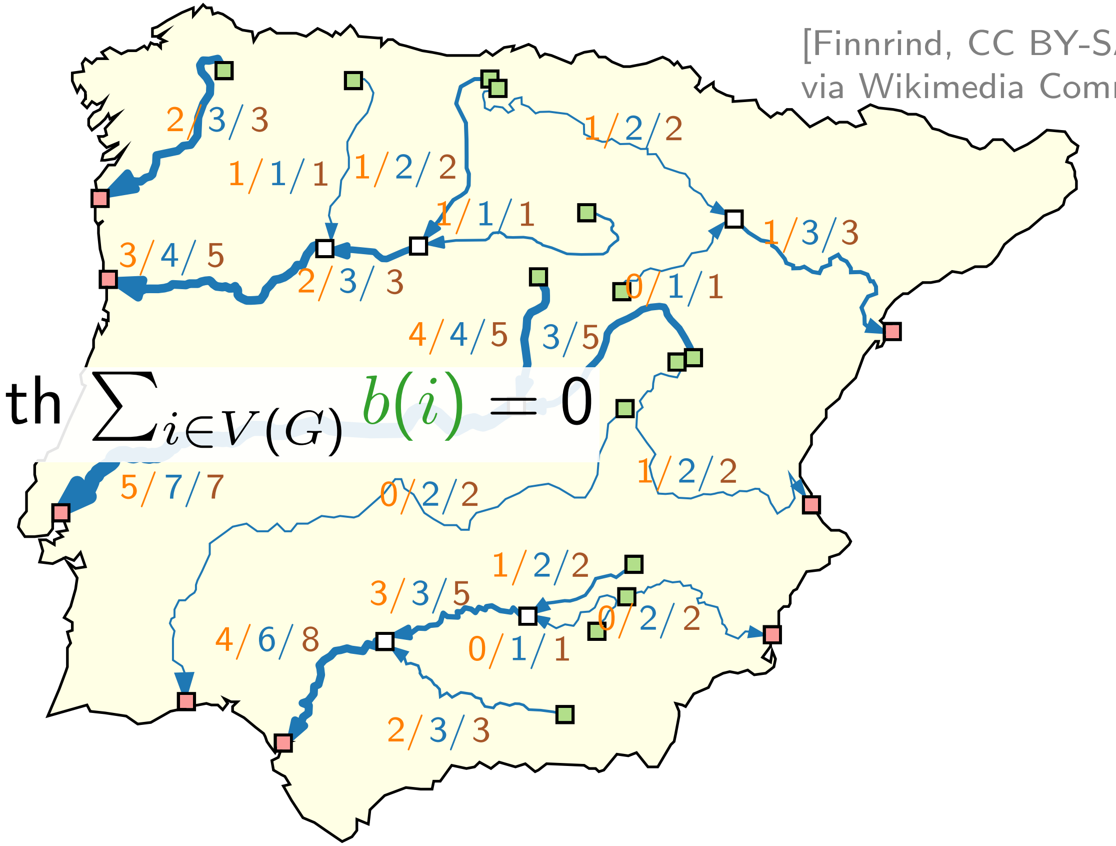

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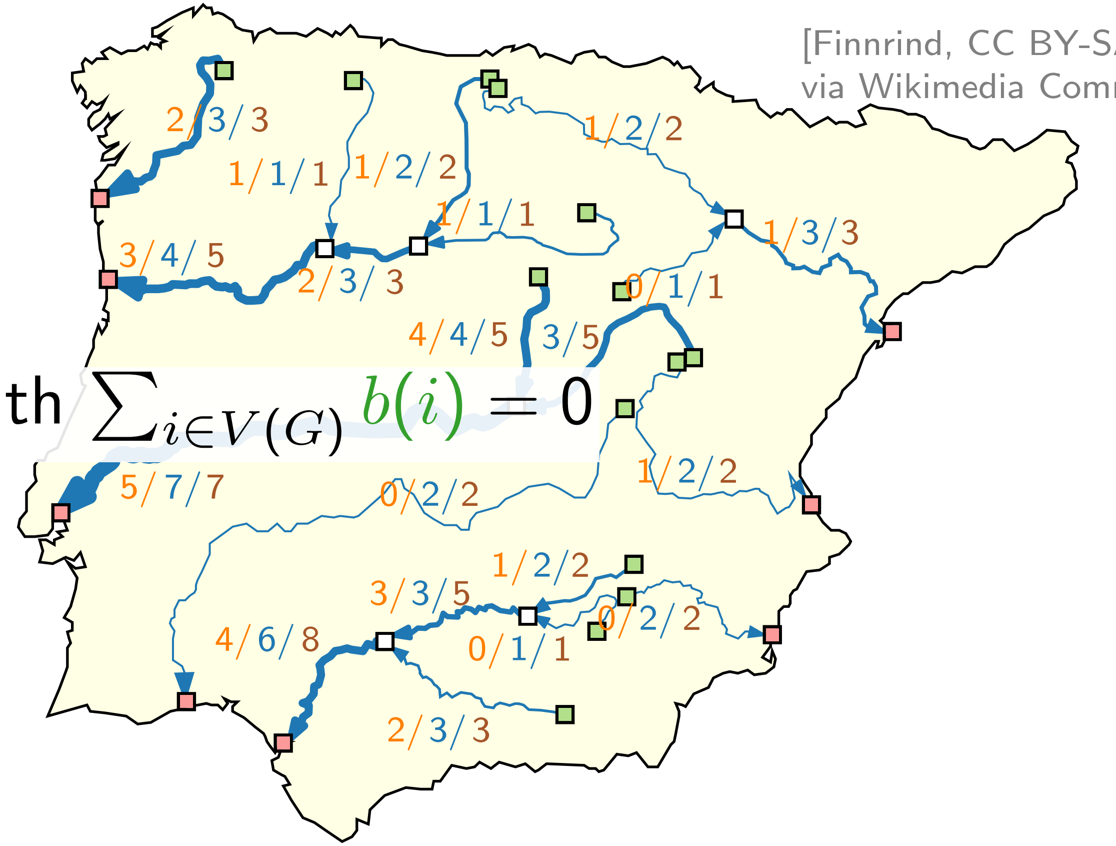

A function $X\colon E(G) \to \mathbb{R}_0^+$ is called **valid flow** if:

$$\ell(i,j) \le X(i,j) \le u(i,j) \qquad \forall (i,j) \in E(G)$$

$$\sum_{(i,j) \in E(G)} X(i,j) - \sum_{(j,i) \in E(G)} X(j,i) = b(i) \qquad \forall i \in V(G)$$

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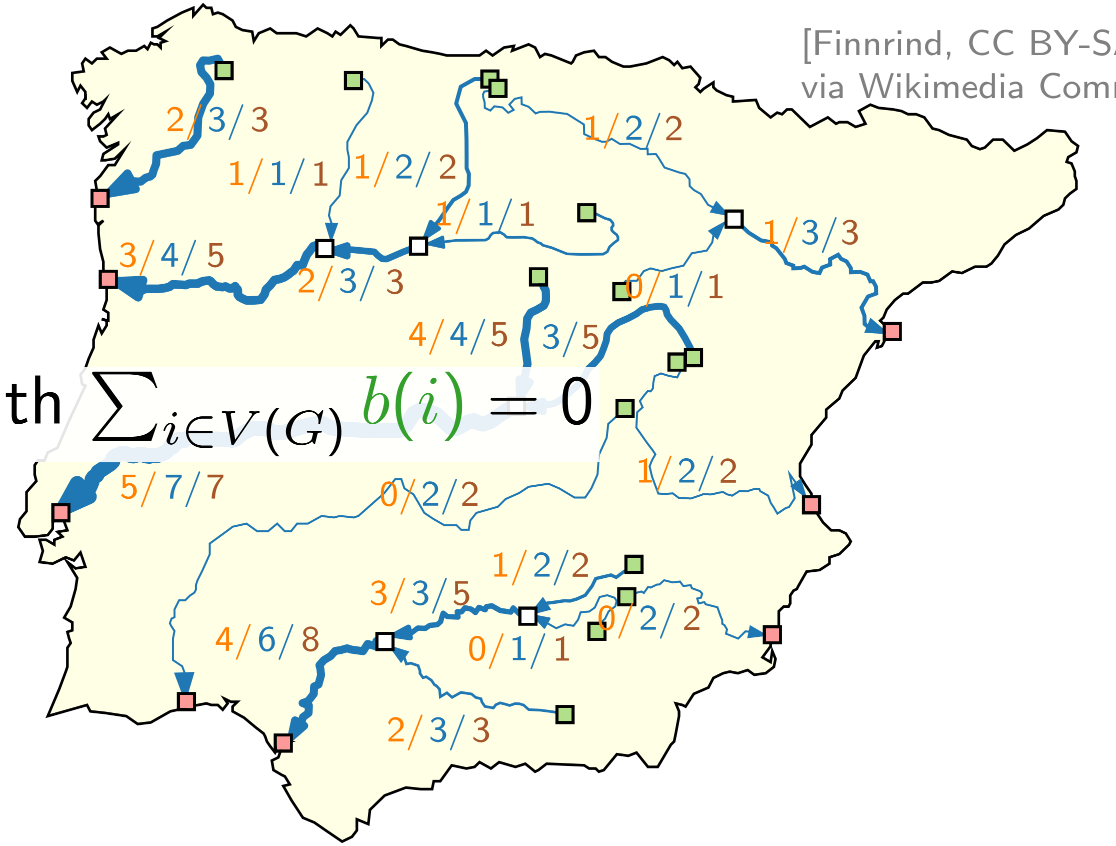
General Flow Network



Flow network $(G; b; \ell; u)$ with

- directed graph G
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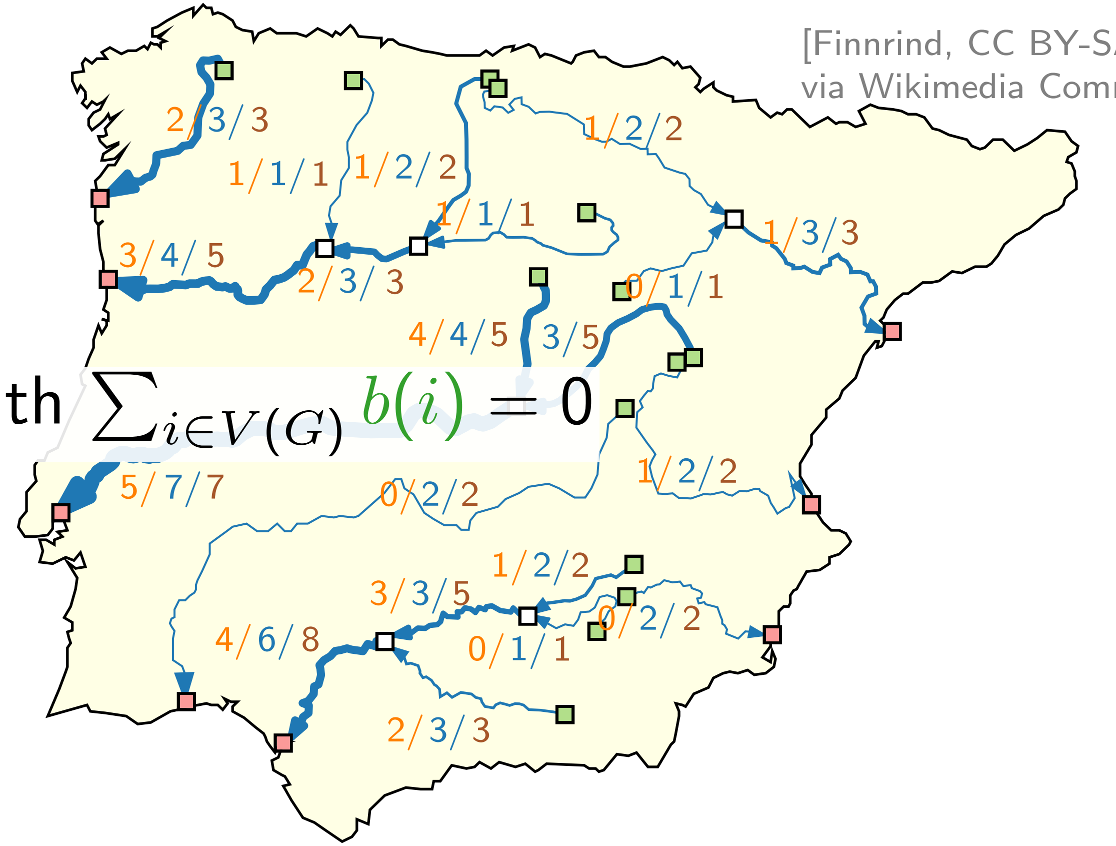

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X is a **minimum-cost flow** if X is a valid flow that minimizes $\text{cost}(X)$.

General Flow Network – Algorithms

n: #vertices
m: #edges

Polynomial Algorithms

#	Due to	Year	Running Time
1	Edmonds and Karp	1972	O((n + m') log U S(n, m, nC))
2	Rock	1980	O((n + m') log U S(n, m, nC))
3	Rock	1980	O(n log C M(n, m, U))
4	Bland and Jensen	1985	O(m log C M(n, m, U))
5	Goldberg and Tarjan	1987	$O(nm \log (n^2/m) \log (nC))$
6	Goldberg and Tarjan	1988	O(nm log n log (nC))
7	Ahuja, Goldberg, Orlin and Tarjan	1988	O(nm log log U log (nC))

Strongly Polynomial Algorithms

#	Due to	Year	Running Time
1	Tardos	1985	$O(m^4)$
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6	Goldberg and Tarjan	1988	$O(nm^2 \log^2 n)$
7	Orlin (this paper)	1988	O((n + m') log n S(n, m))

S(n, m)	= O(m + n log n)	Fredman and Tarjan [1984]
S(n, m, C)	$= O(\text{Min}\ (m + n\sqrt{\log C}),$ $(m \log\log C))$	Ahuja, Mehlhorn, Orlin and Tarjan [1990]; Van Emde Boas, Kaas and Zijlstra[1977]
M(n, m)	$= O(\min\ (nm + n^{2+\varepsilon}, nm \log n)$ where ε is any fixed constant.	King, Rao, and Tarjan [1991]
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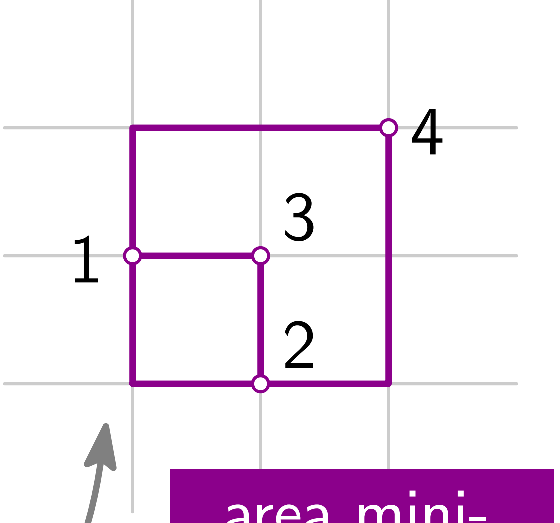
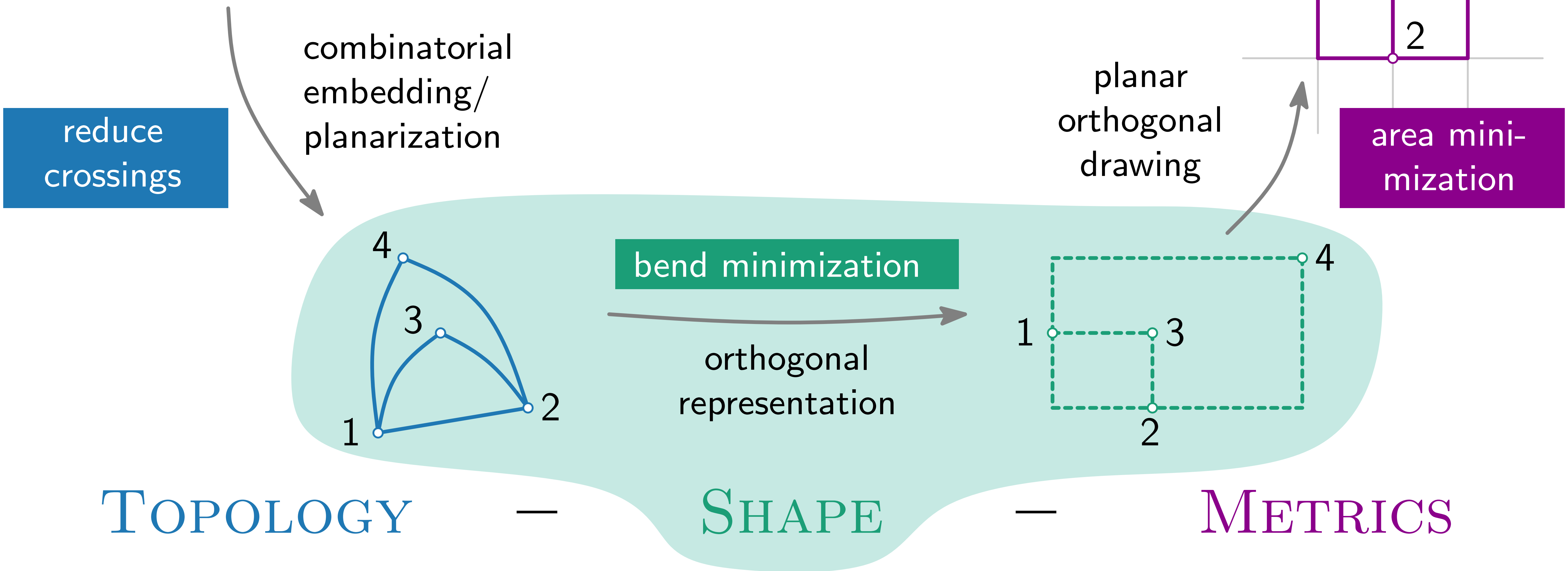
Theorem. [van den Brand, Chen, Kyng, Liu, Peng, Probst, Sachdeva, Sidford 2023]
The minimum-cost flow problem with integral vertex demands, edge capacities, and edge costs can be solved in $O(m^{1+o(1)} \log U \log C)$ time where U is the maximum capacity and C are the maximum costs.

Topology – Shape – Metrics

Three-step approach: [Tamassia 1987]

$V(G) = \{v_1, v_2, v_3, v_4\}$
$E(G) = \{v_1v_2, v_1v_3, v_1v_4, v_2v_3, v_2v_4\}$





Bend Minimization with Given Embedding

Geometric orthogonal bend minimization.

Given:

Find:

Bend Minimization with Given Embedding

Geometric orthogonal bend minimization.

Given: ■ Plane graph G with maximum degree 4

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Given:
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Compare with the following variant:

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Idea.
Formulate as a network-flow problem:

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Flow Network for Bend Minimization

(H1) $H(G)$ corresponds to F, f_0.

(H2) For each **edge** $\{u, v\}$ shared by faces f and g, the sequence δ_1 is reversed and inverted copy of δ_2.

(H3) For each **face** f, it holds that:

$$\sum_{r \in H(f)} C(r) = \begin{cases} -4 & \text{if } f = f_0 \\ +4 & \text{otherwise.} \end{cases}$$

(H4) For each **vertex** v, the sum of incident angles is 2π.

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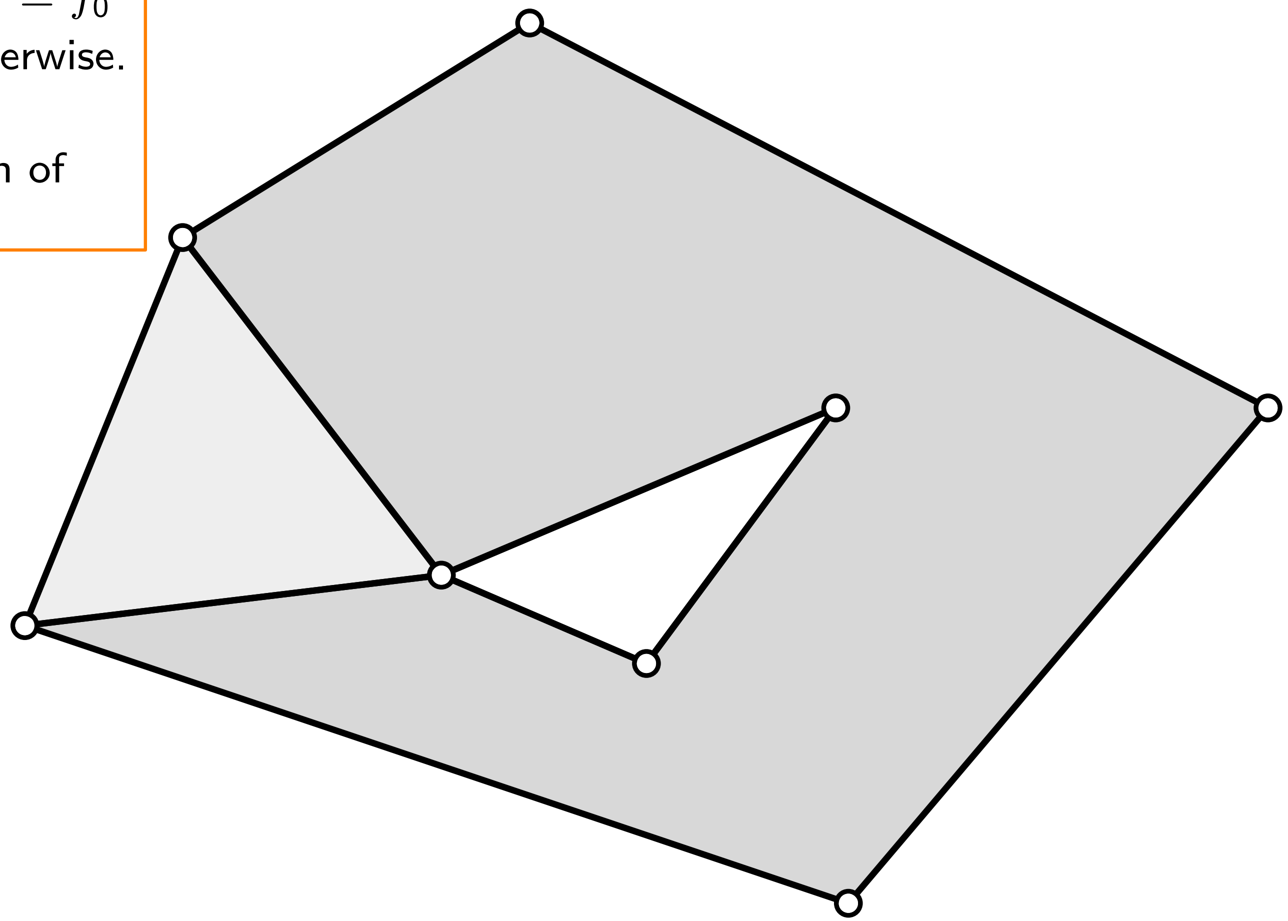

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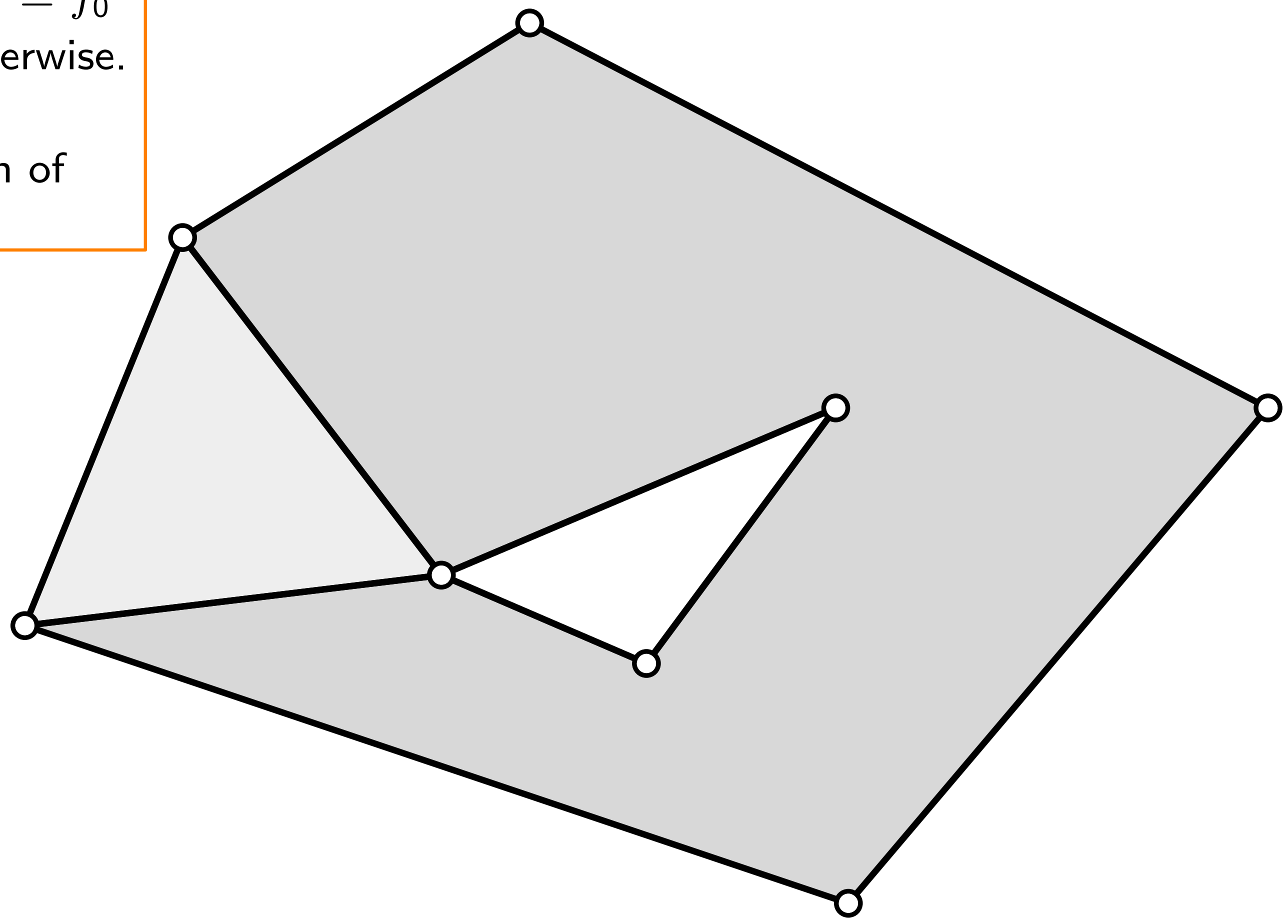

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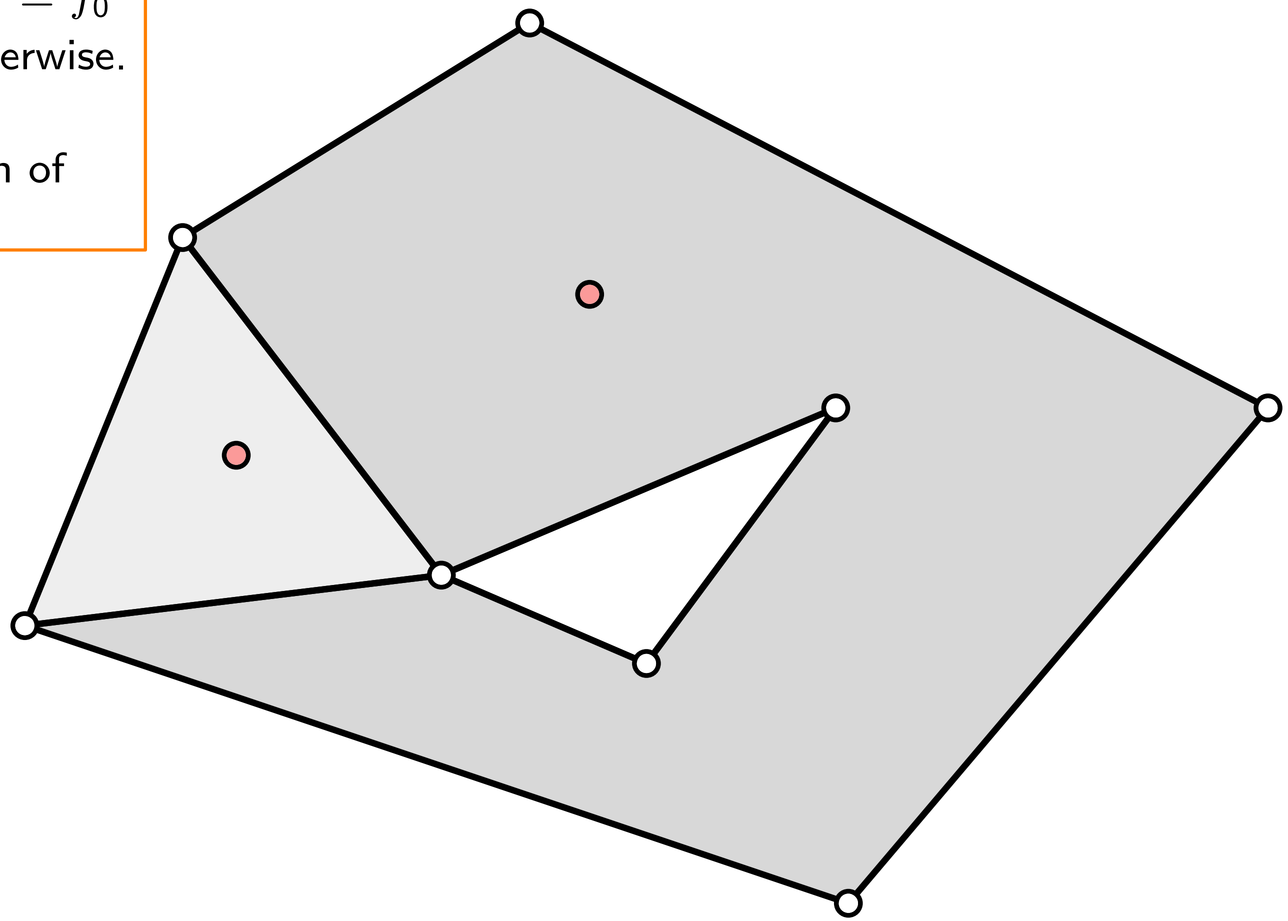

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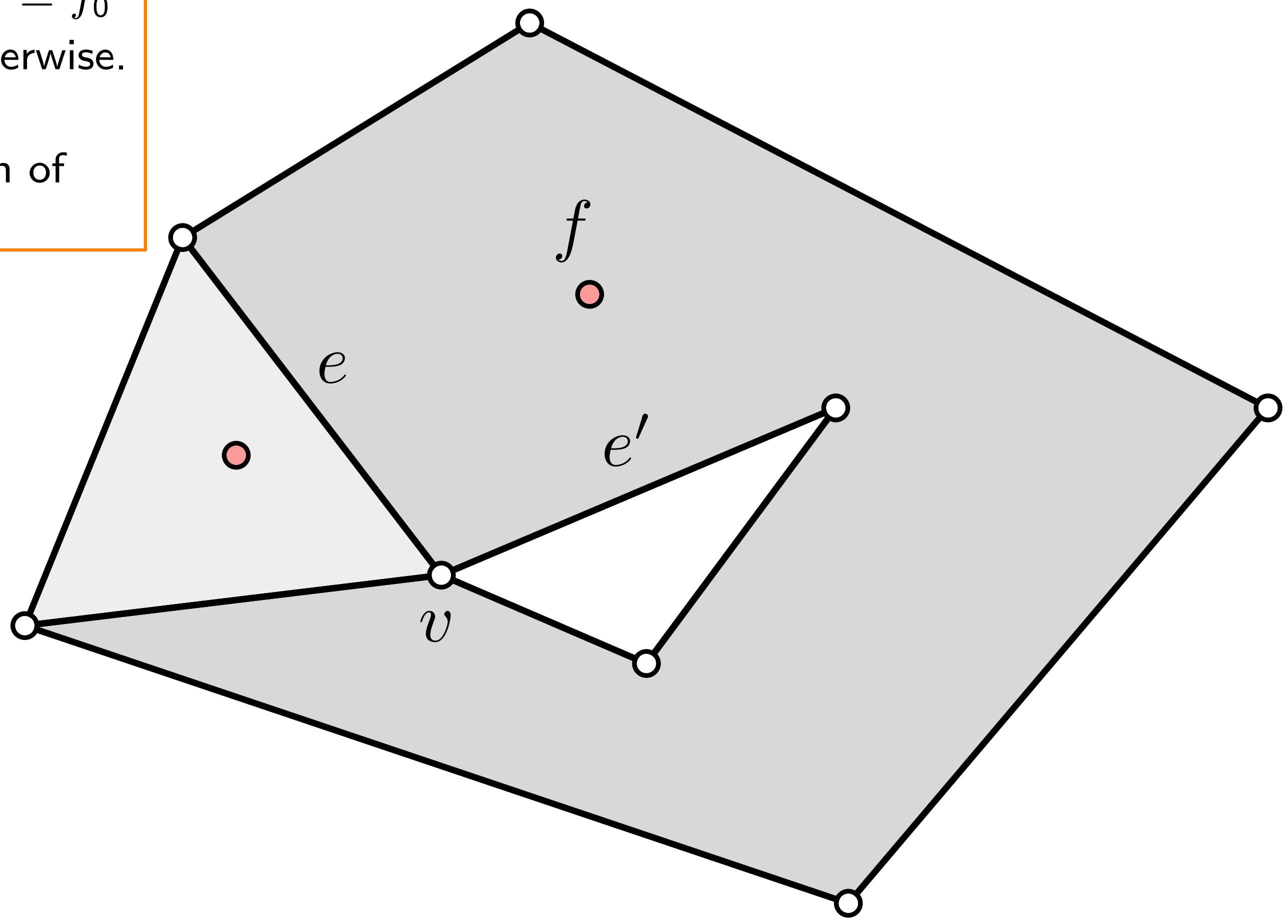

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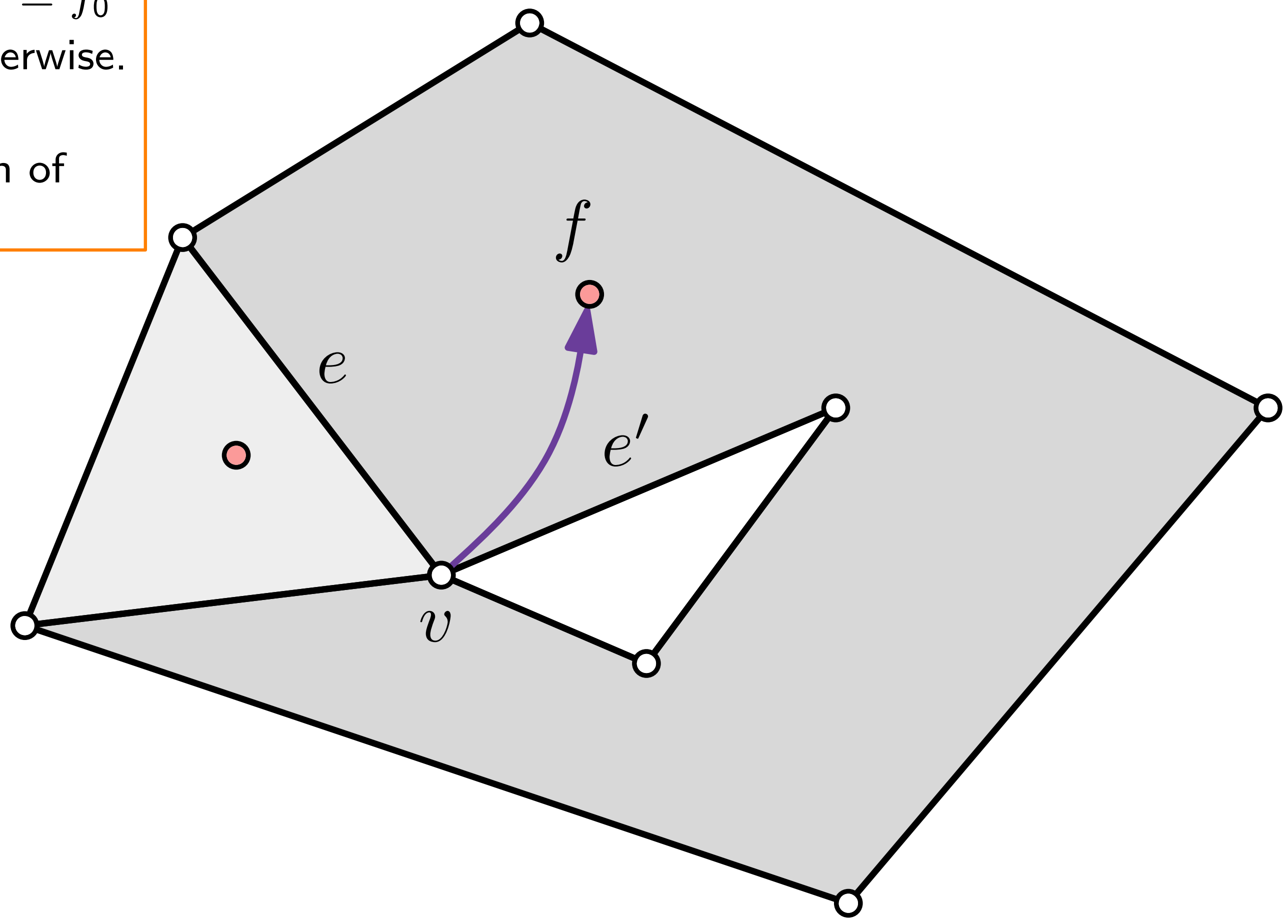

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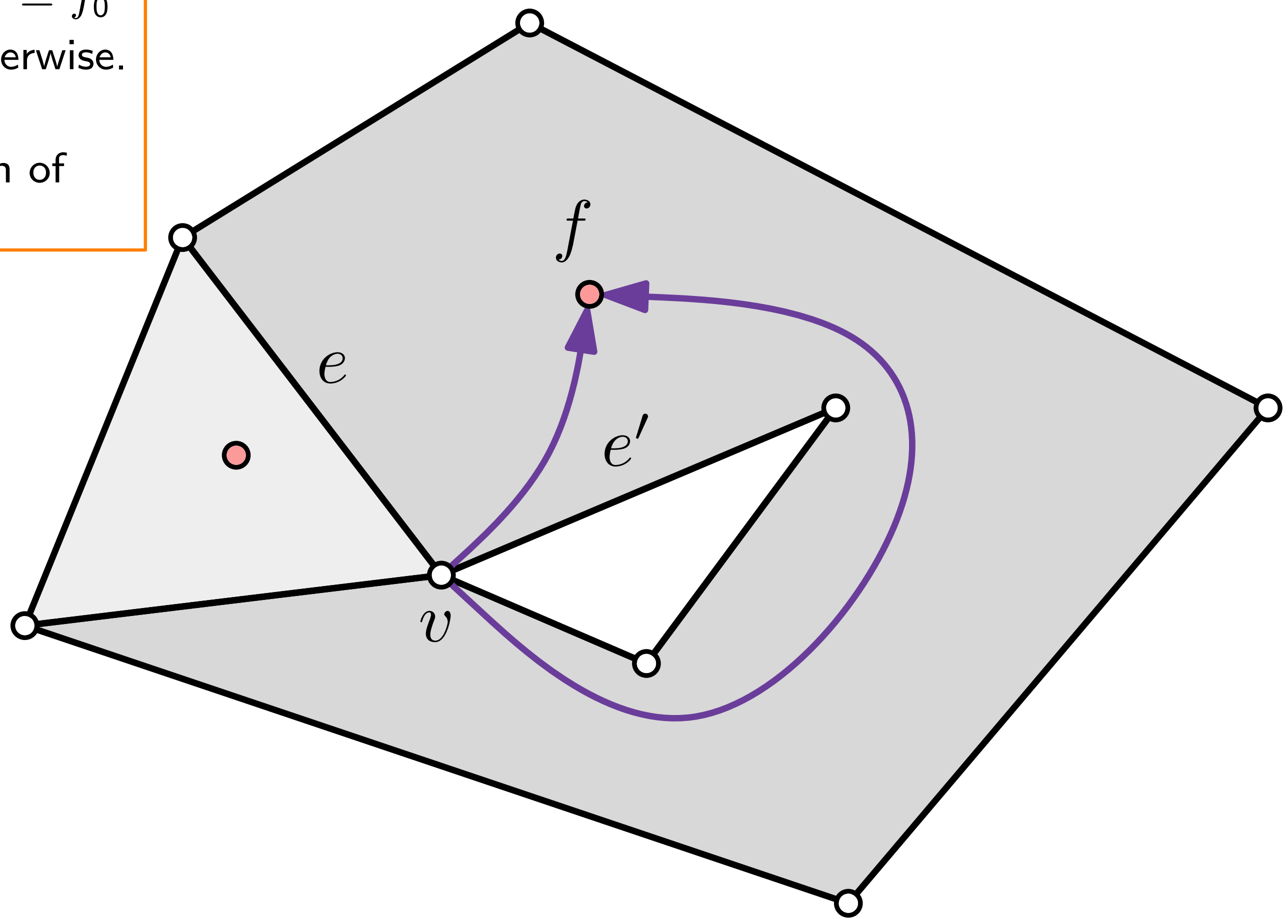

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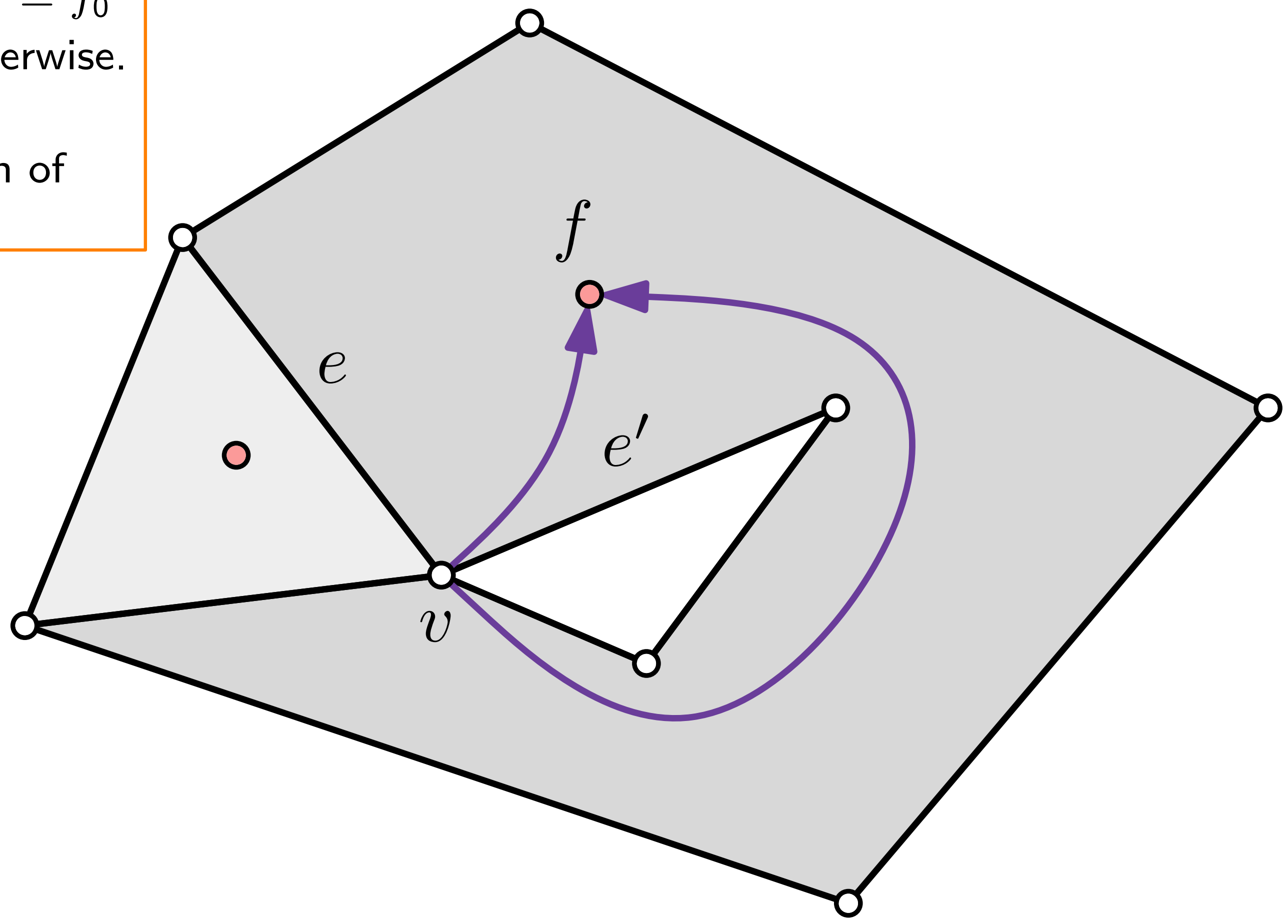

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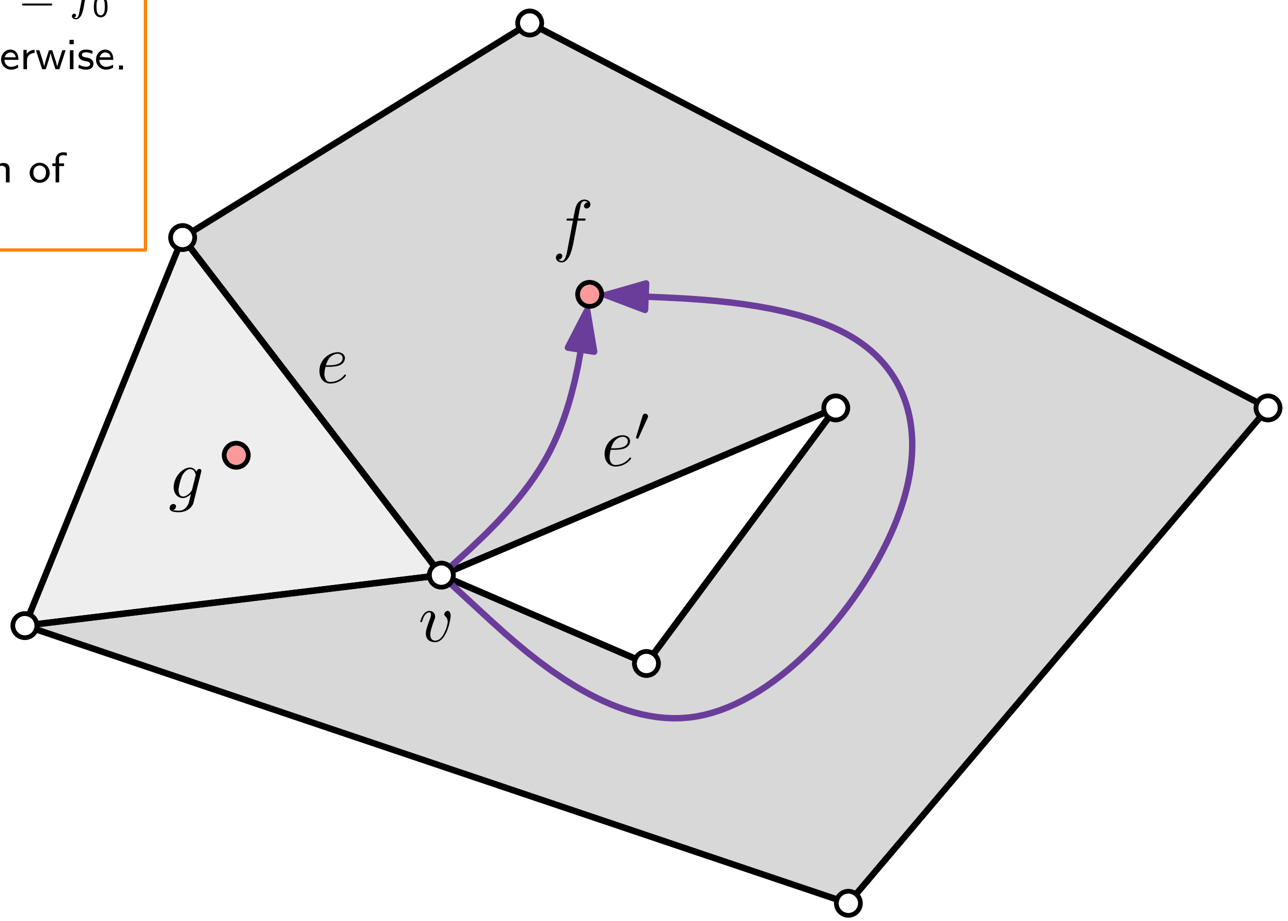

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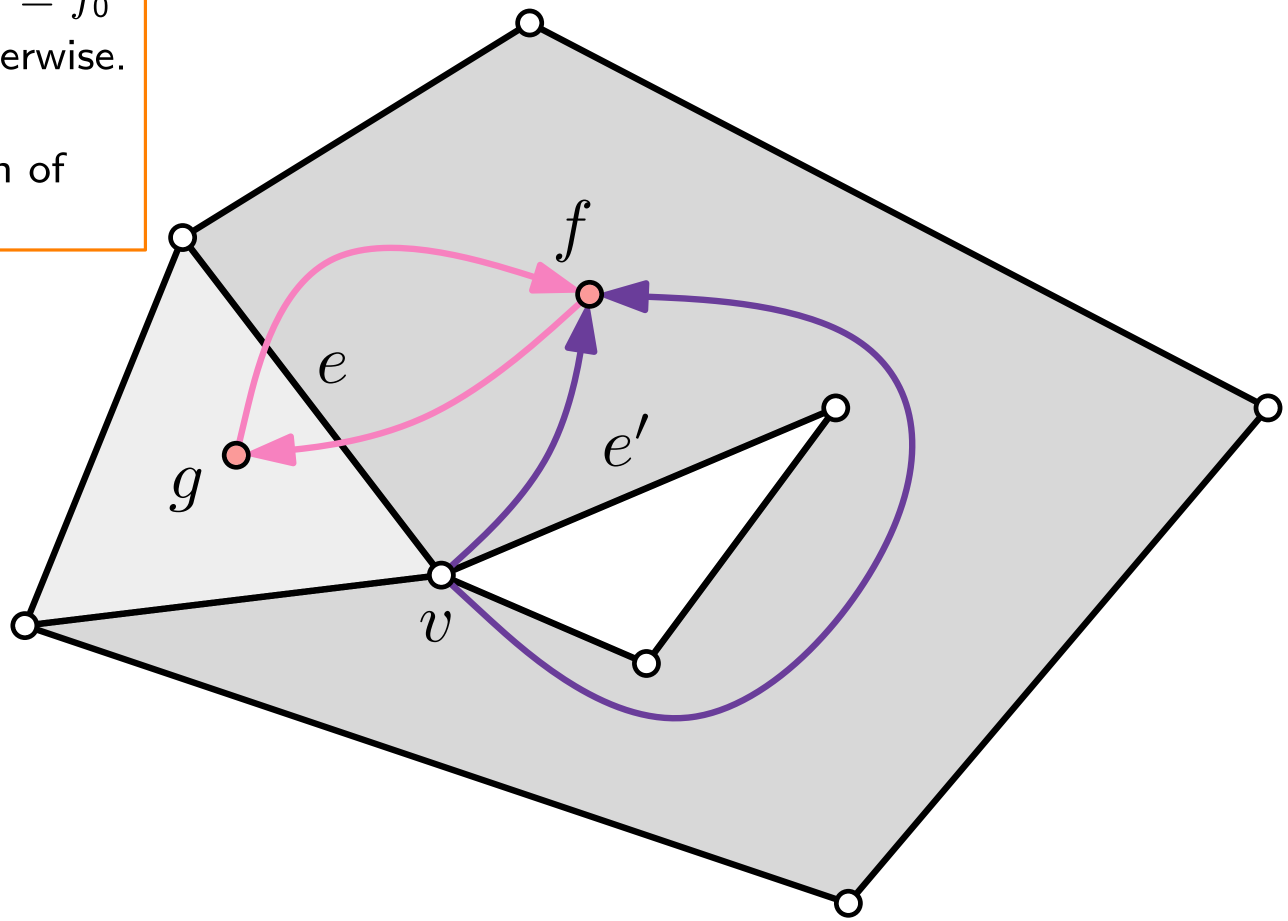

Directed multigraph!

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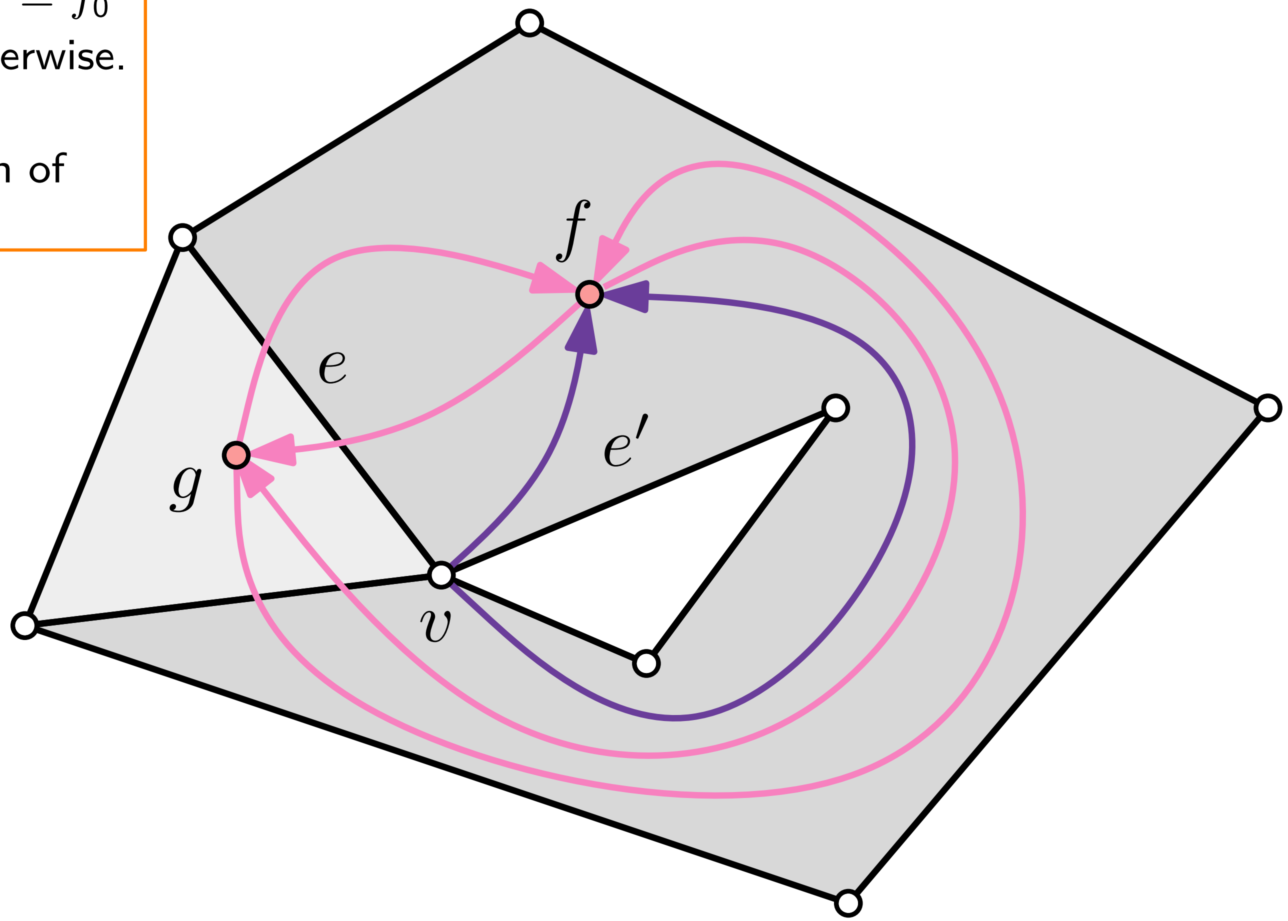

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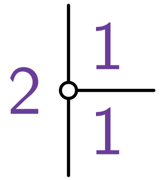

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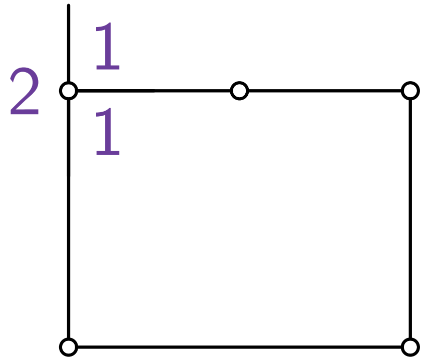

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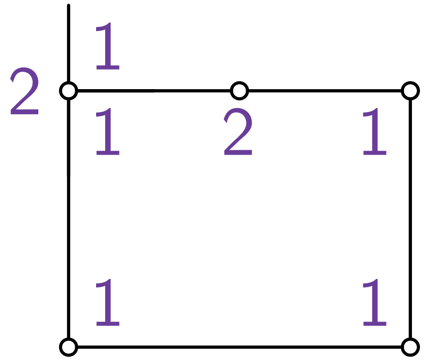

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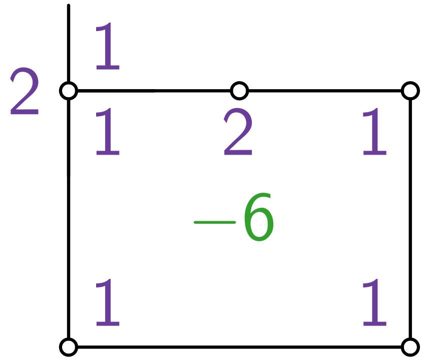

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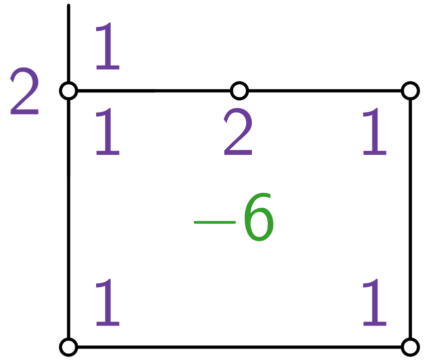

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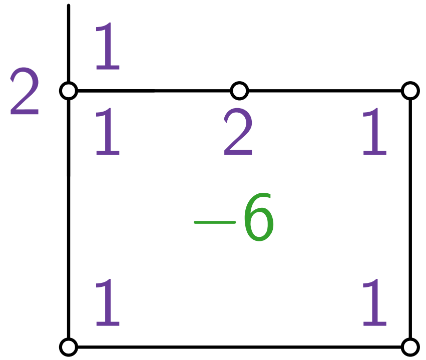

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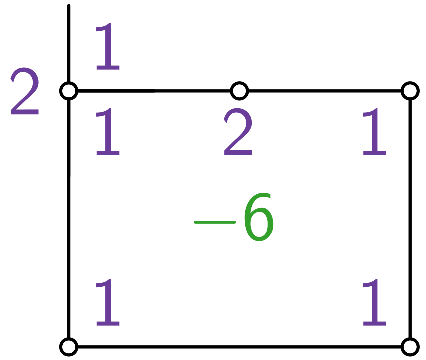

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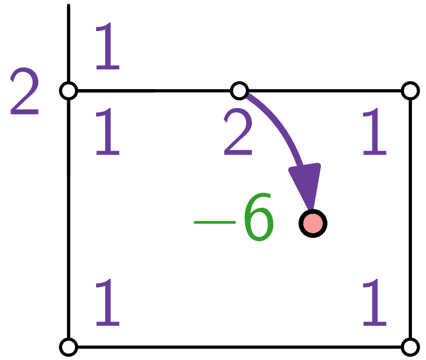

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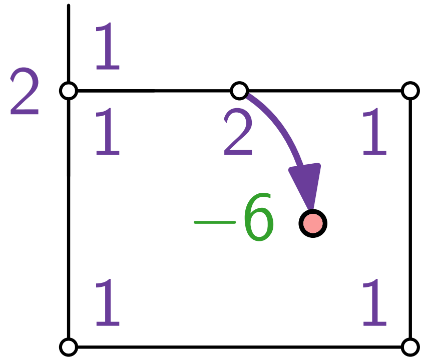

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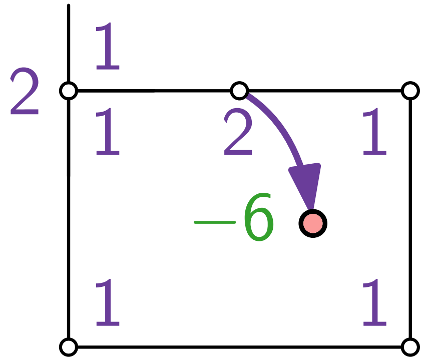

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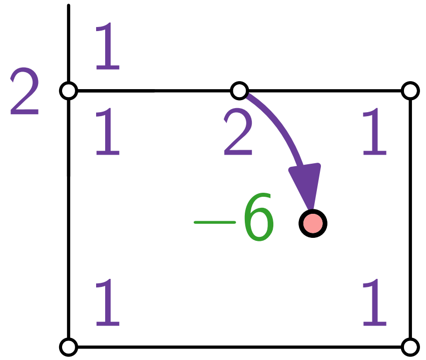

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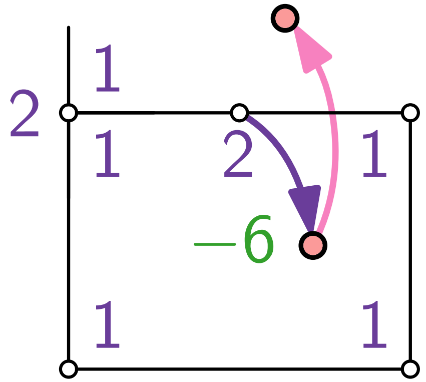

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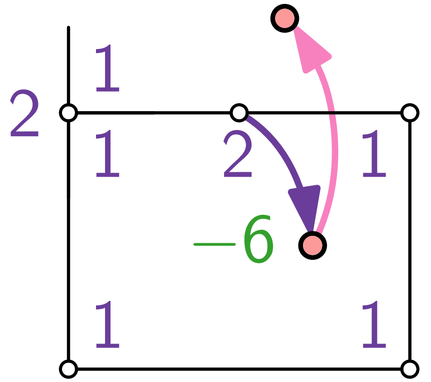

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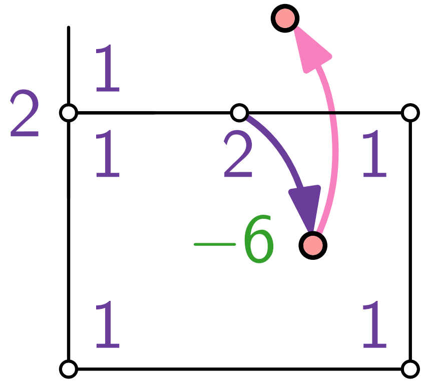

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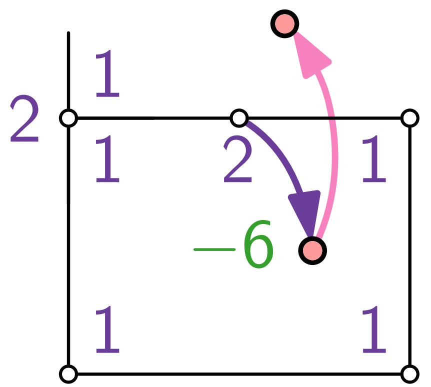
- $b(f) = -2\deg_G(f) + \begin{cases} -4 & \text{if } f = f_0, \\ +4 & \text{otherwise} \end{cases}$

$\Rightarrow \sum_{w \in V(G) \cup F} b(w) = 0$ (Euler)



$\forall (v, f) \in E', v \in V(G), f \in F$ $\quad \ell(v, f) := 1 \leq X(v, f) \leq 4 =: u(v, f)$

$\text{cost}(v, f) = 0$

$\forall (f, g) \in E', f, g \in F$ $\quad \ell(f, g) := 0 \leq X(f, g) \leq \infty =: u(f, g)$

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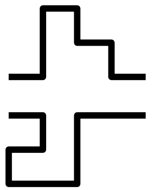

We model only the *number* of bends. Why is it enough?

Flow Network for Bend Minimization

(H1) $H(G)$ corresponds to F, f_0.

(H2) For each **edge** $\{u, v\}$ shared by faces f and g, the sequence δ_1 is reversed and inverted copy of δ_2.

(H3) For each **face** f, it holds that:

$$\sum_{r \in H(f)} C(r) = \begin{cases} -4 & \text{if } f = f_0 \\ +4 & \text{otherwise.} \end{cases}$$

(H4) For each **vertex** v, the sum of incident angles is 2π.

Define flow network $N(G) = ((V(G) \cup F, E'); b; \ell; u; \text{cost})$:

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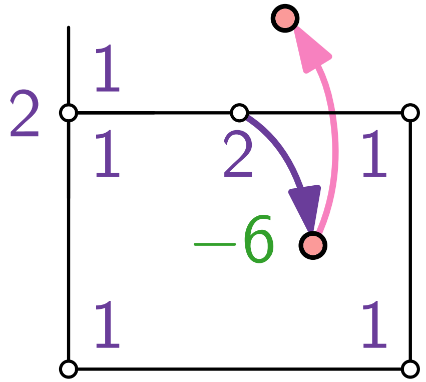

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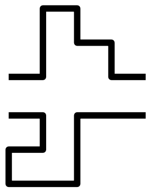
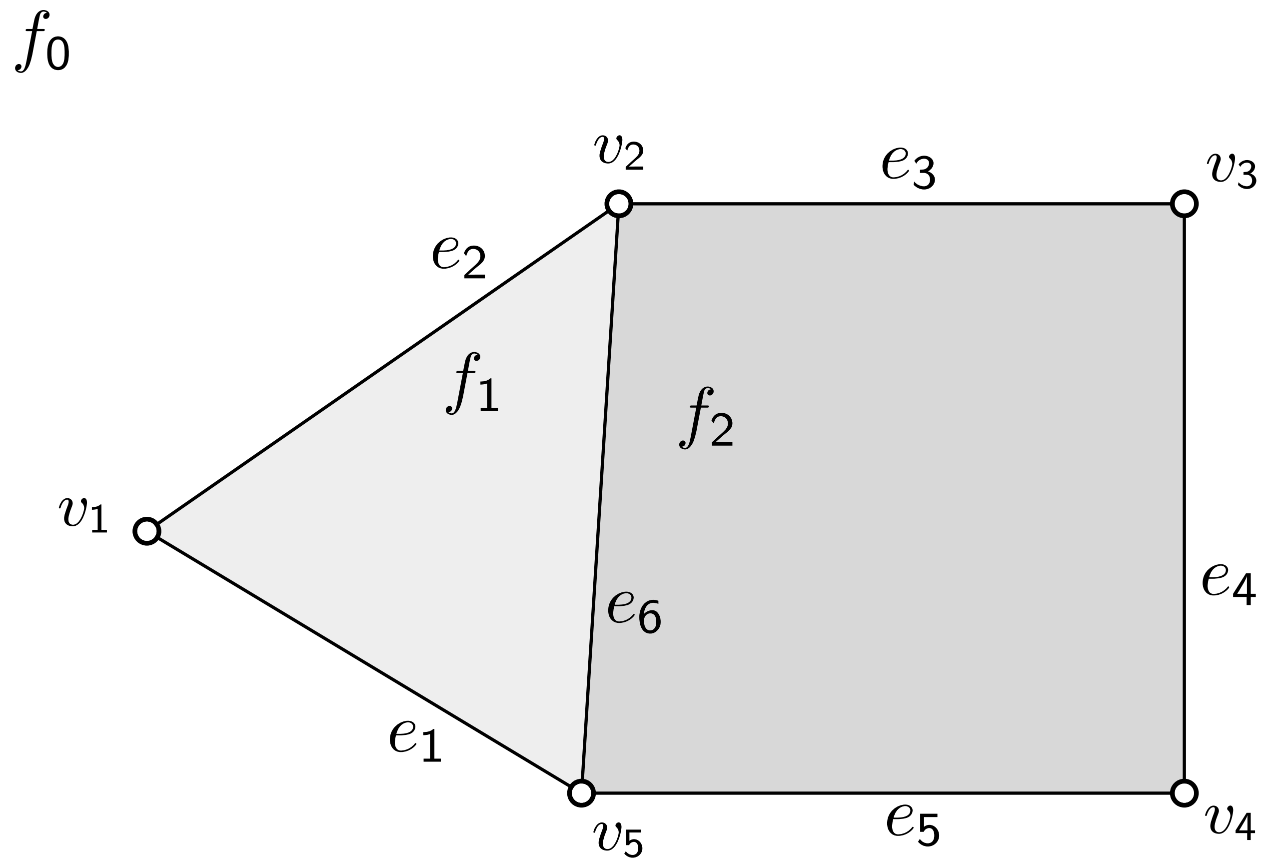


We model only the *number* of bends. Why is it enough?

→ *Exercise!*

Flow Network Example



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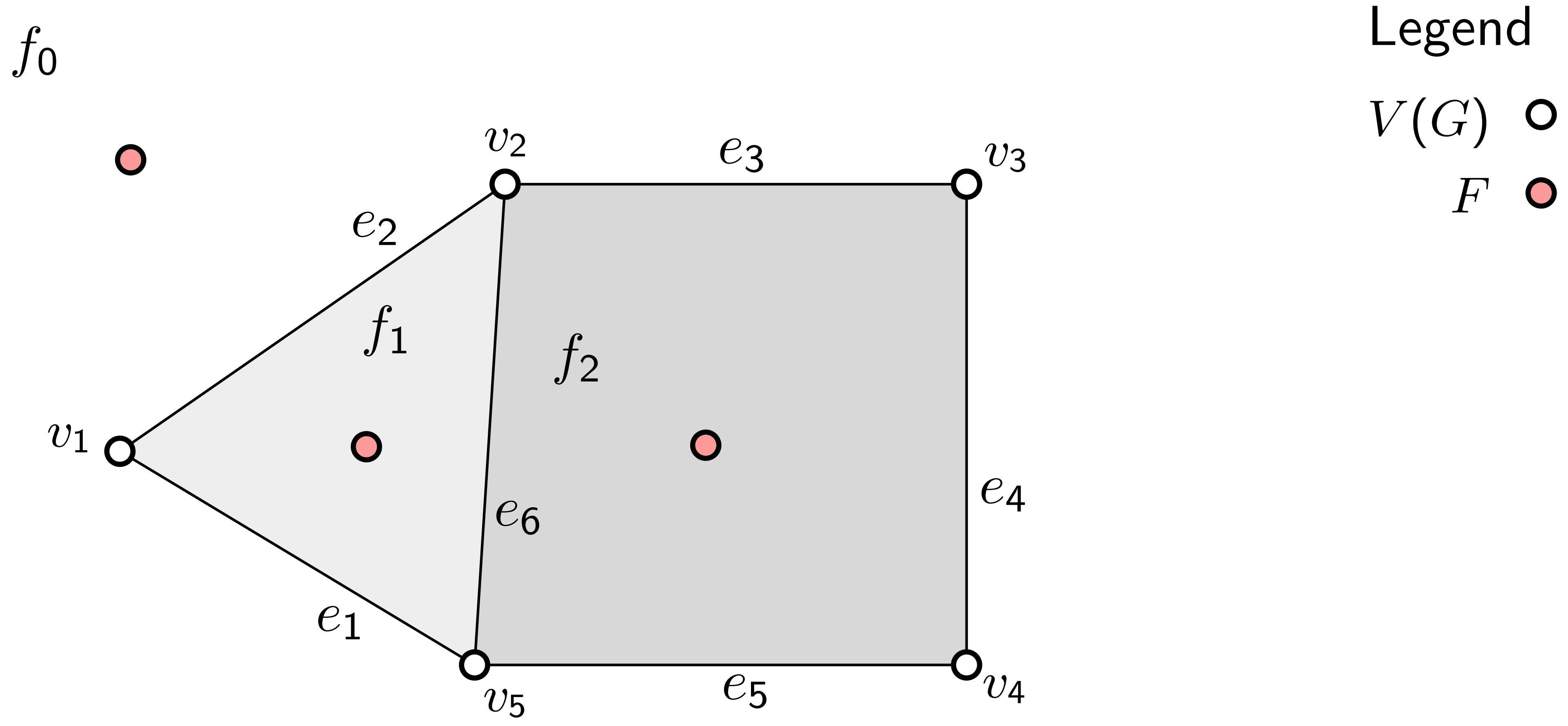

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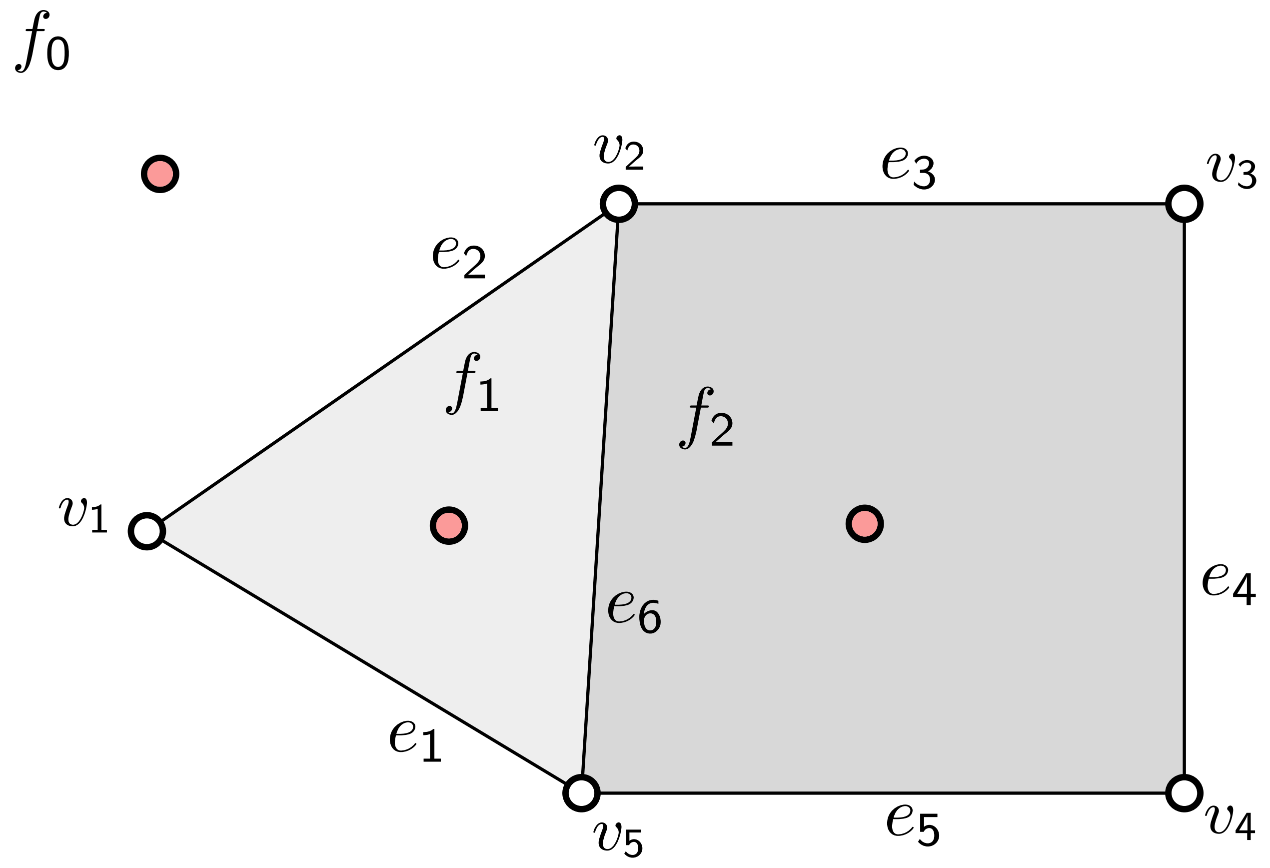

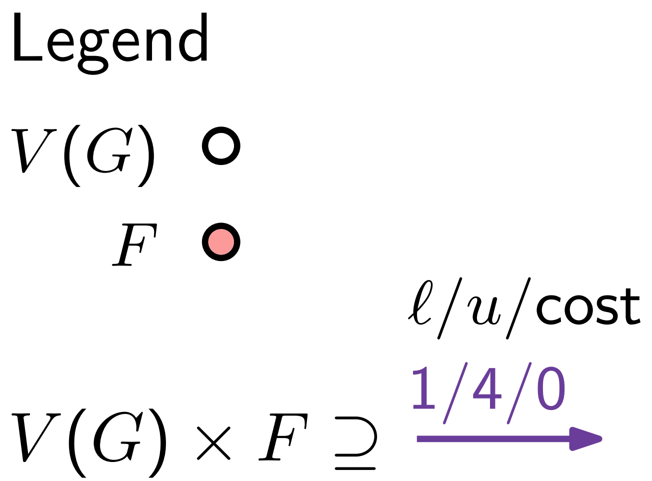

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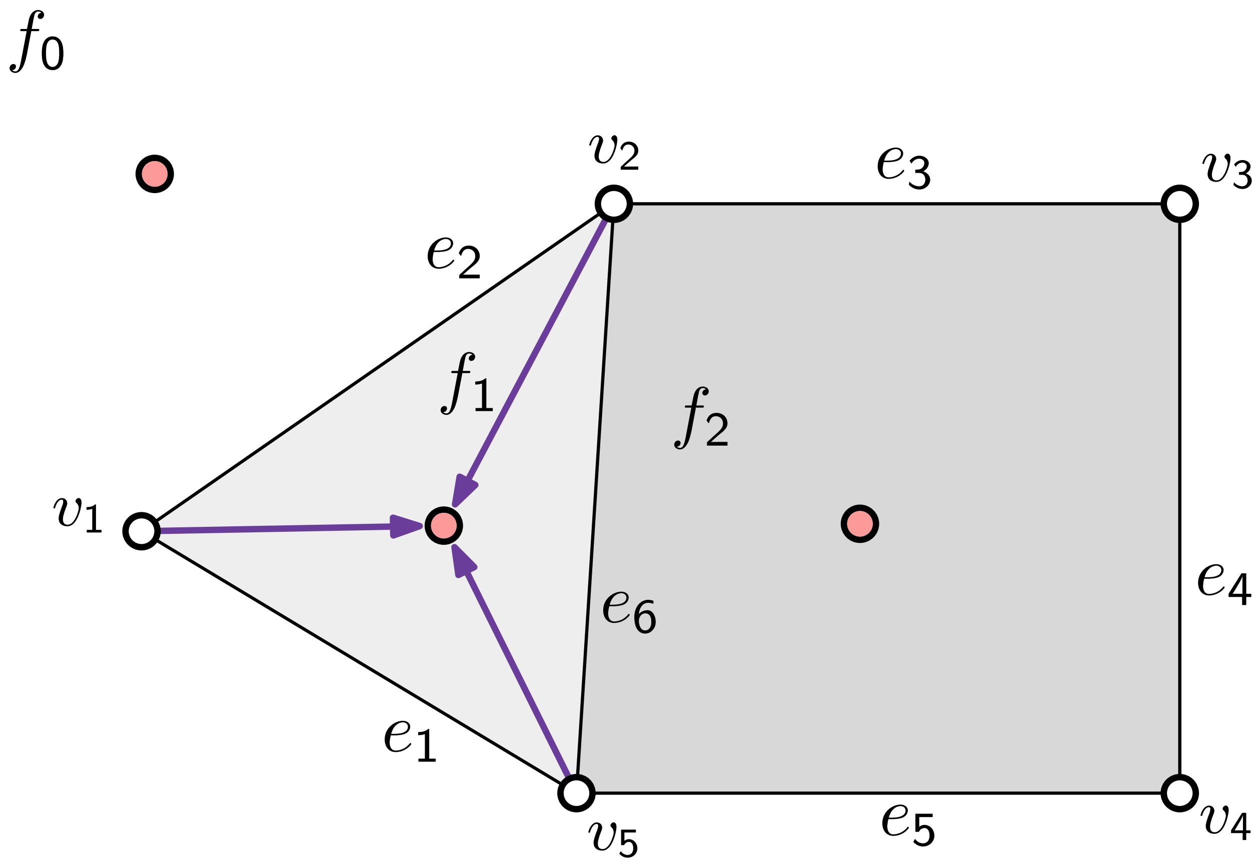


Legend

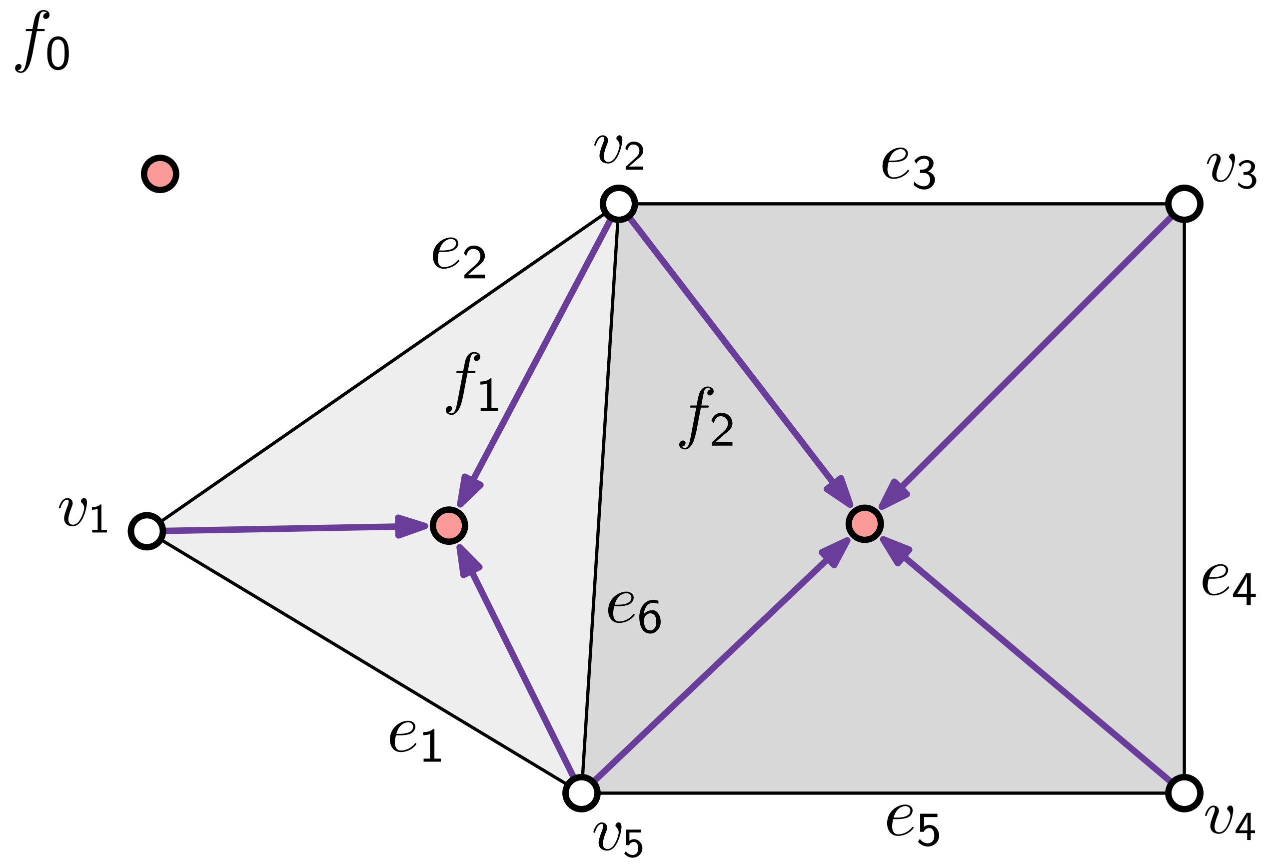
$V(G)$

F

$\ell/u/$cost

$V(G) \times F \supseteq$ 1/4/0

Flow Network Example



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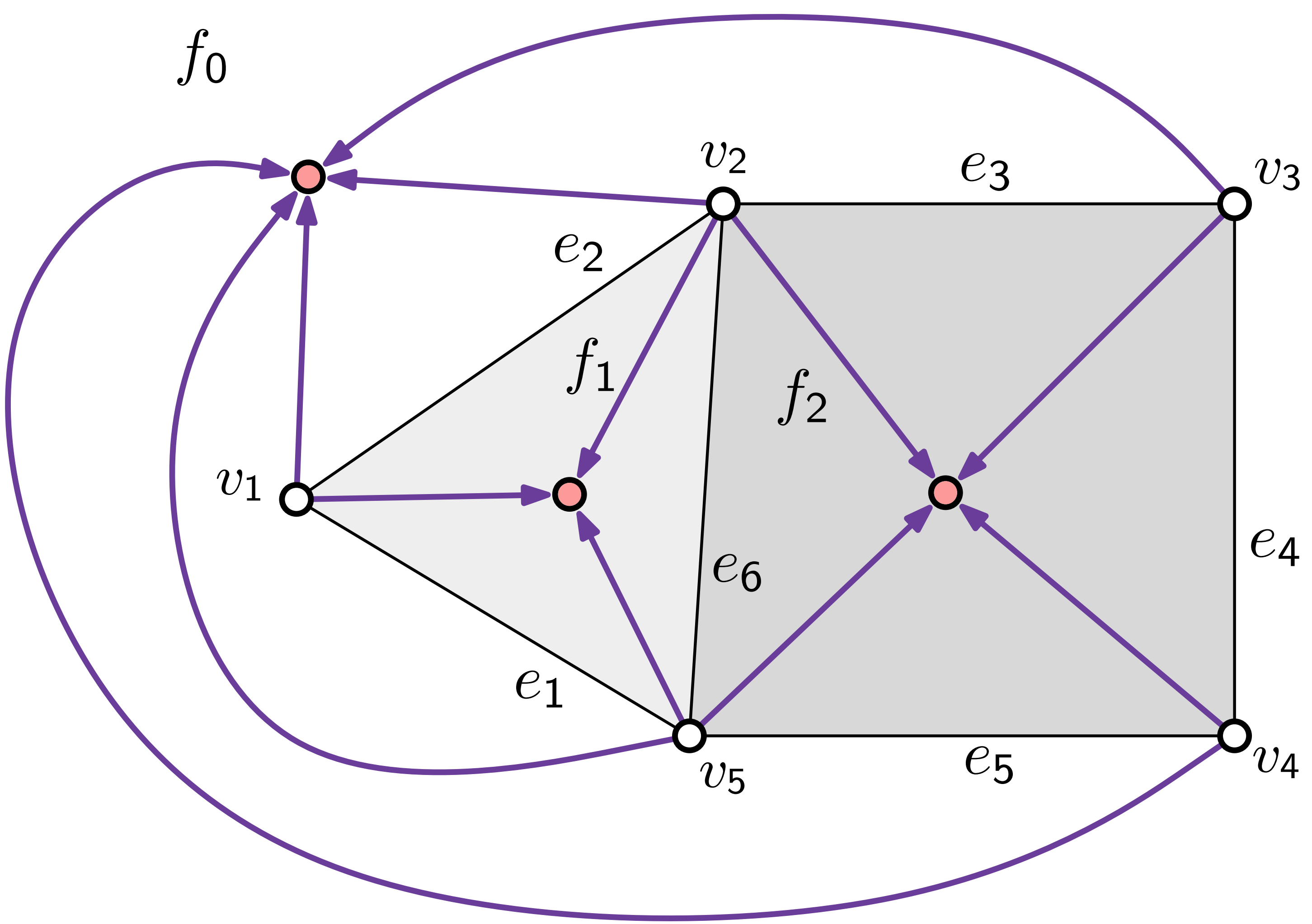

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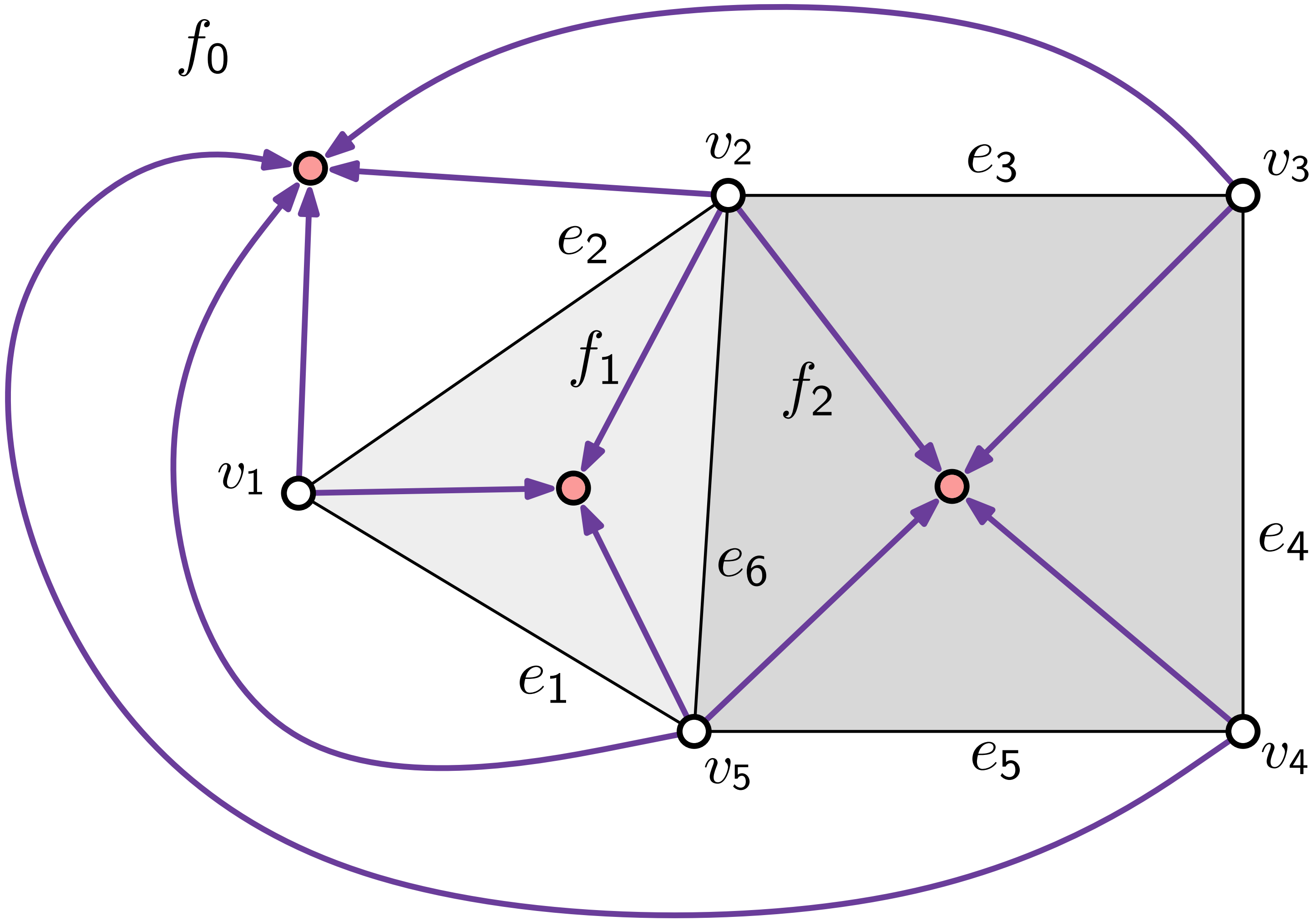
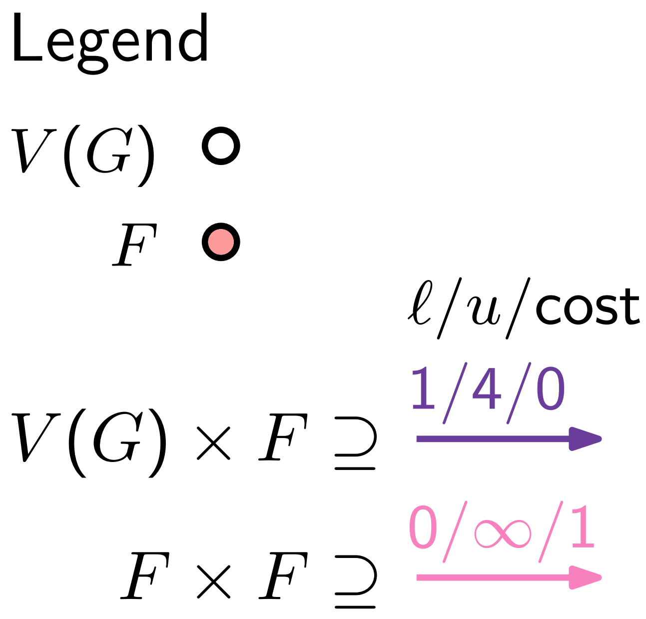
Flow Network Example





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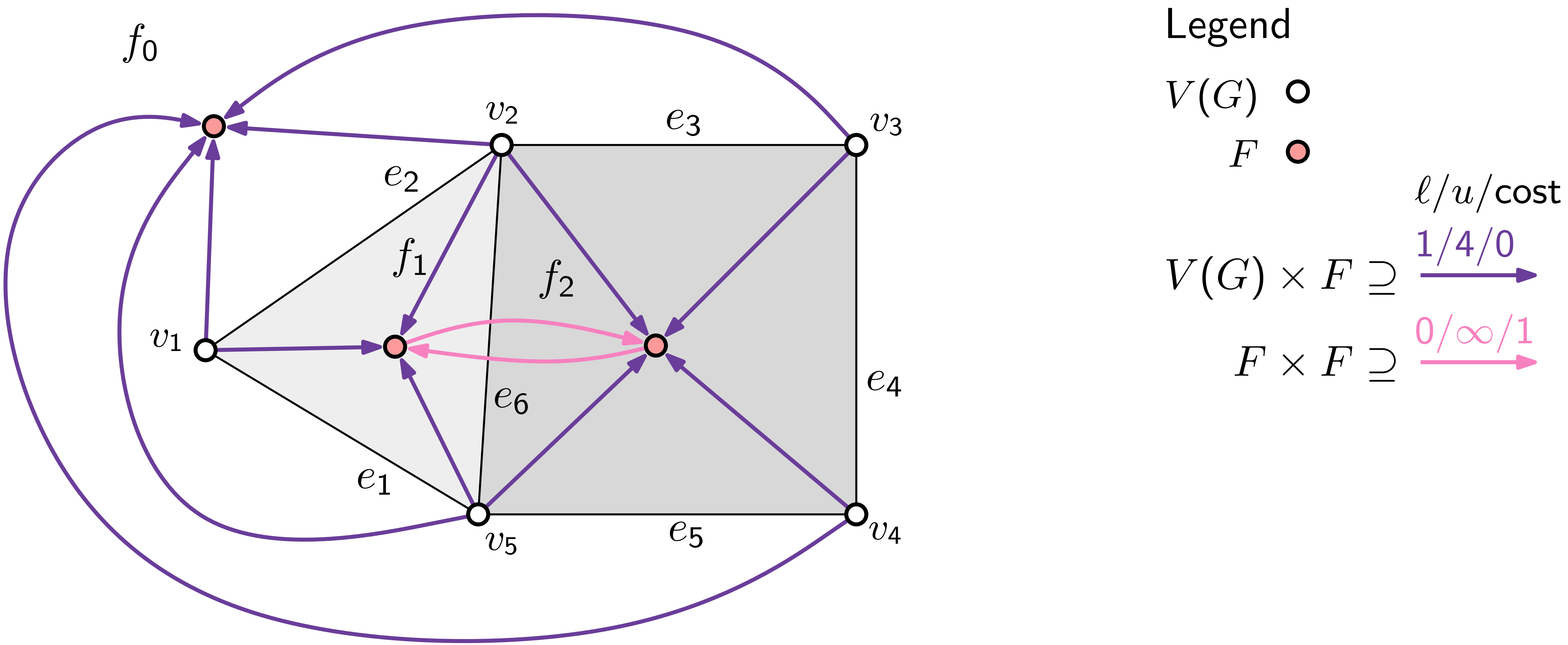

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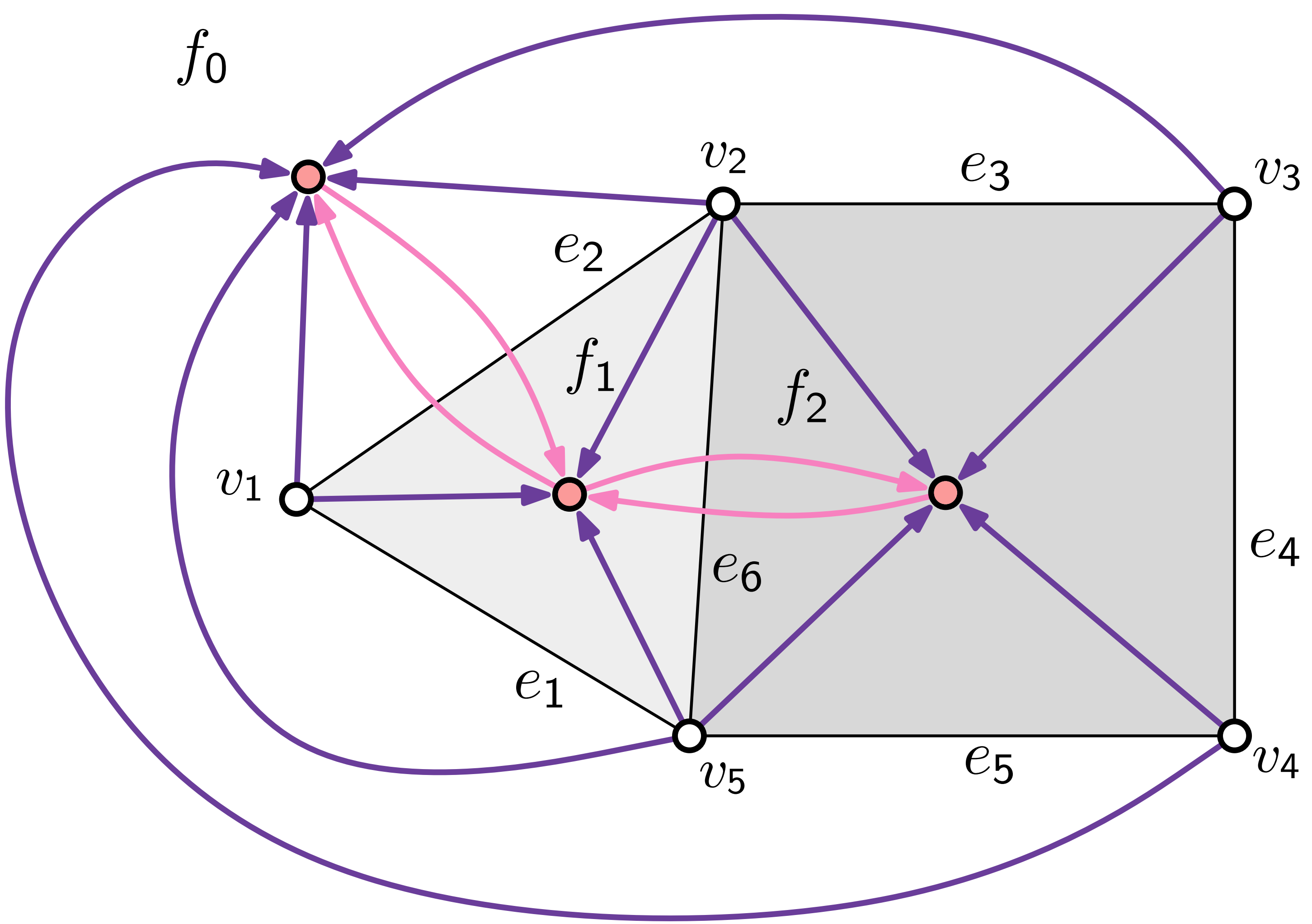

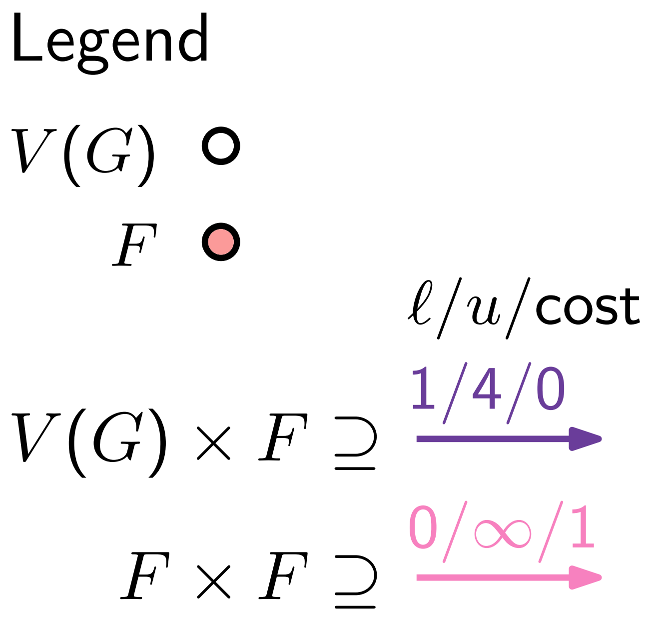

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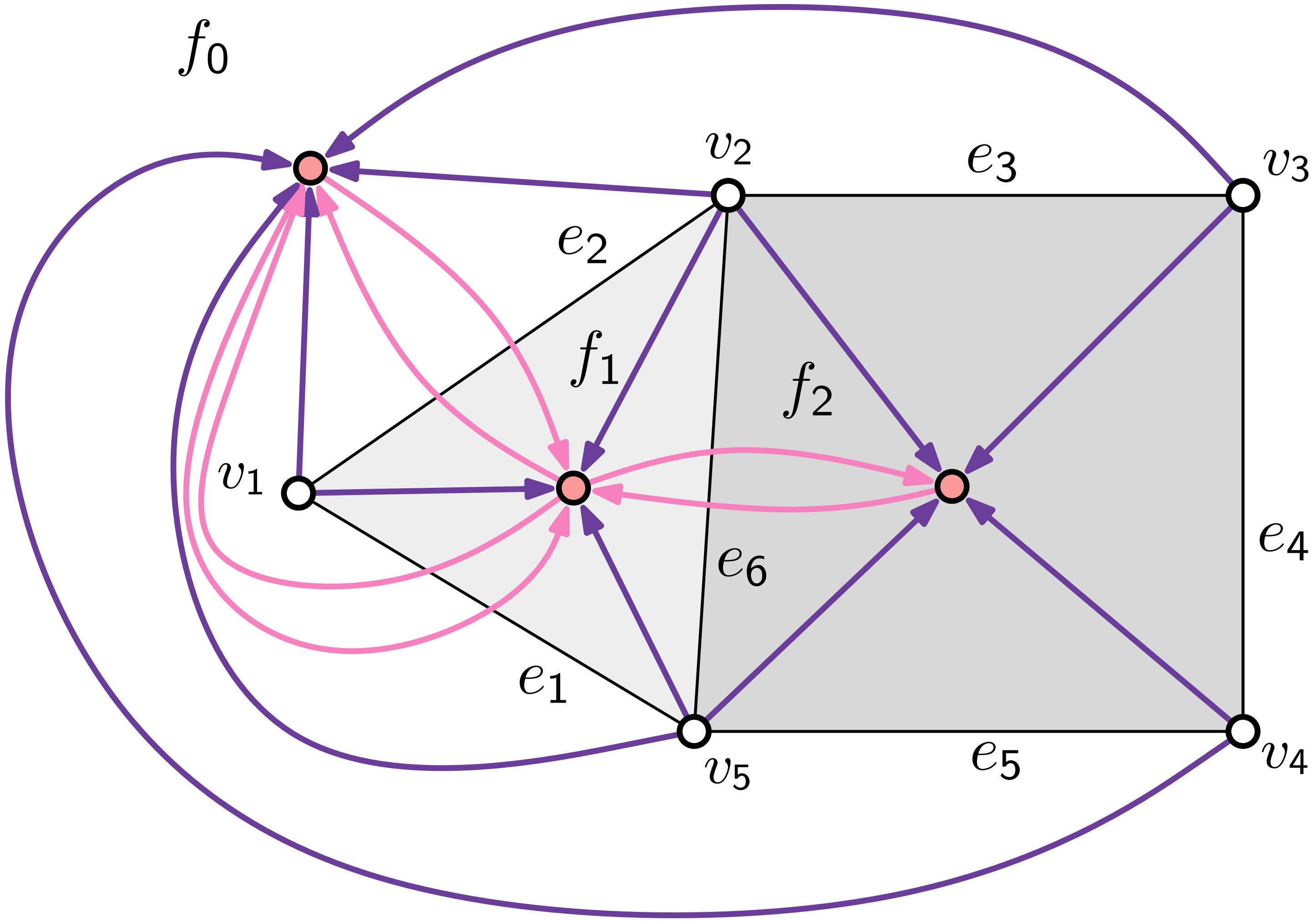

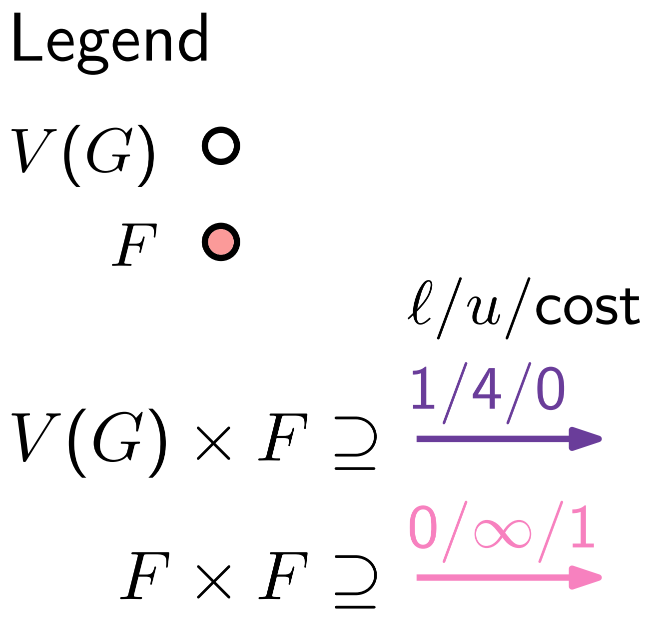

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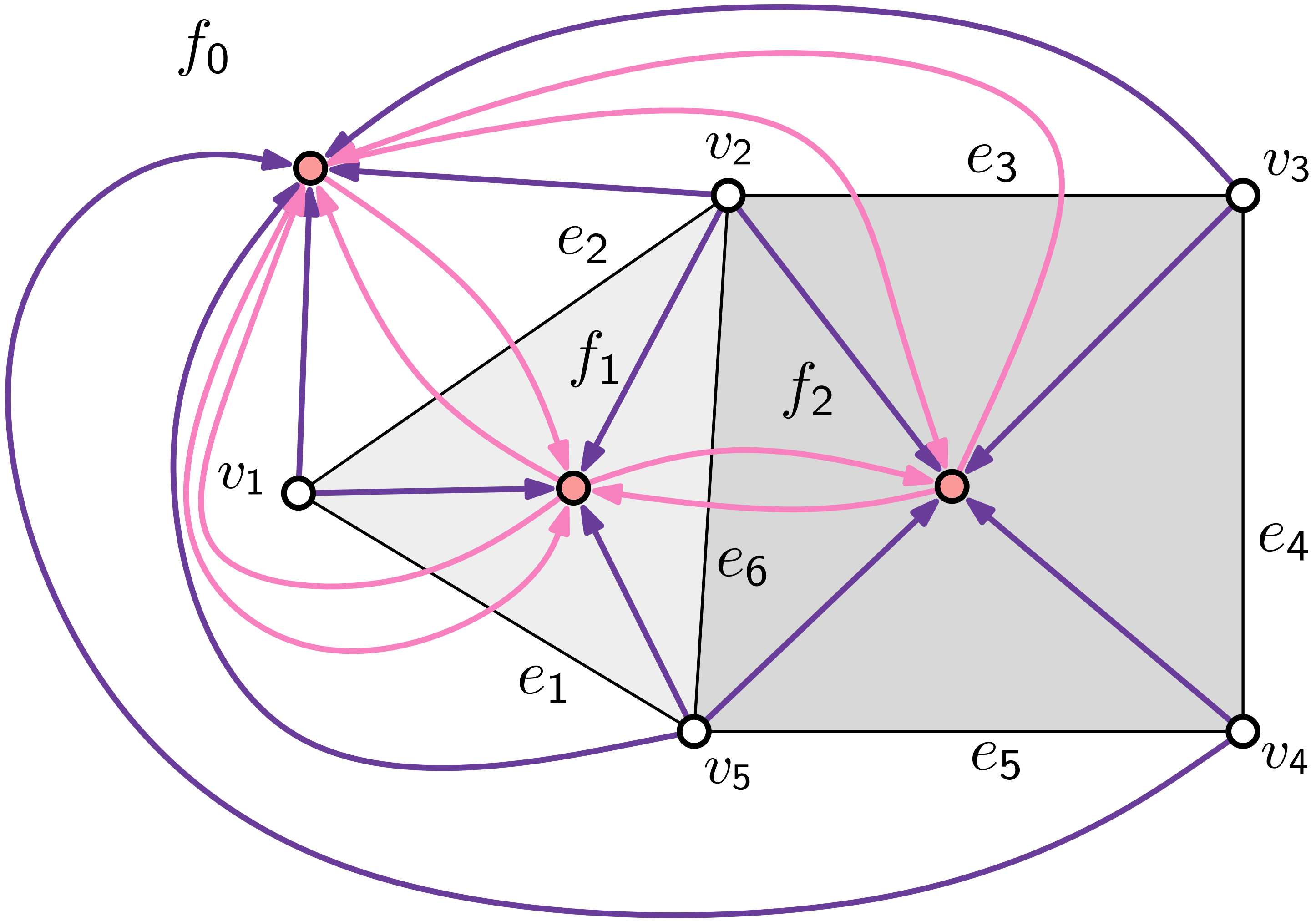


Legend

$V(G)$ ○

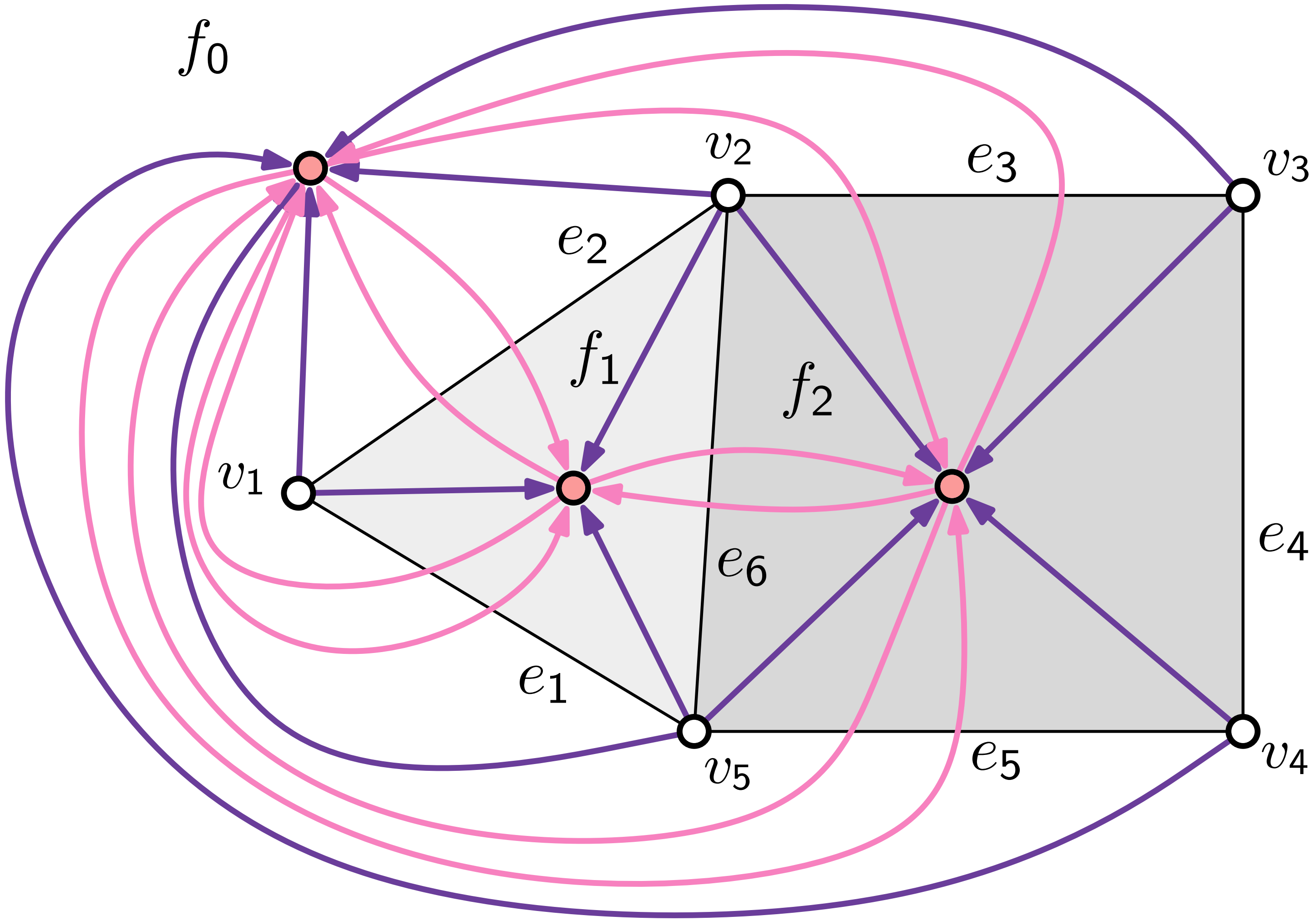
F ●

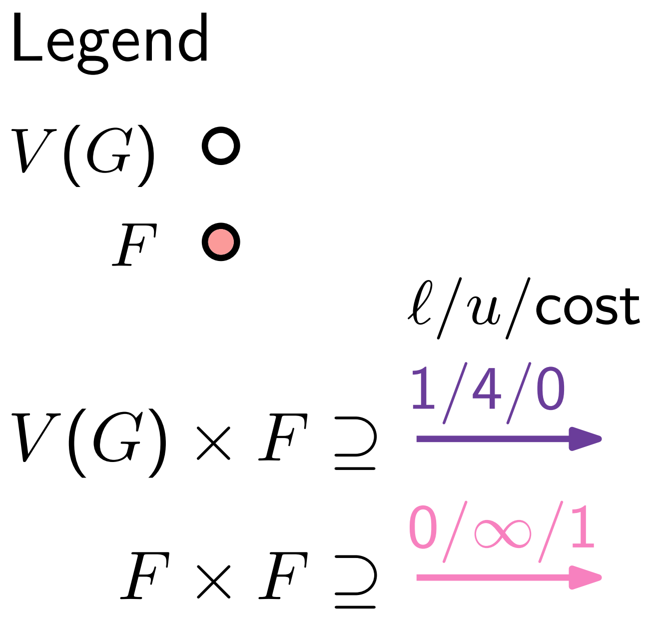
$\ell/u/$cost

$V(G) \times F \supseteq$ 1/4/0 →

$F \times F \supseteq$ 0/∞/1 →

Flow Network Example





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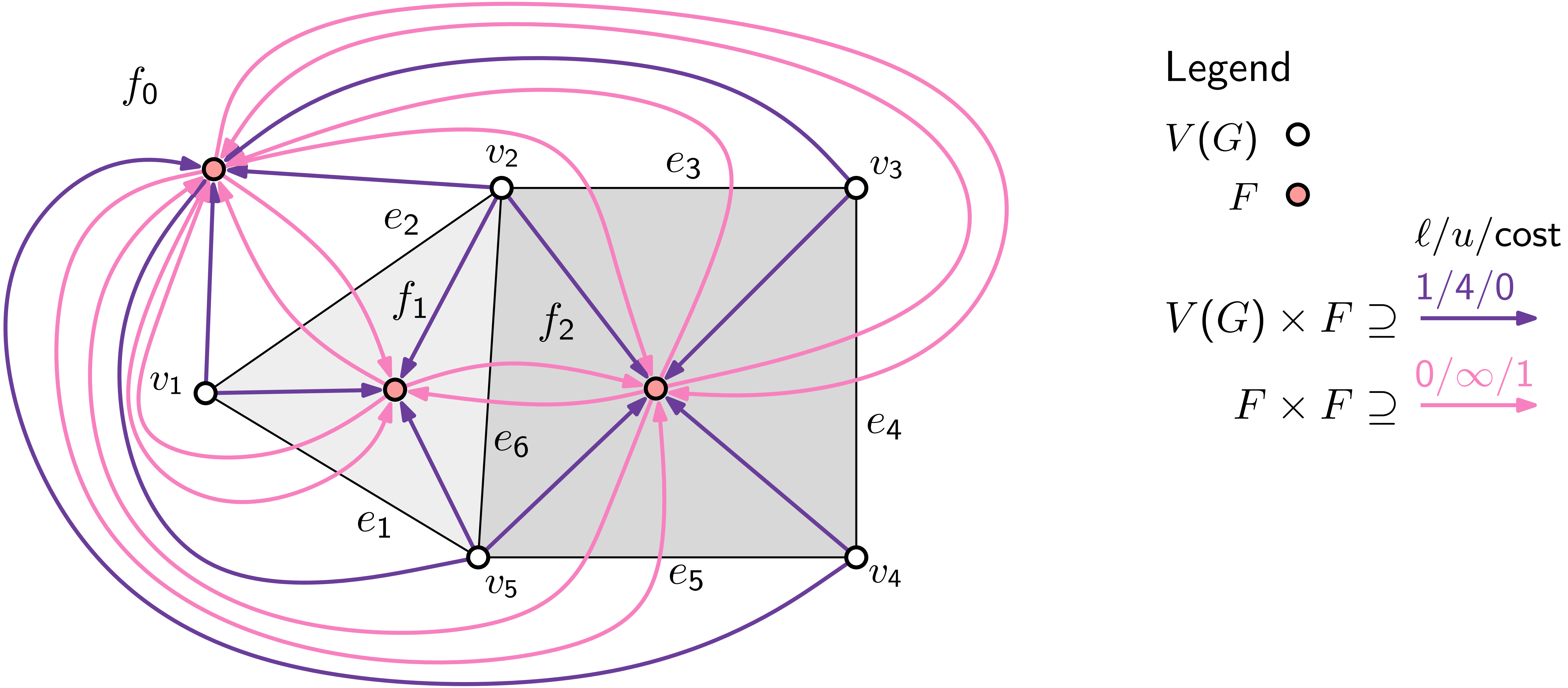

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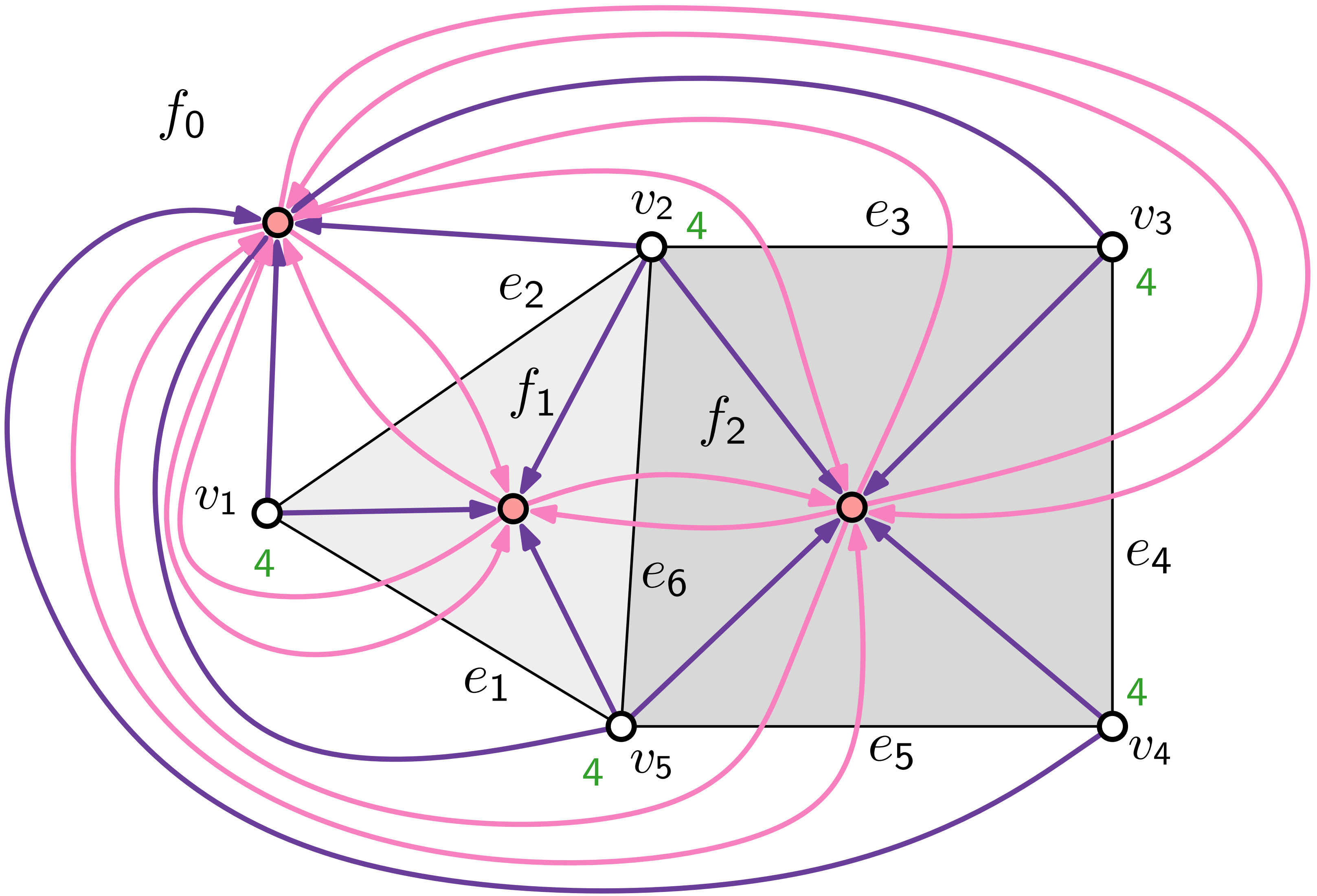

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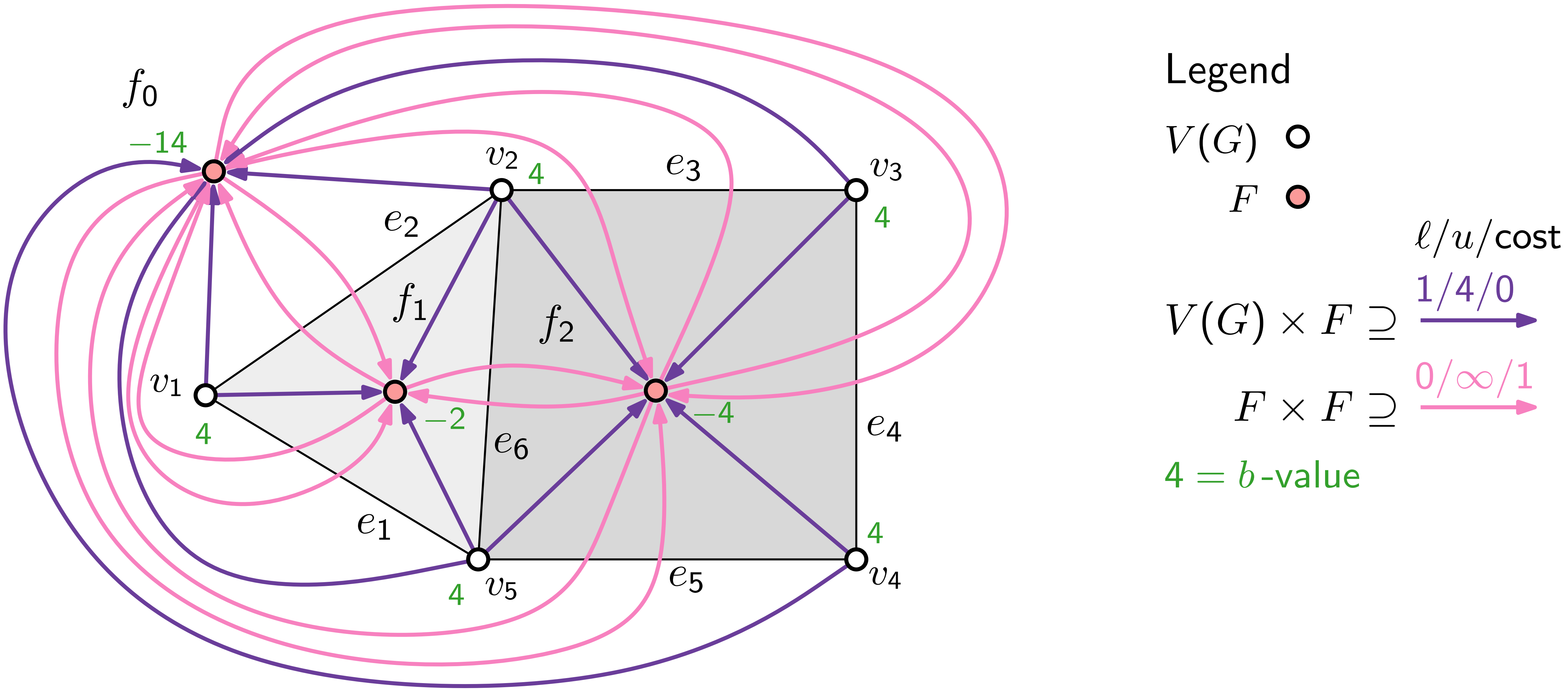
$4 = b$-value

Flow Network Example



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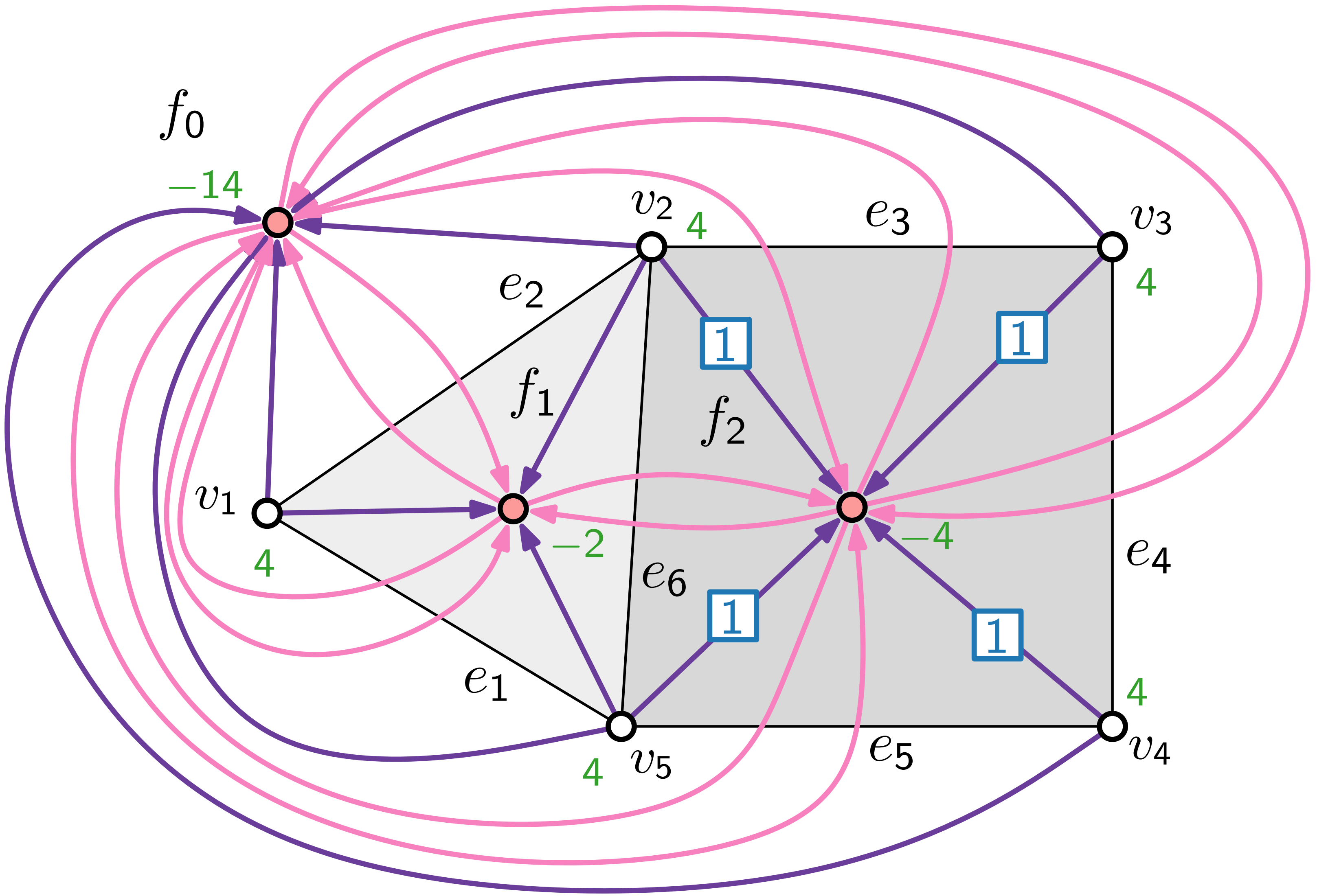

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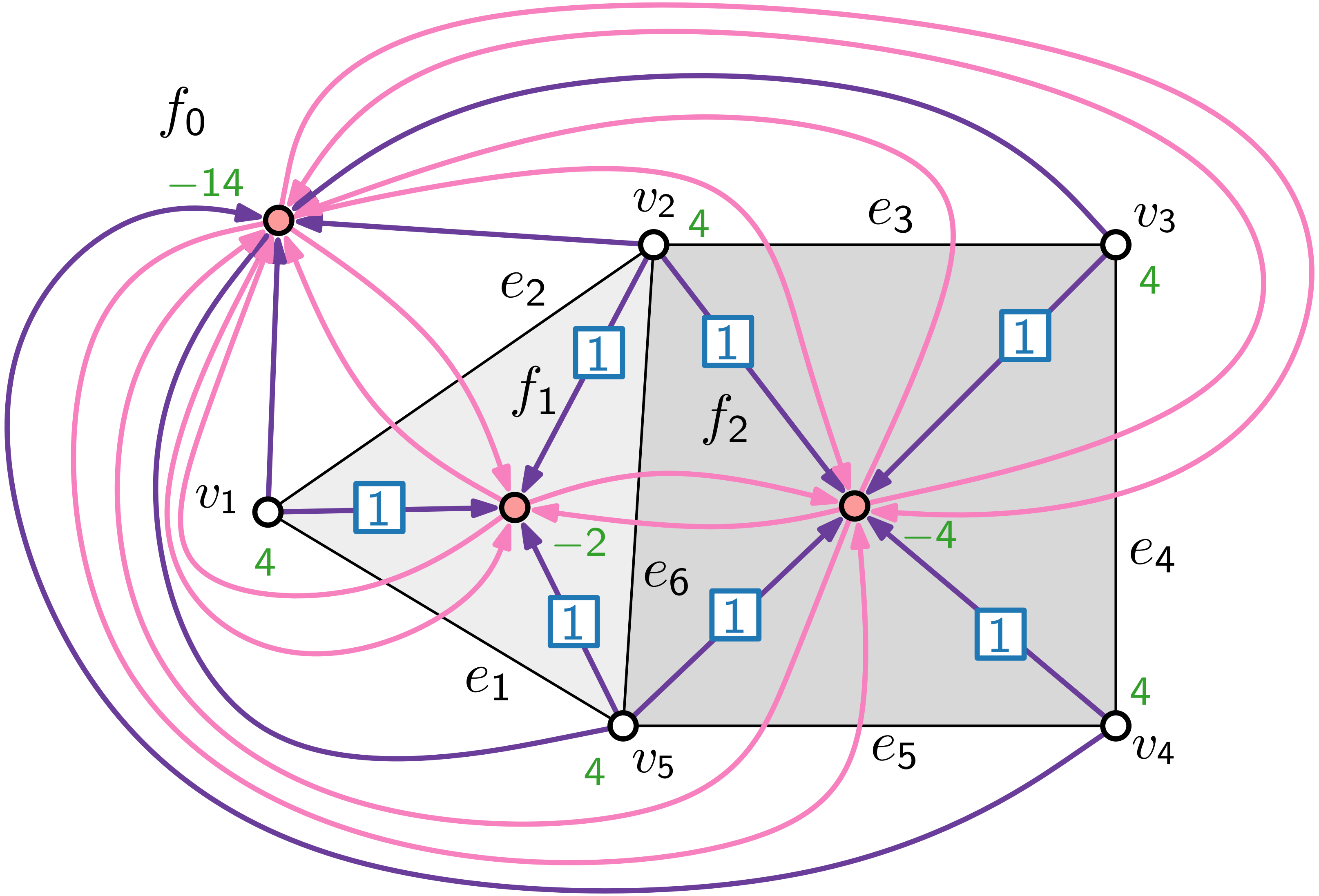
3 flow

Flow Network Example



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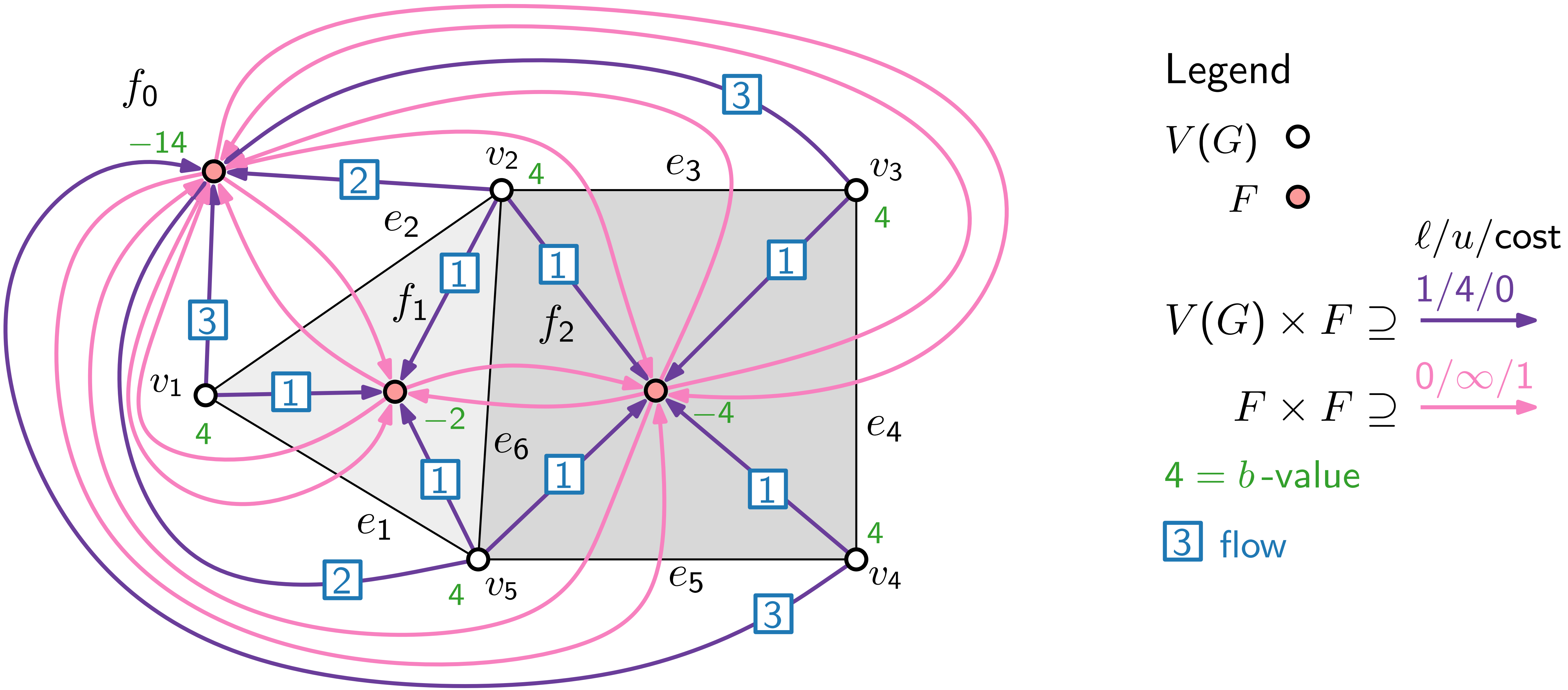

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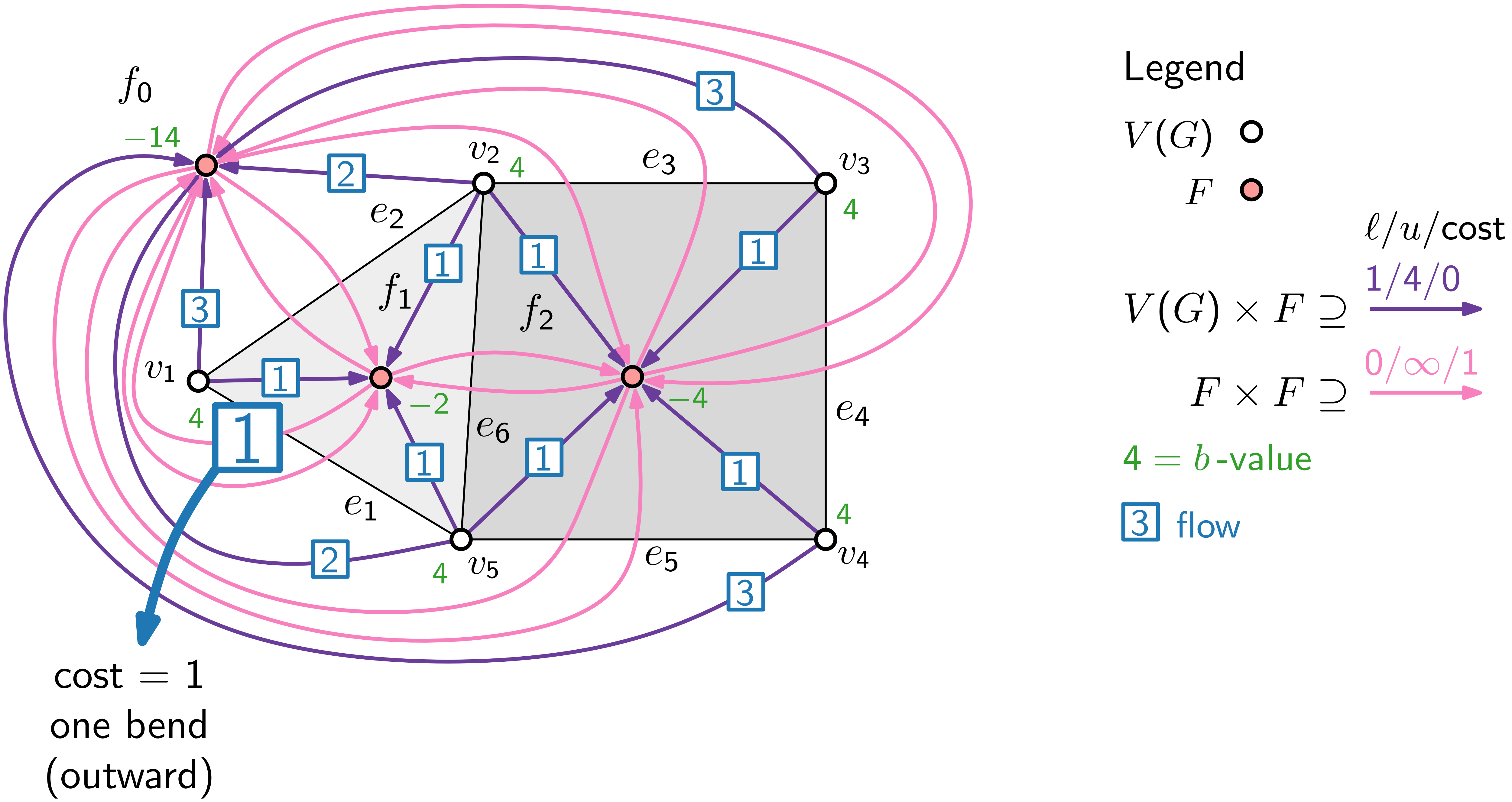

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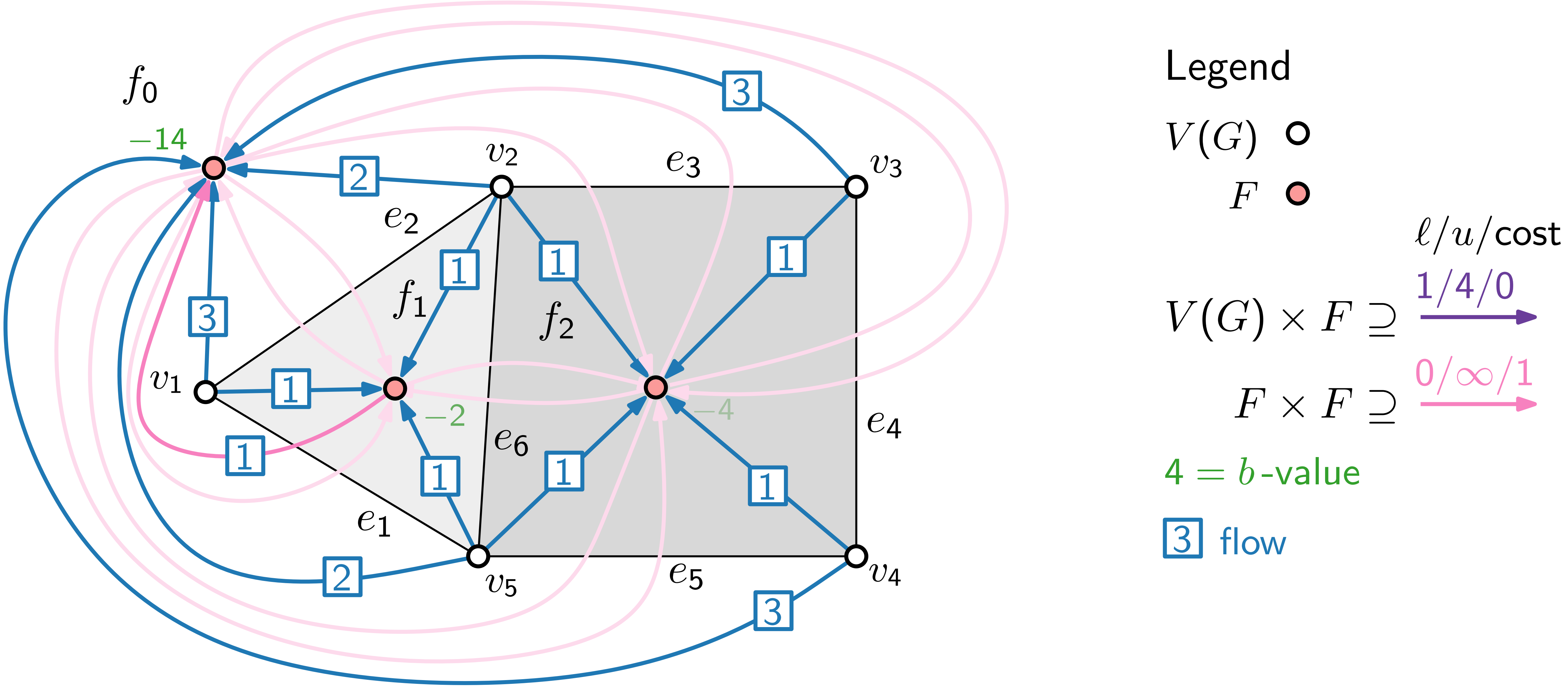

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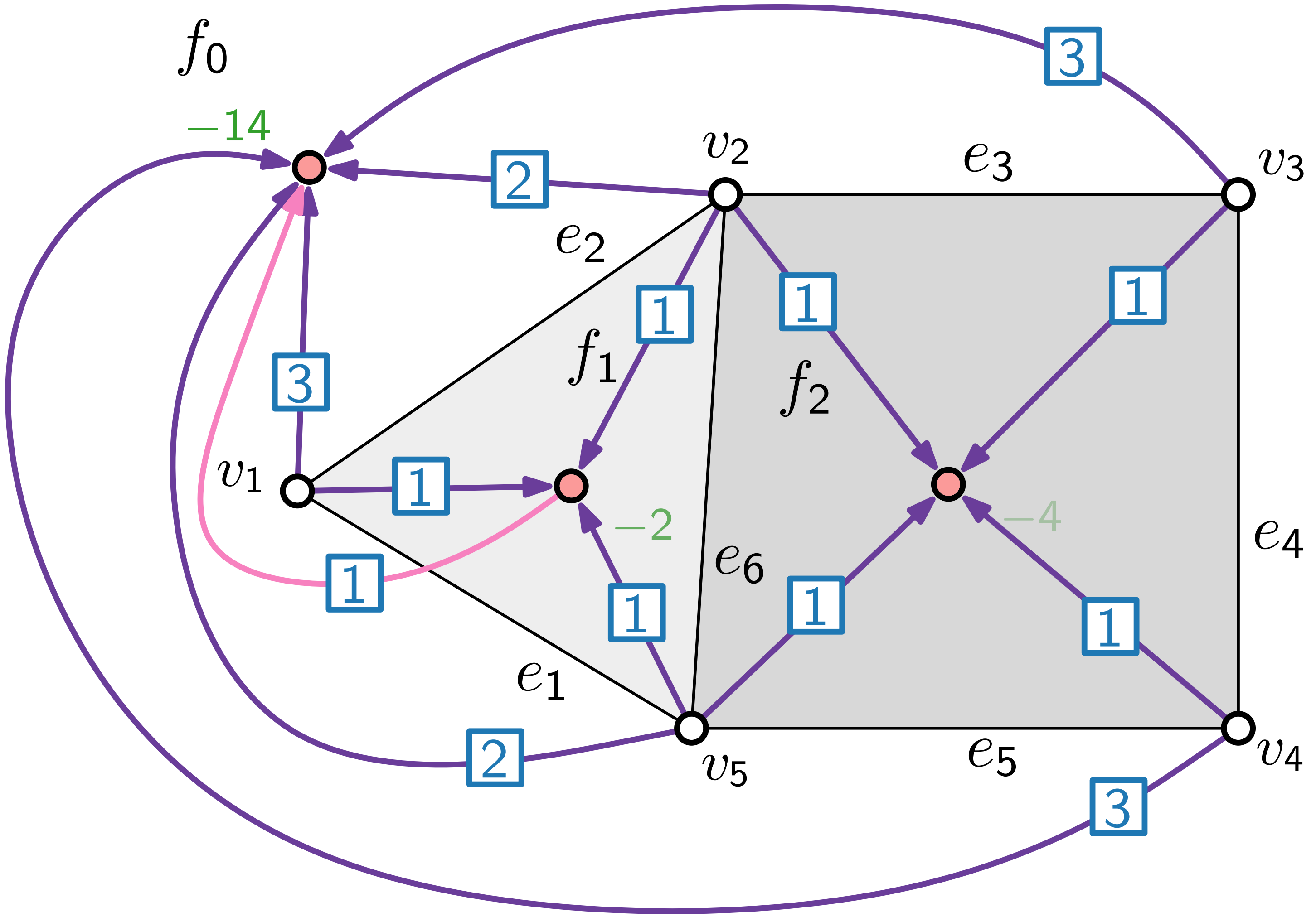

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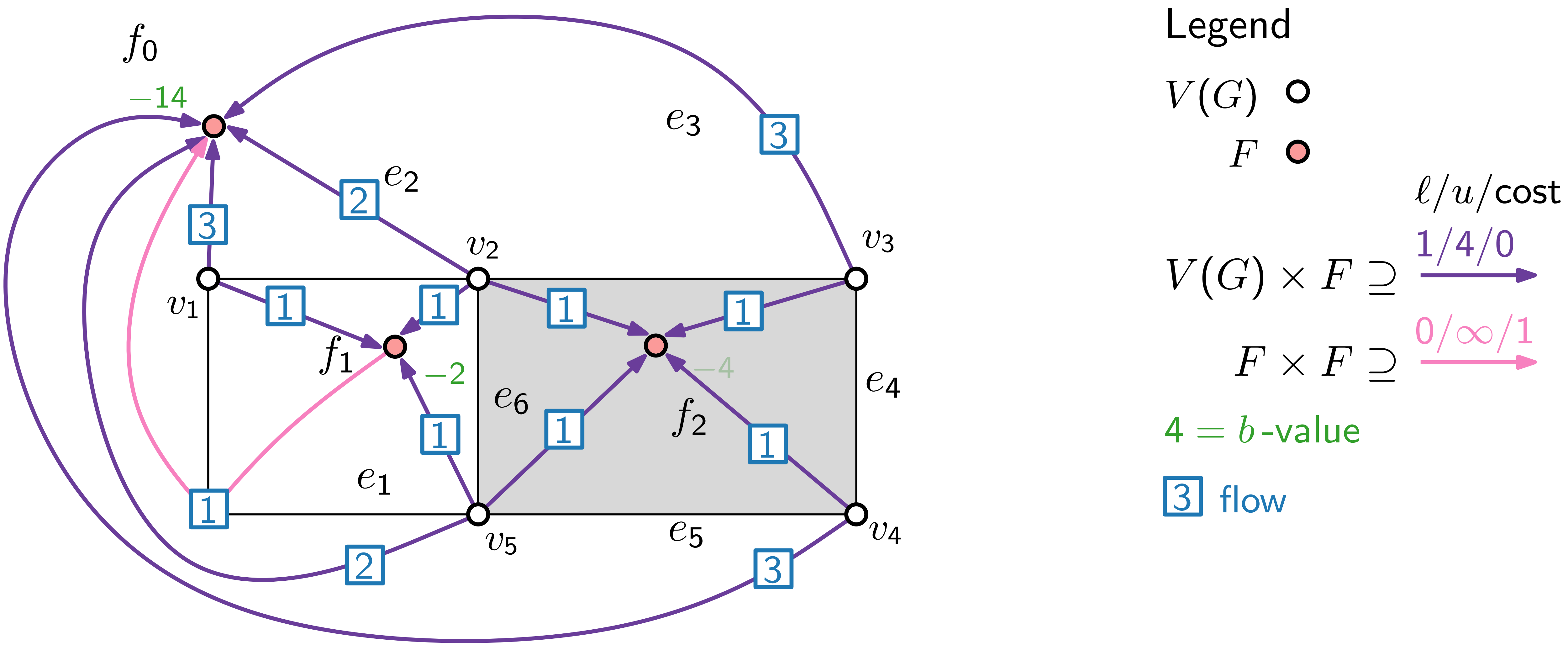

Bend Minimization – Result

Theorem. [Tamassia '87]
A plane graph (G, F, f_0) has a valid orthogonal representation $H(G)$ with k bends. $\Leftrightarrow$

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(H3) Angle sum of $f = \pm 4$ ✓ $\rightarrow$ *Exercise.*

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(N1) $X(vf) = 1/2/3/4$ ✓

(N2) $X((fg)_e) = |\delta|_0$, where (e, δ, x) describes edge e in $H(f)$ ✓

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Proof.

"$\Rightarrow$": Given an orthogonal representation $H(G)$ with k bends, construct a valid flow X in $N(G)$ of cost k.

- Define flow $X \colon E' \to \mathbb{R}_0^+$.
- Show that X is a valid flow and has cost k.

(N1) $X(vf) = 1/2/3/4$ ✓

(N2) $X((fg)_e) = |\delta|_0$, where (e, δ, x) describes edge e in $H(f)$ ✓

(N3) capacities, deficit/demand coverage ✓

Bend Minimization – Result

Theorem. [Tamassia '87]
A plane graph (G, F, f_0) has a valid orthogonal representation $H(G)$ with k bends. $\Leftrightarrow$
The flow network $N(G)$ has a valid flow X with cost k.

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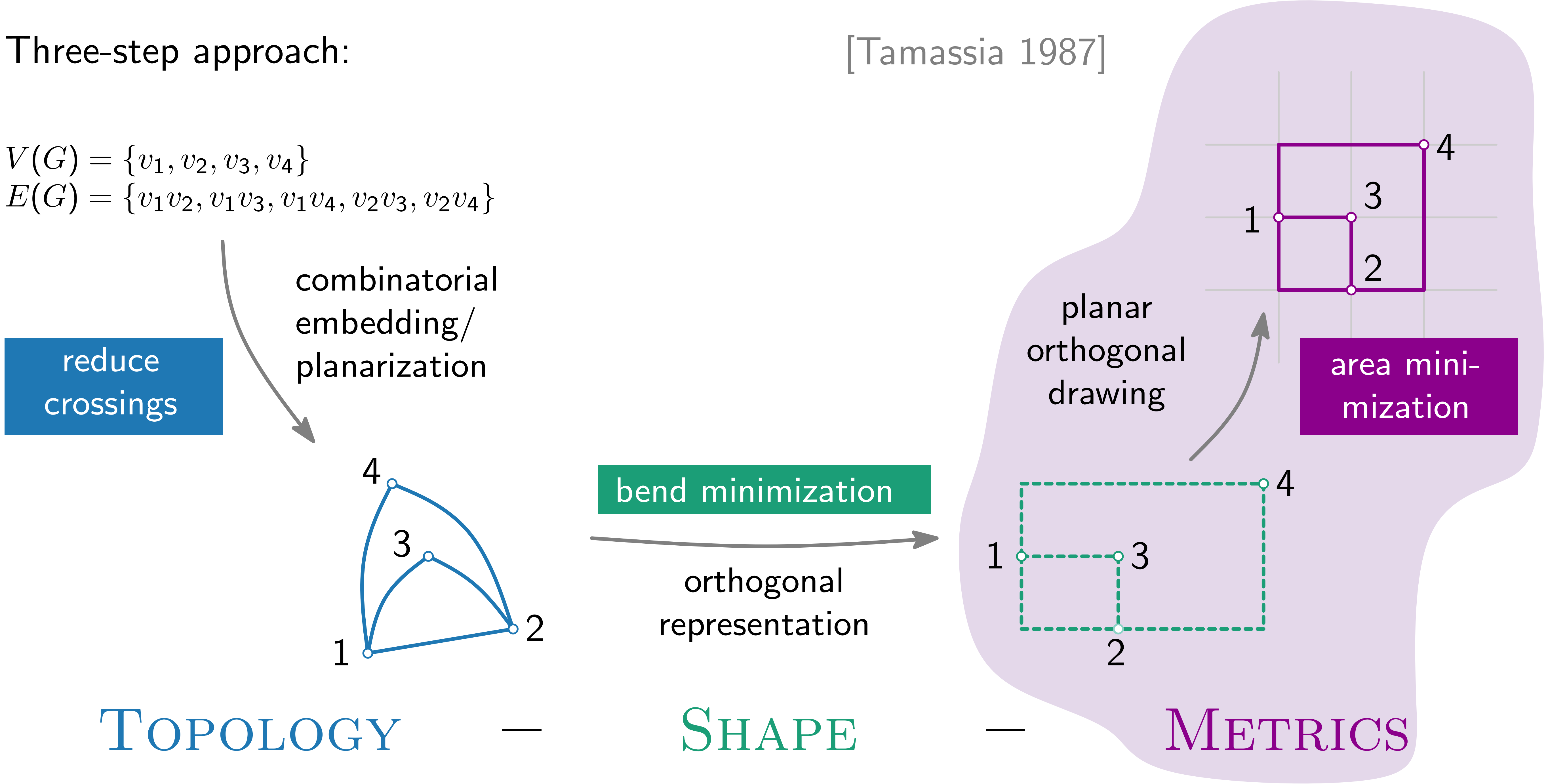
Theorem. [Garg & Tamassia 2001]
Bend minimization without given combinatorial embedding is NP-hard.

Topology – Shape – Metrics

Three-step approach: [Tamassia 1987]

$V(G) = \{v_1, v_2, v_3, v_4\}$
$E(G) = \{v_1v_2, v_1v_3, v_1v_4, v_2v_3, v_2v_4\}$



Compaction

Compaction problem.

Given:

Find:

Compaction

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Given: ■ Plane graph G with maximum degree 4

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All faces are rectangles.

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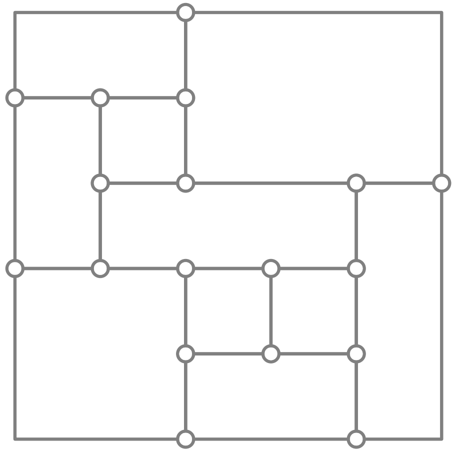
Idea.

- Formulate flow network for horizontal/vertical compaction

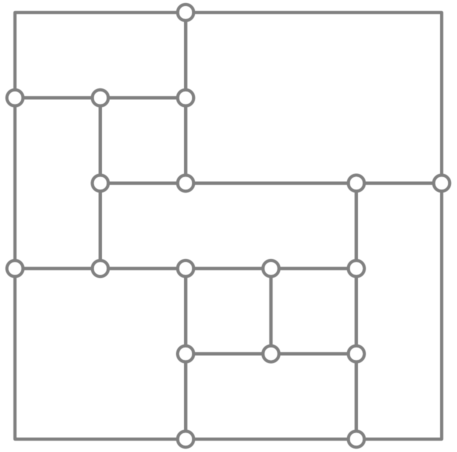
Flow Network for Edge-Length Assignment



Flow Network for Edge-Length Assignment

Definition.

Flow Network $N_{\text{hor}} = ((W_{\text{hor}}, E_{\text{hor}}); b; \ell; u; \text{cost})$



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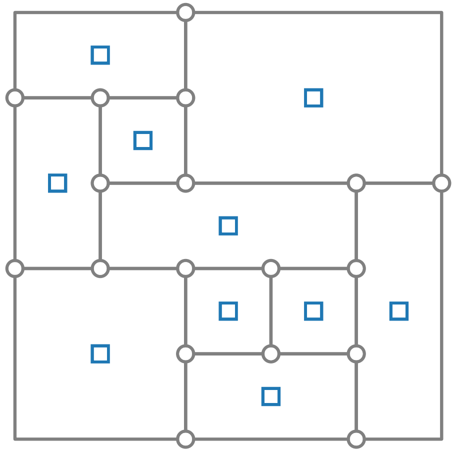

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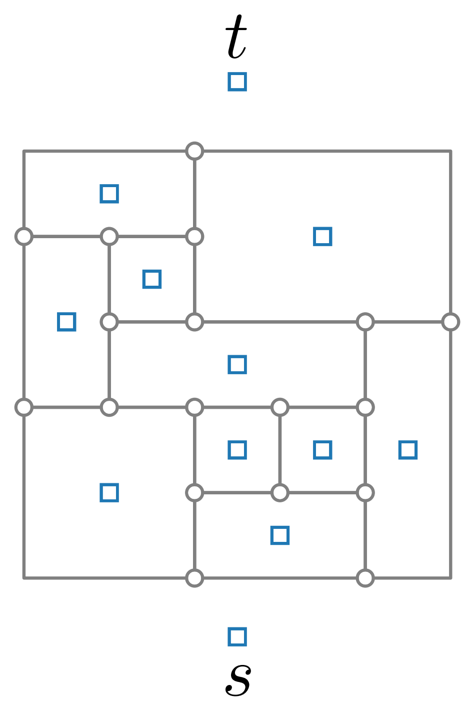

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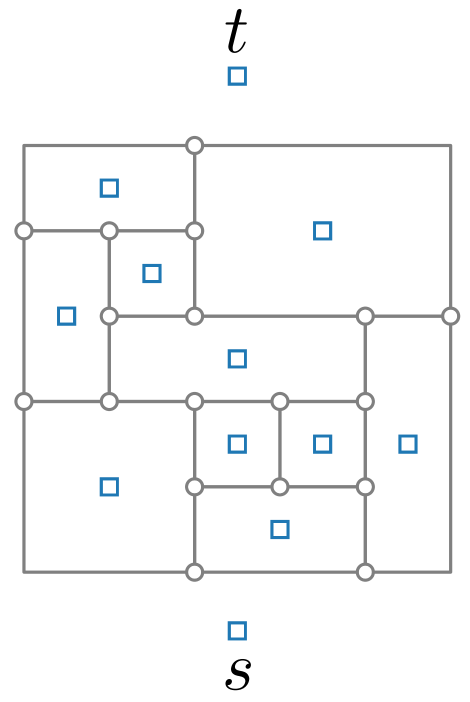

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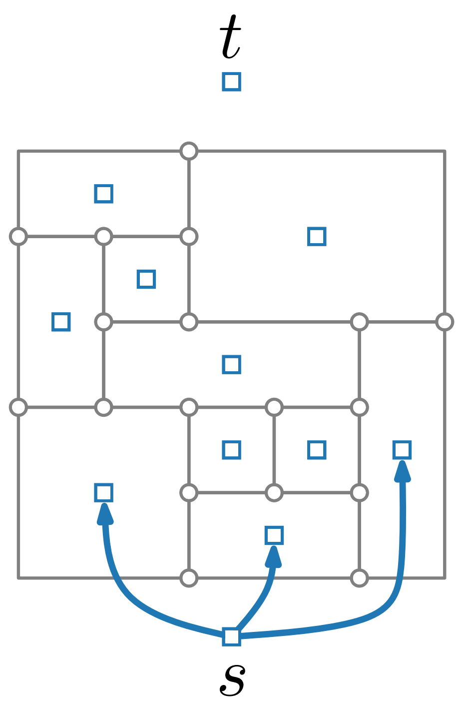

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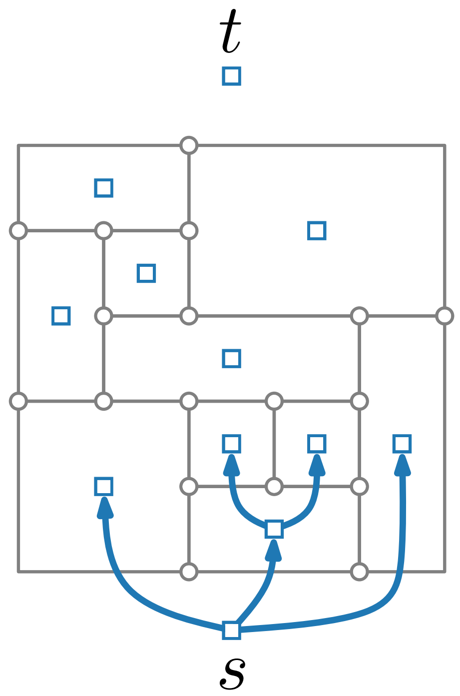

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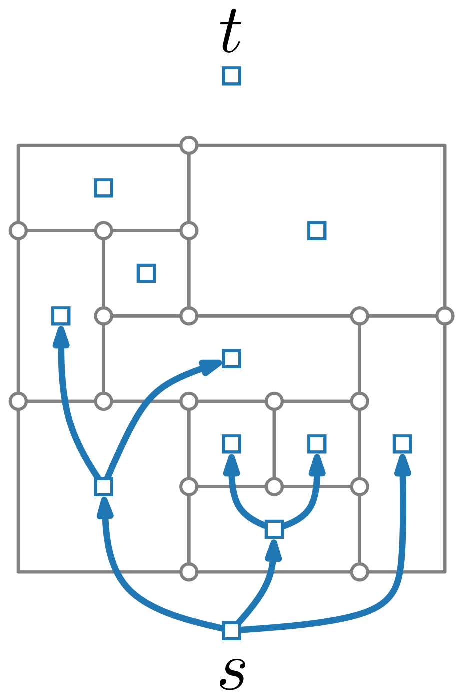

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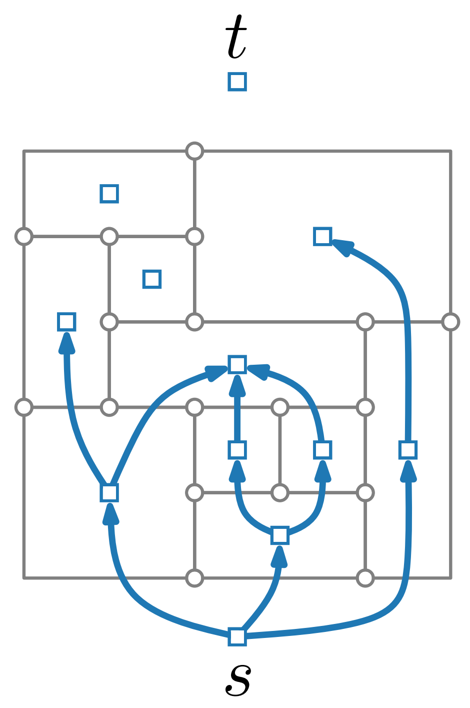

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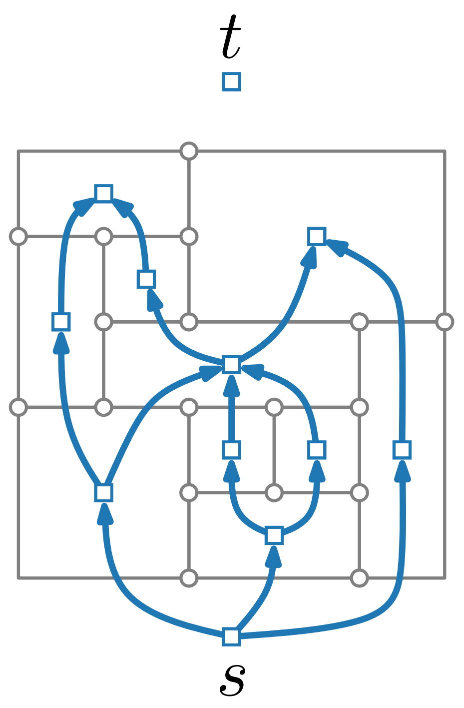

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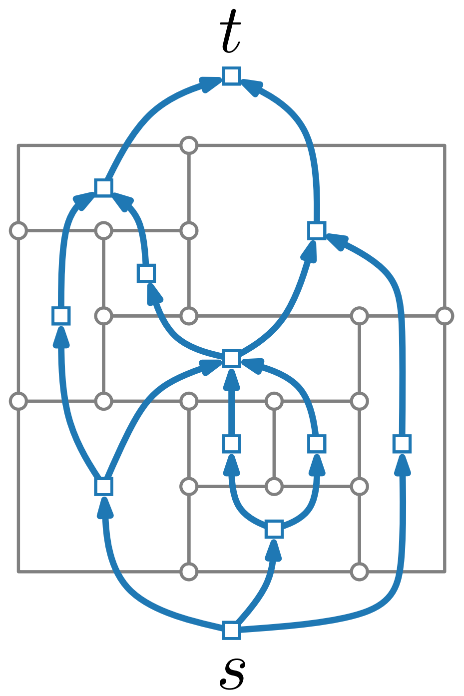

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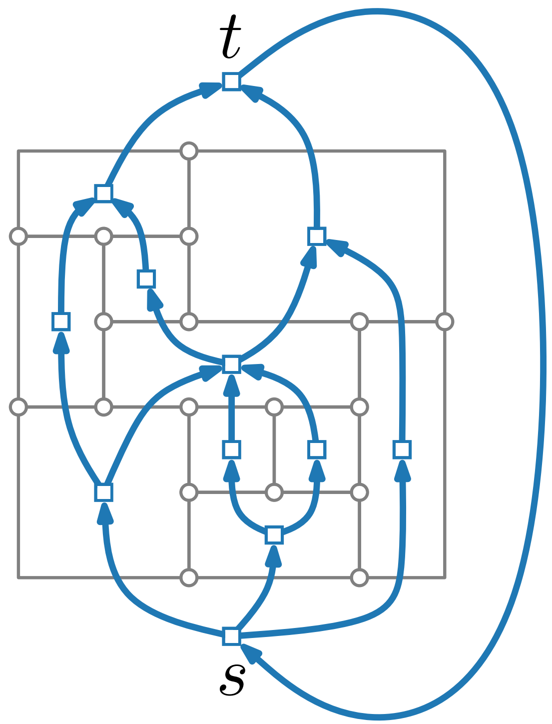

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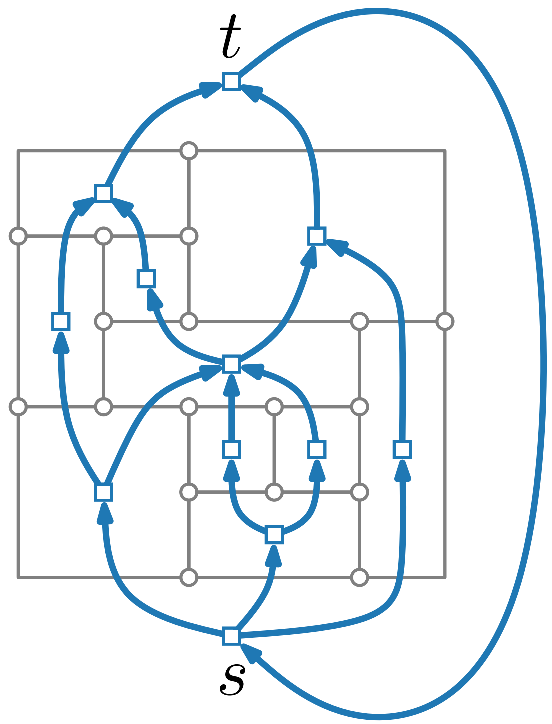

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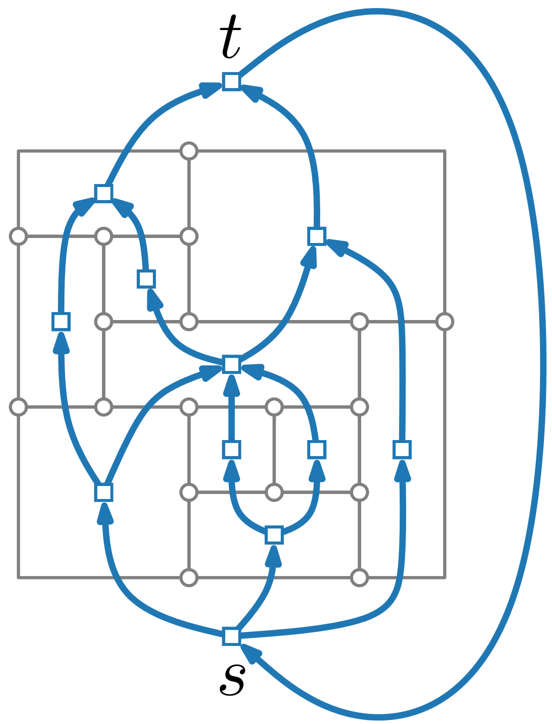

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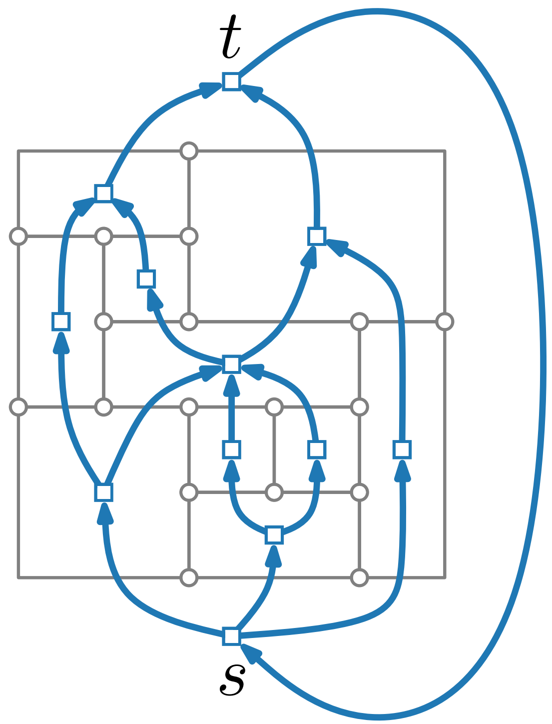

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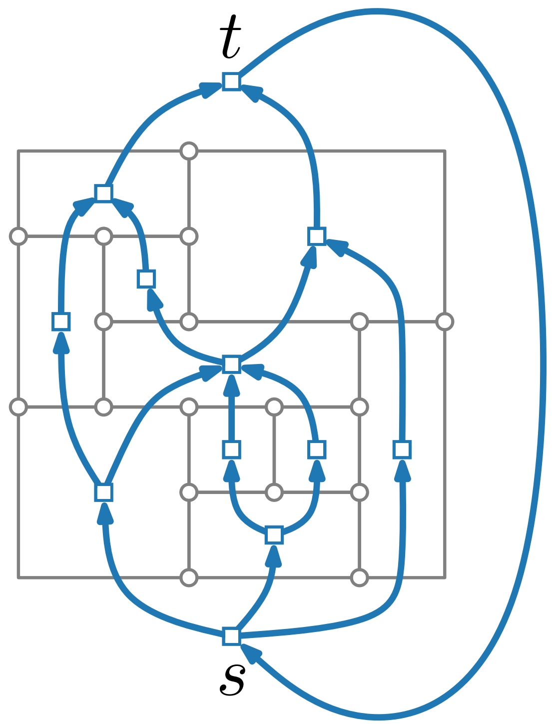

Flow Network for Edge-Length Assignment

Definition.
Flow Network $N_{\text{ver}} = ((W_{\text{ver}}, E_{\text{ver}}); b; \ell; u; \text{cost})$

- $W_{\text{ver}} = F \setminus \{f_0\} \cup \{s, t\}$ □
- $E_{\text{ver}} = \{(f, g) \mid f, g$ share a *vertical* segment and f lies to the *left* of $g\} \cup \{(t, s)\}$
- $\ell(a) = 1 \quad \forall a \in E_{\text{ver}}$
- $u(a) = \infty \quad \forall a \in E_{\text{ver}}$
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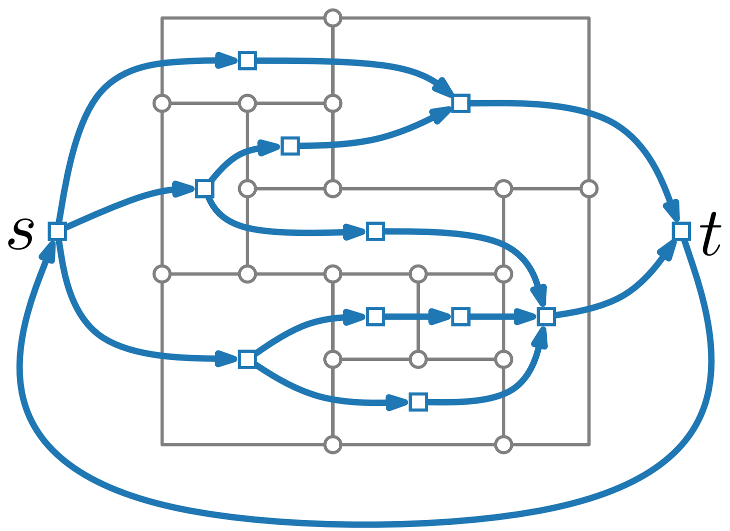
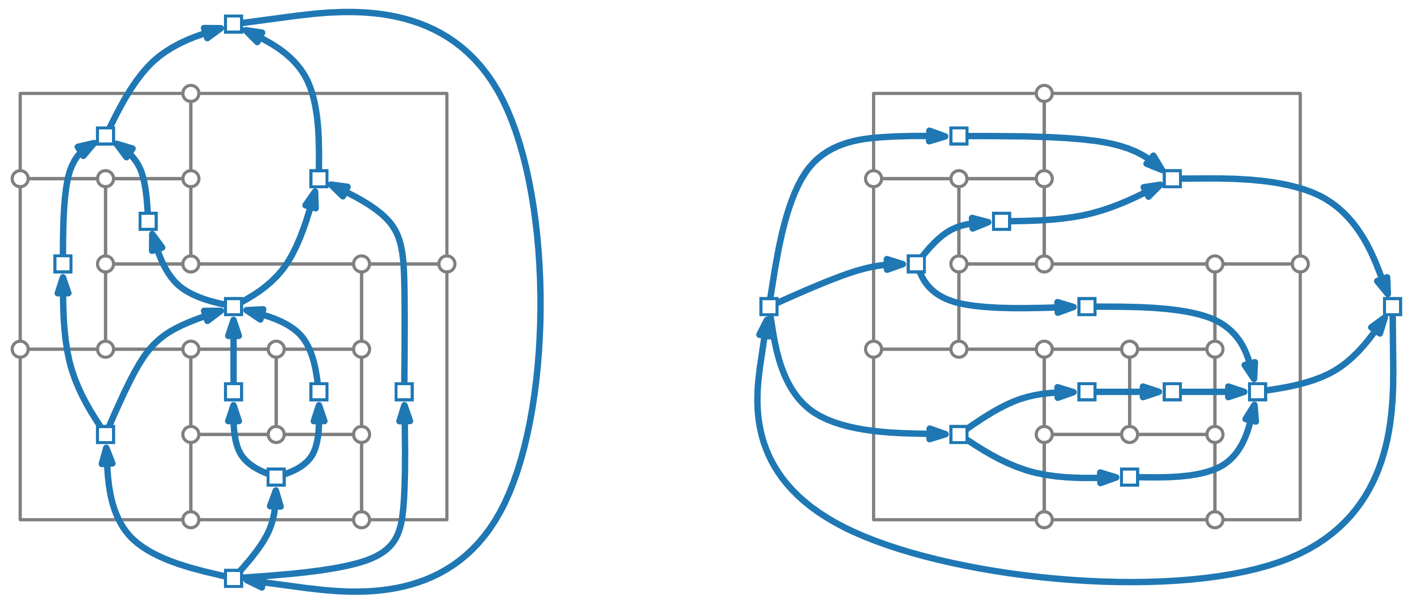


Compaction – Result



Theorem.
A valid flow for N_{hor} and N_{ver} exists $\Leftrightarrow$ corresponding edge lengths induce an orthogonal drawing.

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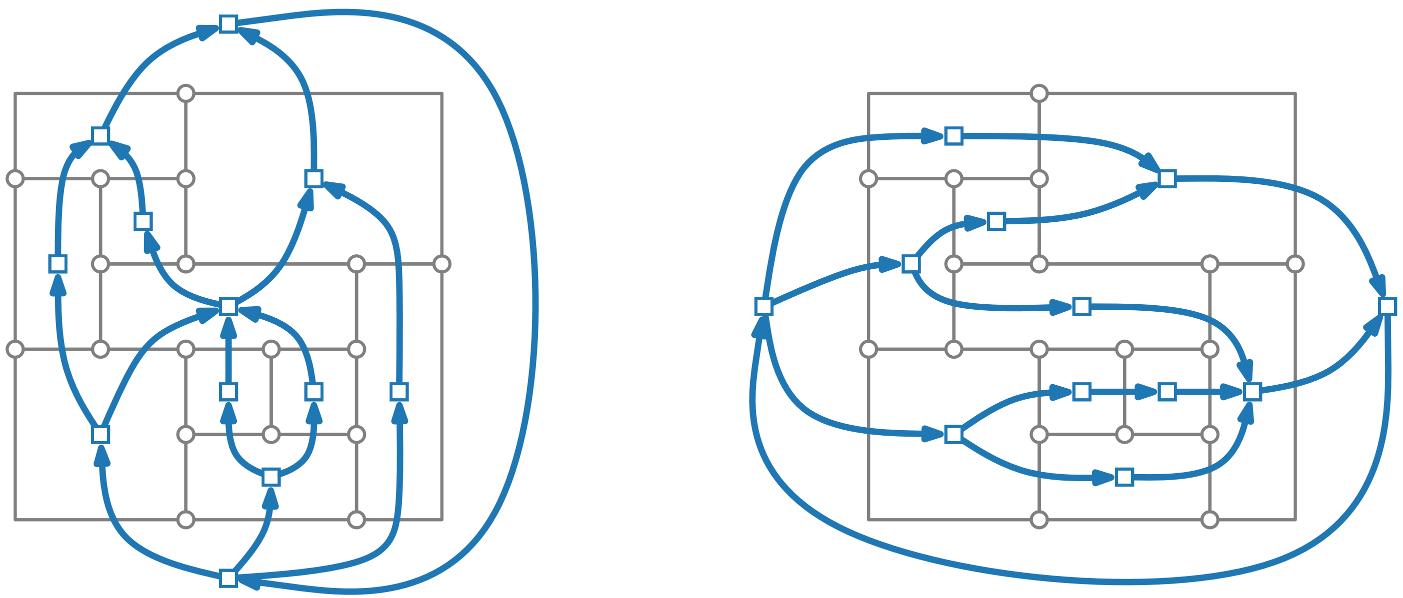

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What values of the drawing do the following quantities represent?

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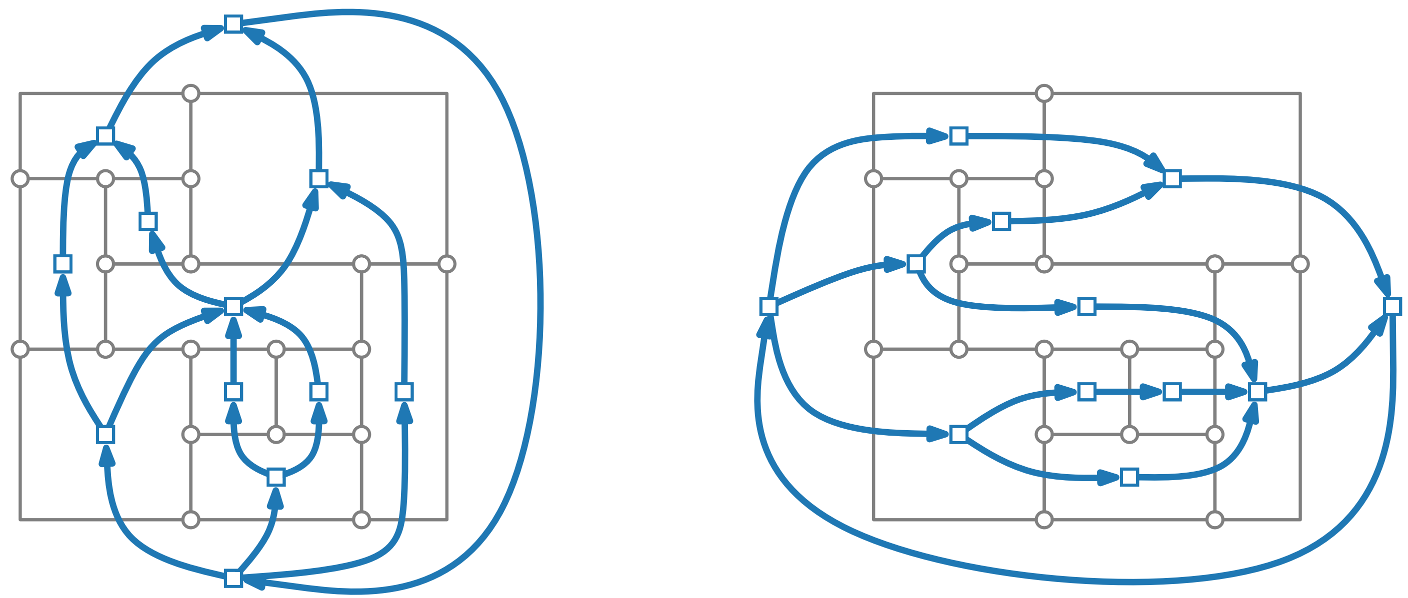

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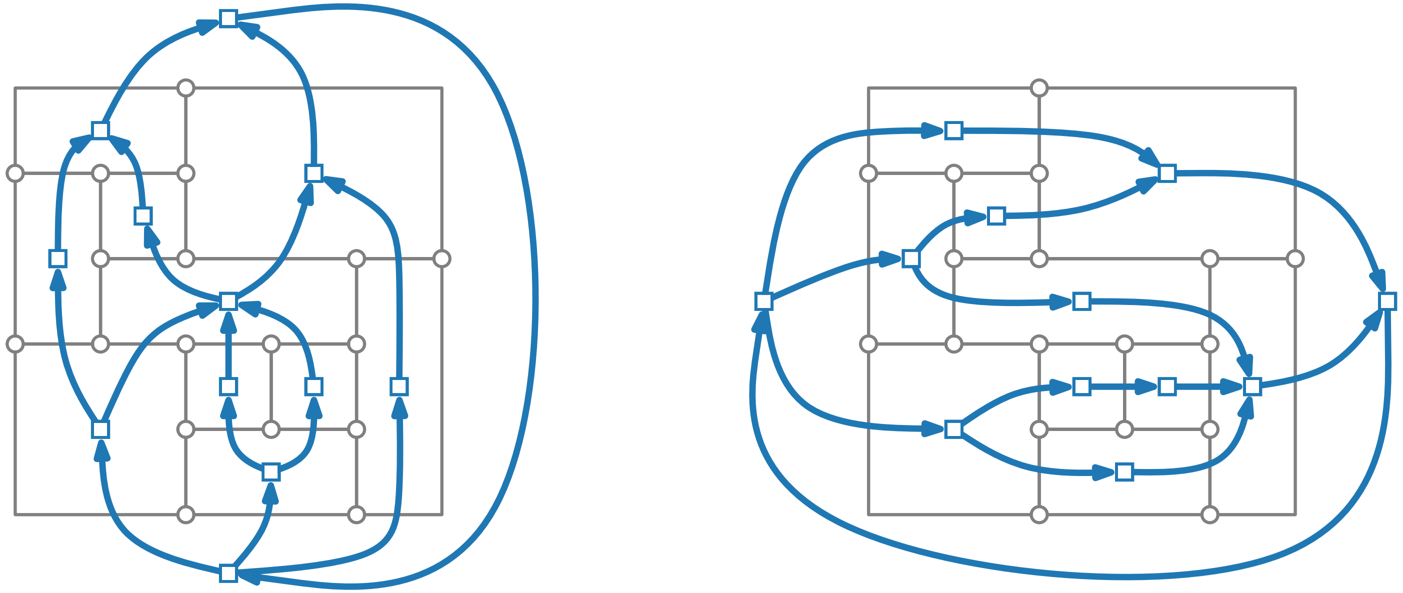

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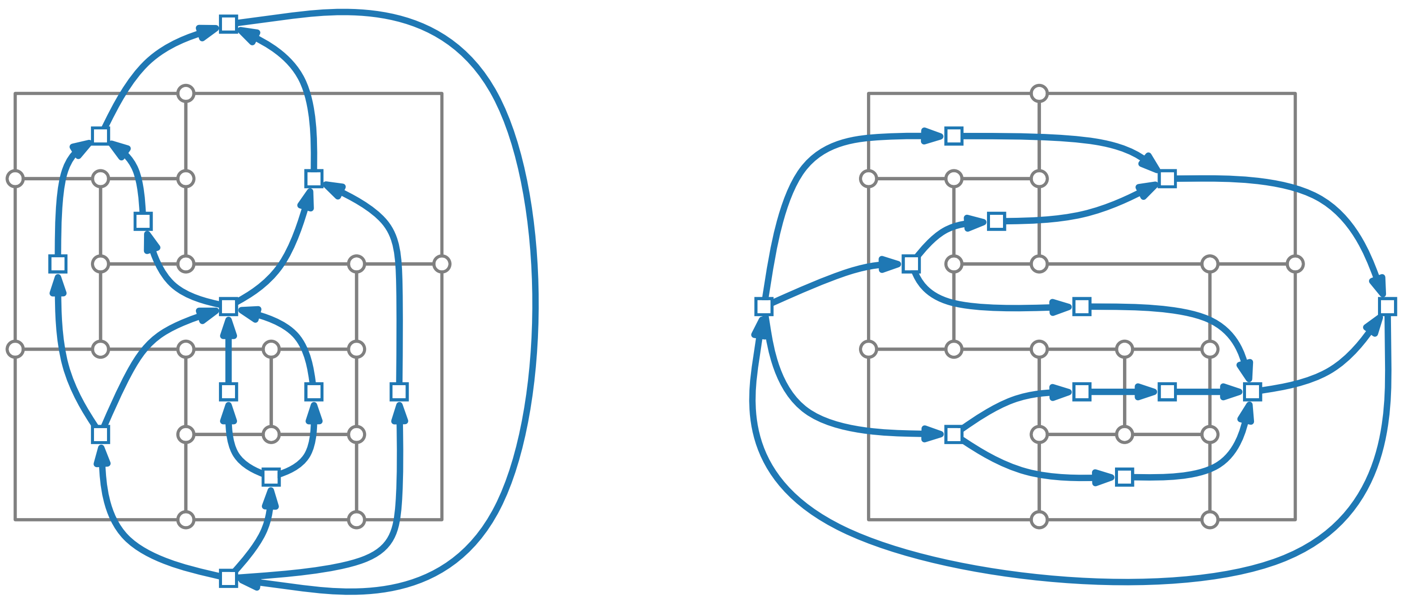
What values of the drawing do the following quantities represent?

- $|X_{\text{hor}}(t,s)|$ and $|X_{\text{ver}}(t,s)|$? width and height of the drawing

Compaction – Result



Theorem.
A valid flow for N_{hor} and N_{ver} exists $\Leftrightarrow$
corresponding edge lengths induce an orthogonal drawing.

What values of the drawing do the following quantities represent?

- $|X_{\text{hor}}(t,s)|$ and $|X_{\text{ver}}(t,s)|$? width and height of the drawing
- $\sum_{e \in E_{\text{hor}}} X_{\text{hor}}(e) + \sum_{e \in E_{\text{ver}}} X_{\text{ver}}(e)$

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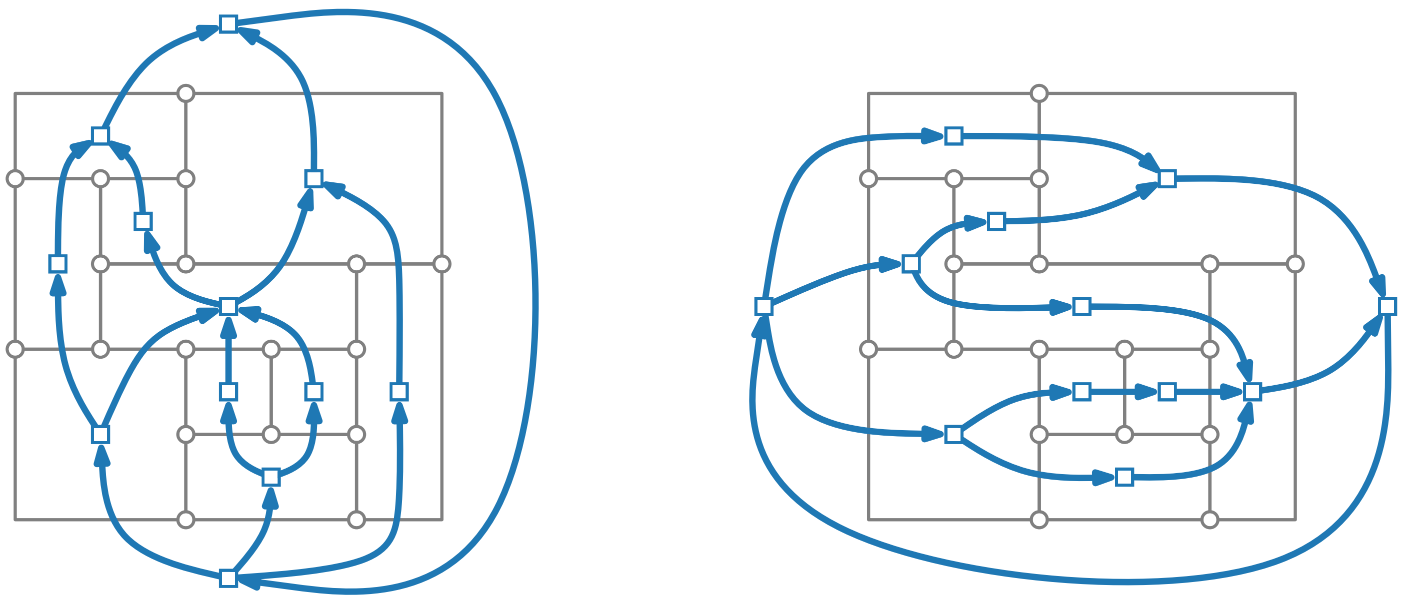

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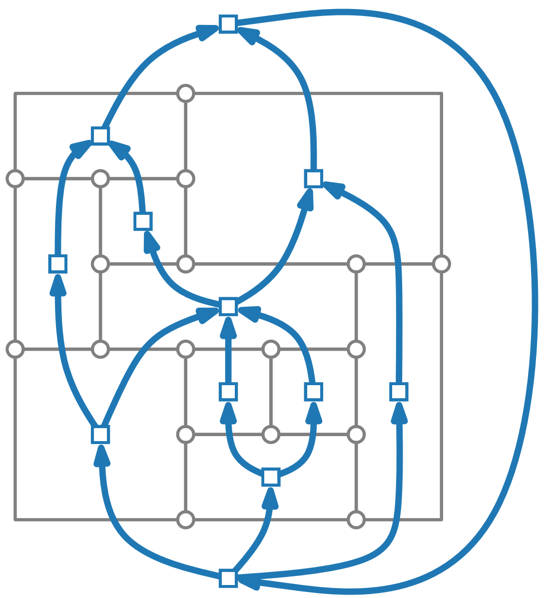

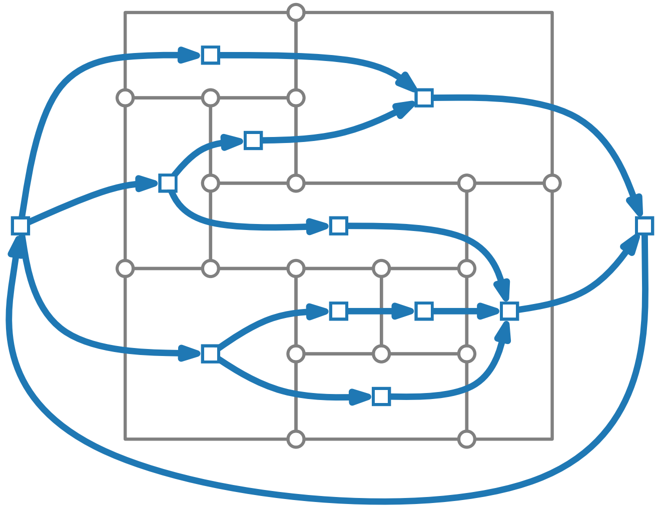

What if not all faces are rectangular?

Theorem.
A valid flow for N_{hor} and N_{ver} exists $\Leftrightarrow$ corresponding edge lengths induce an orthogonal drawing.

What values of the drawing do the following quantities represent?

- $|X_{\text{hor}}(t,s)|$ and $|X_{\text{ver}}(t,s)|$? width and height of the drawing
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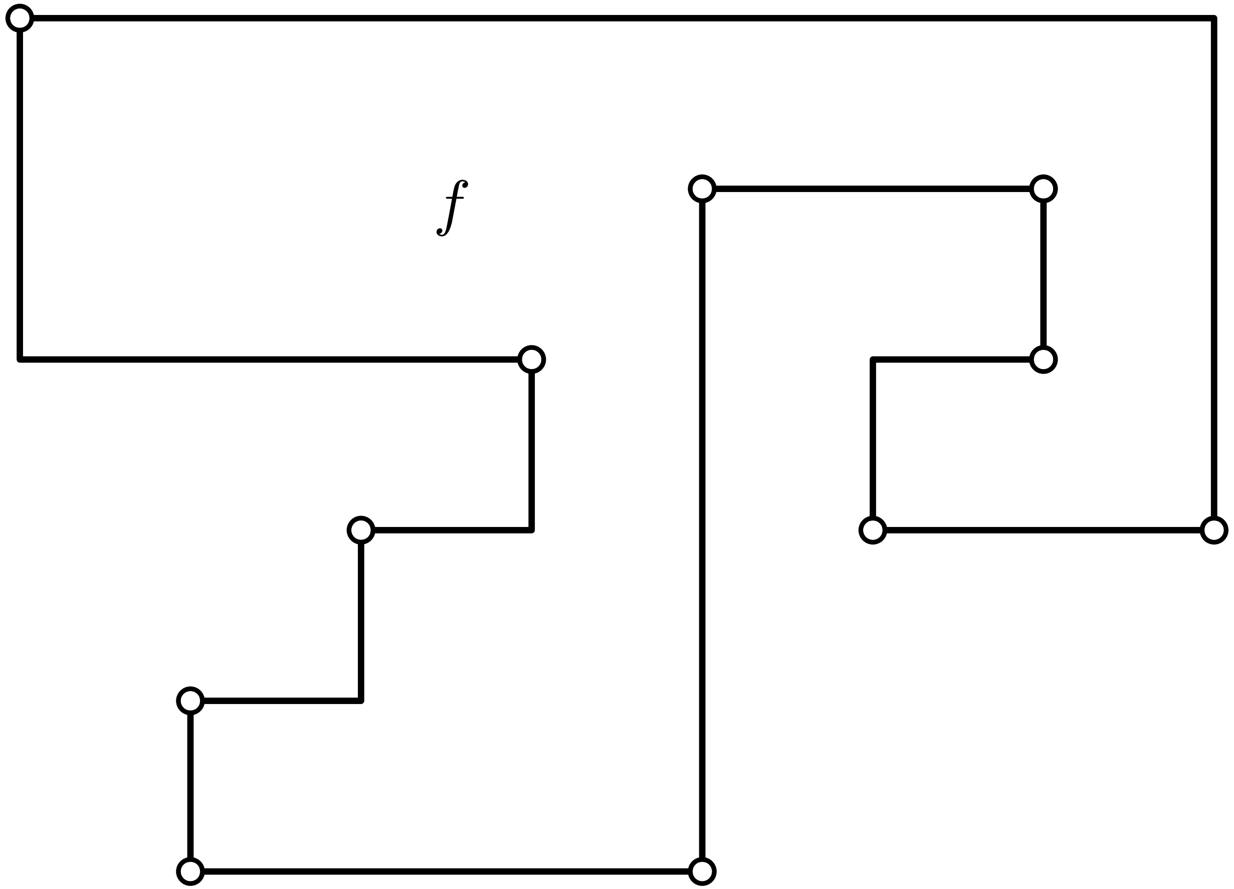
Refinement of G and $H(G)$ – Inner Face



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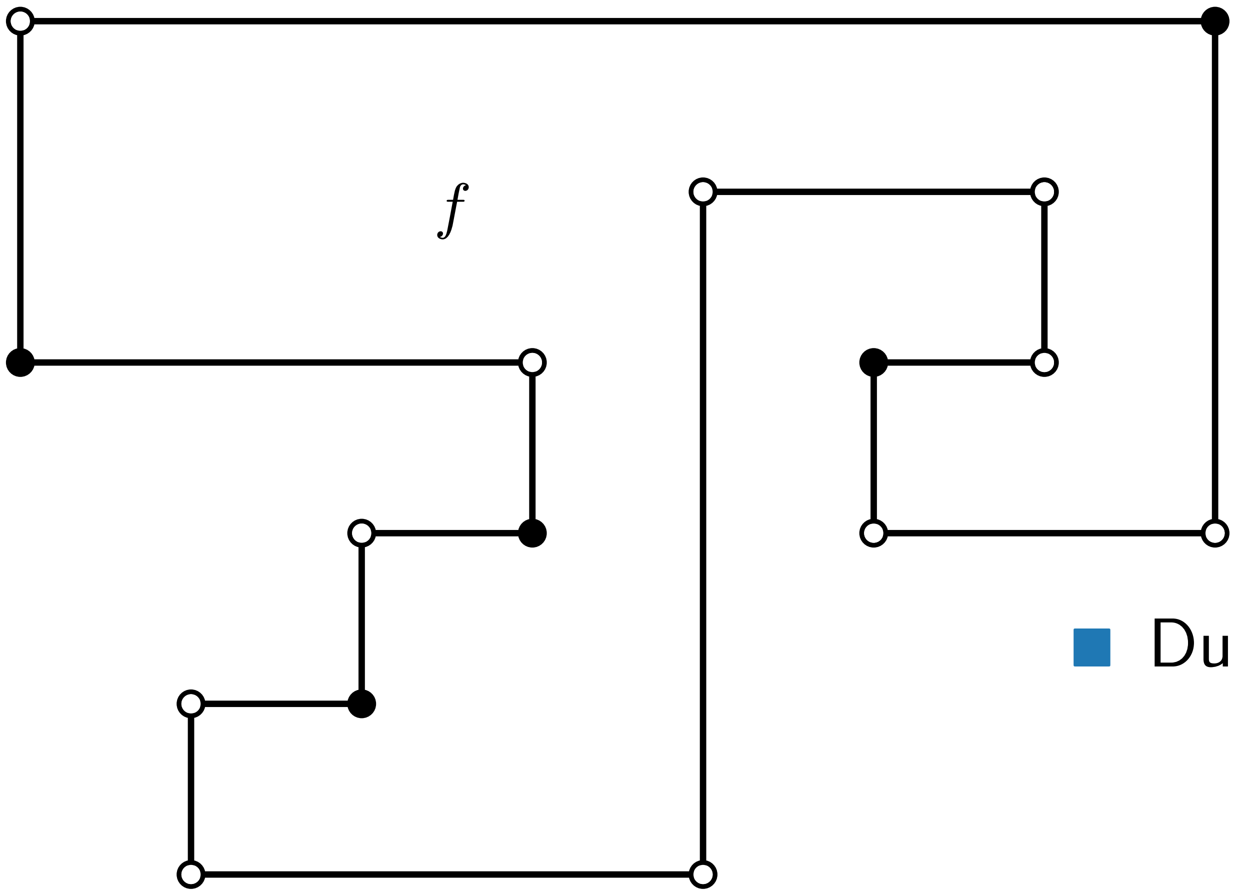

- Dummy vertices for bends

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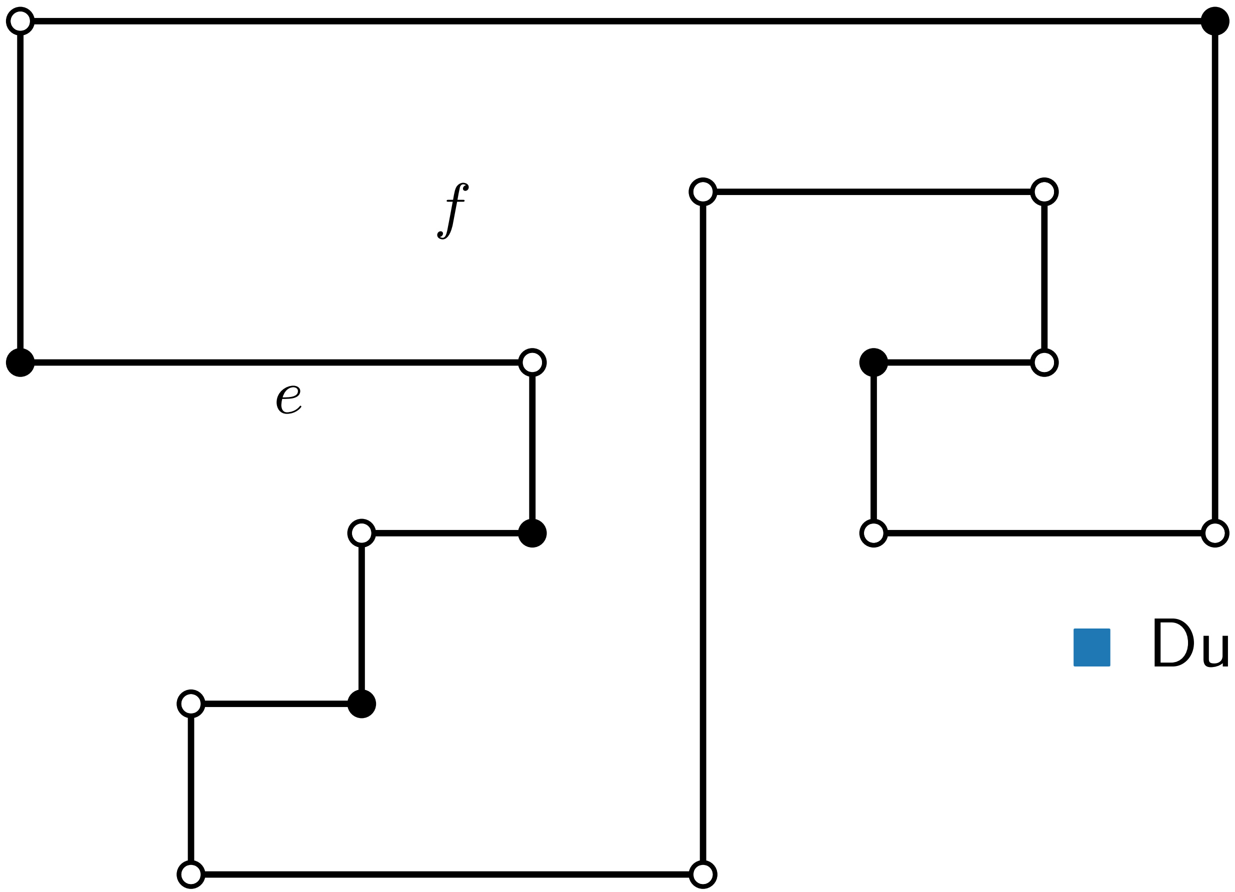

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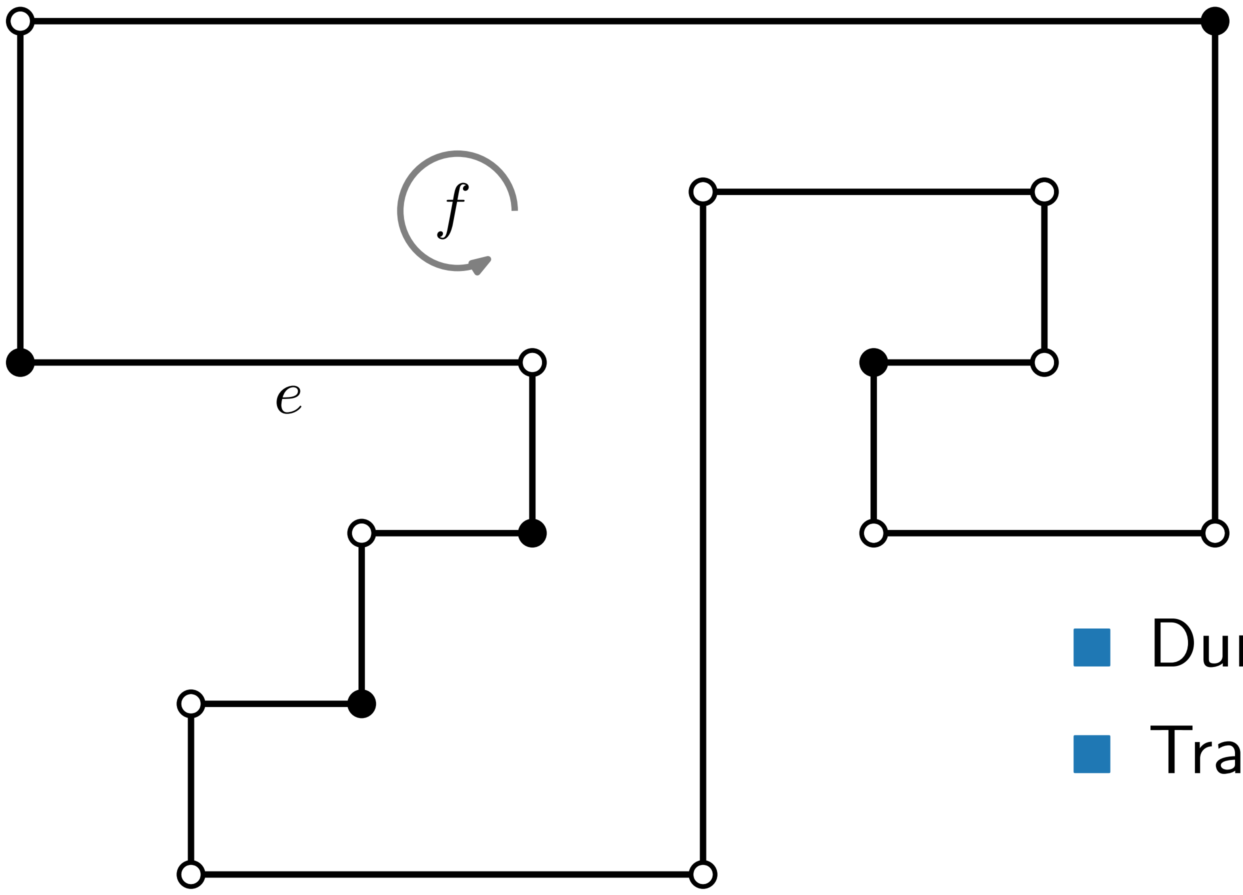

- Dummy vertices for bends
- Traverse counter-clockwise

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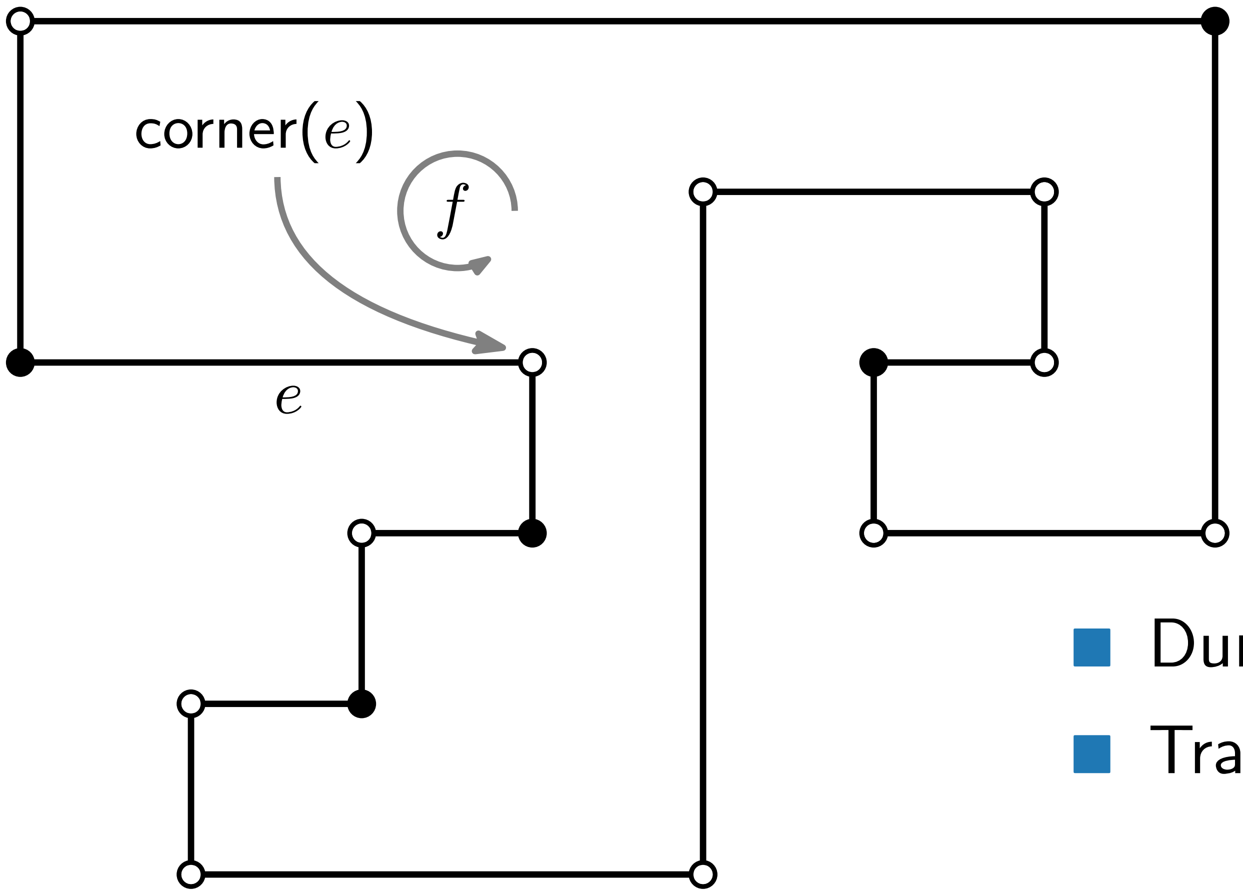

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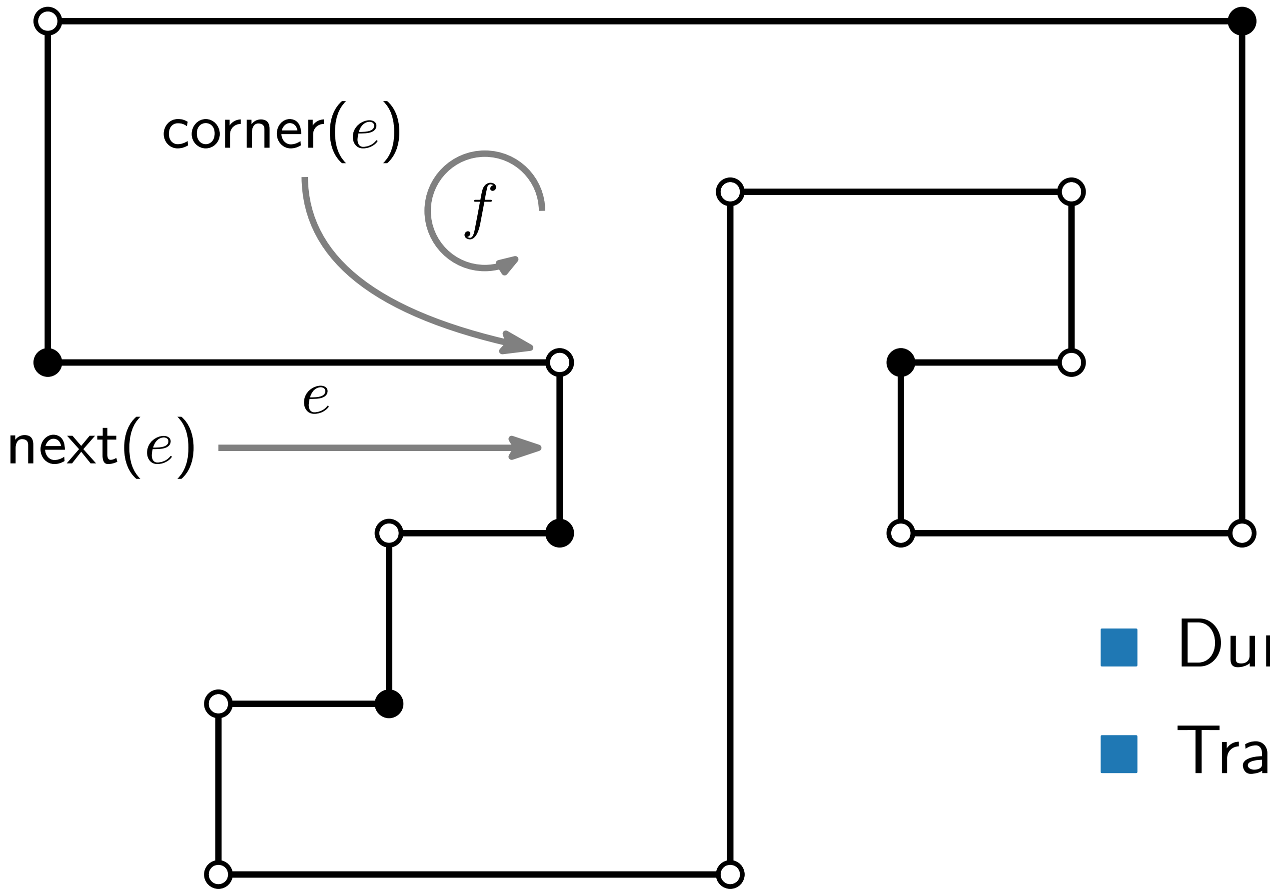

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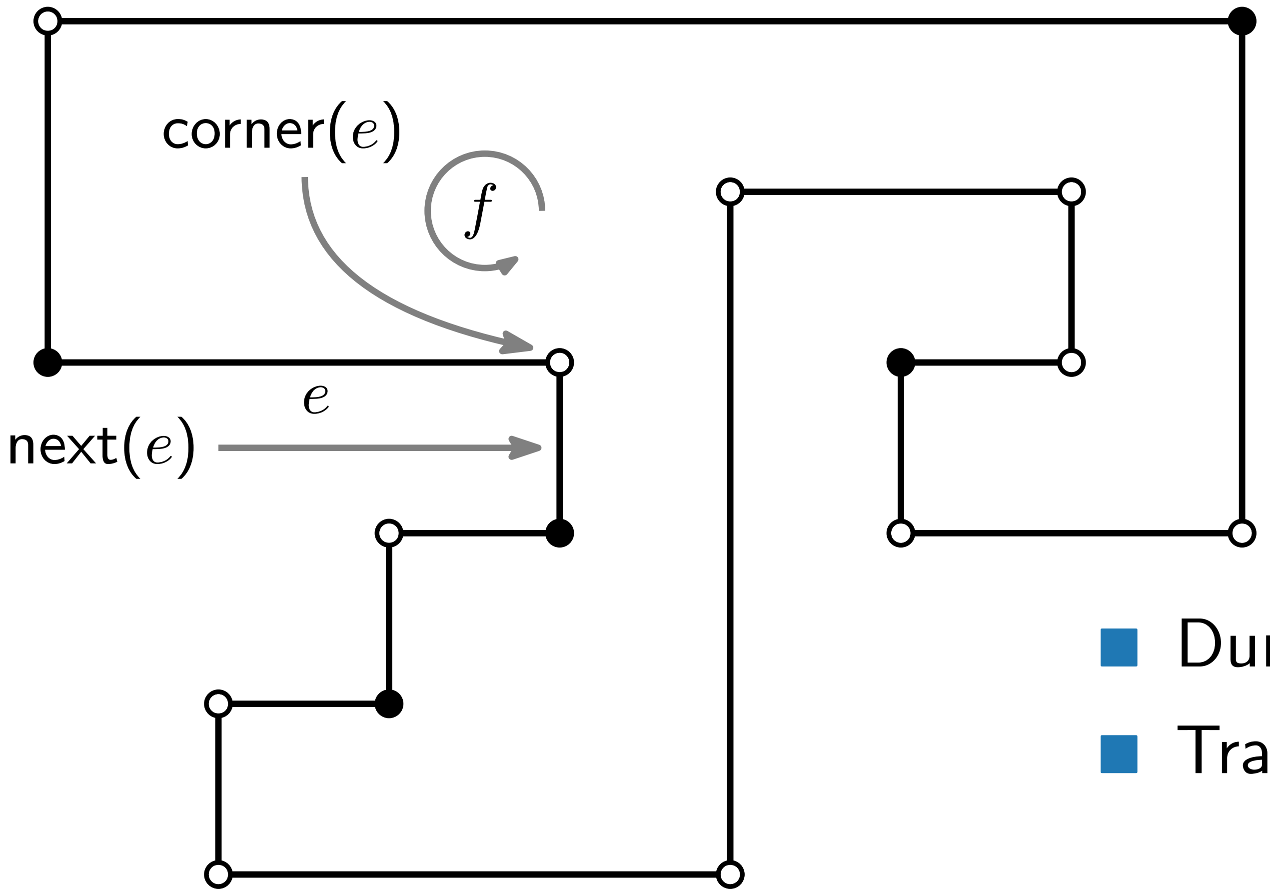

- Dummy vertices for bends
- Traverse counter-clockwise
- $\text{turn}(e) = \begin{cases} 1 & \text{left turn} \\ 0 & \text{no turn} \\ -1 & \text{right turn} \end{cases}$

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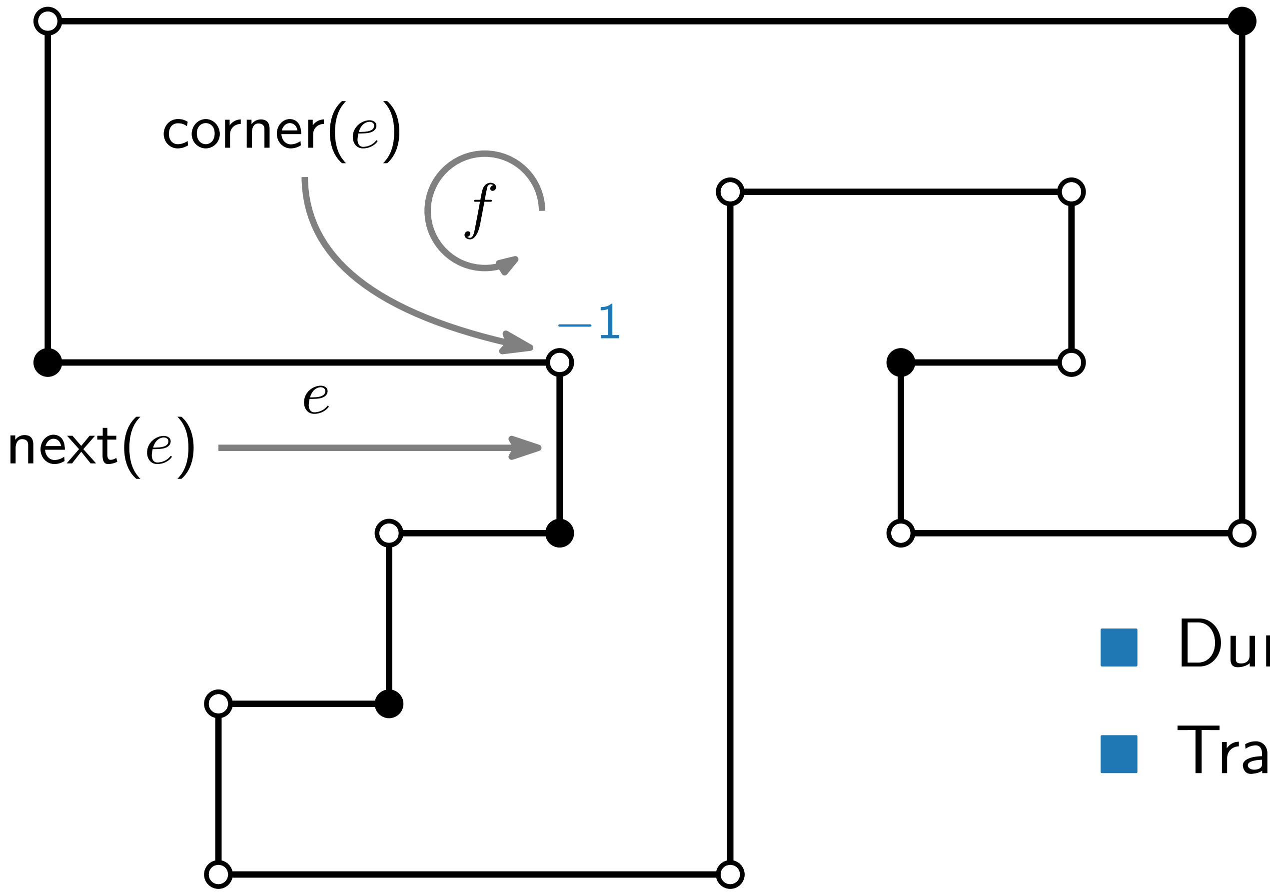

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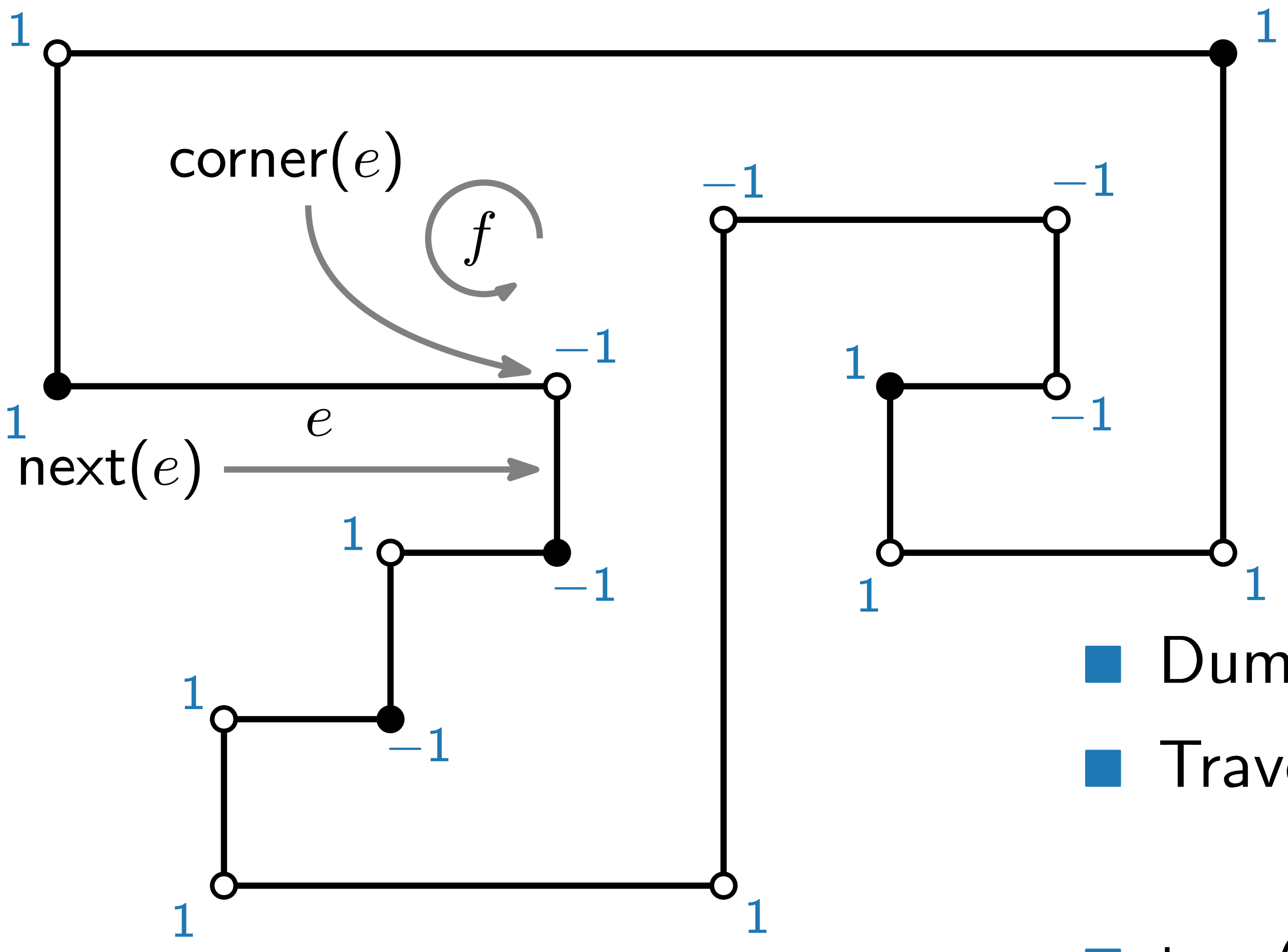

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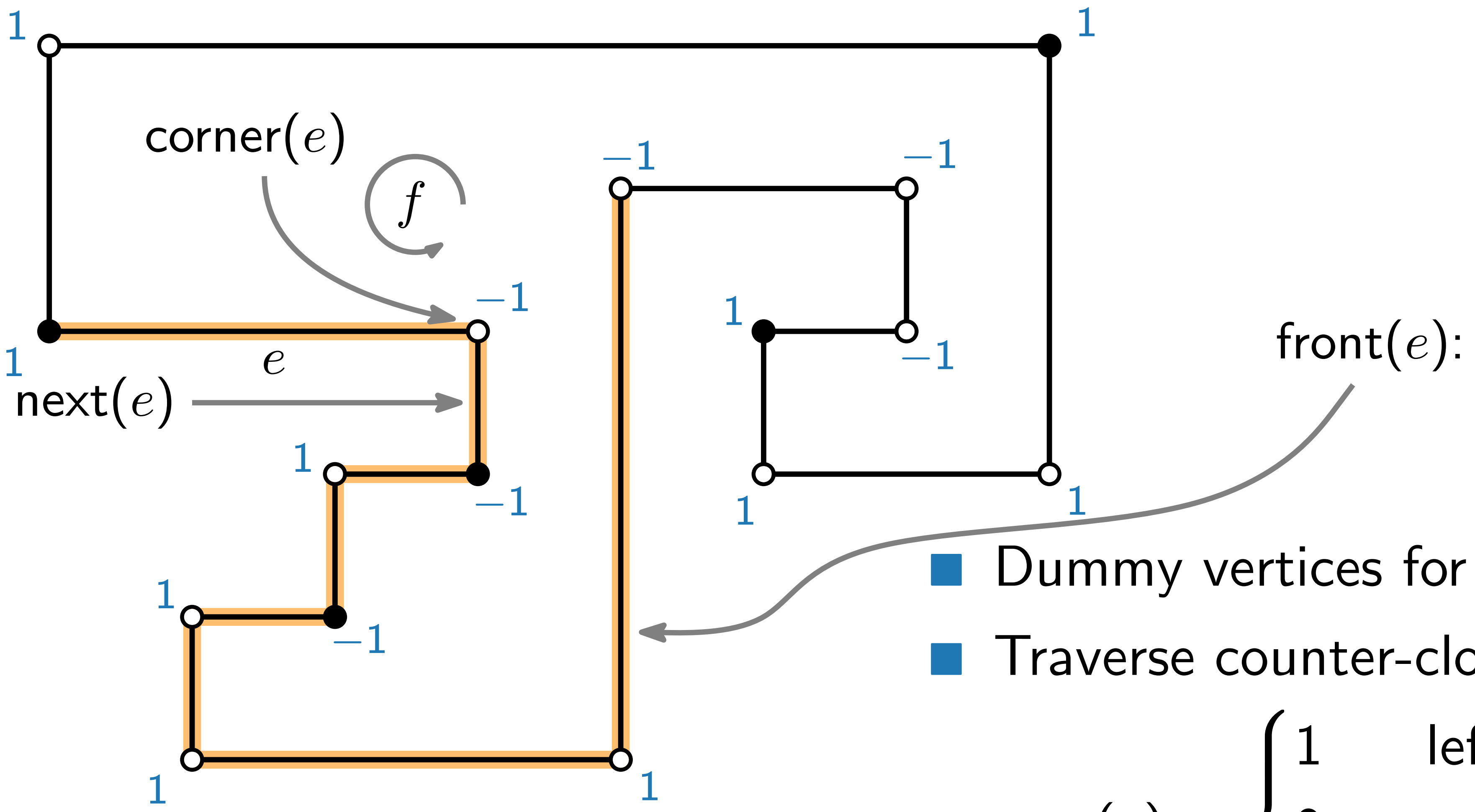

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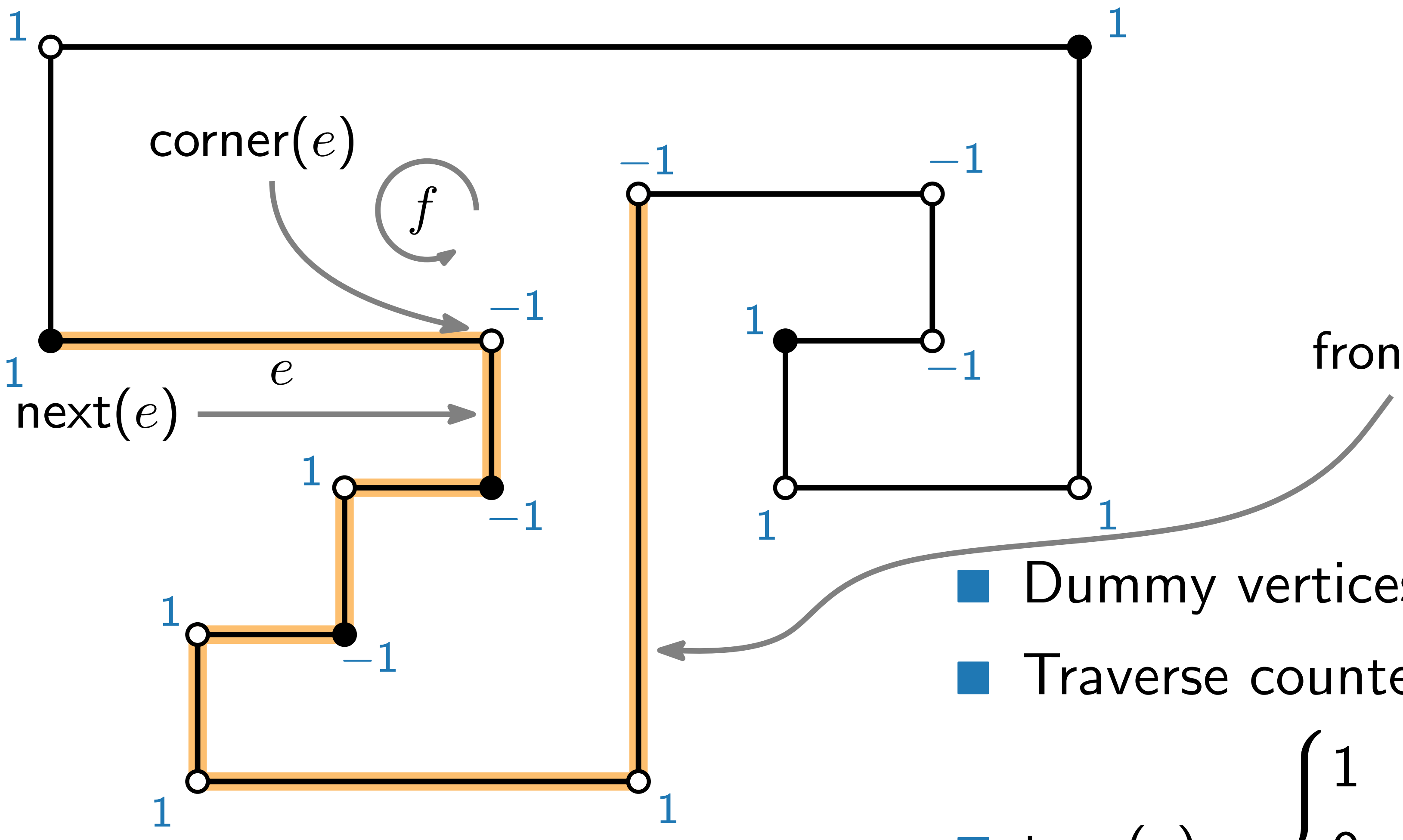

front(e): First edge e after e such that the angle sum between e and e' is positive.

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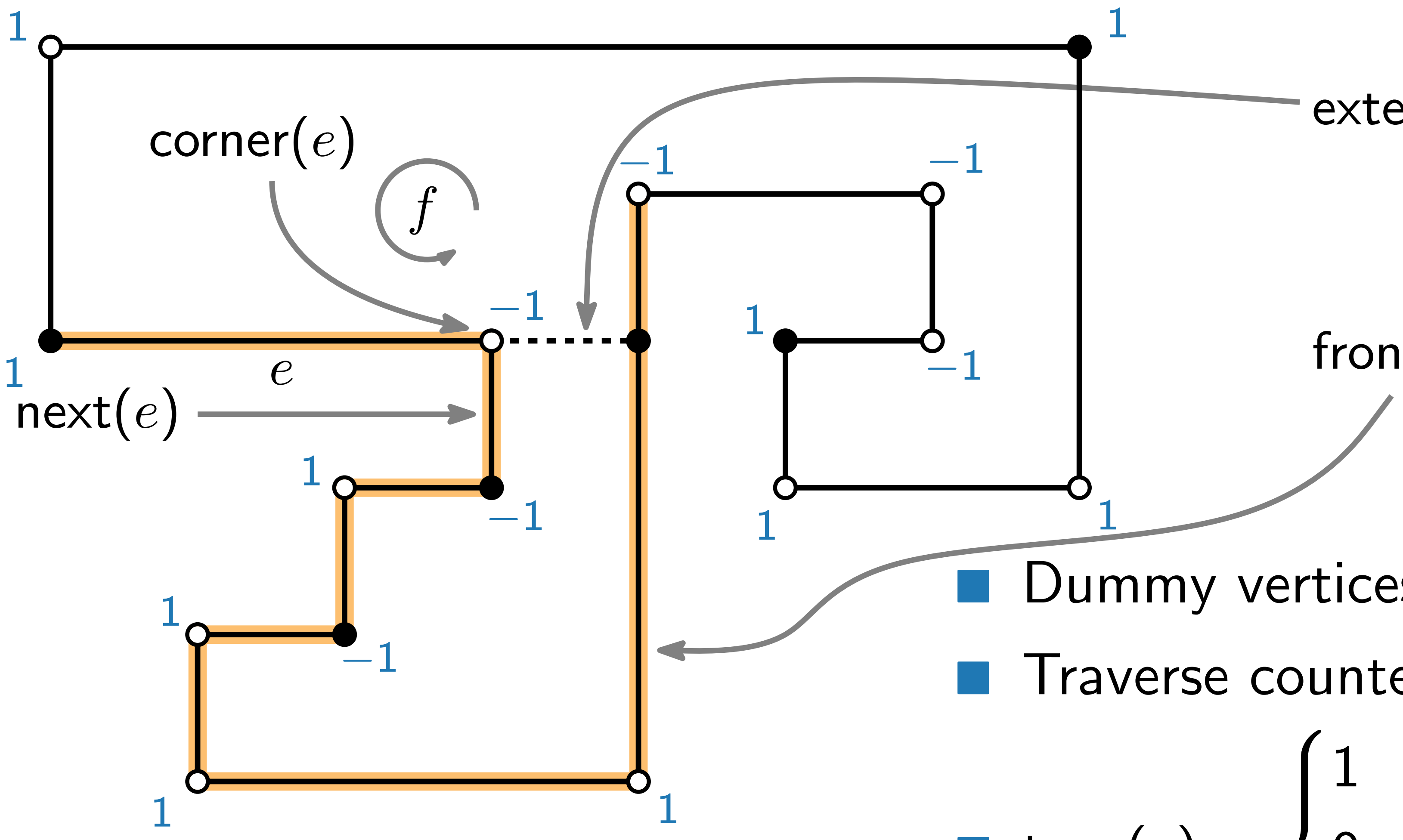

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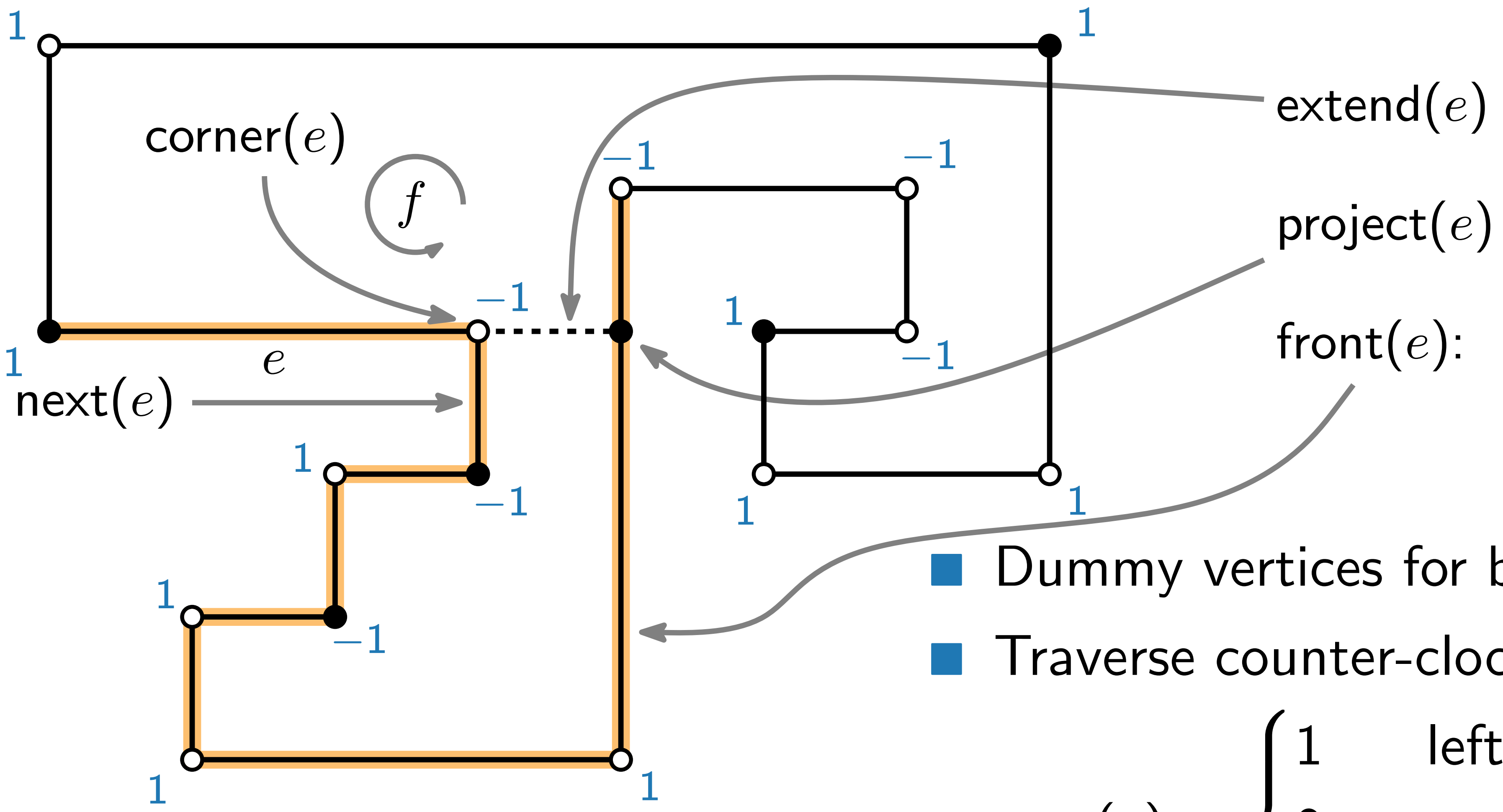

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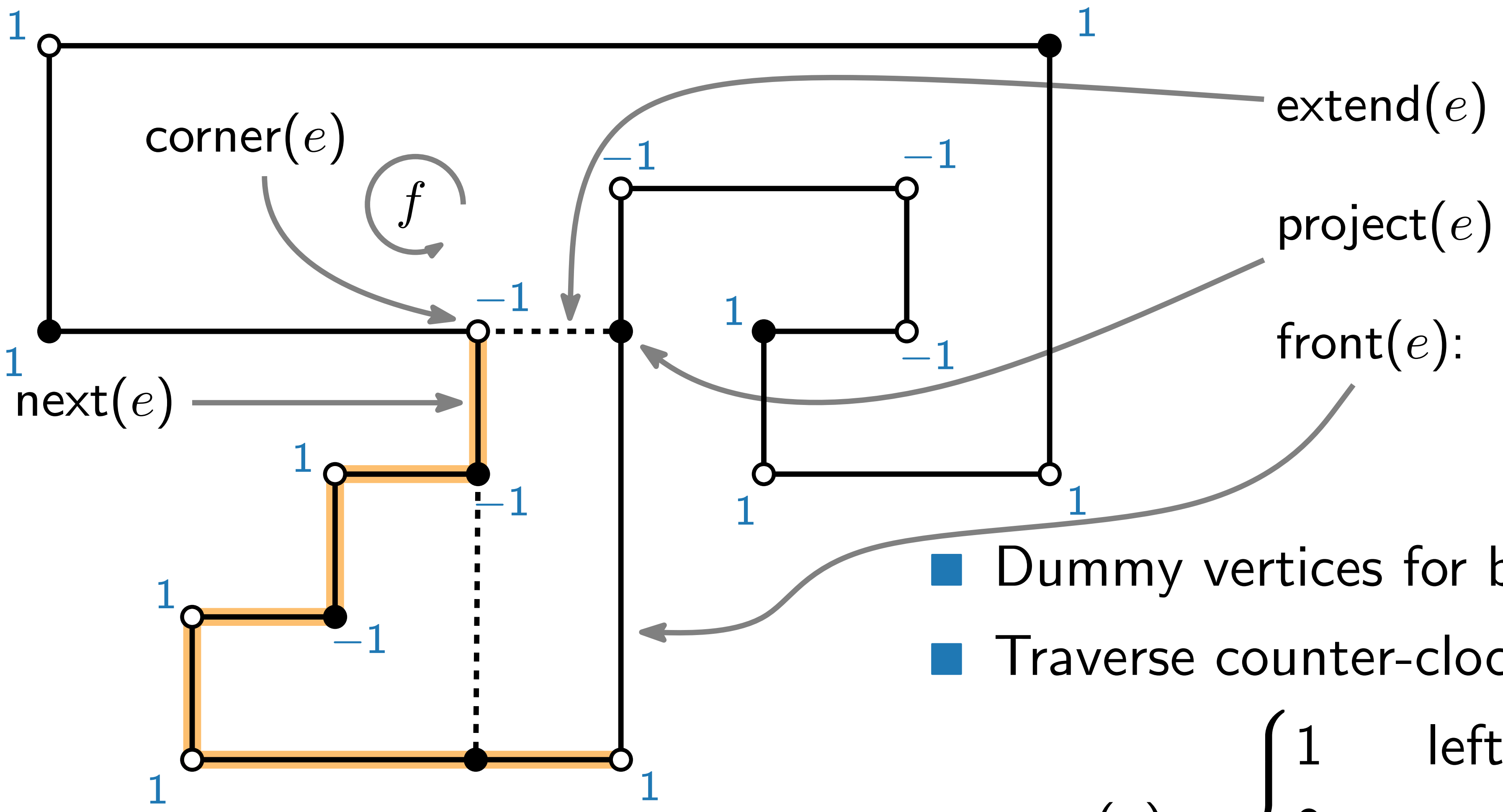

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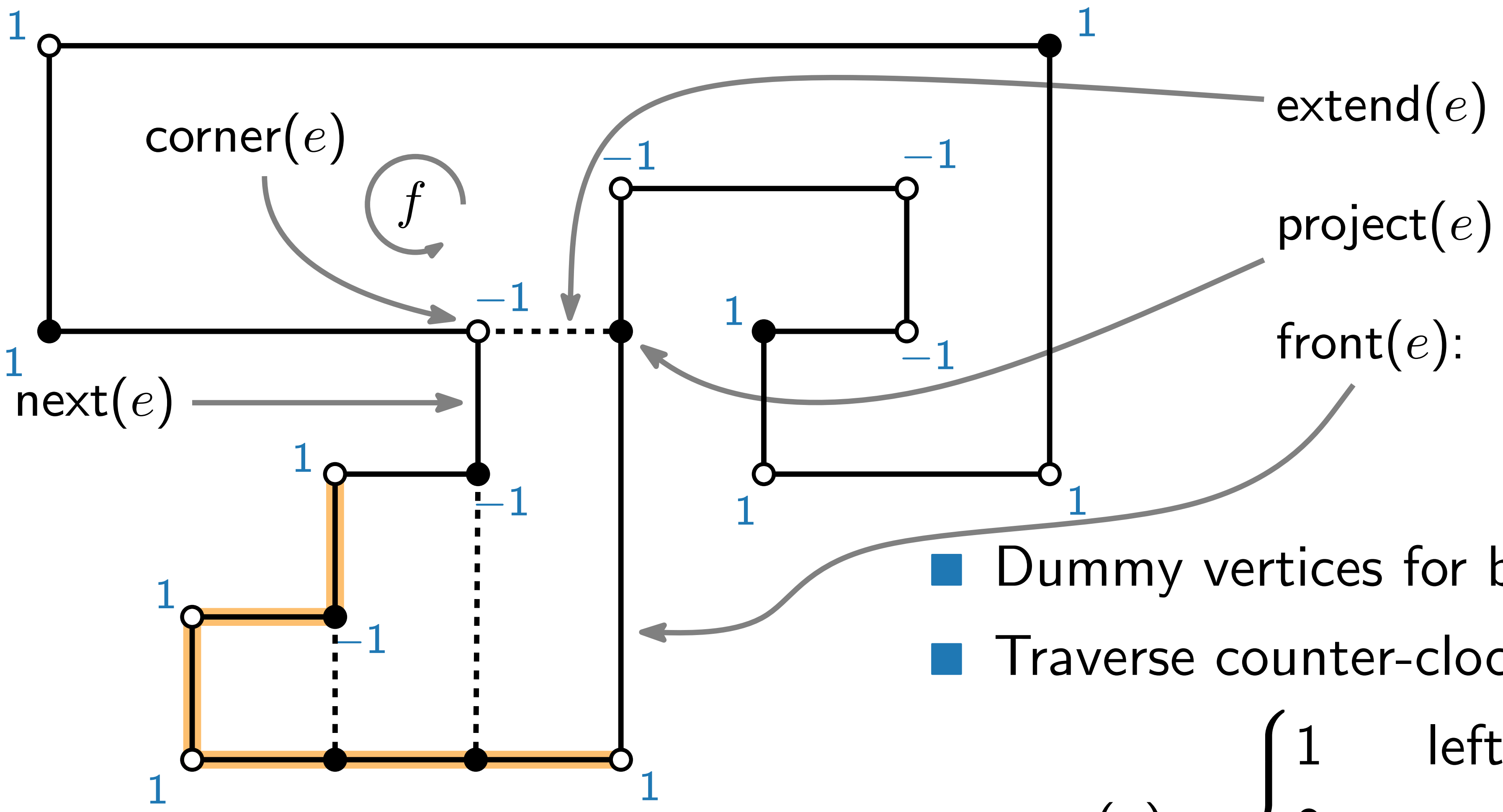

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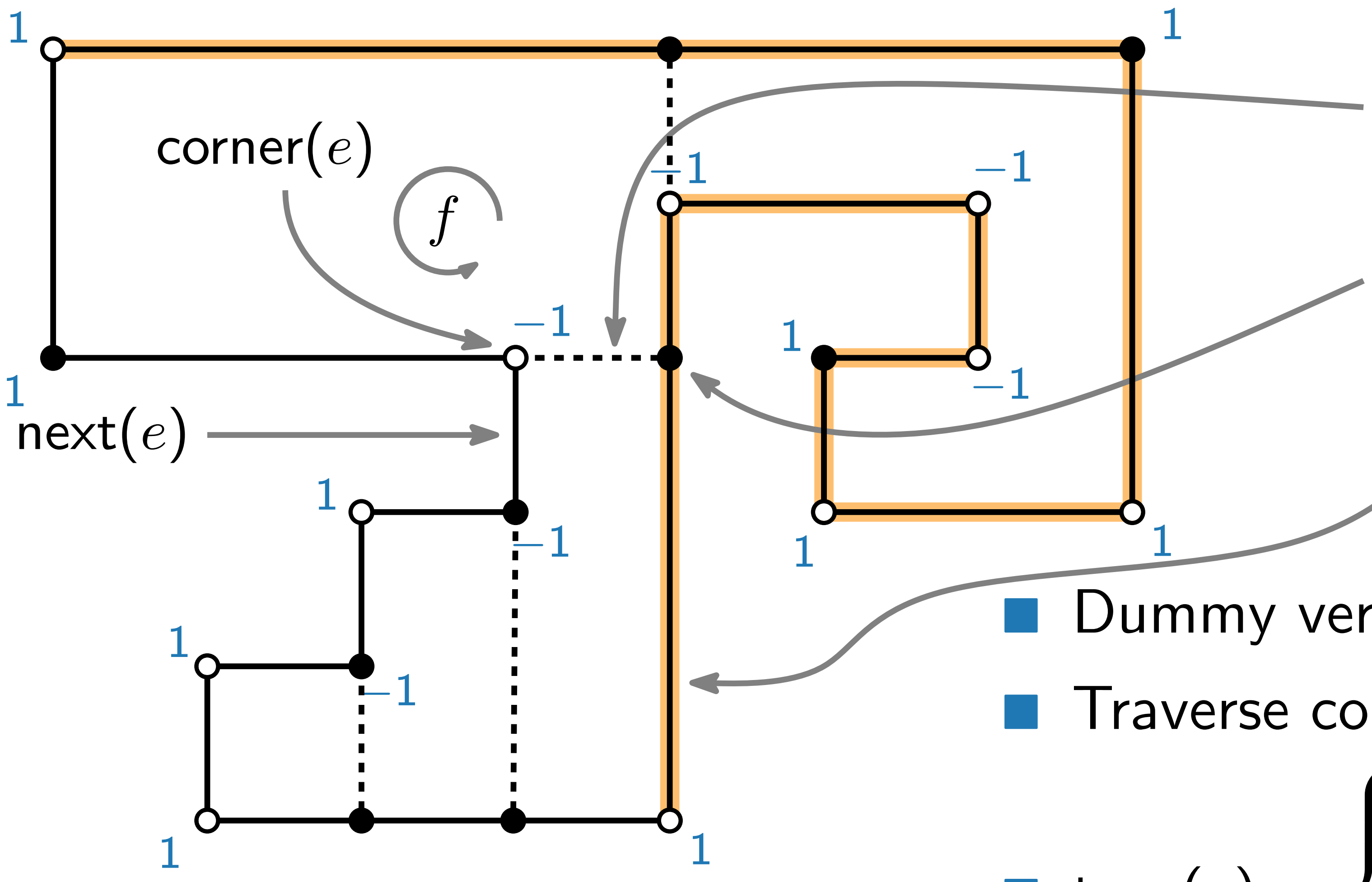

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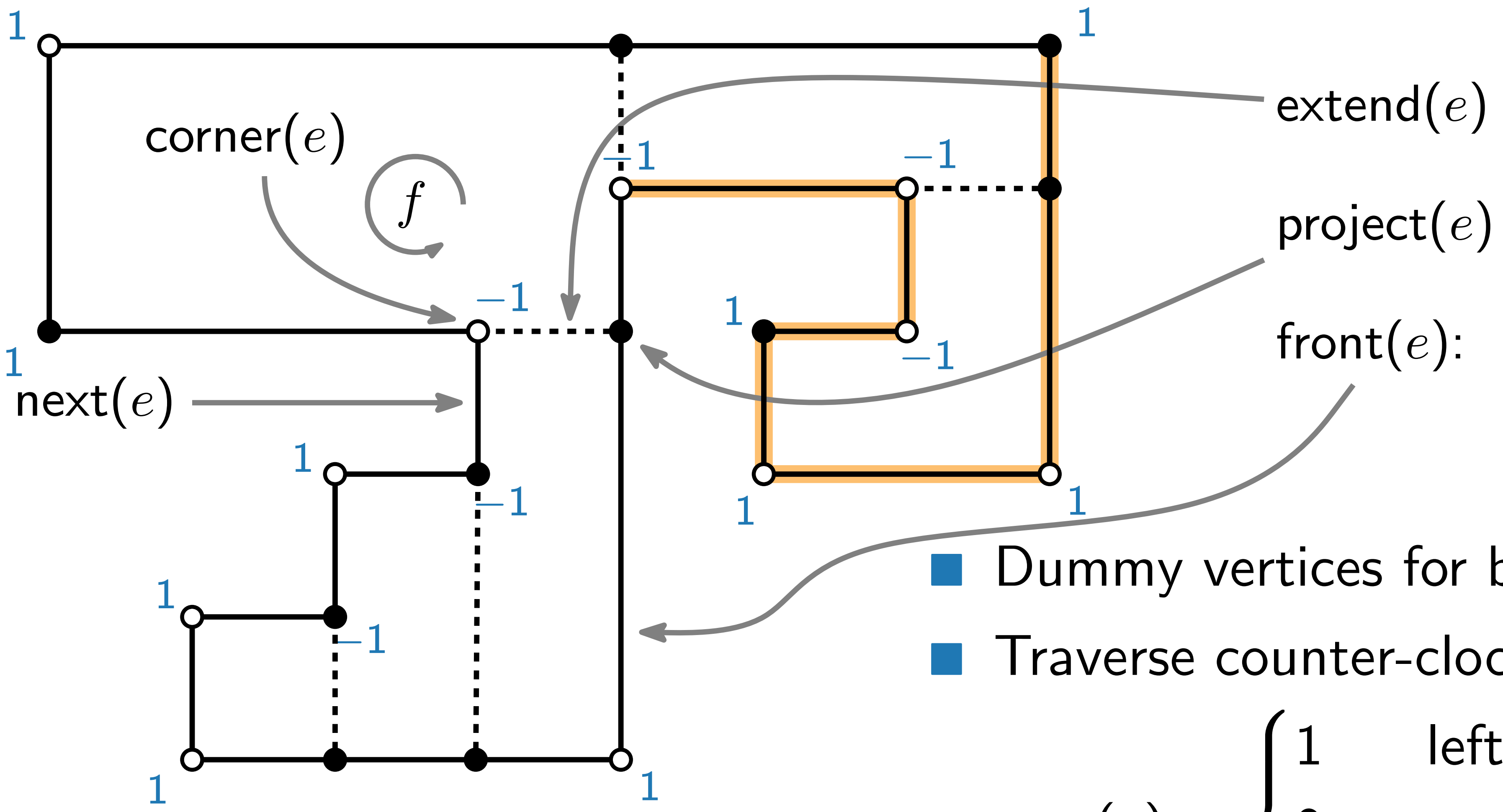

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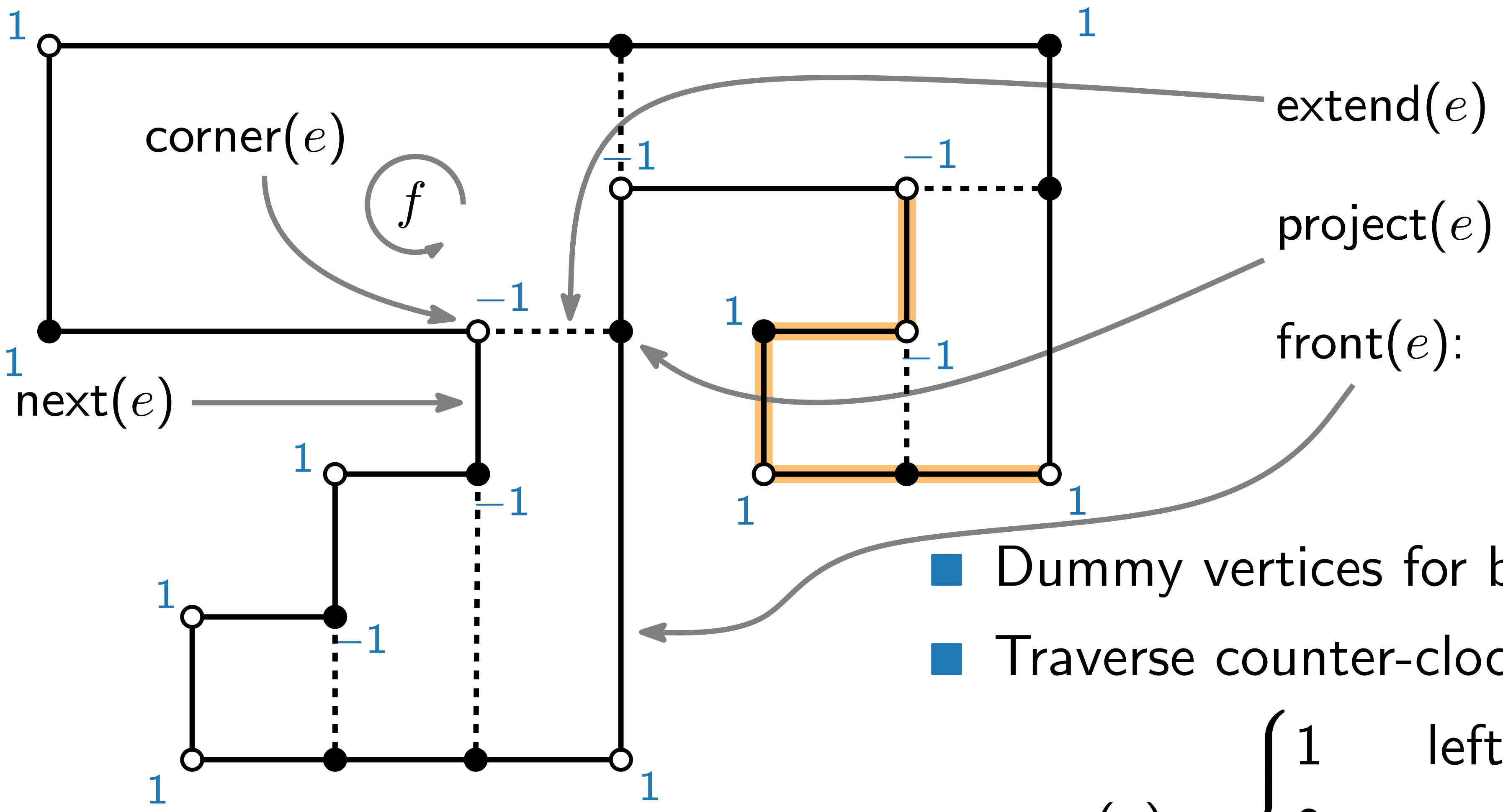

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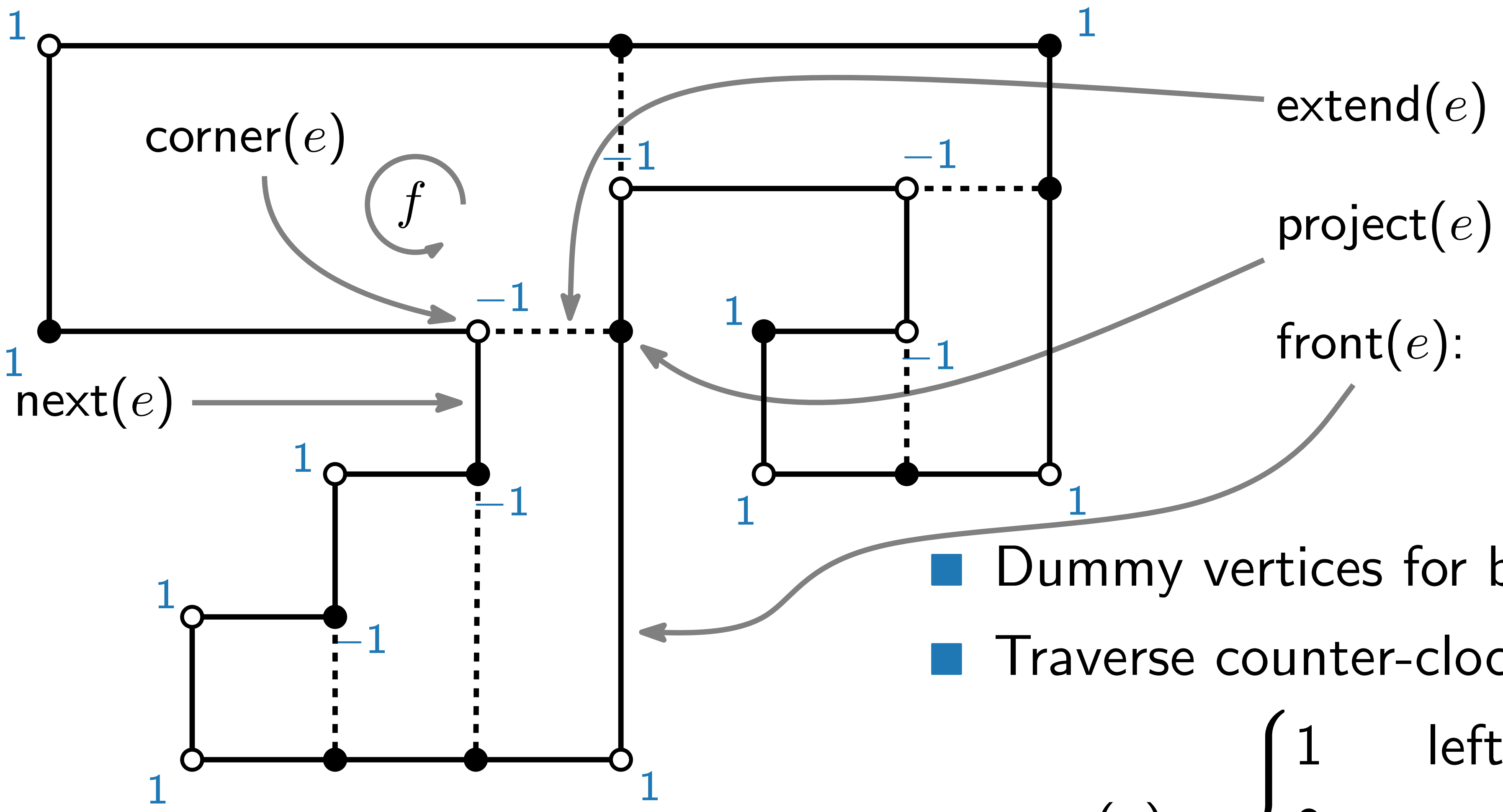

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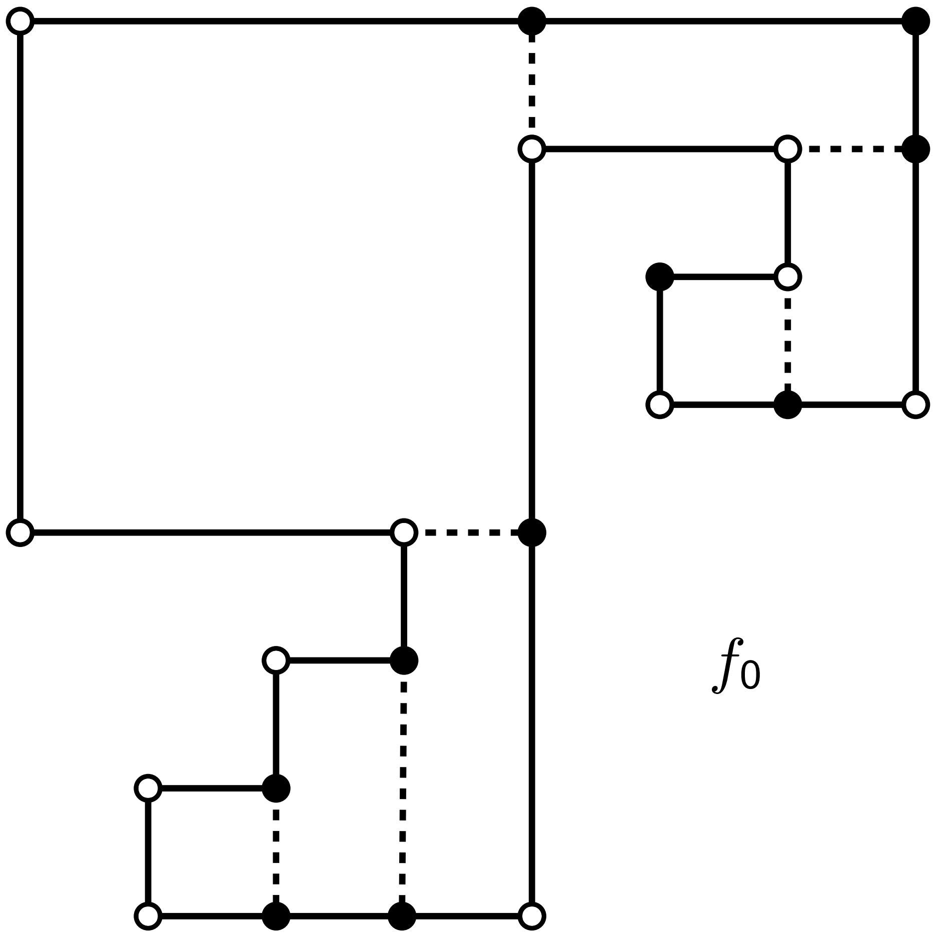

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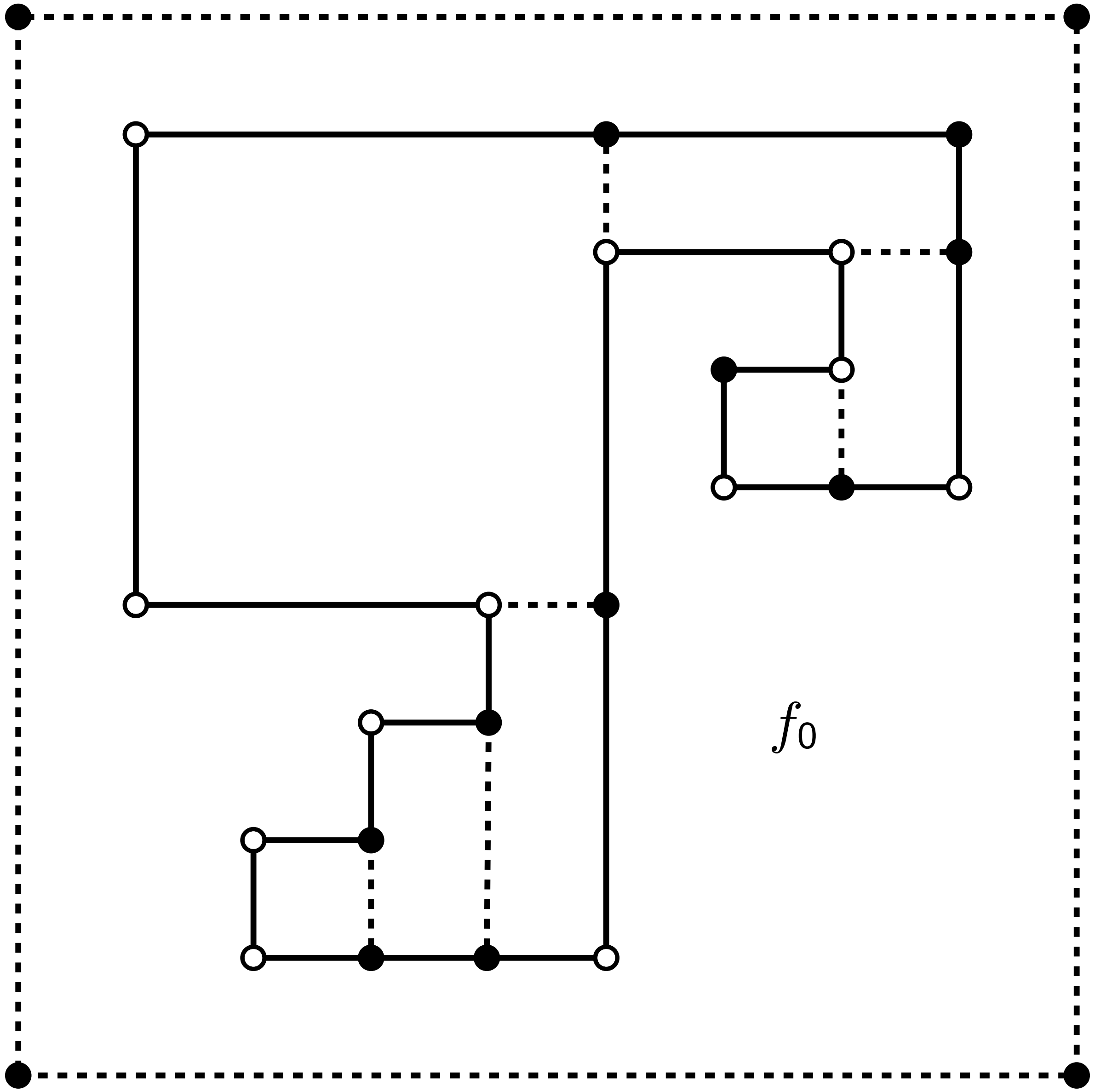

- Add an outer rectangle

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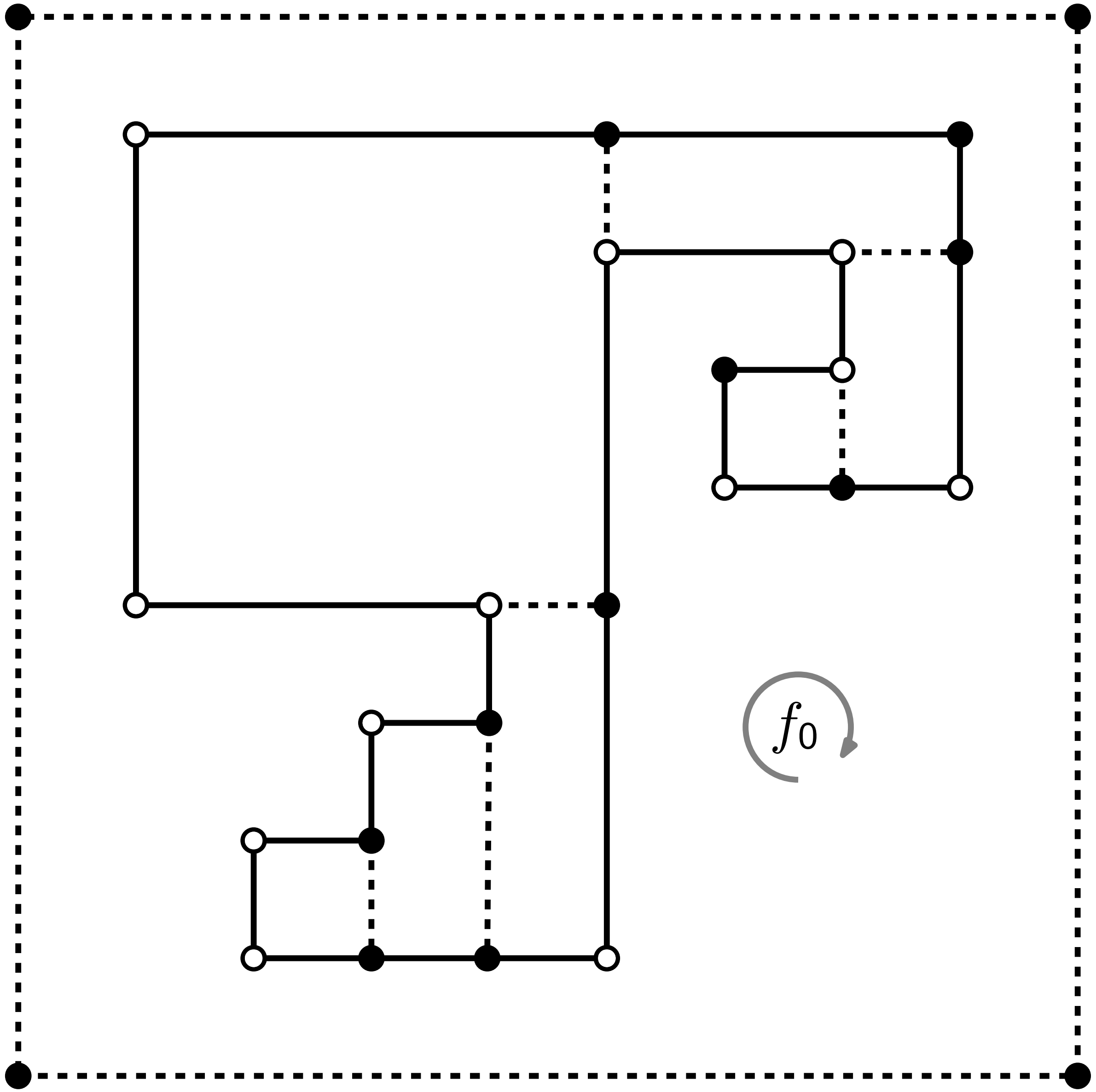

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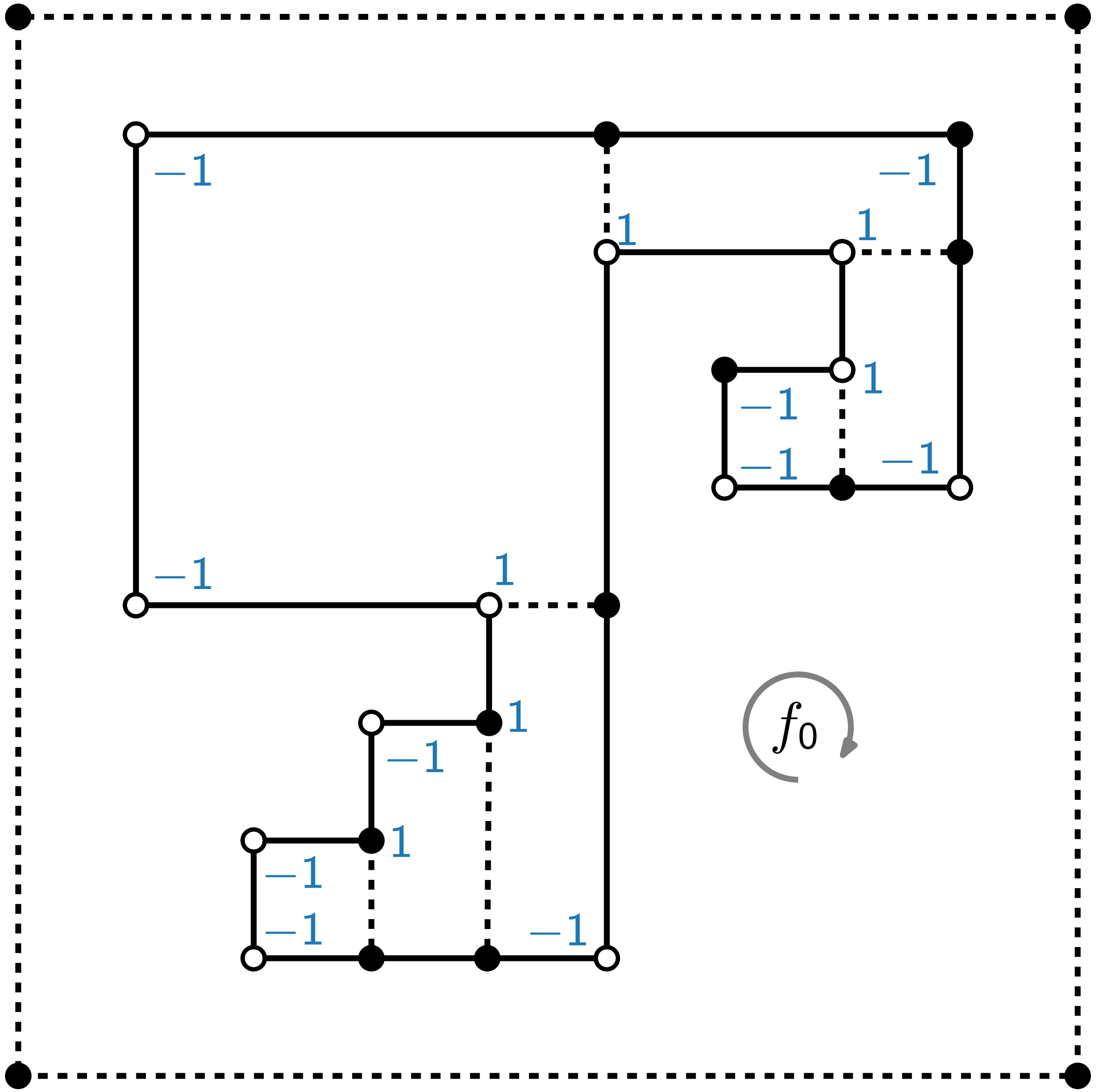

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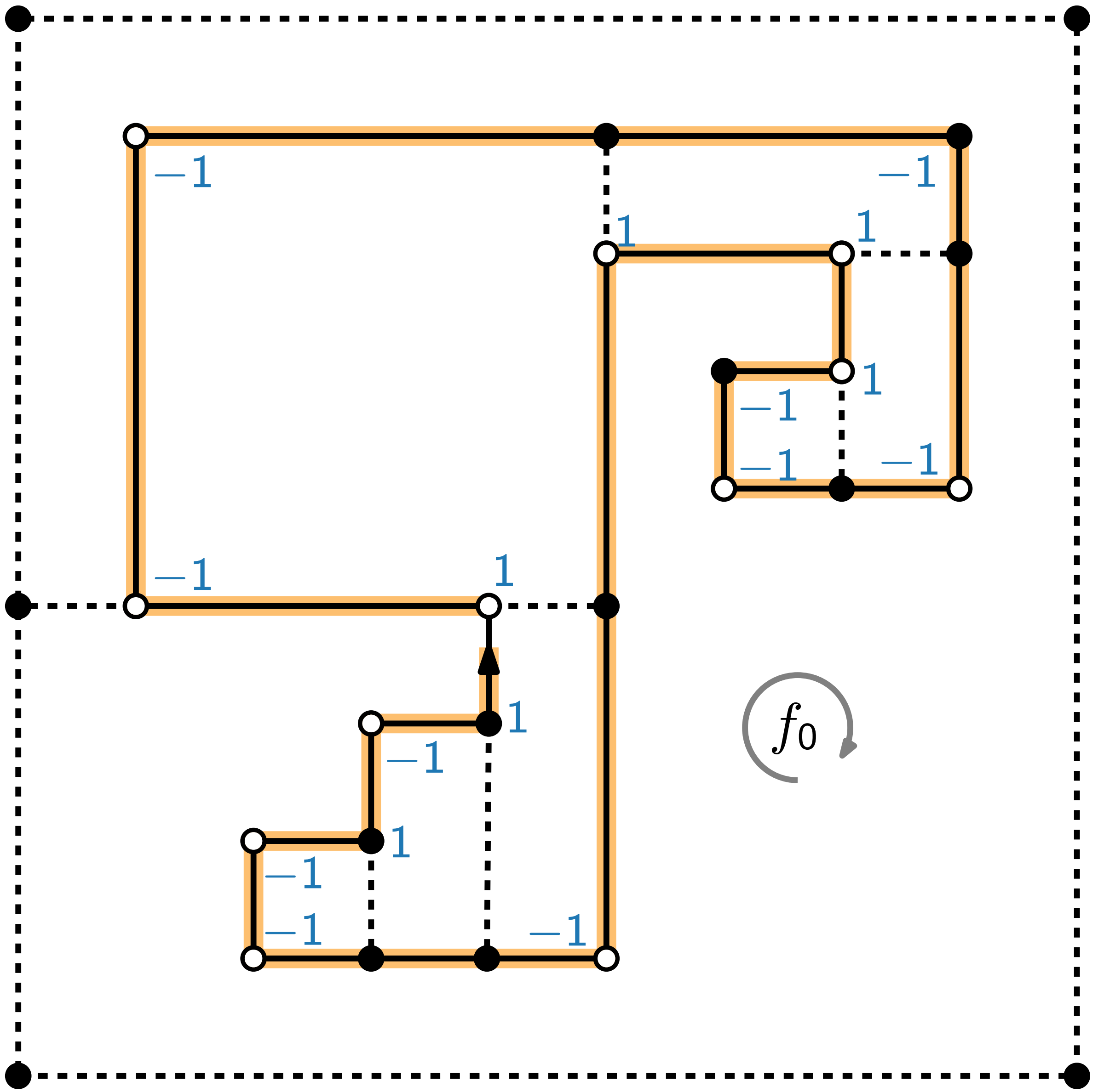

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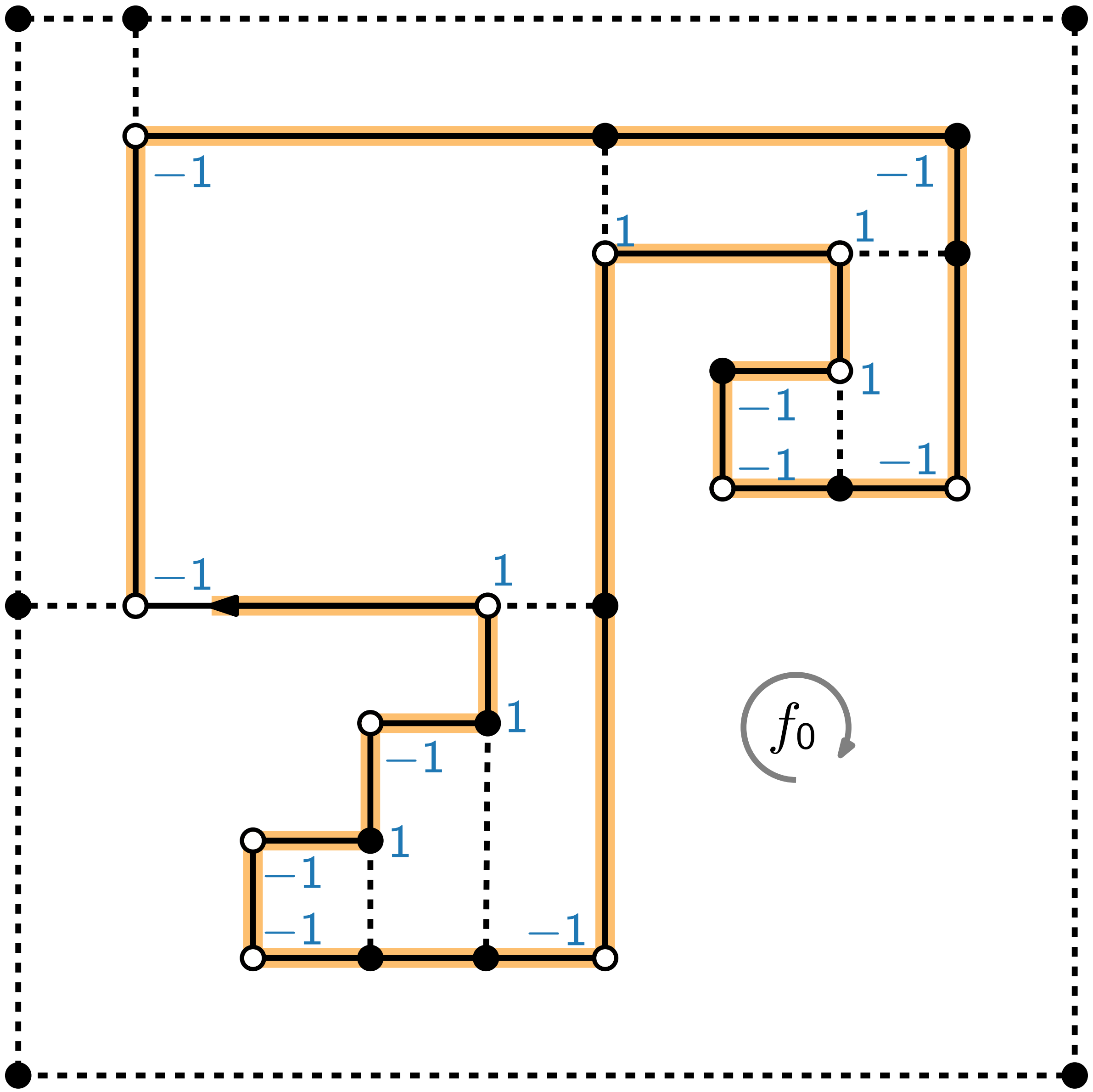

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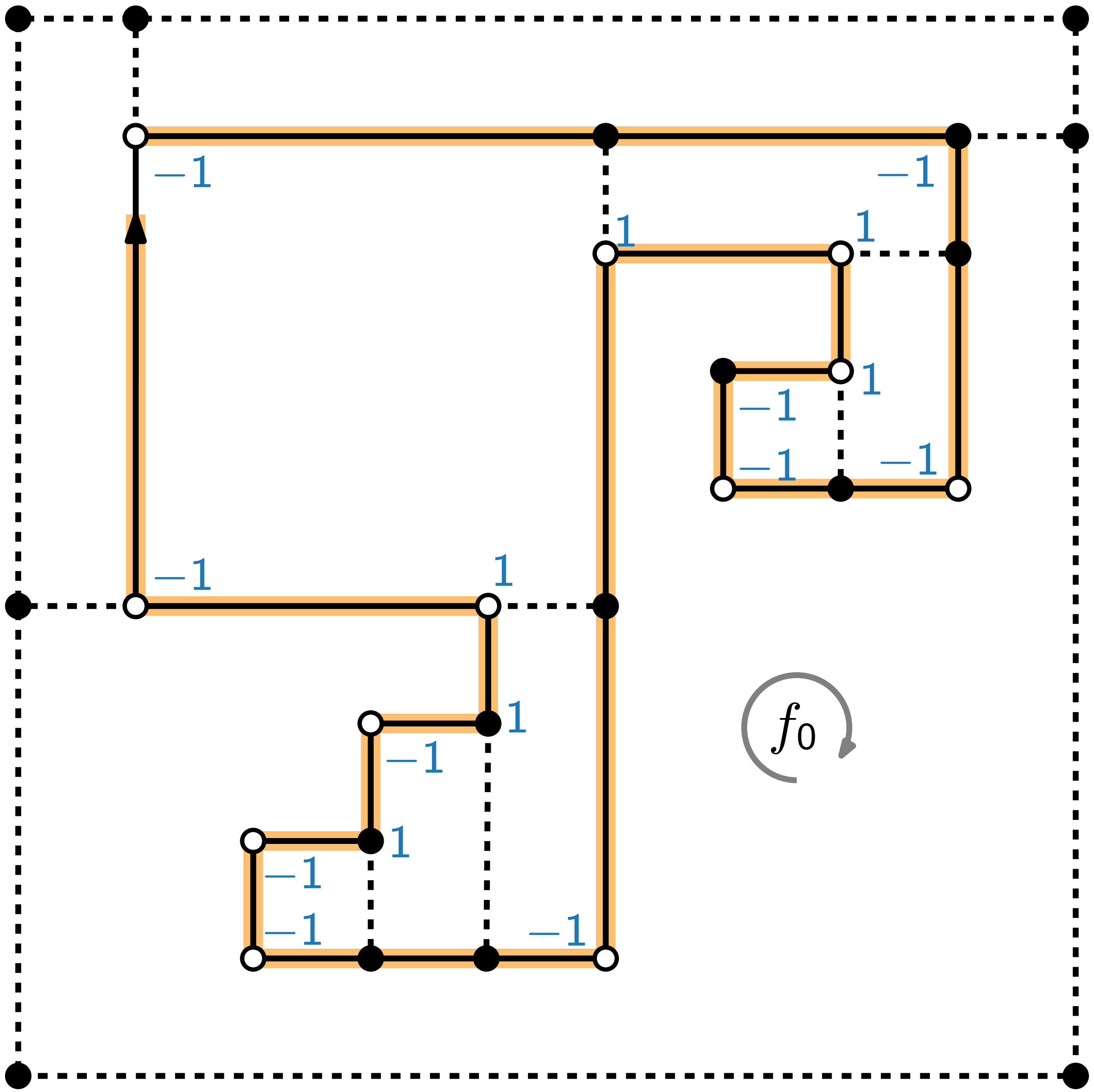

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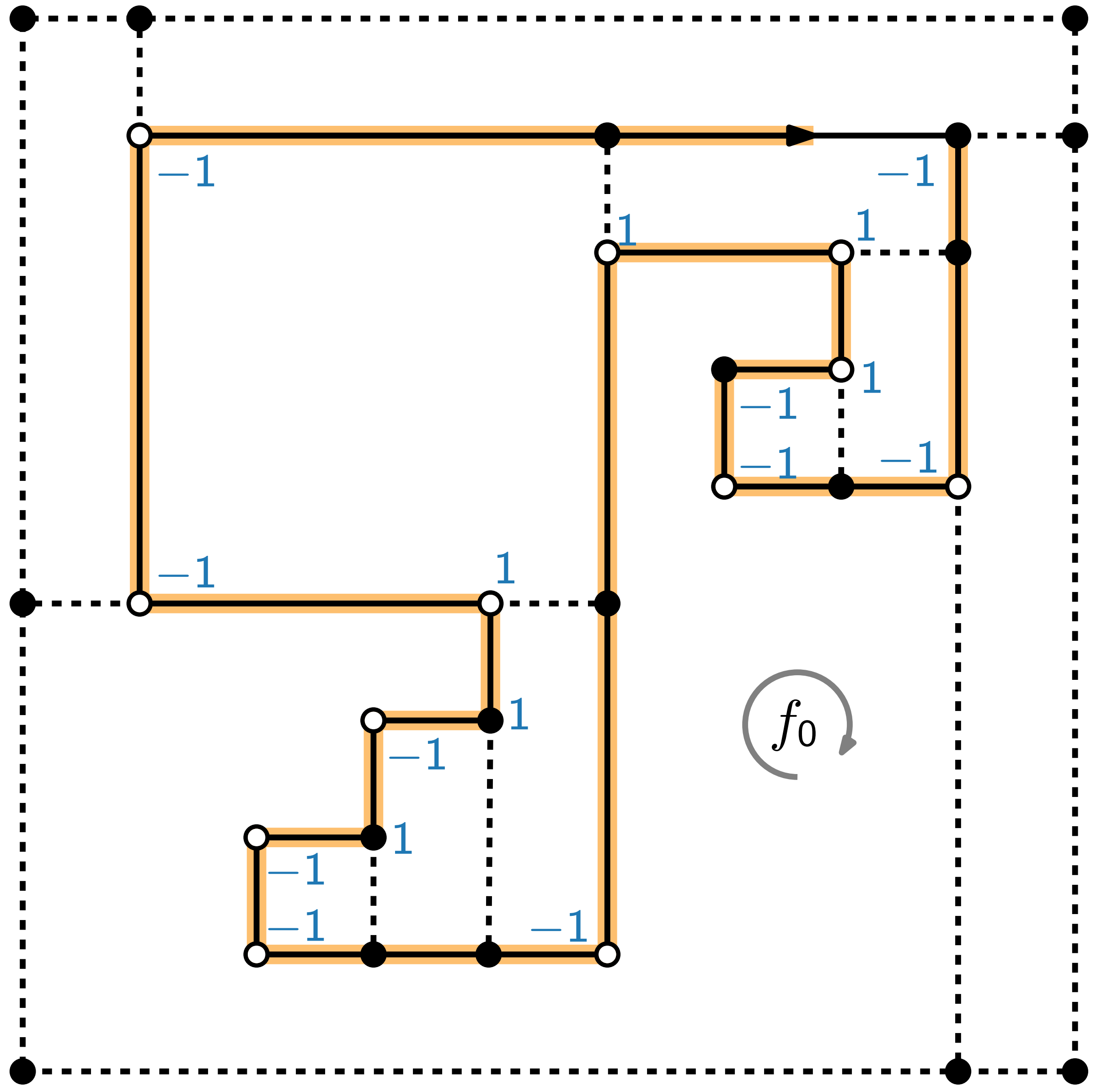

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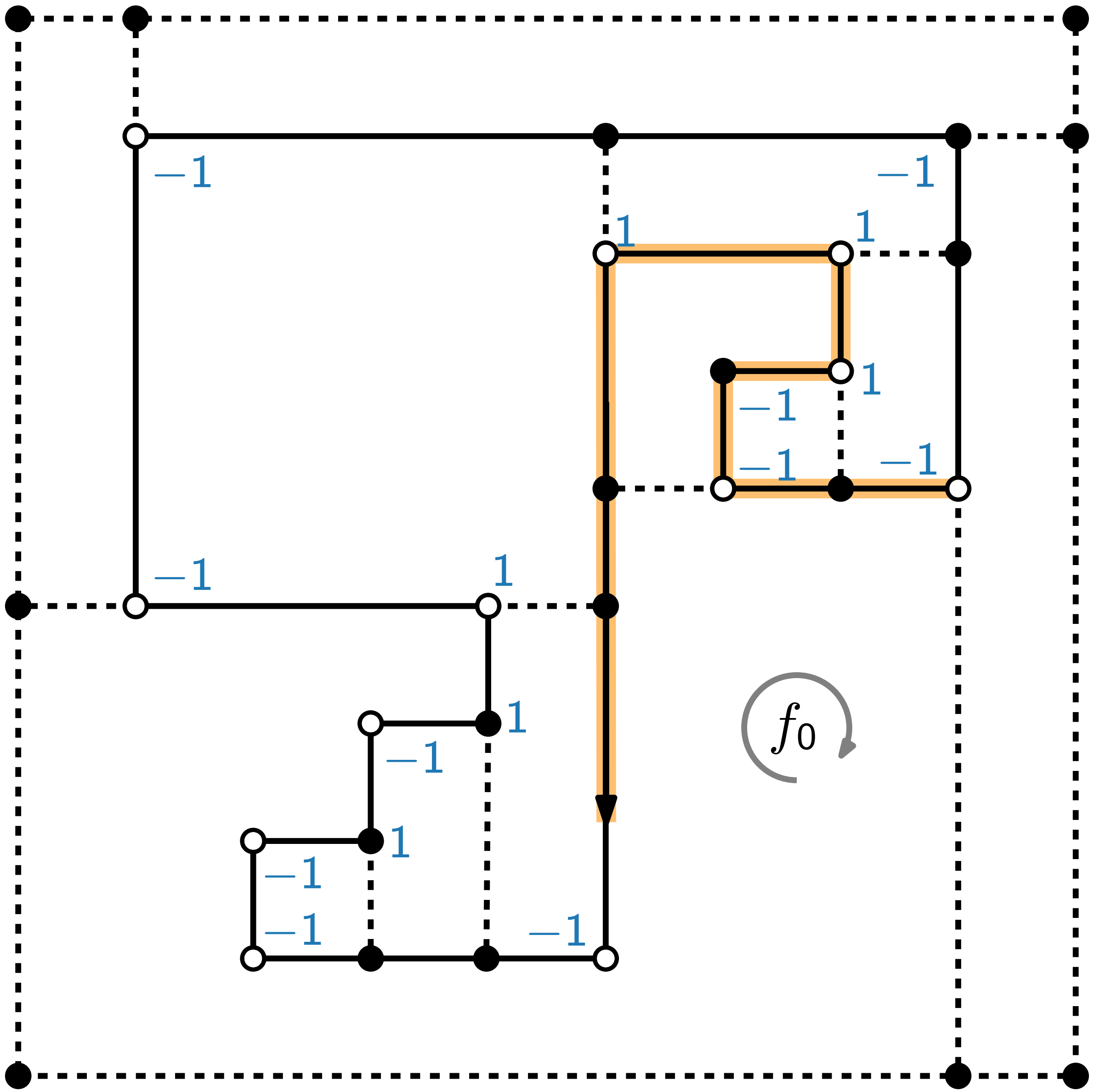

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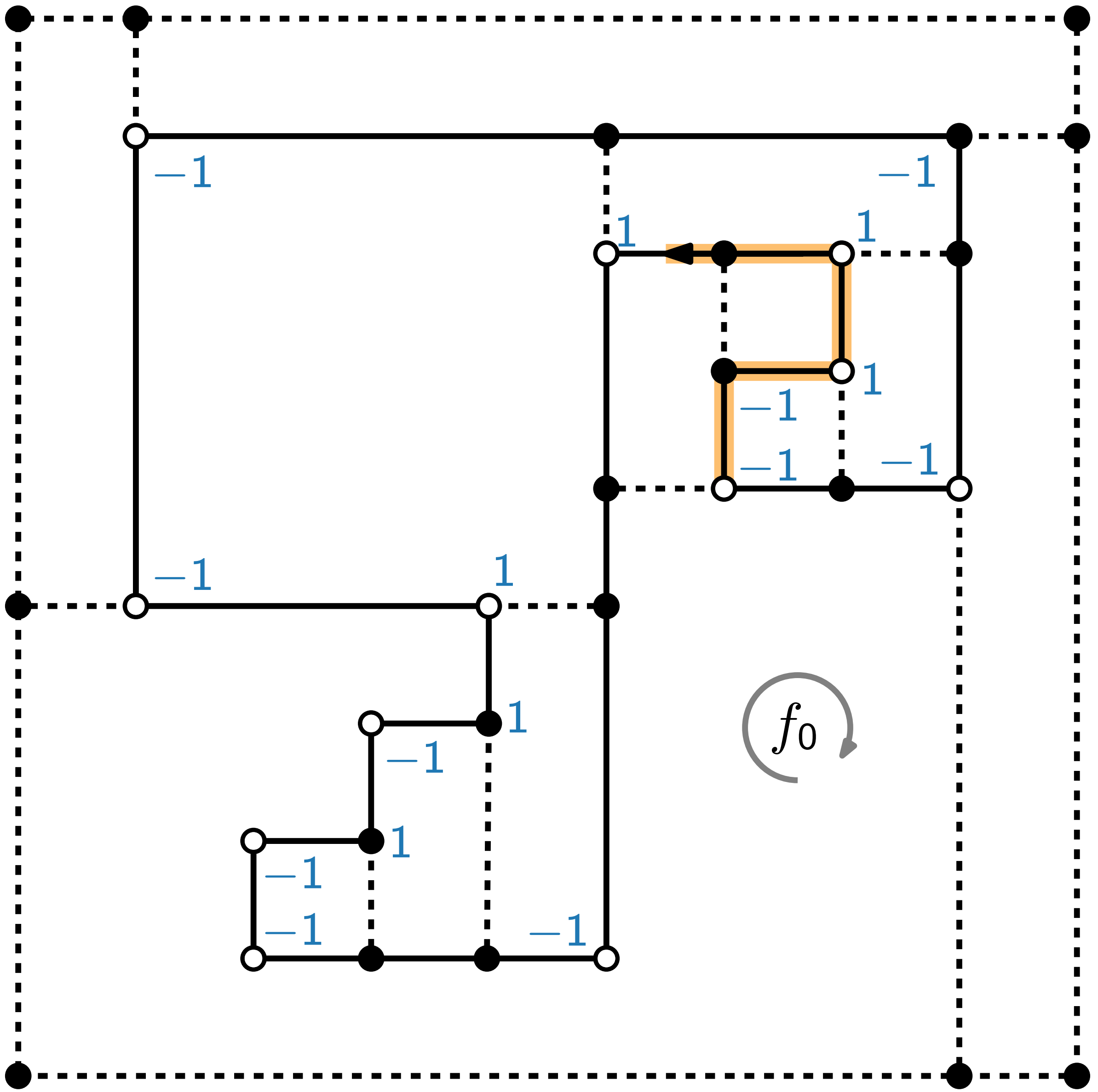

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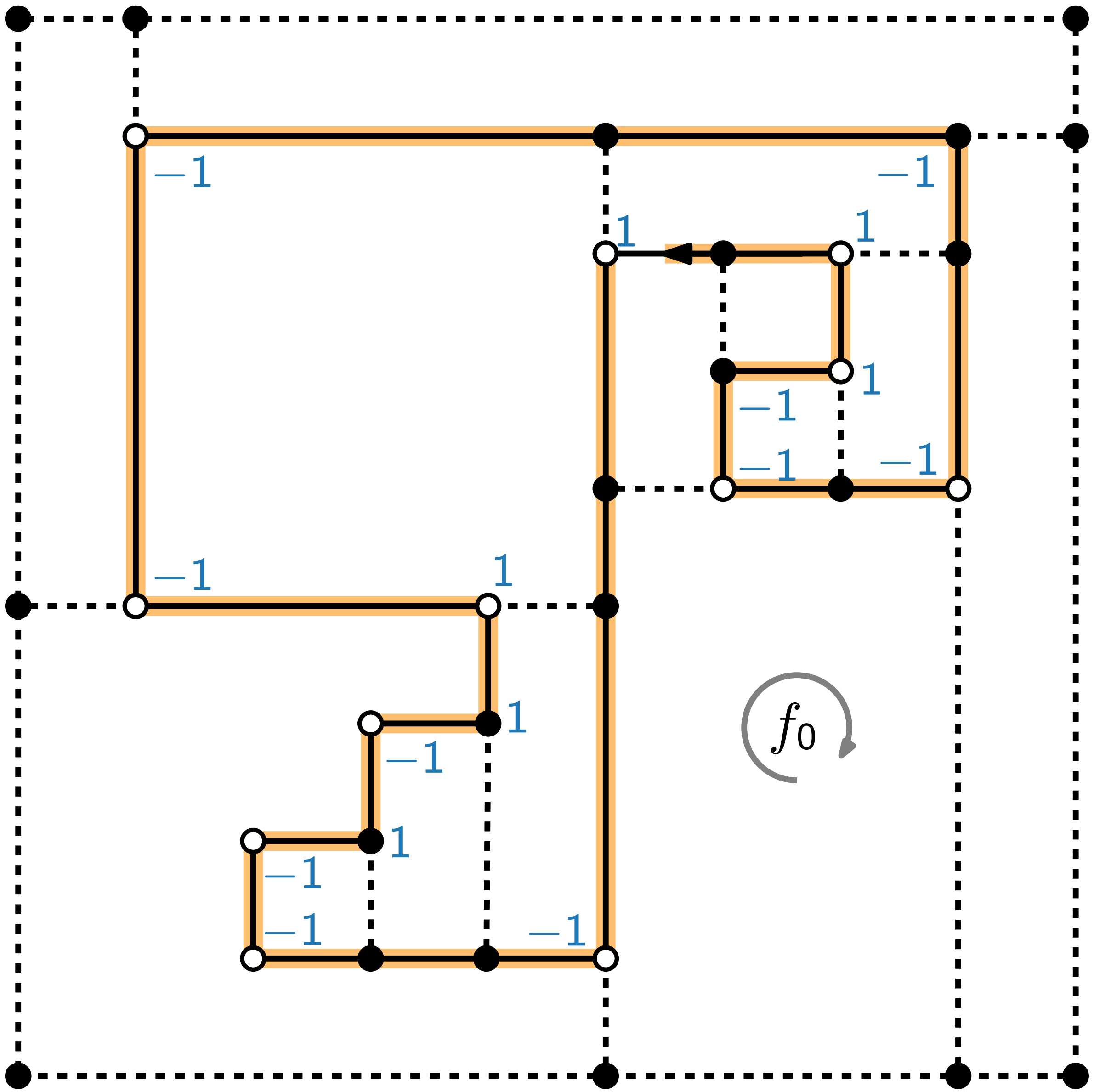

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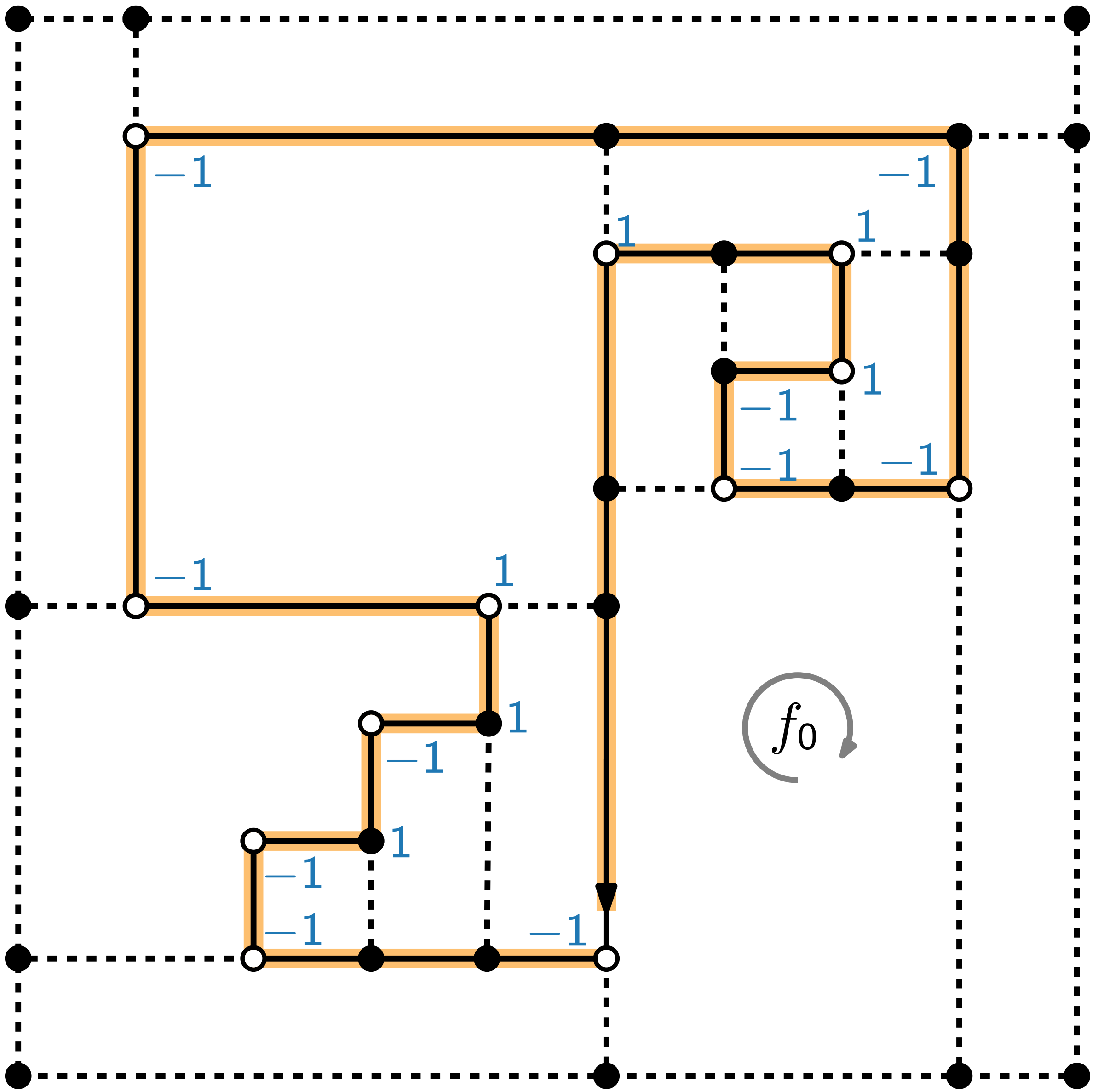

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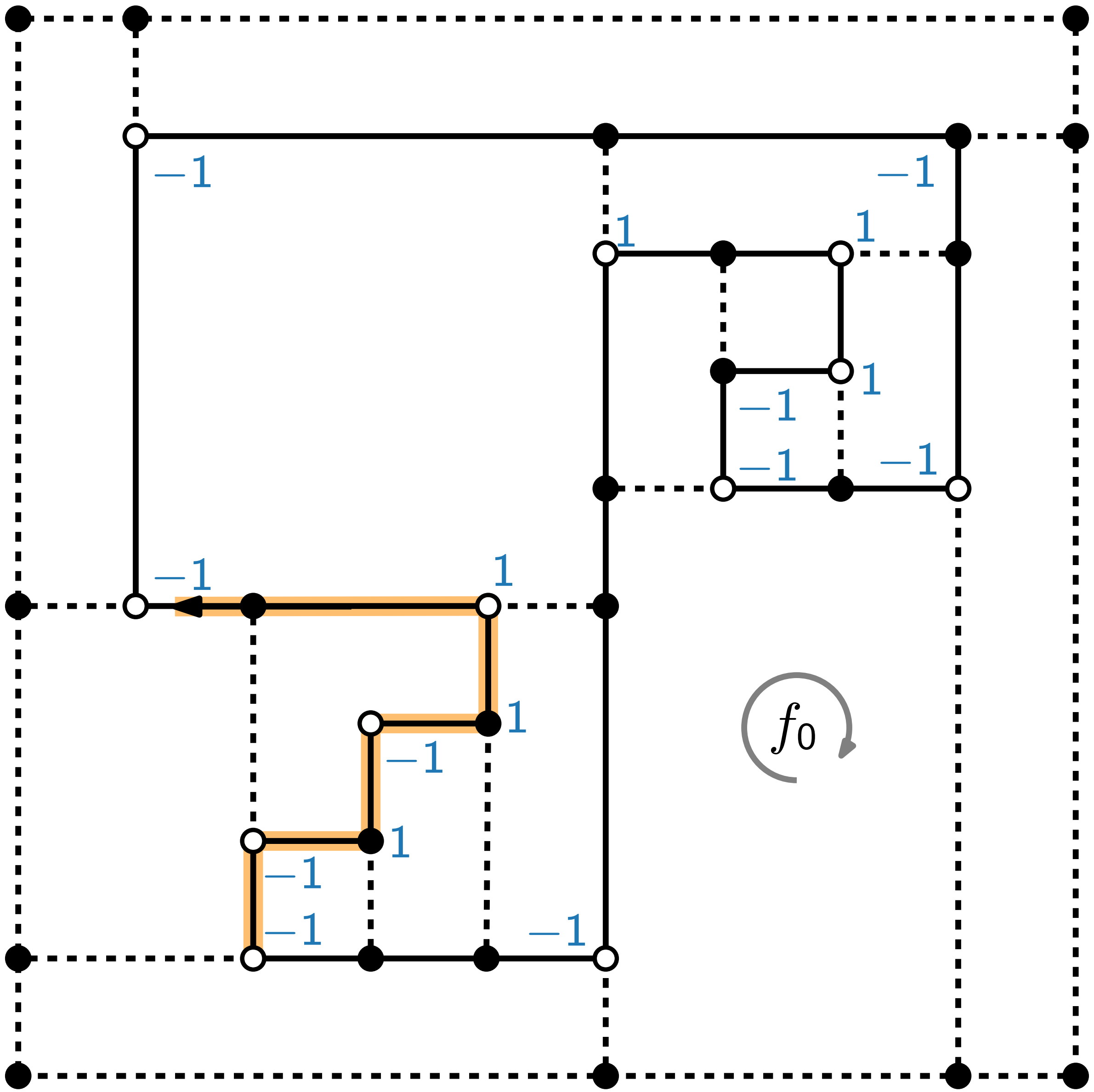

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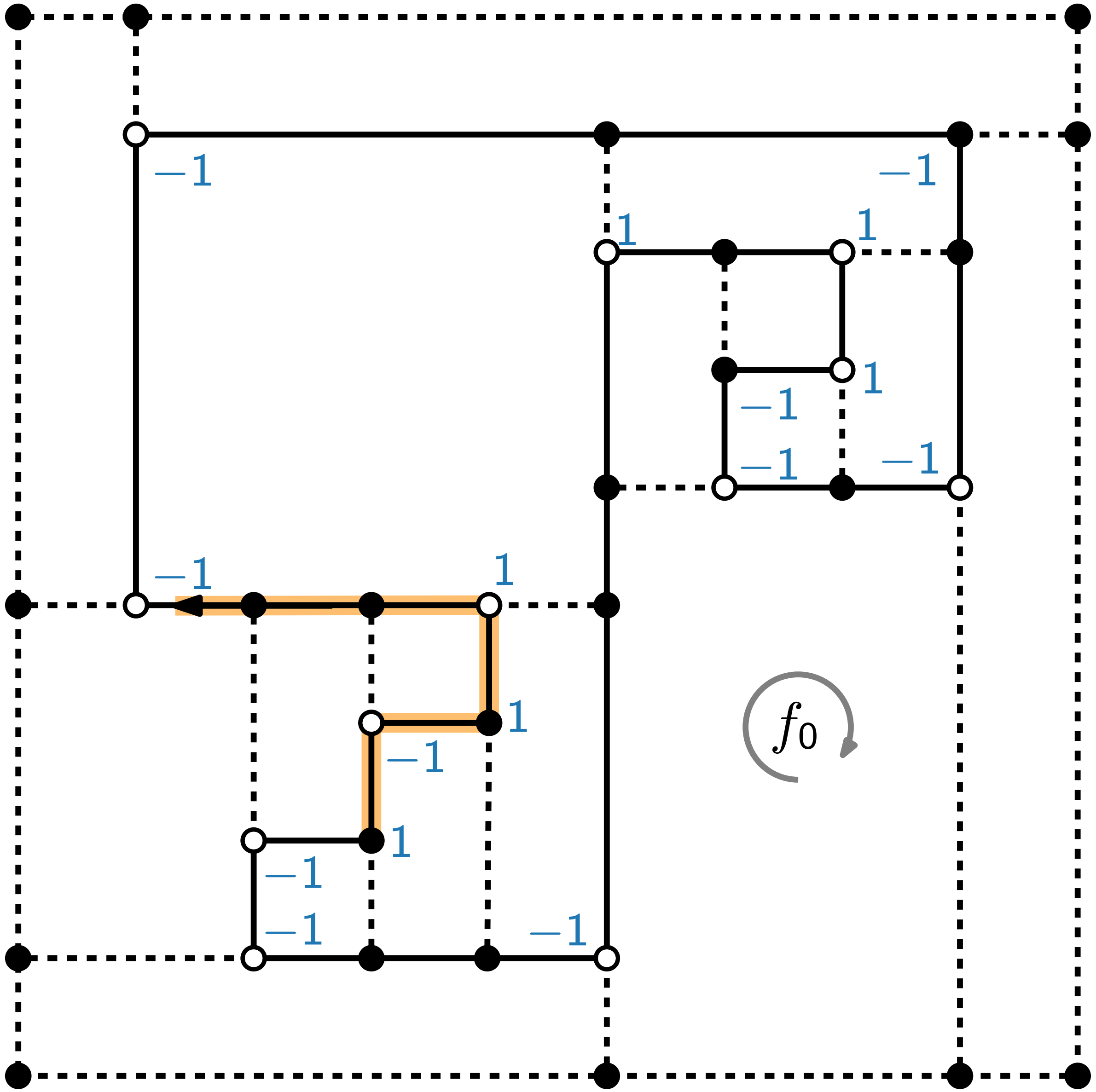

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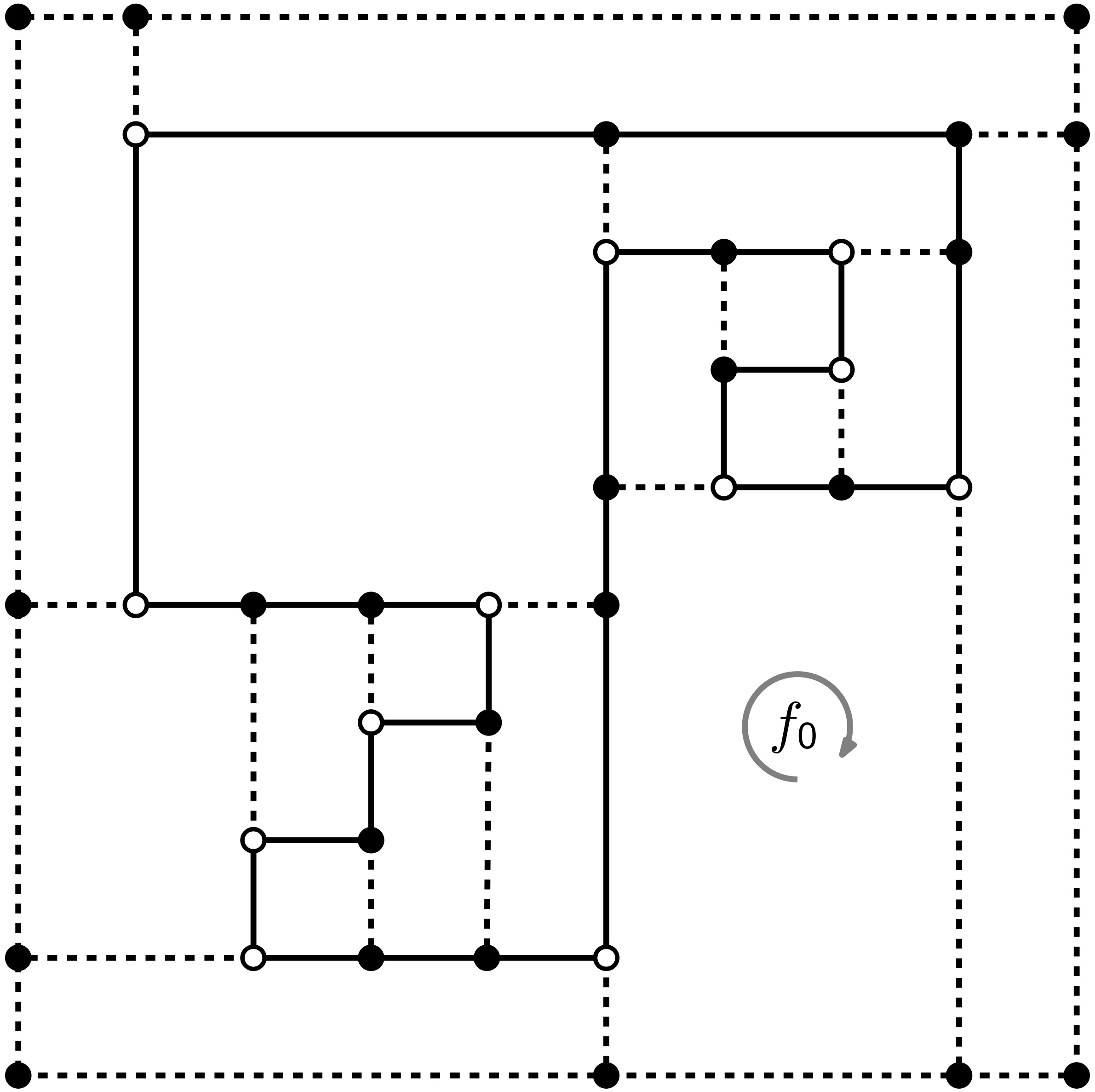

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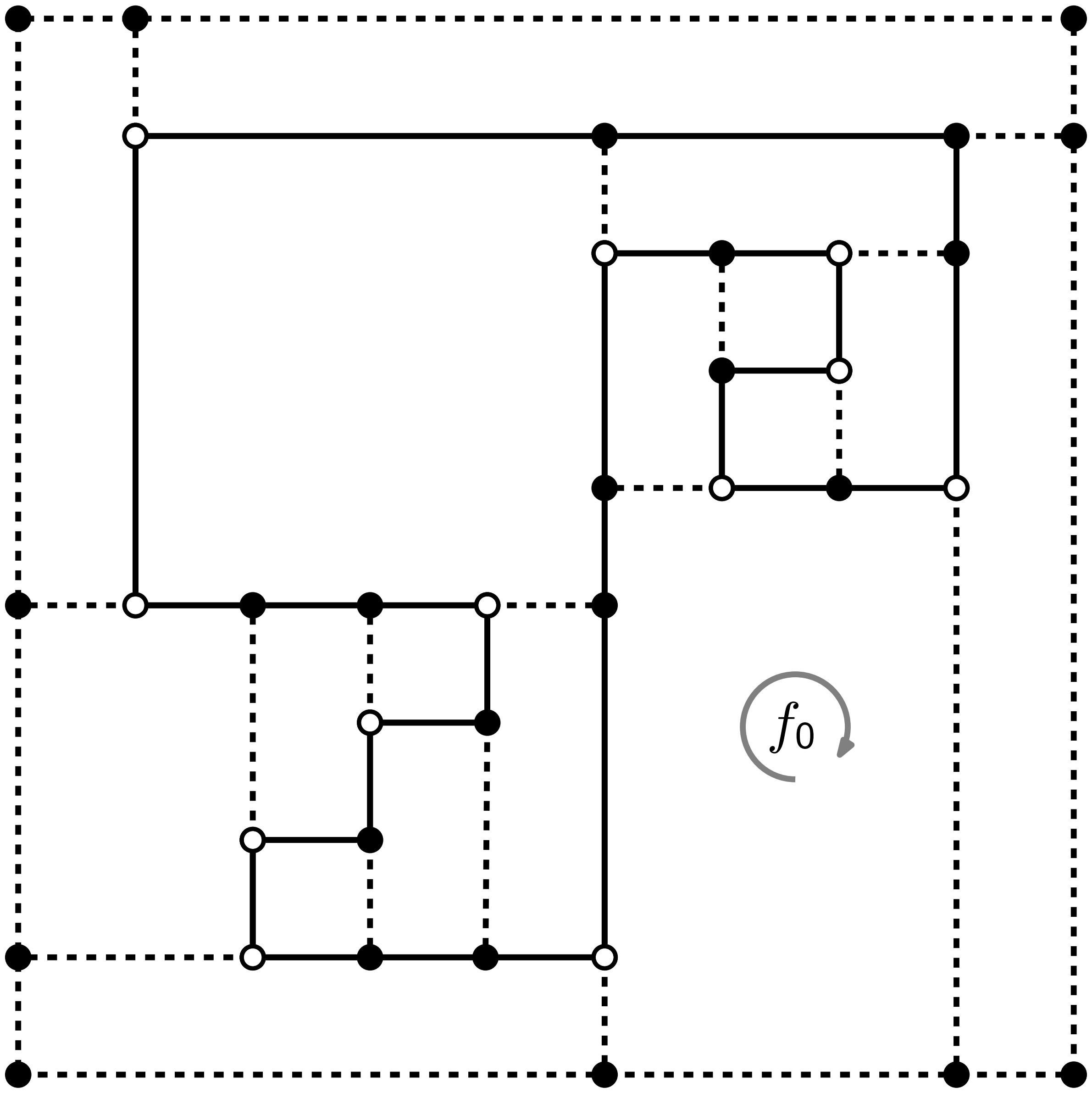

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Area minimized?

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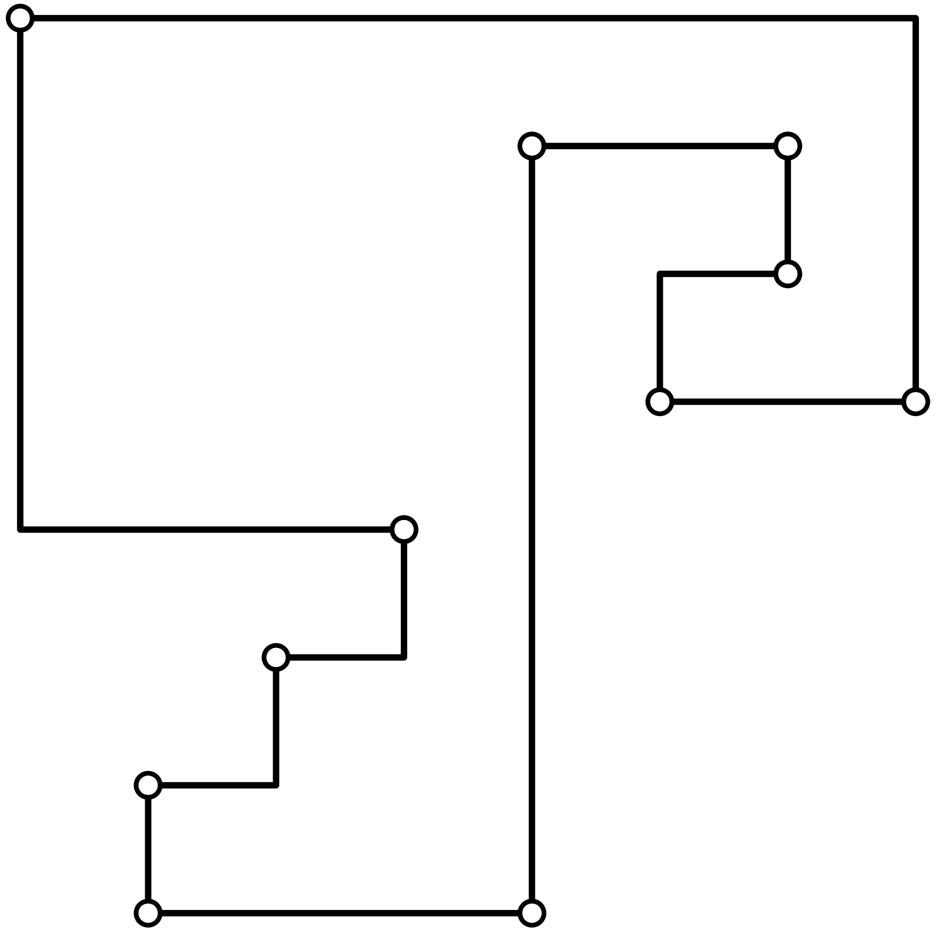

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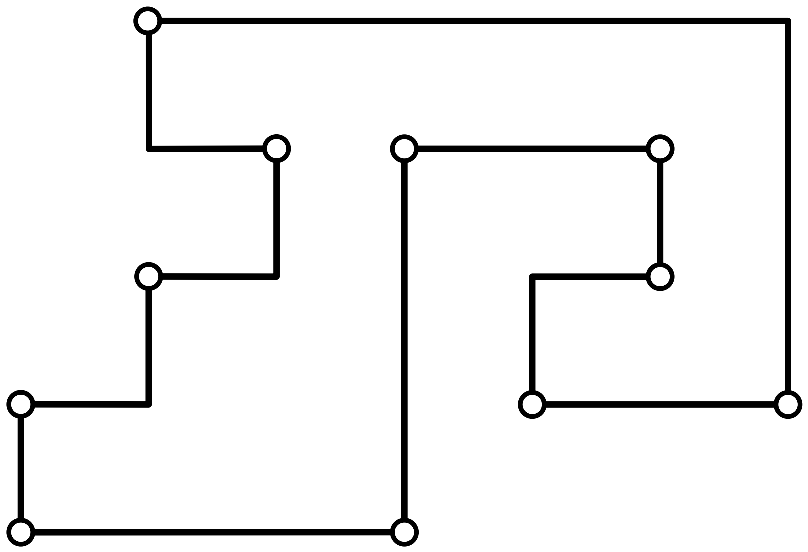

Area minimized? No!

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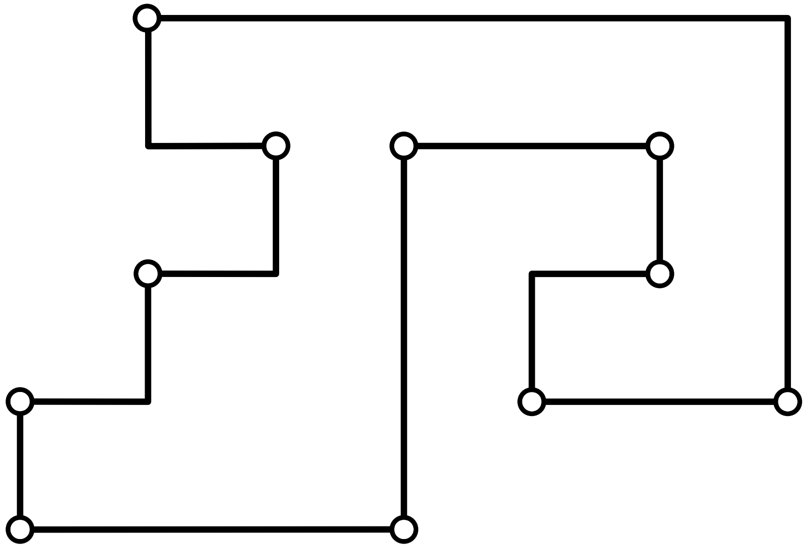

Area minimized? No!

But we get bound $O((n+b)^2)$ on the area.

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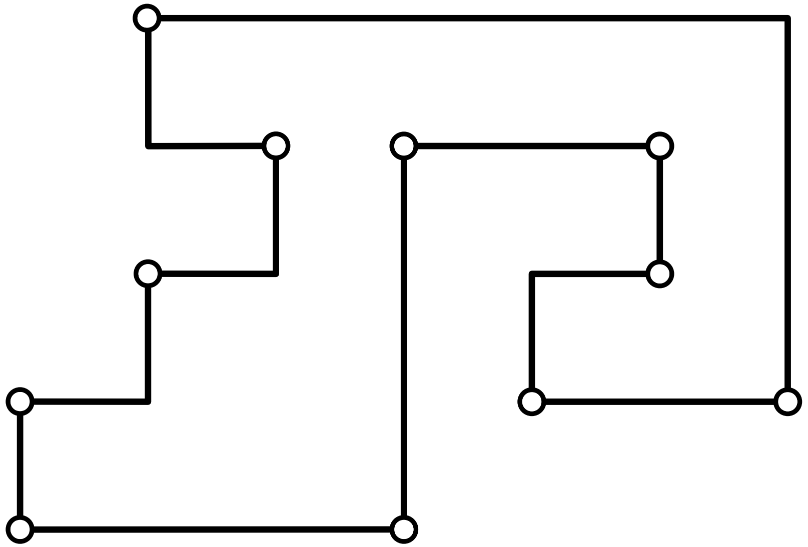

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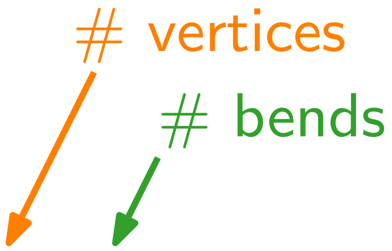

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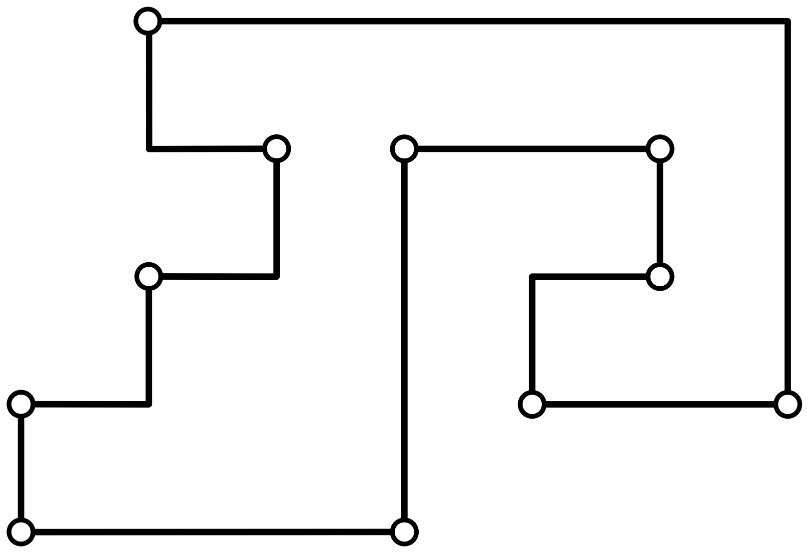

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vertices

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Compaction for a given orthogonal representation is NP-hard in general.

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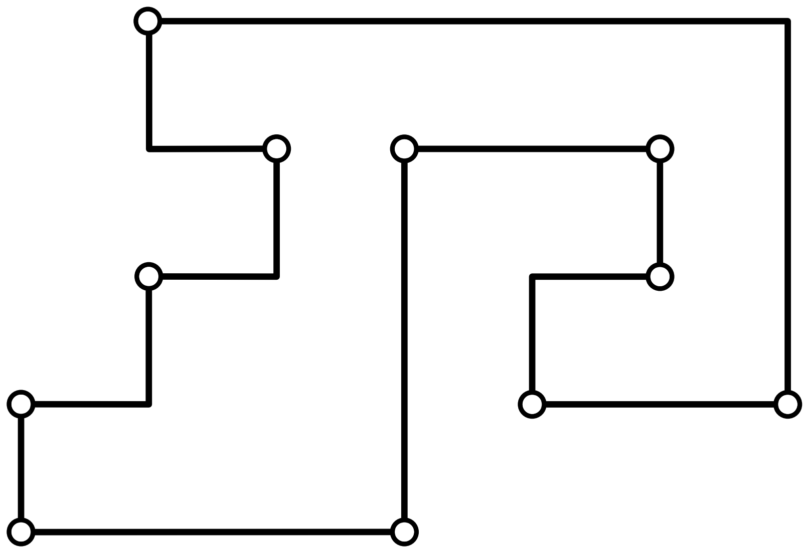

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Compaction is NP-hard even for orthogonal representations of *cycles*.

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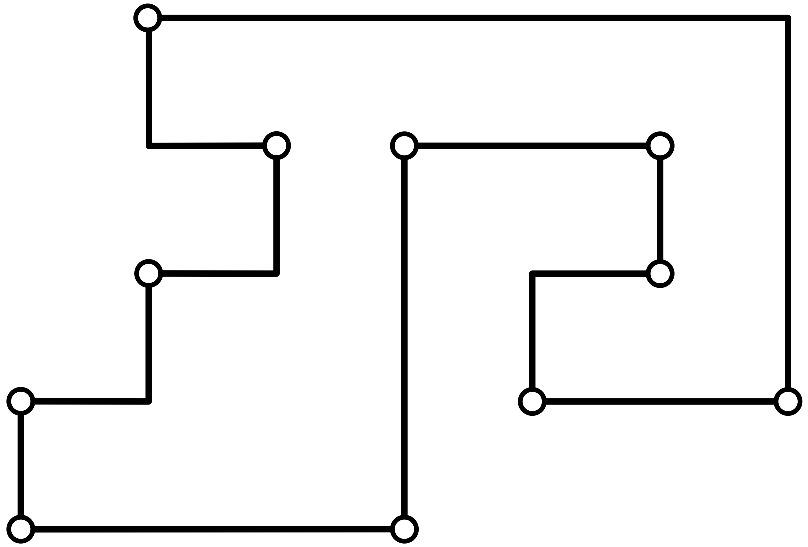

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a literal is a variable x or a negated variable $\neg x$

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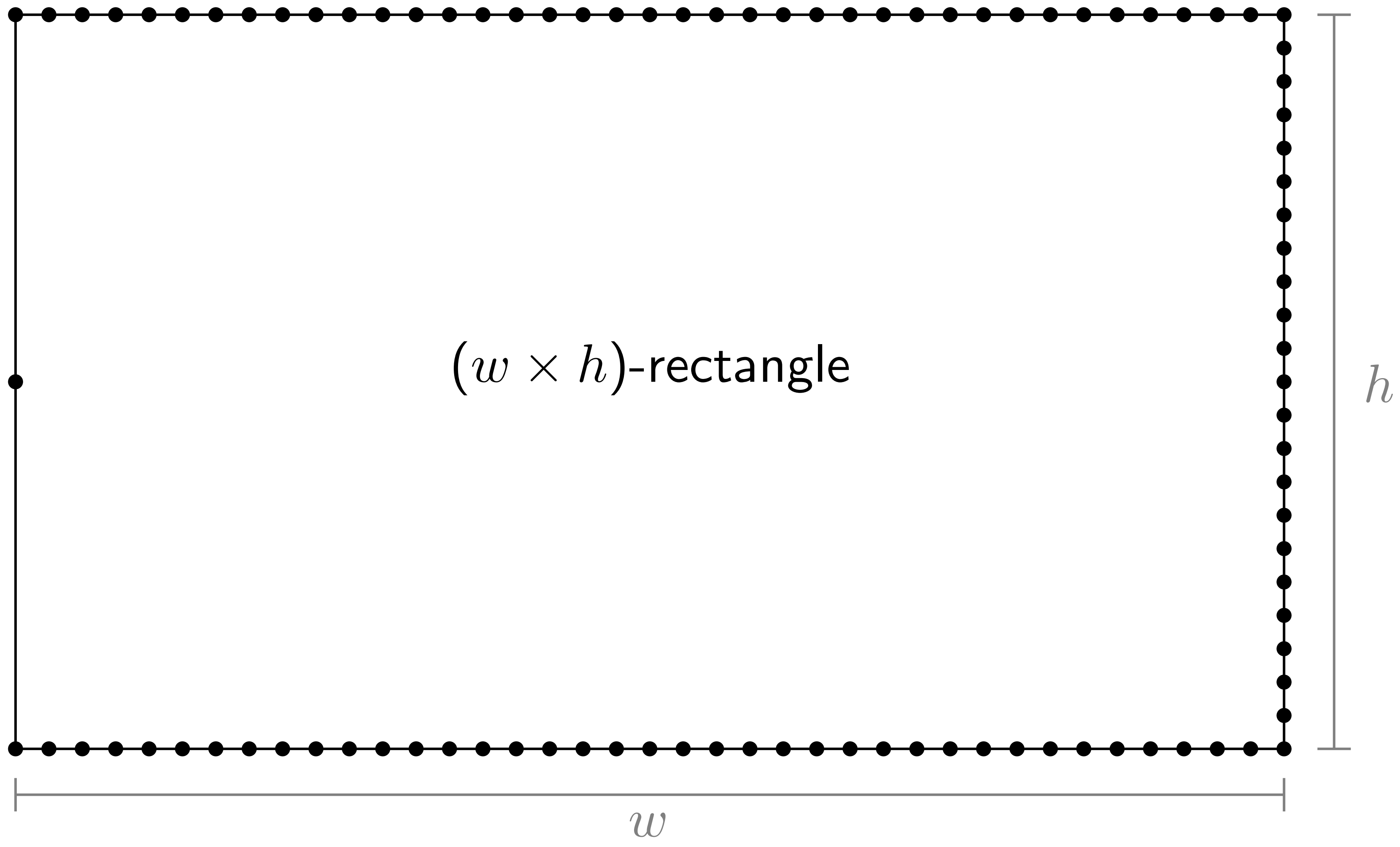
- Given SAT instance $\Phi \Rightarrow$ construct a plane graph G and a orthogonal description $H(G)$
- Φ is satisfiable $\Leftrightarrow$ G can be drawn w.r.t. $H(G)$ in area K for some specific number K

Boundary, Belt, and “Piston” Gadget



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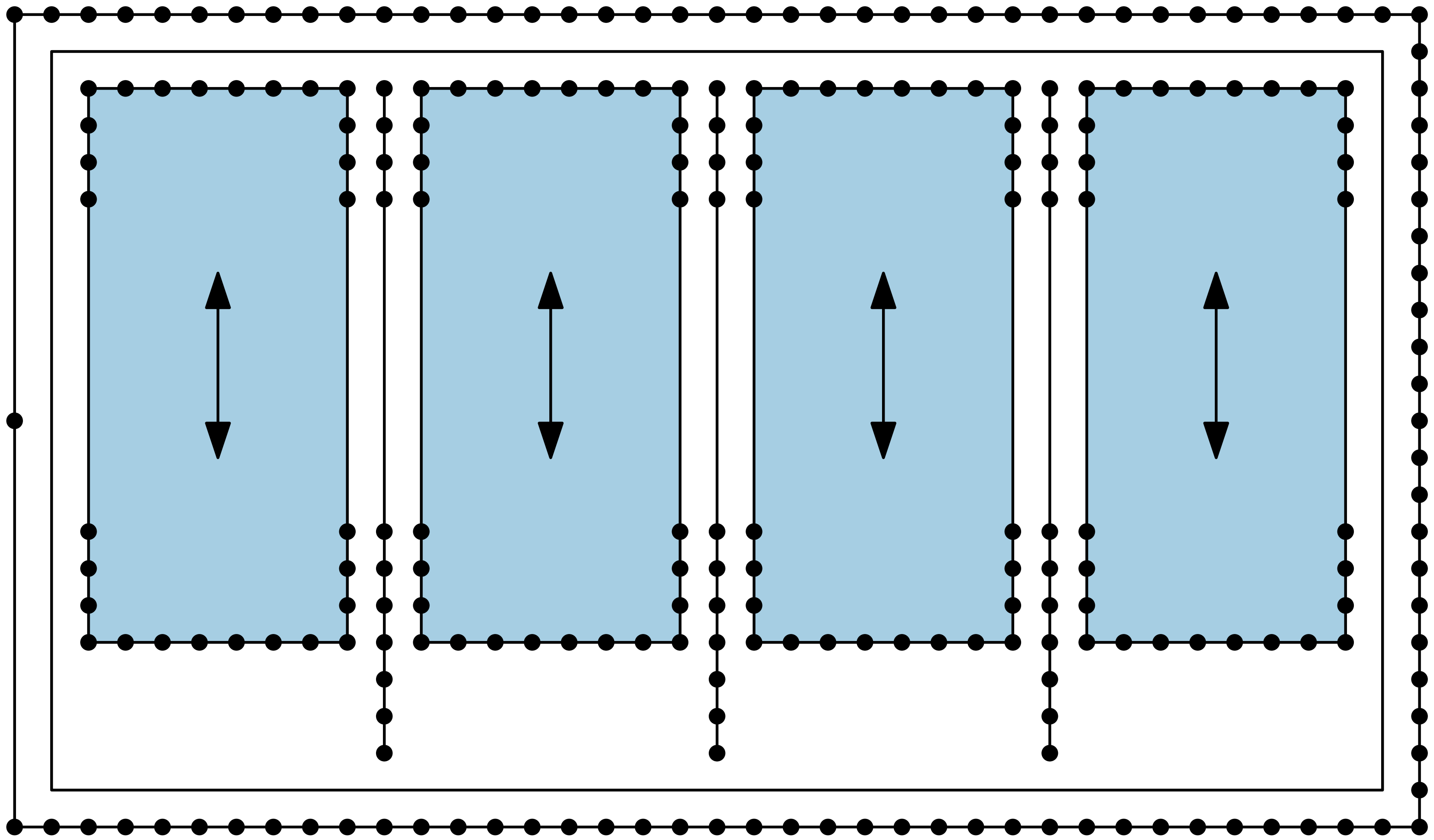

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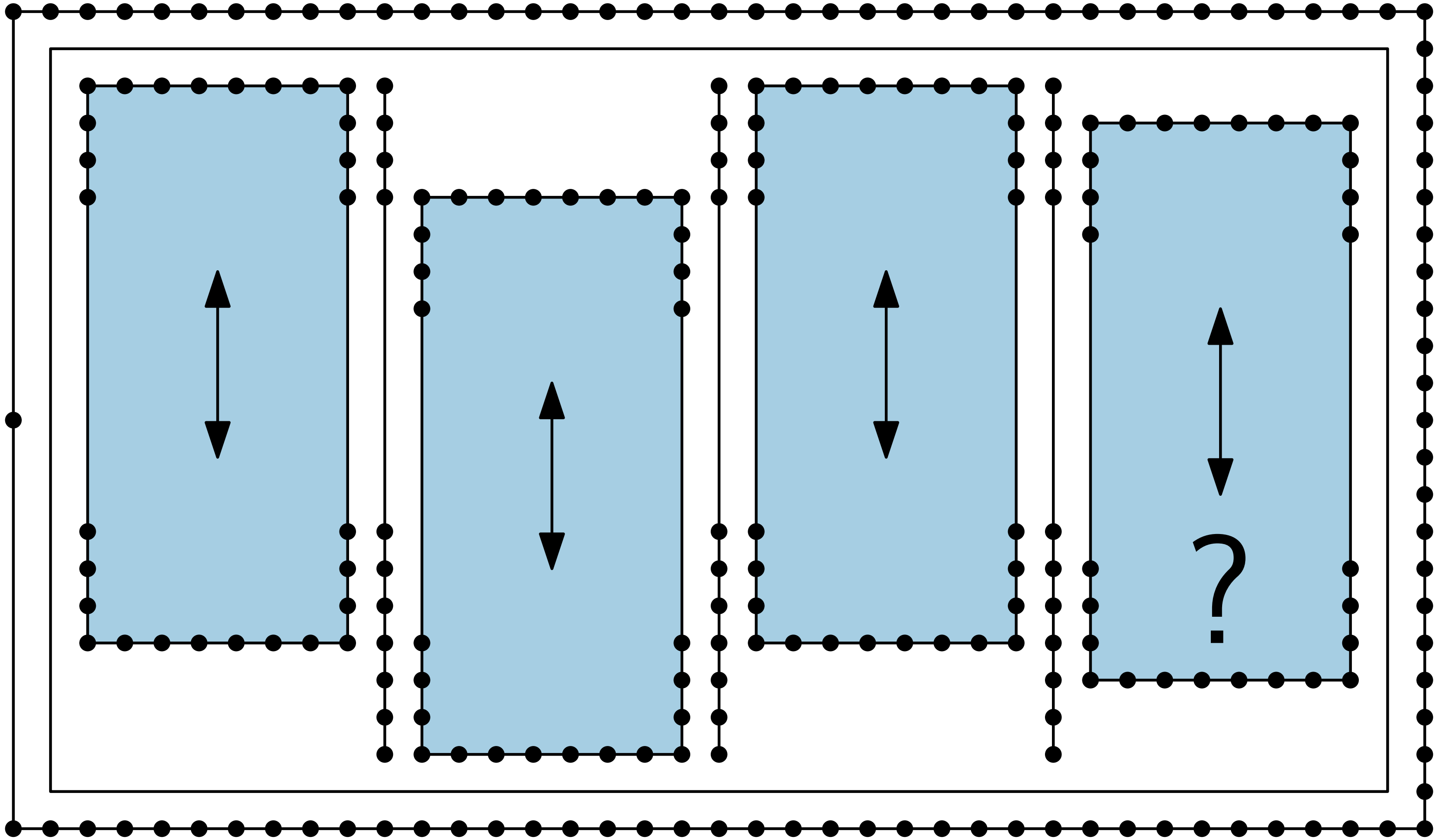

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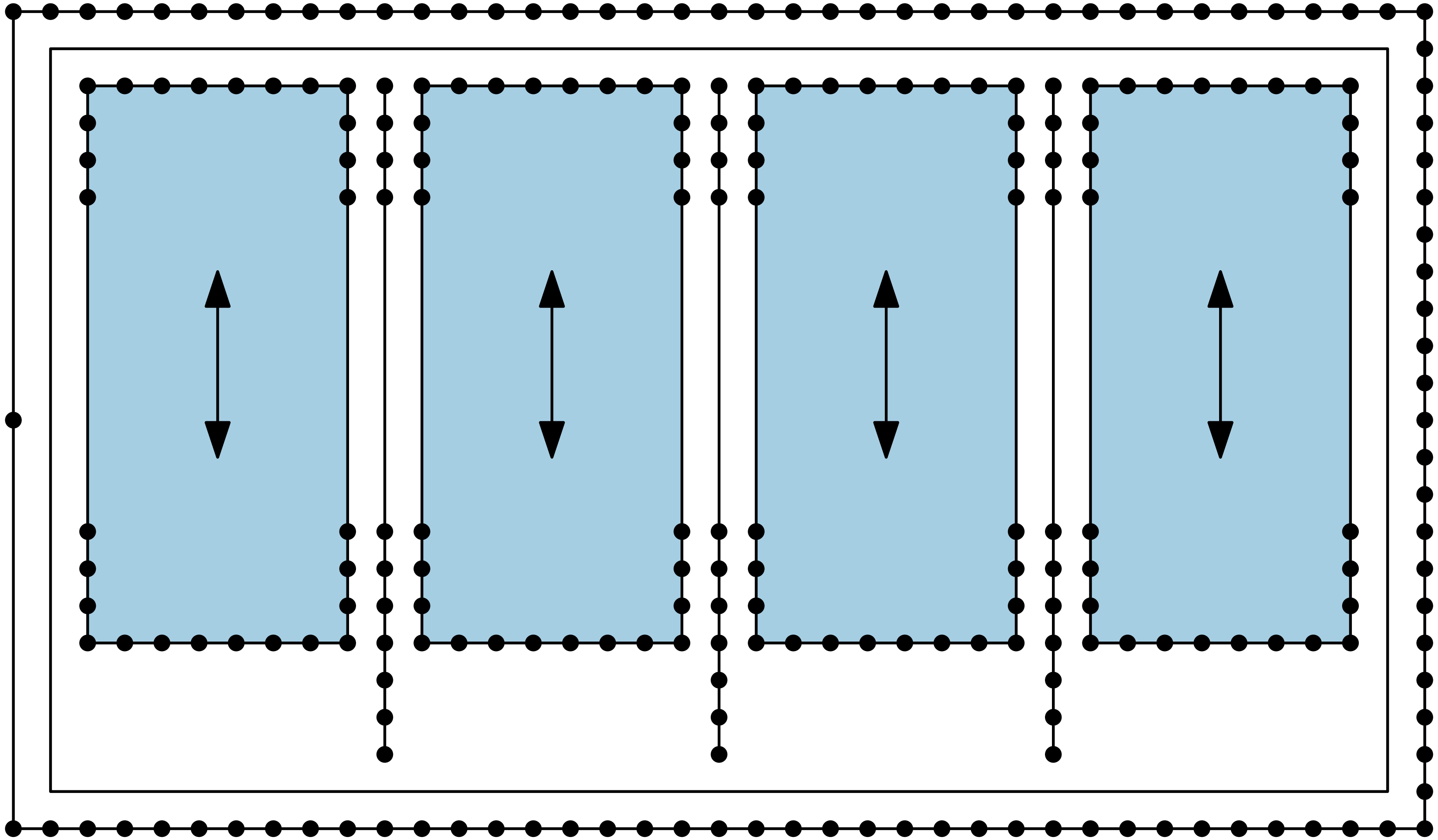

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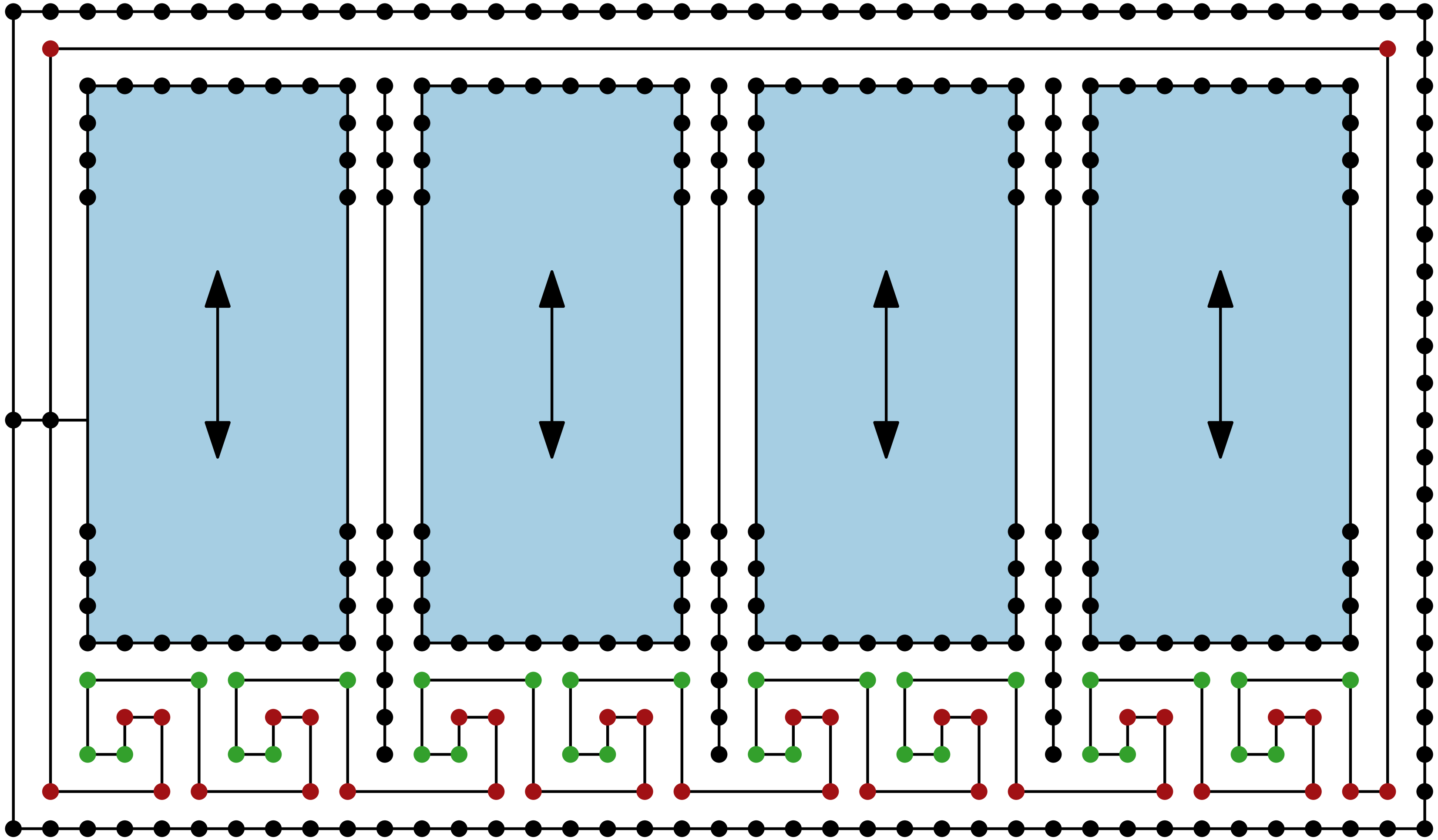

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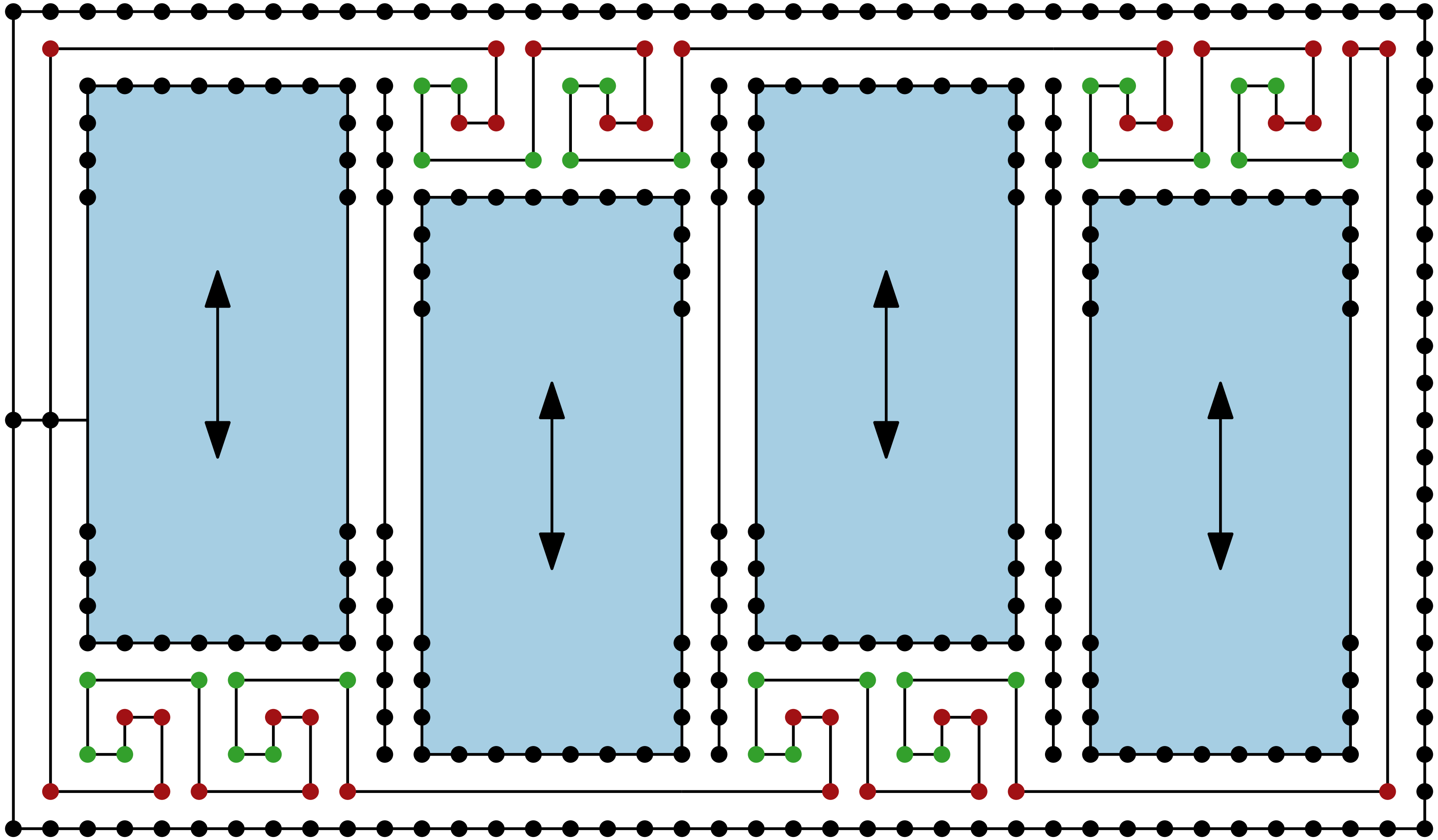

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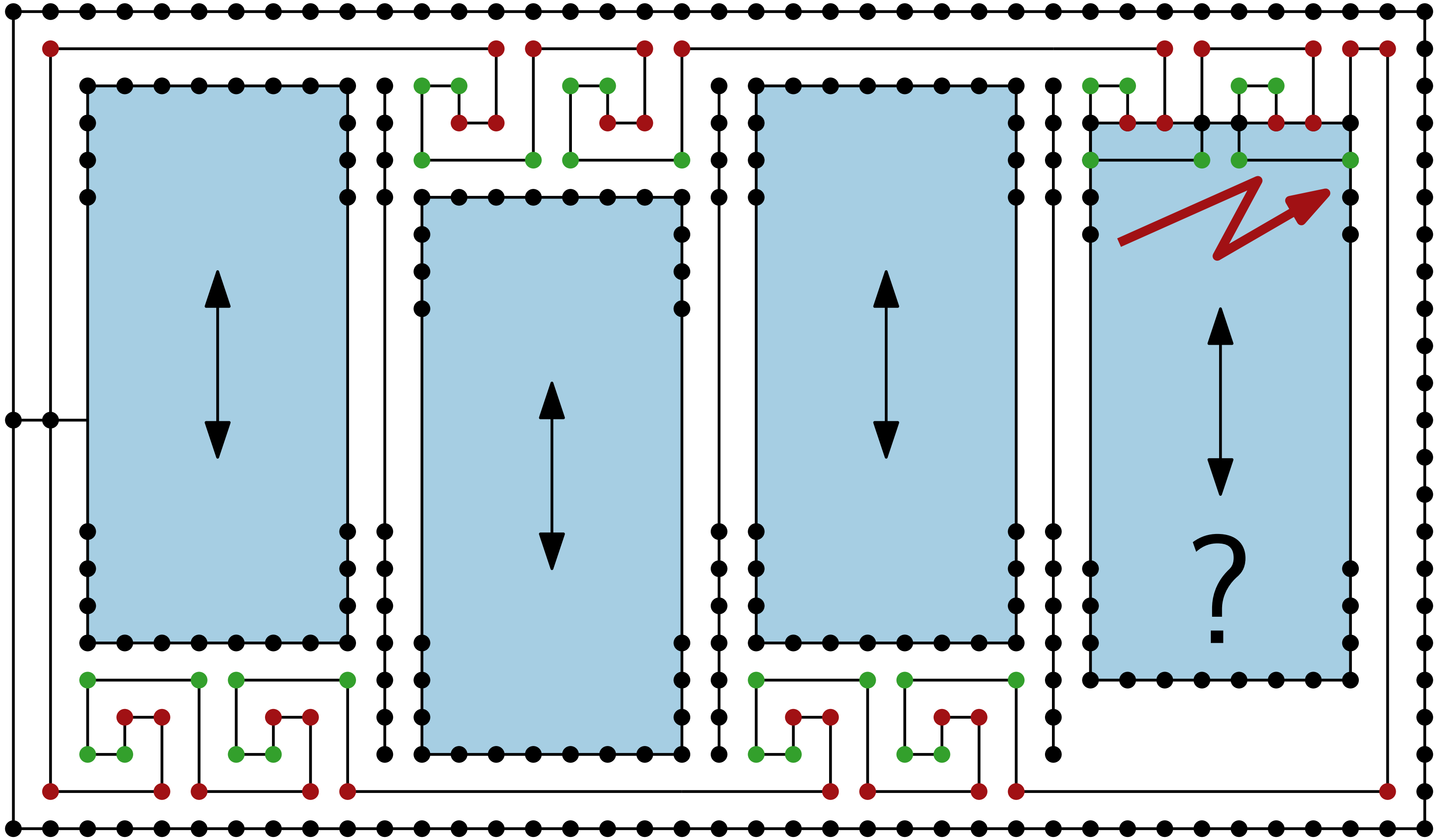

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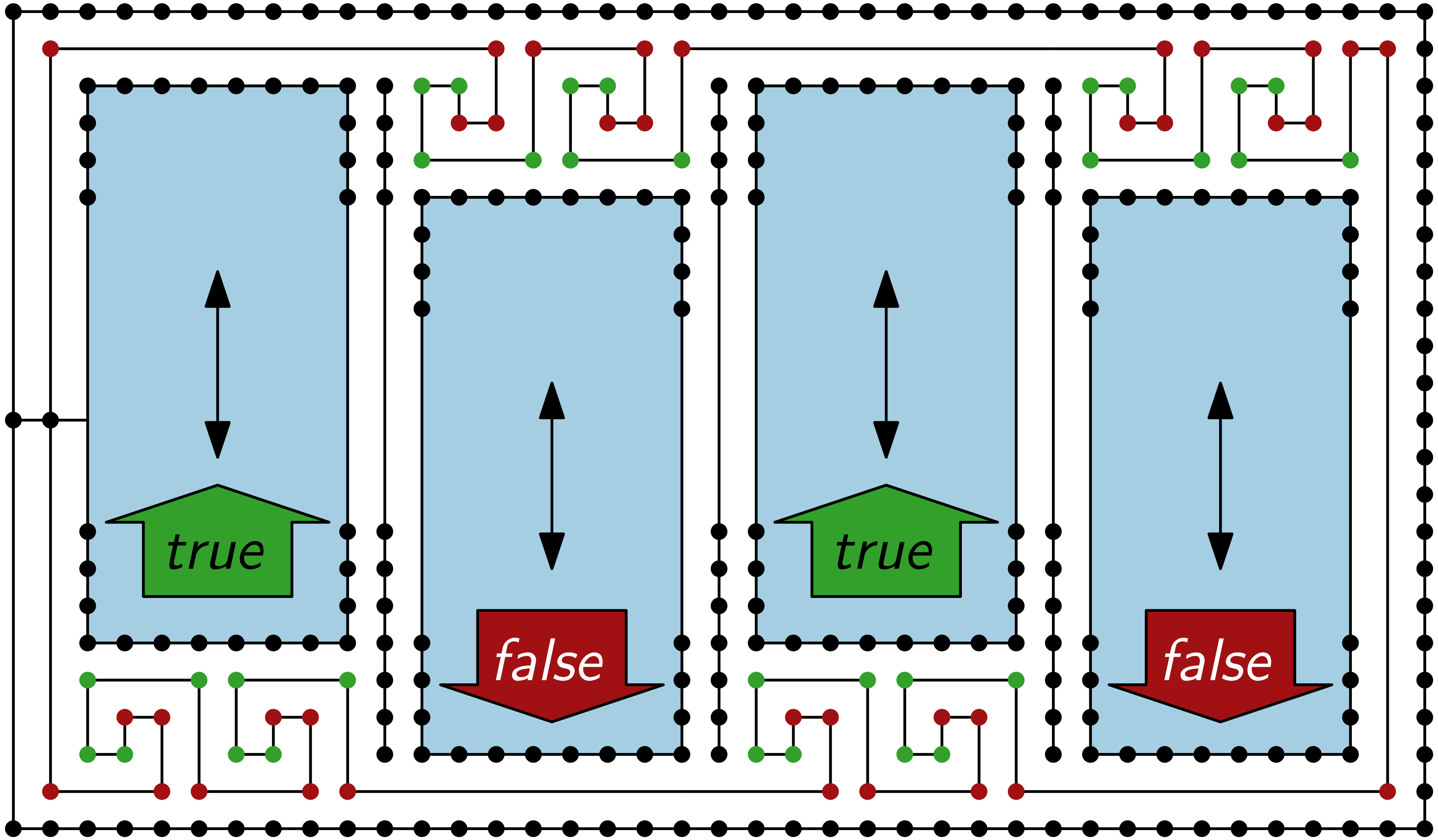
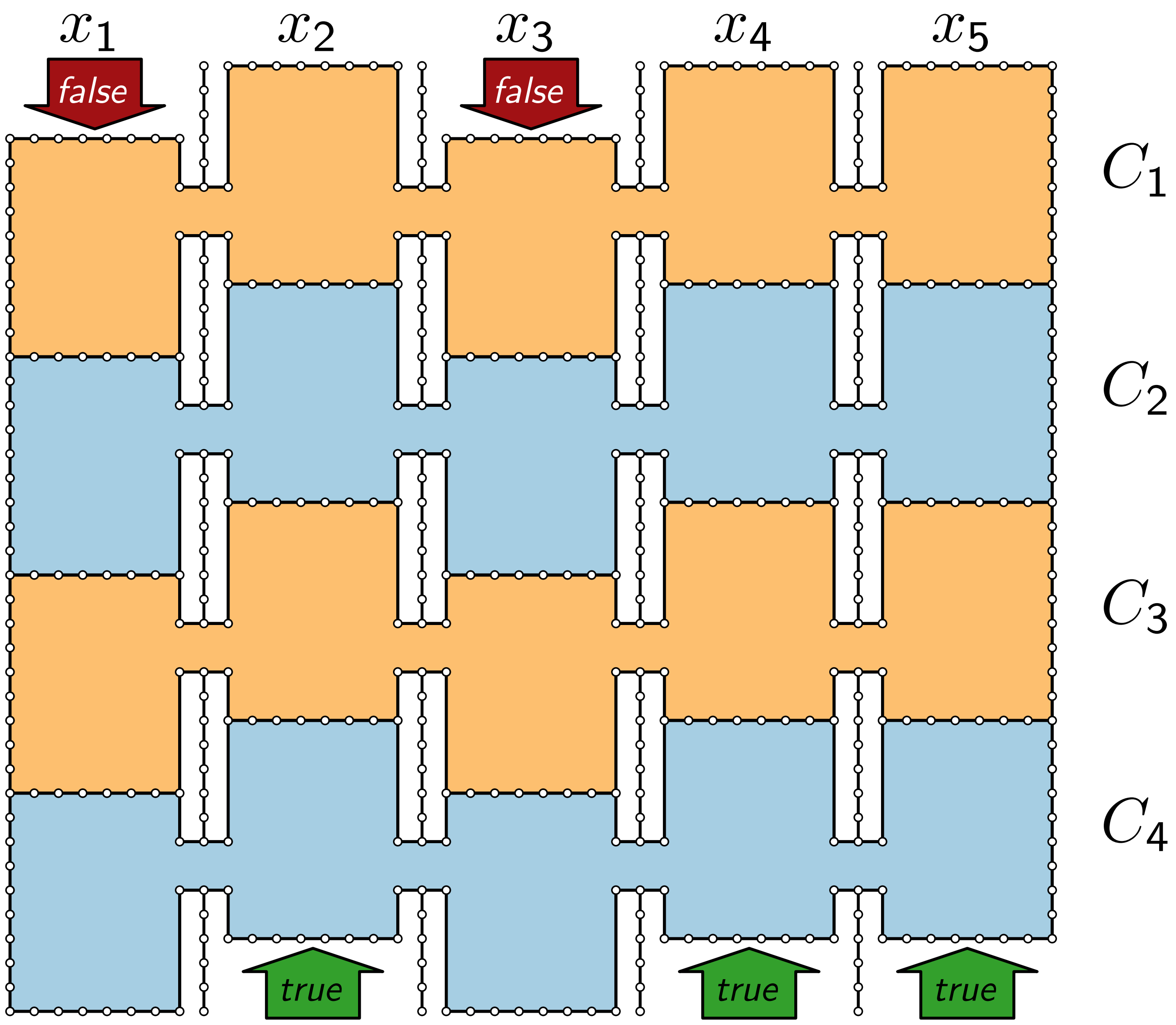


Clause Gadgets



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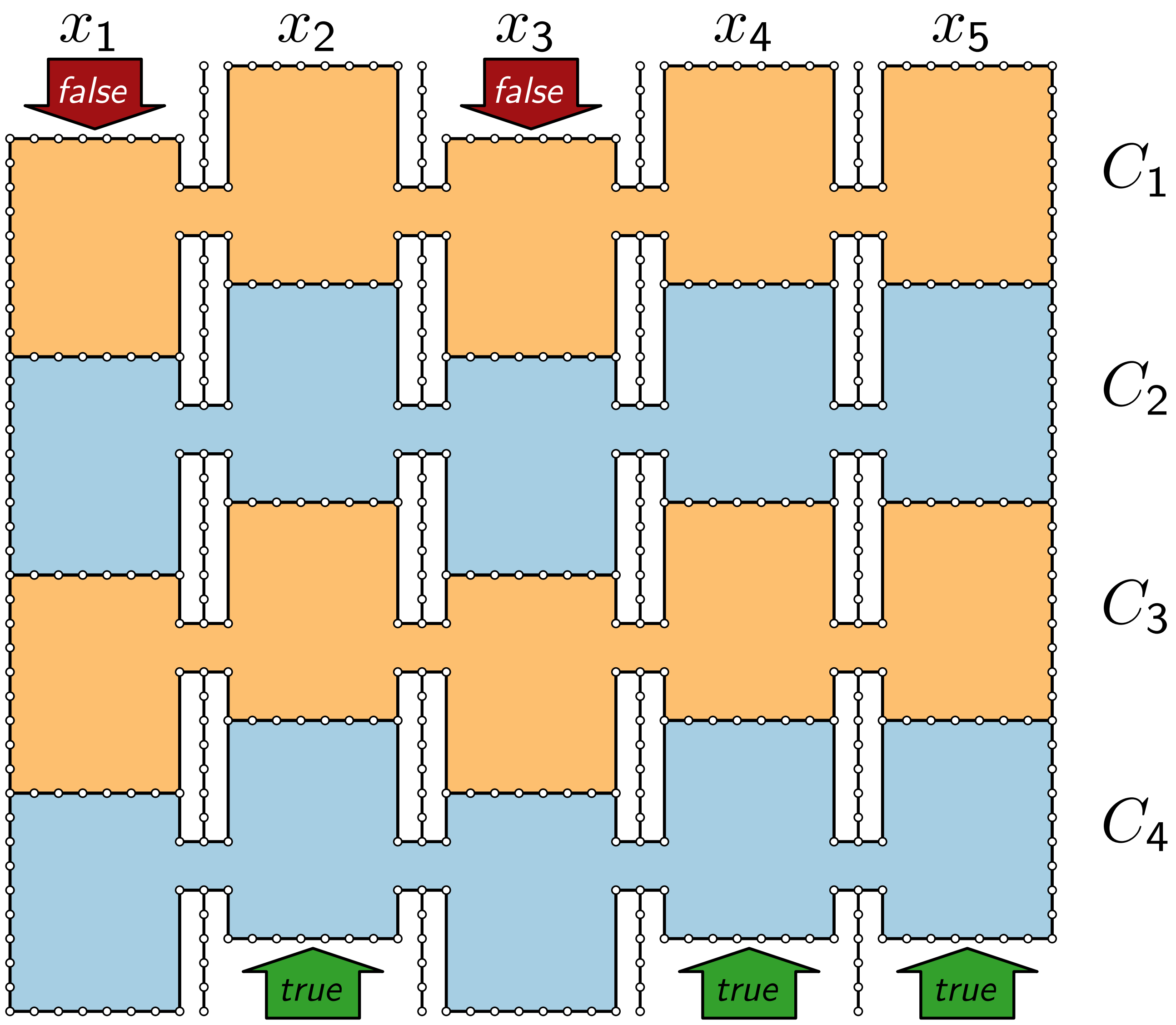

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$C_1 = x_2 \vee \neg x_4$

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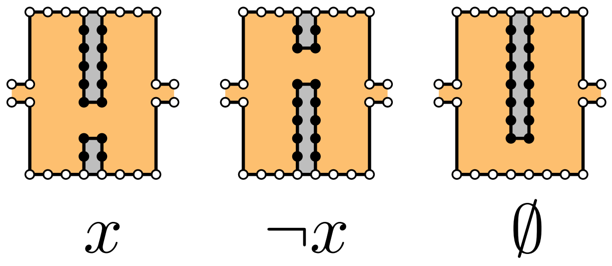

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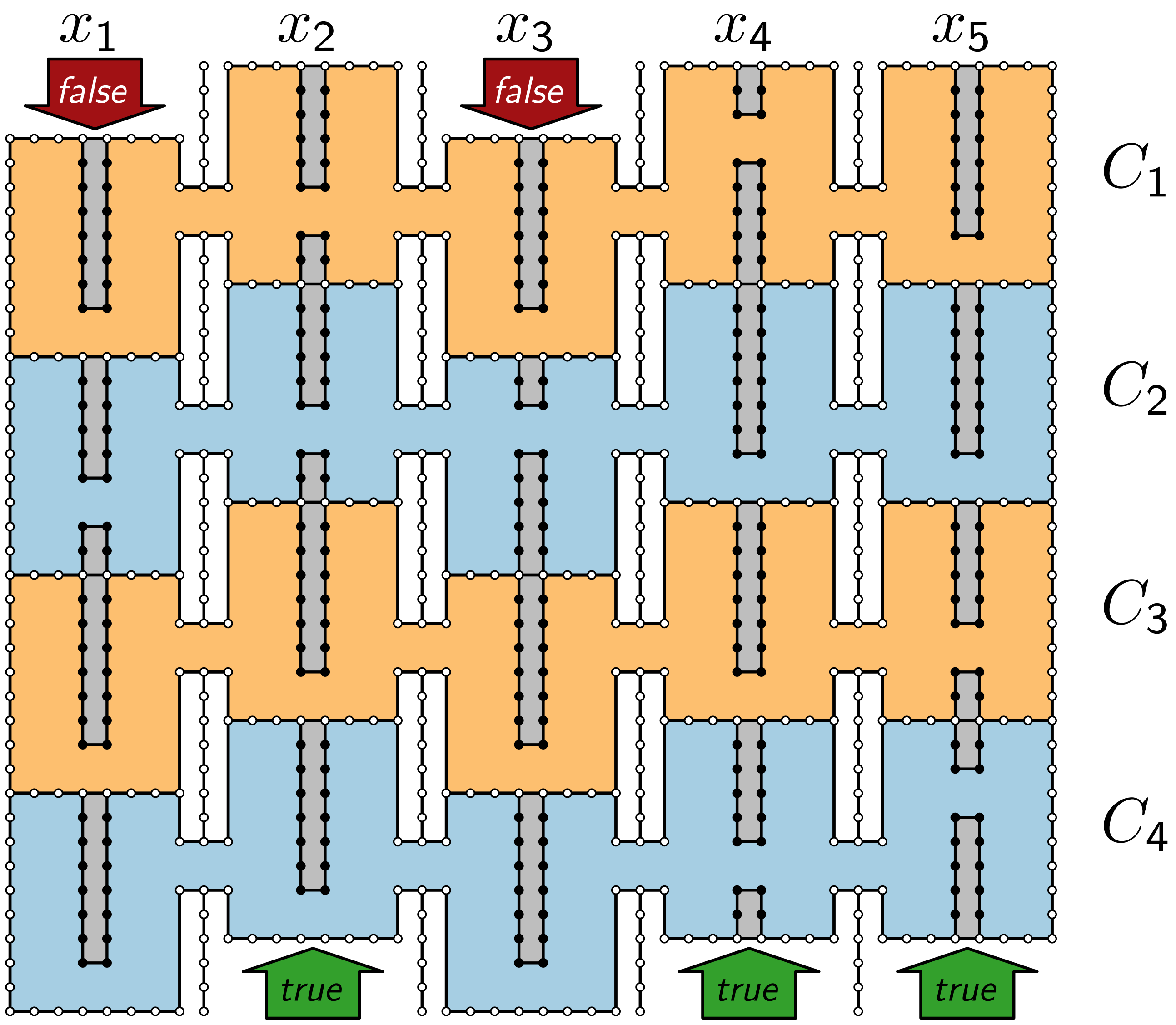

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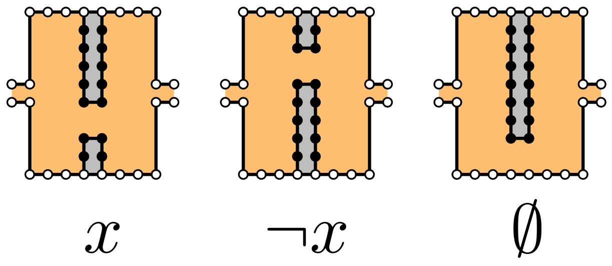

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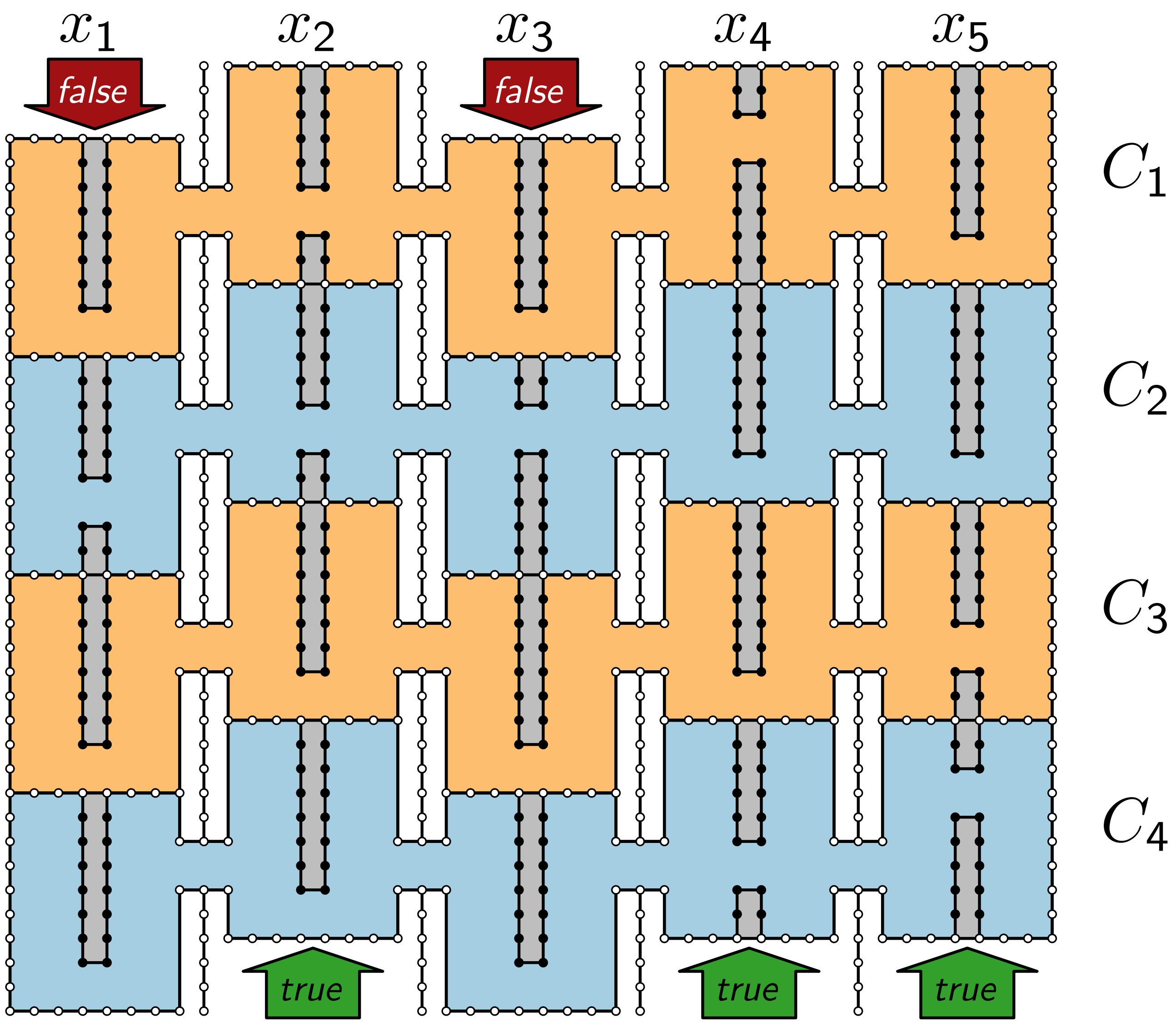

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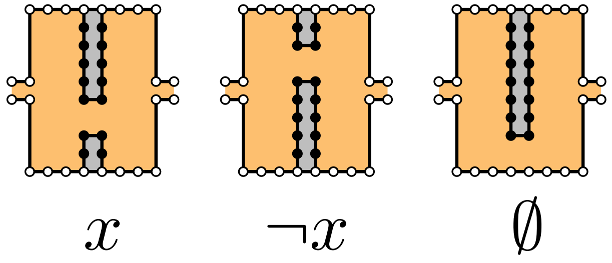

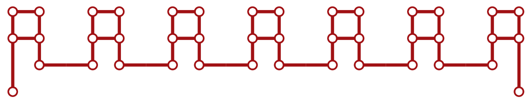

insert $(2n-1)$-chain through each clause

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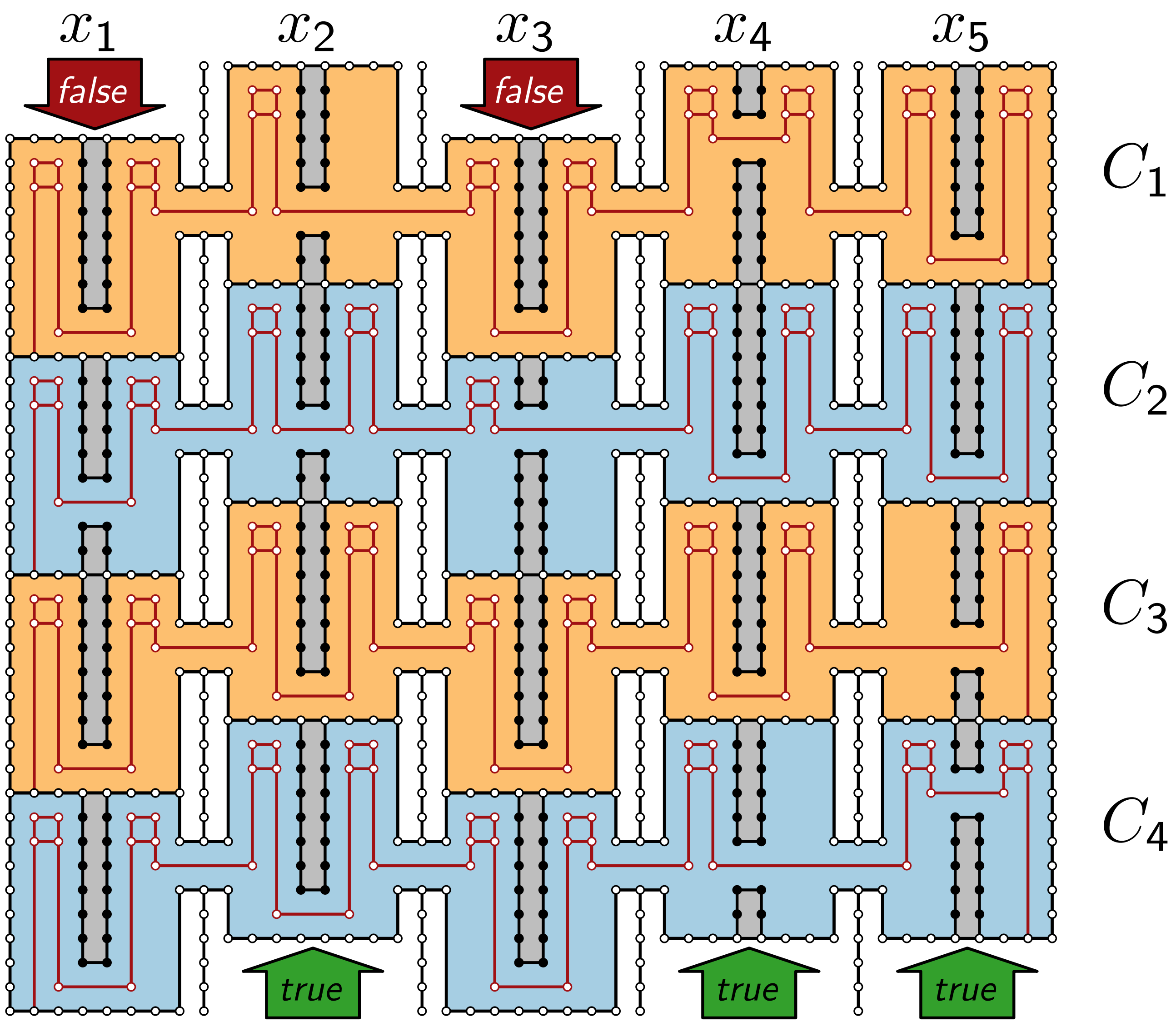

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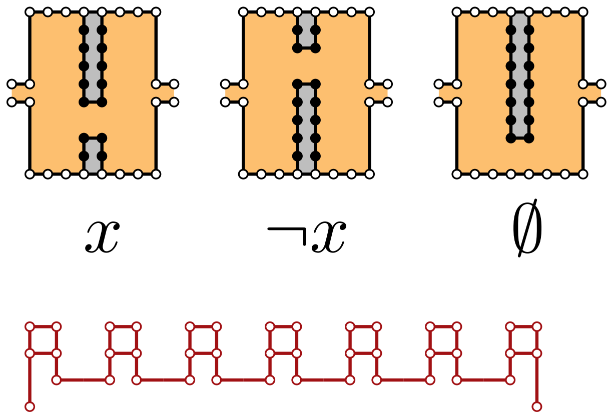

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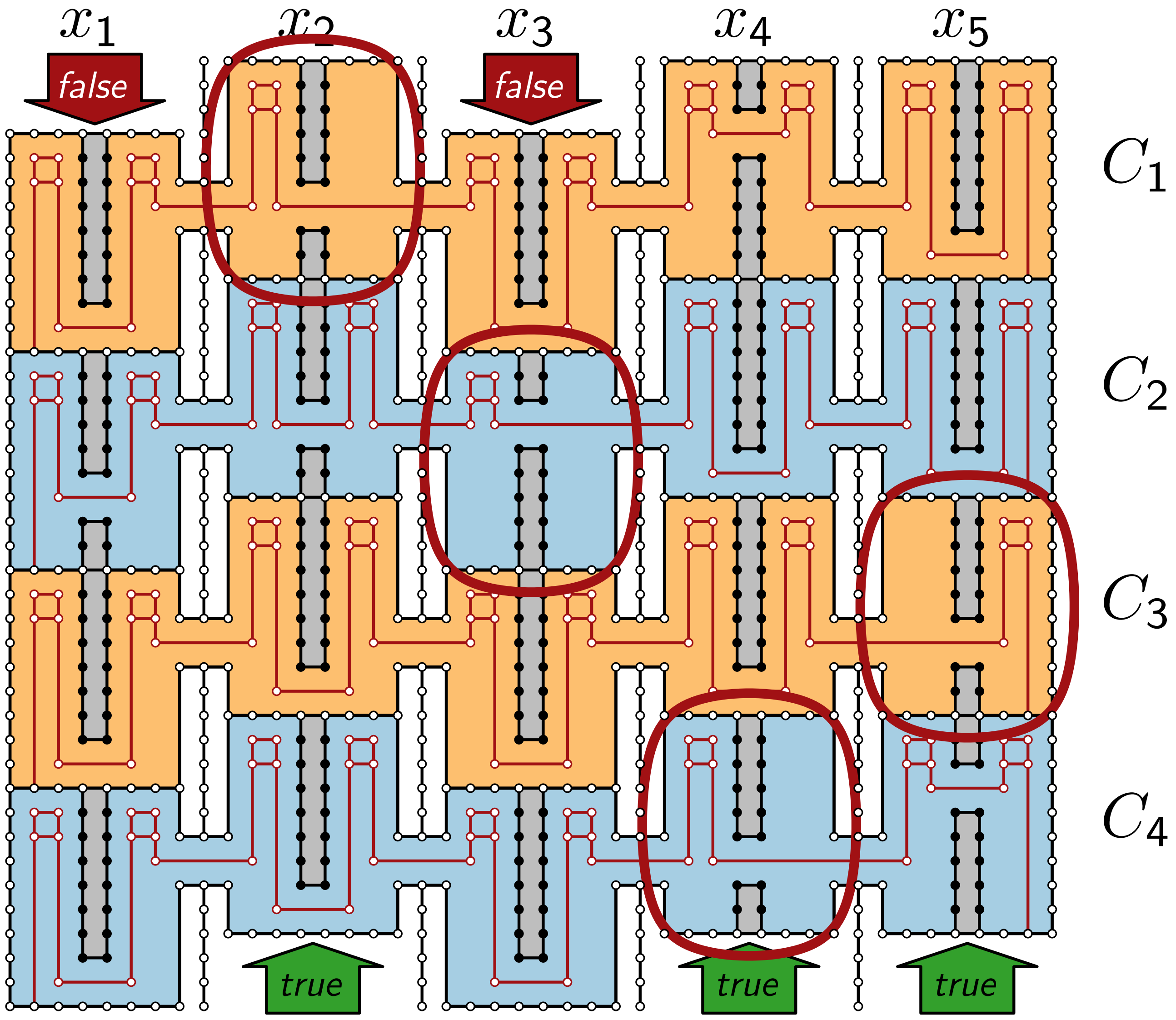

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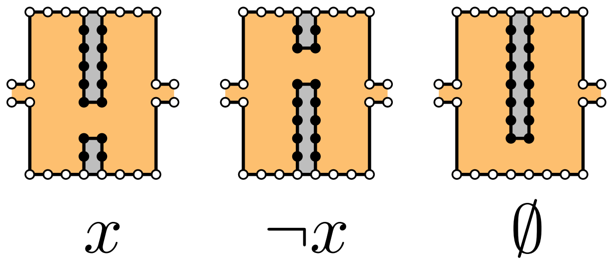
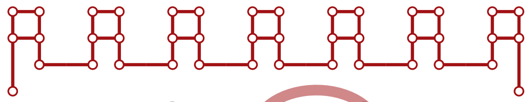
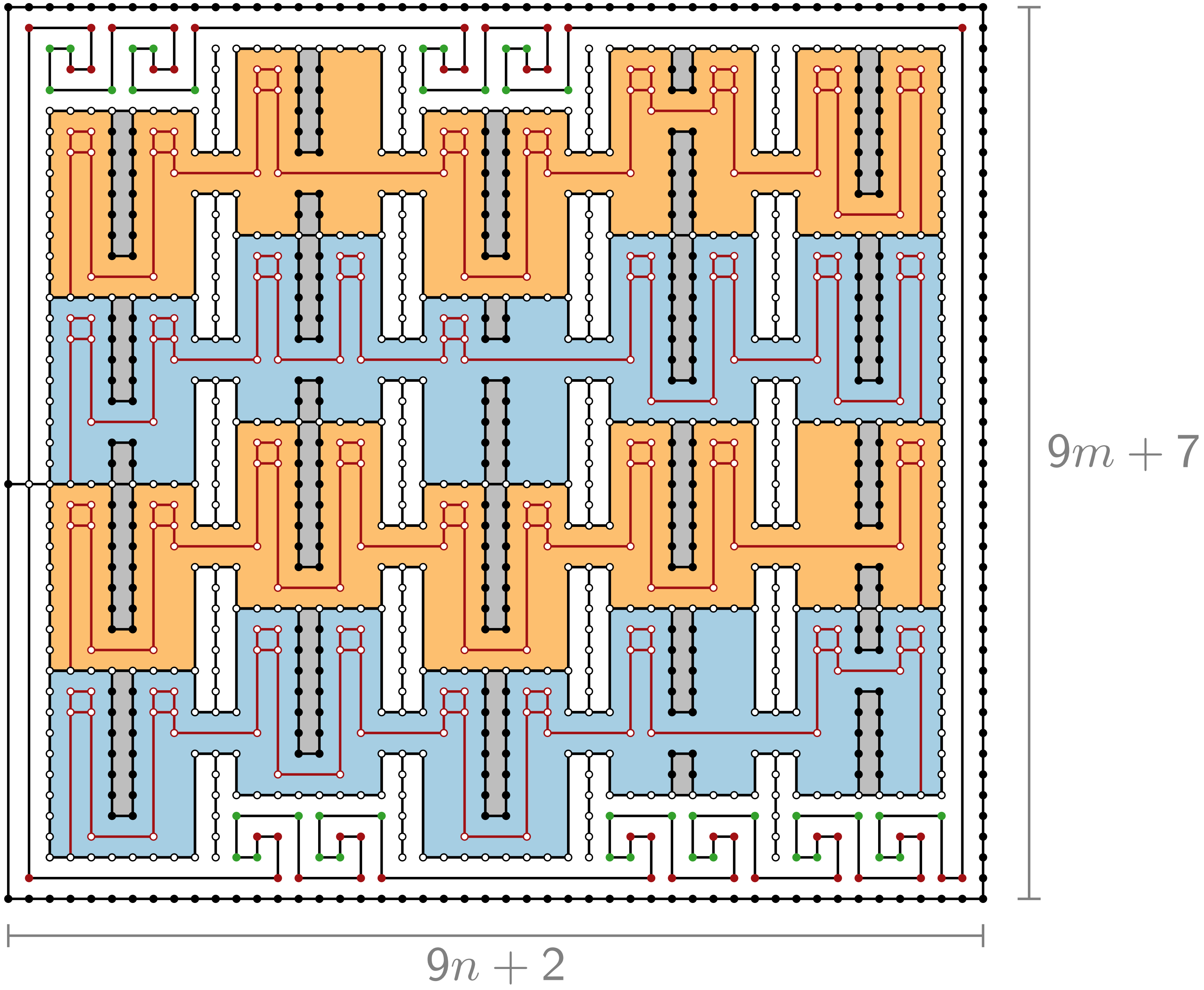




insert $(2n - 1)$-chain through each clause

→ for every clause, there needs to be ≥ 1 "gap of a literal" to be on the same height as the "tunnel" to the next literal

Complete Reduction



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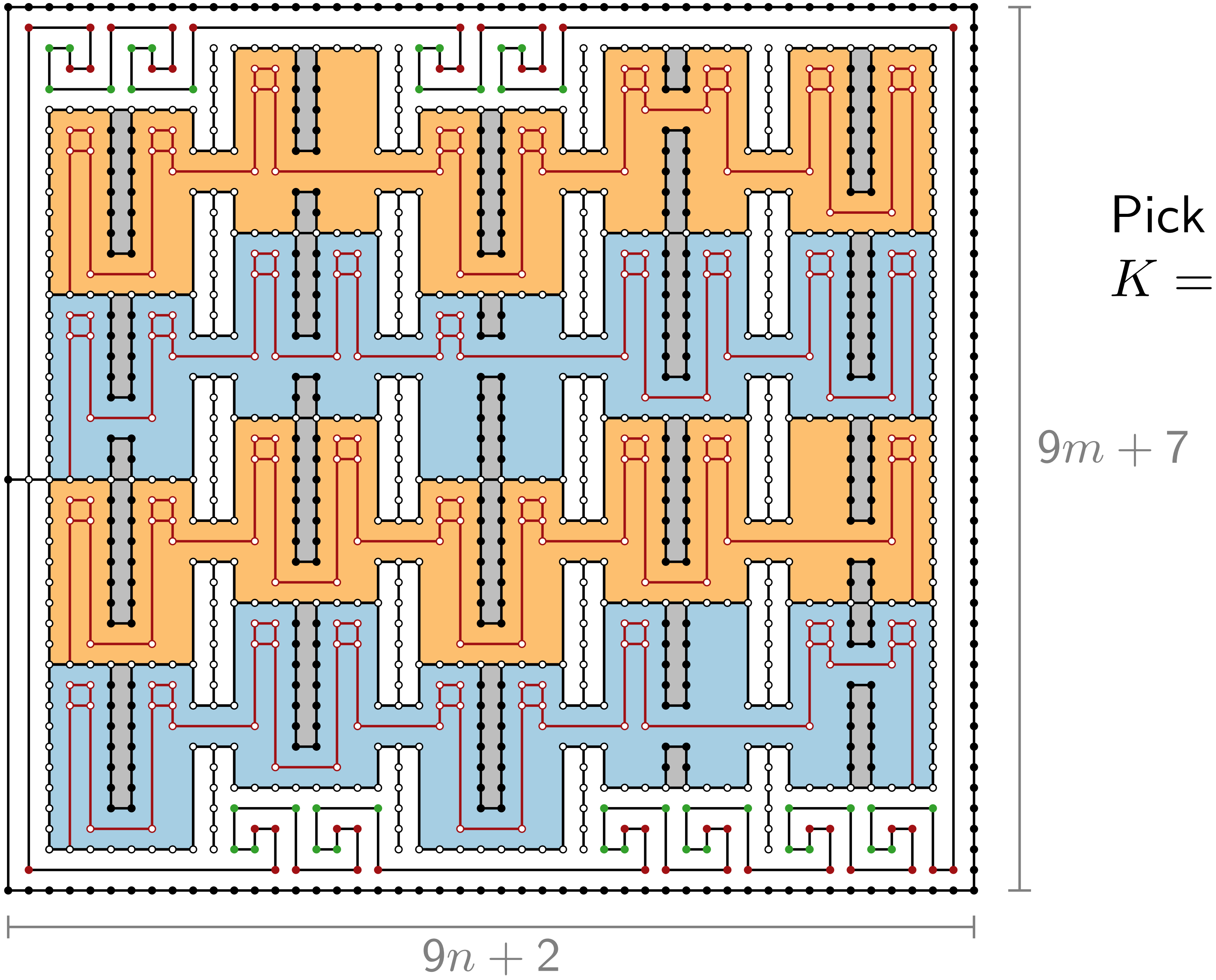

Pick

$K = (9n+2) \times (9m+7)$

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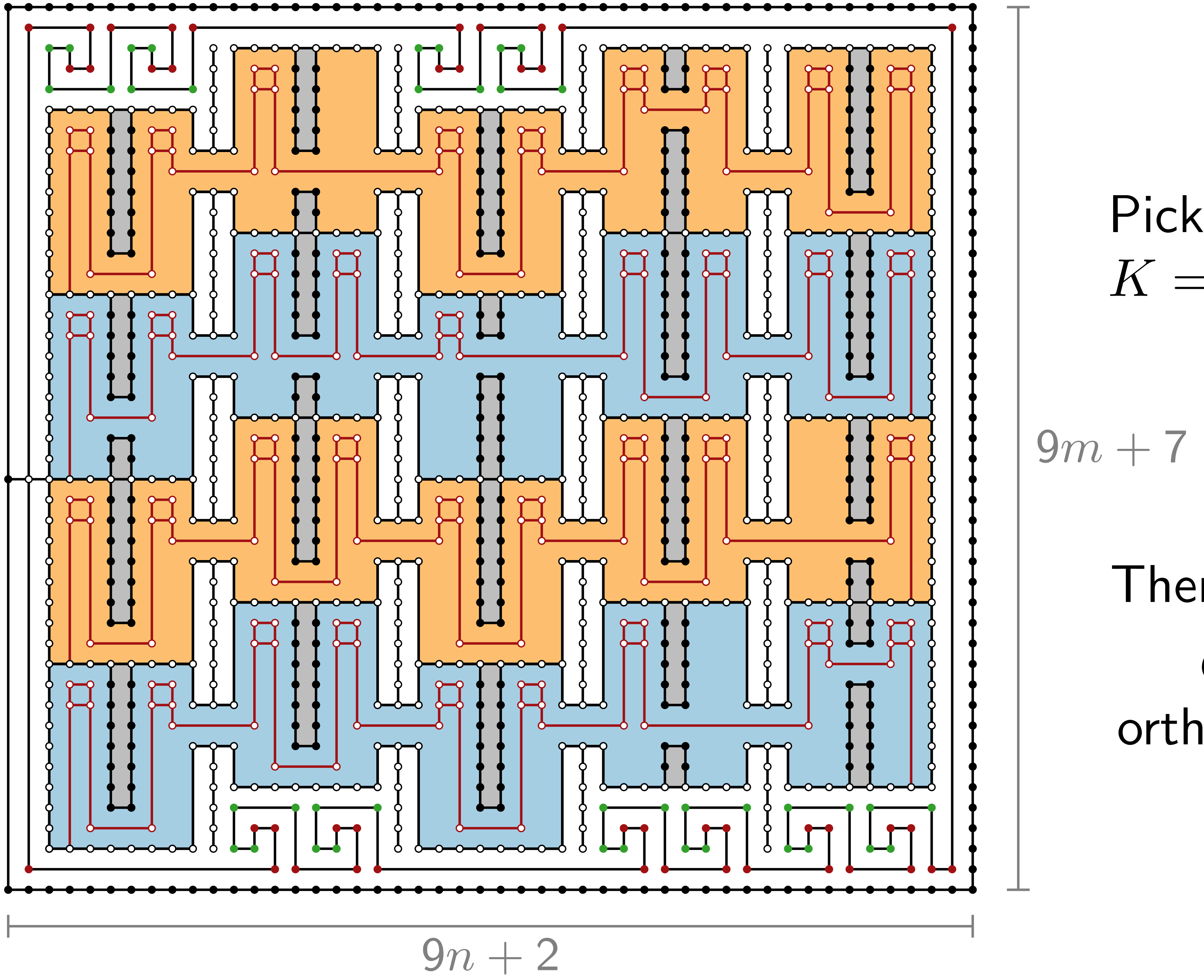

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Then:

G under $H(G)$ has an orthogonal drawing in area K

$\Leftrightarrow$

Φ satisfiable

□

Literature

- [GD Ch. 5] for detailed explanation
- [Tamassia 1987] "On embedding a graph in the grid with the minmum number of bends"
  Original paper on flow for bend minimization.
- [van den Brand, Chen, Kyng, Liu, Peng, Probst, Sachdeva, Sidford 2023]
  "A Deterministic Almost-Linear Time Algorithm for Minimum-Cost Flow"
  State-of-the-art algorithm for solving the minimum-cost flow problem
  (published recently in the proceedings of the FOCS 2023 conference).
- [Patrignani 2001] "On the complexity of orthogonal compaction"
  NP-hardness proof for orthogonal representation of planar max-degree-4 graphs.
- [Evans, Fleszar, Kindermann, Saeedi, Shin, Wolff 2022]
  "Minimum rectilinear polygons for given angle sequences"
  NP-hardness proof for compaction of cycles.