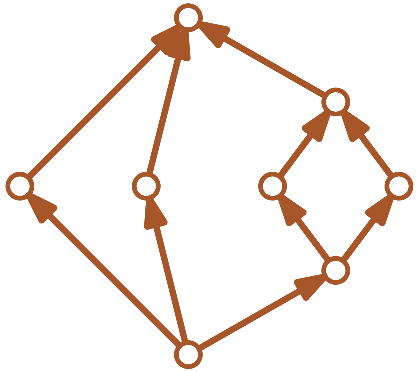
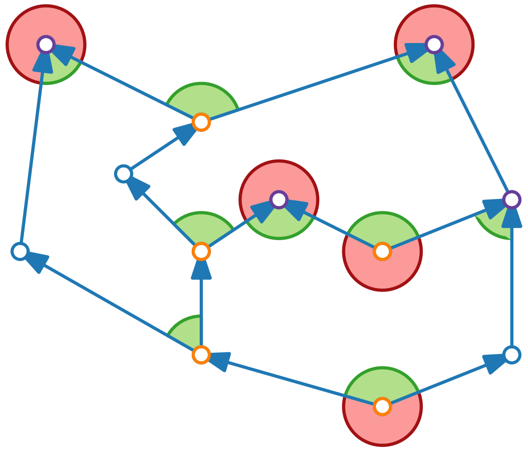
Julius-Maximilians-
UNIVERSITÄT
WÜRZBURG

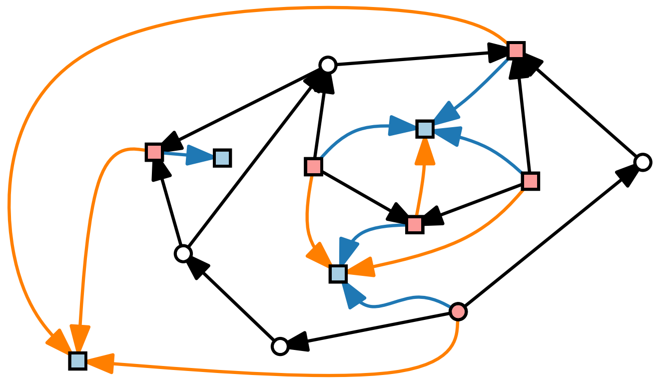
Visualization of Graphs



Lecture 5:
Upward Planar Drawings



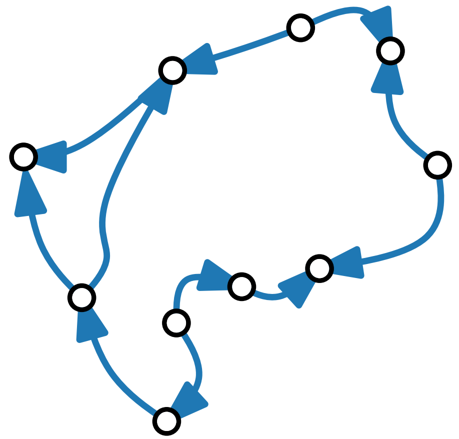
Part I:
Recognition



Johannes Zink

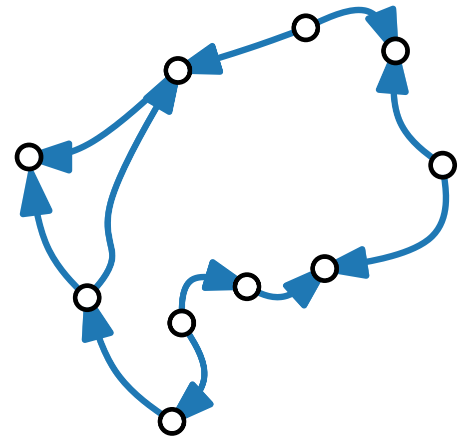
Summer semester 2024

Upward Planar Drawings – Motivation

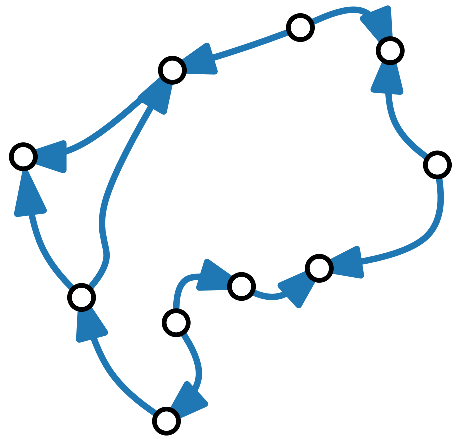


Upward Planar Drawings – Motivation

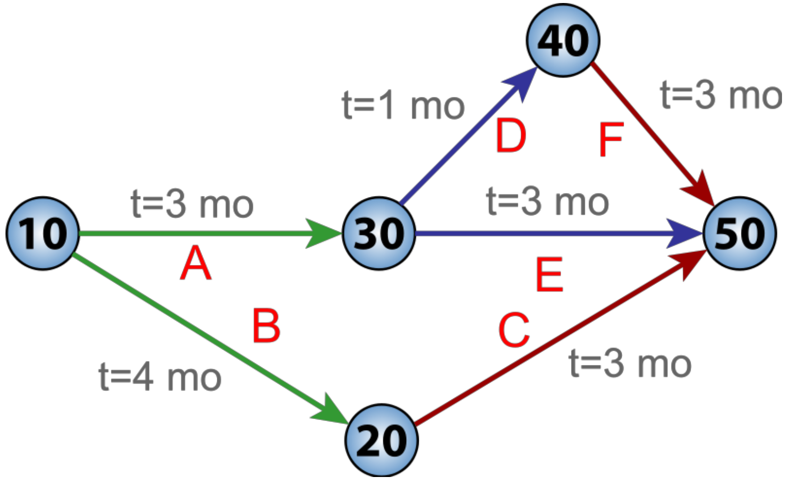
- What may the direction of edges in a directed graph represent?



Upward Planar Drawings – Motivation



- What may the direction of edges in a directed graph represent?
  - Time

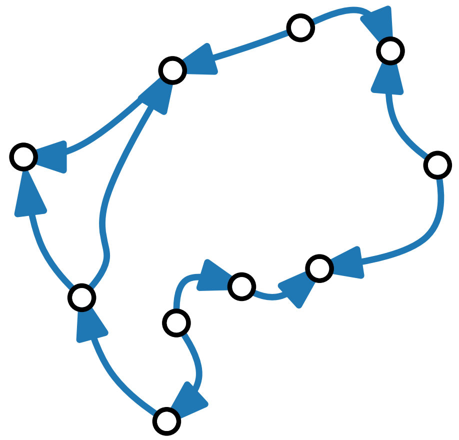


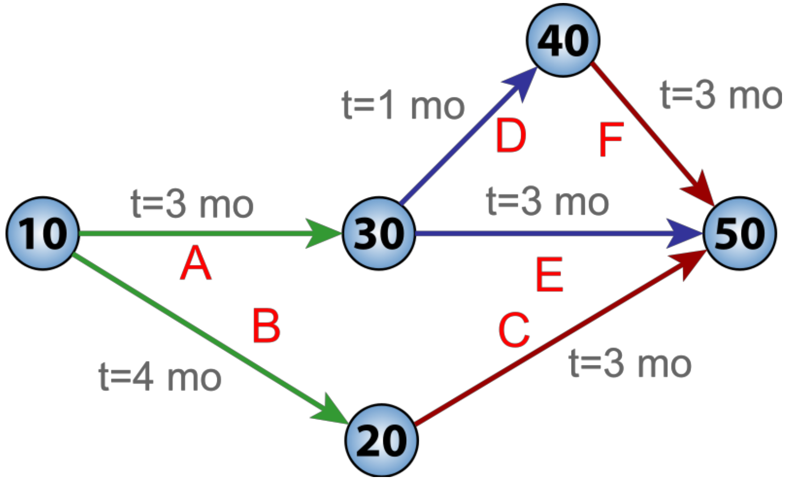
PERT diagram

Program Evaluation and Review Technique
(Project management)

Upward Planar Drawings – Motivation

- What may the direction of edges in a directed graph represent?
  - Time
  - Flow





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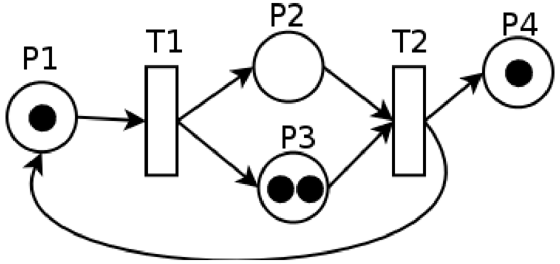
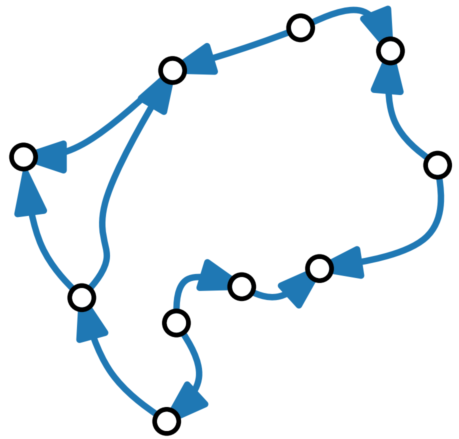


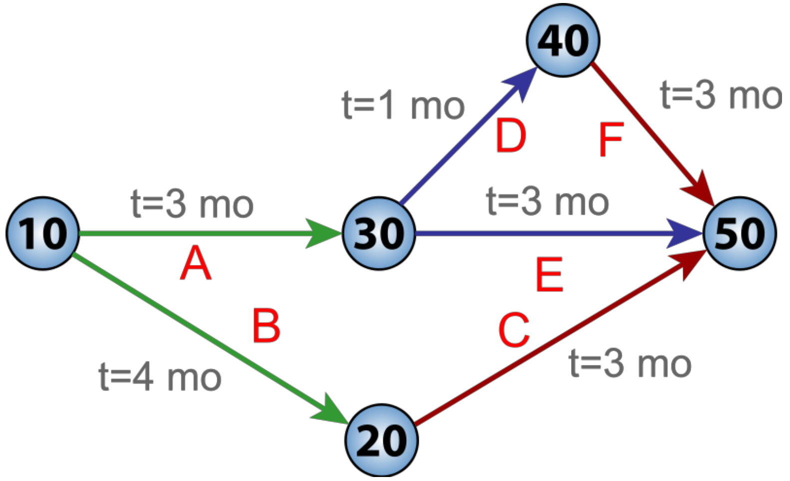
Petri net

Place/Transition net
(Modeling languages for distributed systems)

Upward Planar Drawings – Motivation

- What may the direction of edges in a directed graph represent?
  - Time
  - Flow
  - Hierarchy





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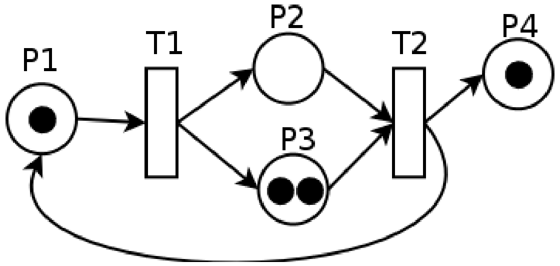

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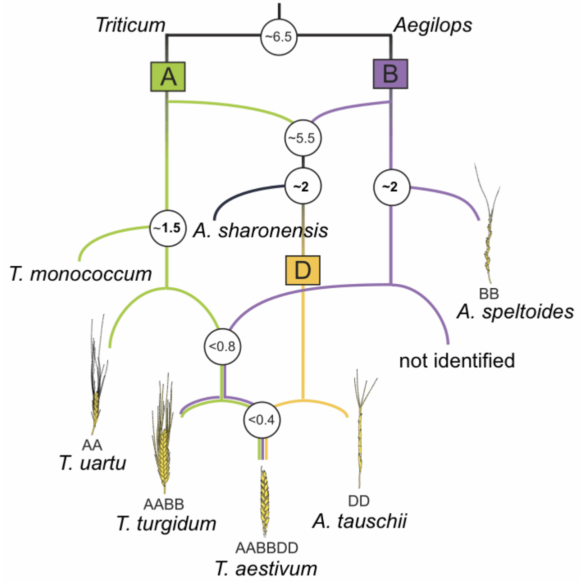
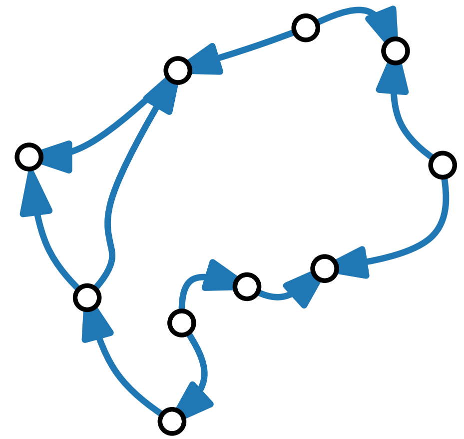


Phylogenetic network

Ancestral trees / networks
(Biology)

Upward Planar Drawings – Motivation



- What may the direction of edges in a directed graph represent?
  - Time
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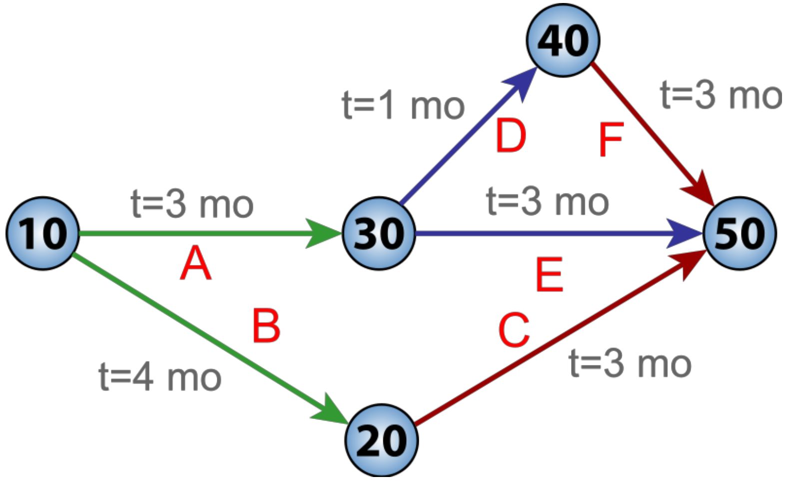
  - Hierarchy
  - ...



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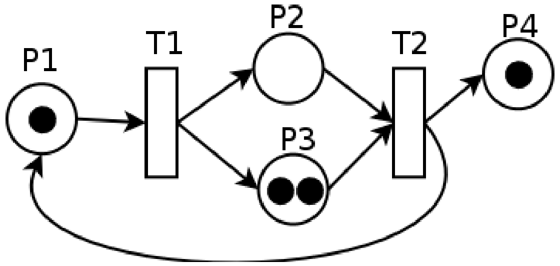

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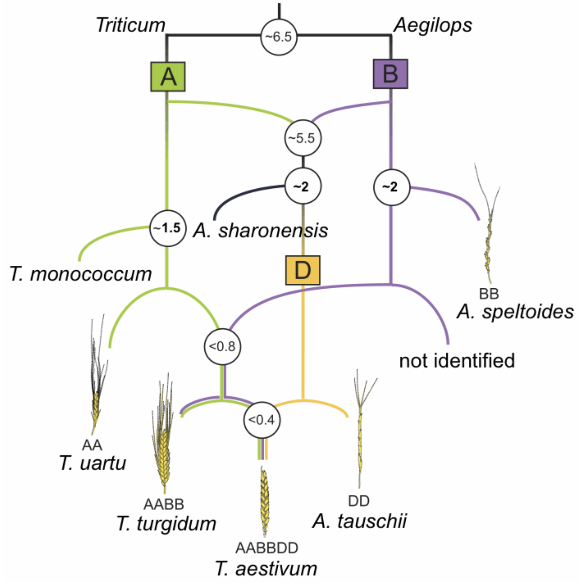

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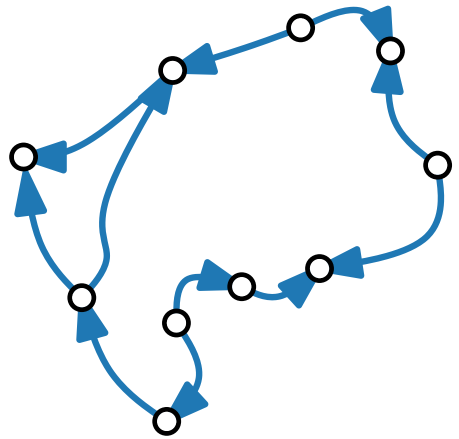
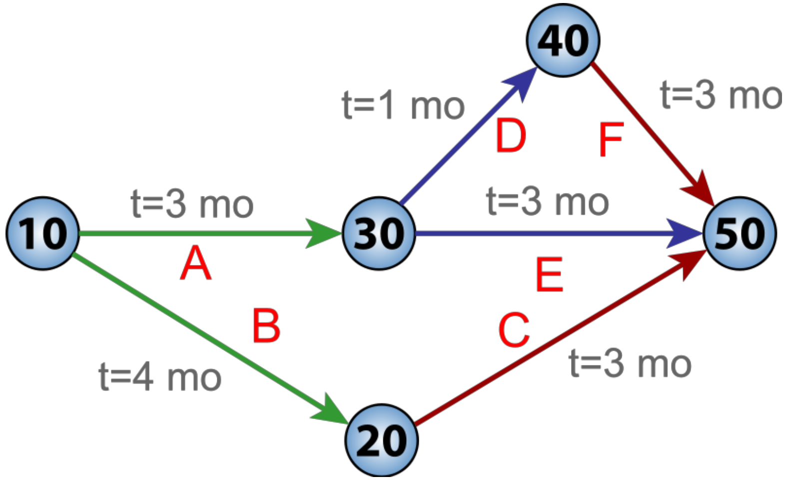


- What may the direction of edges in a directed graph represent?
  - Time
  - Flow
  - Hierarchy
  - . . .
- We aim for drawings where the general direction is preserved.



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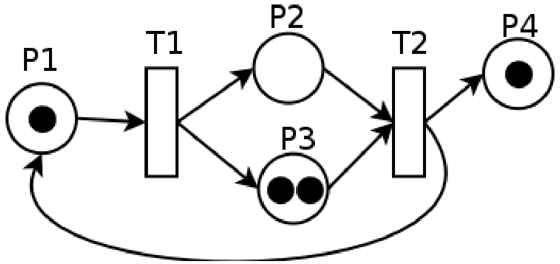

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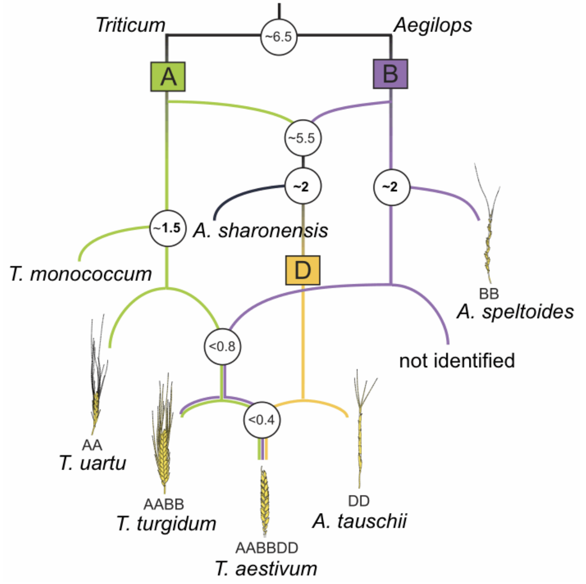

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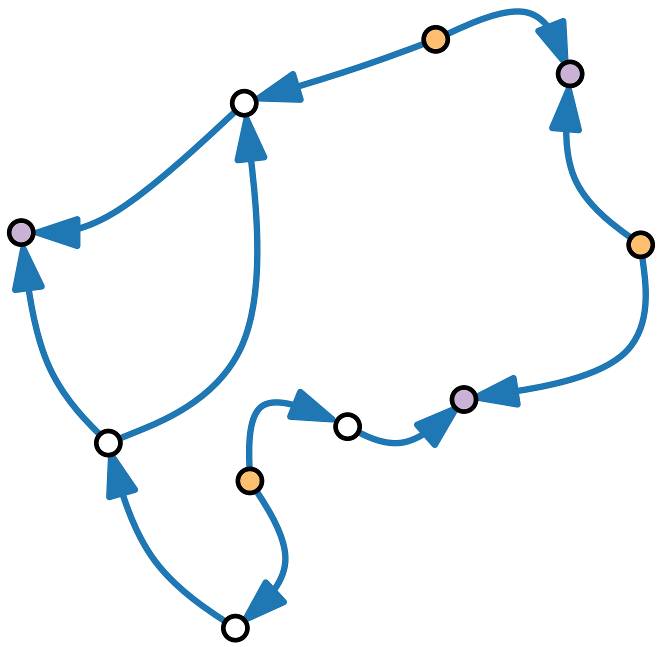
Upward Planar Drawings – Definition

A directed graph (*digraph*) is **upward planar** when it admits a drawing



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A directed graph (*digraph*) is **upward planar** when it admits a drawing

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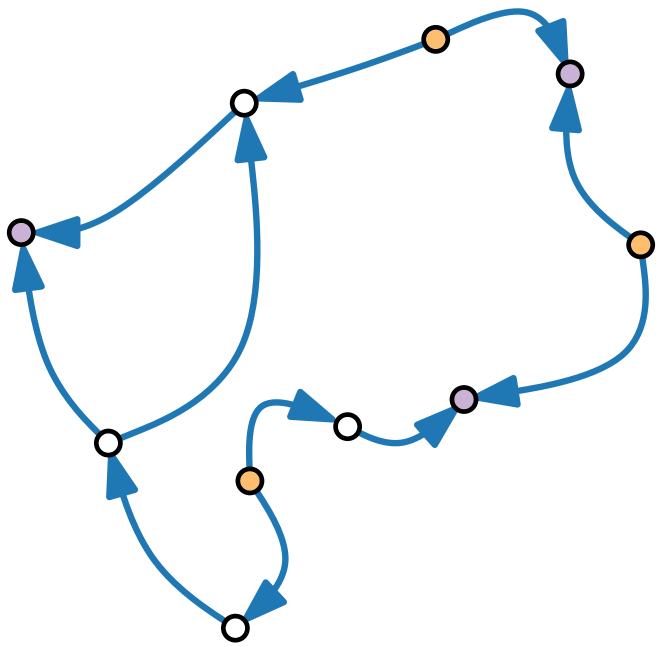
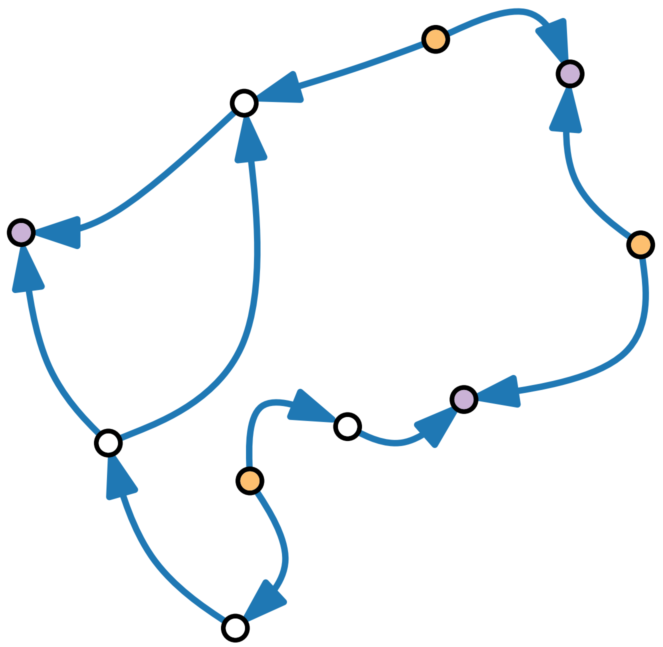


Upward Planar Drawings – Definition

A directed graph (*digraph*) is **upward planar** when it admits a drawing

- that is planar and
- where each edge is drawn as an upward y-monotone curve.



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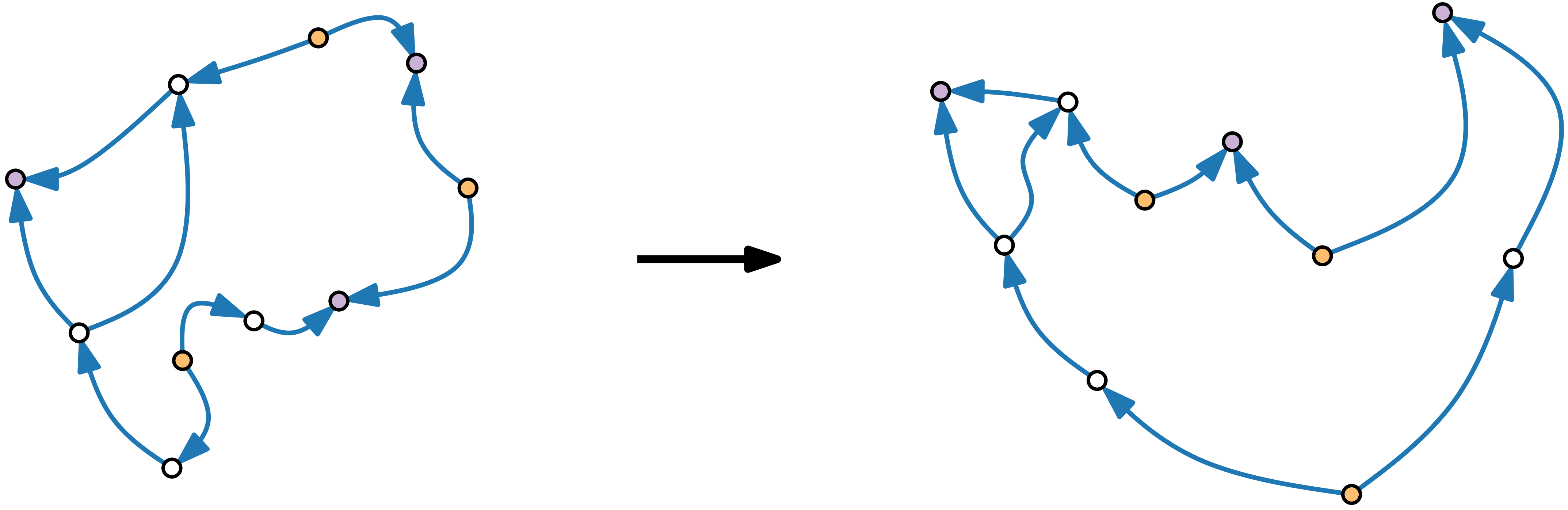

Upward Planarity – Necessary Conditions

- For an (embedded) digraph to be upward planar, it needs to . . .

Upward Planarity – Necessary Conditions

- For an (embedded) digraph to be upward planar, it needs to . . .
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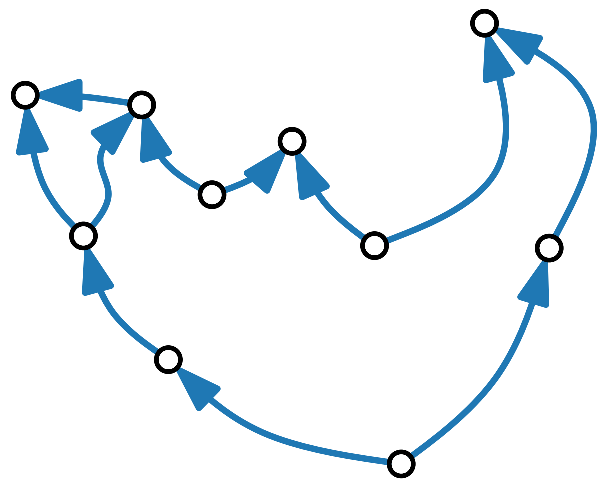

Upward Planarity – Necessary Conditions

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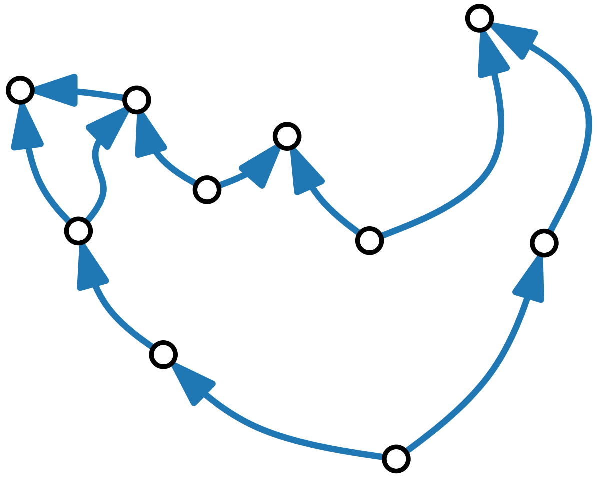

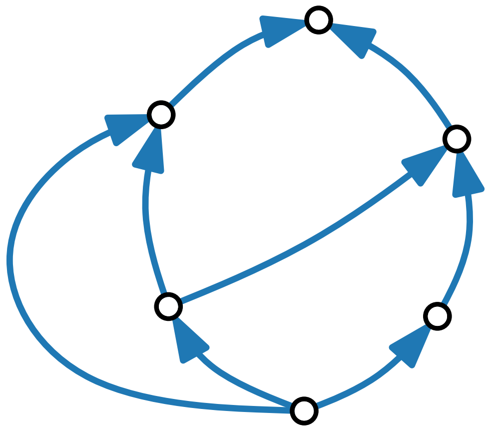

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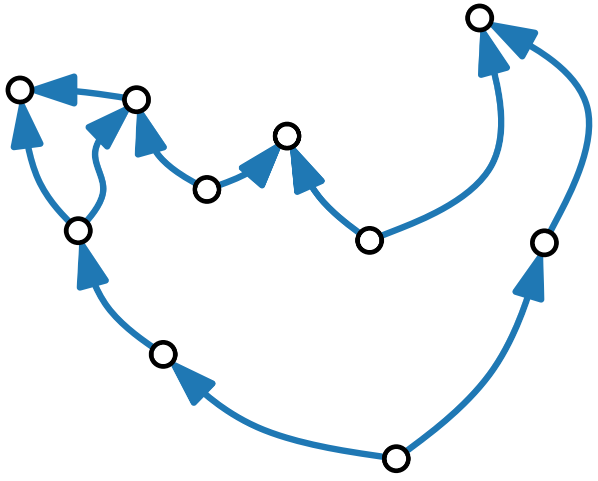

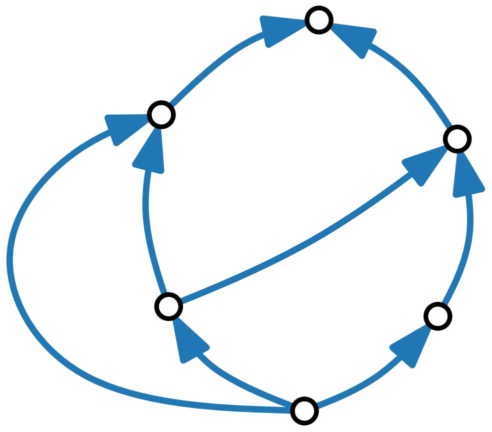

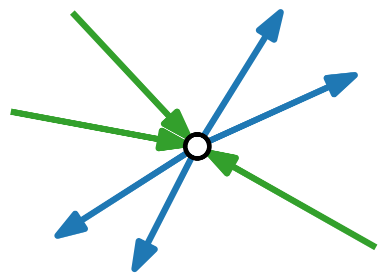

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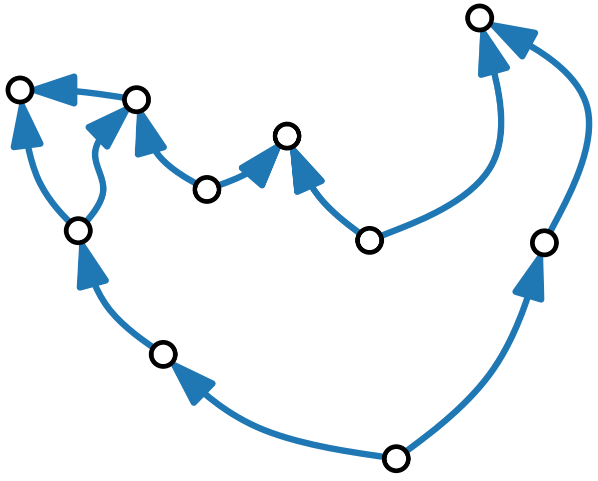

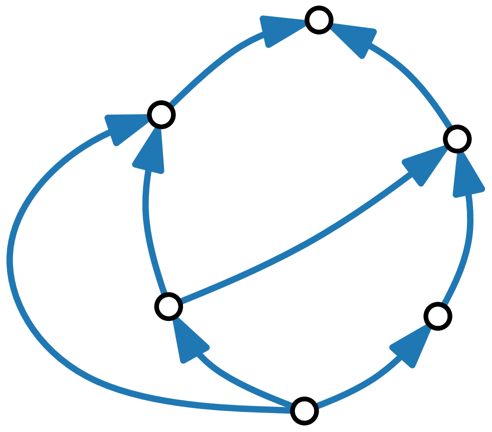

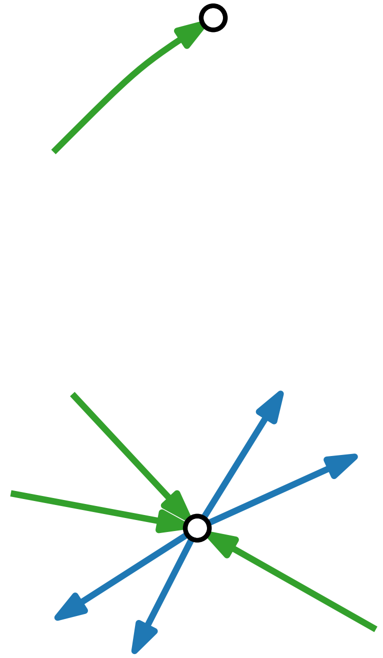

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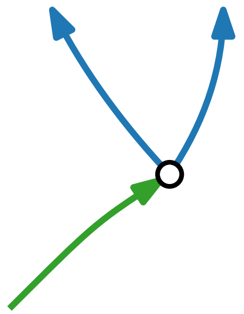

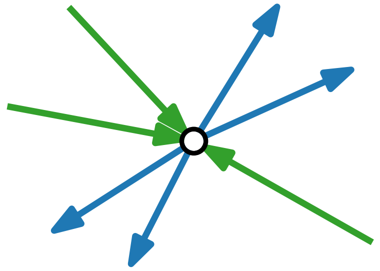

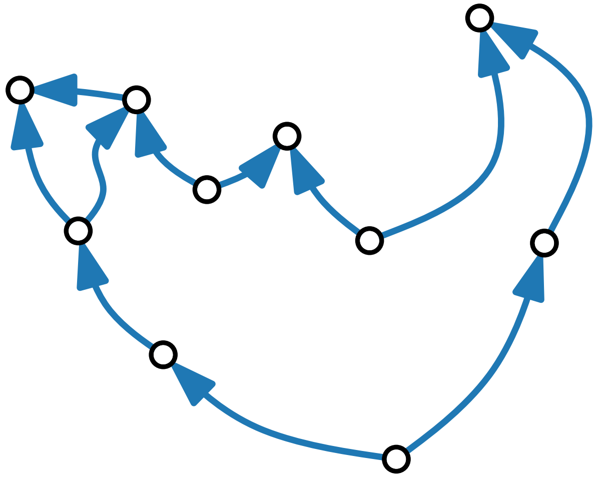

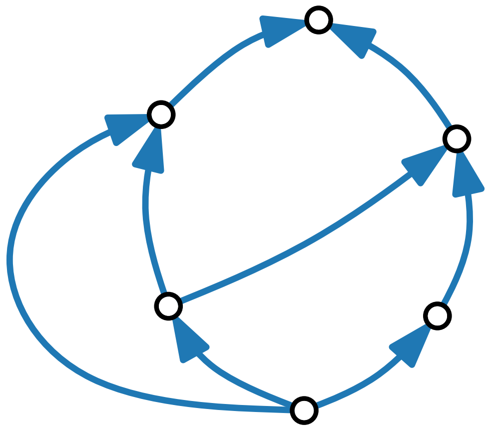

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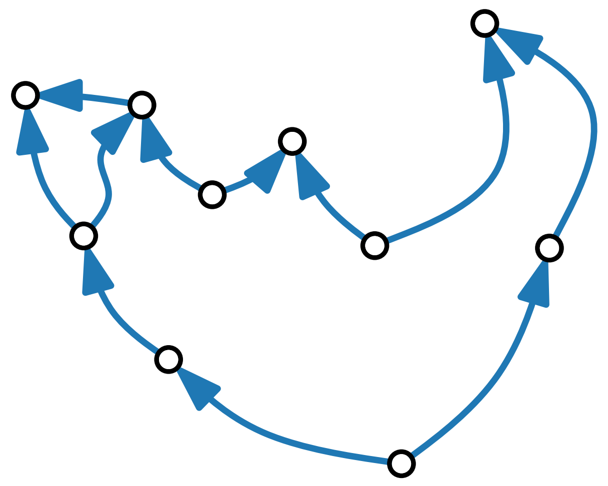

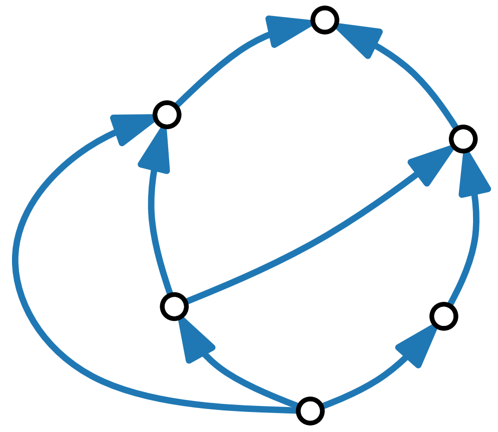

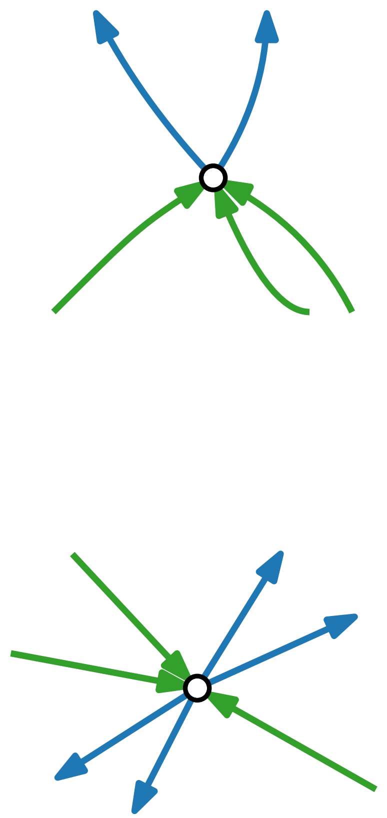

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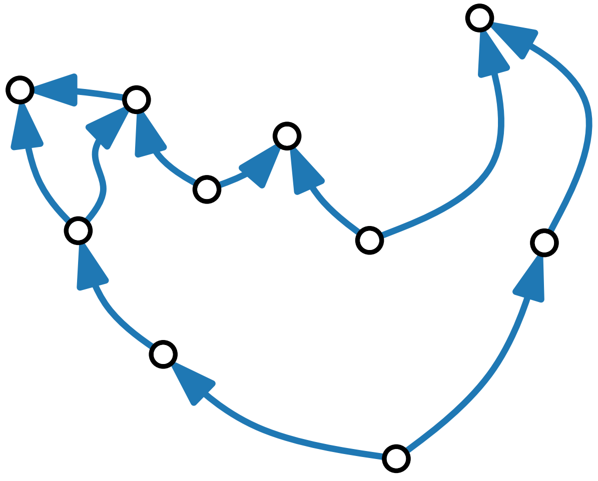

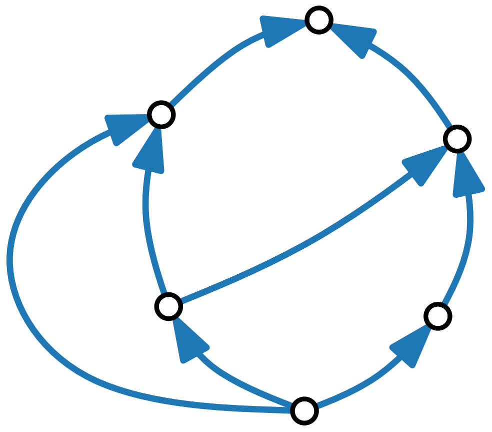

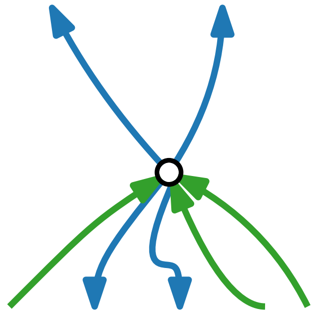

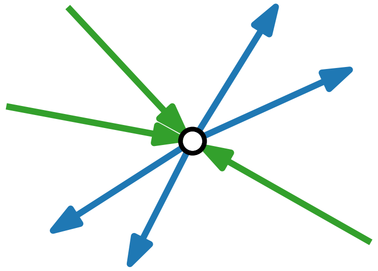

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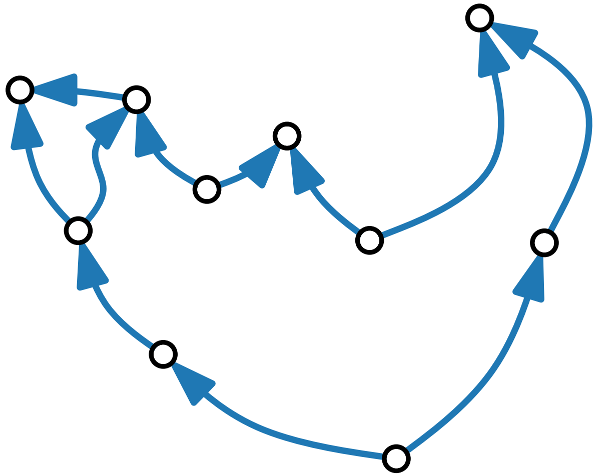

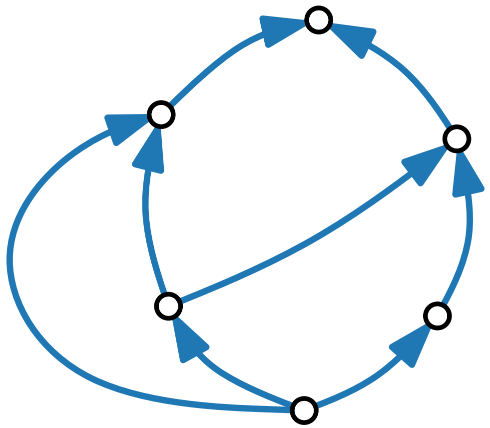

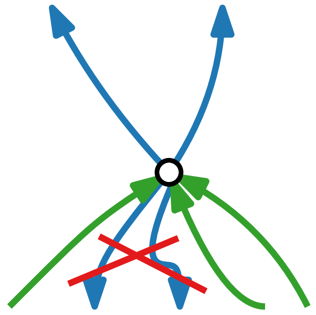

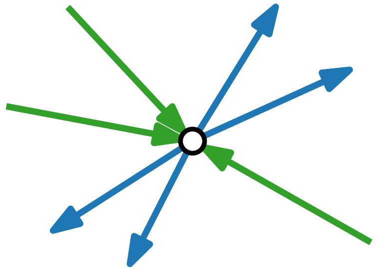

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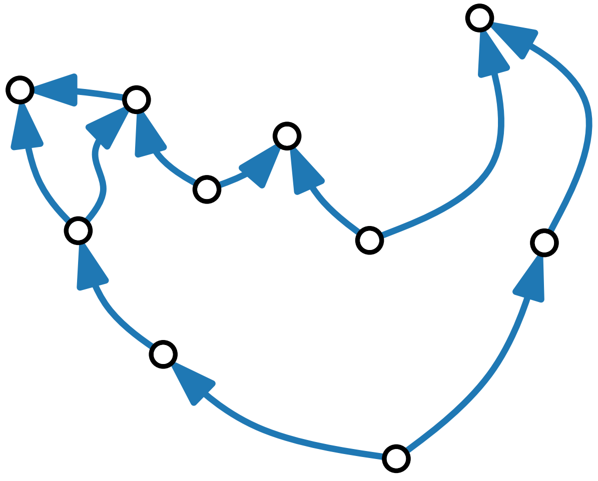

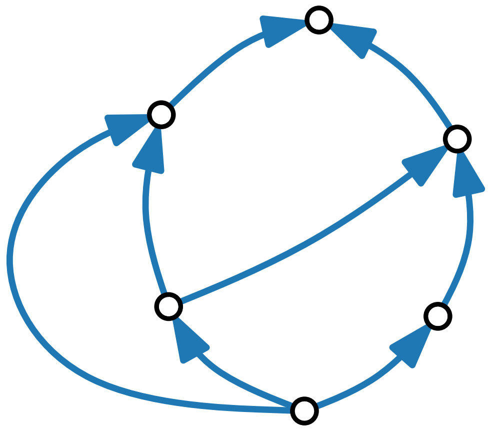

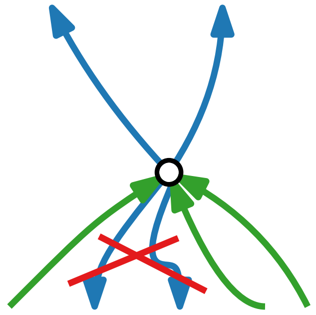

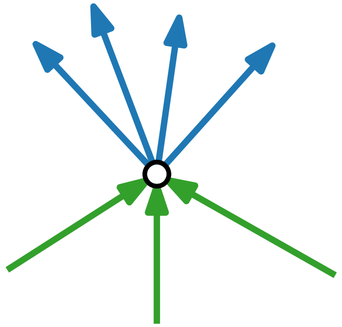

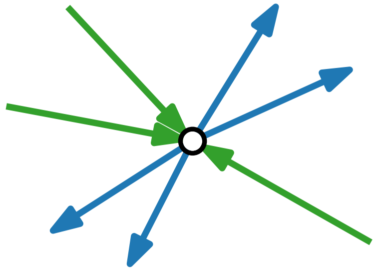

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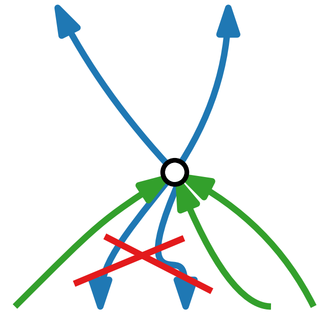
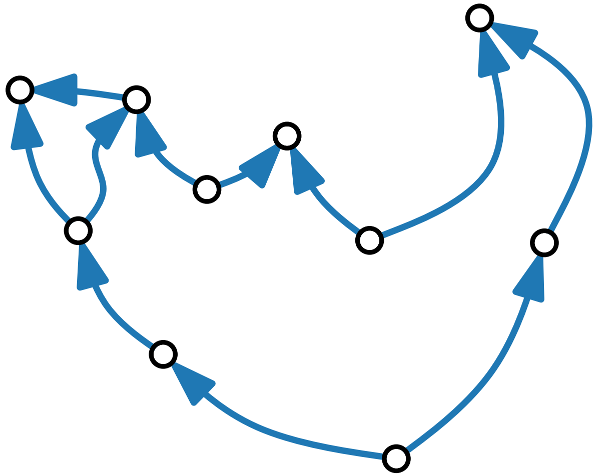
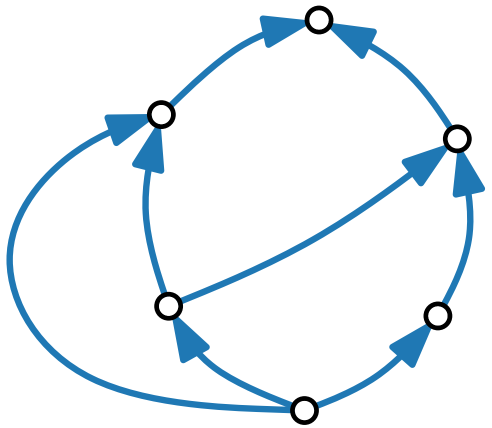
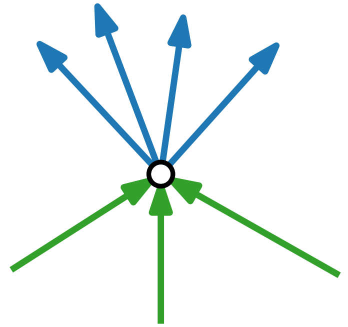


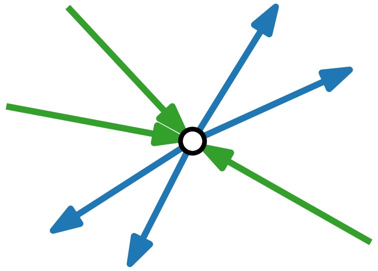




bimodal vertex

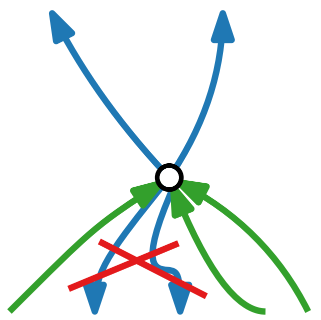


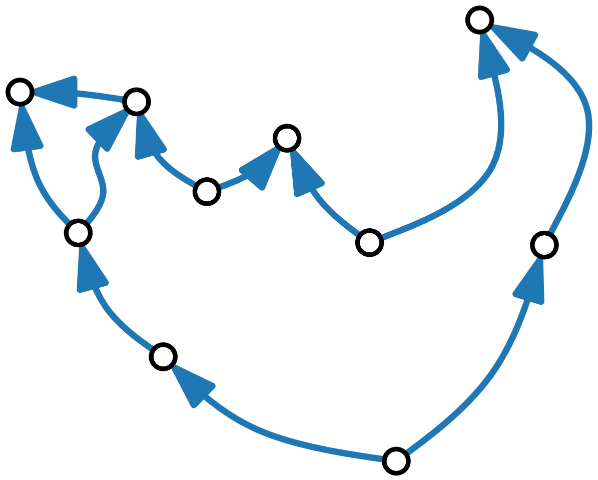
not bimodal

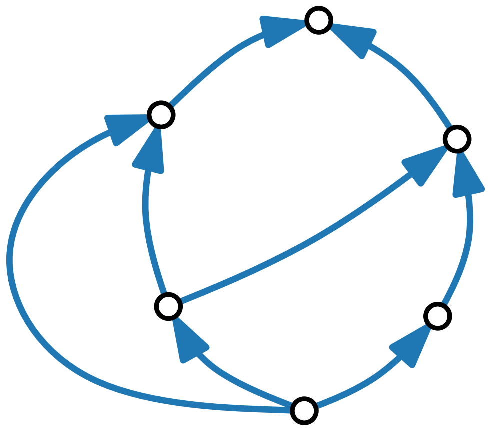


Upward Planarity – Necessary Conditions

- For an (embedded) digraph to be upward planar, it needs to ...
  - be planar
  - be acyclic
  - have a bimodal embedding







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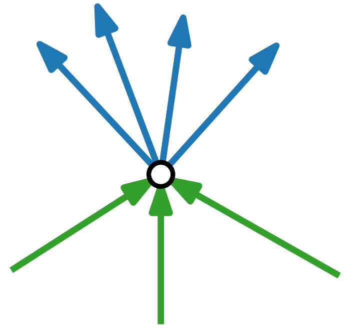

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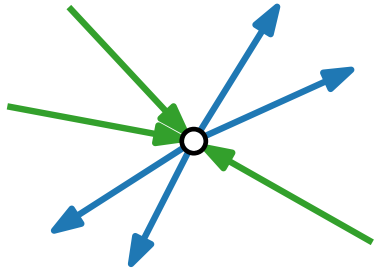

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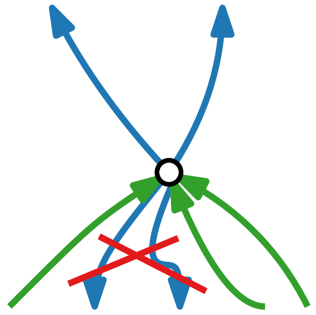

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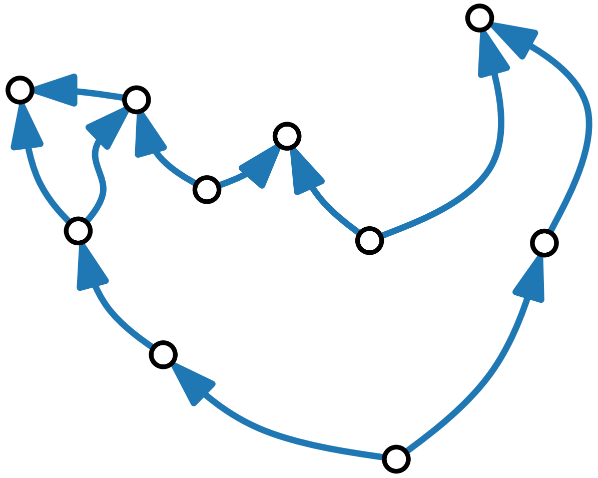
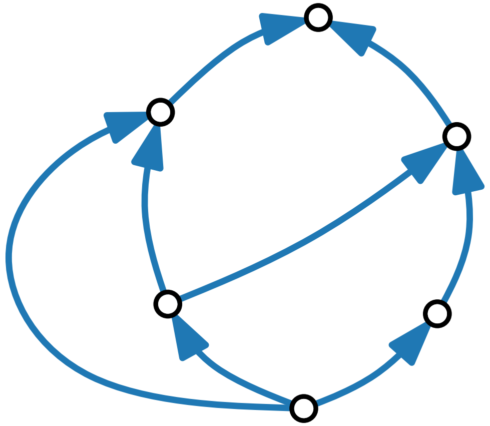
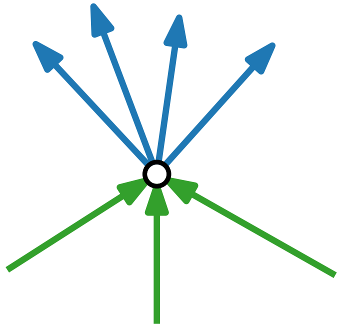




bimodal vertex



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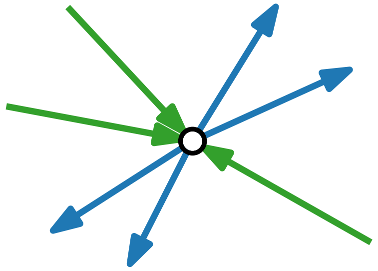
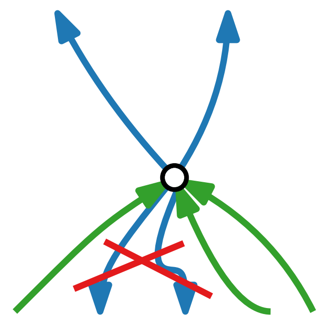
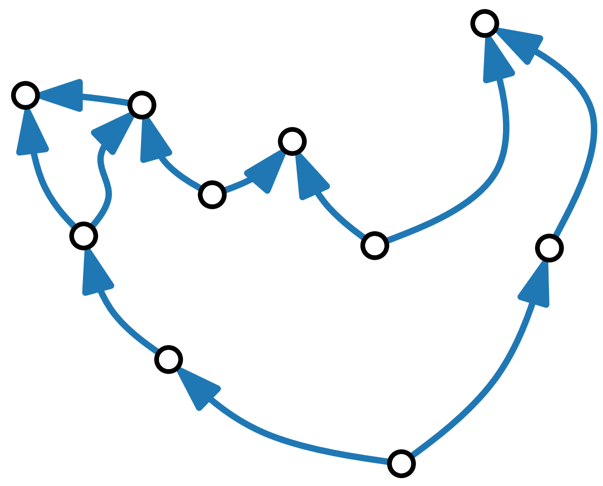


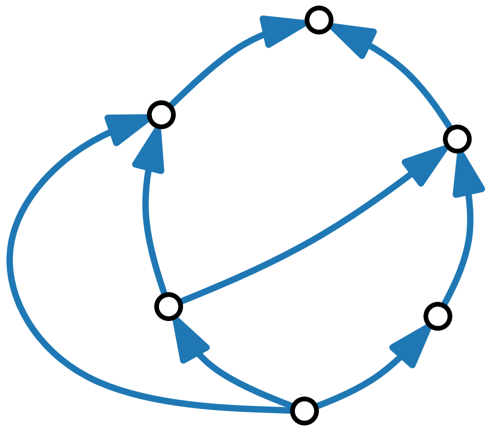
Upward Planarity – Necessary Conditions

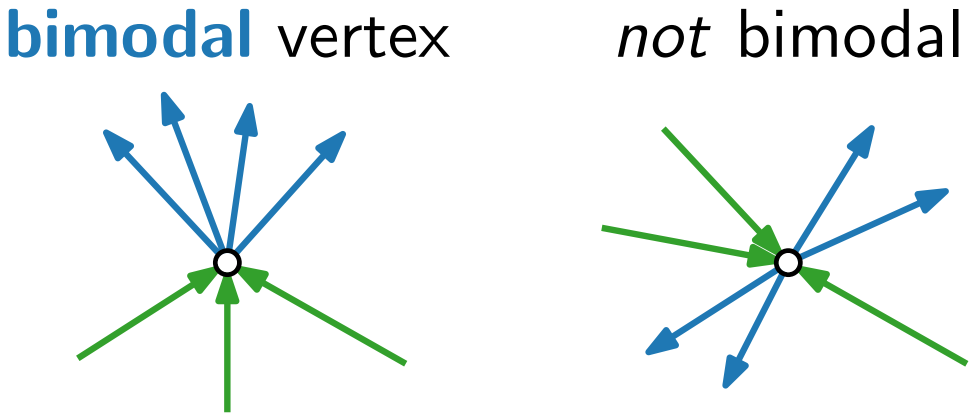
- For an (embedded) digraph to be upward planar, it needs to ...
  - be planar
  - be acyclic
  - have a bimodal embedding



- ... but these conditions are *not sufficient*. → **Exercise**







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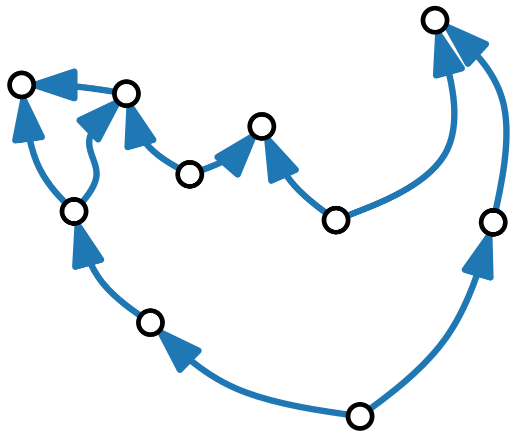

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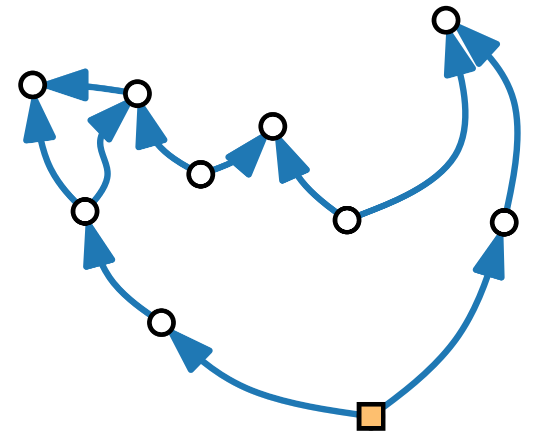

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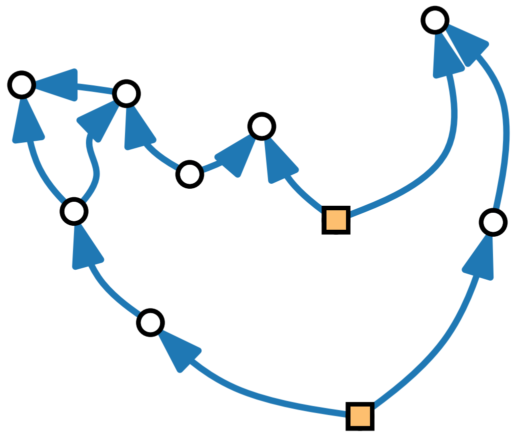

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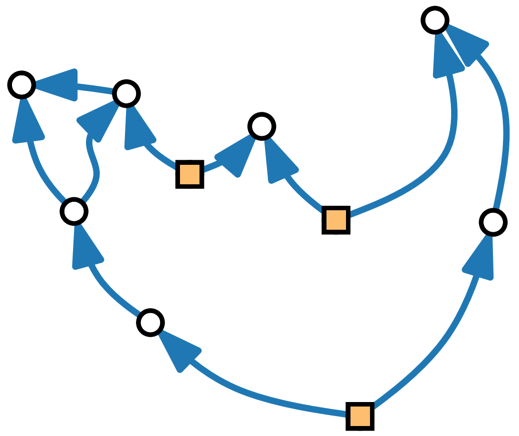

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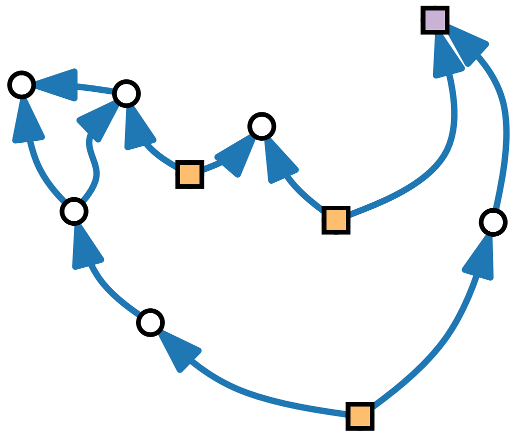

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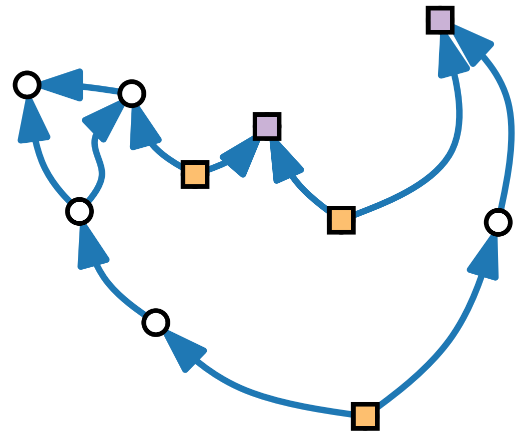

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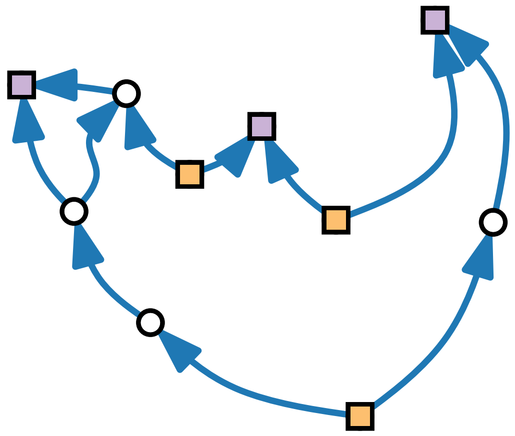

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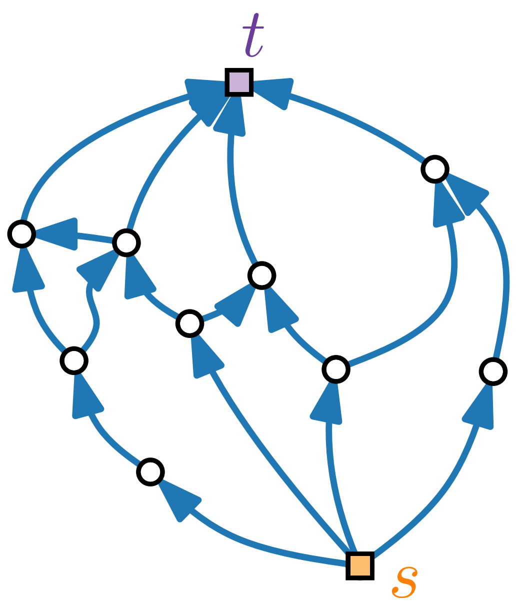

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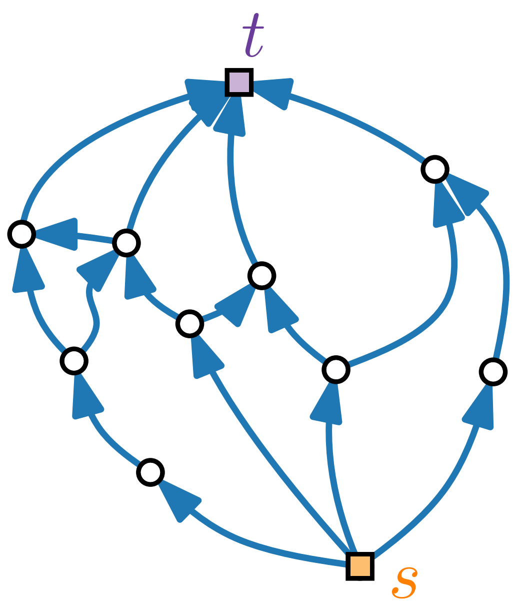

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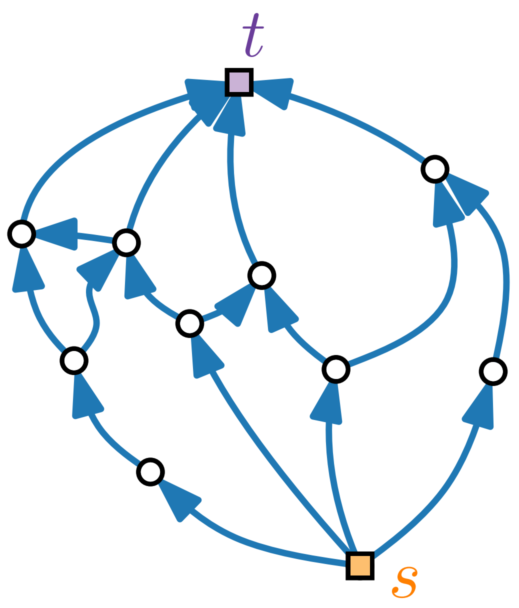

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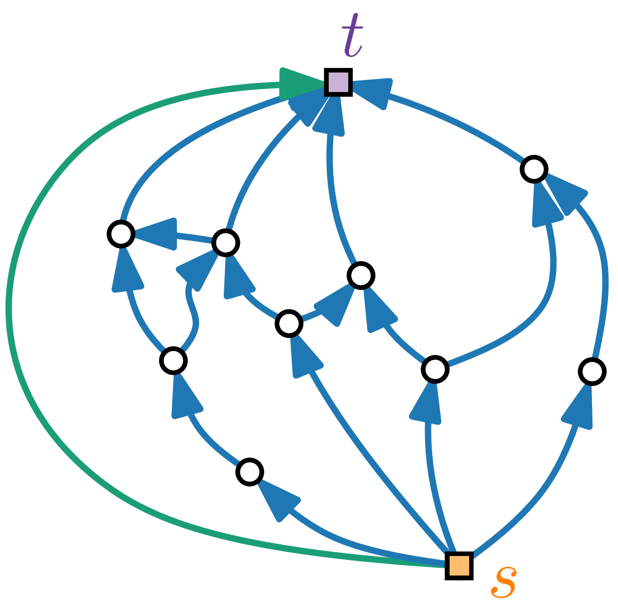

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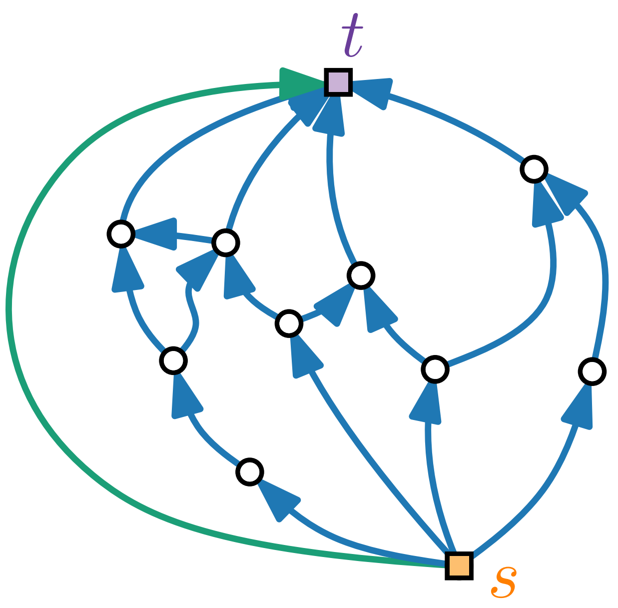

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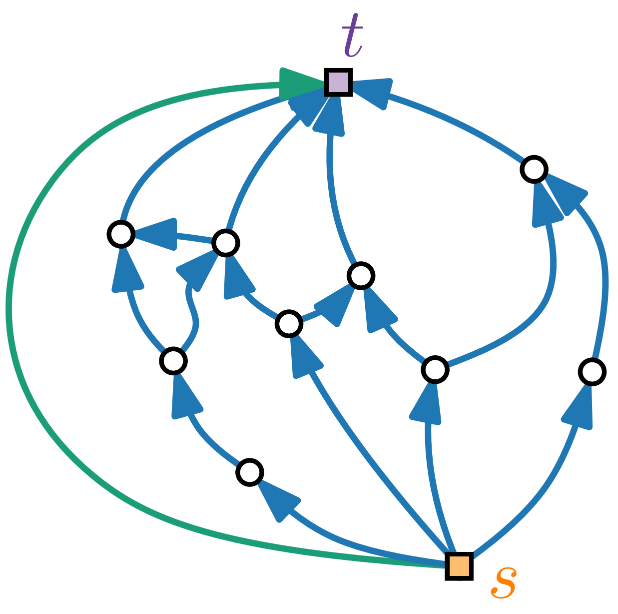

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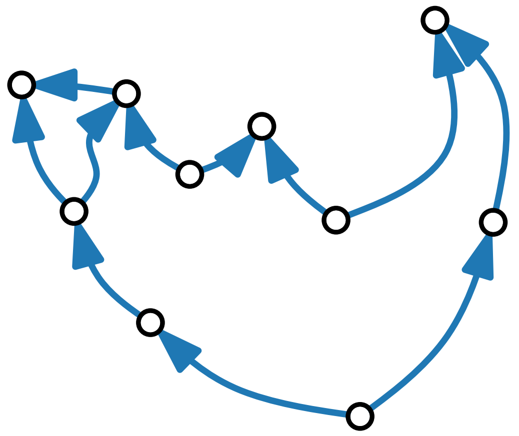

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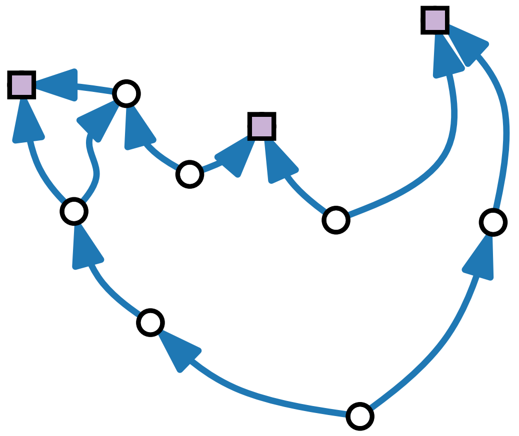

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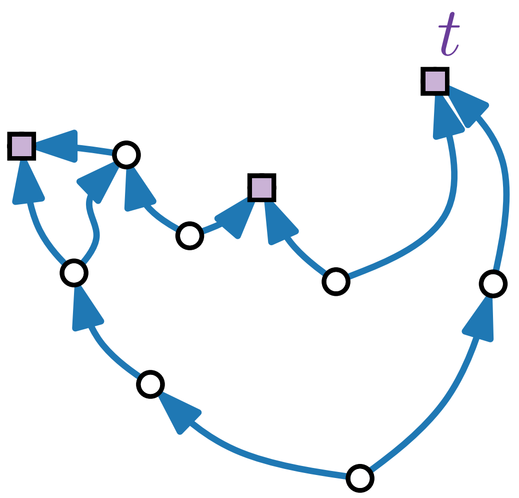

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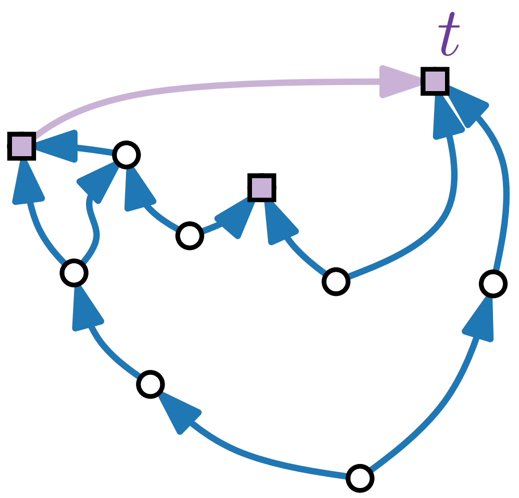

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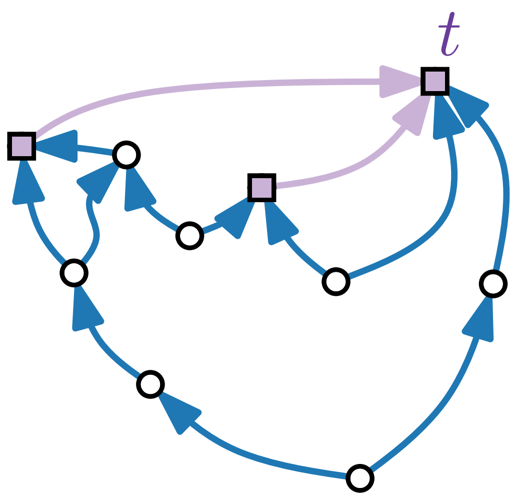

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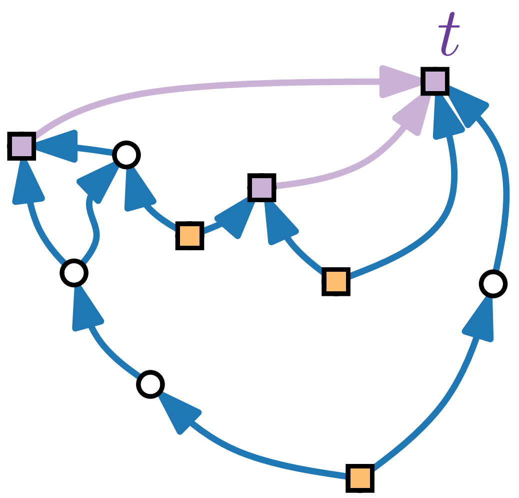

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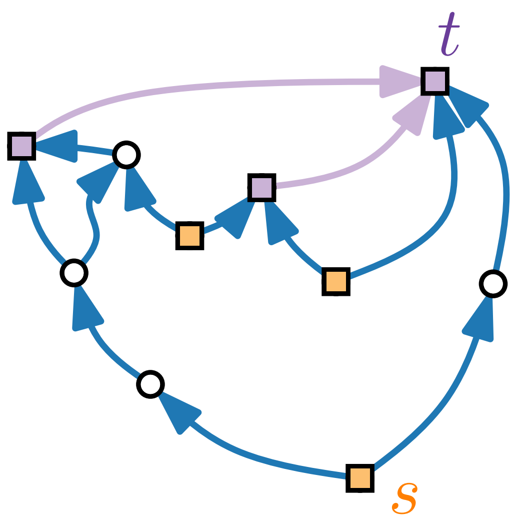

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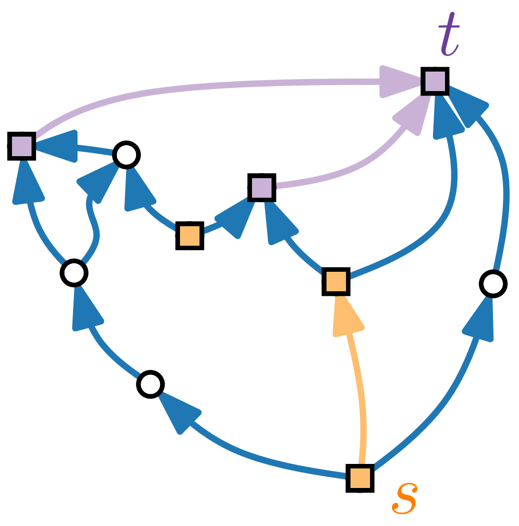

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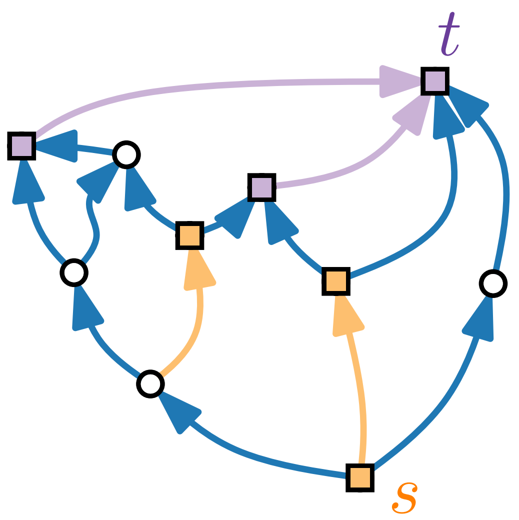

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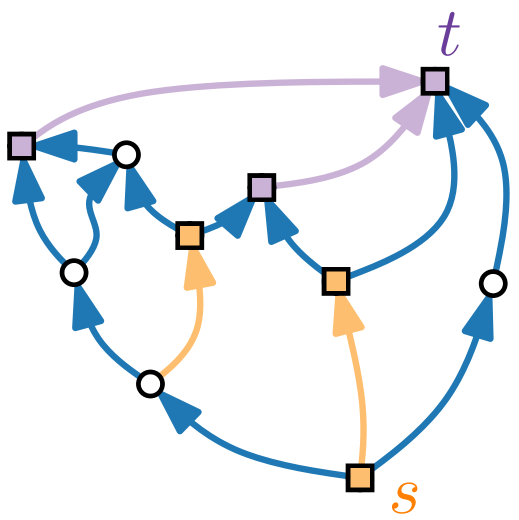

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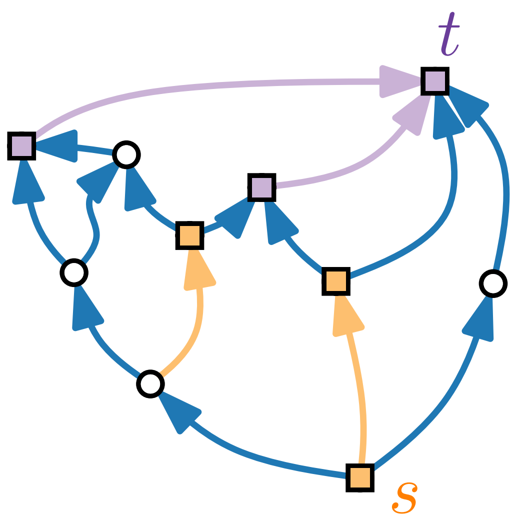

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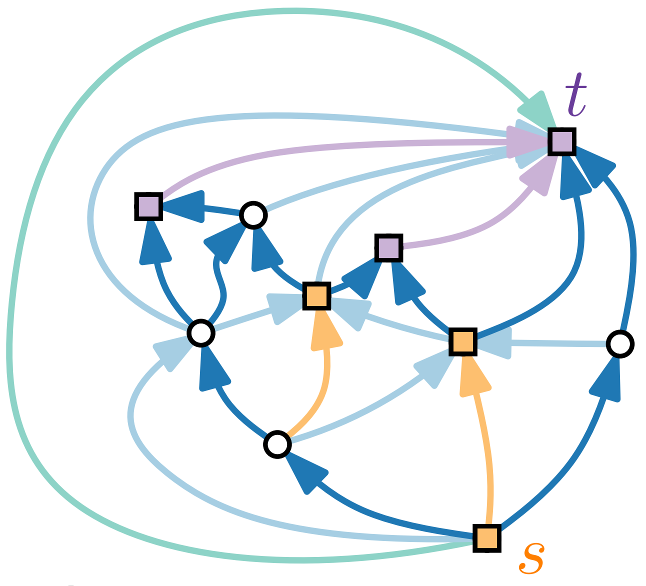

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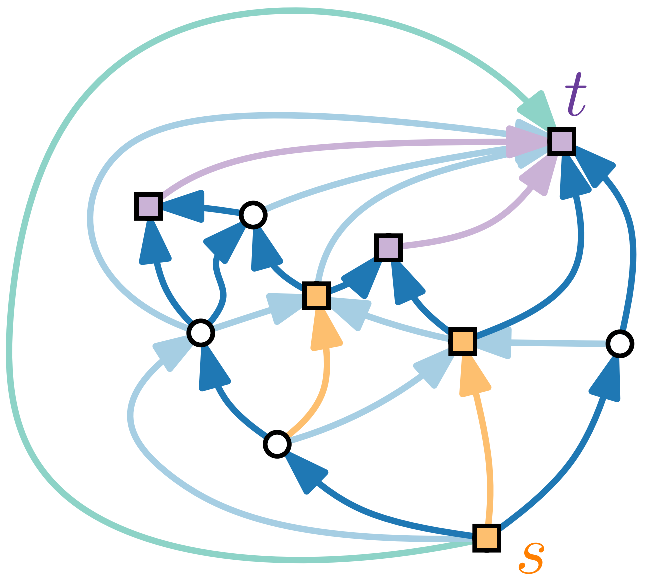

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Claim.
Can be drawn in pre-specified triangle.

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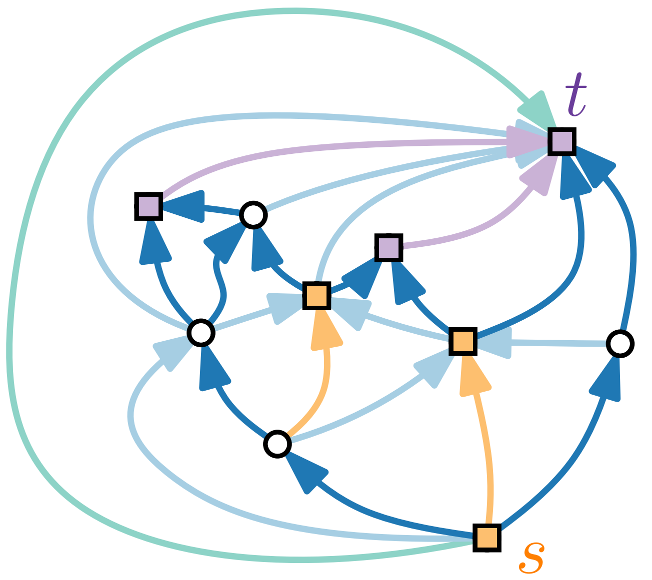

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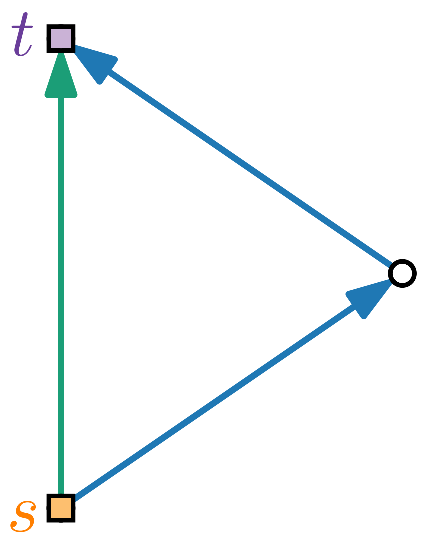

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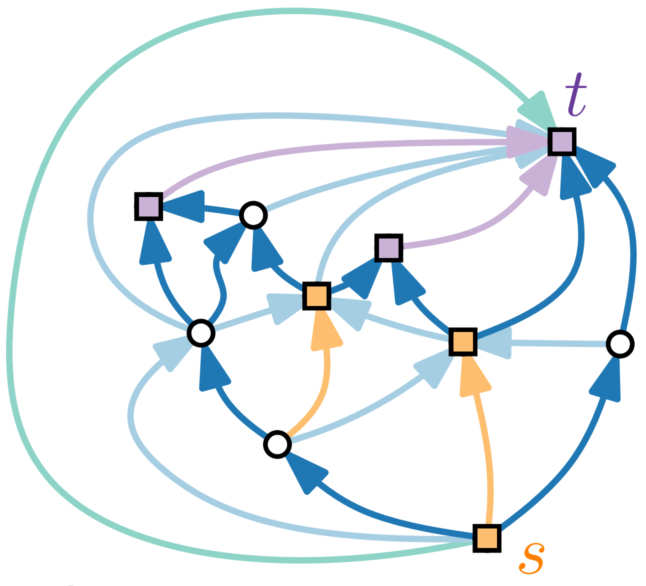

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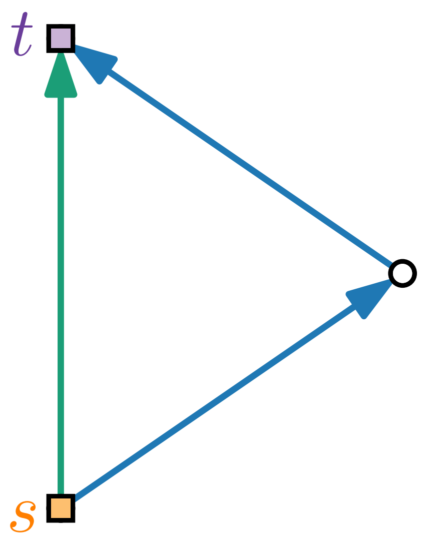
Induction on the number of vertices n.



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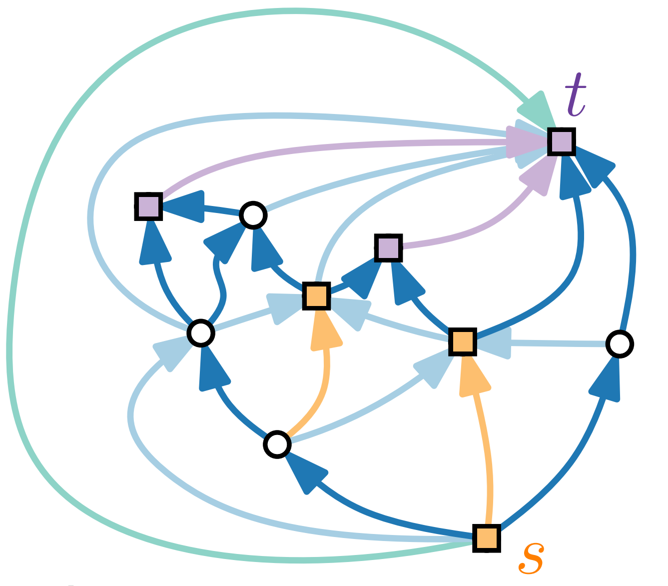

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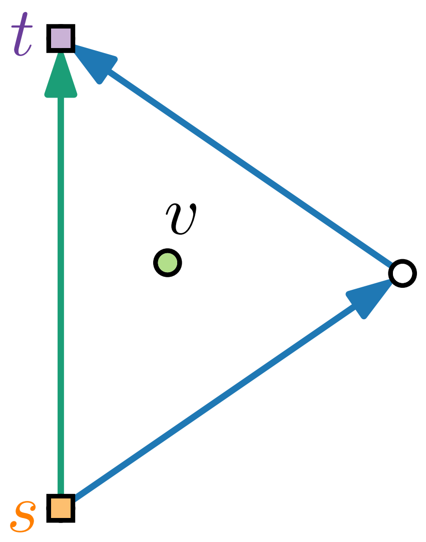

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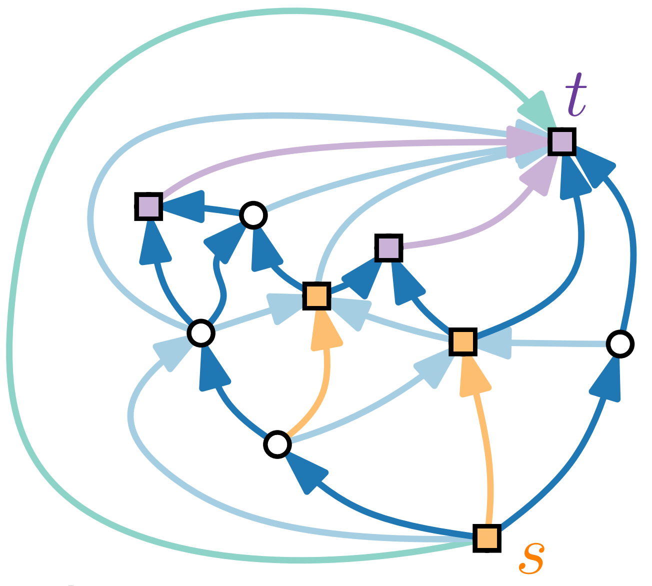

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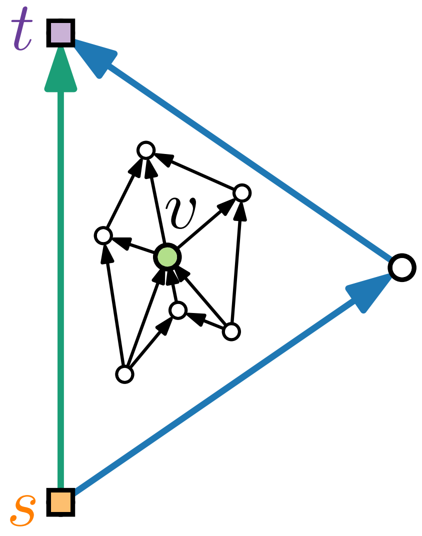

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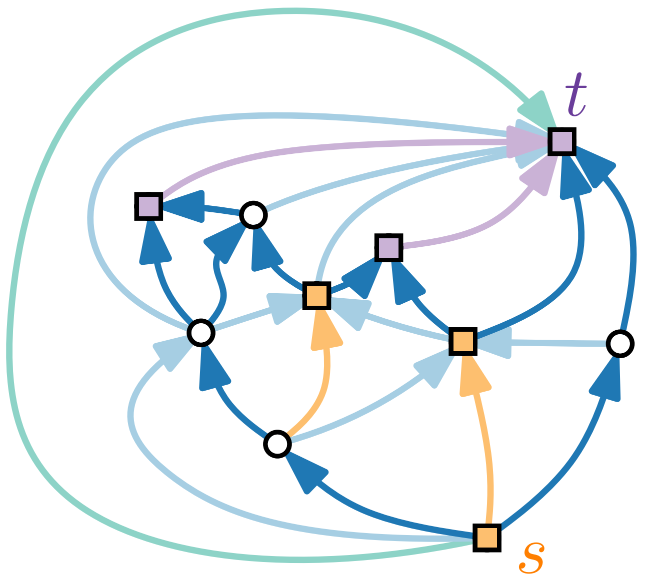

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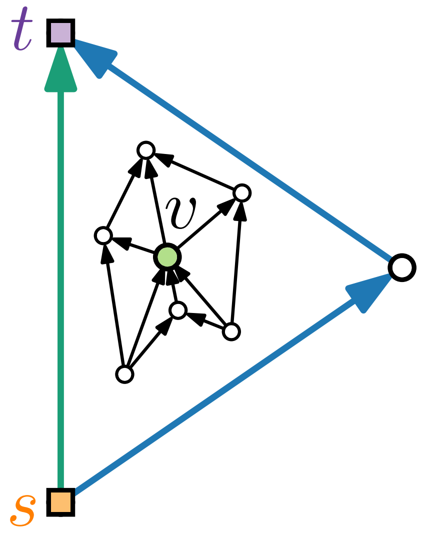

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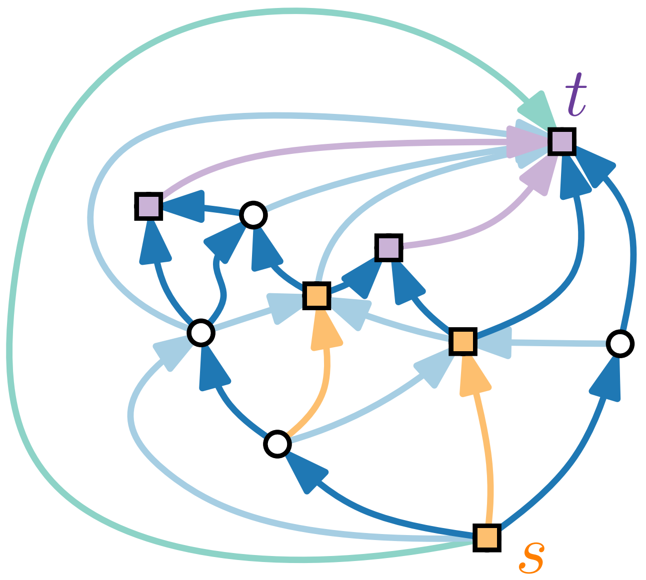

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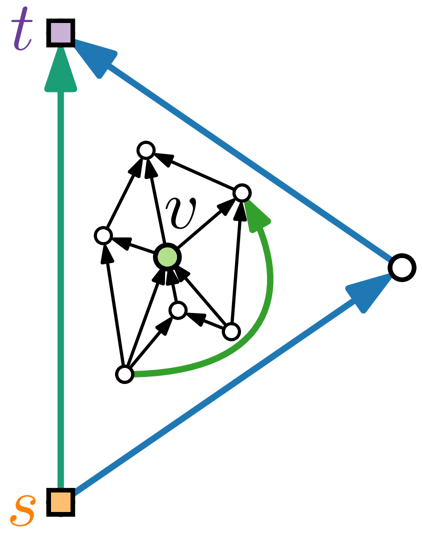

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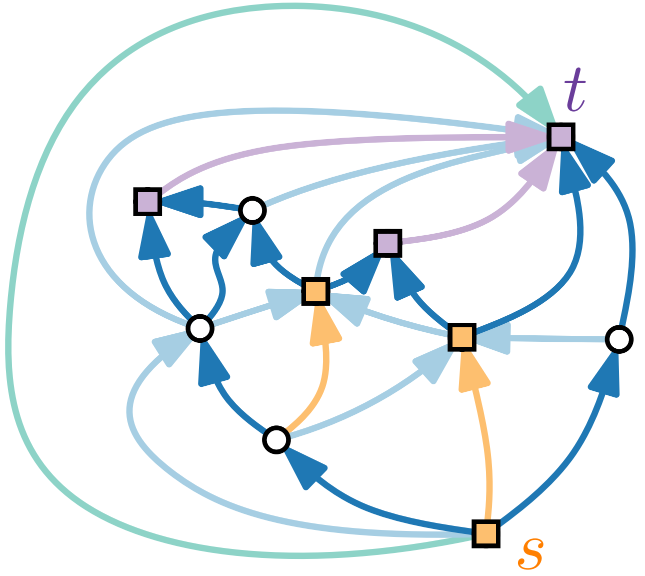

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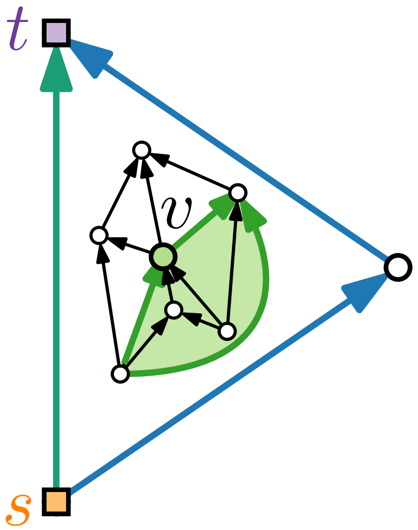

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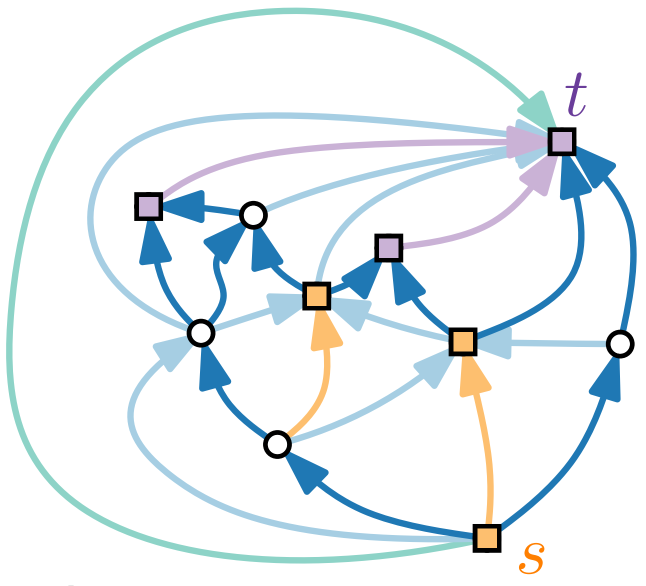

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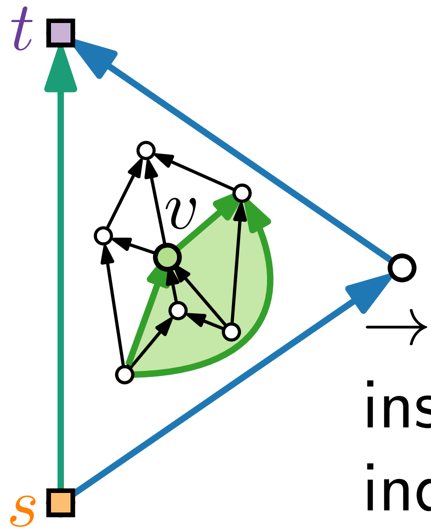

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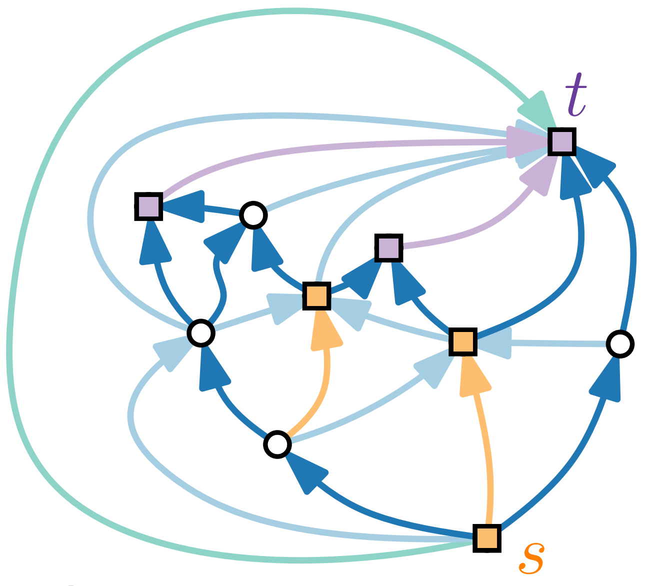

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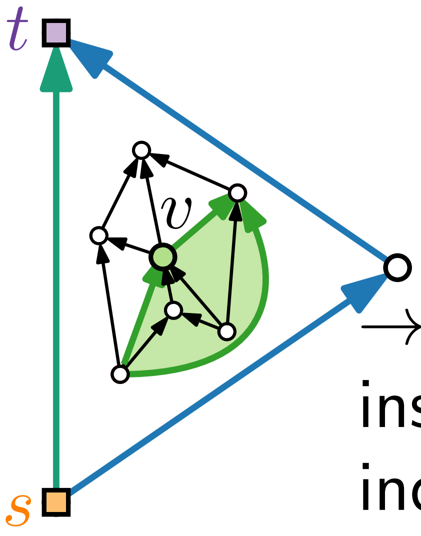

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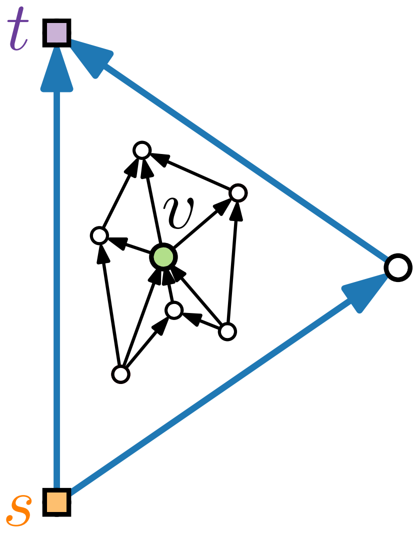

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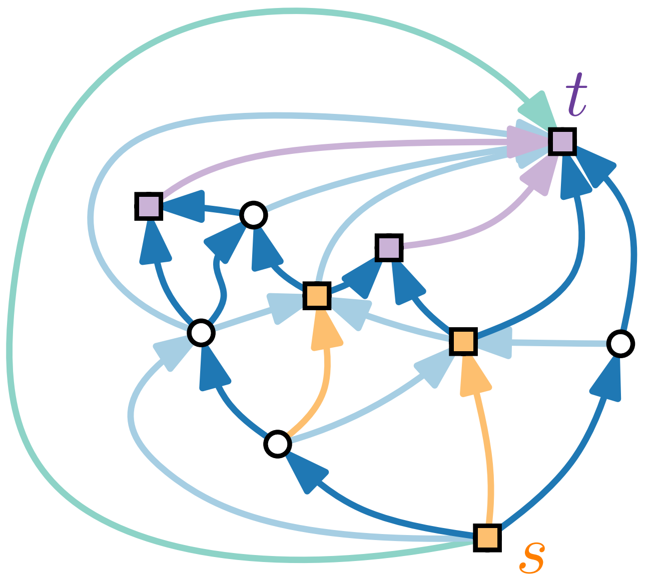

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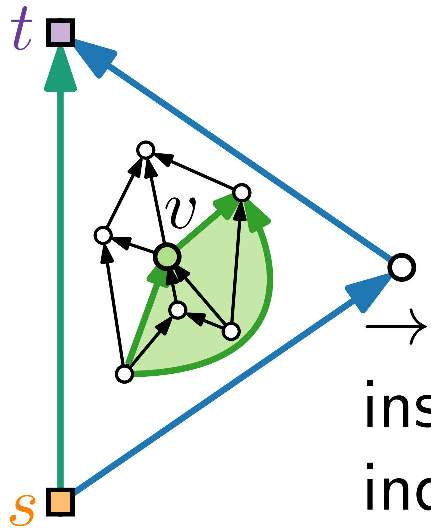

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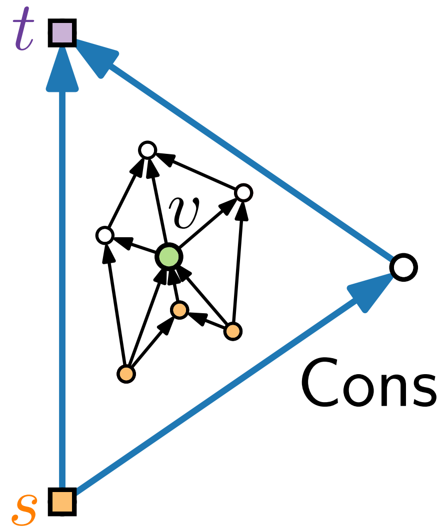

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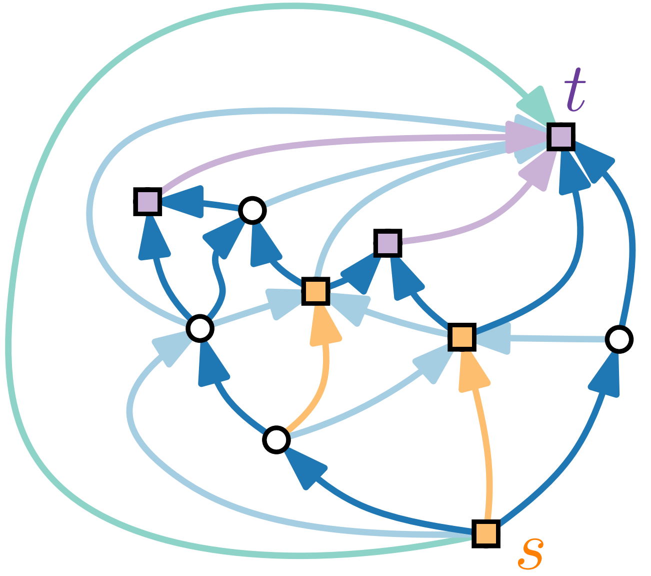

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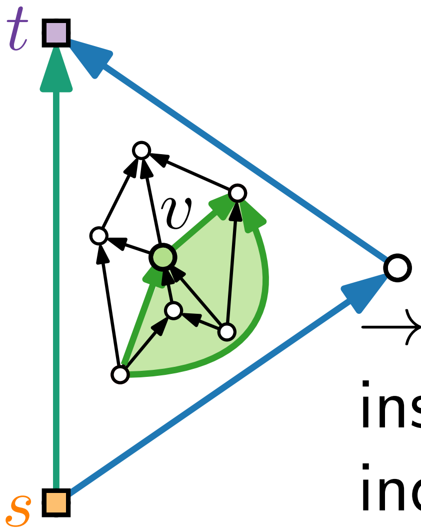

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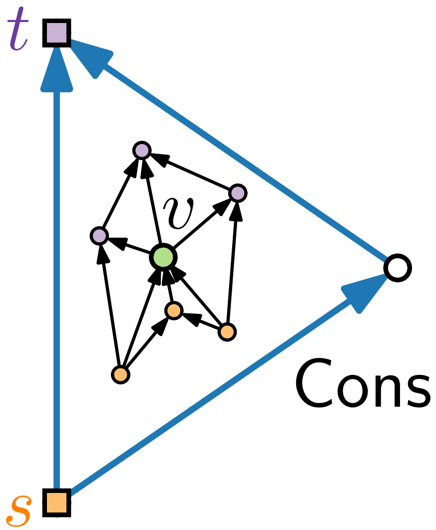

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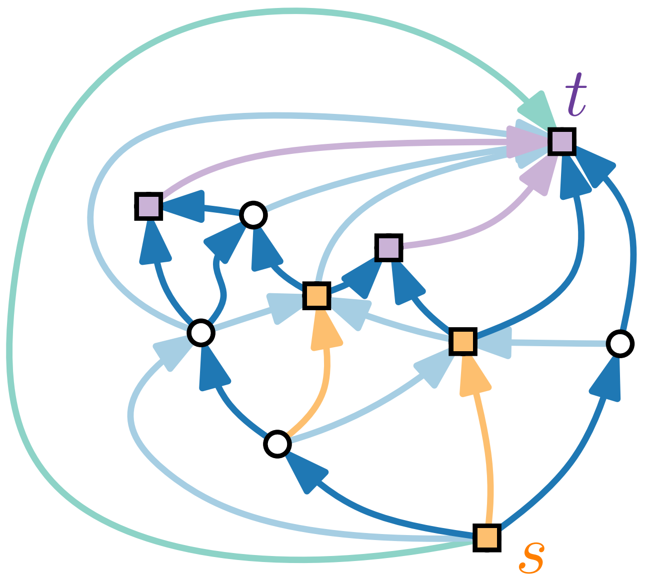

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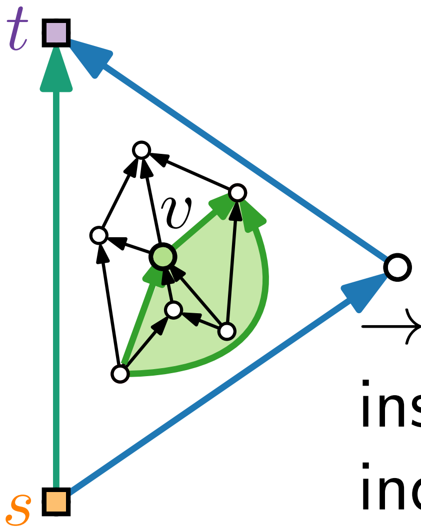

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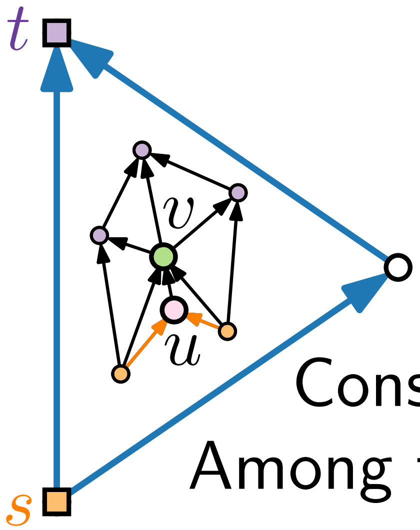

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Among these, take "highest."

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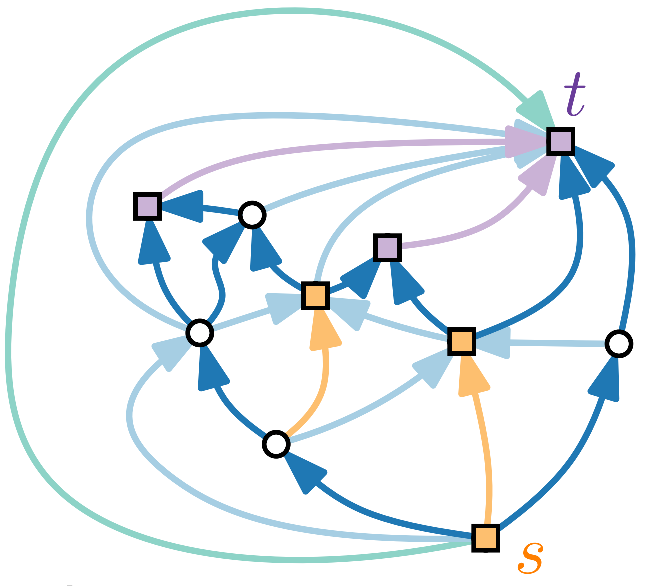

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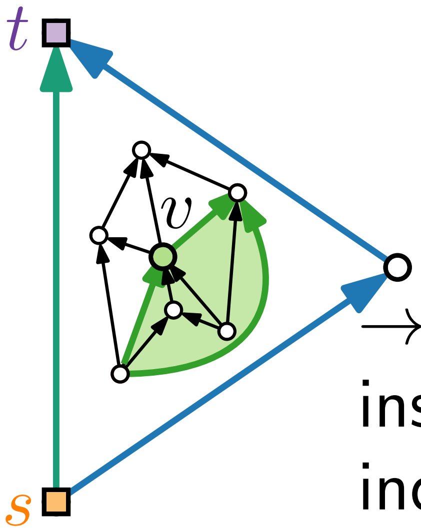

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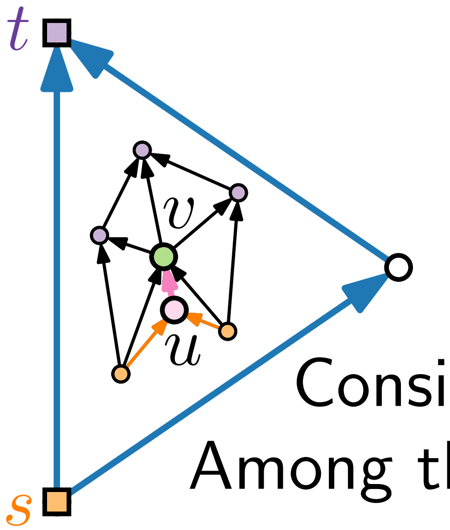
Idea: Contract uv!

Case 2: no chord



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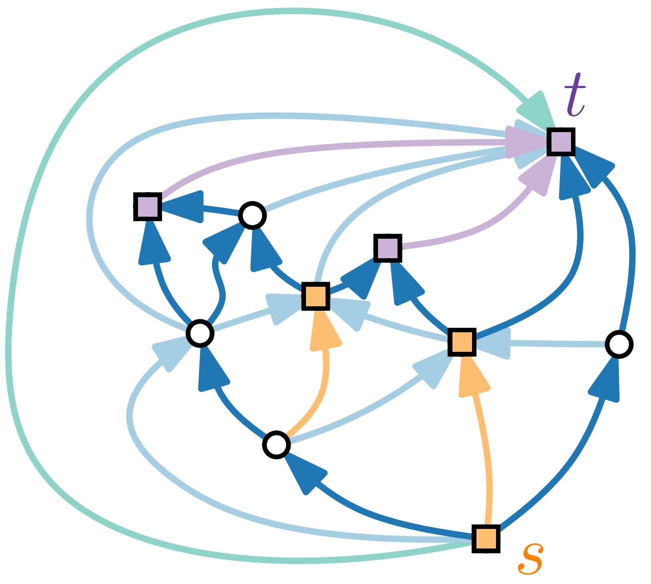

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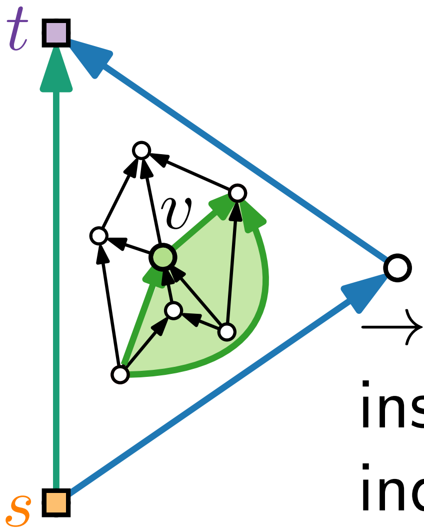

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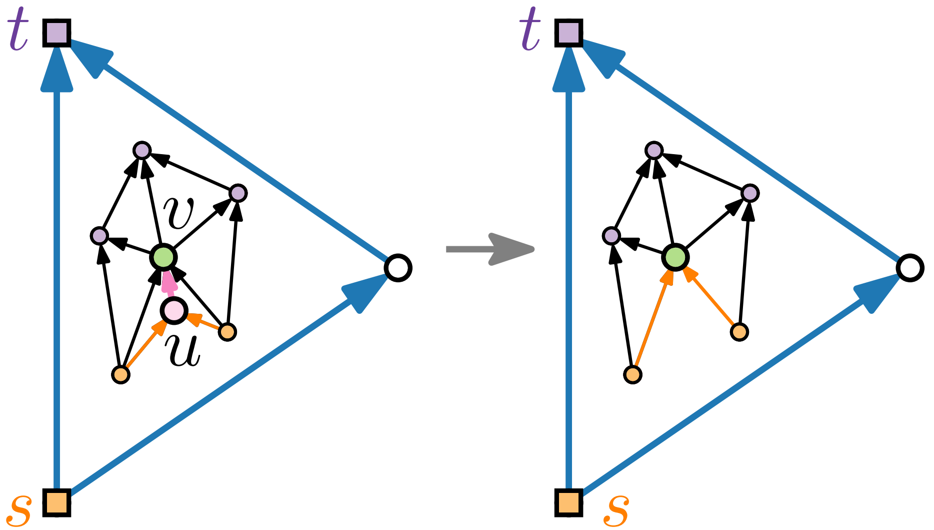

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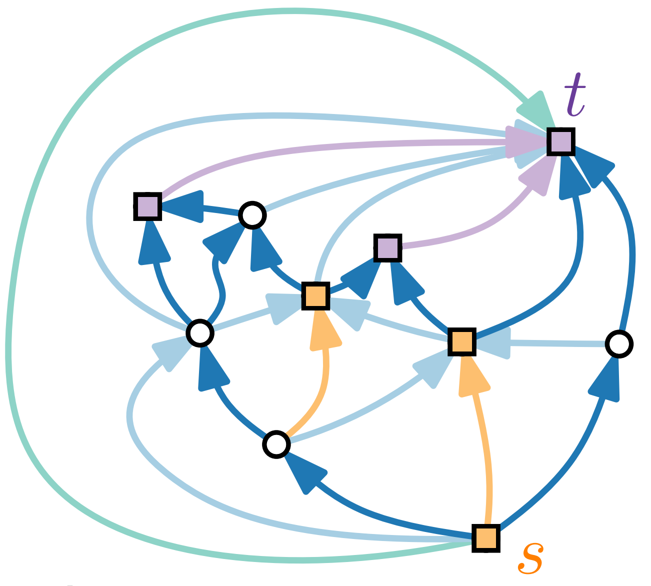

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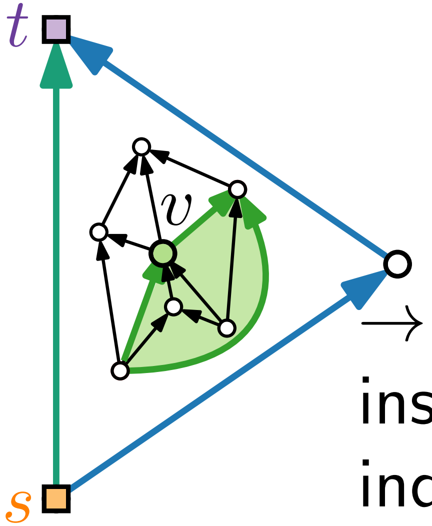

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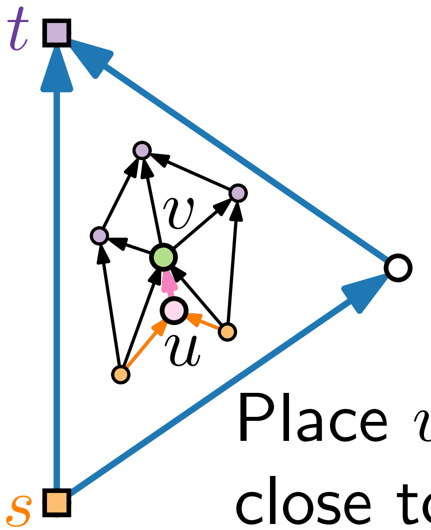

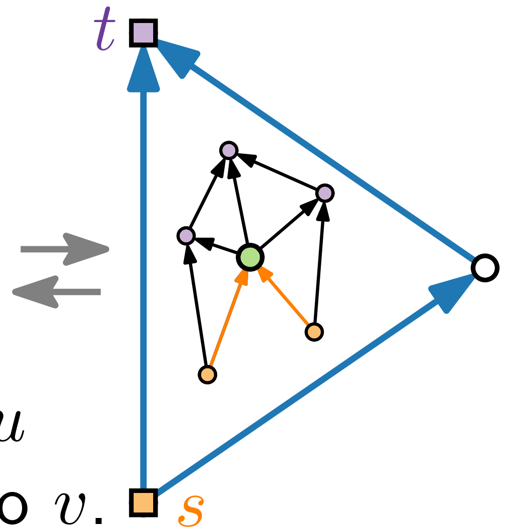

Place u close to v.

Upward Planarity – Complexity

Given a *planar acyclic* digraph G,
decide whether G is upward planar.

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Fixed Embedding Upward Planarity Testing.
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Test whether G is upward planar (w.r.t. to F and f_0).

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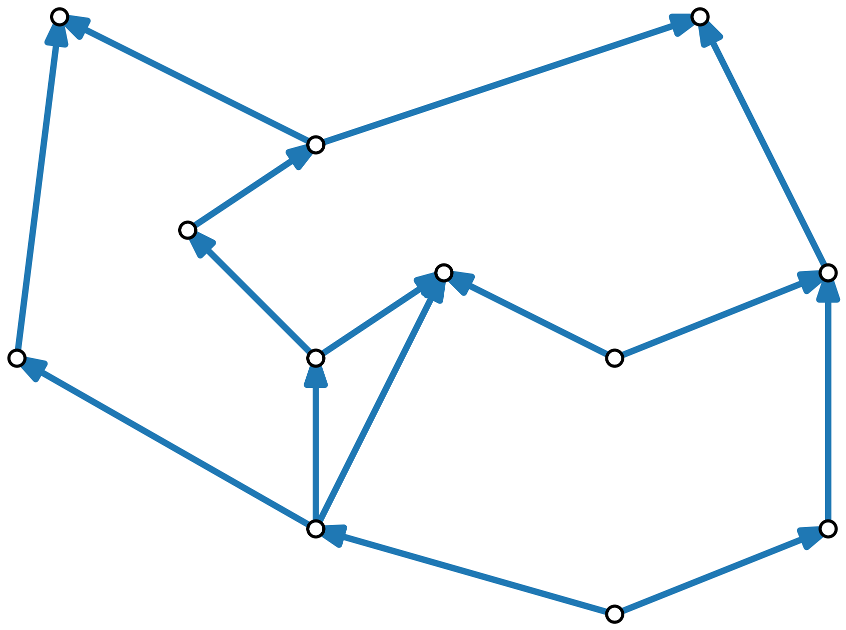
Plan.

- Find a property that any upward planar drawing of G satisfies.
- Formalize this property.
- Specify an algorithm to test this property.

Angles, Local Sources & Sinks

Definitions.



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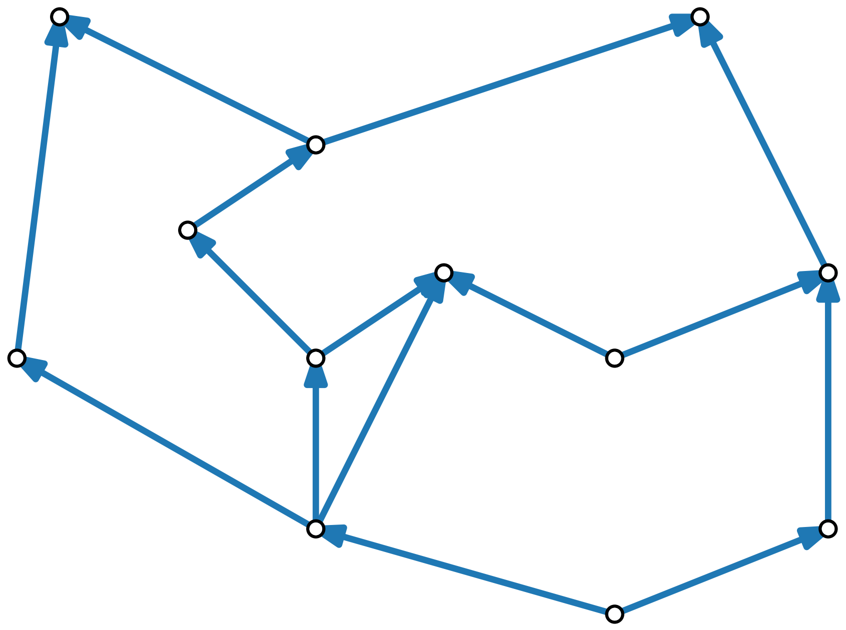

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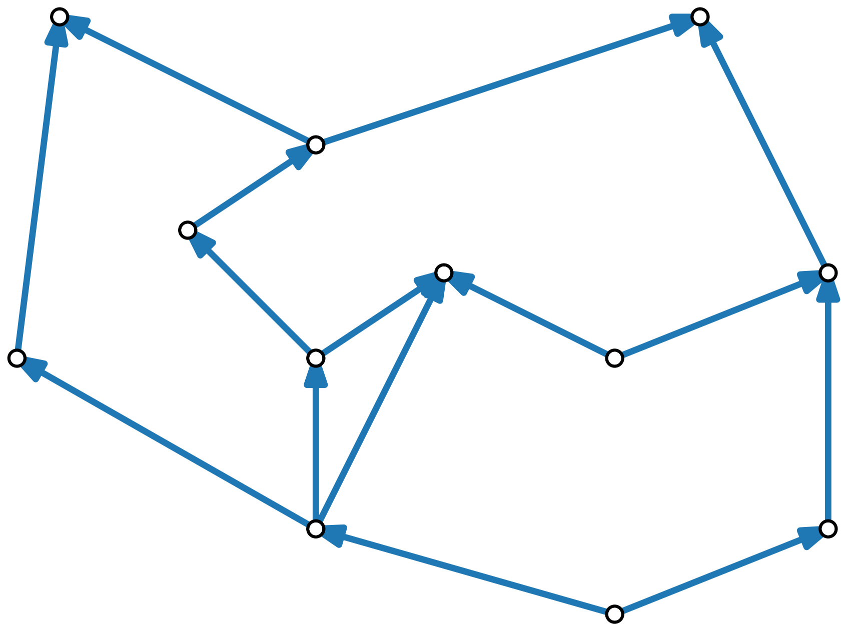

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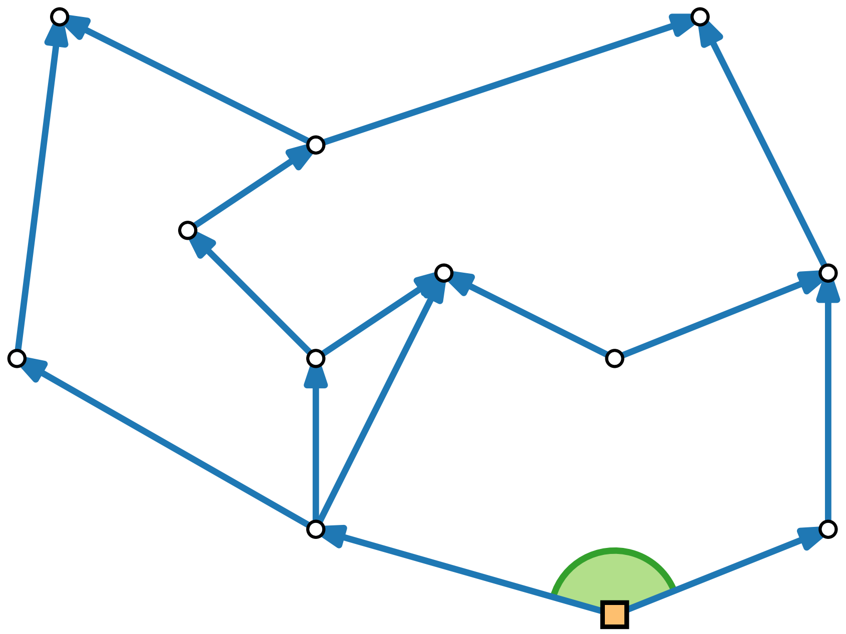

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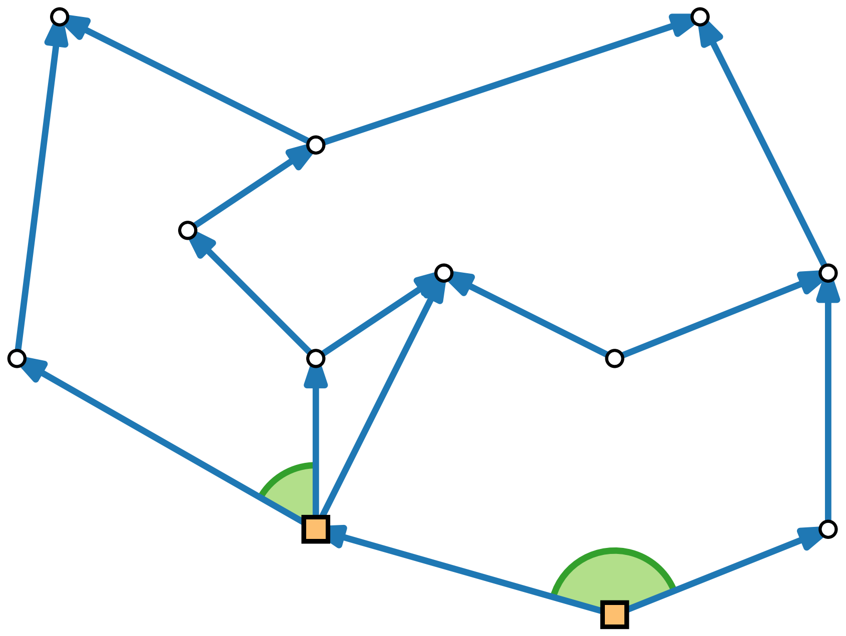

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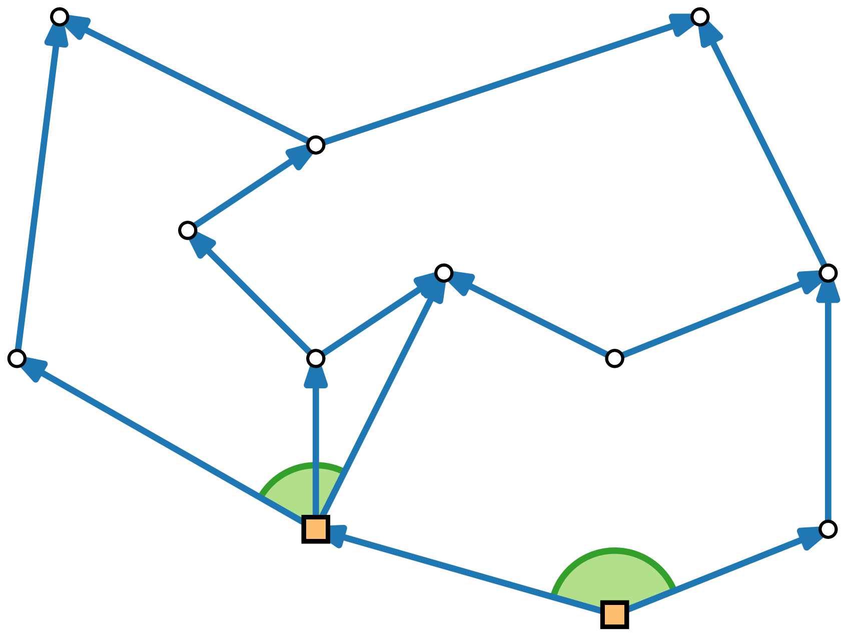

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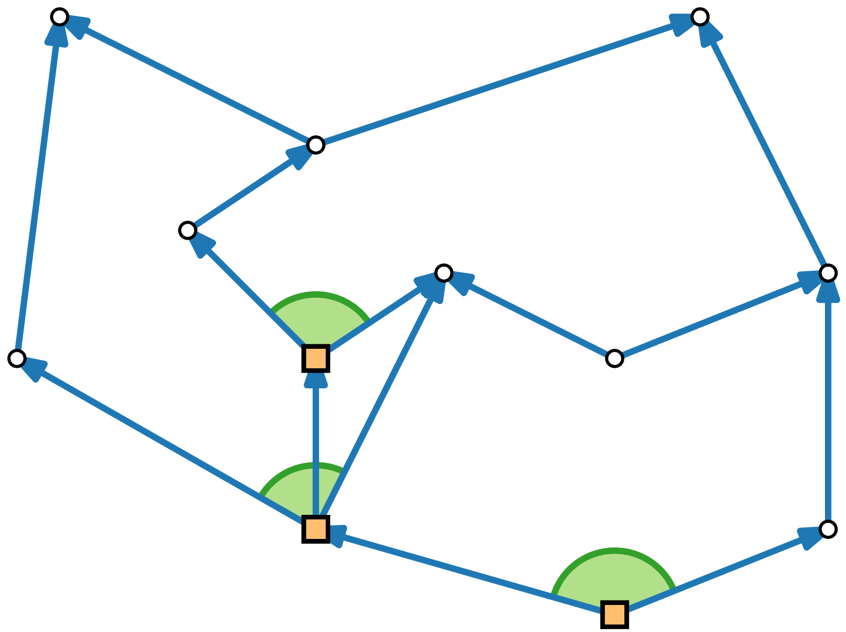

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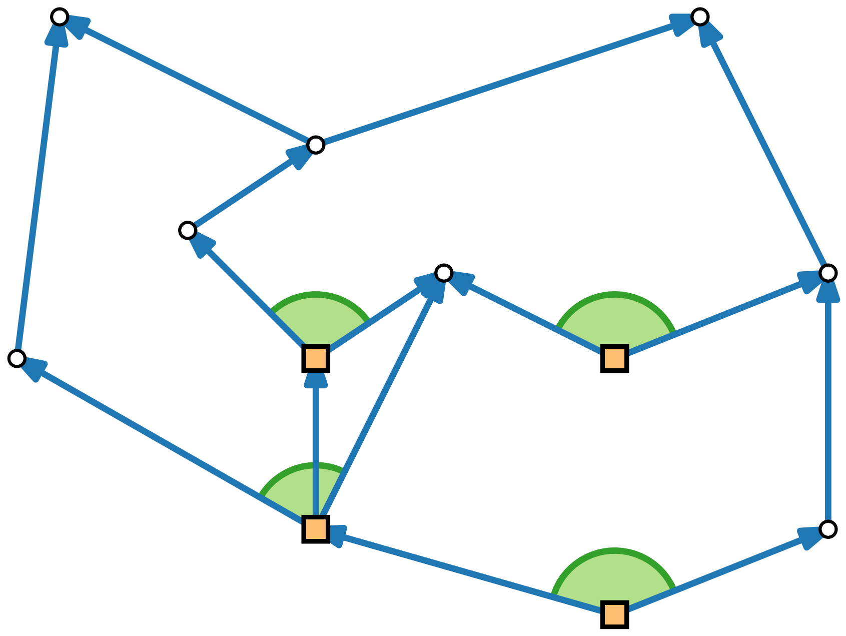

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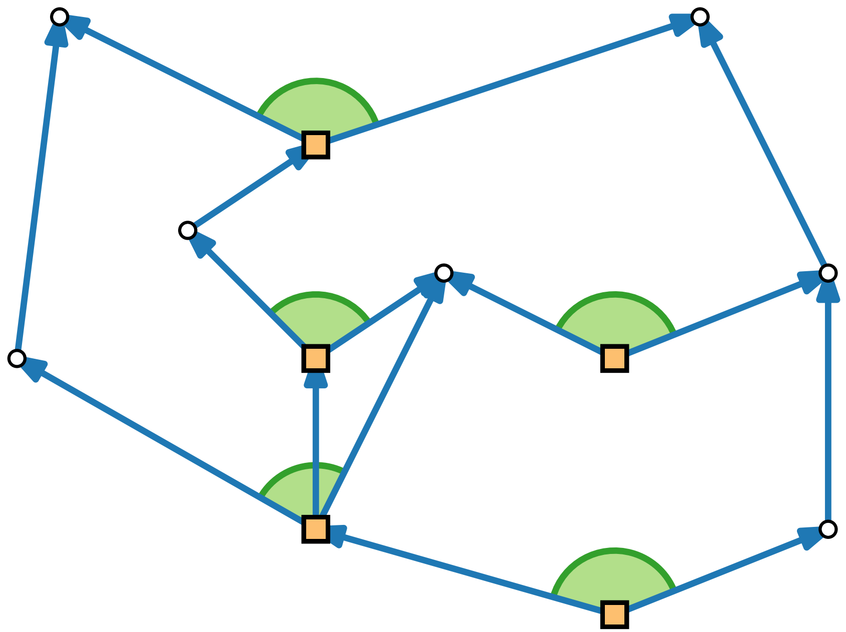

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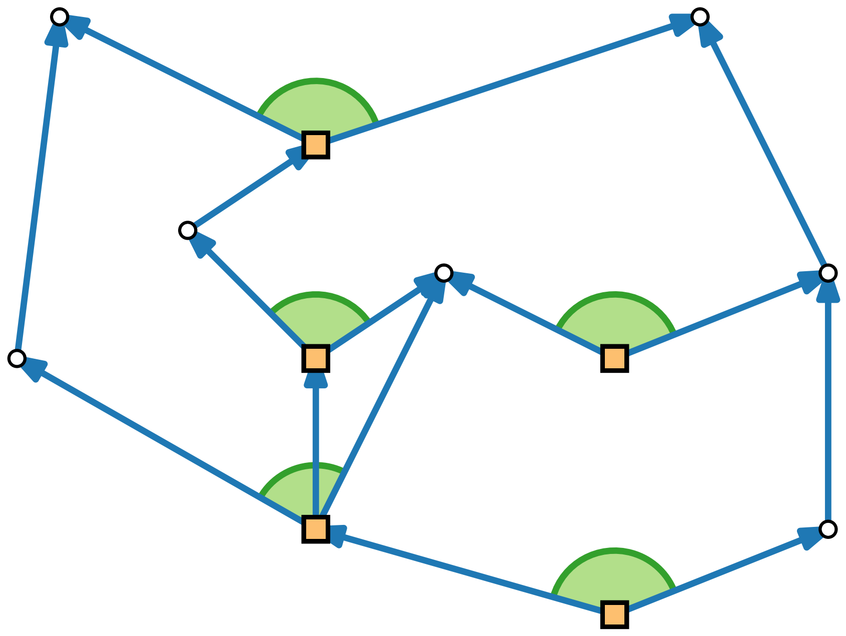

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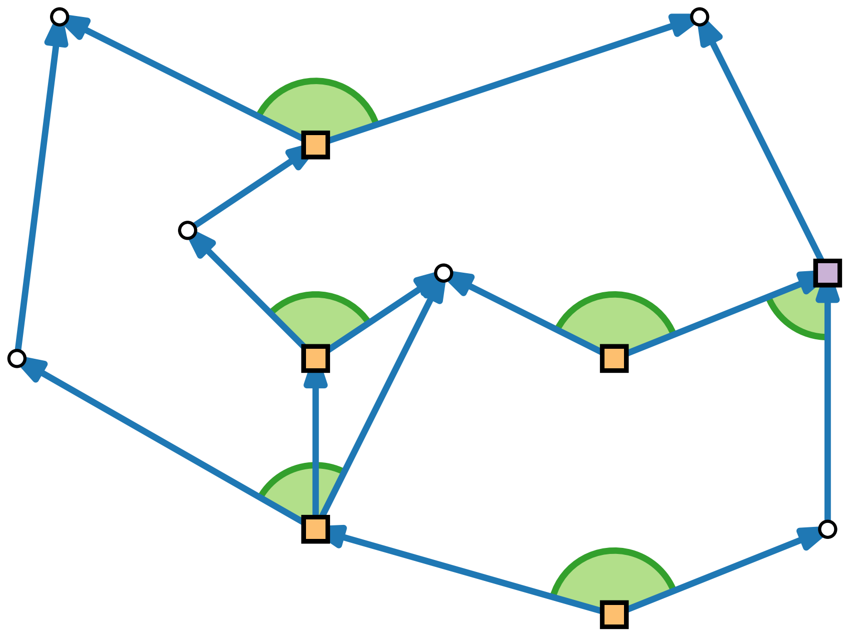

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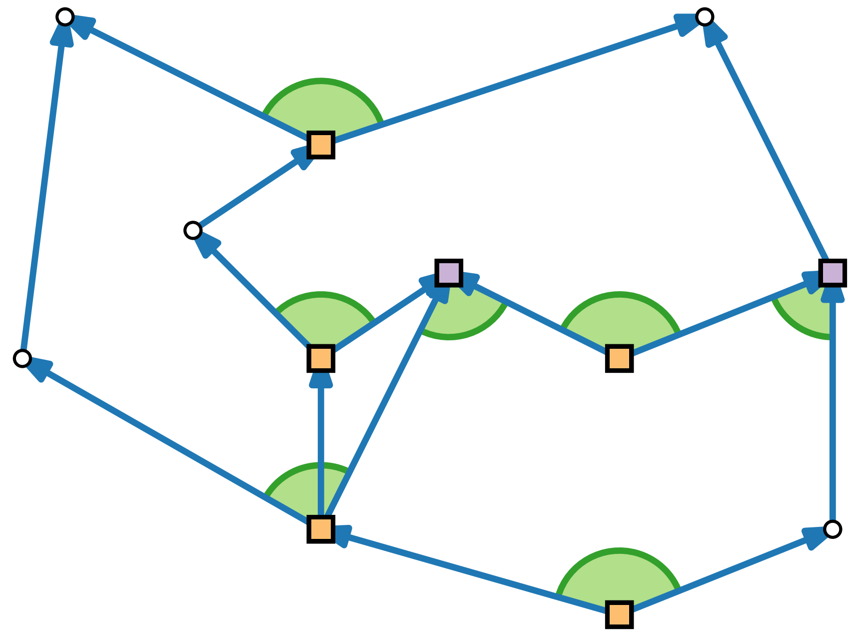

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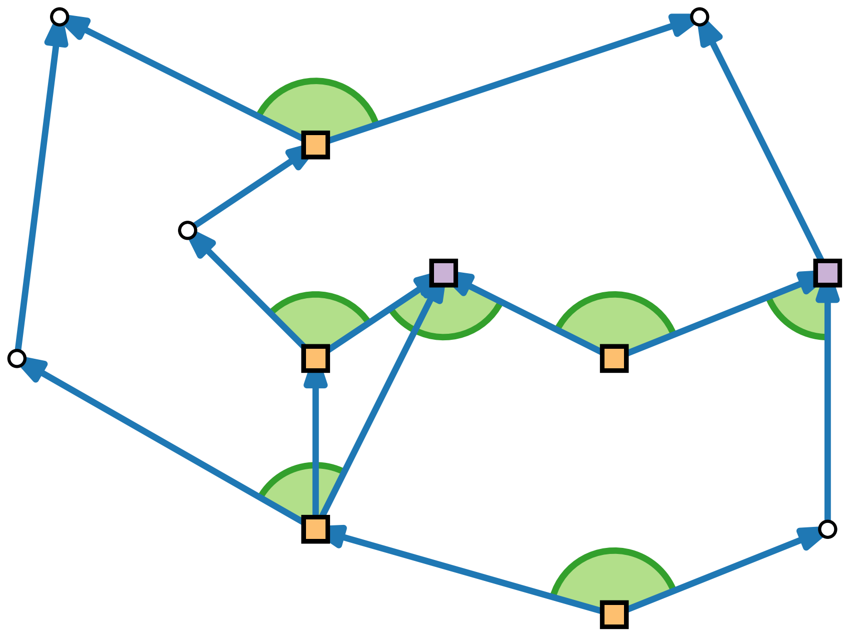

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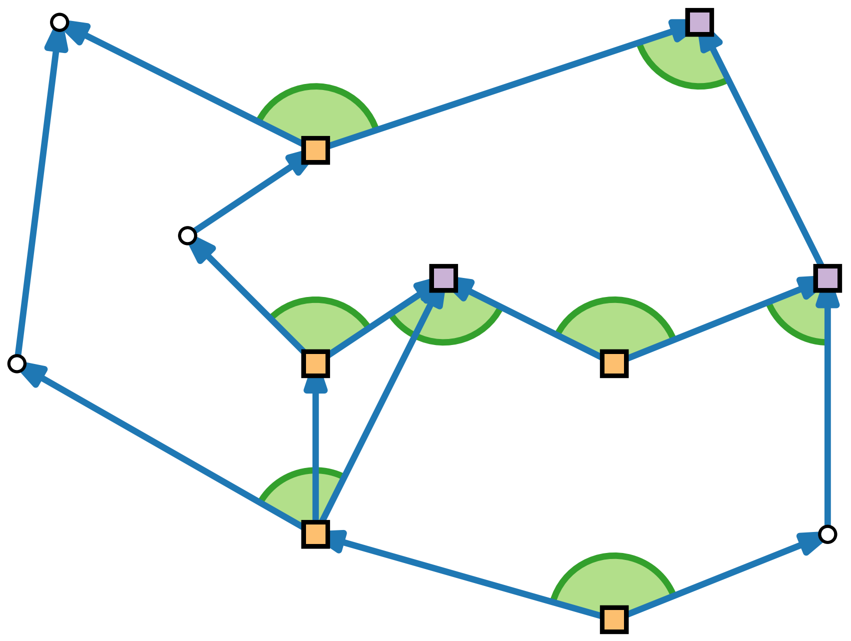

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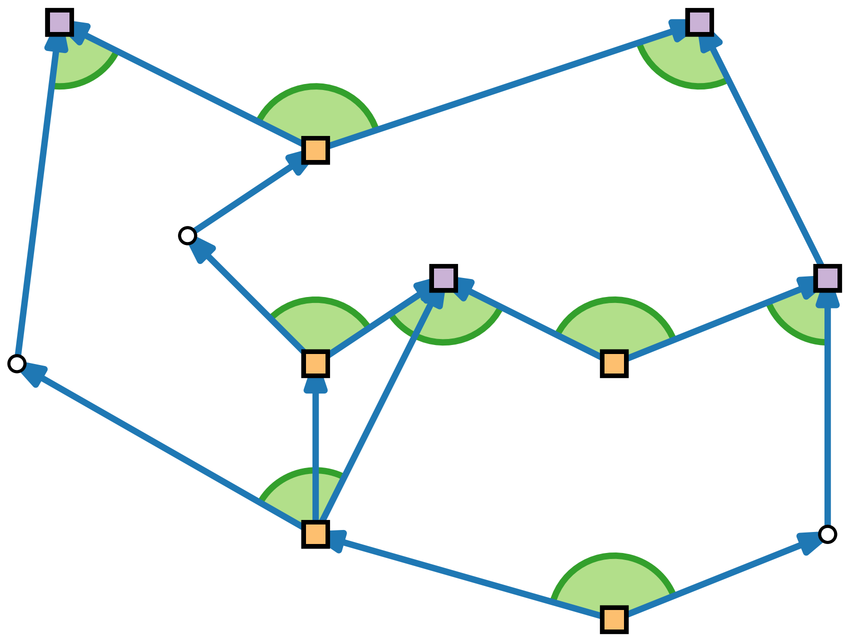

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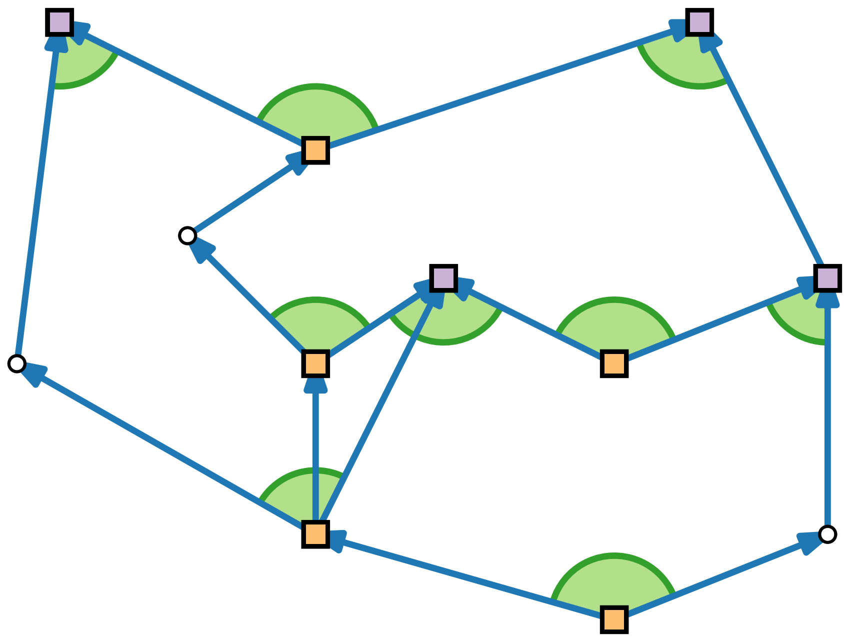

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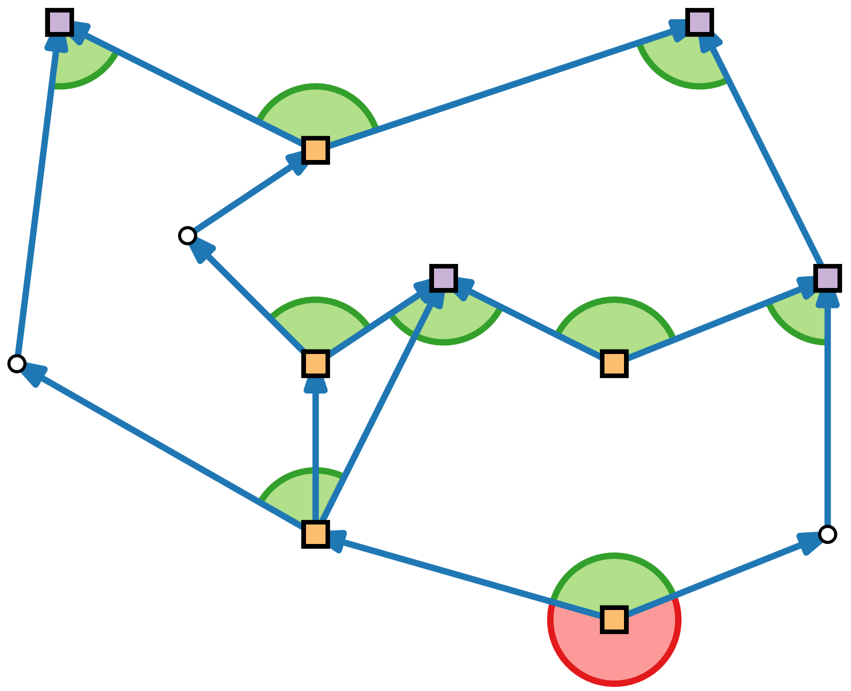

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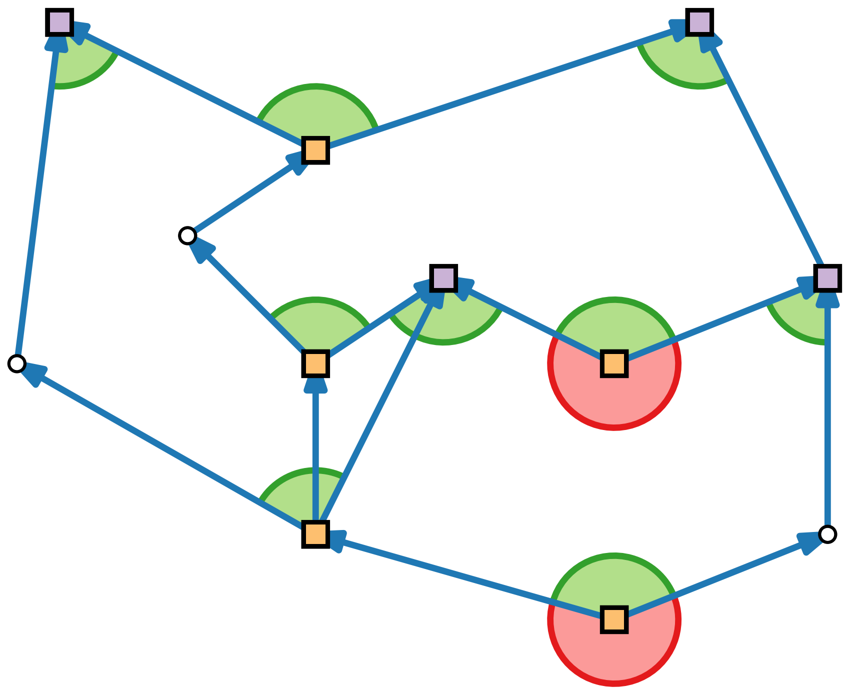

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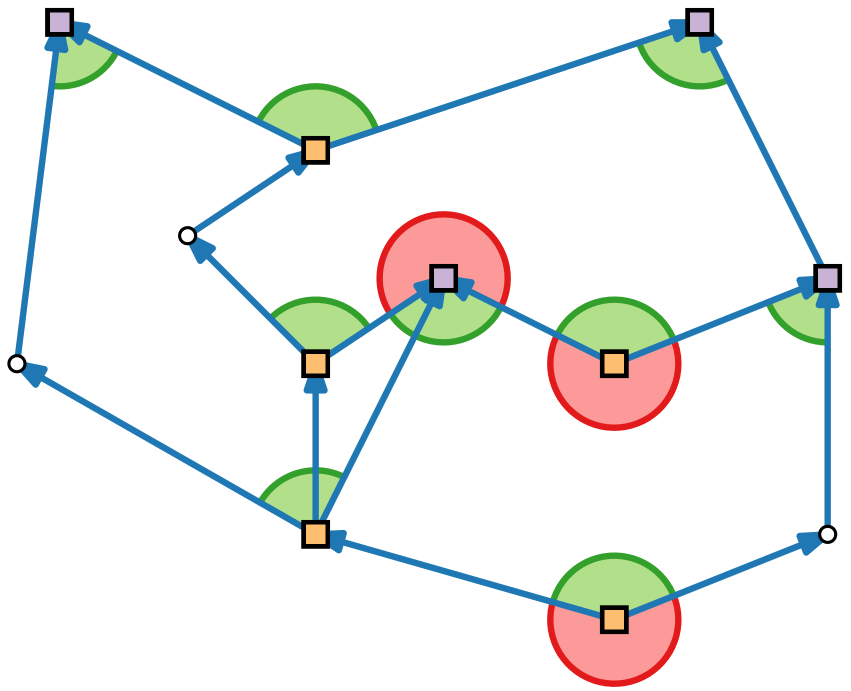

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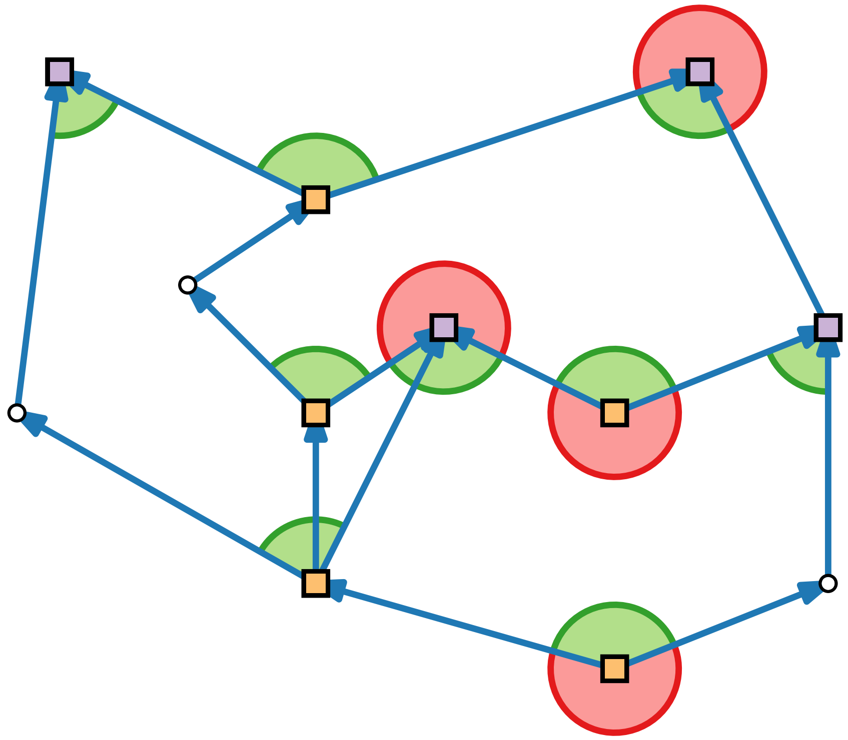

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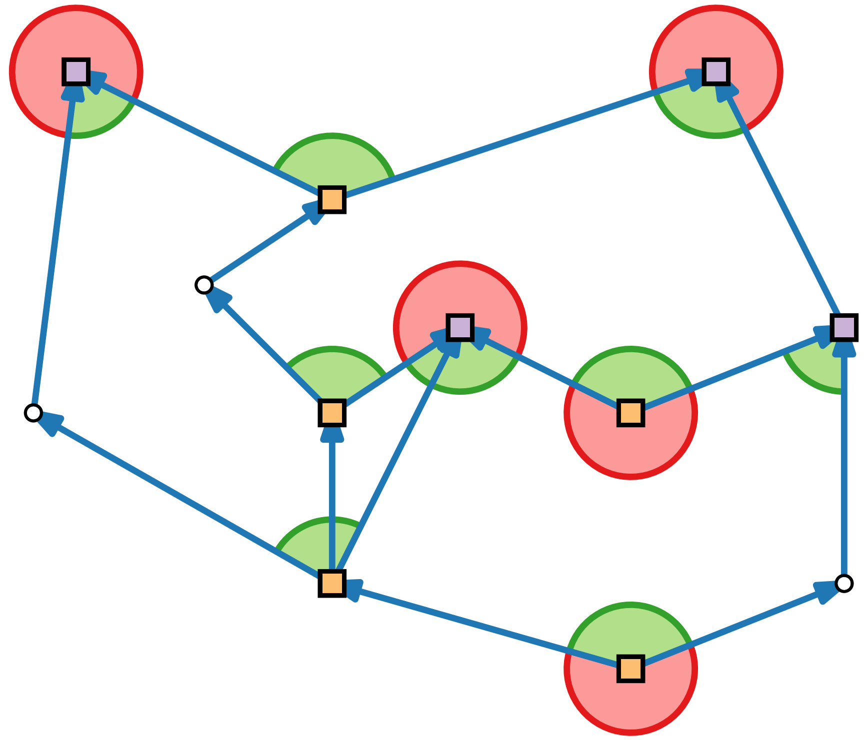

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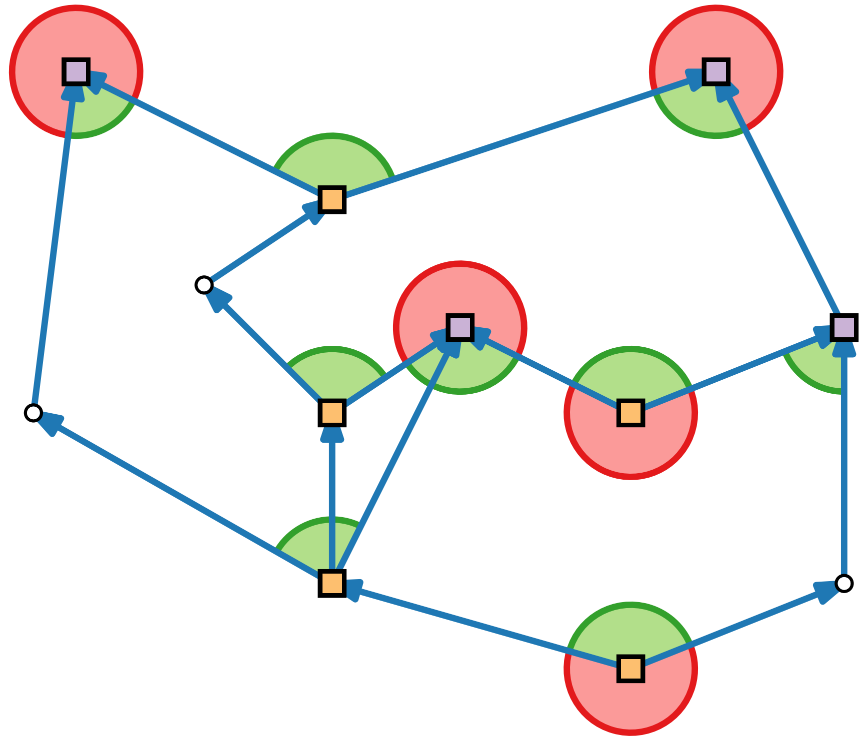

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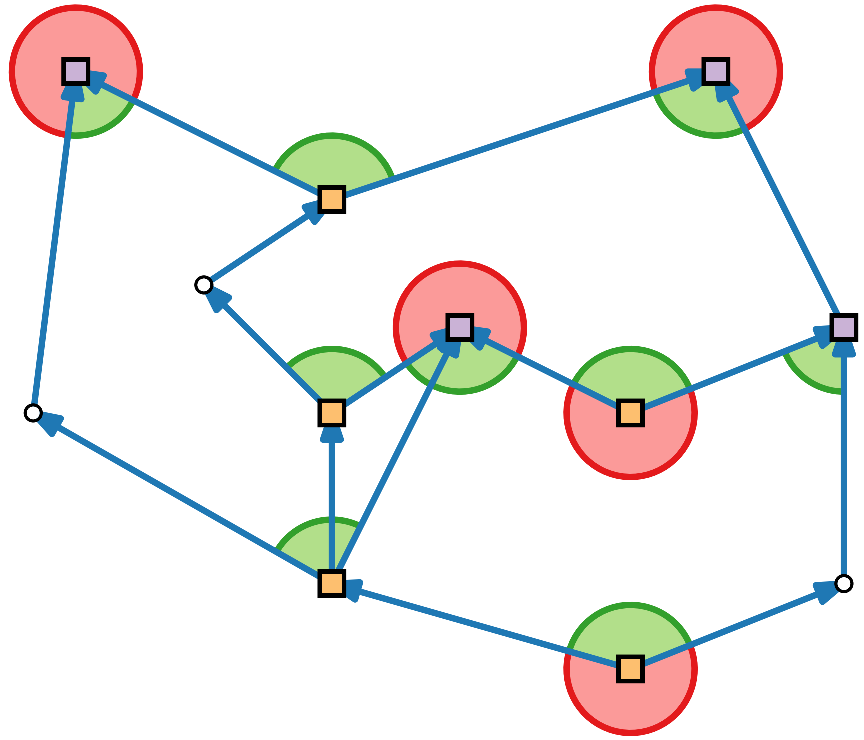

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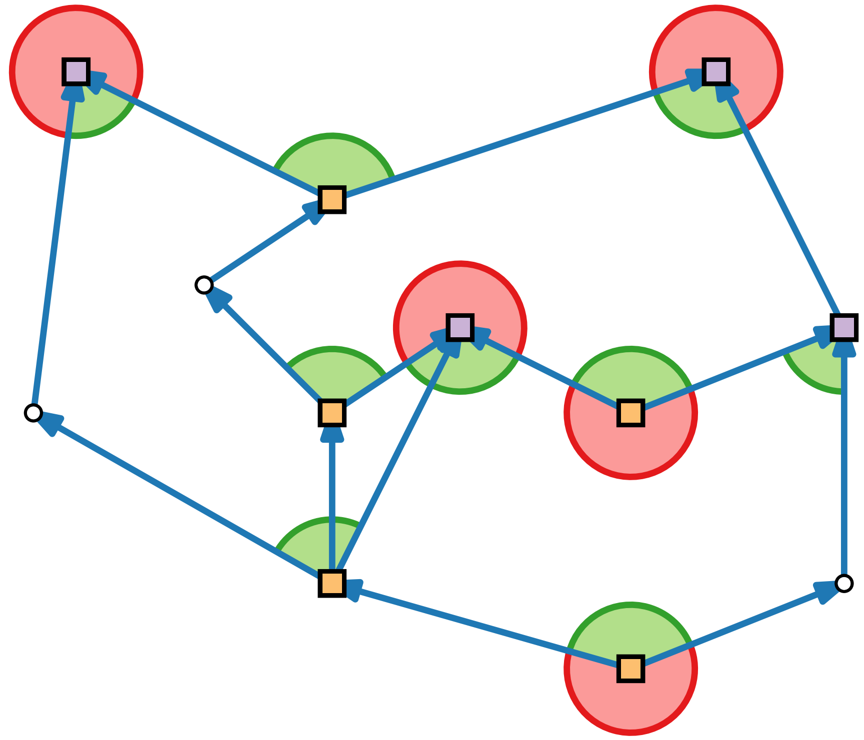

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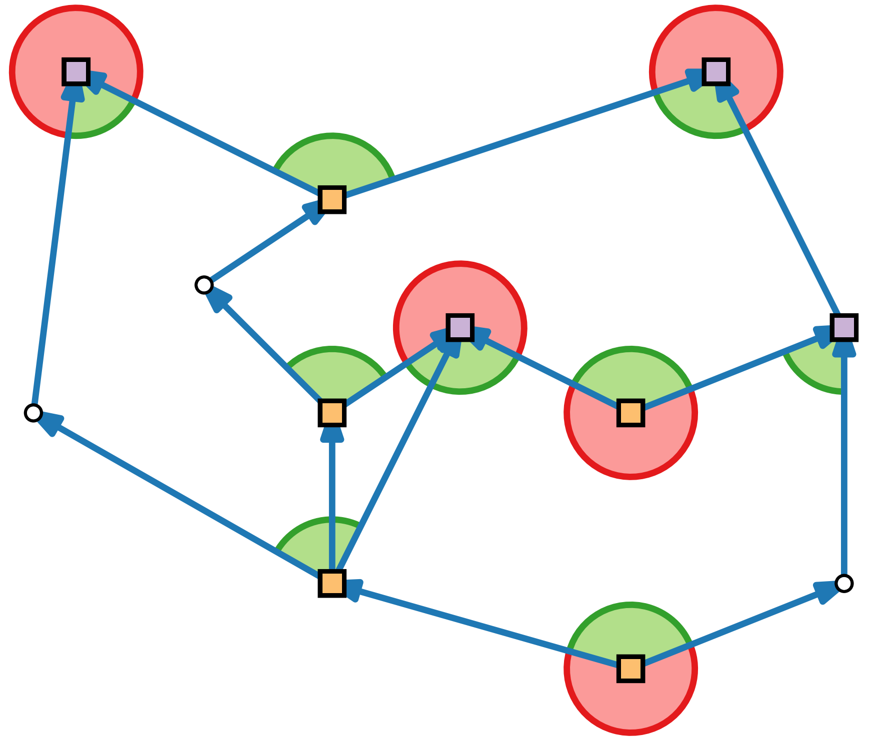

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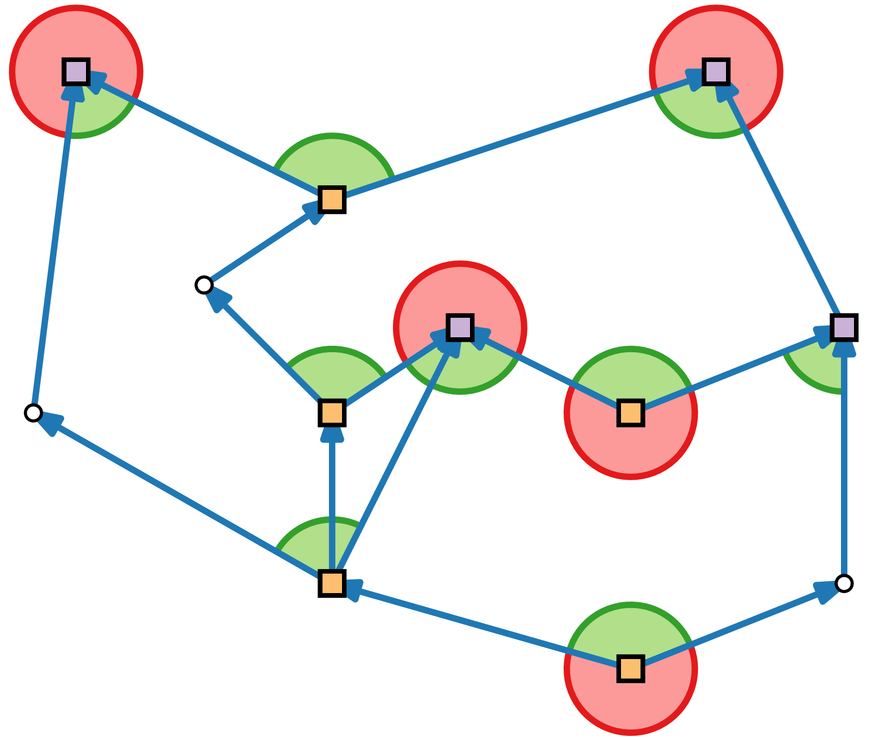

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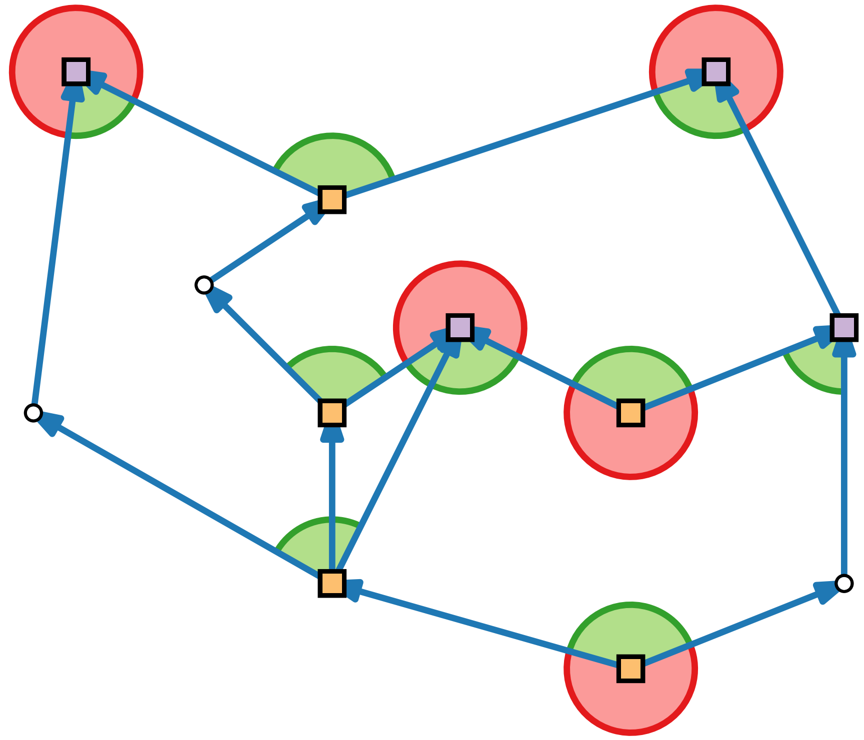

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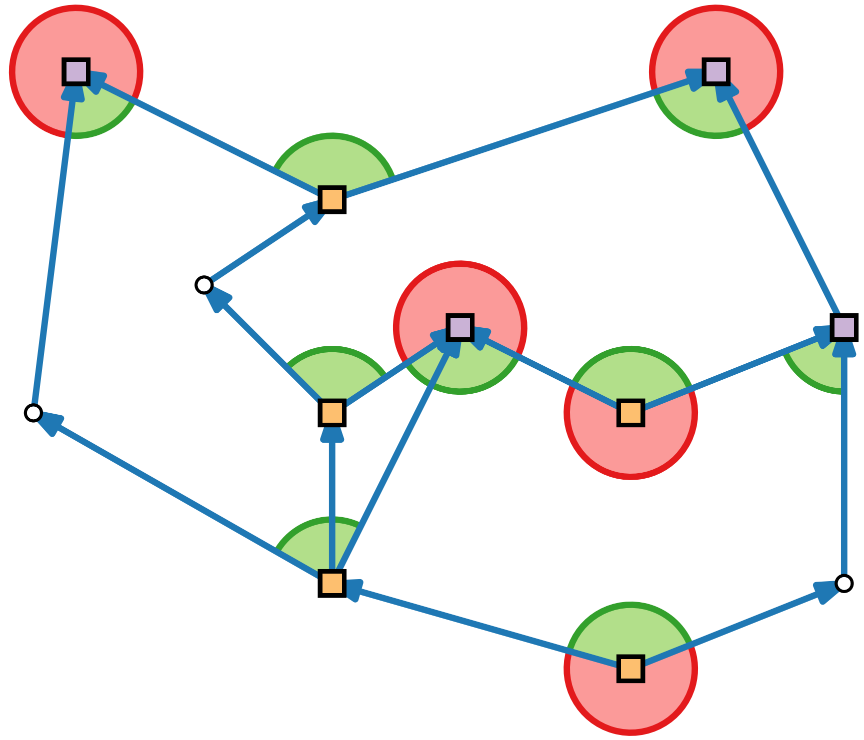

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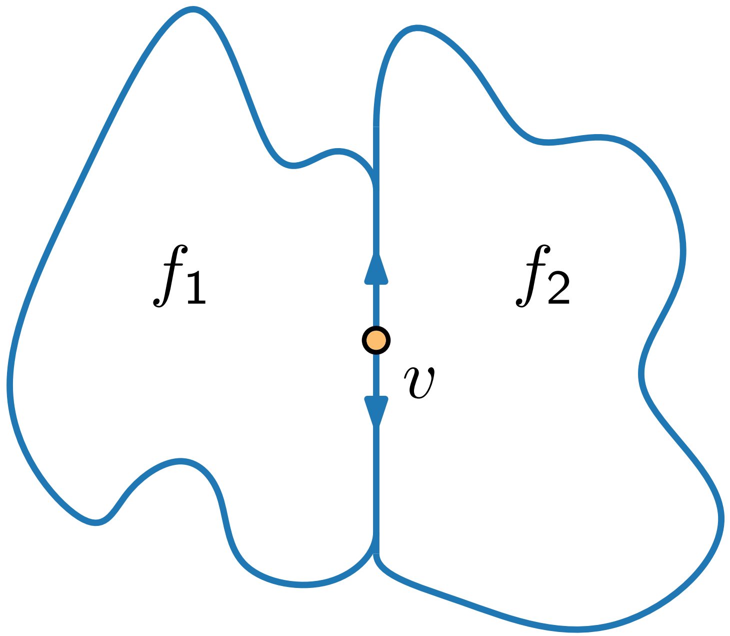

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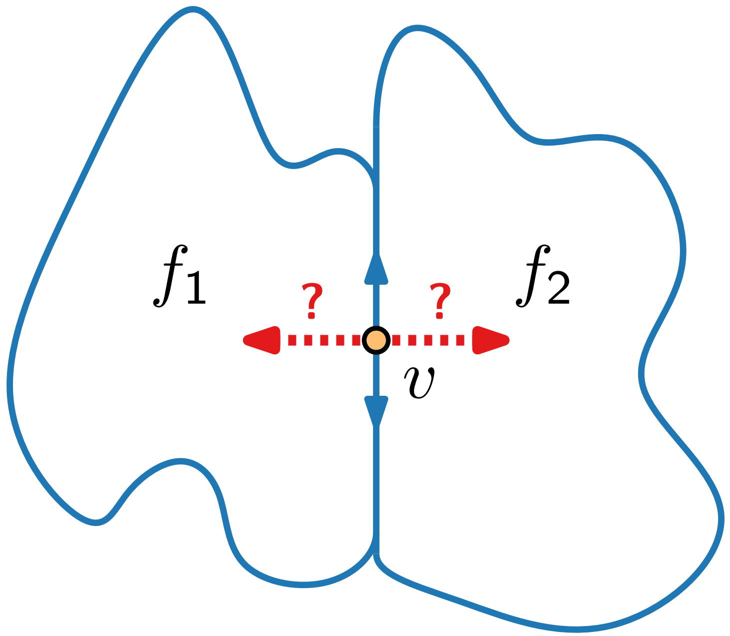

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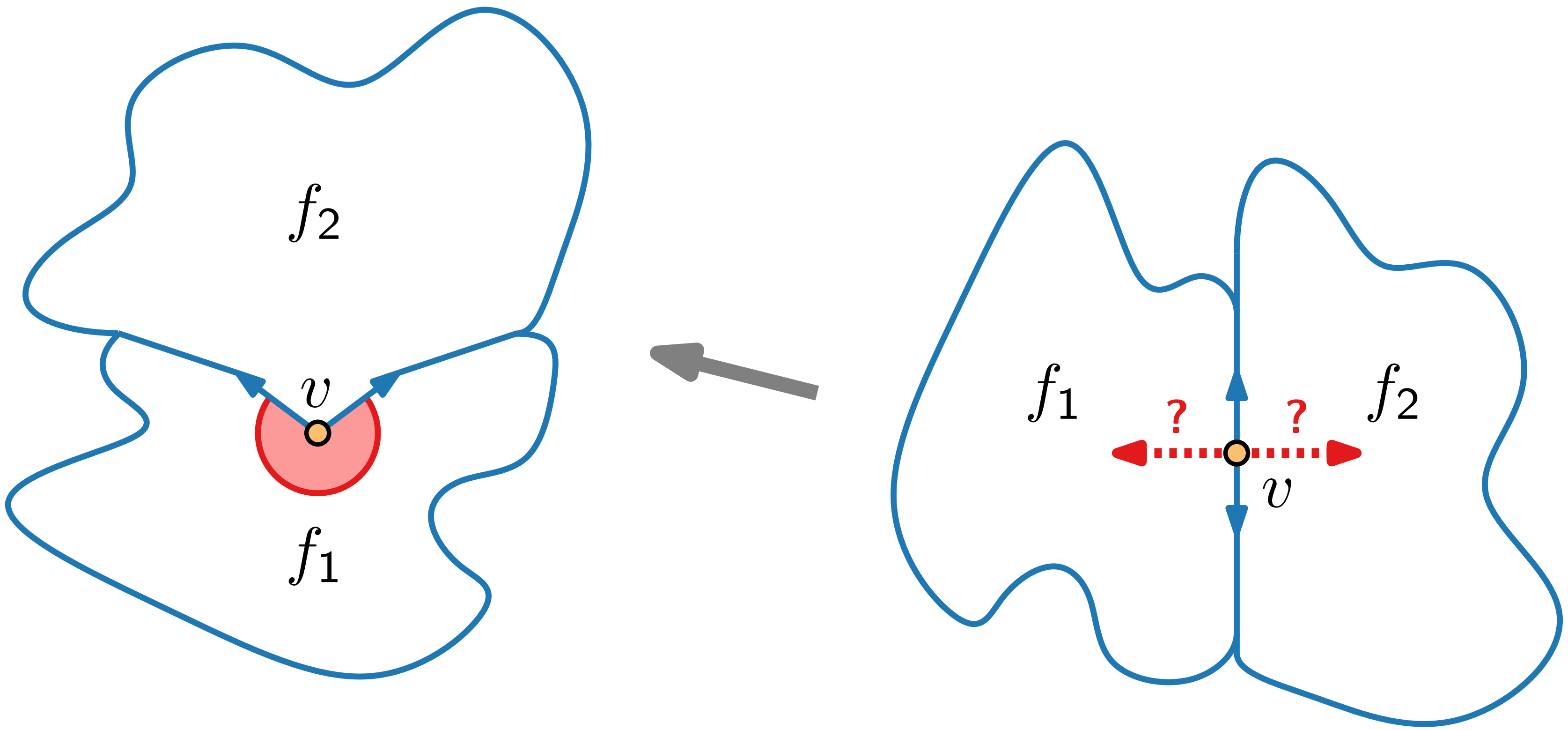

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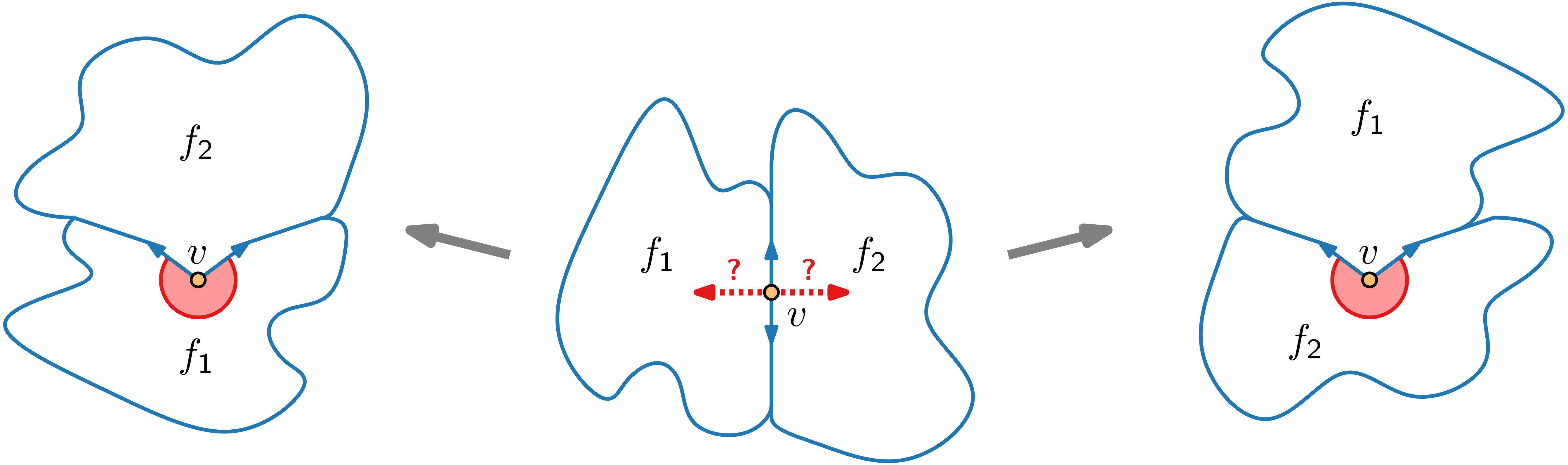

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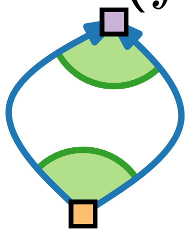

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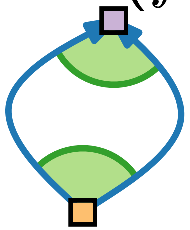
- $L(f) = 0$  $\Rightarrow S(f) = 2$ ✓

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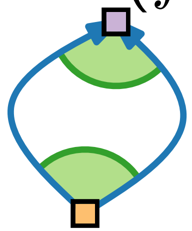

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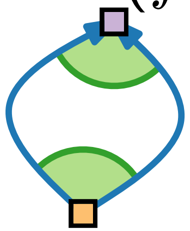
- $L(f) = 0$  $\Rightarrow S(f) = 2$ ✓

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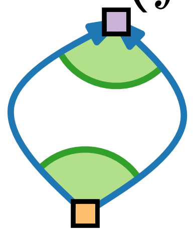
- $L(f) = 0$  $\Rightarrow S(f) = 2$ ✓

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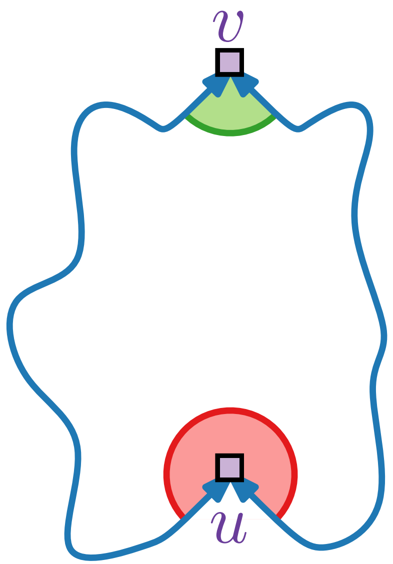

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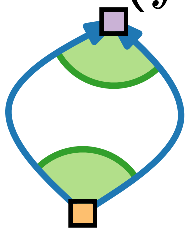
- $L(f) = 0$  $\Rightarrow S(f) = 2$ ✓

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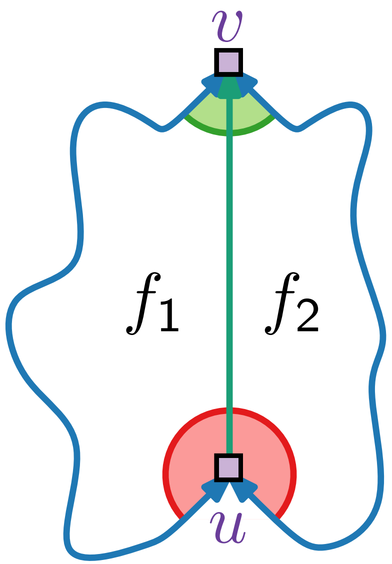

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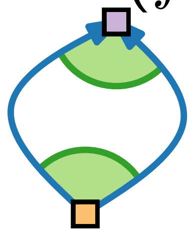
- $L(f) = 0$  $\Rightarrow S(f) = 2$ ✓

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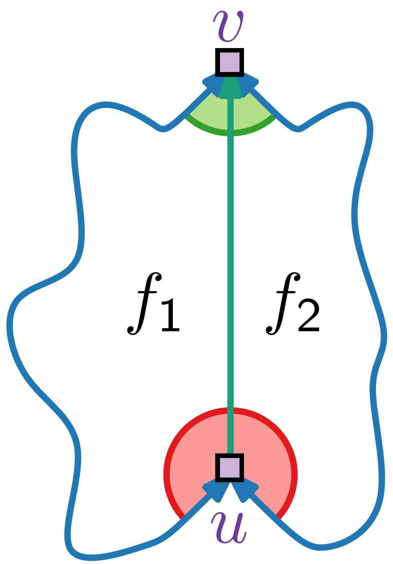

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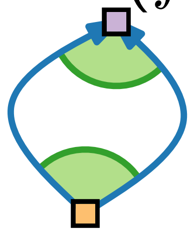
- $L(f) = 0$  $\Rightarrow S(f) = 2$ ✓

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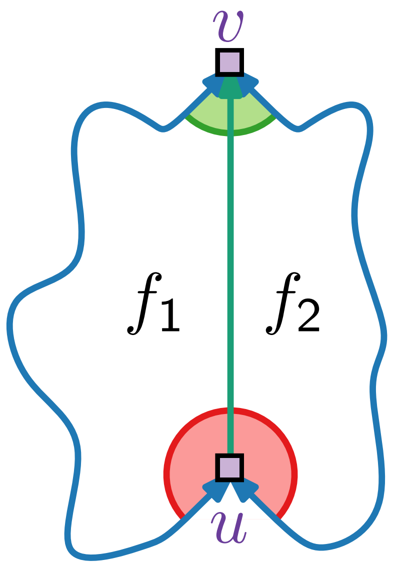

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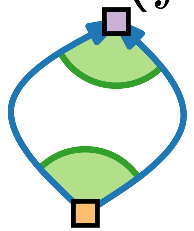
- $L(f) = 0$  $\Rightarrow S(f) = 2$ ✓

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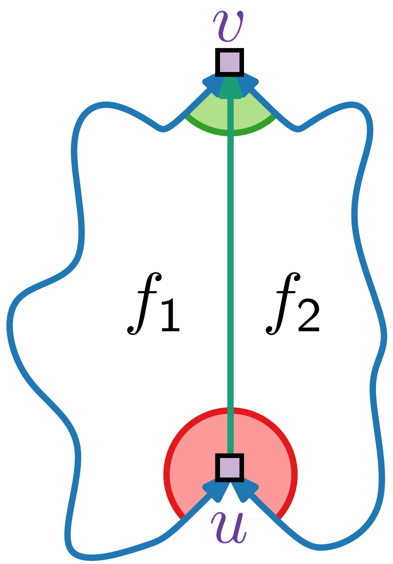

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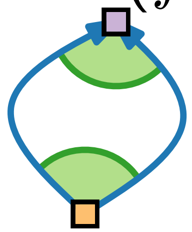
- $L(f) = 0$  $\Rightarrow S(f) = 2$ ✓

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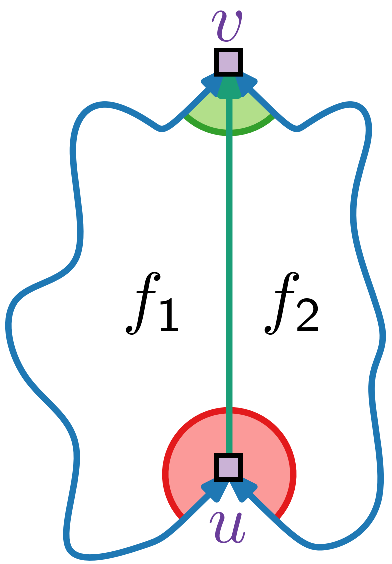

$$\begin{aligned} & \qquad\qquad\quad\;\; -2 \qquad\quad -2 \\ L(f) - S(f) &= L(f_1) + L(f_2) + 1 \\ &\quad - (S(f_1) + S(f_2) - 1) \end{aligned}$$

Angle Relations

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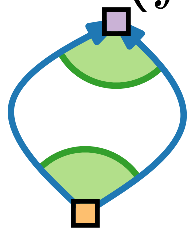

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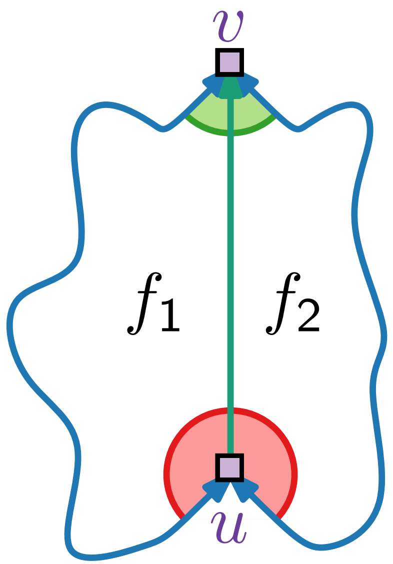

$$\begin{aligned} L(f) - S(f) &= \overbrace{L(f_1)}^{-2} + \overbrace{L(f_2)}^{-2} + 1 \\ &\quad - (S(f_1) + S(f_2) - 1) \\ &= -2 - 2 + 2 = -2 \end{aligned}$$

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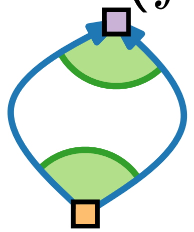
- $L(f) = 0$  $\Rightarrow S(f) = 2$ ✓

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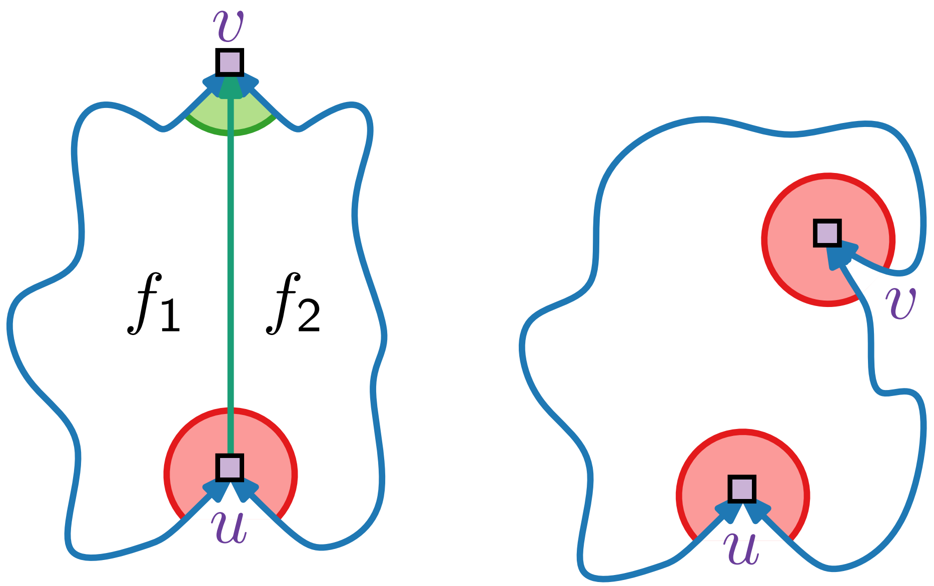

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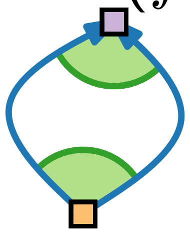

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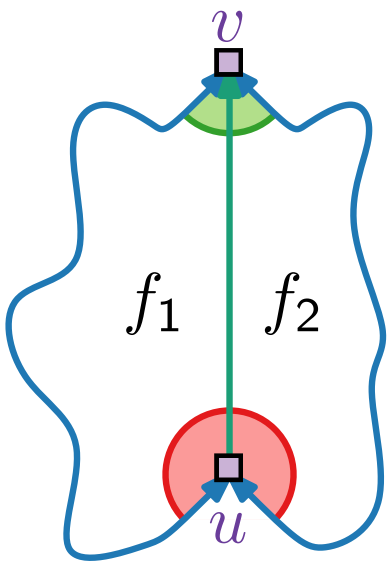

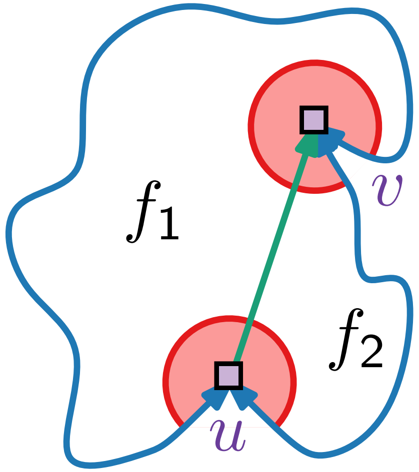

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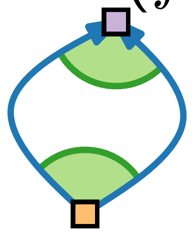

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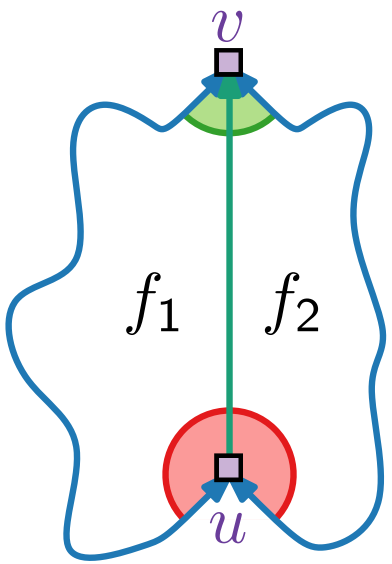

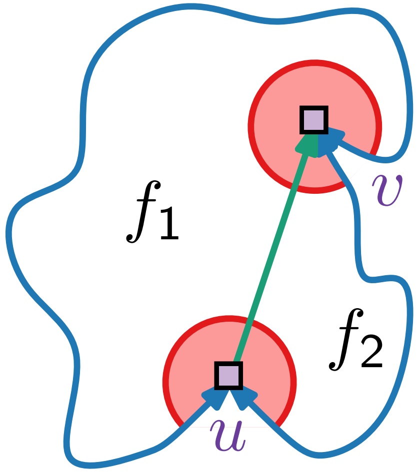

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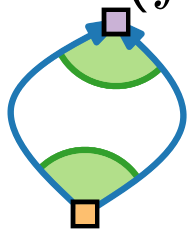
- $L(f) = 0$  $\Rightarrow S(f) = 2$ ✓

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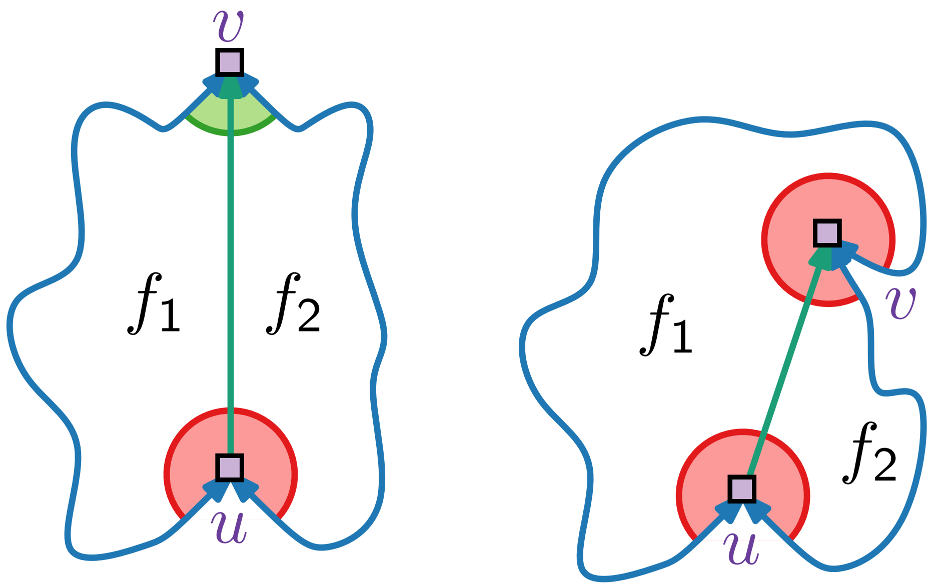

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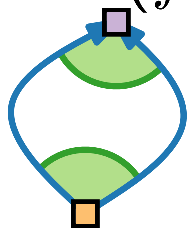
- $L(f) = 0$  $\Rightarrow S(f) = 2$ ✓

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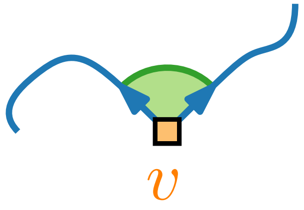

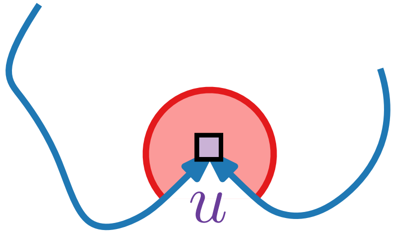

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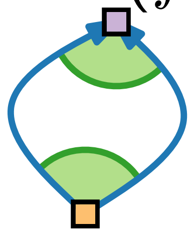
- $L(f) = 0$  $\Rightarrow S(f) = 2$ ✓

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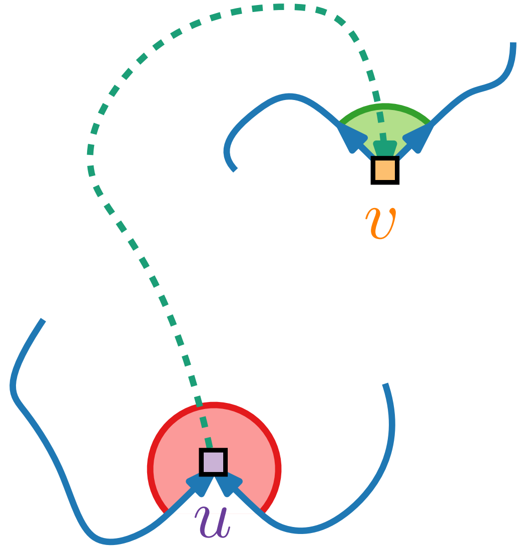

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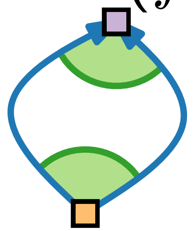
- $L(f) = 0$  $\Rightarrow S(f) = 2$ ✓

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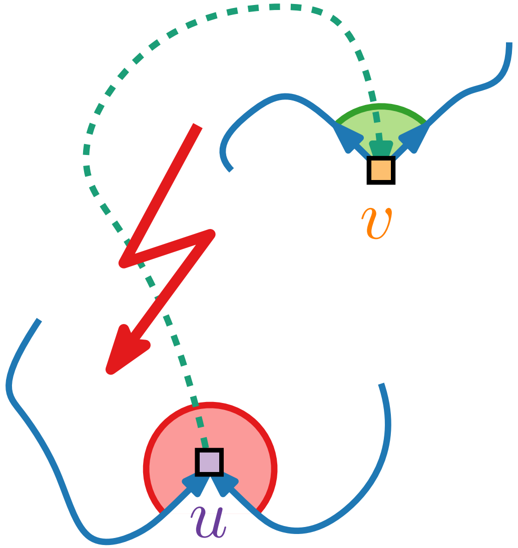

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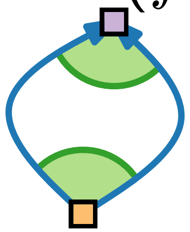
- $L(f) = 0$  $\Rightarrow S(f) = 2$ ✓

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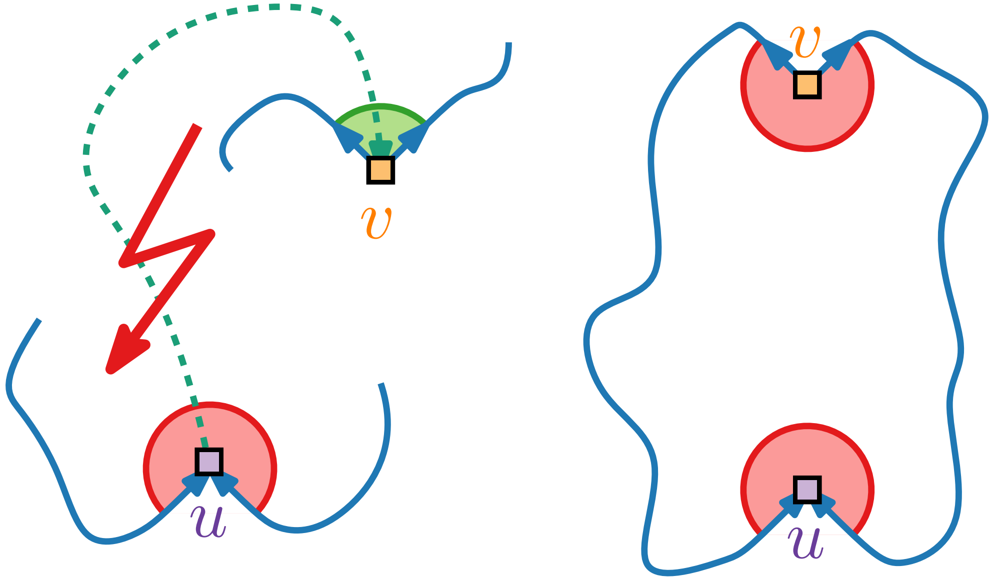

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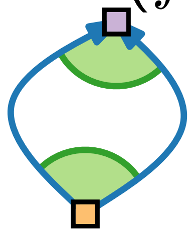
- $L(f) = 0$  $\Rightarrow S(f) = 2$ ✓
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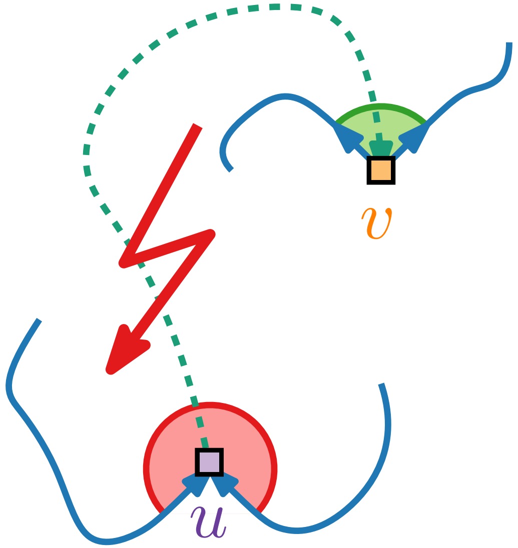

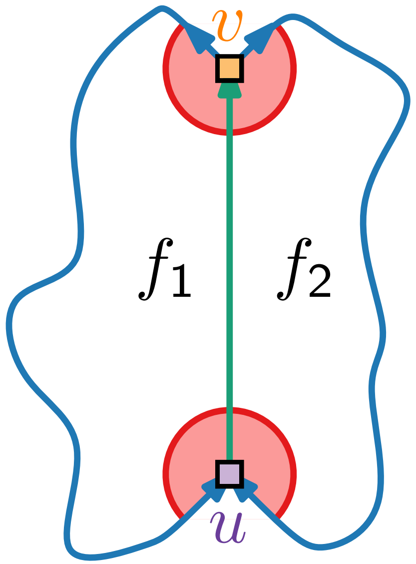

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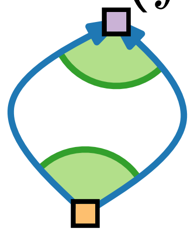
- $L(f) = 0$  $\Rightarrow S(f) = 2$ ✓

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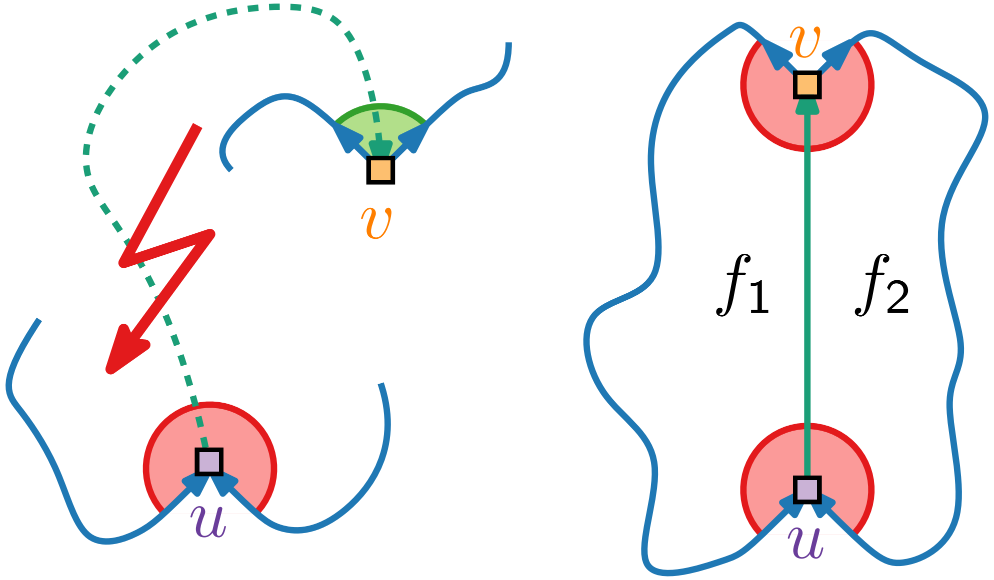

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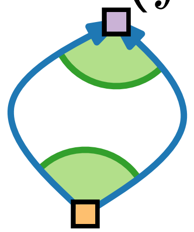

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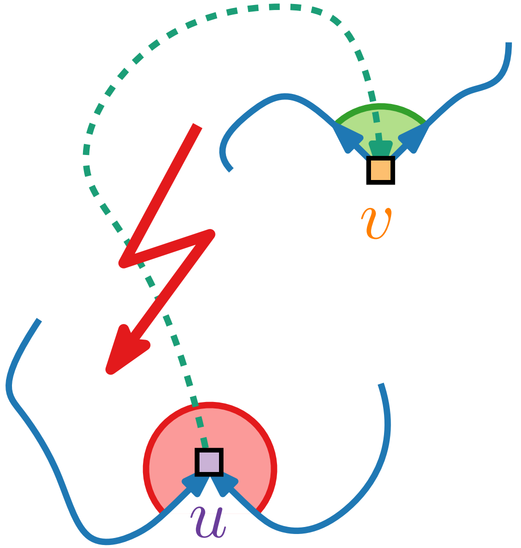

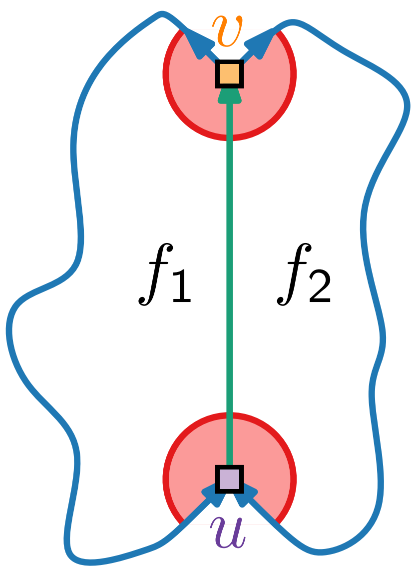

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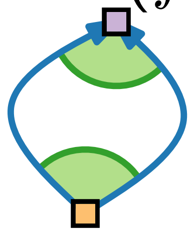
- $L(f) = 0$  $\Rightarrow S(f) = 2$ ✓

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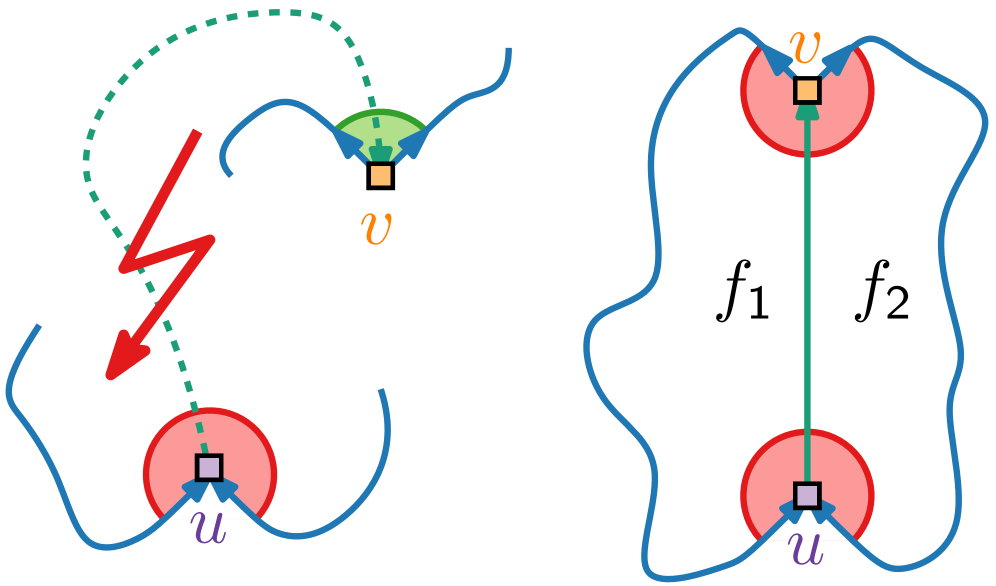

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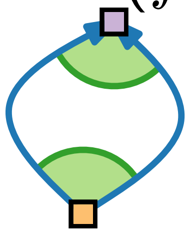

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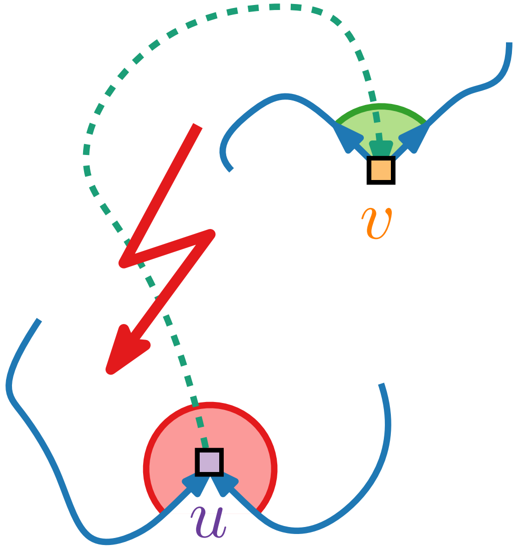

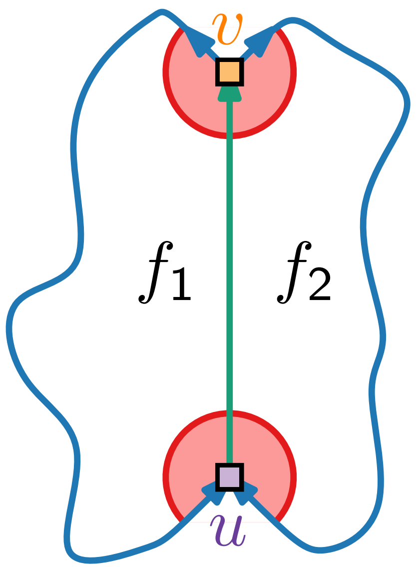

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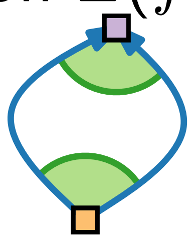

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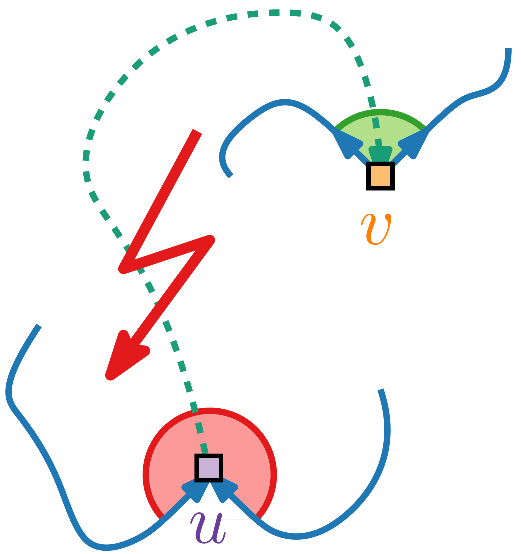

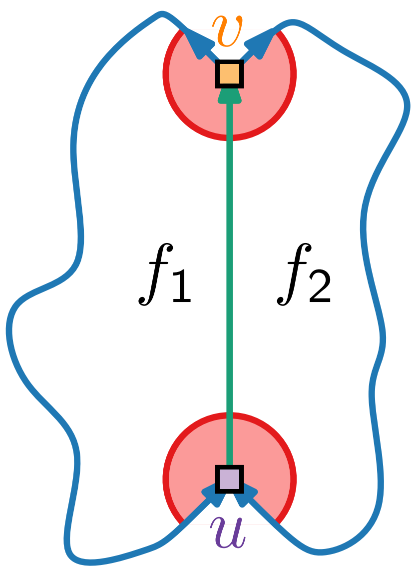

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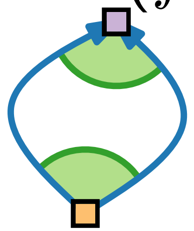

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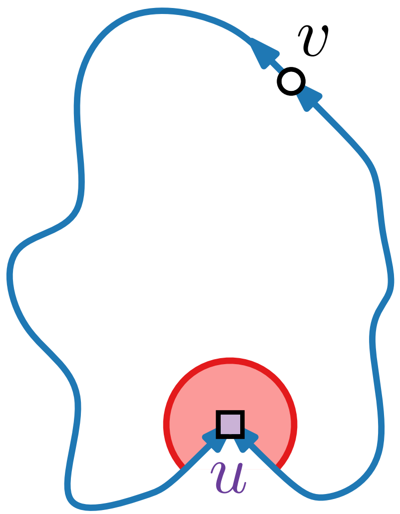

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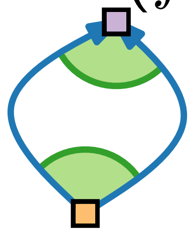

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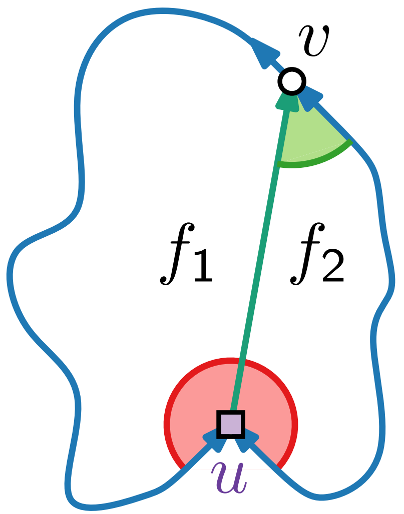

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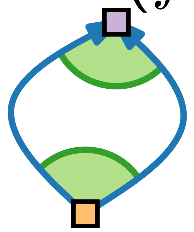
- $L(f) = 0$  $\Rightarrow S(f) = 2$ ✓

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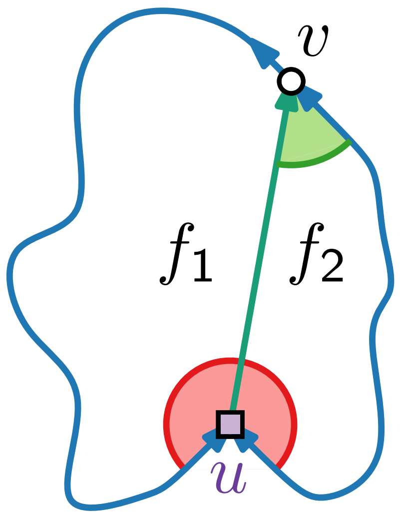

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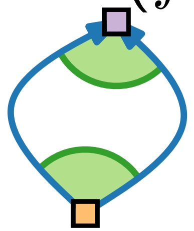

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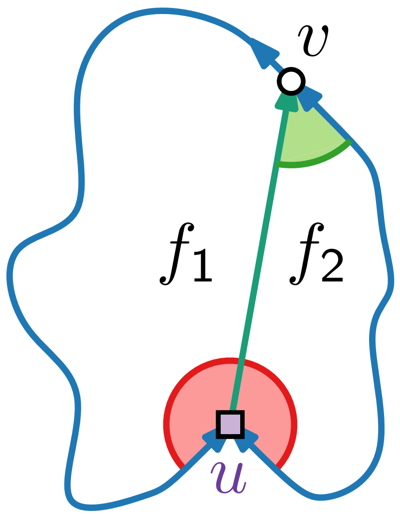

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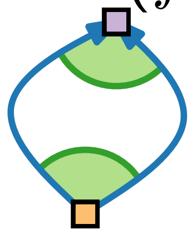
- $L(f) = 0$  $\Rightarrow S(f) = 2$ ✓
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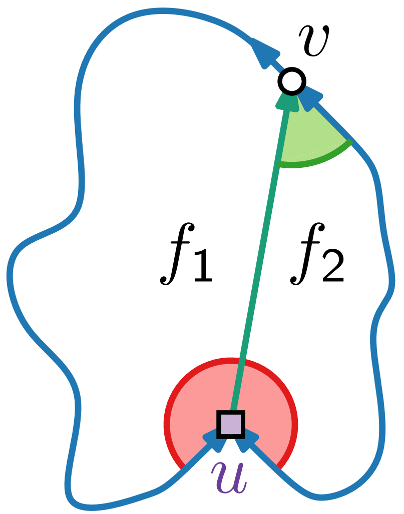

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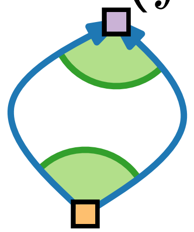
- $L(f) = 0$  $\Rightarrow S(f) = 2$ ✓

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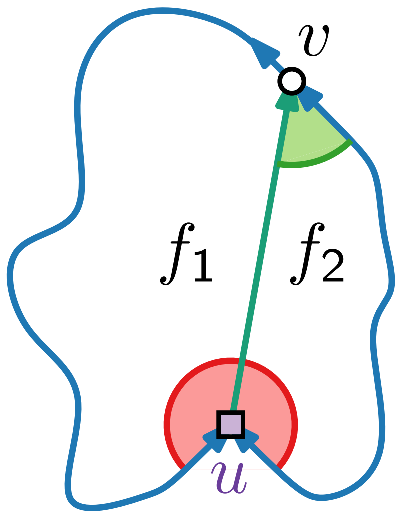

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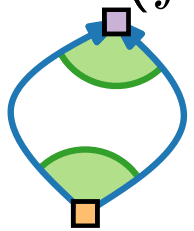
- $L(f) = 0$  $\Rightarrow S(f) = 2$ ✓
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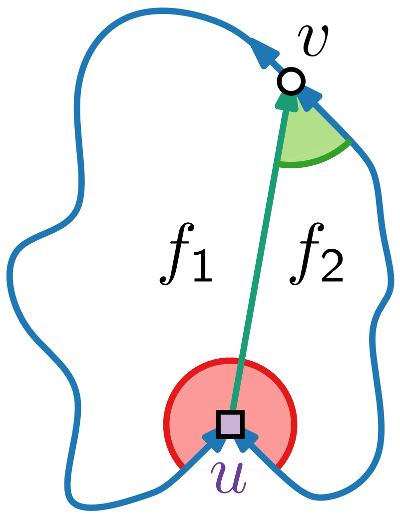

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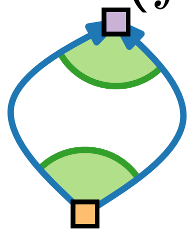
- $L(f) = 0$  $\Rightarrow S(f) = 2$ ✓

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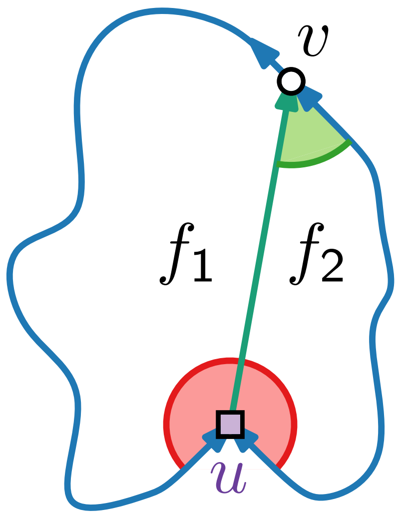

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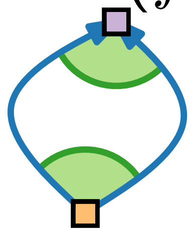
- $L(f) = 0$  $\Rightarrow S(f) = 2$ ✓

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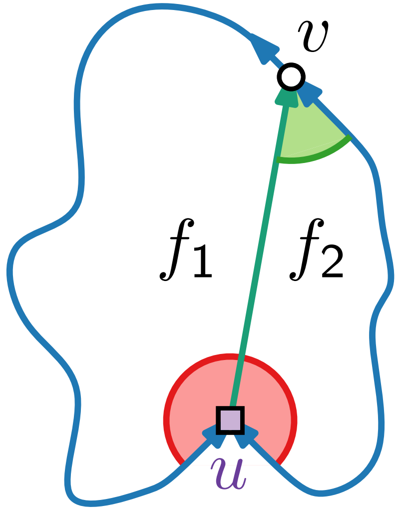

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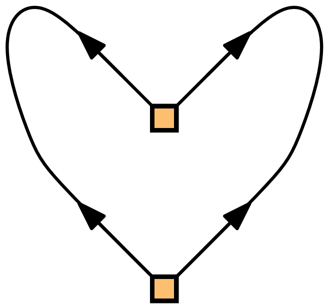

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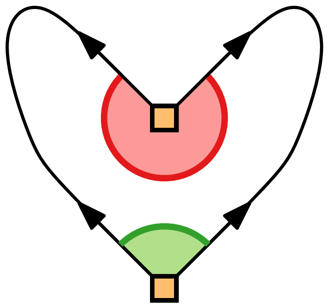

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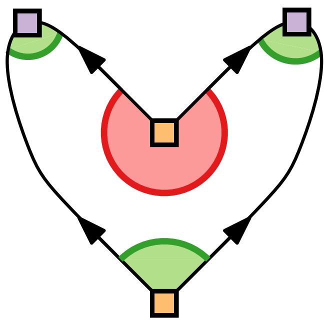

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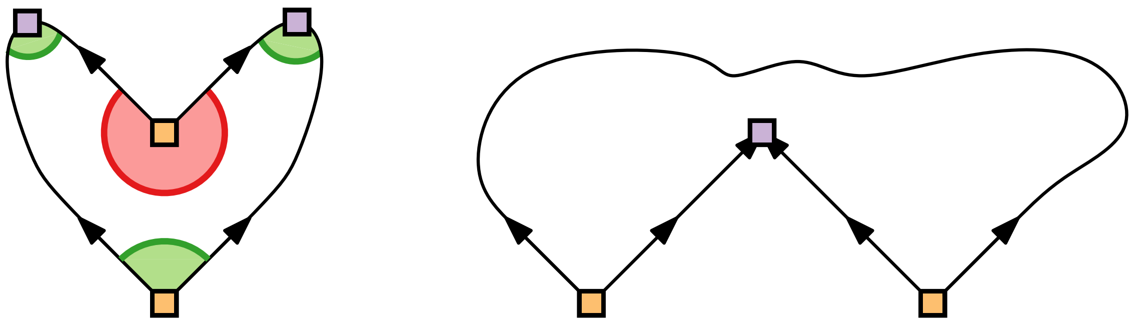

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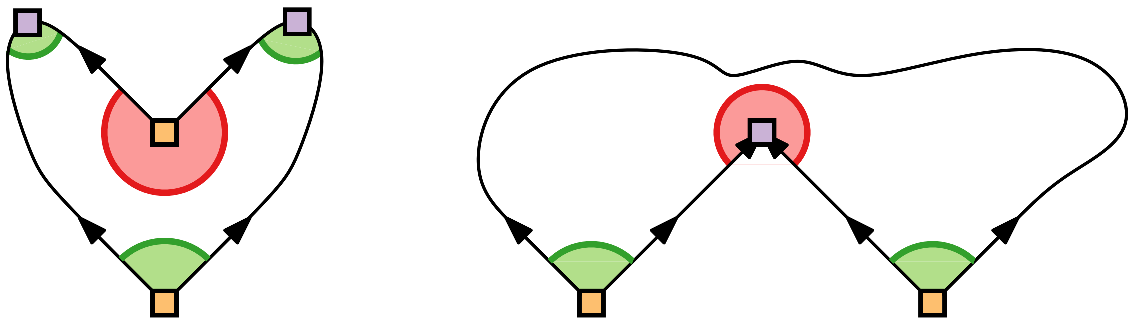

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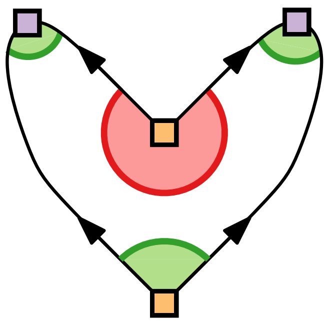

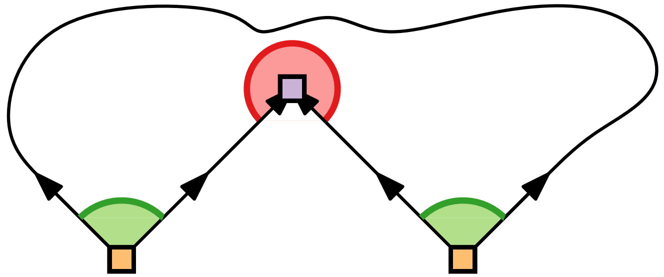

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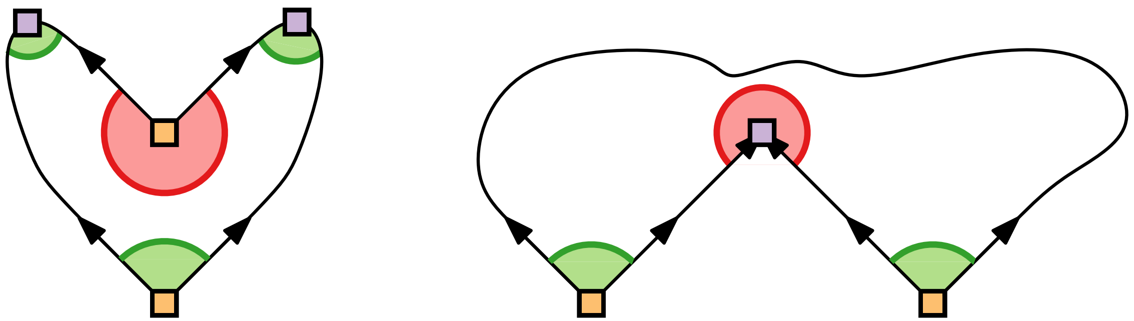

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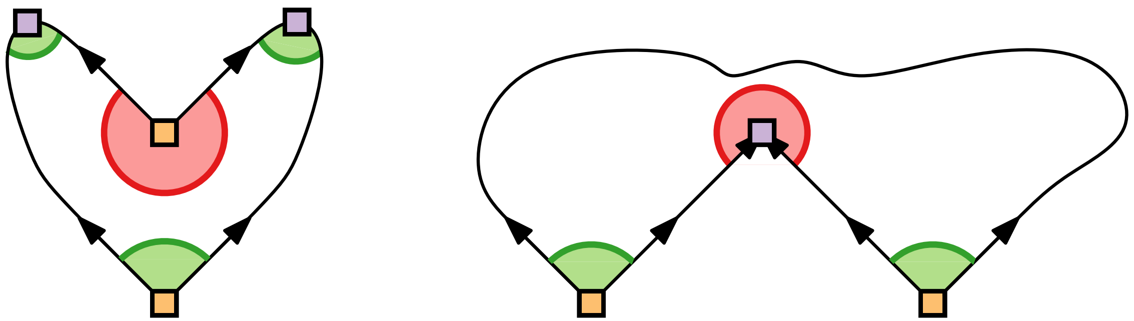

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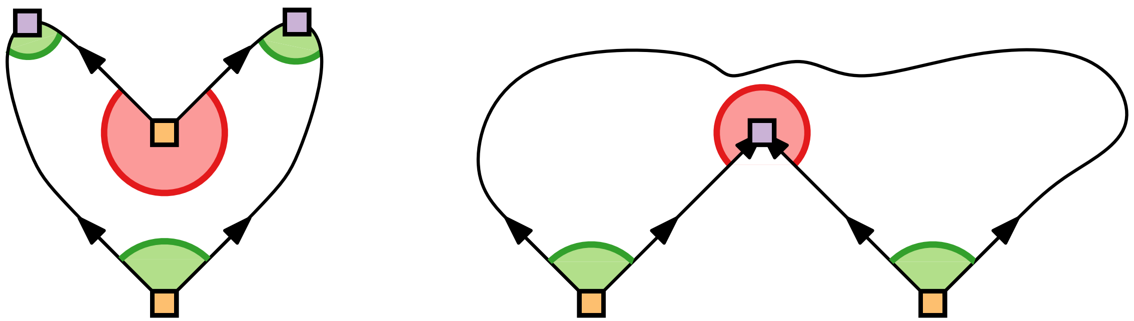

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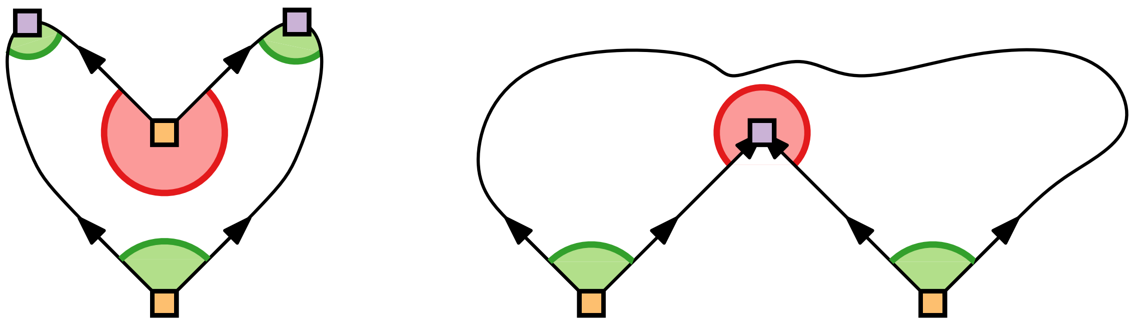

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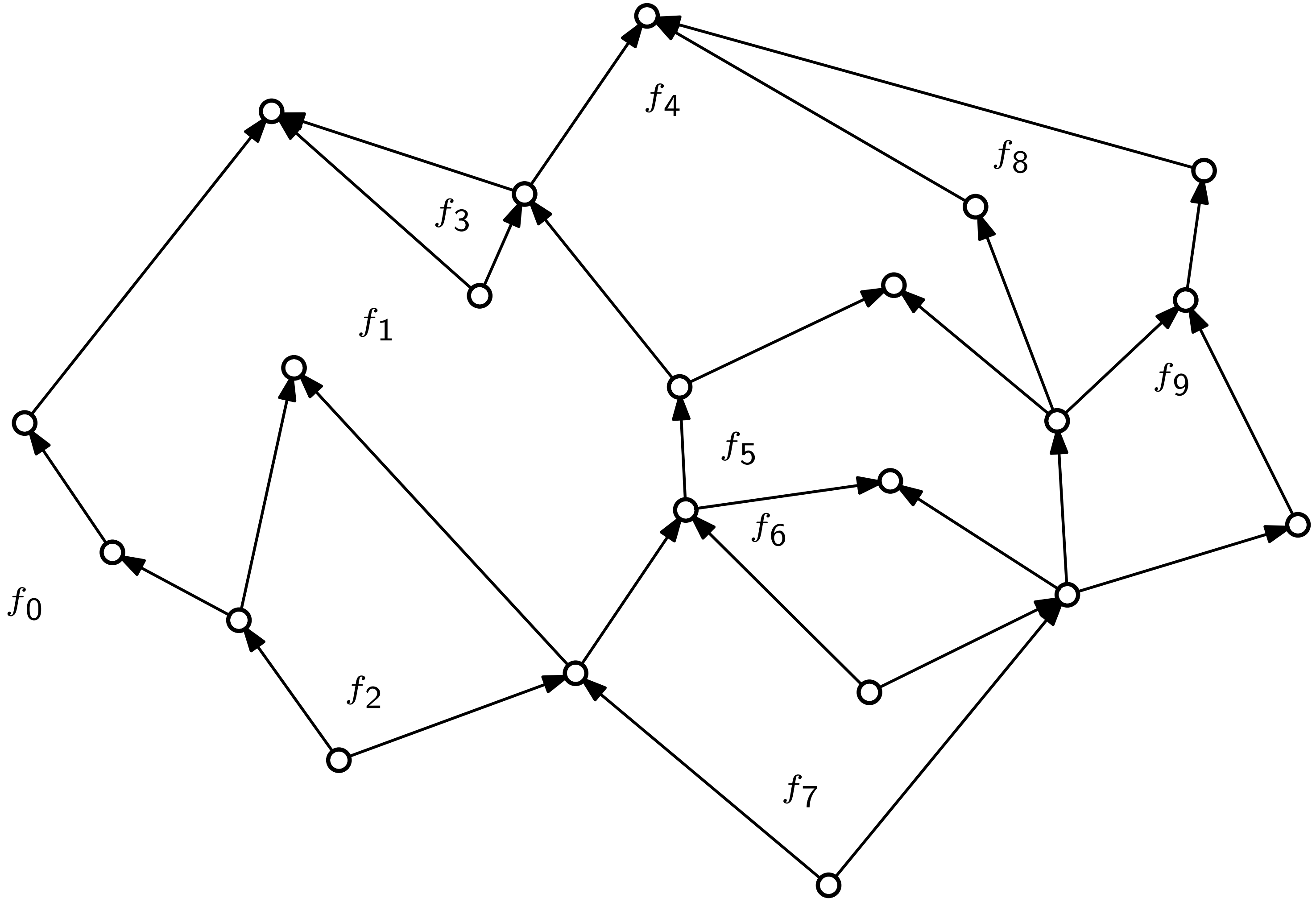

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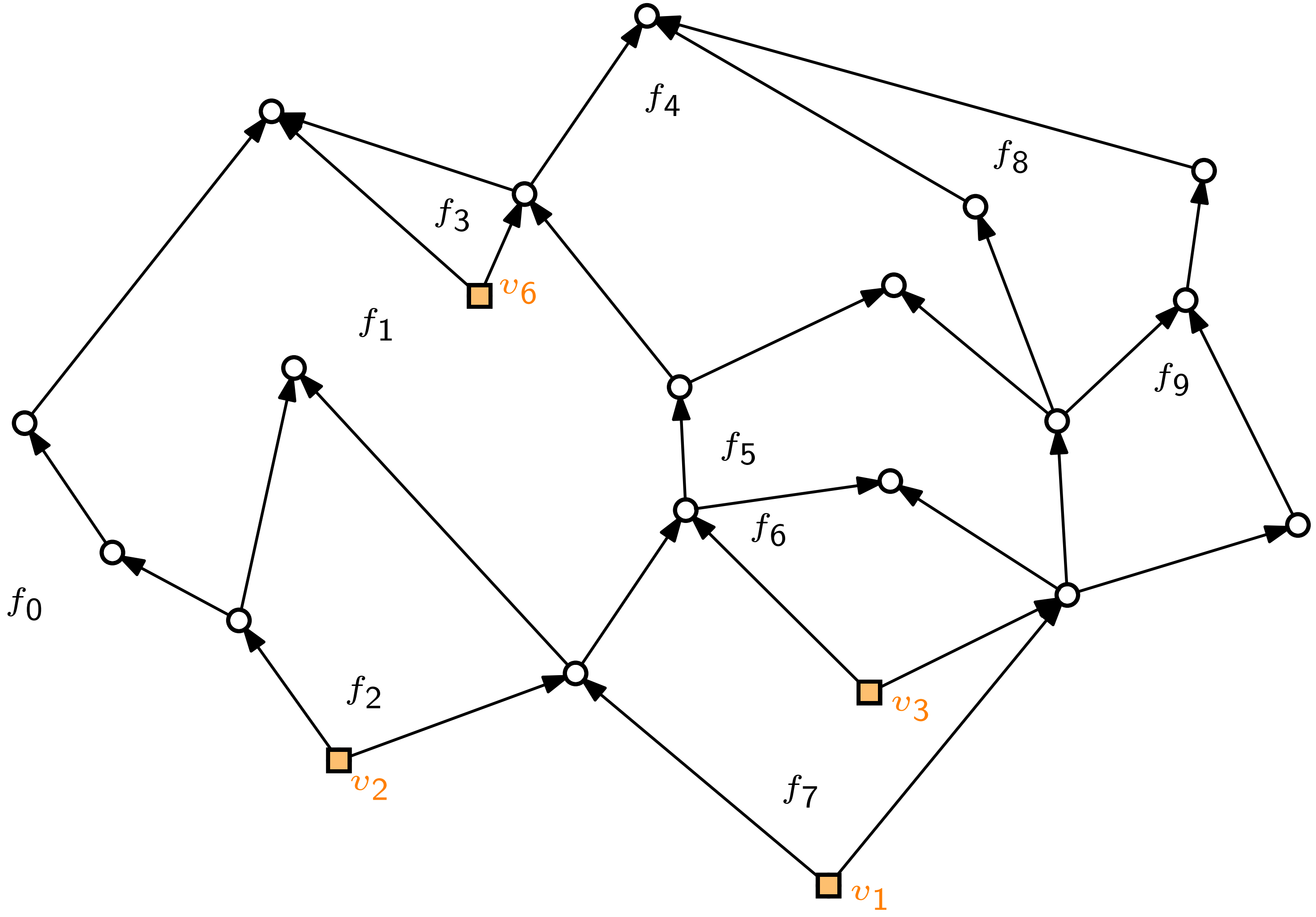

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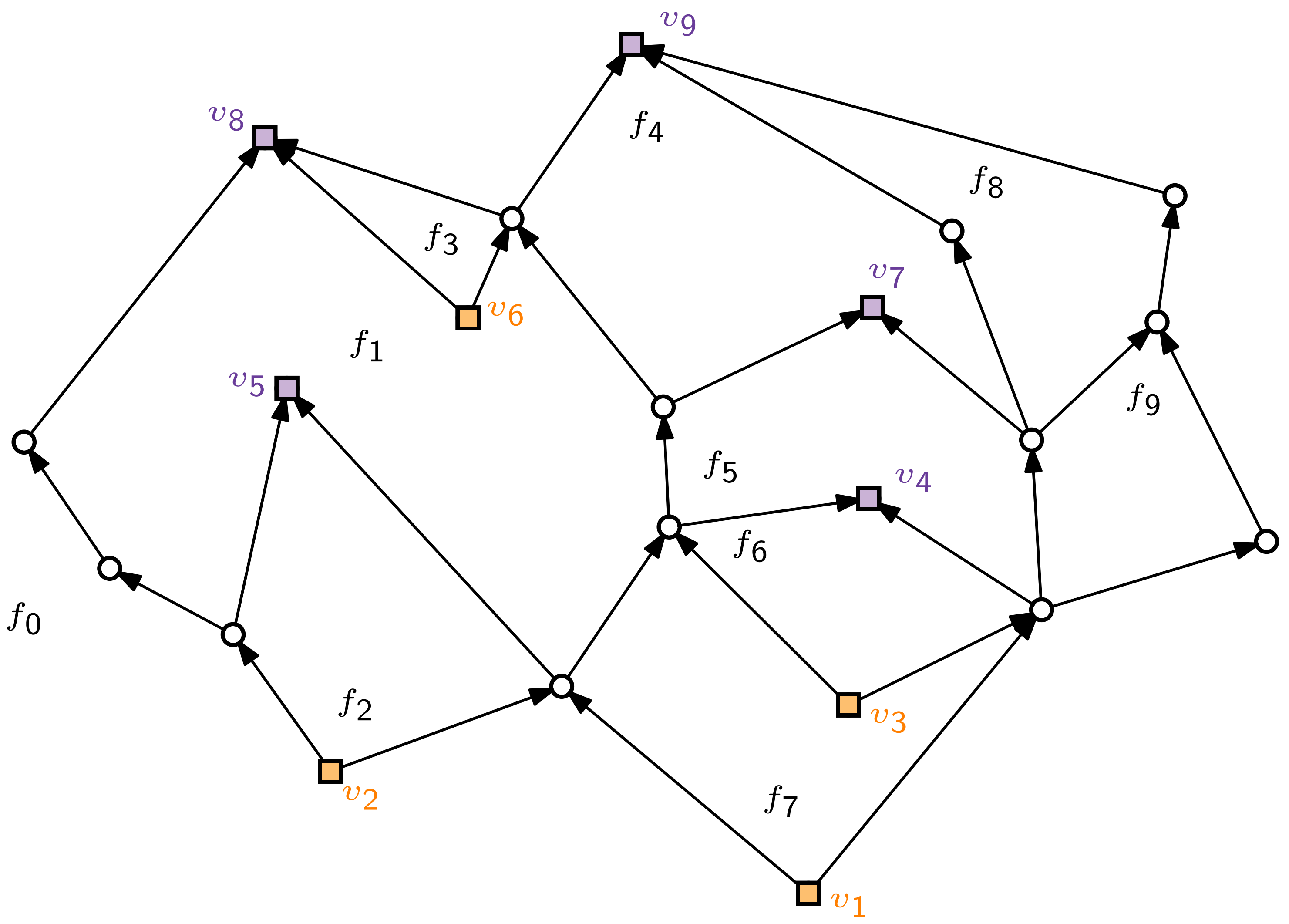

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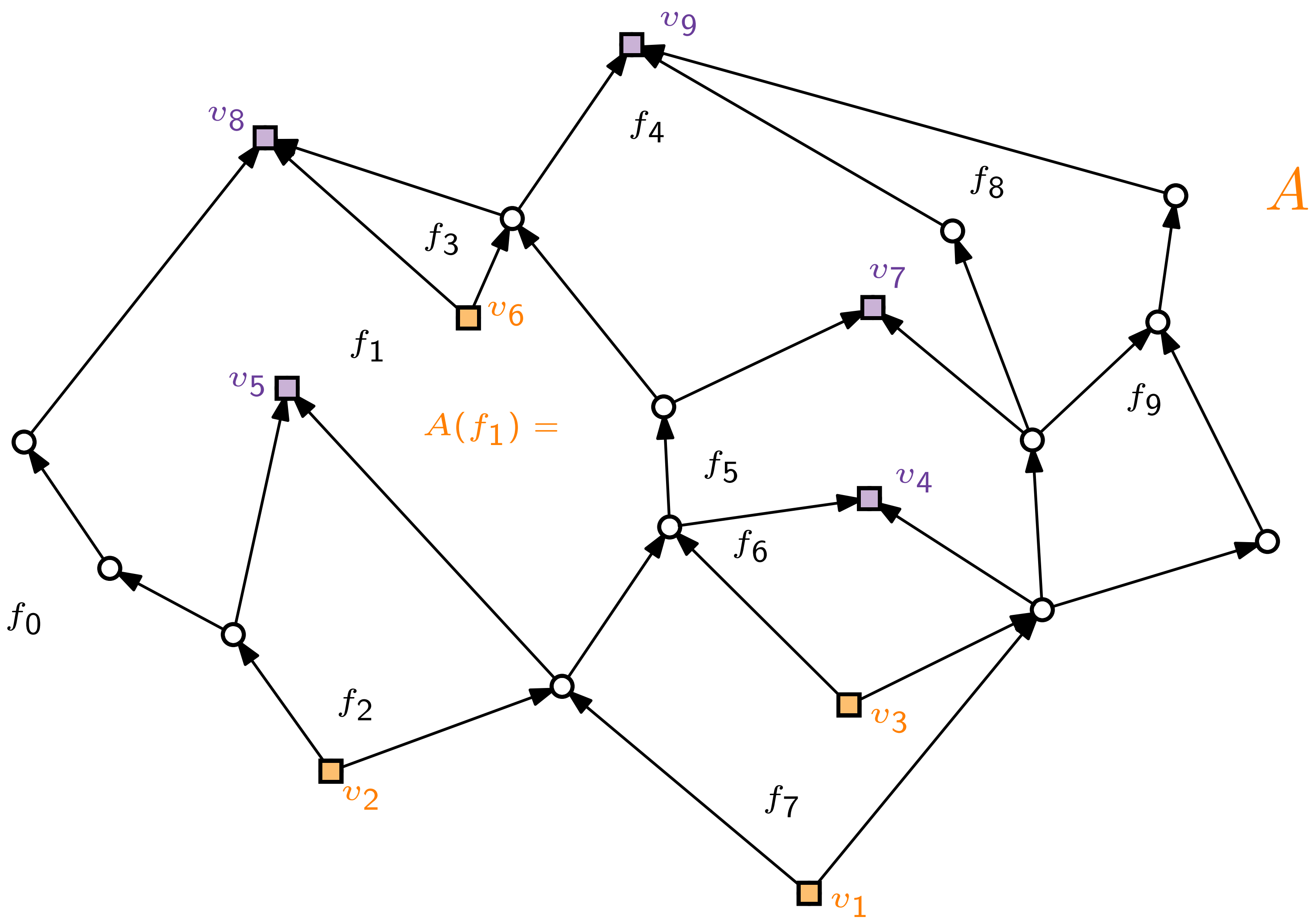

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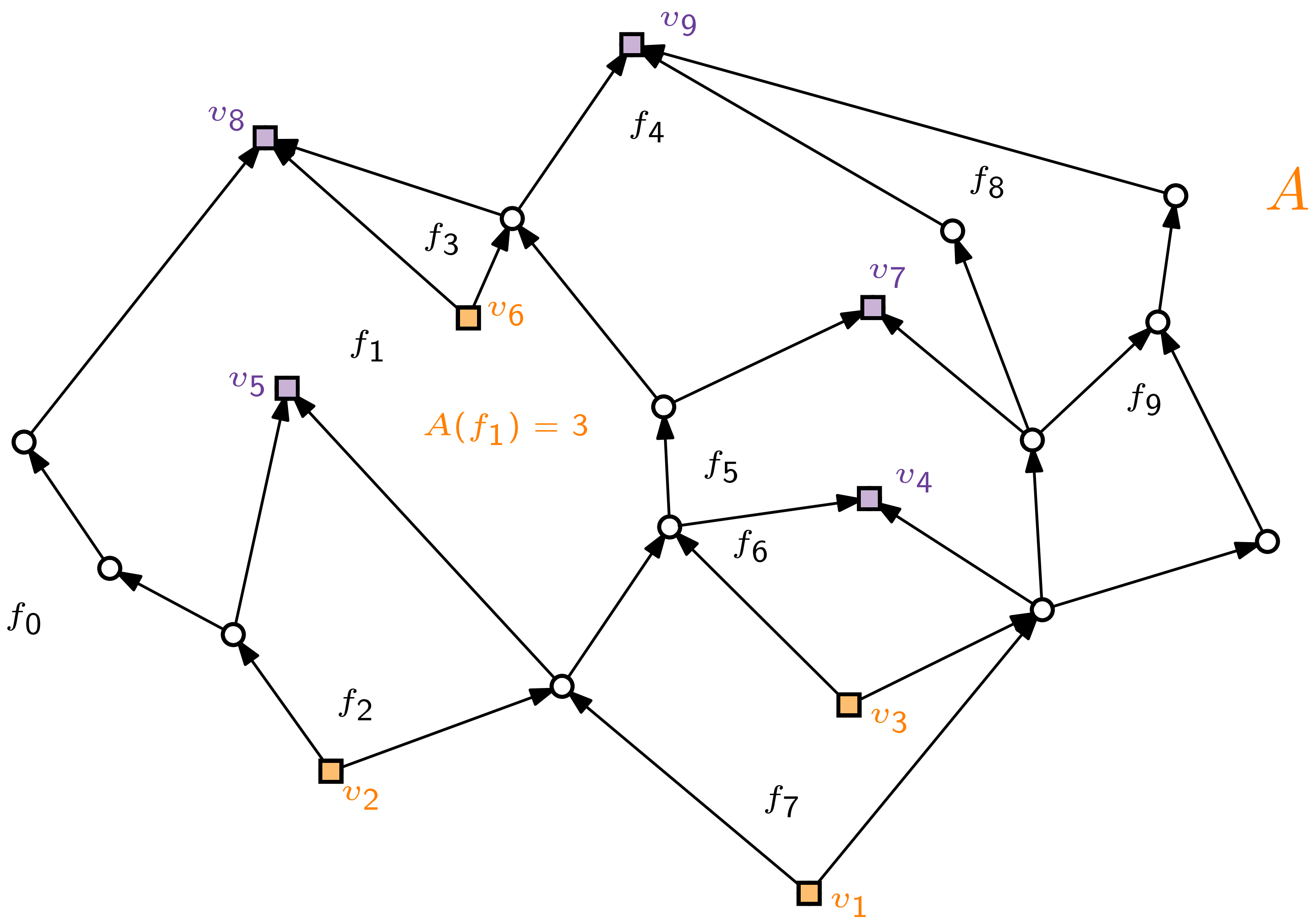

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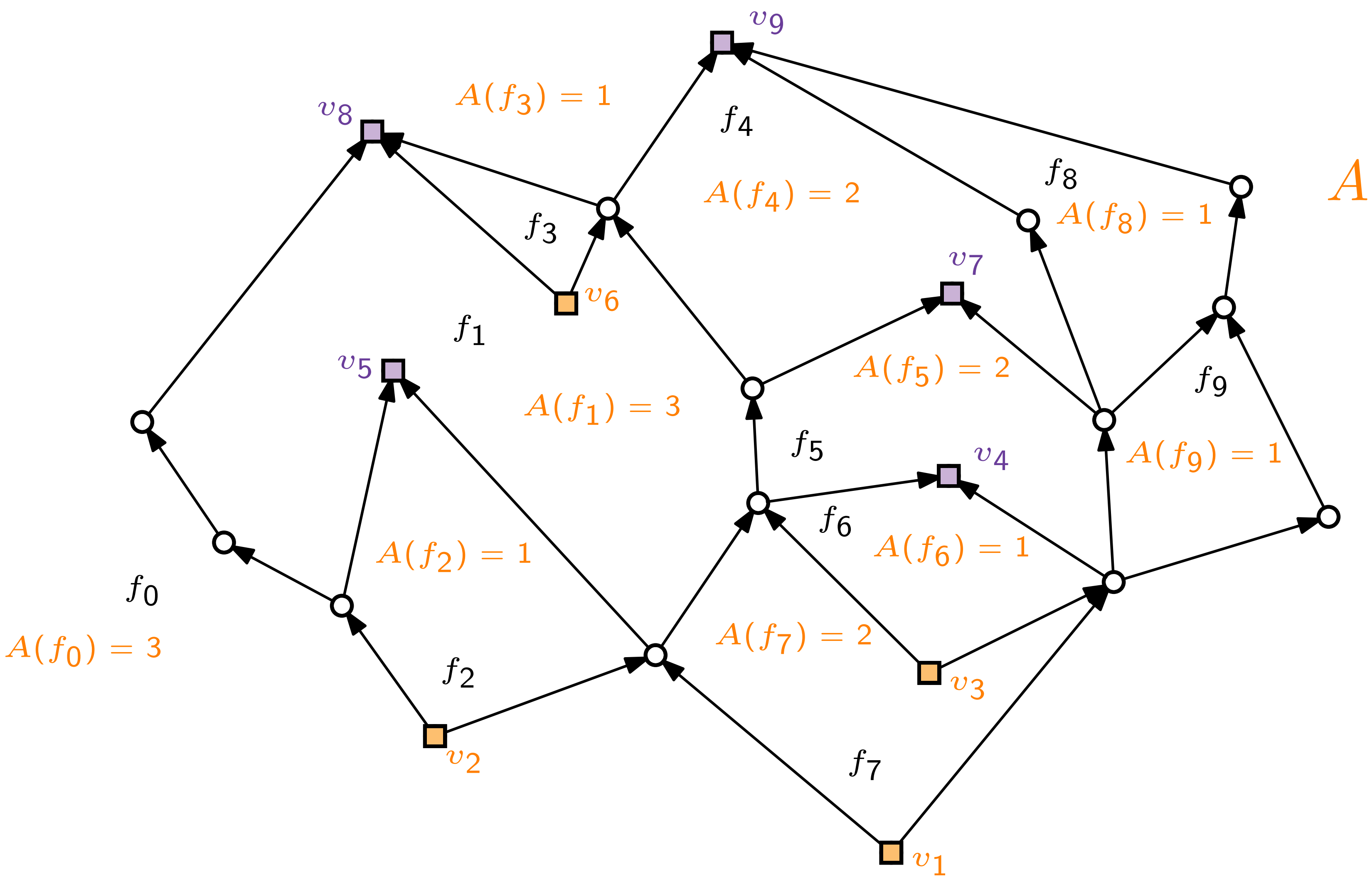

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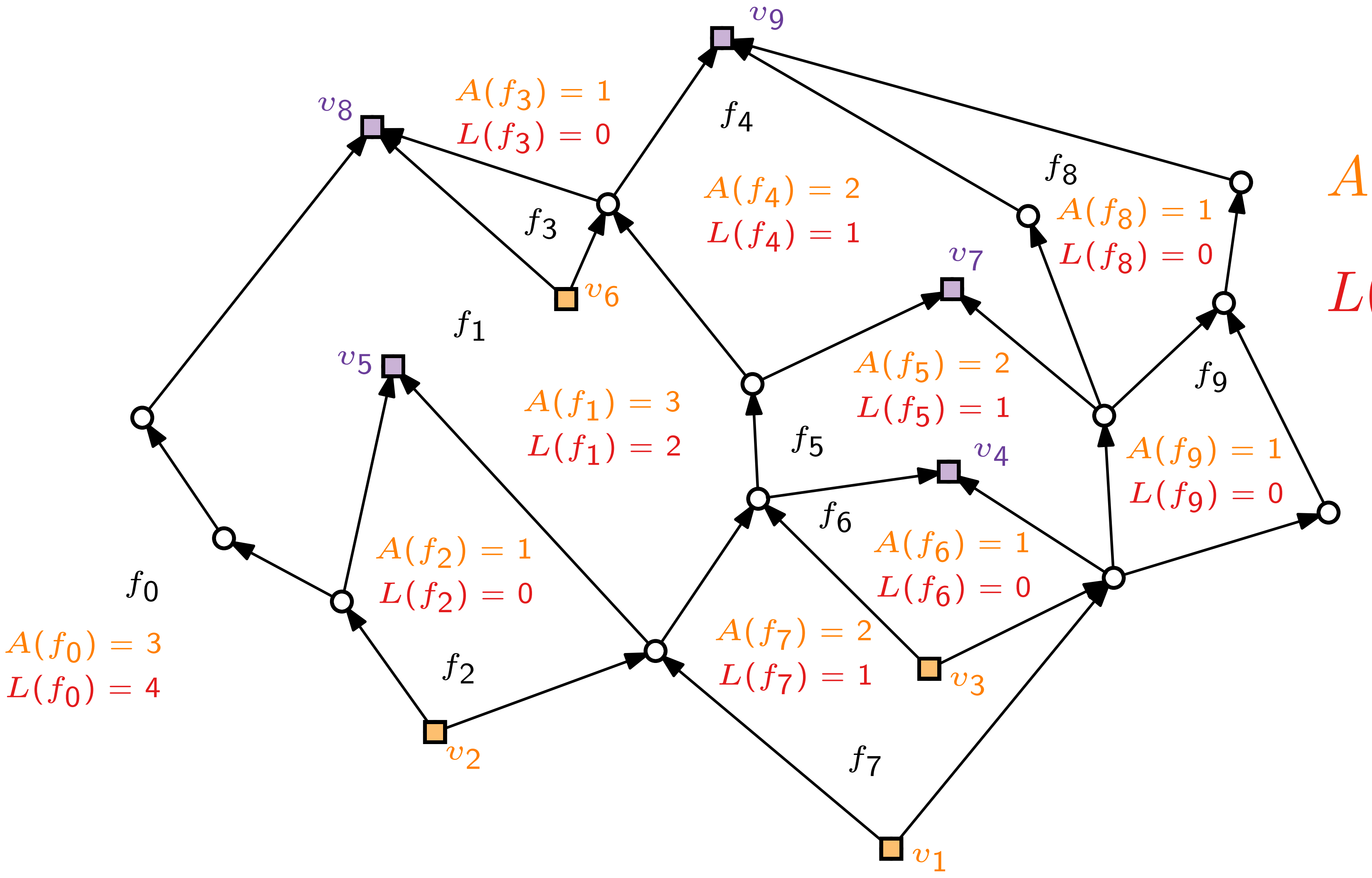

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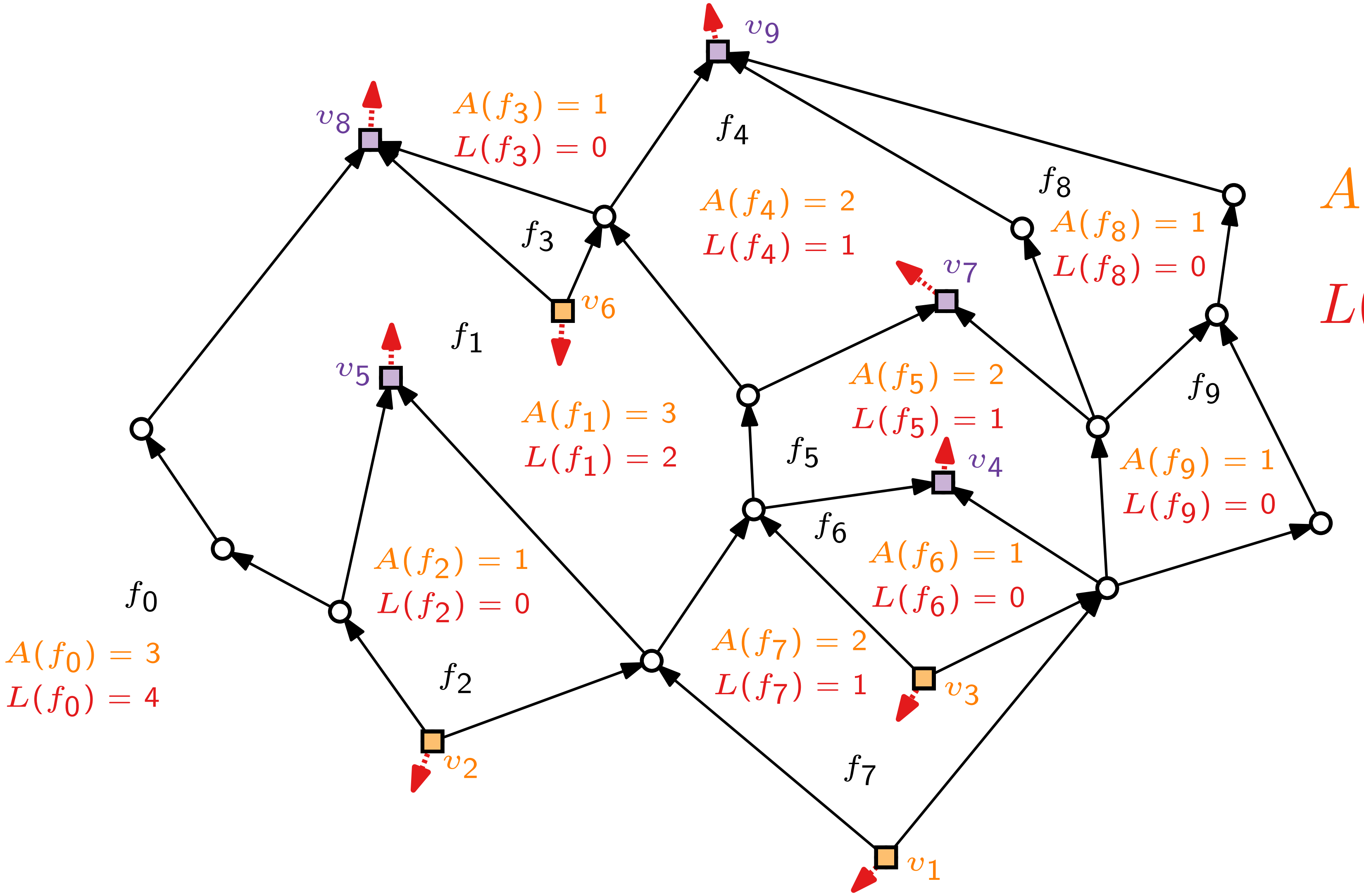

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- Construct planar st-digraph that is a supergraph of G.
- Apply equivalence from Theorem 1.

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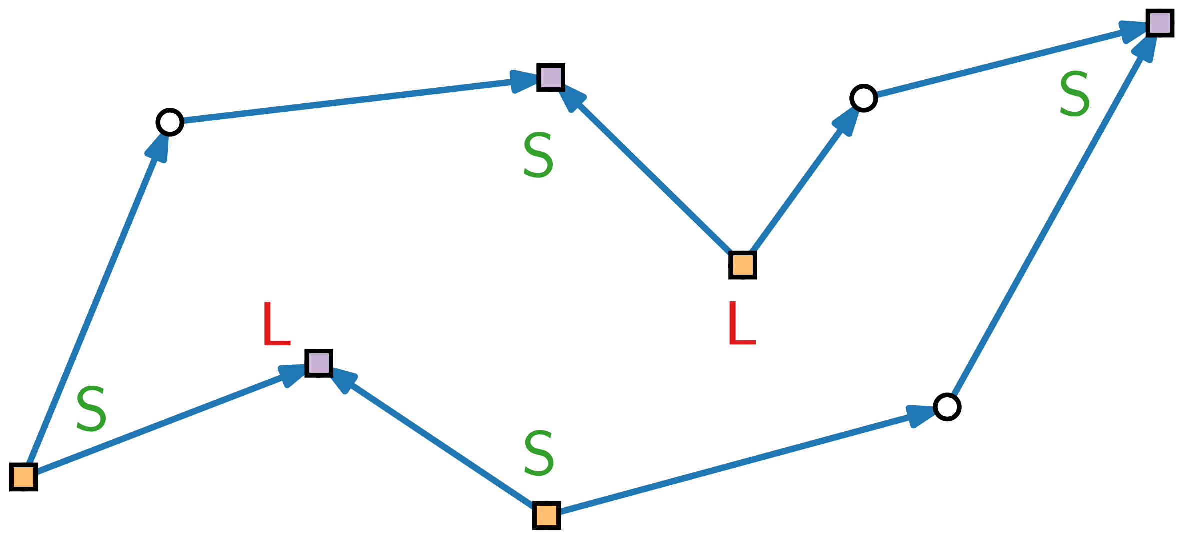
- For $f \neq f_0$ with $|\sigma_f| \geq 2$ containing $\langle$L, S, S$\rangle$ at vertices x, y, z:



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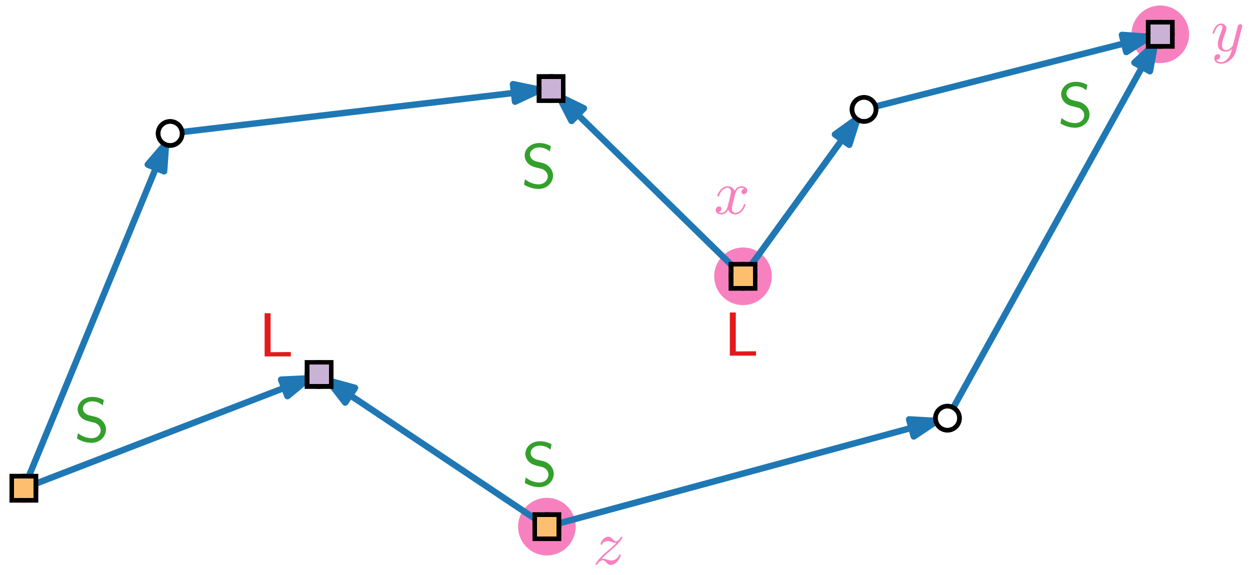

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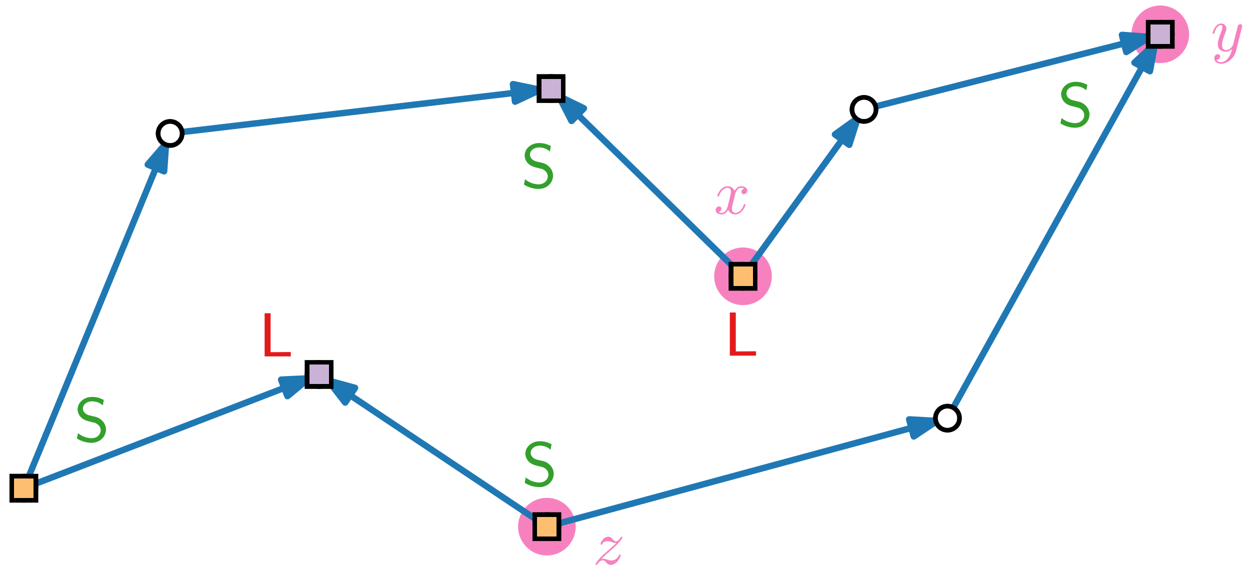

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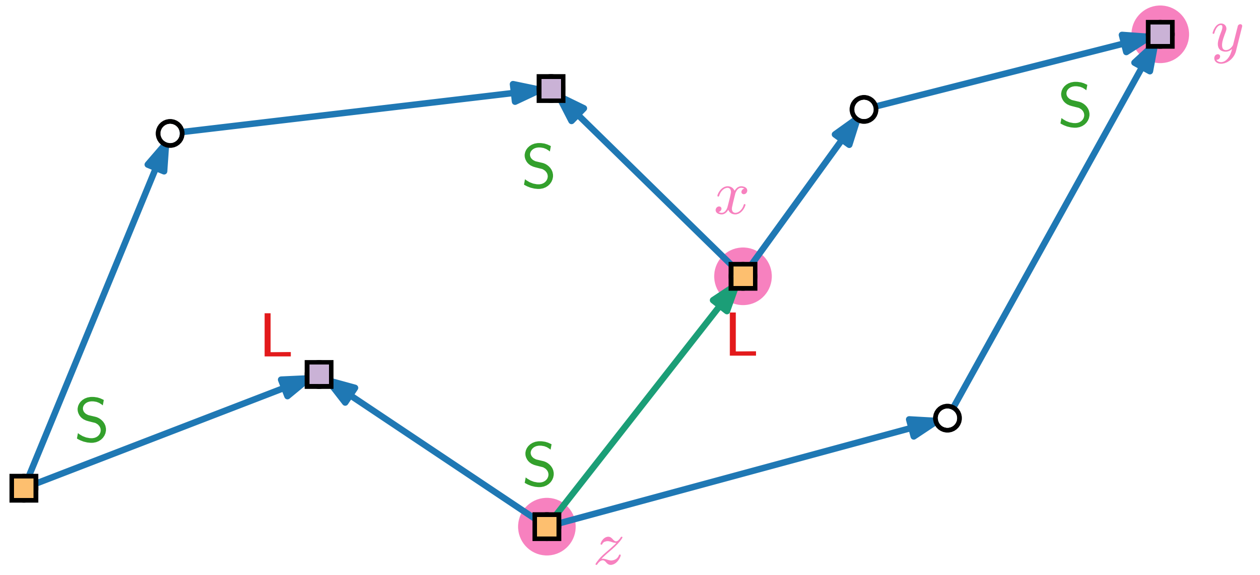

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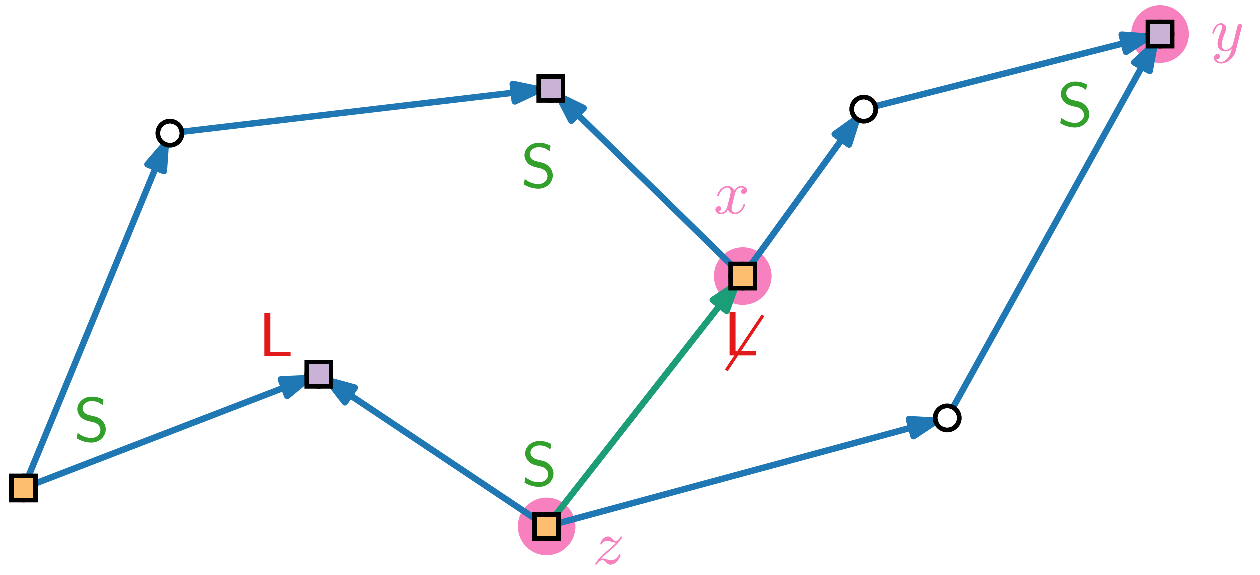

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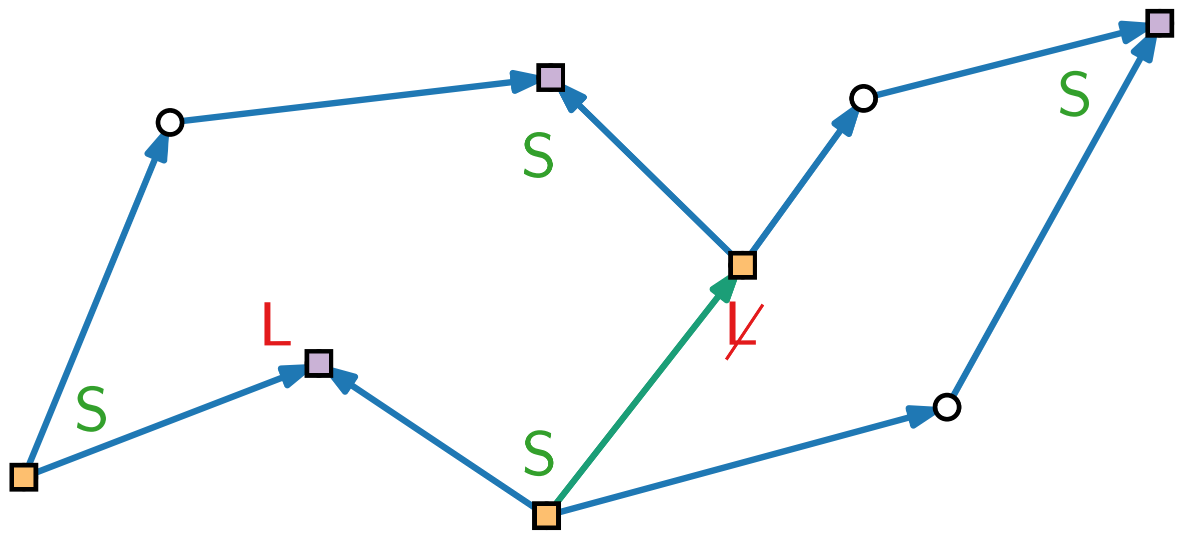

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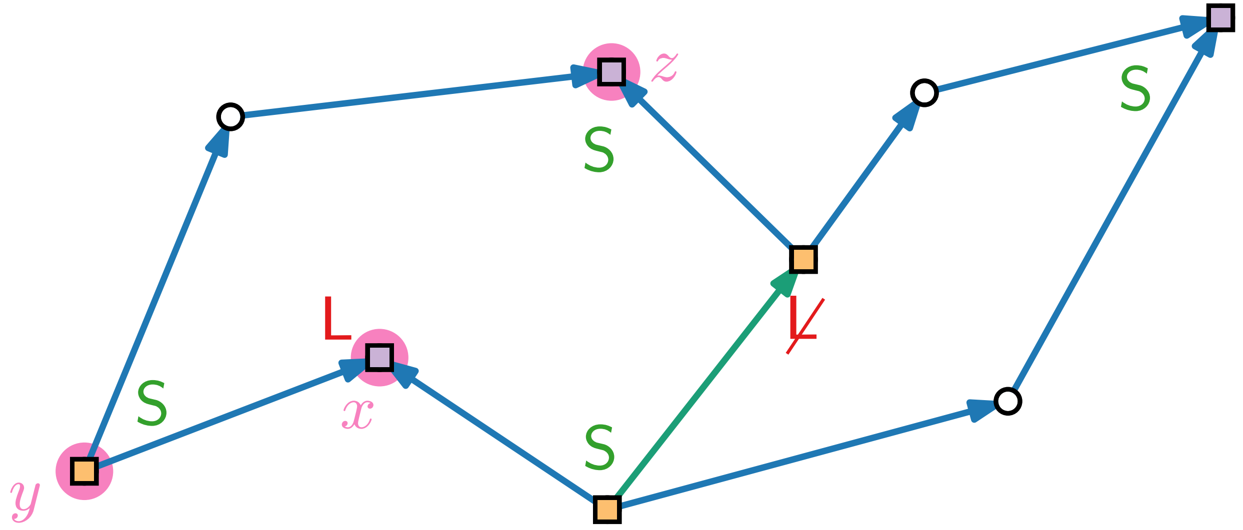

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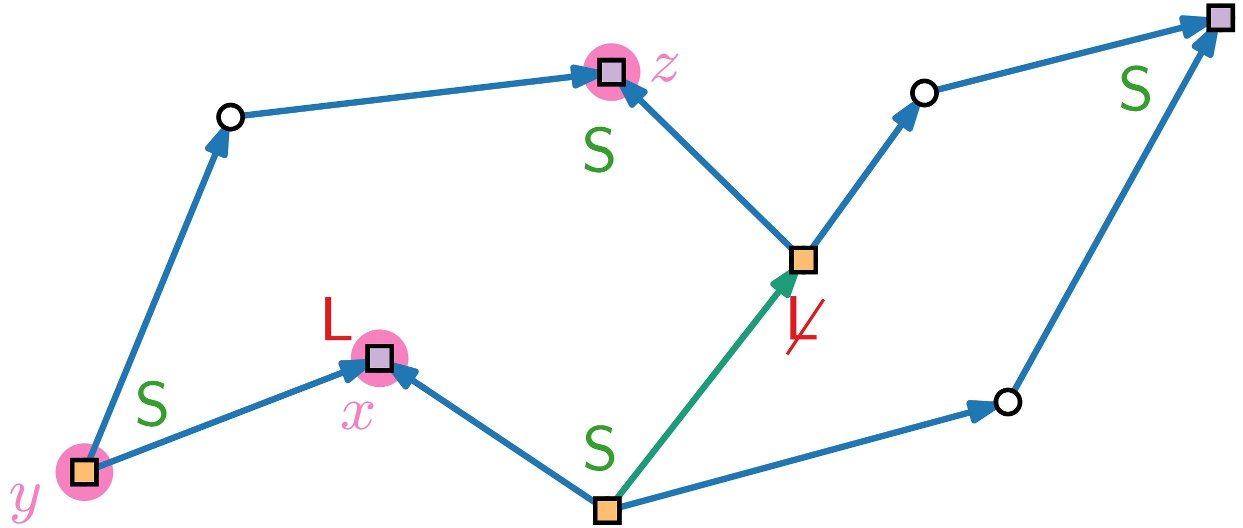

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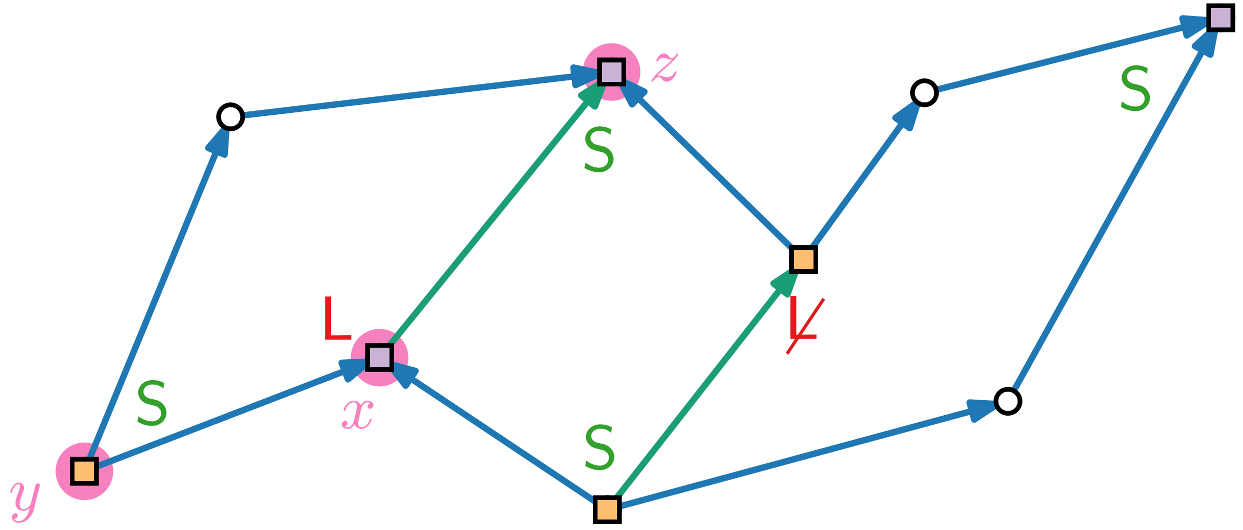

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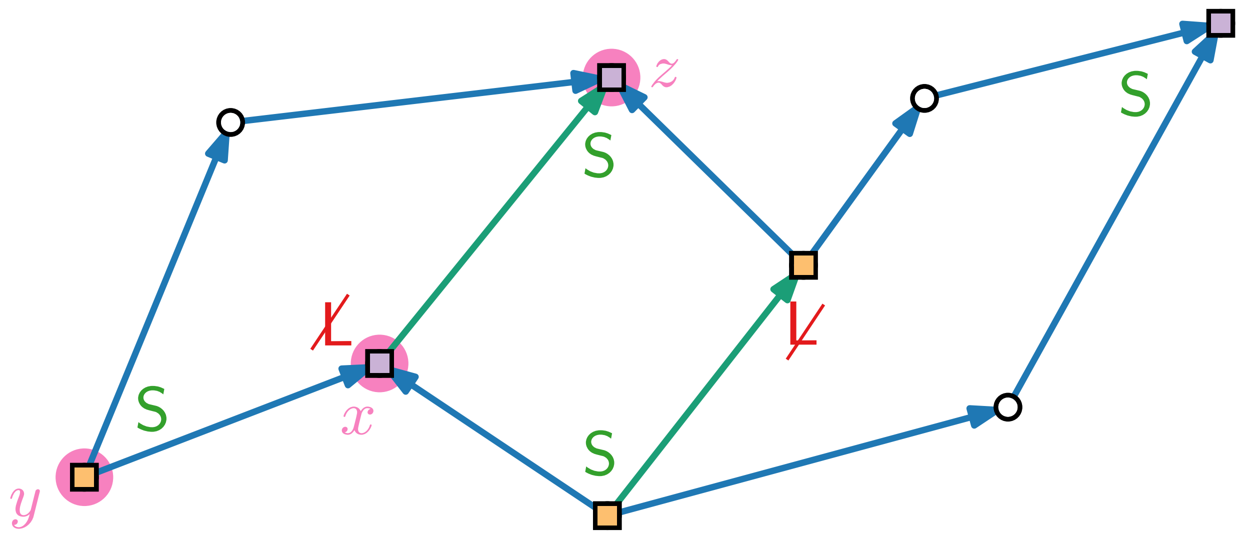

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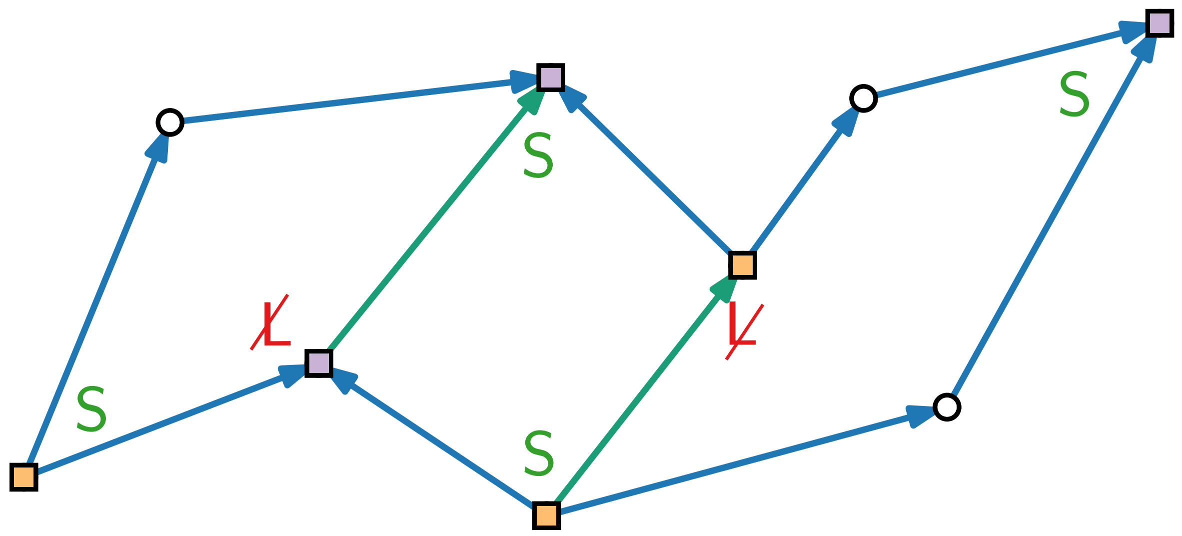

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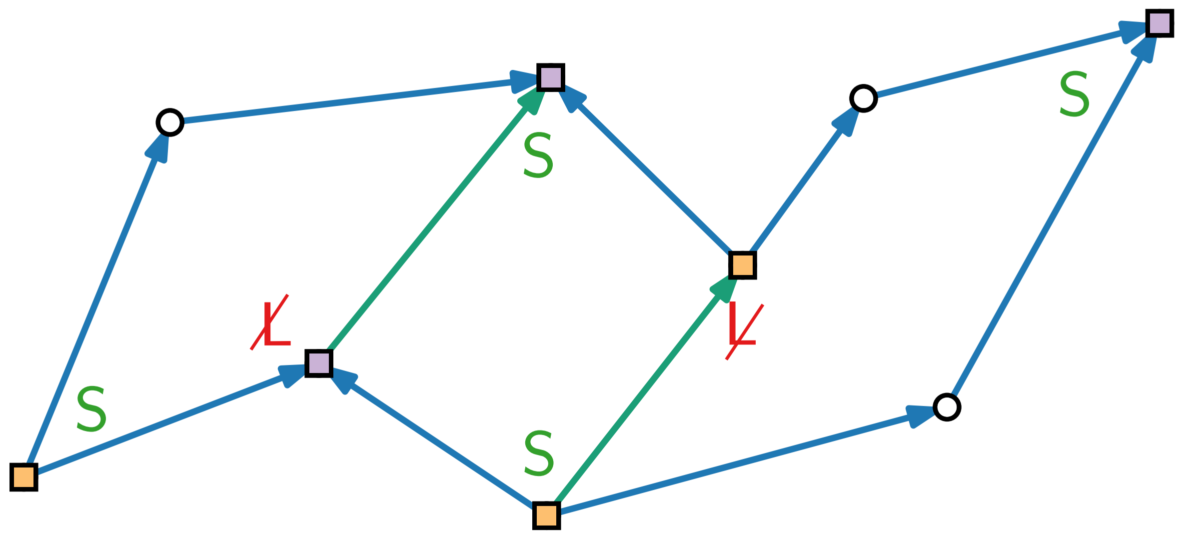

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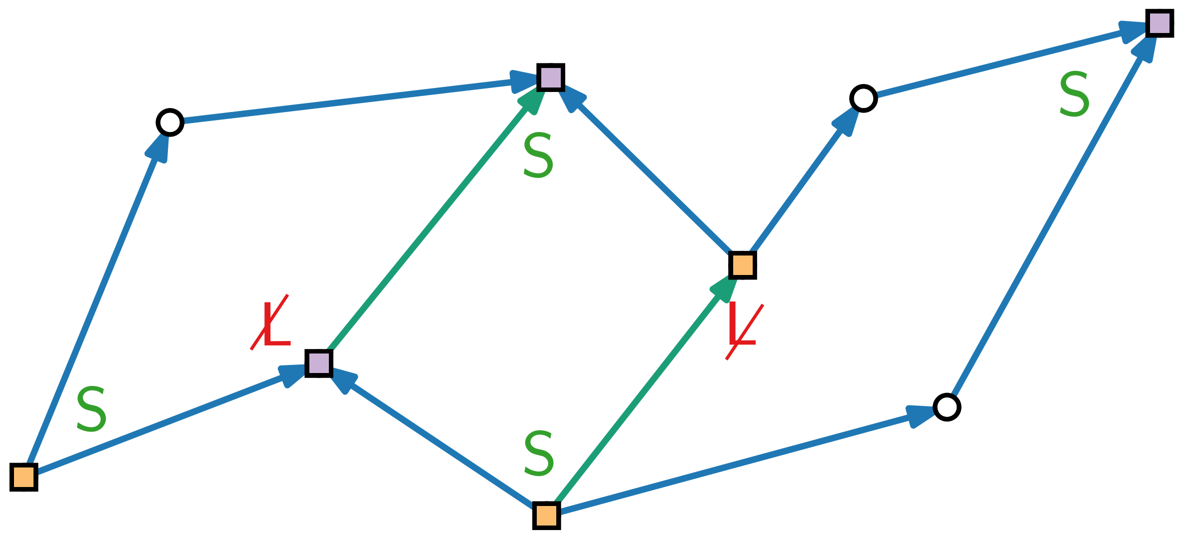

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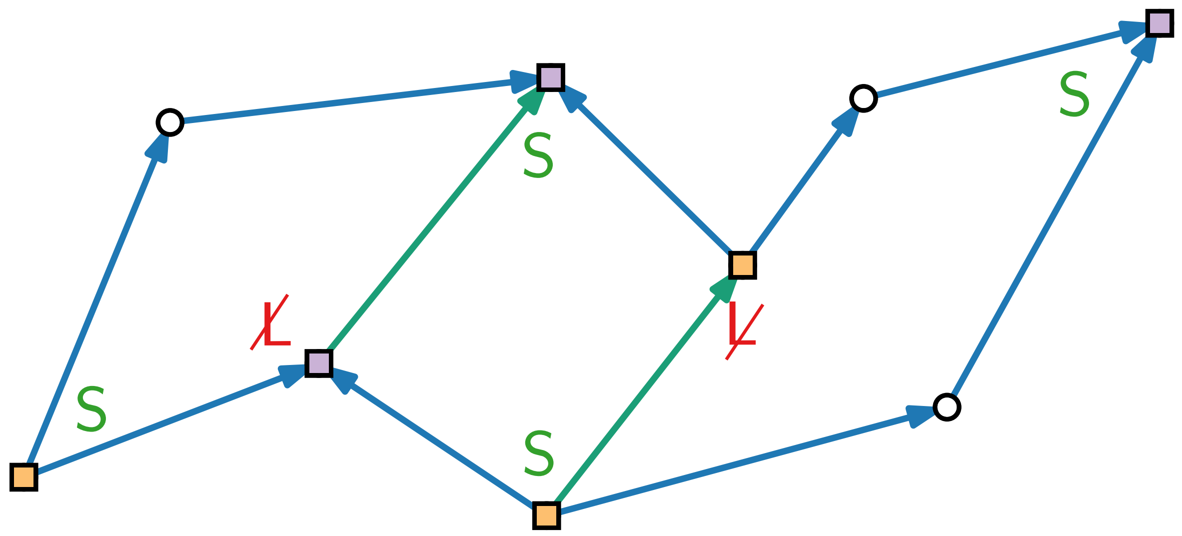

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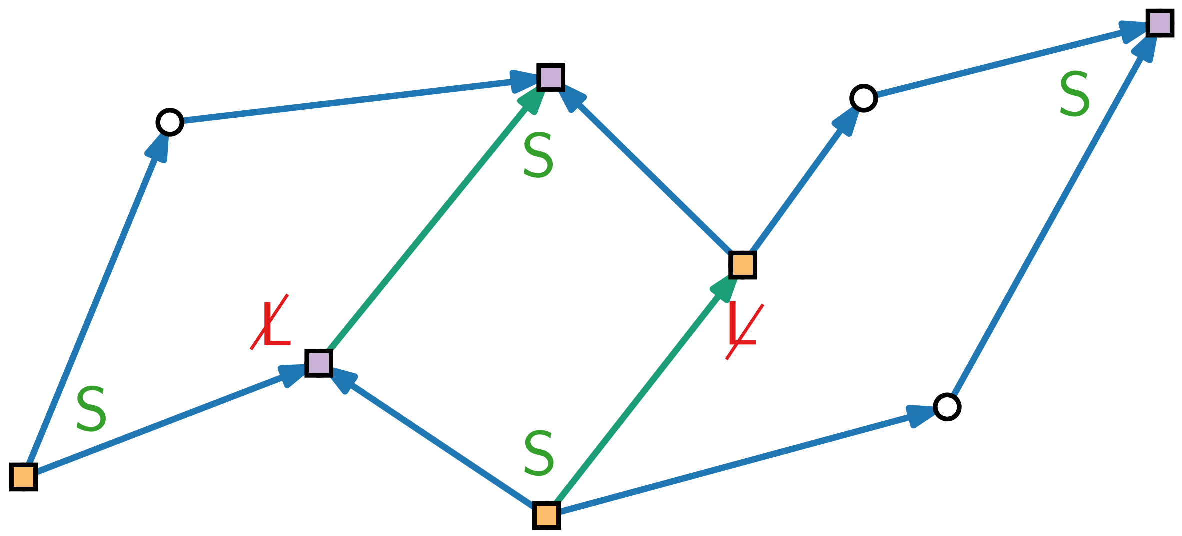

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- Planarity, acyclicity, bimodality are invariants under construction.

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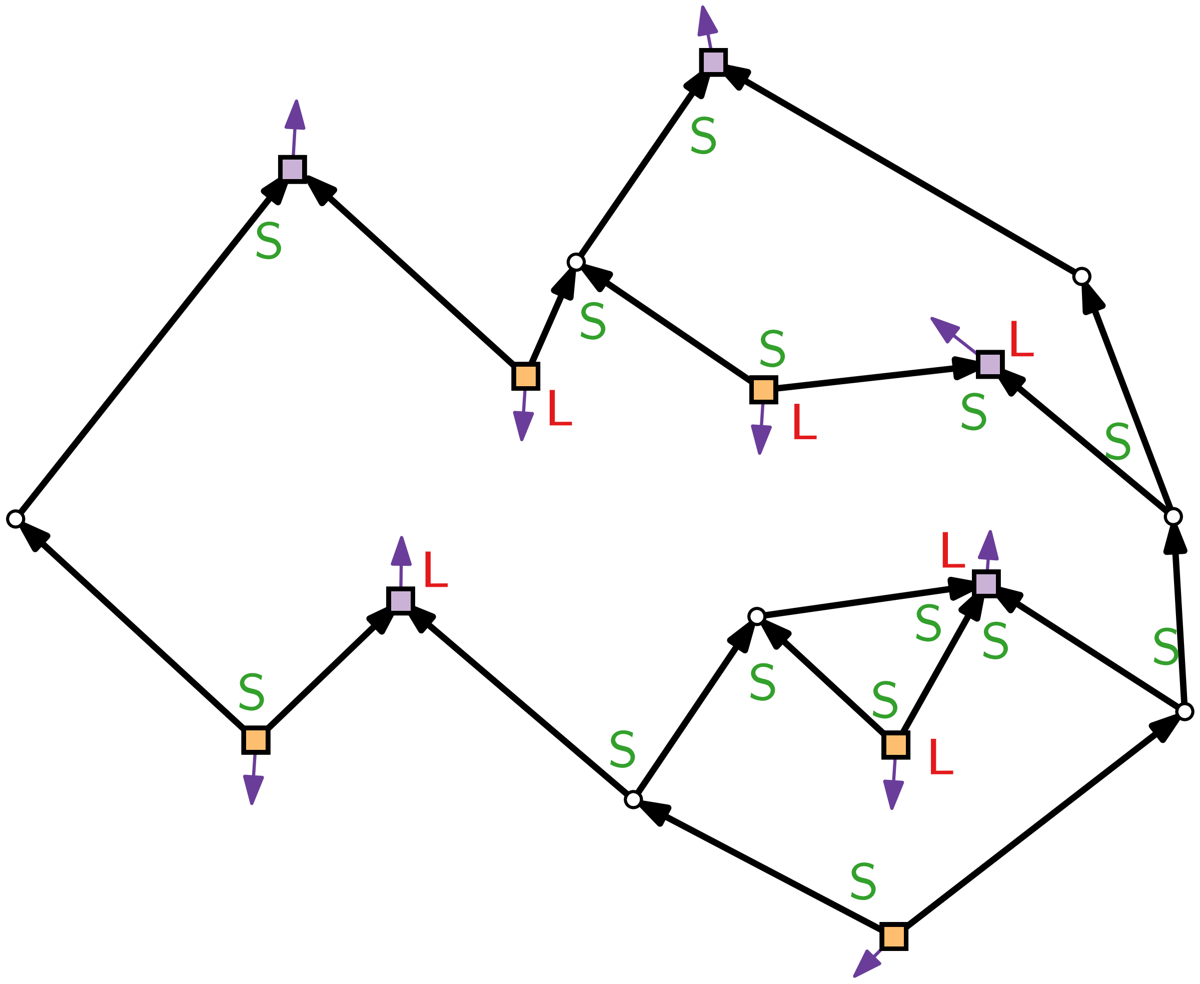

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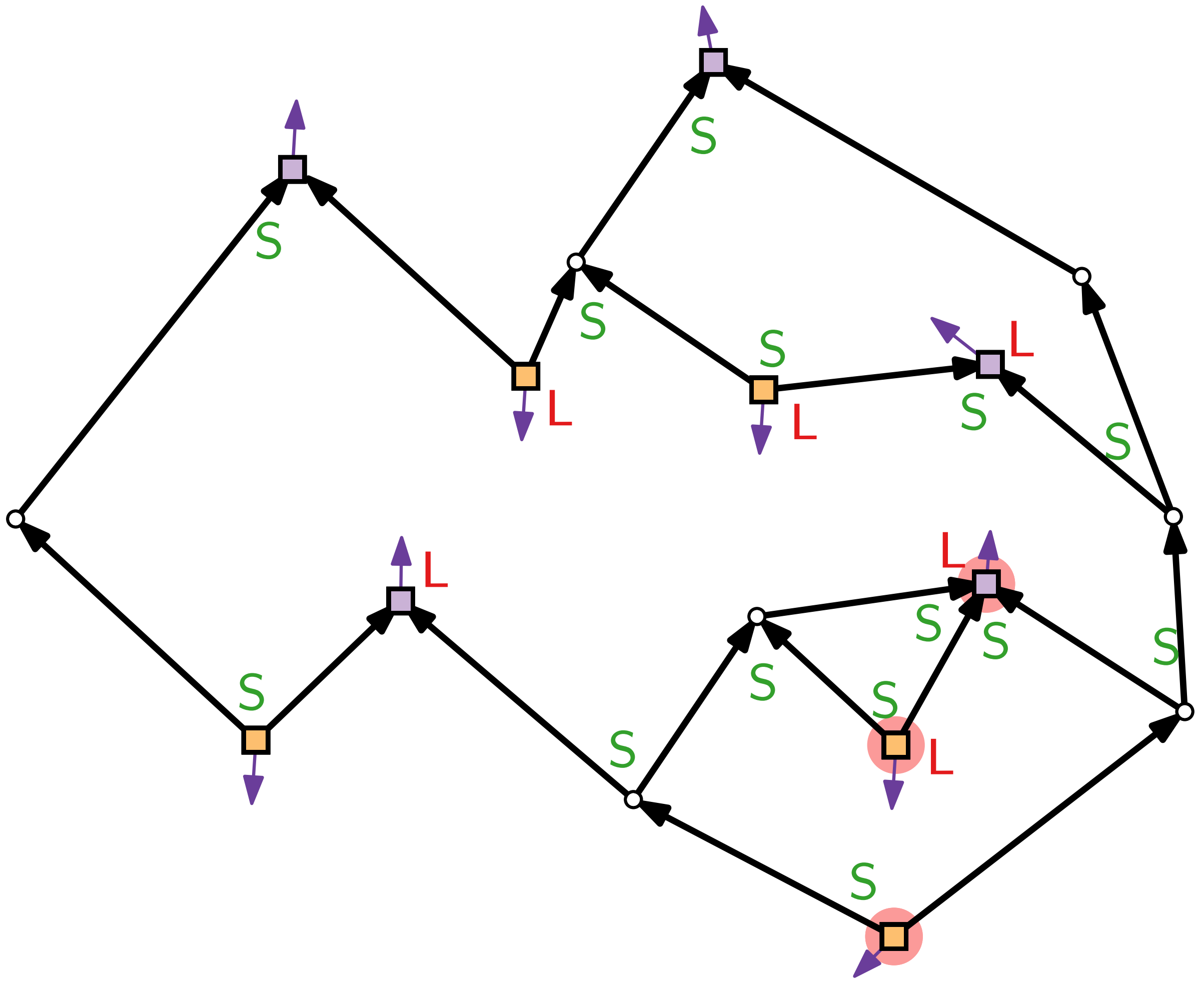

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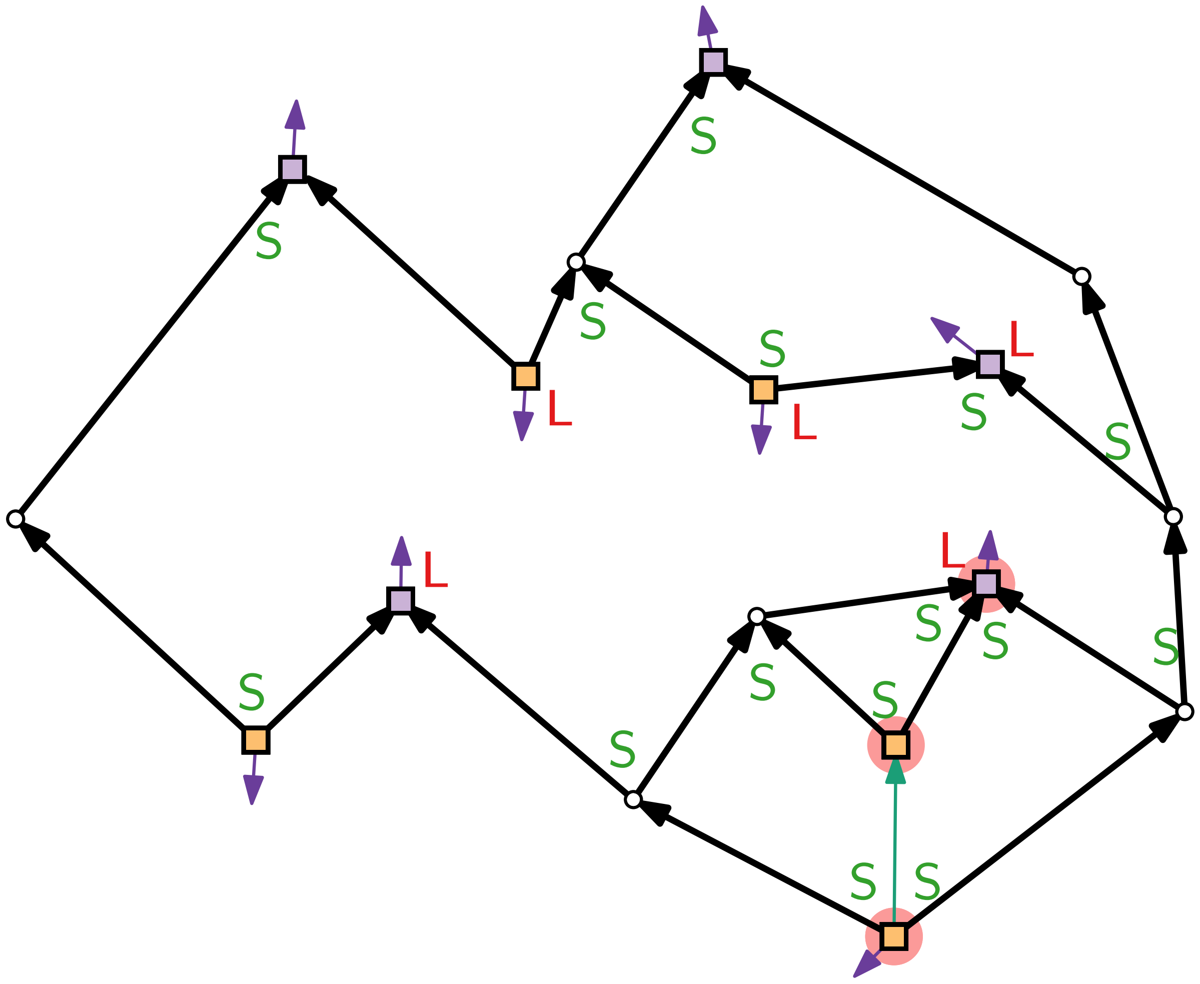

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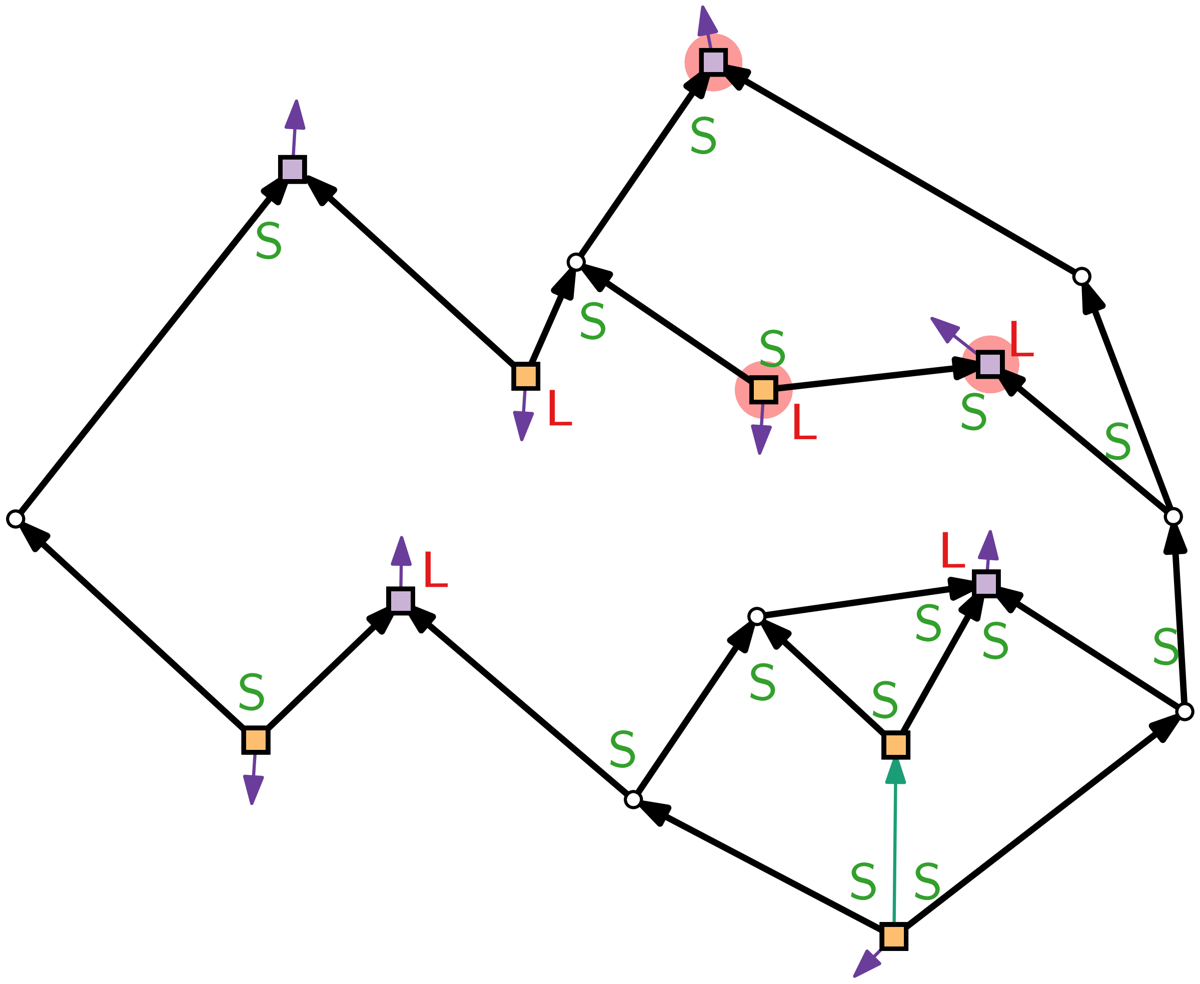

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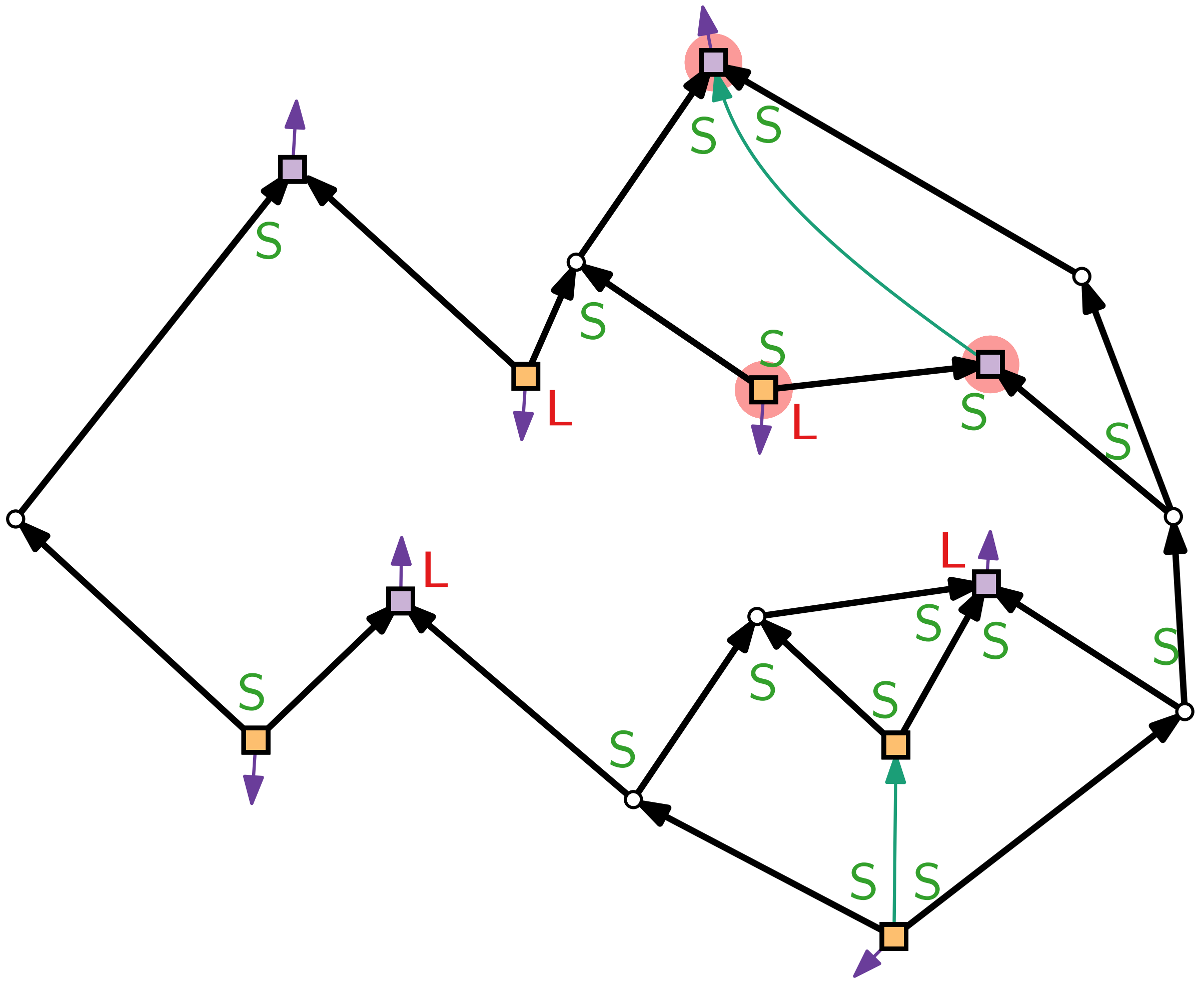

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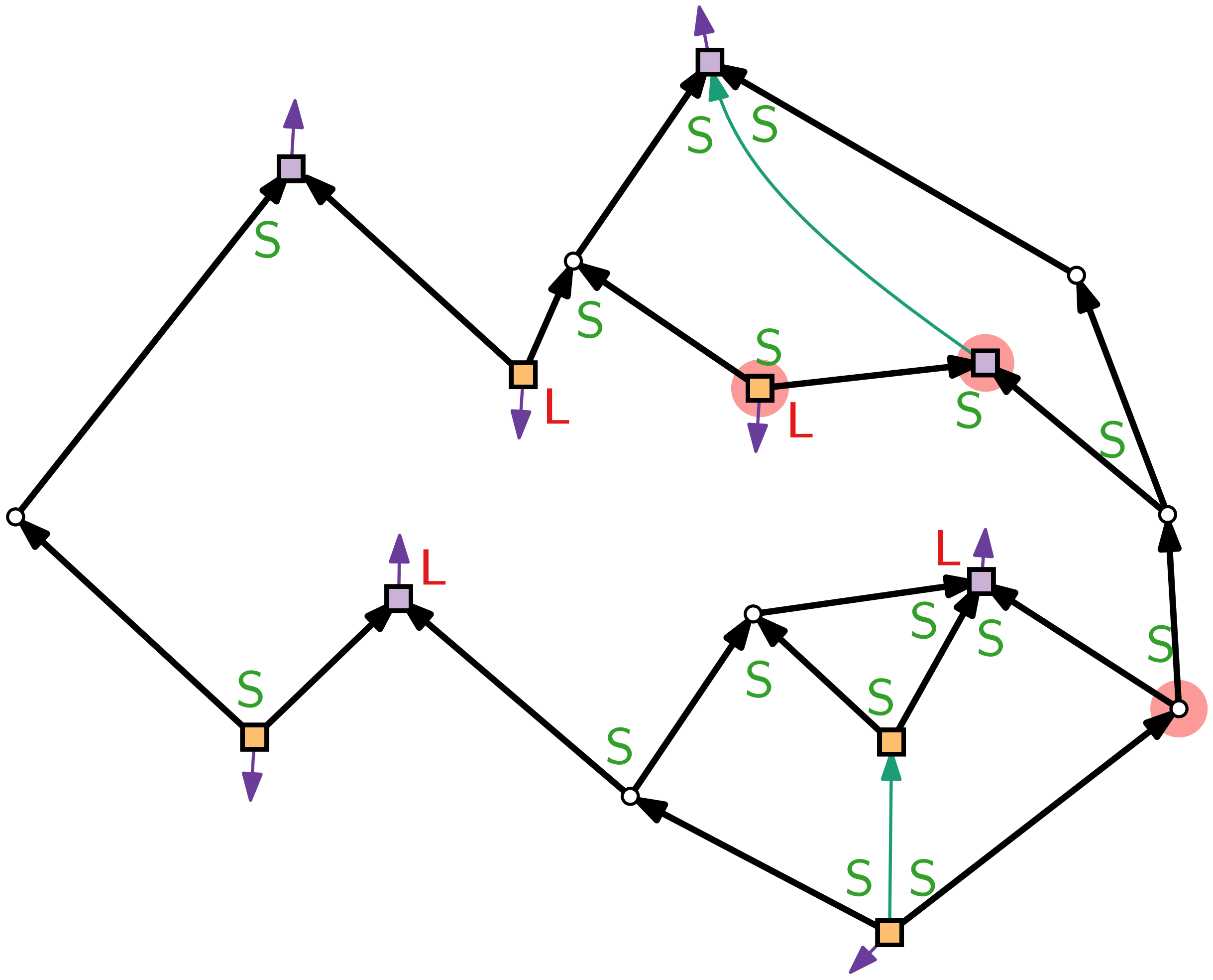

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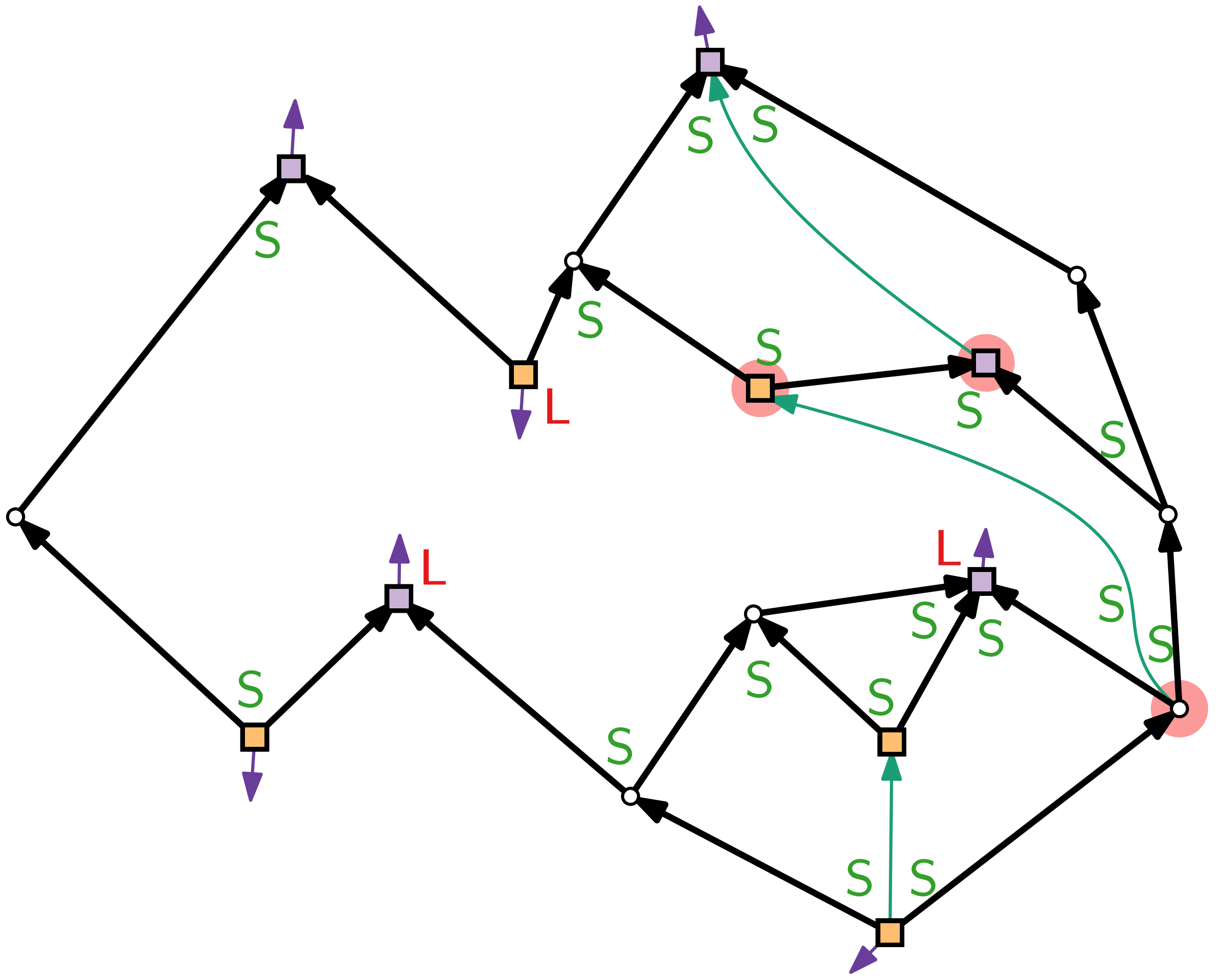

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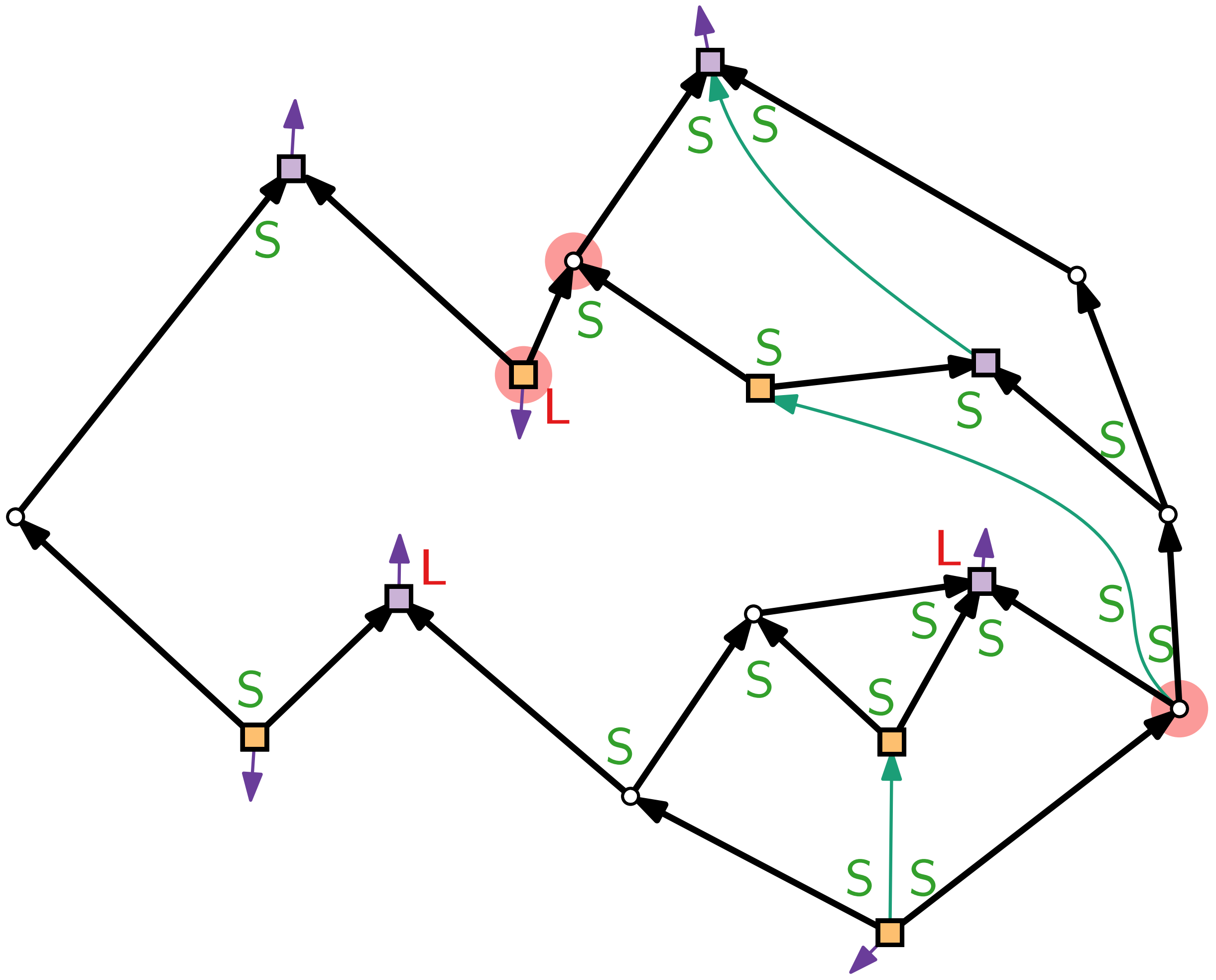

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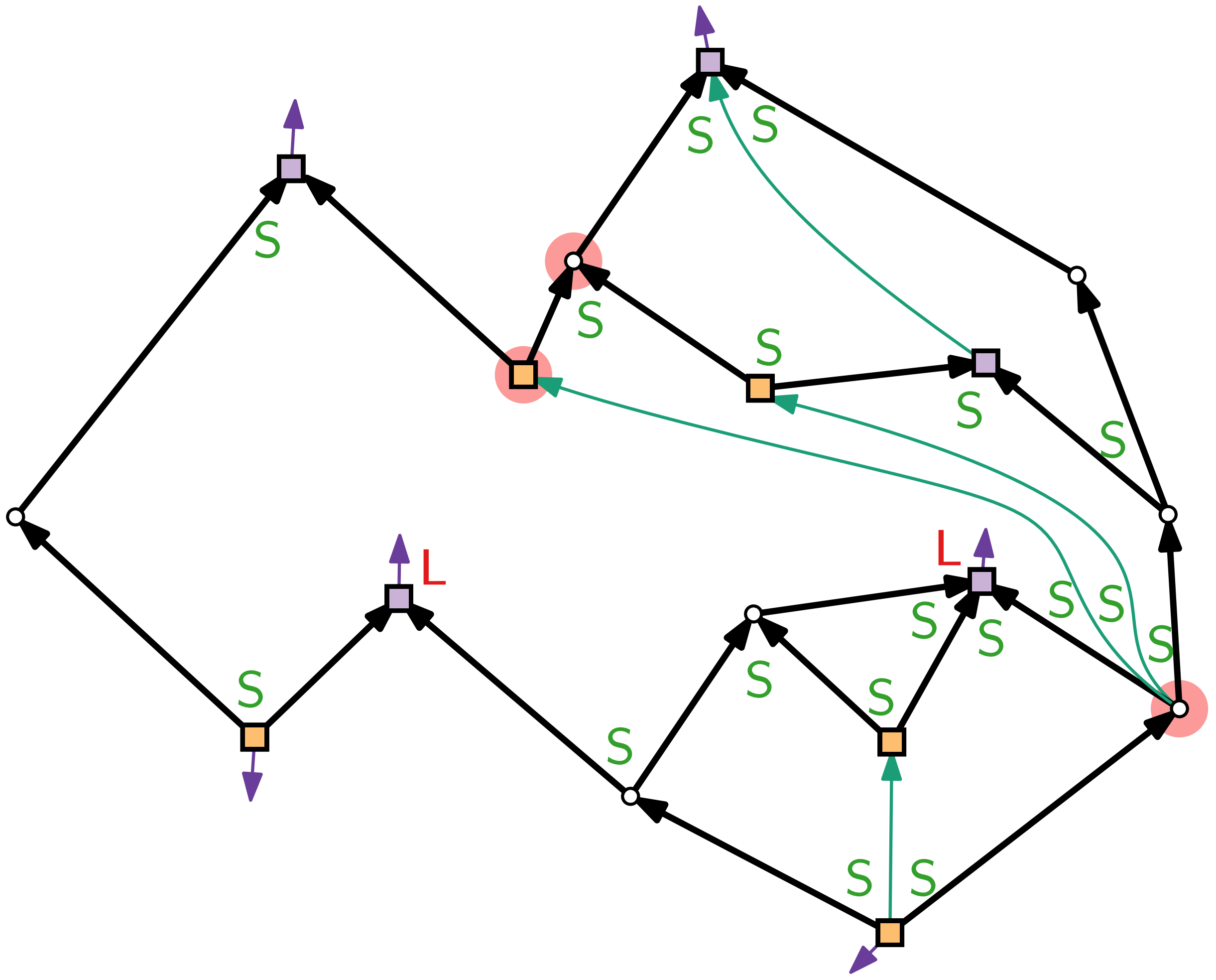

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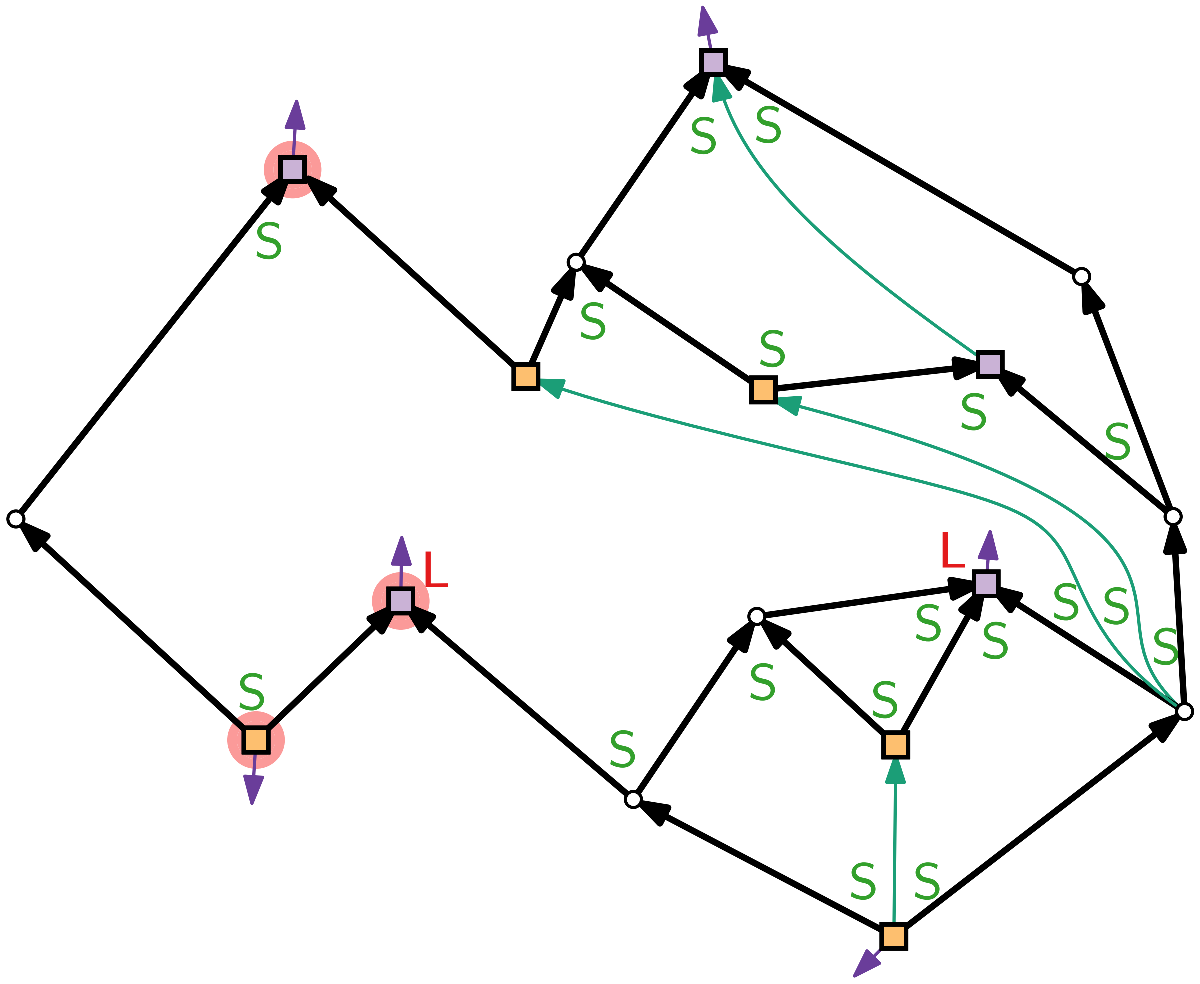

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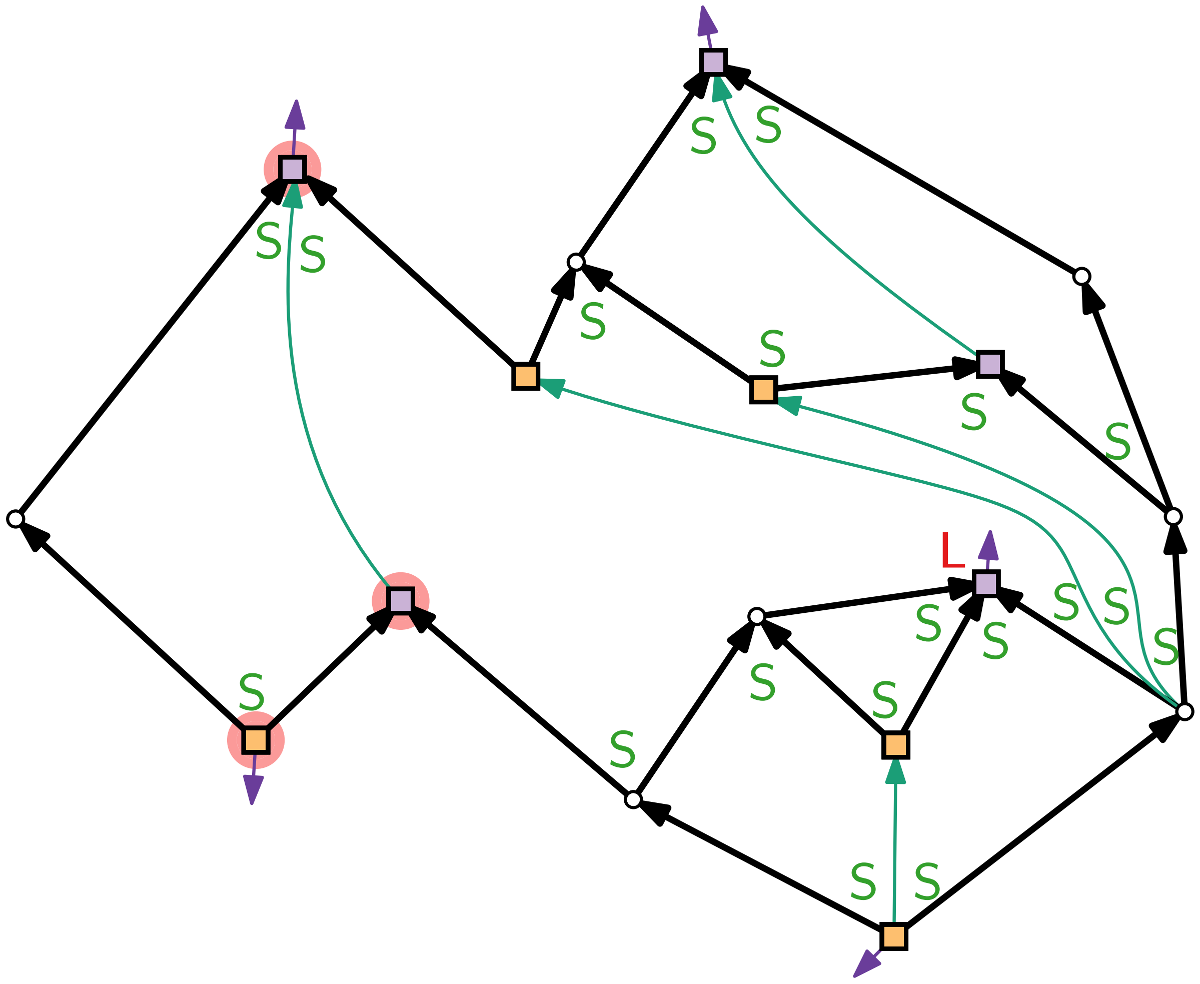

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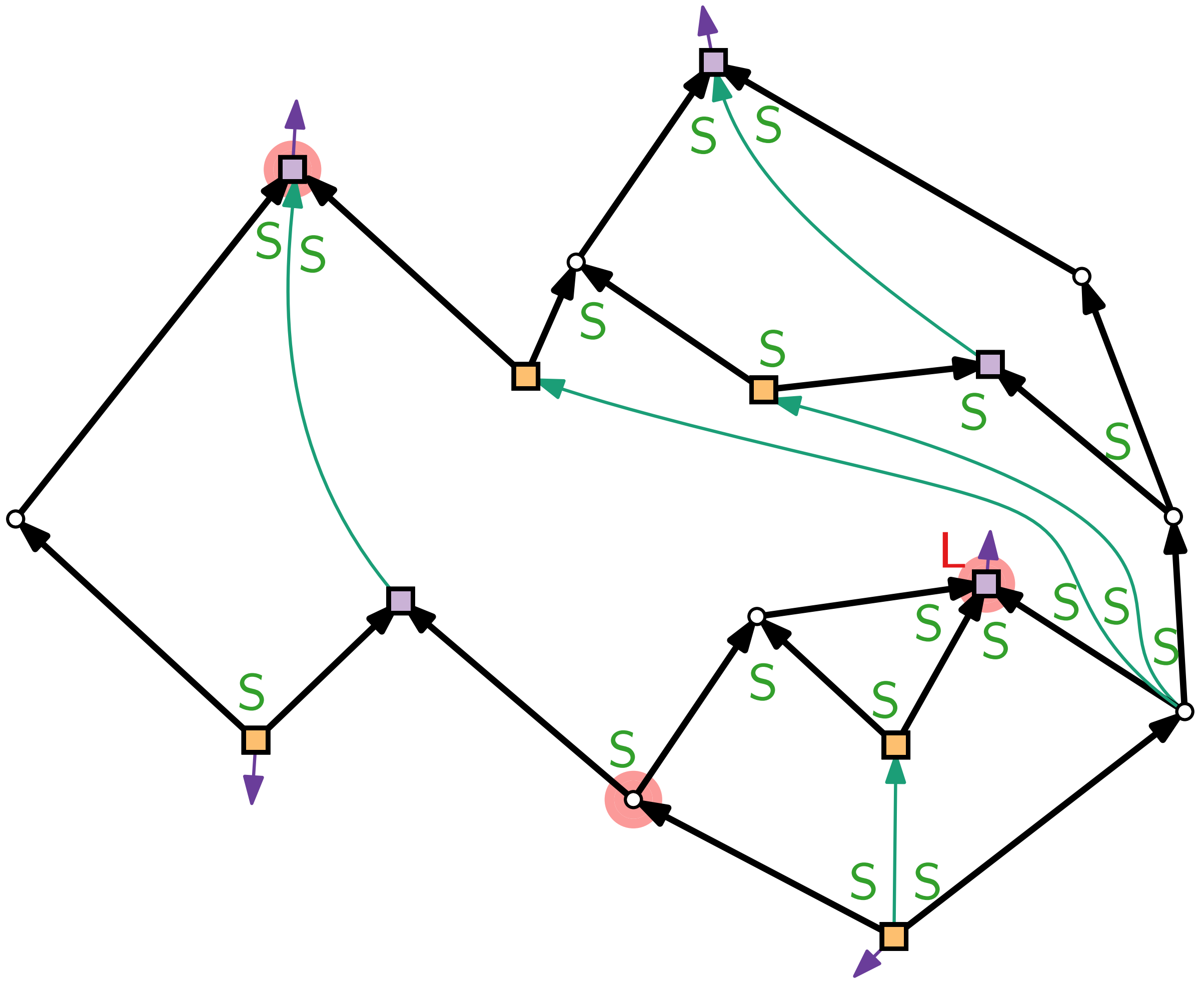

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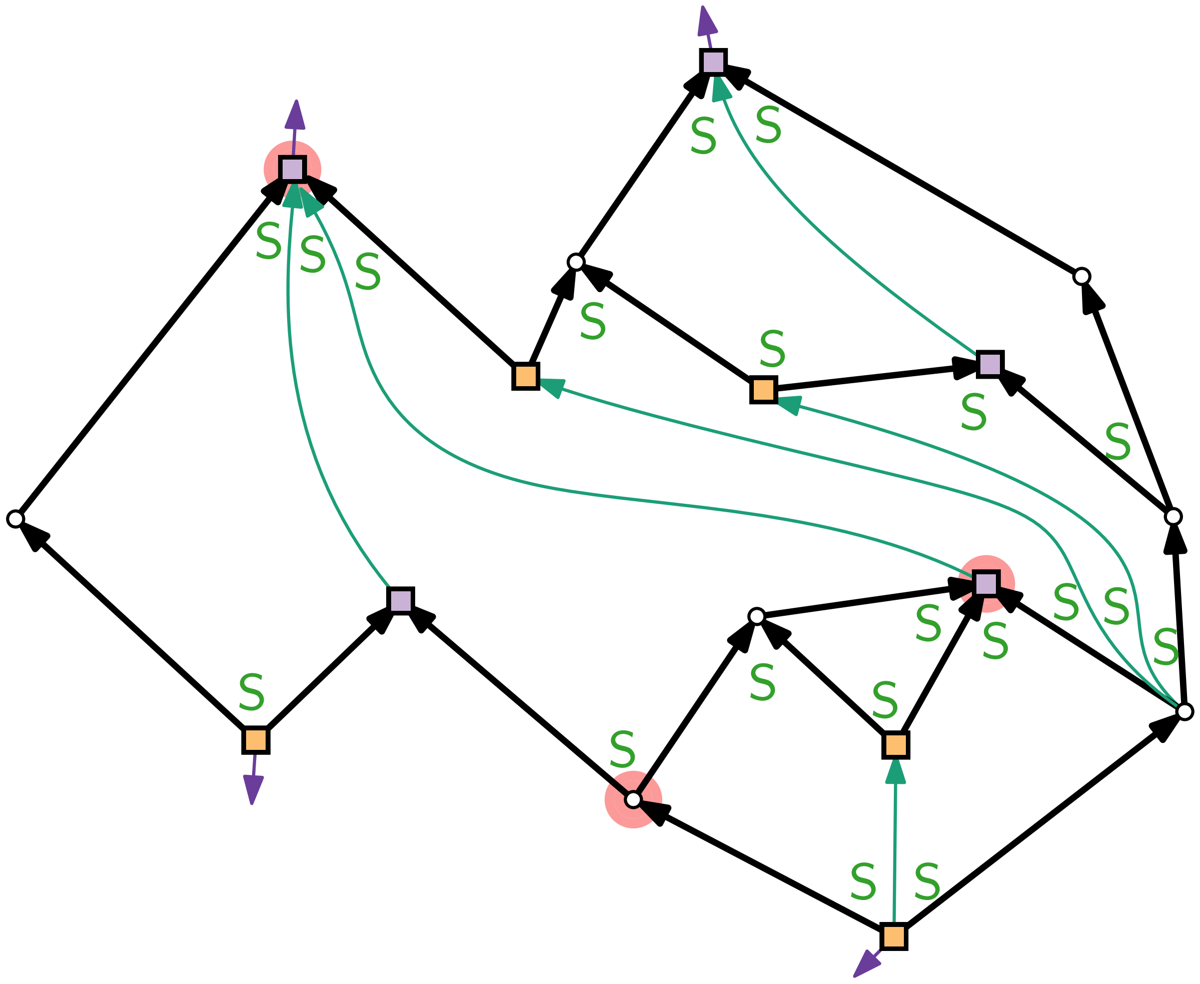

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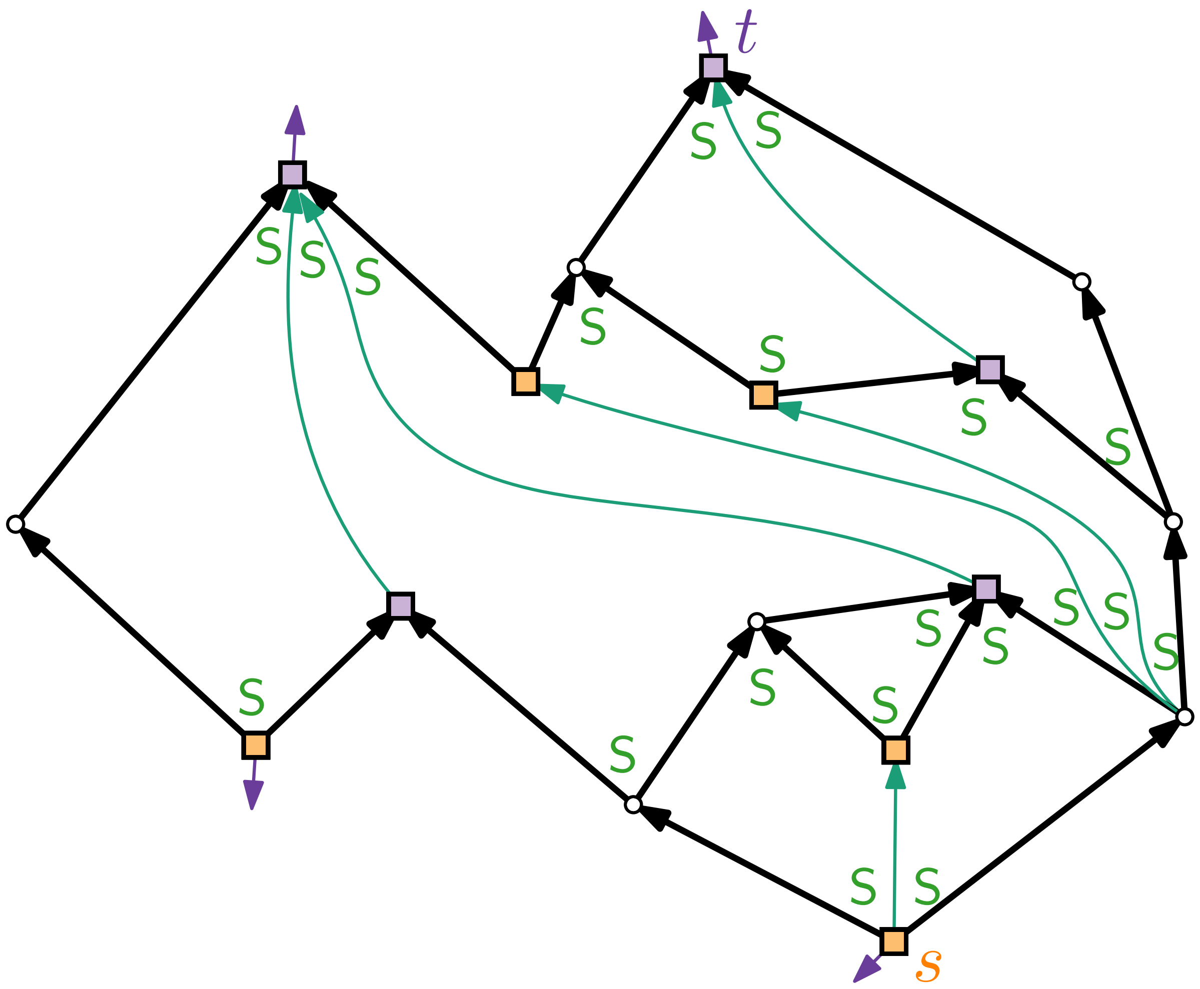

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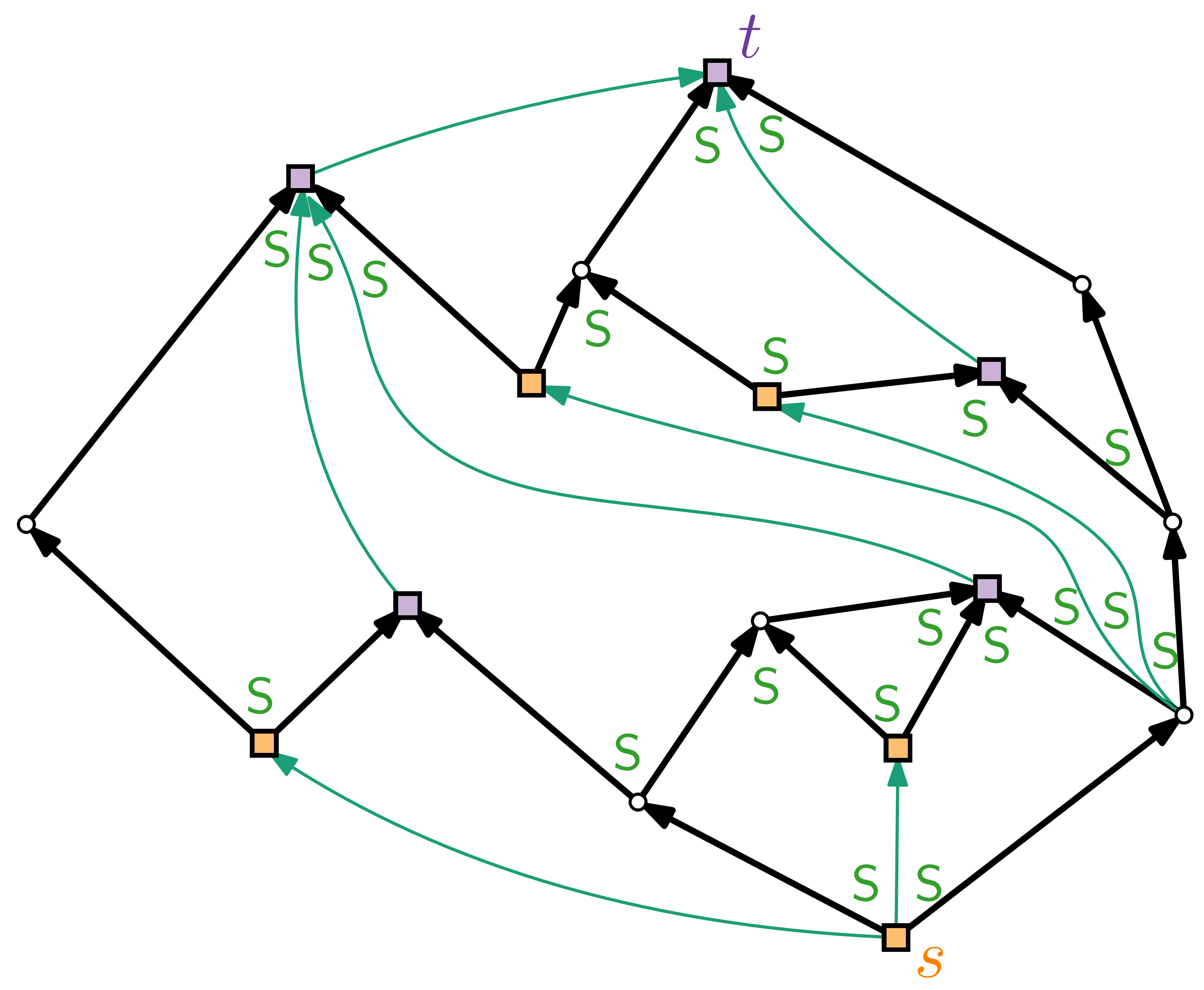

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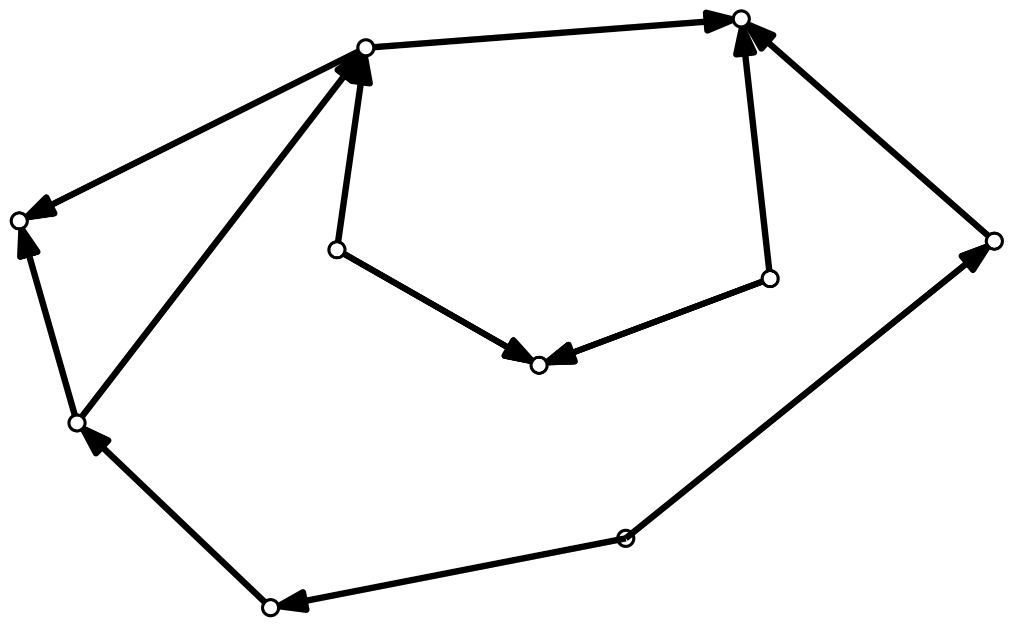

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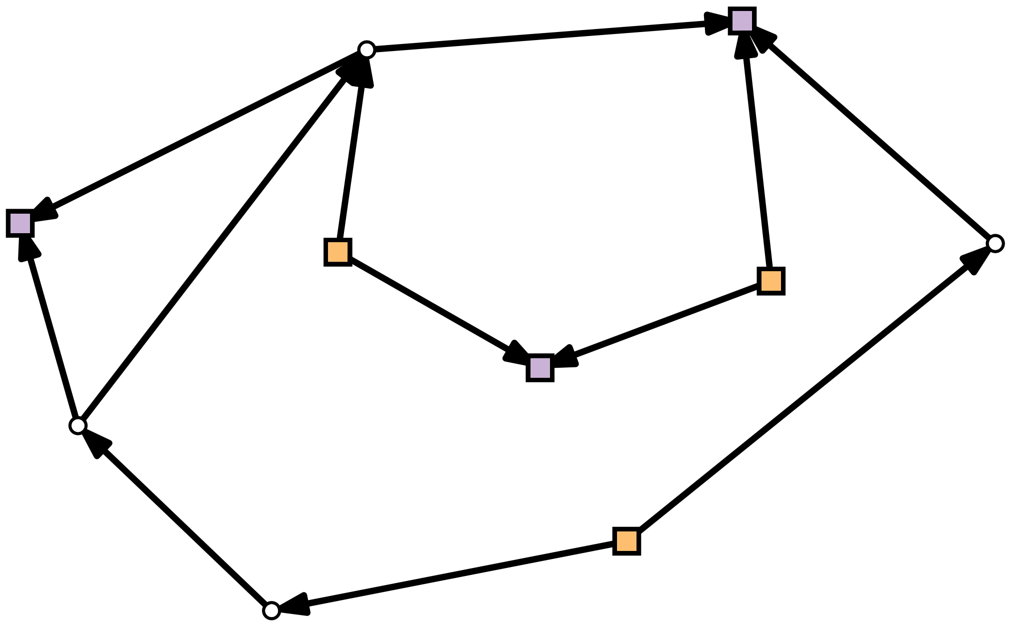

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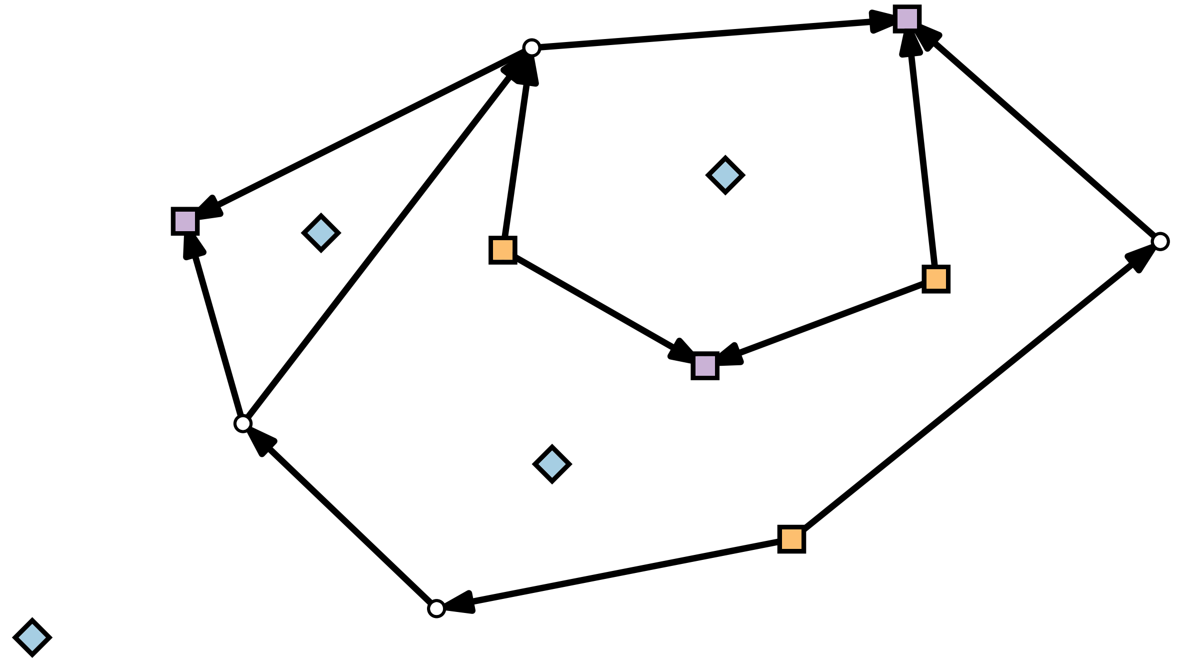

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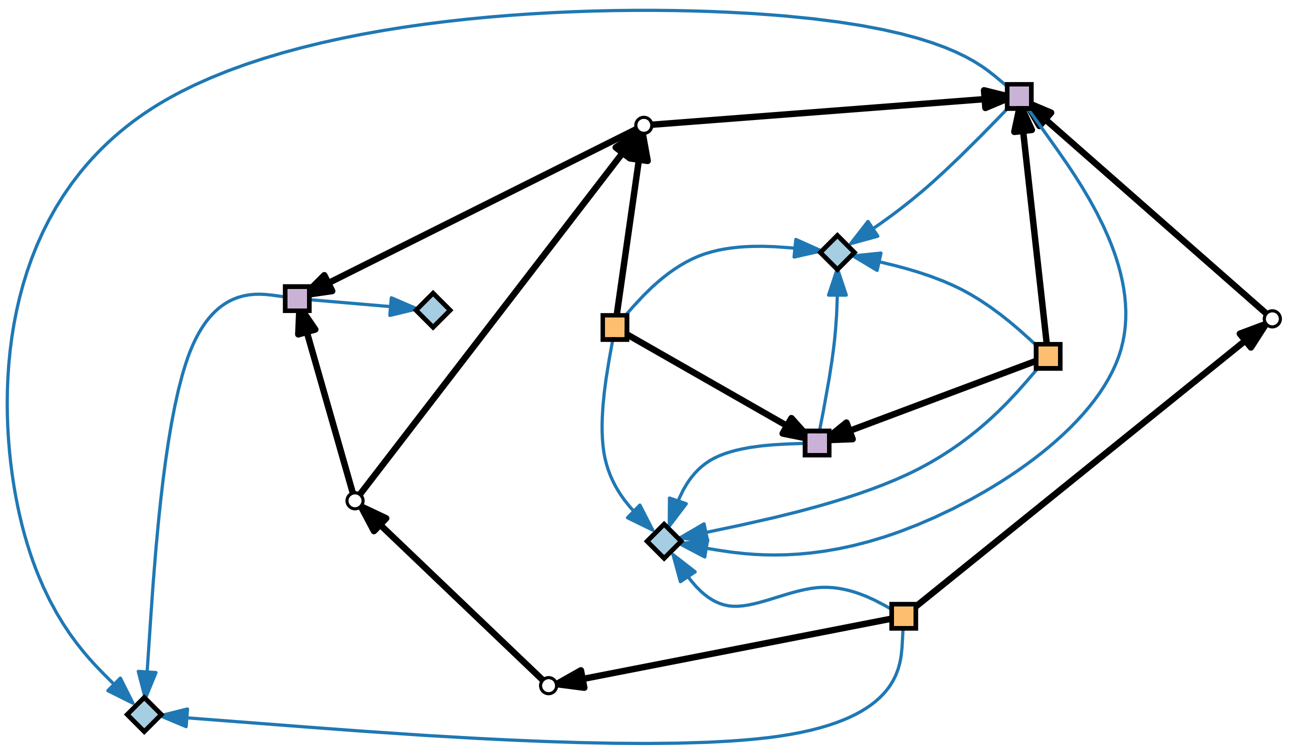

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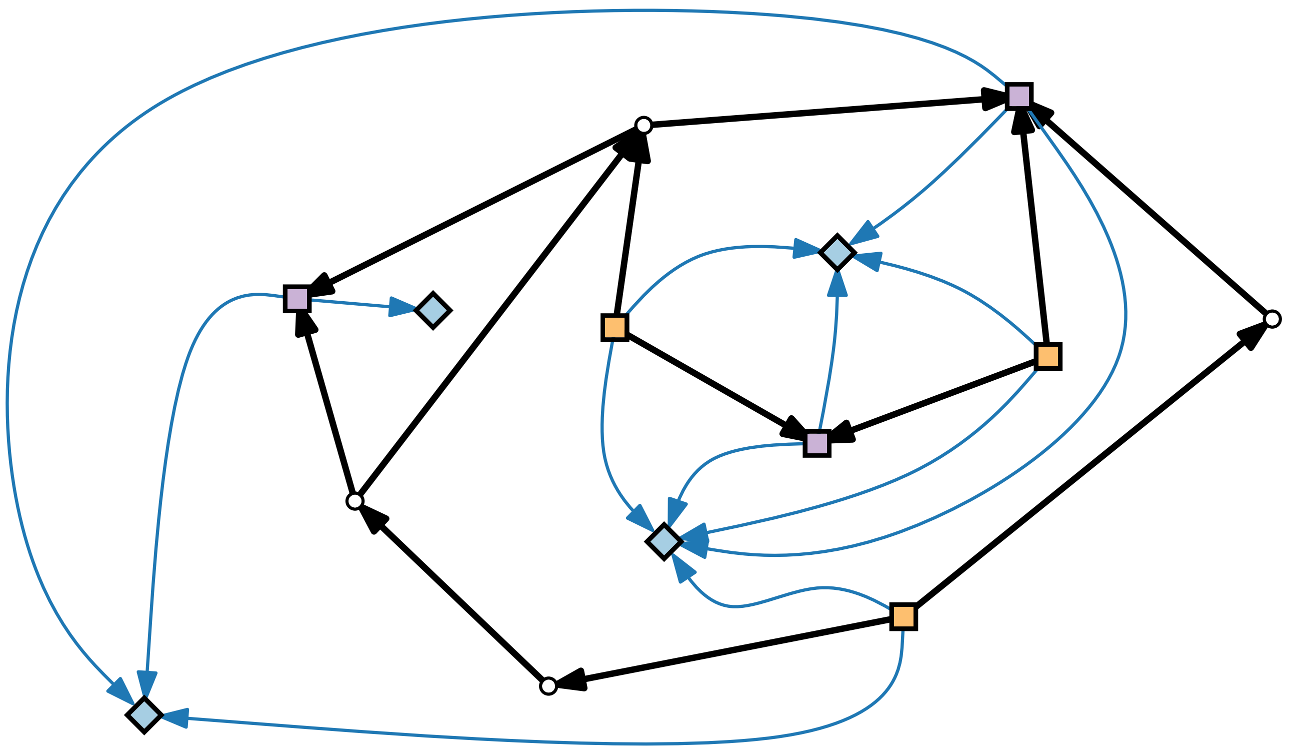

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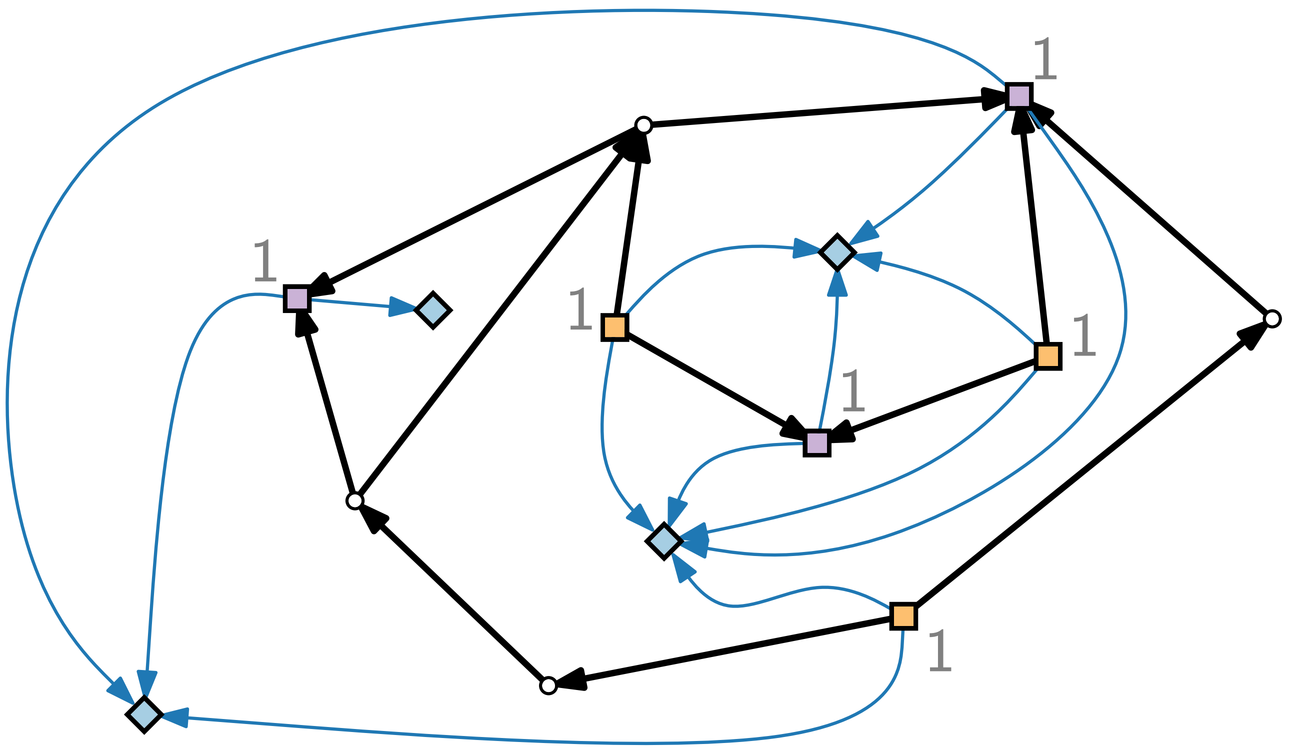

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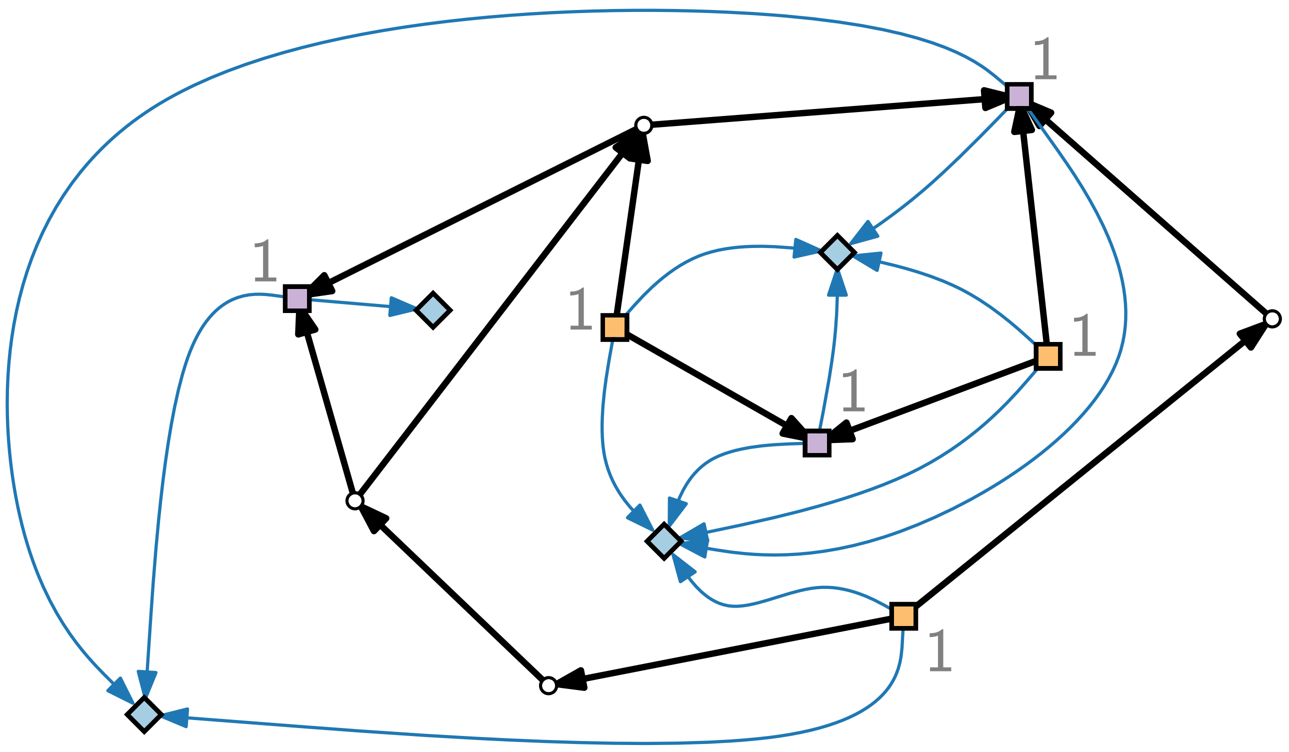

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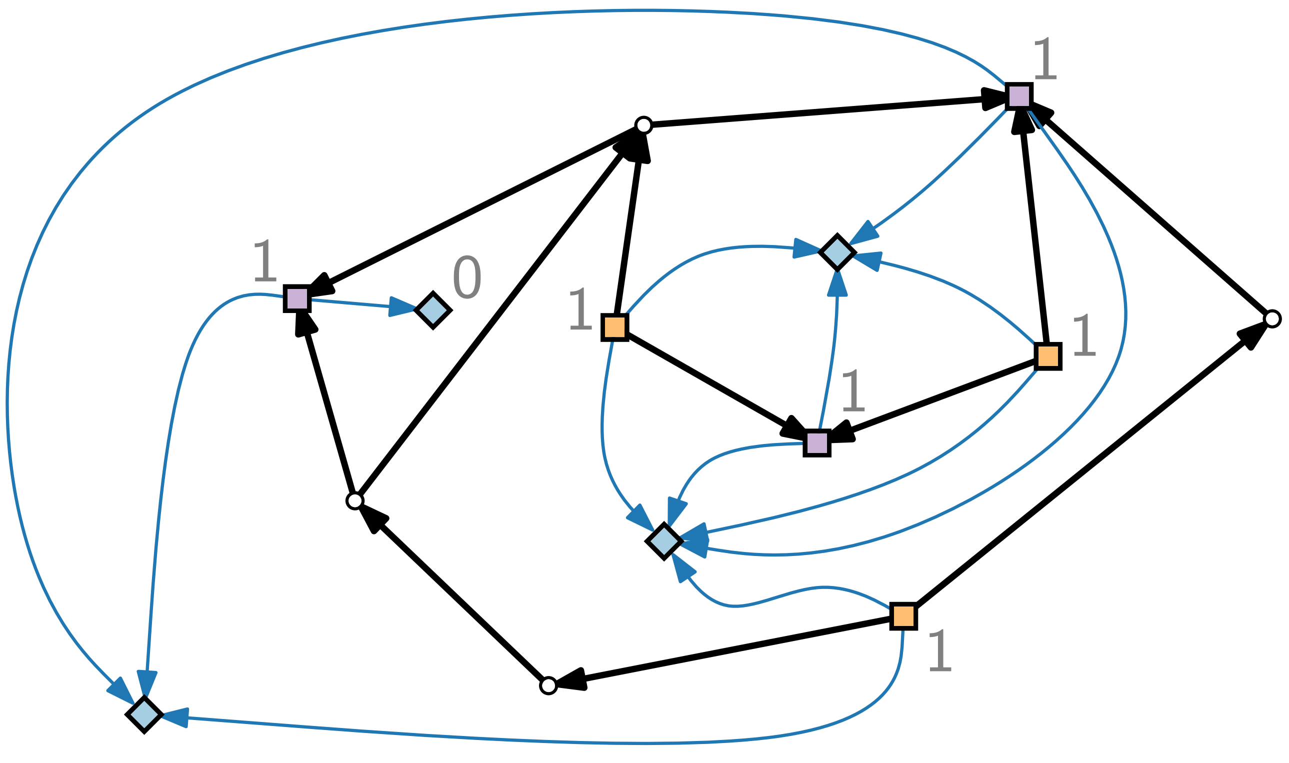
Example.



Finding a Consistent Assignment

Idea. Flow $(v, f) = 1$
from global source / sink v to the incident face f its large angle gets assigned to.

nodes of flow network   edges of flow network
supplies/demands of nodes   lower/upper bounds on edge capcities

Flow network.

$$N_{F,f_0}(G) = ((W, E'); b; \ell; u)$$

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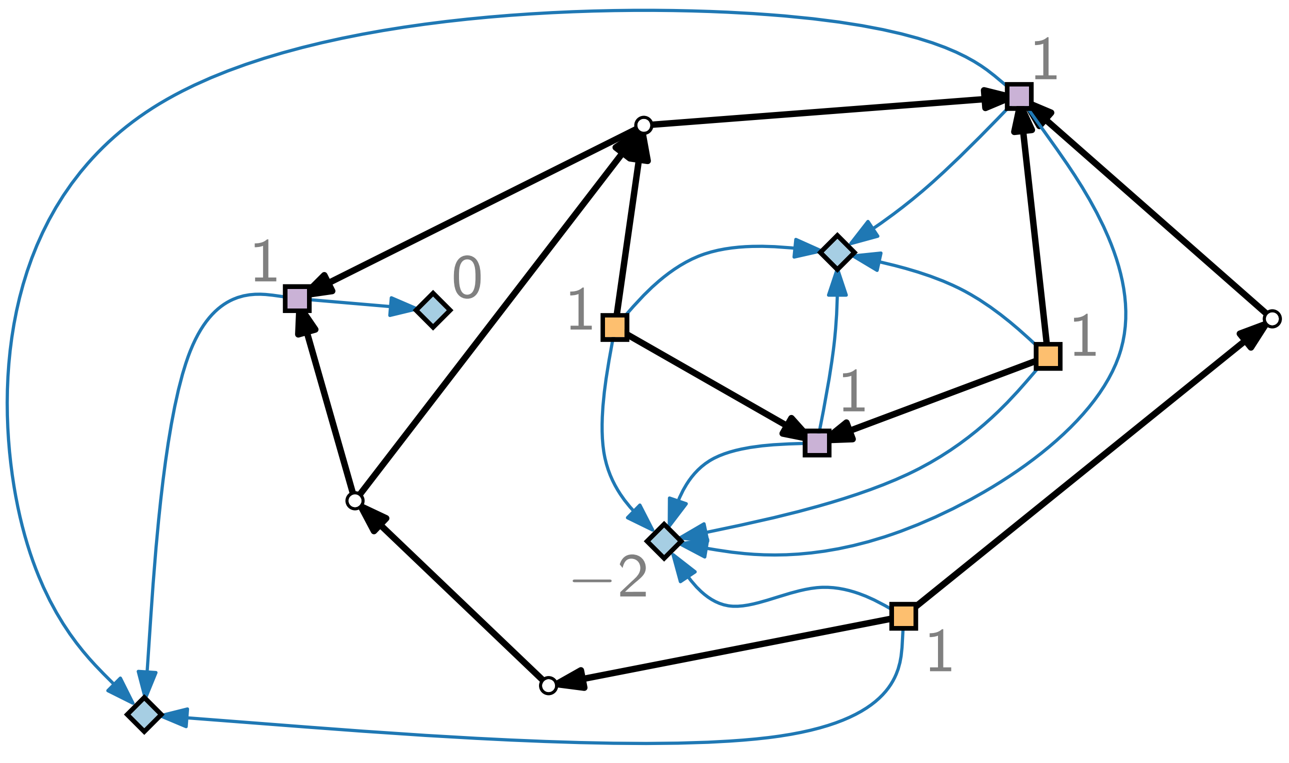

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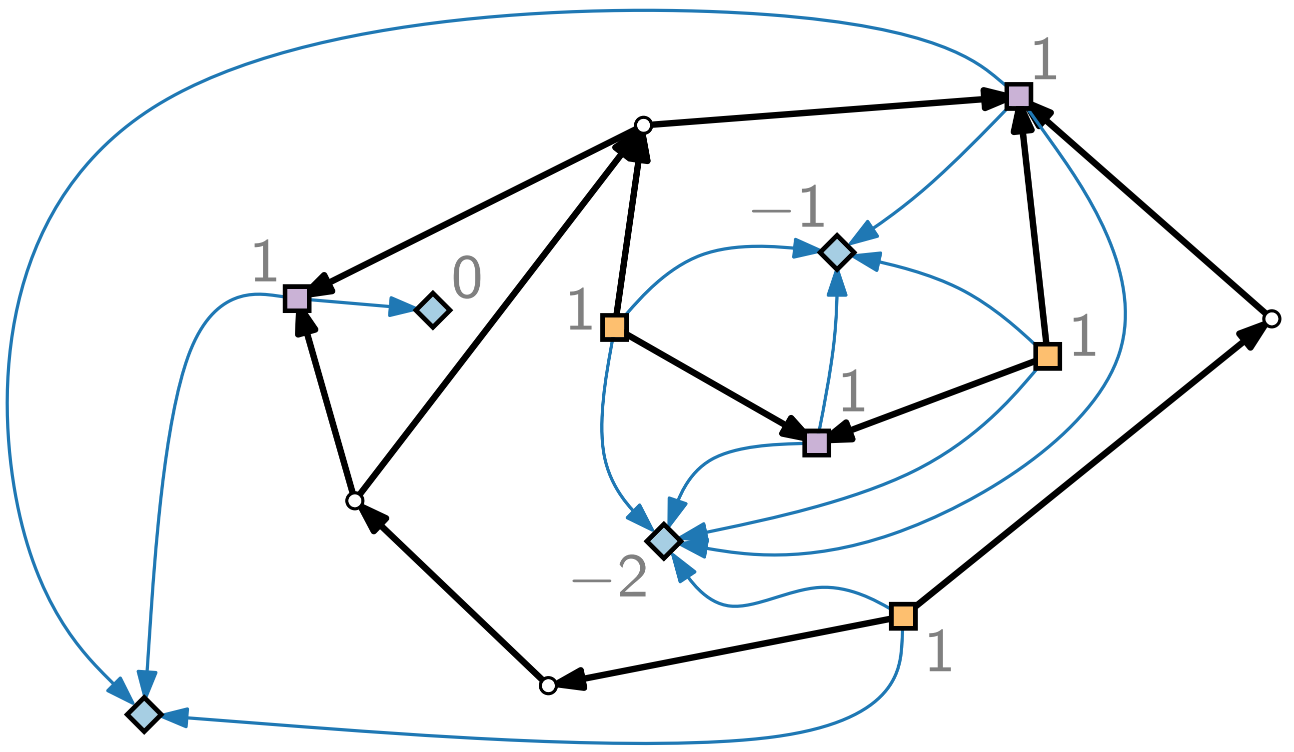

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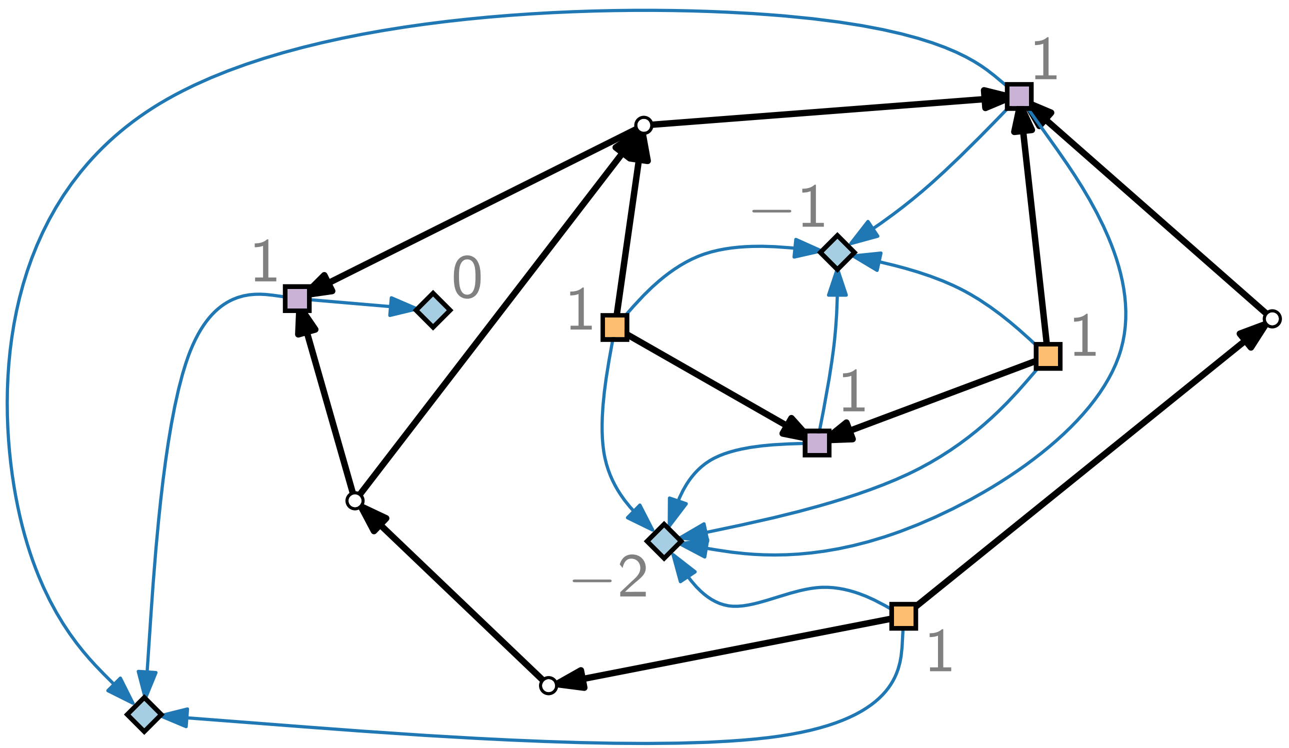

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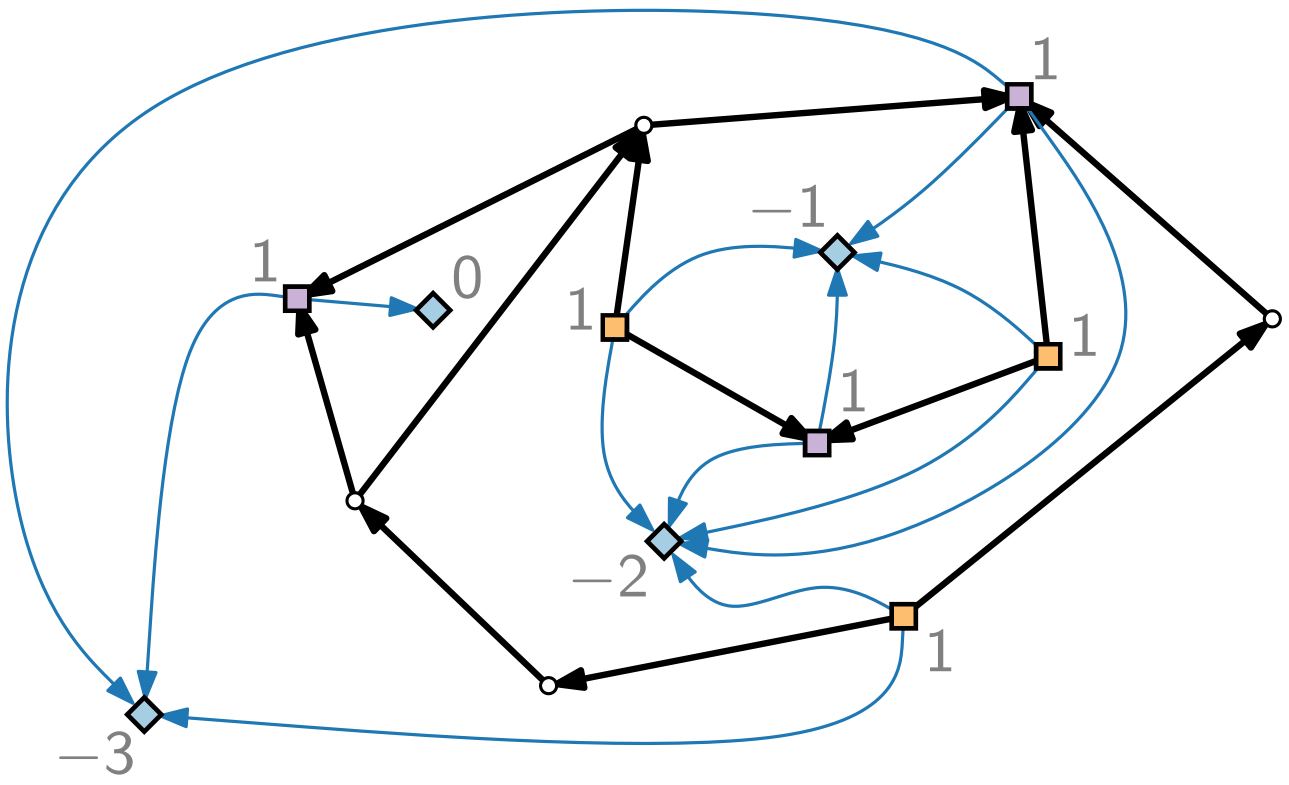

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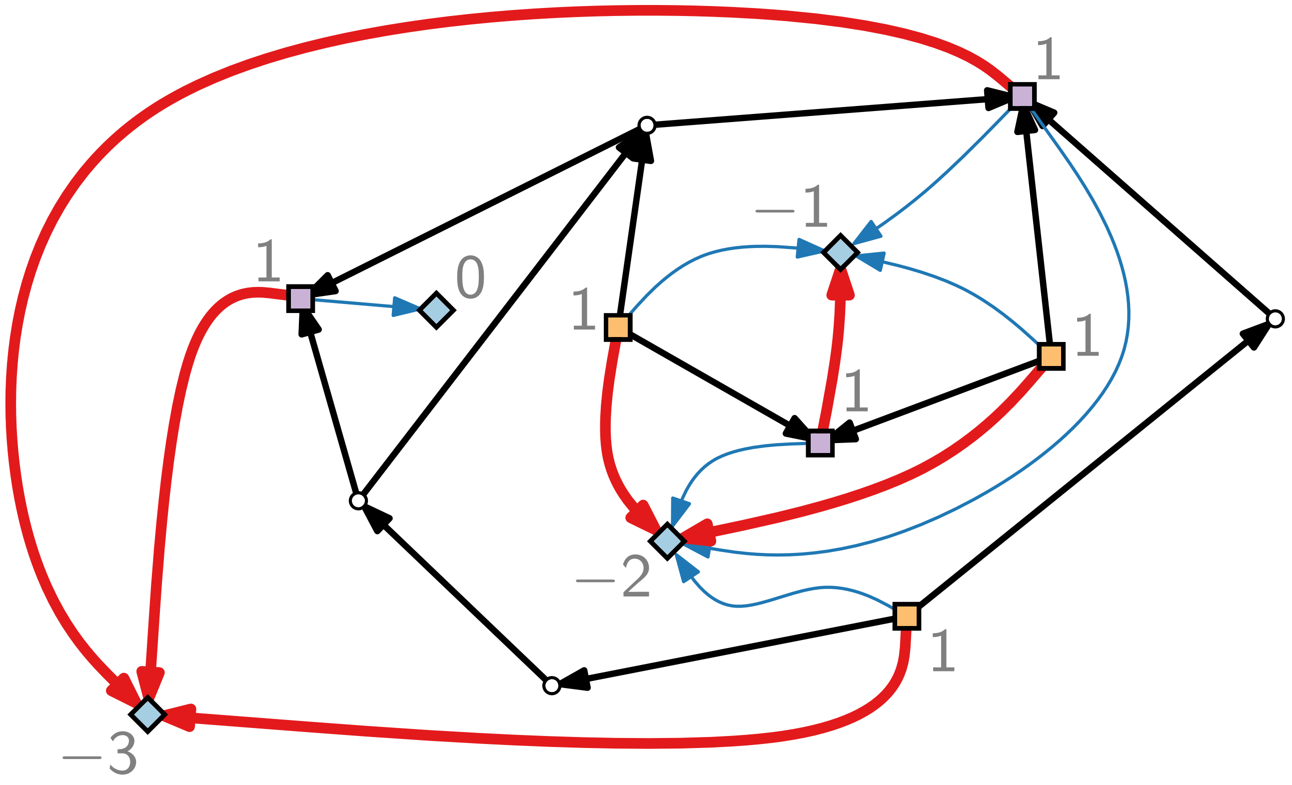
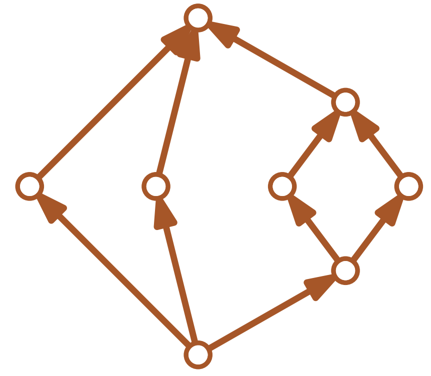
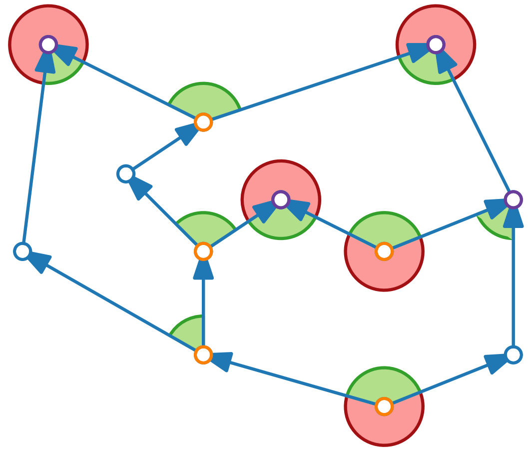


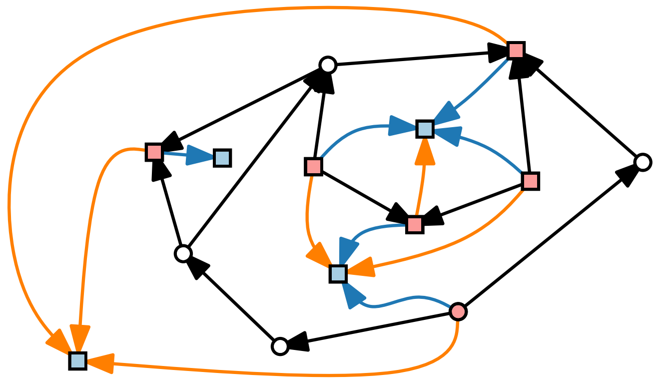
Visualization of Graphs

Lecture 5:
Upward Planar Drawings





Part II:
Series-Parallel Graphs



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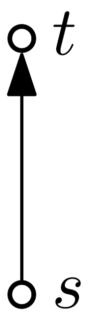

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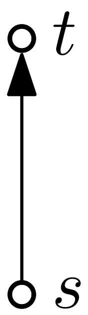

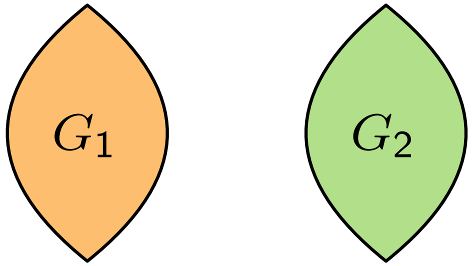

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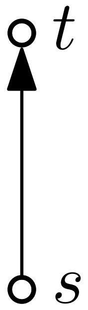

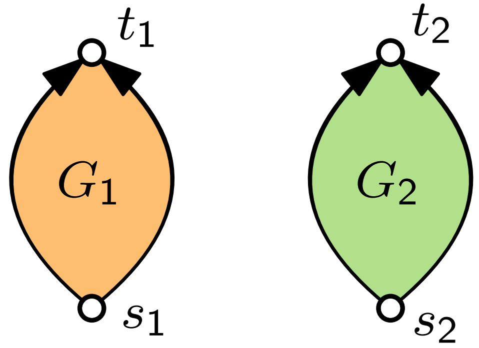

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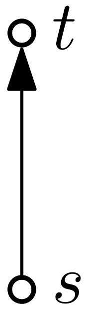

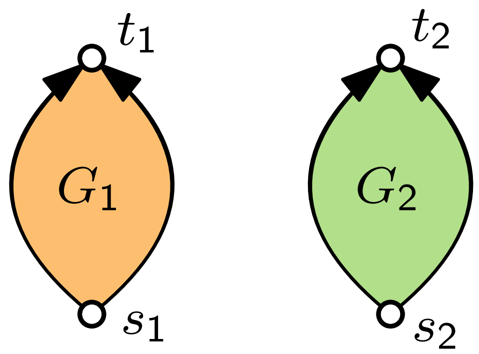

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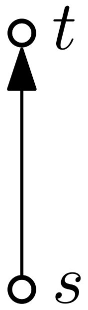
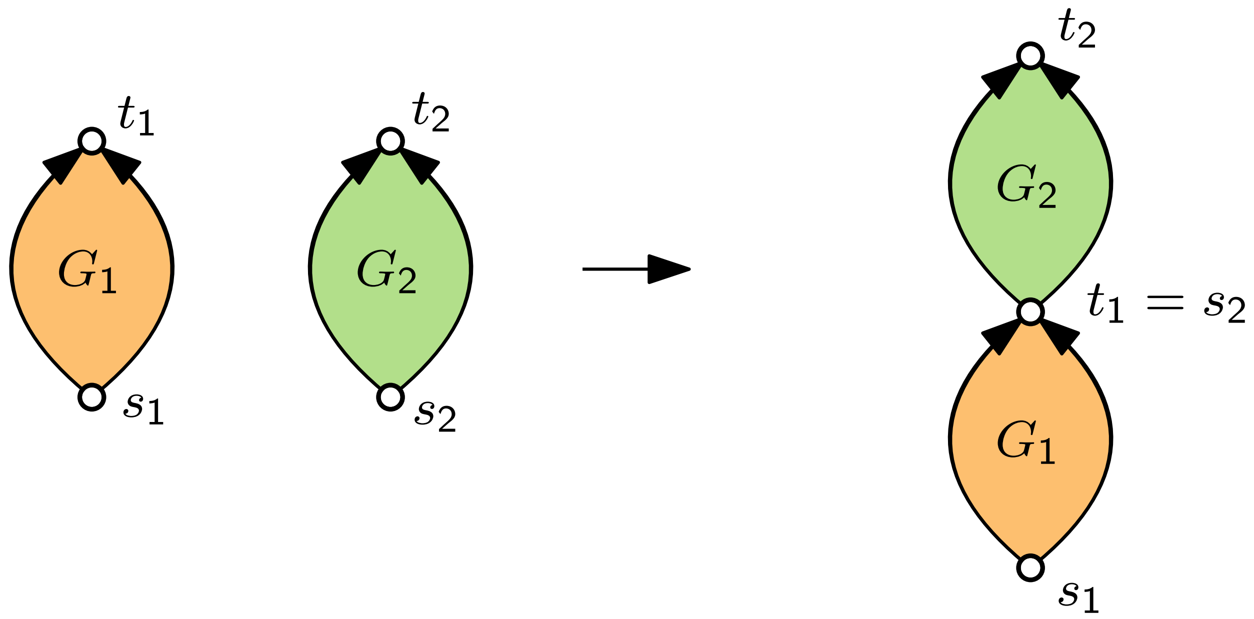


Series composition



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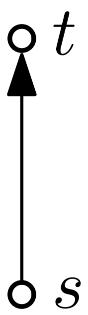

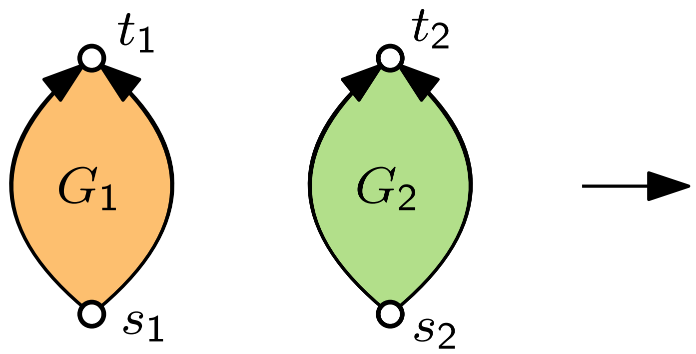

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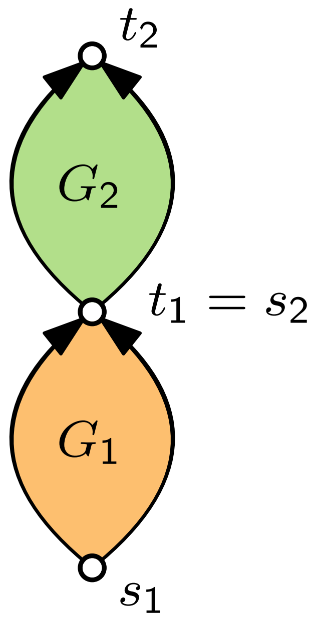
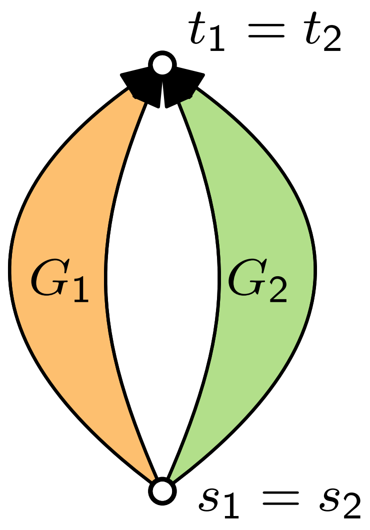


Parallel composition



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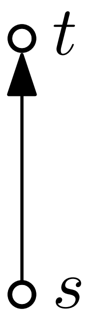
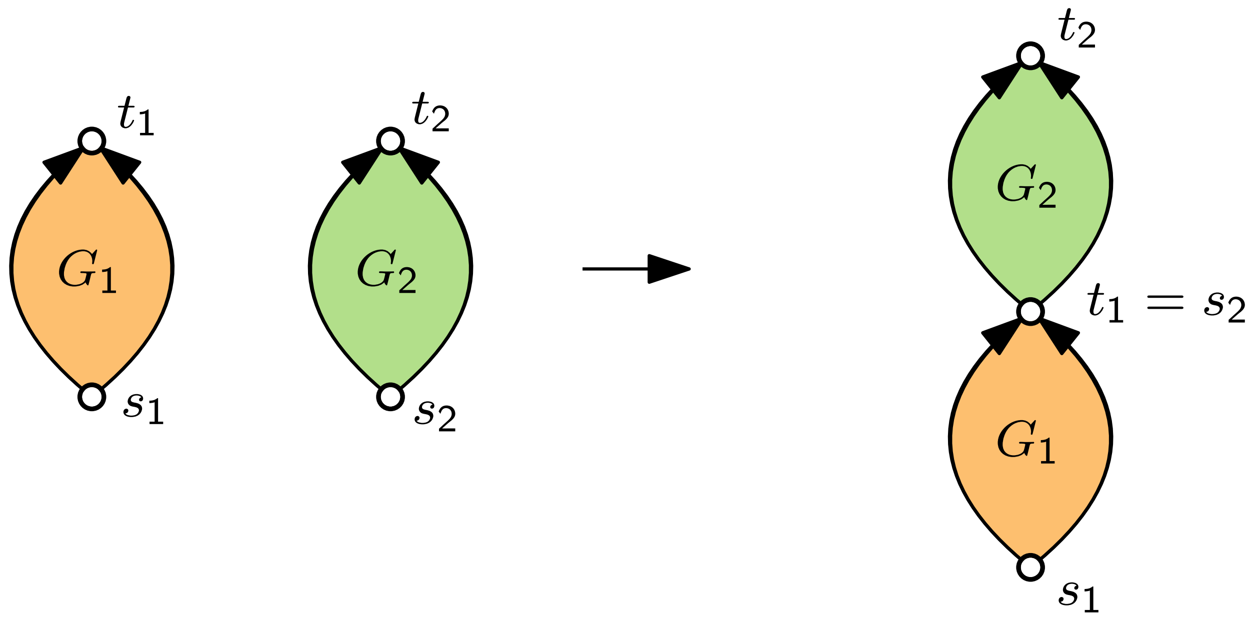


Convince yourself that series-parallel graphs are (upward) planar!

Series composition



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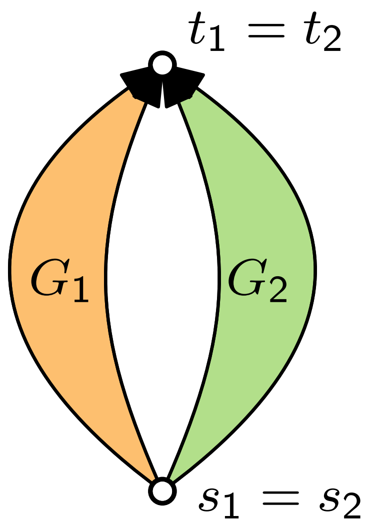

Series-Parallel Graphs – Decomposition Tree

A **decomposition tree** of G is a binary tree T with nodes of three types: **S**, **P** and **Q**.

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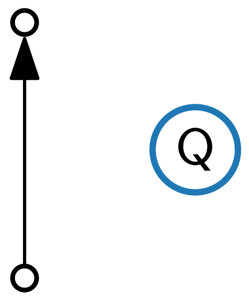
- A **Q**-node represents a single edge.



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A **decomposition tree** of G is a binary tree T with nodes of three types: **S**, **P** and **Q**.

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- An **S**-node represents a series composition;
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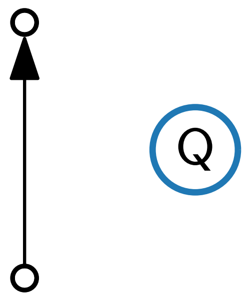

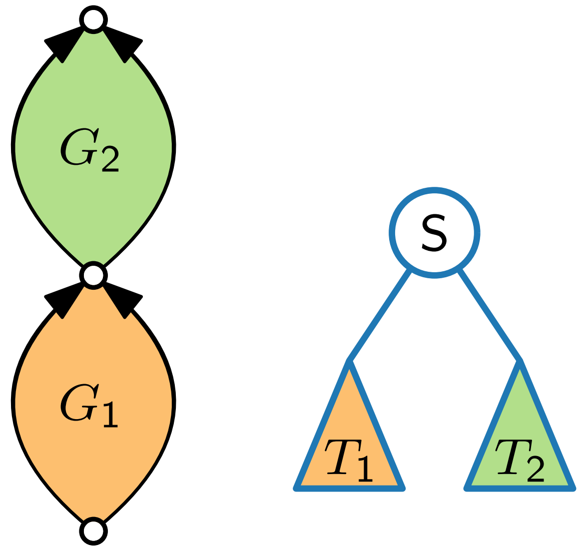

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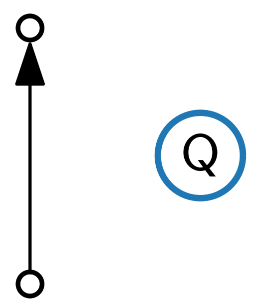
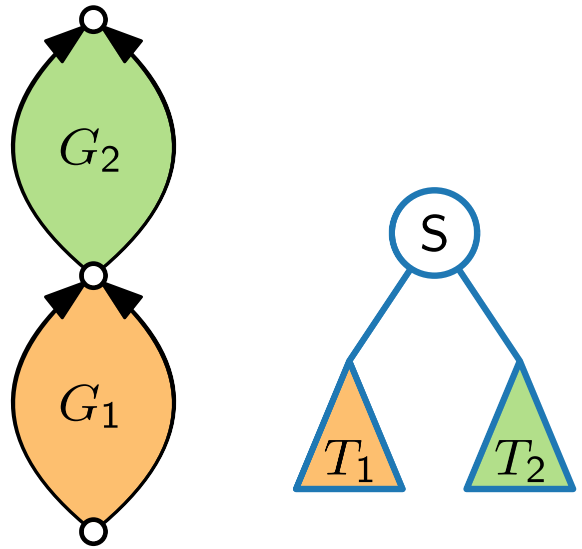
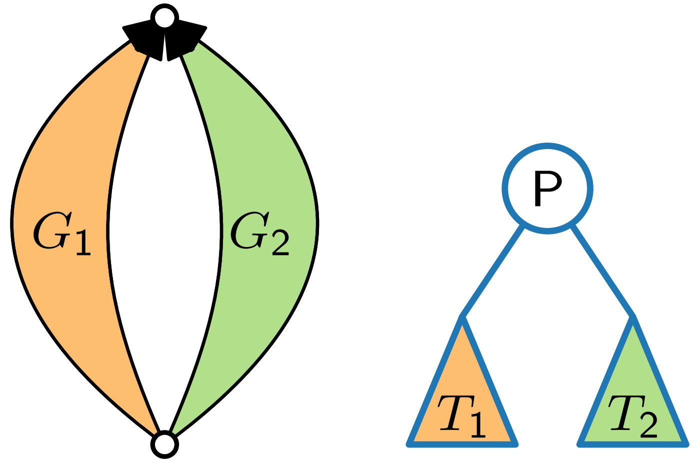
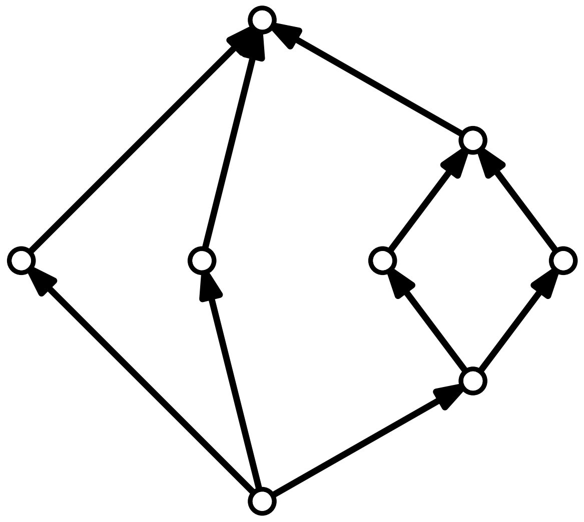
- A **P**-node represents a parallel composition; its children T_1 and T_2 represent G_1 and G_2







Series-Parallel Graphs – Decomposition Example



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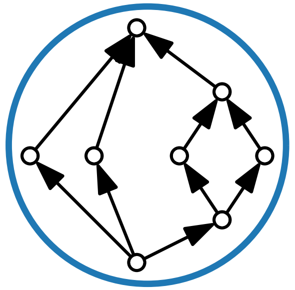

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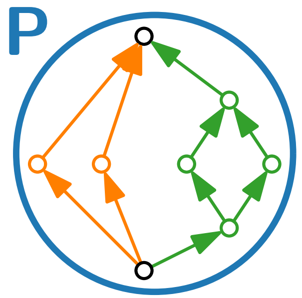

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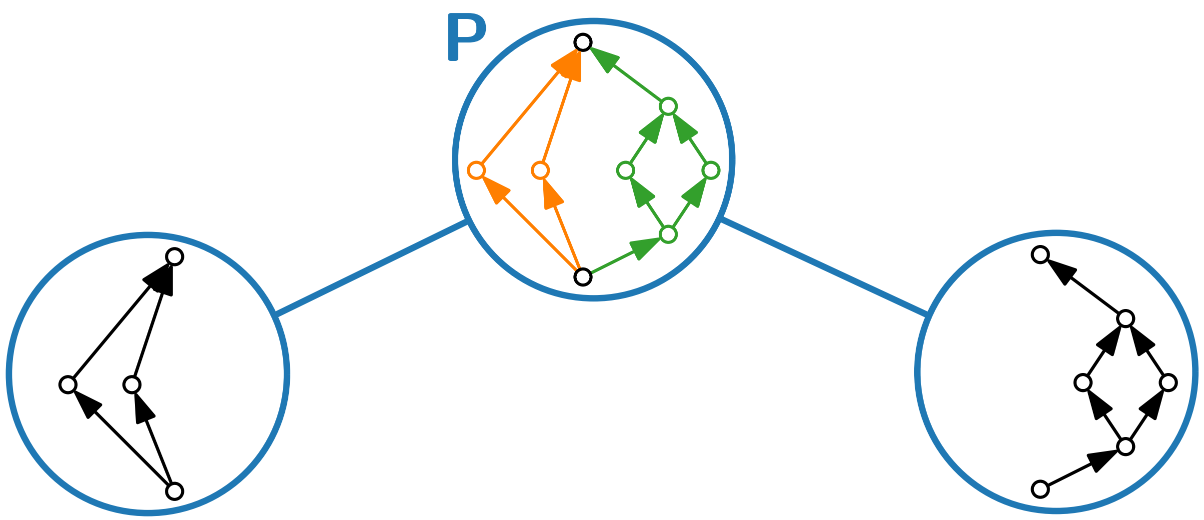

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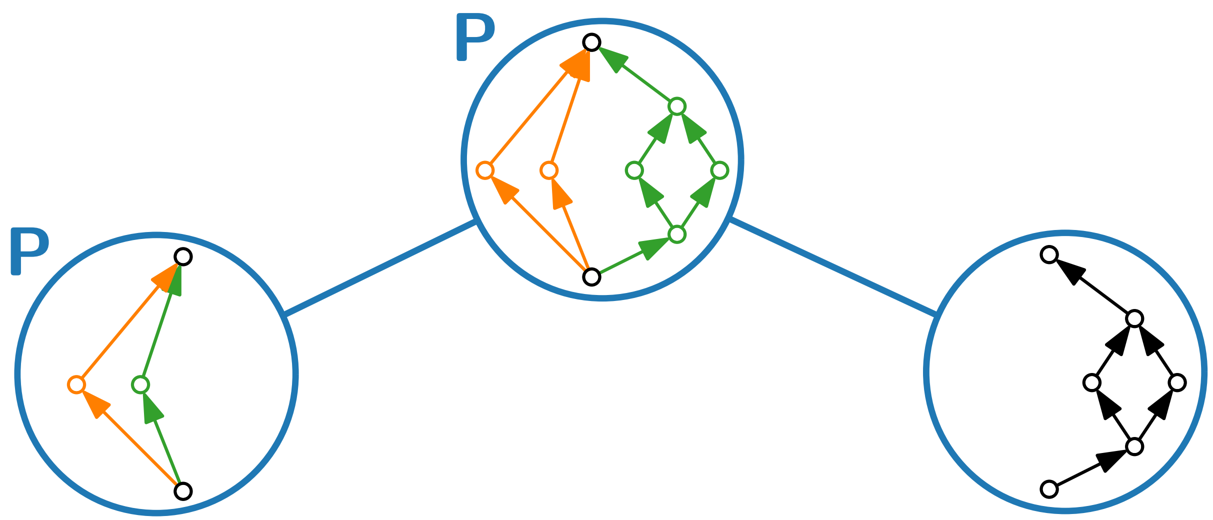

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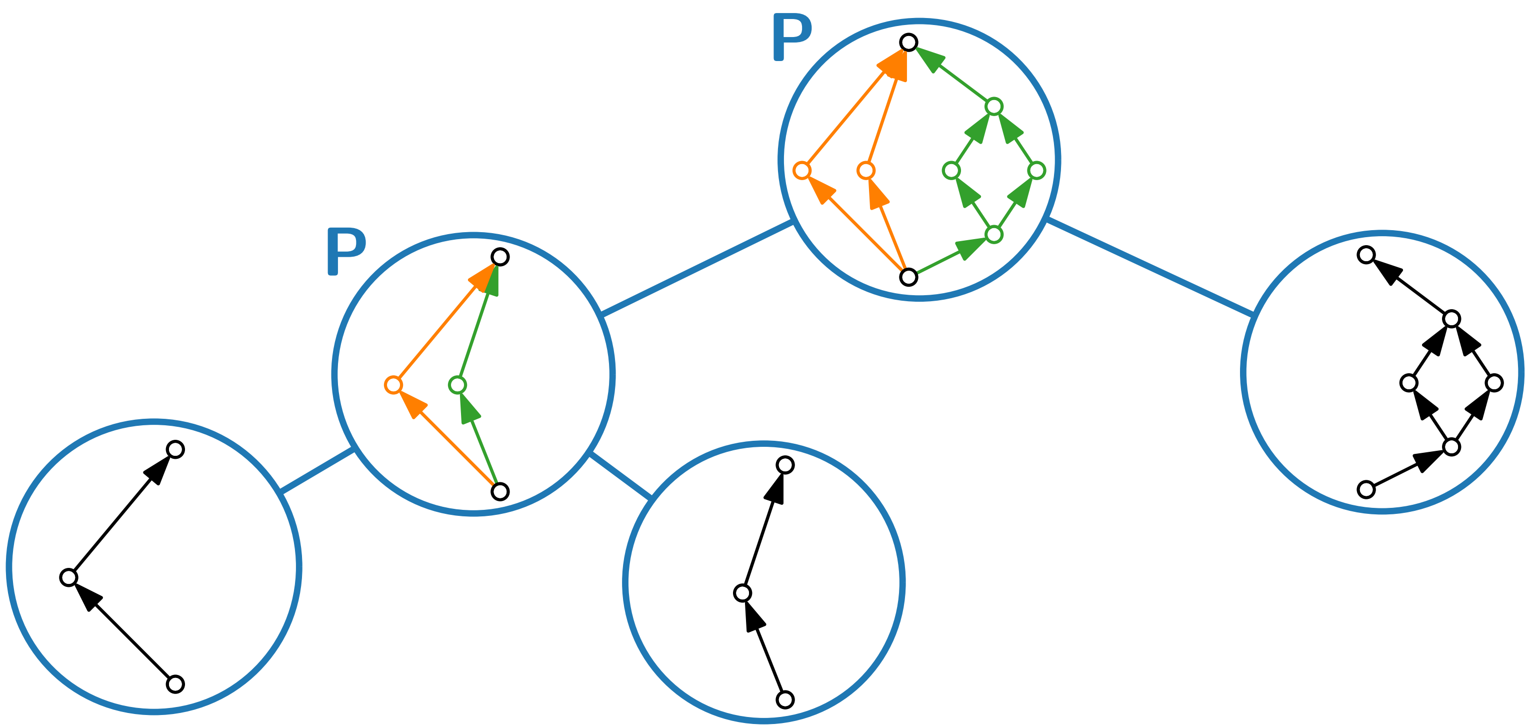

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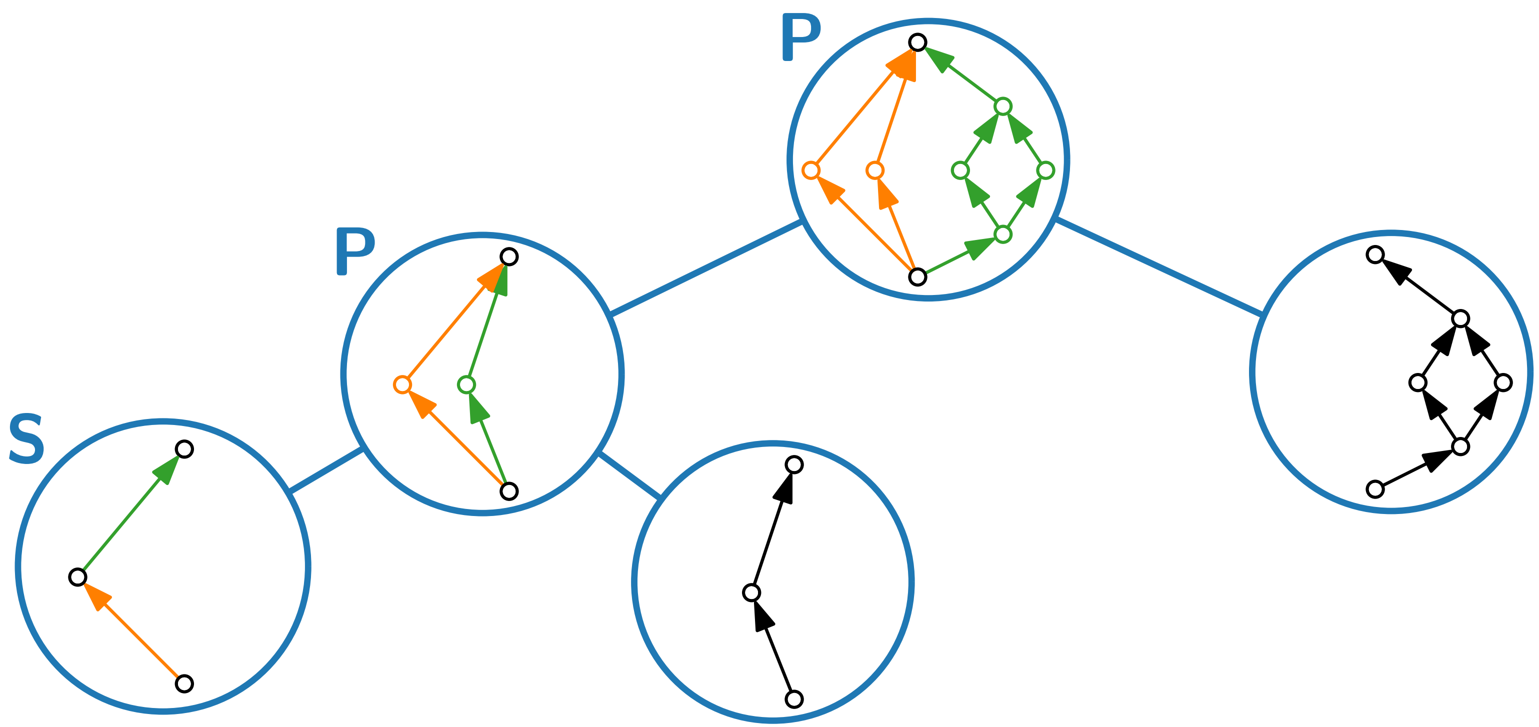

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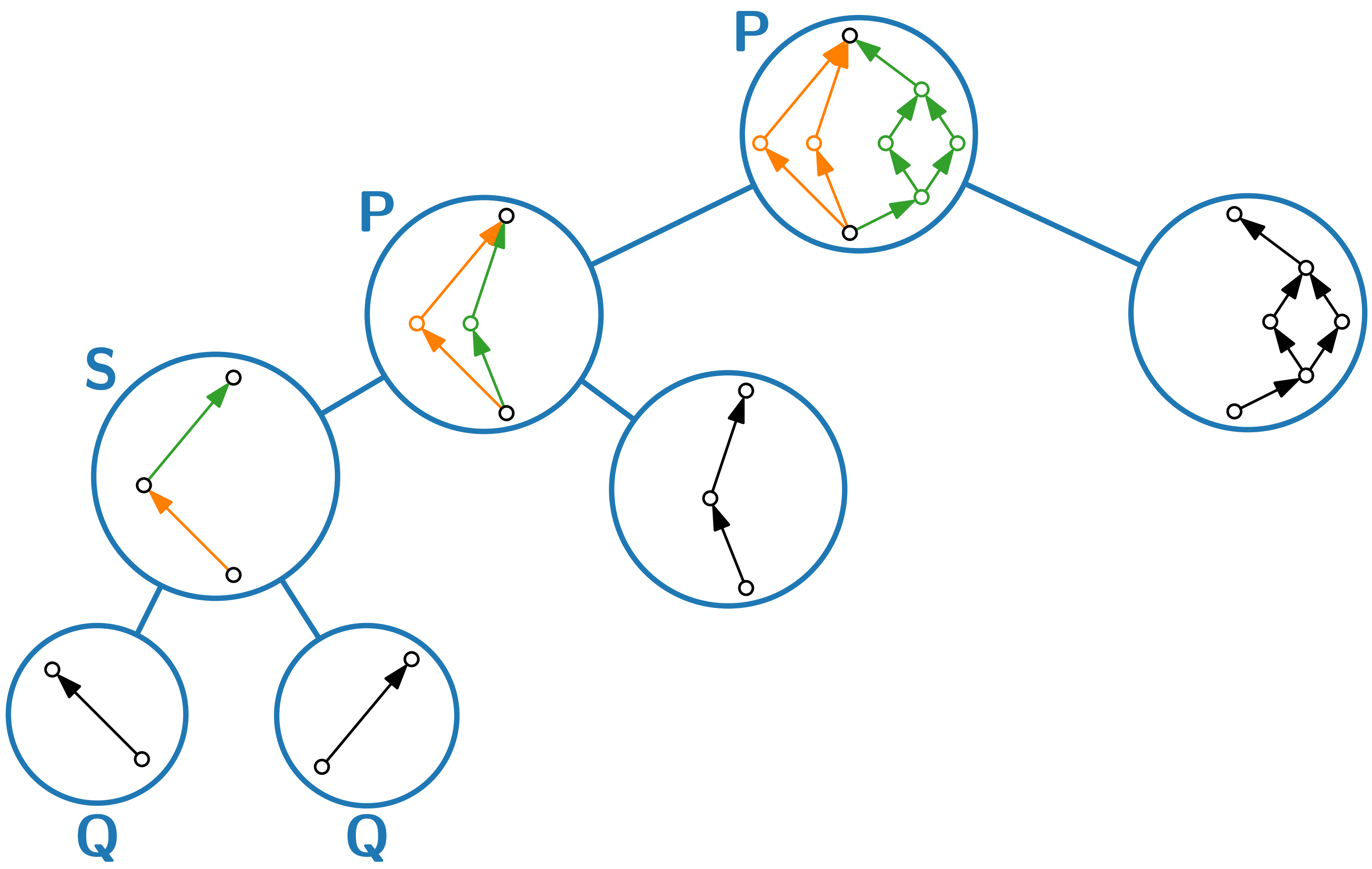

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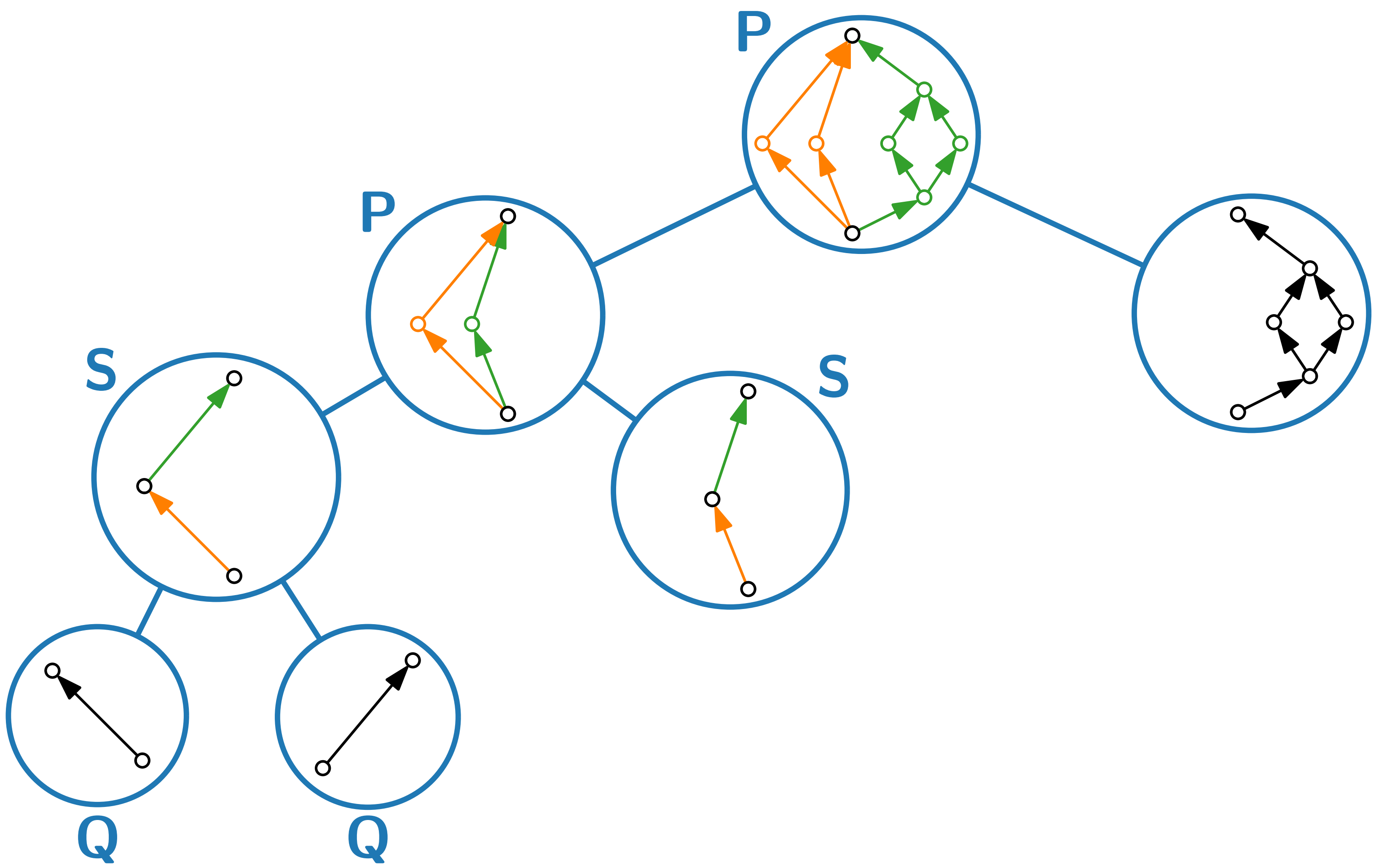

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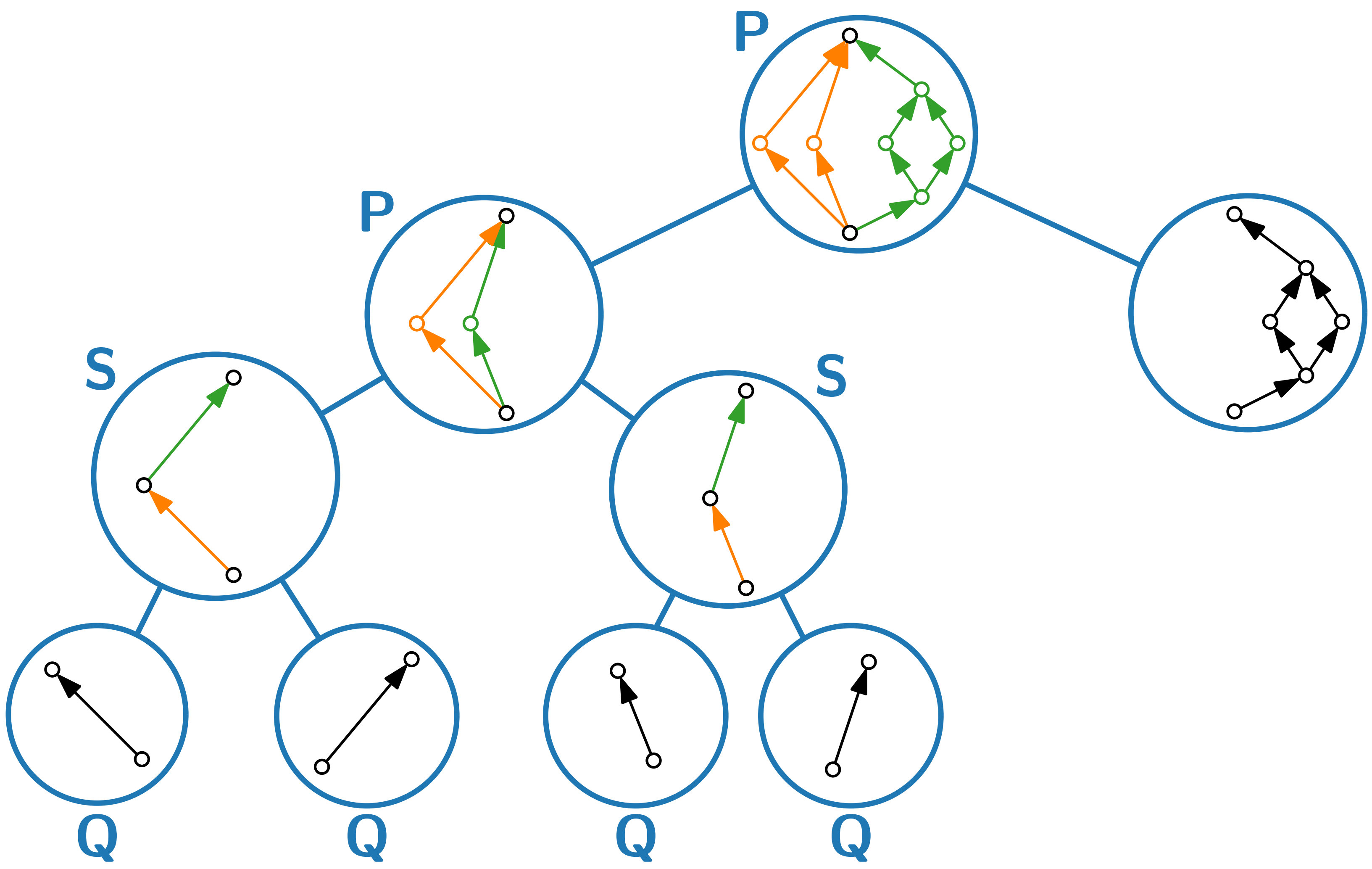

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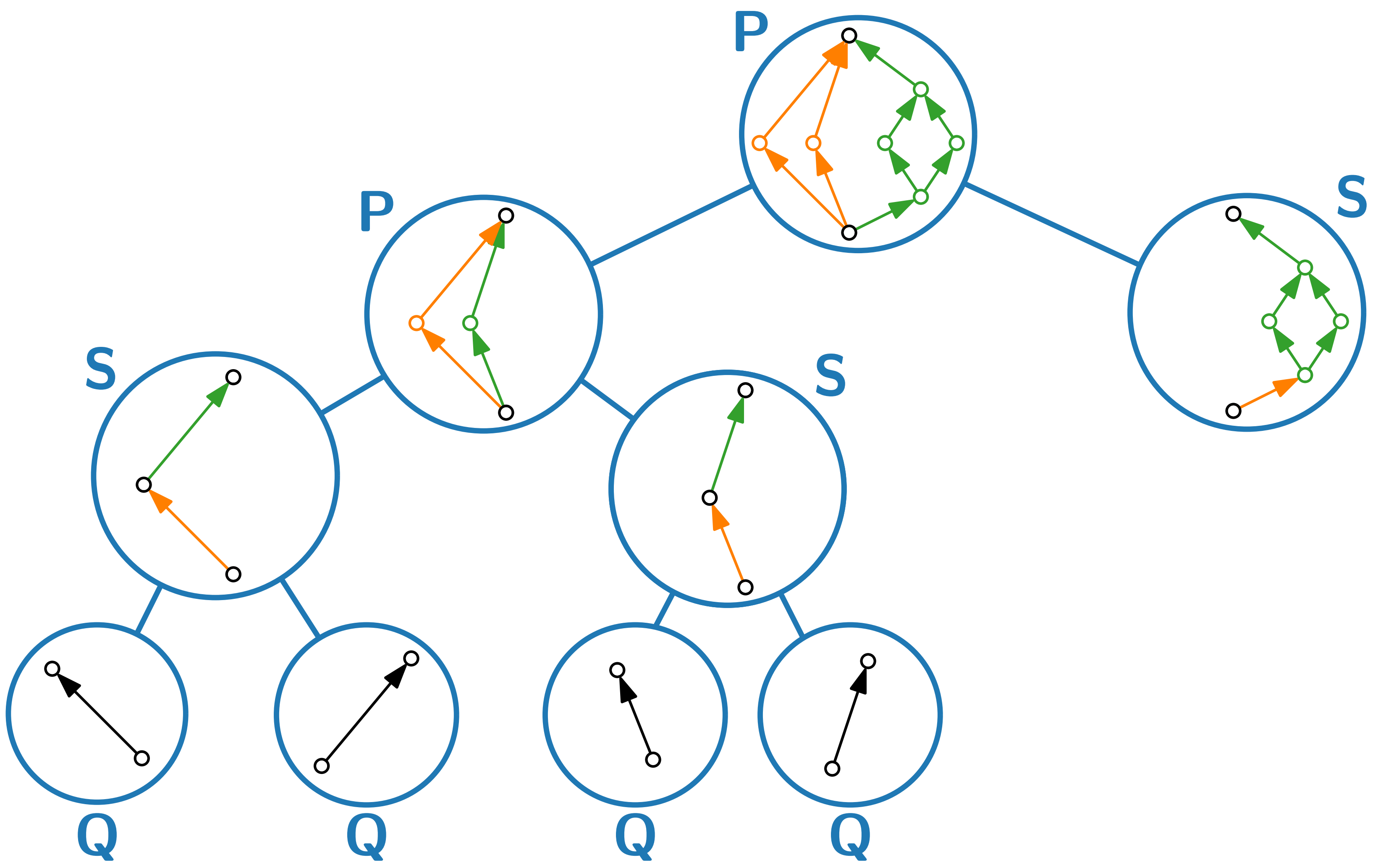

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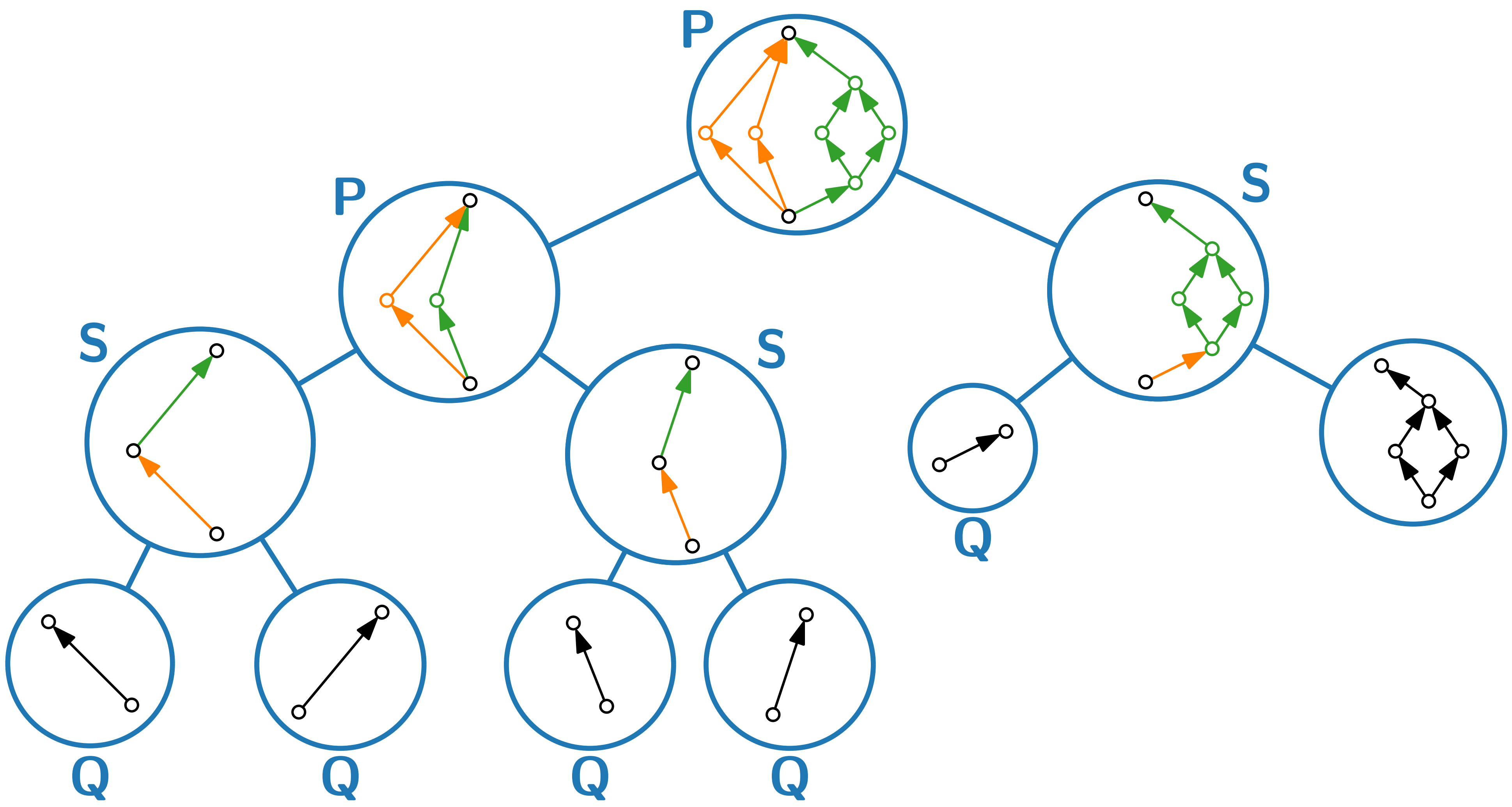

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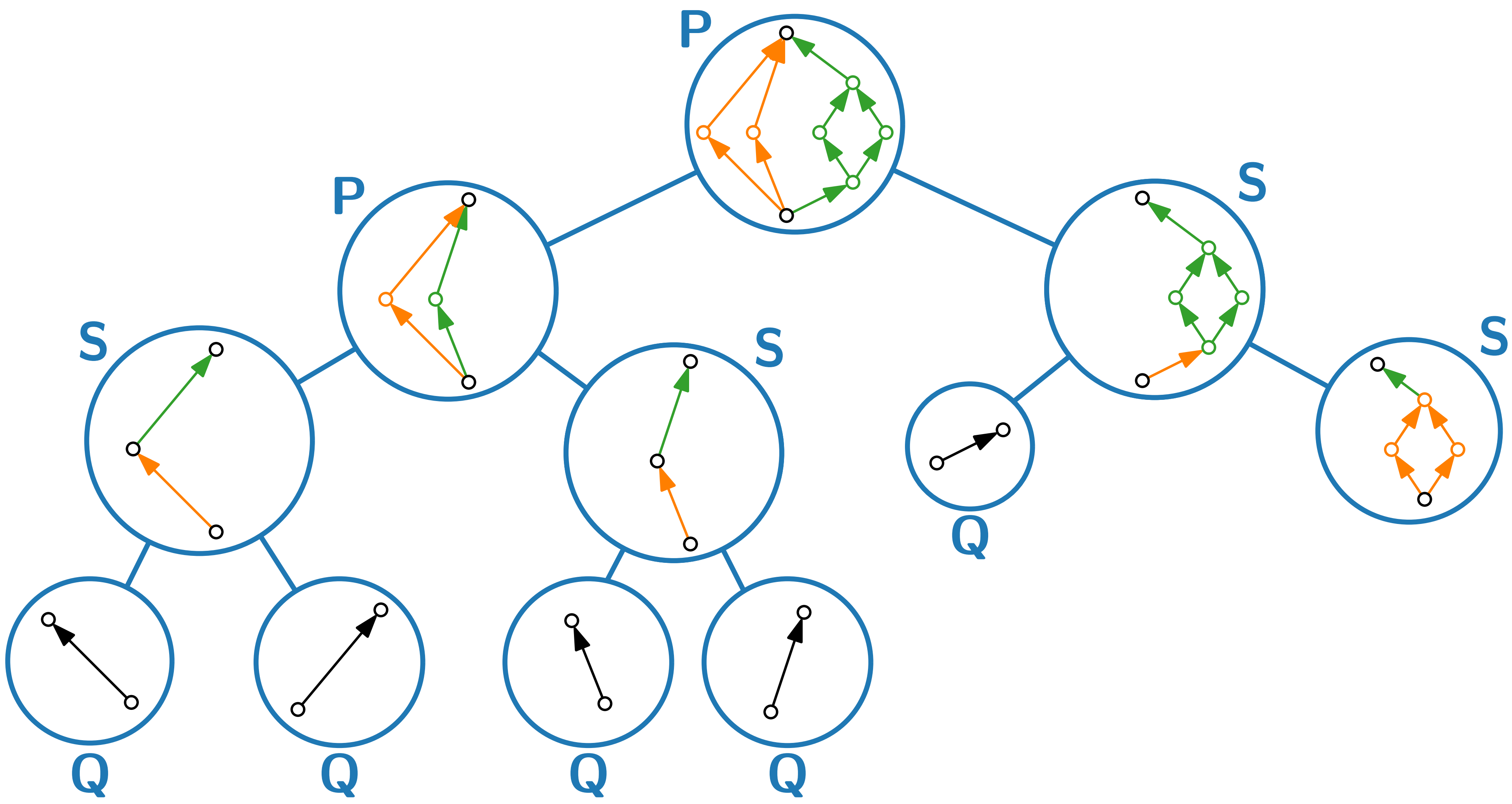

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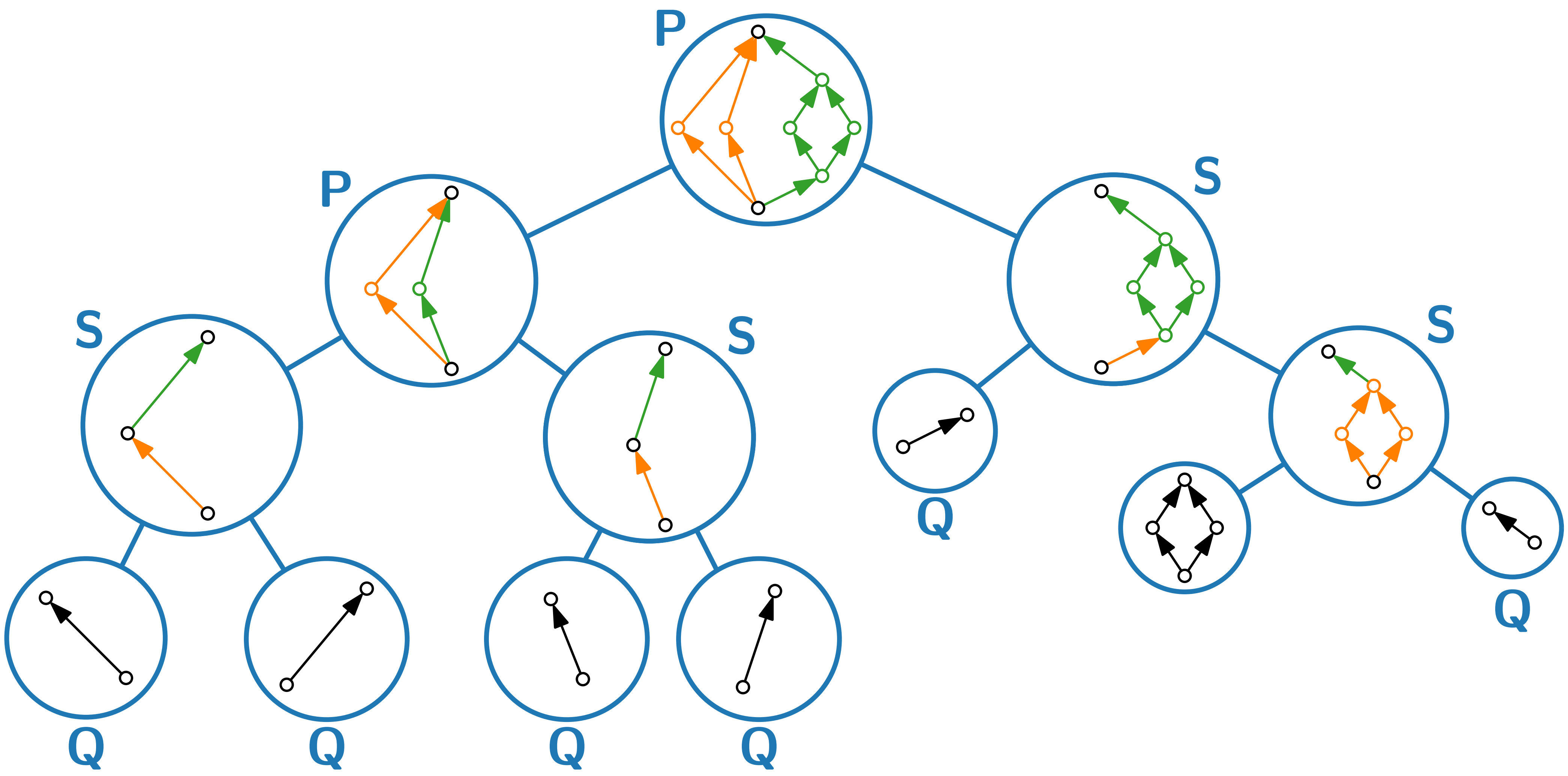

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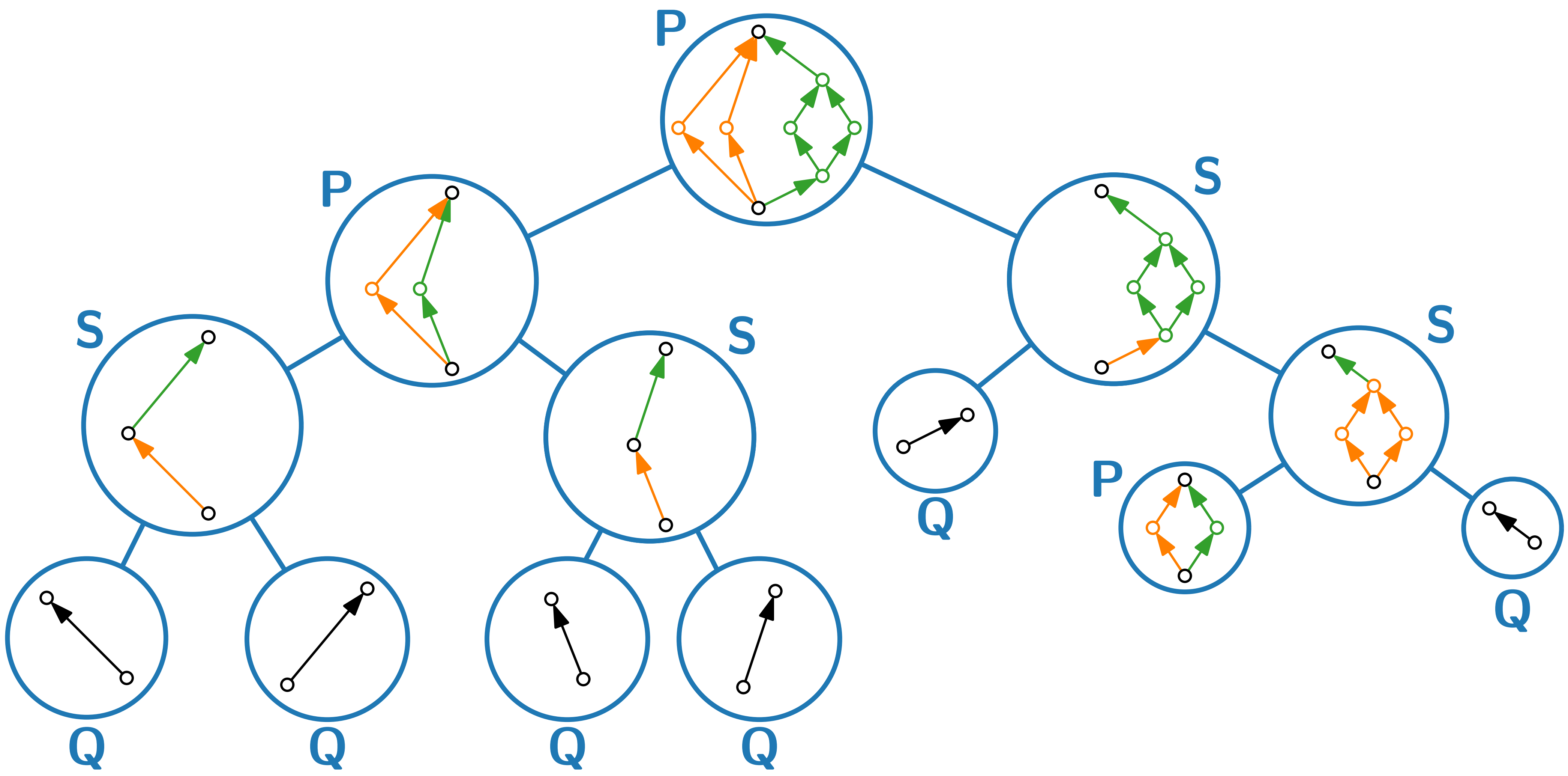

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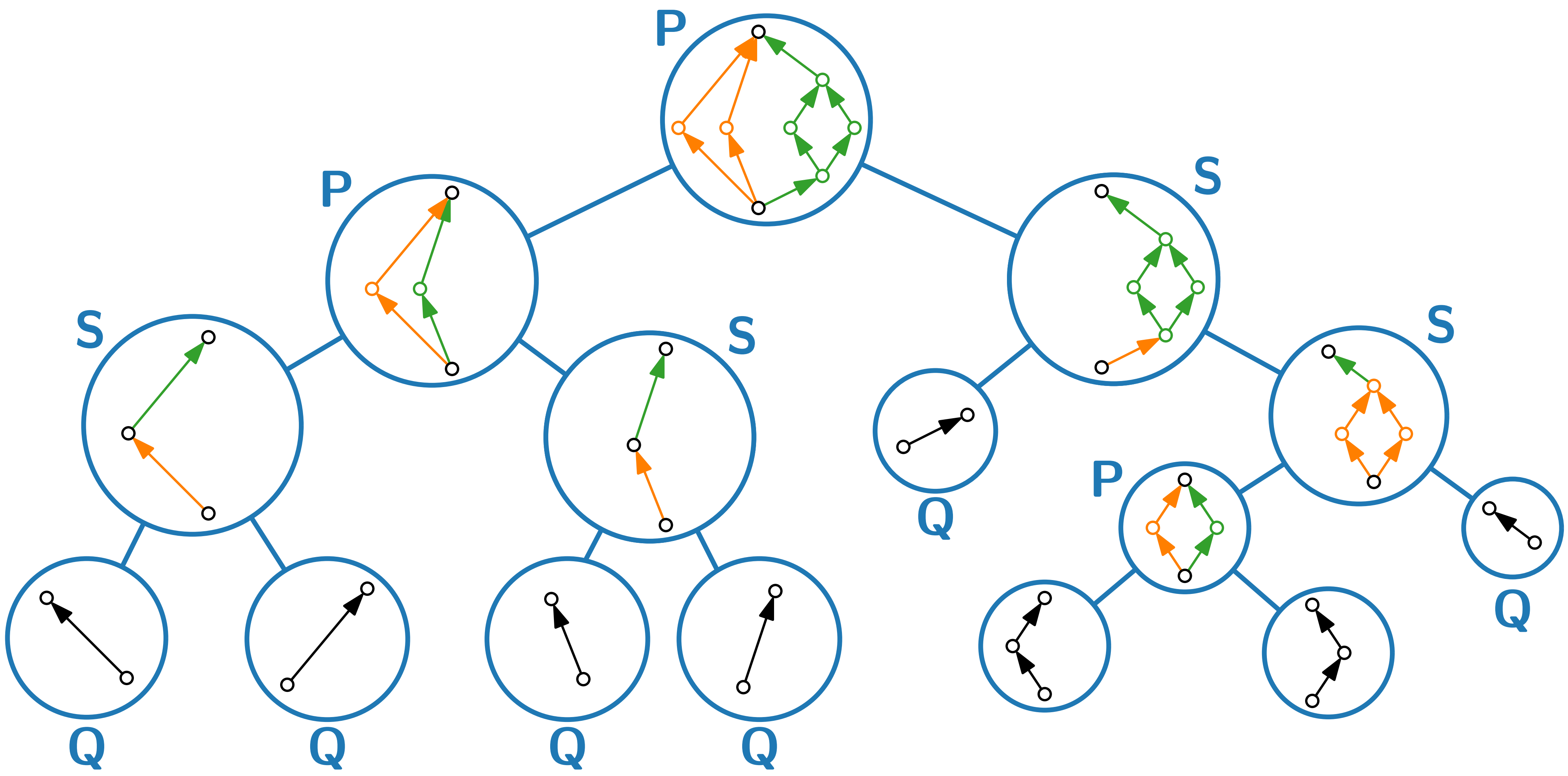

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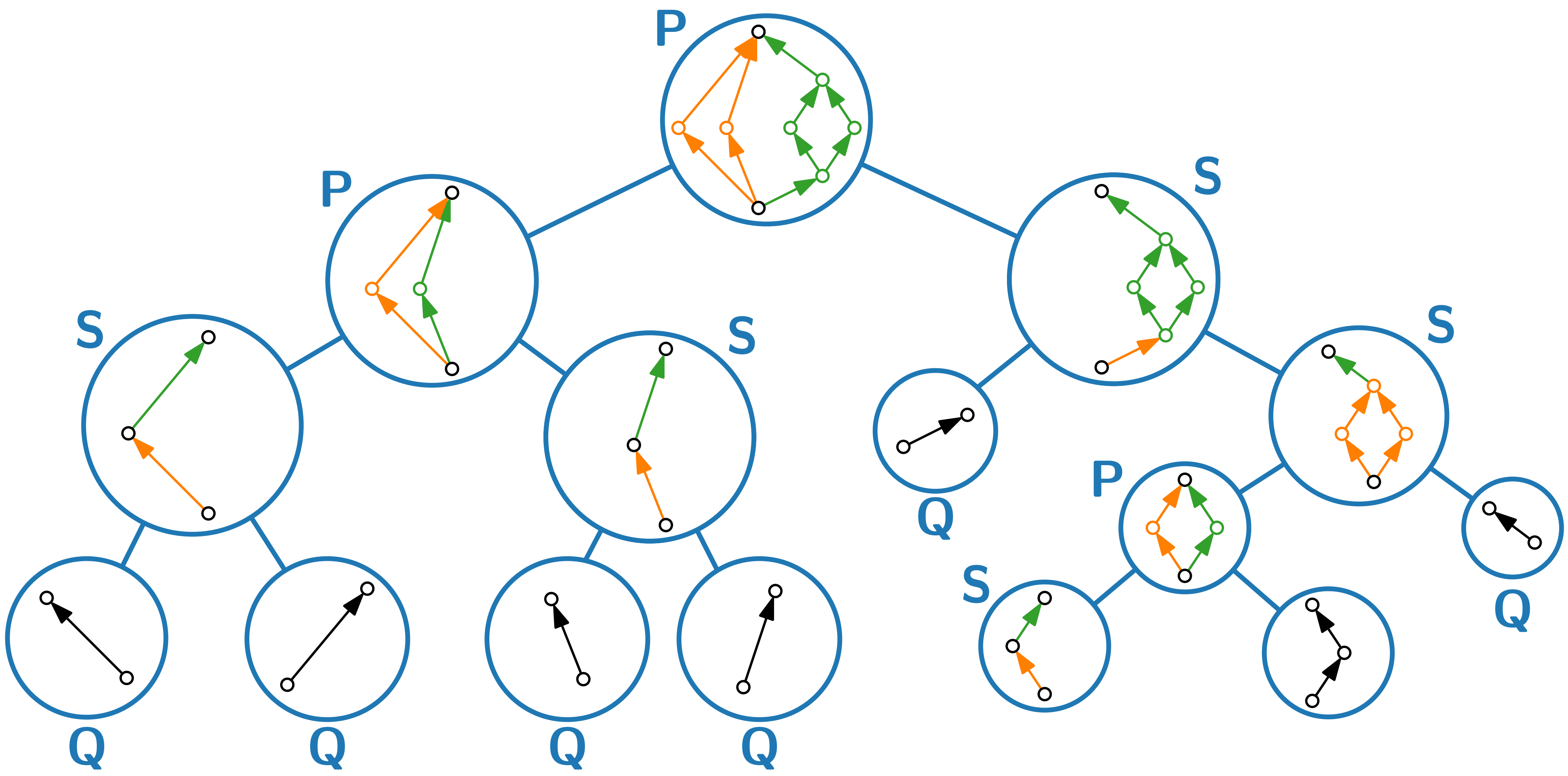

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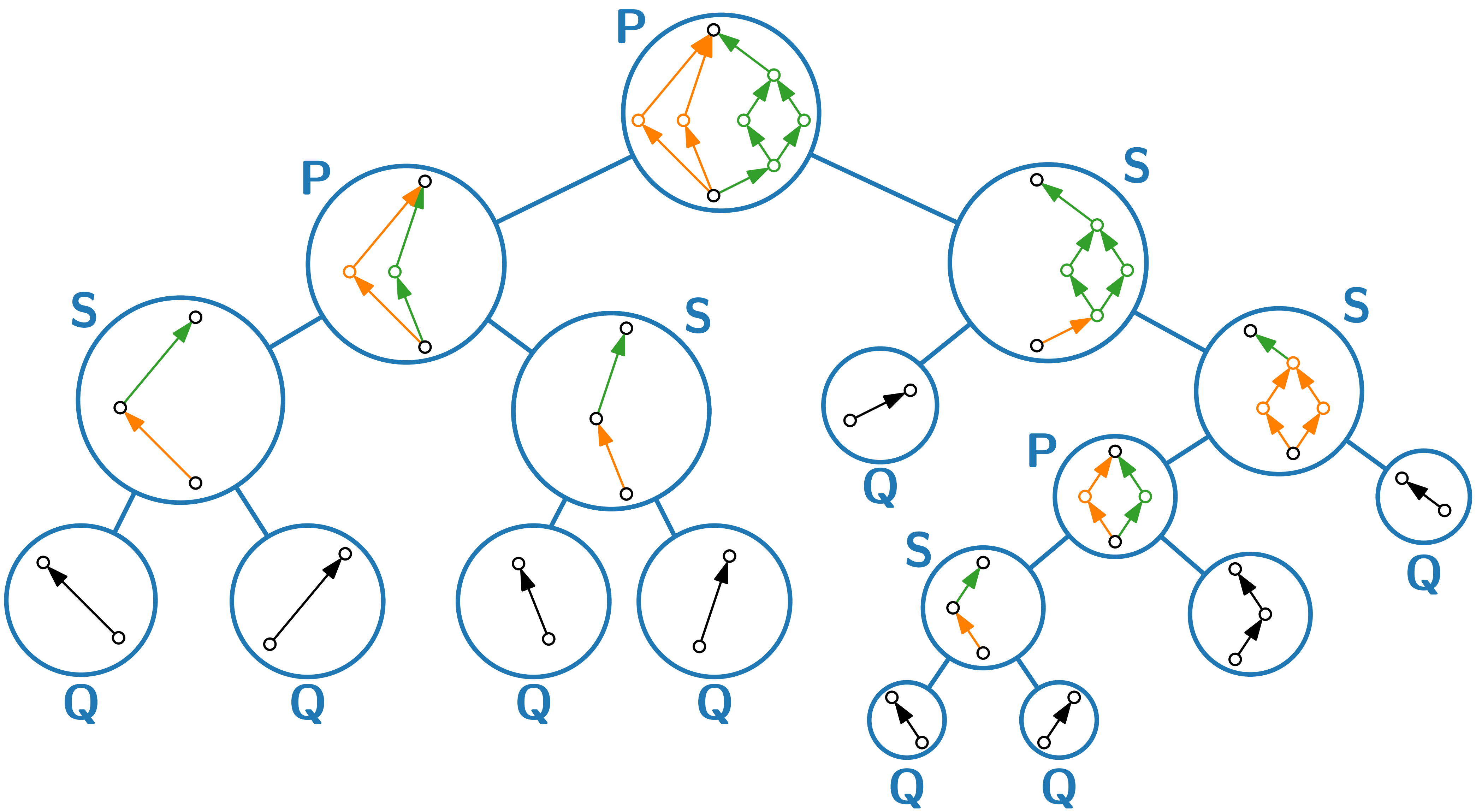

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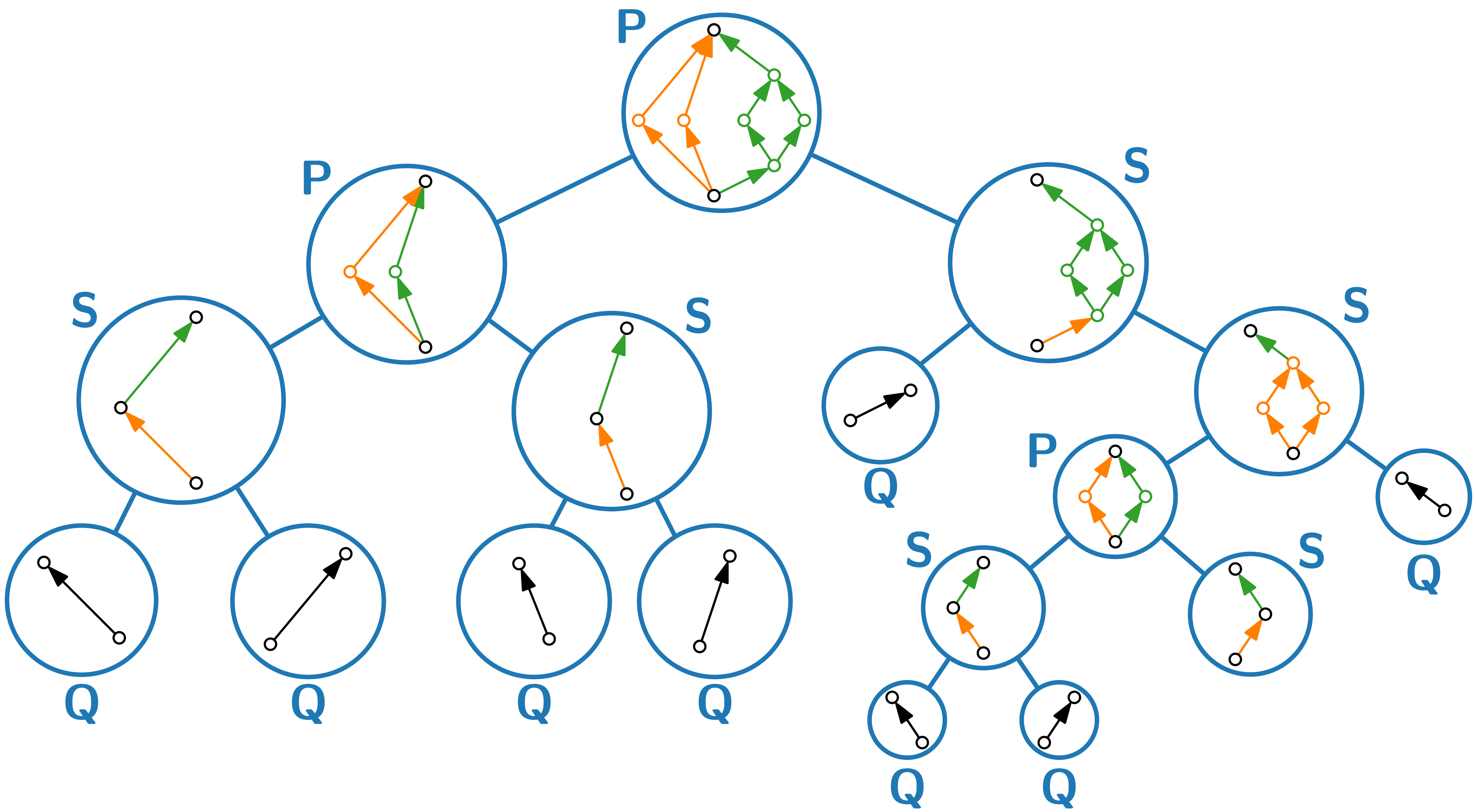

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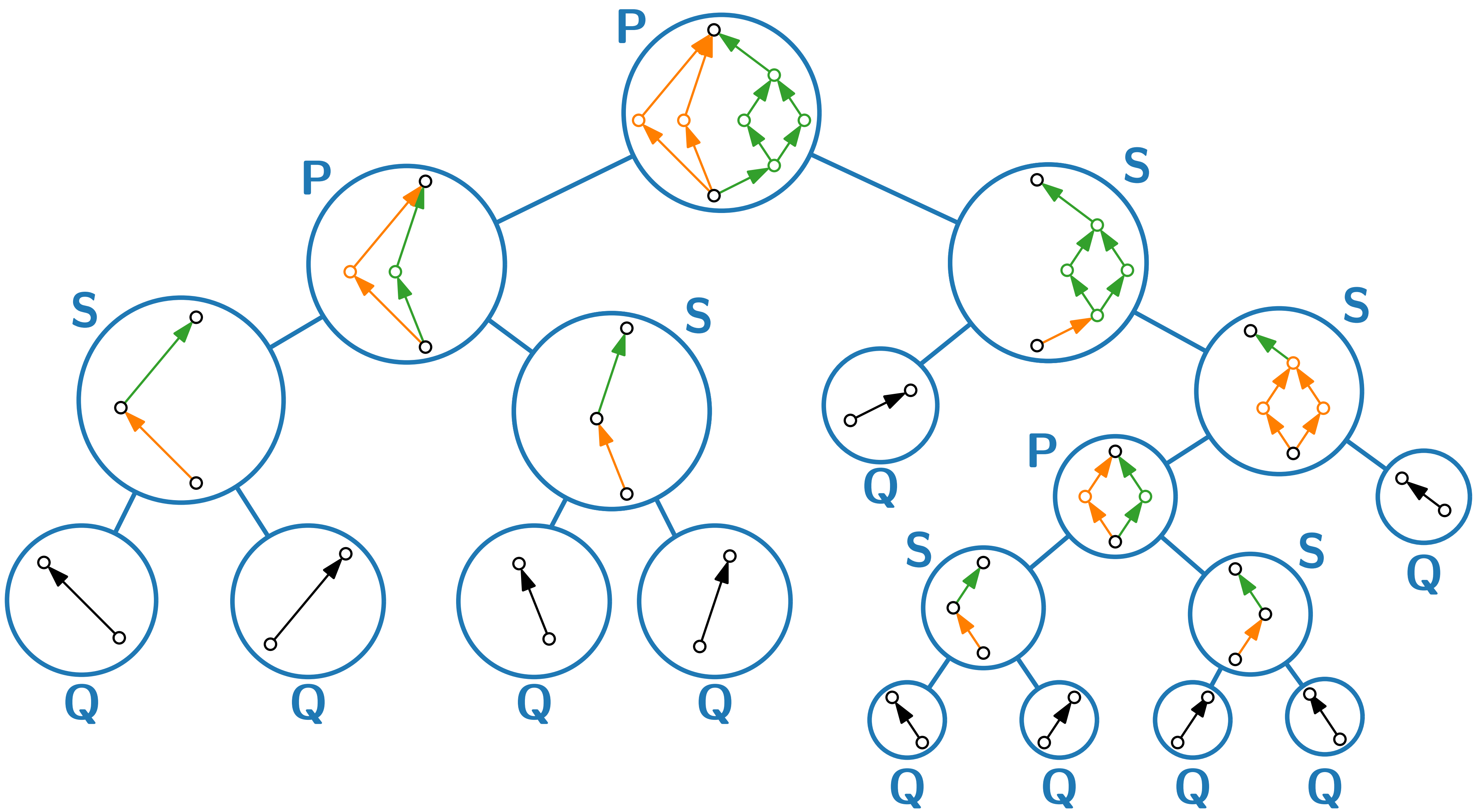
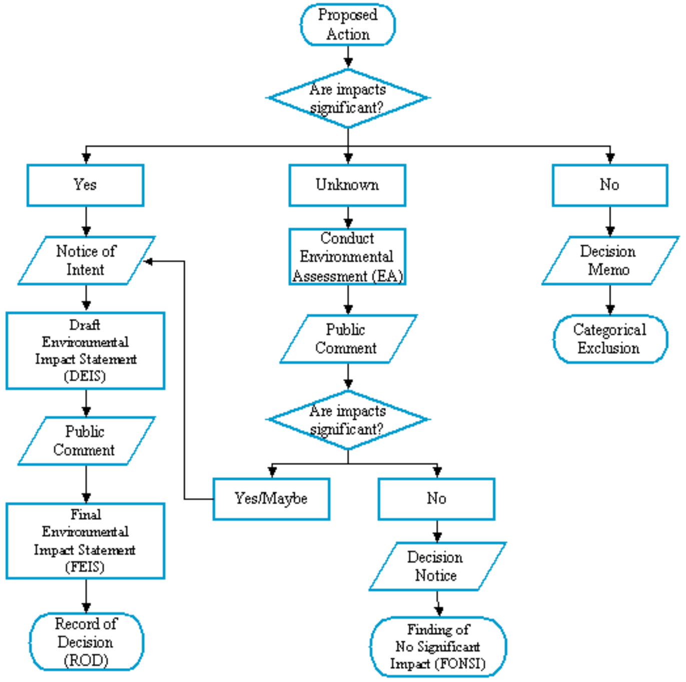
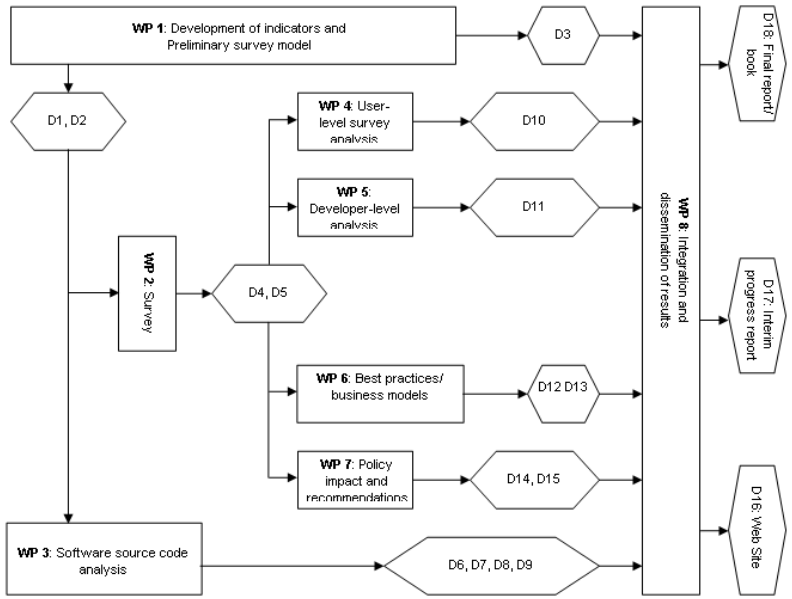


Series-Parallel Graphs – Applications



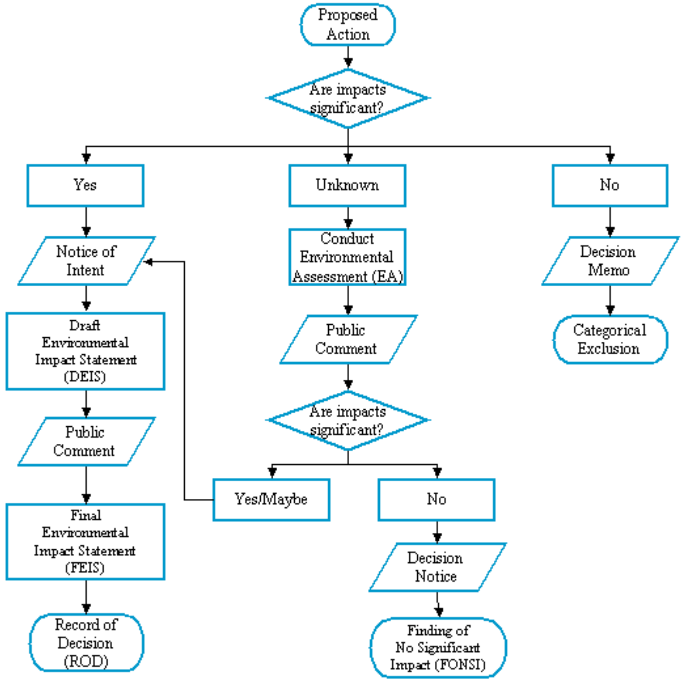
Flowcharts



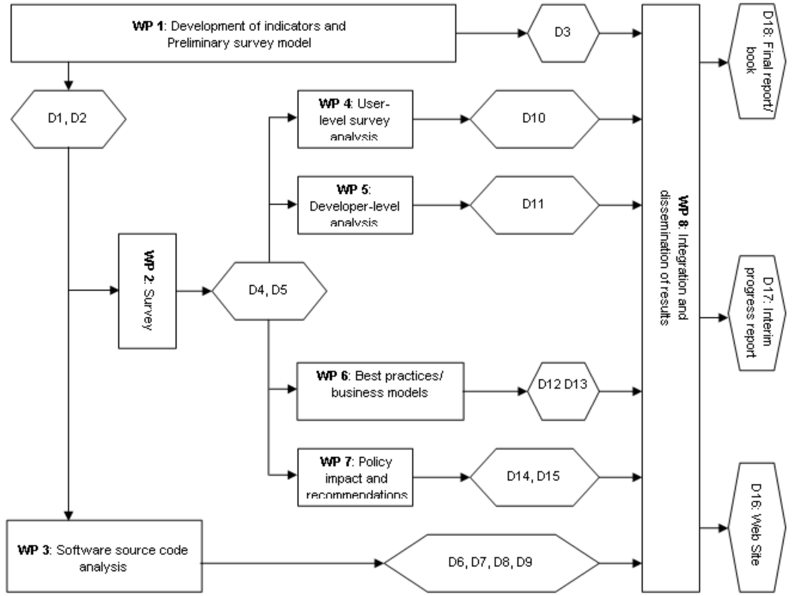
PERT-Diagrams

(Program Evaluation and Review Technique)

Series-Parallel Graphs – Applications



Flowcharts



PERT-Diagrams

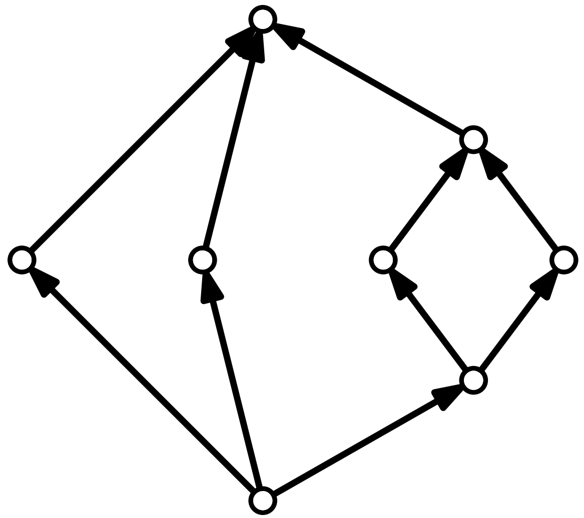
(Program Evaluation and Review Technique)

Computational complexity:

Series-parallel graphs often admit linear-time algorithms for NP-hard problems, e.g., minimum maximal matching, maximum independent set, Hamiltonian completion.

Series-Parallel Graphs – Drawing Style

Drawing conventions

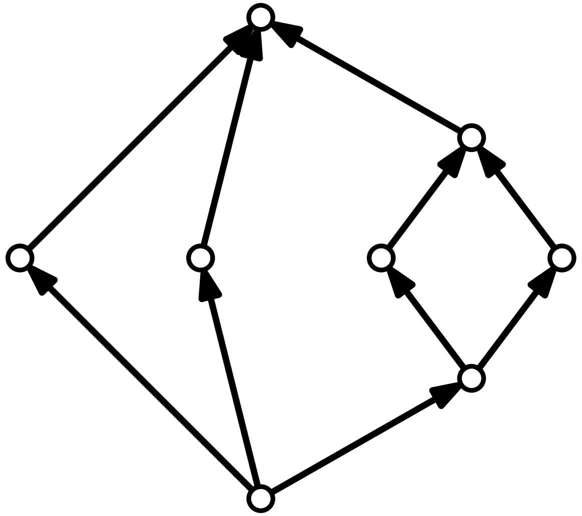


Drawing aesthetics to optimize

Series-Parallel Graphs – Drawing Style

Drawing conventions

- Planarity



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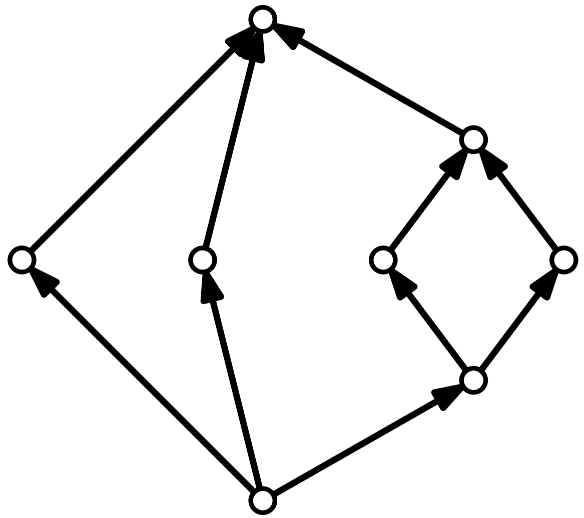
Series-Parallel Graphs – Drawing Style

Drawing conventions

- Planarity
- Straight-line edges



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Series-Parallel Graphs – Drawing Style

Drawing conventions

- Planarity
- Straight-line edges
- Upward

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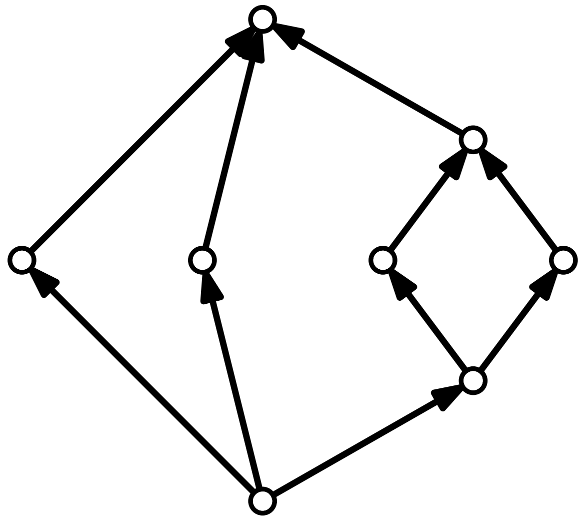


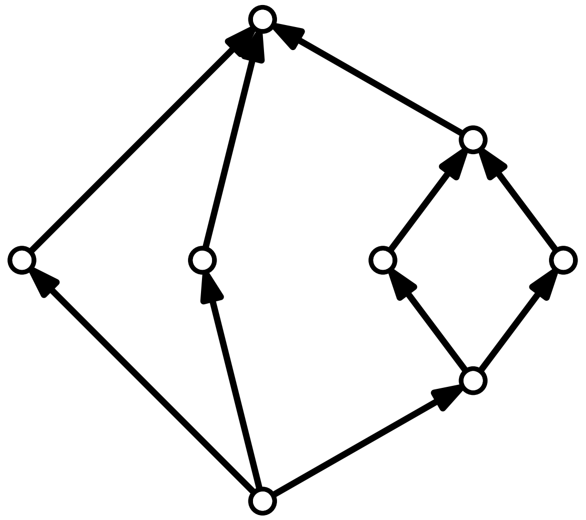
Series-Parallel Graphs – Drawing Style

Drawing conventions

- Planarity
- Straight-line edges
- Upward

Drawing aesthetics to optimize

- Area



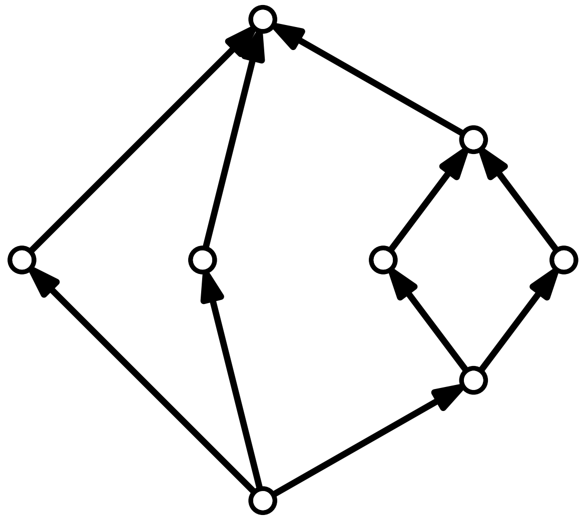
Series-Parallel Graphs – Drawing Style

Drawing conventions

- Planarity
- Straight-line edges
- Upward

Drawing aesthetics to optimize

- Area
- Symmetry



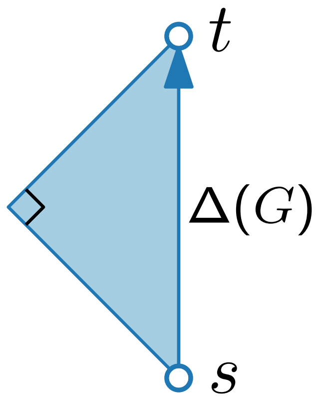
Series-Parallel Graphs – Straight-Line Drawings

Divide & conquer algorithm using the decomposition tree

Series-Parallel Graphs – Straight-Line Drawings

Divide & conquer algorithm using the decomposition tree

- Invariant: draw G inside a right-angled isosceles bounding triangle $\Delta(G)$ with s at the bottom and t at the top



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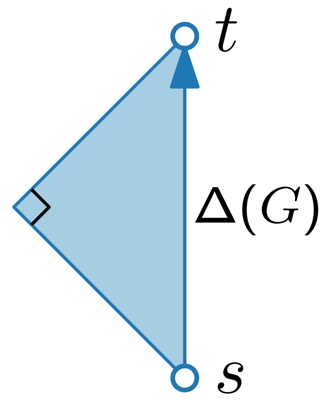

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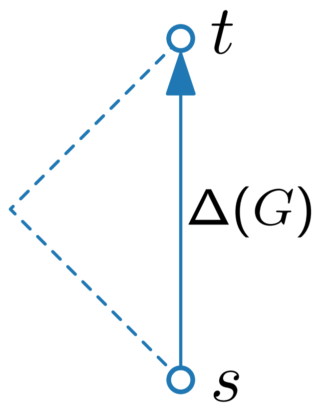

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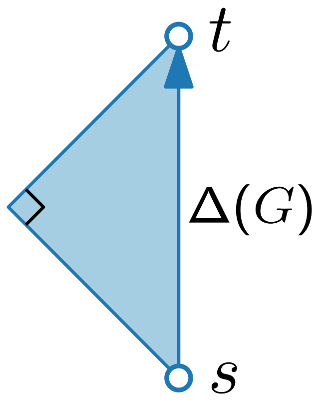

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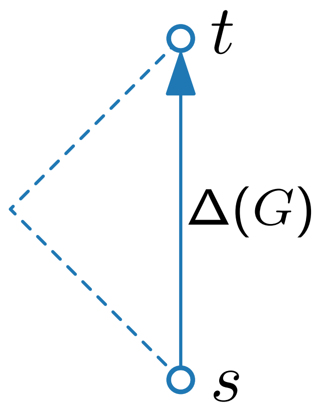

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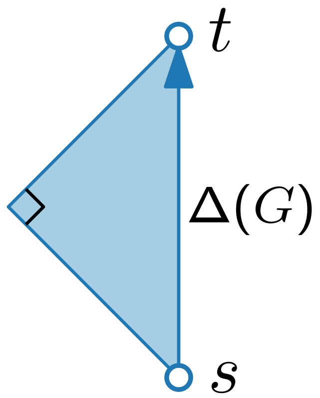

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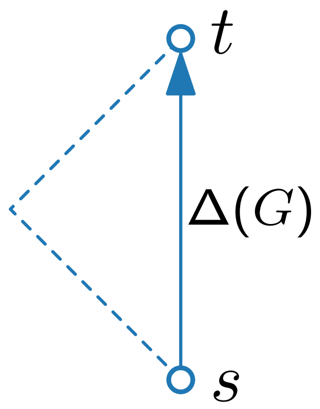

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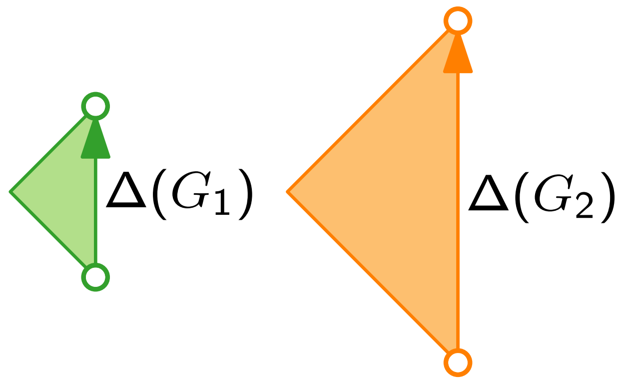

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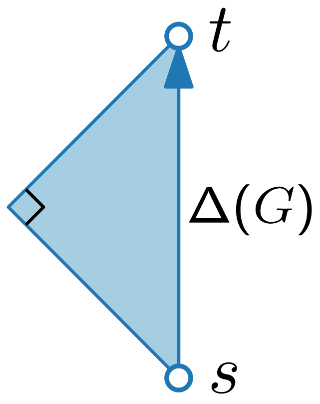

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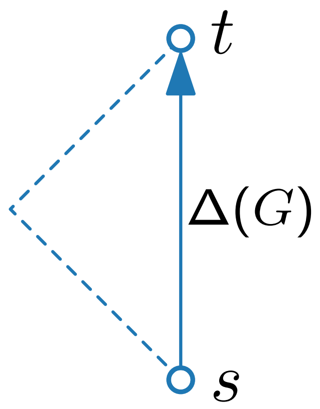

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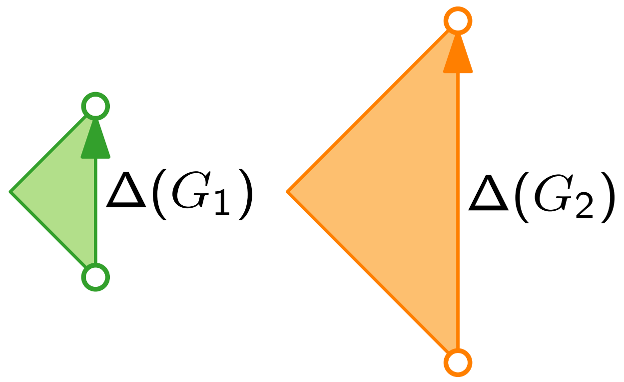

Conquer:

Series-Parallel Graphs – Straight-Line Drawings

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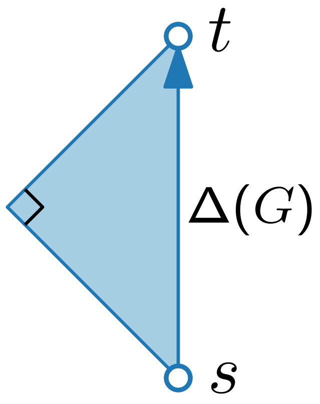

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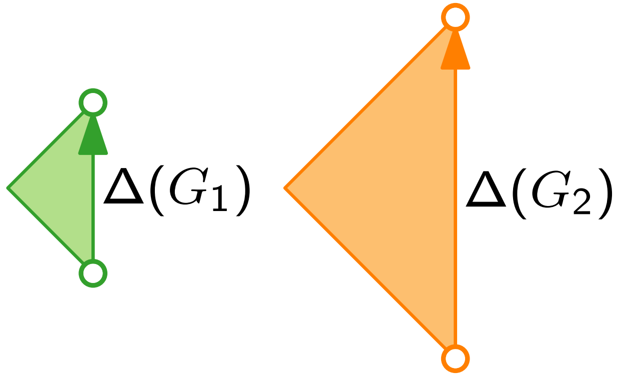
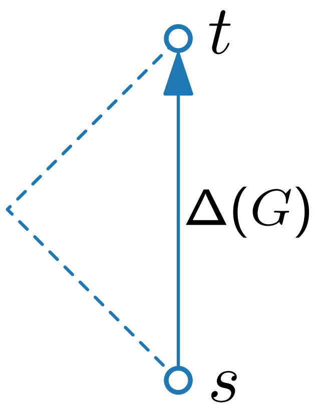


Conquer:

- S-nodes: series compositions



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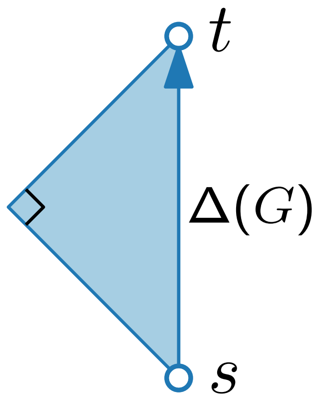

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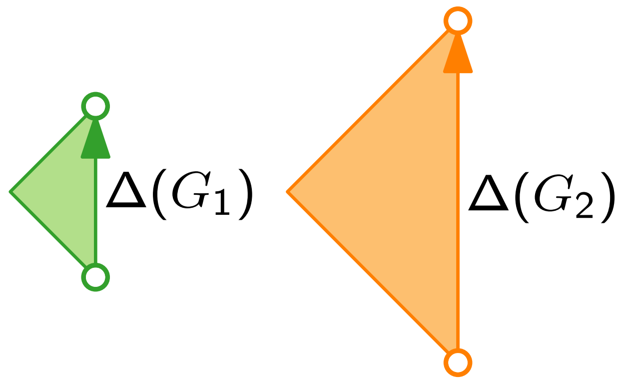

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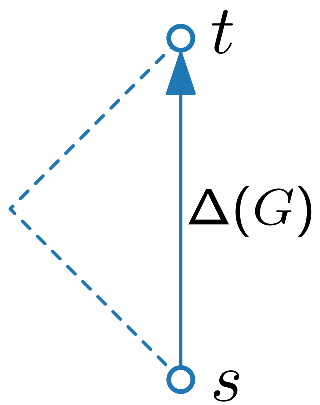

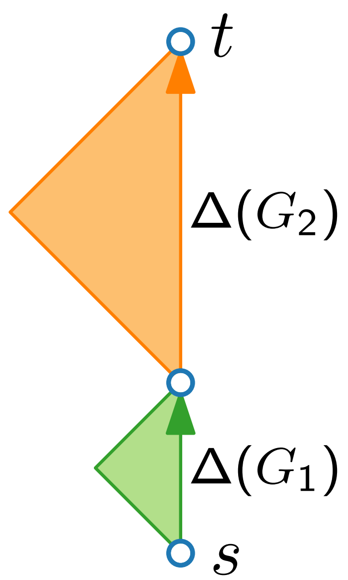

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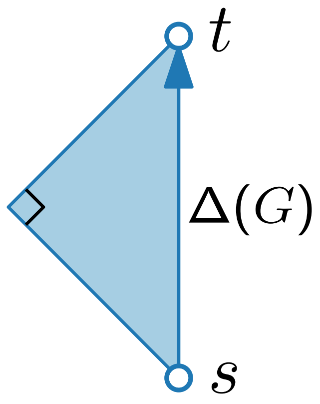

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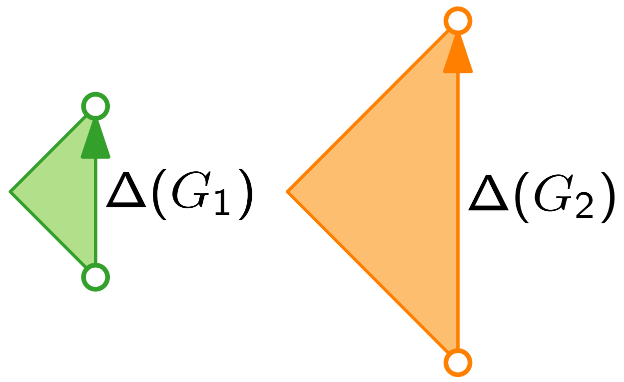

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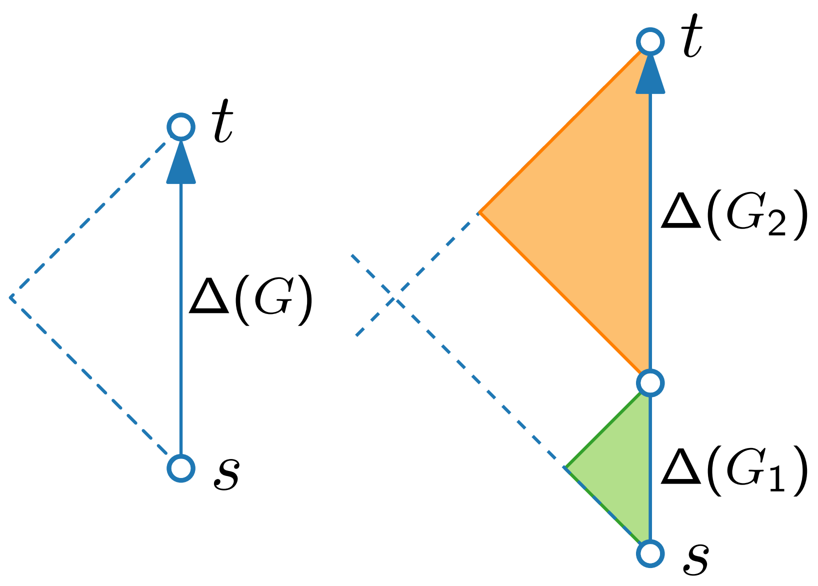

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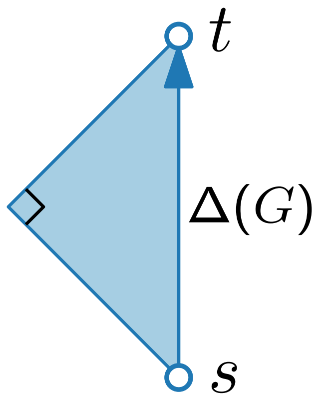

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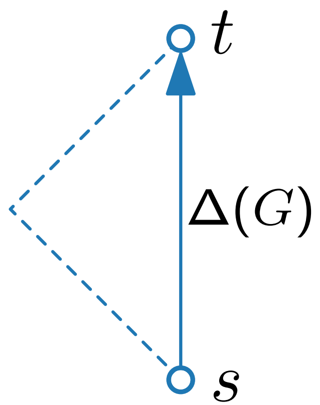

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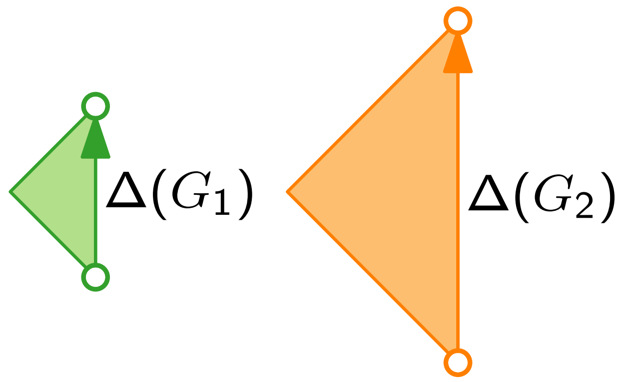

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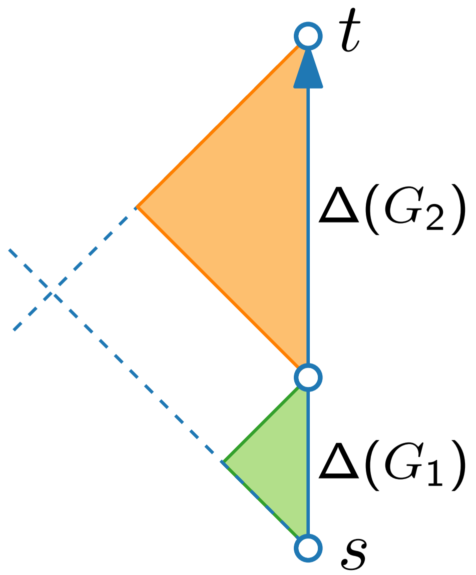

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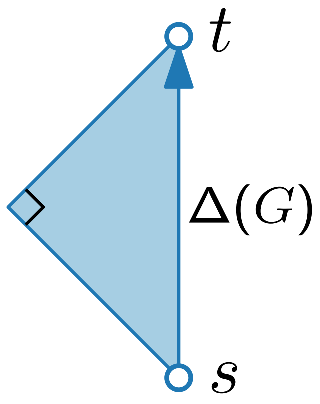

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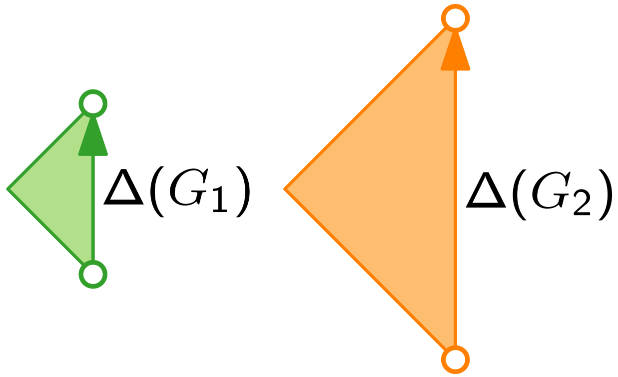

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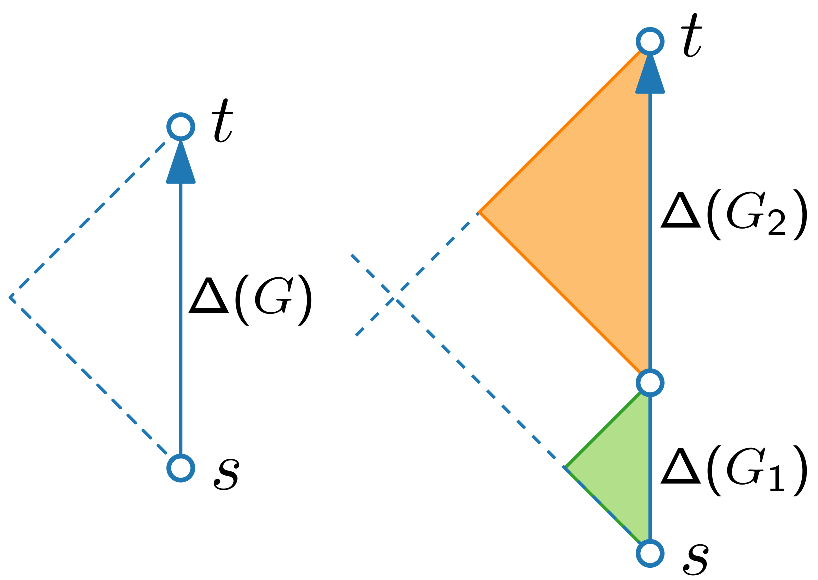

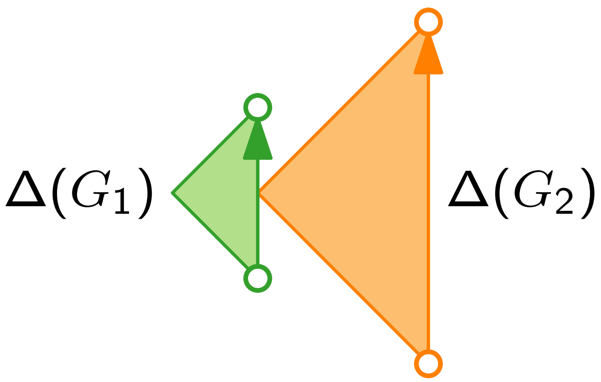

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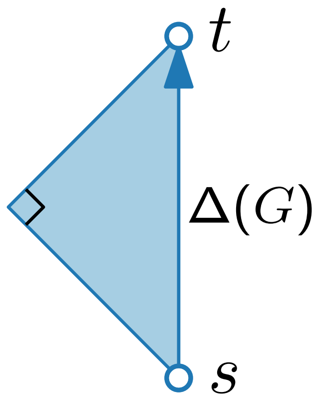

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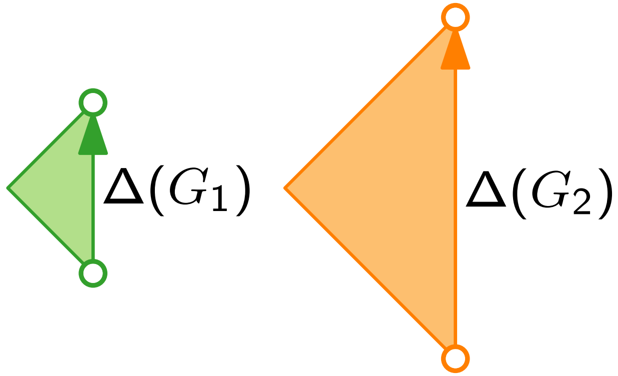

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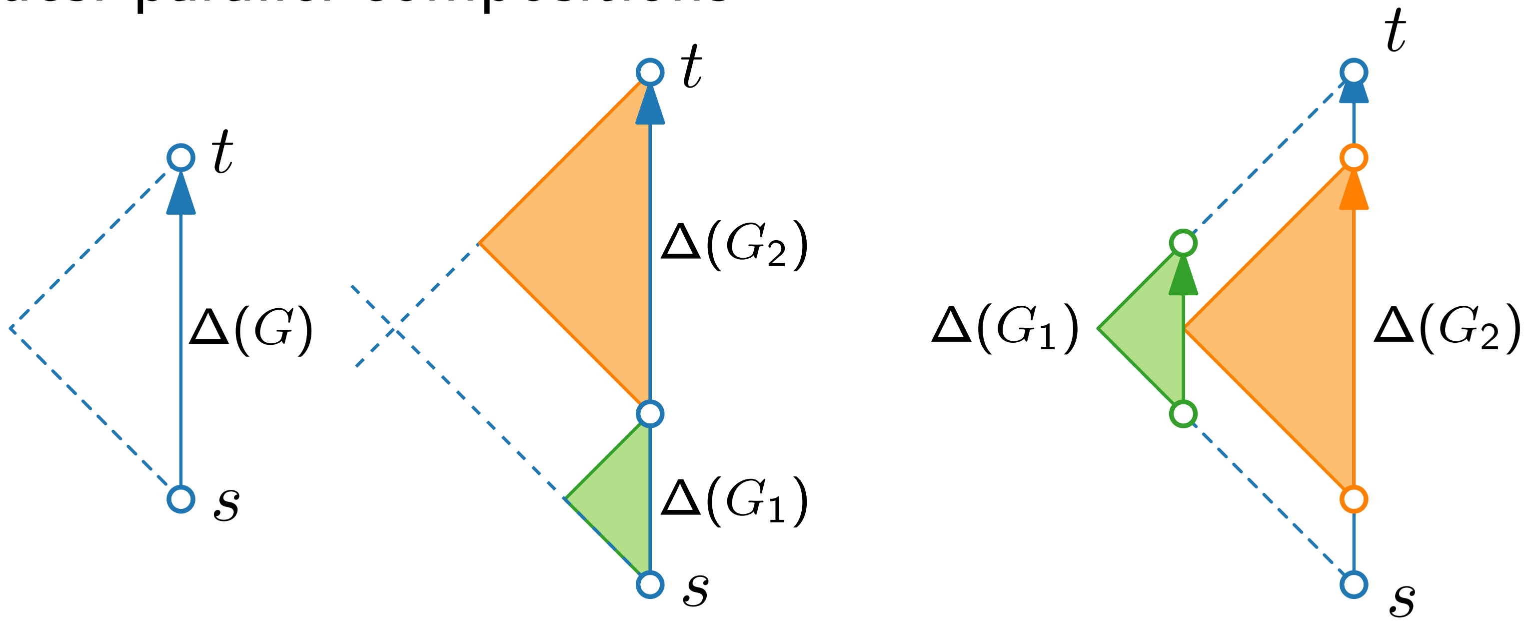

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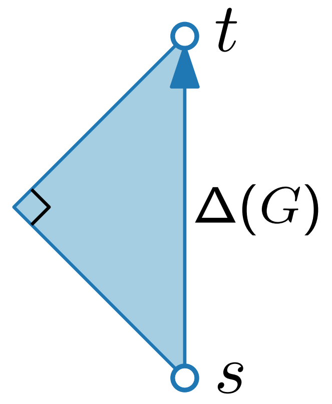

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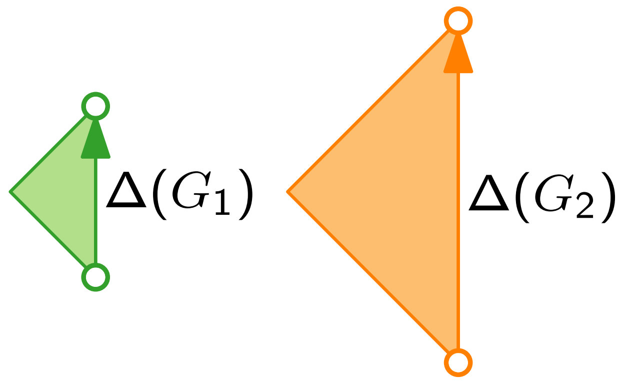

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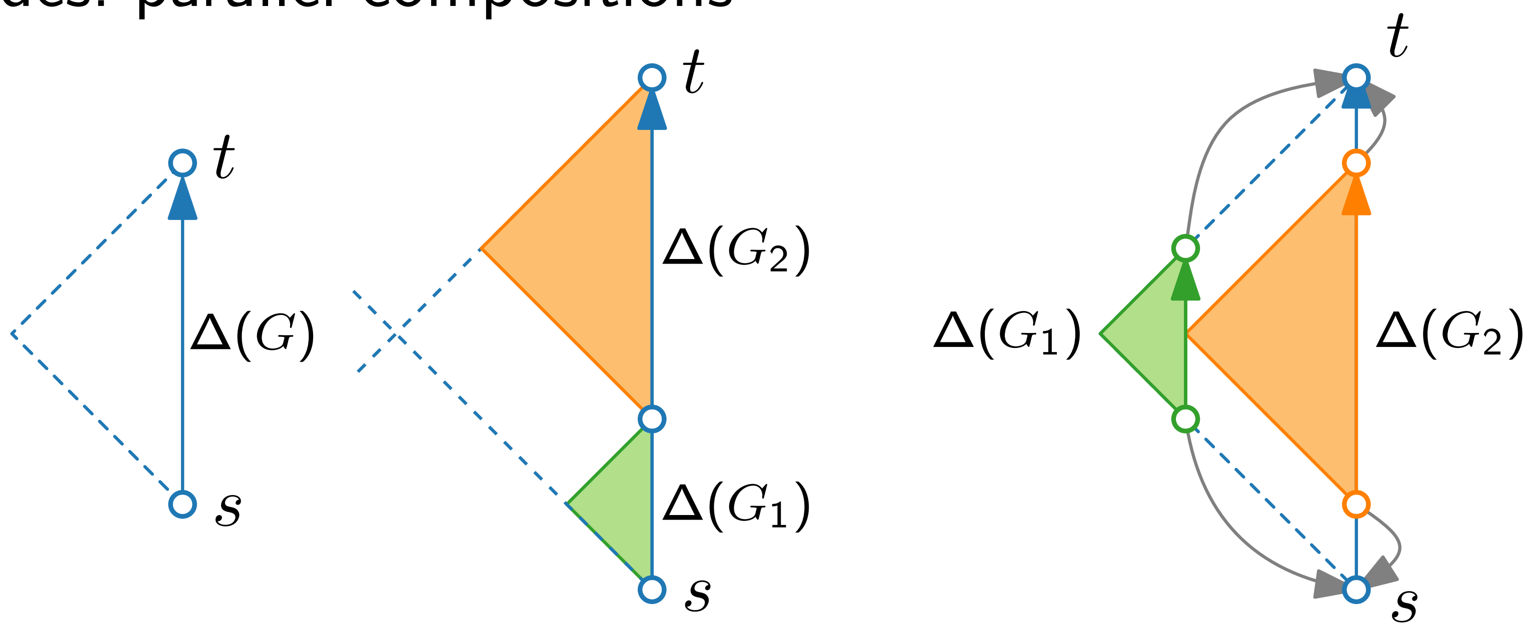

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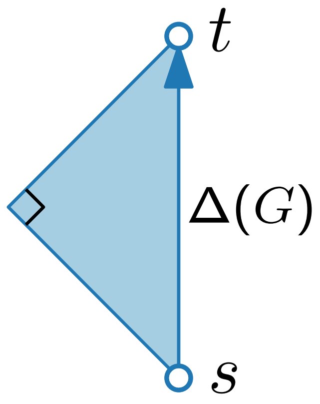

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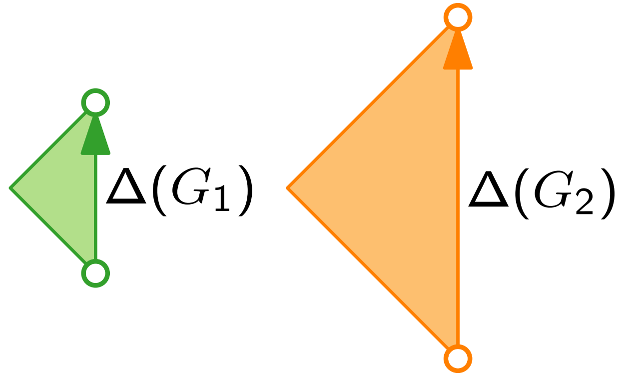

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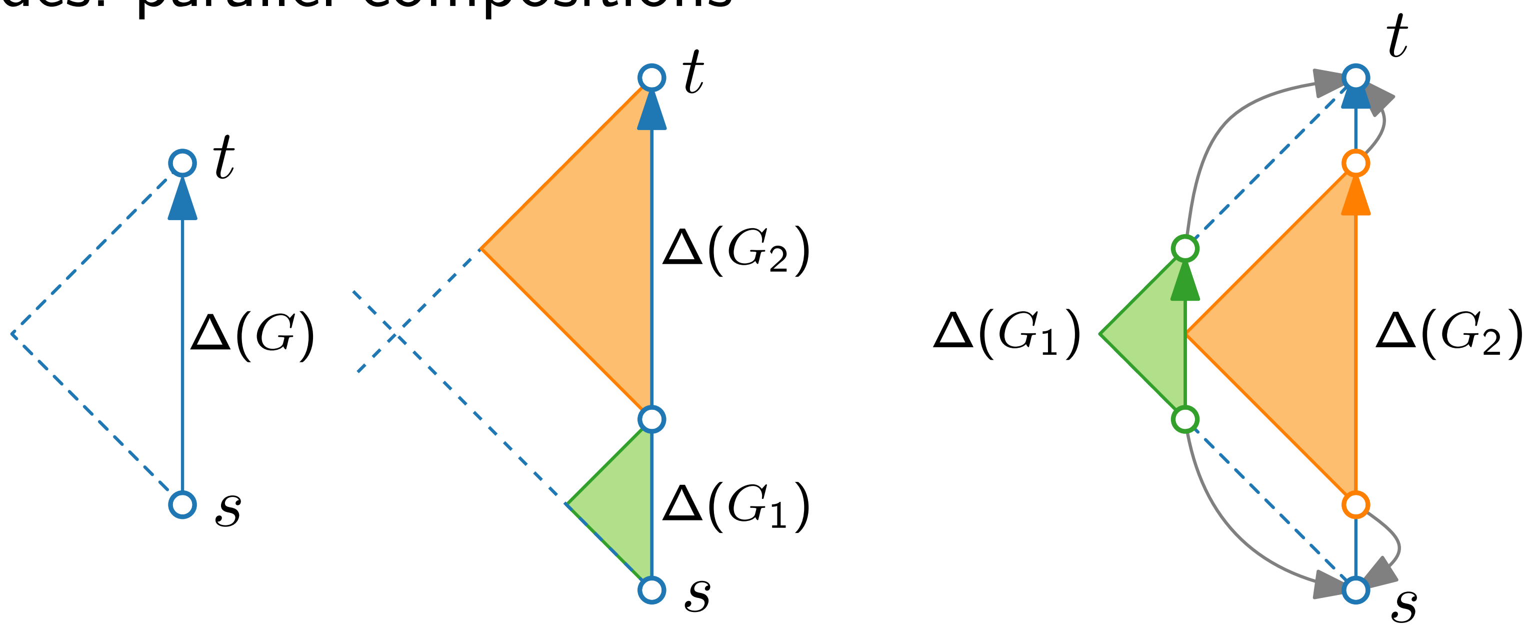
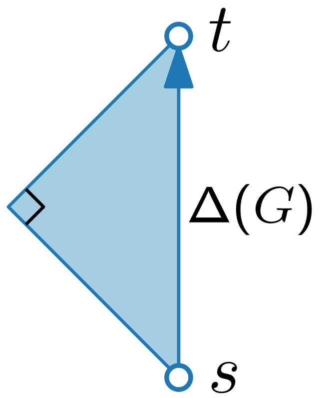


Do you see any problem?

Series-Parallel Graphs – Straight-Line Drawings



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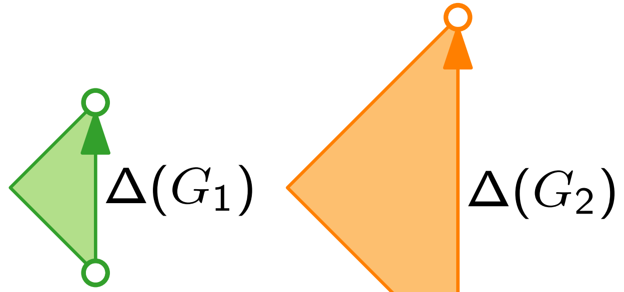

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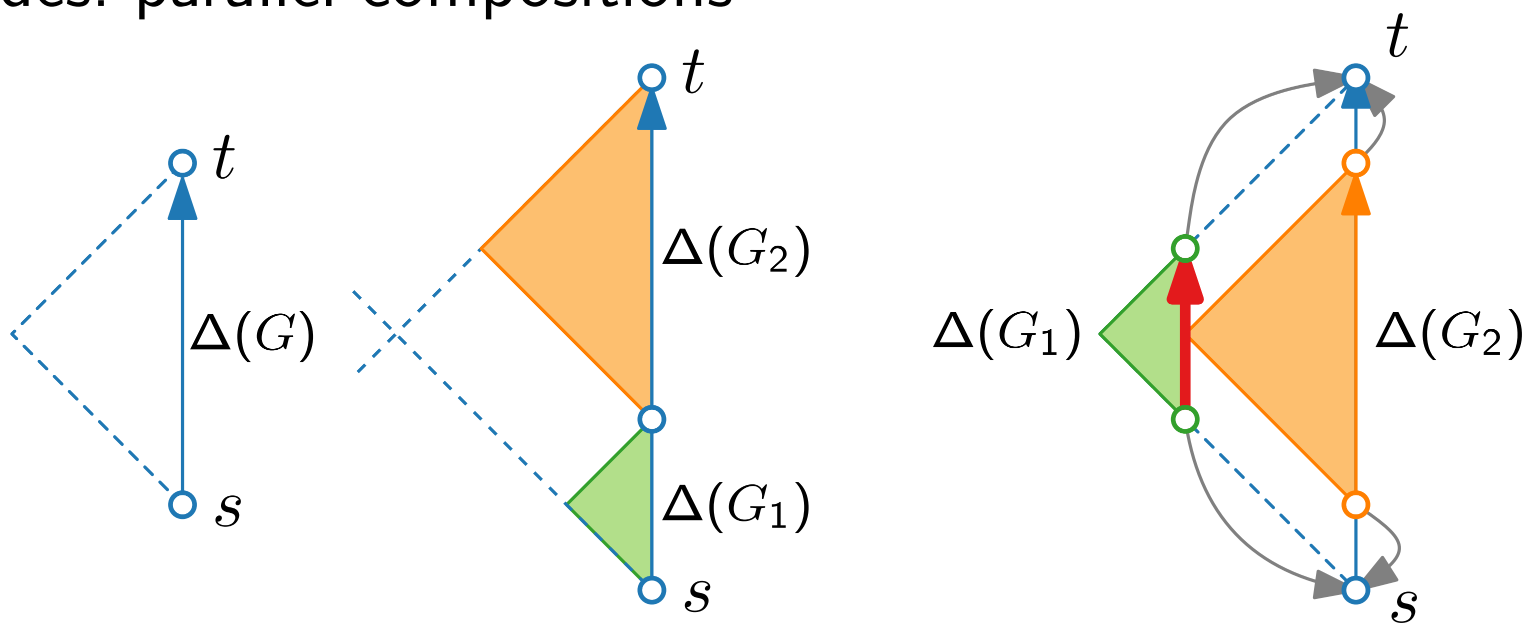
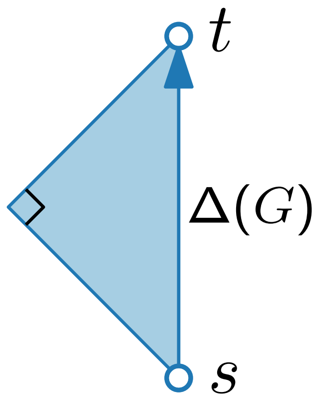


Do you see any problem?

single edge

Series-Parallel Graphs – Straight-Line Drawings



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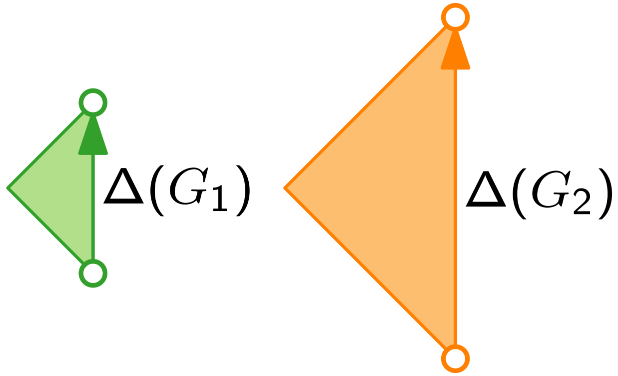

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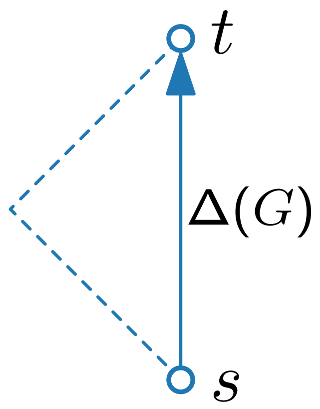
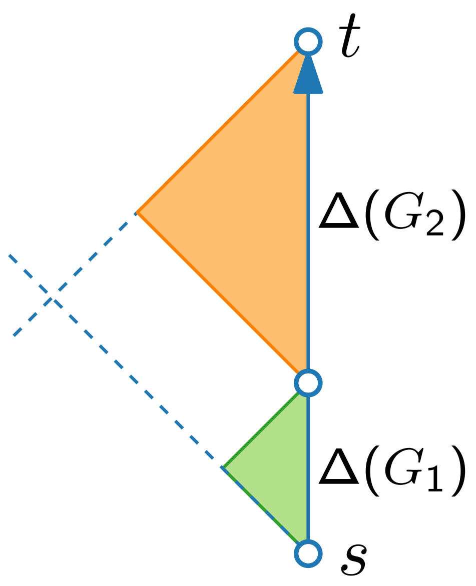
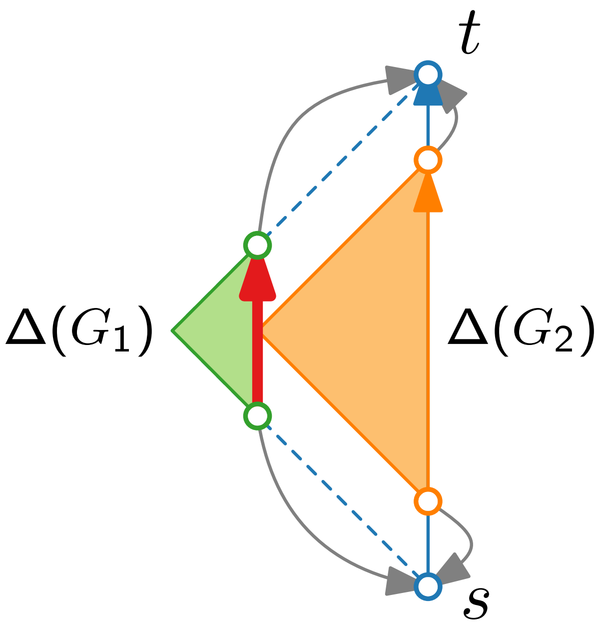




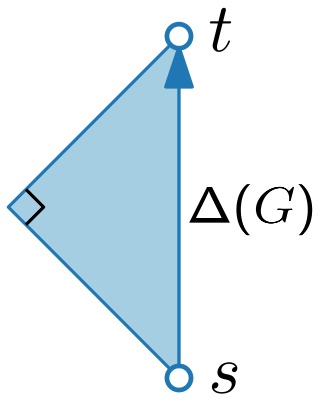


Do you see any problem?

single edge

change embedding!

Series-Parallel Graphs – Straight-Line Drawings



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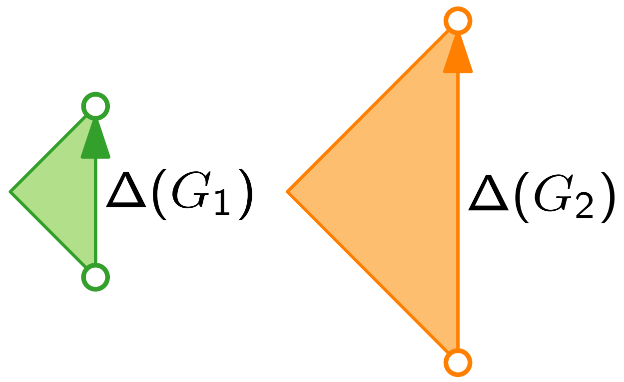

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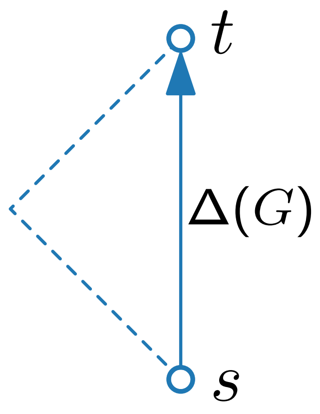

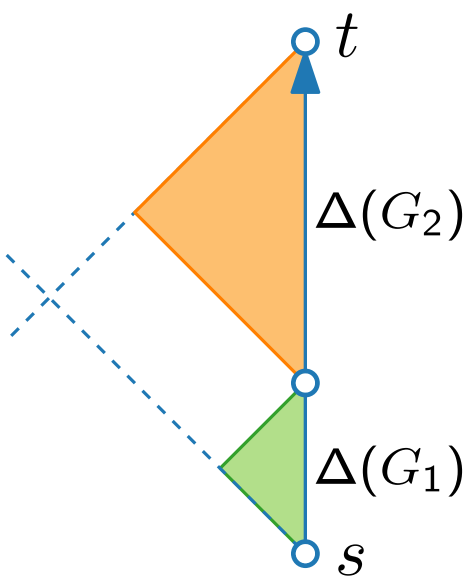

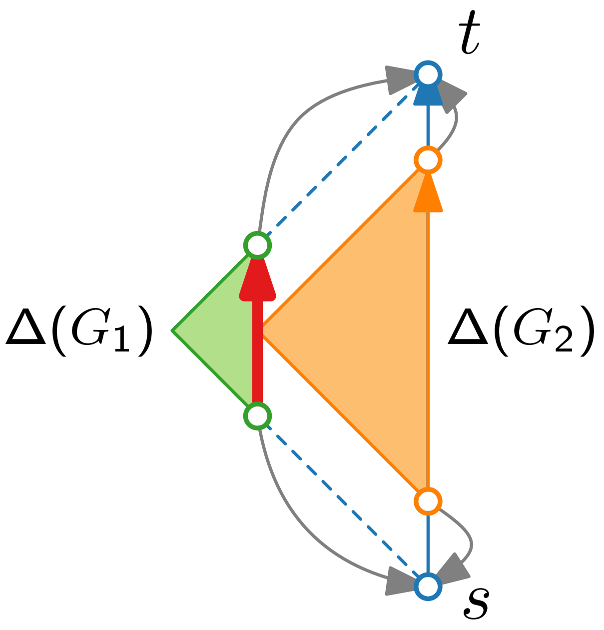

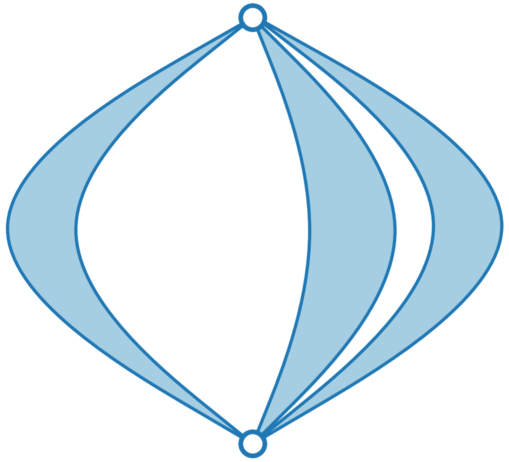

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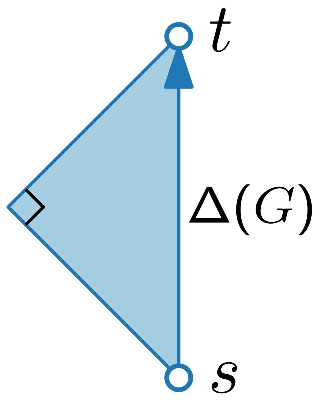
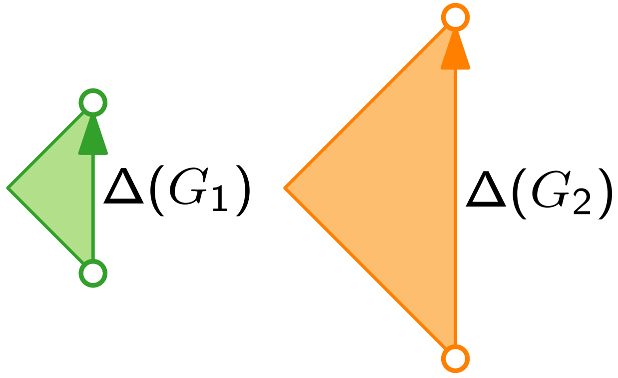


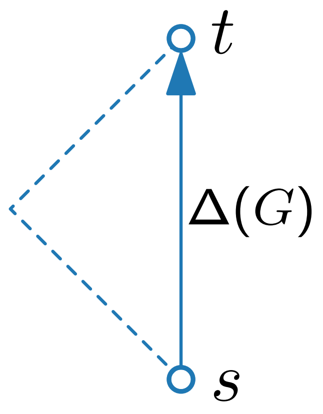
Base case: Q-nodes

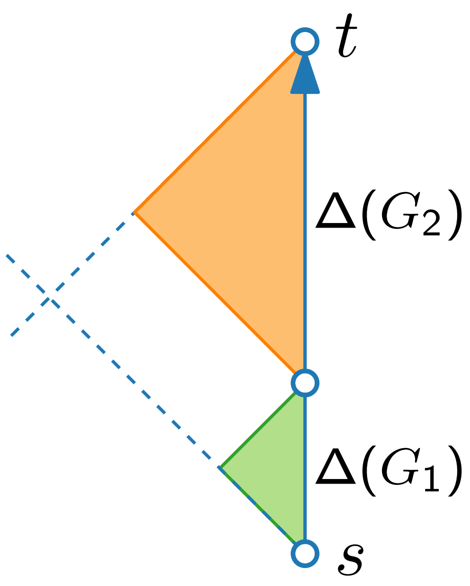
Divide: Draw G_1 and G_2 first

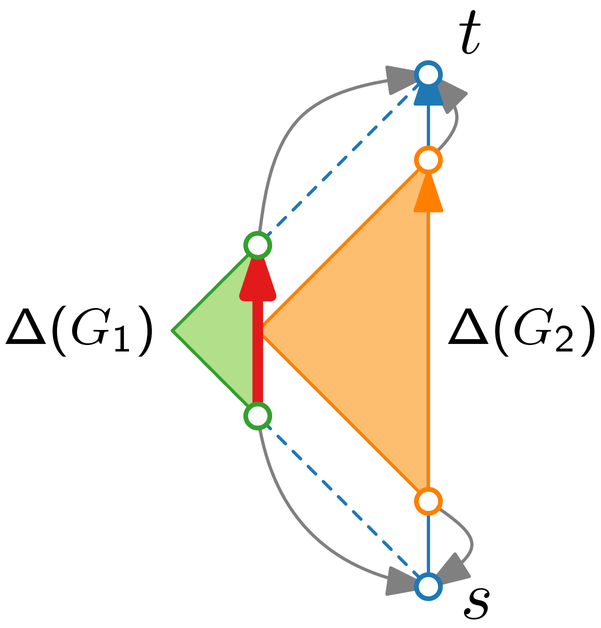


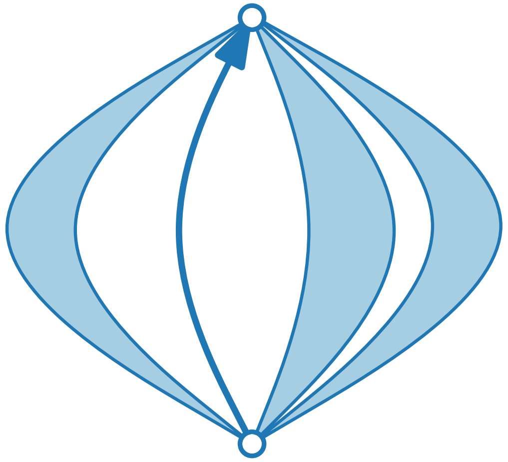
Conquer:

- S-nodes: series compositions
- P-nodes: parallel compositions

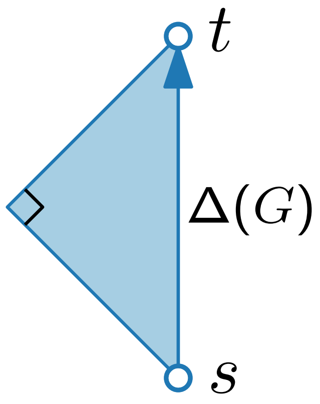








Series-Parallel Graphs – Straight-Line Drawings



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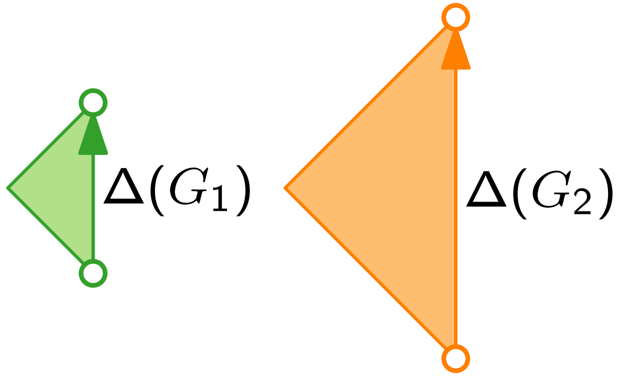

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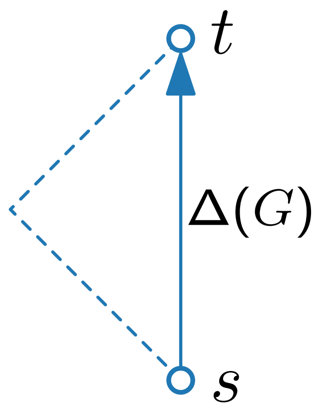

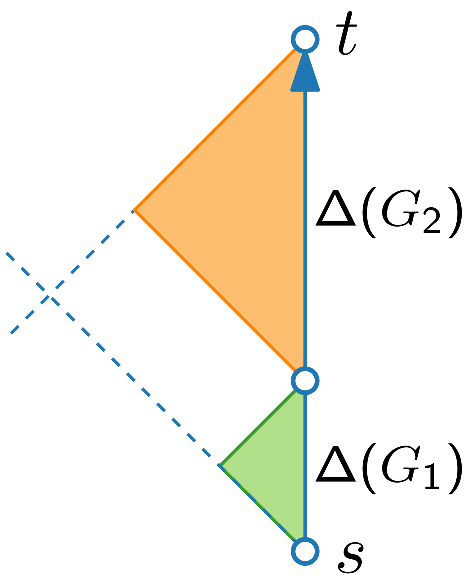

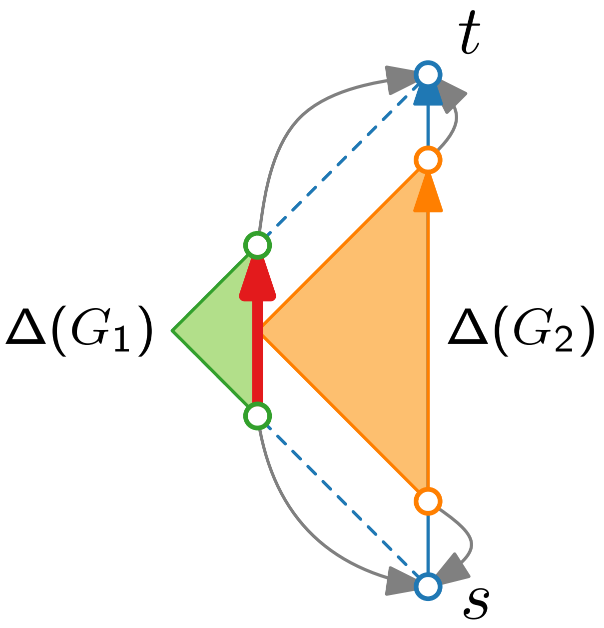

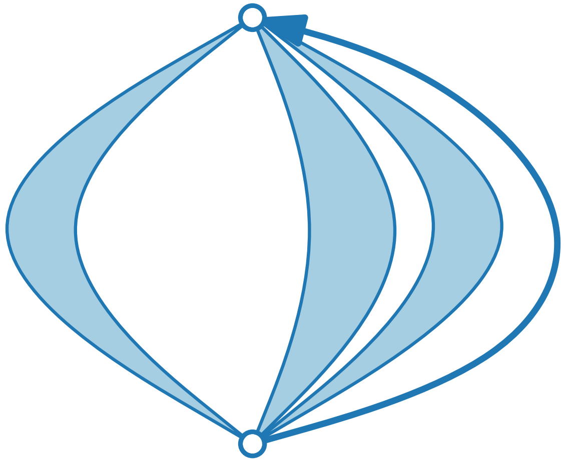

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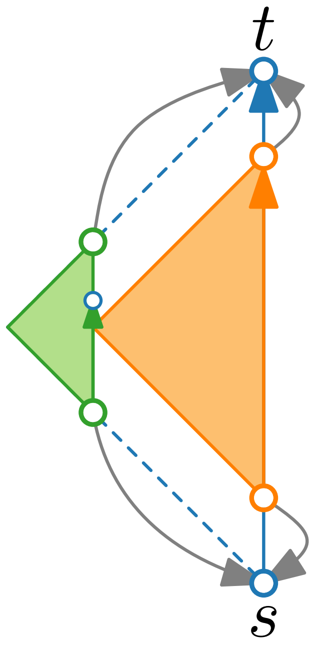

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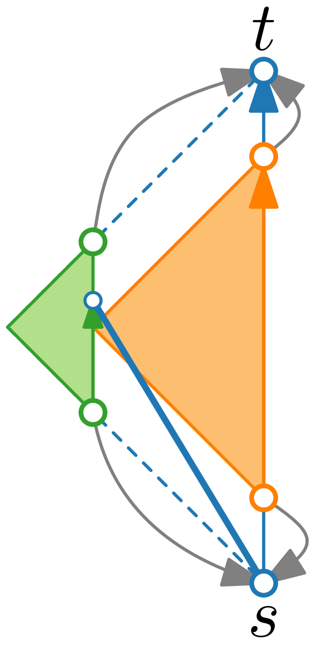

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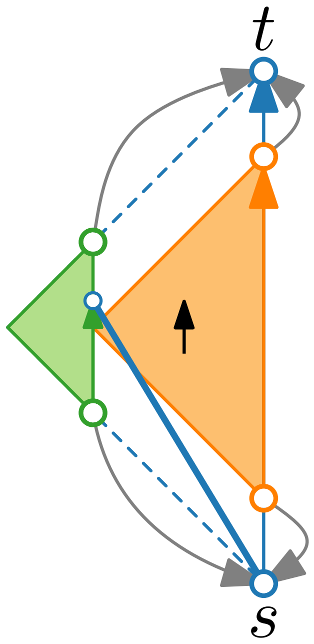

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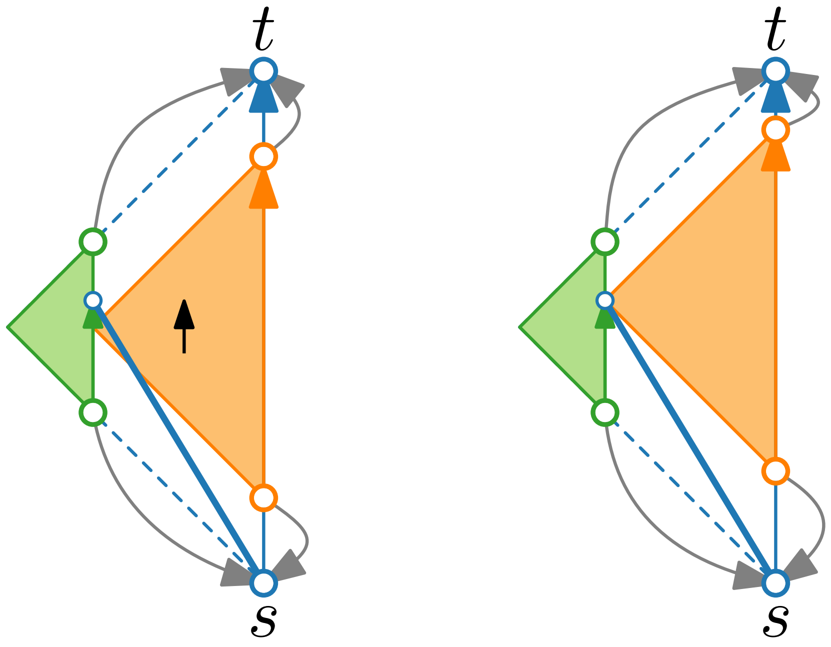

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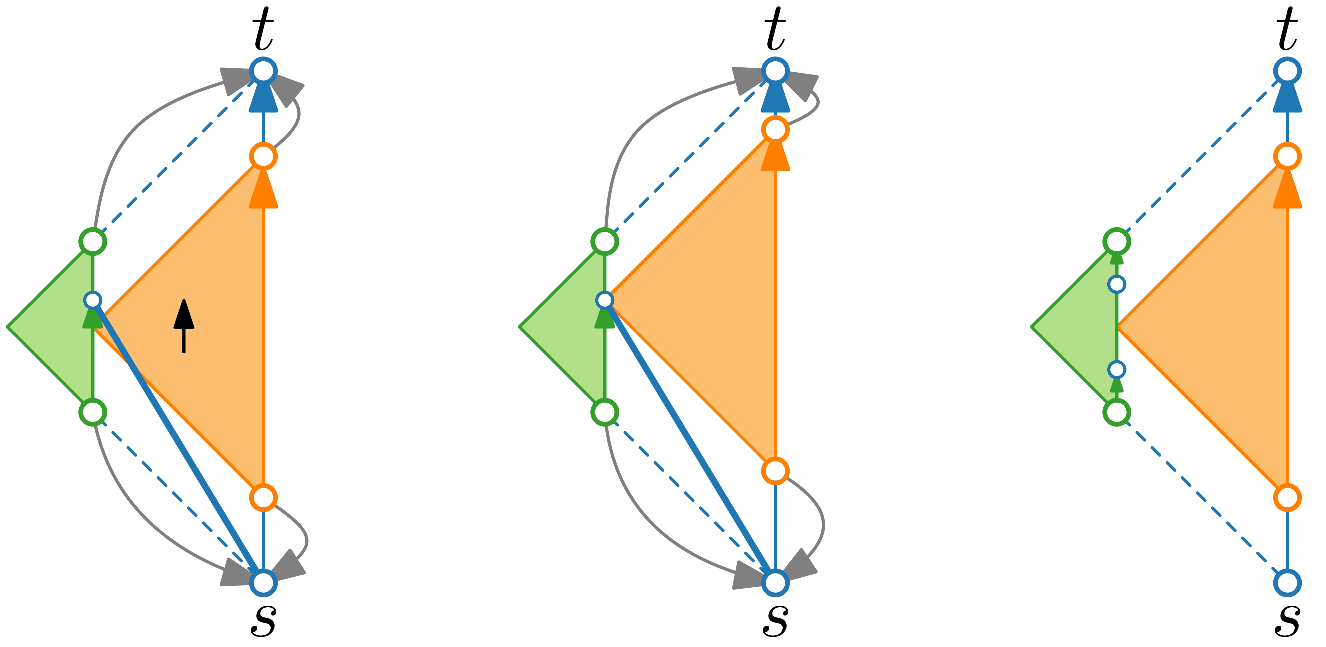

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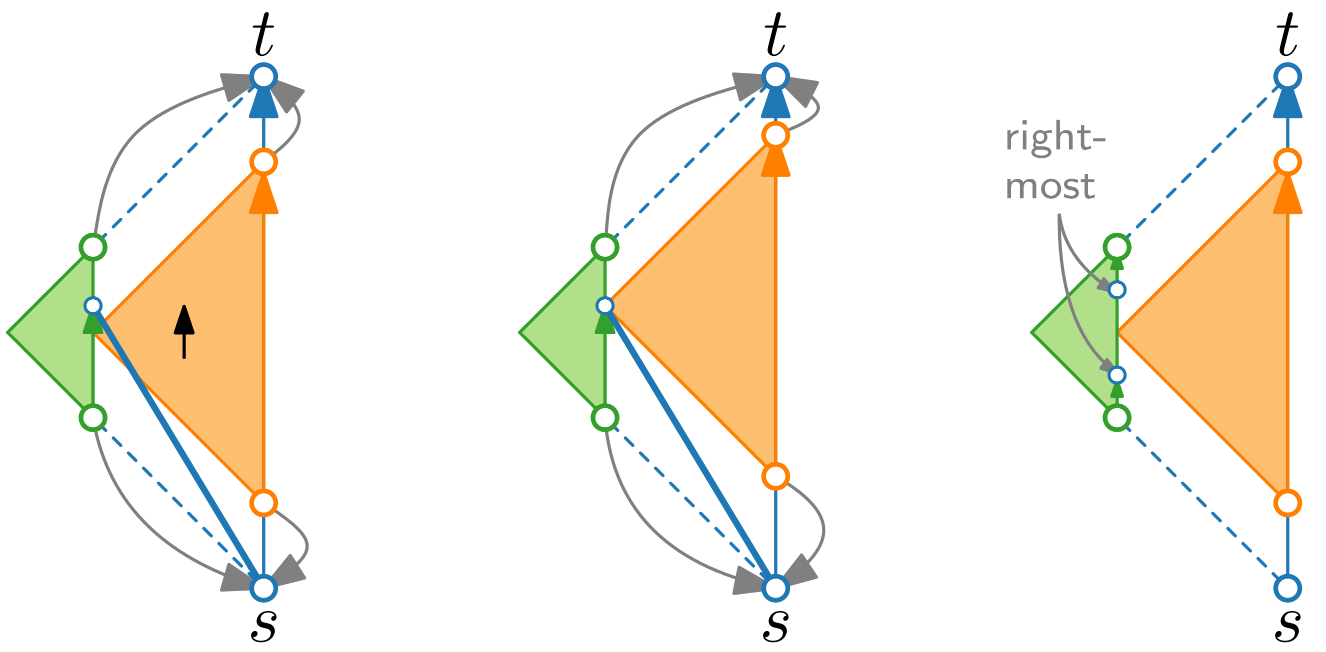

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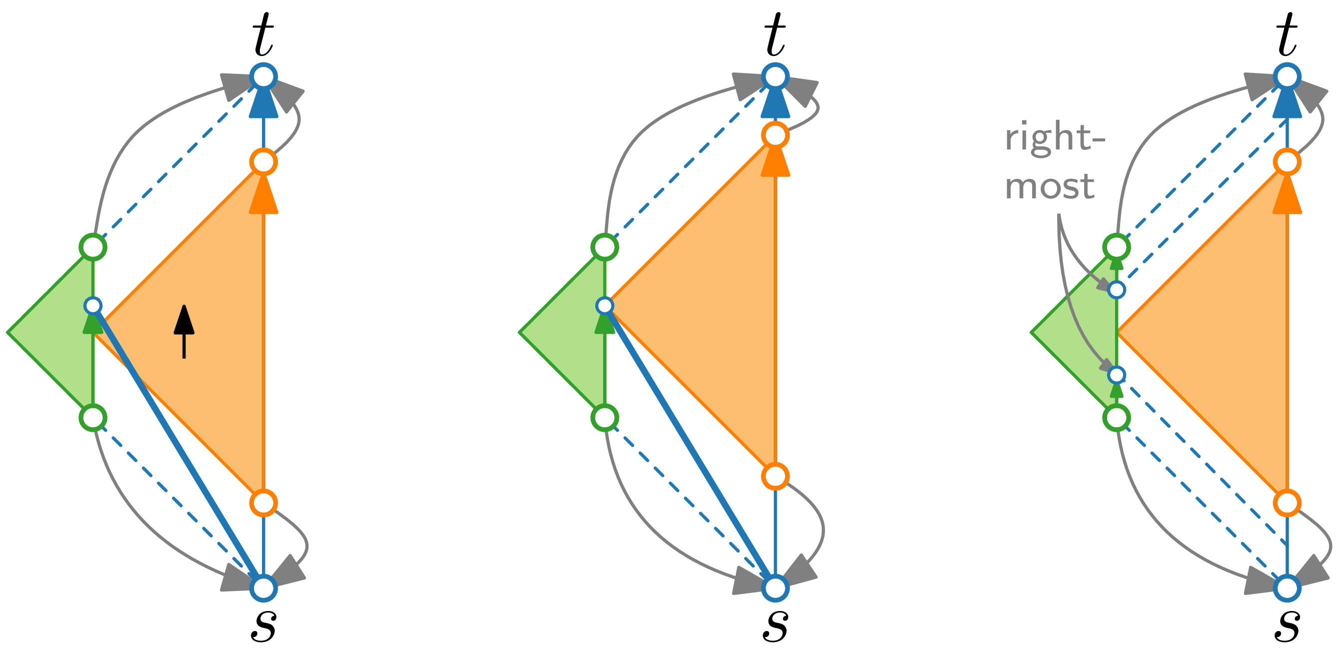

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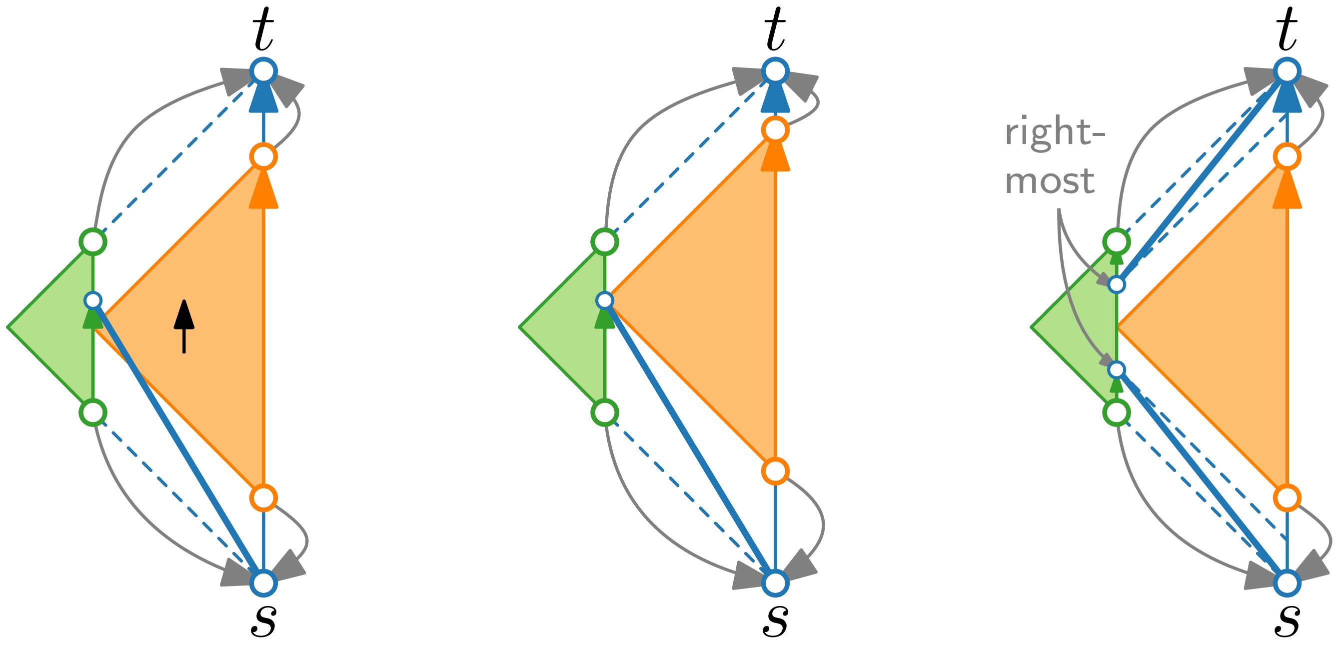

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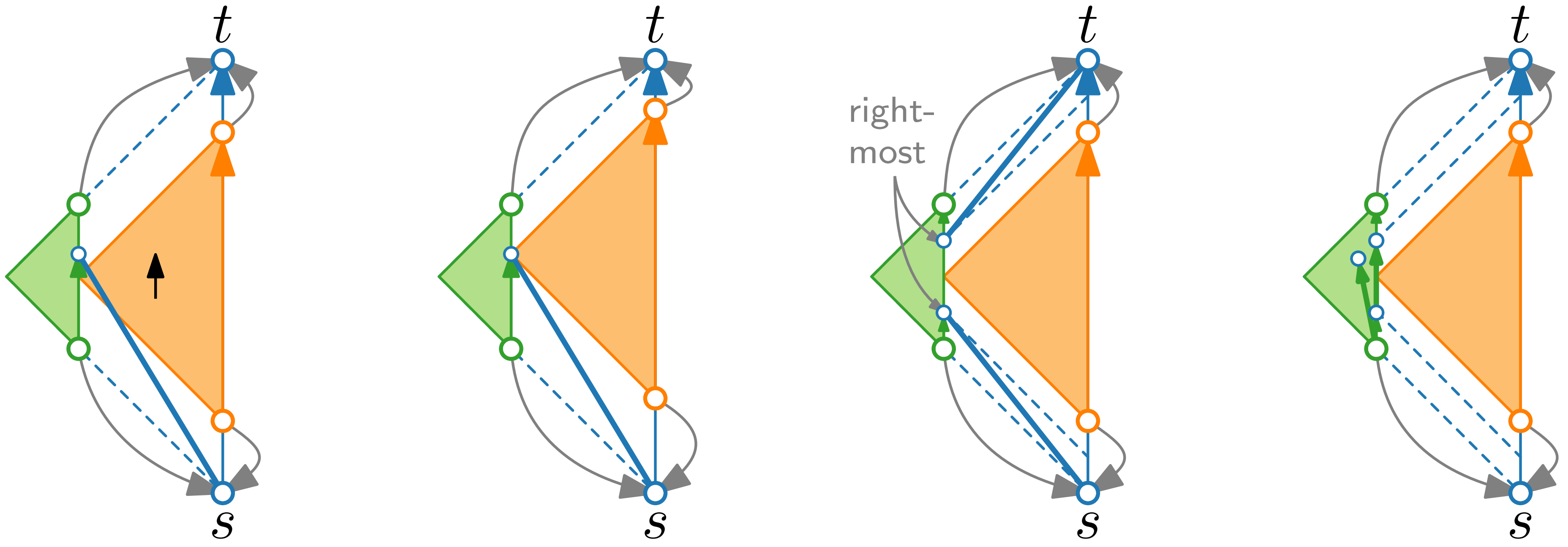

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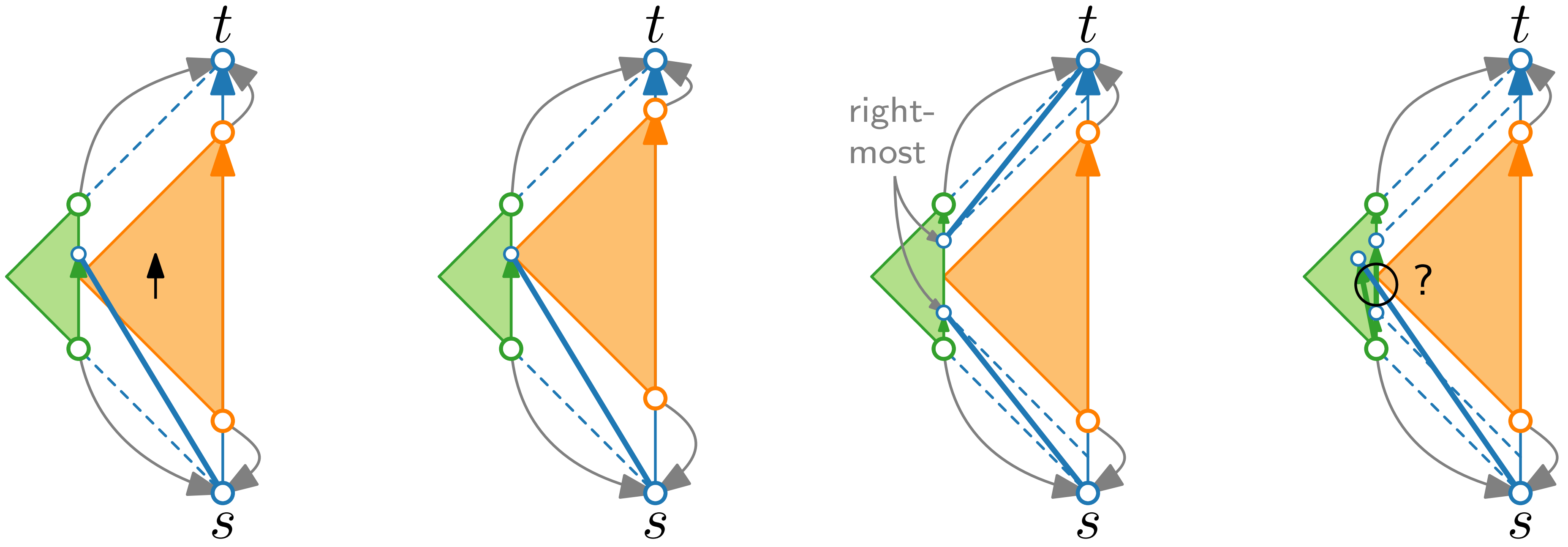

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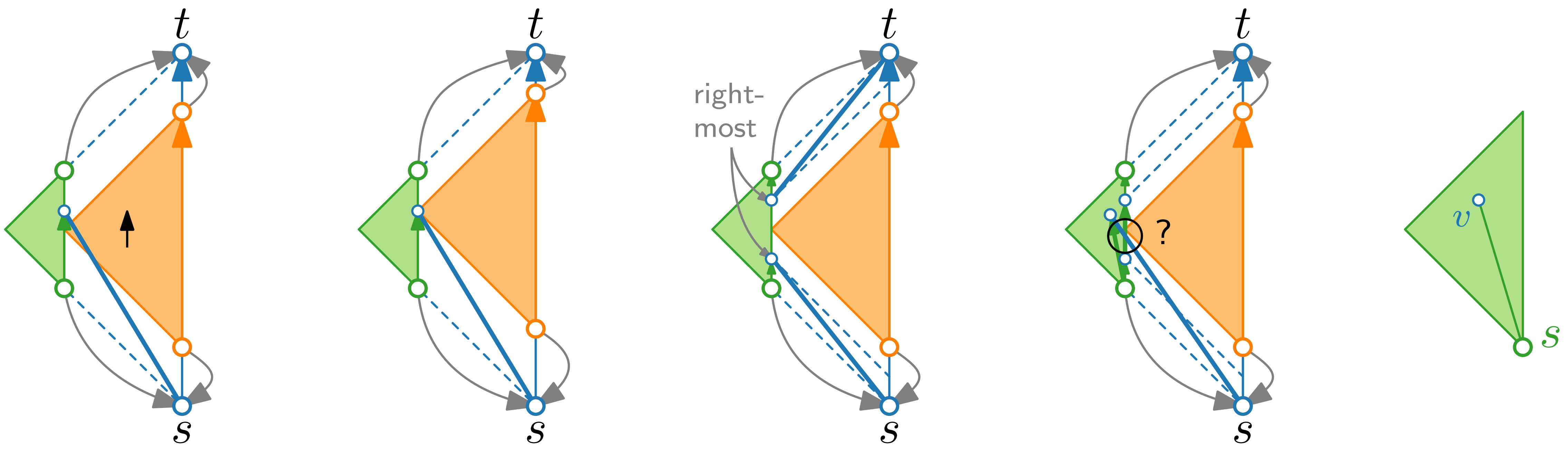

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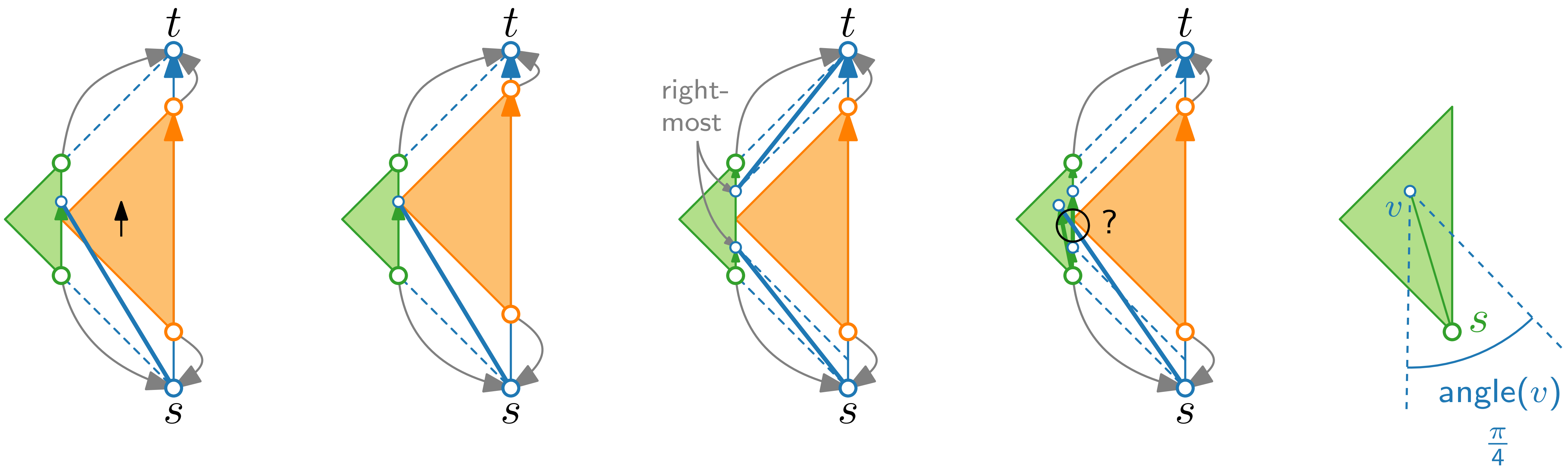

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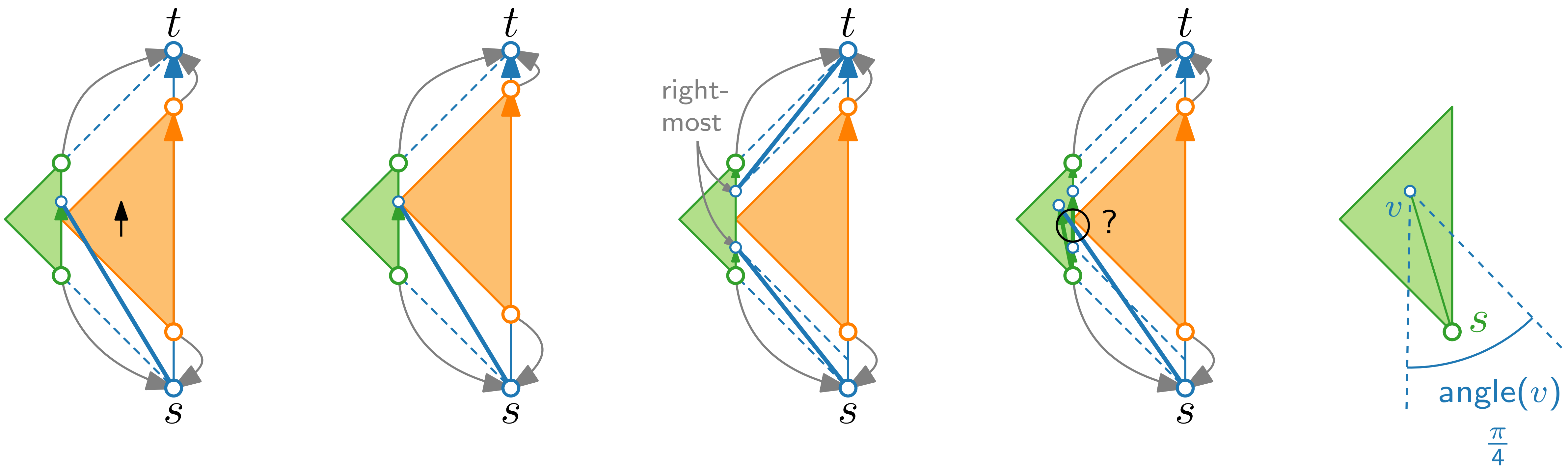

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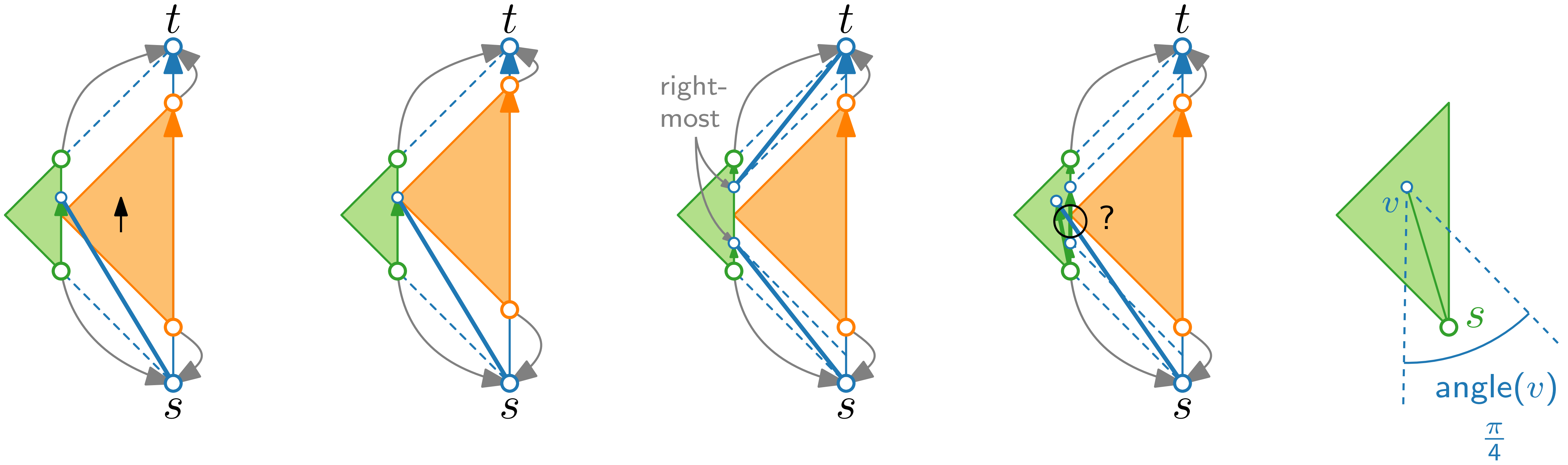

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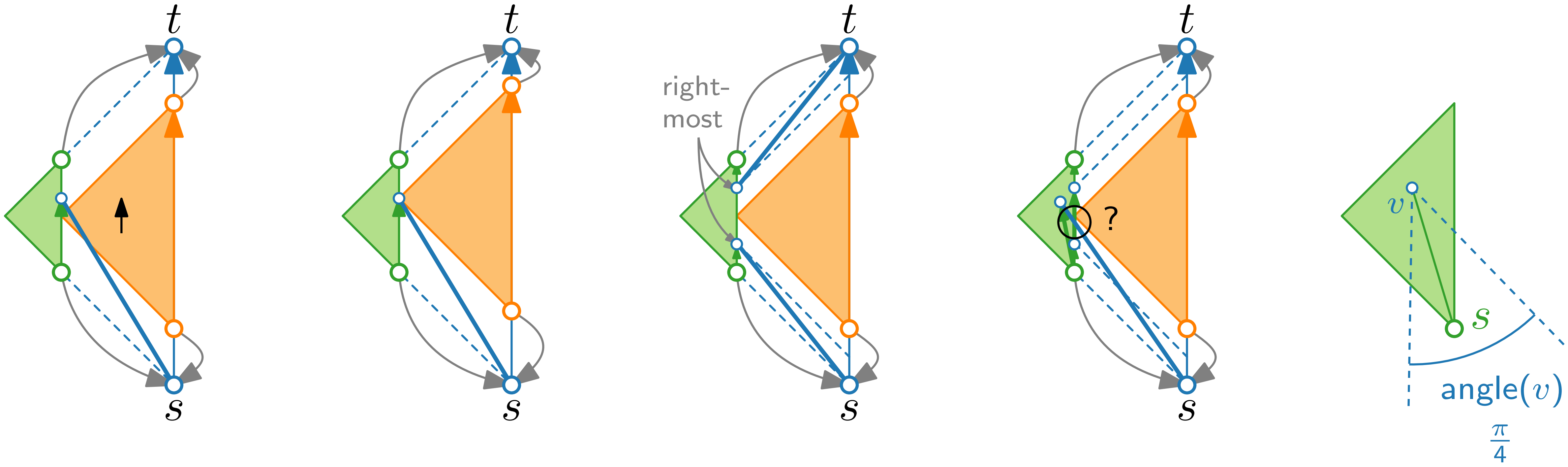

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The drawing produced by the algorithm is planar.

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Let G be a series-parallel graph. Then G (with **variable embedding**) admits a drawing Γ that

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Γ can be computed in linear time.

Series-Parallel Graphs – Fixed Embedding

Theorem. [Bertolazzi, Di Battista, Mannino, Tamassia '94]

For any $n \geq 1$, there exists a $2n$-vertex series-parallel graph G_n in an embedding such that any upward planar straight-line drawing of G_n that respects the given embedding requires $\Omega(4^n)$ area.

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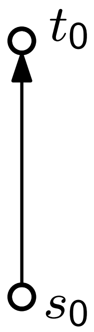

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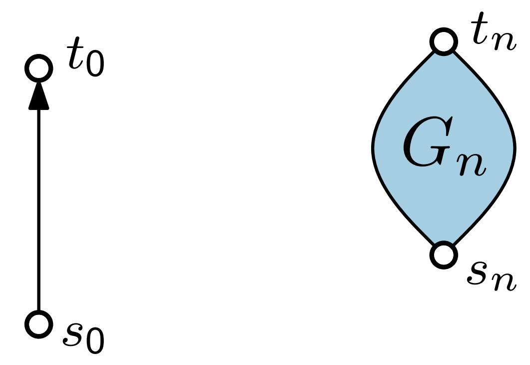

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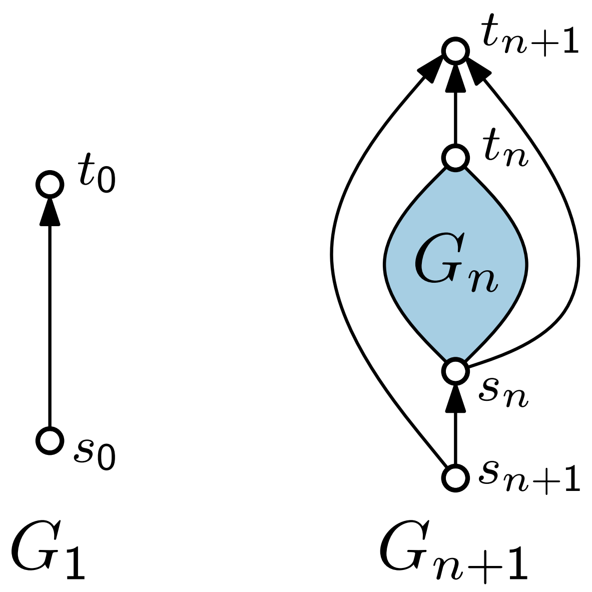

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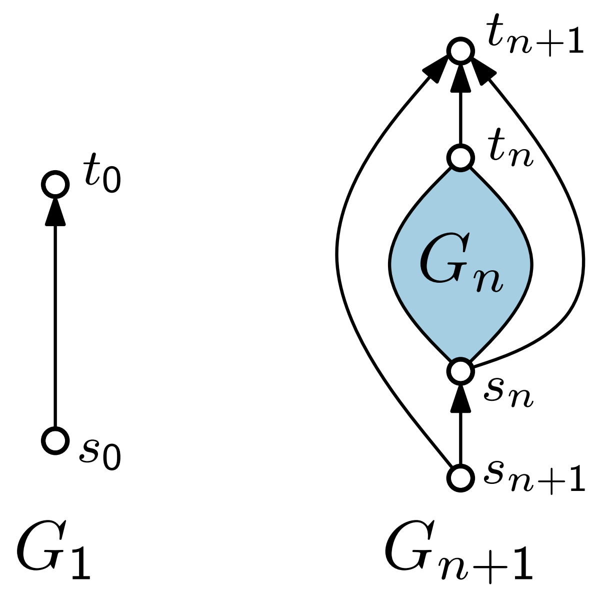

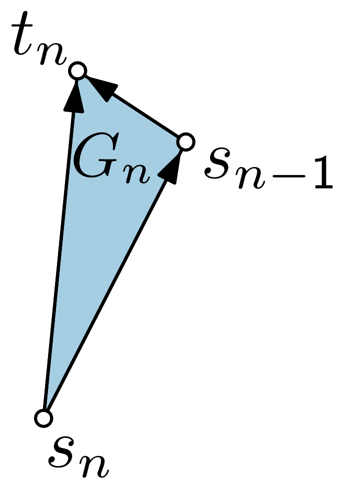

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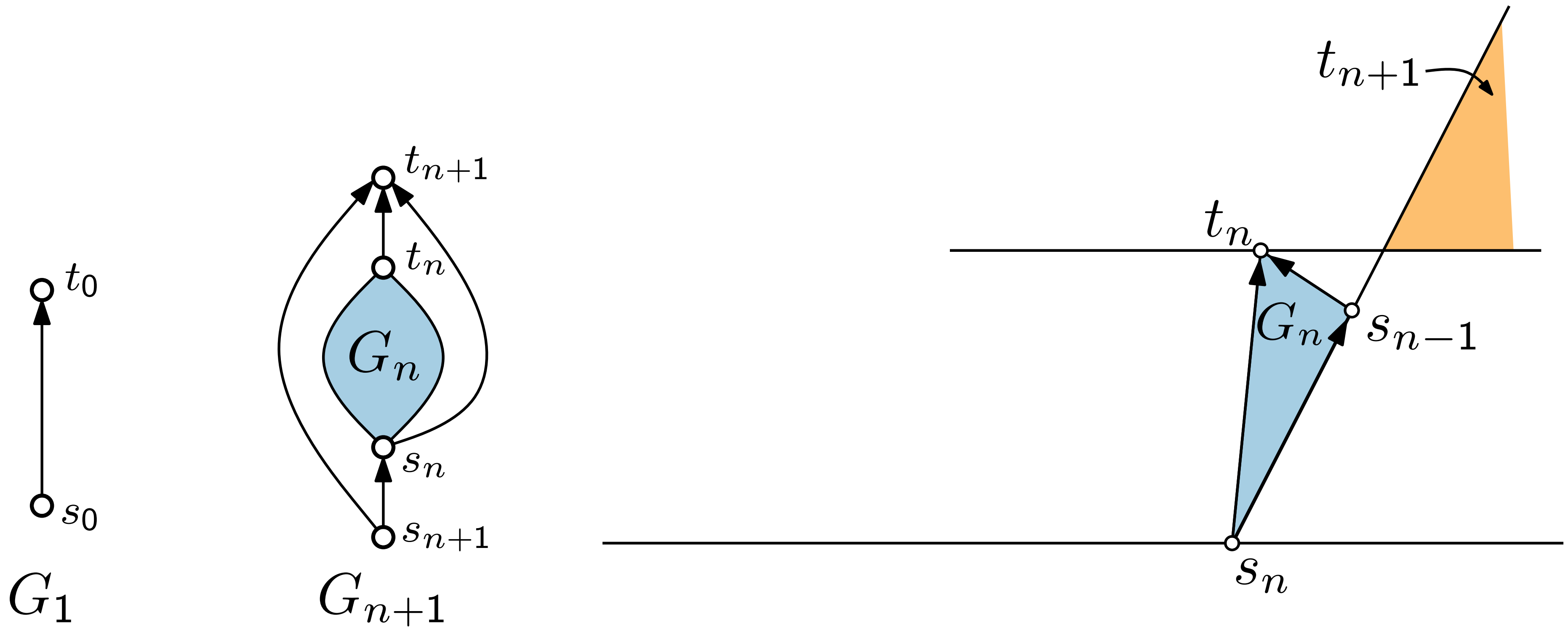

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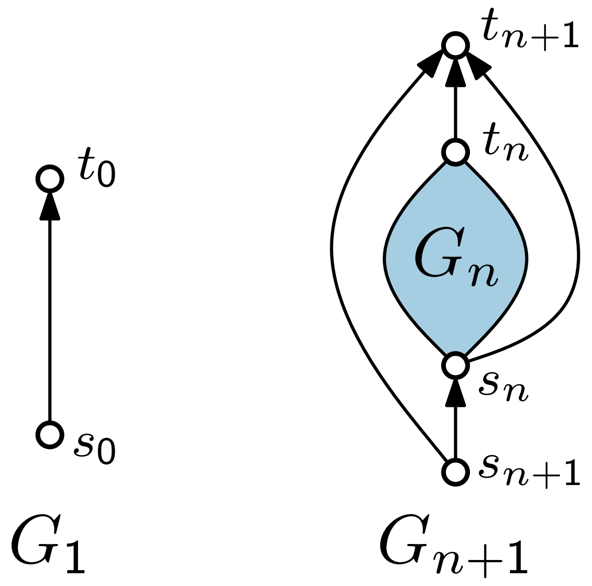

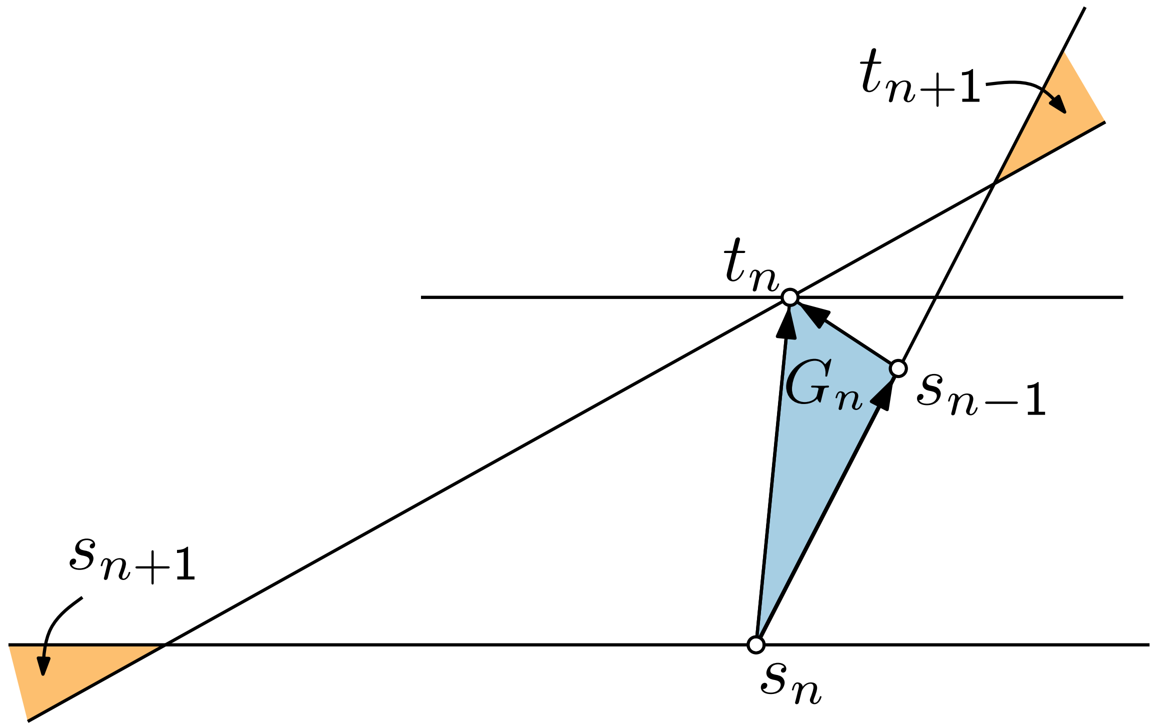

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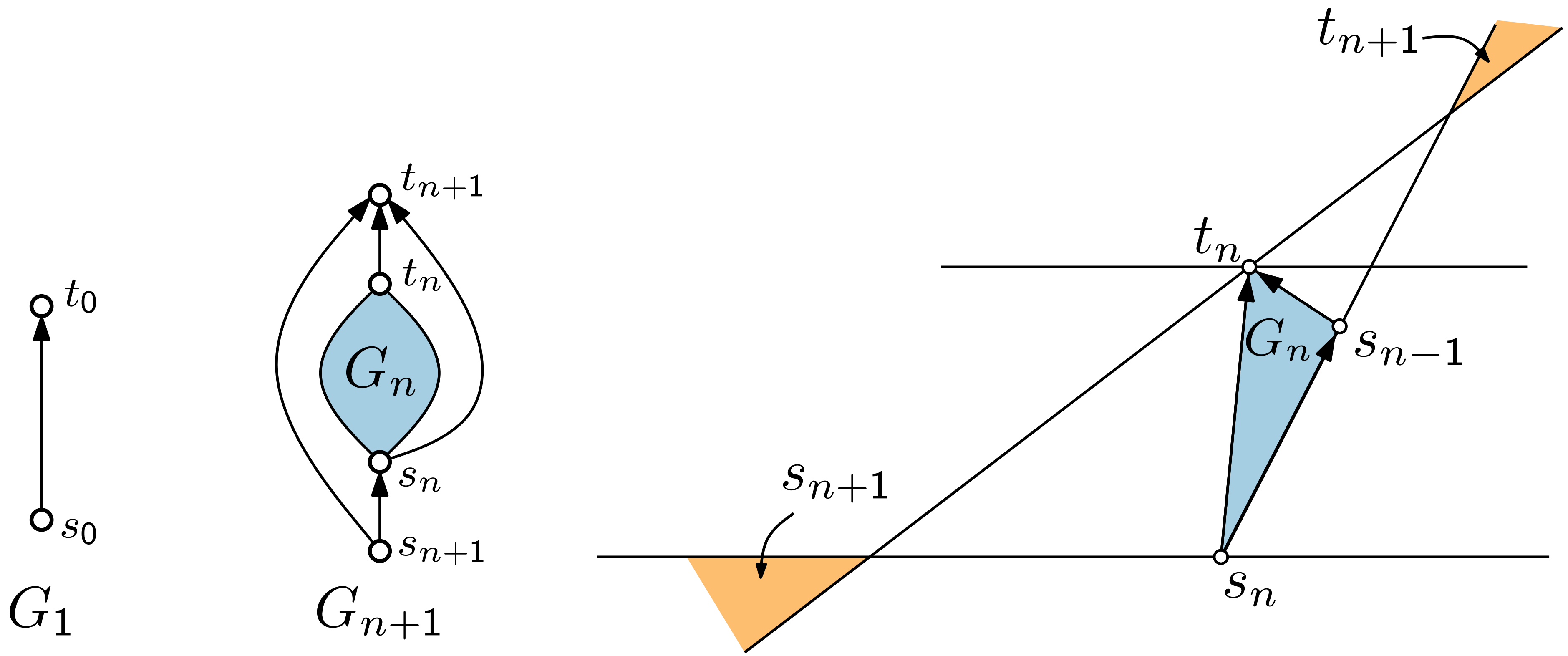

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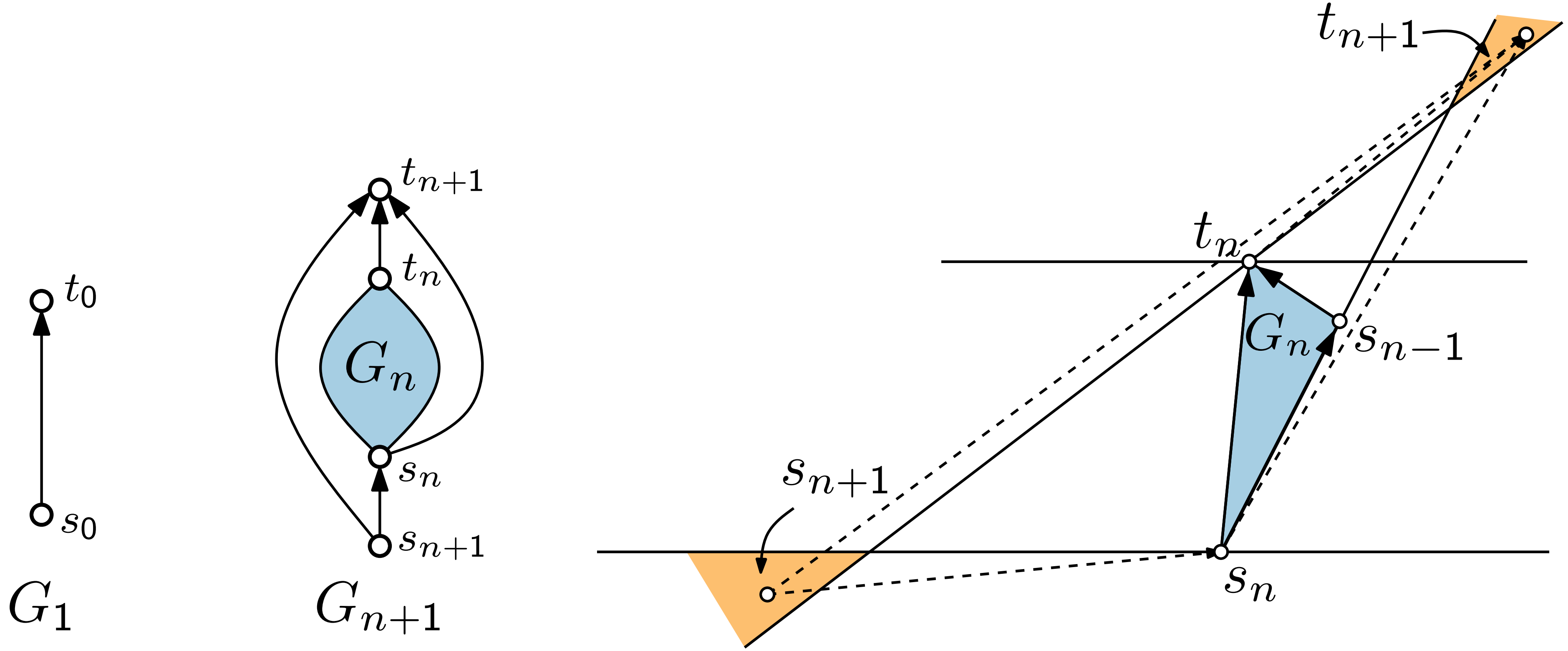

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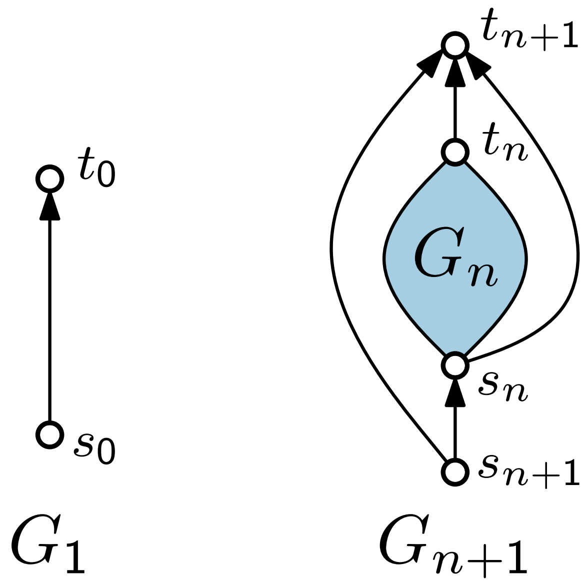

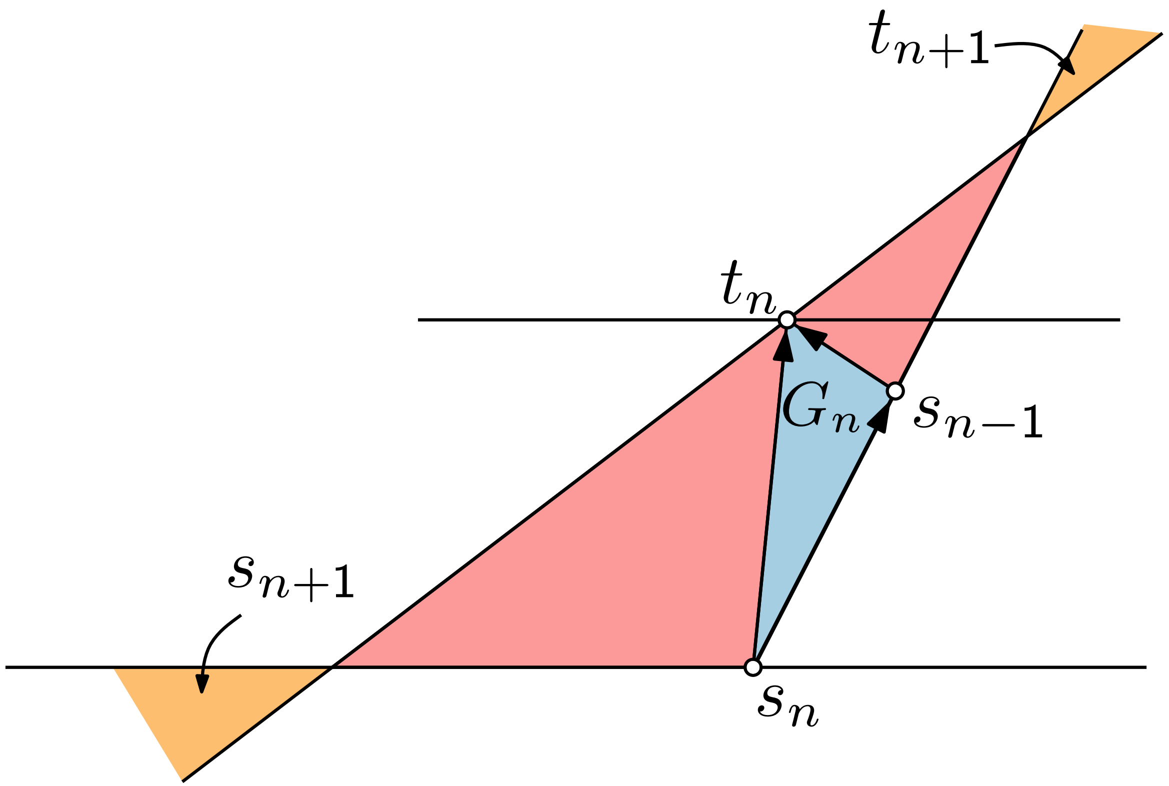

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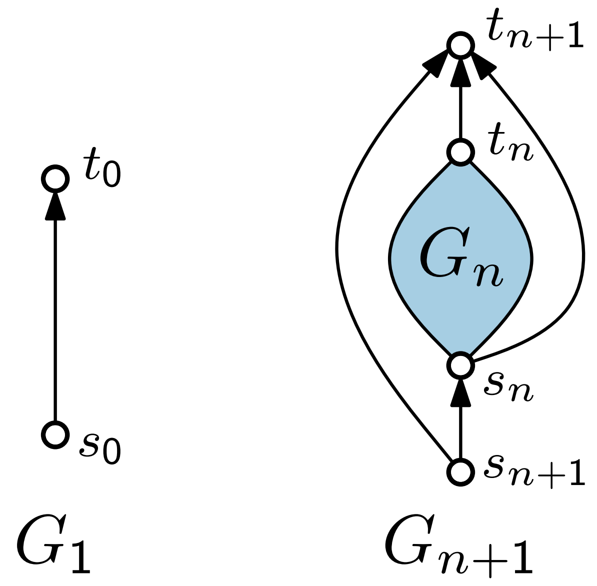

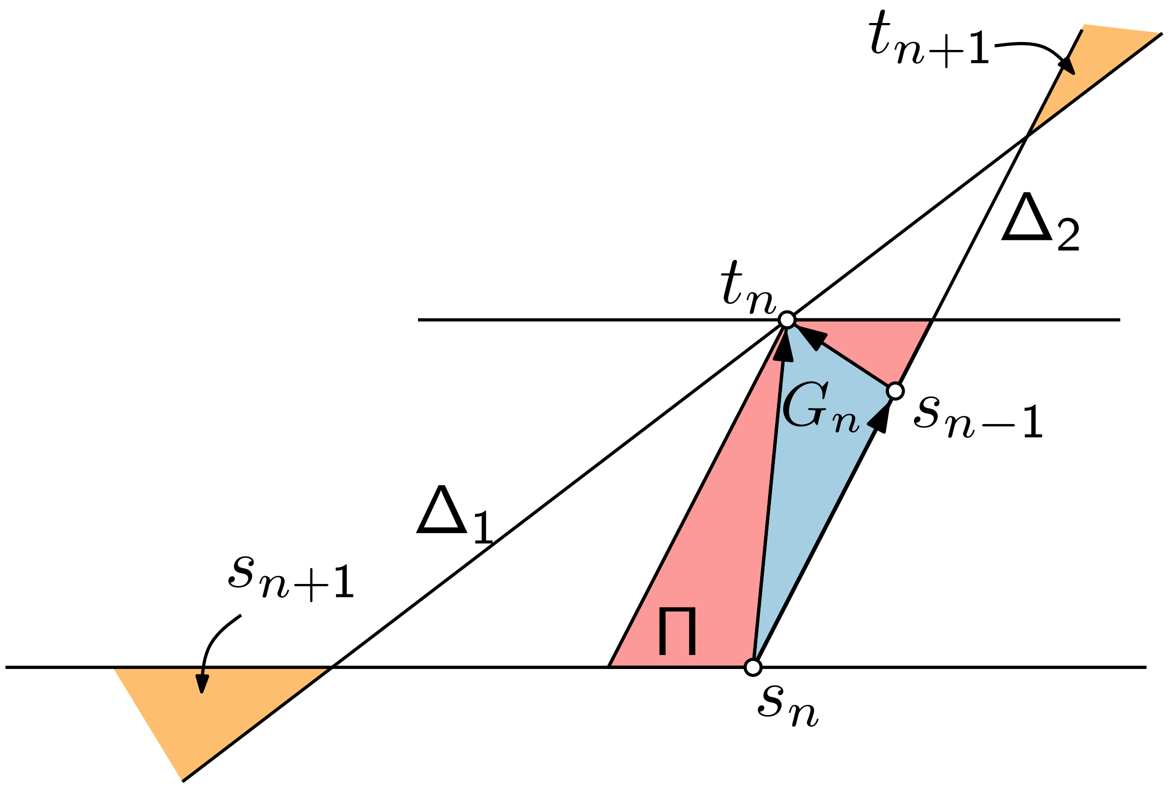

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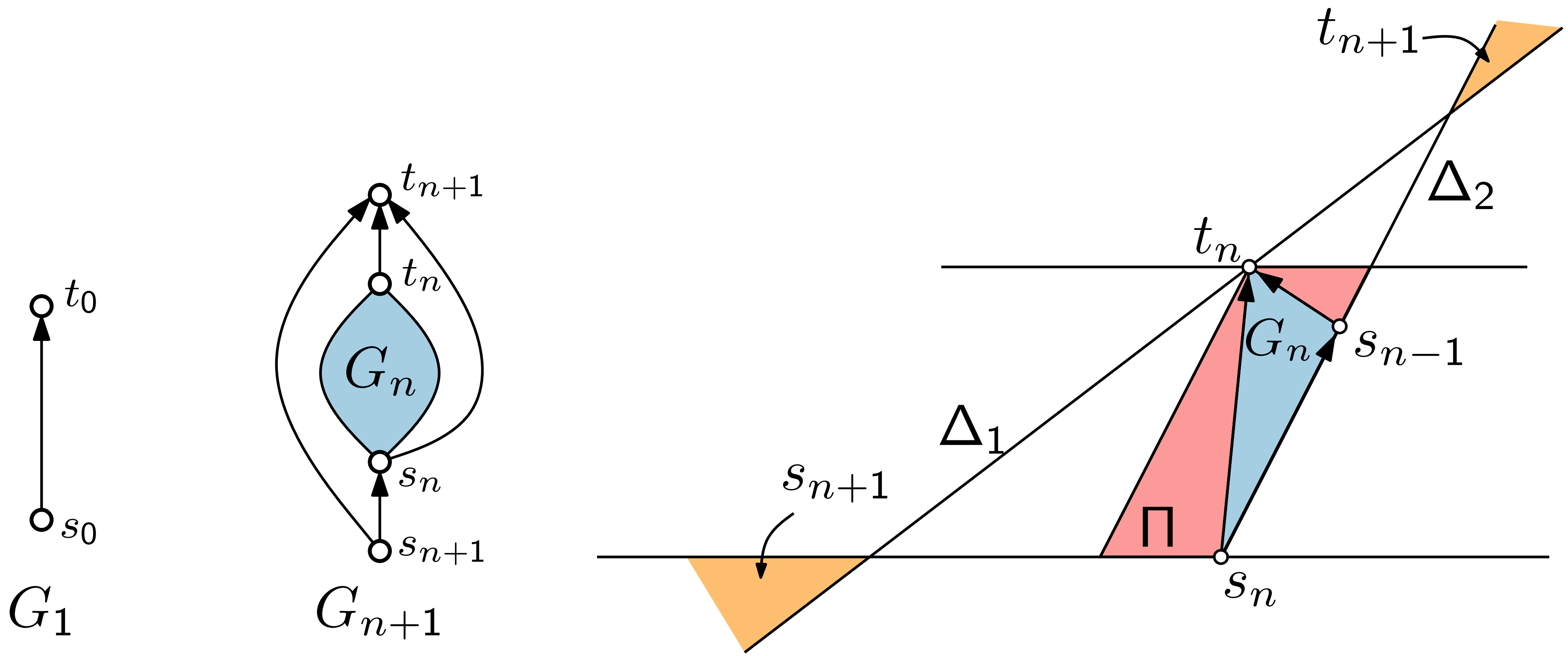

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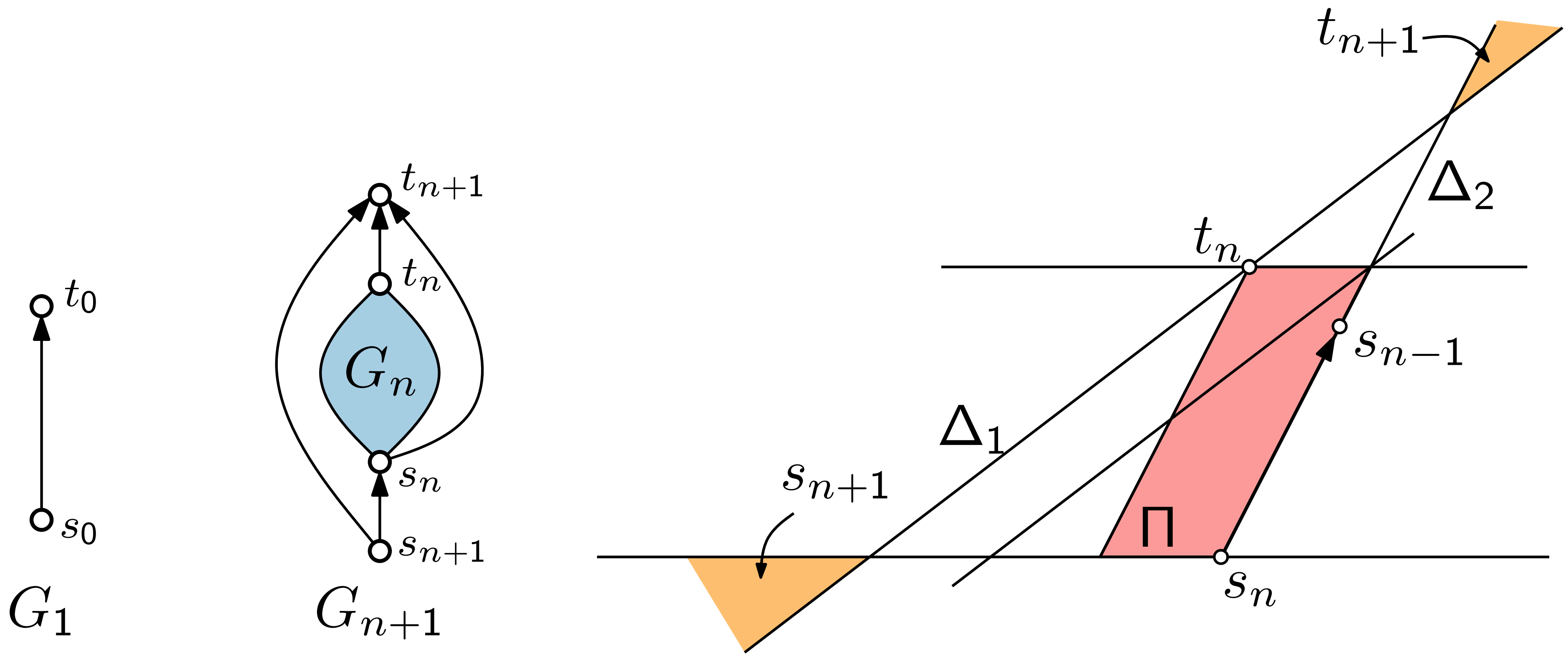

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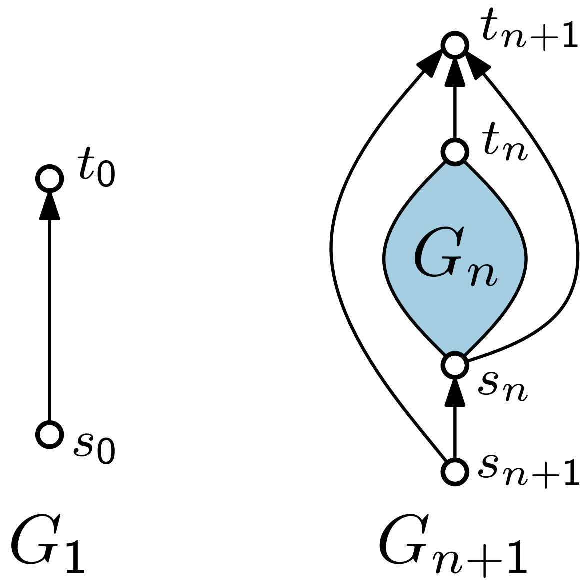

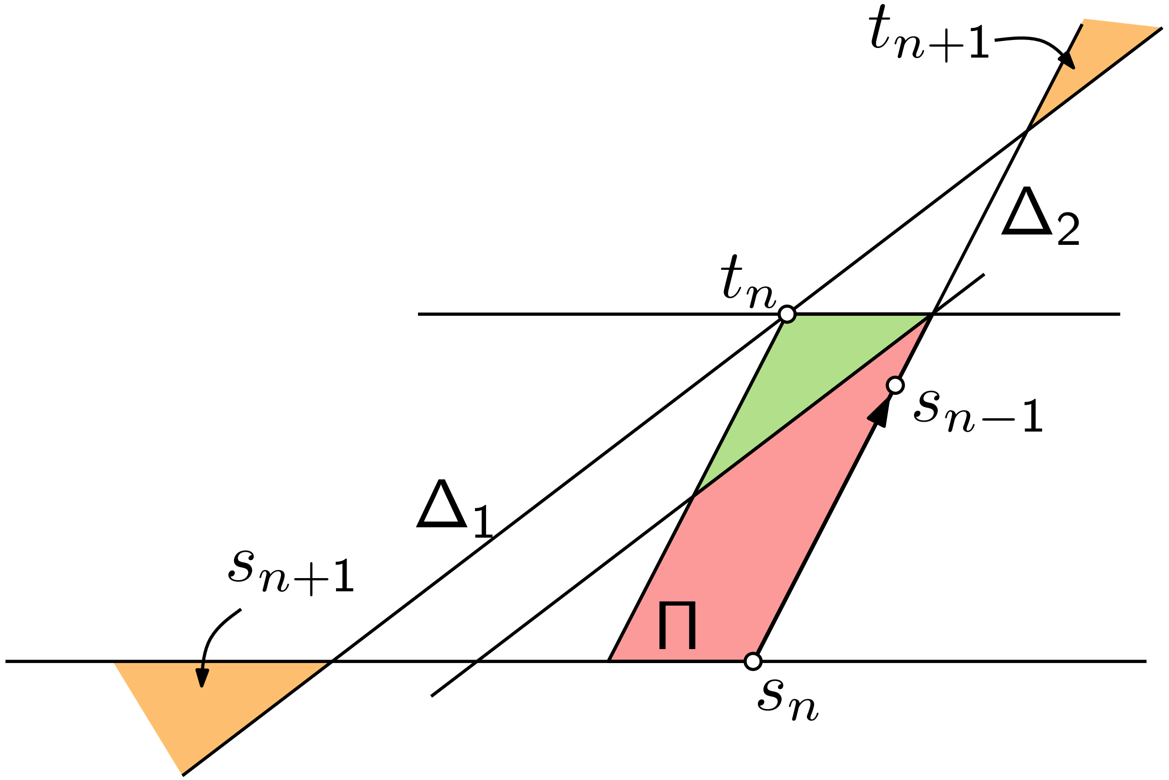

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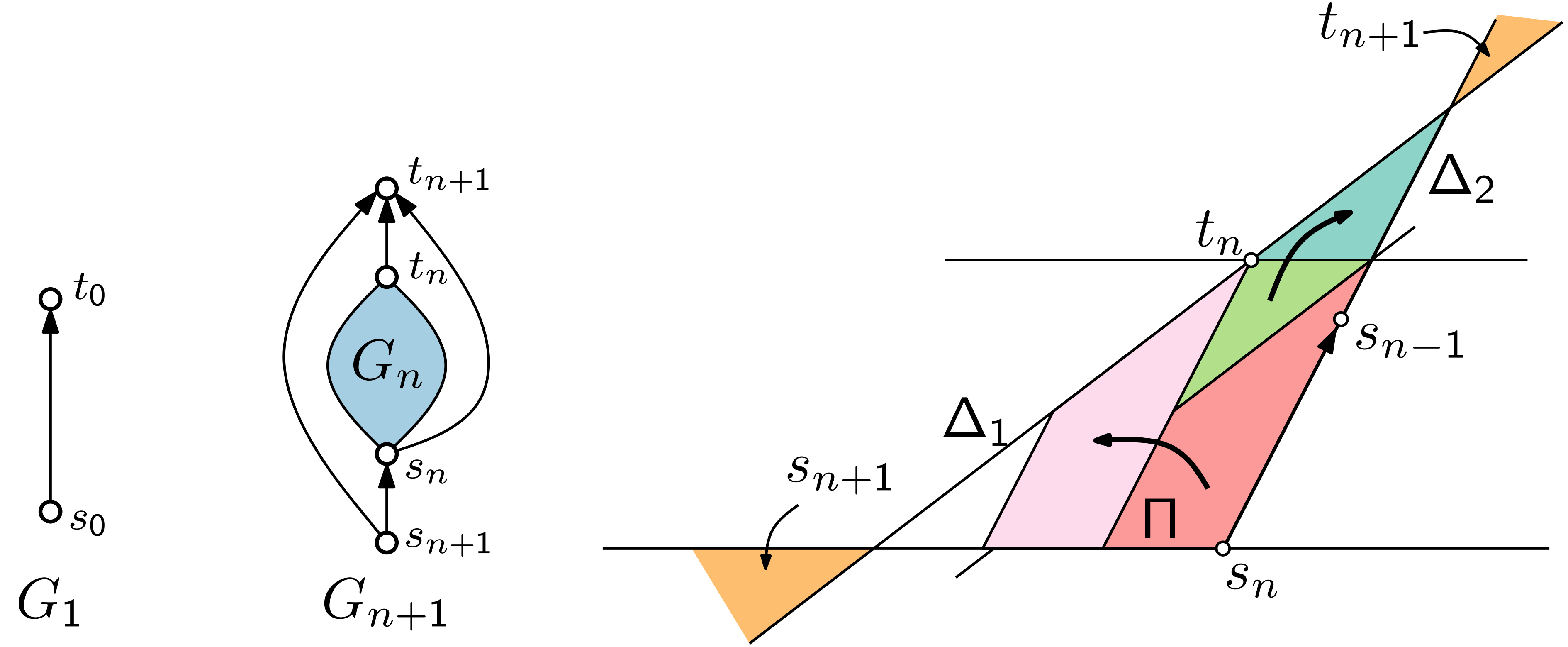

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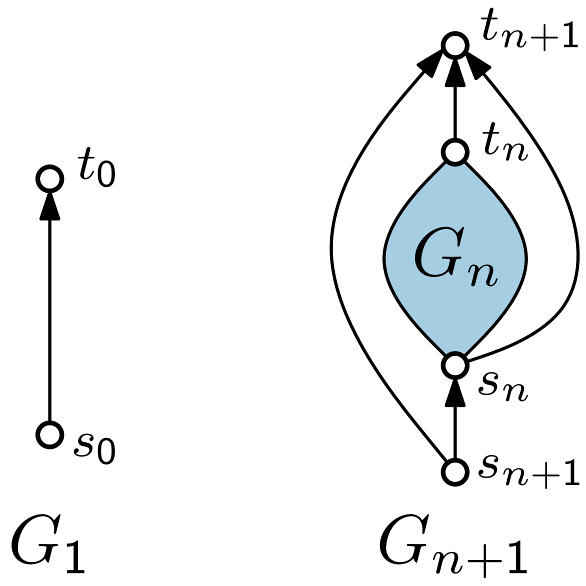

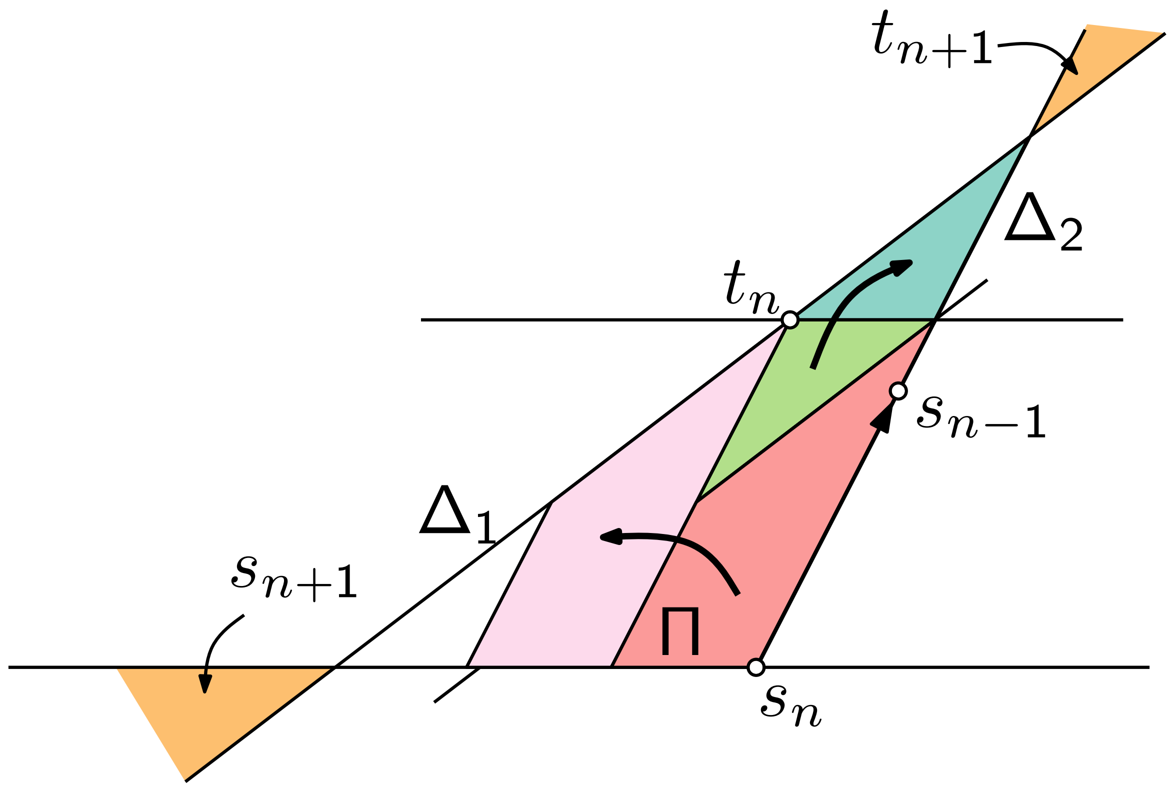

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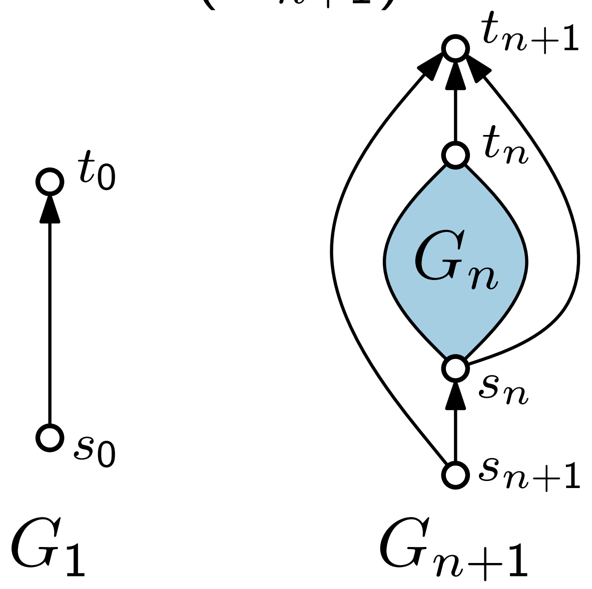
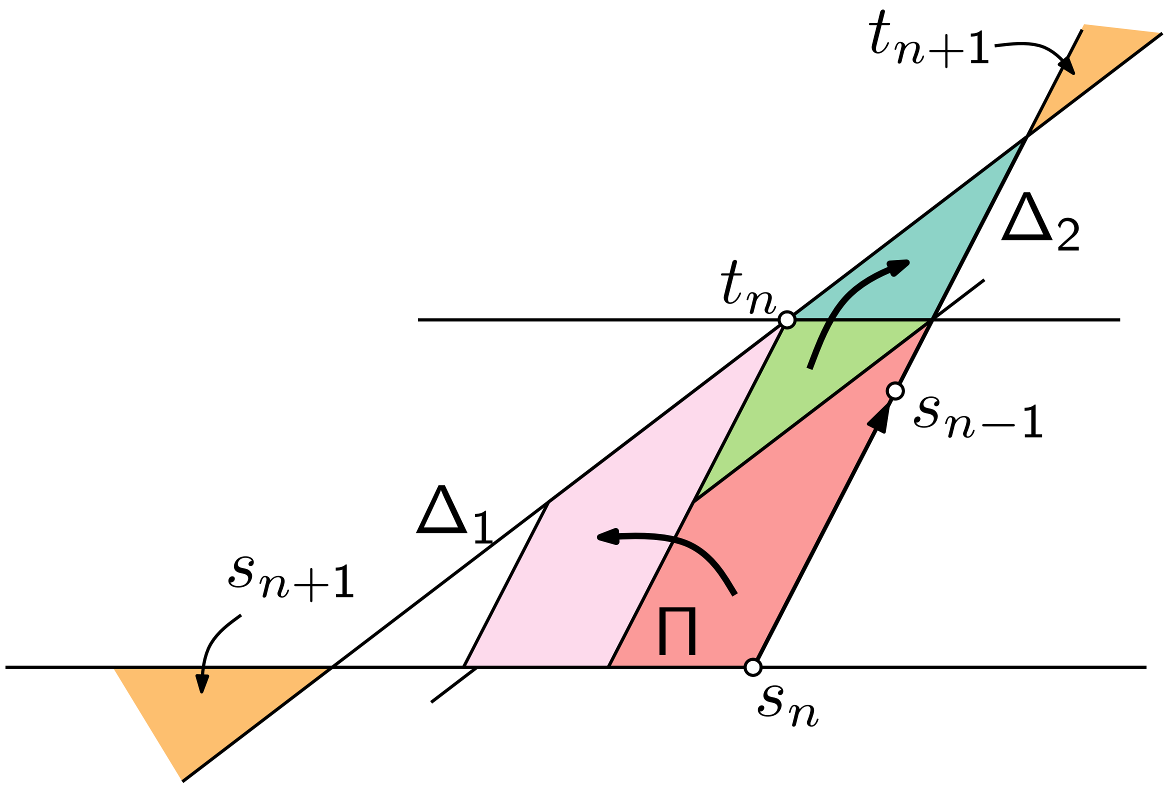
$\Rightarrow 4 \cdot \text{Area}(G_n) < \text{Area}(G_{n+1})$





Discussion

- There exist fixed-parameter (FPT) algorithms to test upward planarity of general digraphs with the parameter being the number of triconnected components.

  [Healy, Lynch 2005, Didimo et al. 2009]

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- There exist fixed-parameter (FPT) algorithms to test upward planarity of general digraphs with the parameter being the number of triconnected components.
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- Many related concepts have been studied:
upward drawings of mixed graphs, upward drawings with layers for the vertices, upward planarity on cylinder/torus, ...

Literature

- See [GD Ch. 6] for detailed explanation on upward planarity.
- See [GD Ch. 3] for divide and conquer methods of series-parallel graphs

Orginal papers referenced:

- [Kelly '87] Fundamentals of Planar Ordered Sets
- [Di Battista &Tamassia '88] Algorithms for Plane Representations of Acyclic Digraphs
- [Garg &Tamassia '95]
  On the Computational Complexity of Upward and Rectilinear Planarity Testing
- [Hutton & Lubiw '96] Upward Planar Drawing of Single-Source Acyclic Digraphs
- [Bertolazzi, Di Battista, Mannino, Tamassia '94]
  Upward Drawings of Triconnected Digraphs
- [Healy & Lynch '05] Building Blocks of Upward Planar Digraphs
- [Didimo, Giordano, Liotta '09] Upward Spirality and Upward Planarity Testing
- [Abbasi, Healy, Rextin '10]
  Improving the running time of embedded upward planarity testing