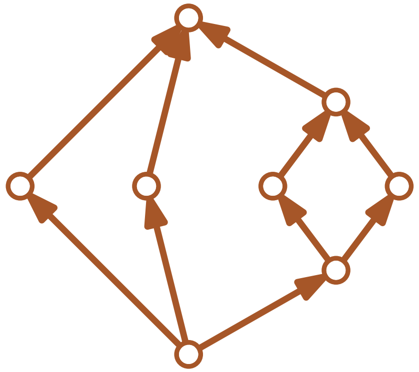
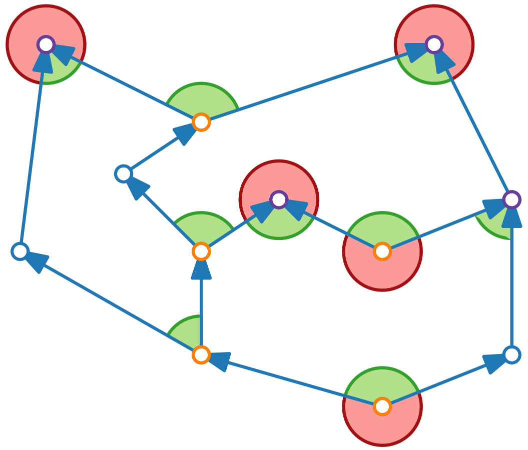
Julius-Maximilians-
UNIVERSITÄT
WÜRZBURG

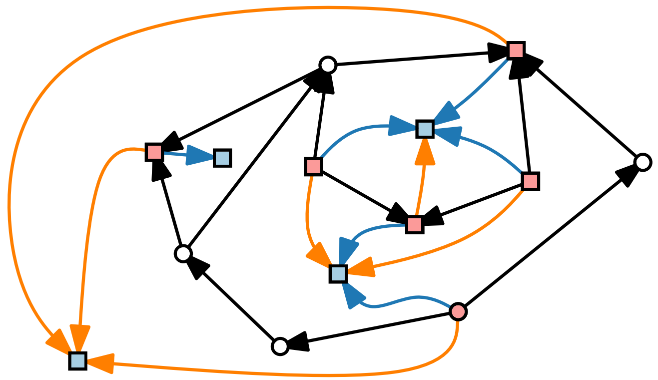
# Visualization of Graphs



## Lecture 5:
## Upward Planar Drawings



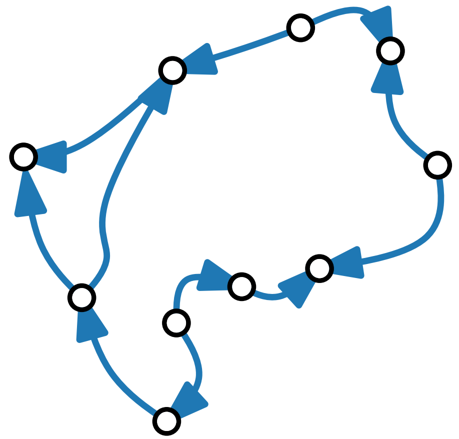
### Part I:
### Recognition



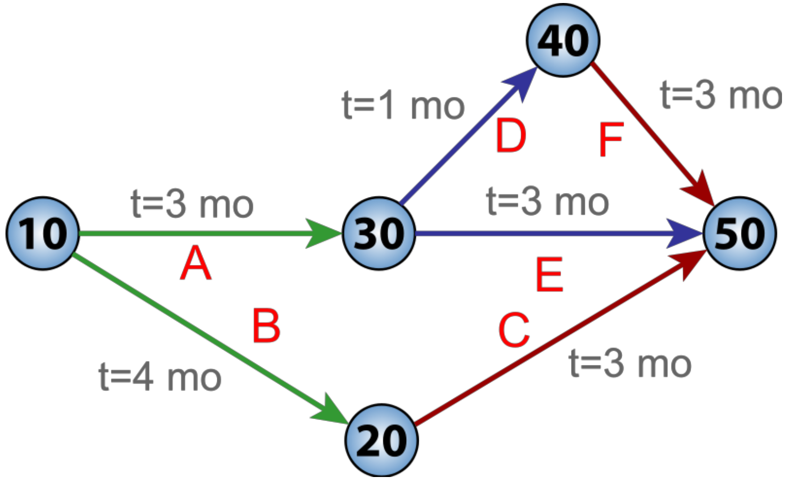
Johannes Zink

Summer semester 2024

# Upward Planar Drawings – Motivation

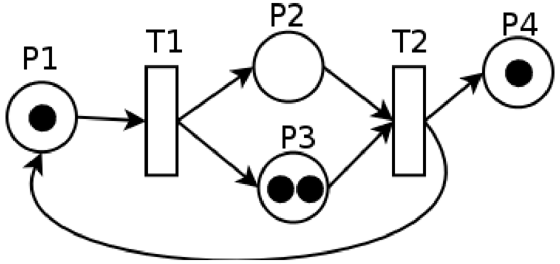


- What may the direction of edges in a directed graph represent?
  - Time
  - Flow
  - Hierarchy
  - ...
- We aim for drawings where the general direction is preserved.



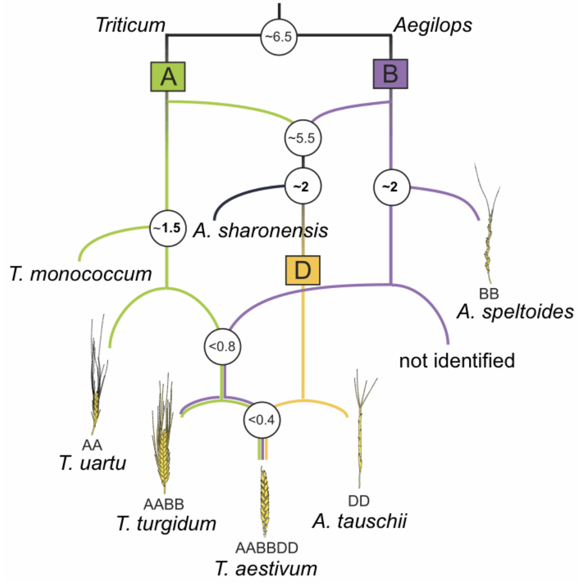

PERT diagram

Program Evaluation and Review Technique
(Project management)




Petri net

Place/Transition net
(Modeling languages for distributed systems)



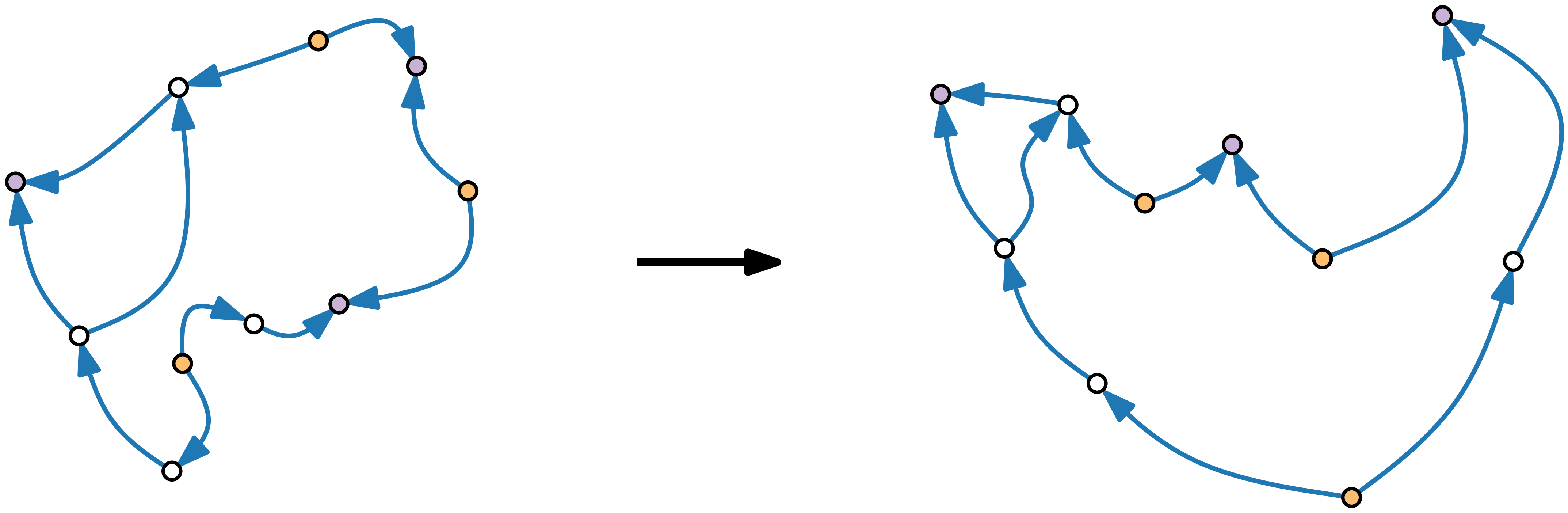

Phylogenetic network

Ancestral trees / networks
(Biology)

# Upward Planar Drawings – Definition

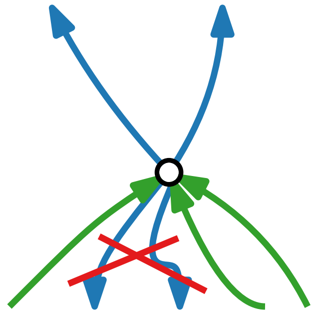
A directed graph (*digraph*) is **upward planar** when it admits a drawing

- that is planar and
- where each edge is drawn as an upward y-monotone curve.

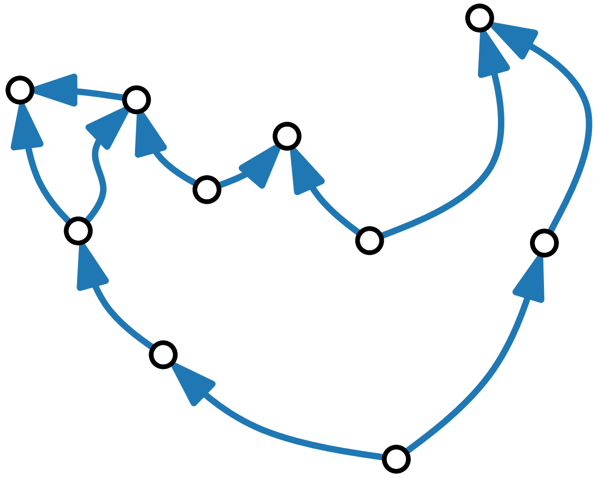


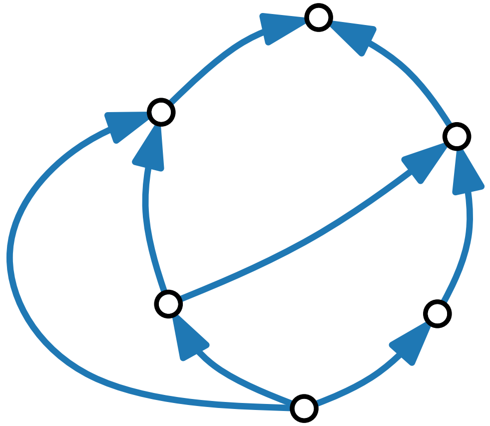
# Upward Planarity – Necessary Conditions

- For an (embedded) digraph to be upward planar, it needs to . . .
  - be planar
  - be acyclic
  - have a bimodal embedding

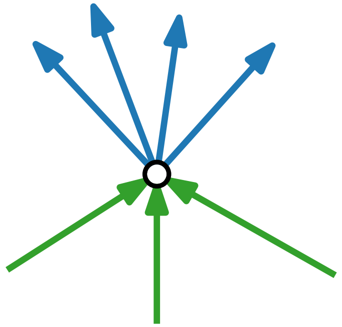


- . . . but these conditions are *not sufficient*. → **Exercise**

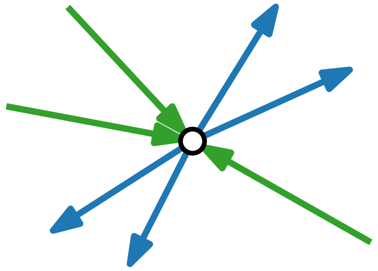




**bimodal** vertex

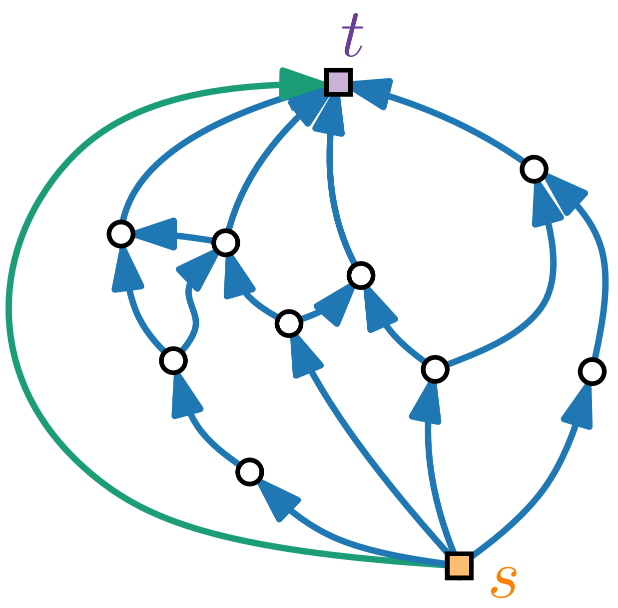


*not* bimodal



# Upward Planarity – Characterization

**Theorem 1.** [Kelly 1987, Di Battista & Tamassia 1988]
For a digraph $G$, the following statements are equivalent:
(1) $G$ is upward planar.
(2) $G$ admits an upward planar straight-line drawing.
(3) $G$ is a spanning subgraph of a planar st-digraph.




*Additionally:*
Embedded such that $s$ and $t$ are on the outer face $f_0$.

*or:*
Edge $(s, t)$ exists.

no crossings

acyclic digraph with a single source $s$ and a single sink $t$

# Upward Planarity – Characterization

**Theorem 1.** [Kelly 1987, Di Battista & Tamassia 1988]
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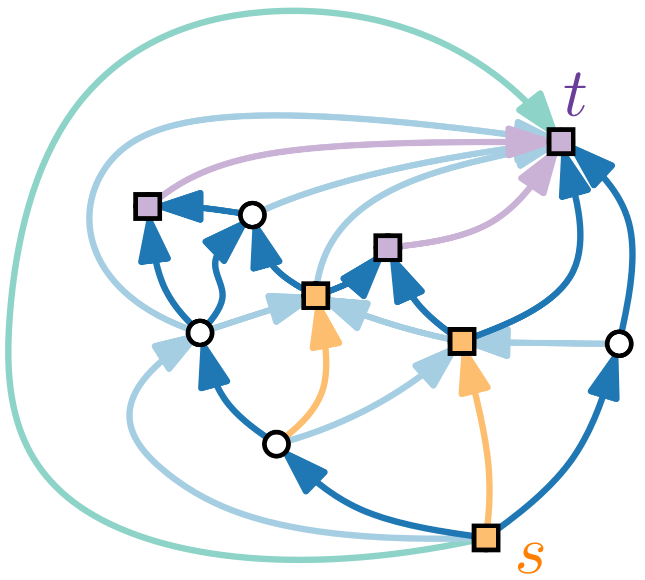
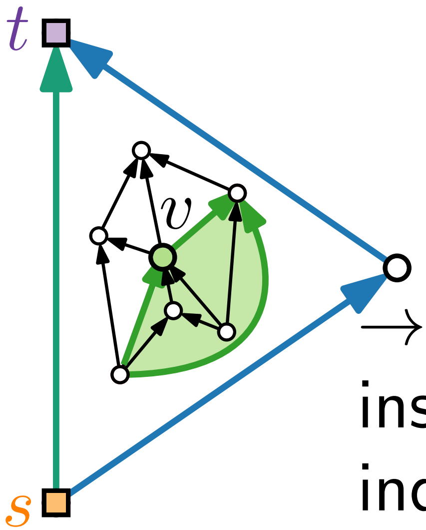



**Proof.**
(2) ⇒ (1) By definition. (1) ⇒ (3) For the proof idea, see the example above.
(3) ⇒ (2) Triangulate & construct drawing:

**Claim.**
Can be drawn in pre-specified triangle.

Induction on the number of vertices $n$.

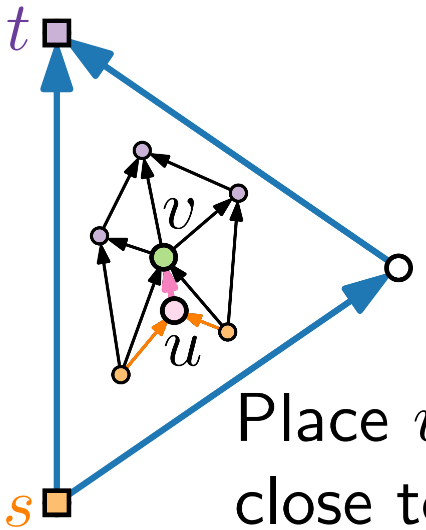
Case 1: chord



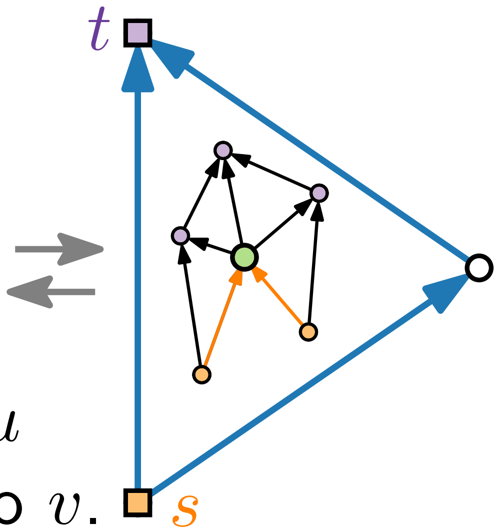

→ two smaller instances; solve inductively

Case 2: no chord

*Idea:* Contract $uv$!




Place $u$ close to $v$.



# Upward Planarity – Complexity

**Theorem.** [Garg & Tamassia, 1995]
Given a *planar acyclic* digraph $G$,
it is NP-hard to decide whether $G$ is upward planar.

**Theorem 2.** [Bertolazzi, Di Battista, Mannino, Tamassia, 1994]
Given an *embedded* planar digraph $G$,
it can be tested in quadratic time whether $G$ is upward planar.

**Corollary.**
Given a *triconnected* planar digraph $G$,
it can be tested in quadratic time whether $G$ is upward planar.

**Theorem.** [Hutton & Lubiw, 1996]
Given an acyclic *single-source* digraph $G$,
it can be tested in linear time whether $G$ is upward planar.

# The Problem

**Fixed Embedding Upward Planarity Testing.**
Let $G$ be a plane digraph, let $F$ be the set of faces of $G$, and let $f_0$ be the outer face of $G$.
Test whether $G$ is upward planar (w.r.t. to $F$ and $f_0$).

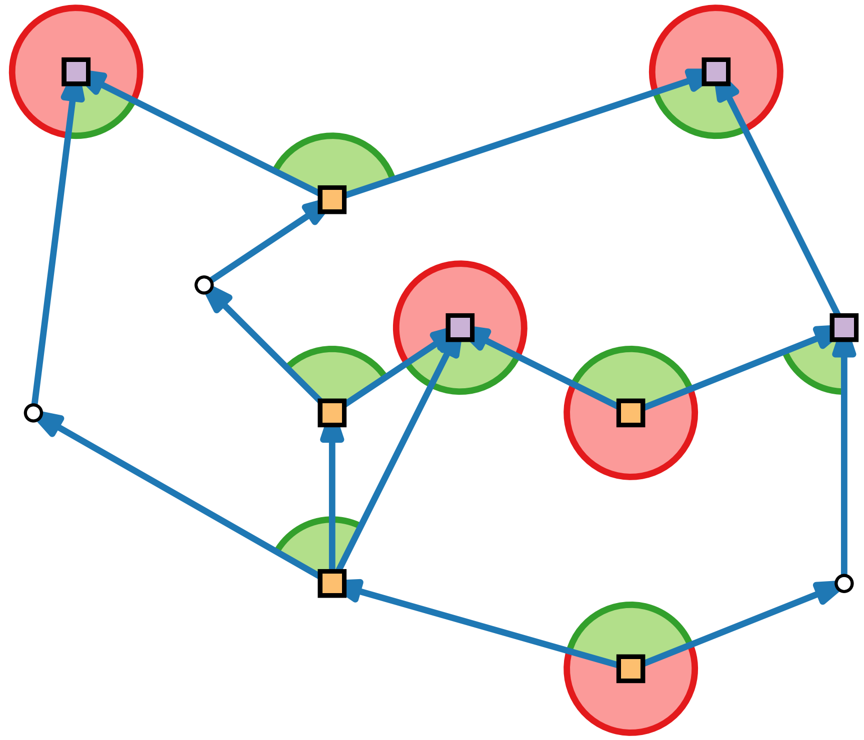
**Plan.**

- Find a property that any upward planar drawing of $G$ satisfies.
- Formalize this property.
- Specify an algorithm to test this property.

# Angles, Local Sources & Sinks

## Definitions.

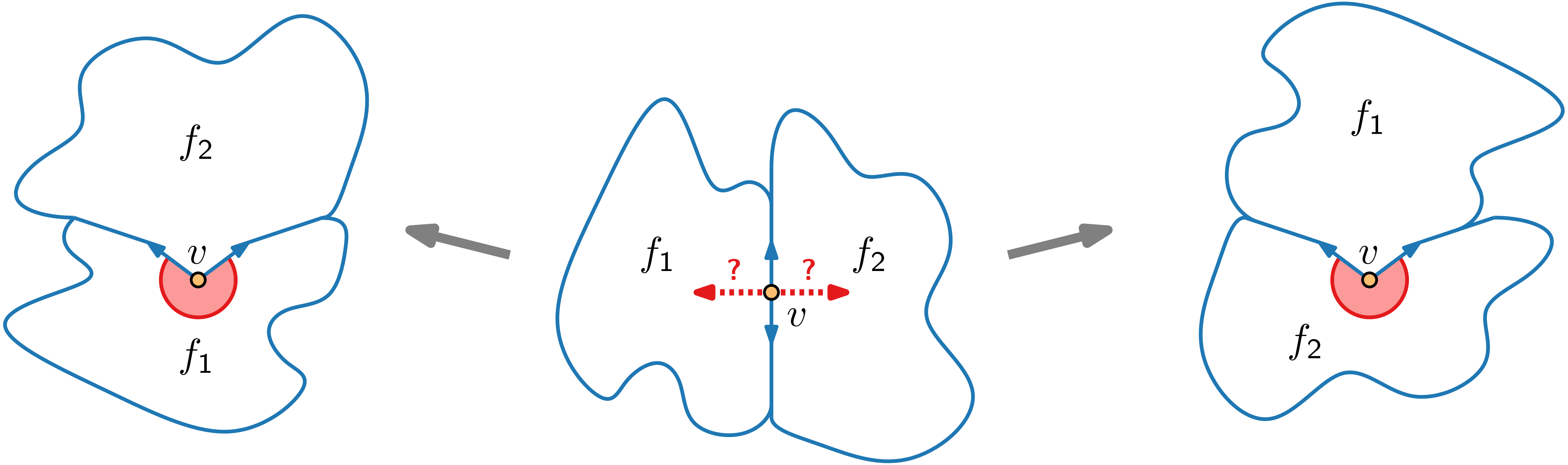
- A vertex $v$ is a **local source** w.r.t. to a face $f$ if $v$ has two outgoing edges on $\partial f$. ← boundary of $f$
- A vertex $v$ is a **local sink** w.r.t. to a face $f$ if $v$ has two incoming edges on $\partial f$.
- An angle $\alpha$ at a local source/sink is **large** if $\alpha > \pi$ and **small** otherwise.
- $L(v) = \#$ large angles at $v$
- $L(f) = \#$ large angles in $f$
- $S(v) = \#$ small angles at $v$
- $S(f) = \#$ small angles at $f$
- $A(f) = \#$ local sources w.r.t. to $f$ $= \#$ local sinks w.r.t. to $f$



**Lemma 1.**
$L(f) + S(f) = 2A(f)$

# Assignment Problem

- Observe that the **global sources** and **global sinks** have precisely one large angle
- All other vertices have only small angles.
- Let $v$ be a global source and let it be incident to faces $f_1$ and $f_2$.
- Does $v$ have a large angle in $f_1$ or $f_2$?

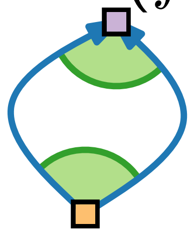


# Angle Relations

**Lemma 2.**

$$L(f) - S(f) = \begin{cases} -2 & \text{if } f \neq f_0, \\ +2 & \text{if } f = f_0. \end{cases}$$

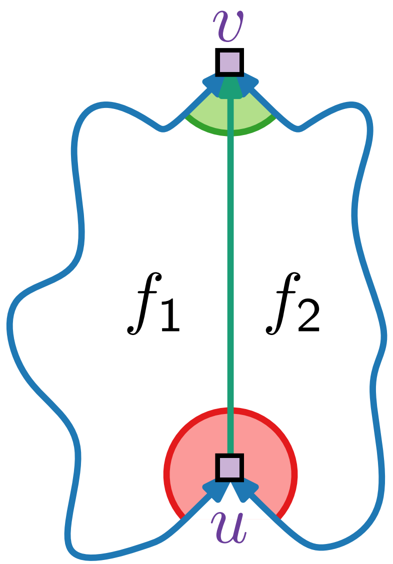
**Proof** by induction on $L(f)$.

- $L(f) = 0$  $\Rightarrow S(f) = 2$ ✓

- $L(f) \geq 1$

Split $f$ with edge from a large angle at a "low" sink $u$ to...

- sink $v$ with small angle:




$$L(f) - S(f) = \underbrace{L(f_1)}_{-2} + \underbrace{L(f_2)}_{-2} + 1 - (S(f_1) + S(f_2) - 1)$$

$$= -2 - 2 + 2 = -2$$

# Angle Relations

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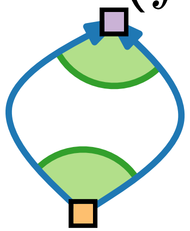
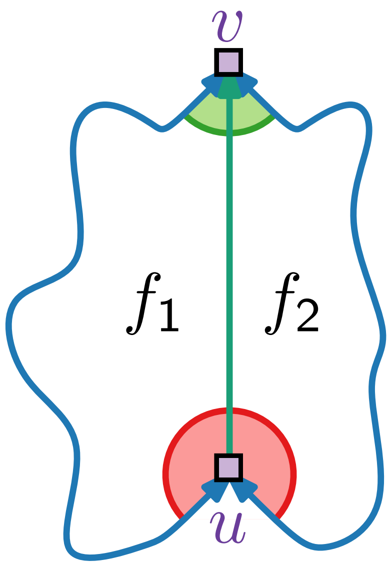


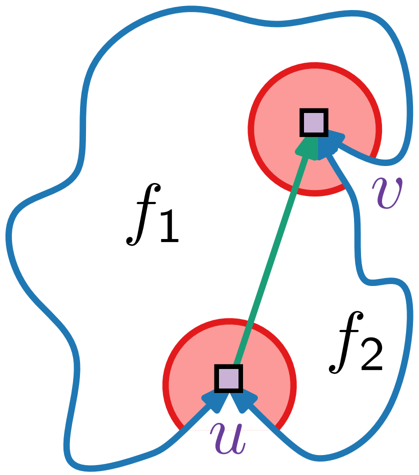
$\Rightarrow S(f) = 2$ ✓

- $L(f) \geq 1$

Split $f$ with edge from a large angle at a "low" sink $u$ to...

- sink $v$ with small/large angle:




$$L(f) - S(f) = \underbrace{L(f_1)}_{-2} + \underbrace{L(f_2)}_{-2} + 1 - (S(f_1) + S(f_2) - 1)$$

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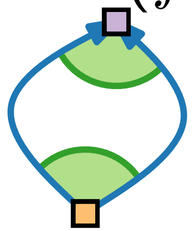
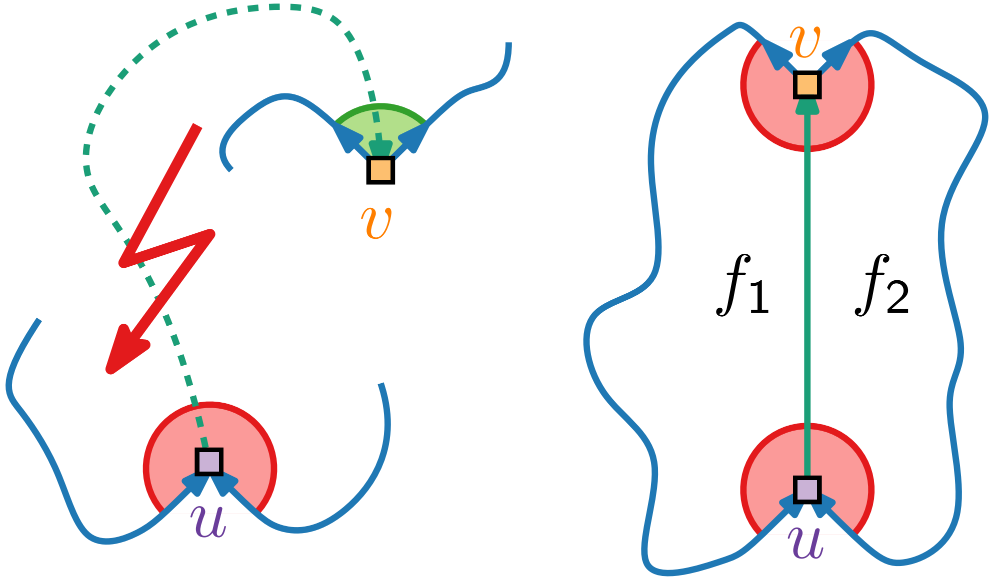
- $L(f) = 0$  $\Rightarrow S(f) = 2$ ✓

- $L(f) \geq 1$

Split $f$ with edge from a large angle at a "low" sink $u$ to...

- source $v$ with ~~small~~/large angle:




$$L(f) - S(f) = \underbrace{L(f_1)}_{-2} + \underbrace{L(f_2)}_{-2} + 2 - \left(S(f_1) + S(f_2)\right)$$

$$= -2 - 2 + 2 = -2$$

# Angle Relations

**Lemma 2.**

$$L(f) - S(f) = \begin{cases} -2 & \text{if } f \neq f_0, \\ +2 & \text{if } f = f_0. \end{cases}$$

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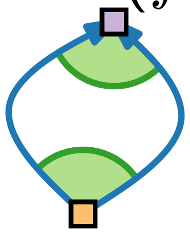
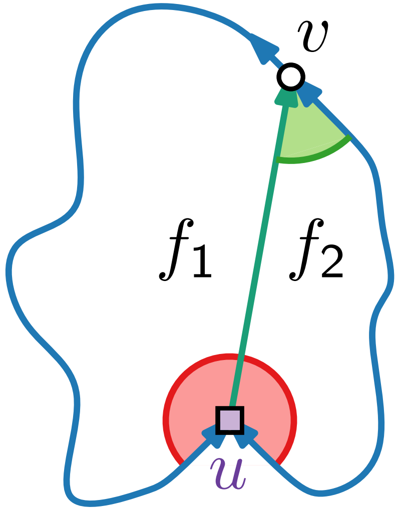
- $L(f) = 0$  $\Rightarrow S(f) = 2$ ✓

- $L(f) \geq 1$

Split $f$ with edge from a large angle at a "low" sink $u$ to...

- vertex $v$ that is neither source nor sink:




$$L(f) - S(f) = \underbrace{L(f_1)}_{-2} + \underbrace{L(f_2)}_{-2} + 1 - (S(f_1) + S(f_2) - 1)$$
$$= -2 - 2 + 2 = -2$$

- Otherwise "high" source $u$ exists. $\rightarrow$ symmetric
- Similar argument for the outer face $f_0$.

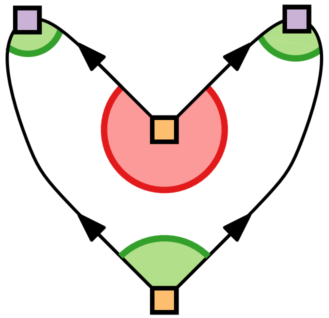
# Number of Large Angles

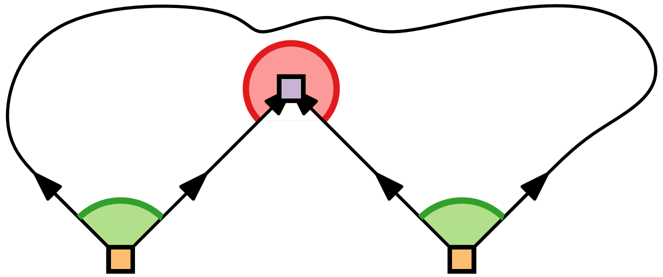
**Lemma 3.**
In every upward planar drawing of $G$, it holds that

- for each vertex $v$: $L(v) = \begin{cases} 0 & \text{if } v \text{ is an inner vertex,} \\ 1 & \text{if } v \text{ is a gobal source / sink;} \end{cases}$
- for each face $f$: $L(f) = \begin{cases} A(f) - 1 & \text{if } f \neq f_0, \\ A(f) + 1 & \text{if } f = f_0. \end{cases}$

**Proof.**
Lemma 1: $L(f) + S(f) = 2A(f)$
Lemma 2: $L(f) - S(f) = \pm 2.$

$$\Rightarrow 2L(f) = 2A(f) \pm 2.$$





# Assignment of Large Angles to Faces

Let $S$ be the set of (global) sources, and let $T$ be the set of (global) sinks.

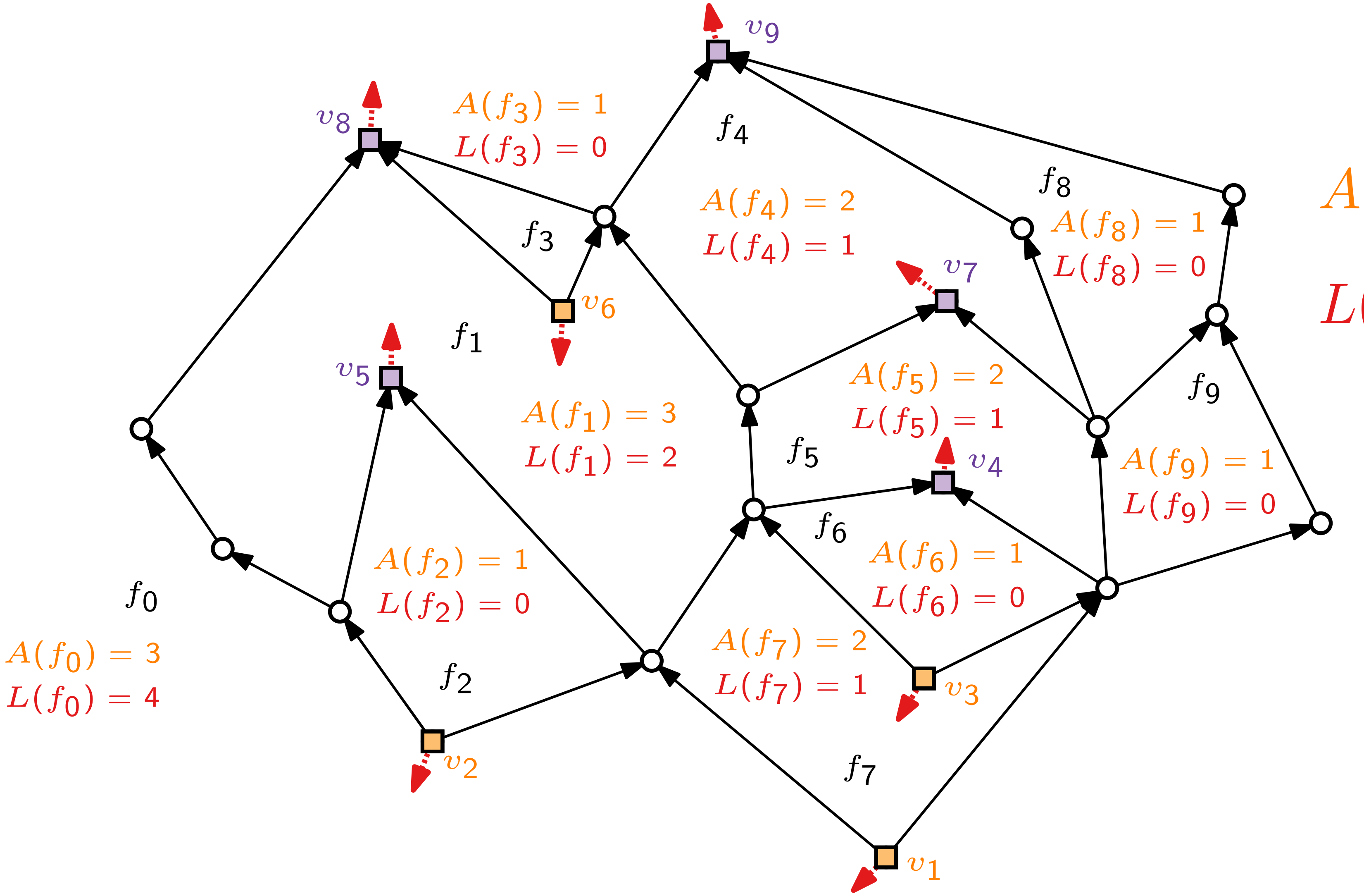
**Definition.**
A **consistent assignment** $\Phi\colon S \cup T \to F$ is a mapping with

$$\Phi\colon v \mapsto \text{ incident face, where } v \text{ forms a large angle}$$

such that

$$|\Phi^{-1}(f)| = L(f) = \begin{cases} A(f) - 1 & \text{if } f \neq f_0, \\ A(f) + 1 & \text{if } f = f_0. \end{cases}$$

# Example of Angle-to-Face Assignment




global sources & sinks

$A(f) =$ # local sources/sinks of $f$

$L(f) =$ # large angles of $f$

assignment
$\Phi\colon S \cup T \to F$

# Result Characterization

**Theorem 3.**
Let $G$ be an acyclic plane digraph with embedding given by $F$ and $f_0$.

Then $G$ is upward planar (respecting $F$ and $f_0$)
$\Leftrightarrow$ $G$ is bimodal and there exists a consistent assignment $\Phi$.

**Proof.**
$\Rightarrow$: As constructed before.

$\Leftarrow$: Idea:

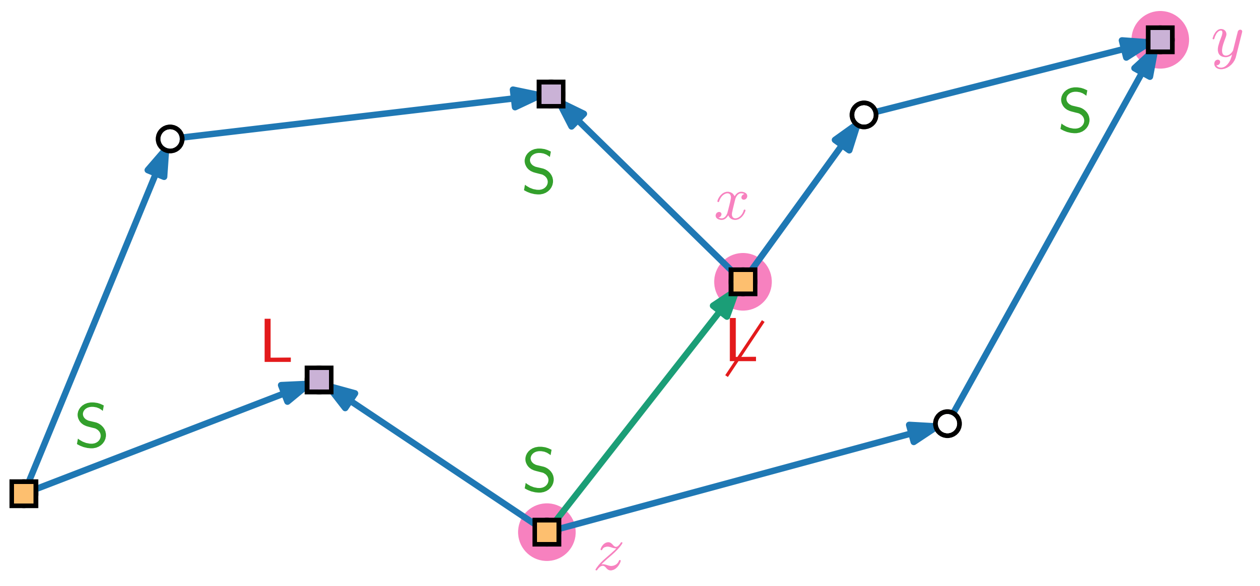
- Construct planar st-digraph that is a supergraph of $G$.
- Apply equivalence from Theorem 1.

$G$ is upward planar. $\Leftrightarrow$ $G$ is a spanning subgraph of a planar st-digraph.

# Refinement Algorithm: $\Phi, F, f_0 \to$ st-digraph

Let $f$ be a face.
Consider the clockwise angle sequence $\sigma_f$ of L / S on local sources and sinks of $f$.

- Goal: Add edges to break large angles (sources and sinks).
- For $f \neq f_0$ with $|\sigma_f| \geq 2$ containing $\langle$L, S, S$\rangle$ at vertices $x, y, z$:
- $x$ source $\Rightarrow$ insert edge $(z, x)$



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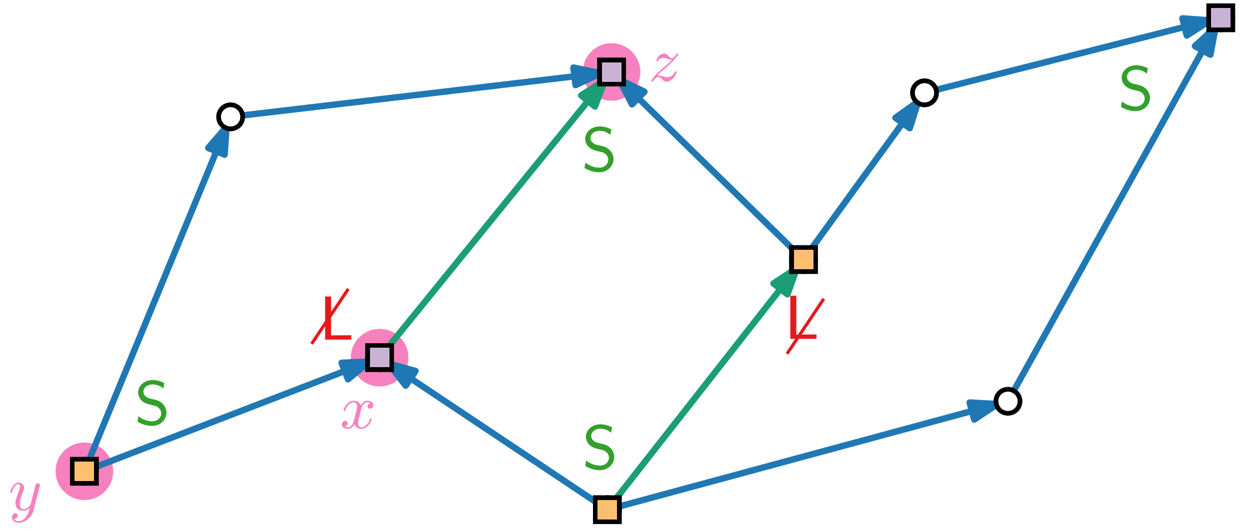
- $x$ sink $\Rightarrow$ insert edge $(x, z)$.



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Let $f$ be a face.
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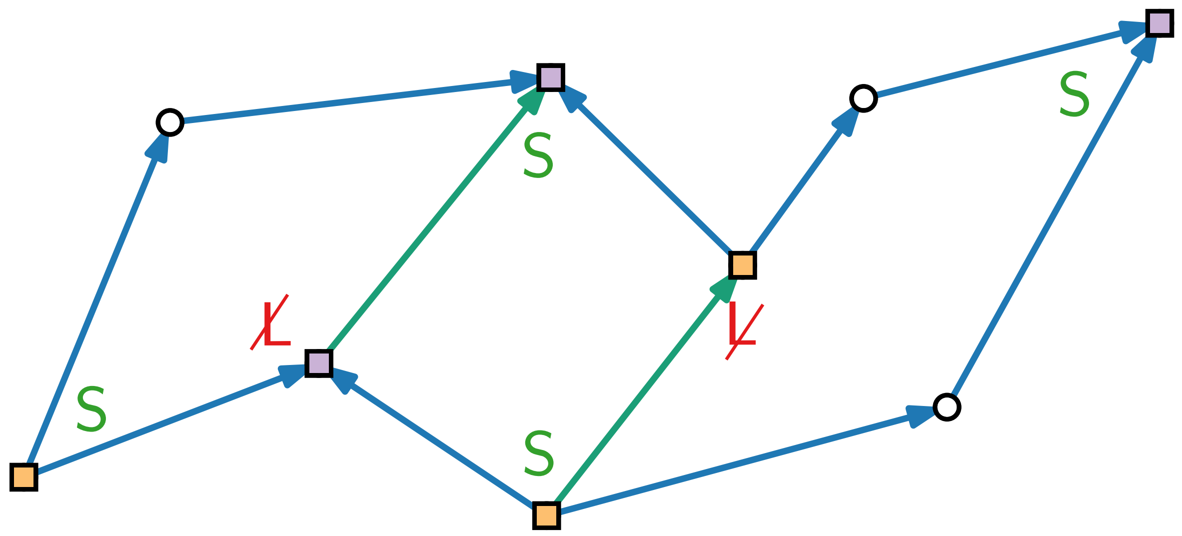
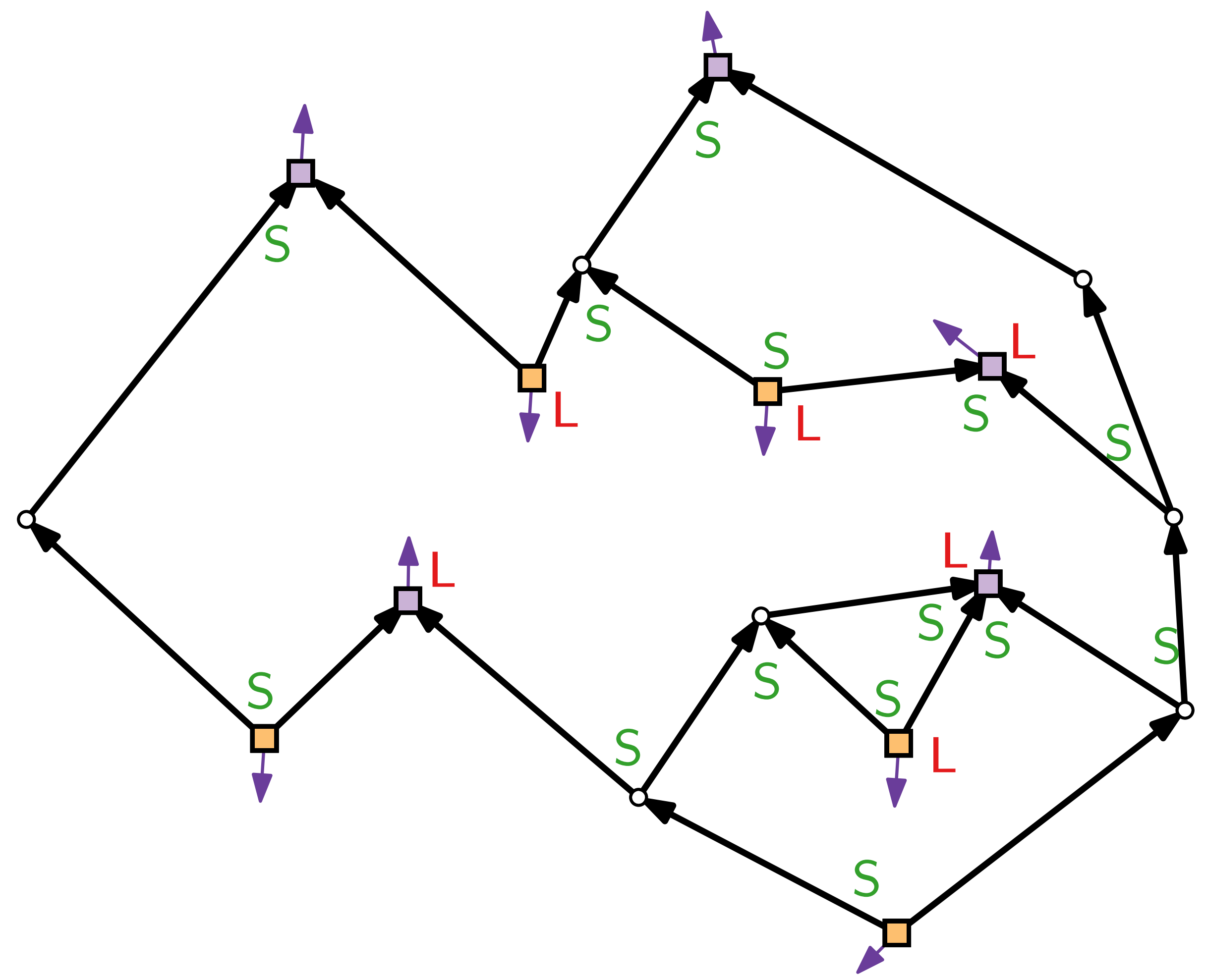
- For $f \neq f_0$ with $|\sigma_f| \geq 2$ containing $\langle$L, S, S$\rangle$ at vertices $x, y, z$:
- $x$ source $\Rightarrow$ insert edge $(z, x)$
- $x$ sink $\Rightarrow$ insert edge $(x, z)$.
- Refine outer face $f_0$ similarly.

  $\rightarrow$ **Exercise**

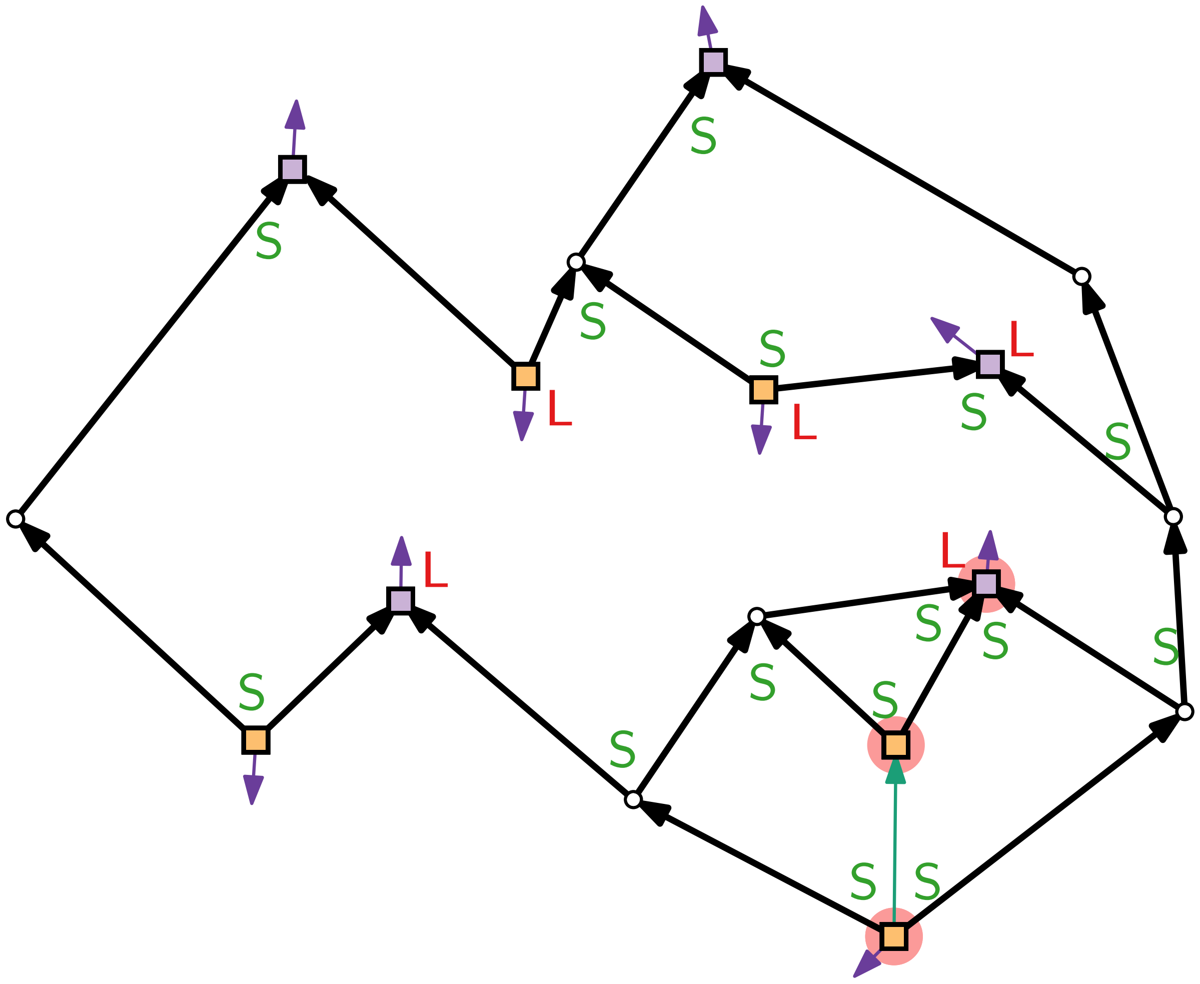



- Refine all faces. $\Rightarrow$ $G$ is contained in a planar st-digraph.
- Planarity, acyclicity, bimodality are invariants under construction.

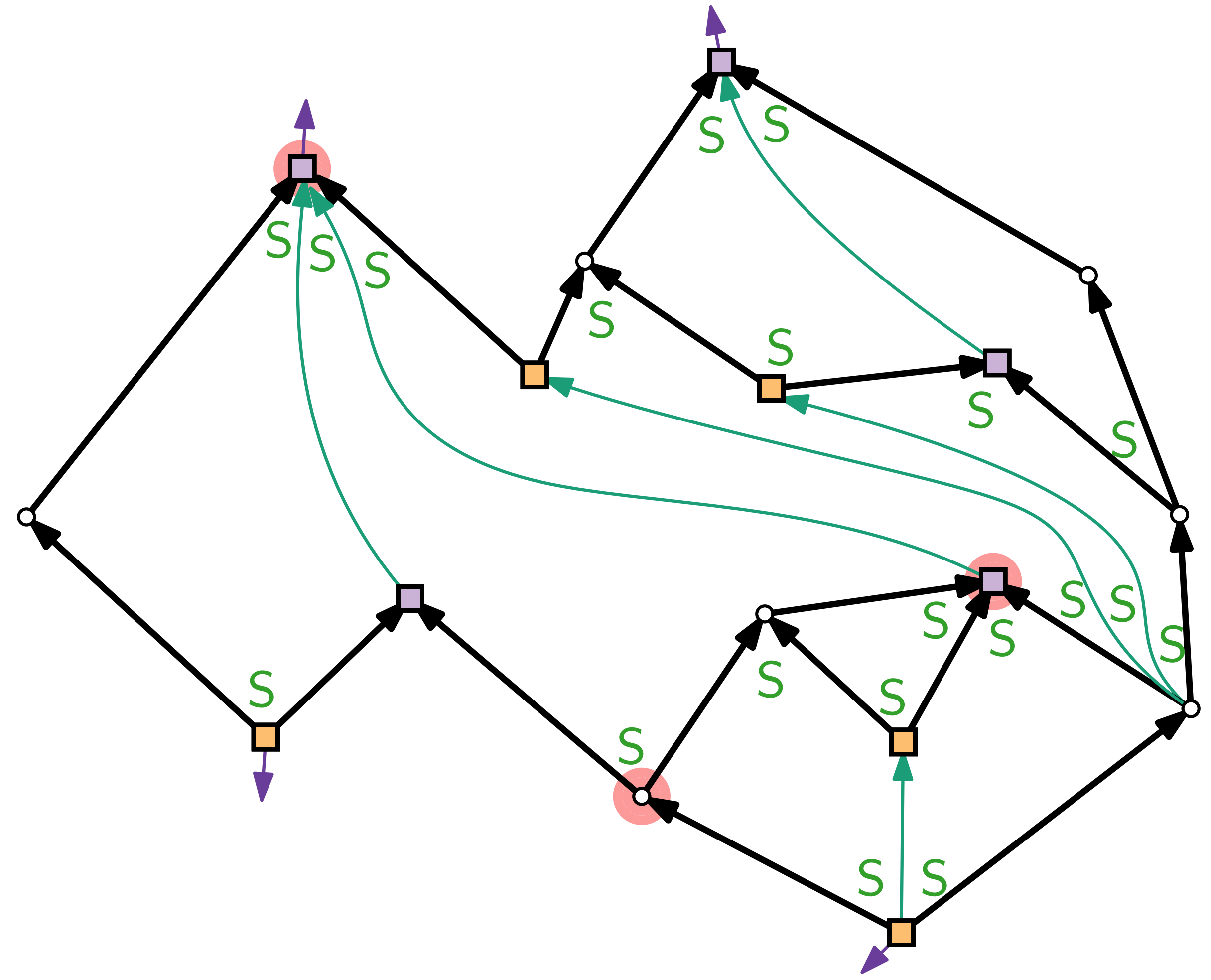
# Refinement Example



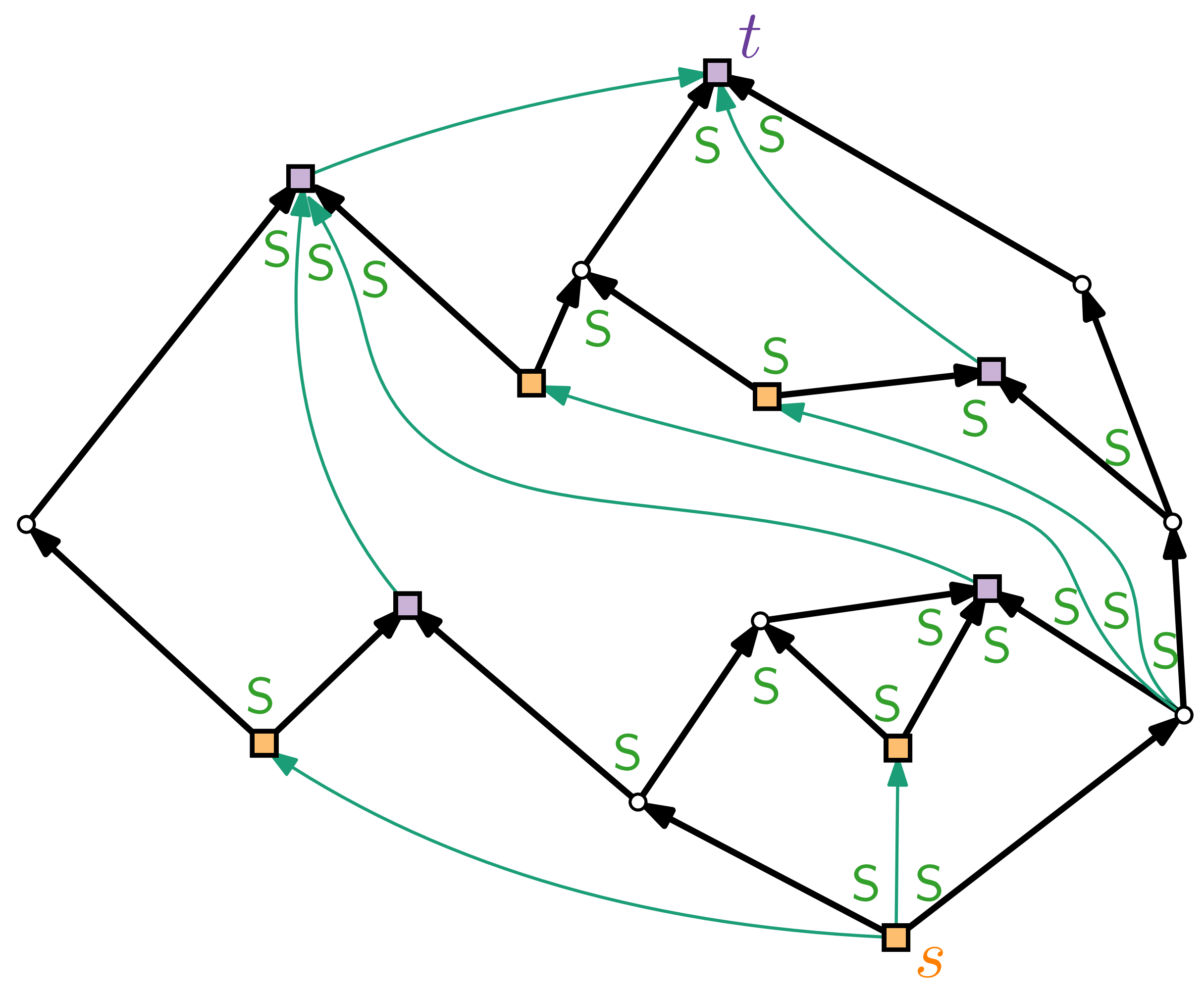
# Refinement Example



# Refinement Example



# Refinement Example



# Result Upward Planarity Test

**Theorem 2.** [Bertolazzi, Di Battista, Mannino, Tamassia '94]
Given an *embedded* planar digraph $G$,
we can test in quadratic time whether $G$ is upward planar.

**Proof.**

- Test for bimodality.
- Test for a consistent assignment $\Phi$ (via flow network).
- If $G$ bimodal and $\Phi$ exists, refine $G$ to plane st-digraph $H$.
- Draw $H$ upward planar.
- Deleted edges added in refinement step.

# Finding a Consistent Assignment

**Idea.** Flow $(v, f) = 1$
from global source / sink $v$ to the incident face $f$ its large angle gets assigned to.

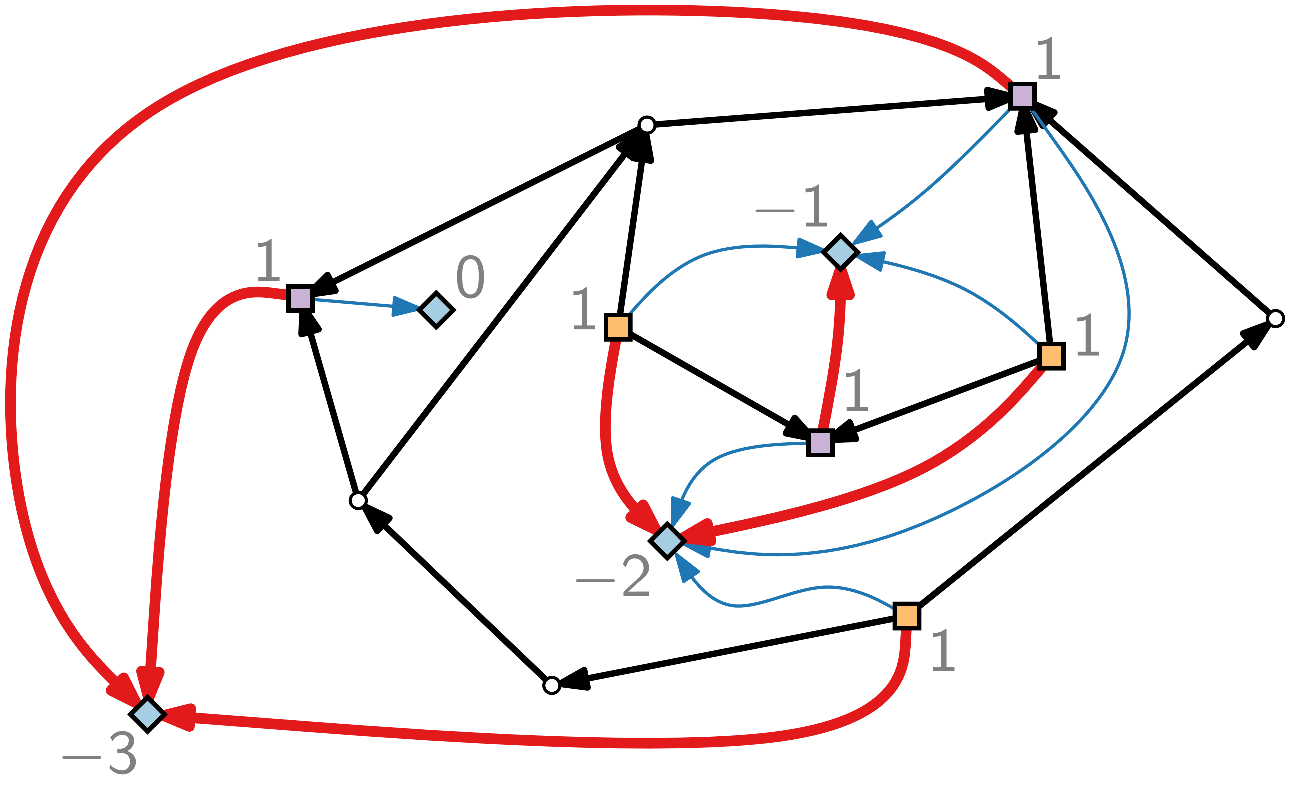
nodes of flow network — edges of flow network — supplies/demands of nodes — lower/upper bounds on edge capcities

**Flow network.**

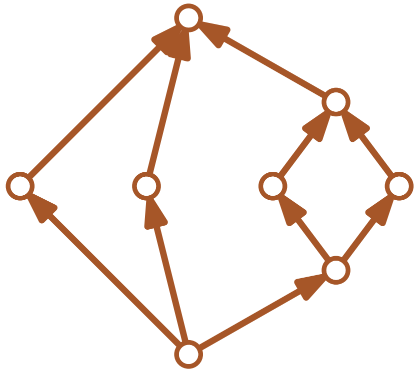
$N_{F,f_0}(G) = ((W, E'); b; \ell; u)$

- $W = \{v \in V(G) \mid v$ source or sink$\} \cup F(G)$
- $E' = \{(v, f) \mid v$ incident to $f\}$
- $\ell(e) = 0 \; \forall e \in E'$
- $u(e) = 1 \; \forall e \in E'$
- $b(w) = \begin{cases} 1 & \forall w \in W \cap V(G) \\ -(A(w) - 1) & \forall w \in F(G) \setminus \{f_0\} \\ -(A(w) + 1) & w = f_0 \end{cases}$

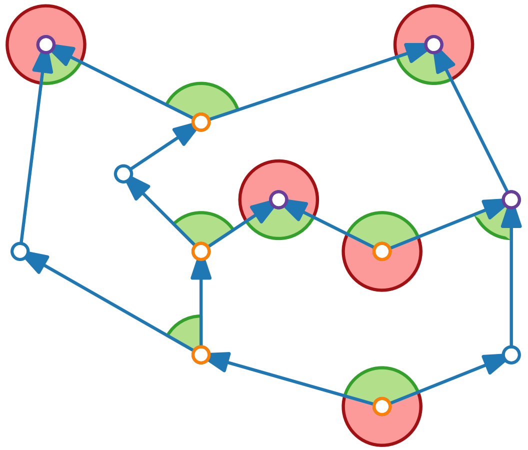
**Example.**



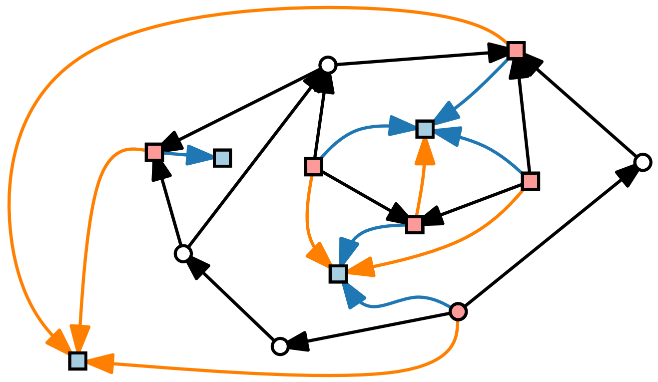
# Visualization of Graphs



## Lecture 5:
## Upward Planar Drawings



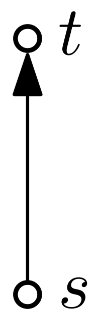
### Part II:
### Series-Parallel Graphs



# Series-Parallel Graphs

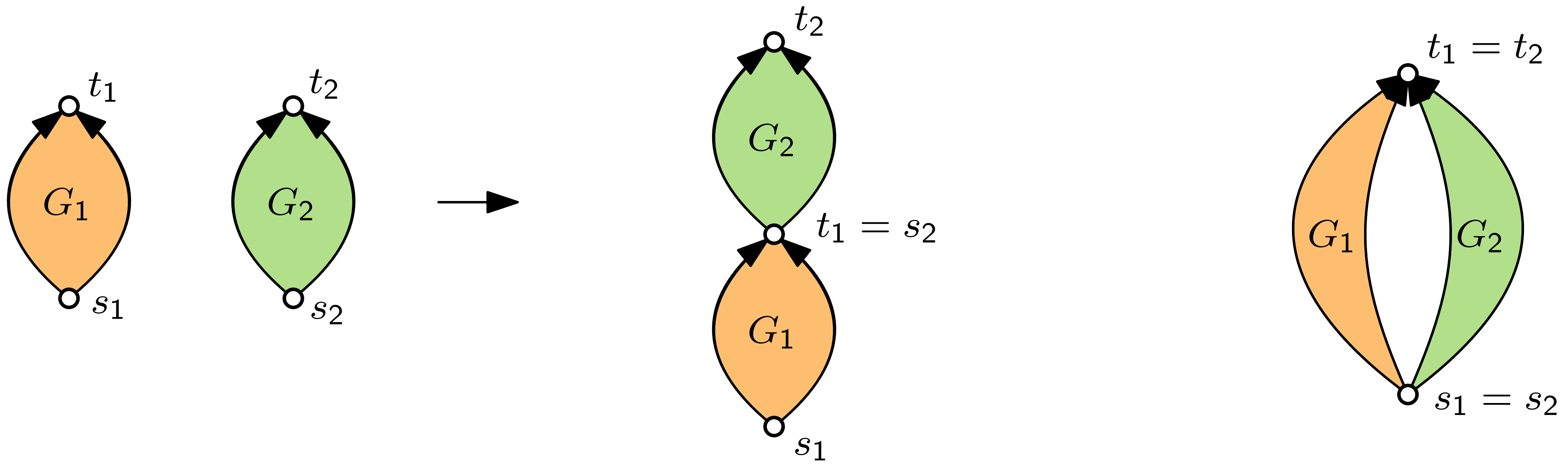
A graph $G$ is **series-parallel** if

- it contains a single (directed) edge $(s, t)$, or
- it consists of two series-parallel graphs $G_1$, $G_2$ with sources $s_1$, $s_2$ and sinks $t_1$, $t_2$ that are combined using one of the following rules:




*Convince yourself that series-parallel graphs are (upward) planar!*

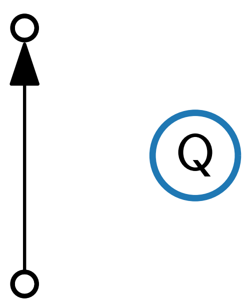
**Series composition** **Parallel composition**

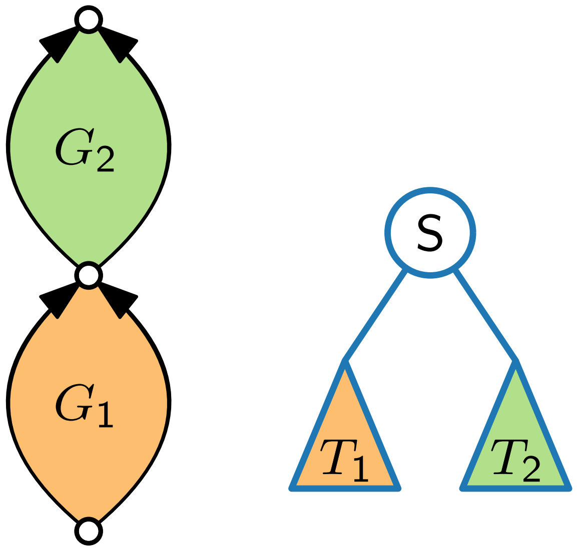


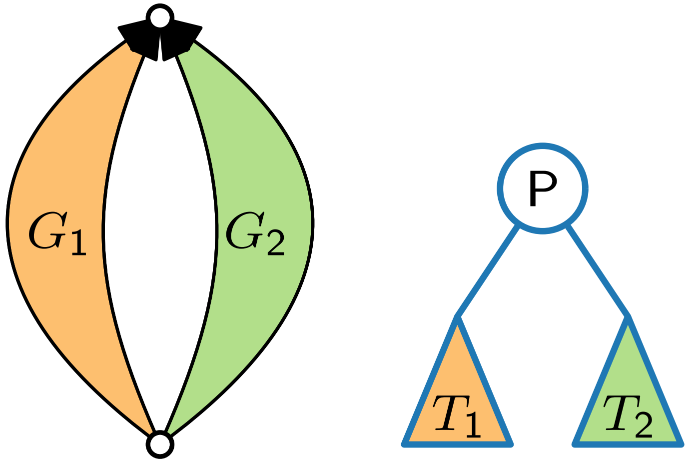
# Series-Parallel Graphs – Decomposition Tree

A **decomposition tree** of $G$ is a binary tree $T$ with nodes of three types: **S**, **P** and **Q**.

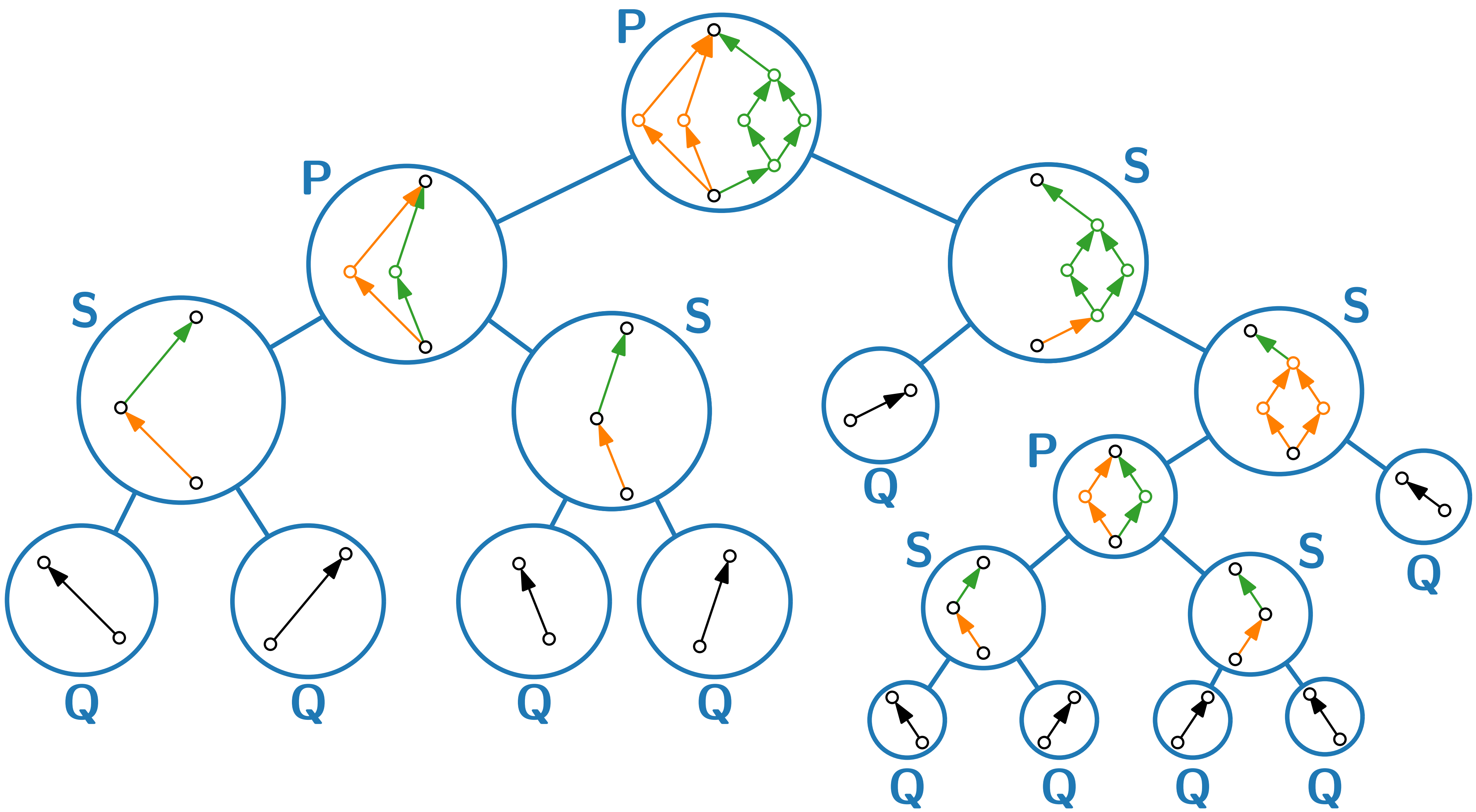
- A **Q**-node represents a single edge.
- An **S**-node represents a series composition; its children $T_1$ and $T_2$ represent $G_1$ and $G_2$.
- A **P**-node represents a parallel composition; its children $T_1$ and $T_2$ represent $G_1$ and $G_2$



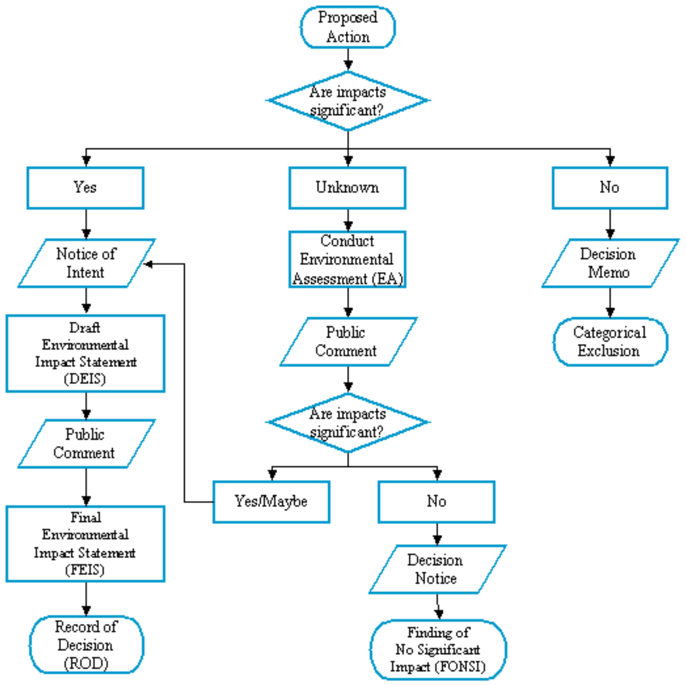



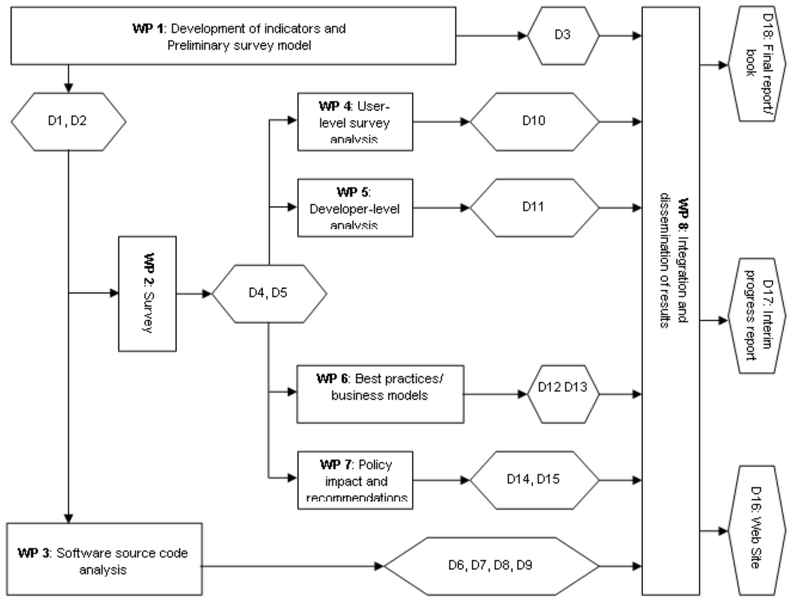
# Series-Parallel Graphs – Decomposition Example



# Series-Parallel Graphs – Applications




Flowcharts




PERT-Diagrams
(Program Evaluation and Review Technique)

**Computational complexity:**
Series-parallel graphs often admit linear-time algorithms for NP-hard problems, e.g., minimum maximal matching, maximum independent set, Hamiltonian completion.

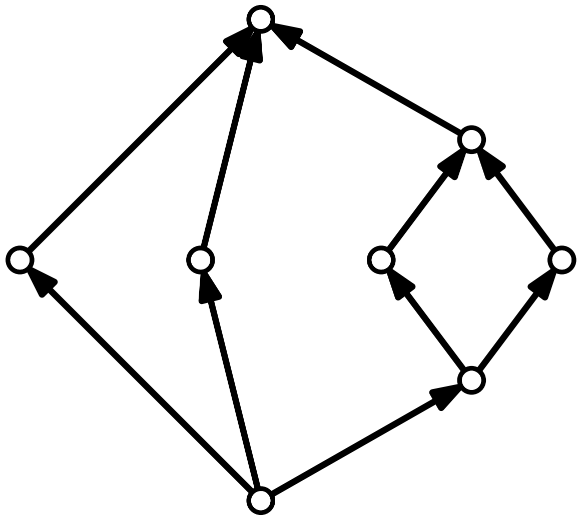
# Series-Parallel Graphs – Drawing Style

## Drawing conventions

- Planarity
- Straight-line edges
- Upward

## Drawing aesthetics to optimize

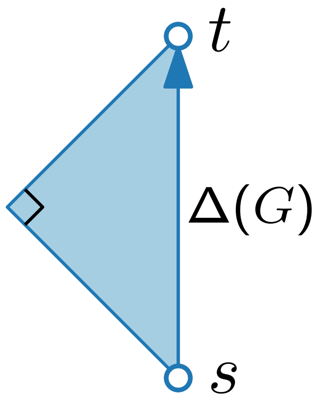
- Area
- Symmetry



# Series-Parallel Graphs – Straight-Line Drawings

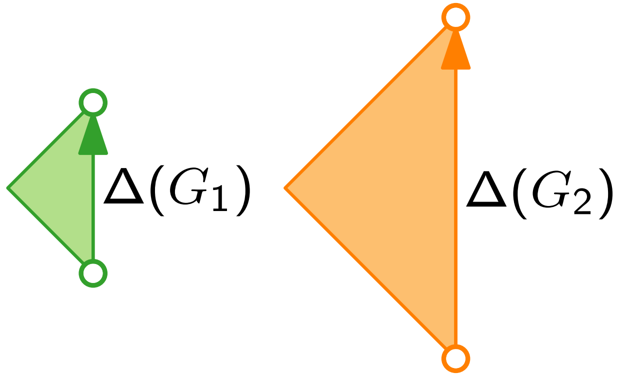
## Divide & conquer algorithm using the decomposition tree

- Invariant: draw $G$ inside a right-angled isosceles bounding triangle $\Delta(G)$ with $s$ at the bottom and $t$ at the top



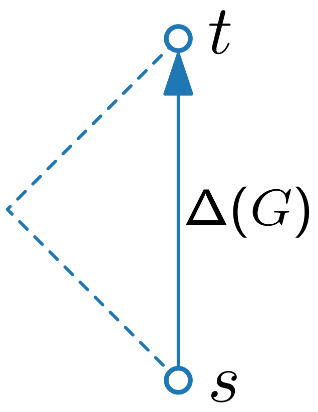

**Base case:** Q-nodes

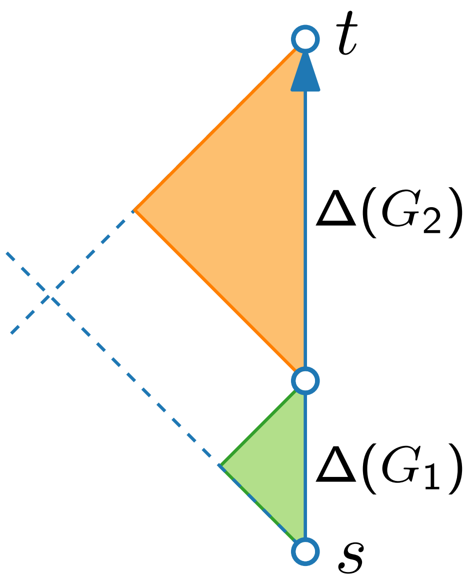
**Divide:** Draw $G_1$ and $G_2$ first

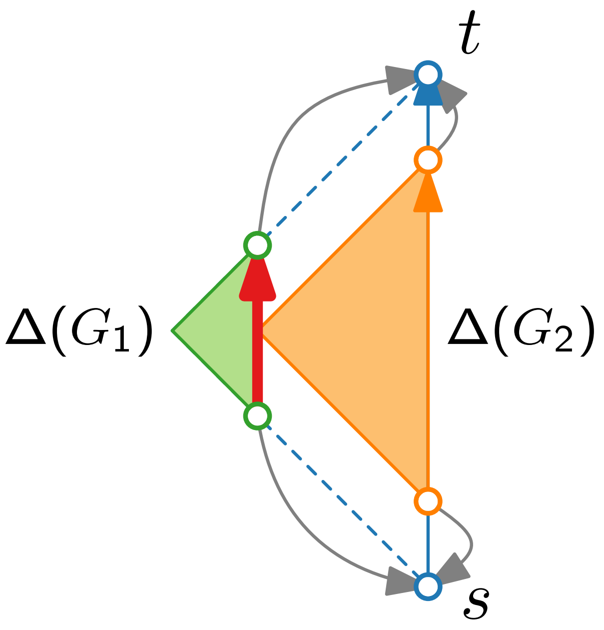



**Conquer:**

- S-nodes: series compositions
- P-nodes: parallel compositions



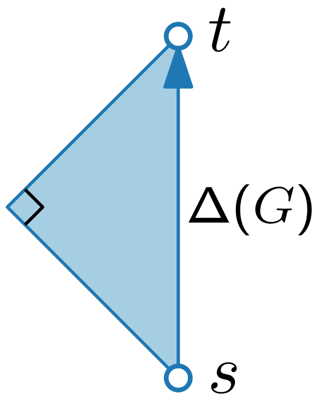




Do you see any problem?

single edge

change embedding!

# Series-Parallel Graphs – Straight-Line Drawings




## Divide & conquer algorithm using the decomposition tree

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**Base case:** Q-nodes

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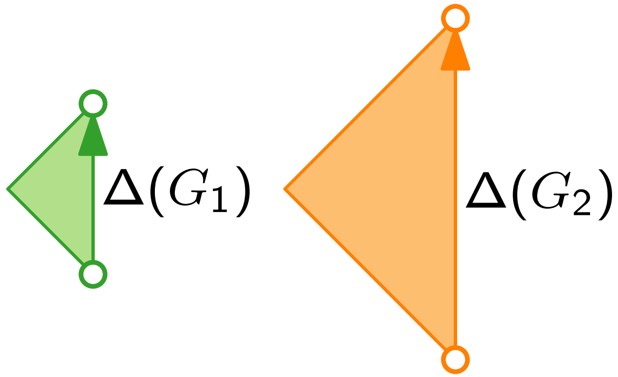


**Conquer:**

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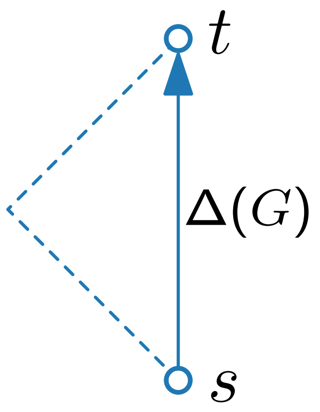
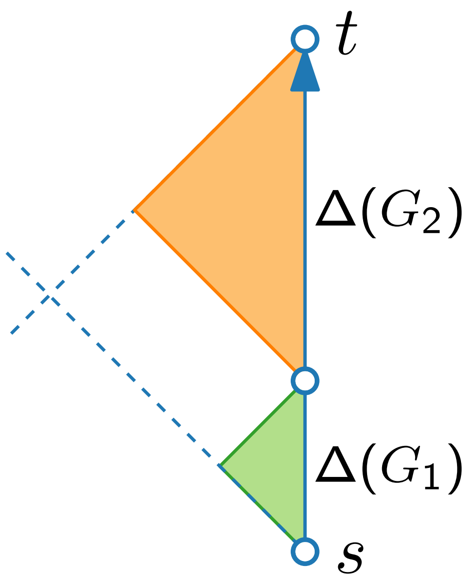
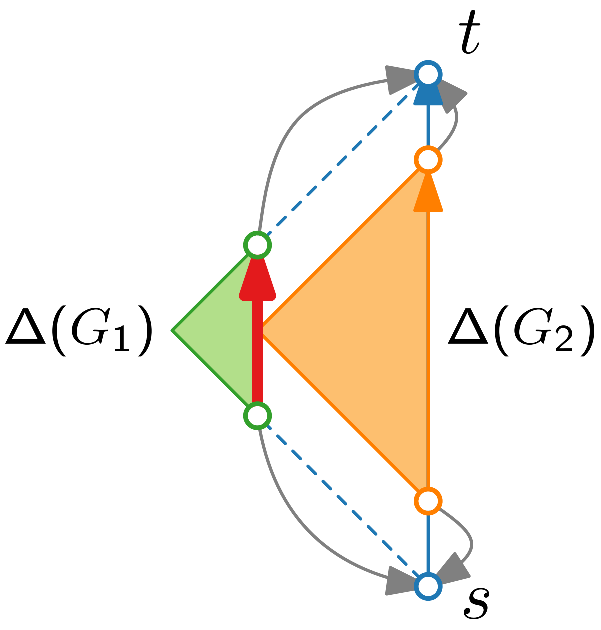
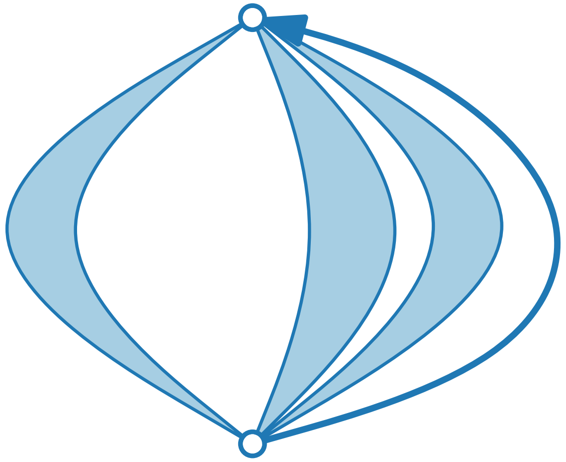


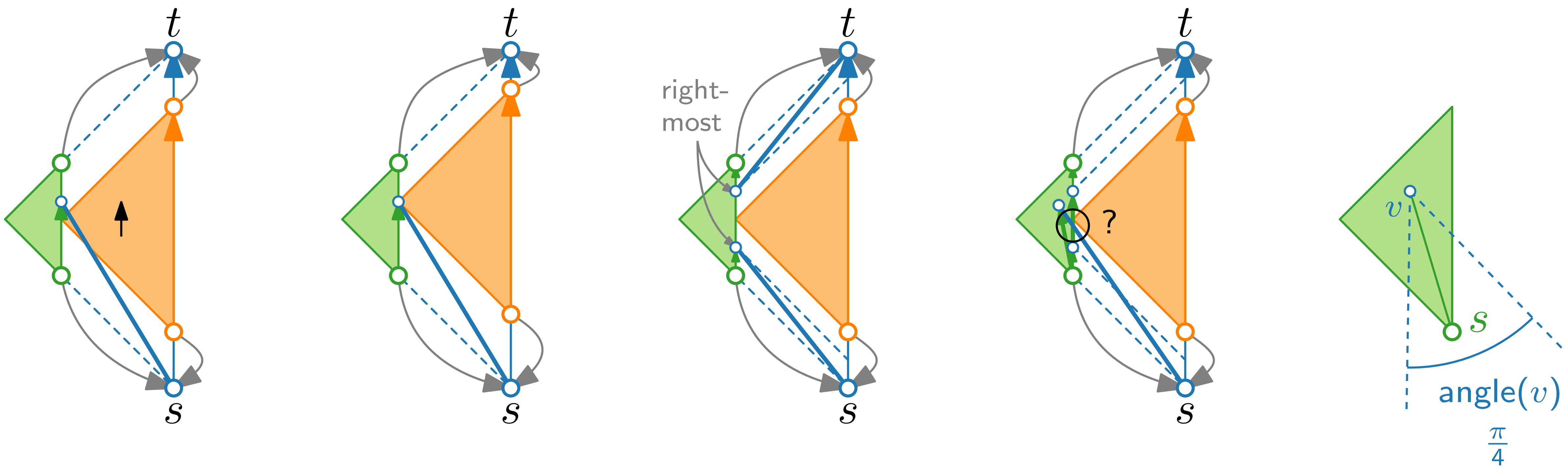



# Series-Parallel Graphs – Straight-Line Drawings

- What makes parallel composition possible without creating crossings?




Assume the following holds:
the only vertex in angle($v$) is $s$

- This condition **is** preserved during the induction step.

**Lemma.**
The drawing produced by the algorithm is planar.

# Series-Parallel Graphs – Result

**Theorem.**
Let $G$ be a series-parallel graph. Then $G$ (with **variable embedding**) admits a drawing $\Gamma$ that

- is upward planar,
- is straight-line, and
- uses quadratic area.
- Isomorphic components of $G$ have congruent drawings up to translation.

$\Gamma$ can be computed in linear time.

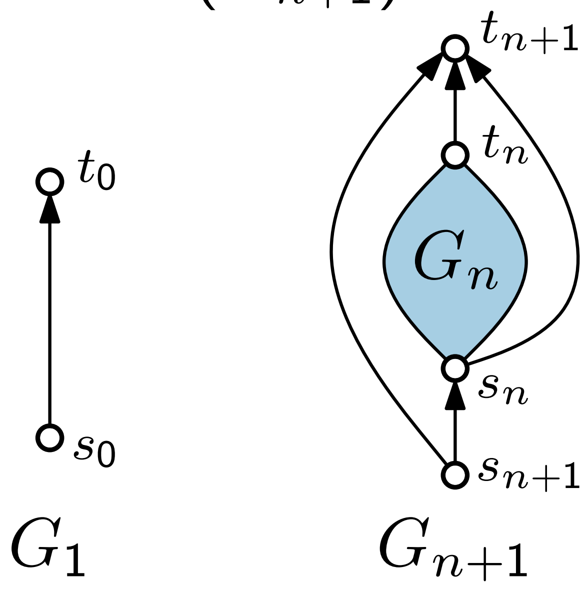
# Series-Parallel Graphs – Fixed Embedding

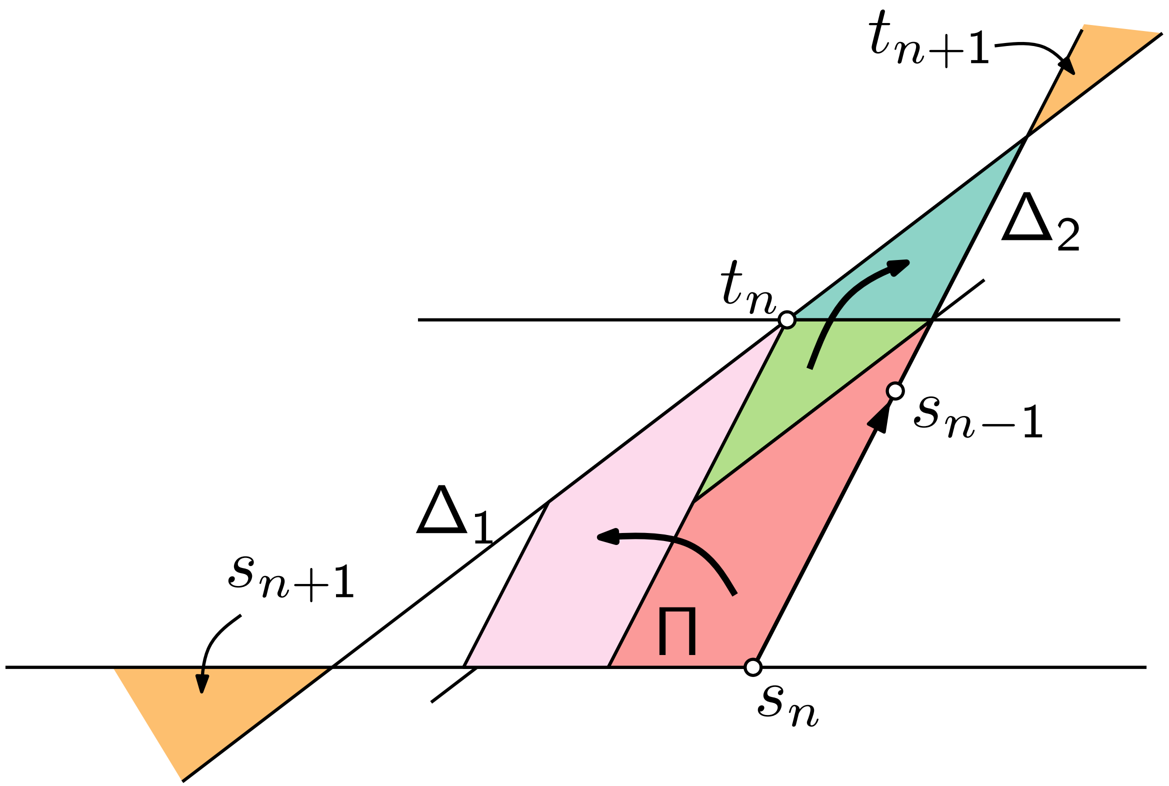
**Theorem.** [Bertolazzi, Di Battista, Mannino, Tamassia '94]

For any $n \geq 1$, there exists a $2n$-vertex series-parallel graph $G_n$ in an embedding such that any upward planar straight-line drawing of $G_n$ that respects the given embedding requires $\Omega(4^n)$ area.

- $2 \cdot \text{Area}(G_n) < \text{Area}(\Pi)$
- $2 \cdot \text{Area}(\Pi) \leq \text{Area}(G_{n+1})$

$\Rightarrow 4 \cdot \text{Area}(G_n) < \text{Area}(G_{n+1})$



# Discussion

- There exist fixed-parameter (FPT) algorithms to test upward planarity of general digraphs with the parameter being the number of triconnected components.
  [Healy, Lynch 2005, Didimo et al. 2009]

- Finding a consistent assignment (Theorem 2) can be sped up to $\mathcal{O}(n + r^{1.5})$, where $r = \#$ sources. [Abbasi, Healy, Rextin 2010]

- Many related concepts have been studied:
  upward drawings of mixed graphs, upward drawings with layers for the vertices, upward planarity on cylinder/torus, ...

# Literature

- See [GD Ch. 6] for detailed explanation on upward planarity.
- See [GD Ch. 3] for divide and conquer methods of series-parallel graphs

Orginal papers referenced:


- [Kelly '87] Fundamentals of Planar Ordered Sets
- [Di Battista &Tamassia '88] Algorithms for Plane Representations of Acyclic Digraphs
- [Garg &Tamassia '95]
  On the Computational Complexity of Upward and Rectilinear Planarity Testing
- [Hutton & Lubiw '96] Upward Planar Drawing of Single-Source Acyclic Digraphs
- [Bertolazzi, Di Battista, Mannino, Tamassia '94]
  Upward Drawings of Triconnected Digraphs
- [Healy & Lynch '05] Building Blocks of Upward Planar Digraphs
- [Didimo, Giordano, Liotta '09] Upward Spirality and Upward Planarity Testing
- [Abbasi, Healy, Rextin '10]
  Improving the running time of embedded upward planarity testing