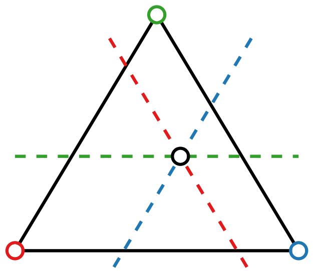
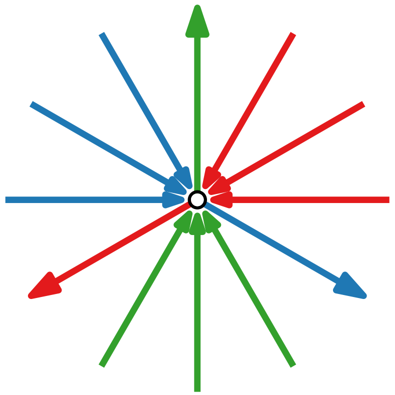
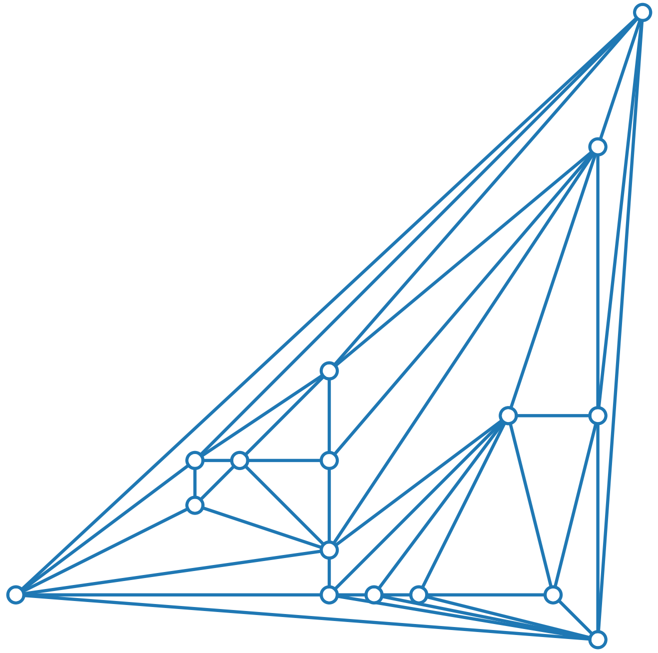
Visualization of Graphs

Lecture 4:
Straight-Line Drawings of Planar Graphs II: Schnyder Woods







Johannes Zink

Summer semester 2024

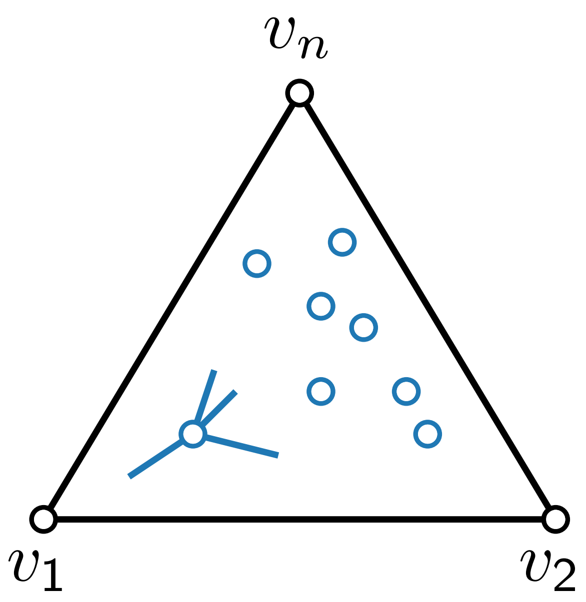
Planar Straight-Line Drawings

Theorem. [De Fraysseix, Pach, Pollack '90]
Every n-vertex planar graph has a planar straight-line drawing of size $(2n-4) \times (n-2)$.

Theorem. [Schnyder '90]
Every n-vertex planar graph has a planar straight-line drawing of size ~~$(n-2) \times (n-2)$~~ $(2n-5) \times (2n-5)$.

Idea. (easier to show)

- Fix outer triangle.
- Compute coordinates of inner vertices
  - based on outer triangle and
  - how much space there should be for other vertices
  - using weighted barycentric coordinates.

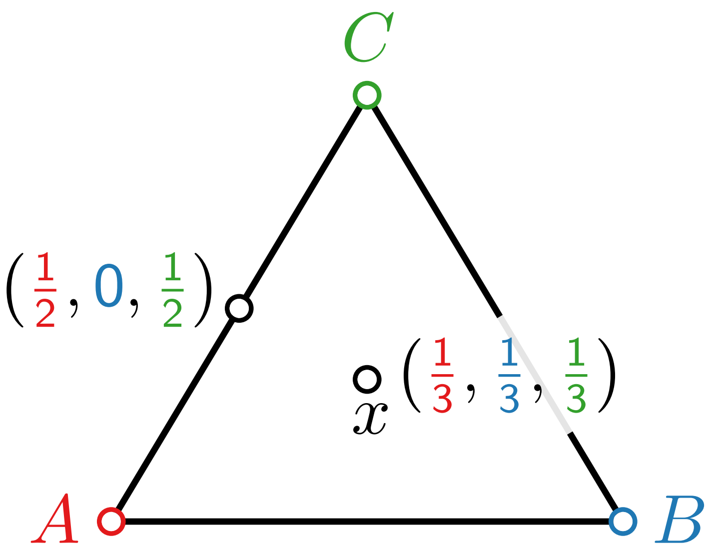


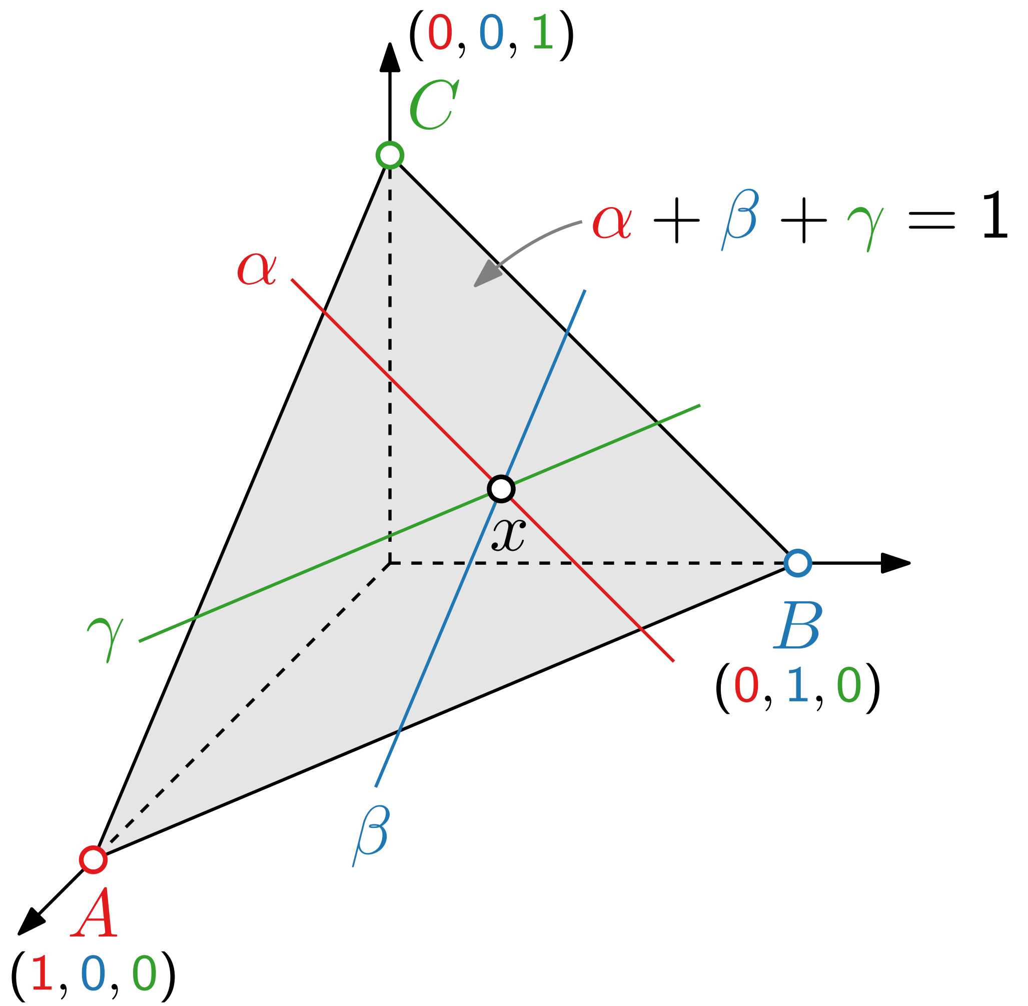
Barycentric Coordinates

Recall: barycenter$(x_1, \dots, x_k) = \sum_{i=1}^{k} x_i / k$

Let A, B, C form a triangle, and let x lie in $\triangle ABC$.
The **barycentric coordinates** of x with respect to $\triangle ABC$ are a triple $(\alpha, \beta, \gamma) \in \mathbb{R}^3_{\geq 0}$ such that

- $\alpha + \beta + \gamma = 1$ and
- $x = \alpha A + \beta B + \gamma C$.





Barycentric Representation

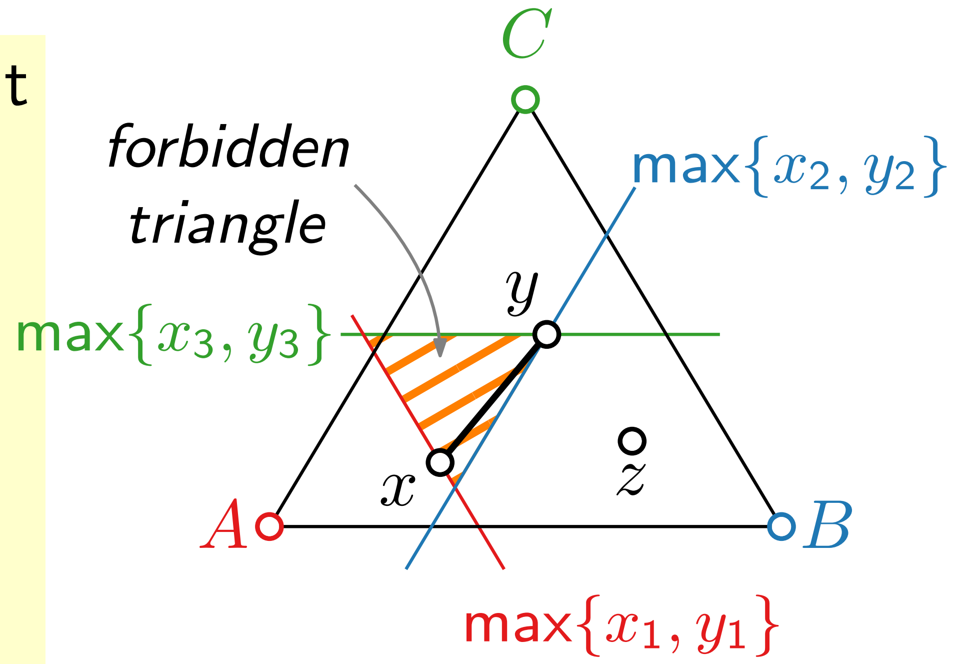
A **barycentric representation** of a graph G is an assignment of barycentric coordinates to the vertices of G:

$$f\colon V(G) \to \mathbb{R}^3_{\geq 0},\ \ v \mapsto (v_1, v_2, v_3)$$

with the following properties:

(B1) $v_1 + v_2 + v_3 = 1$ for all $v \in V(G)$,

(B2) for each $\{x, y\} \in E(G)$ and each $z \in V(G) \setminus \{x, y\}$, there exists a $k \in \{1, 2, 3\}$ with $x_k < z_k$ and $y_k < z_k$.



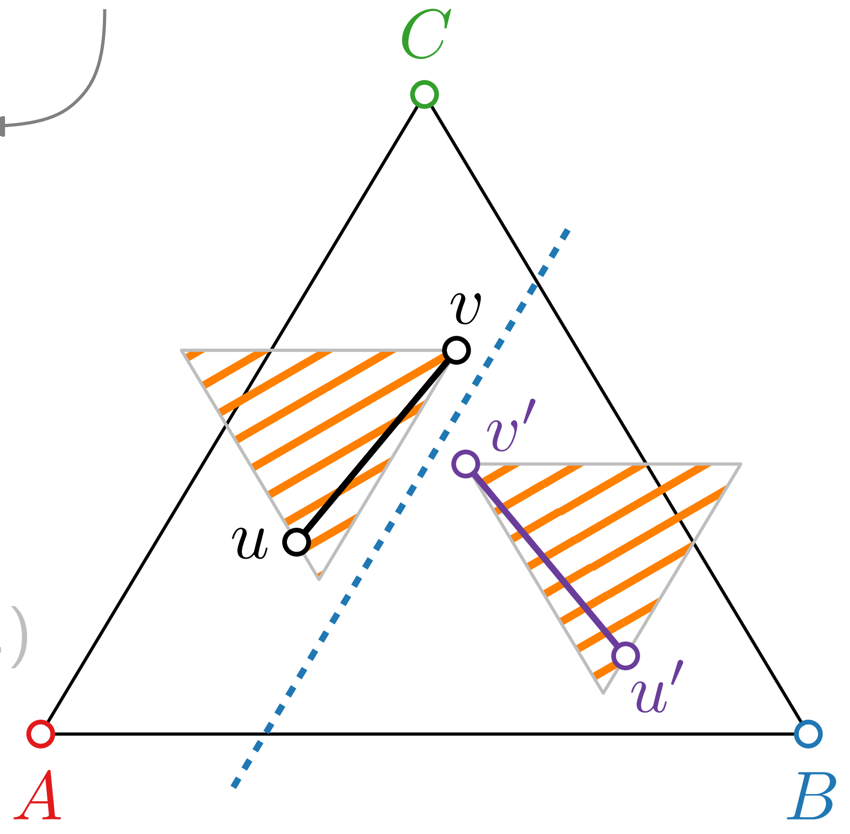
Barycentric Representations of Planar Graphs

Lemma.
Let $f: v \mapsto (v_1, v_2, v_3)$ be a barycentric representation of a planar graph G, and let $A, B, C \in \mathbb{R}^2$ be in general position.
Then the mapping

$$\phi\colon v \in V \mapsto v_1 A + v_2 B + v_3 C$$

yields a **planar straight-line** drawing of G inside $\triangle ABC$.

no three points on a line



- No vertex x can lie on an edge $\{u, v\}$. (Obvious by definition.)
- No pair of edges $\{u, v\}$ and $\{u', v'\}$ crosses:

$u'_i > u_i, v_i, \quad v'_j > u_j, v_j, \quad u_k > u'_k, v'_k, \quad v_l > u'_l, v'_l$

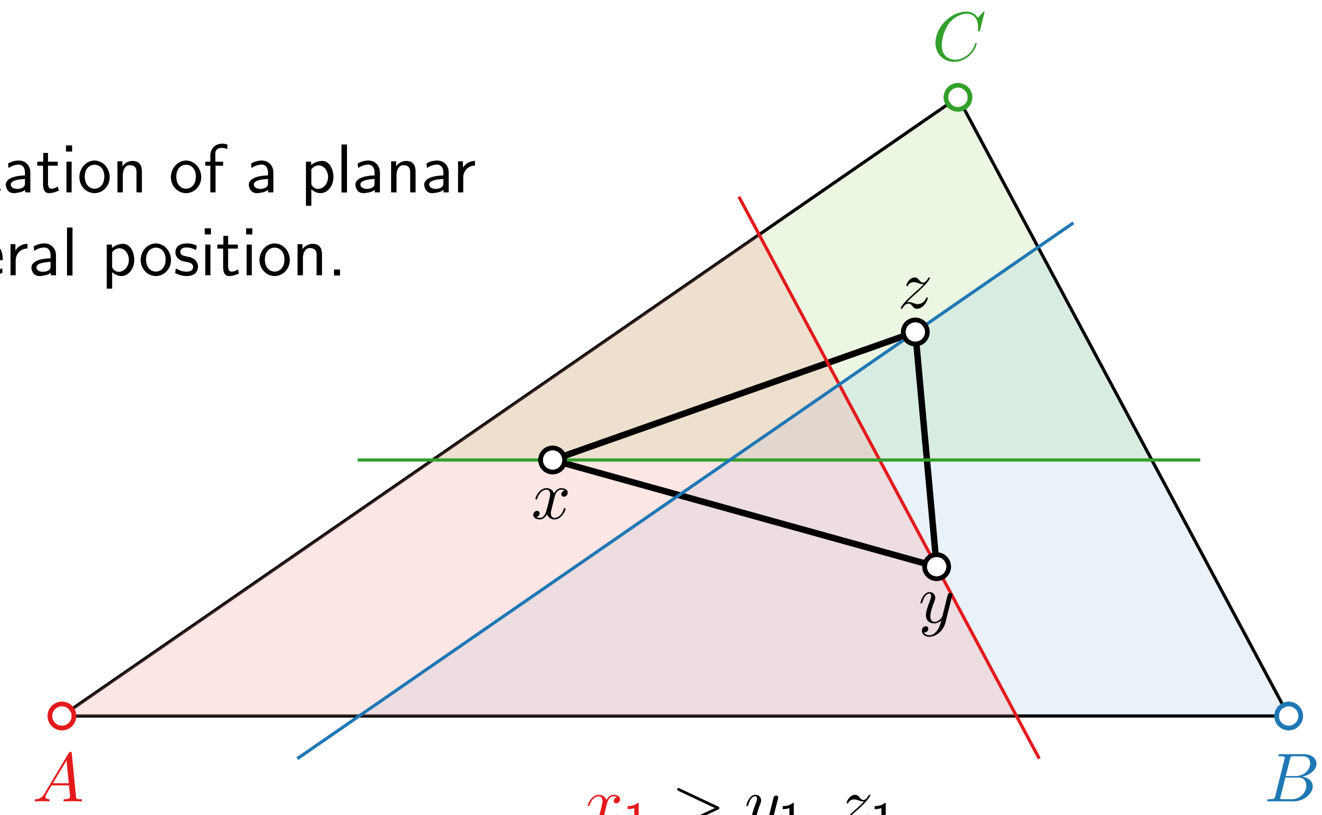
$\Rightarrow \{i, j\} \cap \{k, l\} = \emptyset$

w.l.o.g. $i = j = 2 \Rightarrow u'_2, v'_2 > u_2, v_2 \;\Rightarrow$ separated by a straight line

How to find a barycentric representation?

Schnyder Labeling

Let $f\colon v \mapsto (v_1, v_2, v_3)$ be a barycentric representation of a planar triangulation G, and let $A, B, C \in \mathbb{R}^2$ be in general position.



$x_1 > y_1, z_1$

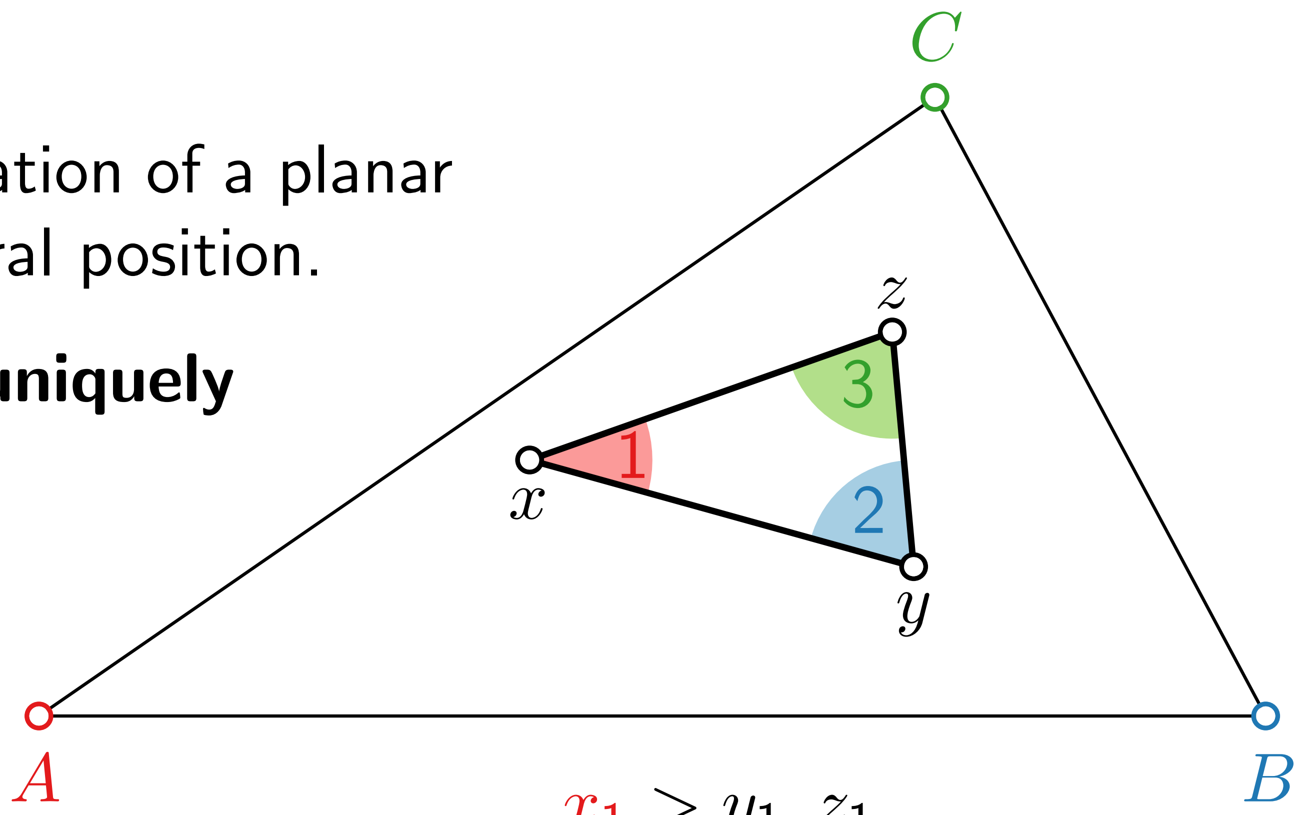
$y_2 > x_2, z_2$

$z_3 > x_3, y_3$

Schnyder Labeling

Let $f\colon v \mapsto (v_1, v_2, v_3)$ be a barycentric representation of a planar triangulation G, and let $A, B, C \in \mathbb{R}^2$ be in general position.

We can label each angle in each triangle $\triangle xyz$ **uniquely** with $k \in \{1, 2, 3\}$.



$$x_1 > y_1, z_1$$

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Schnyder Labeling

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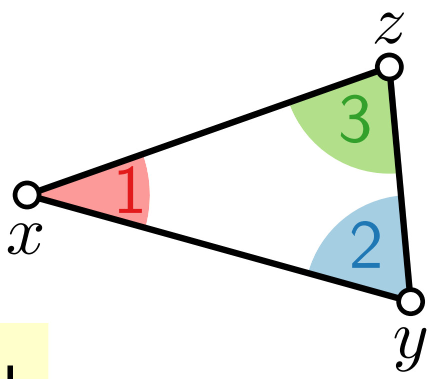
We can label each angle in each triangle $\triangle xyz$ **uniquely** with $k \in \{1, 2, 3\}$.

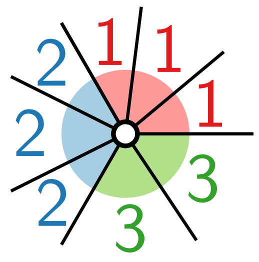


A **Schnyder labeling** of a plane triangulation G is a labeling of all internal angles with labels 1, 2, and 3 such that:

Faces: The three angles of an internal face are labeled 1, 2, and 3 in counterclockwise (ccw) order.

Vertices: The ccw order of labels around each vertex consists of

- a non-empty interval of 1s,
- followed by a non-empty interval of 2s,
- followed by a non-empty interval of 3s.

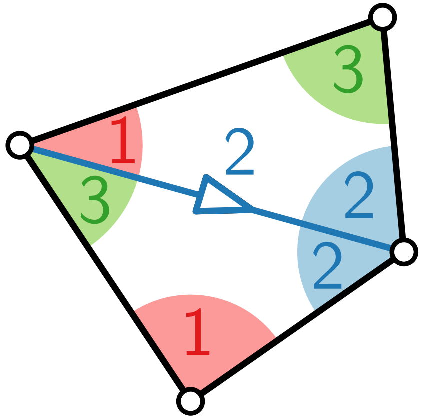


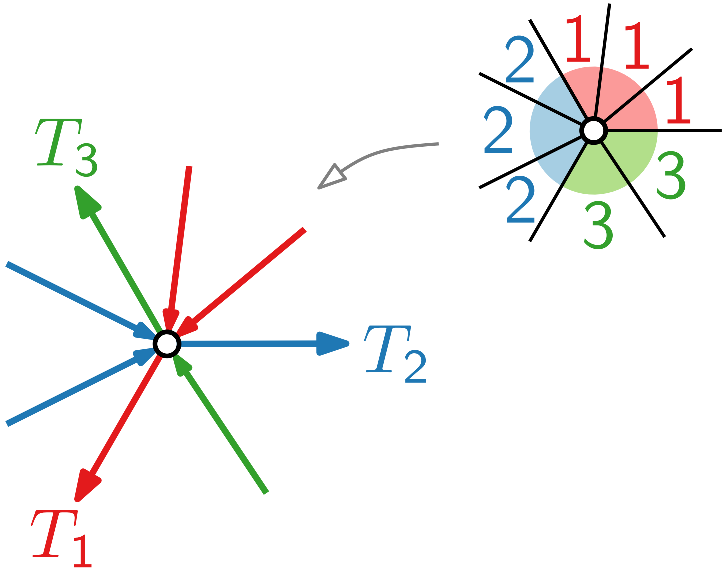
Schnyder Wood

A Schnyder labeling induces an edge labeling.

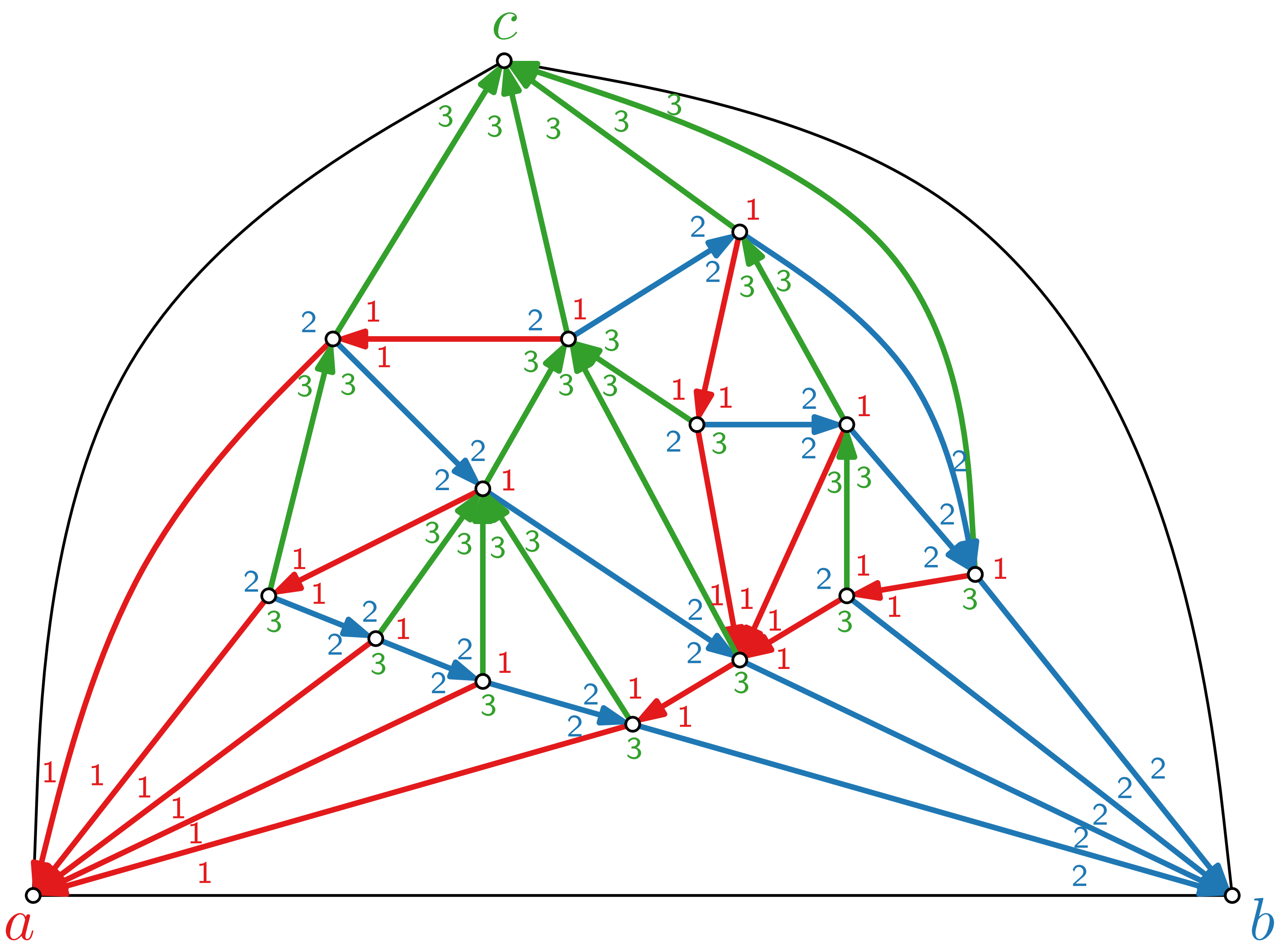
A **Schnyder wood** (or **Schnyder realizer**) of a plane triangulation G is a partition of the inner edges of G into three sets of oriented edges T_1, T_2, T_3 such that, for each inner vertex v of G, it holds that

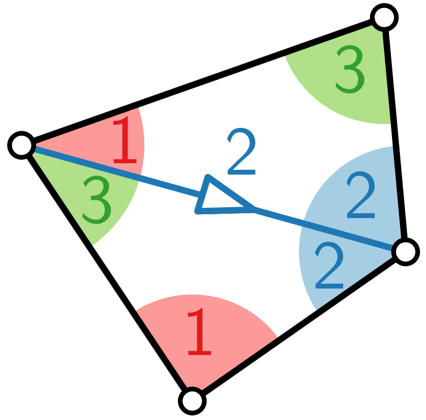
- v has one outgoing edge in each of T_1, T_2, and T_3.
- The ccw order of edges around v is:
  leaving in T_1, entering in T_3, leaving in T_2,
  entering in T_1, leaving in T_3, entering in T_2.

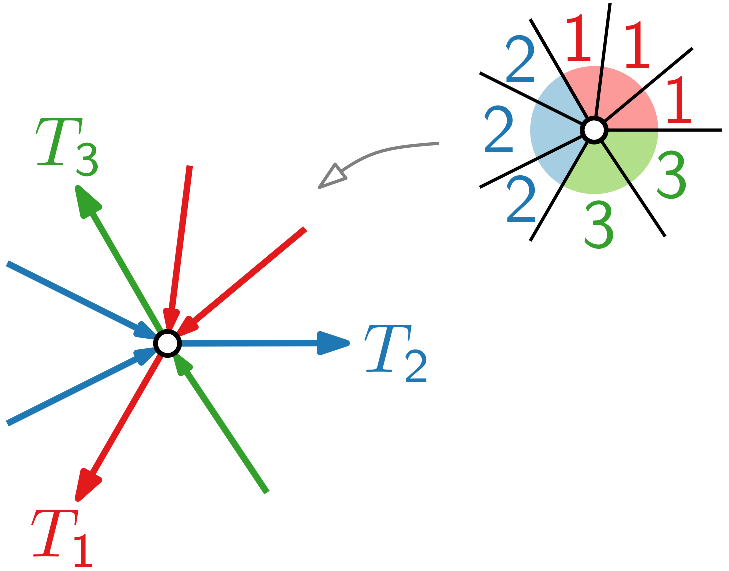




Schnyder Wood – Example and Properties







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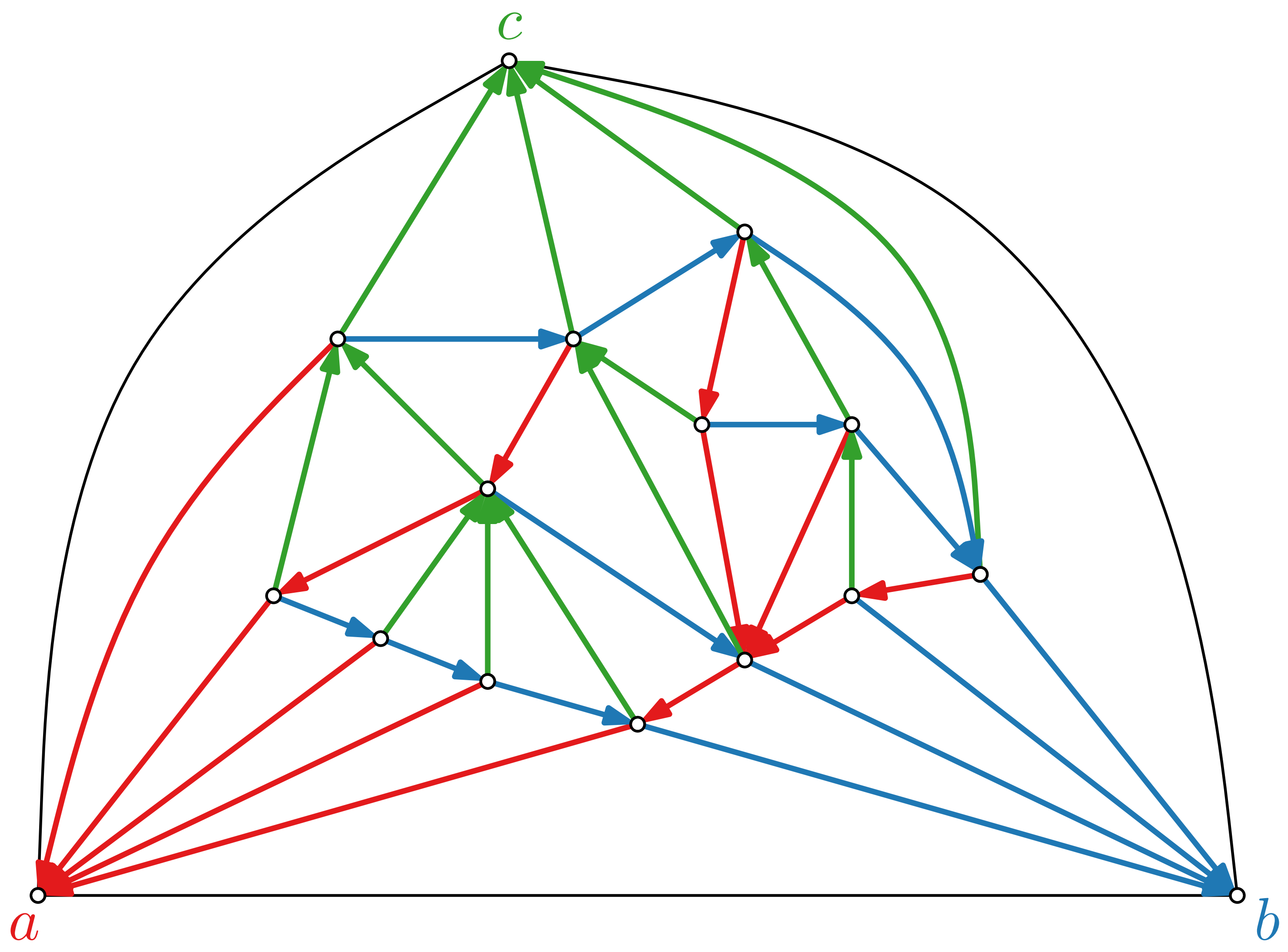
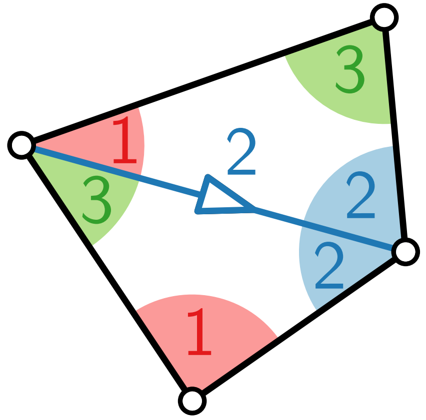
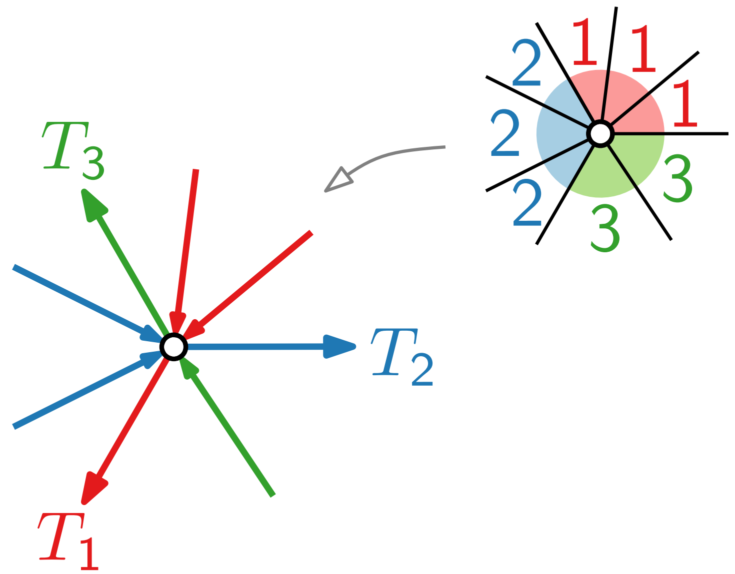
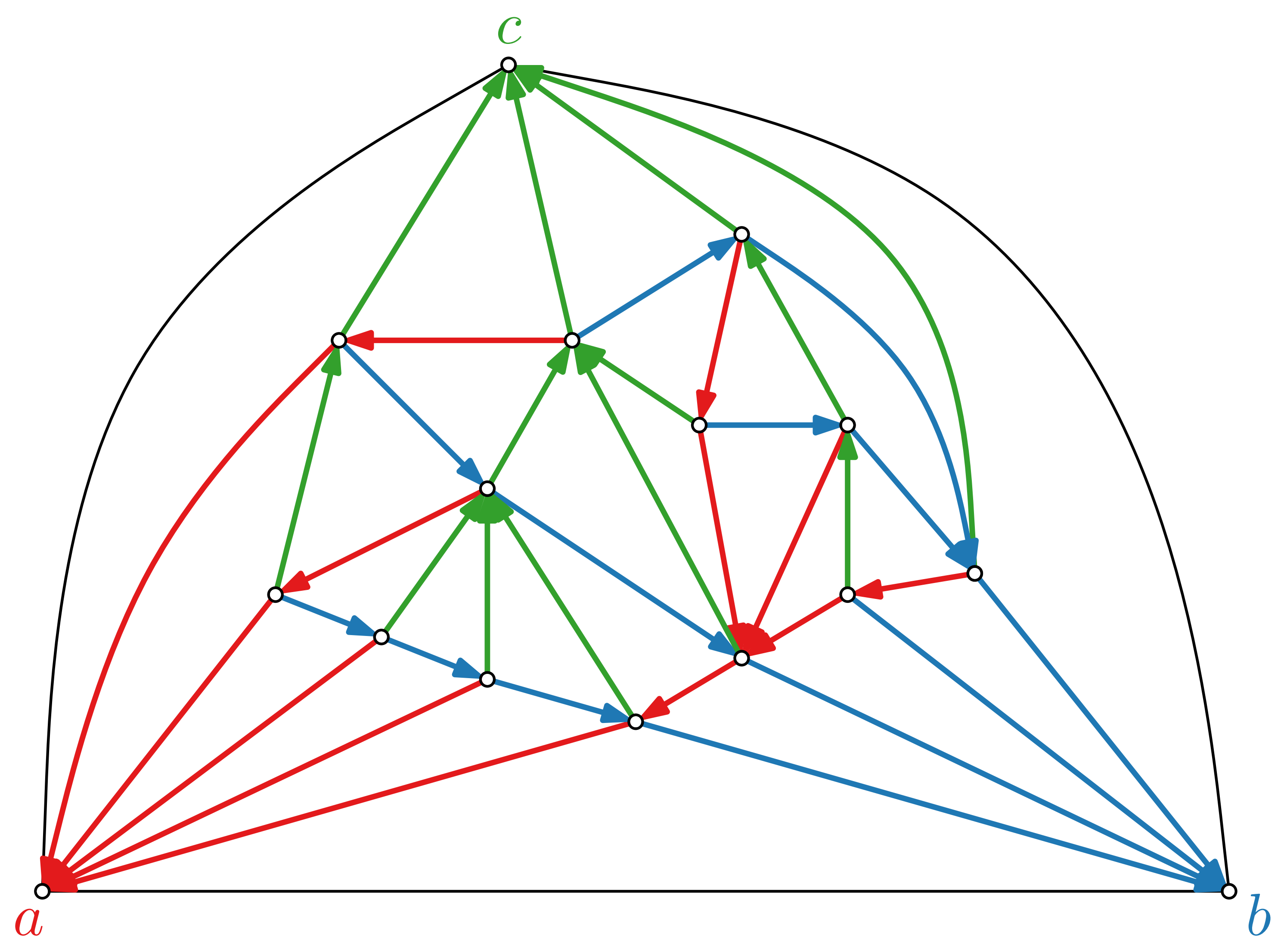


(a Schnyder labeling is not unique)

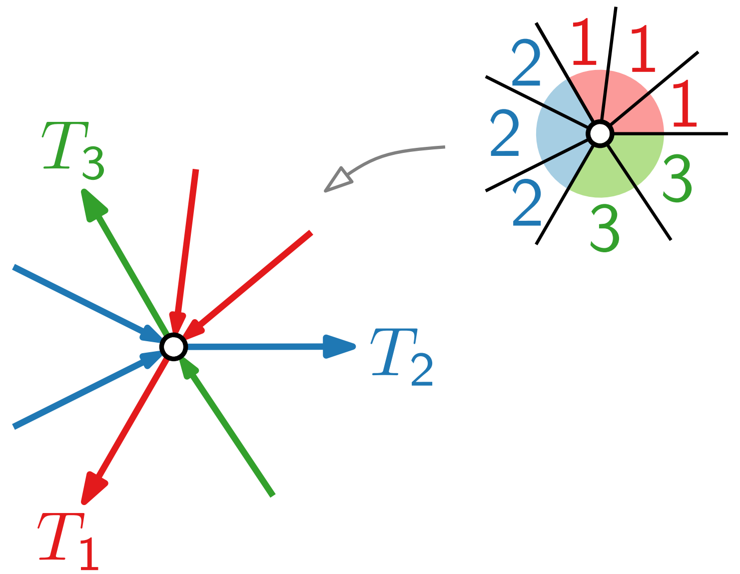




Schnyder Wood – Example and Properties

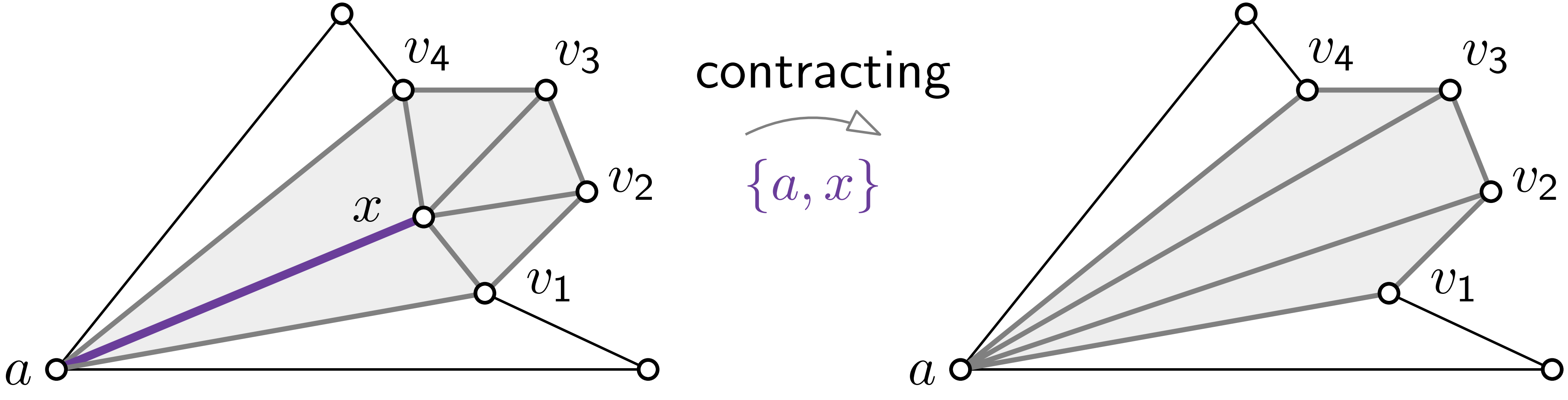


- All inner edges incident to a, b, and c are incoming in the same set (color).
- T_1, T_2, and T_3 are trees. Each spans all inner vertices and one outer vertex (its root).



Schnyder Wood – Existence

Lemma. [Kampen 1976]
Let G be a plane triangulation with vertices a, b, c on the outer face.
Then there exists a **contractible edge** $\{a, x\}$ in G with $x \notin \{b, c\}$.



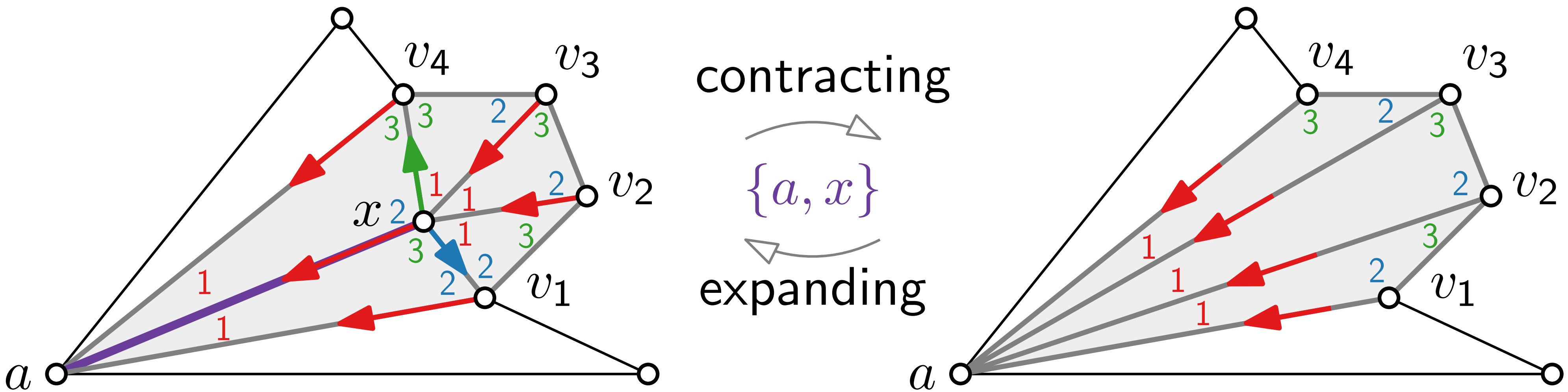
. . . requires that a and x have exactly two common neighbors.

Schnyder Wood – Existence

Lemma. [Kampen 1976]
Let G be a plane triangulation with vertices a, b, c on the outer face.
Then there exists a **contractible edge** $\{a, x\}$ in G with $x \notin \{b, c\}$.

Theorem.
Every plane triangulation has a Schnyder labeling and a Schnyder wood.

Proof by induction on # vertices via edge contractions.

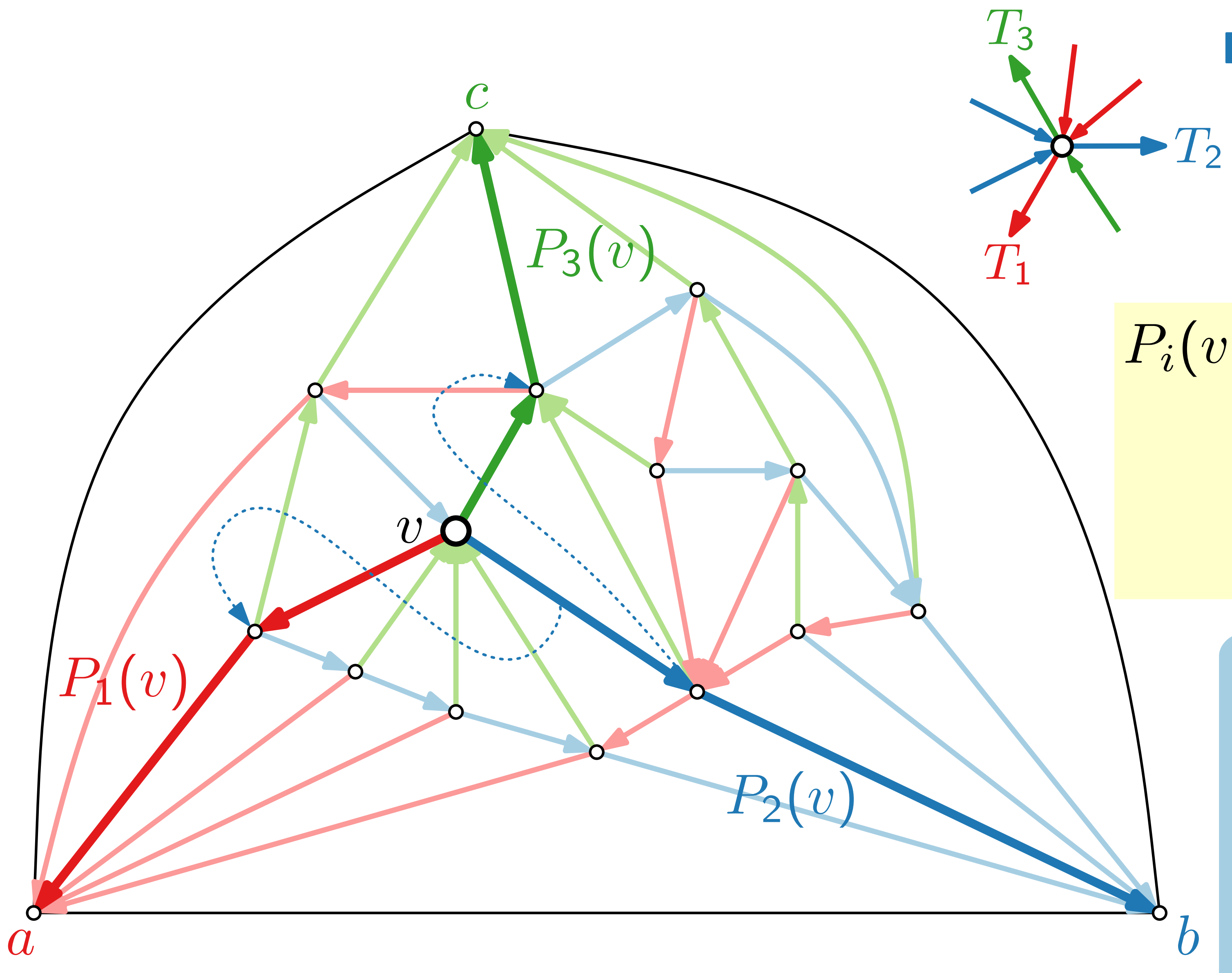


…requires that a and x have exactly two common neighbors.

This constructive proof yields an algorithm for computing a Schnyder labeling. It can be implemented to run in $\mathcal{O}(n)$ time.

→ *Exercise*

Schnyder Wood – More Properties



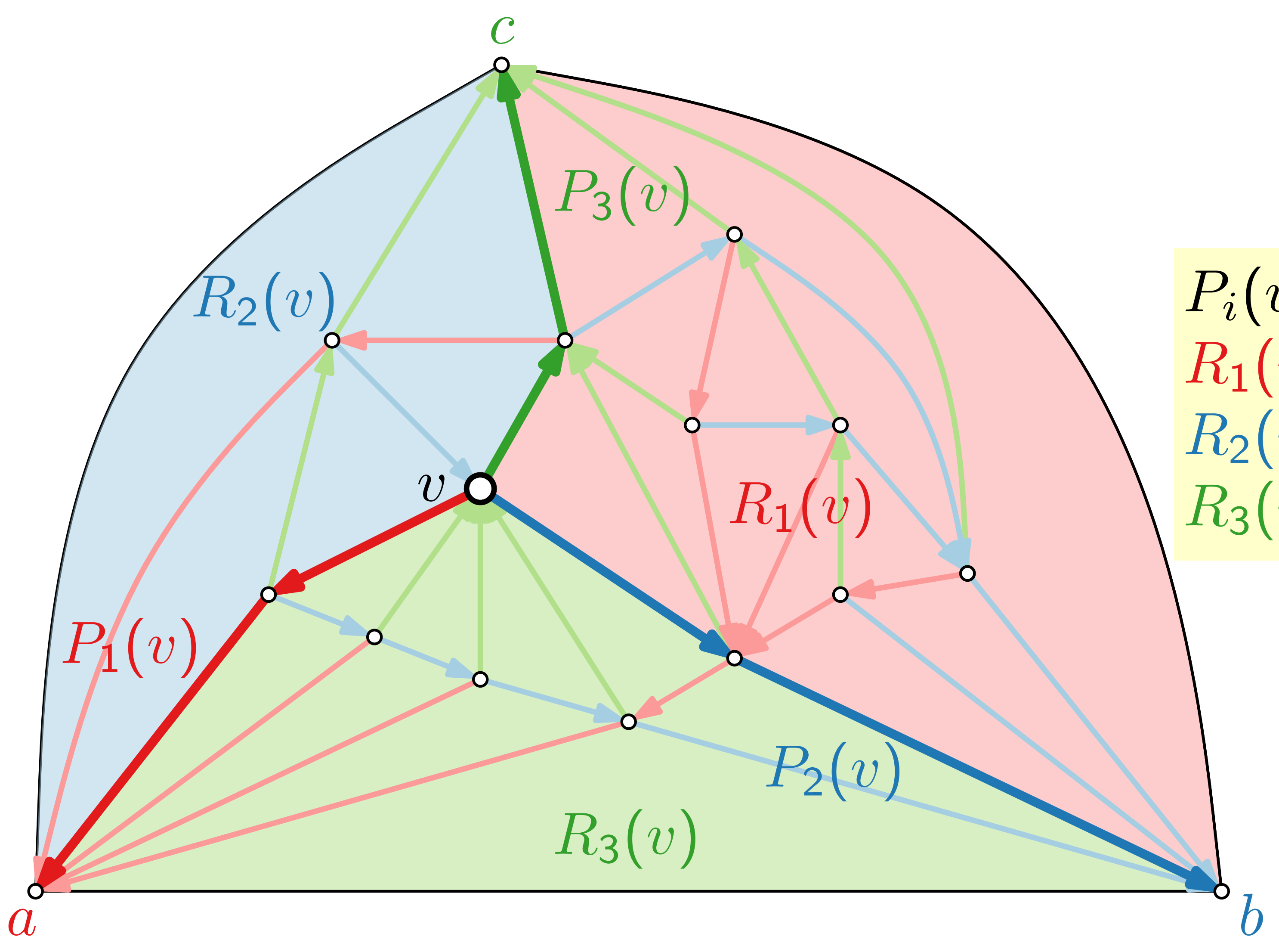
- From each vertex v there exists a unique
  - directed red path $P_1(v)$ to a,
  - directed blue path $P_2(v)$ to b, and
  - directed green path $P_3(v)$ to c.

$P_i(v)$: unique path from v to root of T_i

Lemma.

- $P_1(v)$, $P_2(v)$, $P_3(v)$ cross only at v.

Schnyder Wood – More Properties



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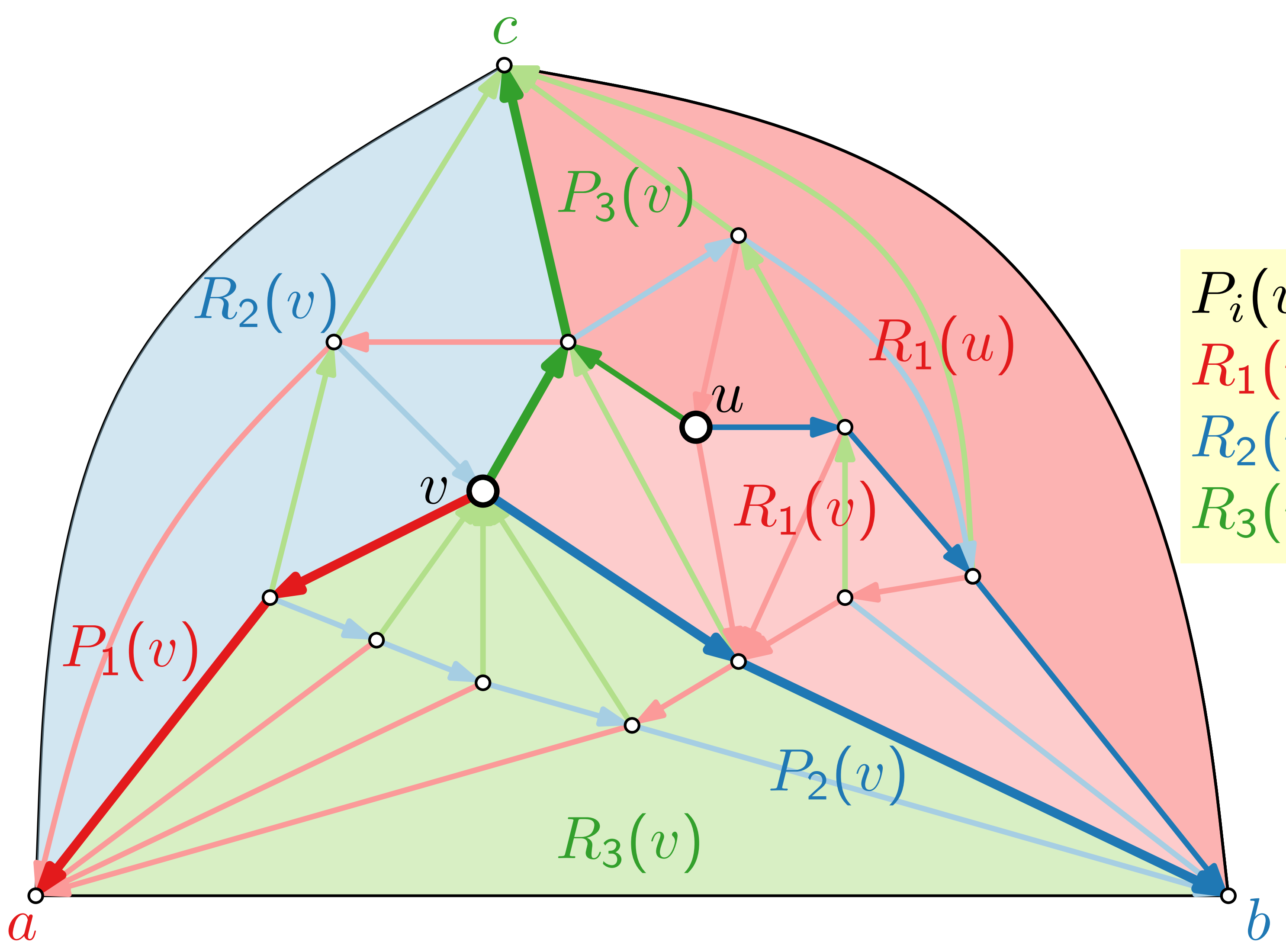
$P_i(v)$: unique path from v to root of T_i
$R_1(v)$: set of faces bounded by $\langle P_2(v), bc, P_3(v)\rangle$
$R_2(v)$: set of faces bounded by $\langle P_3(v), ca, P_1(v)\rangle$
$R_3(v)$: set of faces bounded by $\langle P_1(v), ab, P_2(v)\rangle$

Lemma.

- $P_1(v)$, $P_2(v)$, $P_3(v)$ cross only at v.
- For inner vertices $u \neq v$, it holds that $u \in R_i(v) \Rightarrow R_i(u) \subsetneq R_i(v)$.

Schnyder Wood – More Properties



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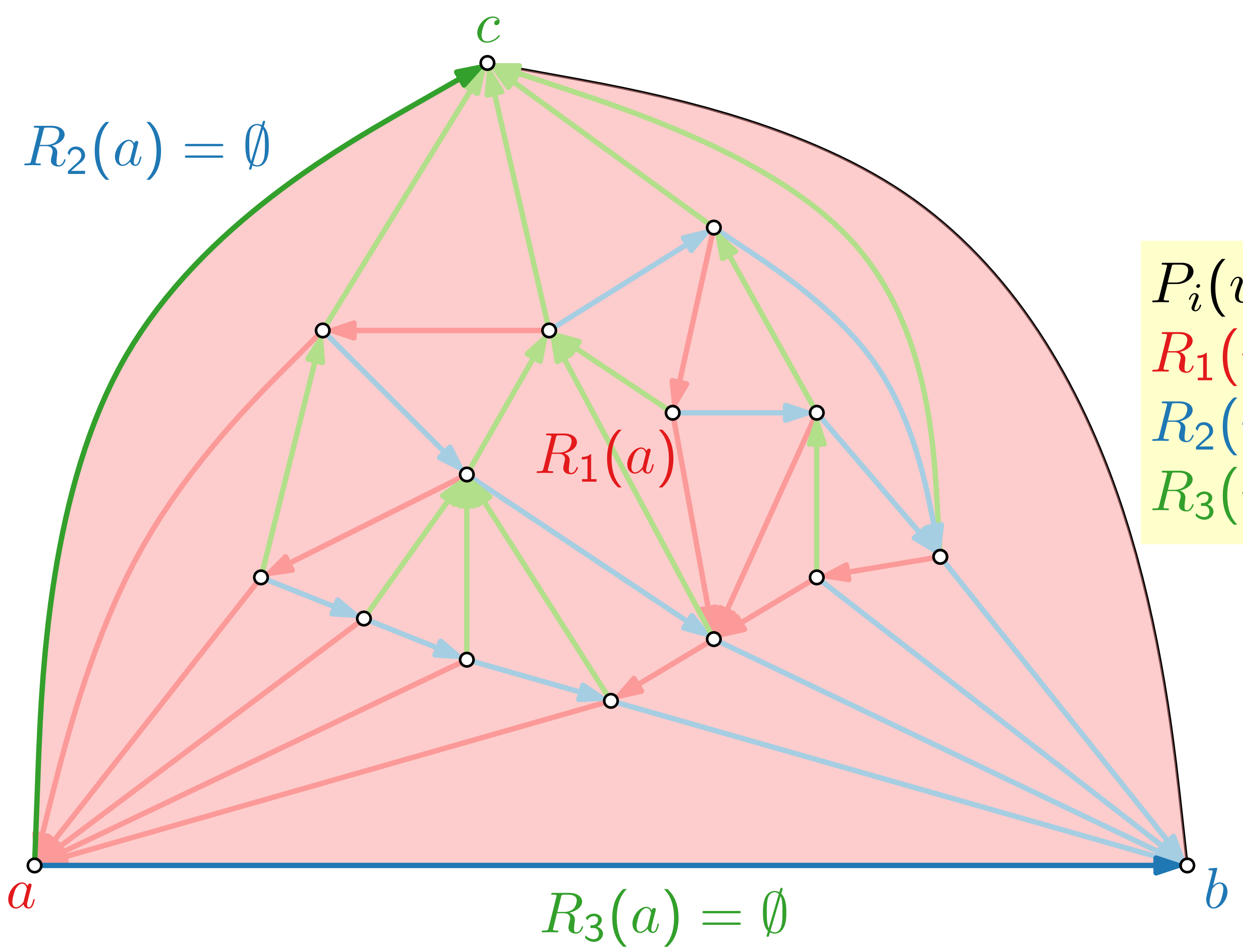
Schnyder Wood – More Properties



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Lemma.
- $P_1(v)$, $P_2(v)$, $P_3(v)$ cross only at v.
- For inner vertices $u \neq v$, it holds that $u \in R_i(v) \Rightarrow R_i(u) \subsetneq R_i(v)$.
- $|R_1(v)| + |R_2(v)| + |R_3(v)| = 2n - 5$

Schnyder Drawing

Theorem. [Schnyder '90]
For a plane triangulation G, the mapping

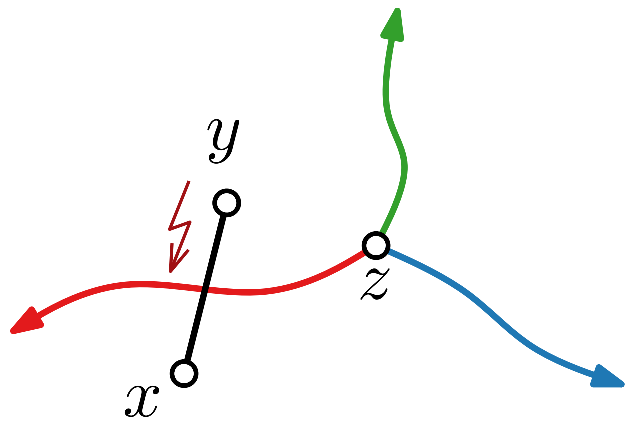
$$f\colon v \mapsto (v_1, v_2, v_3) = \frac{1}{2n-5}(|R_1(v)|, |R_2(v)|, |R_3(v)|)$$

is a barycentric representation of G and, thus, yields a planar straight-line drawing of G

(B1) $v_1 + v_2 + v_3 = 1$ for all $v \in V(G)$ ✓

(B2) for each $\{x, y\} \in E(G)$ and each $z \in V(G) \setminus \{x, y\}$ there exists $k \in \{1, 2, 3\}$ with $x_k < z_k$ and $y_k < z_k$

- $\{x, y\}$ must lie in $R_i(z)$ for some $i \in \{1, 2, 3\}$



Schnyder Drawing

Set $A = (0,0)$, $B = (2n-5, 0)$, and $C = (0, 2n-5)$.

Theorem. [Schnyder '90]

For a plane triangulation G, the mapping

$$f \colon v \mapsto (v_1, v_2, v_3) = \frac{1}{2n-5}(|R_1(v)|, |R_2(v)|, |R_3(v)|)$$

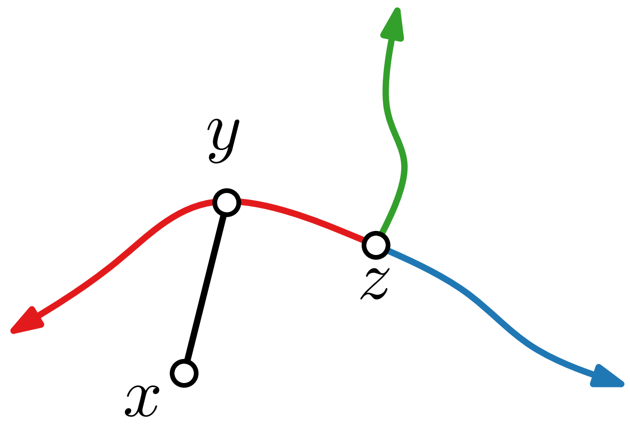
is a barycentric representation of G and, thus, yields a planar straight-line drawing of G on the $(2n-5) \times (2n-5)$ grid.

(B1) $v_1 + v_2 + v_3 = 1$ for all $v \in V(G)$ ✓

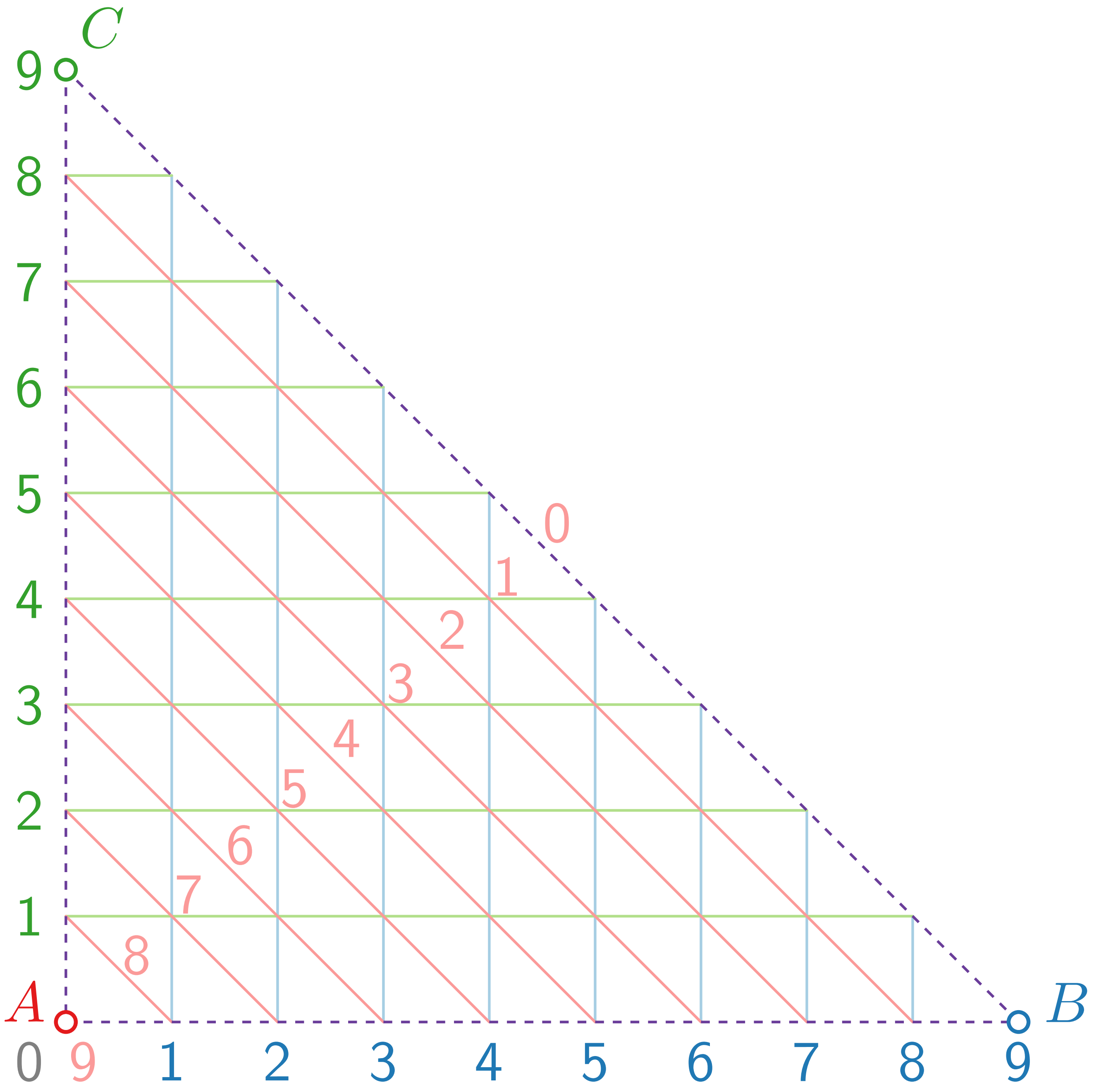
(B2) for each $\{x, y\} \in E(G)$ and each $z \in V(G) \setminus \{x, y\}$ there exists $k \in \{1, 2, 3\}$ with $x_k < z_k$ and $y_k < z_k$ ✓

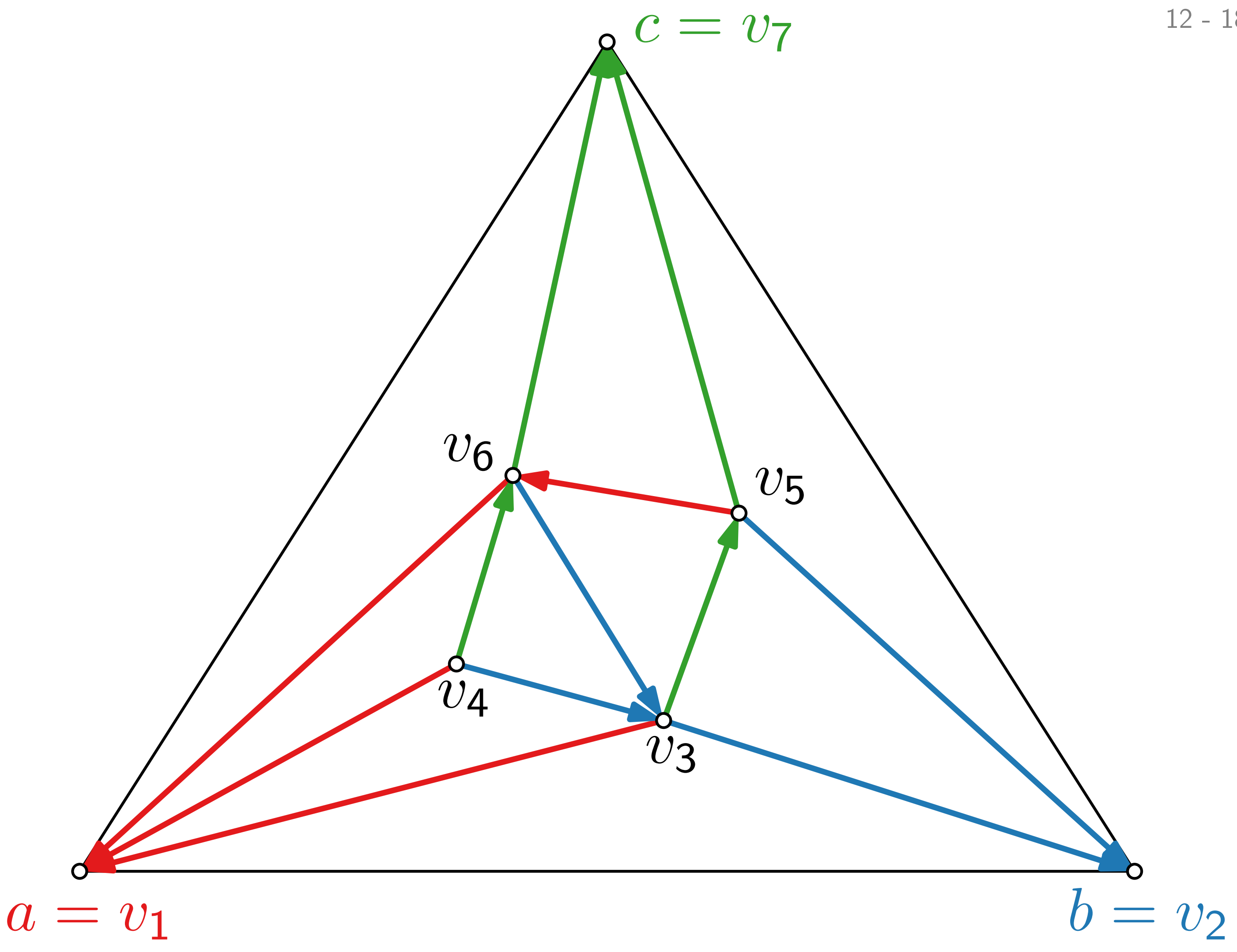
- $\{x, y\}$ must lie in $R_i(z)$ for some $i \in \{1, 2, 3\}$
- $x, y \in R_i(z) \Rightarrow R_i(x), R_i(y) \subsetneq R_i(z)$

  $\Rightarrow |R_i(x)|, |R_i(y)| < |R_i(z)|$



Schnyder Drawing – Example





$n = 7; \quad 2n - 5 = 9$

$f(v_1) = (9, 0, 0)$

$f(v_2) = (0, 9, 0)$

$f(v_3) = (2, 6, 1)$

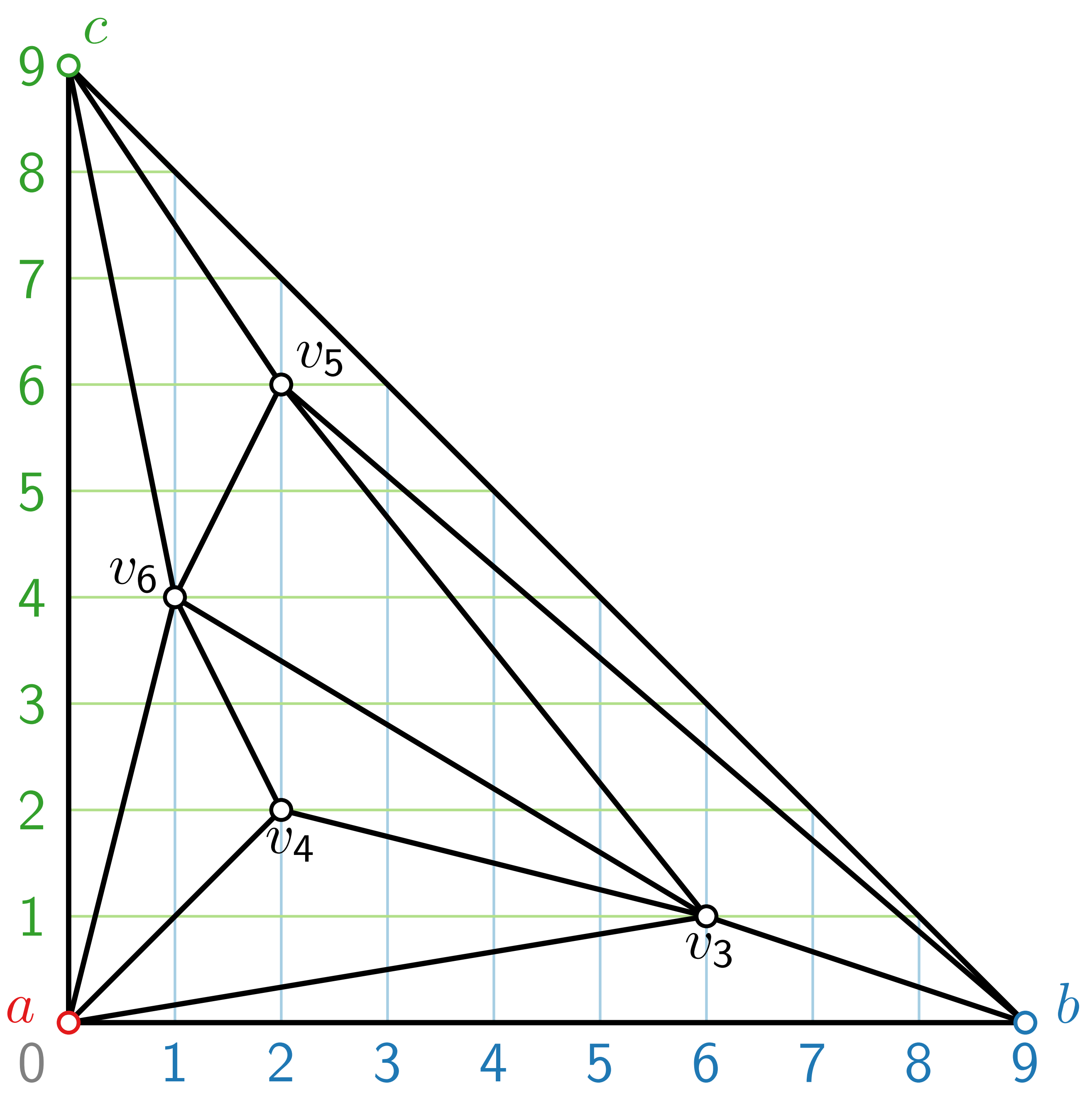
$f(v_4) = (5, 2, 2)$

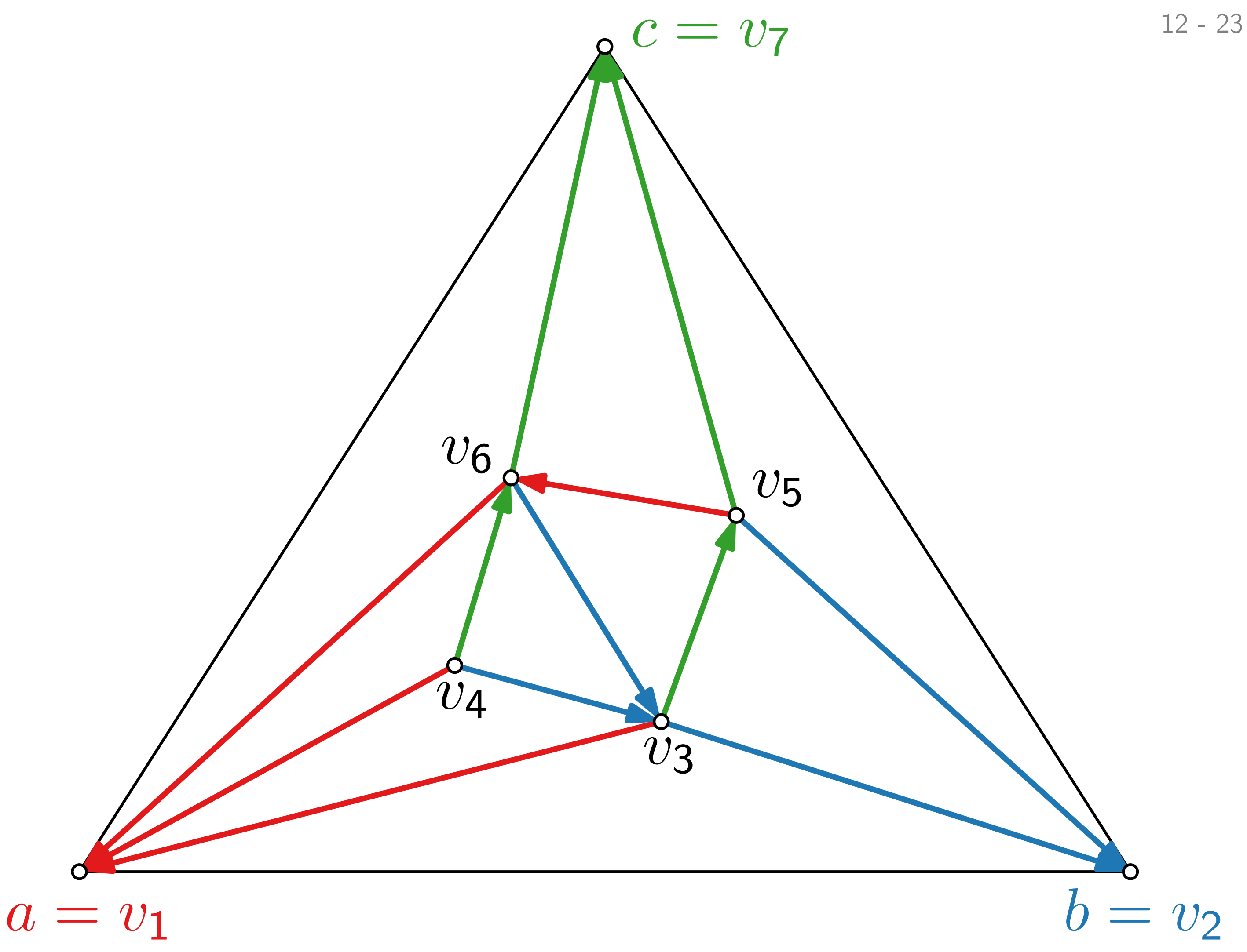
$f(v_5) = (1, 2, 6)$

$f(v_6) = (4, 1, 4)$

$f(v_7) = (0, 0, 9)$

Schnyder Drawing – Example





$n = 7; \quad 2n - 5 = 9$

$f(v_1) = (9, 0, 0)$

$f(v_2) = (0, 9, 0)$

$f(v_3) = (2, 6, 1)$

$f(v_4) = (5, 2, 2)$

$f(v_5) = (1, 2, 6)$

$f(v_6) = (4, 1, 4)$

$f(v_7) = (0, 0, 9)$

Weak Barycentric Representation

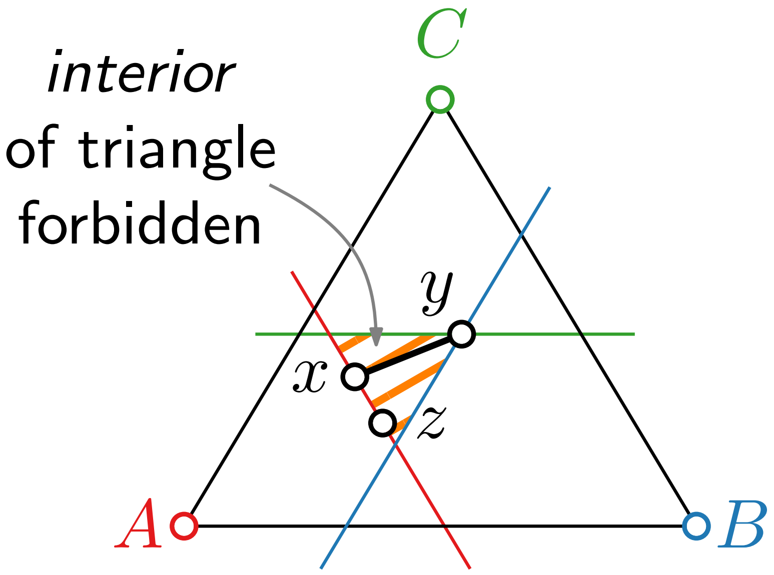
A **weak barycentric representation** of a graph G is an assignment of barycentric coordinates to $V(G)$:

$$f\colon V(G) \to \mathbb{R}^3_{\geq 0},\ \ v \mapsto (v_1, v_2, v_3)$$

with the following properties:

(W1) $v_1 + v_2 + v_3 = 1$ for all $v \in V(G)$,

(W2) for each $\{x, y\} \in E(G)$ and each $z \in V(G) \setminus \{x, y\}$, there exists a $k \in \{1, 2, 3\}$ with $(x_k, x_{k+1}) <_{\text{lex}} (z_k, z_{k+1})$ and $(y_k, y_{k+1}) <_{\text{lex}} (z_k, z_{k+1})$.



i.e., either $y_k < z_k$ or $y_k = z_k$ and $y_{k+1} < z_{k+1}$

indices modulo 3

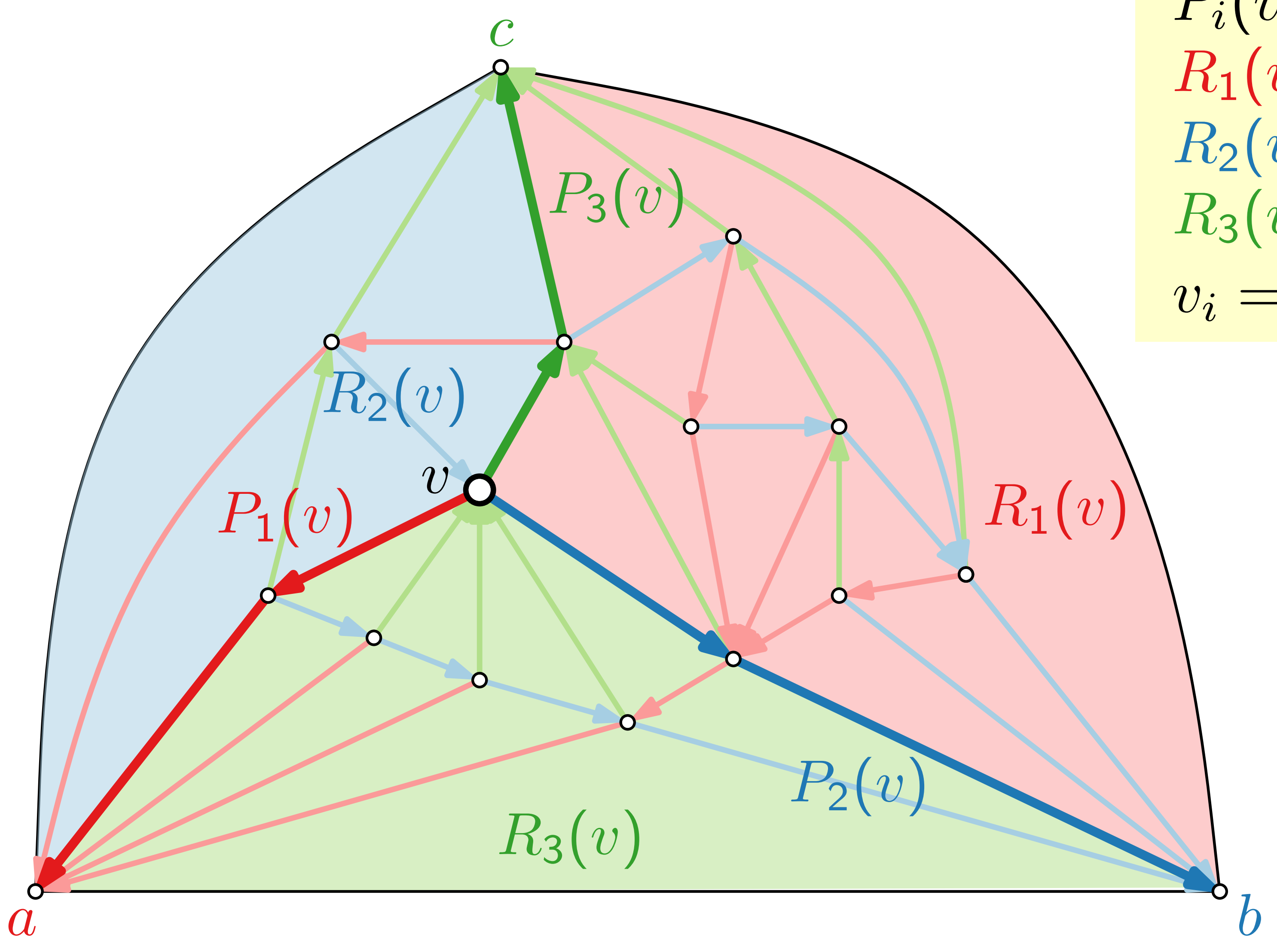
Lemma.
For a weak barycentric representation $f\colon v \mapsto (v_1, v_2, v_3)$ and a triangle $\triangle ABC$, the mapping $\phi\colon V(G) \to \mathbb{R}^3$ with

$$v \mapsto v_1 A + v_2 B + v_3 C$$

yields a **planar** drawing of G inside $\triangle ABC$.

Proof. $\rightarrow$ *Exercise!*

Counting Vertices



$P_i(v)$: unique path from v to root of T_i

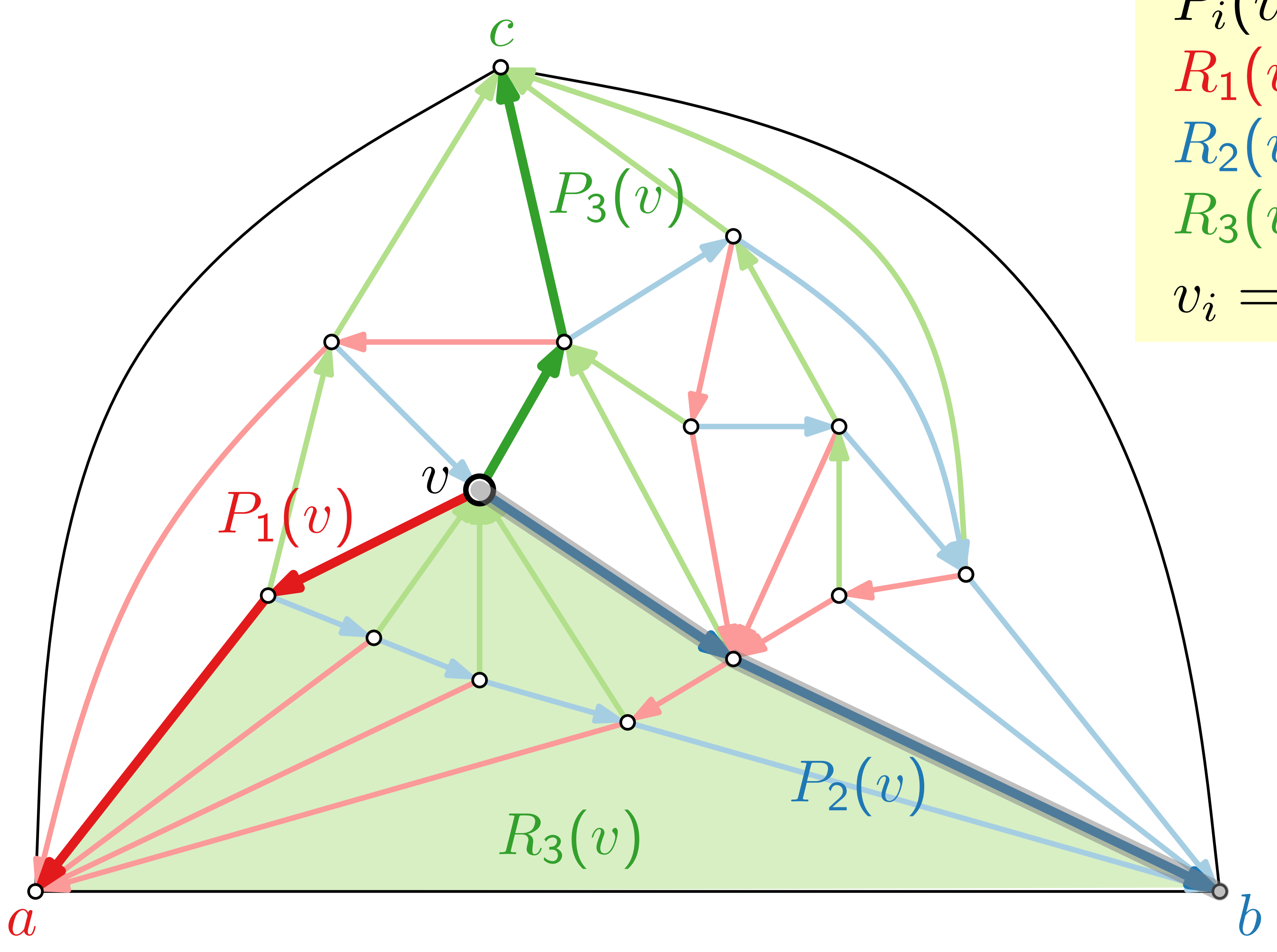
$R_1(v)$: subgraph bounded by $\langle P_2(v), bc, P_3(v)\rangle$

$R_2(v)$: subgraph bounded by $\langle P_3(v), ca, P_1(v)\rangle$

$R_3(v)$: subgraph bounded by $\langle P_1(v), ab, P_2(v)\rangle$

$v_i = |V(R_i(v))| - |V(P_{i-1}(v))|$ (indices modulo 3)

Counting Vertices



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$R_3(v)$: subgraph bounded by $\langle P_1(v), ab, P_2(v)\rangle$

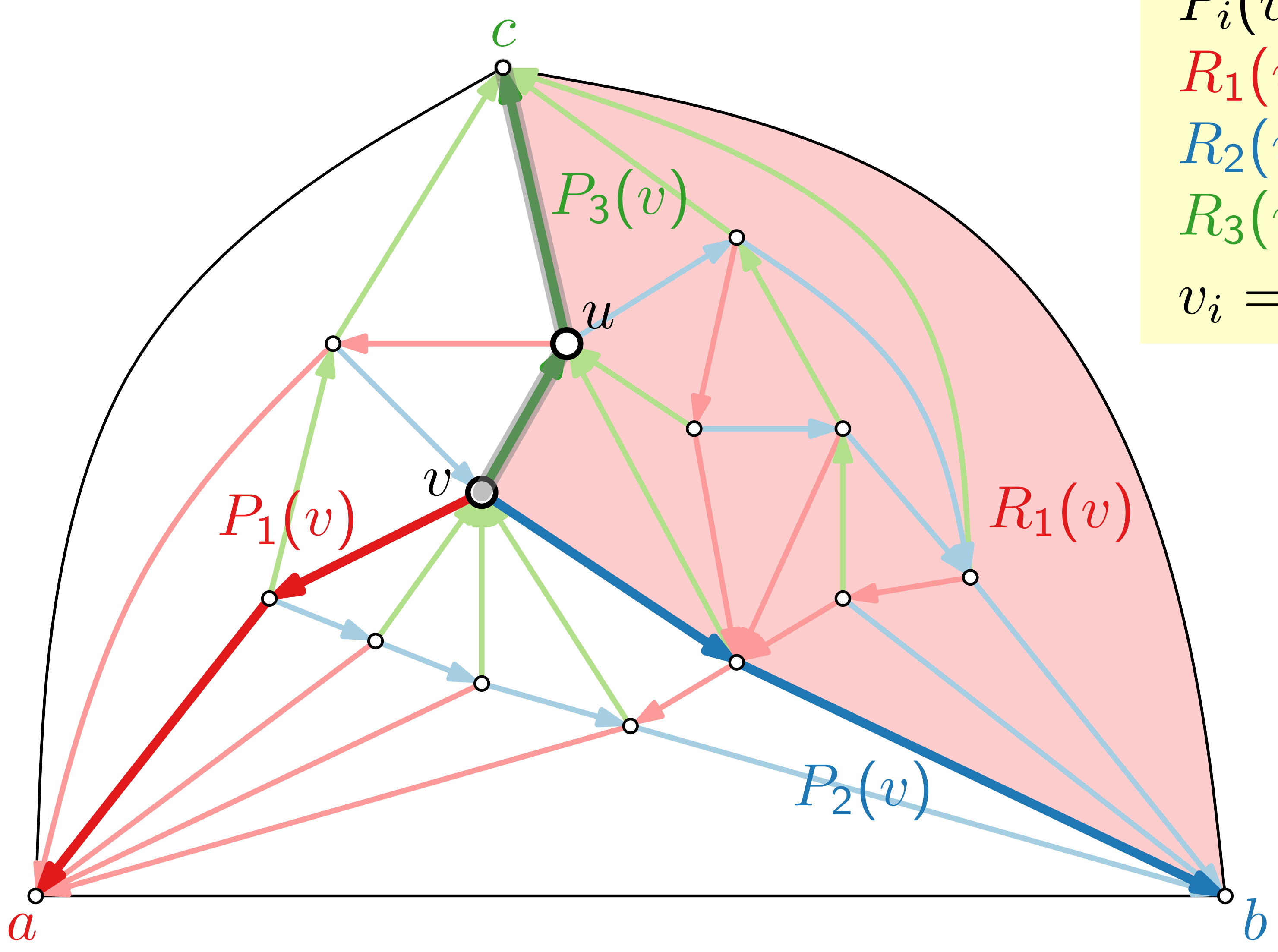
$v_i = |V(R_i(v))| - |V(P_{i-1}(v))|$ (indices modulo 3)

$v_1 = 10 - 3 = 7$

$v_2 = 6 - 3 = 3$

$v_3 = 8 - 3 = 5$

Counting Vertices



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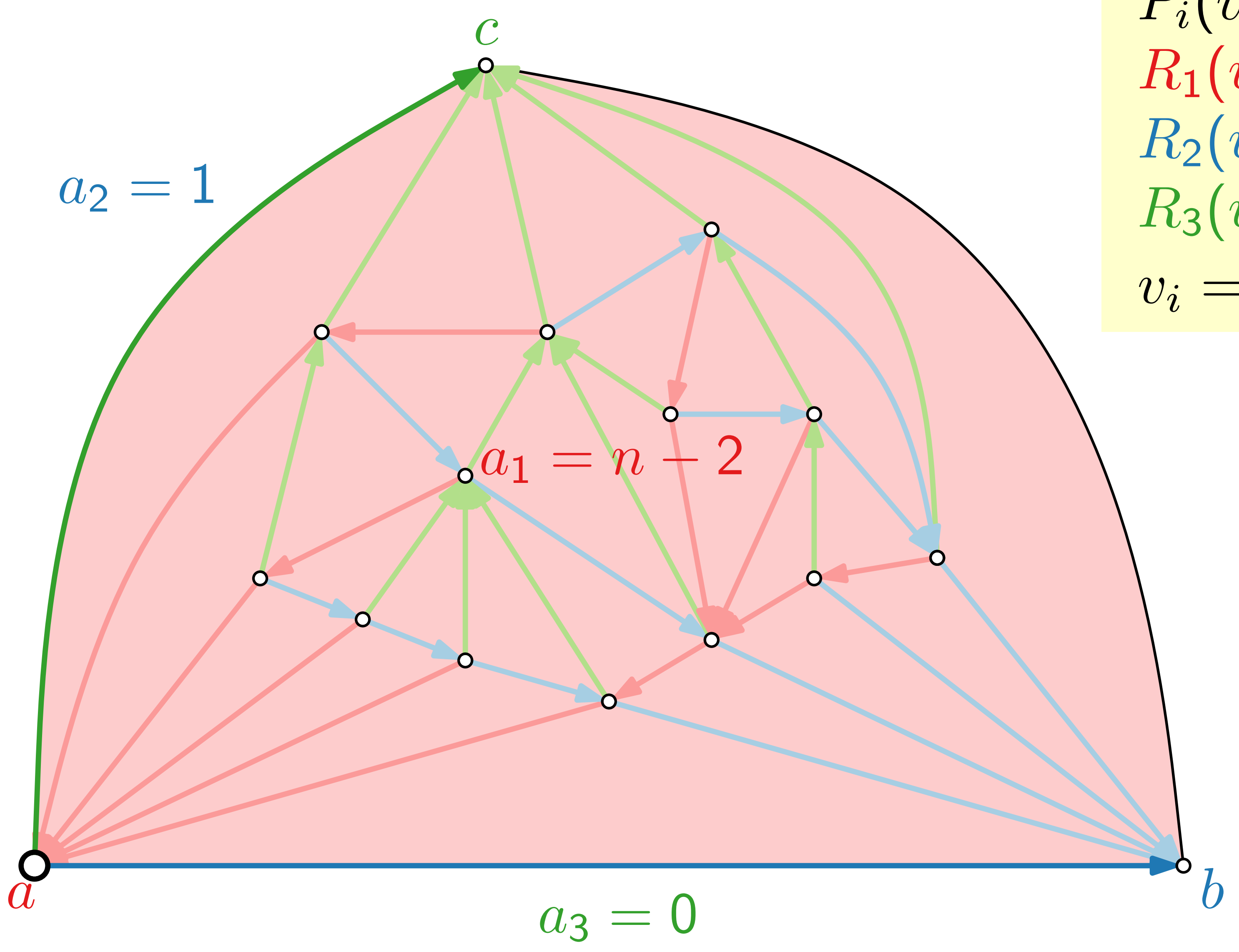
Lemma.

- For inner vertices $u \neq v$, it holds that $u \in R_i(v) \Rightarrow (u_i, u_{i+1}) <_{\text{lex}} (v_i, v_{i+1})$.

Counting Vertices



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Lemma.

- For inner vertices $u \neq v$, it holds that $u \in R_i(v) \Rightarrow (u_i, u_{i+1}) <_{\text{lex}} (v_i, v_{i+1})$.
- $v_1 + v_2 + v_3 = n - 1$

Schnyder Drawing*

Set $A = (0, 0)$, $B = (n - 1, 0)$, and $C = (0, n - 1)$.

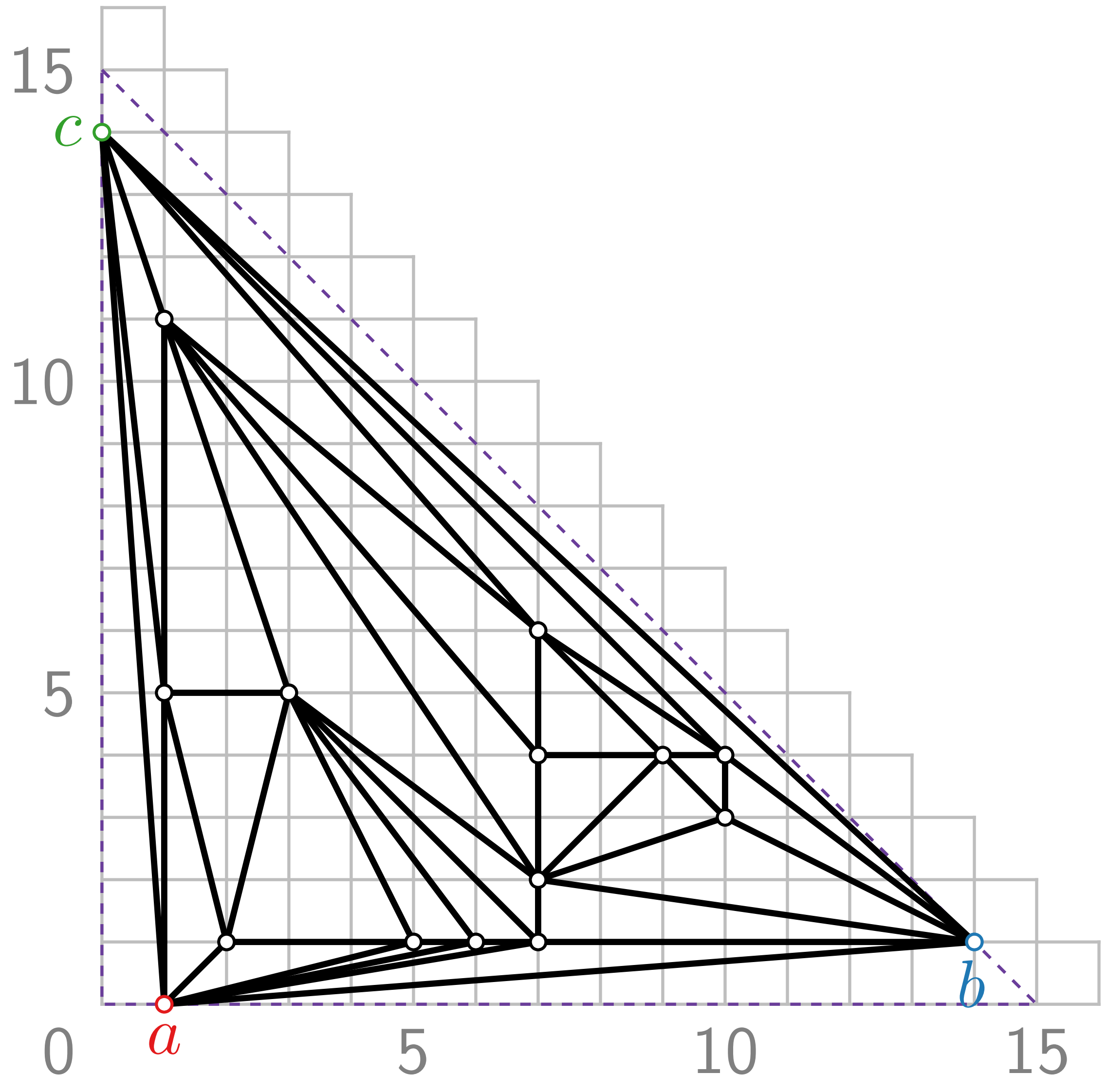
Theorem. [Schnyder '90]

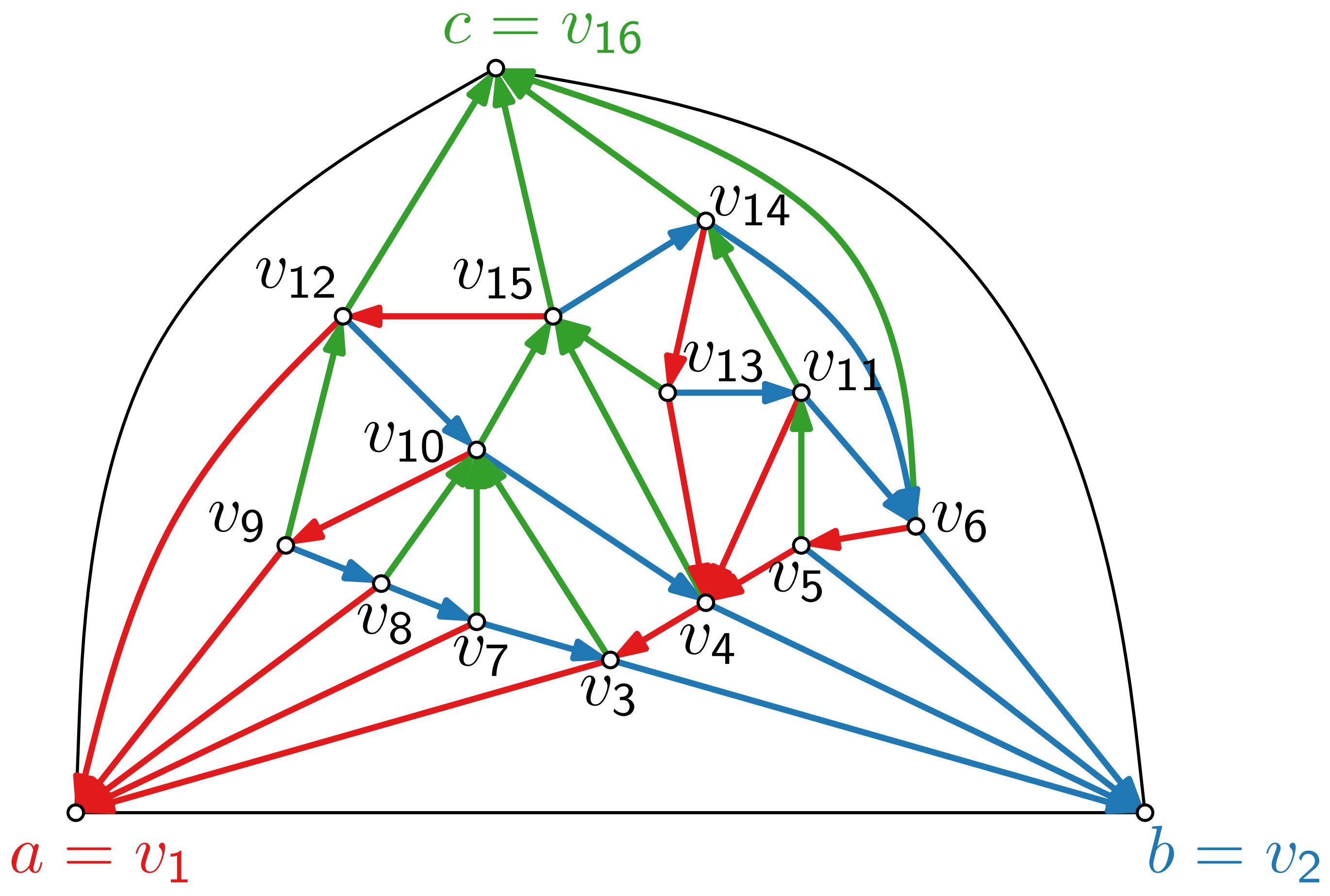
For a plane triangulation G, the mapping

$$f : v \mapsto \frac{1}{n-1}(v_1, v_2, v_3)$$

is a weak barycentric representation of G and, thus, yields a planar straight-line drawing of G on the $(n-2) \times (n-2)$ grid.

Schnyder Drawing* – Example





$n = 16,\ n - 2 = 14$

$f(a) = (14, 1, 0)$

$f(b) = (0, 14, 1)$

$f(c) = (1, 0, 14)$

$f(v_3) = (7, 7, 1)$

$f(v_4) = (6, 7, 2)$

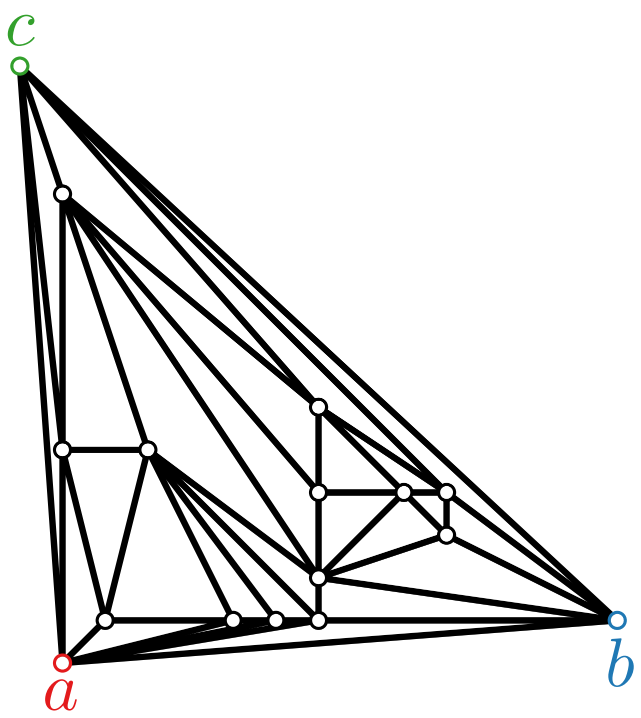
$f(v_5) = (2, 10, 3)$

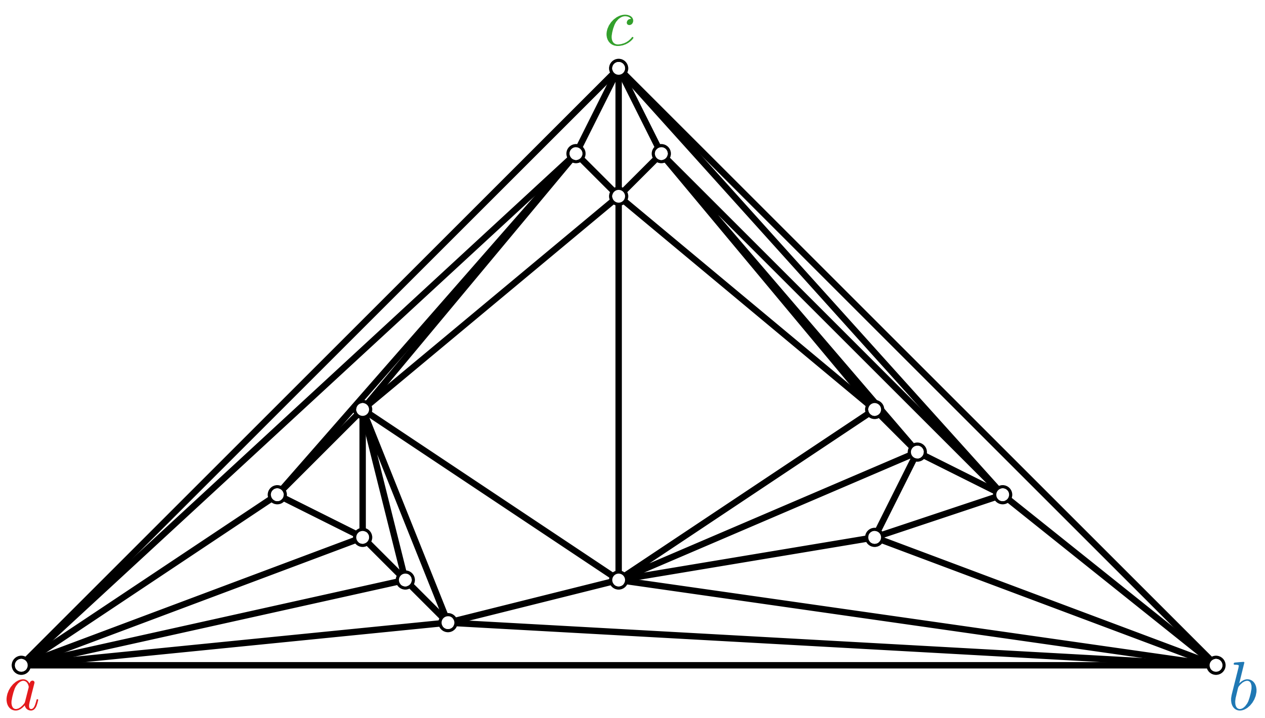
$\vdots$

Results & Variations

Theorem. [De Fraysseix, Pach, Pollack '90]
Every n-vertex planar graph has a planar straight-line drawing of size $(2n-4) \times (n-2)$. Such a drawing can be computed in $O(n)$ time.

Theorem. [Schnyder '90]
Every n-vertex planar graph has a planar straight-line drawing of size $(n-2) \times (n-2)$. Such a drawing can be computed in $O(n)$ time. ← *Exercise!*





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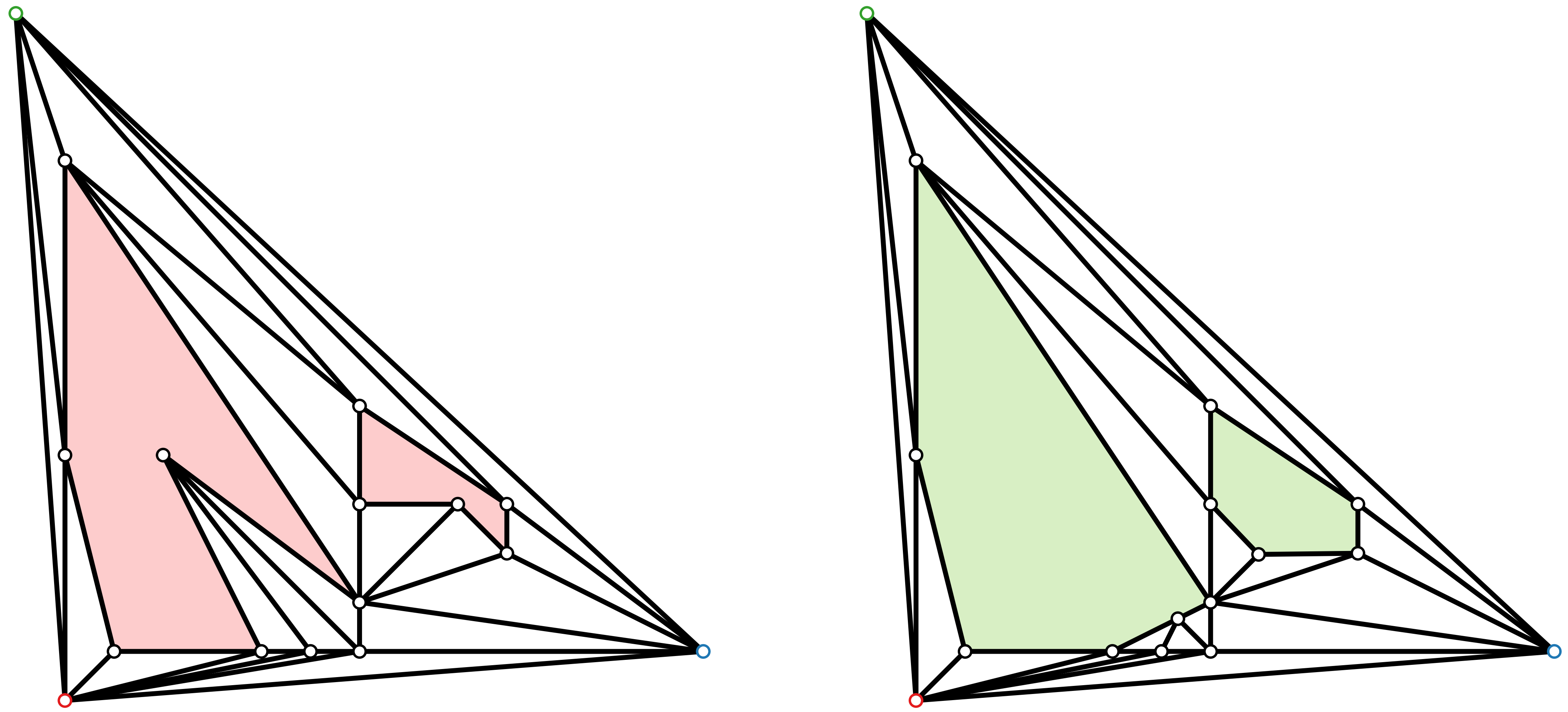
Theorem. [Brandenburg '08]
Every n-vertex planar graph has a planar straight-line drawing of size $4n/3 \times 2n/3$. Such a drawing can be computed in $O(n)$ time.

Theorem. [Dolev, Leighton, Trickey '84]
There exist n-vertex plane graphs such that any planar straight-line drawing of them has an area of at least $(2n/3-1) \times (2n/3-1)$.

[Frati, Patrignani '07] Area at least $n^2/9 + \Omega(n)$ in the variable-embedding setting.

Results & Variations



Results & Variations

Theorem. [Kant '96]
Every n-vertex 3-connected planar graph has a planar straight-line drawing of size $(2n-4) \times (n-2)$ where all faces are drawn convex.
Such a drawing can be computed in $O(n)$ time.

Theorem. [Chrobak & Kant '97]
Every n-vertex 3-connected planar graph has a planar straight-line drawing of size $(n-2) \times (n-2)$ where all faces are drawn convex.
Such a drawing can be computed in $O(n)$ time.

Theorem. [Felsner '01]
Every 3-connected planar graph with f faces has a planar straight-line drawing of size $(f-1) \times (f-1)$ where all faces are drawn convex.
Such a drawing can be computed in $O(n)$ time.

Literature

- [PGD Ch. 4.3] for detailed explanation of Schnyder woods etc.
- [Sch90] "Embedding planar graphs on the grid", Walter Schnyder, SoCG 1990 – original paper on Schnyder realizer method.