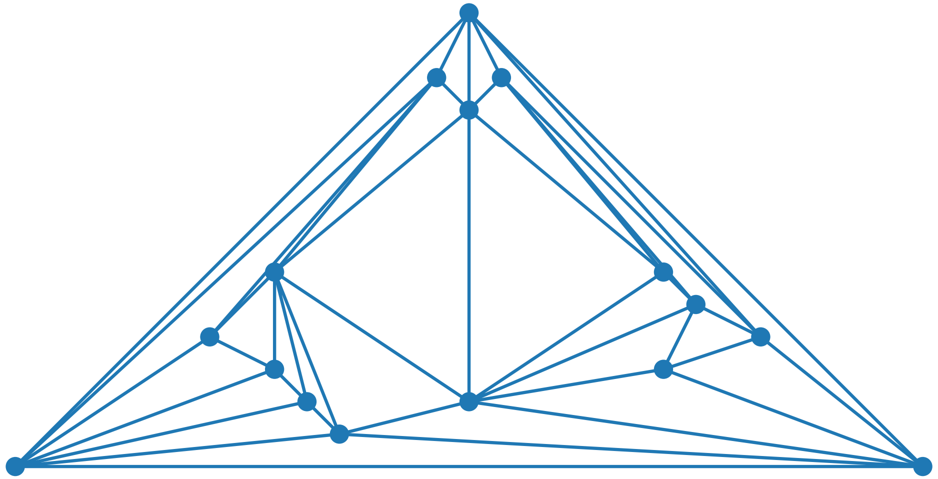
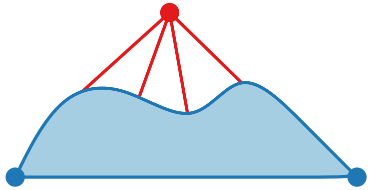
Julius-Maximilians-UNIVERSITÄT WÜRZBURG

Visualization of Graphs

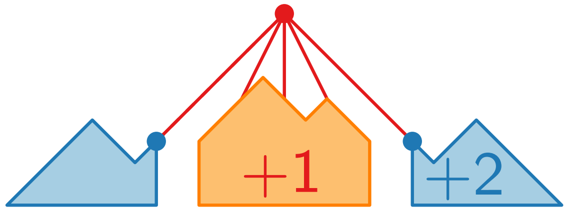
Lecture 3:
Straight-Line Drawings of Planar Graphs I: Canonical Orderings and the Shift Method



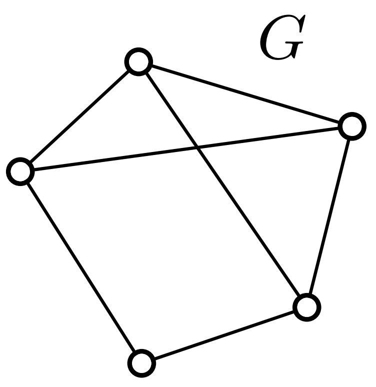


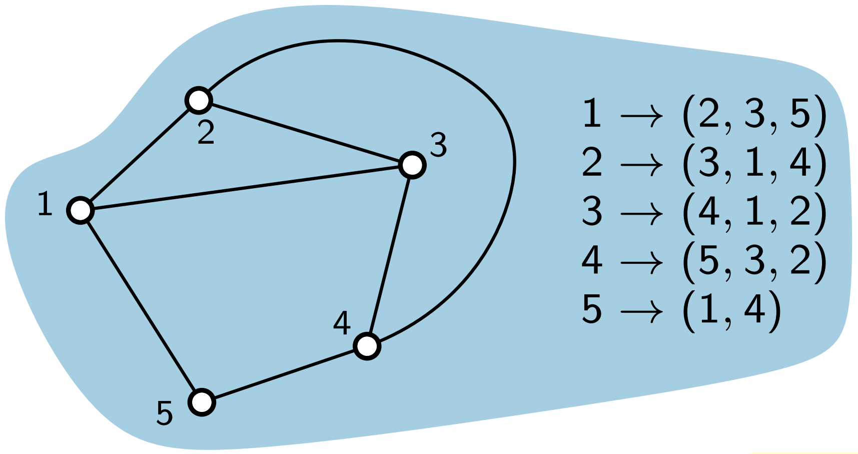
Johannes Zink

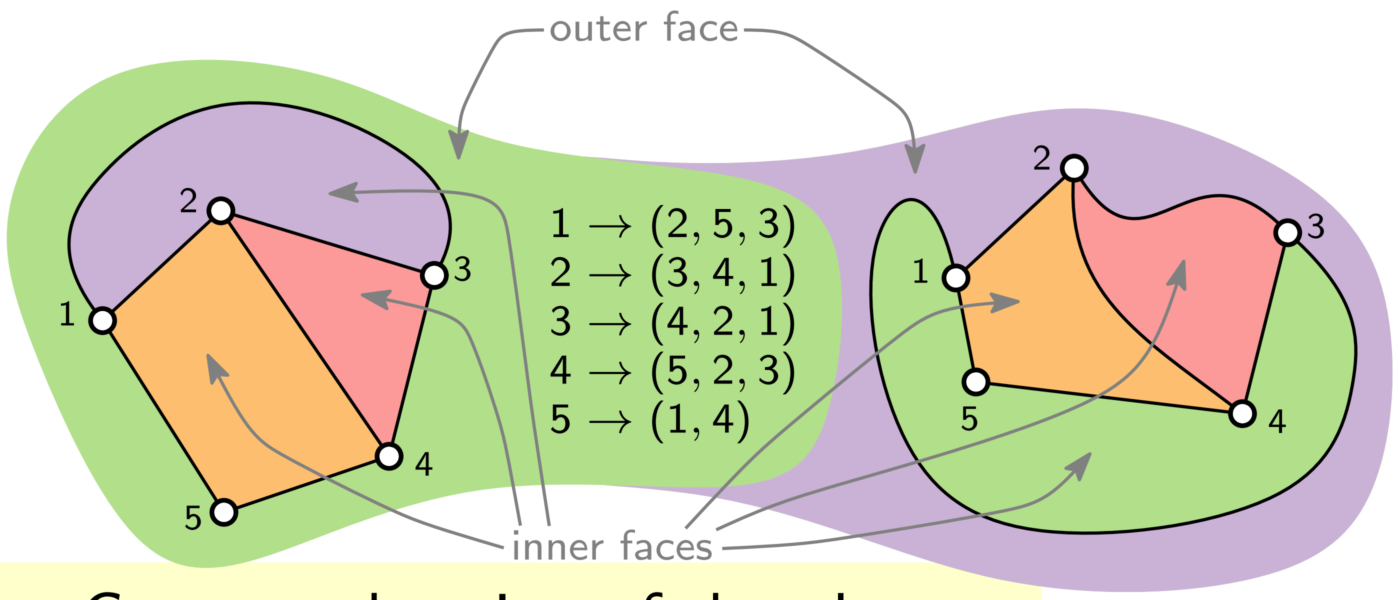
Summer semester 2024



Planar Graphs







G is **planar**:
it can be drawn in such a way that no two edges intersect each other.

planar embedding:
clockwise orientation of adjacent vertices around each vertex

A planar graph can have many planar embeddings.

A planar embedding can have many planar drawings!

faces: Connected region of the plane bounded by edges

Euler's polyhedra formula.

$$\#\text{faces} - \#\text{edges} + \#\text{vertices} = \#\text{conn.comp.} + 1$$
$$f - m + n = c + 1$$

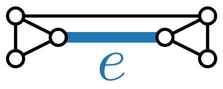
Proof. By induction on m:

Induction hypothesis in G': $f' - m' + n' = c' + 1$

$m = 0 \Rightarrow f = 1$ and $c = n$ ✓

$m \geq 1 \Rightarrow$ delete some edge e $\Rightarrow m' = m - 1$

$\Rightarrow c' = c + 1$

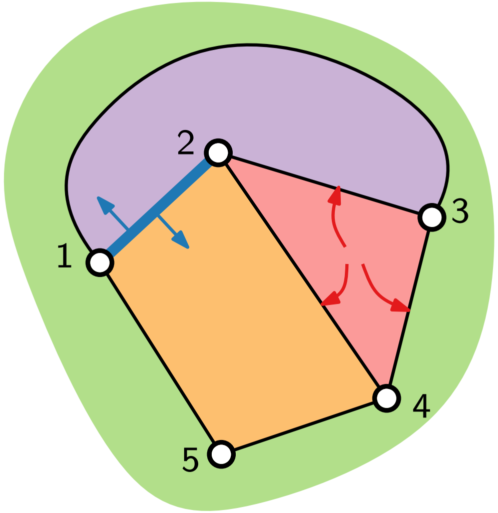


$\Rightarrow f' = f - 1$ ✓

Properties of Planar Graphs

Euler's polyhedra formula.
$$\#\text{faces} - \#\text{edges} + \#\text{vertices} = \#\text{conn.comp.} + 1$$
$$f - m + n = c + 1$$

Theorem. G simple planar graph with $n \geq 3$ vtc.
1. $m \leq 3n - 6$
2. $f \leq 2n - 4$
3. There is a vertex of degree at most 5.



Proof. 1. Every edge incident to ≤ 2 faces
Every face incident to ≥ 3 edges
$\Rightarrow 3f \leq \#\text{incidences} \leq 2m$
$\Rightarrow 6 \leq 3c + 3 = 3f - 3m + 3n \leq 2m - 3m + 3n = 3n - m$
$\Rightarrow m \leq 3n - 6$

idea: count edge–face incidences

2. $3f \leq 2m \leq 6n - 12 \Rightarrow f \leq 2n - 4$

Handshaking lemma.
$\sum_{v \in V(G)} \deg(v) = 2|E|$.

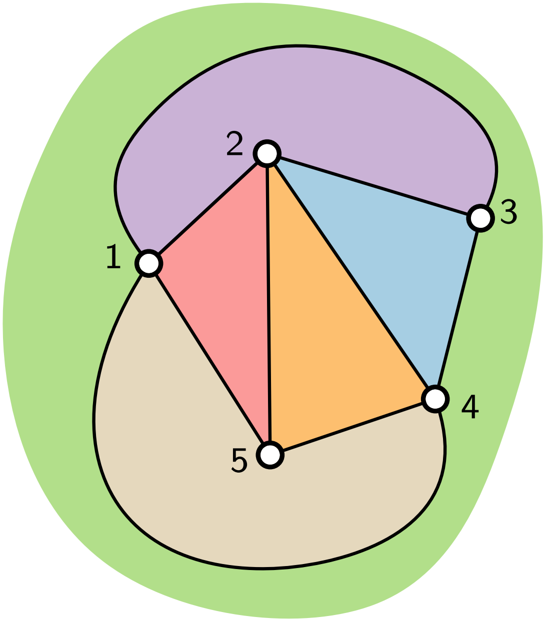
3. $\sum_{v \in V(G)} \deg(v) = 2m \leq 6n - 12$
$\Rightarrow \min_{v \in V(G)} \deg(v) \leq \text{average degree}(G) = \frac{1}{n} \sum_{v \in V(G)} \deg(v) \leq \frac{6n-12}{n} < 6$

Triangulations

planar graph given with a planar embedding

A **plane (inner) triangulation** is a plane graph where every (inner) face is a triangle.

A **maximal planar graph** is a planar graph where adding any edge would violate planarity.

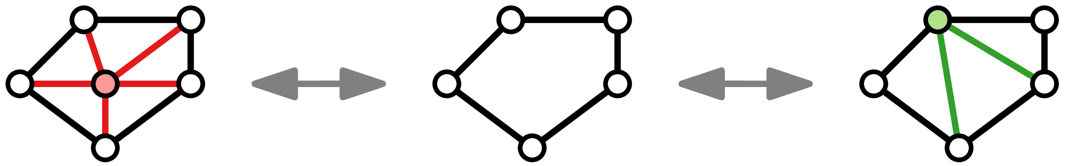


Observation.
Any maximal plane graph is a plane triangulation (and vice versa).

Lemma.
Any plane triangulation is 3-connected and thus has a unique planar embedding (up to mirroring).

We focus on plane triangulations:

Lemma.
Every plane graph is subgraph of a plane triangulation.



Motivation

- Why planar and straight-line?

[Bennett, Ryall, Spaltzeholz and Gooch '07]

The Aesthetics of Graph Visualization

3.2. Edge Placement Heuristics

By far the most agreed-upon edge placement heuristic is to *minimize the number of edge crossings* in a graph [BMRW98, Har98, DH96, Pur02, TR05, TBB88]. The importance of avoiding edge crossings has also been extensively validated in terms of user preference and performance (see Section 4). Similarly, based on perceptual principles, it is beneficial to *minimize the number of edge bends* within a graph [Pur02, TR05, TBB88]. Edge bends make edges more difficult to follow because an edge with a sharp bend is more likely to be perceived as two separate objects. This leads to the heuristic of *keeping edge bends uniform* with respect to the bend's position on the edge and its angle [TR05]. If an edge must be bent to satisfy other aesthetic criteria, the angle of the bend should be as little as possible, and the bend placement should evenly divide the edge.

Drawing conventions

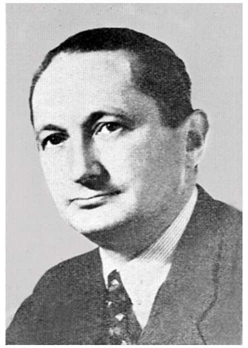
- No crossings $\Rightarrow$ planar
- No bends $\Rightarrow$ straight-line

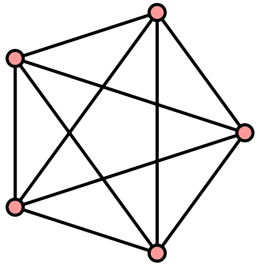
Drawing aesthetics to optimize

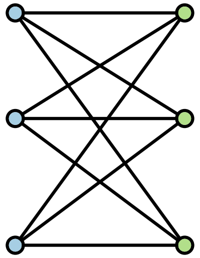
- Area

Towards Straight-Line Drawings

Theorem. [Kuratowski 1930]
G planar $\Leftrightarrow$
neither K_5 nor $K_{3,3}$ minor of G


Kazimierz Kuratowski (1896–1980)

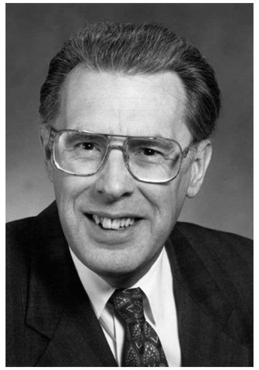

K_5


$K_{3,3}$

Characterization

Theorem. [Hopcroft & Tarjan 1974]
Let G be a graph with n vertices. There is an $\mathcal{O}(n)$-time algorithm to test whether G is planar.

Also computes a planar embedding in $\mathcal{O}(n)$ time.


John Edward Hopcroft (1939–)
en.wikipedia.org/wiki/User:Shakespeare


Robert Endre Tarjan (1948–)
Renatokeshet, GFDL via Wikimedia

Recognition

Theorem. [Wagner 1936, Fáry 1948, Stein 1951]
Every planar graph has a planar drawing where the edges are straight-line segments.

The algorithms implied by these theorems produce drawings whose area is **not** bounded by any polynomial in n.


Klaus Wagner (1910–2000)
Autor: Konrad Jacobs, wikipedia

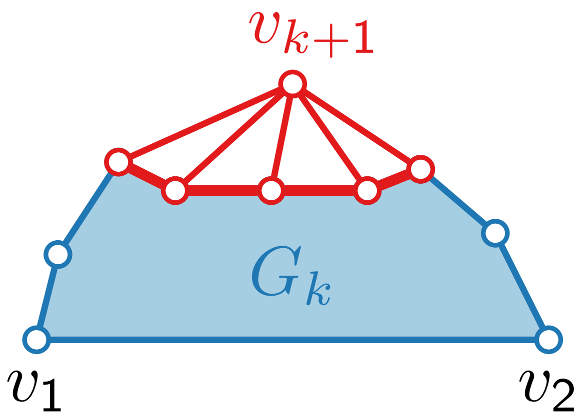
Drawing

Planar Straight-Line Drawings

Theorem. [De Fraysseix, Pach, Pollack '90]
Every n-vertex planar graph has a planar straight-line drawing of size $(2n-4) \times (n-2)$.

Idea.

- Find a *canonical order* $(v_1, \ldots, v_n)$ of the vertices of a triangulation.
- Start with the single edge (v_1, v_2). Let this be the graph G_2.
- Let $k \in \{3, \ldots, n\}$. To obtain G_{k+1}, add v_{k+1} to G_k so that the neighbors of v_{k+1} are on the outer face of G_k.
- The neighbors of v_{k+1} in G_k form a path of length at least two.



Theorem. [Schnyder '90]
Every n-vertex planar graph has a planar straight-line drawing of size $(n-2) \times (n-2)$.

(next lecture)

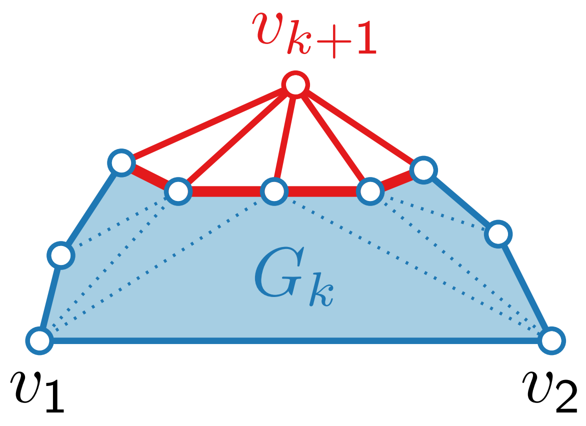
Canonical Order – Definition

Definition.
Let G be a plane triangulation on $n \geq 3$ vertices.
An ordering $\pi = (v_1, v_2, \ldots, v_n)$ of $V(G)$ is a **canonical order** if the following conditions hold for each $k \in \{3, 4, \ldots, n\}$:

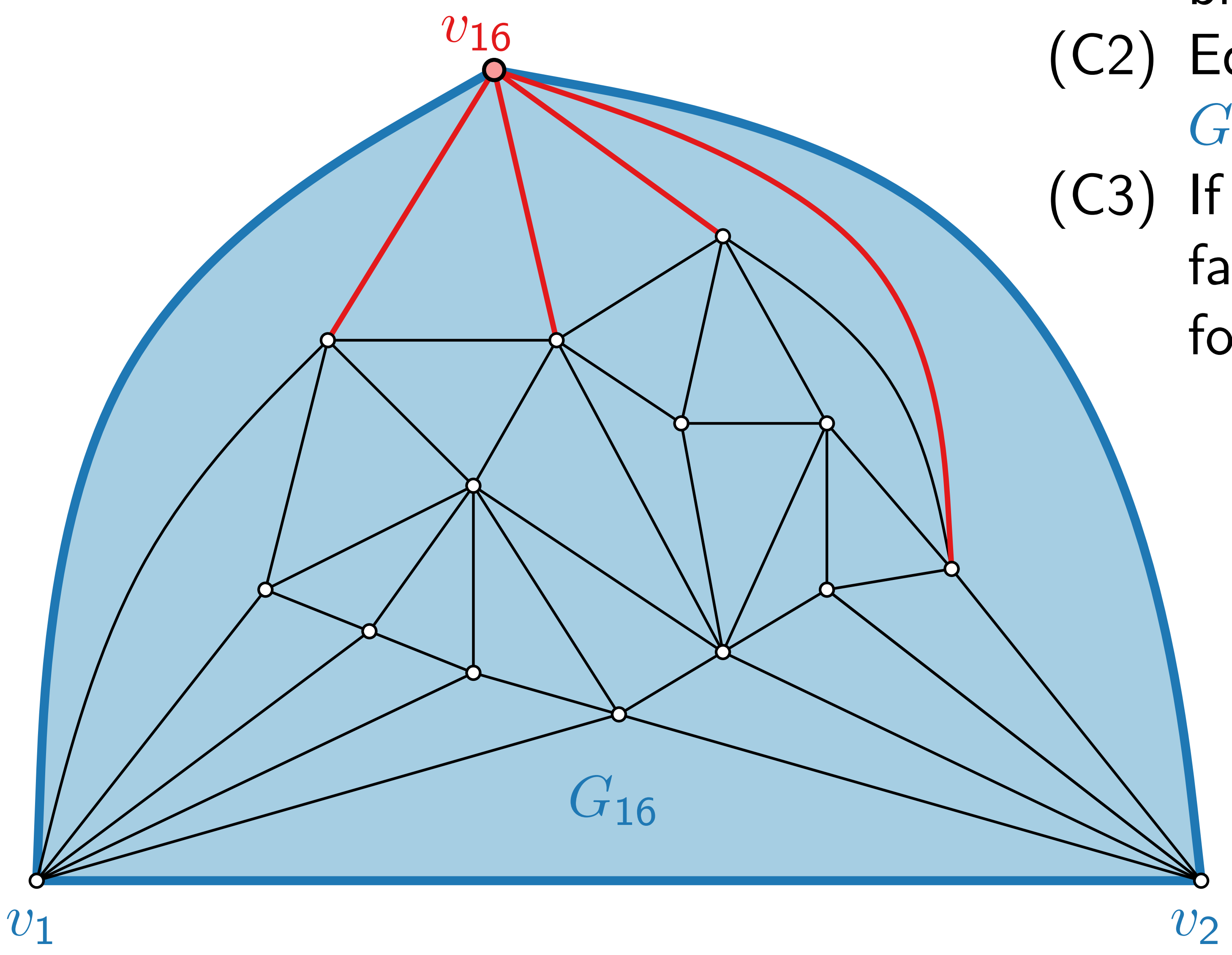
(C1) Vertices $\{v_1, \ldots, v_k\}$ induce a biconnected inner triangulation; call it G_k.

(C2) Edge (v_1, v_2) belongs to the outer face of G_k.

(C3) If $k < n$ then vertex v_{k+1} lies in the outer face of G_k, and the neighbors of v_{k+1} form a path on the boundary of G_k.



Canonical Order – Example



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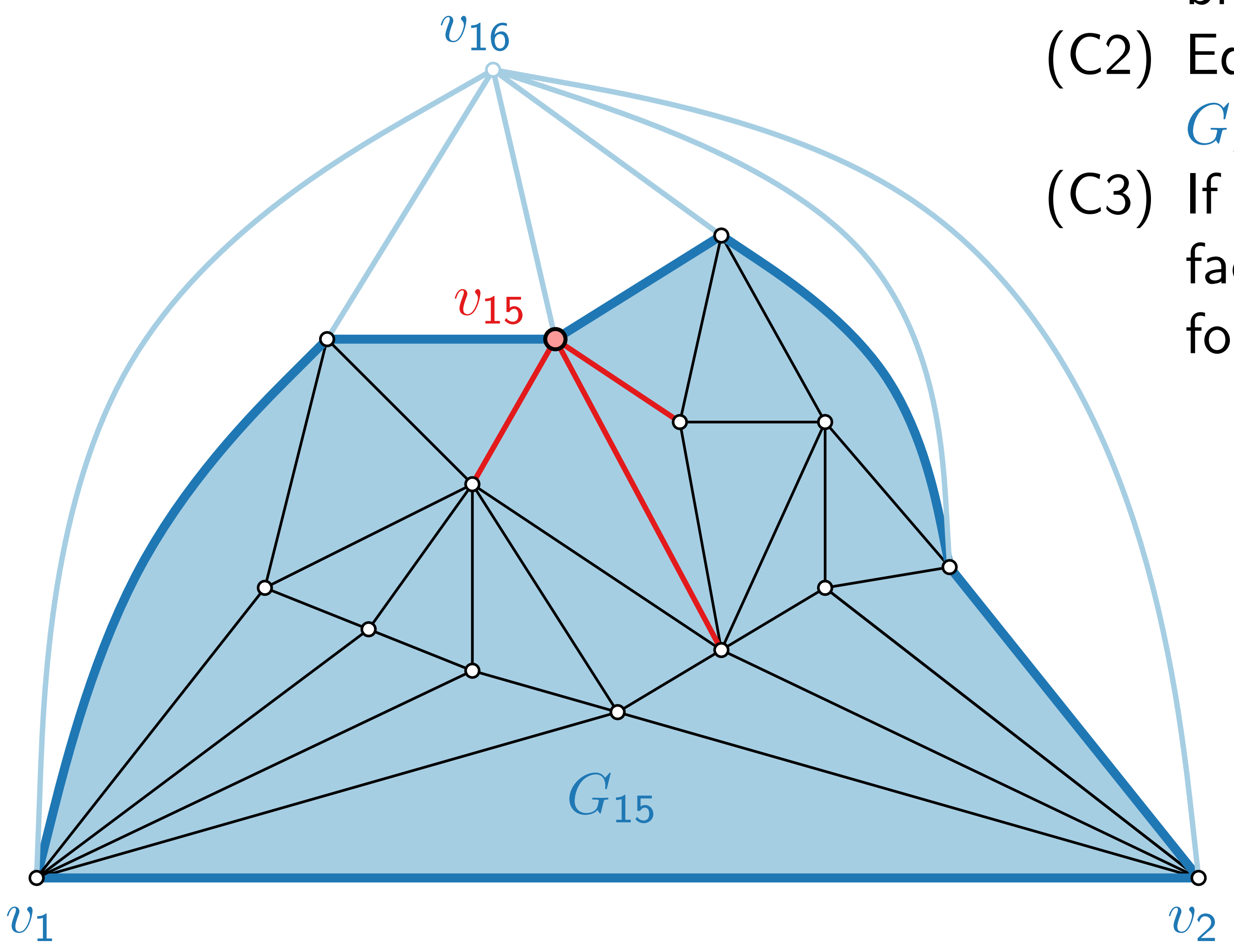

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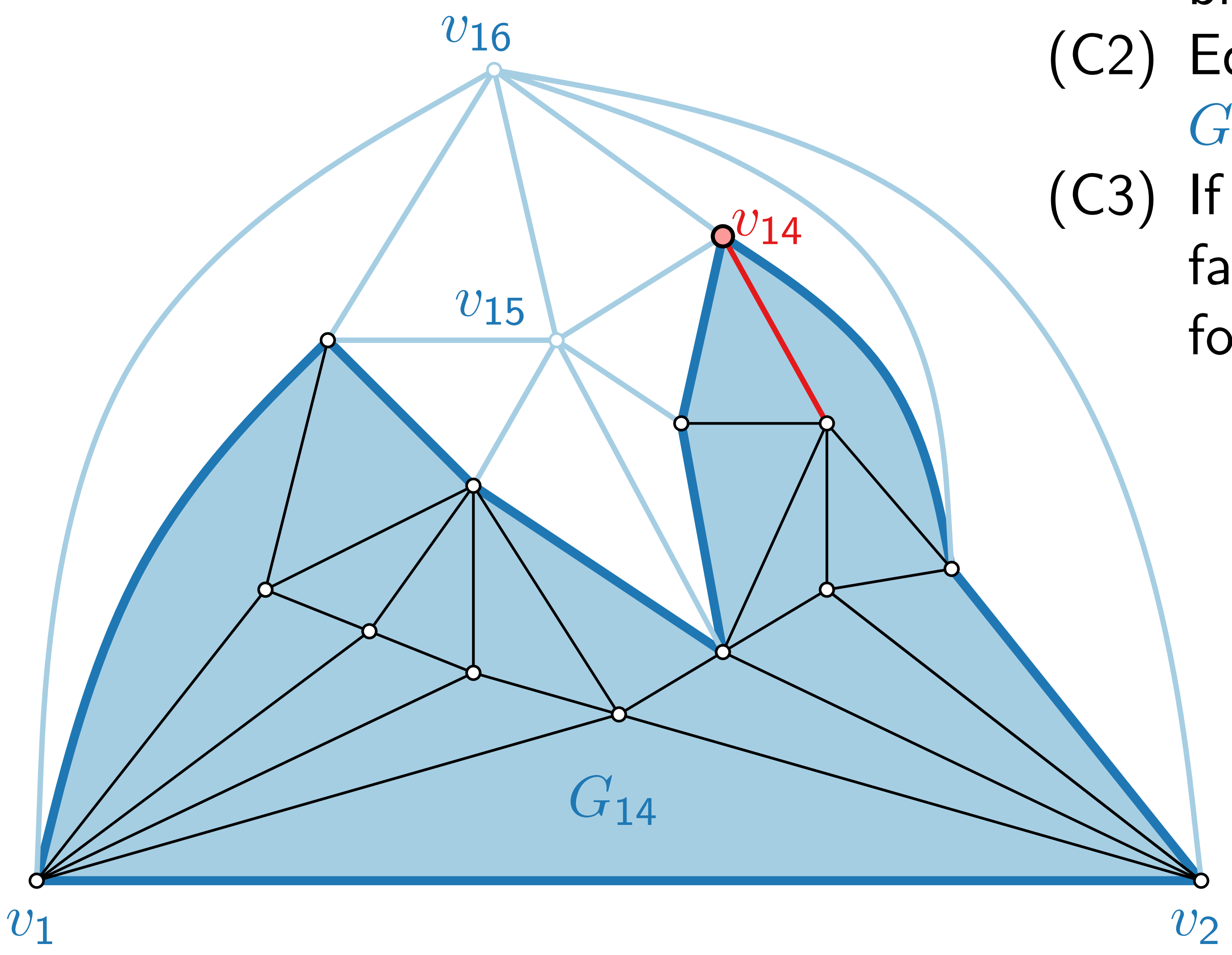

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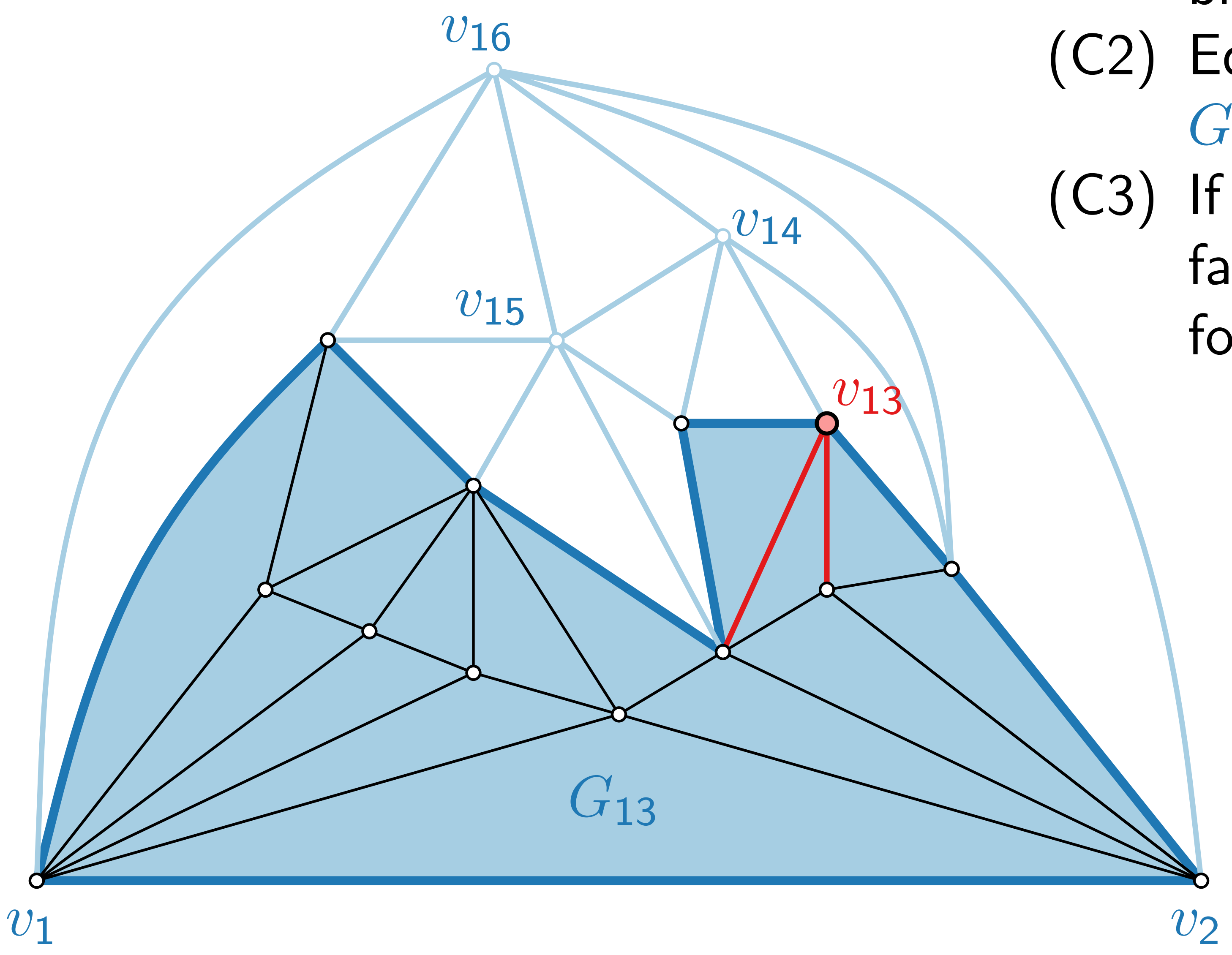

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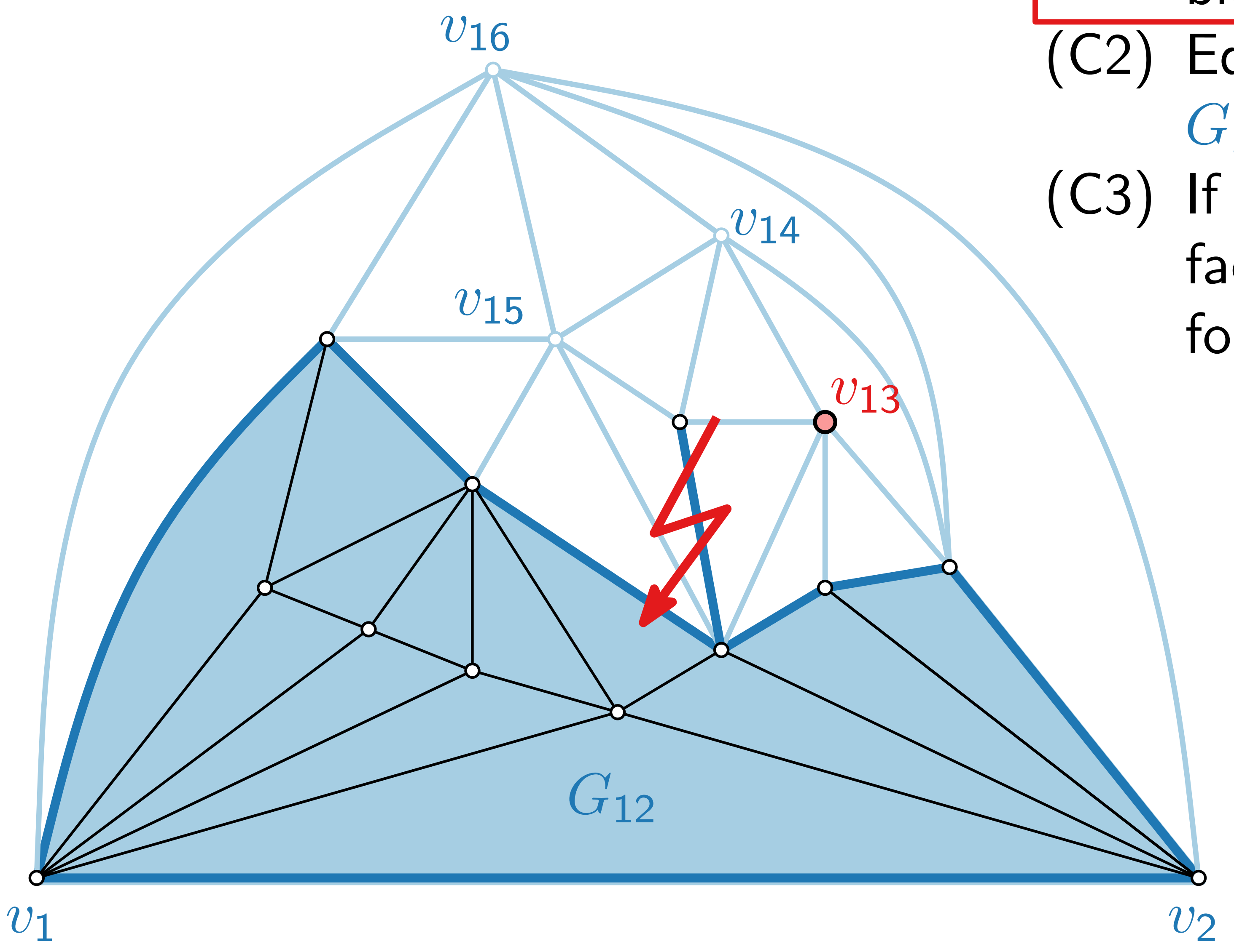

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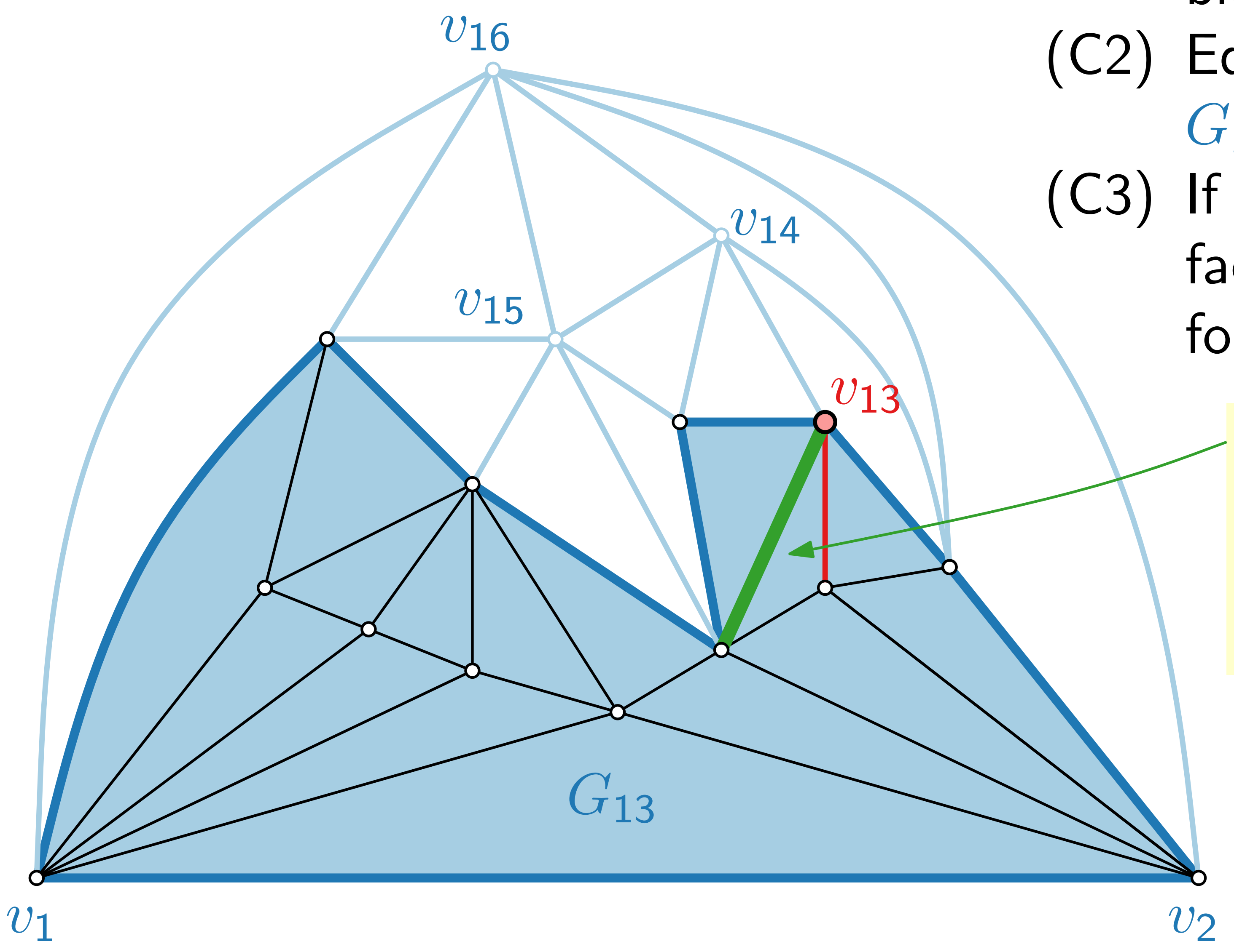

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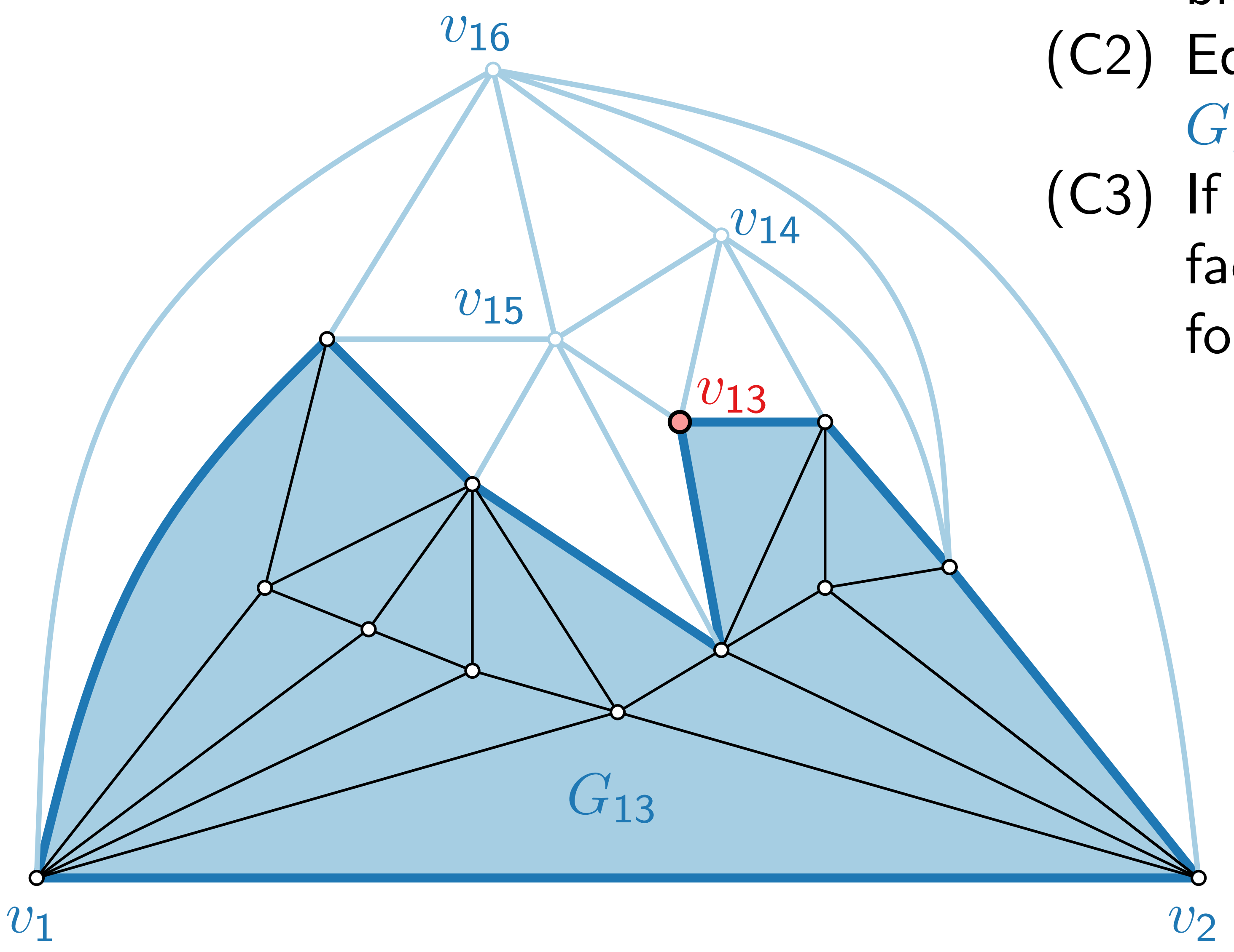
chord:
edge joining two non-adjacent vertices in a cycle

Canonical Order – Example



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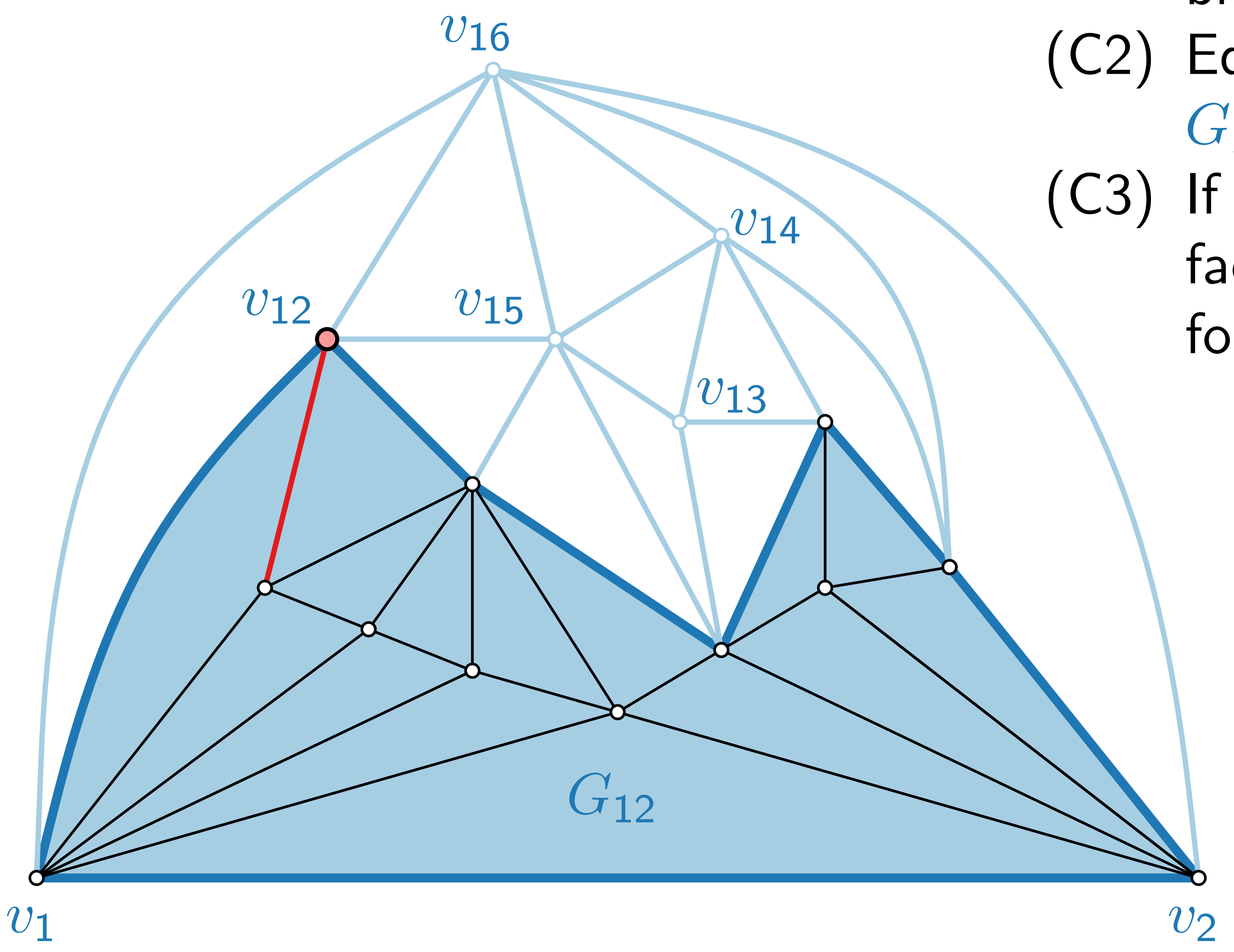

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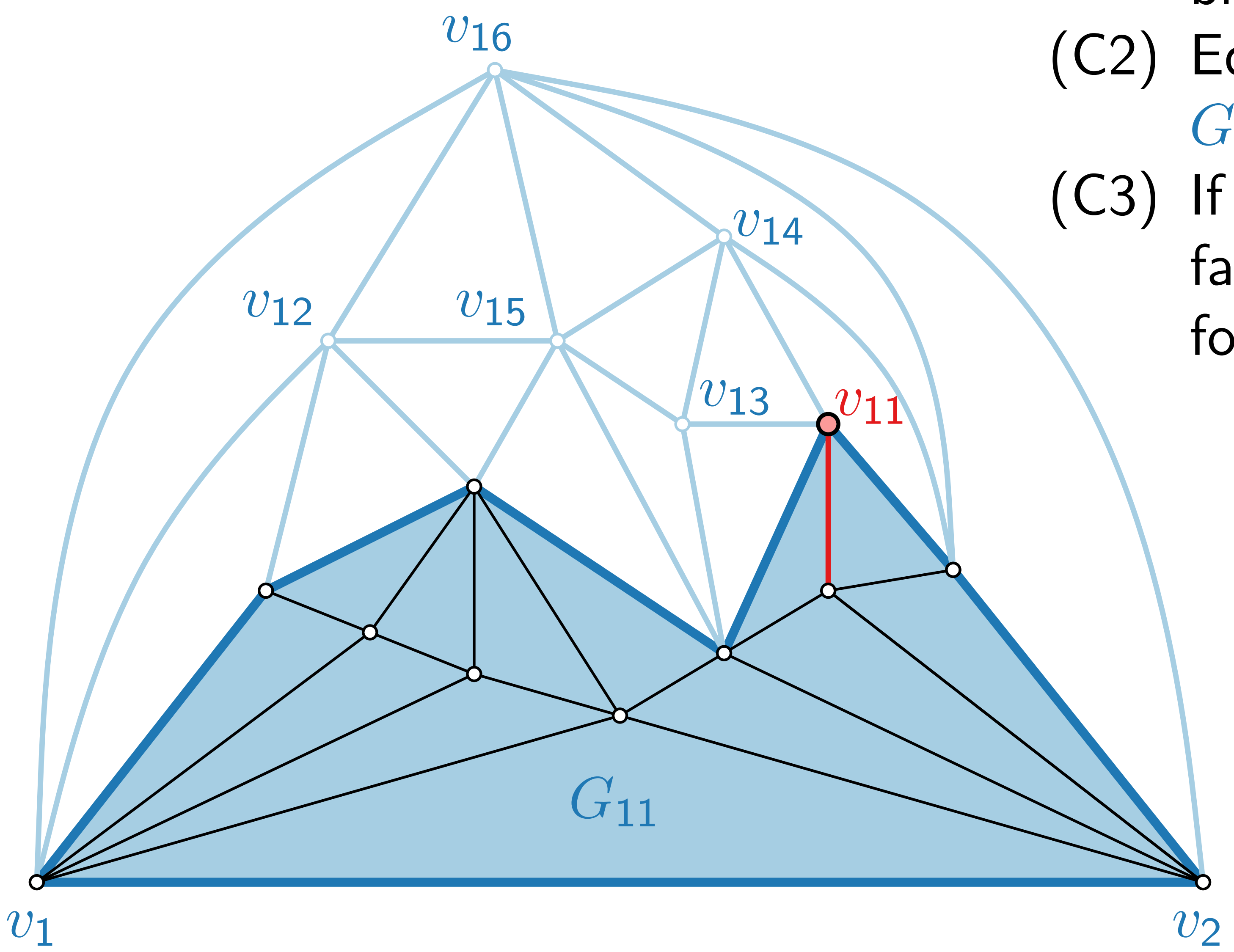

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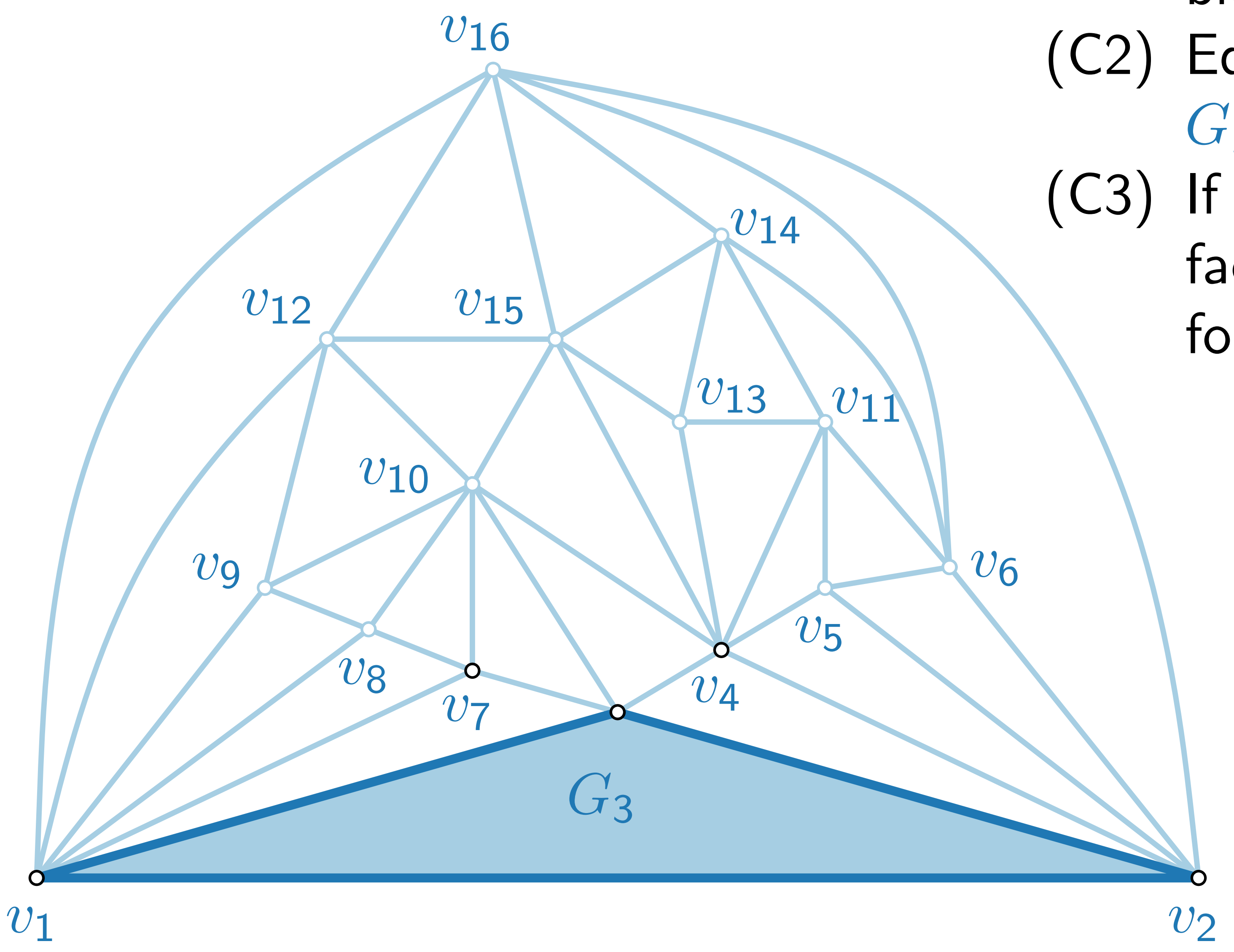

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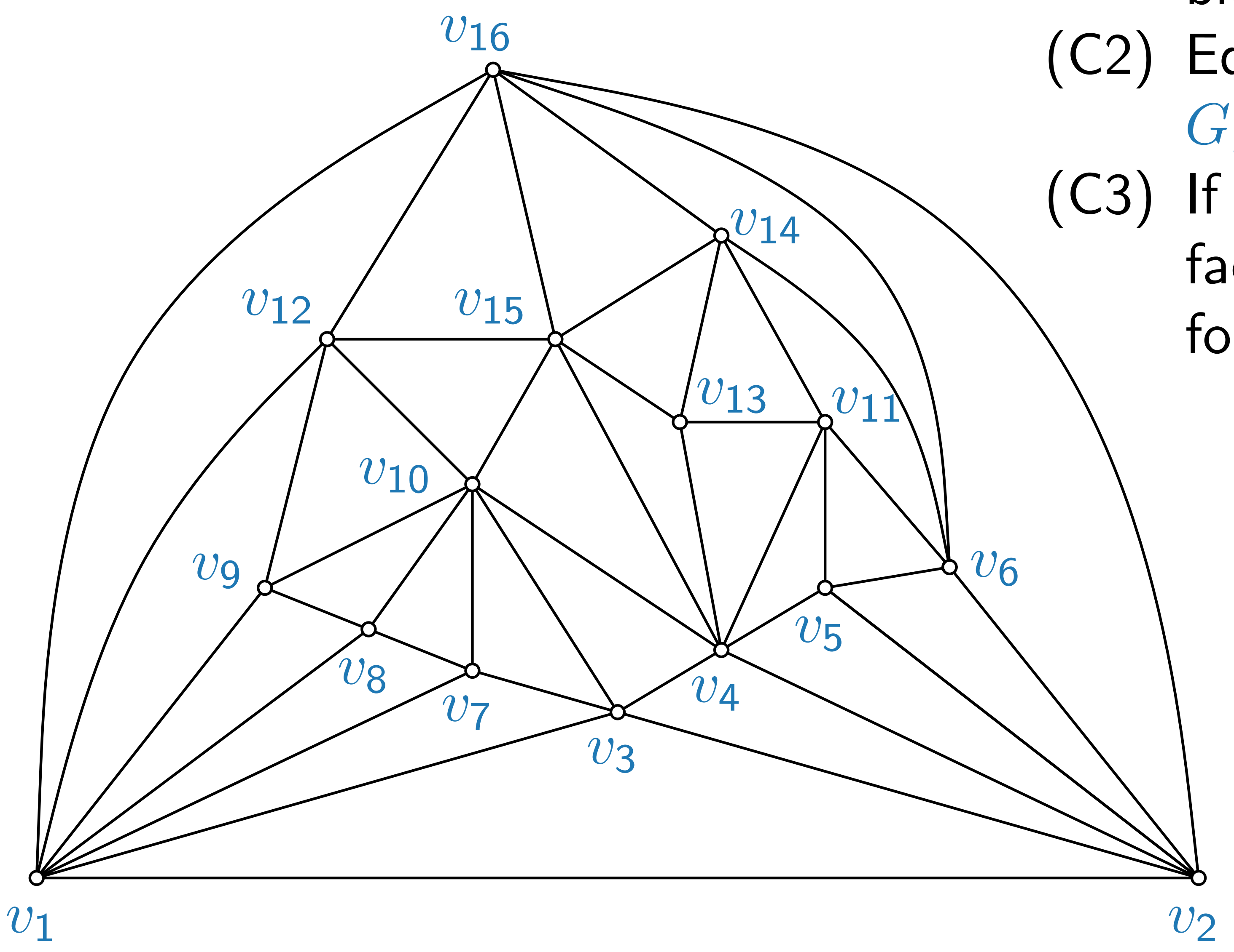

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(C3) If $k < n$ then vertex v_{k+1} lies in the outer face of G_k, and the neighbors of v_{k+1} form a path on the boundary of G_k.

Canonical Order – Existence

(C1) G_k biconnected inner triangulation

(C2) (v_1, v_2) on outer face of G_k

(C3) $k < n \Rightarrow v_{k+1}$ in outer face of G_k, neighbors of v_{k+1} form path on boundary of G_k

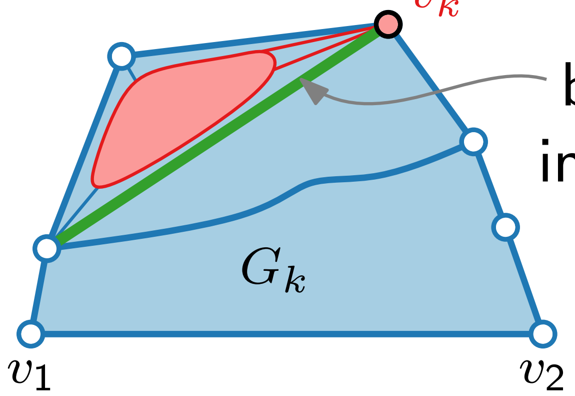
Lemma.
Every plane triangulation has a canonical order.

Consider any n-vertex plane triangulation. We show this statement by induction on k from n down to 3.

Induction base $(k = n)$: Let $G_n = G$, and let v_1, v_2, v_n be the vertices of the outer face of G_n. Conditions (C1)–(C3) hold.

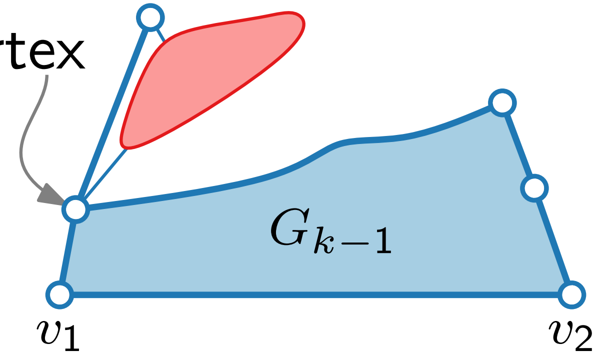
Induction hypothesis: Vertices $v_{n-1}, \ldots, v_{k+1}$ have been chosen such that conditions (C1)–(C3) hold for every $i \in \{k+1, \ldots, n\}$.

Induction step: Consider G_k. We search for v_k.



because v_k incident to a chord

cut vertex

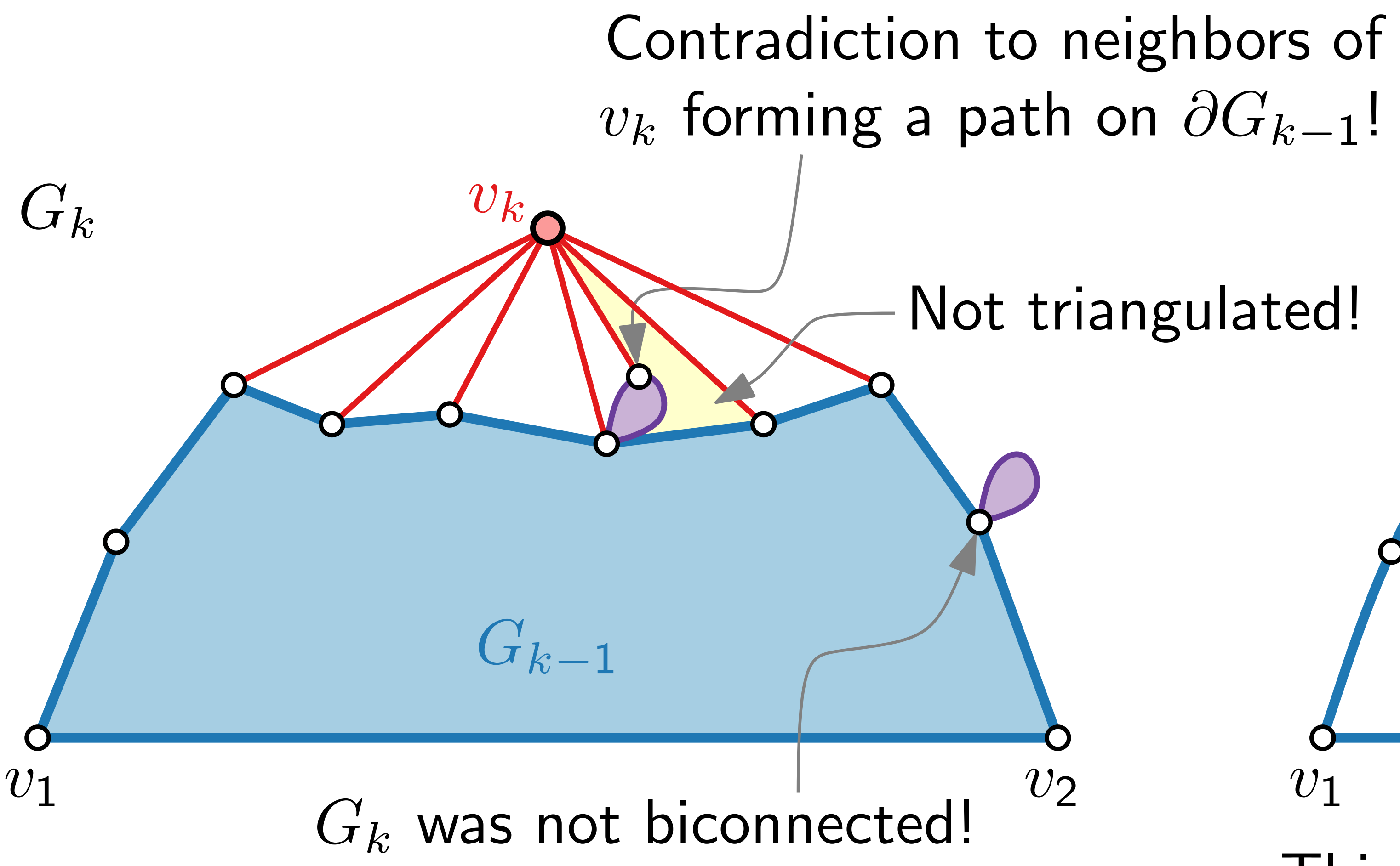


We need to show:

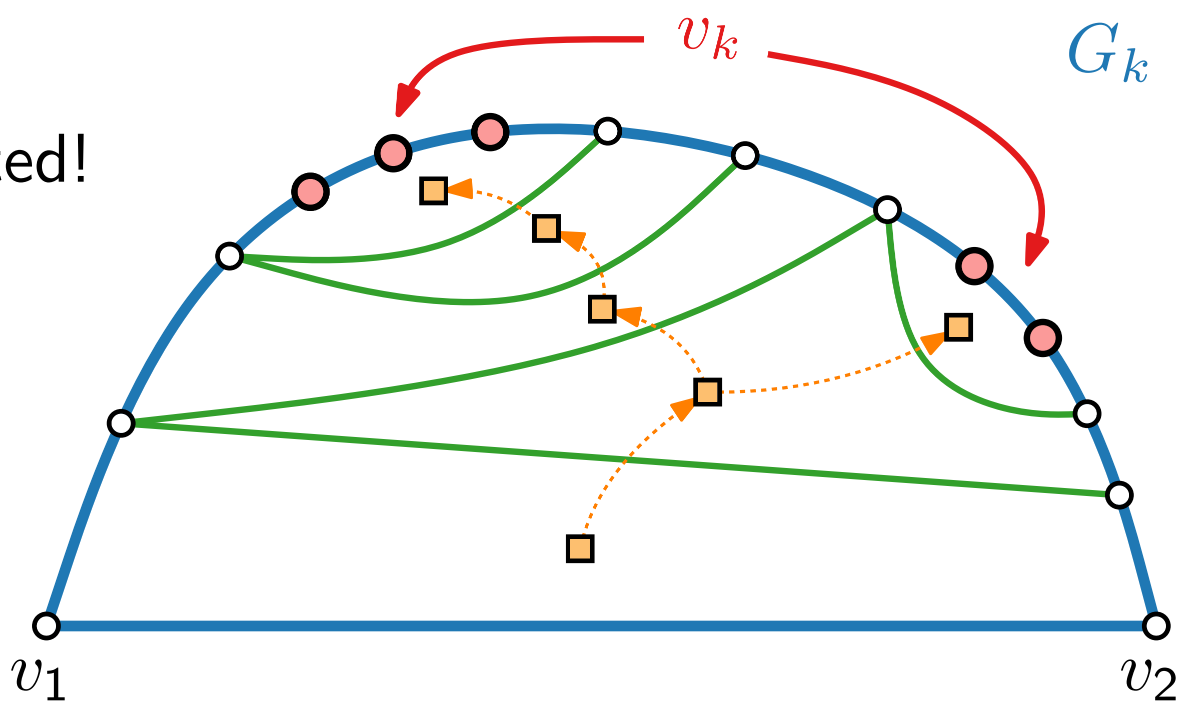
1. v_k not incident to chord is sufficient.
2. Such v_k exists.

Canonical Order – Existence

Claim 1.
If v_k is not incident to a chord, then G_{k-1} is biconnected.



Claim 2.
There exists a vertex in G_k that is not incident to a chord as choice for v_k.



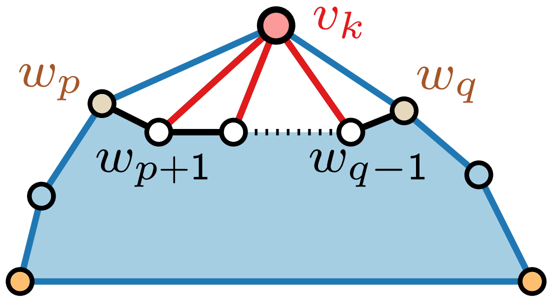
This completes the proof of the lemma. □

Canonical Order – Implementation

outer face

```
CanonicalOrder(G, ⟨v_1, v_2, v_n⟩)
foreach v ∈ V(G) do
    chords(v) ← 0; out(v) ← false; mark(v) ← false
out(v_1), out(v_2), out(v_n) ← true
for k = n downto 3 do
    choose v ∈ V(G) \ {v_1, v_2} such that mark(v) = false,
     out(v) = true, chords(v) = 0  // use list of candidates
    v_k ← v; mark(v_k) ← true; out(v_k) ← false
    let w_p, ..., w_q be the ordered unmarked neighbors of v_k
    for i = p + 1 to q − 1 do   // O(n) time in total
        out(w_i) ← true           // O(m) = O(n) in total
        foreach u ∈ Adj[w_i] \ {w_{i−1}, w_{i+1}} do
            if out(u) then chords(w_i)++, chords(u)++
    if p + 1 = q then chords(w_p)−−, chords(w_q)−−
```

- chord(v)= # chords incident to v
- out(v) $=$ true iff v on boundary of current outer face
- mark(v) $=$ true iff v has received a number $\geq k$



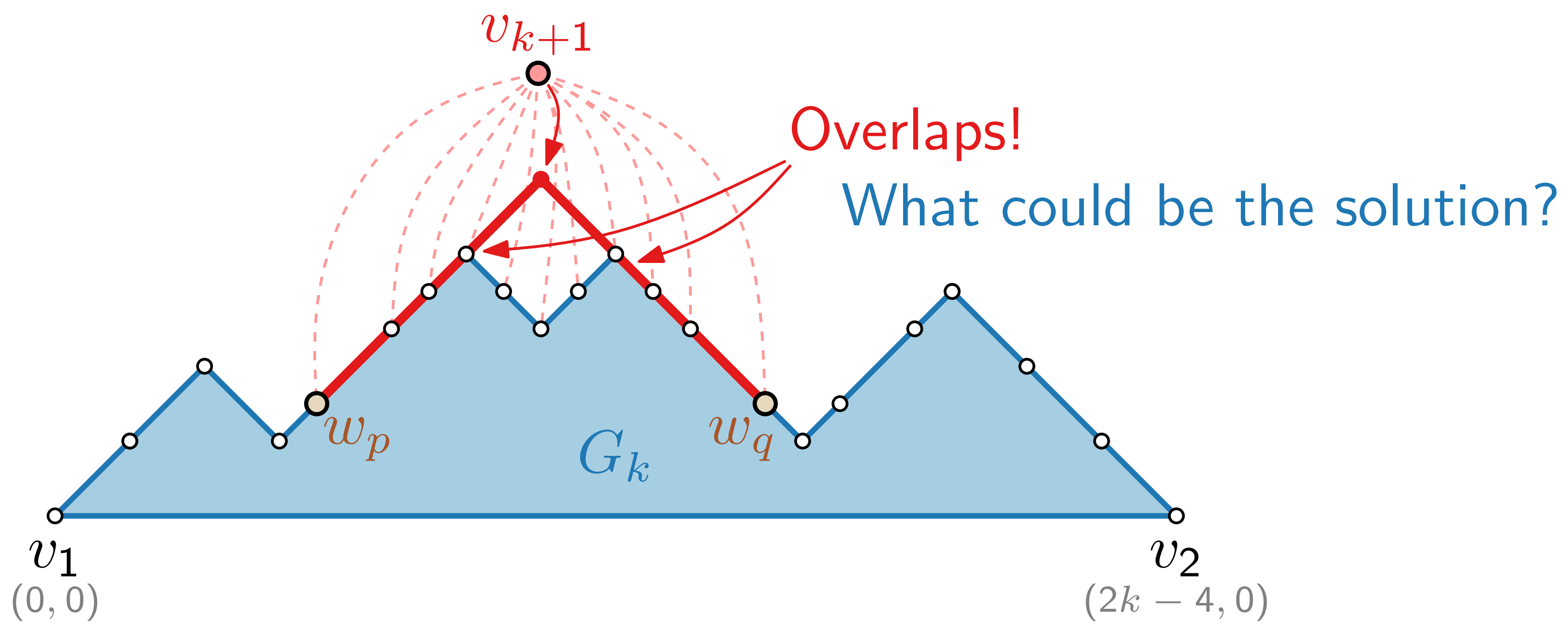
Lemma.
Algorithm CanonicalOrder computes a canonical order of a plane graph in $\mathcal{O}(n)$ time.

Shift Method – Idea

Drawing invariants:

G_k is drawn such that

- v_1 is at $(0, 0)$, v_2 is at $(2k - 4, 0)$,
- boundary of G_k (minus edge $\{v_1, v_2\}$) is drawn x-monotone,
- each edge on the boundary of G_k (except $\{v_1, v_2\}$) is drawn with slopes ± 1.



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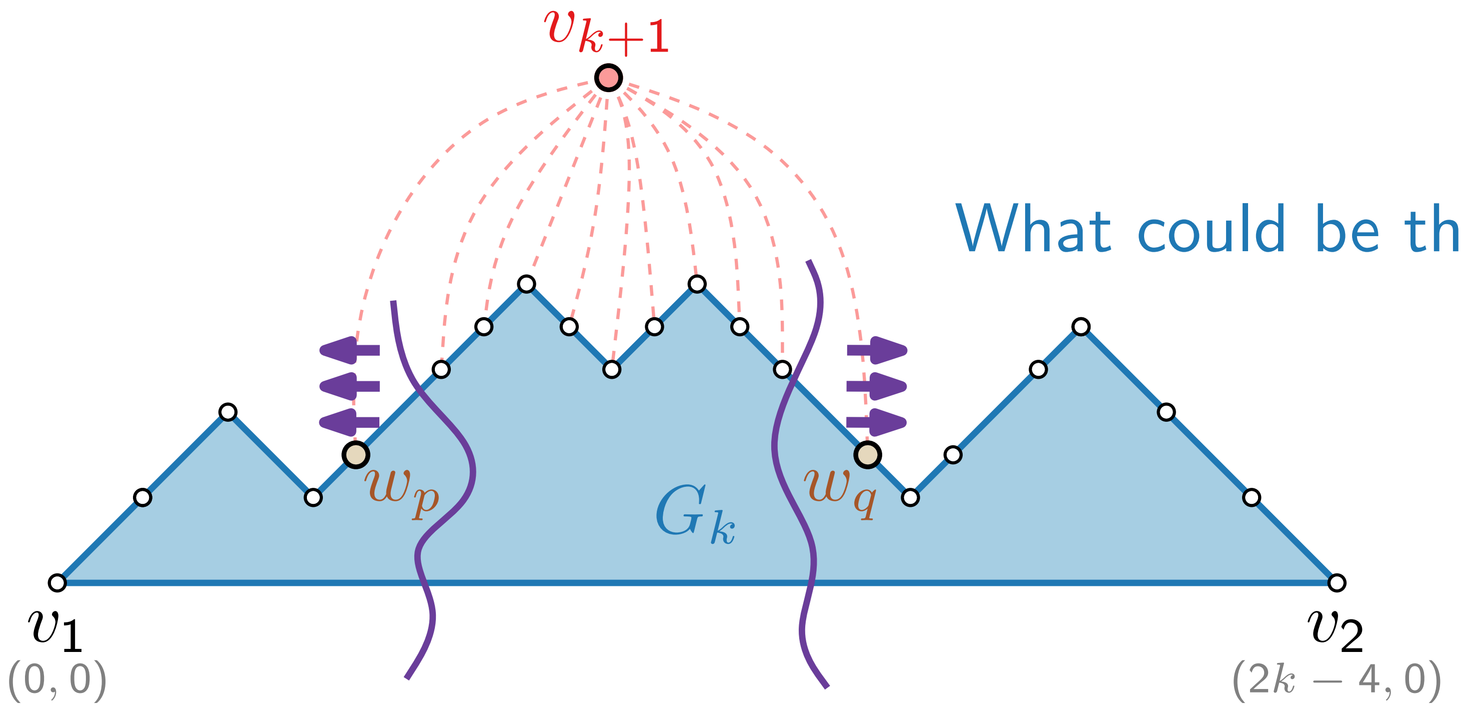

What could be the solution?

Shift Method – Idea

Drawing invariants:

G_k is drawn such that

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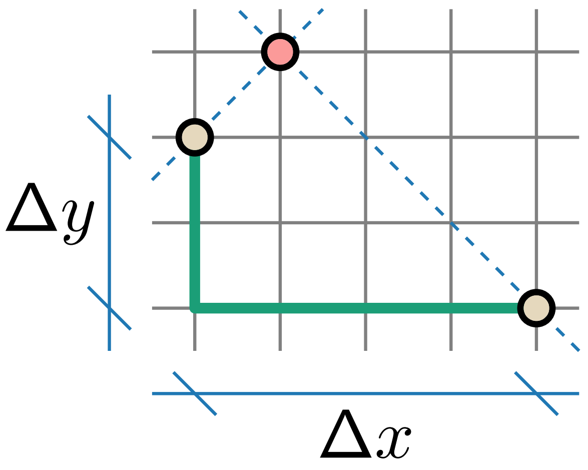
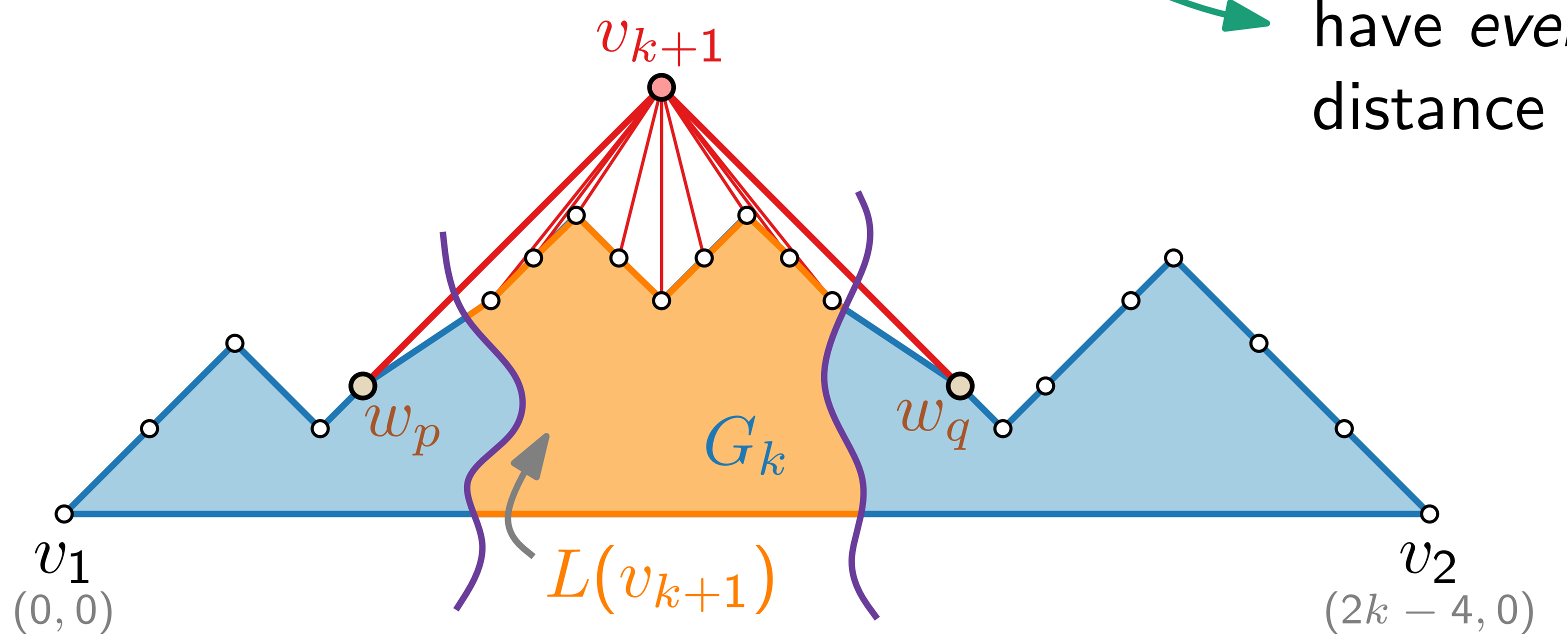
- each edge on the boundary of G_k (except $\{v_1, v_2\}$) is drawn with slopes ± 1.

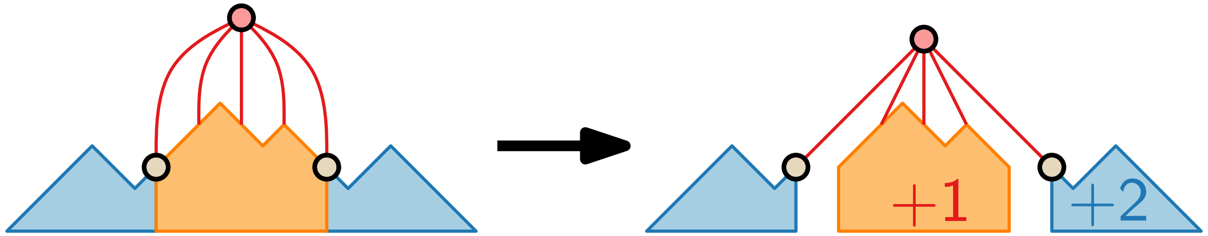
Will v_{k+1} lie on the grid?

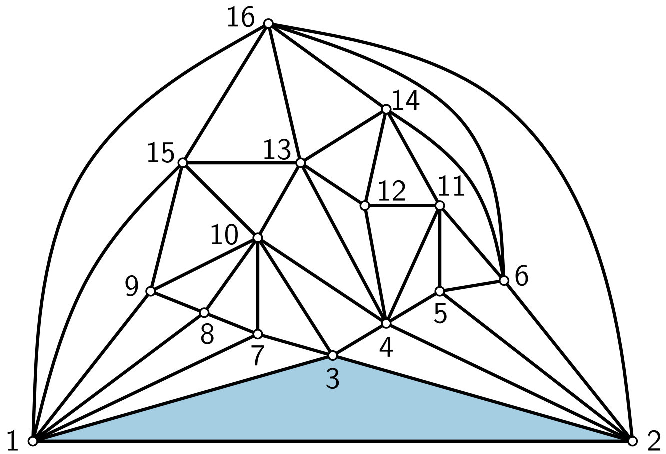


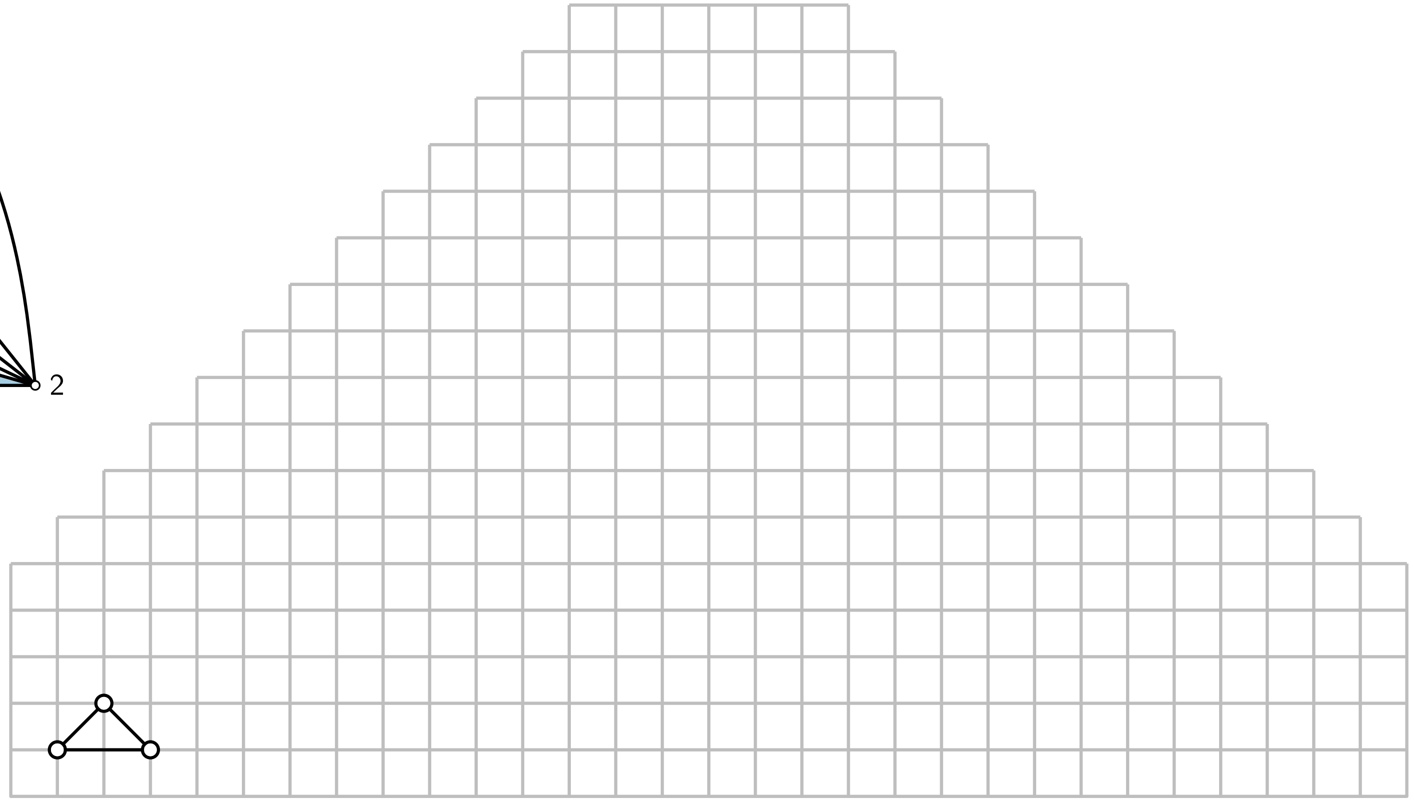
Yes, because w_p and w_q have *even* Manhattan distance $\Delta x + \Delta y$.



Shift Method – Example







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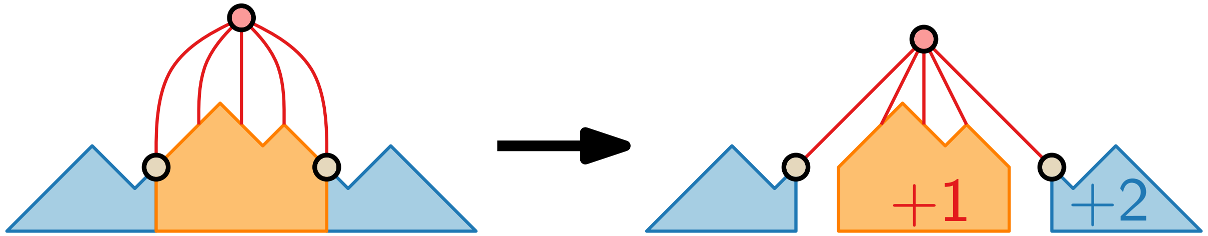
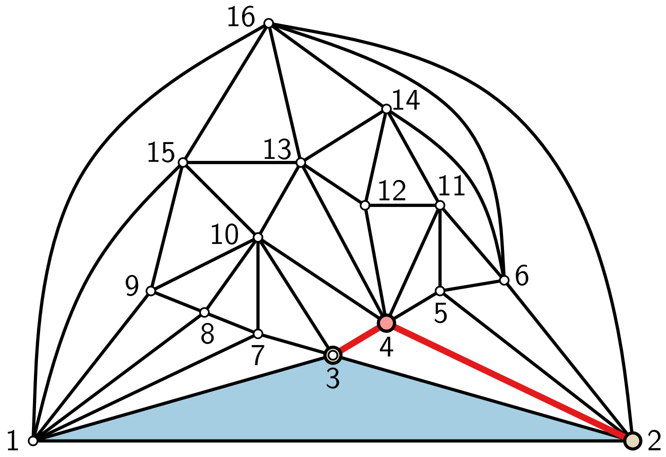
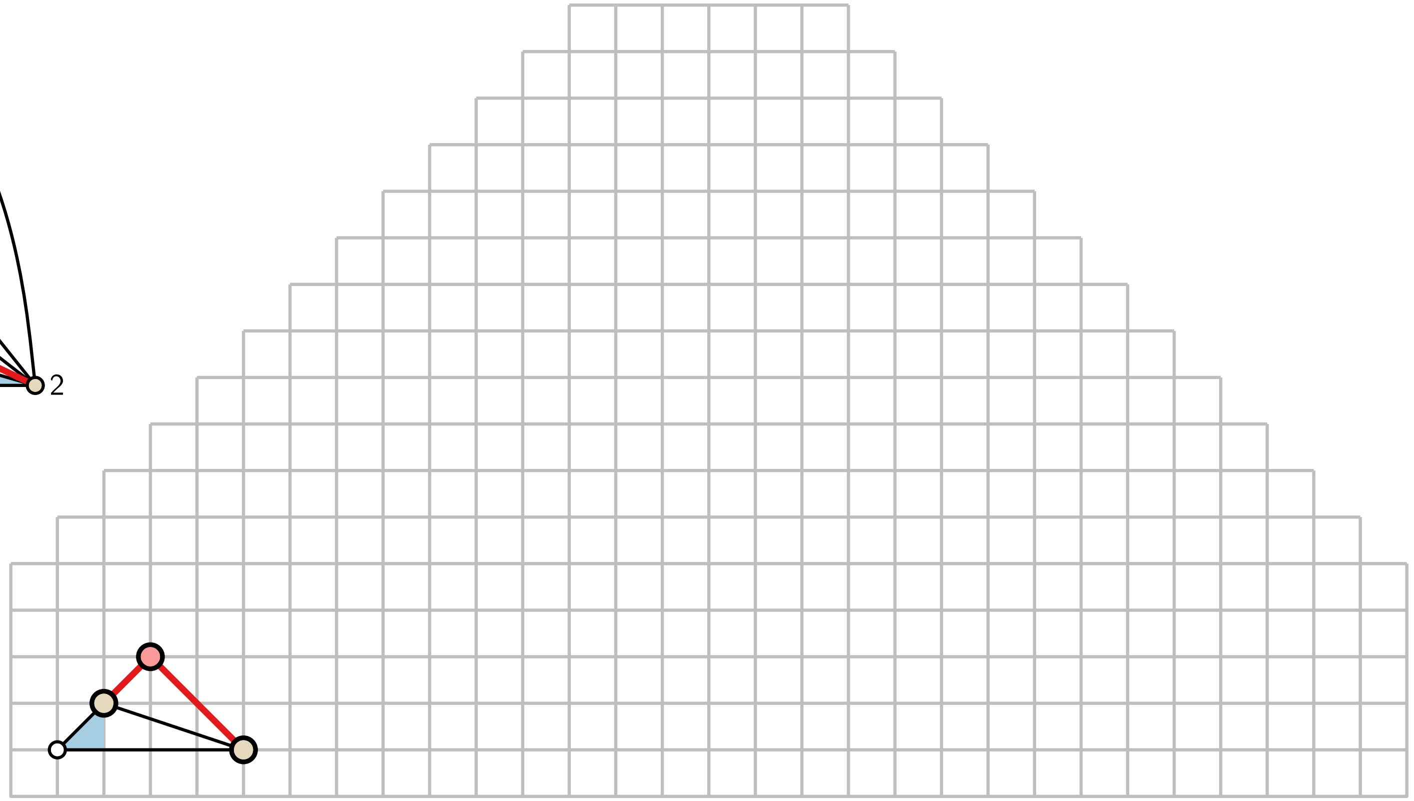
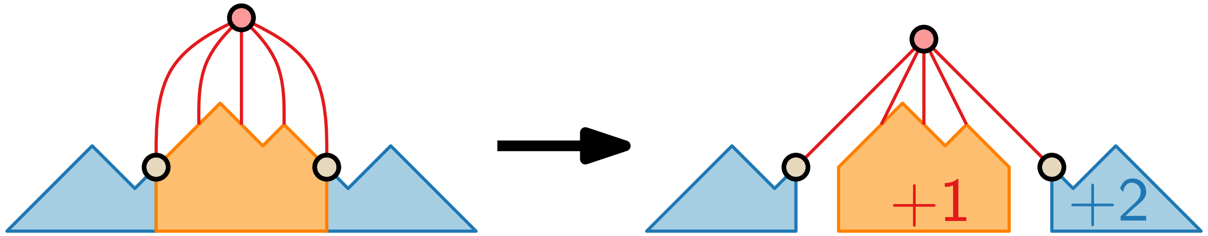
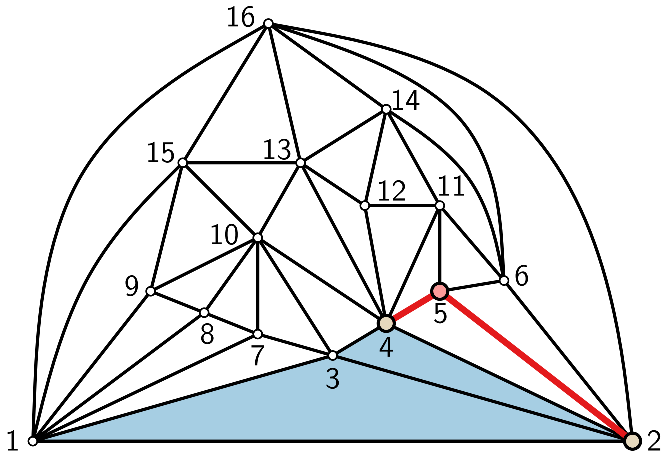


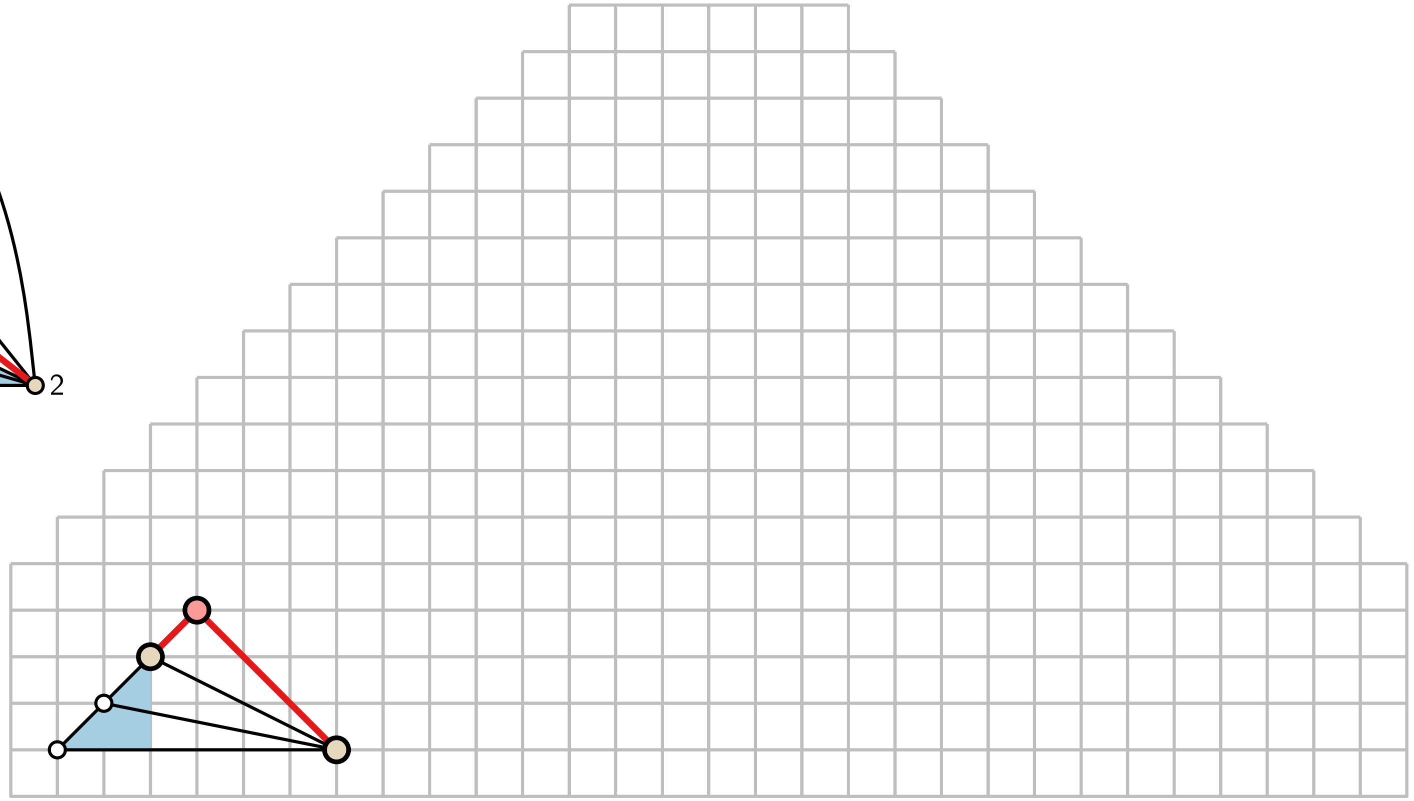




Shift Method – Example







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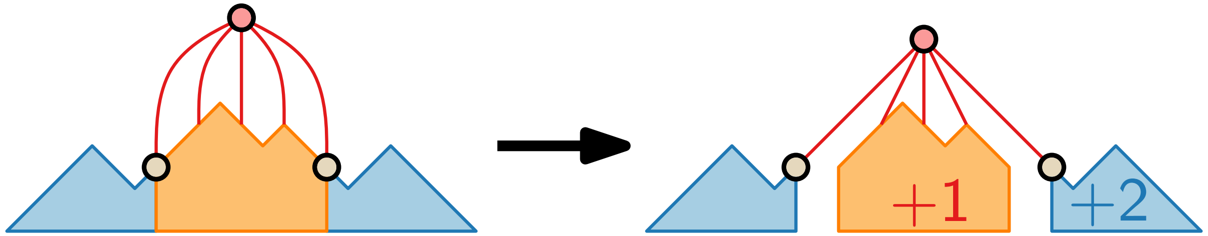

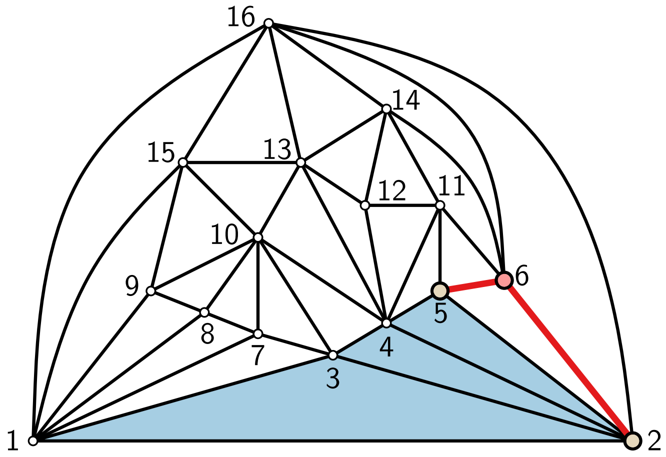

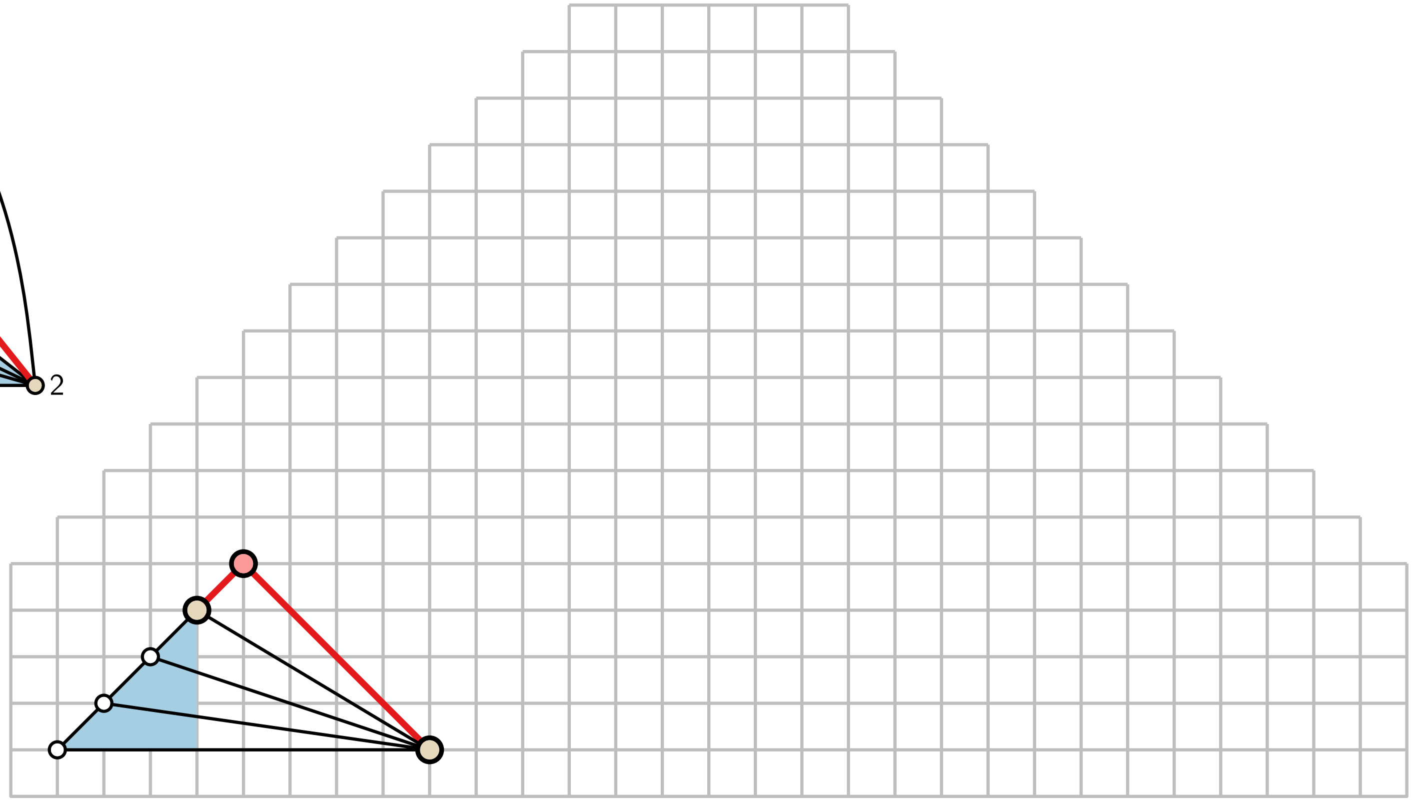

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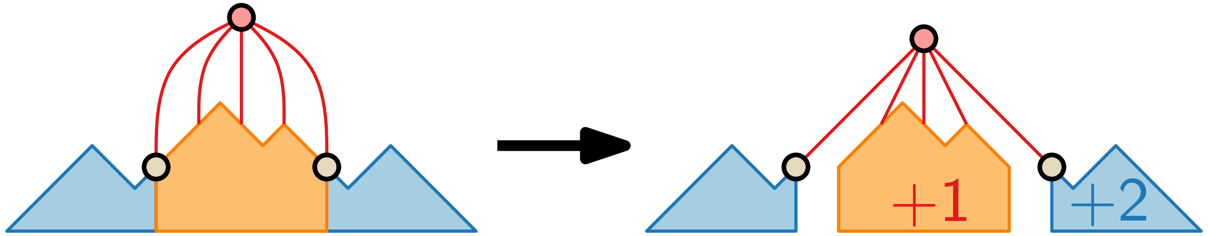

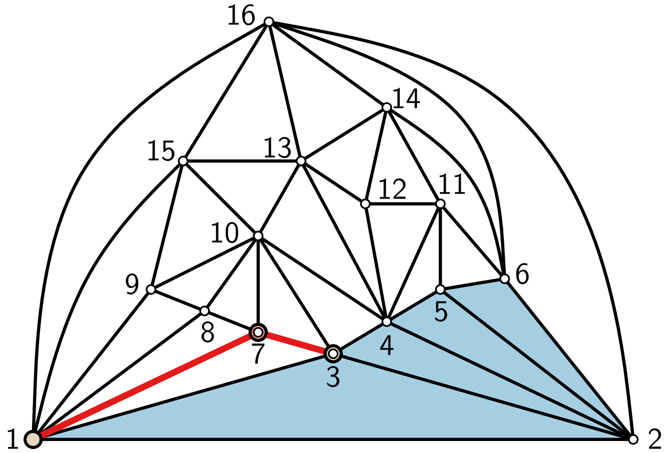

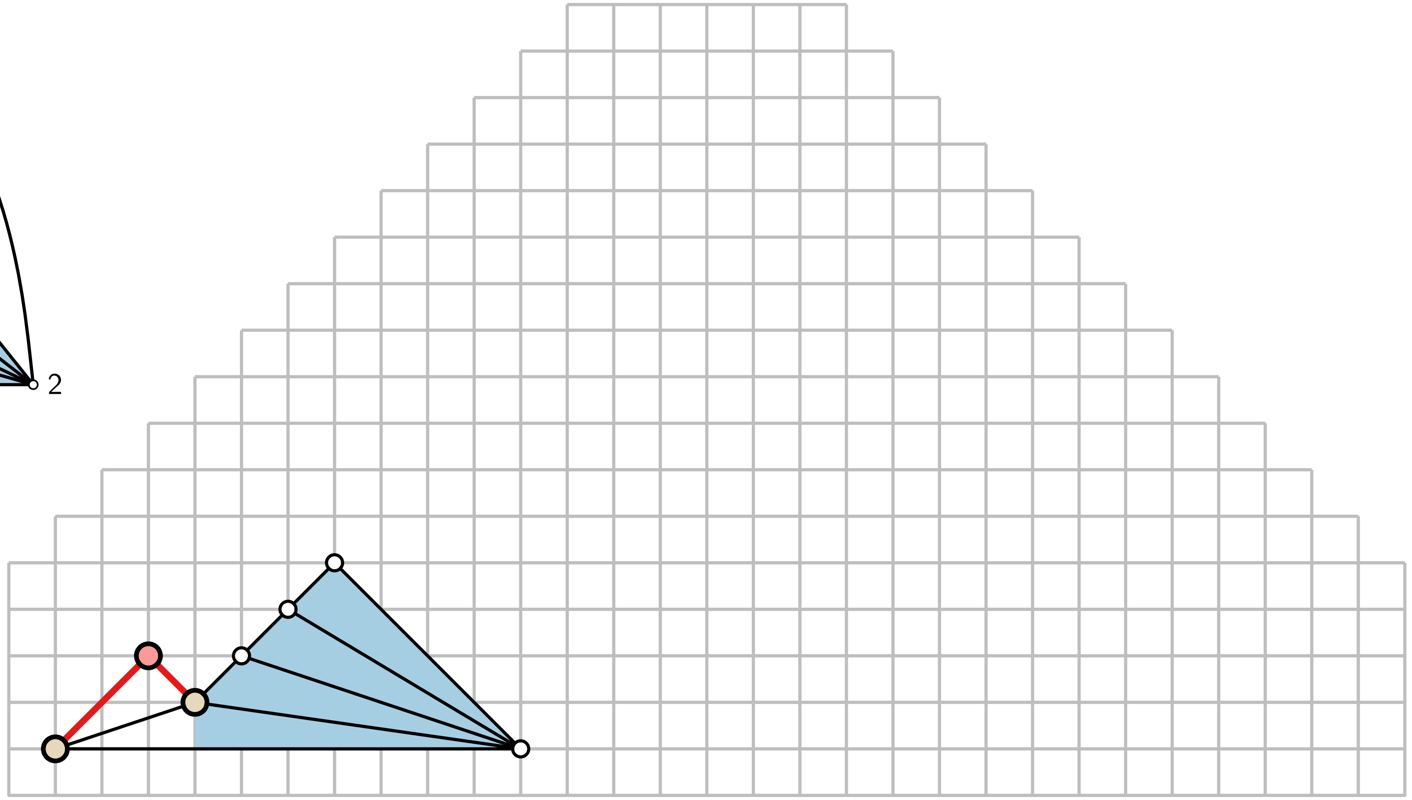

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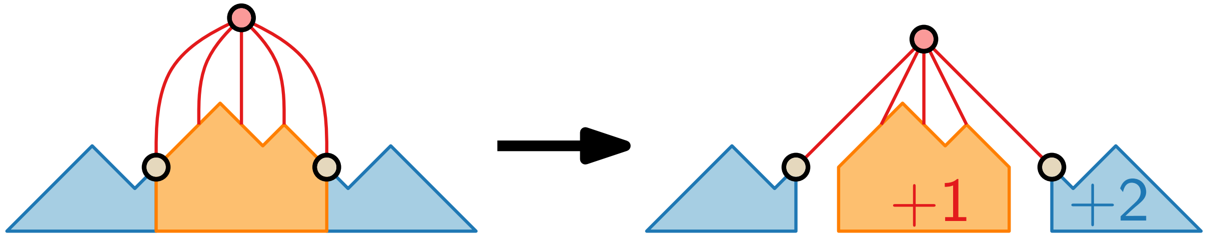
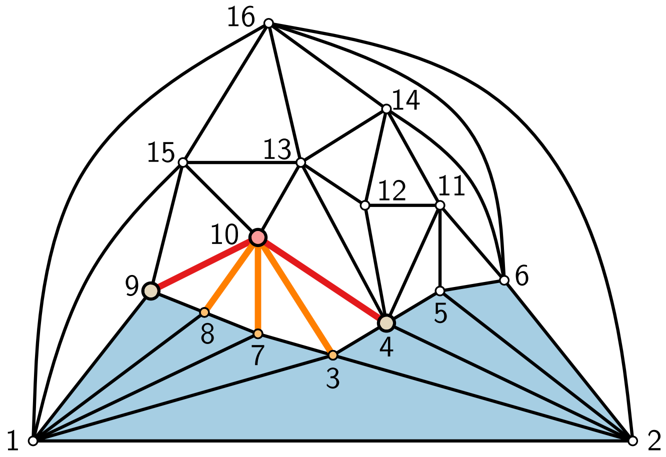
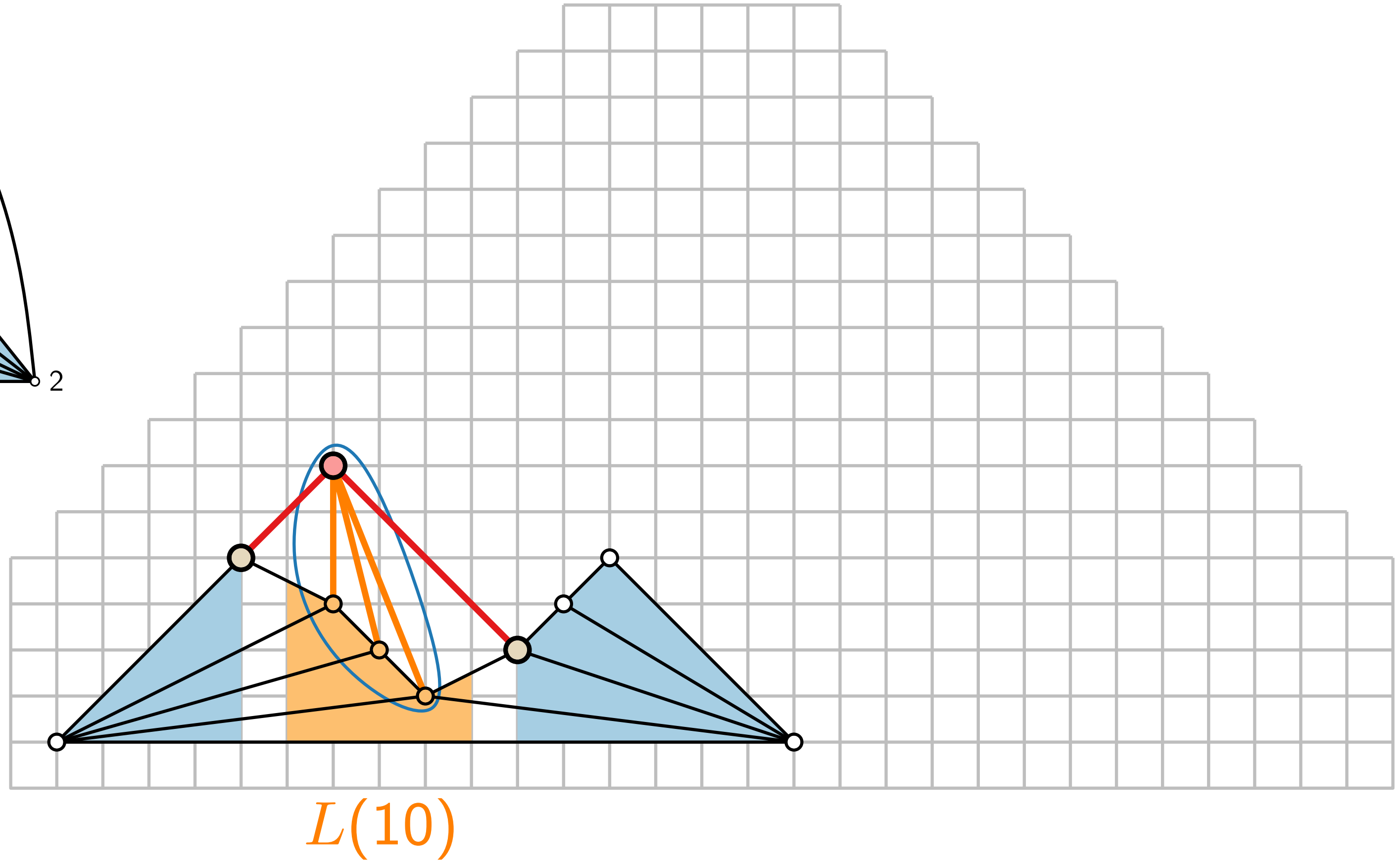
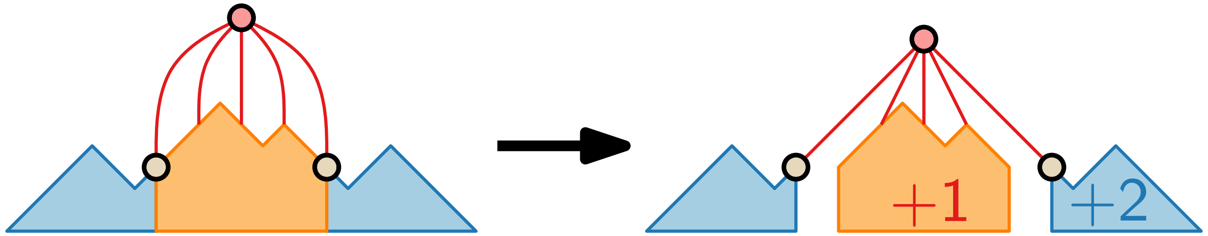
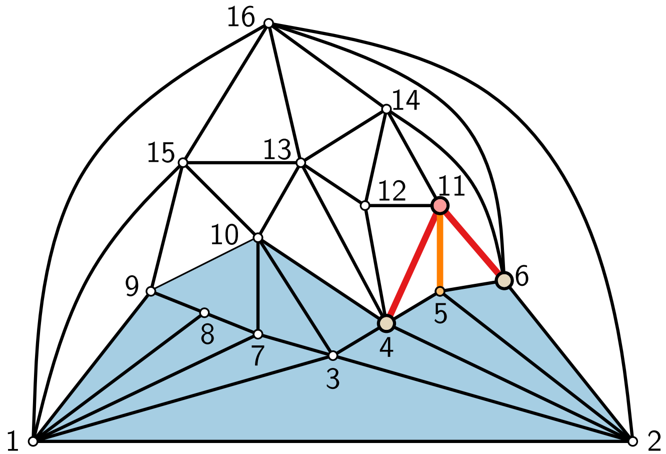


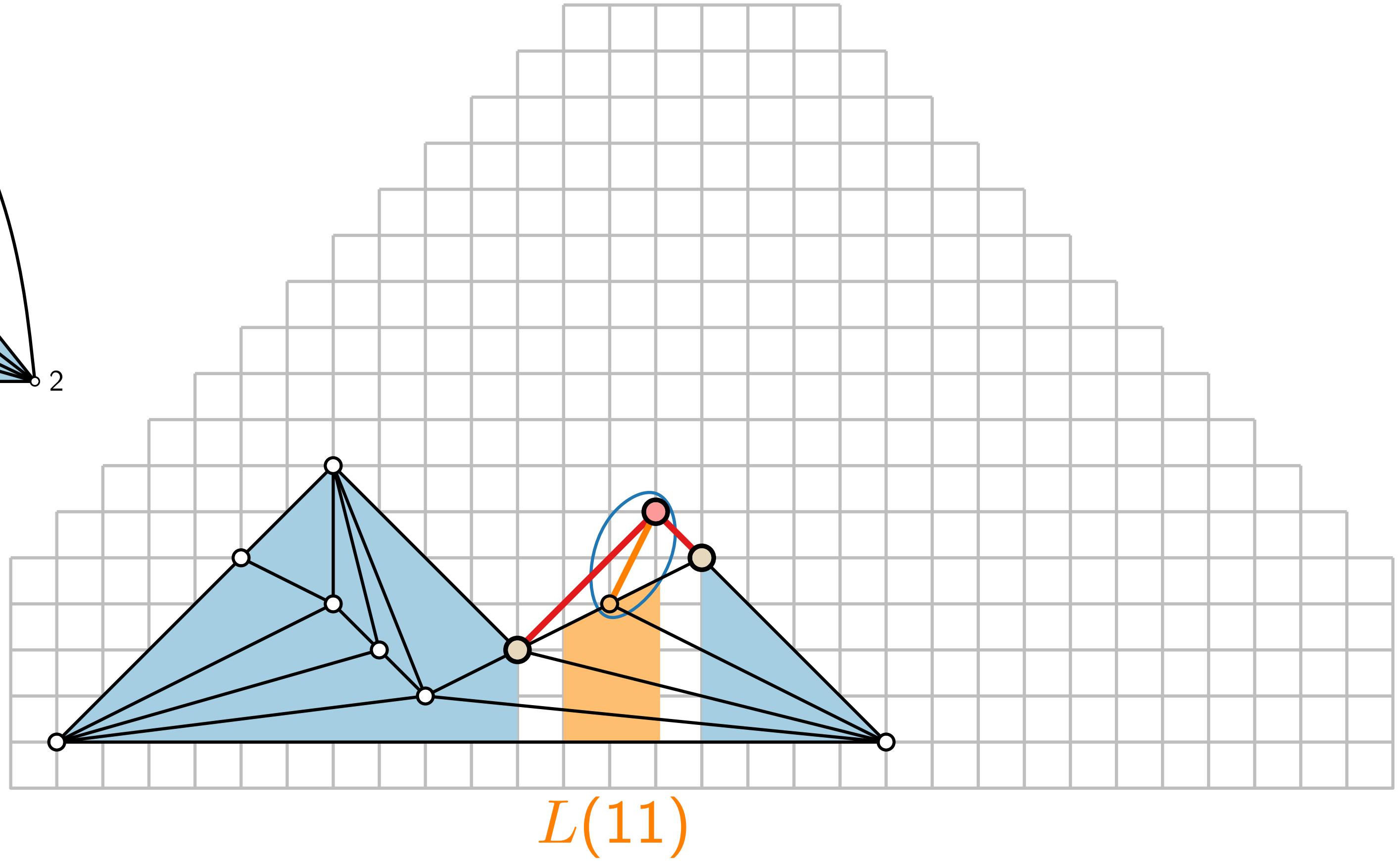




Shift Method – Example







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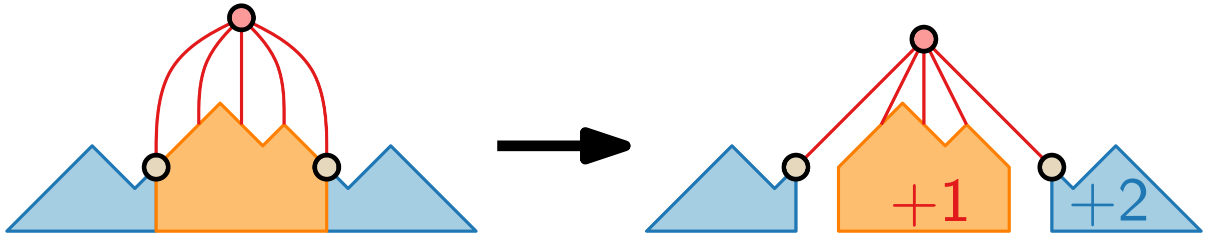

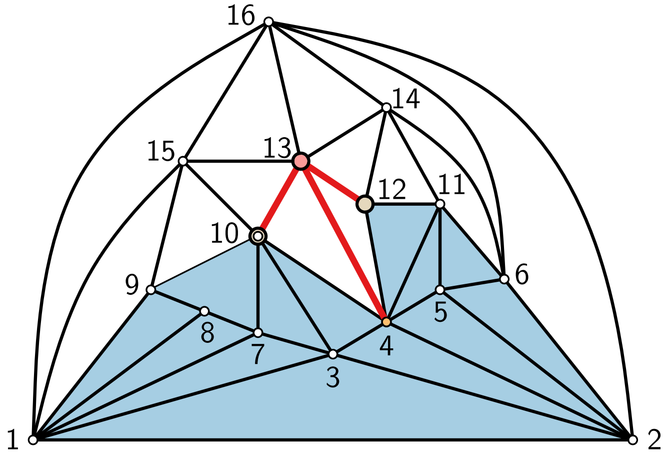

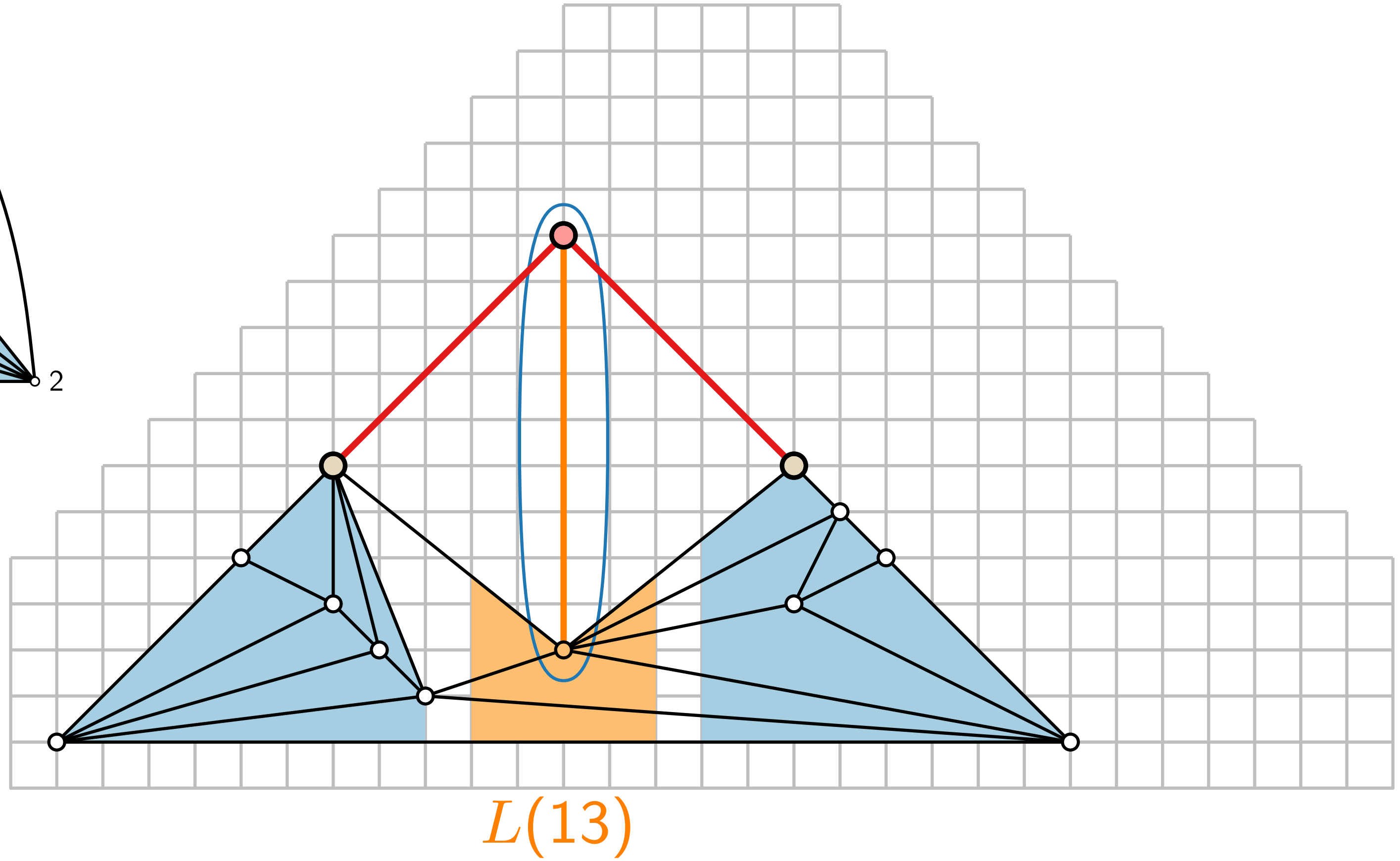

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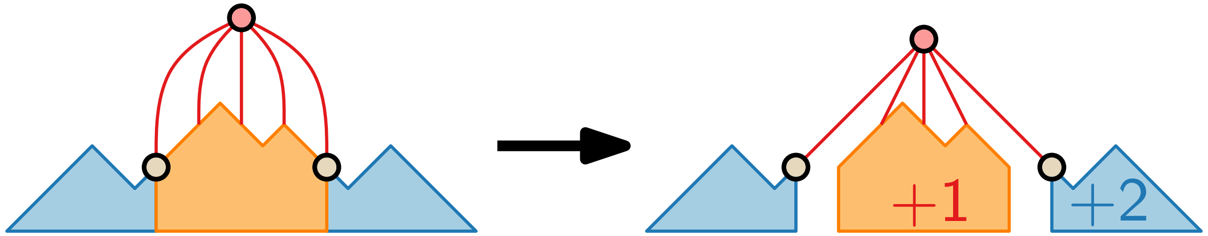
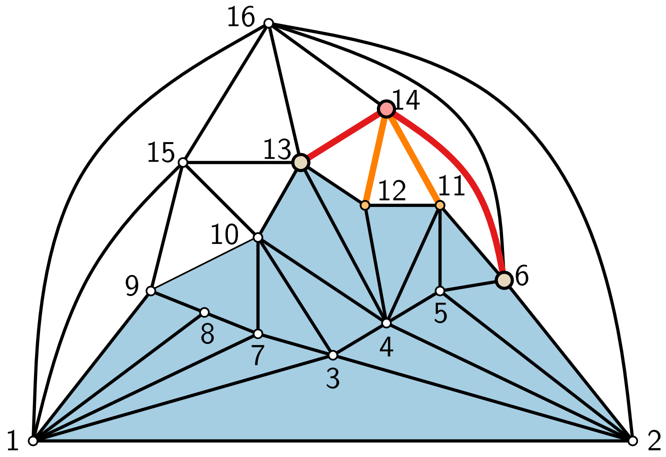
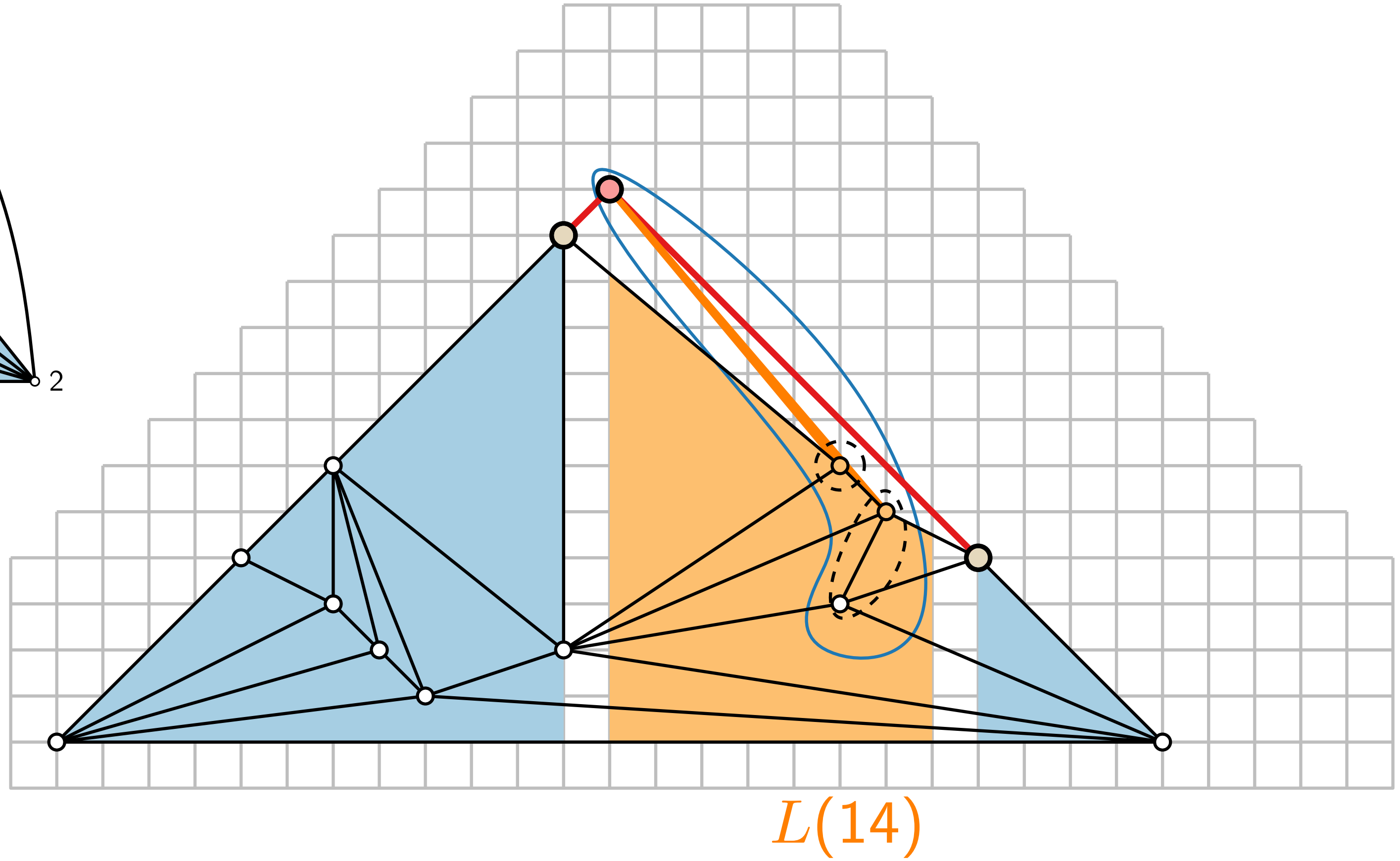
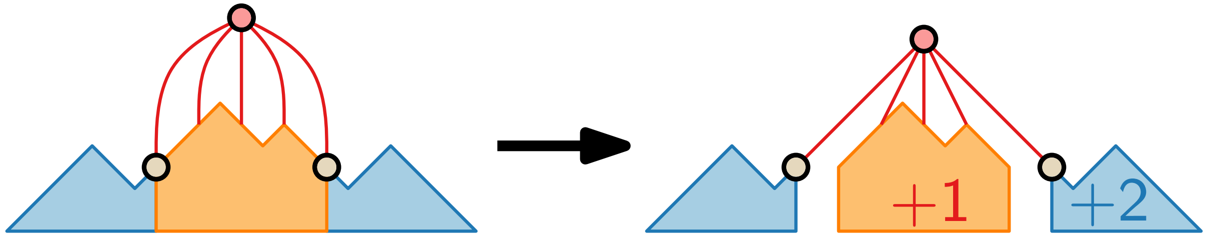
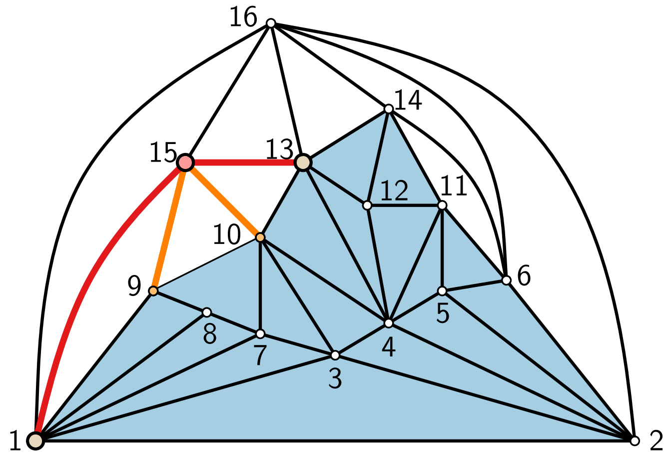


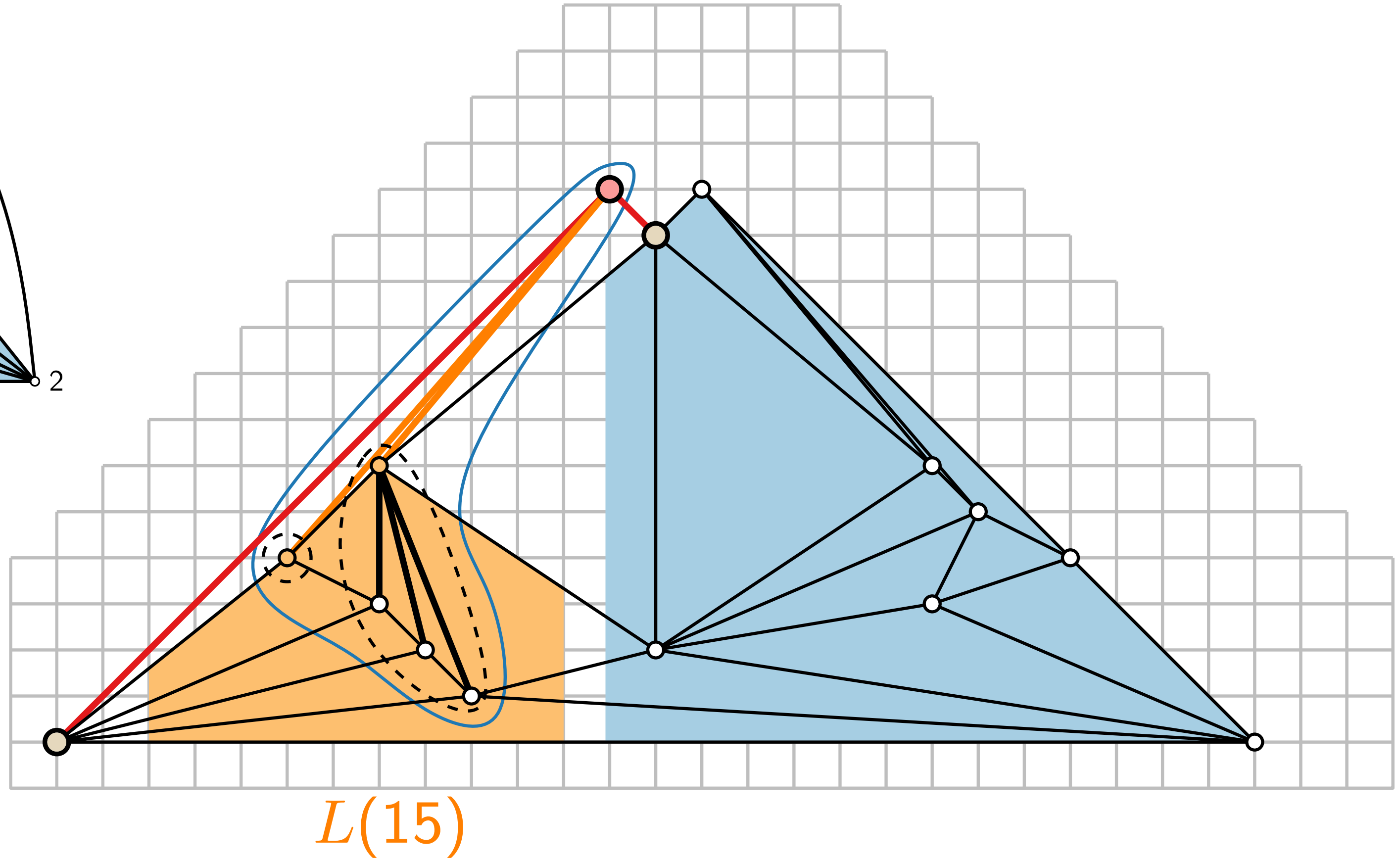




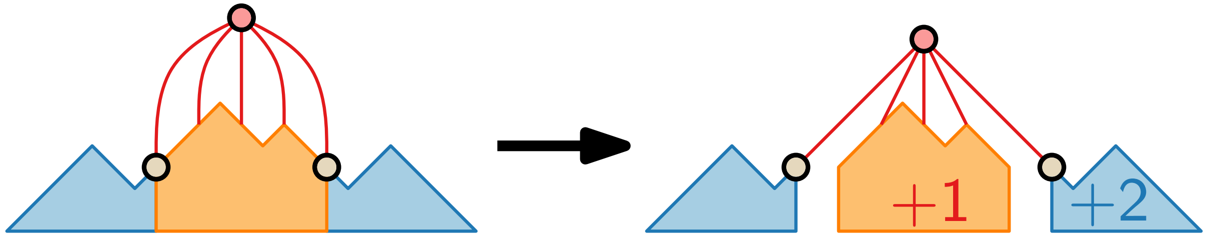
Shift Method – Example

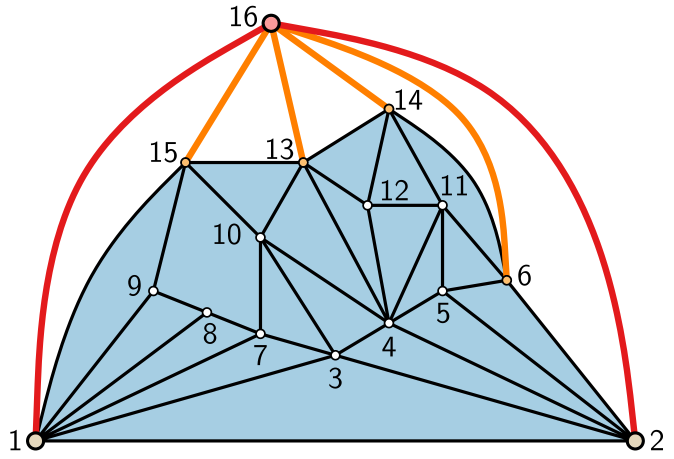


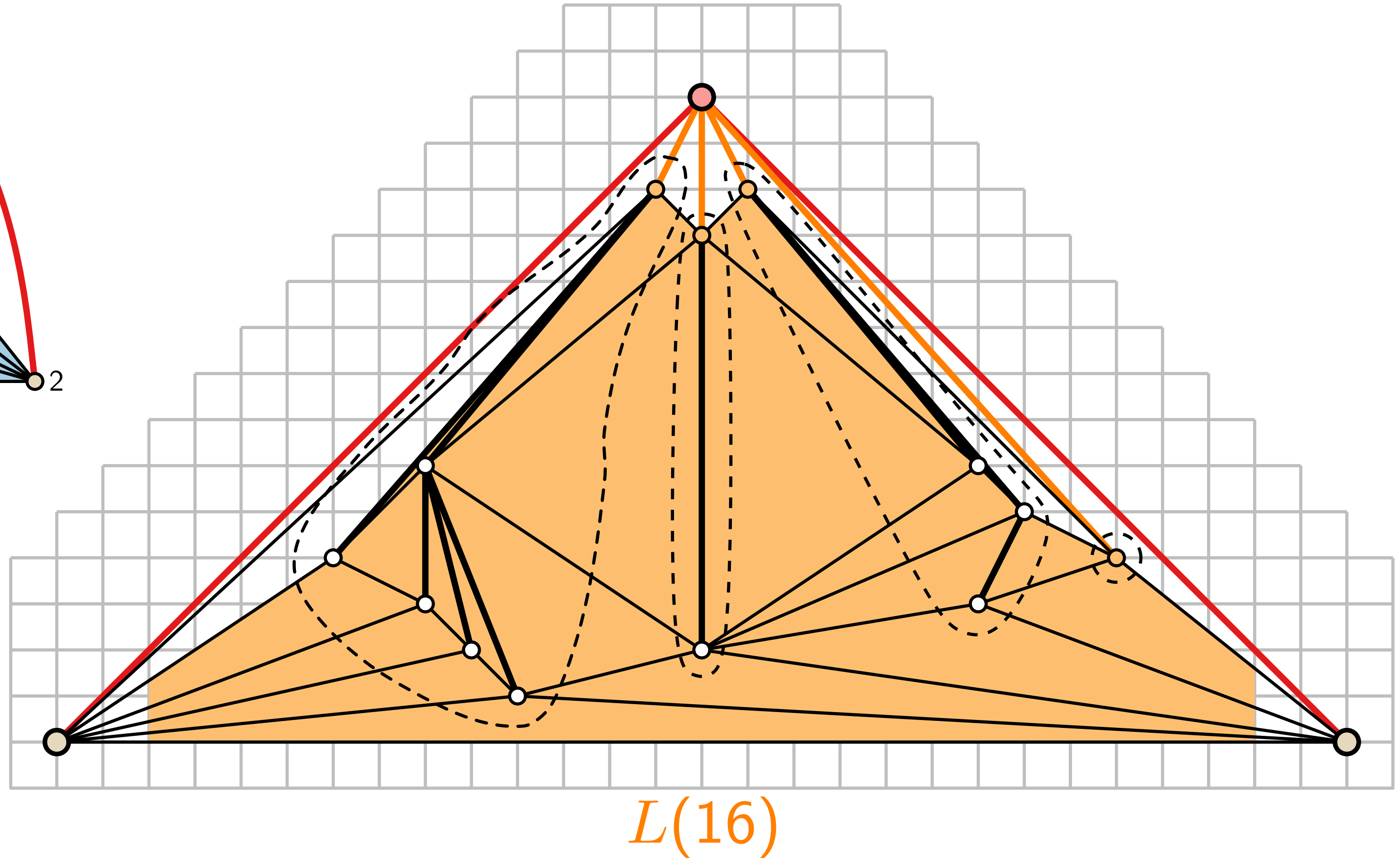




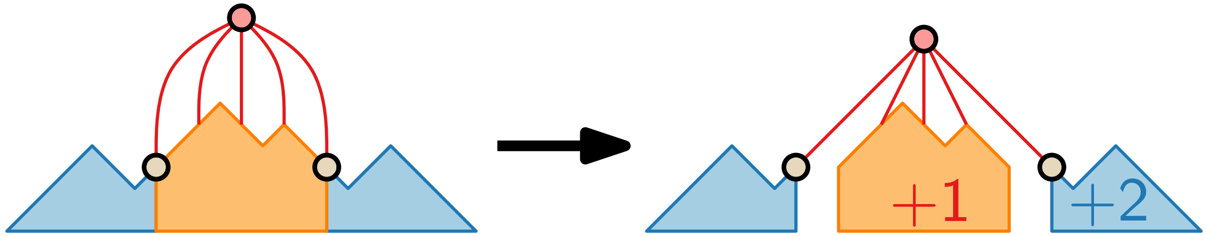
Shift Method – Example

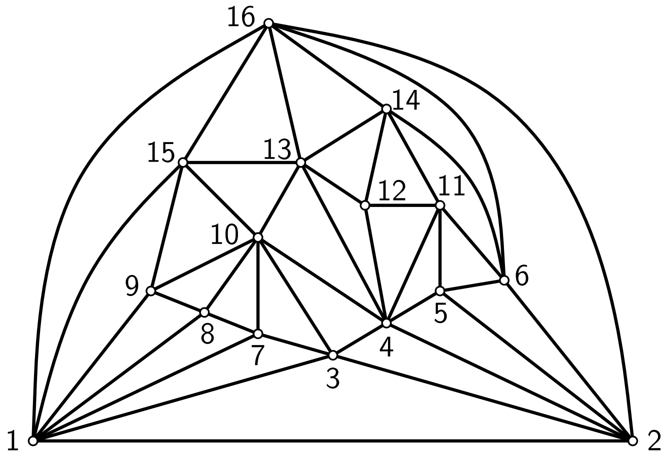


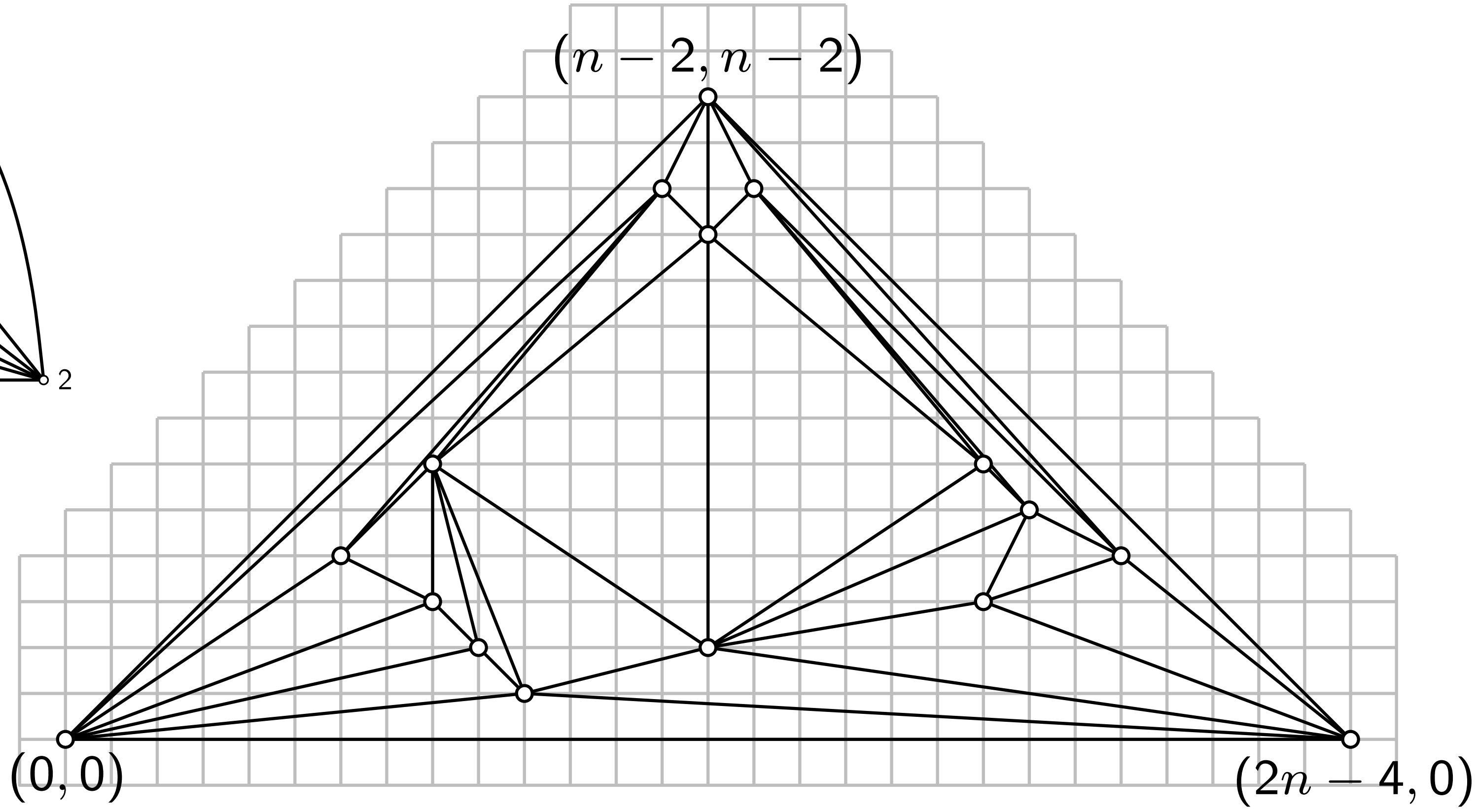




Shift Method – Example





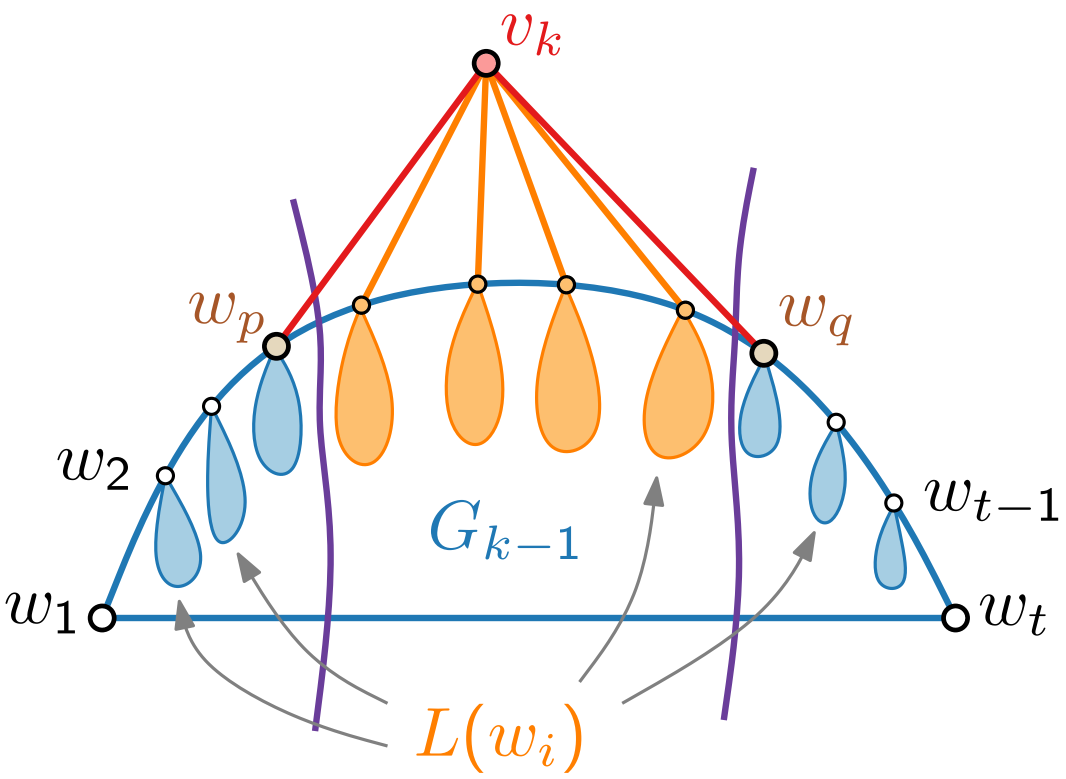


Shift Method – Planarity

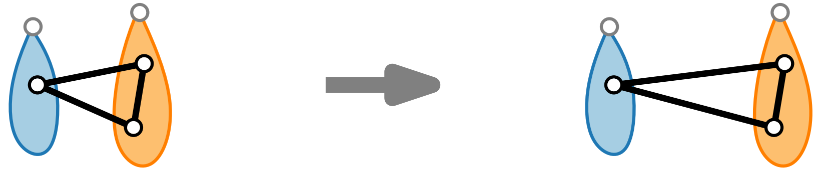
Observations.

- Each internal vertex is covered exactly once.
- Covering relation defines a tree in G
- and a forest in G_i, $1 \leq i \leq n-1$.

Lemma.
Let $0 \leq \delta_1 \leq \delta_2 \leq \cdots \leq \delta_t \in \mathbb{N}$, s.t. $\delta_{p+1} - \delta_p \geq 1$, $\delta_q - \delta_{q-1} \geq 1$, $\delta_q - \delta_p \geq 2$ and even. If we shift $L(w_i)$ by δ_i to the right, then we get a planar straight-line drawing.



Proof by induction:
If G_{k-1} is drawn planar and straight-line, then so is G_k.
Ideas:
- New edges don't intersect other edges ($\rightarrow$ invariants).
- Edges within each $L(w_i)$ do not change.
- Other edges lie within triangles that only become flatter without causing new intersections.



Shift Method – Pseudocode

canonical order of $V(G)$

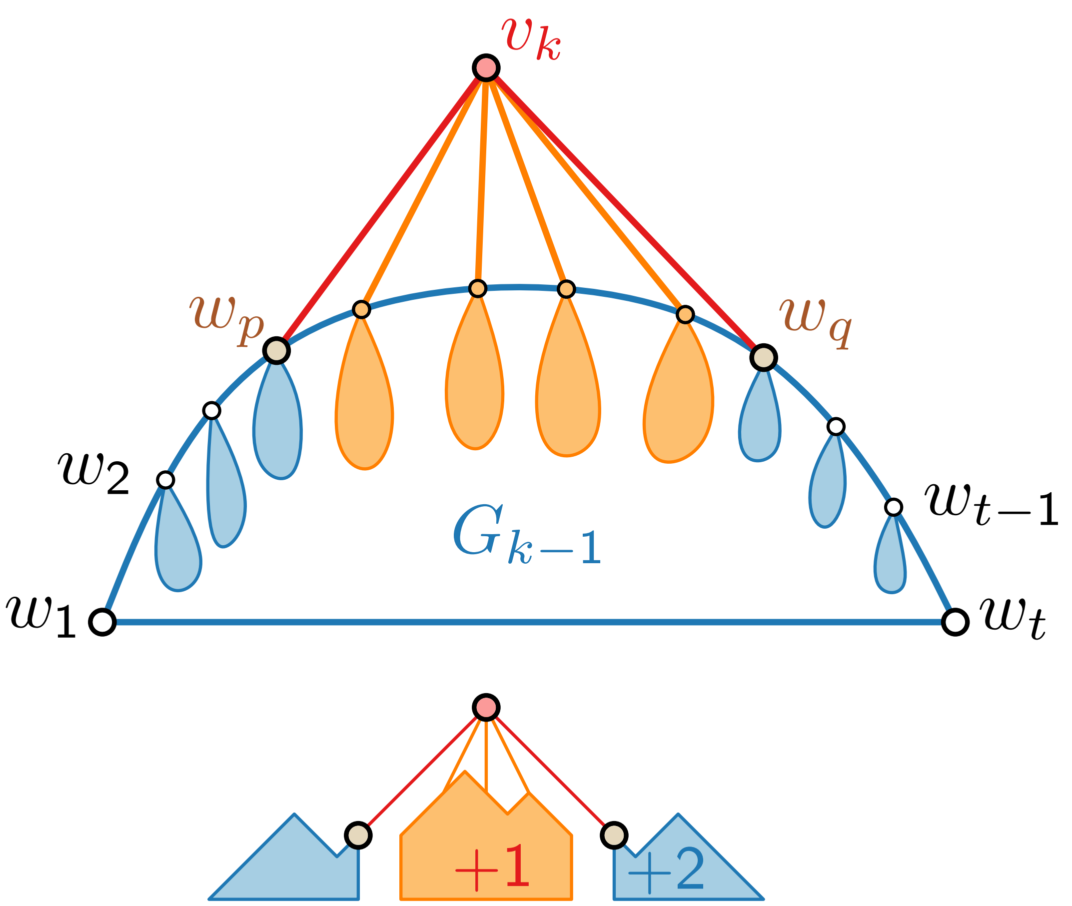
ShiftMethod(G, $(v_1, v_2, \dots, v_n)$)

for $k = 1$ to 3 **do**
  $L(v_k) \leftarrow \{v_k\}$

$P(v_1) \leftarrow (0,0)$; $P(v_2) \leftarrow (2,0)$, $P(v_3) \leftarrow (1,1)$

for $k = 4$ to n **do**
  Let ∂G_{k-1} be $v_1 = w_1, w_2, \dots, w_{t-1}, w_t = v_2$.
  Let $w_p, \dots, w_q$ be the neighbors of v_k.
  foreach $v \in \bigcup_{i=p+1}^{q-1} L(w_i)$ **do** // $\mathcal{O}(n^2)$ *in total*
    $x(v) \leftarrow x(v) + 1$
  foreach $v \in \bigcup_{i=q}^{t} L(w_i)$ **do** // $\mathcal{O}(n^2)$ *in total*
    $x(v) \leftarrow x(v) + 2$
  $P(v_k) \leftarrow$ intersection of slope-± 1 diagonals through $P(w_p)$ and $P(w_q)$
  $L(v_k) \leftarrow \bigcup_{i=p+1}^{q-1} L(w_i) \cup \{v_k\}$

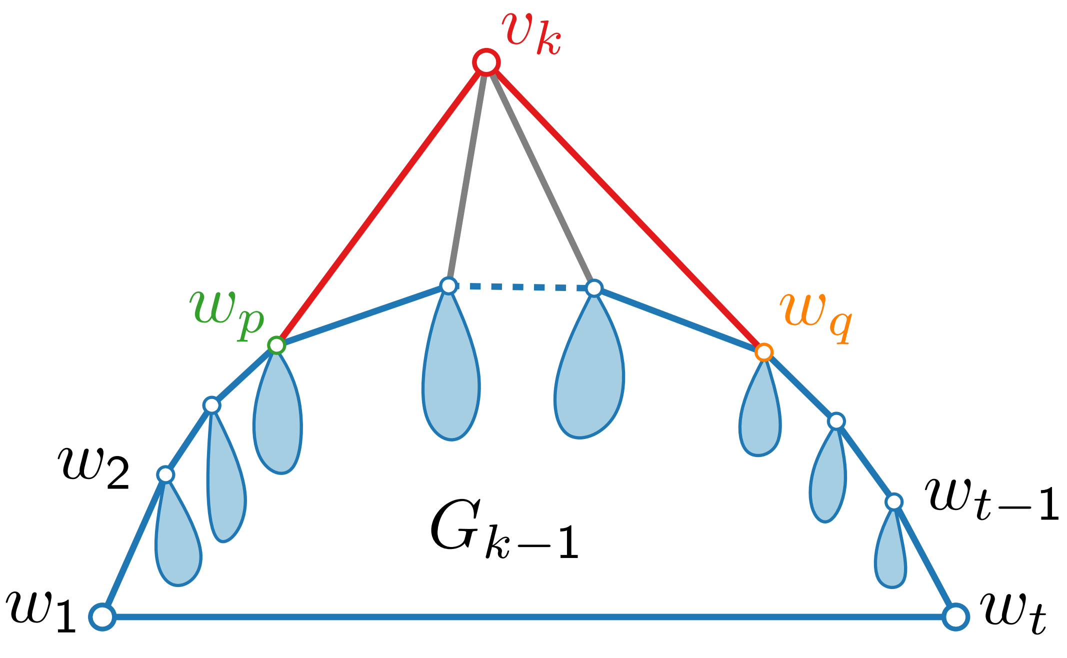
return $P(v_1), \dots, P(v_n)$



Running Time?

Shift Method – Linear-Time Implementation

Idea 1.
To compute $x(v_k)$ and $y(v_k)$,
we need only $y(w_p)$, $y(w_q)$, and $x(w_q) - x(w_p)$



Idea 2.
Instead of storing explicit x-coordinates,
we store, for each vertex within a specific spanning tree,
the x-distance to its parent (v_1 is the root).

After an x-distance is computed for each v_k,
use preorder traversal to compute all x-coordinates.

(1) $x(v_k) = \frac{1}{2}(x(w_q) + x(w_p) + y(w_q) - y(w_p))$

(2) $y(v_k) = \frac{1}{2}(x(w_q) - x(w_p) + y(w_q) + y(w_p))$

(3) $x(v_k) - x(w_p) = \frac{1}{2}(x(w_q) - x(w_p) + y(w_q) - y(w_p))$

Shift Method – Linear-Time Implementation

Relative x-distance tree.

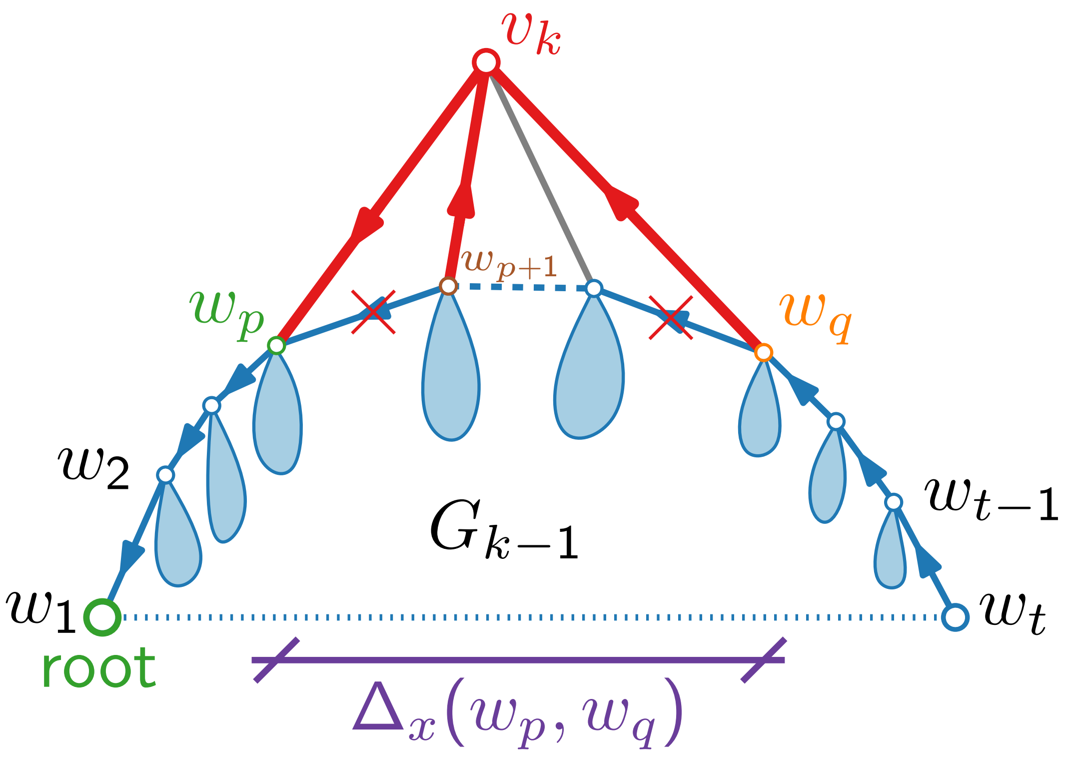
For each vertex v store

- x-offset $\Delta_x(v)$ from parent
- y-coordinate $y(v)$

Calculations.

- $\Delta_x(w_{p+1})$++, $\Delta_x(w_q)$++
- $\Delta_x(w_p, w_q) = \Delta_x(w_{p+1}) + \ldots + \Delta_x(w_q)$
- $\Delta_x(v_k)$ by (3)
- $y(v_k)$ by (2)
- $\Delta_x(w_q) = \Delta_x(w_p, w_q) - \Delta_x(v_k)$
- $\Delta_x(w_{p+1}) = \Delta_x(w_{p+1}) - \Delta_x(v_k)$

takes $\mathcal{O}(n)$ time in total ☺



(1) $x(v_k) = \frac{1}{2}(x(w_q) + x(w_p) + y(w_q) - y(w_p))$

(2) $y(v_k) = \frac{1}{2}(x(w_q) - x(w_p) + y(w_q) + y(w_p))$

(3) $\underbrace{x(v_k) - x(w_p)}_{\Delta_x(v_k)} = \frac{1}{2}(\underbrace{x(w_q) - x(w_p)}_{\Delta_x(w_p, w_q)} + y(w_q) - y(w_p))$

Discussion

- The shift method by de Fraysseix, Pach, and Pollack provides an algorithmic tool to efficiently draw a plane graph onto a polynomial-size grid using only straight-line edges.
- The linear-time implementation was later proposed by Chrobak and Payne.
- Although we are guaranteed to get a very small grid, only straight-line edges, and no edge crossings, the resulting drawings are not always visually pleasing: the drawings tend to have very small angles and a big variance in the size of the triangular faces.
- A quite different approach yielding similar results is by Schnyder ($\rightarrow$ next lecture).

Literature

- [PGD Ch. 4.2] for detailed explanation of the shift method
- [de Fraysseix, Pach, Pollack 1990] “How to draw a planar graph on a grid”
  – original paper introducing the shift method
- [Chrobak, Payne 1995] “A linear-time algorithm for drawing a planar graph on a grid”
  – original paper on how to implement the shift method in linear time