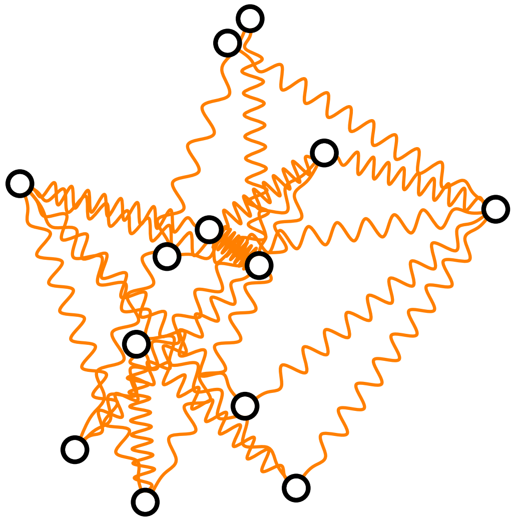
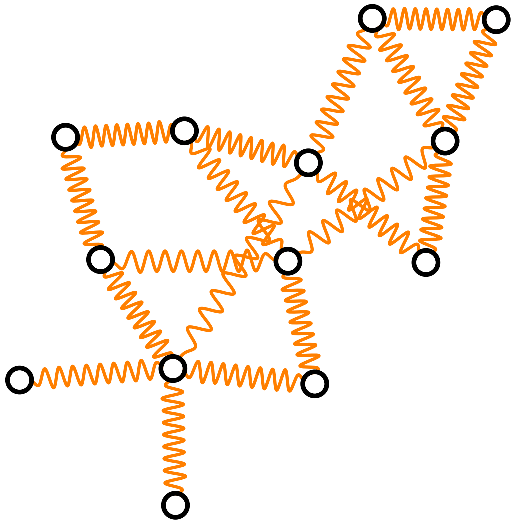
Julius-Maximilians-UNIVERSITÄT WÜRZBURG

Visualization of Graphs

Lecture 2:
Force-Directed Drawing Algorithms



Part I:
Spring Embedders

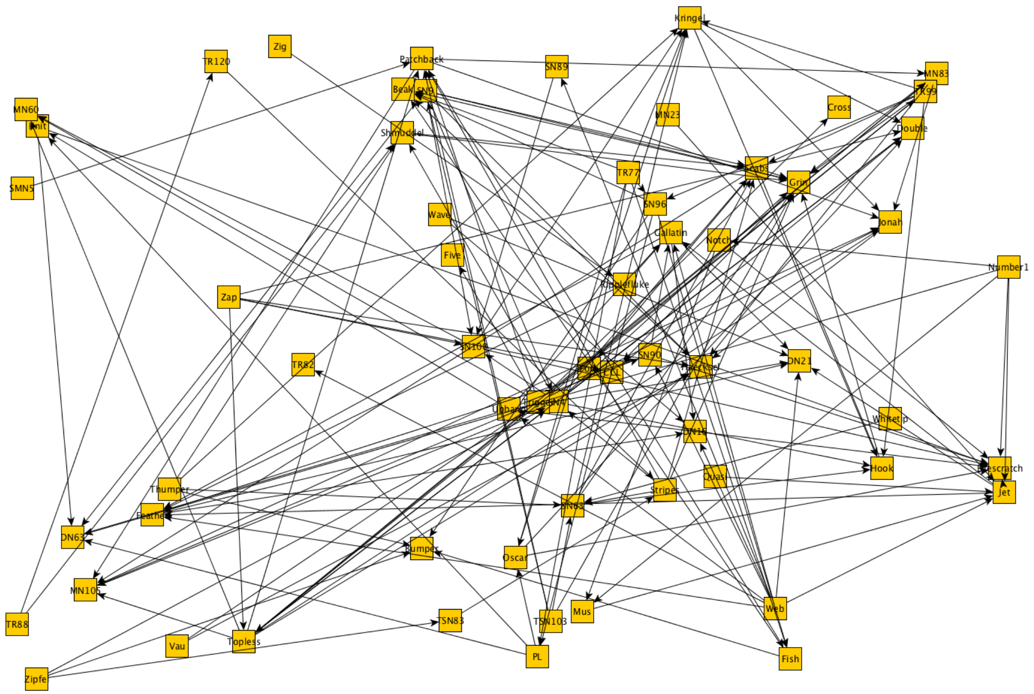


Johannes Zink

Summer semester 2024

General Layout Problem

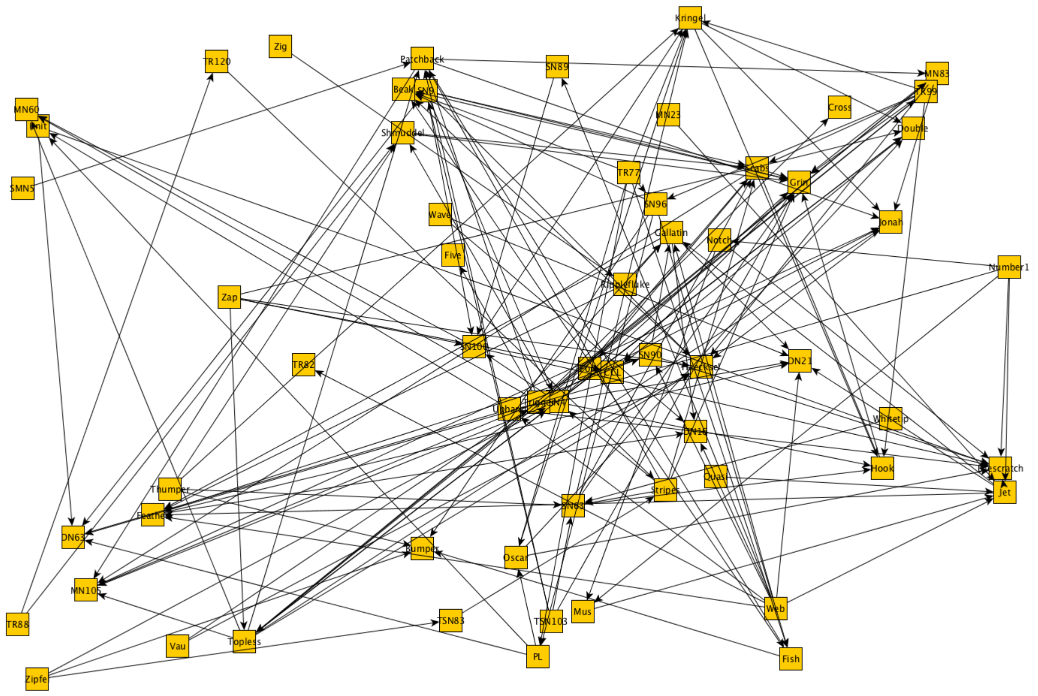
Input: Graph G

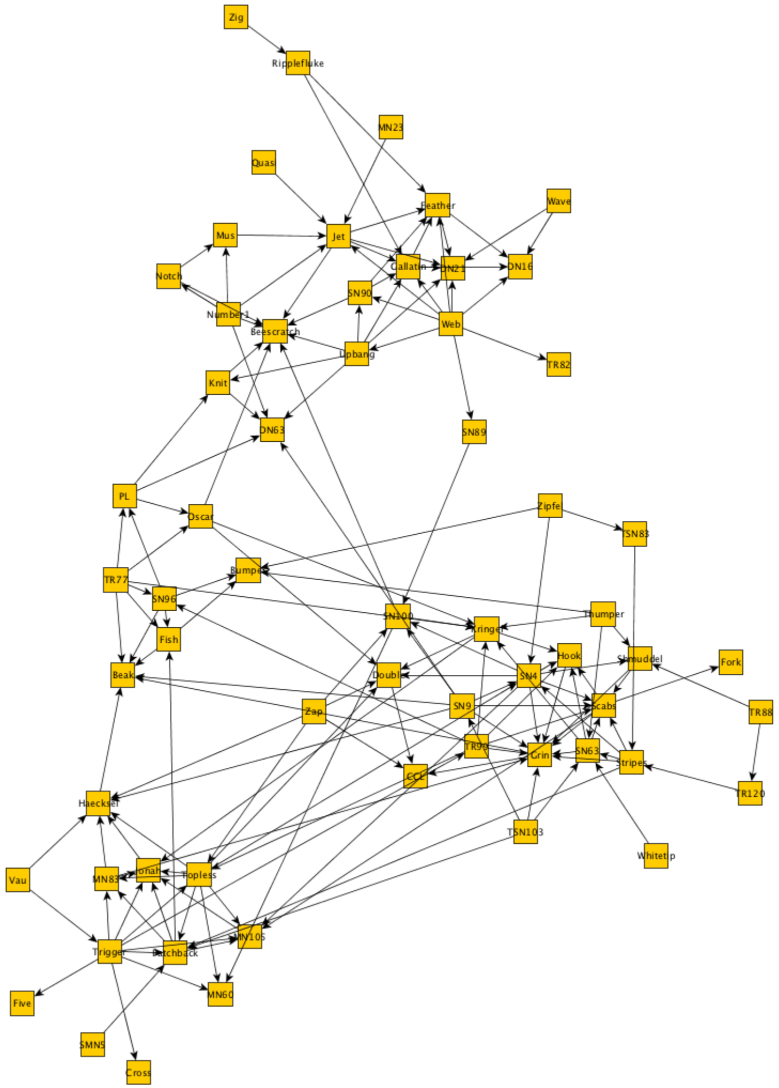


General Layout Problem

Input: Graph G

Output: Clear and readable straight-line drawing of G





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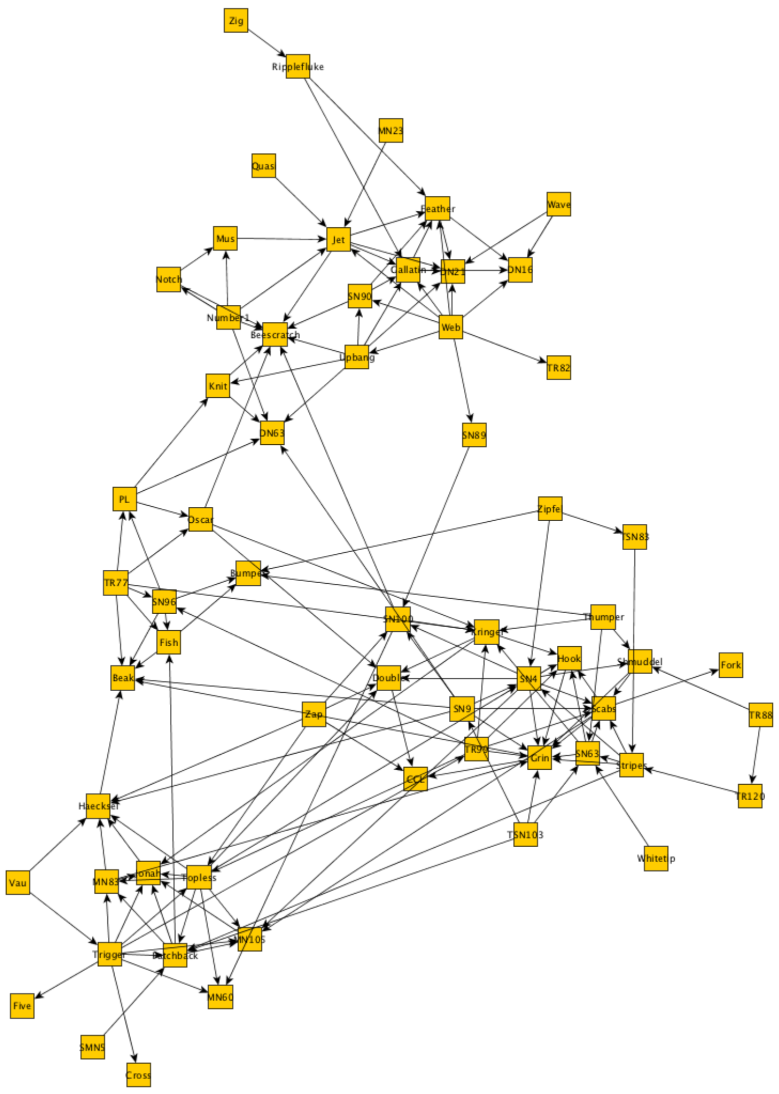
Drawing aesthetics to optimize:



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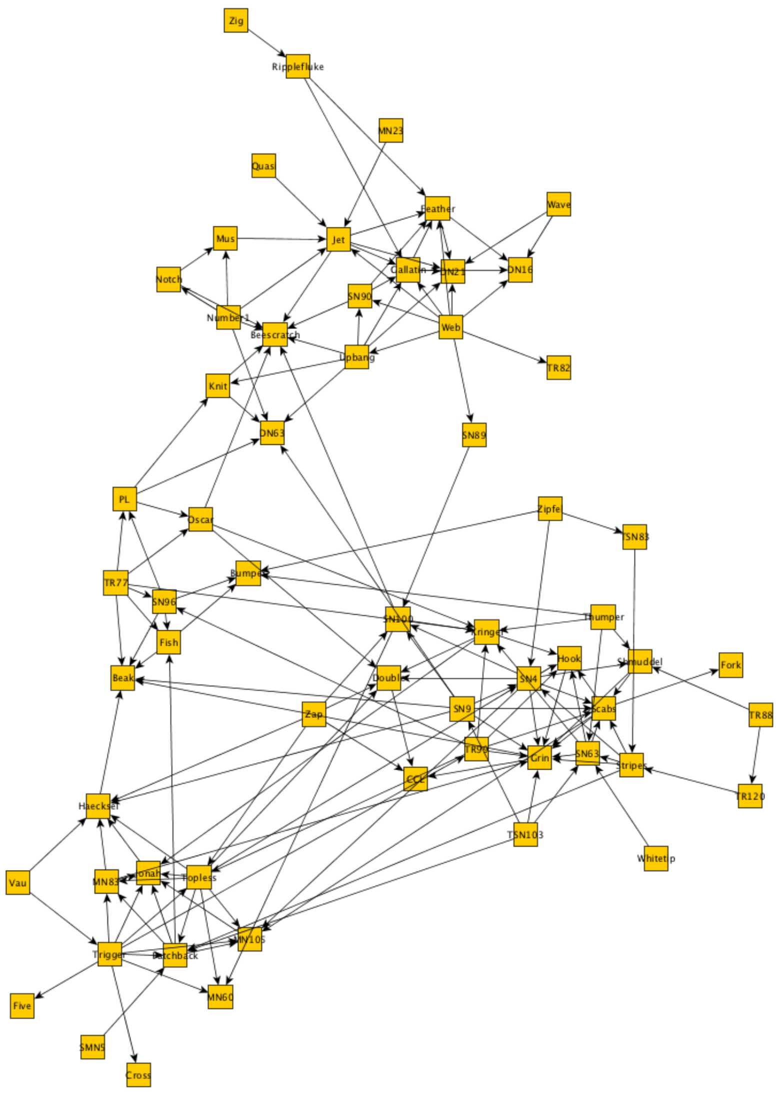
- adjacent vertices are close



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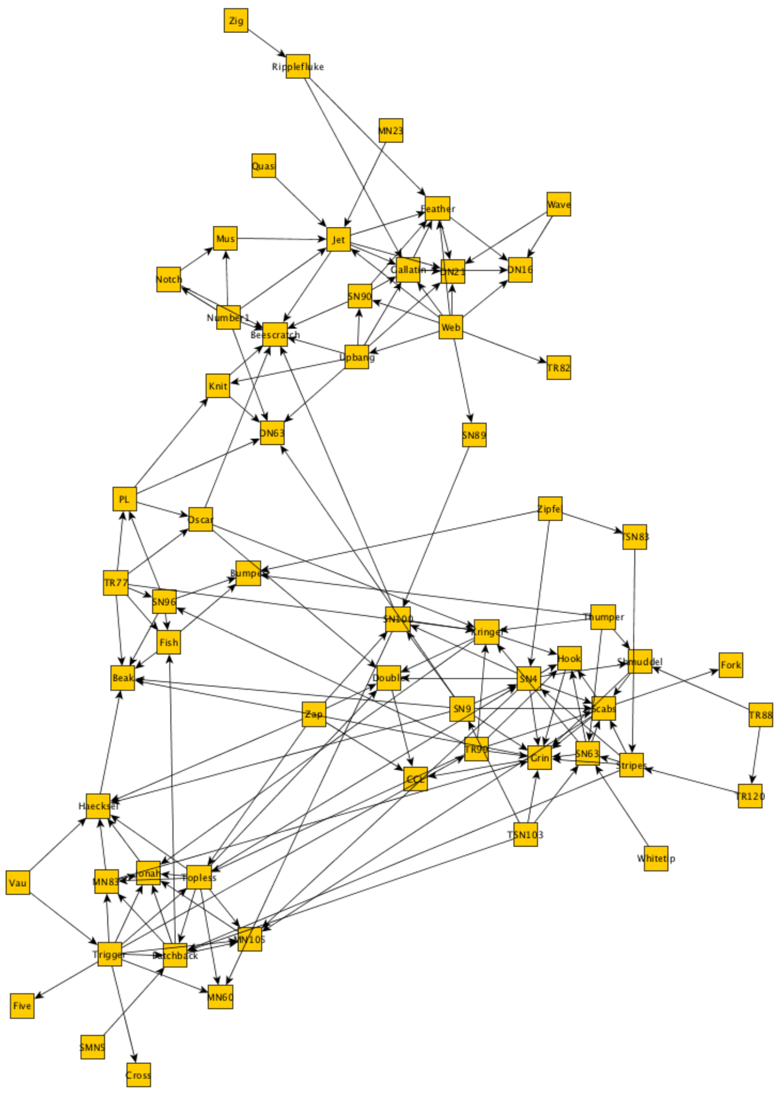

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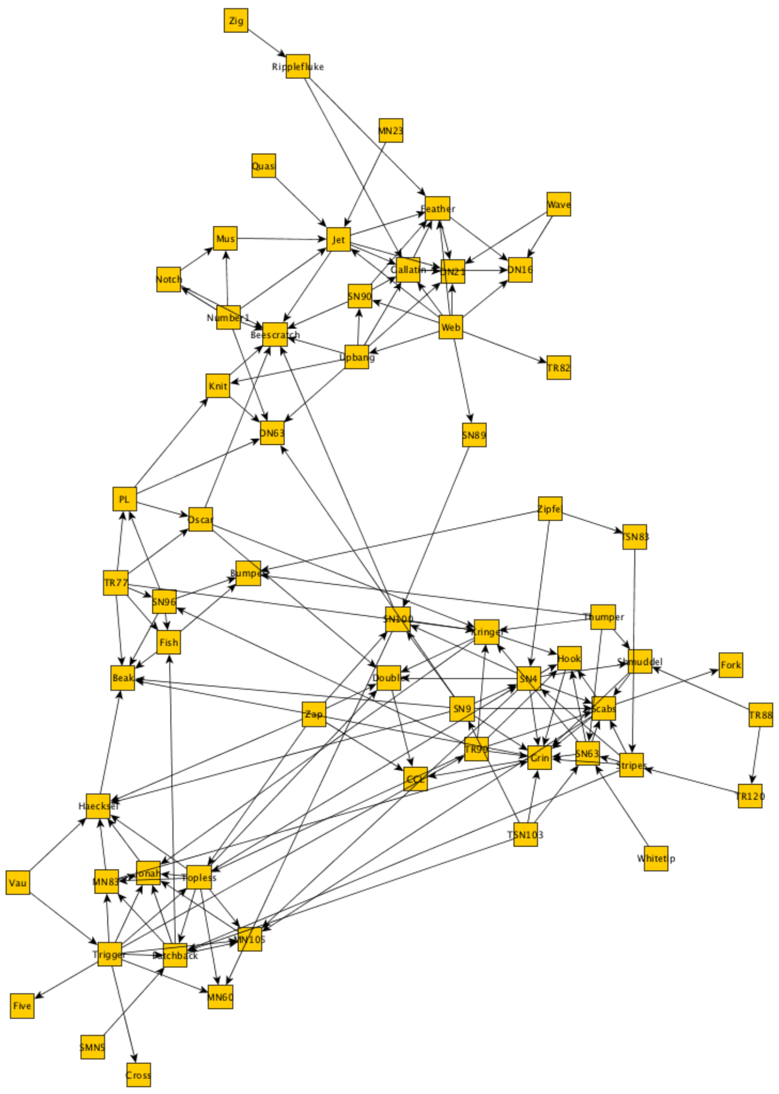
Drawing aesthetics to optimize:

- adjacent vertices are close
- non-adjacent vertices are far apart
- edges short, straight-line, **similar length**



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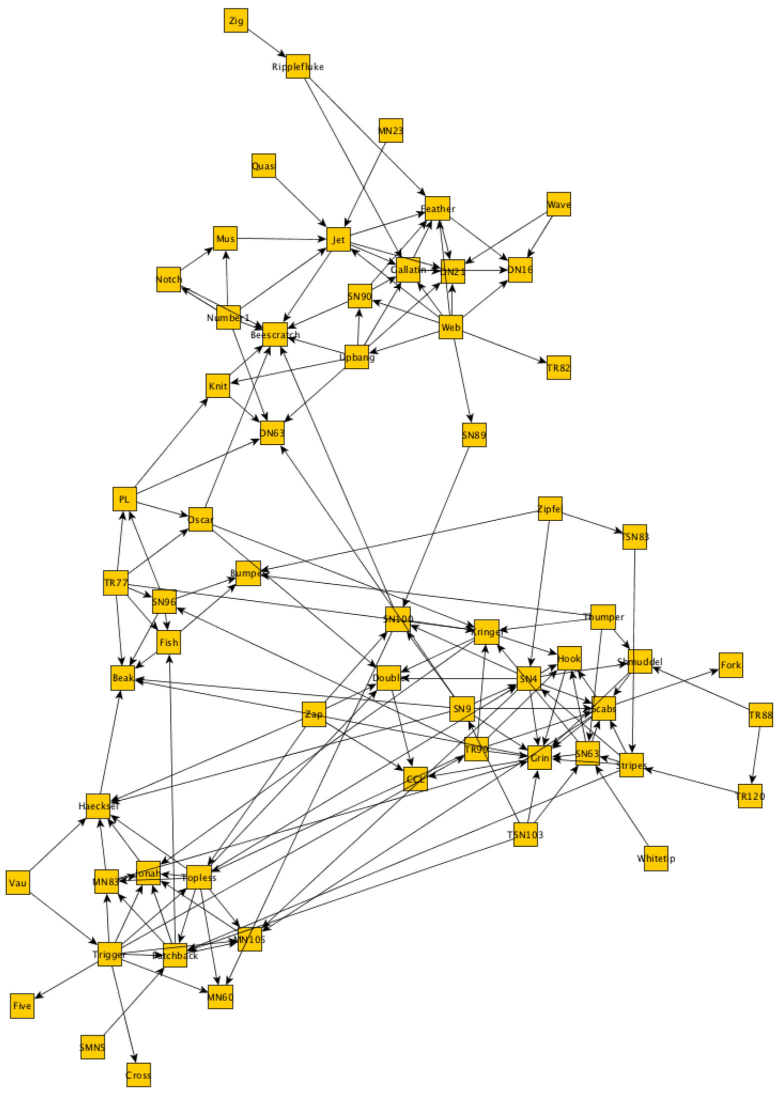
- densely connected parts (clusters) form communities



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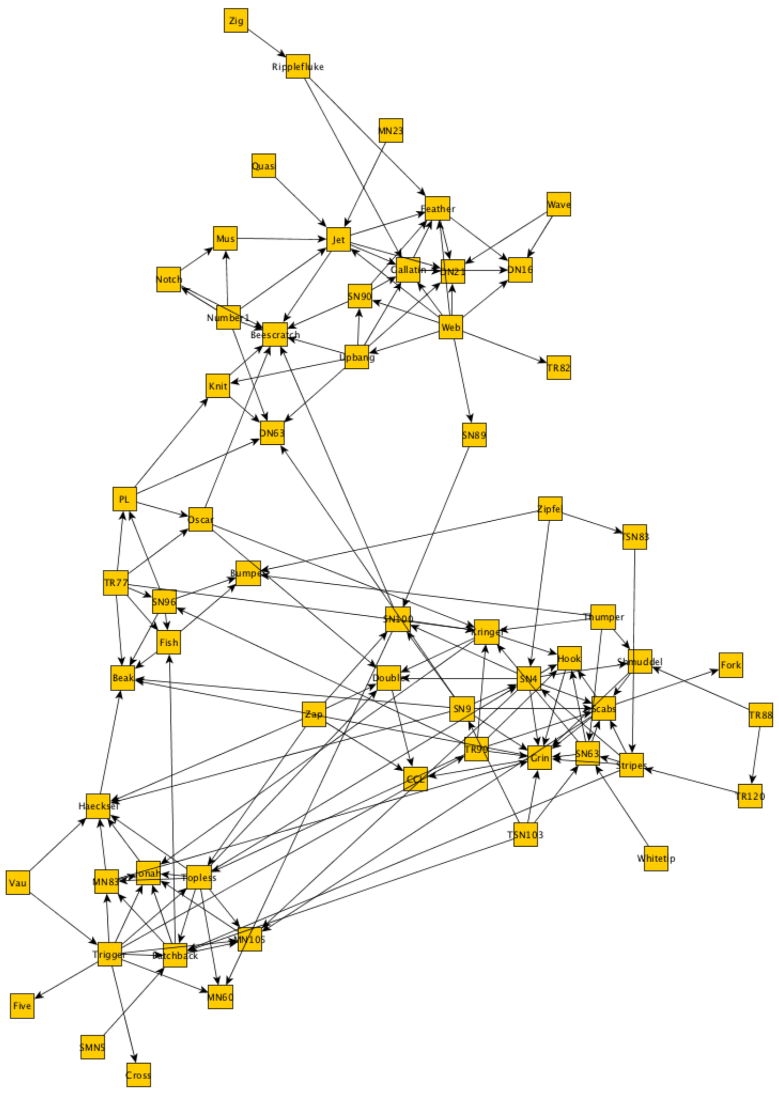

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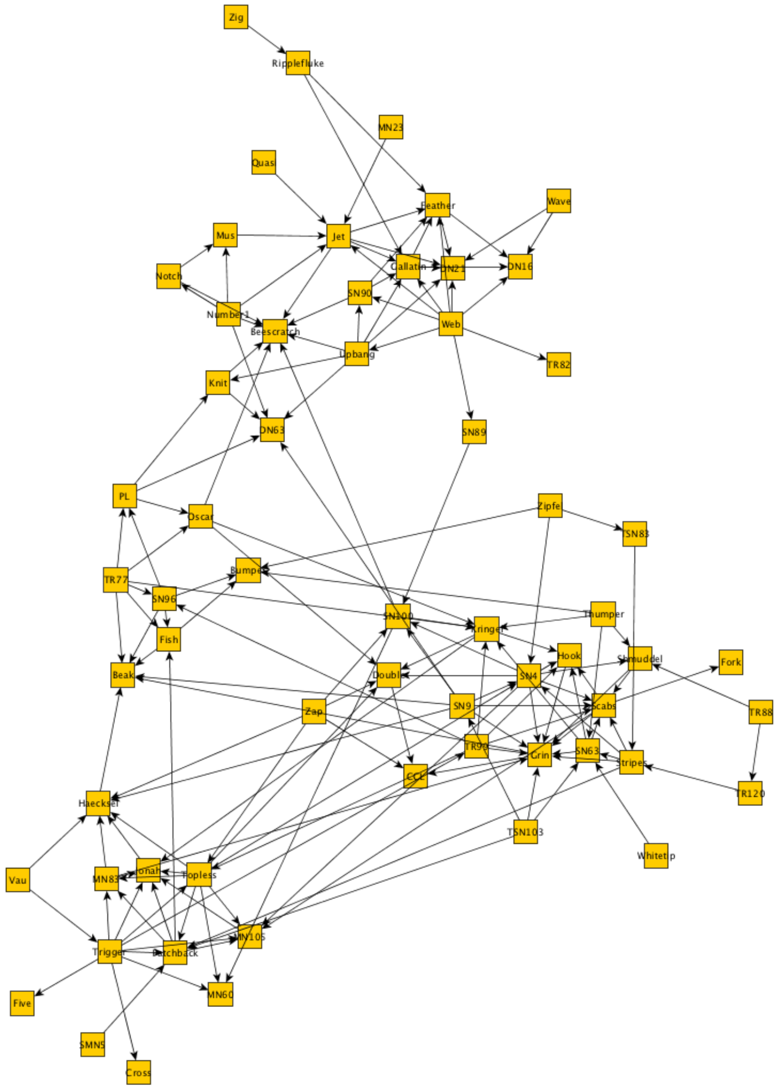
- nodes distributed evenly



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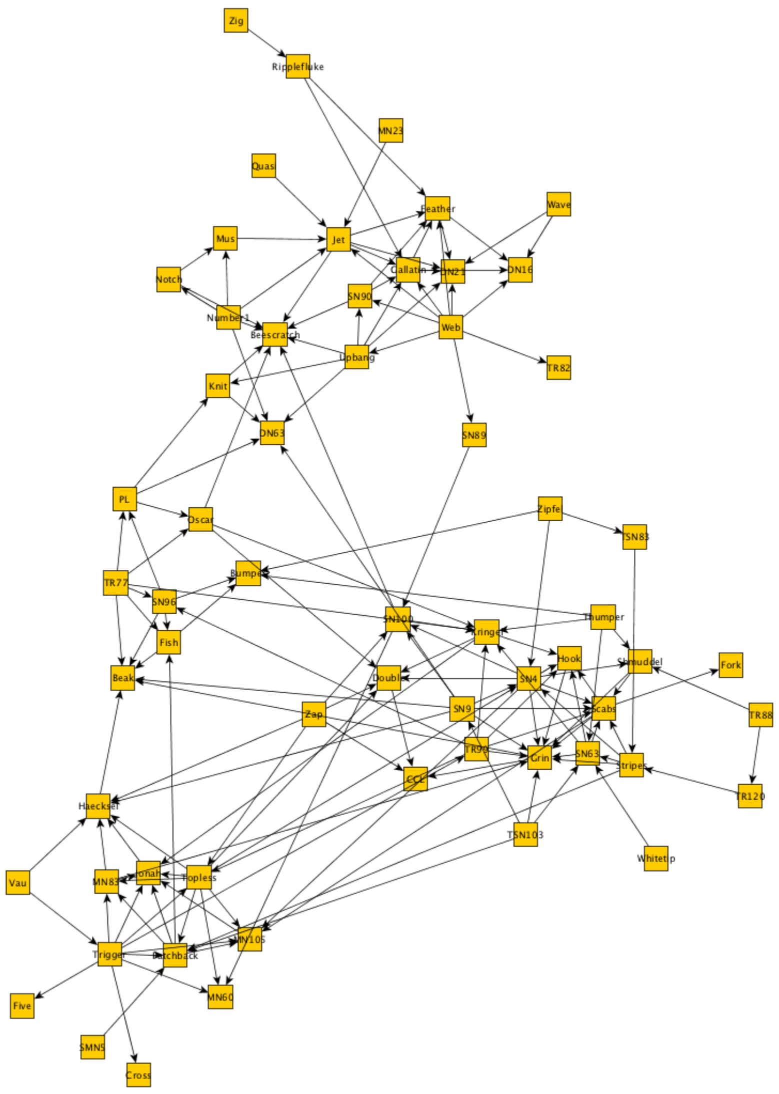

Optimization criteria partially contradict each other.

Fixed Edge Lengths?

Input: Graph G, required length $\ell(e)$ for each edge $e \in E(G)$.

Output: Drawing of G that realizes the given edge lengths.

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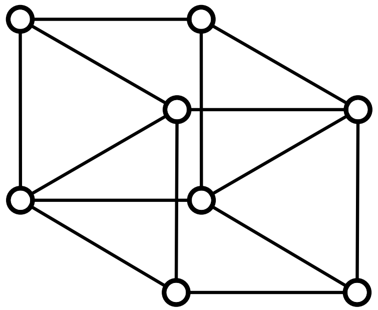

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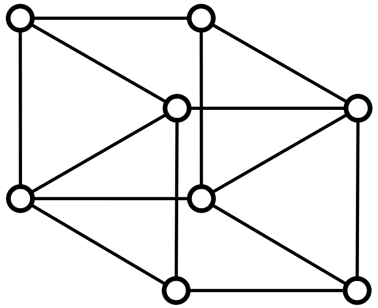

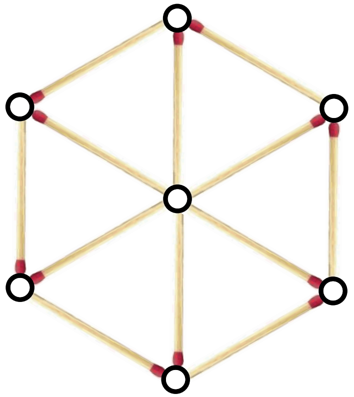

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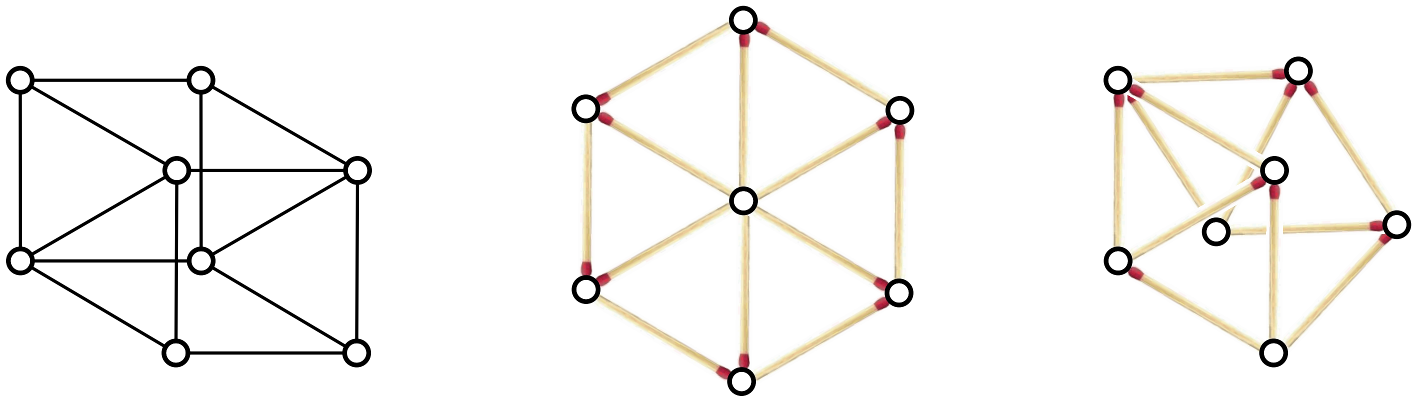

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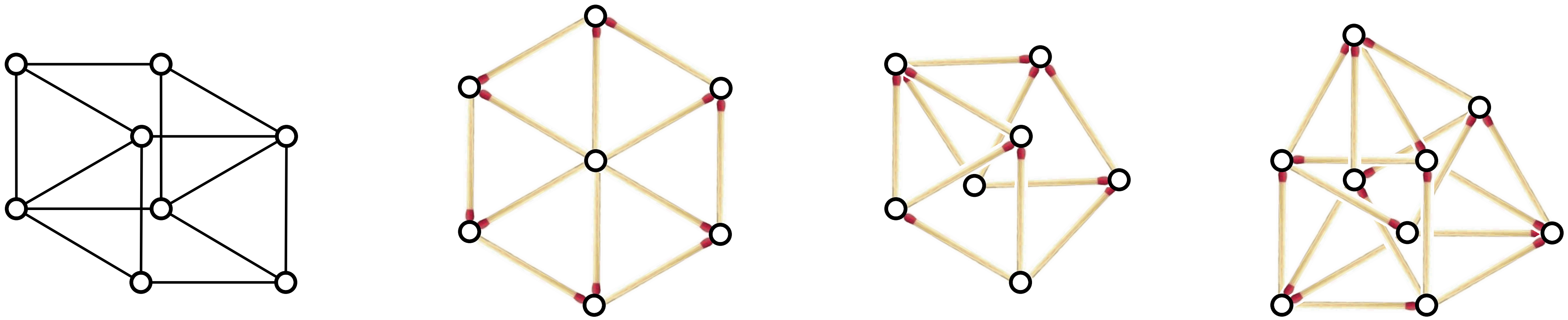

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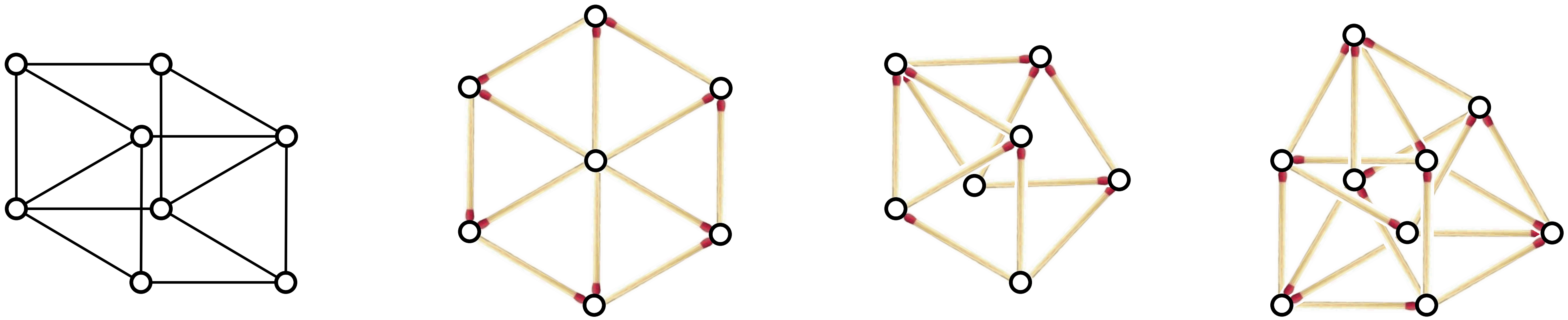

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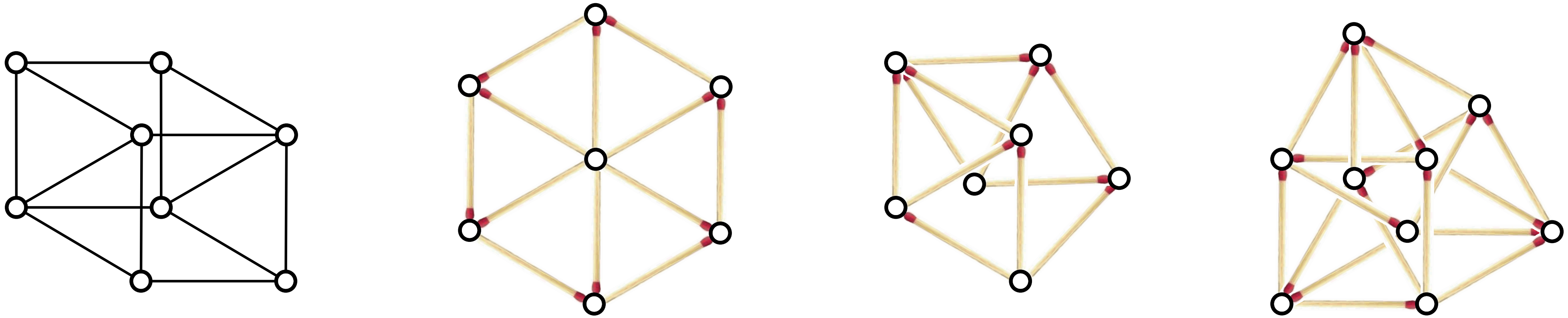

NP-hard for

- uniform edge lengths in any dimension [Johnson '82]

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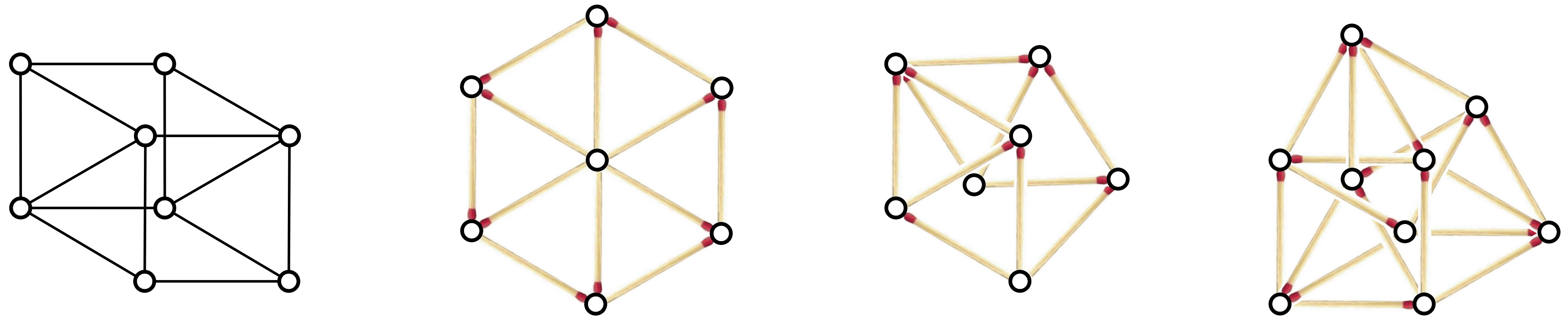

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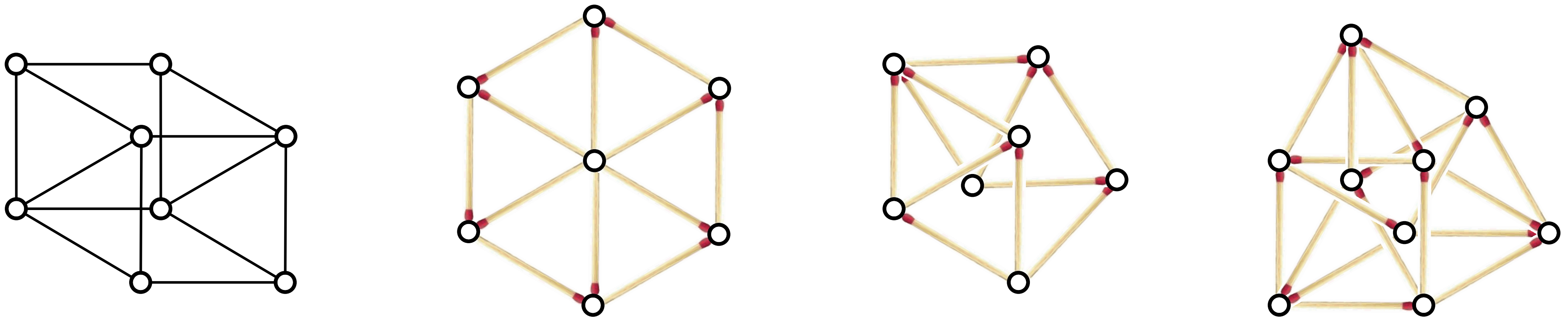

NP-hard for

- uniform edge lengths in any dimension [Johnson '82]
- uniform edge lengths in planar drawings [Eades, Wormald '90]
- edge lengths in $\{1, 2\}$ [Saxe '80]

Physical Analogy

Idea. [Eades '84]

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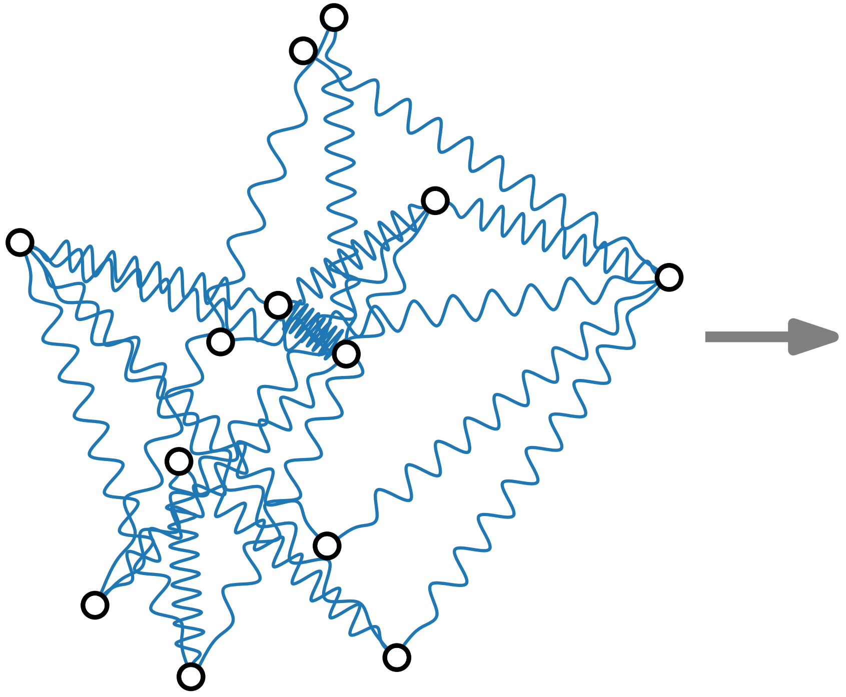

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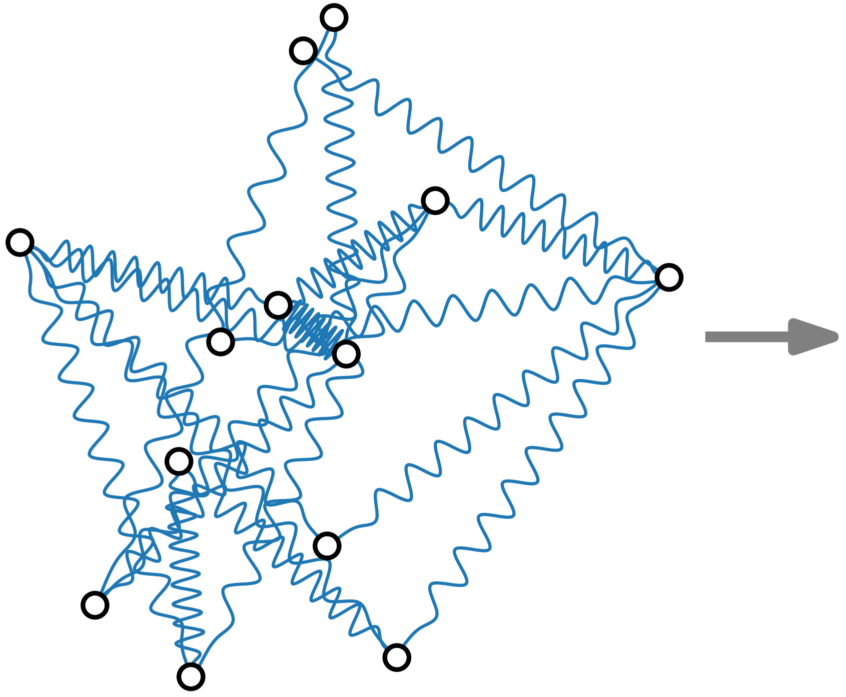

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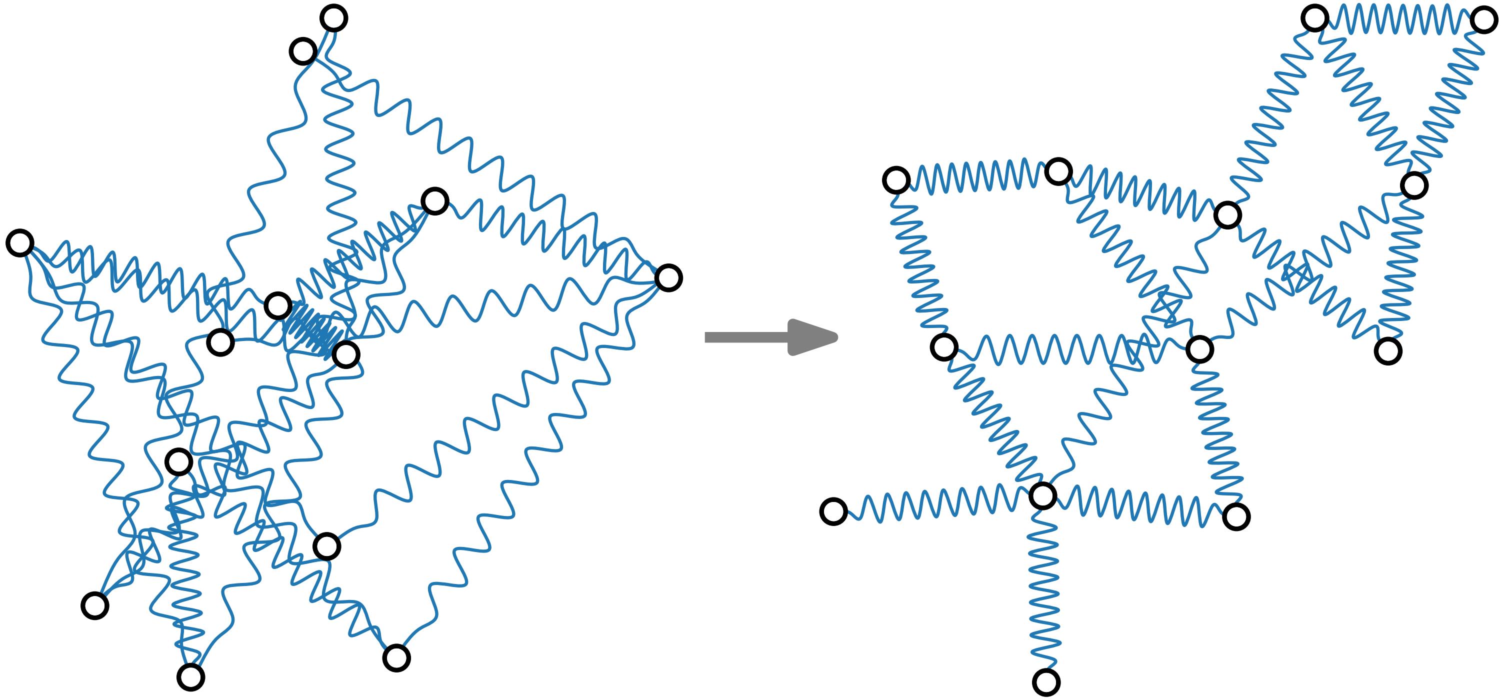

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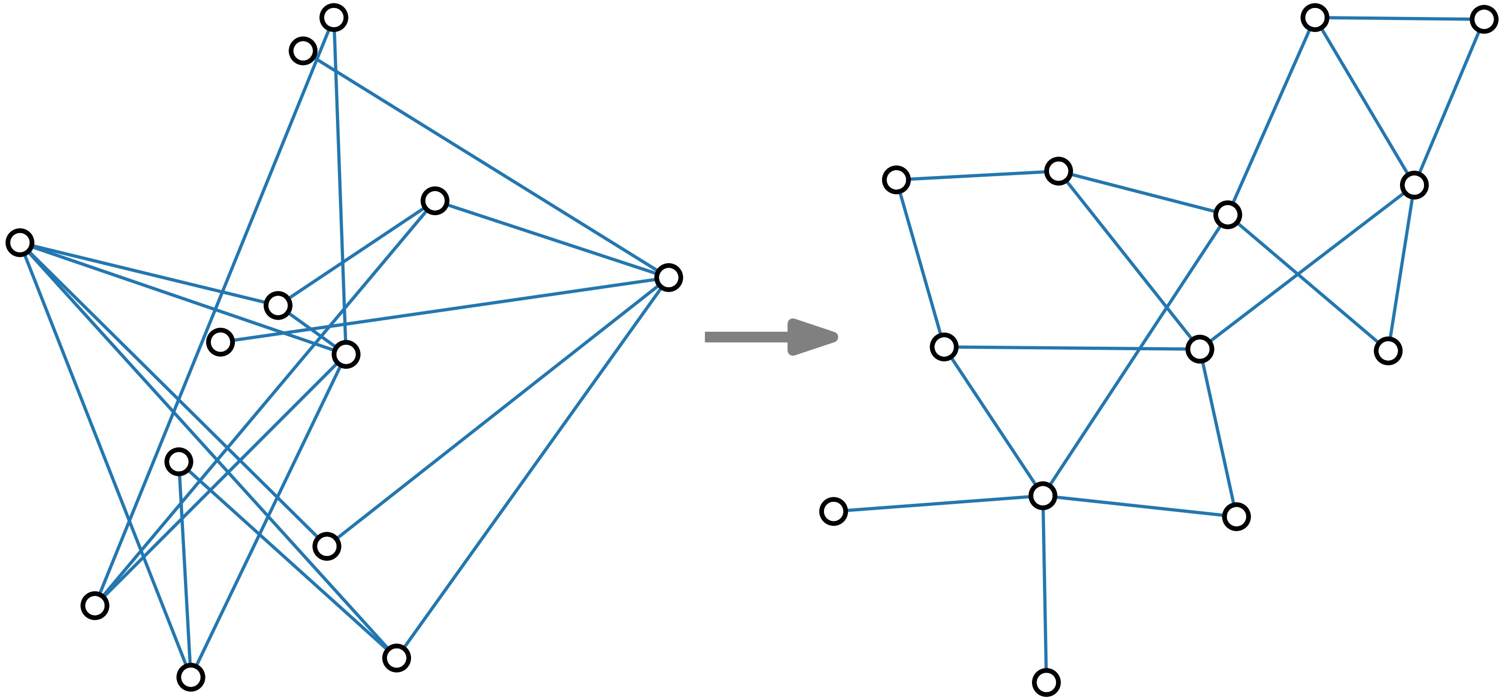

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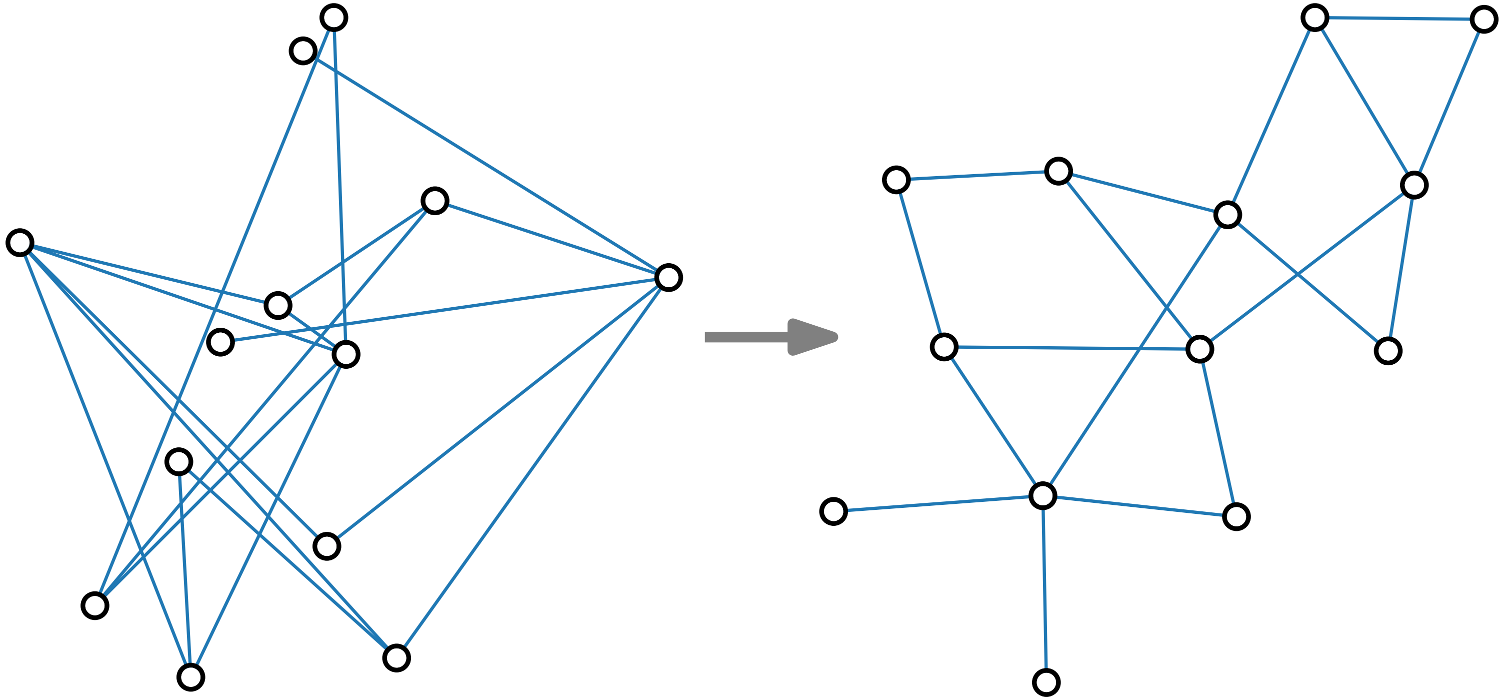
Attractive forces.



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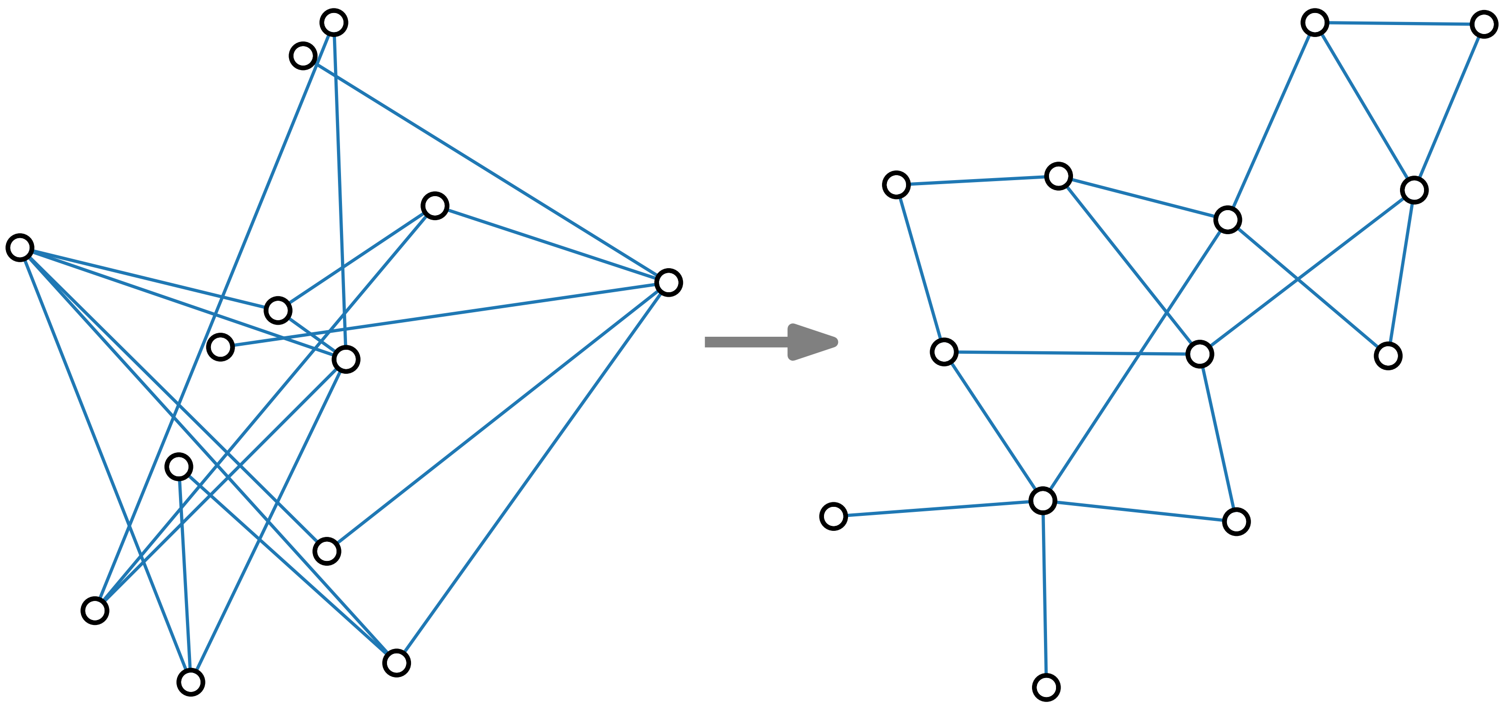

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pairs $\{u, v\}$ of adjacent vertices:

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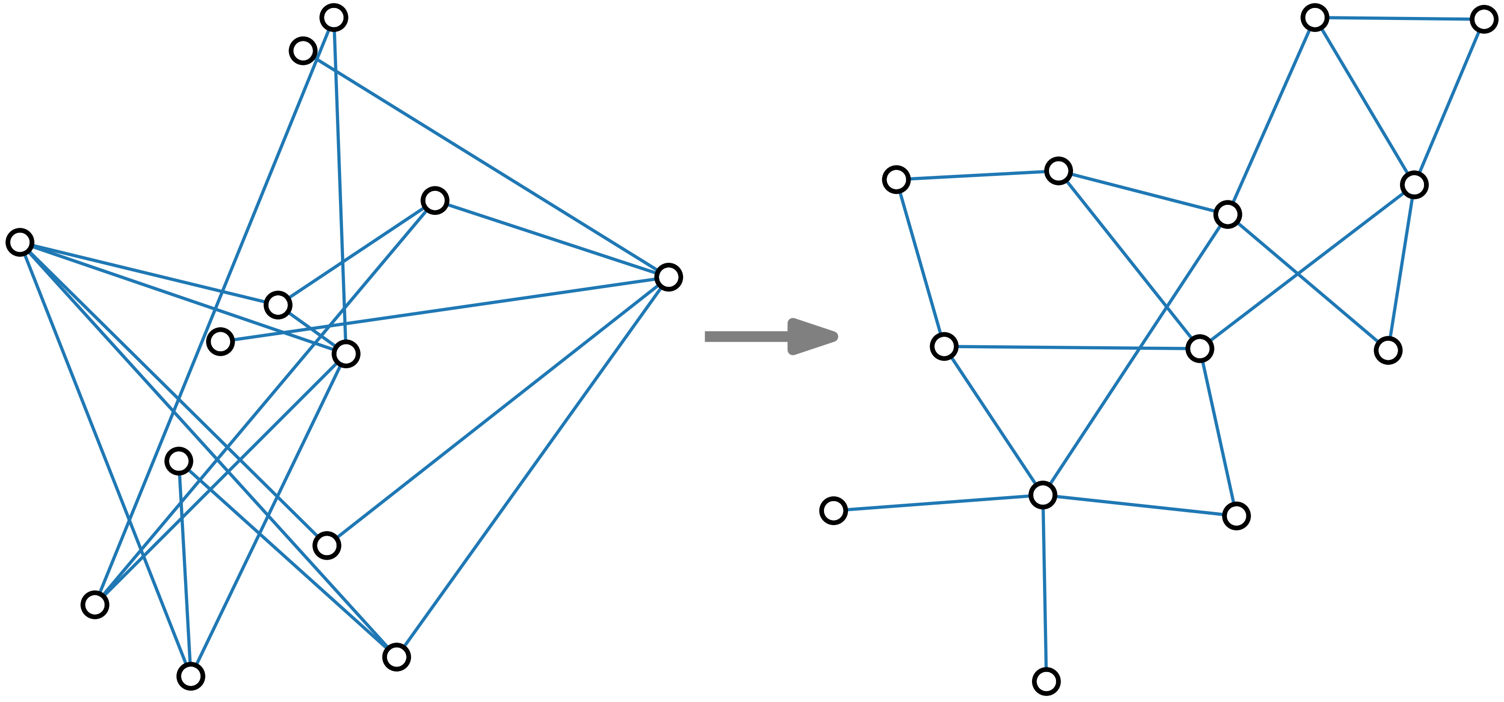

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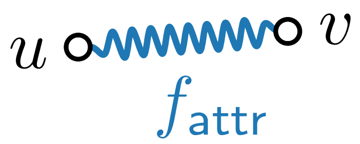

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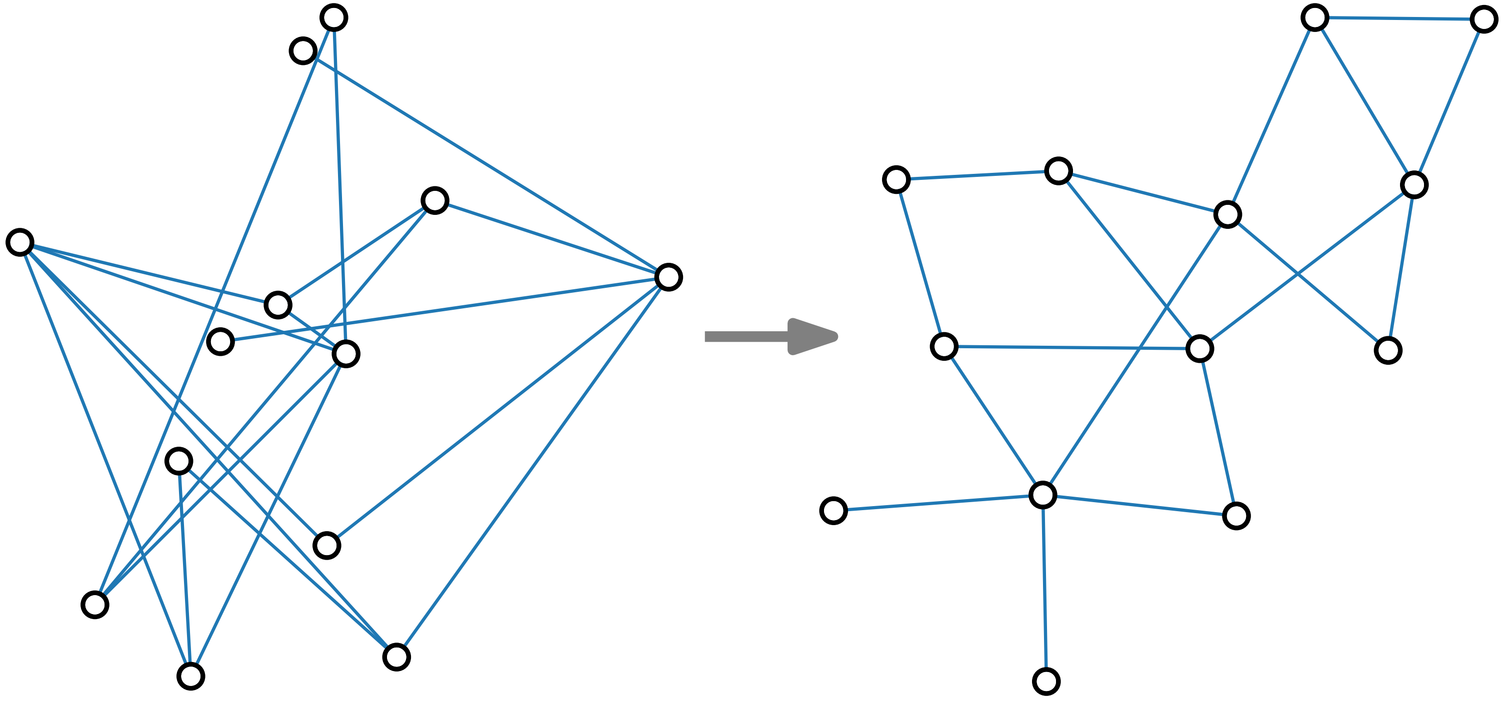

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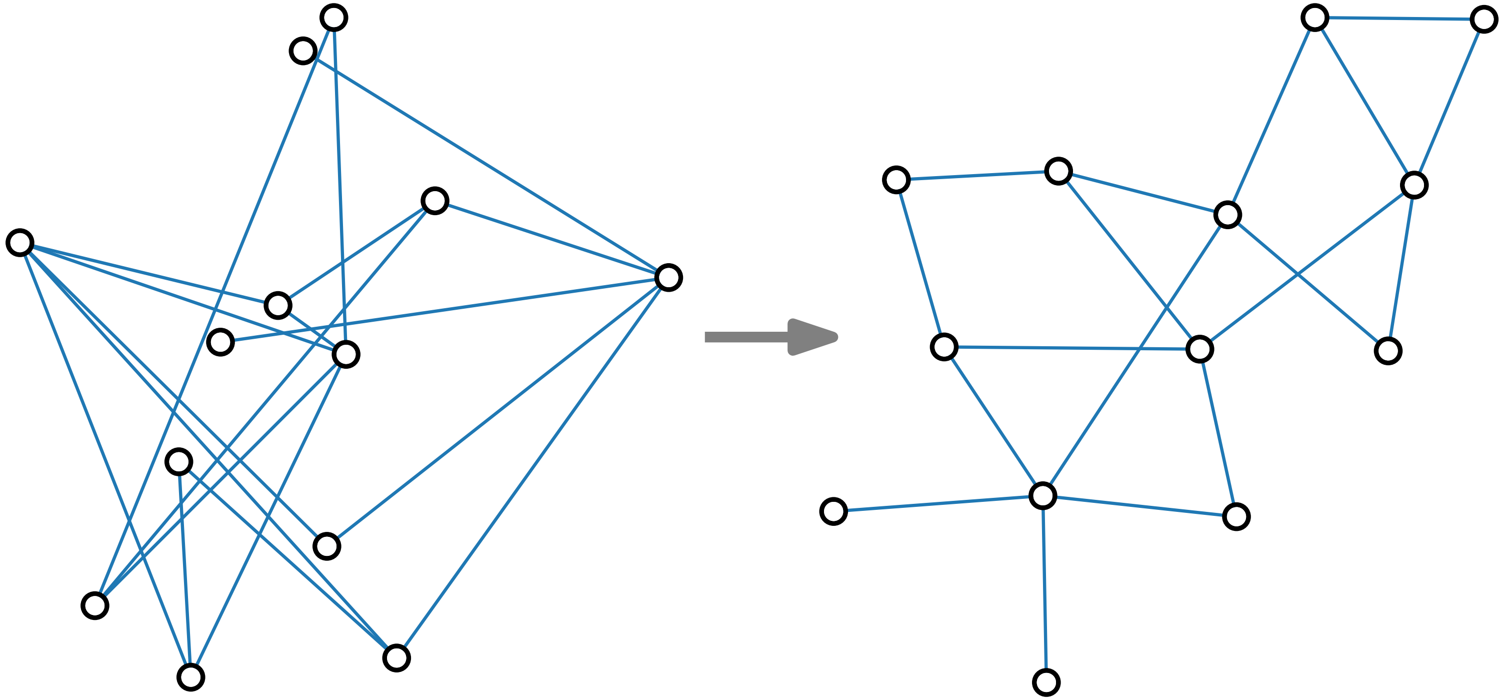

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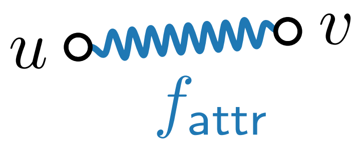

Repulsive forces.

any pair $\{x, y\}$ of vertices:

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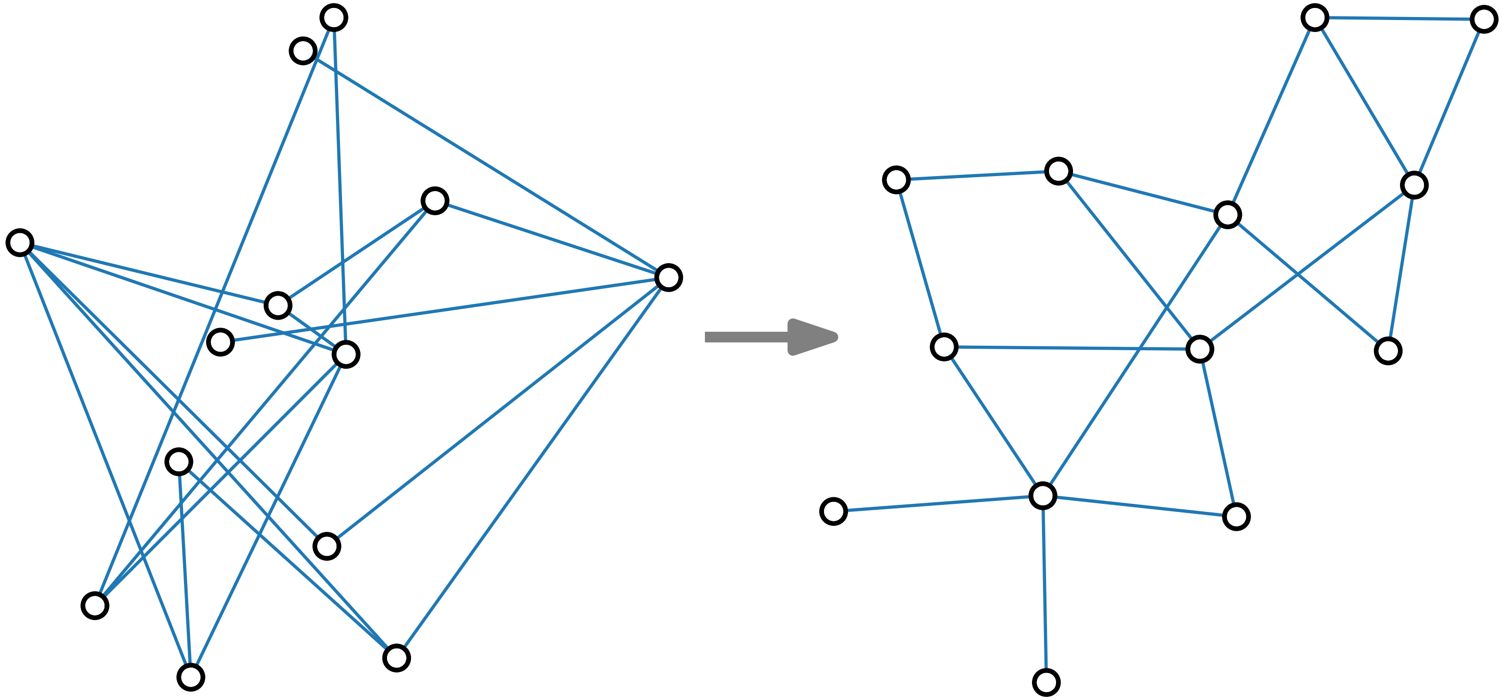

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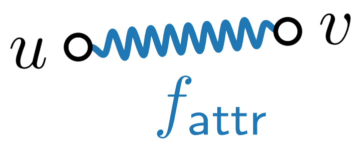

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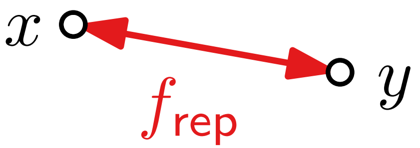

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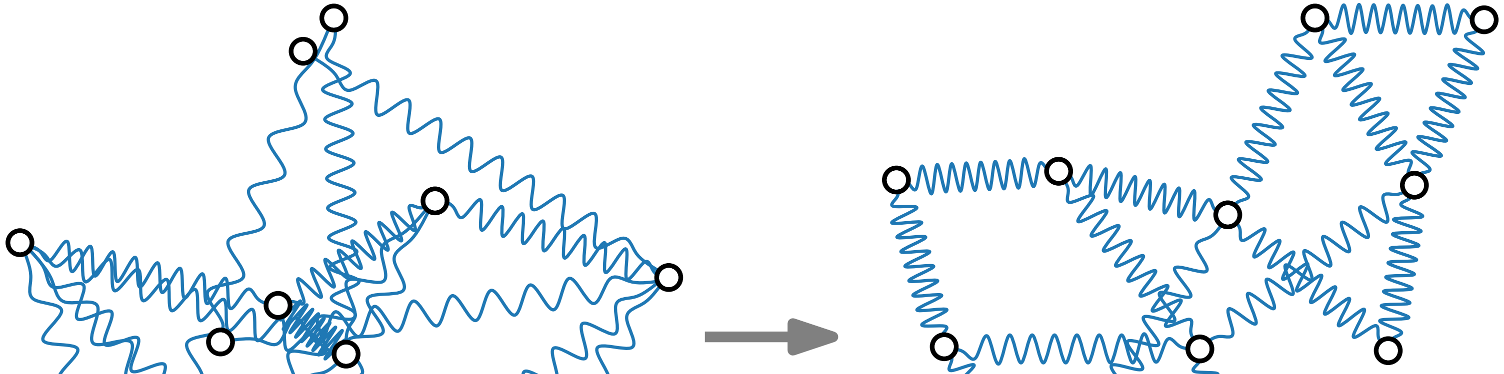

So-called **spring-embedder** algorithms that work according to this or similar principles are among the most frequently used graph-drawing methods in practice.

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Force-Directed Algorithms

ForceDirected(graph G, $p = (p_v)_{v \in V}$, $\varepsilon > 0$, $K \in \mathbb{N}$)

return p

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initial layout; may be randomly chosen positions

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end layout

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initial layout; may be randomly chosen positions

ForceDirected(graph G, $p = (p_v)_{v \in V}$, $\varepsilon > 0$, $K \in \mathbb{N}$)

threshold

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Force-Directed Algorithms

initial layout; may be randomly chosen positions

max # iterations

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ForceDirected(graph G, $p = (p_v)_{v \in V}$, $\varepsilon > 0$, $K \in \mathbb{N}$)

$t \leftarrow 1$

while $t \leq K$ **and** $\max_{v \in V(G)} \|F_v(t-1)\| > \varepsilon$ **do**

$t \leftarrow t + 1$

return p

threshold (assume $F_v(0) = \infty$)

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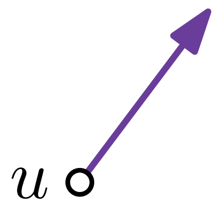

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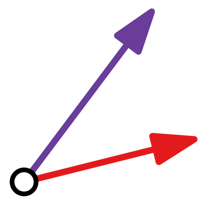

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vertices adjacent to u

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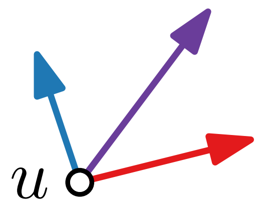
  return p

end layout



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    $p_u \leftarrow p_u + \delta(t) \cdot F_u(t)$

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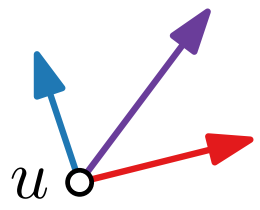
return p

end layout



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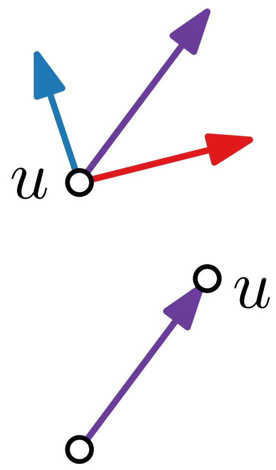
vertices adjacent to u

end layout



Force-Directed Algorithms

initial layout; may be randomly chosen positions

max # iterations

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threshold (assume $F_v(0) = \infty$)

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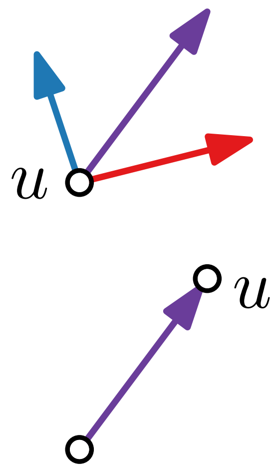

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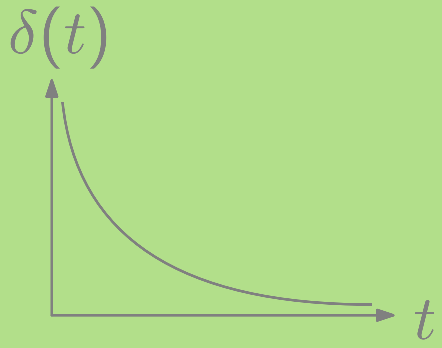

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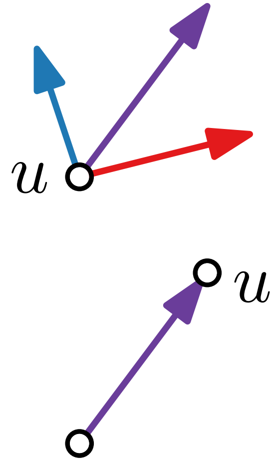

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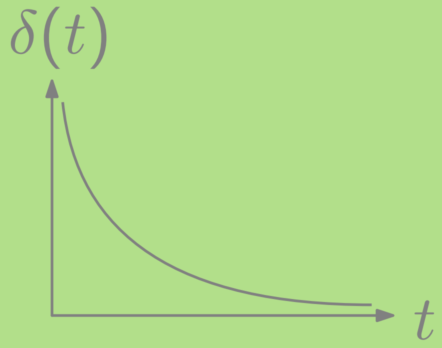

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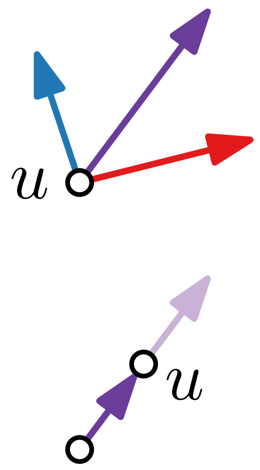

Spring Embedder by Eades – Model

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- **Resulting displacement vector**

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Spring Embedder by Eades – Model

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$$\boxed{f_{\text{rep}}(p_u, p_v) = \frac{c_{\text{rep}}}{\|p_v - p_u\|^2} \cdot \overrightarrow{p_v p_u}}$$

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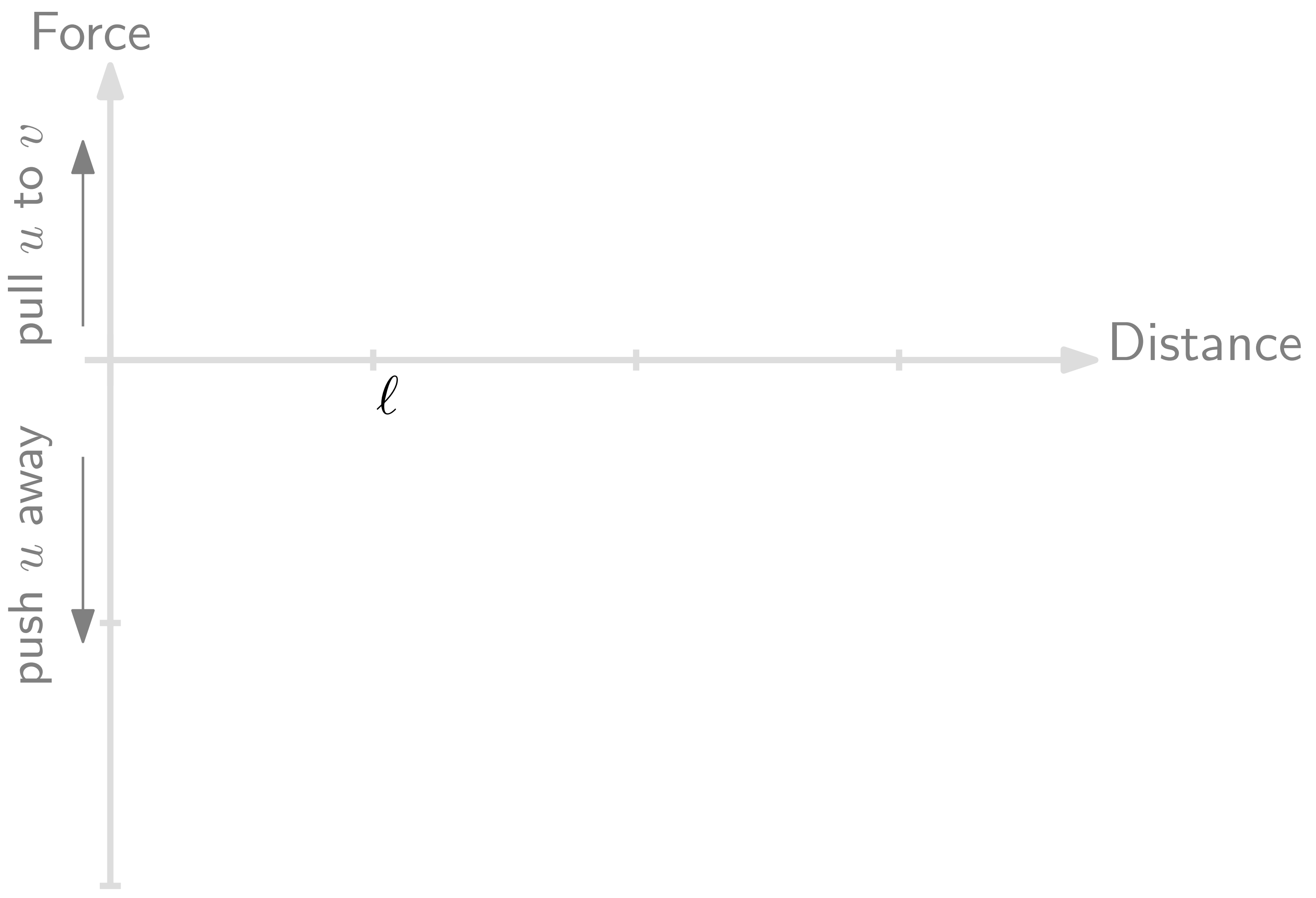

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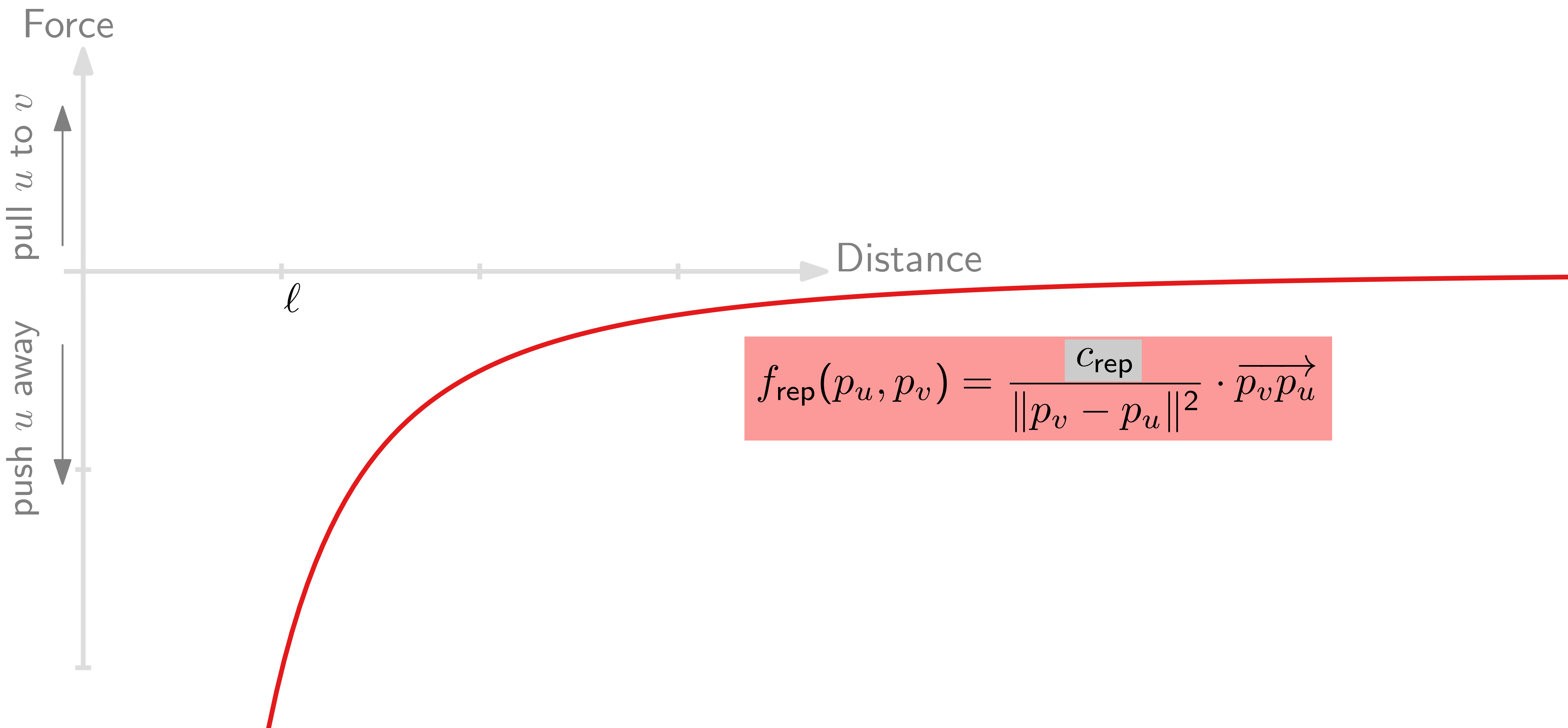

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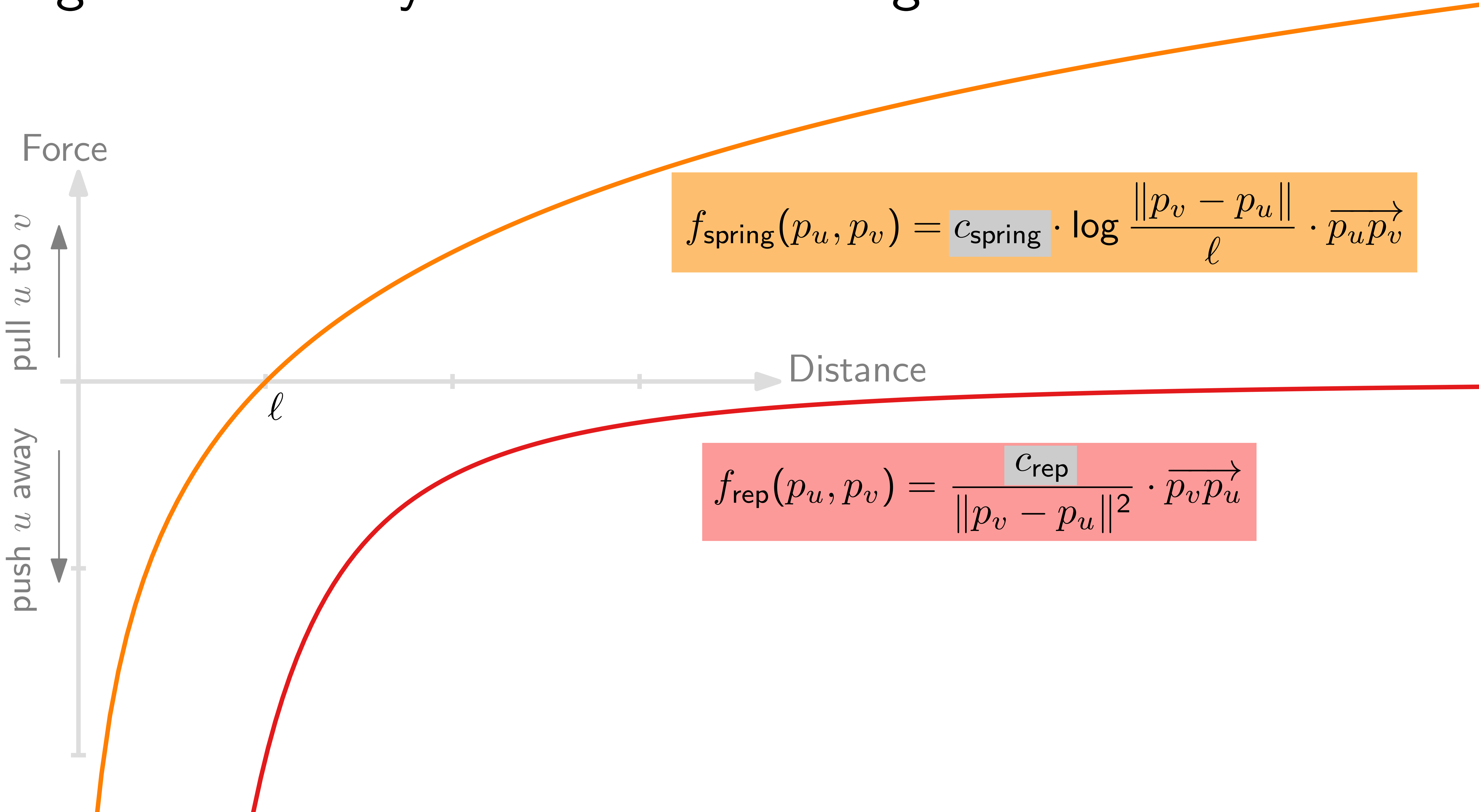

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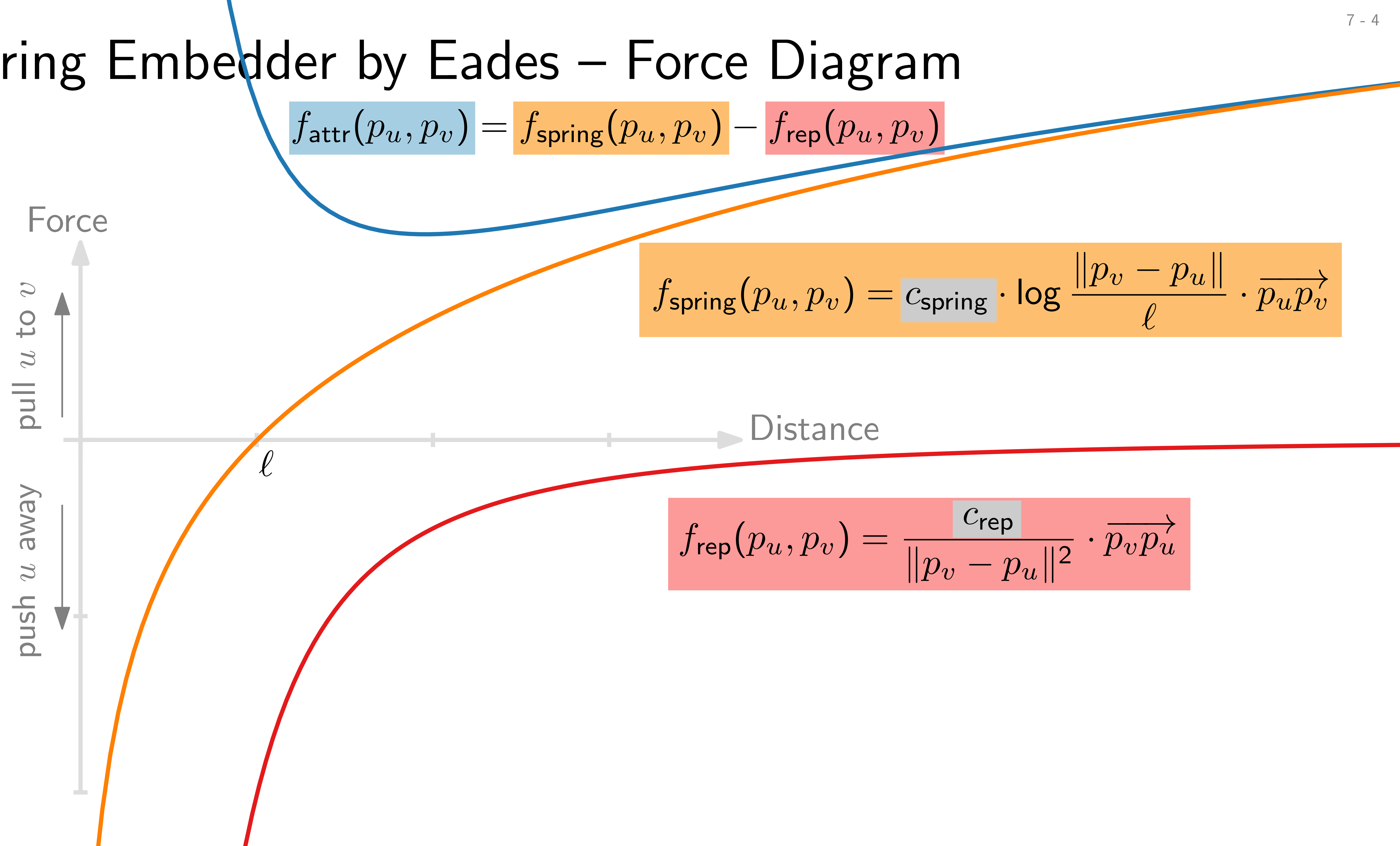

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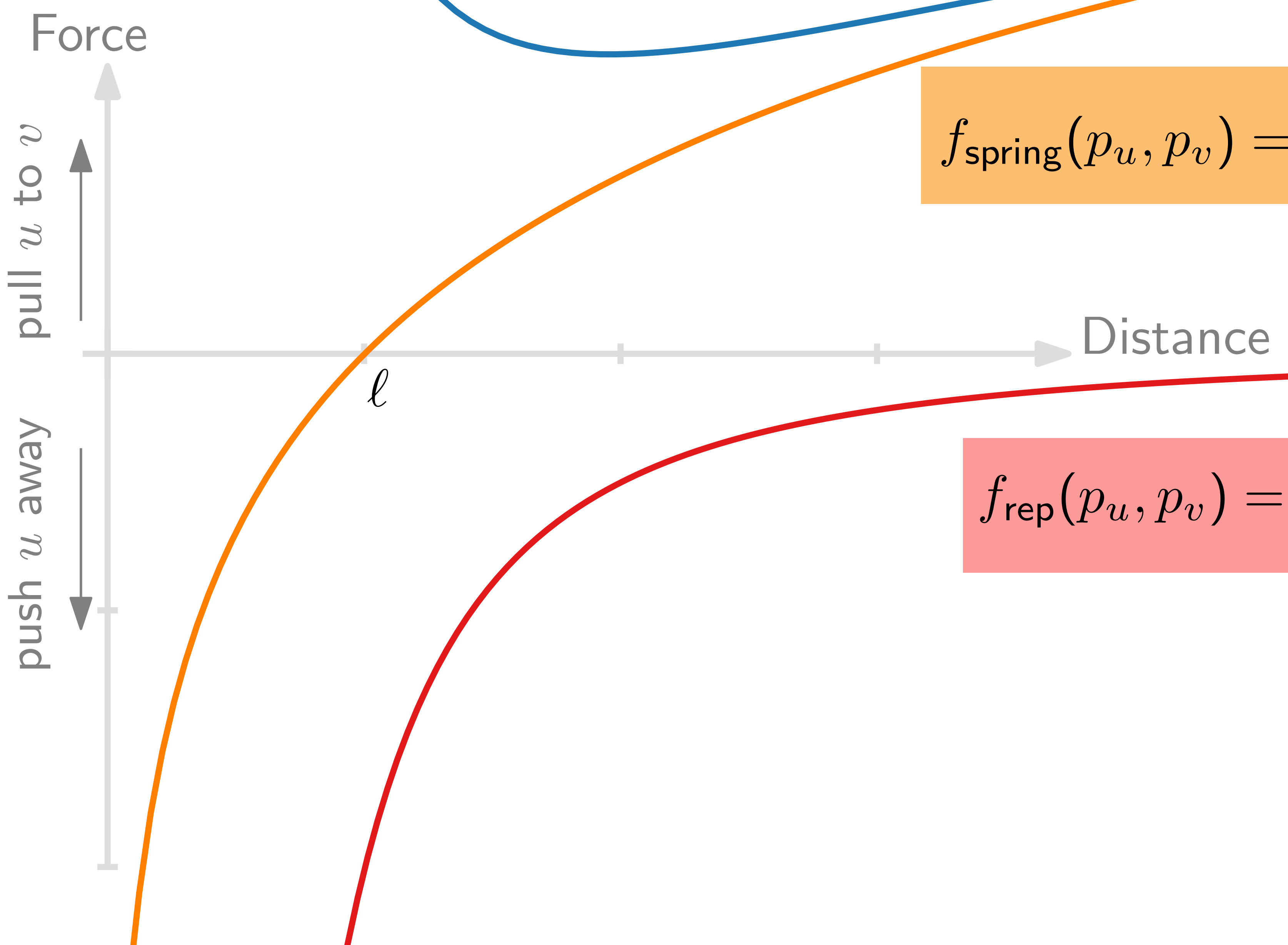

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Variant by Fruchterman & Reingold

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- **Repulsive forces**

$$f_{\text{rep}}(p_u, p_v) = \frac{\ell^2}{\|p_v - p_u\|} \cdot \overrightarrow{p_v p_u}$$

- **Attractive forces**

spring constant (e.g., 1.0)

$$f_{\text{spring}}(p_u, p_v) = c_{\text{spring}} \cdot \log \frac{\|p_v - p_u\|}{\ell} \cdot \overrightarrow{p_u p_v}$$

$$f_{\text{attr}}(p_u, p_v) = f_{\text{spring}}(p_u, p_v) - f_{\text{rep}}(p_u, p_v)$$

- **Resulting displacement vector**

$$F_u = \sum_{v \in V(G)} f_{\text{rep}}(p_u, p_v) + \sum_{v \in \text{Adj}[u]} f_{\text{attr}}(p_u, p_v)$$

Notation.

- $\|p_u - p_v\|$ = Euclidean distance between u and v
- $\overrightarrow{p_u p_v}$ = unit vector pointing from u to v
- ℓ = ideal spring length for edges

Variant by Fruchterman & Reingold

ForceDirected(graph G, $p = (p_v)_{v \in V}$, $\varepsilon > 0$, $K \in \mathbb{N}$)
  $t \leftarrow 1$
  while $t \leq K$ **and** $\max_{v \in V(G)} \|F_v(t-1)\| > \varepsilon$ **do**
    foreach $u \in V(G)$ **do**
      $F_u(t) \leftarrow \sum_{v \in V(G)} f_{\text{rep}}(p_u, p_v) + \sum_{v \in \text{Adj}[u]} f_{\text{attr}}(p_u, p_v)$
    foreach $u \in V(G)$ **do**
      $p_u \leftarrow p_u + \delta(t) \cdot F_u(t)$
    $t \leftarrow t + 1$
  return p

- **Repulsive forces**

$$f_{\text{rep}}(p_u, p_v) = \frac{\ell^2}{\|p_v - p_u\|} \cdot \overrightarrow{p_v p_u}$$

- **Attractive forces**

$$f_{\text{attr}}(p_u, p_v) = \frac{\|p_v - p_u\|^2}{\ell} \cdot \overrightarrow{p_u p_v}$$

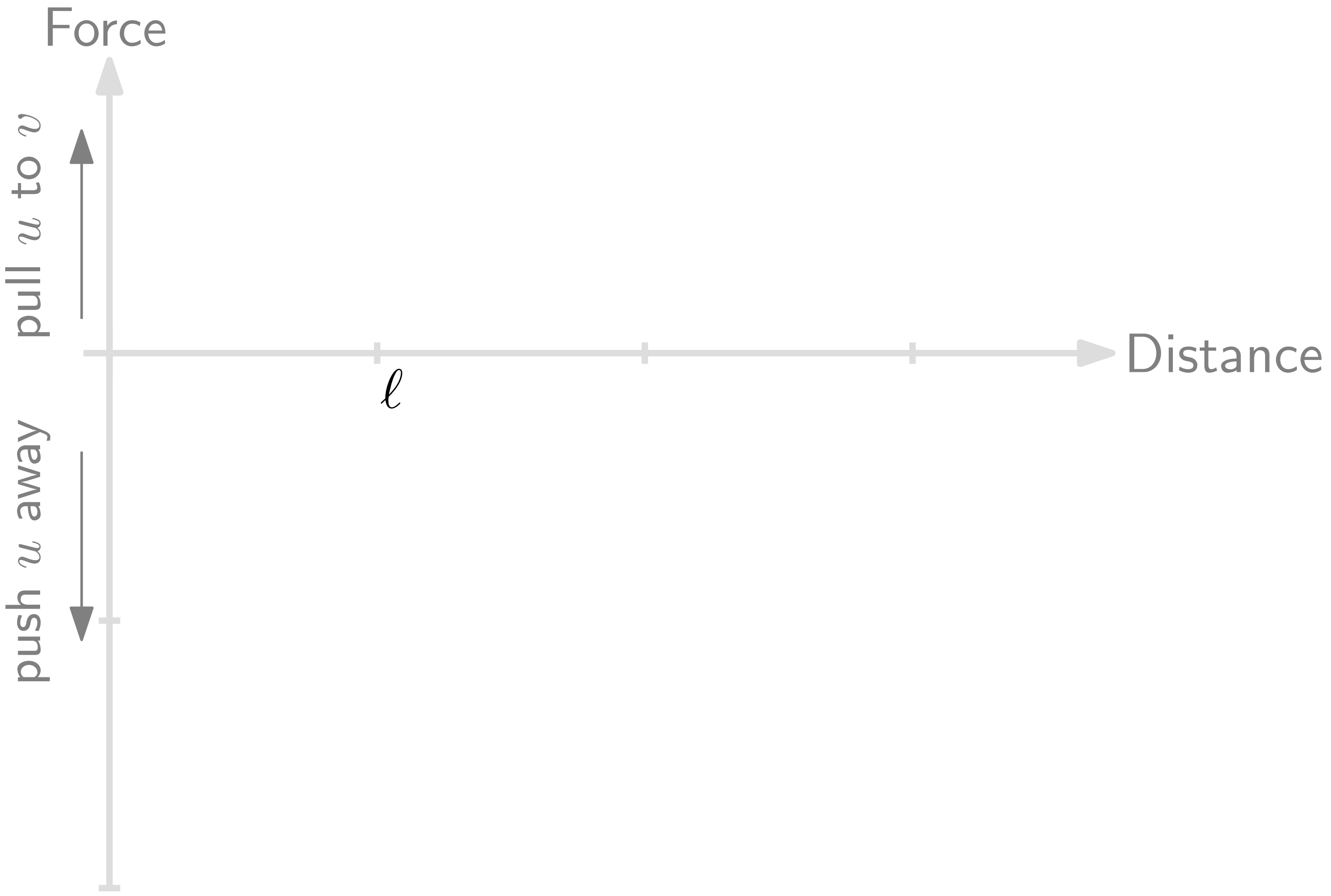
- **Resulting displacement vector**

$$F_u = \sum_{v \in V(G)} f_{\text{rep}}(p_u, p_v) + \sum_{v \in \text{Adj}[u]} f_{\text{attr}}(p_u, p_v)$$

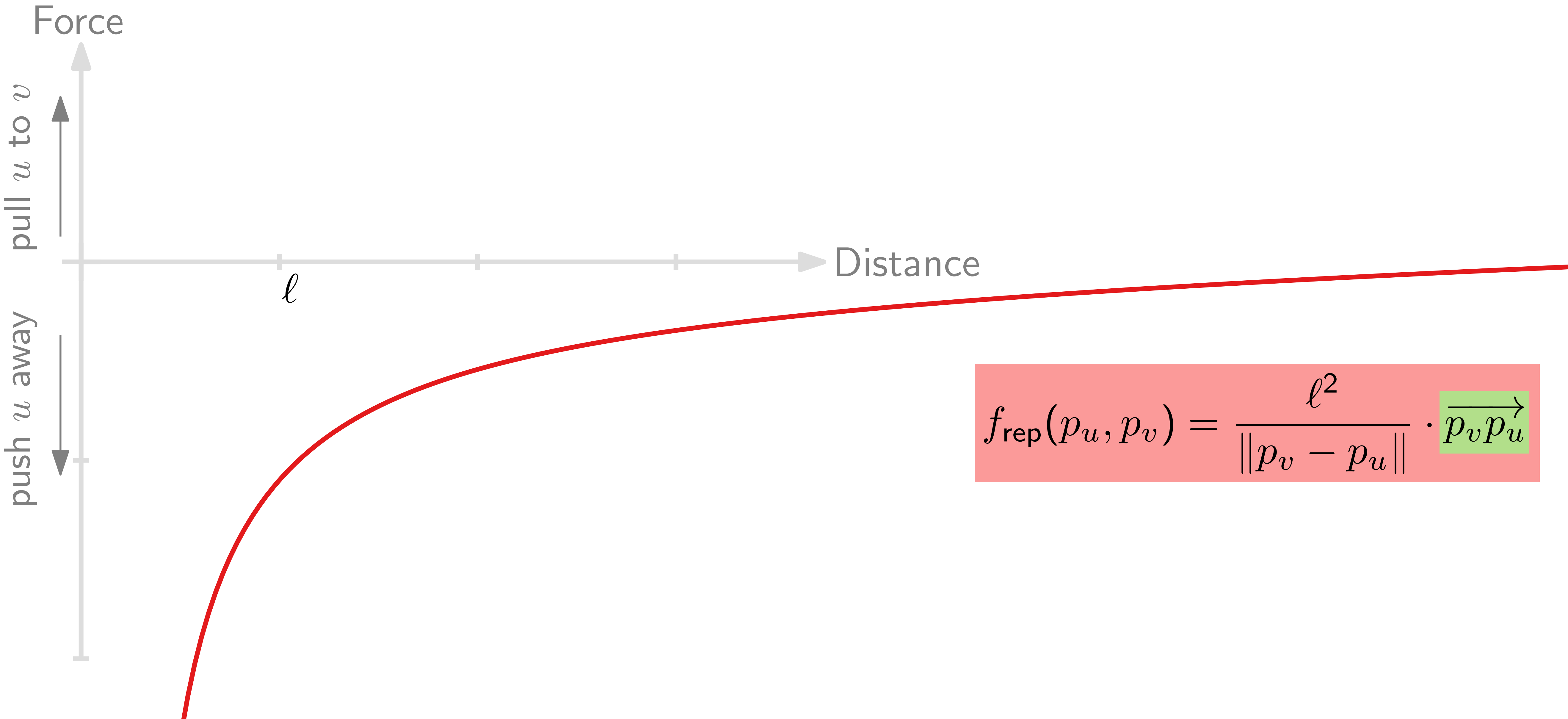
Notation.

- $\|p_u - p_v\|$ = Euclidean distance between u and v
- $\overrightarrow{p_u p_v}$ = unit vector pointing from u to v
- ℓ = ideal spring length for edges

Fruchterman & Reingold – Force Diagram



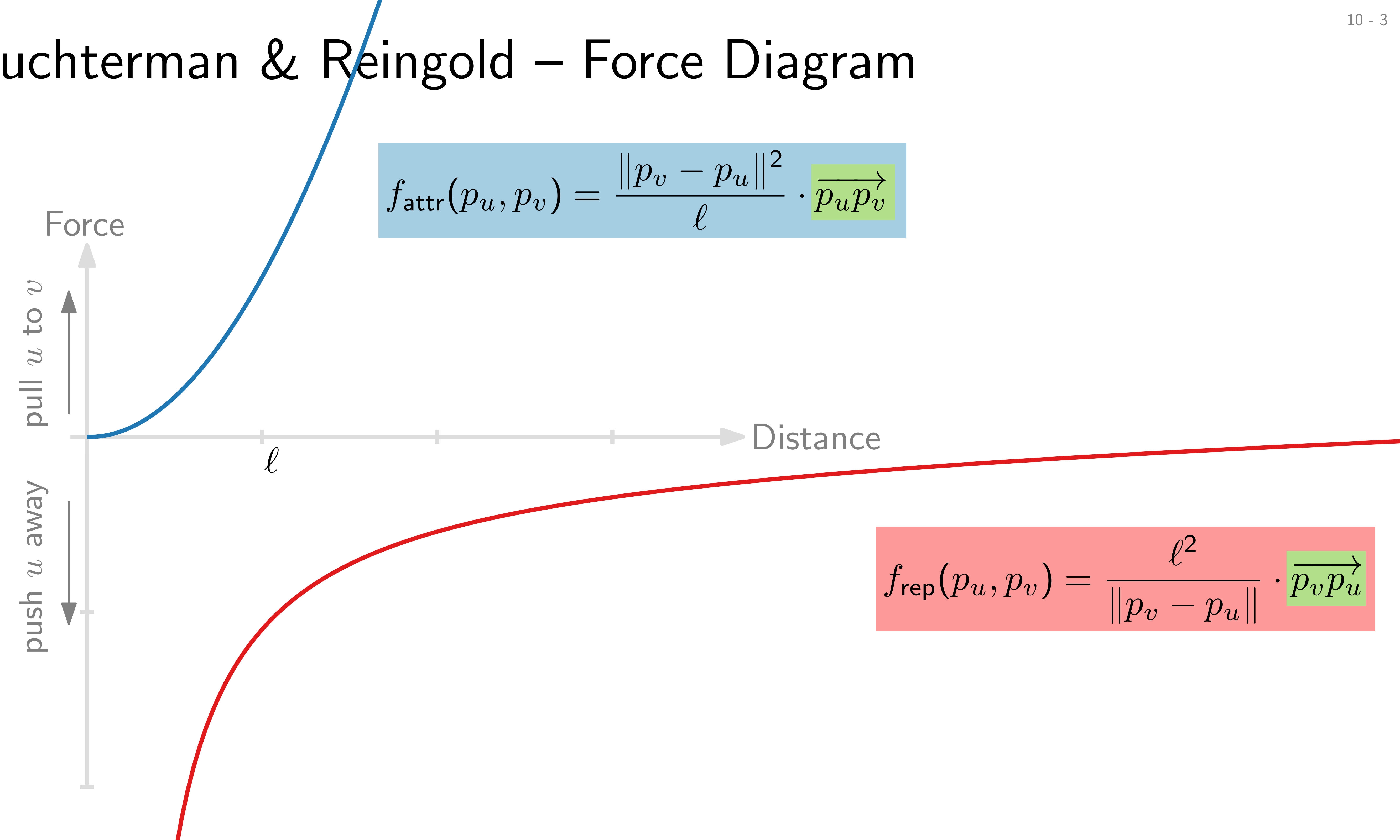
Fruchterman & Reingold – Force Diagram



$$f_{\text{rep}}(p_u, p_v) = \frac{\ell^2}{\|p_v - p_u\|} \cdot \overrightarrow{p_v p_u}$$

Fruchterman & Reingold – Force Diagram

$$f_{\text{attr}}(p_u, p_v) = \frac{\|p_v - p_u\|^2}{\ell} \cdot \overrightarrow{p_u p_v}$$

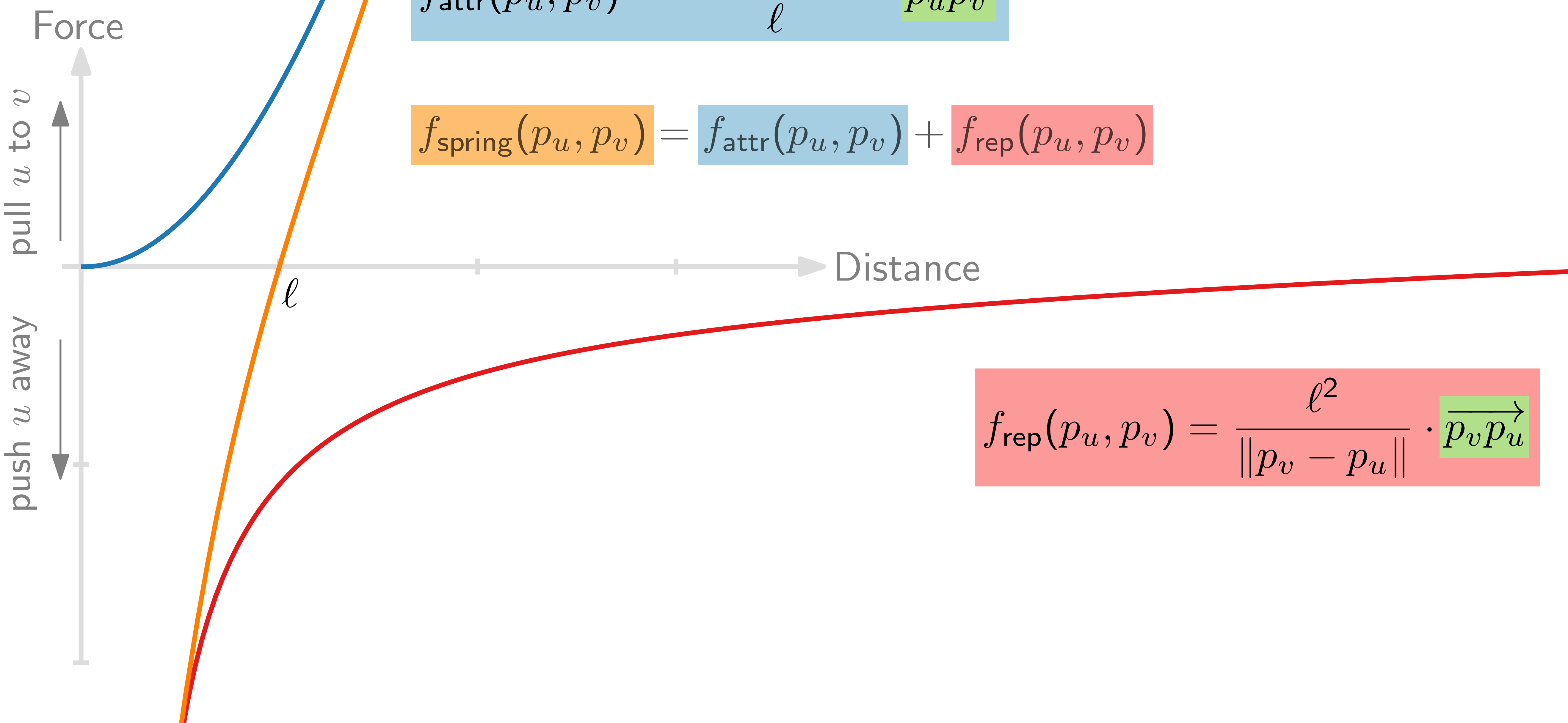


$$f_{\text{rep}}(p_u, p_v) = \frac{\ell^2}{\|p_v - p_u\|} \cdot \overrightarrow{p_v p_u}$$

Fruchterman & Reingold – Force Diagram

$$f_{\mathsf{attr}}(p_u, p_v) = \frac{\|p_v - p_u\|^2}{\ell} \cdot \overrightarrow{p_u p_v}$$

$$f_{\mathsf{spring}}(p_u, p_v) = f_{\mathsf{attr}}(p_u, p_v) + f_{\mathsf{rep}}(p_u, p_v)$$



$$f_{\mathsf{rep}}(p_u, p_v) = \frac{\ell^2}{\|p_v - p_u\|} \cdot \overrightarrow{p_v p_u}$$

Adaptability

degree of vertex u, i.e., $|\mathrm{Adj}[u]|$

Inertia. (“Trägheit”)

- Define vertex mass $\Phi(u) = 1 + \deg(u)/2$
- Set $f_{\mathrm{attr}}(u, p_v) = f_{\mathrm{attr}}(p_u, p_v) \cdot 1/\Phi(u)$

Adaptability

degree of vertex u, i.e., $|\mathrm{Adj}[u]|$

Inertia. ("Trägheit")

- Define vertex mass $\Phi(u) = 1 + \deg(u)/2$
- Set $f_{\mathrm{attr}}(u, p_v) = f_{\mathrm{attr}}(p_u, p_v) \cdot 1/\Phi(u)$

Gravitation.

- Define centroid $\sigma_V = 1/|V(G)| \cdot \sum_{v \in V(G)} p_v$
- Add force $f_{\mathrm{grav}}(v) = c_{\mathrm{grav}} \cdot \Phi(v) \cdot \overrightarrow{p_v \sigma_V}$

Adaptability

degree of vertex u, i.e., $|\mathrm{Adj}[u]|$

Inertia. ("Trägheit")

- Define vertex mass $\Phi(u) = 1 + \deg(u)/2$
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Gravitation.

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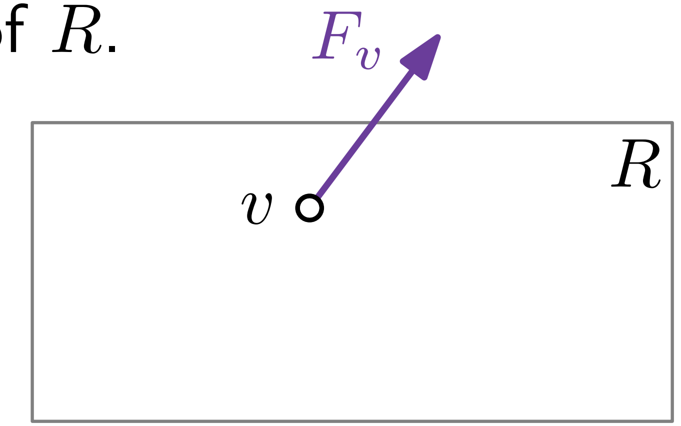
- Add force $f_{\text{grav}}(v) = c_{\text{grav}} \cdot \Phi(v) \cdot \overrightarrow{p_v \sigma_V}$

Restricted drawing area.

If F_v points beyond area R, clip vector appropriately at the border of R.



Adaptability

degree of vertex u, i.e., $|\mathrm{Adj}[u]|$

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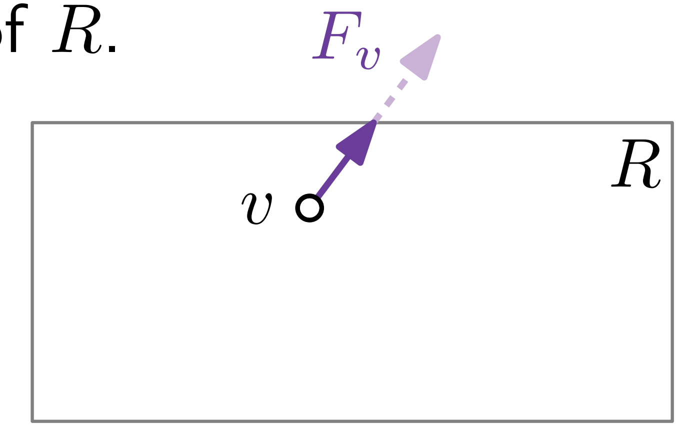

Adaptability

degree of vertex u, i.e., $|\mathrm{Adj}[u]|$

Inertia. ("Trägheit")

- Define vertex mass $\Phi(u) = 1 + \deg(u)/2$
- Set $f_{\mathrm{attr}}(u, p_v) = f_{\mathrm{attr}}(p_u, p_v) \cdot 1/\Phi(u)$

Gravitation.

- Define centroid $\sigma_V = 1/|V(G)| \cdot \sum_{v \in V(G)} p_v$
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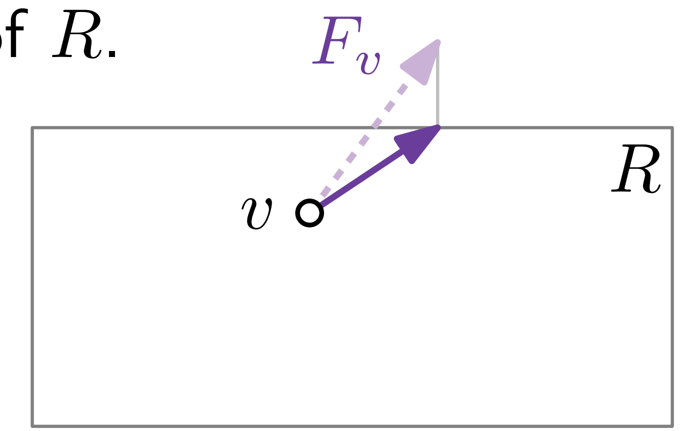

Adaptability

degree of vertex u, i.e., $|\mathrm{Adj}[u]|$

Inertia. (“Trägheit”)

- Define vertex mass $\Phi(u) = 1 + \deg(u)/2$
- Set $f_{\text{attr}}(u, p_v) = f_{\text{attr}}(p_u, p_v) \cdot 1/\Phi(u)$

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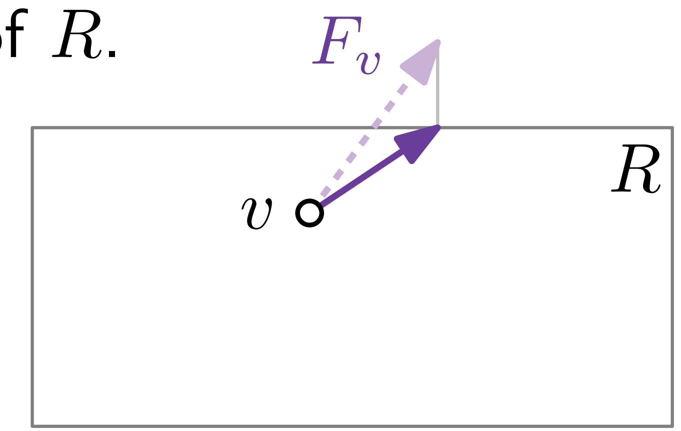
Restricted drawing area.

If F_v points beyond area R, clip vector appropriately at the border of R.



And many more...

- magnetic orientation of edges [GD Ch. 10.4]
- other energy models
- planarity preserving
- speed-ups

Speeding up "Convergence" by Adaptive Displacement $\delta_v(t)$

ForceDirected(graph G, $p = (p_v)_{v \in V}$, $\varepsilon > 0$, $K \in \mathbb{N}$)

$t \leftarrow 1$

while $t \leq K$ **and** $\max_{v \in V(G)} \|F_v(t-1)\| > \varepsilon$ **do**

  foreach $u \in V(G)$ **do**

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return p

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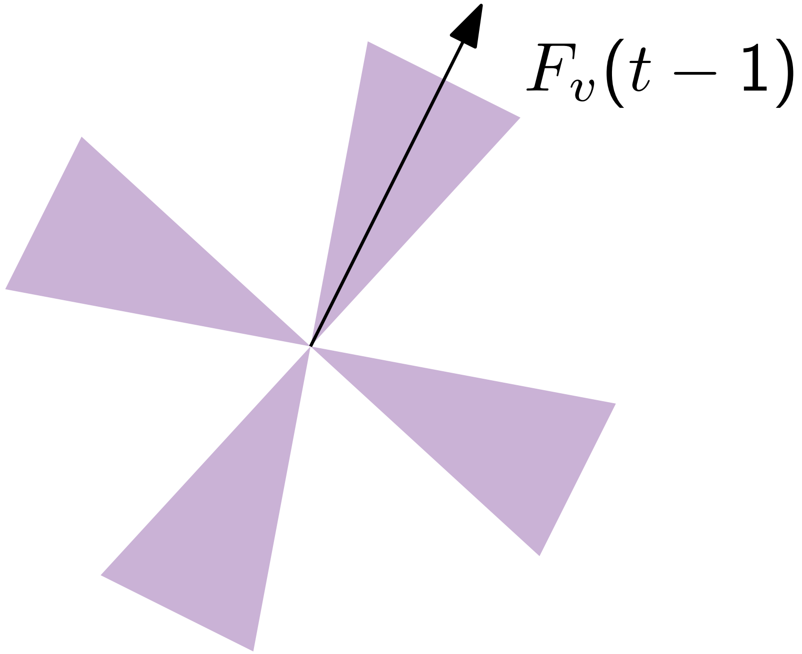
    foreach $u \in V(G)$ **do**
      $p_u \leftarrow p_u + \cancel{\delta(t)} \cdot F_u(t)$
                 $\delta_v(t)$
    $t \leftarrow t + 1$
  return p

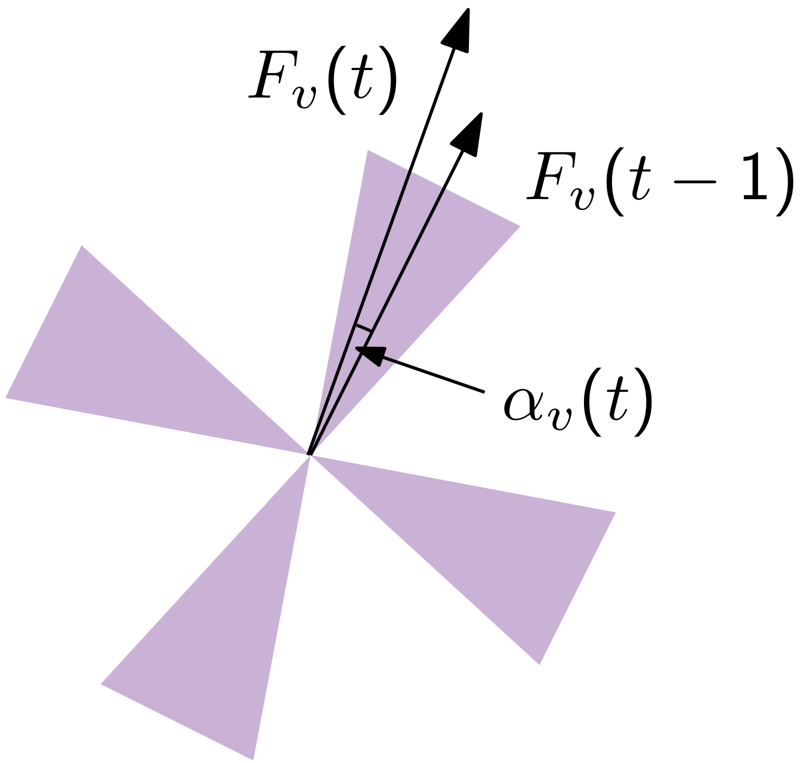
Speeding up "Convergence" by Adaptive Displacement $\delta_v(t)$

[Frick, Ludwig, Mehldau '95]



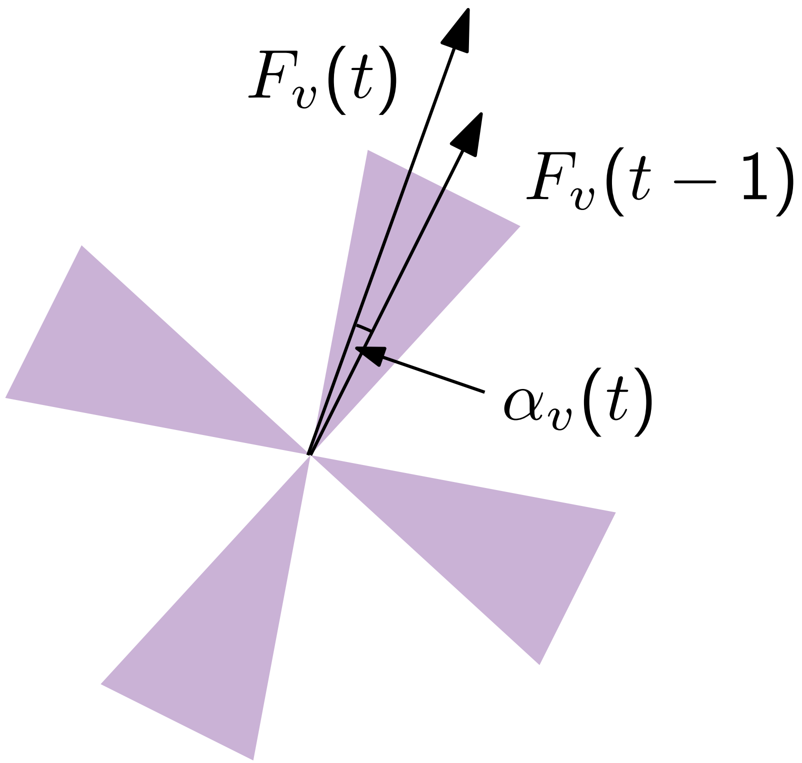
Speeding up "Convergence" by Adaptive Displacement $\delta_v(t)$

[Frick, Ludwig, Mehldau '95]



Speeding up "Convergence" by Adaptive Displacement $\delta_v(t)$

[Frick, Ludwig, Mehldau '95]

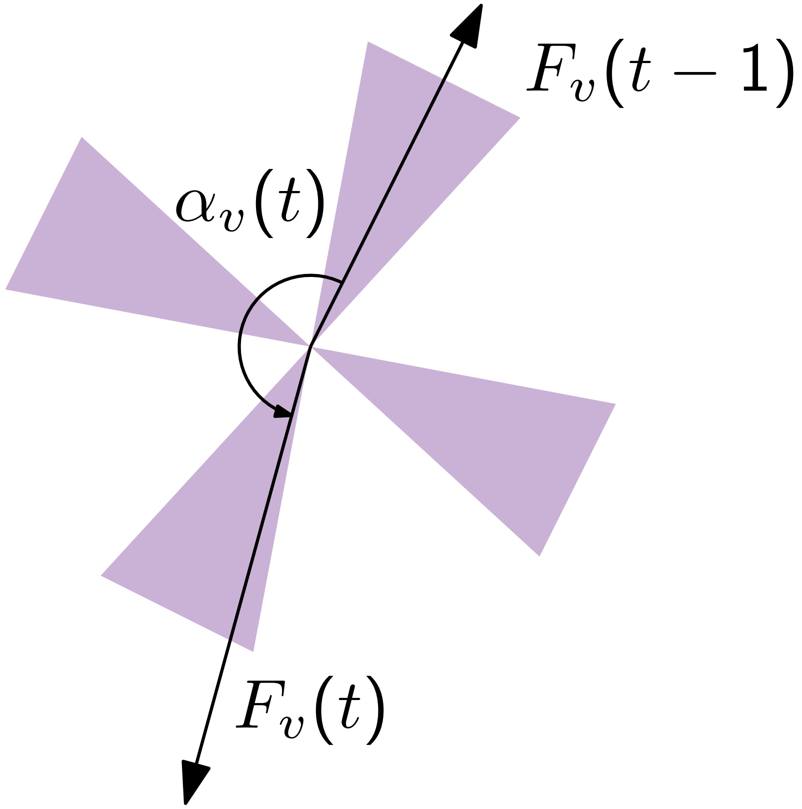


Same direction.

$\rightarrow$ increase temperature $\delta_v(t)$

Speeding up "Convergence" by Adaptive Displacement $\delta_v(t)$

[Frick, Ludwig, Mehldau '95]

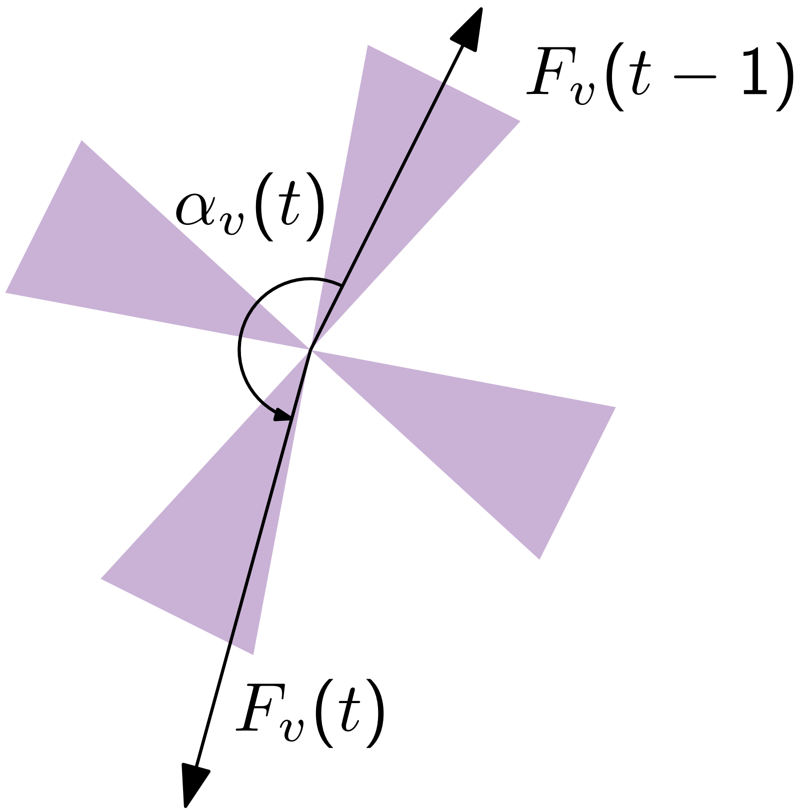


Same direction.

$\rightarrow$ increase temperature $\delta_v(t)$

Speeding up "Convergence" by Adaptive Displacement $\delta_v(t)$

[Frick, Ludwig, Mehldau '95]



Same direction.

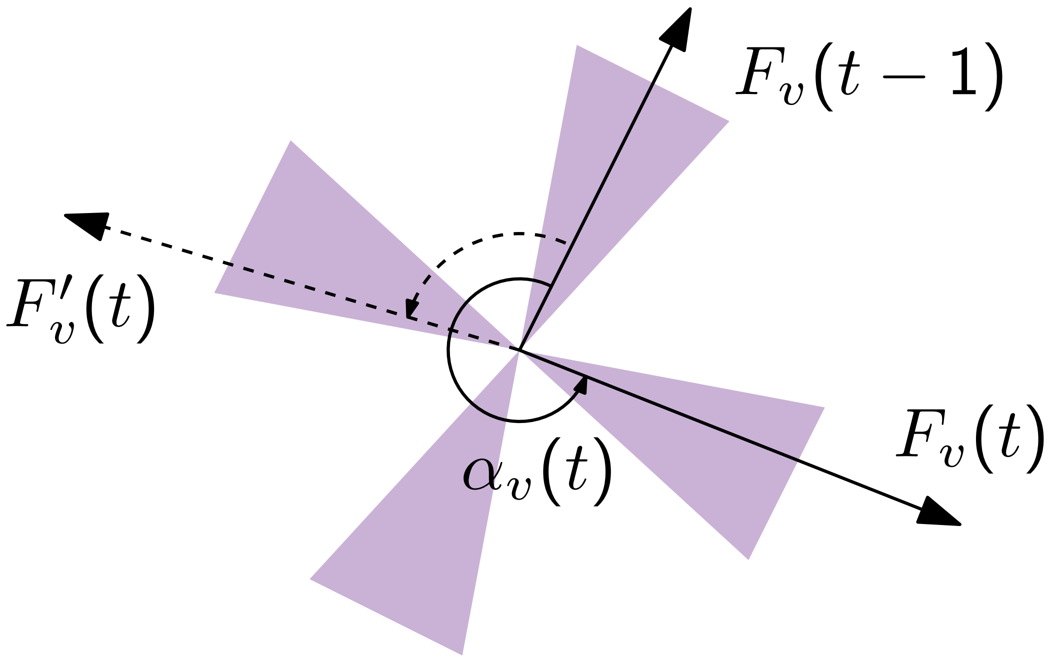
$\rightarrow$ increase temperature $\delta_v(t)$

Oscillation.

$\rightarrow$ decrease temperature $\delta_v(t)$

Speeding up "Convergence" by Adaptive Displacement $\delta_v(t)$

[Frick, Ludwig, Mehldau '95]

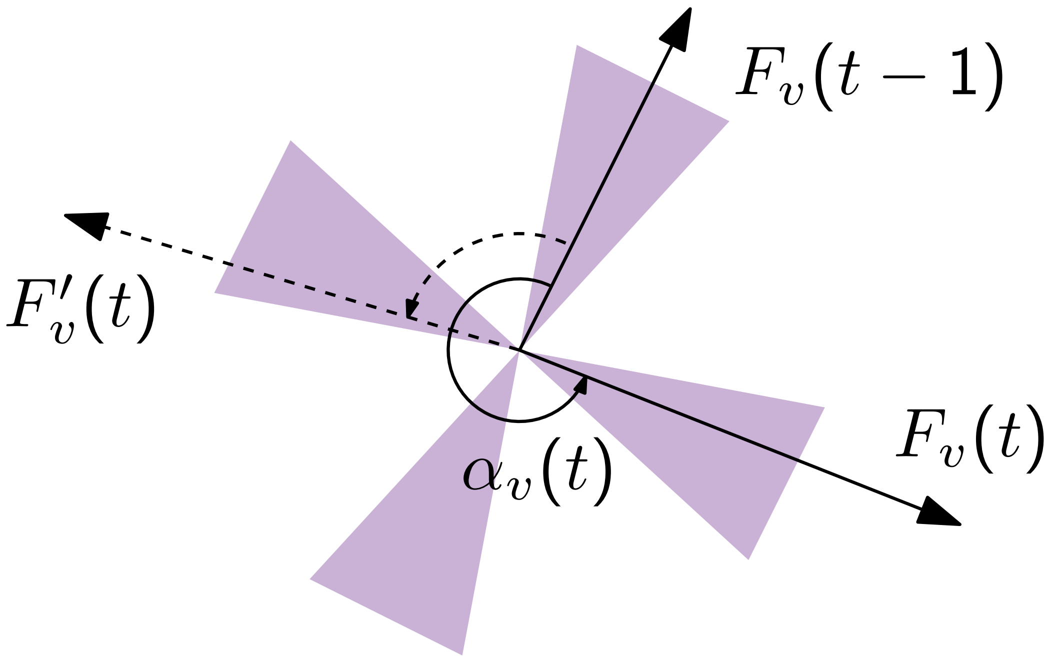


Same direction.
$\rightarrow$ increase temperature $\delta_v(t)$

Oscillation.
$\rightarrow$ decrease temperature $\delta_v(t)$

Speeding up "Convergence" by Adaptive Displacement $\delta_v(t)$

[Frick, Ludwig, Mehldau '95]



Same direction.
$\rightarrow$ increase temperature $\delta_v(t)$

Oscillation.
$\rightarrow$ decrease temperature $\delta_v(t)$

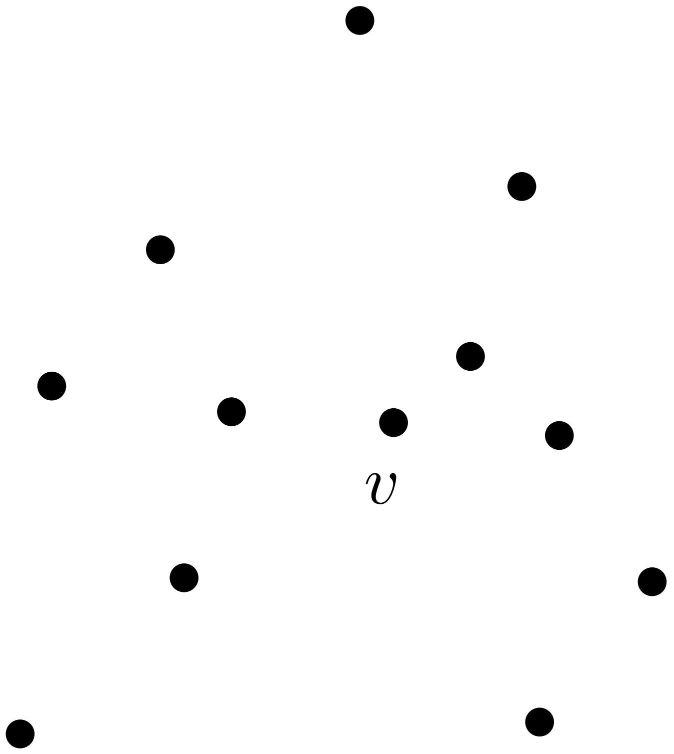
Rotation.

- count rotations
- if applicable

$\rightarrow$ decrease temperature $\delta_v(t)$

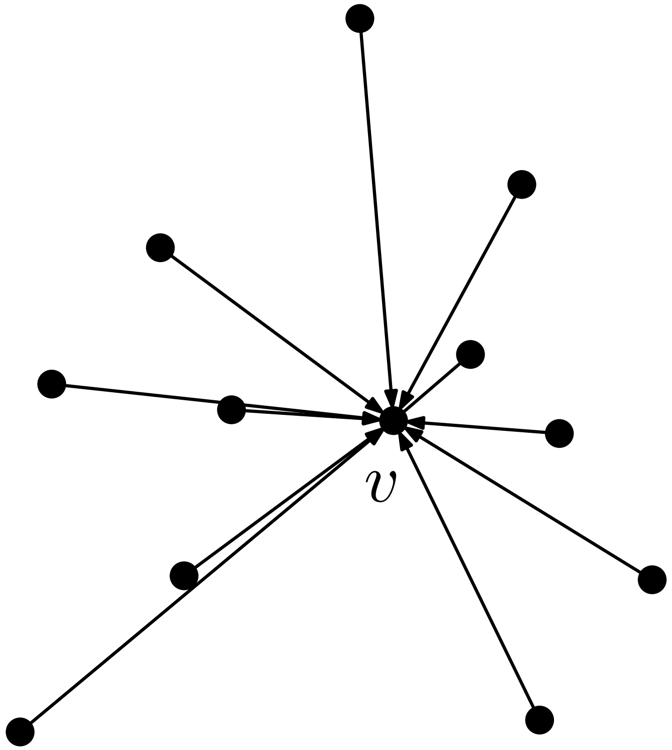
Speeding up “Convergence” via Grids

[Fruchterman & Reingold ’91]



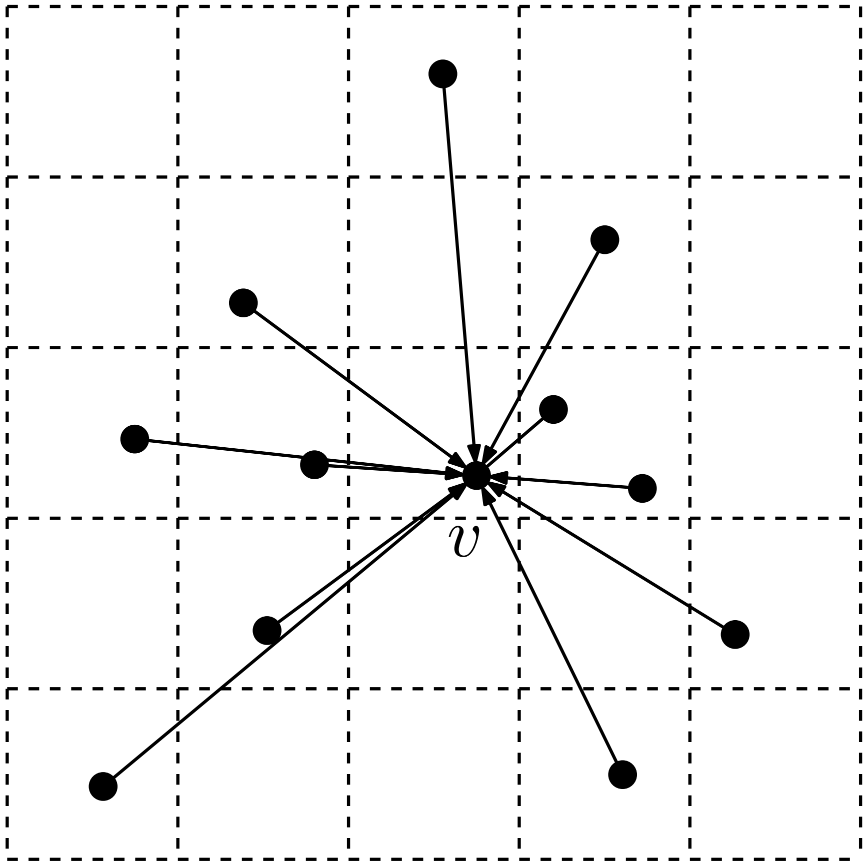
Speeding up “Convergence” via Grids

[Fruchterman & Reingold ’91]



Speeding up "Convergence" via Grids

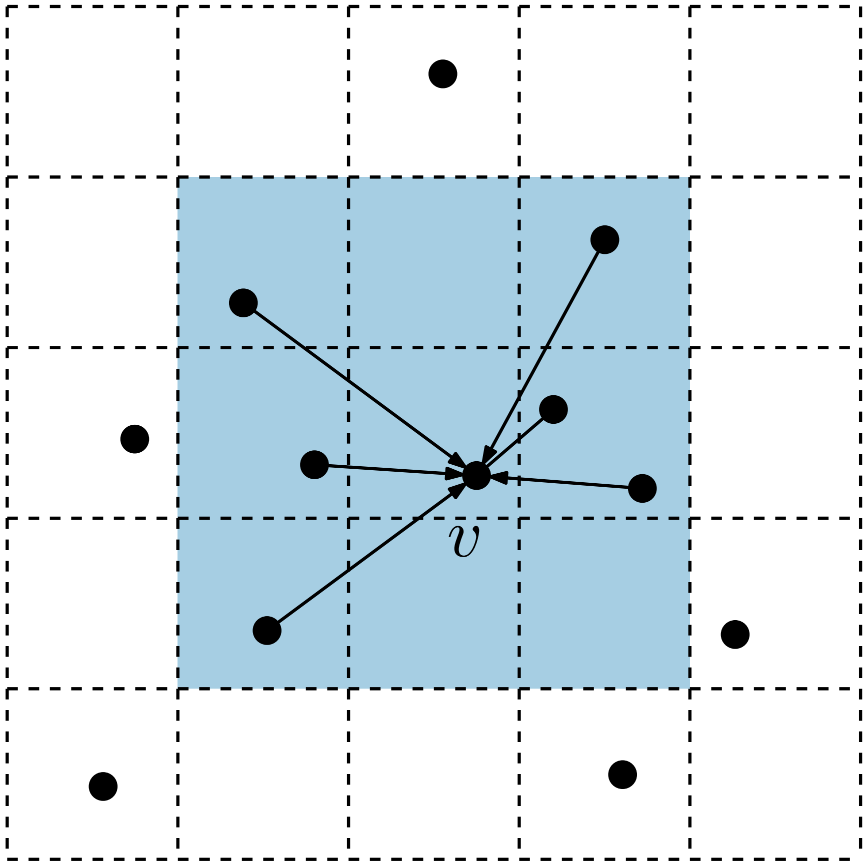
[Fruchterman & Reingold '91]



- divide plane into a grid

Speeding up "Convergence" via Grids

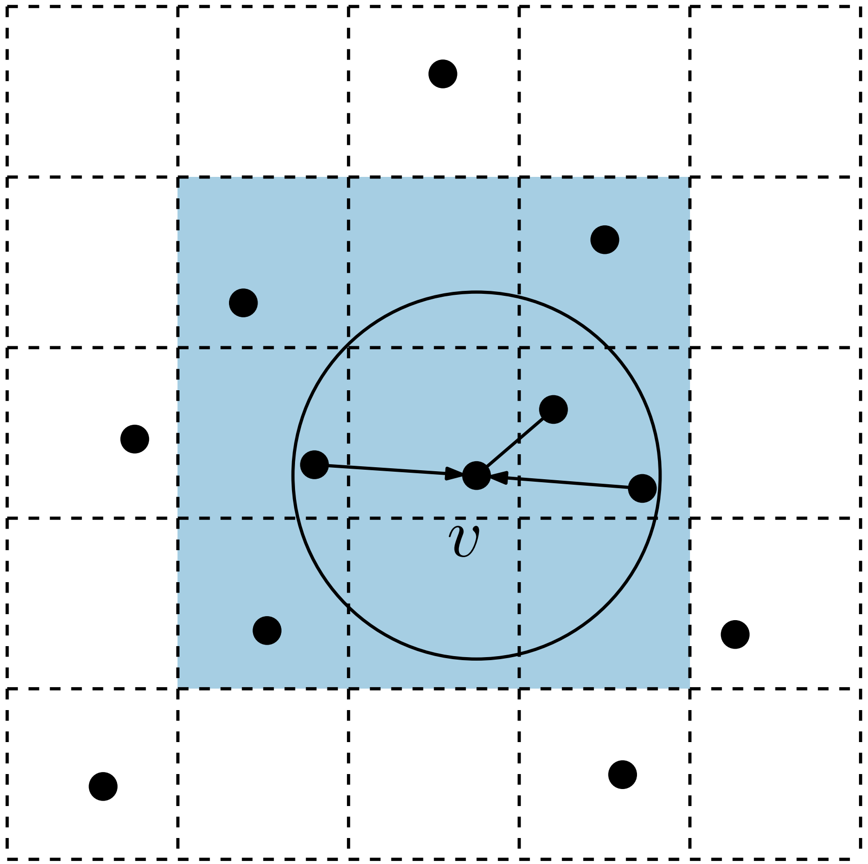
[Fruchterman & Reingold '91]



- divide plane into a grid
- consider repulsive forces only to vertices in neighboring cells

Speeding up "Convergence" via Grids

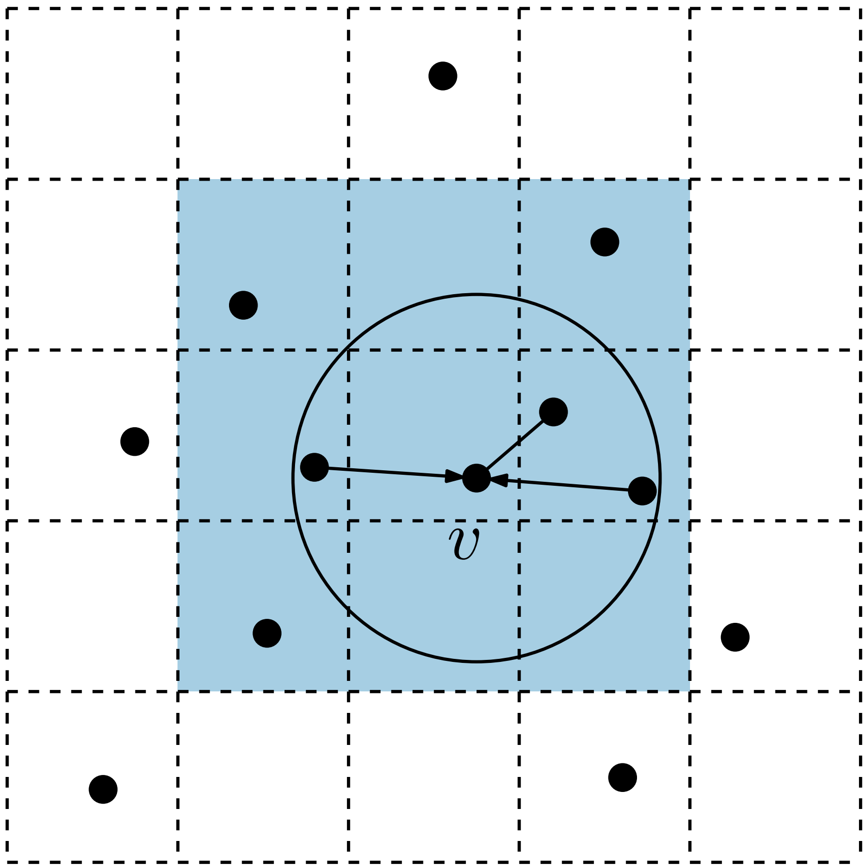
[Fruchterman & Reingold '91]



- divide plane into a grid
- consider repulsive forces only to vertices in neighboring cells
- and only if the distance is less than some threshold

Speeding up “Convergence” via Grids

[Fruchterman & Reingold ’91]



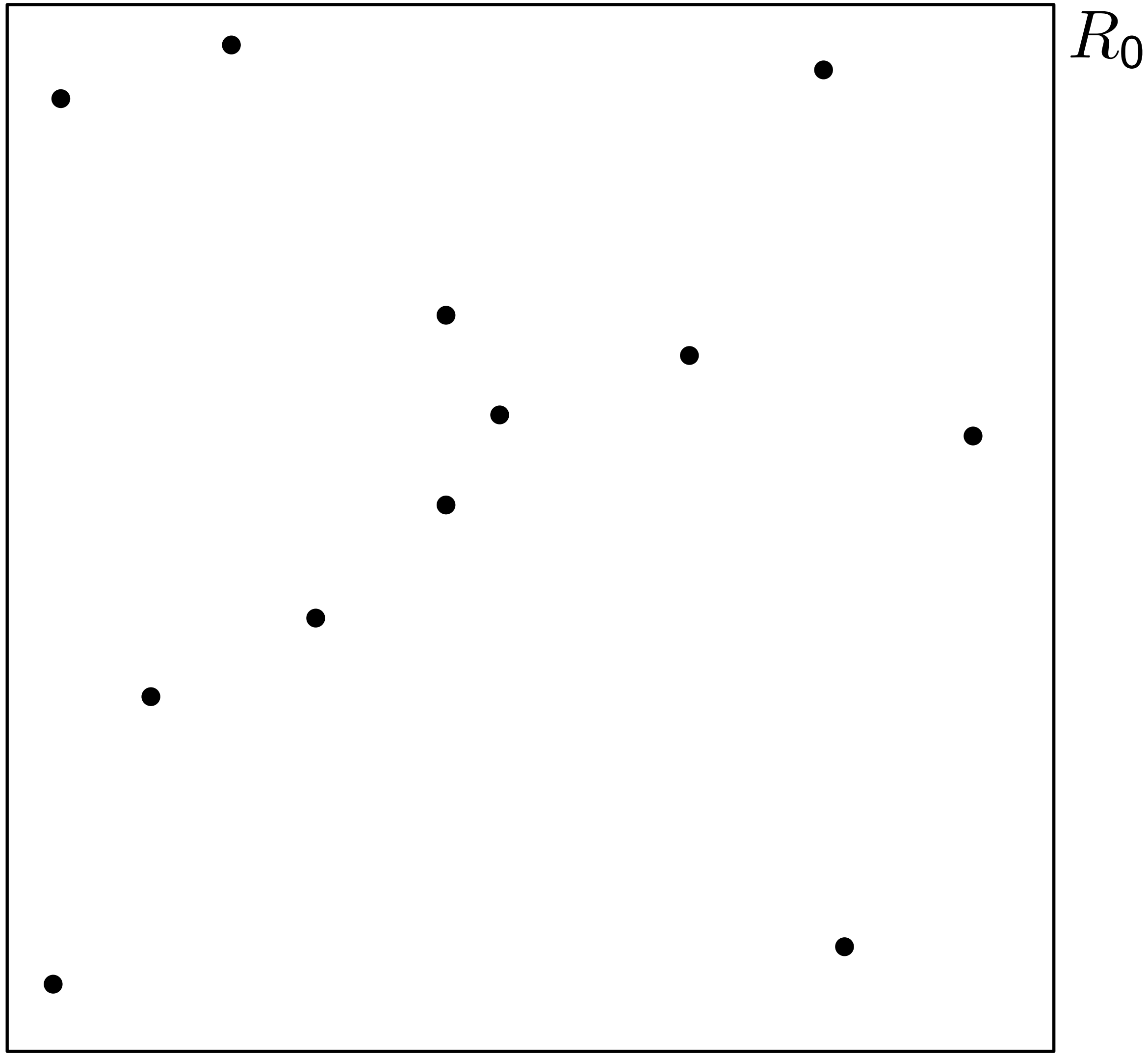
- divide plane into a grid
- consider repulsive forces only to vertices in neighboring cells
- and only if the distance is less than some threshold

Discussion.

- good idea to improve actual runtime
- asymptotic runtime does not improve
- might introduce oscillation and thus a quality loss

Speeding up Repulsive-Force Computation with Quad Trees

[Barnes, Hut '86]

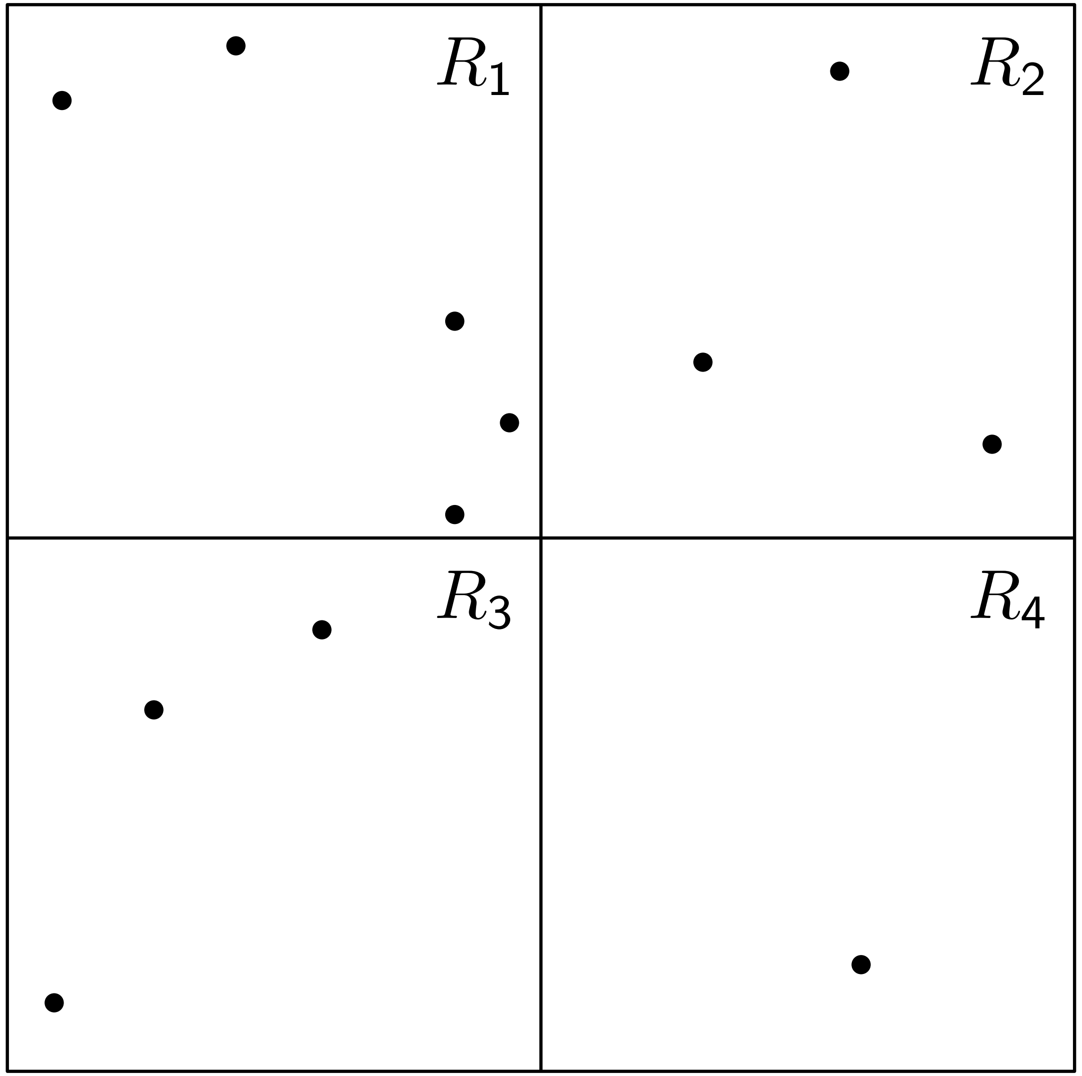


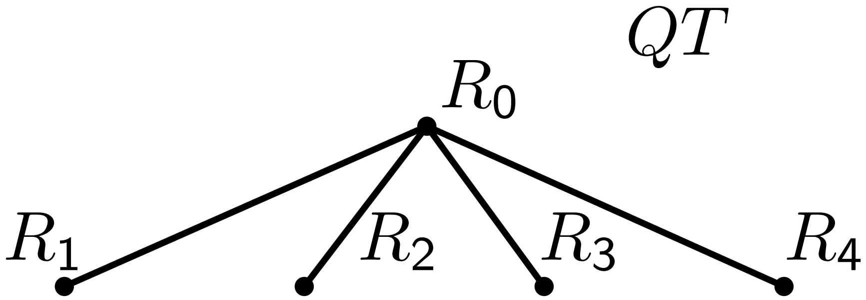
QT

R_0

Speeding up Repulsive-Force Computation with Quad Trees

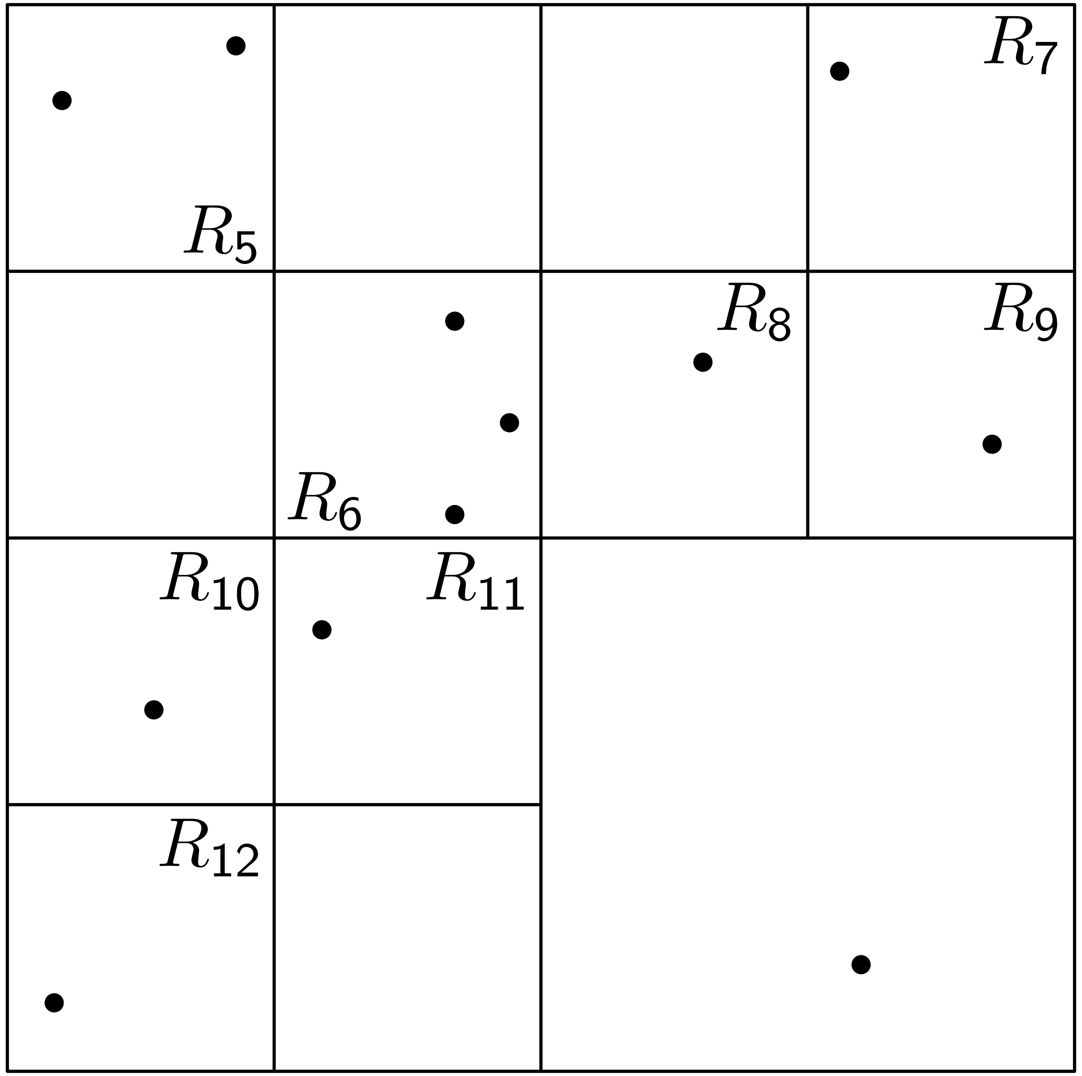
[Barnes, Hut ’86]

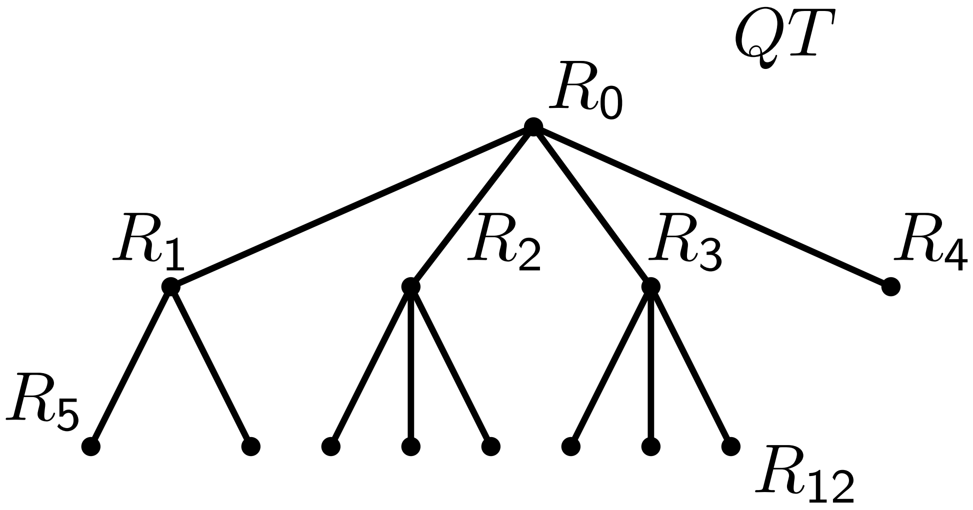




Speeding up Repulsive-Force Computation with Quad Trees

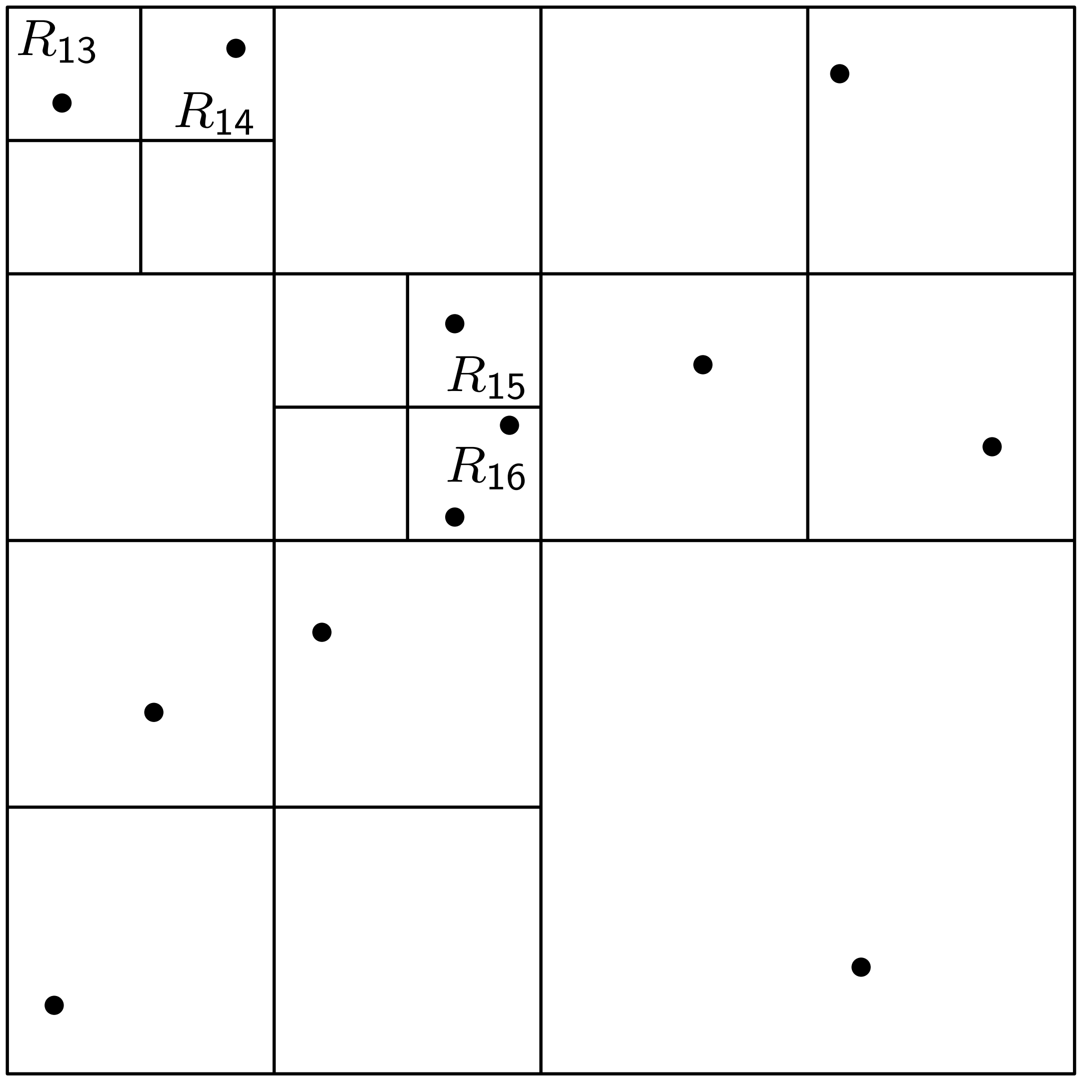
[Barnes, Hut '86]

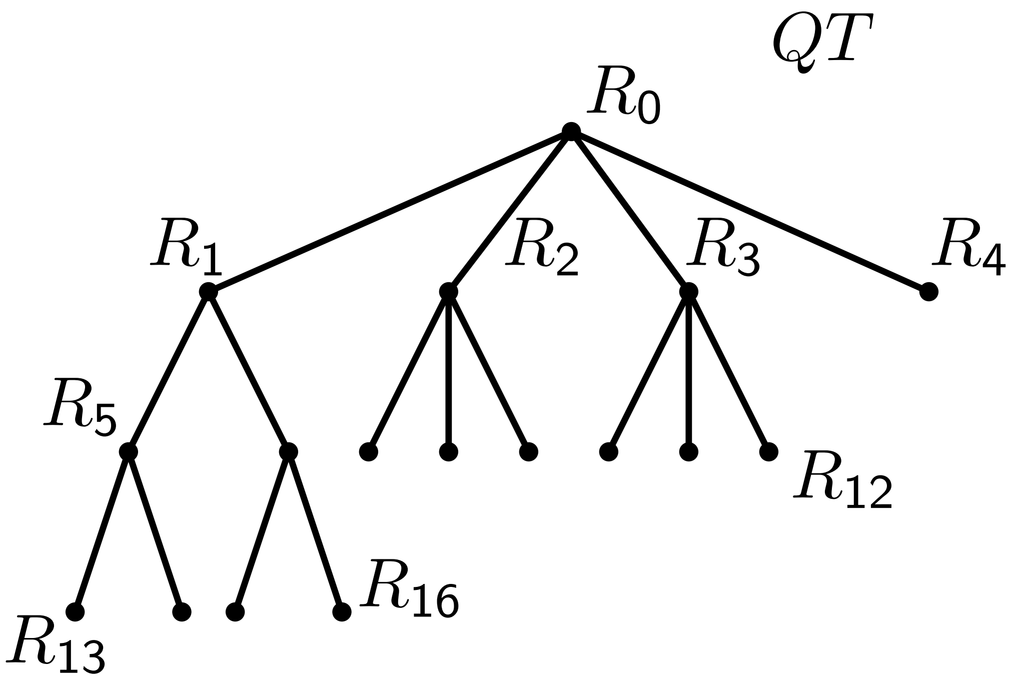




Speeding up Repulsive-Force Computation with Quad Trees

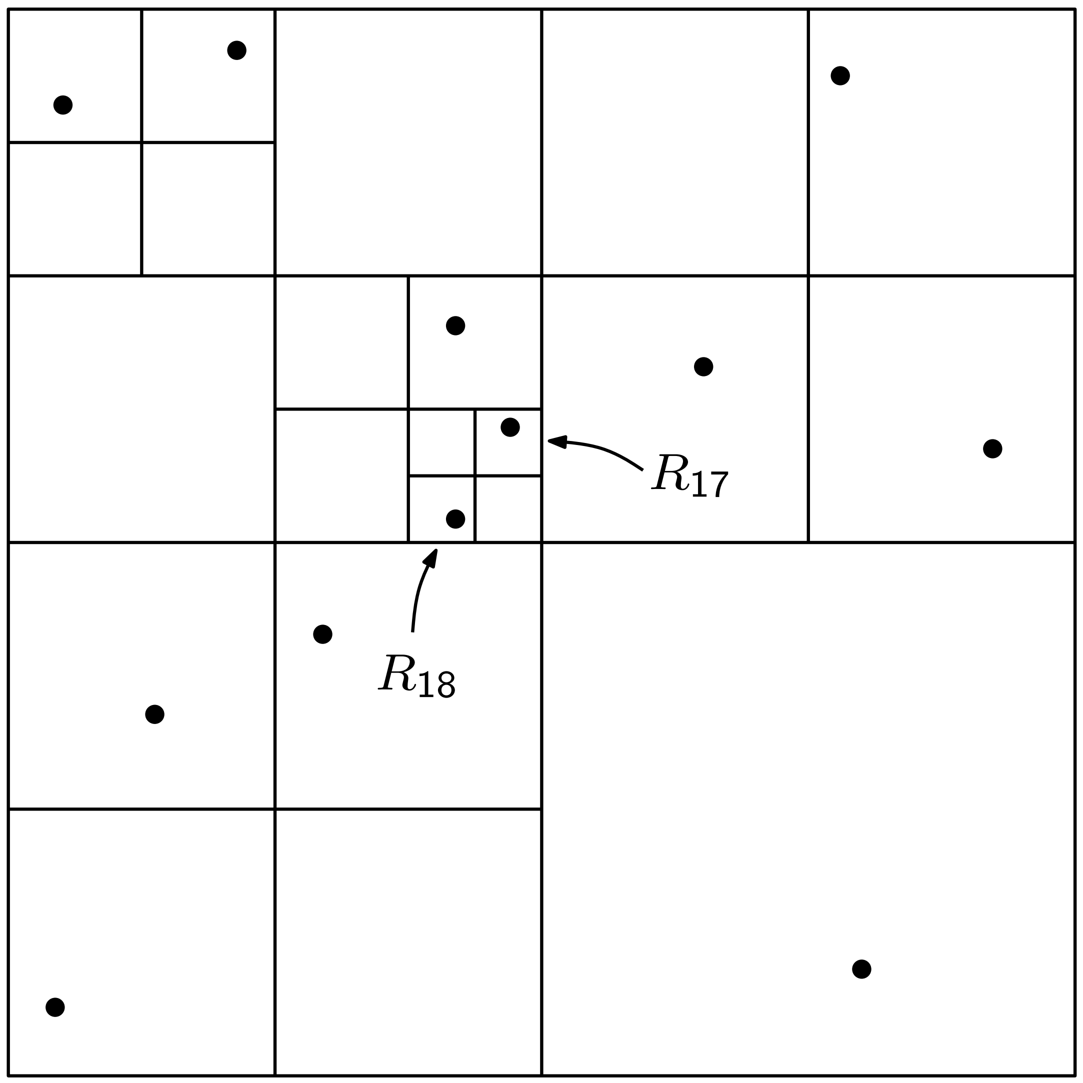
[Barnes, Hut '86]

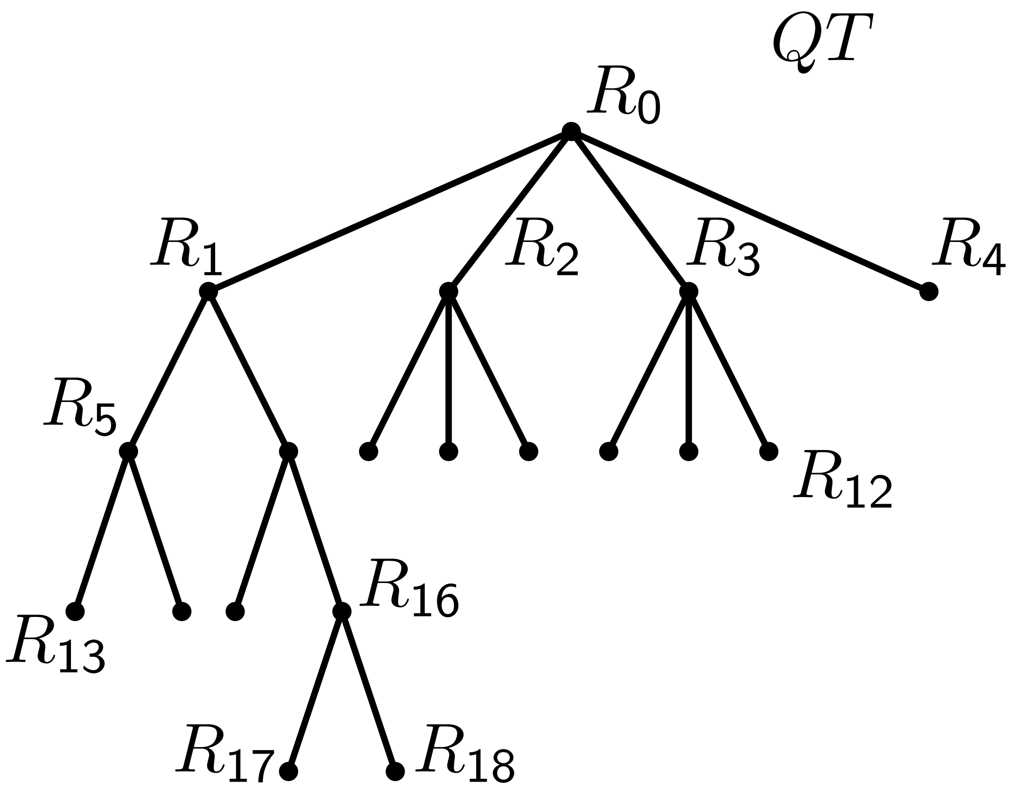




Speeding up Repulsive-Force Computation with Quad Trees

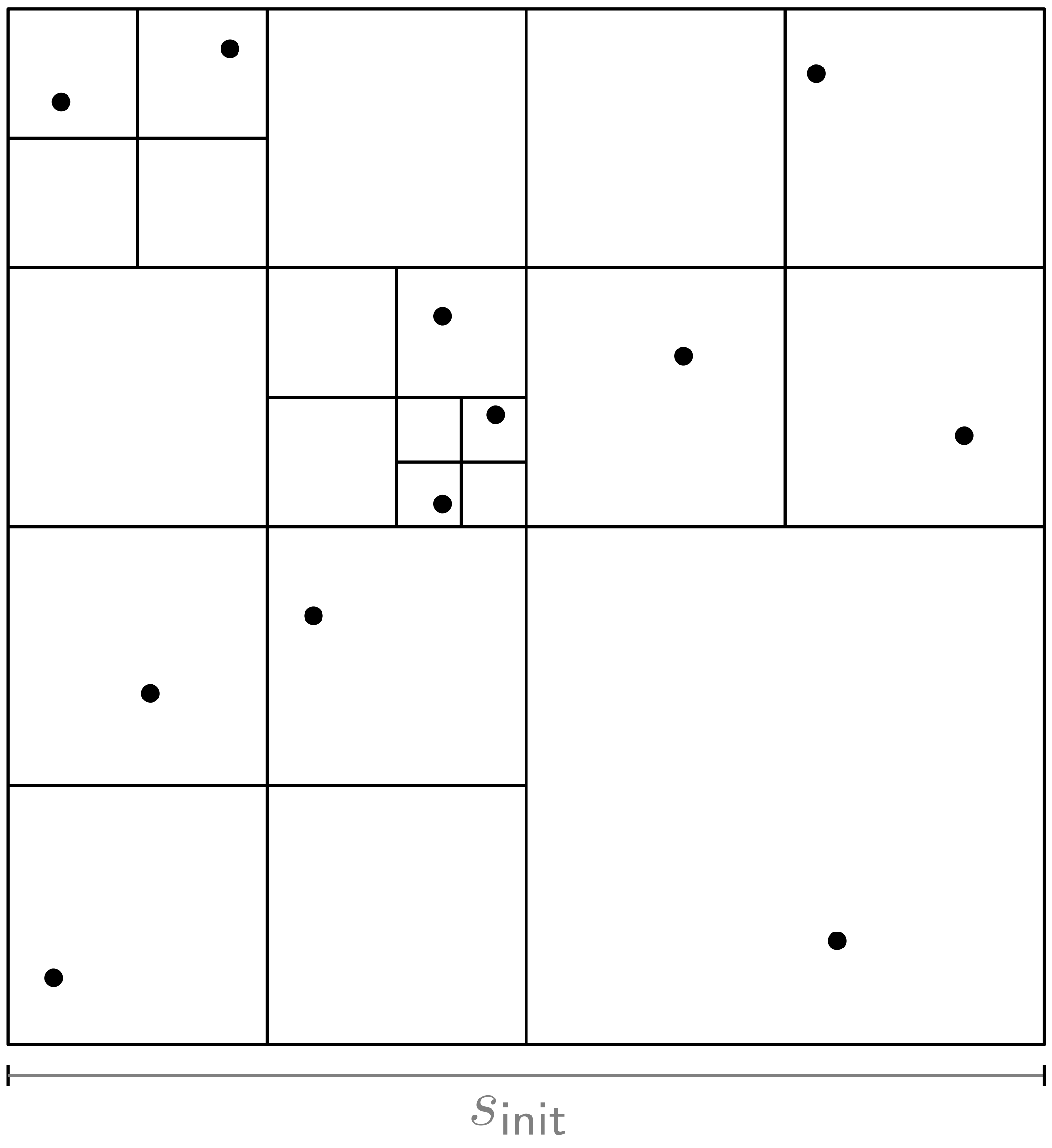
[Barnes, Hut '86]

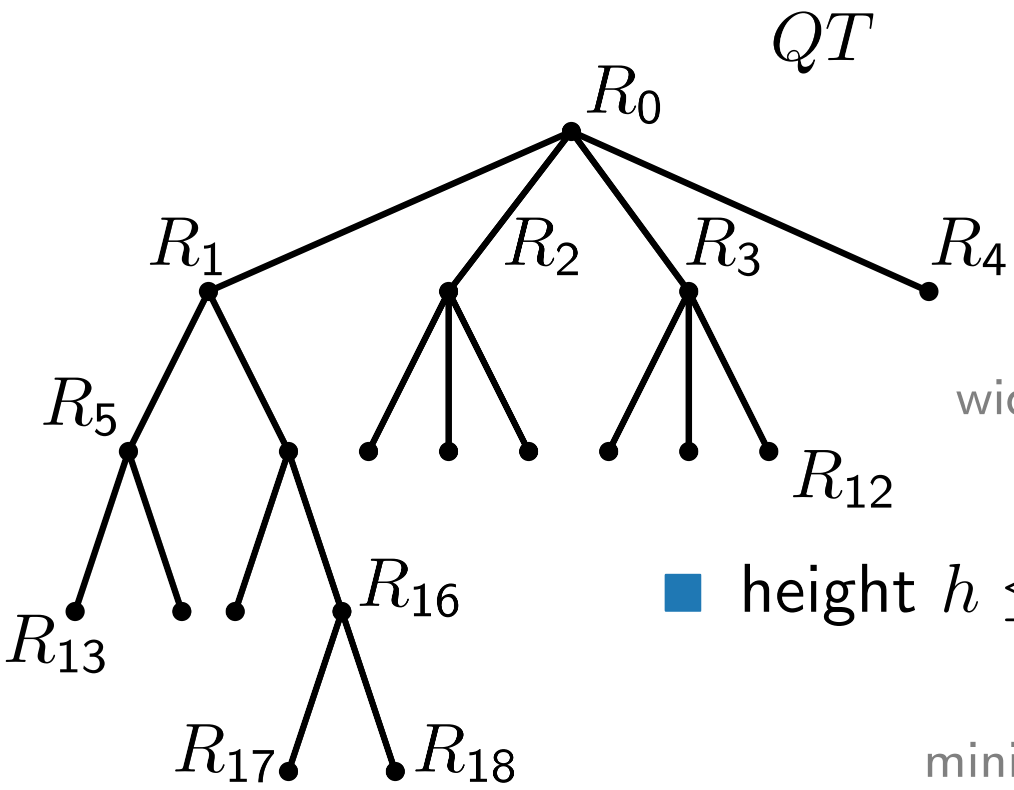




Speeding up Repulsive-Force Computation with Quad Trees

[Barnes, Hut '86]





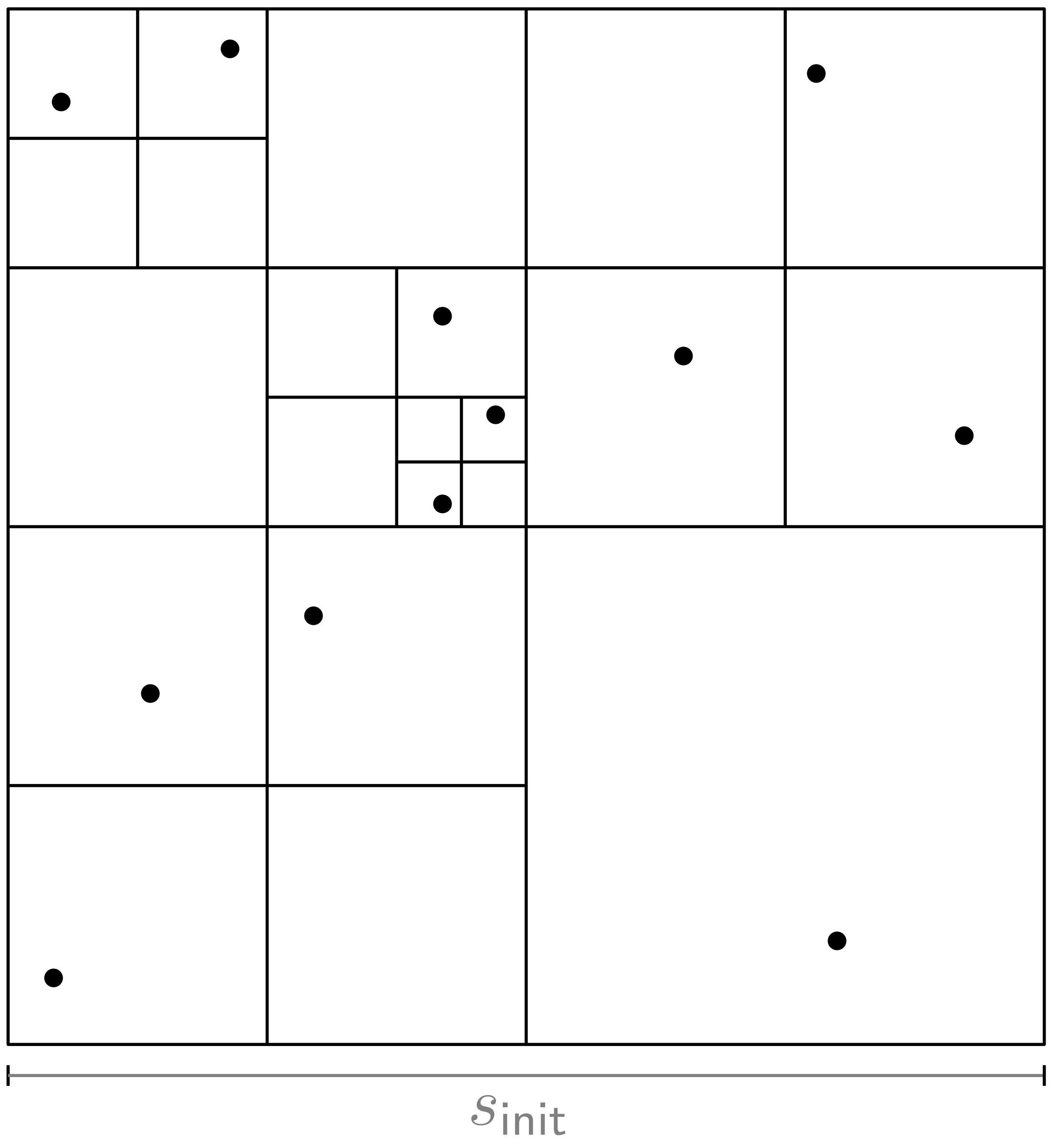
width/height of outer square

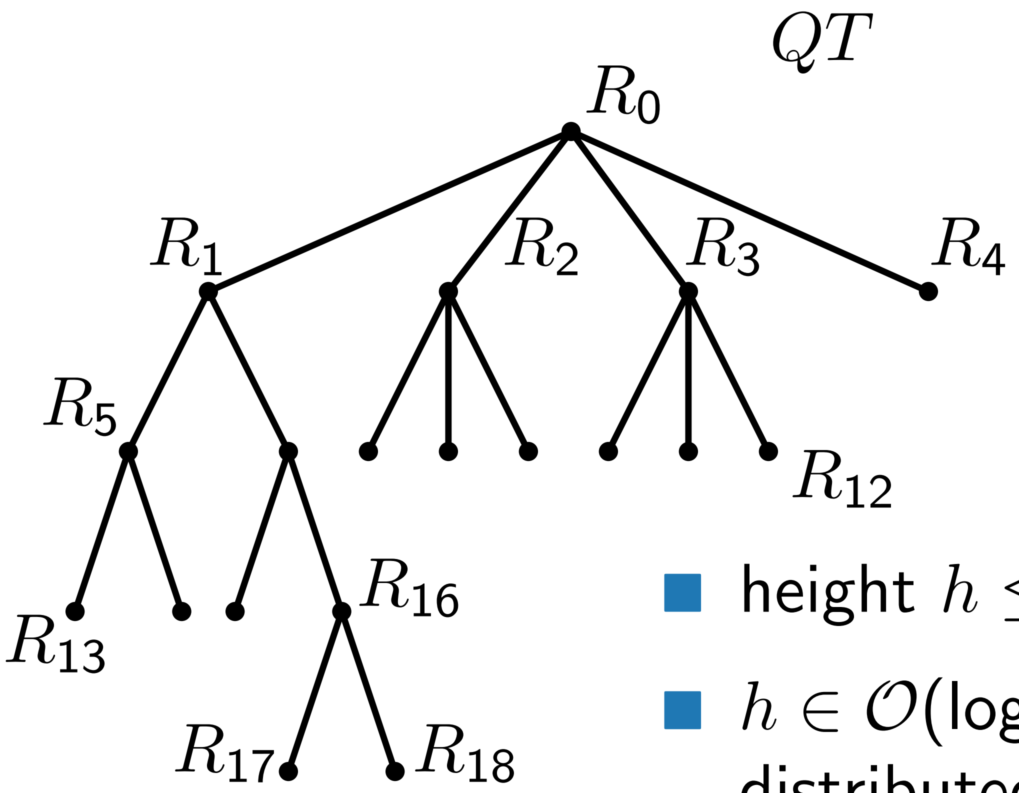
- height $h \leq \log_2\left(\frac{s_{\text{init}}}{d_{\min}}\right) + \frac{3}{2}$

minimum distance btw. two pts.

Speeding up Repulsive-Force Computation with Quad Trees

[Barnes, Hut '86]

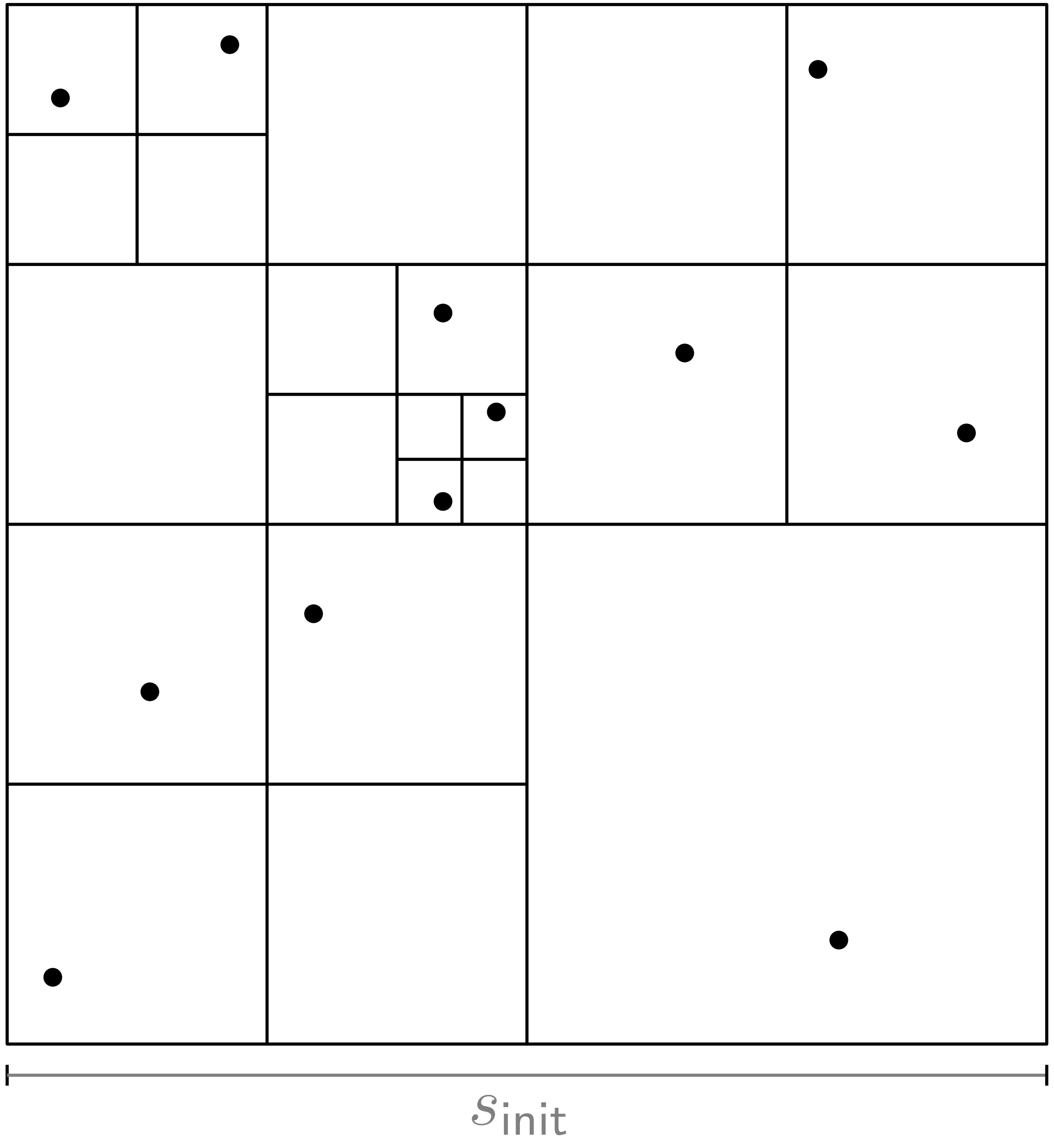


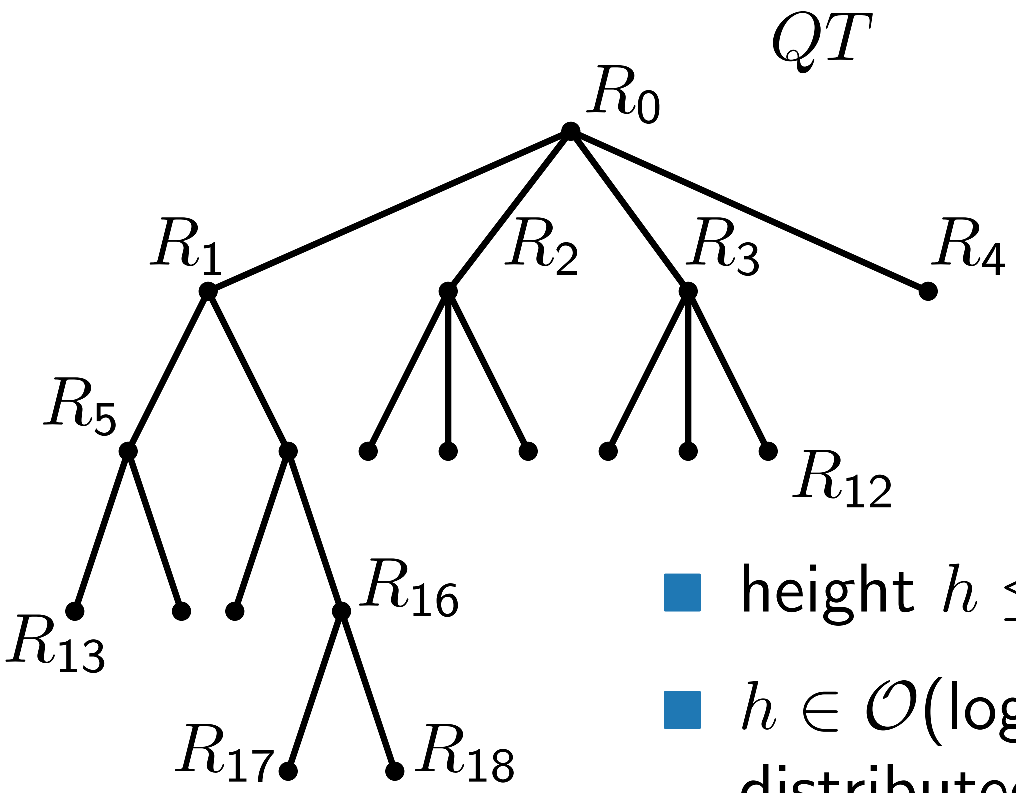


- height $h \leq \log_2\left(\frac{s_{\text{init}}}{d_{\min}}\right) + \frac{3}{2}$
- $h \in \mathcal{O}(\log n)$ if vertices evenly distributed in the initial box

Speeding up Repulsive-Force Computation with Quad Trees

[Barnes, Hut '86]

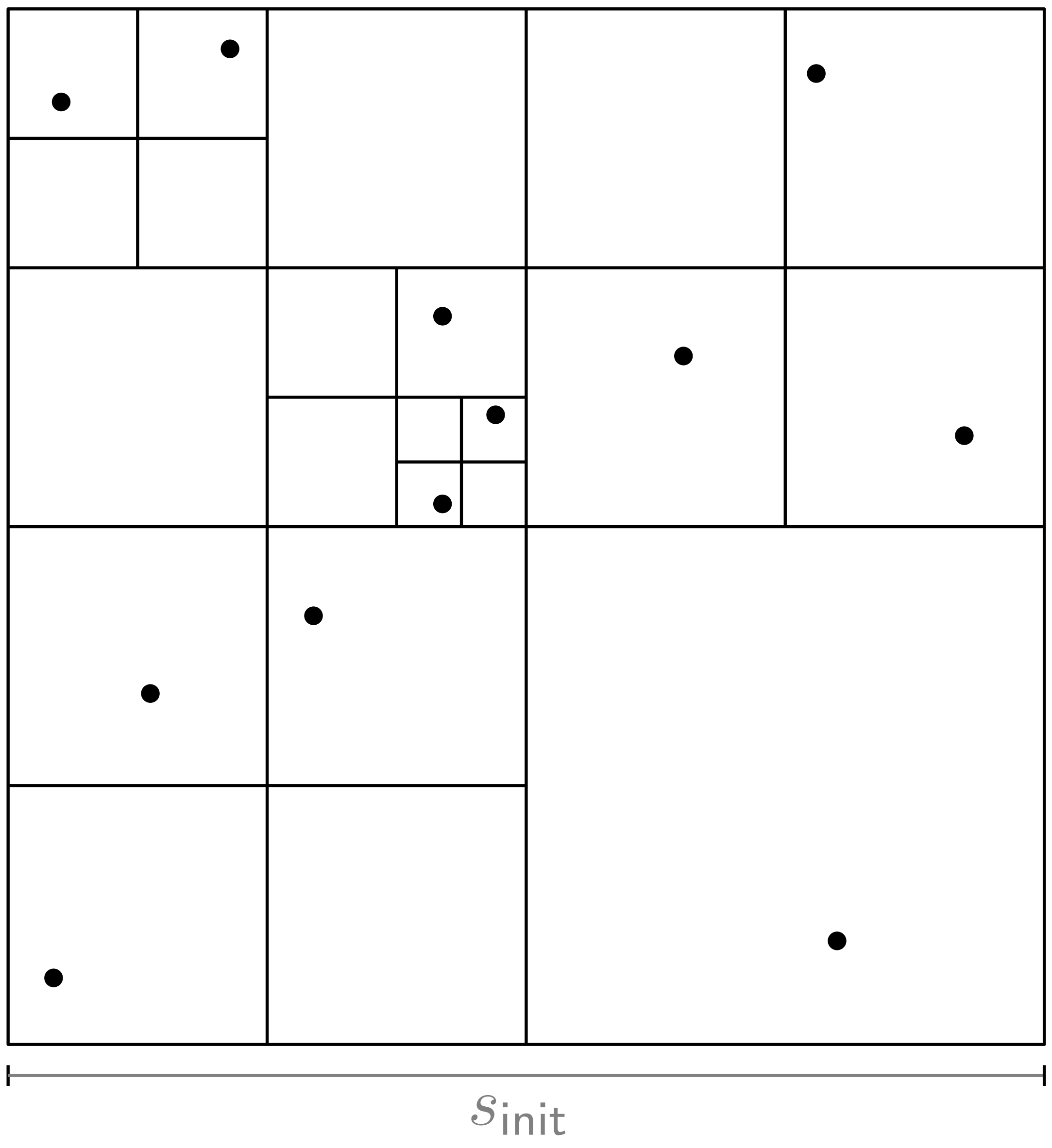


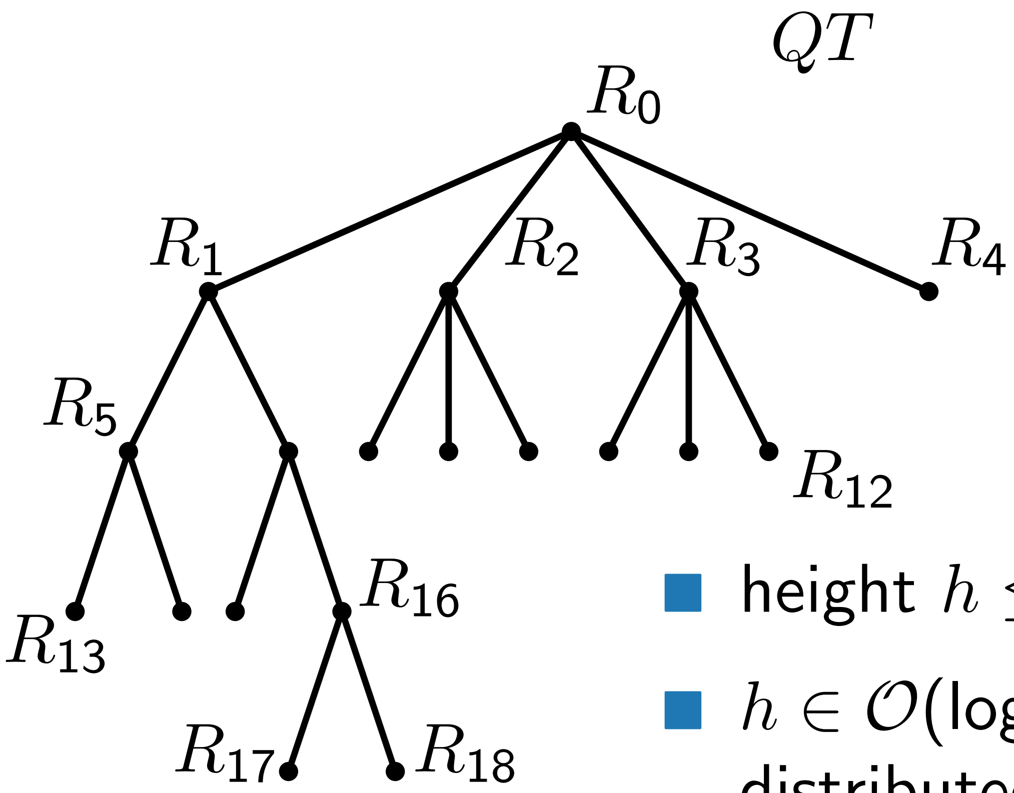


- height $h \le \log_2\left(\frac{s_{\text{init}}}{d_{\text{min}}}\right) + \frac{3}{2}$
- $h \in \mathcal{O}(\log n)$ if vertices evenly distributed in the initial box
- time/space in $\mathcal{O}(hn)$

Speeding up Repulsive-Force Computation with Quad Trees

[Barnes, Hut '86]

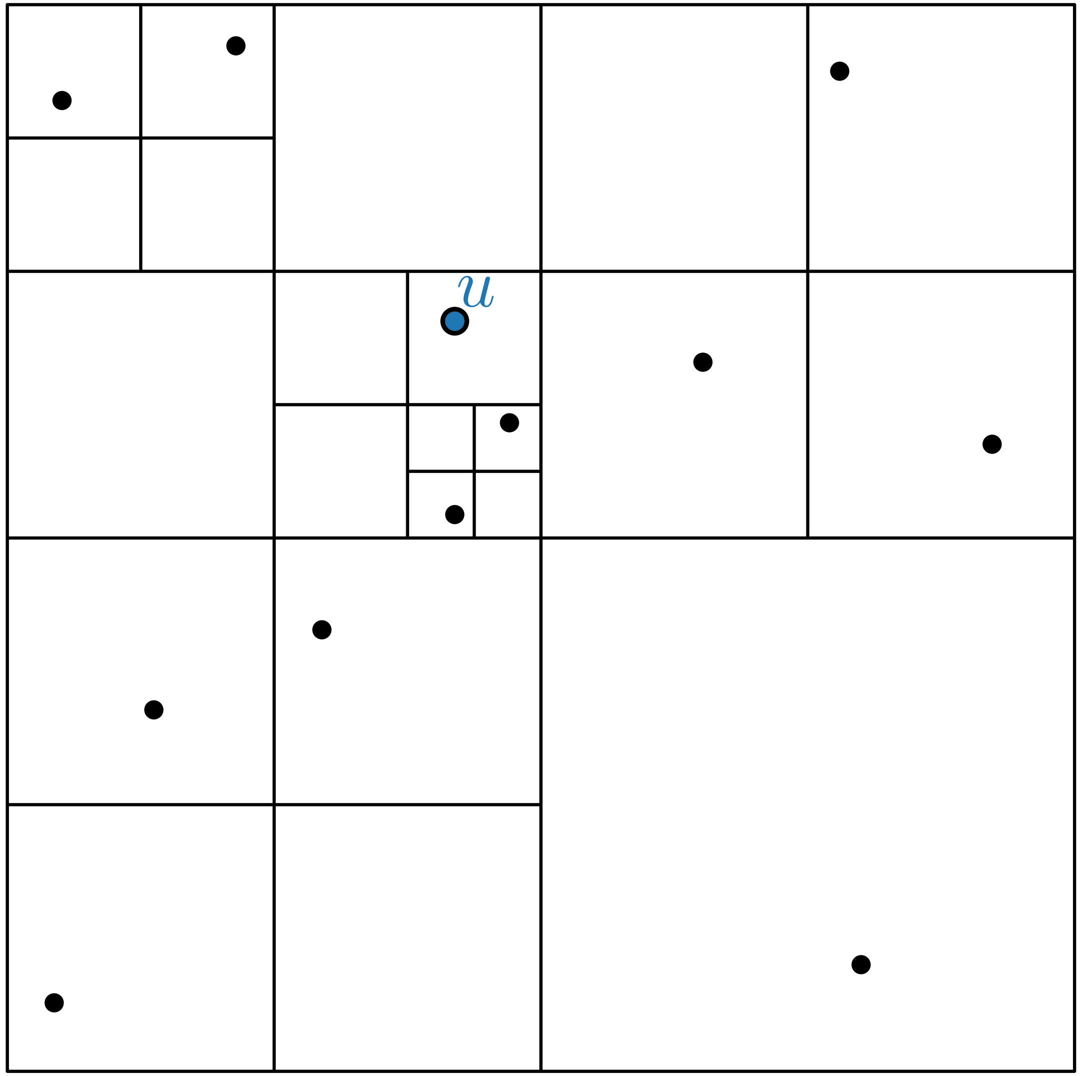


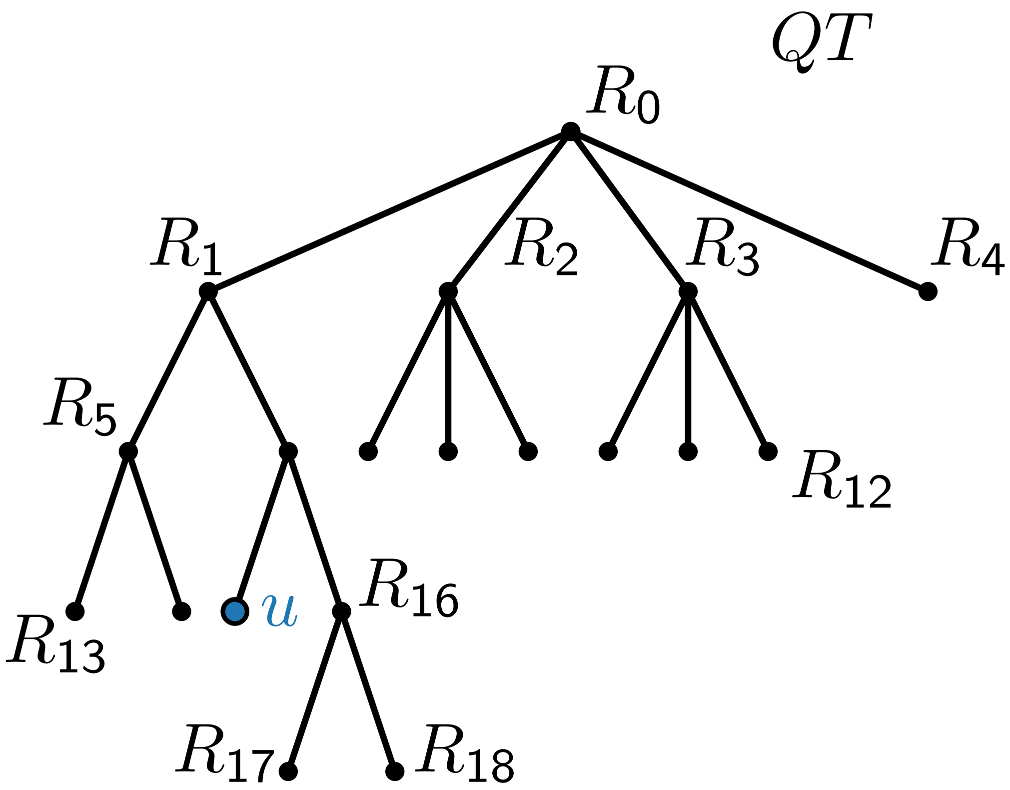


- height $h \leq \log_2\left(\frac{s_{\text{init}}}{d_{\min}}\right) + \frac{3}{2}$
- $h \in \mathcal{O}(\log n)$ if vertices evenly distributed in the initial box
- time/space in $\mathcal{O}(hn)$
- compressed quad tree can be computed in $\mathcal{O}(n \log n)$ time

Speeding up Repulsive-Force Computation with Quad Trees

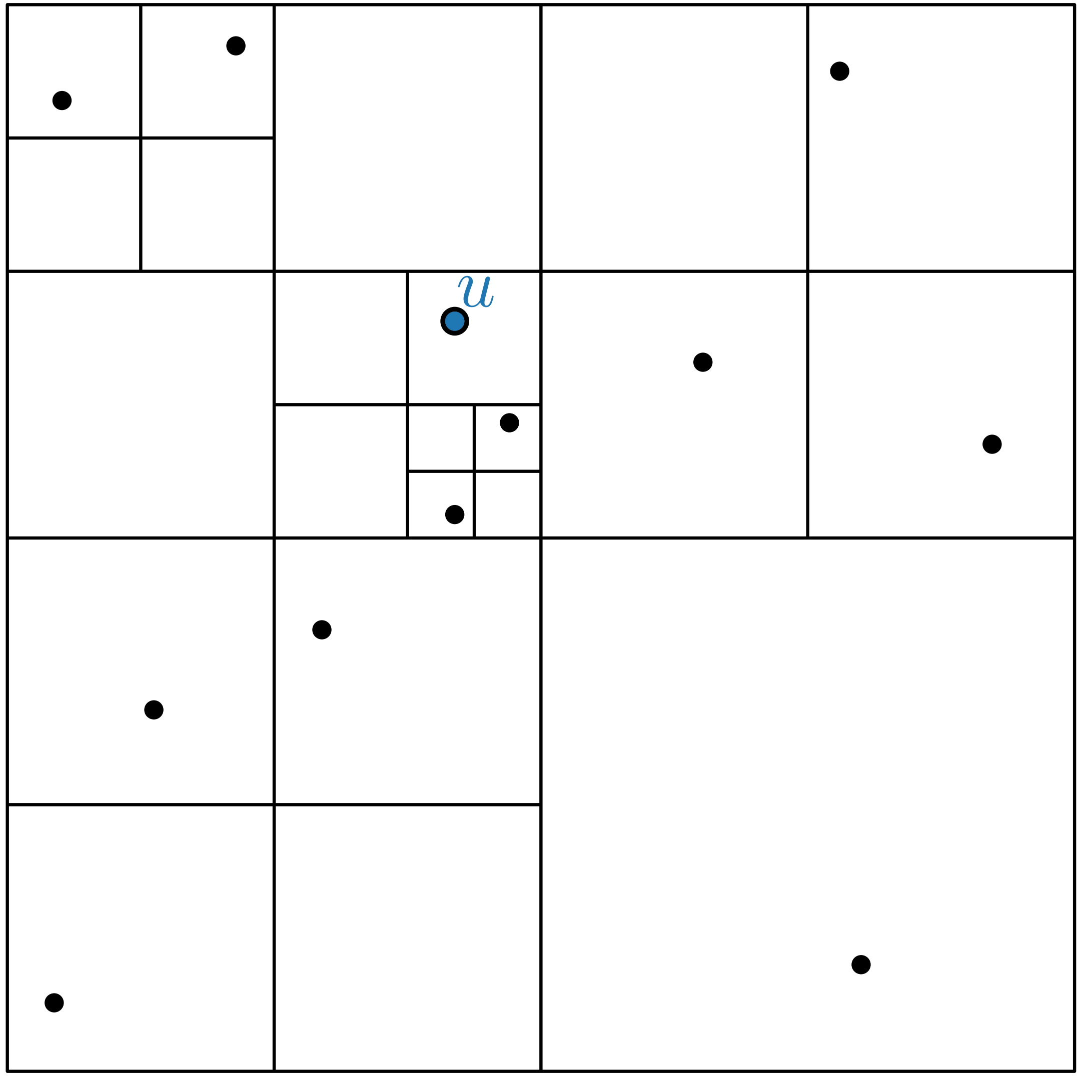
[Barnes, Hut '86]

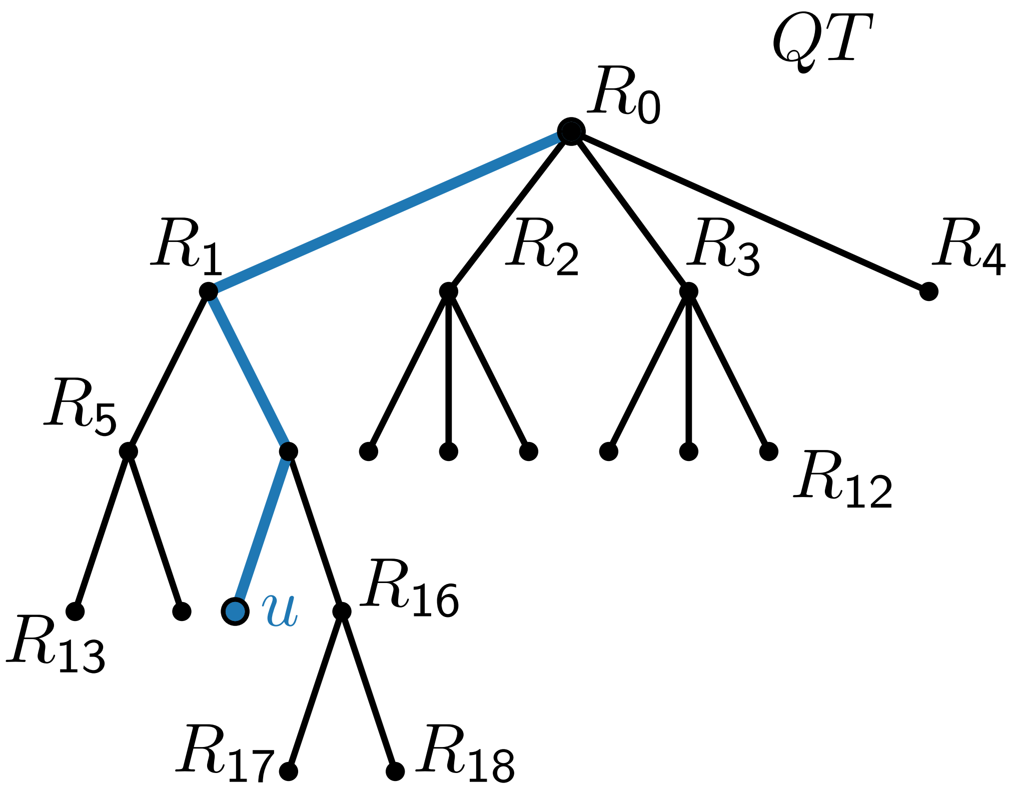




Speeding up Repulsive-Force Computation with Quad Trees

[Barnes, Hut '86]





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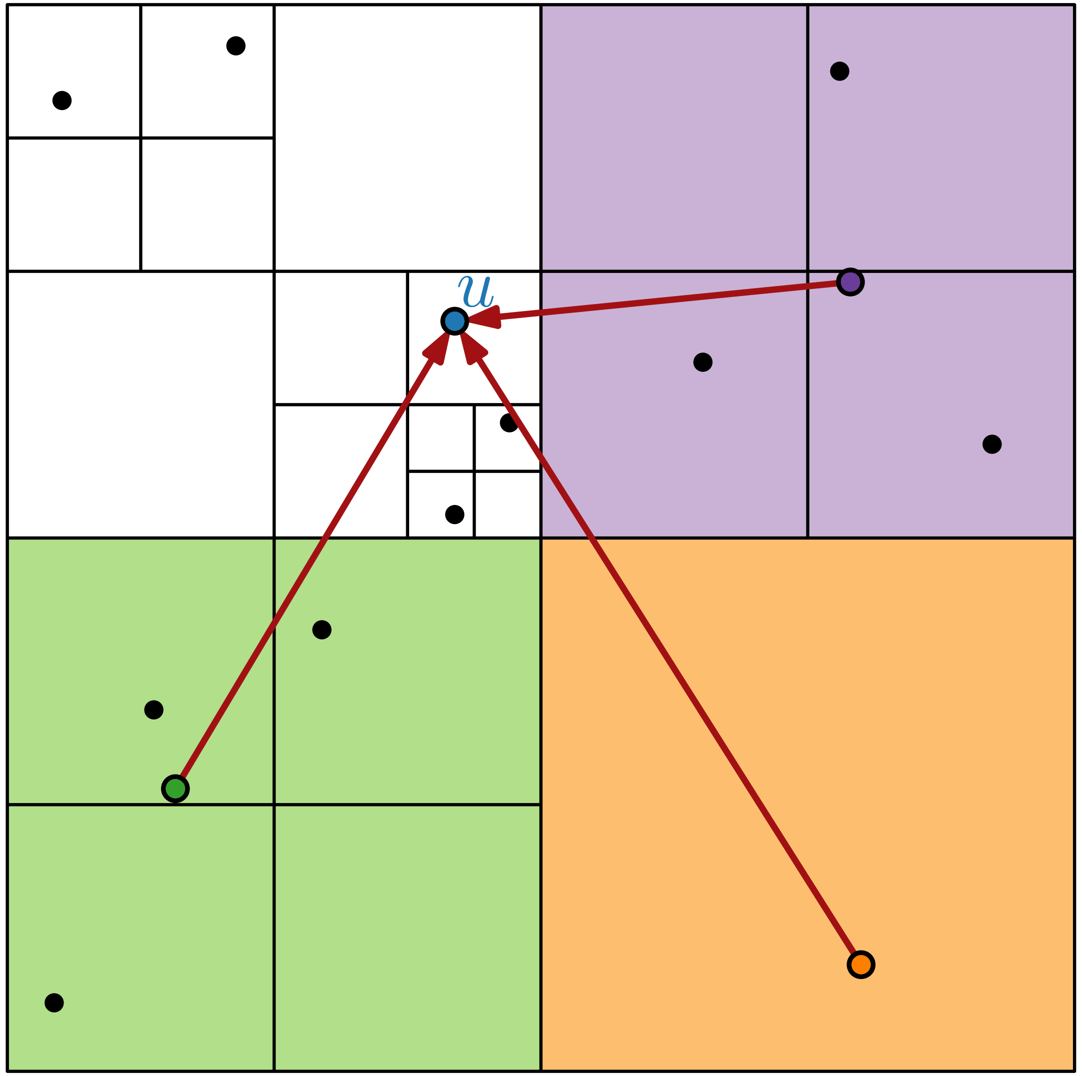

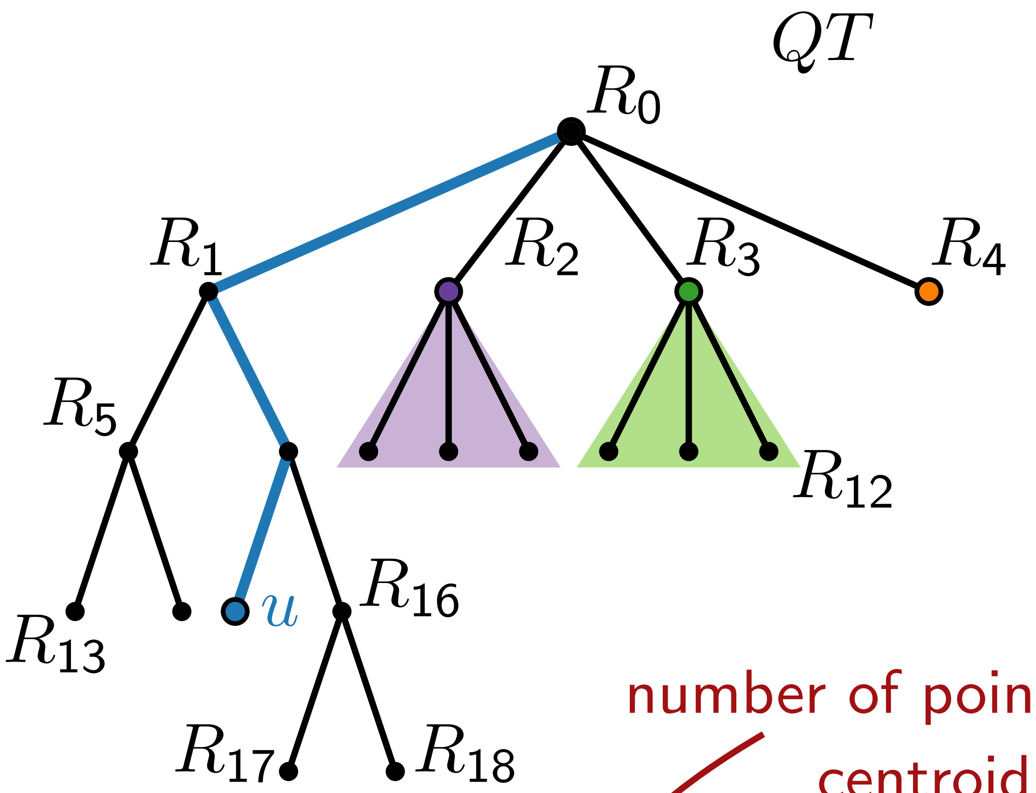

number of points in the subtree R_i

centroid of R_i (pre-computed)

$$f_{\text{rep}}(R_i, p_u) = \boxed{|R_i|} \cdot f_{\text{rep}}(\boxed{\sigma_{R_i}}, p_u)$$

for each child R_i of a vertex on path from root to u.

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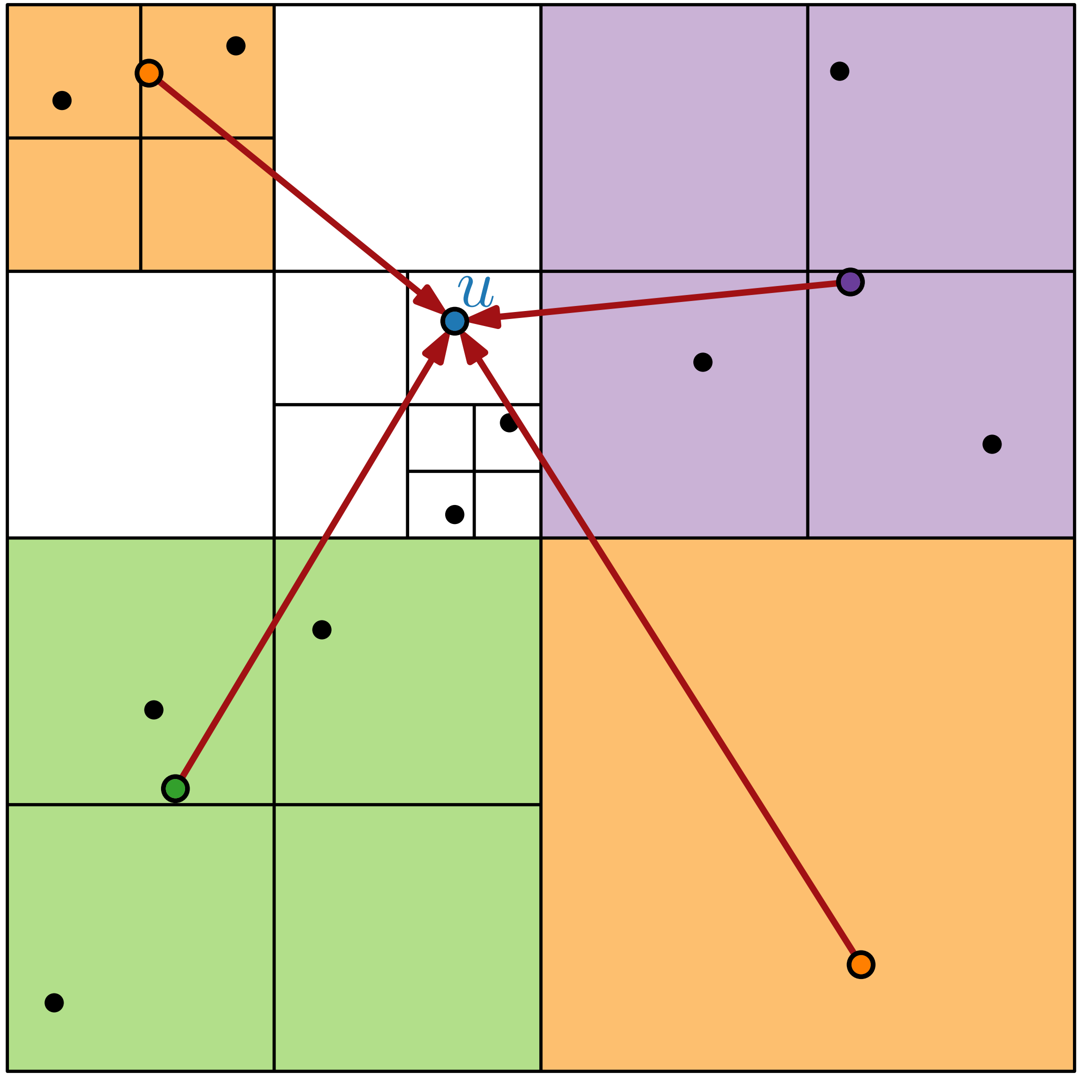

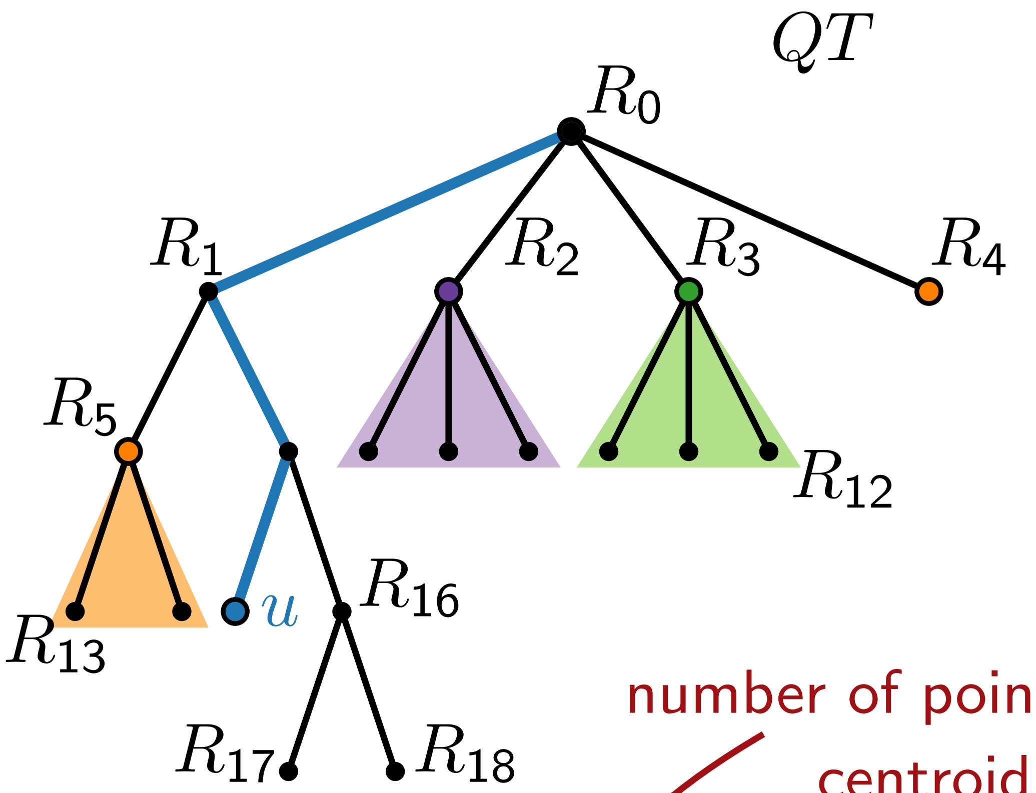

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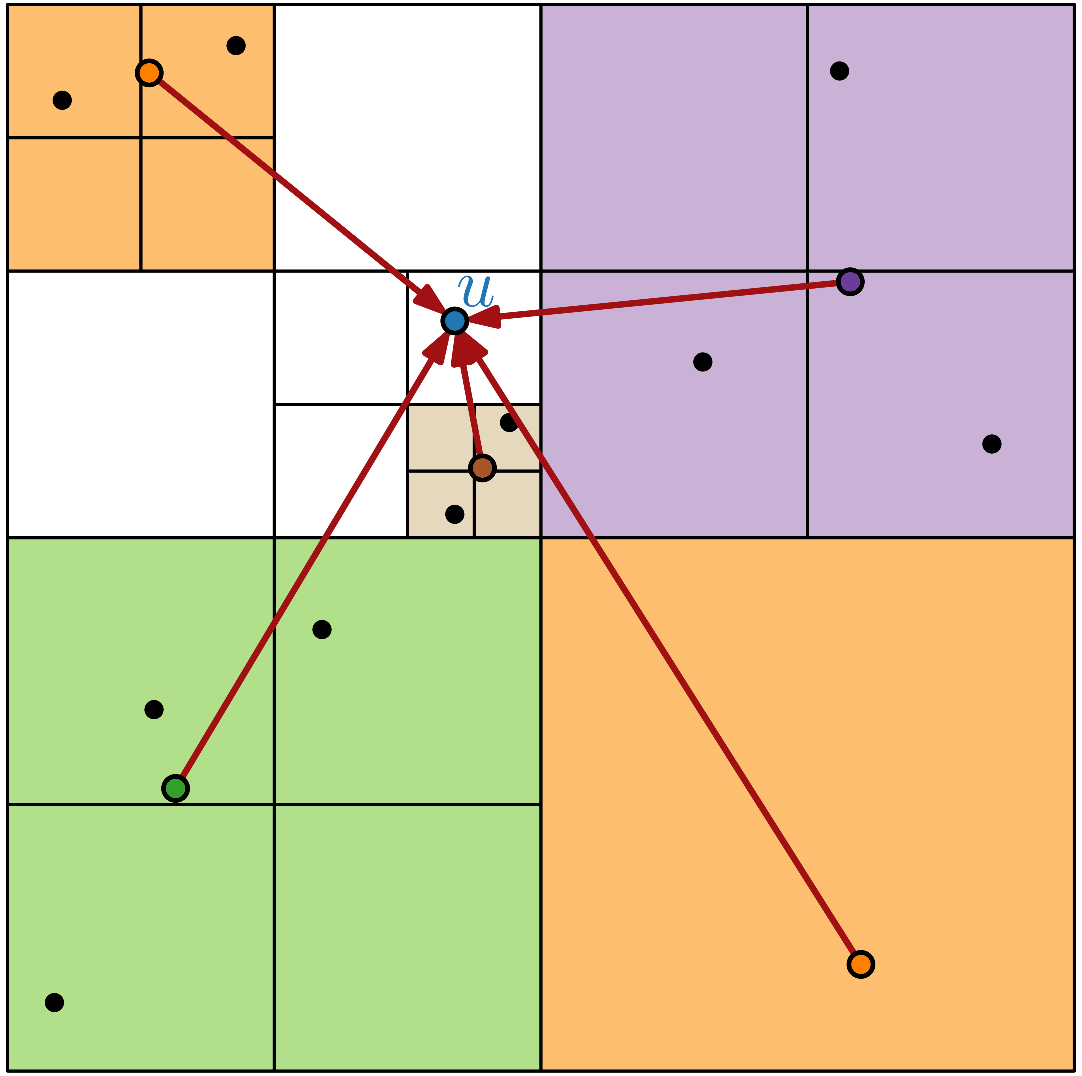

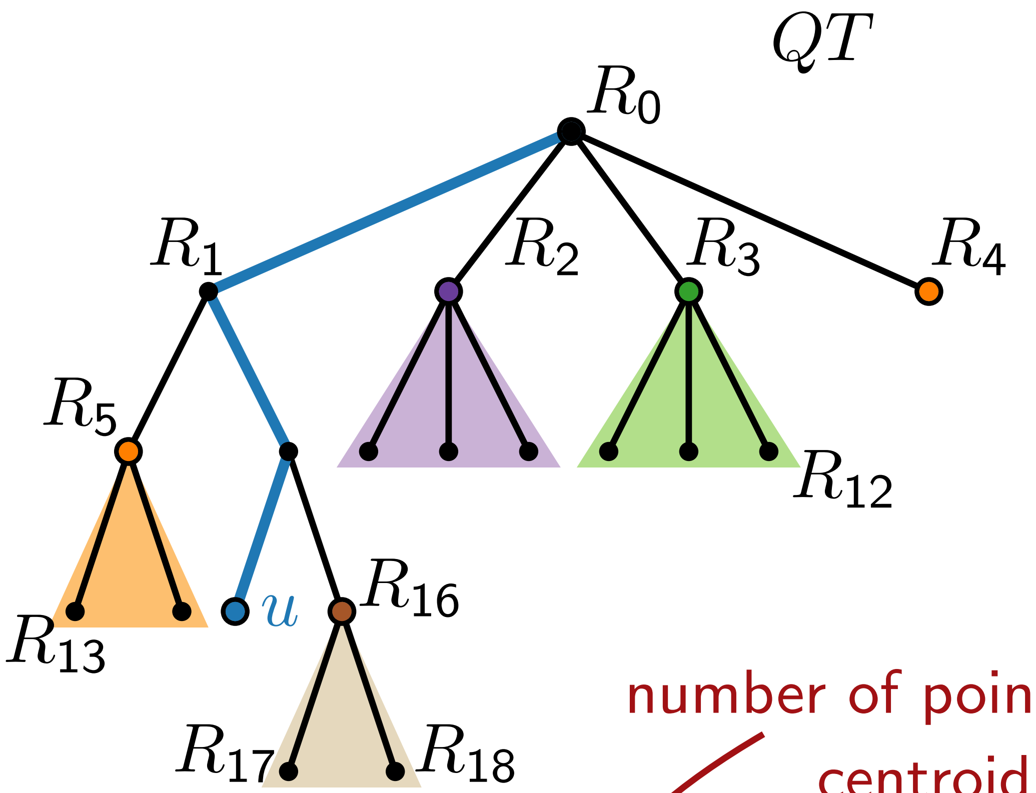

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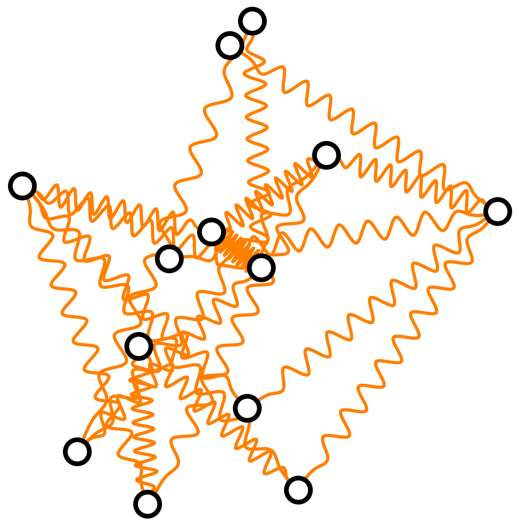
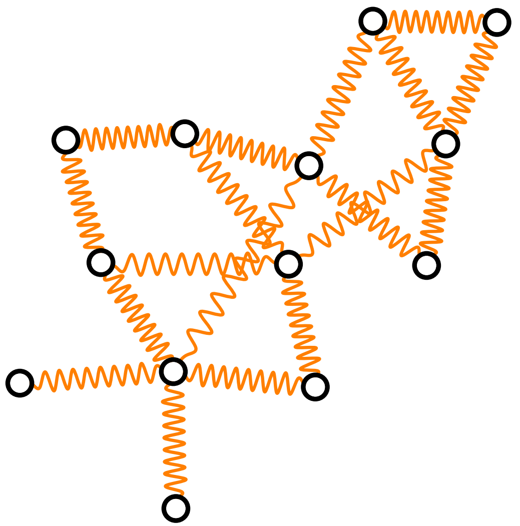
Visualization of Graphs

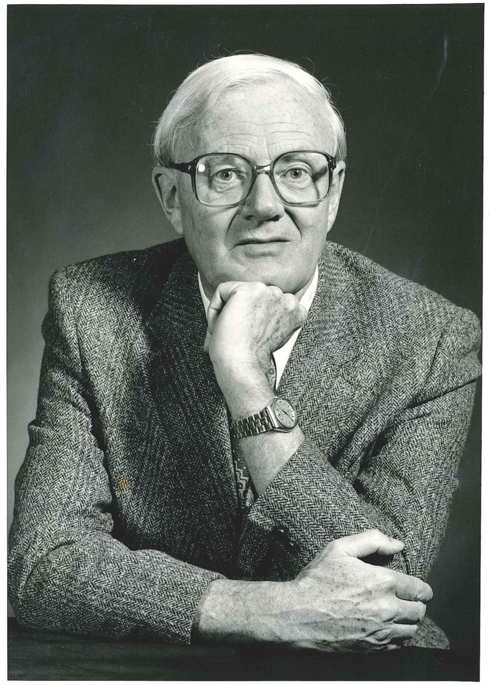
Lecture 2:
Force-Directed Drawing Algorithms



Part II:
Tutte Embeddings



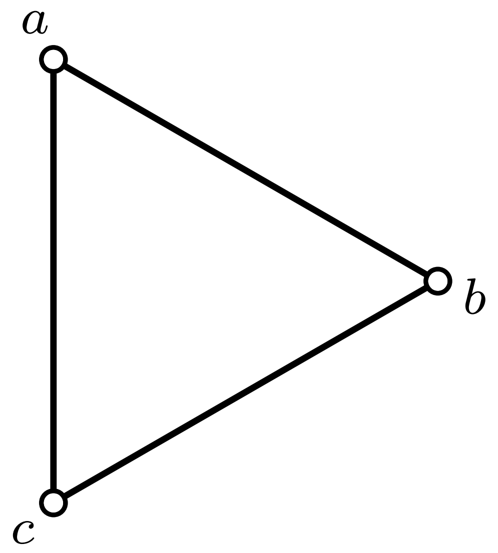
Idea

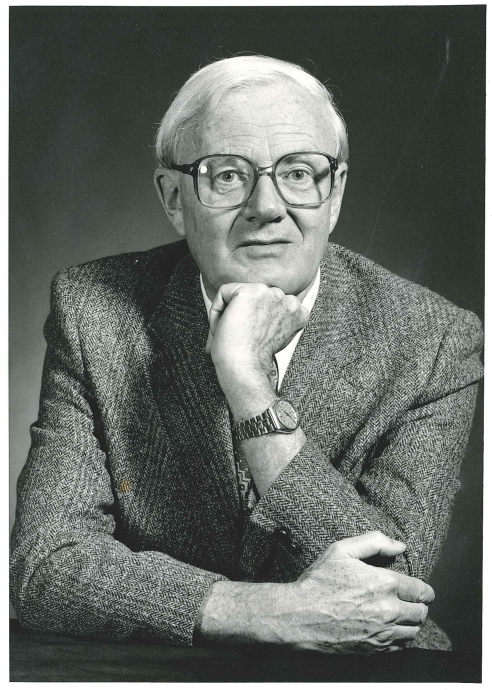


William T. Tutte
1917 – 2002

Idea

Consider a fixed triangle (a, b, c)

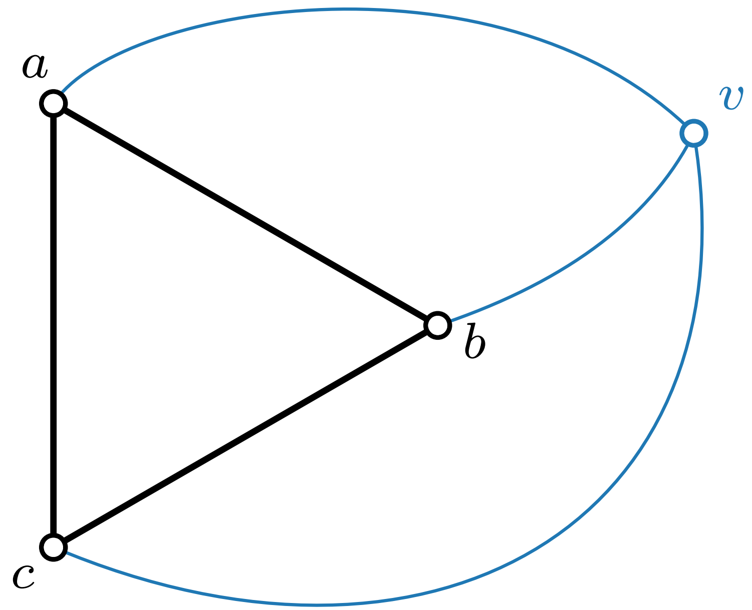


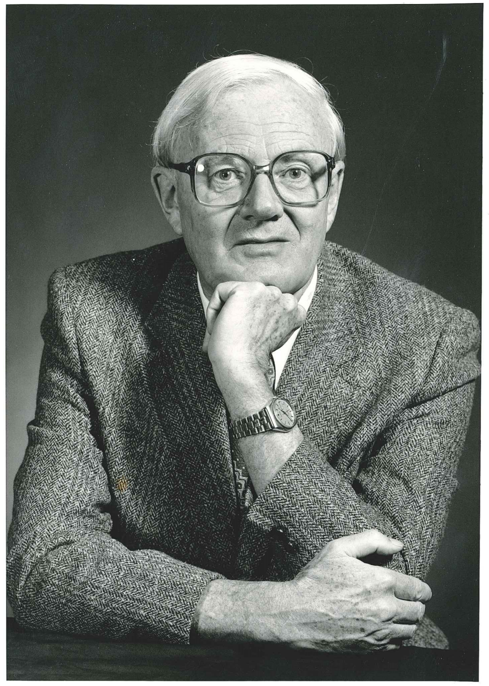


William T. Tutte
1917 – 2002

Idea

Consider a fixed triangle (a, b, c) with a common neighbor v



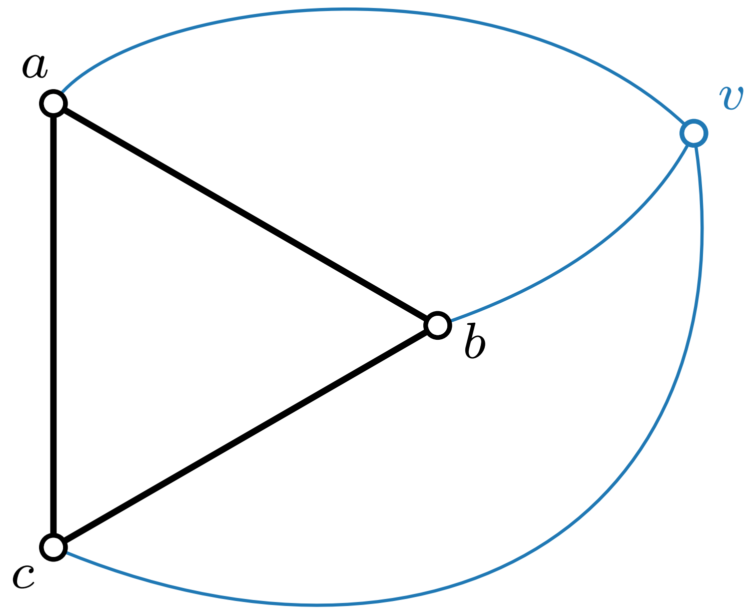


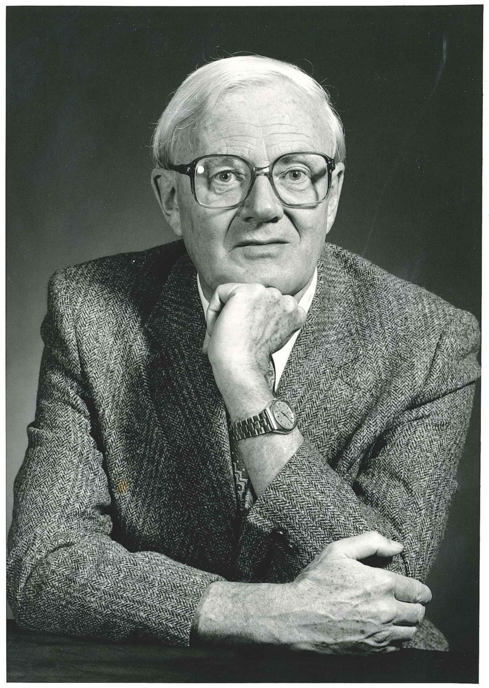
William T. Tutte
1917 – 2002

Idea

Consider a fixed triangle (a, b, c) with a common neighbor v

Where would you place v?





William T. Tutte
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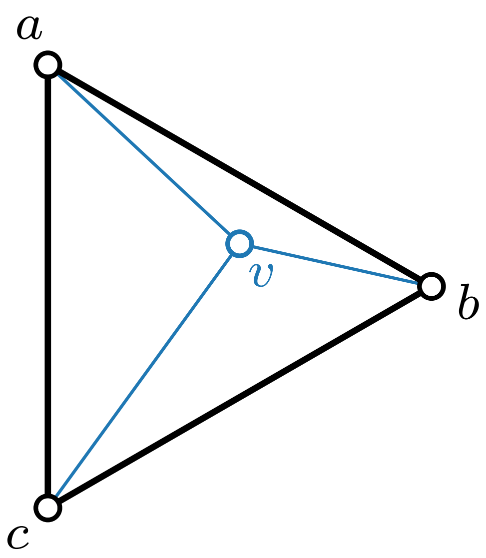

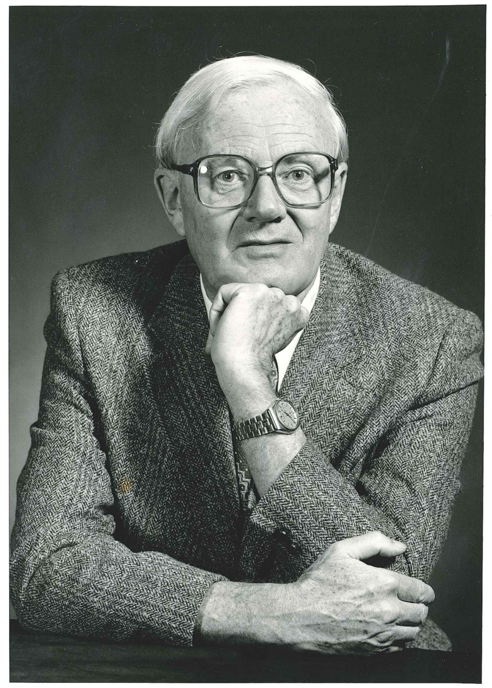

William T. Tutte
1917 – 2002

Where would you place v?

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Consider a fixed triangle (a, b, c) with a common neighbor v

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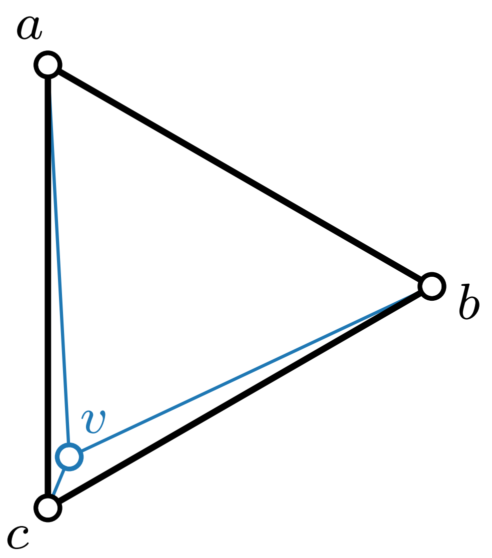

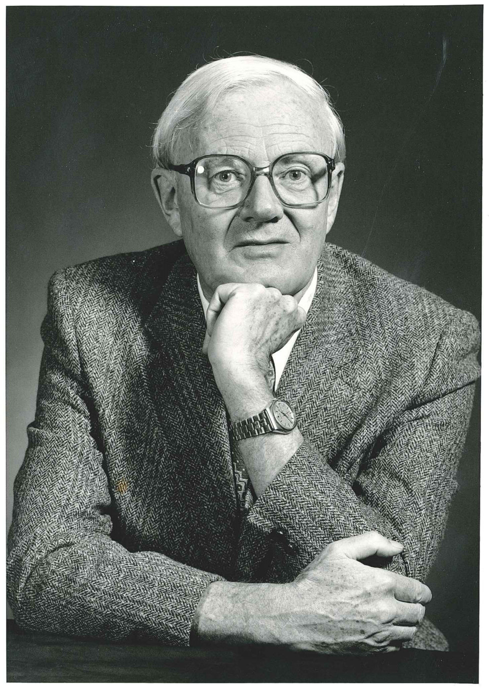

William T. Tutte
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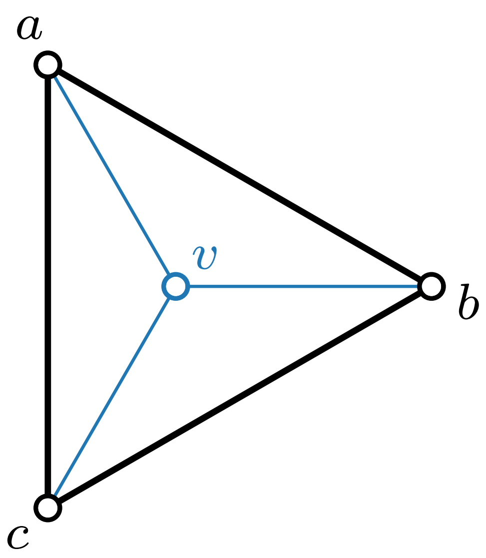

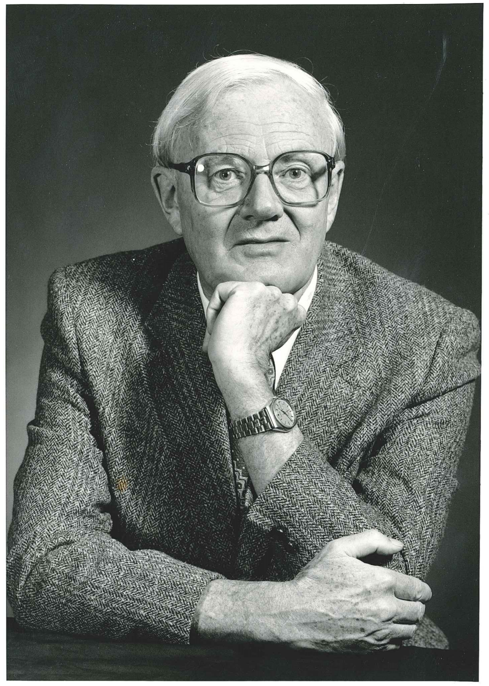

William T. Tutte
1917 – 2002

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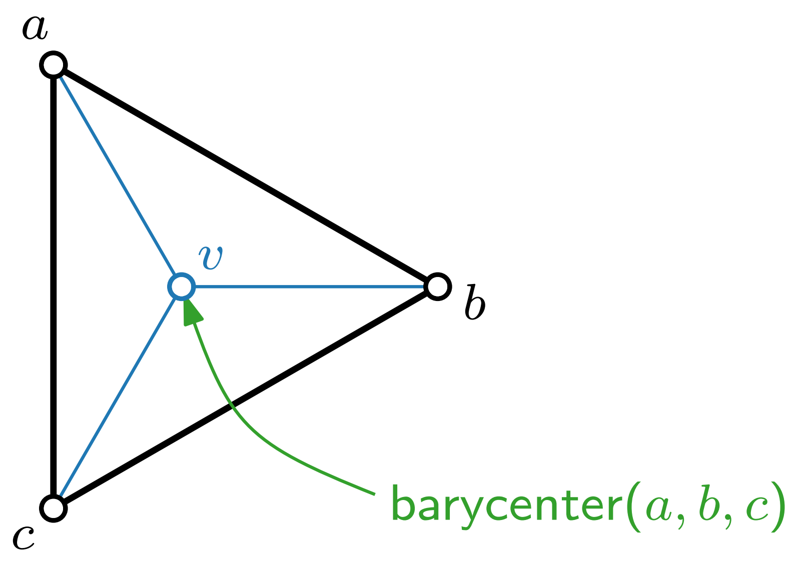

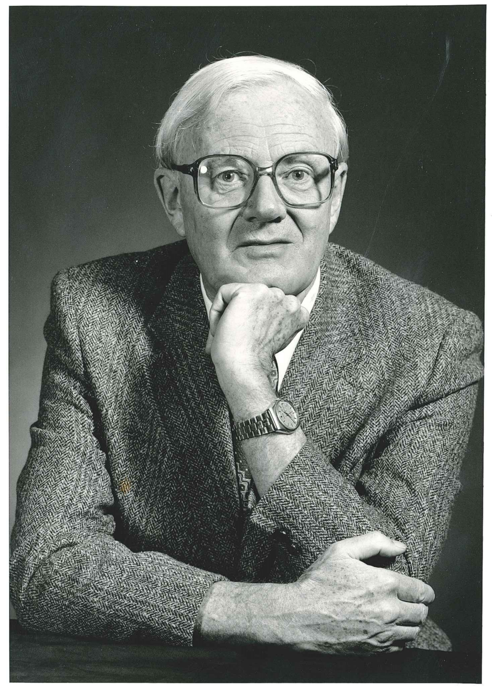

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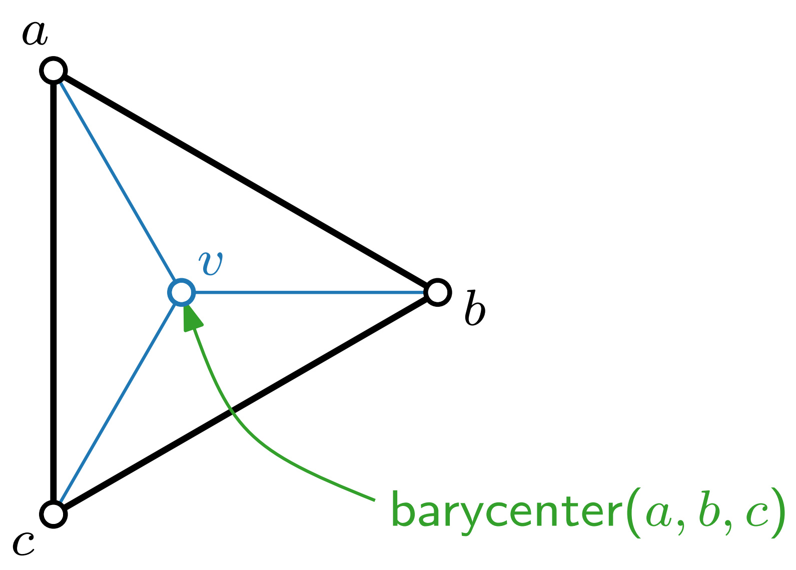
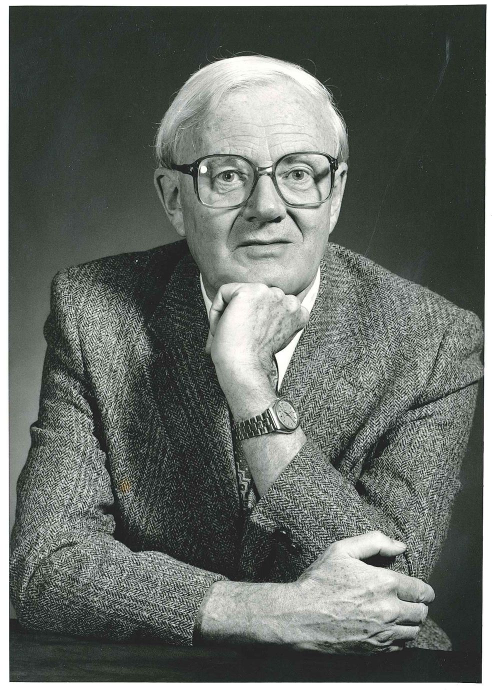


barycenter$(x_1, \ldots, x_k) =$?



William T. Tutte
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Idea

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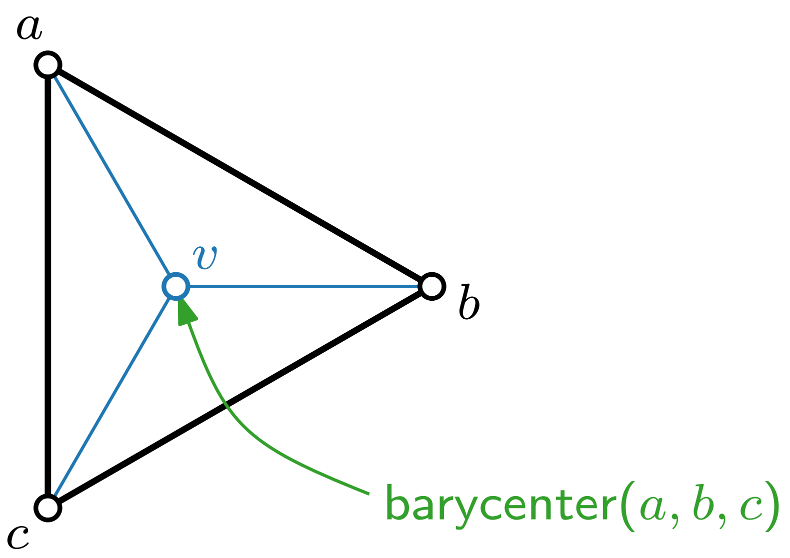
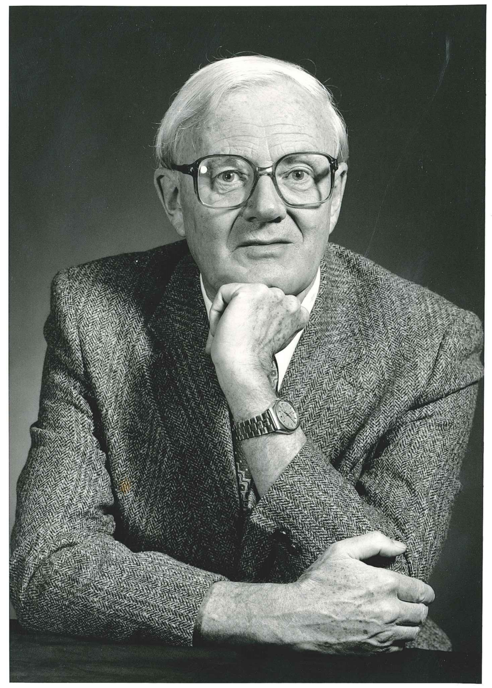


barycenter$(x_1, \ldots, x_k) = \sum_{i=1}^{k} x_i / k$



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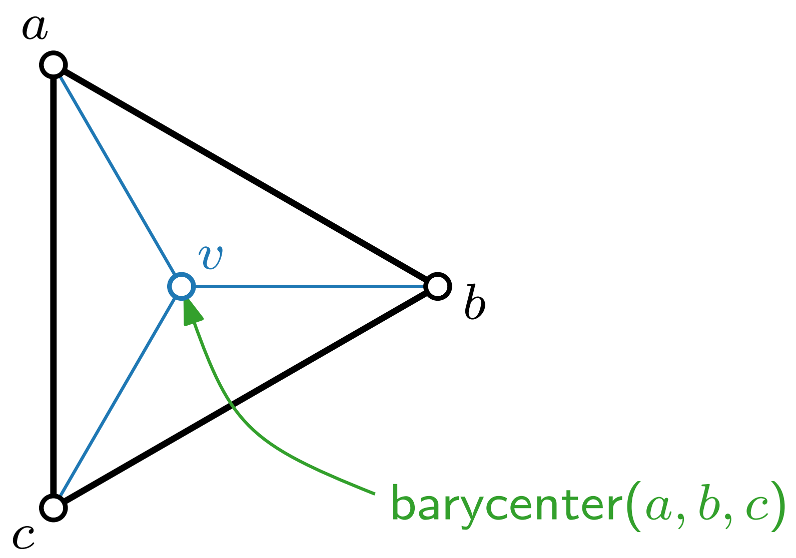

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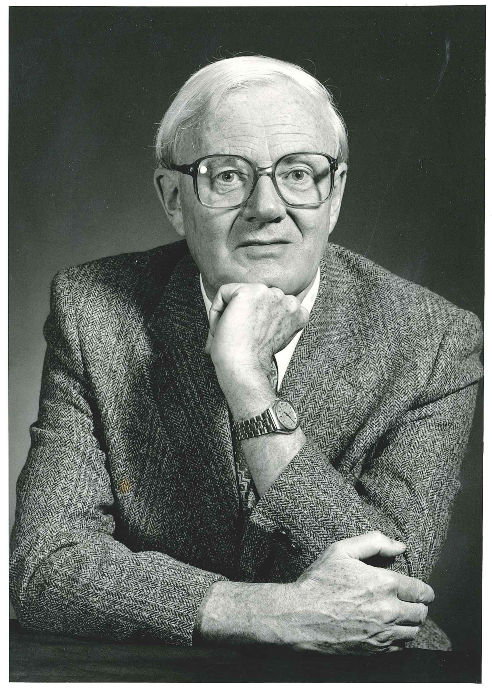

William T. Tutte
1917 – 2002

Idea.
Repeatedly place every vertex at barycenter of neighbors.

Tutte's Forces

ForceDirected(graph G, $p = (p_v)_{v \in V}$, $\varepsilon > 0$, $K \in \mathbb{N}$)

$t \leftarrow 1$

while $t \leq K$ **and** $\max_{v \in V(G)} \|F_v(t-1)\| > \varepsilon$ **do**

  foreach $u \in V(G)$ **do**

    $F_u(t) \leftarrow \sum_{v \in V(G)} f_{\mathsf{rep}}(p_u, p_v) + \sum_{v \in \mathsf{Adj}[u]} f_{\mathsf{attr}}(p_u, p_v)$

  foreach $u \in V(G)$ **do**

    $p_u \leftarrow p_u + \delta(t) \cdot F_u(t)$

  $t \leftarrow t + 1$

return p

Tutte's Forces

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Tutte's Forces

Goal.

$p_u = \text{barycenter}(\text{Adj}[u])$

ForceDirected(graph G, $p = (p_v)_{v \in V}$, $\varepsilon > 0$, $K \in \mathbb{N}$)

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$\text{barycenter}(x_1, \ldots, x_k) = \sum_{i=1}^{k} x_i / k$

Tutte's Forces

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$\quad = \sum_{v \in \text{Adj}[u]} p_v /$

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barycenter$(x_1, \ldots, x_k) = \sum_{i=1}^{k} x_i / k$

Tutte's Forces

Goal.

$$p_u = \text{barycenter}(\text{Adj}[u]) = \textstyle\sum_{v \in \text{Adj}[u]} p_v / \deg(u)$$

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$$\text{barycenter}(x_1, \ldots, x_k) = \textstyle\sum_{i=1}^{k} x_i / k$$

Tutte's Forces

Goal.

$$p_u = \text{barycenter}(\text{Adj}[u]) = \sum_{v \in \text{Adj}[u]} p_v / \deg(u)$$

$$F_u(t) = \sum_{v \in \text{Adj}[u]} p_v / \deg(u) - p_u$$

ForceDirected(graph G, $p = (p_v)_{v \in V}$, $\varepsilon > 0$, $K \in \mathbb{N}$)

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$$\text{barycenter}(x_1, \ldots, x_k) = \sum_{i=1}^{k} x_i / k$$

Tutte's Forces

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$$p_u = \text{barycenter}(\text{Adj}[u]) = \sum_{v \in \text{Adj}[u]} p_v / \deg(u)$$

$$F_u(t) = \sum_{v \in \text{Adj}[u]} p_v / \deg(u) - p_u = \sum_{v \in \text{Adj}[u]} (p_v - p_u) / \deg(u)$$

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$$\text{barycenter}(x_1, \ldots, x_k) = \sum_{i=1}^{k} x_i / k$$

$\overrightarrow{p_u p_v}$ = unit vector pointing from u to v

$\|p_u - p_v\|$ = Euclidean distance between u and v

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Tutte's Forces

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$$\boxed{F_u(t)} = \sum_{v \in \text{Adj}[u]} p_v / \deg(u) - p_u = \sum_{v \in \text{Adj}[u]} (p_v - p_u) / \deg(u) = \sum_{v \in \text{Adj}[u]} \frac{\|p_u - p_v\|}{\deg(u)} \overrightarrow{p_u p_v}$$

- **Repulsive forces** $f_{\text{rep}}(p_u, p_v) = 0$

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Goal.

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$$F_u(t) = \sum_{v \in \text{Adj}[u]} p_v / \deg(u) - p_u = \sum_{v \in \text{Adj}[u]} (p_v - p_u) / \deg(u) = \sum_{v \in \text{Adj}[u]} \frac{\|p_u - p_v\|}{\deg(u)} \overrightarrow{p_u p_v}$$

- **Repulsive forces** $f_{\text{rep}}(p_u, p_v) = 0$
- **Attractive forces**

$$f_{\text{attr}}(p_u, p_v) = \frac{\|p_u - p_v\|}{\deg(u)} \overrightarrow{p_u p_v}$$

ForceDirected(graph G, $p = (p_v)_{v \in V}$, $\varepsilon > 0$, $K \in \mathbb{N}$)
  $t \leftarrow 1$
  while $t \leq K$ **and** $\max_{v \in V(G)} \|F_v(t-1)\| > \varepsilon$ **do**
    foreach $u \in V(G)$ **do**
      $F_u(t) \leftarrow \sum_{v \in V(G)} f_{\text{rep}}(p_u, p_v) + \sum_{v \in \text{Adj}[u]} f_{\text{attr}}(p_u, p_v)$
    foreach $u \in V(G)$ **do**
      $p_u \leftarrow p_u + \cancel{\delta(t)}\ 1 \cdot F_u(t)$
    $t \leftarrow t + 1$
  return p

$$\text{barycenter}(x_1, \ldots, x_k) = \sum_{i=1}^{k} x_i / k$$

$\overrightarrow{p_u p_v}$ = unit vector pointing from u to v

$\|p_u - p_v\|$ = Euclidean distance between u and v

Tutte's Forces

Goal.

$$p_u = \text{barycenter}(\text{Adj}[u]) = \sum_{v \in \text{Adj}[u]} p_v / \deg(u)$$

$$F_u(t) = \sum_{v \in \text{Adj}[u]} p_v / \deg(u) - p_u = \sum_{v \in \text{Adj}[u]} (p_v - p_u) / \deg(u) = \sum_{v \in \text{Adj}[u]} \frac{\|p_u - p_v\|}{\deg(u)} \overrightarrow{p_u p_v}$$

- **Repulsive forces** $f_{\text{rep}}(p_u, p_v) = 0$
- **Attractive forces**

$$f_{\text{attr}}(p_u, p_v) = \frac{\|p_u - p_v\|}{\deg(u)} \overrightarrow{p_u p_v}$$

ForceDirected(graph G, $p = (p_v)_{v \in V}$, $\varepsilon > 0$, $K \in \mathbb{N}$)
 $t \leftarrow 1$
 while $t \leq K$ **and** $\max_{v \in V(G)} \|F_v(t-1)\| > \varepsilon$ **do**
  foreach $u \in V(G)$ **do**
   $F_u(t) \leftarrow \sum_{v \in V(G)} f_{\text{rep}}(p_u, p_v) + \sum_{v \in \text{Adj}[u]} f_{\text{attr}}(p_u, p_v)$
  foreach $u \in V(G)$ **do**
   $p_u \leftarrow p_u + \cancel{\delta(t)}\ 1 \cdot F_u(t)$
  $t \leftarrow t + 1$
 return p

$$\text{barycenter}(x_1, \ldots, x_k) = \sum_{i=1}^{k} x_i / k$$

Global minimum: $p_u = (0,0)\ \forall u \in V(G)$ ☹

$\overrightarrow{p_u p_v}$ = unit vector pointing from u to v

$\|p_u - p_v\|$ = Euclidean distance between u and v

Tutte's Forces

Goal.

$$p_u = \text{barycenter}(\text{Adj}[u]) = \sum_{v \in \text{Adj}[u]} p_v / \deg(u)$$

$$F_u(t) = \sum_{v \in \text{Adj}[u]} p_v / \deg(u) - p_u = \sum_{v \in \text{Adj}[u]} (p_v - p_u) / \deg(u) = \sum_{v \in \text{Adj}[u]} \frac{\|p_u - p_v\|}{\deg(u)} \overrightarrow{p_u p_v}$$

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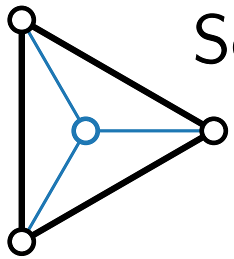
Global minimum: $p_u = (0, 0)\ \forall u \in V(G)$



Solution: fix coordinates of outer face!

$\overrightarrow{p_u p_v}$ = unit vector pointing from u to v

$\|p_u - p_v\|$ = Euclidean distance between u and v

Tutte's Forces

Goal.

$$p_u = \text{barycenter}(\text{Adj}[u])$$
$$= \textstyle\sum_{v \in \text{Adj}[u]} p_v / \deg(u)$$

$$\boxed{F_u(t)} = \textstyle\sum_{v \in \text{Adj}[u]} p_v / \deg(u) - p_u$$
$$= \textstyle\sum_{v \in \text{Adj}[u]} (p_v - p_u) / \deg(u)$$
$$= \boxed{\sum_{v \in \text{Adj}[u]} \frac{\|p_u - p_v\|}{\deg(u)} \overrightarrow{p_u p_v}}$$

- **Repulsive forces** $f_{\text{rep}}(p_u, p_v) = 0$
- **Attractive forces**

$$f_{\text{attr}}(p_u, p_v) = \begin{cases} 0 & \text{if } u \text{ fixed,} \\ \frac{\|p_u - p_v\|}{\deg(u)} \overrightarrow{p_u p_v} & \text{otherwise.} \end{cases}$$

ForceDirected(graph G, $p = (p_v)_{v \in V}$, $\varepsilon > 0$, $K \in \mathbb{N}$)

$t \leftarrow 1$
while $t \le K$ **and** $\max_{v \in V(G)} \|F_v(t-1)\| > \varepsilon$ **do**
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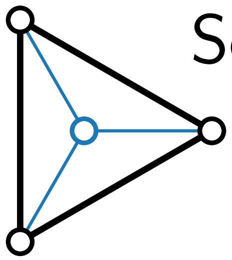

Solution: fix coordinates of outer face!

$\overrightarrow{p_u p_v}$ = unit vector pointing from u to v

$\|p_u - p_v\|$ = Euclidean distance between u and v

System of Linear Equations

Goal.

$p_u = \text{barycenter}(\text{Adj}[u]) \;\; = \sum_{v \in \text{Adj}[u]} p_v / \deg(u)$

System of Linear Equations

Goal. $p_u = (x_u, y_u)$

$p_u = \text{barycenter}(\text{Adj}[u]) \;\; = \sum_{v \in \text{Adj}[u]} p_v / \deg(u)$

System of Linear Equations

Goal. $p_u = (x_u, y_u)$

$p_u = \text{barycenter}(\mathsf{Adj}[u]) \ = \sum_{v \in \mathsf{Adj}[u]} p_v / \deg(u)$

$x_u = \sum_{v \in \mathsf{Adj}[u]} x_v / \deg(u)$

$y_u = \sum_{v \in \mathsf{Adj}[u]} y_v / \deg(u)$

System of Linear Equations

Goal. $p_u = (x_u, y_u)$

$p_u = \text{barycenter}(\text{Adj}[u]) = \sum_{v \in \text{Adj}[u]} p_v / \deg(u)$

$x_u = \sum_{v \in \text{Adj}[u]} x_v / \deg(u) \Leftrightarrow \deg(u) \cdot x_u = \sum_{v \in \text{Adj}[u]} x_v$

$y_u = \sum_{v \in \text{Adj}[u]} y_v / \deg(u) \Leftrightarrow \deg(u) \cdot y_u = \sum_{v \in \text{Adj}[u]} y_v$

System of Linear Equations

Goal. $p_u = (x_u, y_u)$

$p_u = \text{barycenter}(\text{Adj}[u]) = \sum_{v \in \text{Adj}[u]} p_v / \deg(u)$

$$x_u = \sum_{v \in \text{Adj}[u]} x_v / \deg(u) \Leftrightarrow \deg(u) \cdot x_u = \sum_{v \in \text{Adj}[u]} x_v \Leftrightarrow \deg(u) \cdot x_u - \sum_{v \in \text{Adj}[u]} x_v = 0$$
$$y_u = \sum_{v \in \text{Adj}[u]} y_v / \deg(u) \Leftrightarrow \deg(u) \cdot y_u = \sum_{v \in \text{Adj}[u]} y_v \Leftrightarrow \deg(u) \cdot y_u - \sum_{v \in \text{Adj}[u]} y_v = 0$$

System of Linear Equations

Goal. $p_u = (x_u, y_u)$

$p_u = \text{barycenter}(\text{Adj}[u]) = \sum_{v \in \text{Adj}[u]} p_v / \deg(u)$

Two systems of linear equations:

$$x_u = \sum_{v \in \text{Adj}[u]} x_v / \deg(u) \Leftrightarrow \deg(u) \cdot x_u = \sum_{v \in \text{Adj}[u]} x_v \Leftrightarrow \boxed{\deg(u) \cdot x_u - \sum_{v \in \text{Adj}[u]} x_v = 0}$$

$$y_u = \sum_{v \in \text{Adj}[u]} y_v / \deg(u) \Leftrightarrow \deg(u) \cdot y_u = \sum_{v \in \text{Adj}[u]} y_v \Leftrightarrow \boxed{\deg(u) \cdot y_u - \sum_{v \in \text{Adj}[u]} y_v = 0}$$

System of Linear Equations

Goal. $p_u = (x_u, y_u)$

$p_u = \text{barycenter}(\text{Adj}[u]) = \sum_{v \in \text{Adj}[u]} p_v / \deg(u)$

$$Ax = b$$

Two systems of linear equations:

$$x_u = \sum_{v \in \text{Adj}[u]} x_v / \deg(u) \Leftrightarrow \deg(u) \cdot x_u = \sum_{v \in \text{Adj}[u]} x_v \Leftrightarrow \boxed{\deg(u) \cdot x_u - \sum_{v \in \text{Adj}[u]} x_v = 0}$$

$$y_u = \sum_{v \in \text{Adj}[u]} y_v / \deg(u) \Leftrightarrow \deg(u) \cdot y_u = \sum_{v \in \text{Adj}[u]} y_v \Leftrightarrow \boxed{\deg(u) \cdot y_u - \sum_{v \in \text{Adj}[u]} y_v = 0}$$

System of Linear Equations

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$p_u = \text{barycenter}(\text{Adj}[u]) = \sum_{v \in \text{Adj}[u]} p_v / \deg(u)$

$Ax = b \qquad Ay = b$

Two systems of linear equations:

$$x_u = \sum_{v \in \text{Adj}[u]} x_v / \deg(u) \;\Leftrightarrow\; \deg(u) \cdot x_u = \sum_{v \in \text{Adj}[u]} x_v \;\Leftrightarrow\; \deg(u) \cdot x_u - \sum_{v \in \text{Adj}[u]} x_v = 0$$

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System of Linear Equations

Goal. $p_u = (x_u, y_u)$

$p_u = \text{barycenter}(\text{Adj}[u]) = \sum_{v \in \text{Adj}[u]} p_v / \deg(u)$

$Ax = b \quad Ay = b \quad b = (0)_n$

Two systems of linear equations:

$$x_u = \sum_{v \in \text{Adj}[u]} x_v / \deg(u) \Leftrightarrow \deg(u) \cdot x_u = \sum_{v \in \text{Adj}[u]} x_v \Leftrightarrow \deg(u) \cdot x_u - \sum_{v \in \text{Adj}[u]} x_v = 0$$

$$y_u = \sum_{v \in \text{Adj}[u]} y_v / \deg(u) \Leftrightarrow \deg(u) \cdot y_u = \sum_{v \in \text{Adj}[u]} y_v \Leftrightarrow \deg(u) \cdot y_u - \sum_{v \in \text{Adj}[u]} y_v = 0$$

System of Linear Equations

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Two systems of linear equations:

$$x_u = \sum_{v \in \text{Adj}[u]} x_v / \deg(u) \Leftrightarrow \deg(u) \cdot x_u = \sum_{v \in \text{Adj}[u]} x_v \Leftrightarrow \deg(u) \cdot x_u - \sum_{v \in \text{Adj}[u]} x_v = 0$$

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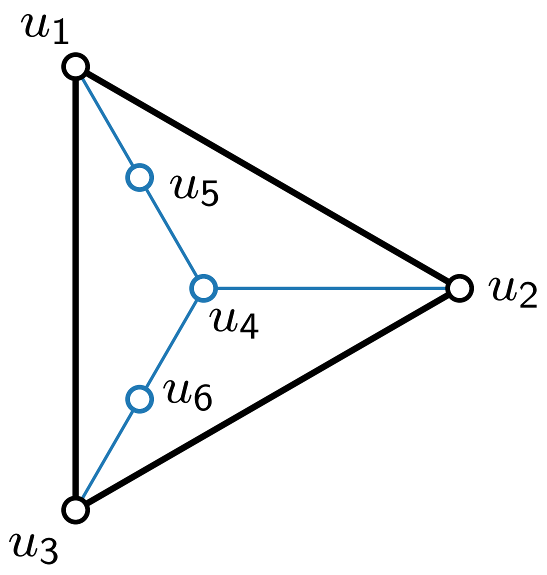


System of Linear Equations

Goal. $p_u = (x_u, y_u)$

$p_u = \text{barycenter}(\text{Adj}[u]) = \sum_{v \in \text{Adj}[u]} p_v / \deg(u)$

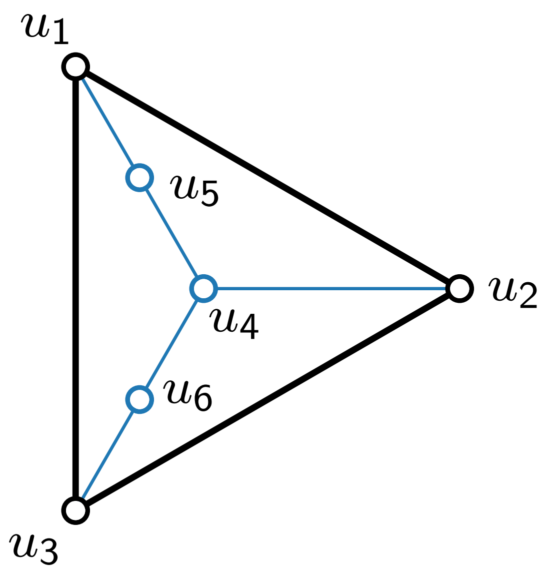
$Ax = b \quad Ay = b \quad b = (0)_n$

Two systems of linear equations:

$$x_u = \sum_{v \in \text{Adj}[u]} x_v / \deg(u) \Leftrightarrow \deg(u) \cdot x_u = \sum_{v \in \text{Adj}[u]} x_v \Leftrightarrow \deg(u) \cdot x_u - \sum_{v \in \text{Adj}[u]} x_v = 0$$

$$y_u = \sum_{v \in \text{Adj}[u]} y_v / \deg(u) \Leftrightarrow \deg(u) \cdot y_u = \sum_{v \in \text{Adj}[u]} y_v \Leftrightarrow \deg(u) \cdot y_u - \sum_{v \in \text{Adj}[u]} y_v = 0$$

A



$$\begin{pmatrix} \\ \\ \\ \end{pmatrix}$$

System of Linear Equations

Goal. $p_u = (x_u, y_u)$

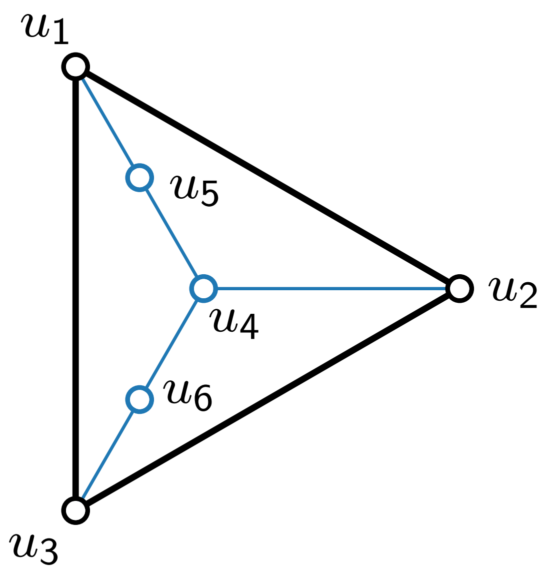
$$Ax = b \qquad Ay = b \qquad b = (0)_n$$

$p_u =$ barycenter(Adj$[u]$) $= \sum_{v \in \mathrm{Adj}[u]} p_v / \deg(u)$

Two systems of linear equations:

$$x_u = \sum_{v \in \mathrm{Adj}[u]} x_v / \deg(u) \Leftrightarrow \deg(u) \cdot x_u = \sum_{v \in \mathrm{Adj}[u]} x_v \Leftrightarrow \boxed{\deg(u) \cdot x_u - \sum_{v \in \mathrm{Adj}[u]} x_v = 0}$$

$$y_u = \sum_{v \in \mathrm{Adj}[u]} y_v / \deg(u) \Leftrightarrow \deg(u) \cdot y_u = \sum_{v \in \mathrm{Adj}[u]} y_v \Leftrightarrow \boxed{\deg(u) \cdot y_u - \sum_{v \in \mathrm{Adj}[u]} y_v = 0}$$



$$A = \begin{array}{c} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{array} \begin{pmatrix} \\ \\ \\ \\ \\ \\ \end{pmatrix}$$

System of Linear Equations

Goal. $p_u = (x_u, y_u)$

$p_u = \text{barycenter}(\text{Adj}[u]) = \sum_{v \in \text{Adj}[u]} p_v / \deg(u)$

$Ax = b \quad Ay = b \quad b = (0)_n$

Two systems of linear equations:

$$x_u = \sum_{v \in \text{Adj}[u]} x_v / \deg(u) \Leftrightarrow \deg(u) \cdot x_u = \sum_{v \in \text{Adj}[u]} x_v \Leftrightarrow \deg(u) \cdot x_u - \sum_{v \in \text{Adj}[u]} x_v = 0$$

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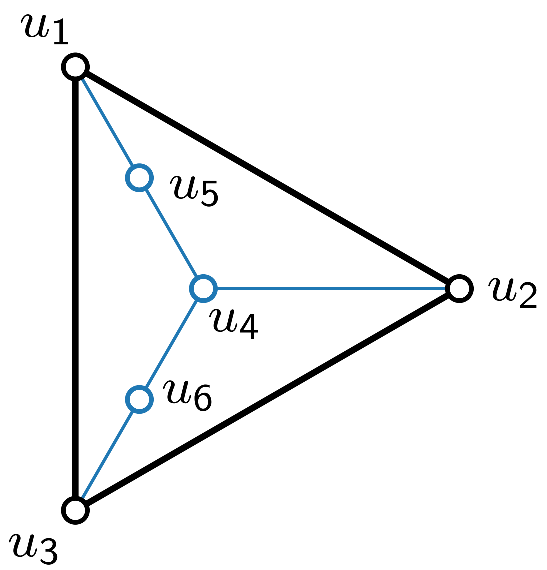


$$A \quad \begin{array}{c|c} & u_1 \quad u_2 \quad u_3 \quad u_4 \quad u_5 \quad u_6 \\ \hline \begin{array}{c} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{array} & \begin{pmatrix} \; & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \end{pmatrix} \end{array}$$

System of Linear Equations

Goal. $p_u = (x_u, y_u)$

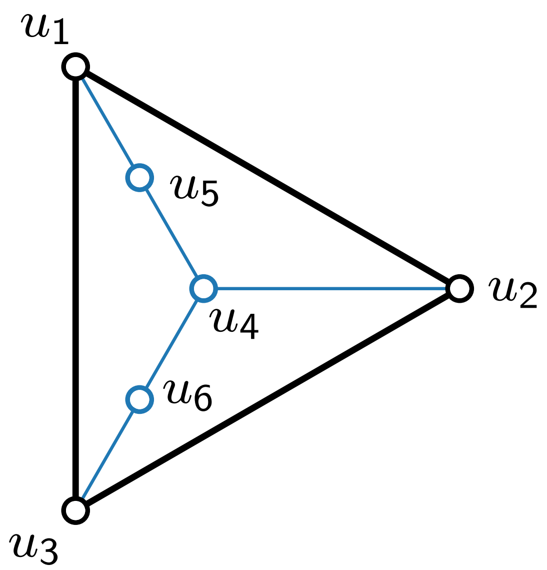
$p_u =$ barycenter$(\mathsf{Adj}[u])$ $= \sum_{v \in \mathsf{Adj}[u]} p_v / \deg(u)$

$Ax = b \quad Ay = b \quad b = (0)_n$

Two systems of linear equations:

$$x_u = \sum_{v \in \mathsf{Adj}[u]} x_v / \deg(u) \Leftrightarrow \deg(u) \cdot x_u = \sum_{v \in \mathsf{Adj}[u]} x_v \Leftrightarrow \boxed{\deg(u) \cdot x_u - \sum_{v \in \mathsf{Adj}[u]} x_v = 0}$$

$$y_u = \sum_{v \in \mathsf{Adj}[u]} y_v / \deg(u) \Leftrightarrow \deg(u) \cdot y_u = \sum_{v \in \mathsf{Adj}[u]} y_v \Leftrightarrow \boxed{\deg(u) \cdot y_u - \sum_{v \in \mathsf{Adj}[u]} y_v = 0}$$



$$A = \begin{array}{c|cccccc} & u_1 & u_2 & u_3 & u_4 & u_5 & u_6 \\ \hline u_1 & \multicolumn{6}{c}{\left(\begin{array}{cccccc} 3 & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \end{array}\right)} \\ u_2 & & & & & & \\ u_3 & & & & & & \\ u_4 & & & & & & \\ u_5 & & & & & & \\ u_6 & & & & & & \end{array}$$

System of Linear Equations

Goal. $p_u = (x_u, y_u)$

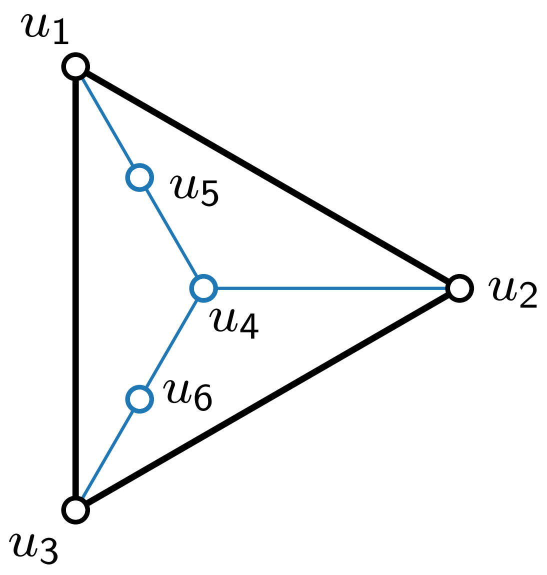
$p_u = \text{barycenter}(\mathrm{Adj}[u]) = \sum_{v \in \mathrm{Adj}[u]} p_v / \deg(u)$

$Ax = b \quad Ay = b \quad b = (0)_n$

Two systems of linear equations:

$x_u = \sum_{v \in \mathrm{Adj}[u]} x_v / \deg(u) \Leftrightarrow \deg(u) \cdot x_u = \sum_{v \in \mathrm{Adj}[u]} x_v \Leftrightarrow \deg(u) \cdot x_u - \sum_{v \in \mathrm{Adj}[u]} x_v = 0$

$y_u = \sum_{v \in \mathrm{Adj}[u]} y_v / \deg(u) \Leftrightarrow \deg(u) \cdot y_u = \sum_{v \in \mathrm{Adj}[u]} y_v \Leftrightarrow \deg(u) \cdot y_u - \sum_{v \in \mathrm{Adj}[u]} y_v = 0$



A

$$\begin{array}{c|cccccc} & u_1 & u_2 & u_3 & u_4 & u_5 & u_6 \\ \hline u_1 & 3 & -1 & & & & \\ u_2 & & & & & & \\ u_3 & & & & & & \\ u_4 & & & & & & \\ u_5 & & & & & & \\ u_6 & & & & & & \end{array}$$

System of Linear Equations

Goal. $p_u = (x_u, y_u)$

$p_u = \text{barycenter}(\mathsf{Adj}[u]) = \sum_{v \in \mathsf{Adj}[u]} p_v / \deg(u)$

$Ax = b \quad Ay = b \quad b = (0)_n$

Two systems of linear equations:

$$x_u = \sum_{v \in \mathsf{Adj}[u]} x_v / \deg(u) \Leftrightarrow \deg(u) \cdot x_u = \sum_{v \in \mathsf{Adj}[u]} x_v \Leftrightarrow \deg(u) \cdot x_u - \sum_{v \in \mathsf{Adj}[u]} x_v = 0$$

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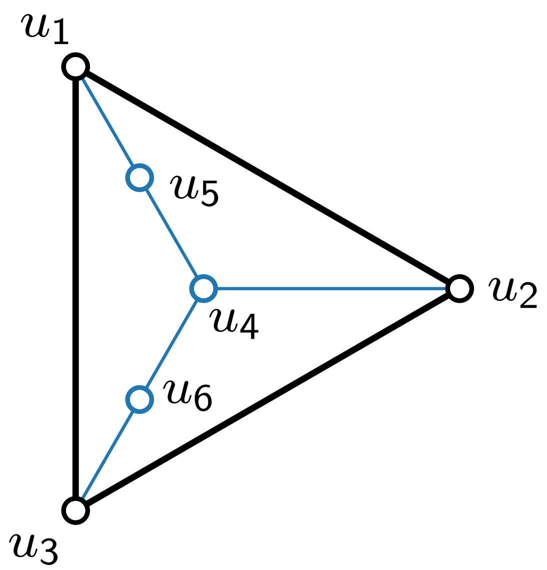


A

$$\begin{array}{c} \\ u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{array} \begin{array}{c} \begin{array}{cccccc} u_1 & u_2 & u_3 & u_4 & u_5 & u_6 \end{array} \\ \begin{pmatrix} 3 & -1 & -1 & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \end{pmatrix} \end{array}$$

System of Linear Equations

Goal. $p_u = (x_u, y_u)$

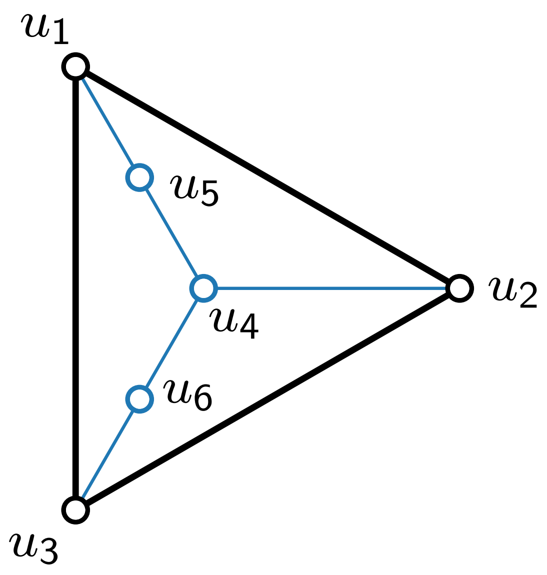
$Ax = b \quad Ay = b \quad b = (0)_n$

$p_u = \text{barycenter}(\text{Adj}[u]) = \sum_{v \in \text{Adj}[u]} p_v / \deg(u)$

Two systems of linear equations:

$$x_u = \sum_{v \in \text{Adj}[u]} x_v / \deg(u) \Leftrightarrow \deg(u) \cdot x_u = \sum_{v \in \text{Adj}[u]} x_v \Leftrightarrow \deg(u) \cdot x_u - \sum_{v \in \text{Adj}[u]} x_v = 0$$

$$y_u = \sum_{v \in \text{Adj}[u]} y_v / \deg(u) \Leftrightarrow \deg(u) \cdot y_u = \sum_{v \in \text{Adj}[u]} y_v \Leftrightarrow \deg(u) \cdot y_u - \sum_{v \in \text{Adj}[u]} y_v = 0$$



$$A$$

$$\begin{array}{c c}
 & \begin{array}{cccccc} u_1 & u_2 & u_3 & u_4 & u_5 & u_6 \end{array} \\
\begin{array}{c} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{array} & \begin{pmatrix} 3 & -1 & -1 & 0 & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \end{pmatrix}
\end{array}$$

System of Linear Equations

Goal. $p_u = (x_u, y_u)$

$p_u = \text{barycenter}(\text{Adj}[u]) = \sum_{v \in \text{Adj}[u]} p_v / \deg(u)$

$Ax = b \quad Ay = b \quad b = (0)_n$

Two systems of linear equations:

$$x_u = \sum_{v \in \text{Adj}[u]} x_v / \deg(u) \Leftrightarrow \deg(u) \cdot x_u = \sum_{v \in \text{Adj}[u]} x_v \Leftrightarrow \deg(u) \cdot x_u - \sum_{v \in \text{Adj}[u]} x_v = 0$$

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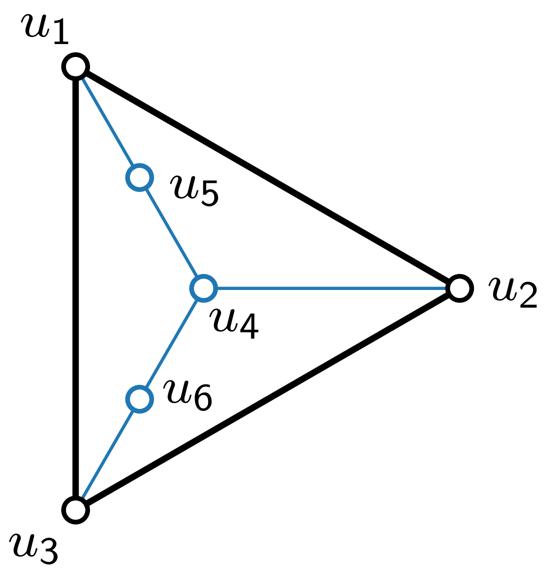

A

$$\begin{array}{c c} & \begin{array}{cccccc} u_1 & u_2 & u_3 & u_4 & u_5 & u_6 \end{array} \\ \begin{array}{c} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{array} & \begin{pmatrix} 3 & -1 & -1 & 0 & -1 & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \end{pmatrix} \end{array}$$

System of Linear Equations

Goal. $p_u = (x_u, y_u)$

$$p_u = \text{barycenter}(\mathrm{Adj}[u]) = \sum_{v \in \mathrm{Adj}[u]} p_v / \deg(u)$$

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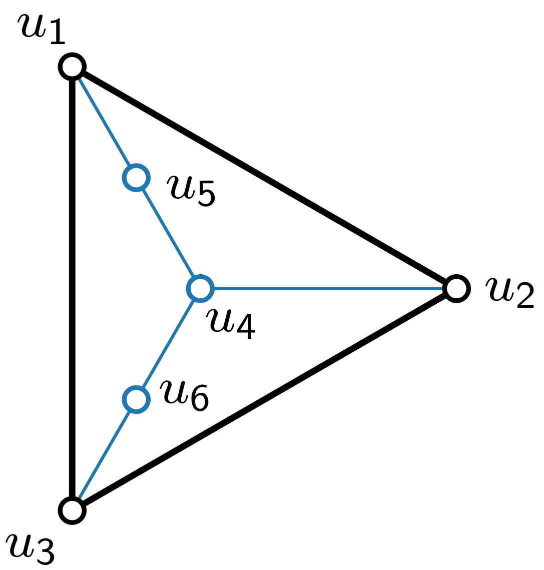

A

$$\begin{array}{c} \\ u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{array} \begin{array}{c} \begin{array}{cccccc} u_1 & u_2 & u_3 & u_4 & u_5 & u_6 \end{array} \\ \begin{pmatrix} 3 & -1 & -1 & 0 & -1 & 0 \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \end{pmatrix} \end{array}$$

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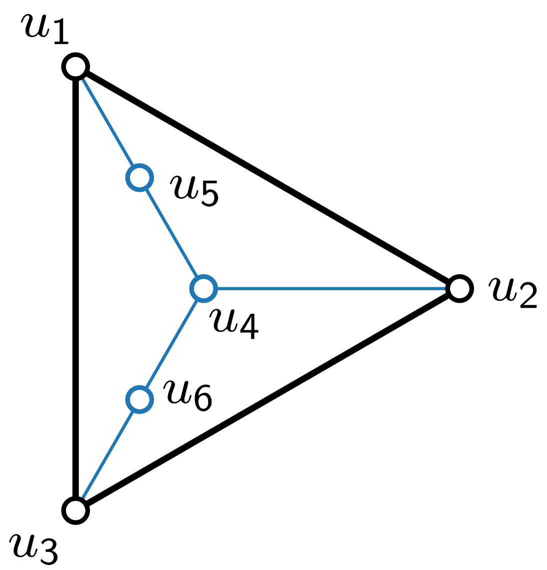

$$A = \begin{array}{c|c} & \begin{matrix} u_1 & u_2 & u_3 & u_4 & u_5 & u_6 \end{matrix} \\ \begin{matrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{matrix} & \begin{pmatrix} 3 & -1 & -1 & 0 & -1 & 0 \\ & 3 & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \end{pmatrix} \end{array}$$

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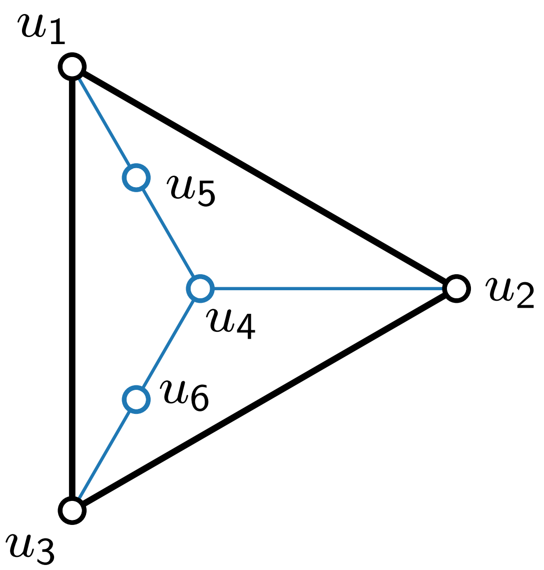

$$A$$

$$\begin{array}{c} \\ u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{array} \begin{array}{c} \begin{array}{cccccc} u_1 & u_2 & u_3 & u_4 & u_5 & u_6 \end{array} \\ \begin{pmatrix} 3 & -1 & -1 & 0 & -1 & 0 \\ -1 & 3 & -1 & -1 & 0 & 0 \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \end{pmatrix} \end{array}$$

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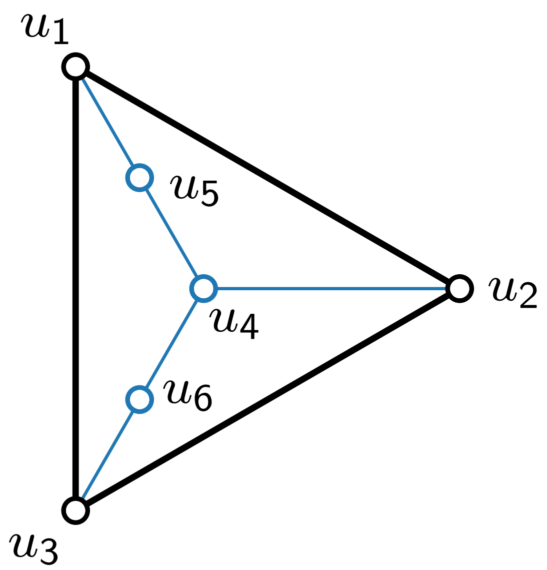

$$A = \begin{array}{c|cccccc} & u_1 & u_2 & u_3 & u_4 & u_5 & u_6 \\ \hline u_1 & 3 & -1 & -1 & 0 & -1 & 0 \\ u_2 & -1 & 3 & -1 & -1 & 0 & 0 \\ u_3 & -1 & -1 & 3 & 0 & 0 & -1 \\ u_4 & & & & & & \\ u_5 & & & & & & \\ u_6 & & & & & & \end{array}$$

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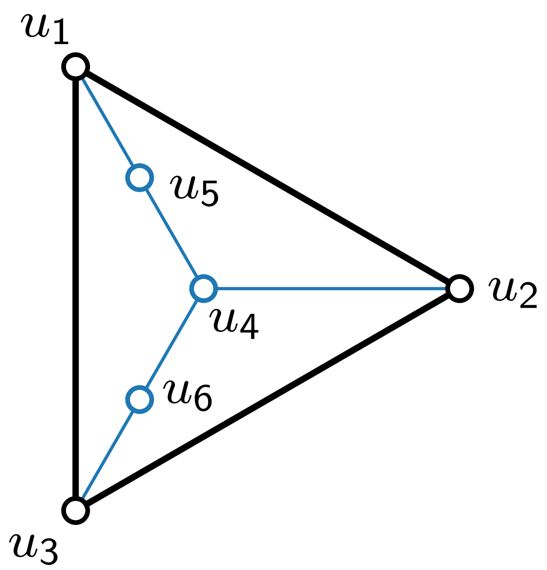

$$A \qquad \begin{array}{c c} & \begin{array}{cccccc} u_1 & u_2 & u_3 & u_4 & u_5 & u_6 \end{array} \\ \begin{array}{c} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{array} & \begin{pmatrix} 3 & -1 & -1 & 0 & -1 & 0 \\ -1 & 3 & -1 & -1 & 0 & 0 \\ -1 & -1 & 3 & 0 & 0 & -1 \\ 0 & -1 & 0 & 3 & -1 & -1 \\ & & & & & \\ & & & & & \end{pmatrix} \end{array}$$

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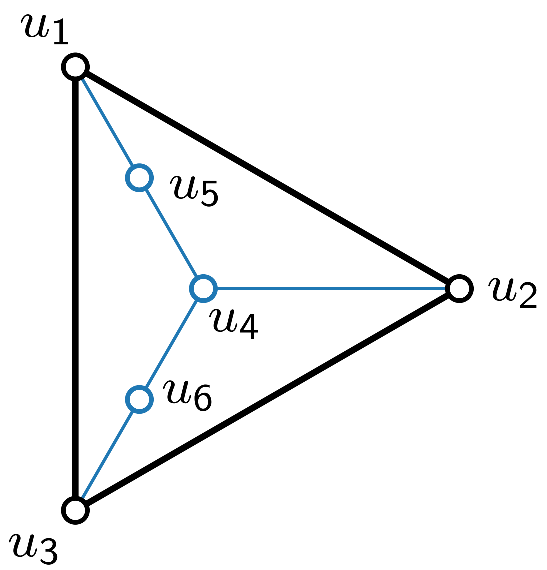

$$A$$

$$\begin{array}{c|c}
 & \begin{array}{cccccc} u_1 & u_2 & u_3 & u_4 & u_5 & u_6 \end{array} \\
\begin{array}{c} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{array} &
\begin{pmatrix}
3 & -1 & -1 & 0 & -1 & 0 \\
-1 & 3 & -1 & -1 & 0 & 0 \\
-1 & -1 & 3 & 0 & 0 & -1 \\
0 & -1 & 0 & 3 & -1 & -1 \\
-1 & 0 & 0 & -1 & 2 & 0 \\
 & & & & &
\end{pmatrix}
\end{array}$$

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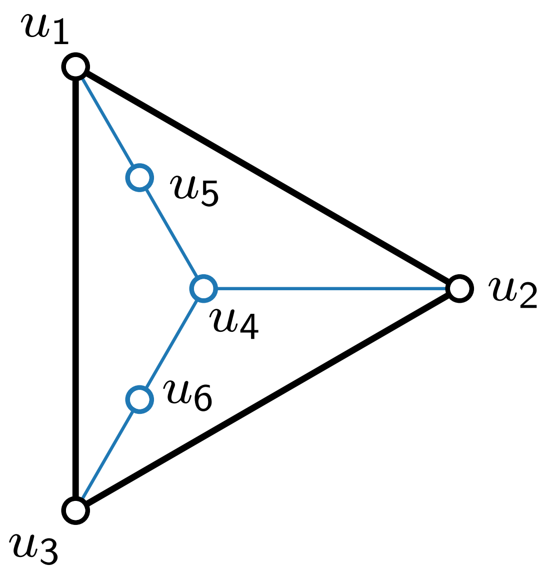

$$A \qquad \begin{array}{c} \\ u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{array} \begin{array}{c} \begin{array}{cccccc} u_1 & u_2 & u_3 & u_4 & u_5 & u_6 \end{array} \\ \begin{pmatrix} 3 & -1 & -1 & 0 & -1 & 0 \\ -1 & 3 & -1 & -1 & 0 & 0 \\ -1 & -1 & 3 & 0 & 0 & -1 \\ 0 & -1 & 0 & 3 & -1 & -1 \\ -1 & 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & -1 & -1 & 0 & 2 \end{pmatrix} \end{array}$$

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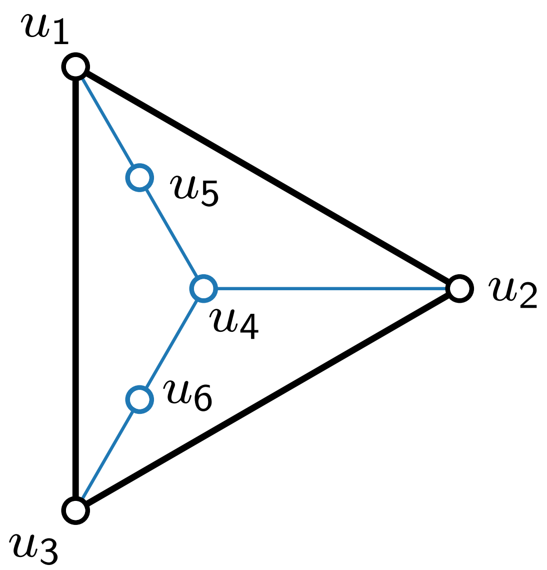

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$A_{ii} = \deg(u_i)$

System of Linear Equations

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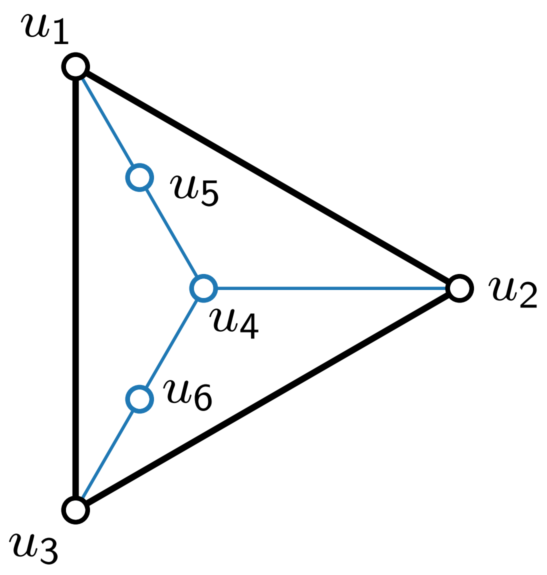

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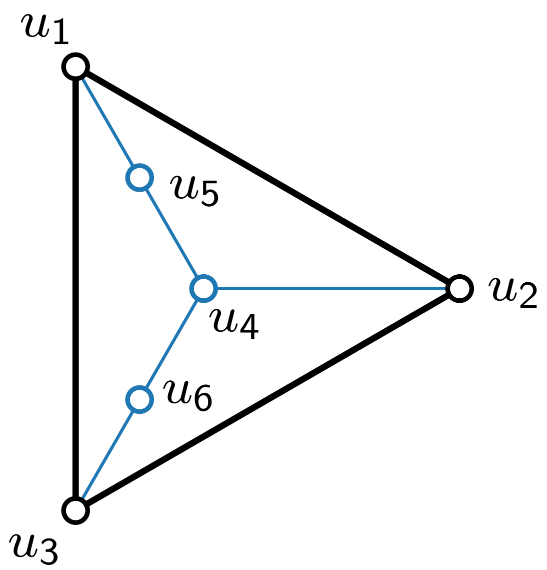

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Laplacian matrix of G

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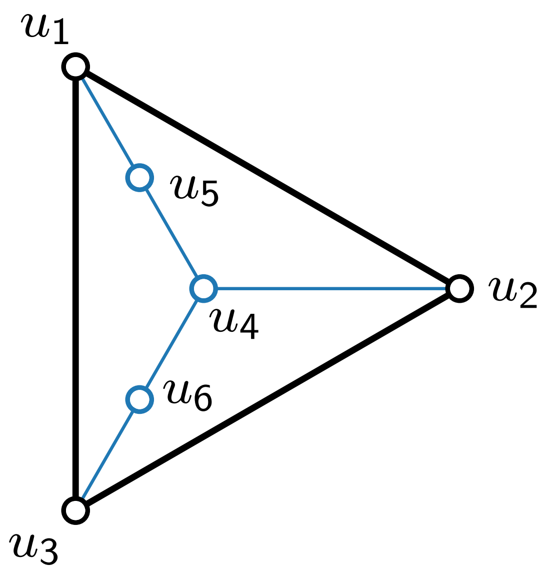

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unique solution

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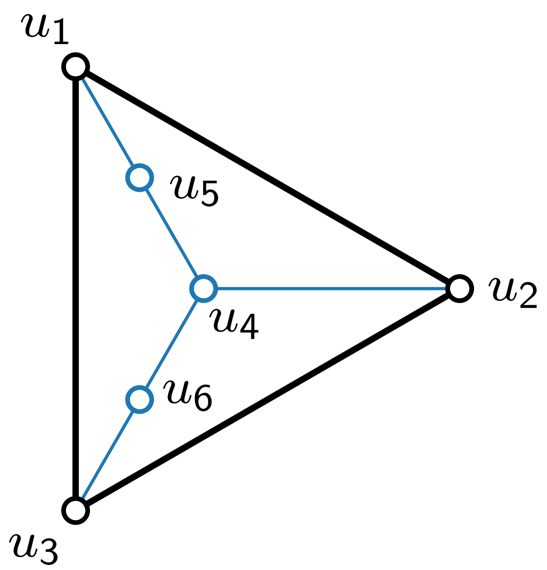

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variables, constraints, $\det(A) =$

unique solution

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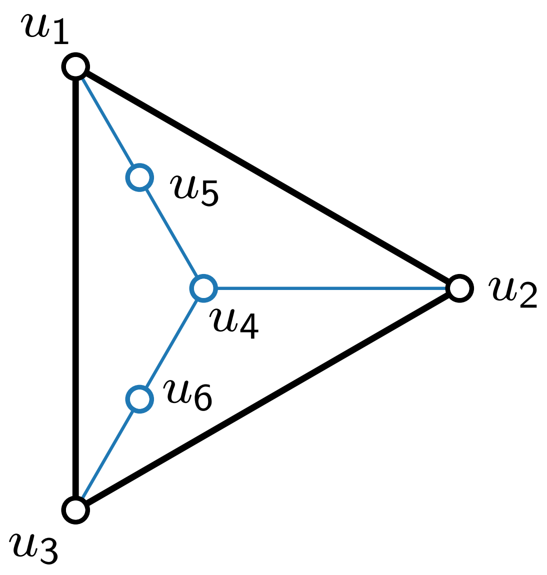

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n variables, constraints, $\det(A) =$

unique solution

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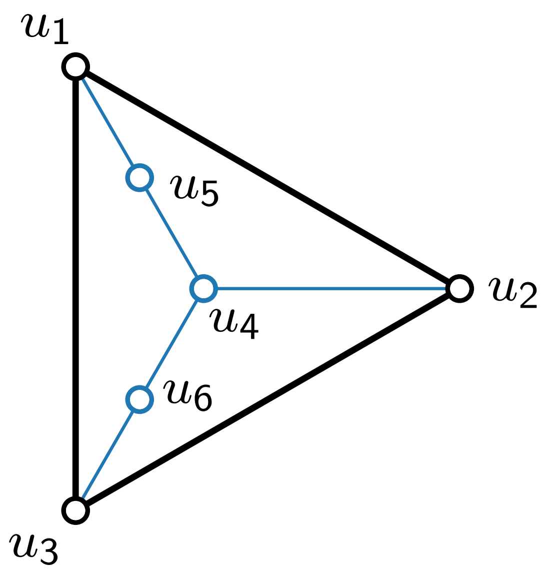

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Laplacian matrix of G

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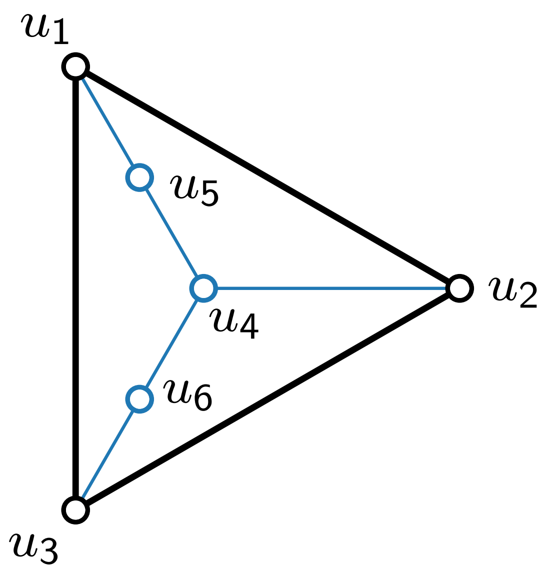

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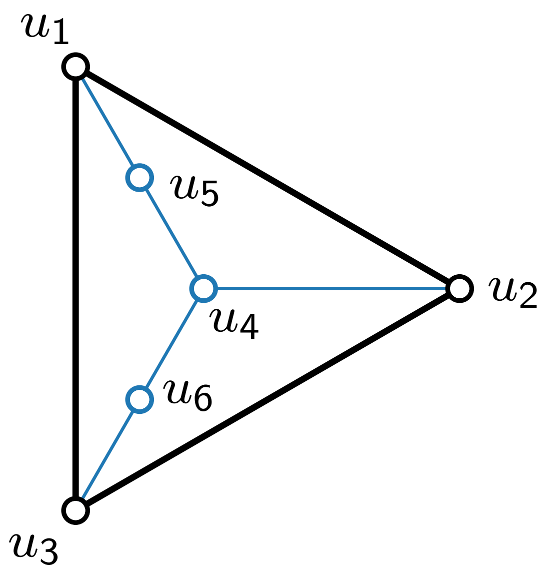

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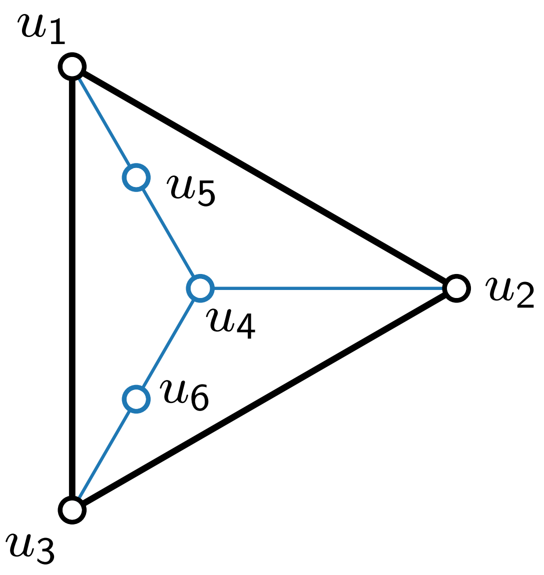

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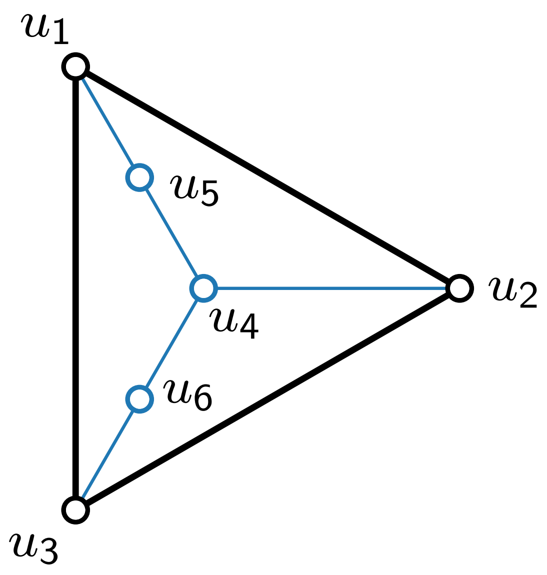

$$
\begin{array}{c} \\ u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{array}
\begin{array}{c}
\begin{array}{cccccc} u_1 & u_2 & u_3 & u_4 & u_5 & u_6 \end{array} \\
\begin{pmatrix}
3 & -1 & -1 & 0 & -1 & 0 \\
-1 & 3 & -1 & -1 & 0 & 0 \\
-1 & -1 & 3 & 0 & 0 & -1 \\
0 & -1 & 0 & 3 & -1 & -1 \\
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0 & 0 & -1 & -1 & 0 & 2
\end{pmatrix}
\end{array}
\quad A
$$

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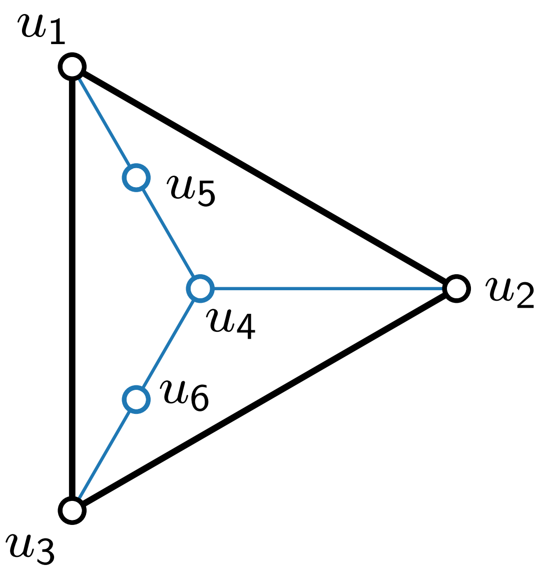

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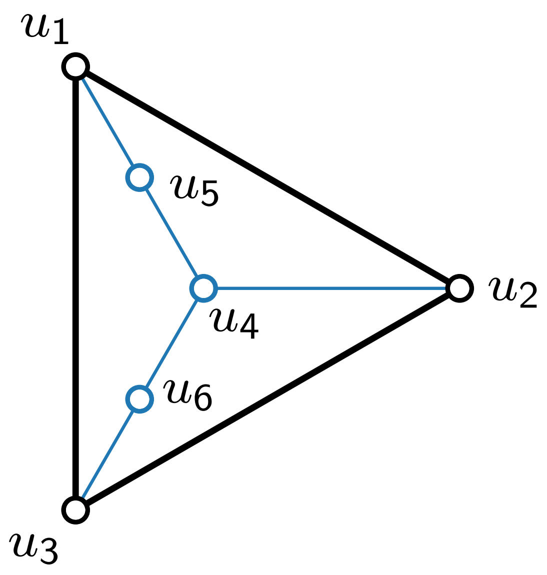

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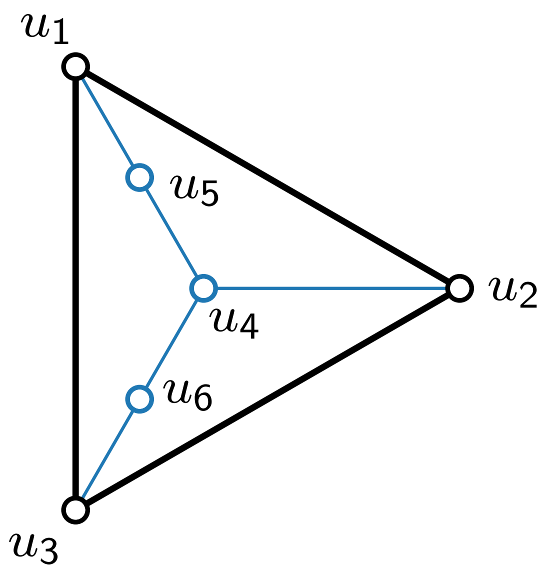

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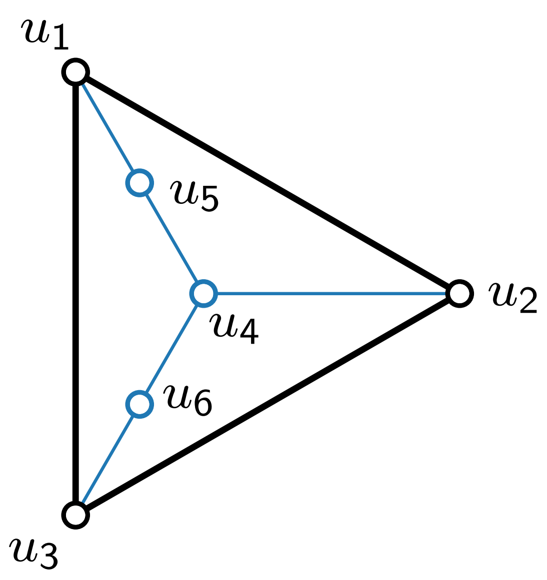

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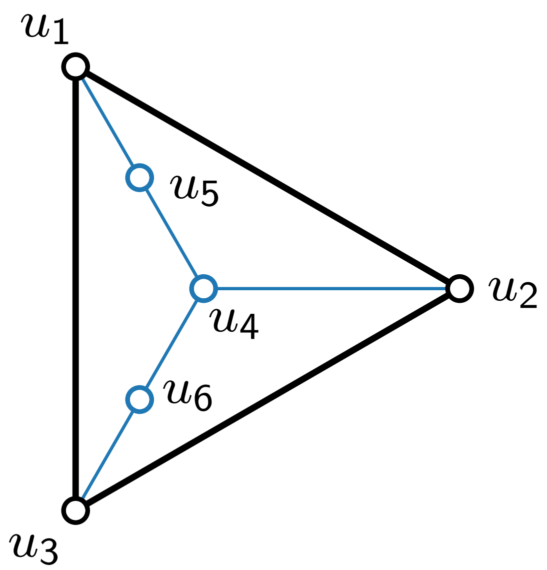

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Theorem.
Tutte's barycentric algorithm admits a unique solution.
It can be computed in polynomial time.

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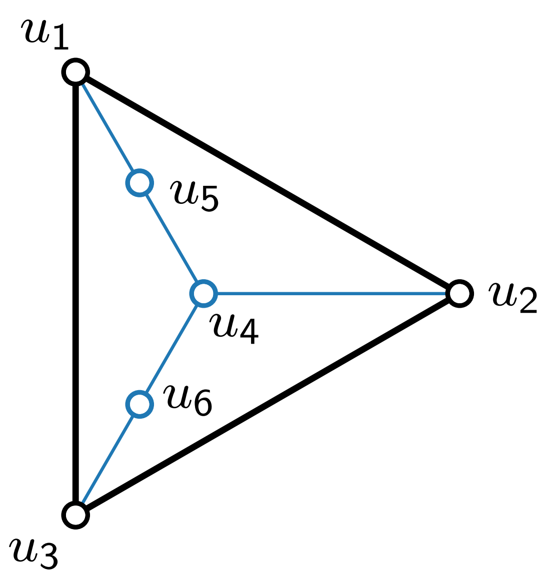

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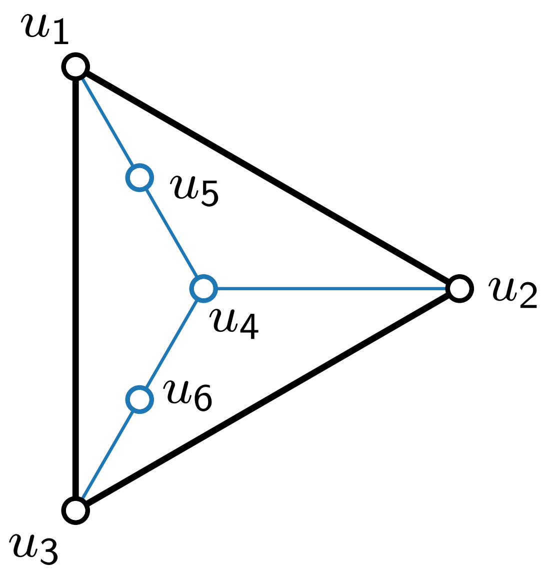

$$A = \begin{array}{c|cccccc} & u_1 & u_2 & u_3 & u_4 & u_5 & u_6 \\ \hline u_1 & 3 & -1 & -1 & 0 & -1 & 0 \\ u_2 & -1 & 3 & -1 & -1 & 0 & 0 \\ u_3 & -1 & -1 & 3 & 0 & 0 & -1 \\ u_4 & 0 & -1 & 0 & 3 & -1 & -1 \\ u_5 & -1 & 0 & 0 & -1 & 2 & 0 \\ u_6 & 0 & 0 & -1 & -1 & 0 & 2 \end{array}$$

Laplacian matrix of G

$A_{ii} = \deg(u_i)$

$$A_{ij, i \neq j} = \begin{cases} -1 & u_i u_j \in E \\ 0 & u_i u_j \notin E \end{cases}$$

Solution:
1. No need to change fixed vertices.
2. Constraints that depend on fixed vertices are constant and can be moved into b.

k variables, k constraints, $\det(A) > 0$

$k = \#$free vertices $\Rightarrow$ unique solution ☺

System of Linear Equations

solve two systems of linear equations

Theorem. **= Tutte drawing**
Tutte's barycentric algorithm admits a unique solution.
It can be computed in polynomial time.

Goal. $p_u = (x_u, y_u)$

$p_u =$ barycenter(Adj[u]) $=$

$$x_u = \sum_{v \in \mathrm{Adj}[u]} x_v / \deg(u) \Leftrightarrow \deg(u) \cdot x_u = \sum_{v \in \mathrm{Adj}[u]} x_v \Leftrightarrow \deg(u) \cdot x_u - \sum_{v \in \mathrm{Adj}[u]} x_v = 0$$

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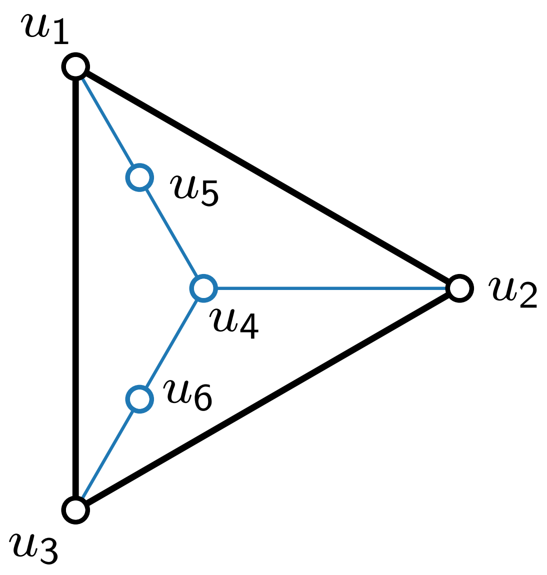

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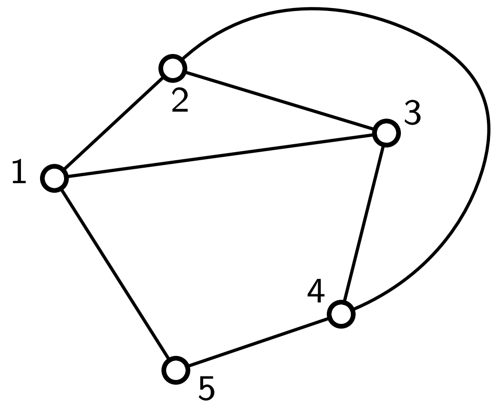
G **planar**: G can be drawn such that no two edges cross each other.

G **connected**: $\exists\ u$–v path for every vertex pair $\{u, v\}$.



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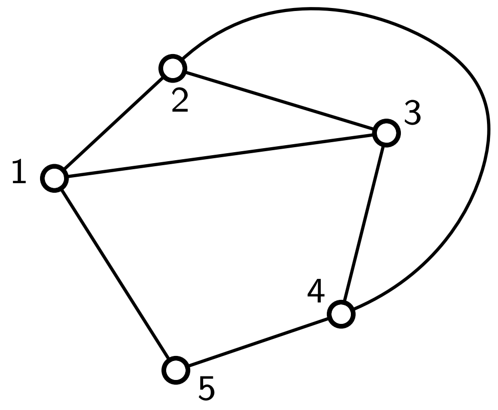

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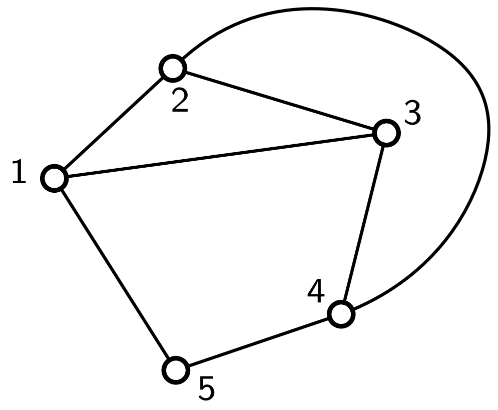

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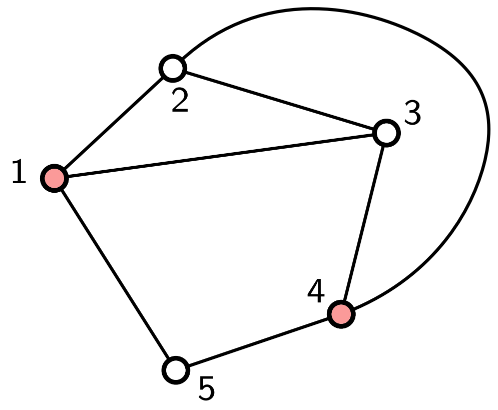

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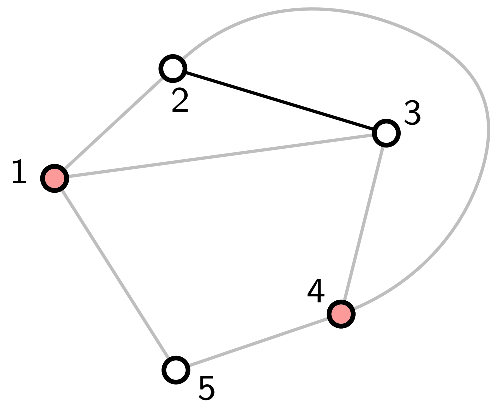

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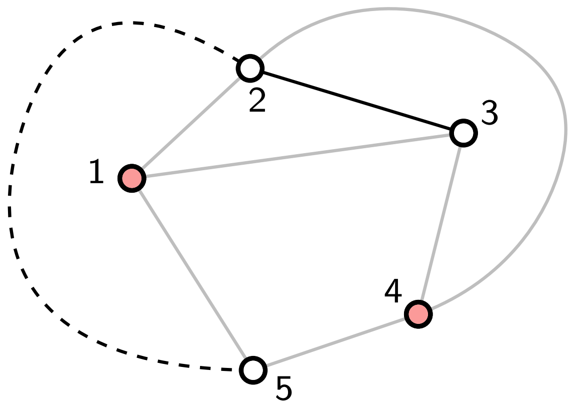

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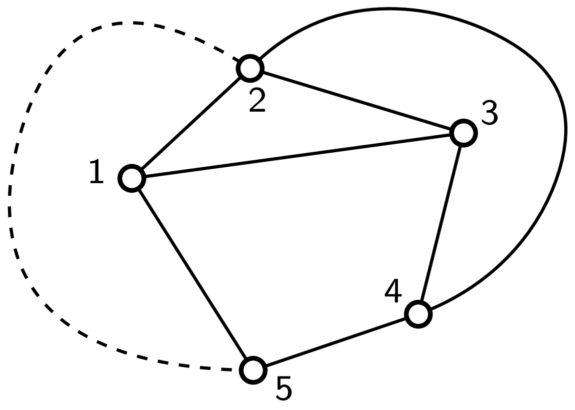

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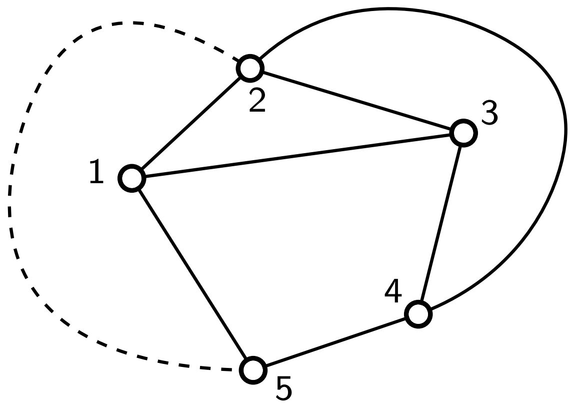

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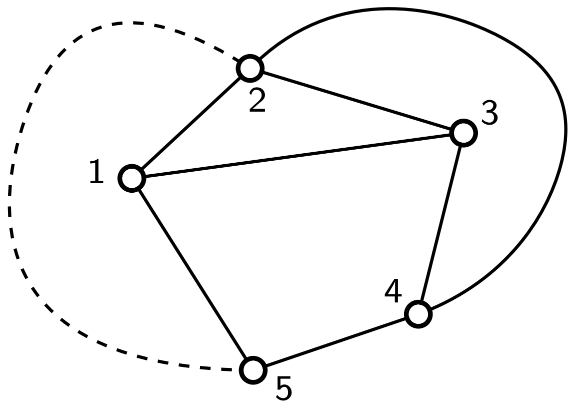

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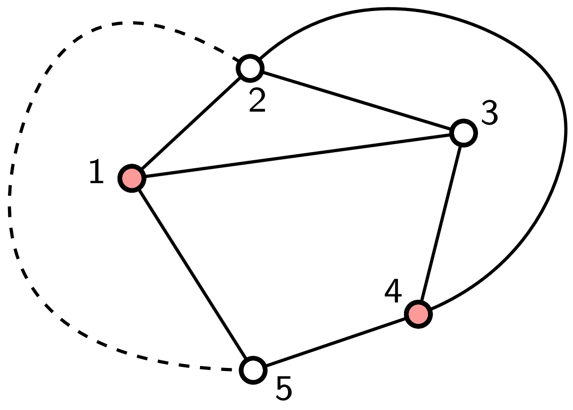

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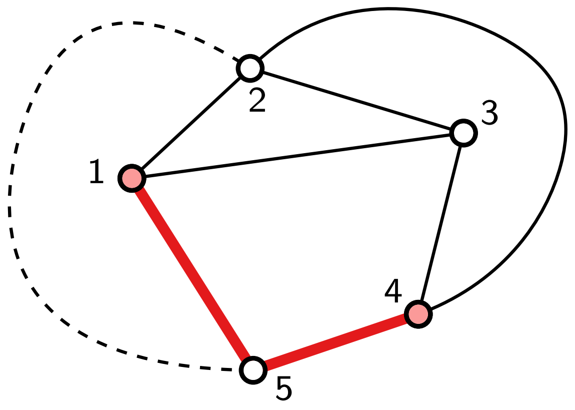

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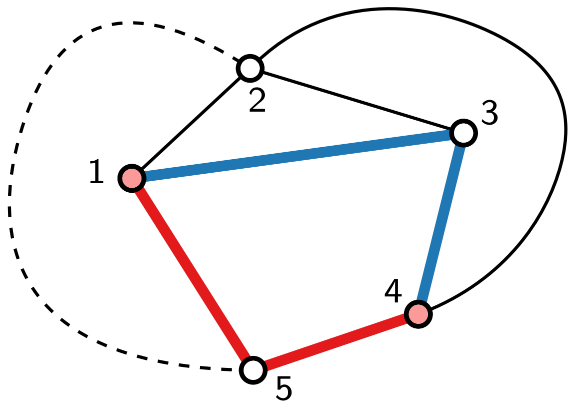

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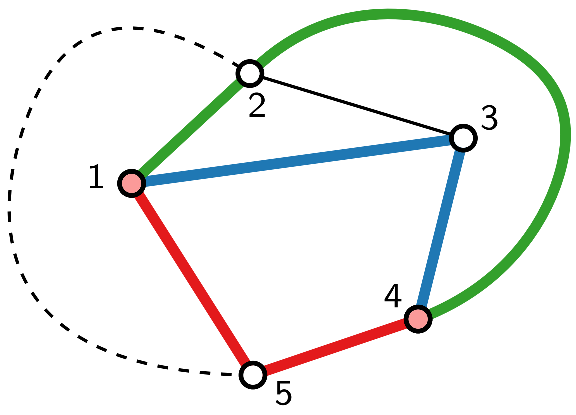

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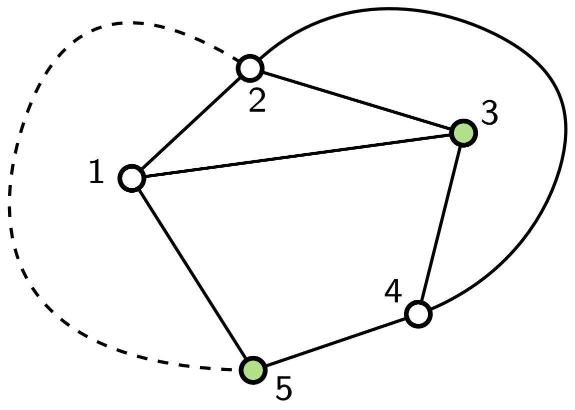

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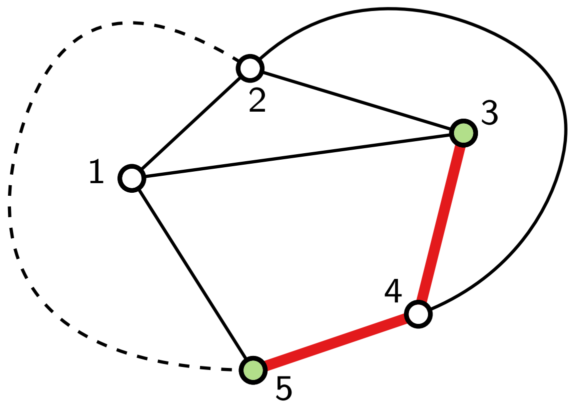

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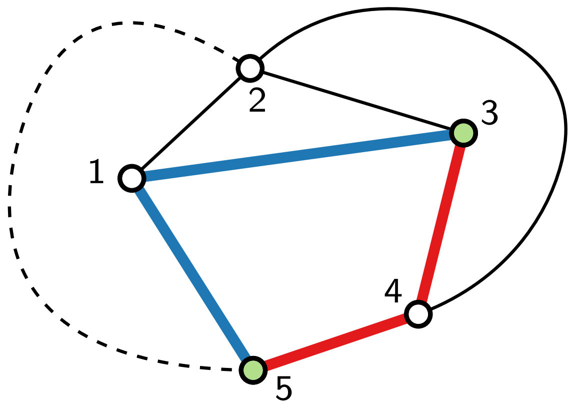

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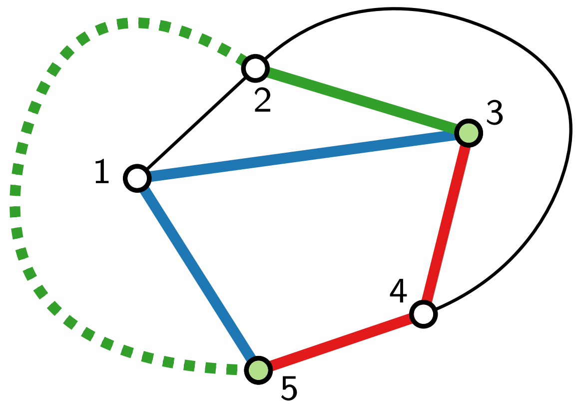

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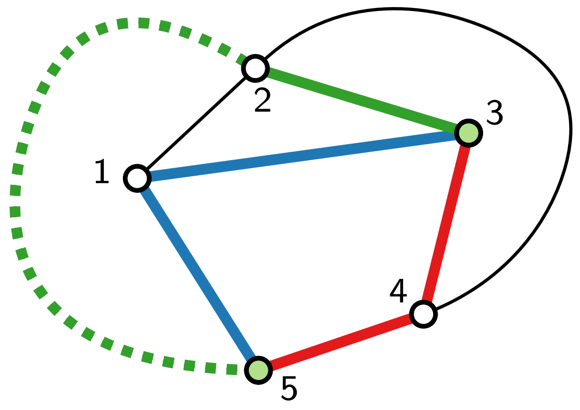
Every 3-connected planar graph has a unique planar embedding.



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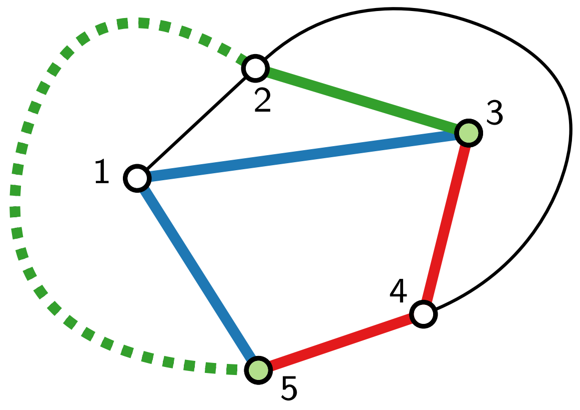

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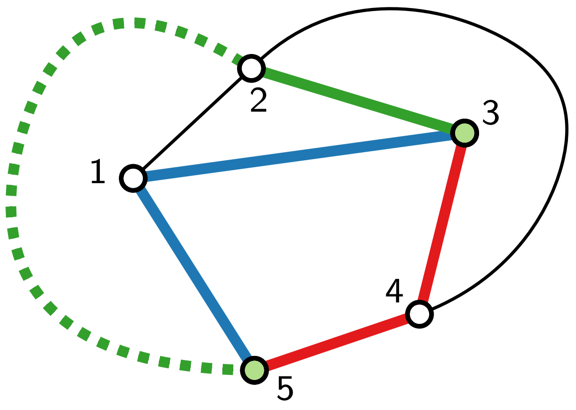

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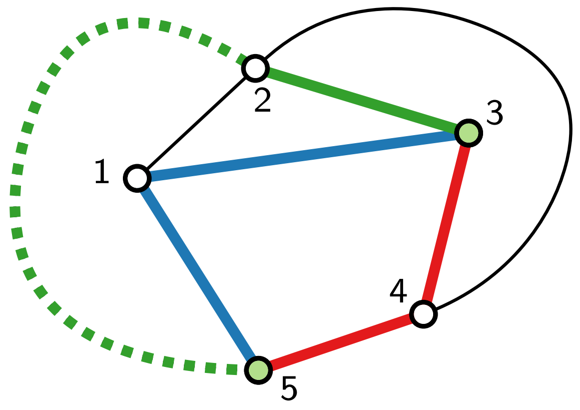

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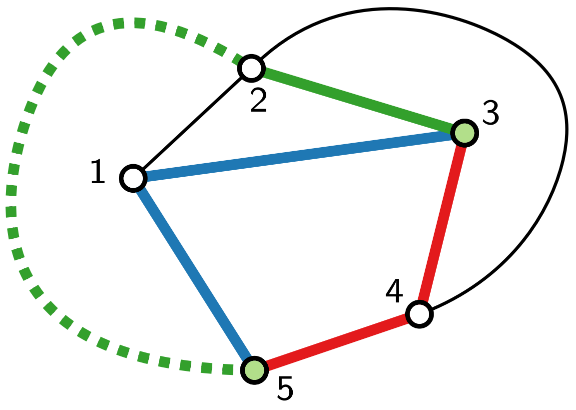

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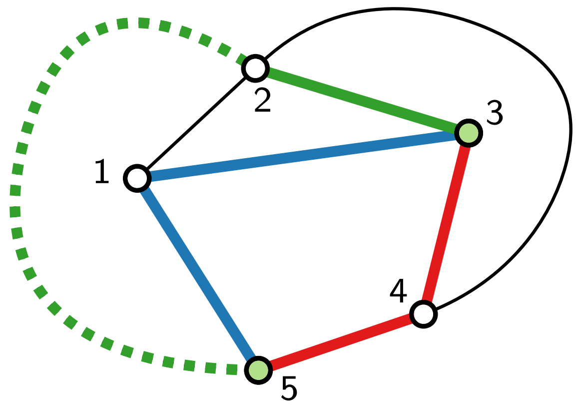

Γ_1

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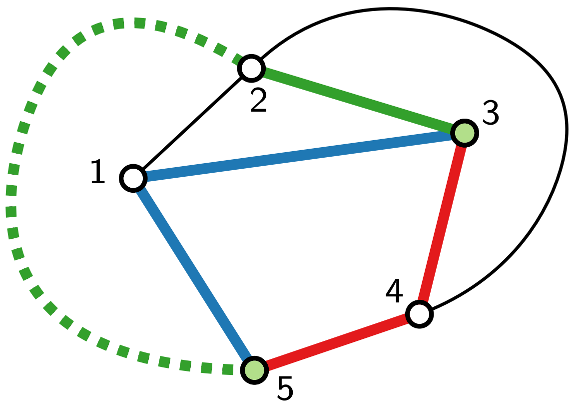

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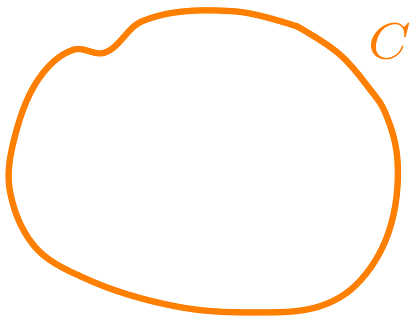

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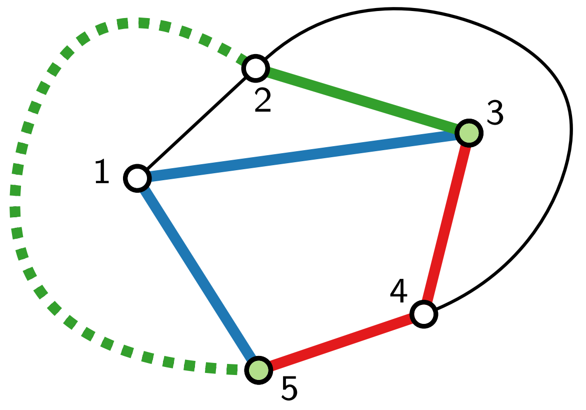

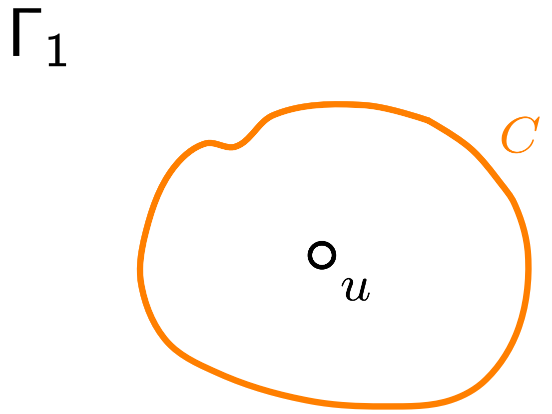

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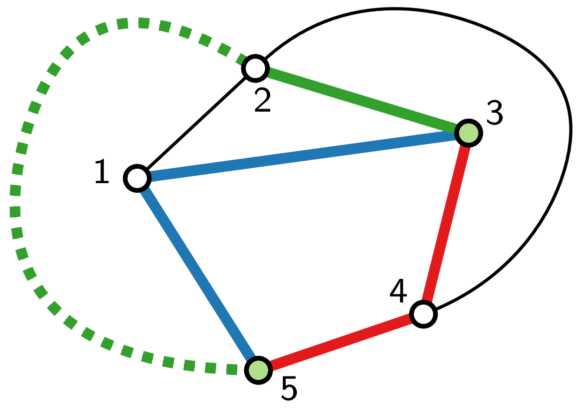

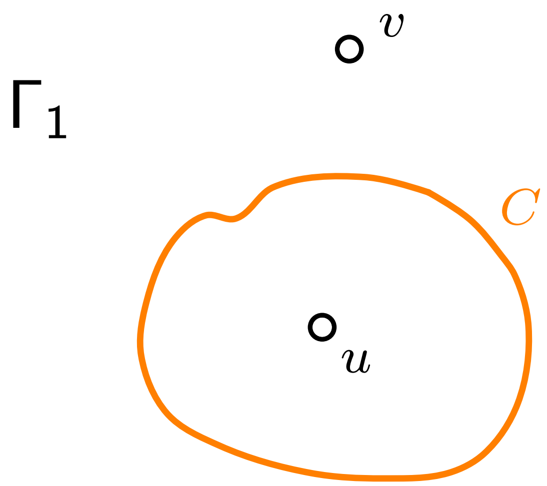

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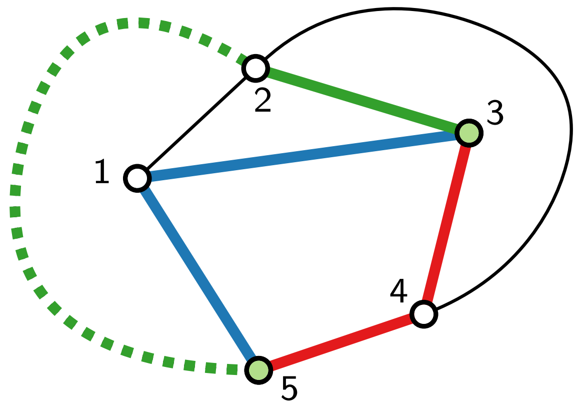

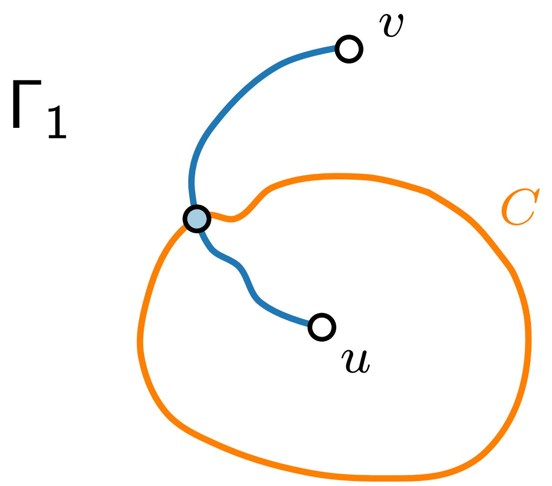

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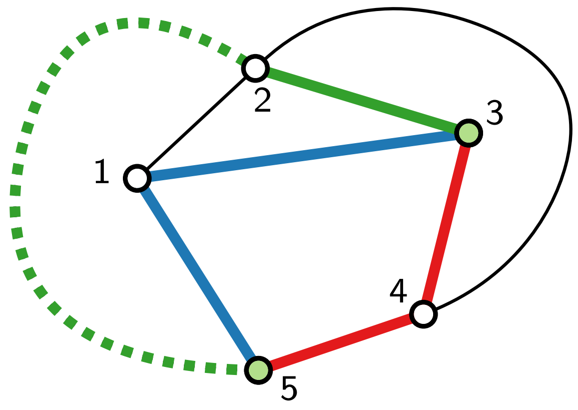

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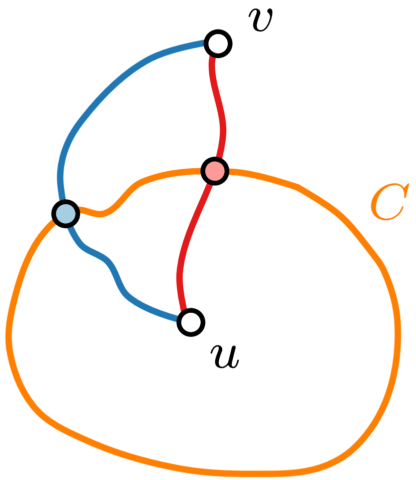

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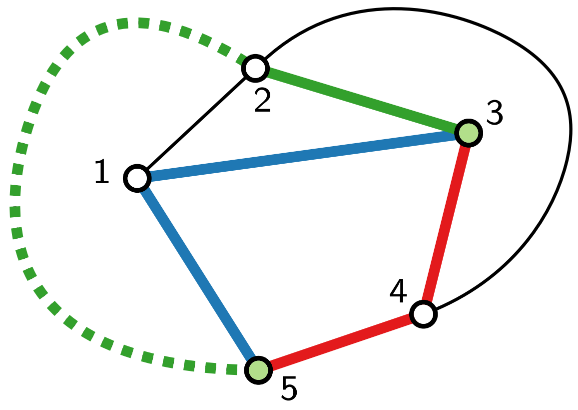

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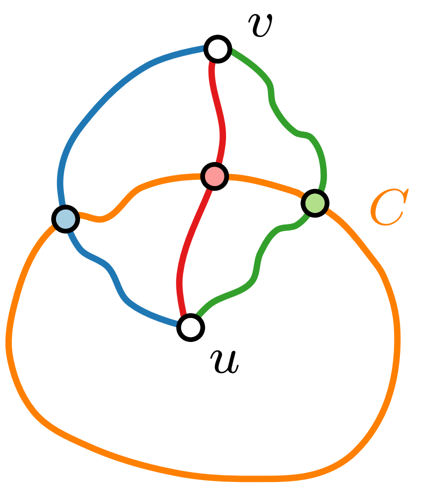

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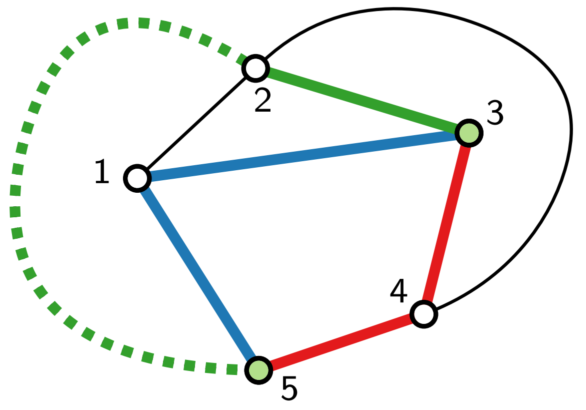

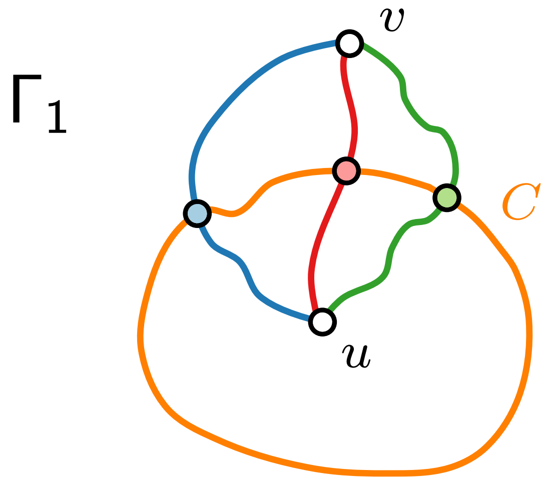

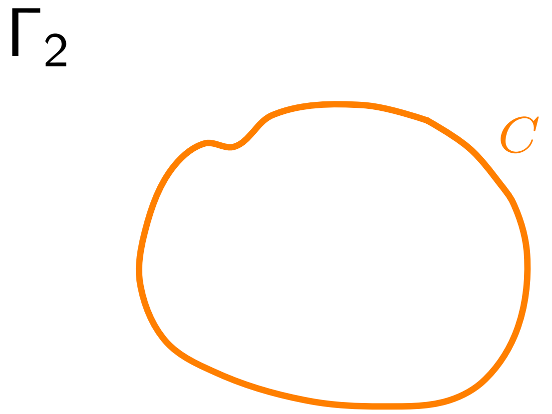

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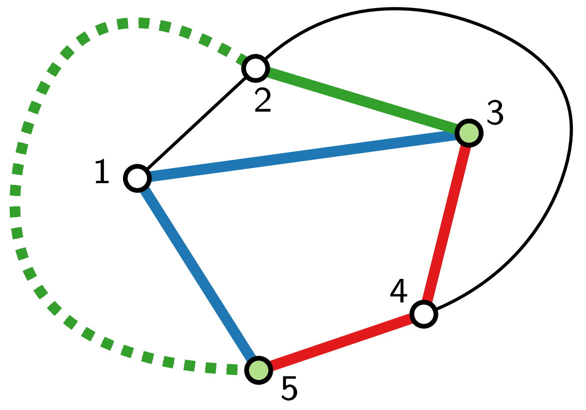

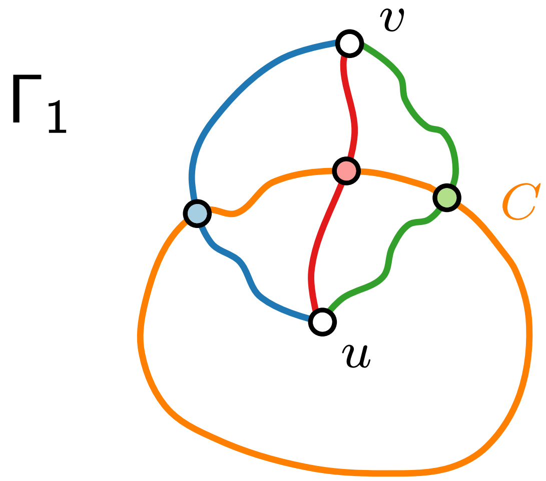

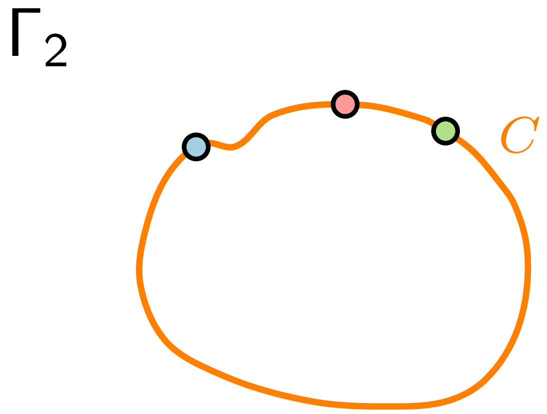

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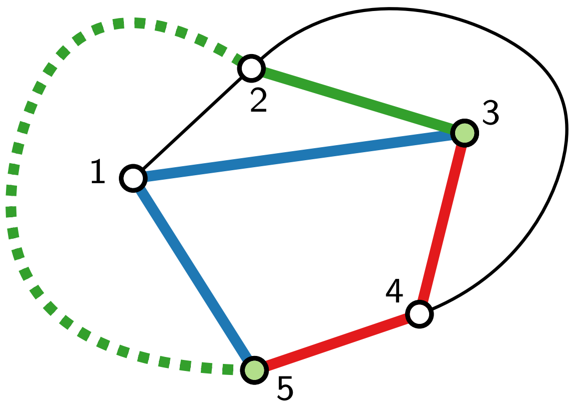

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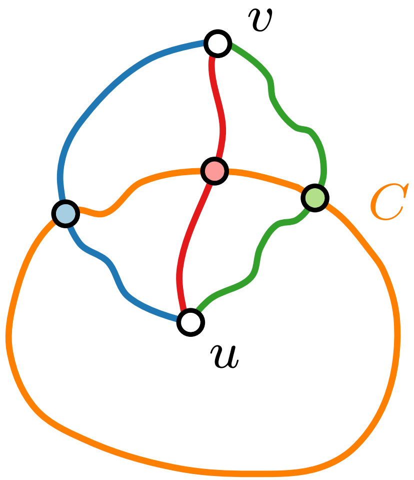

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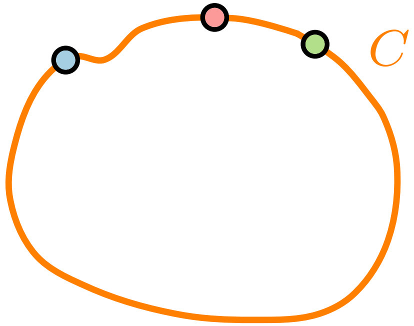

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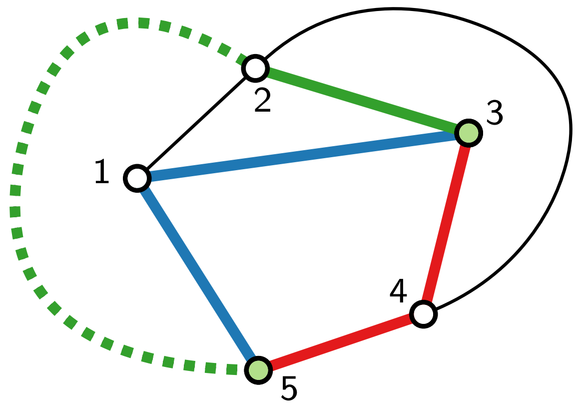

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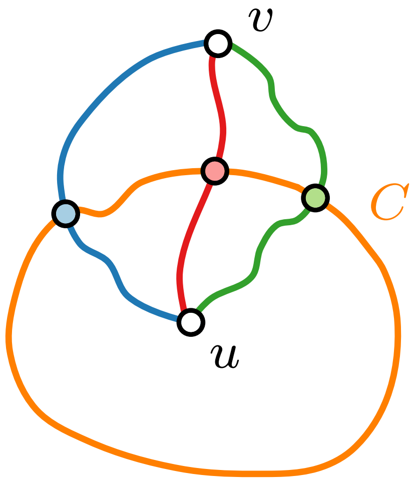

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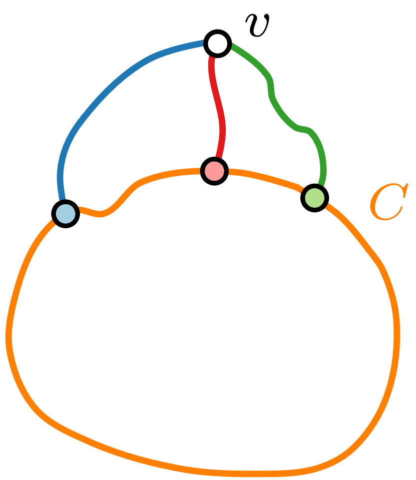

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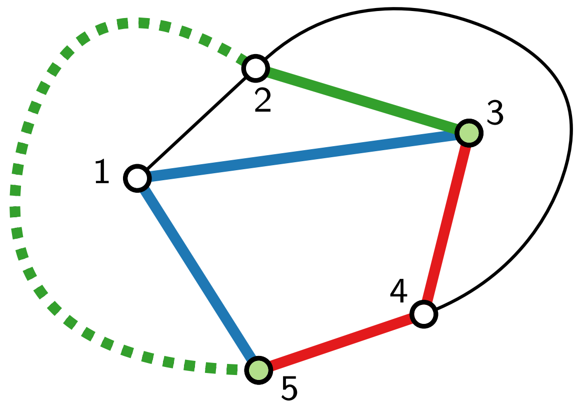

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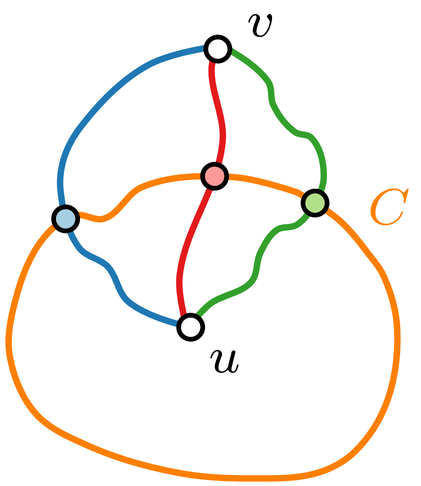

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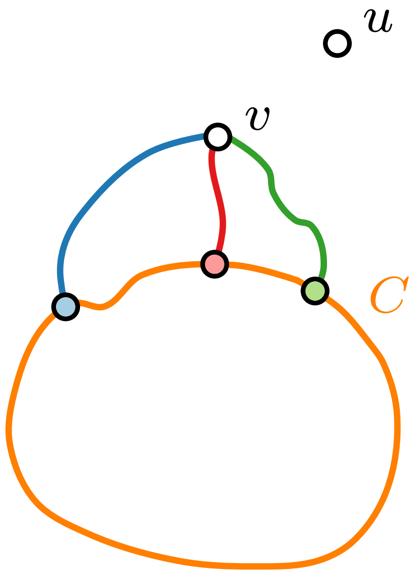

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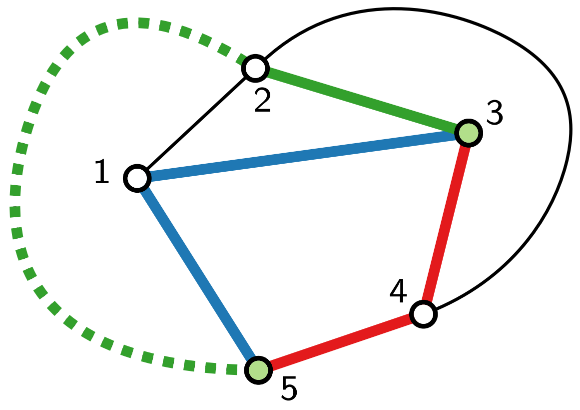

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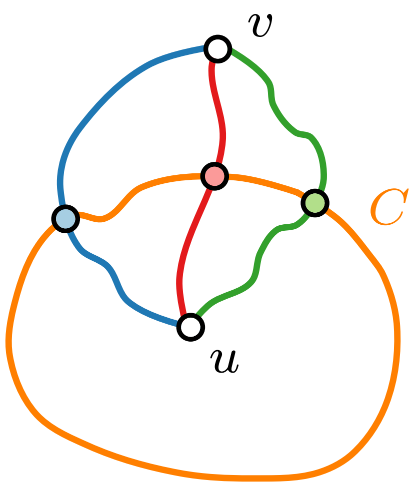

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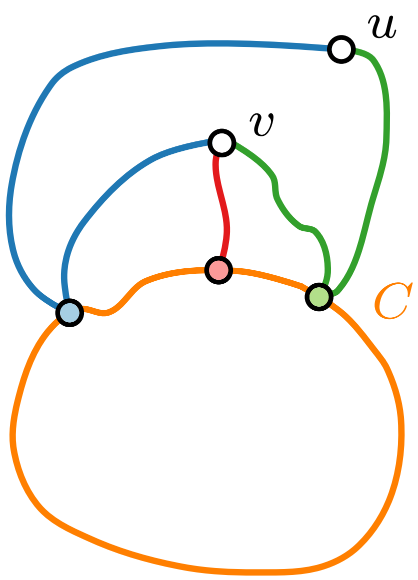

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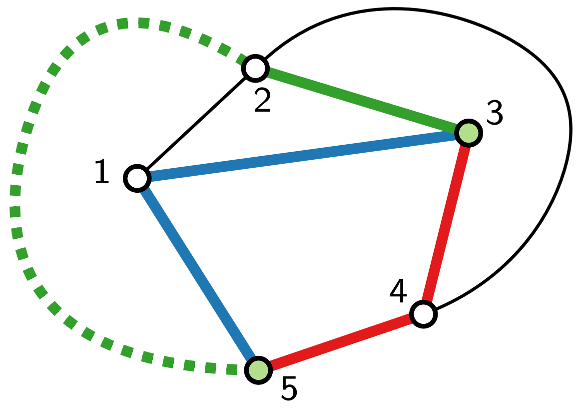

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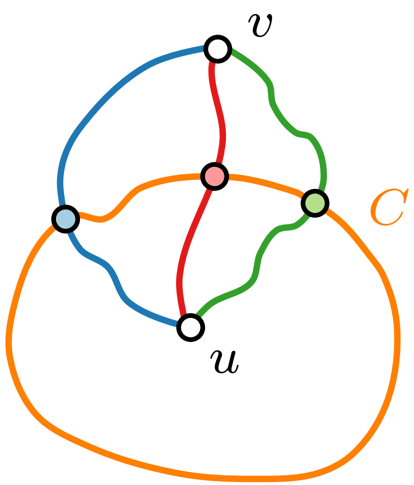

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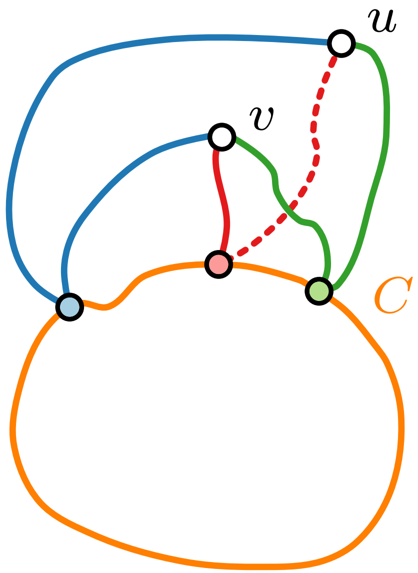

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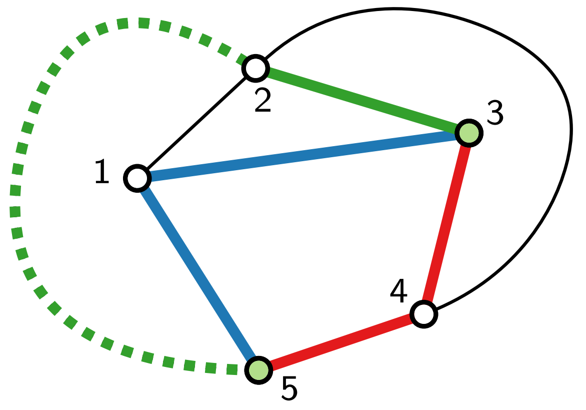

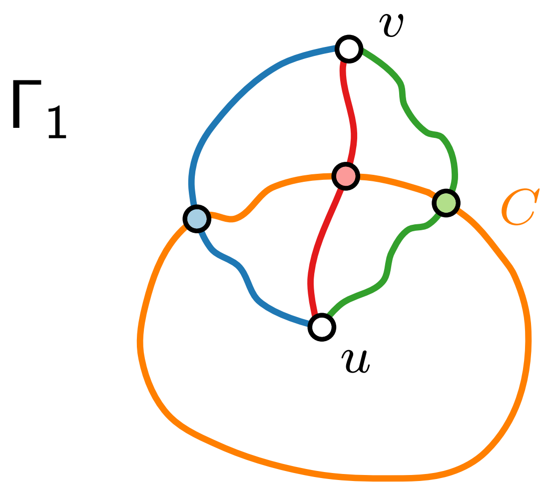

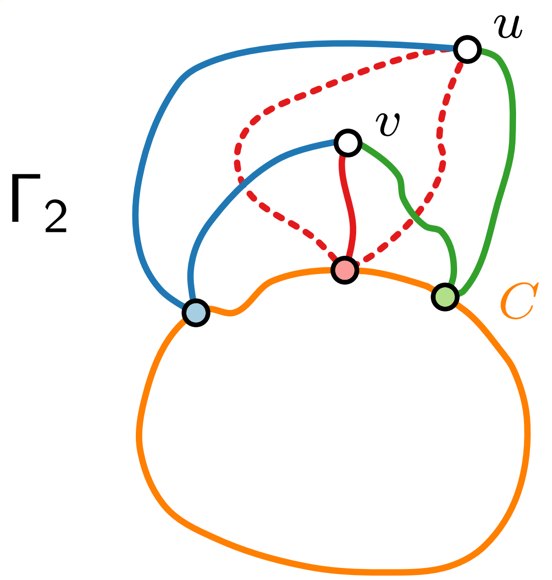

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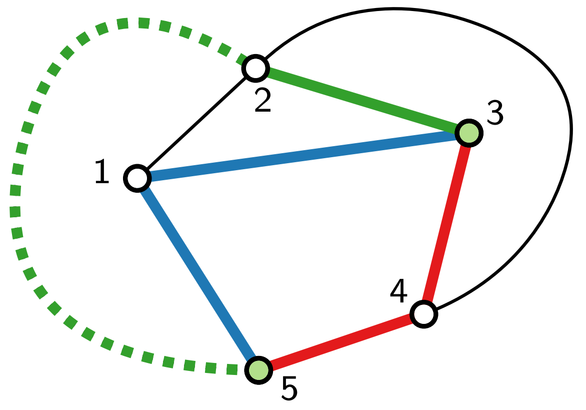

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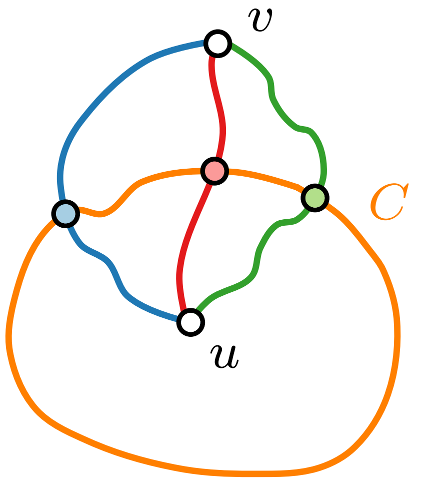

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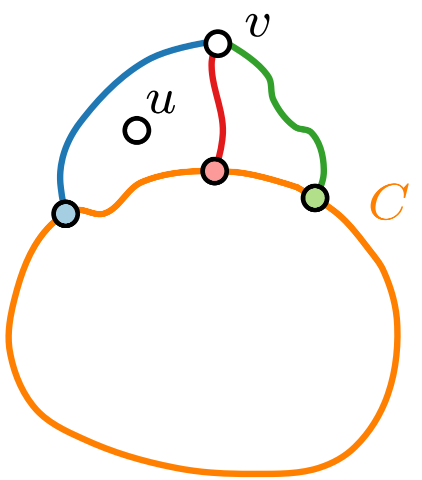

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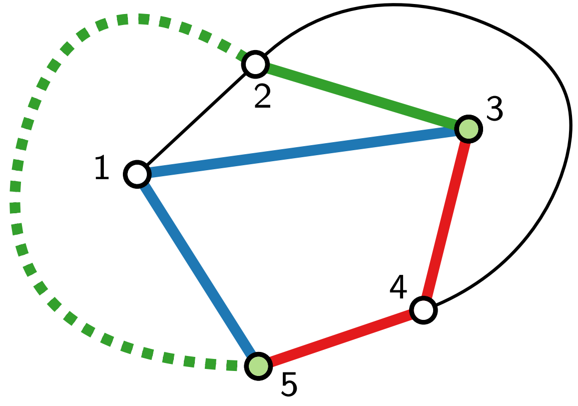

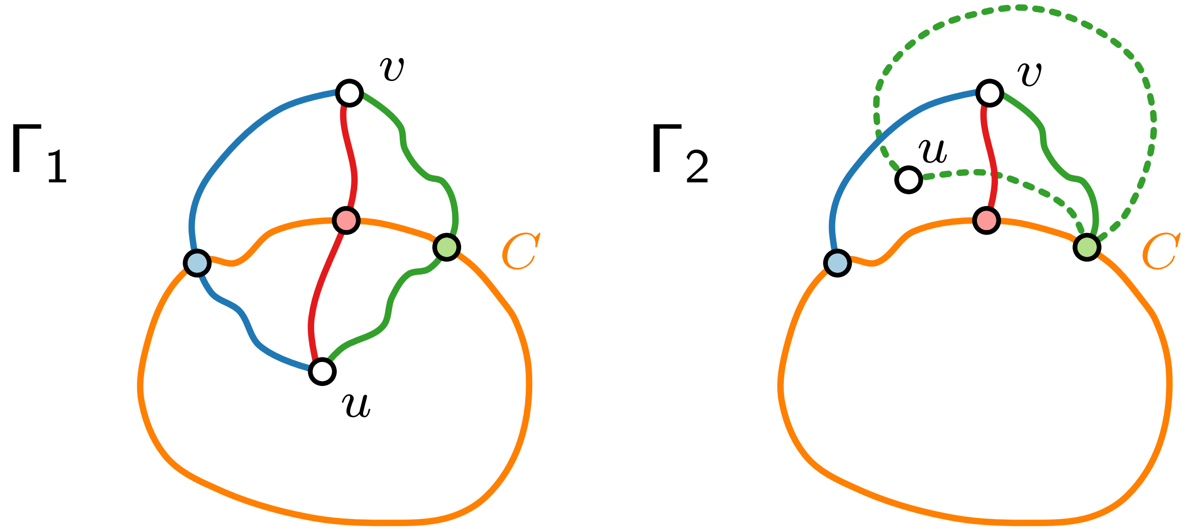

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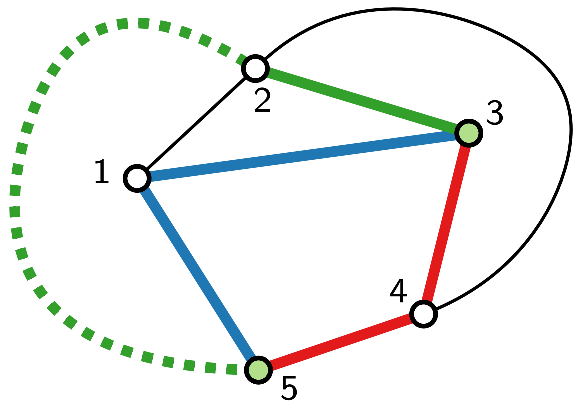

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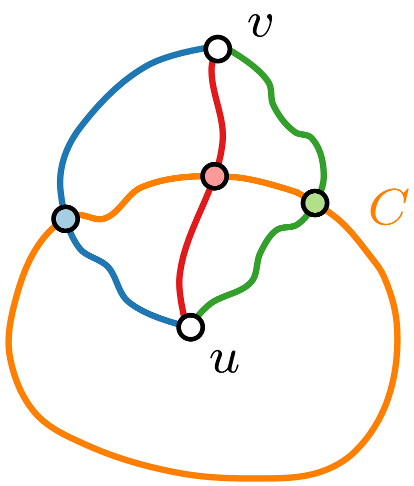

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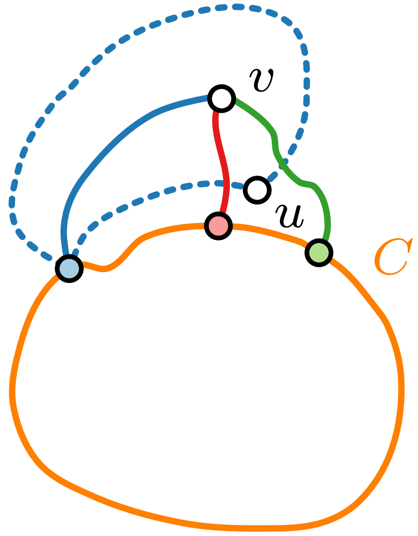

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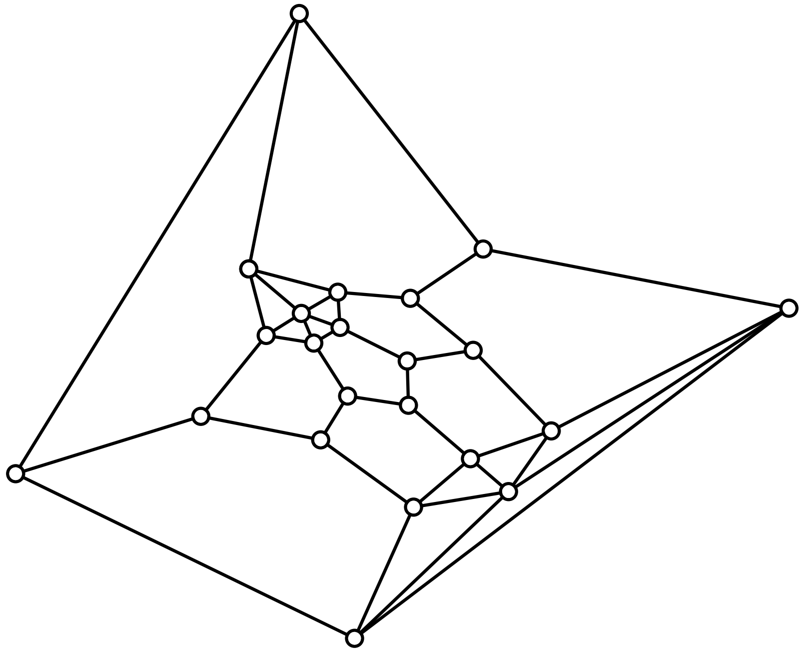

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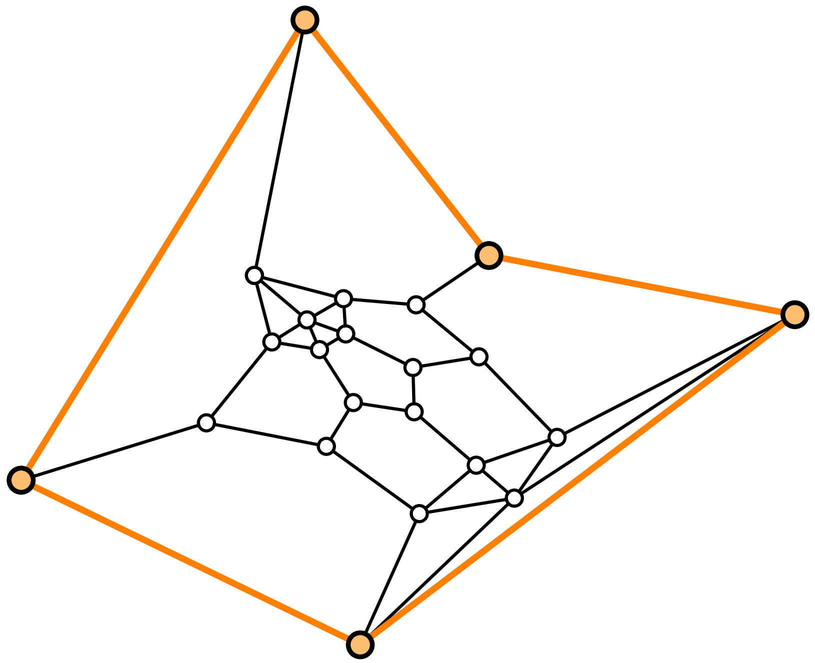
Theorem. [Tutte 1963]

Let G be a 3-connected planar graph, and let C be a face of its unique embedding.



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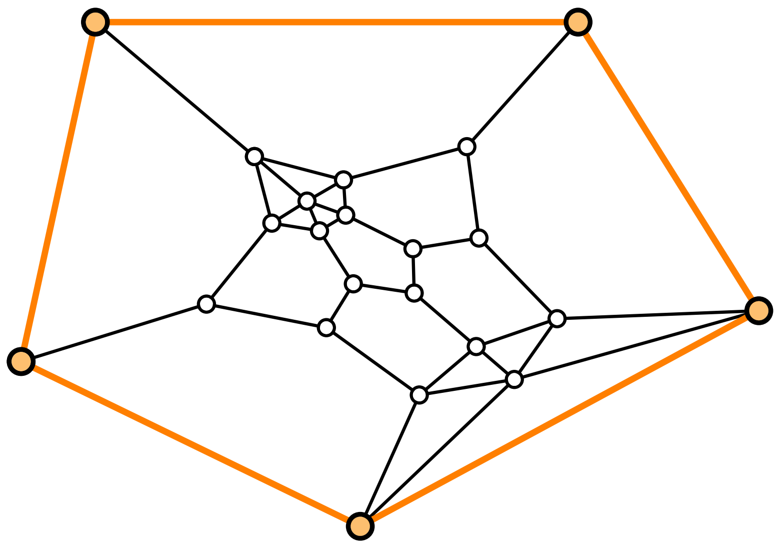
If we fix C on a strictly convex polygon,



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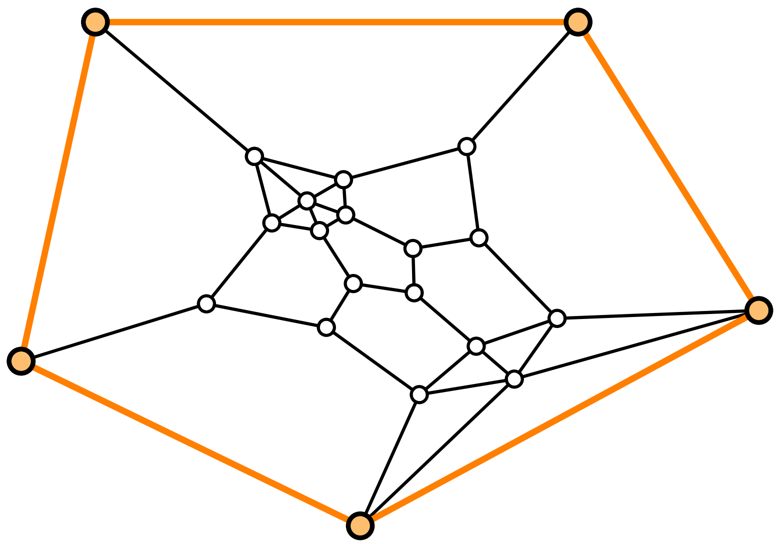
Theorem. [Tutte 1963]
Let G be a 3-connected planar graph, and let C be a face of its unique embedding.
If we fix C on a strictly convex polygon, then the Tutte drawing of G is planar



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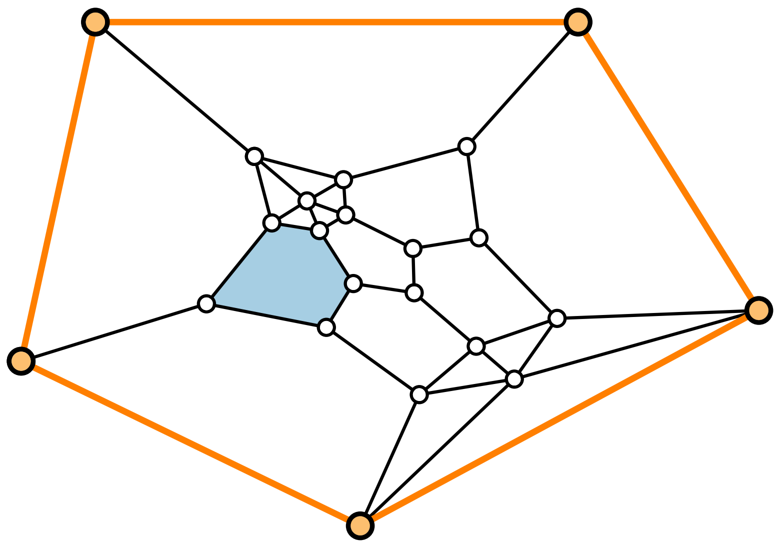
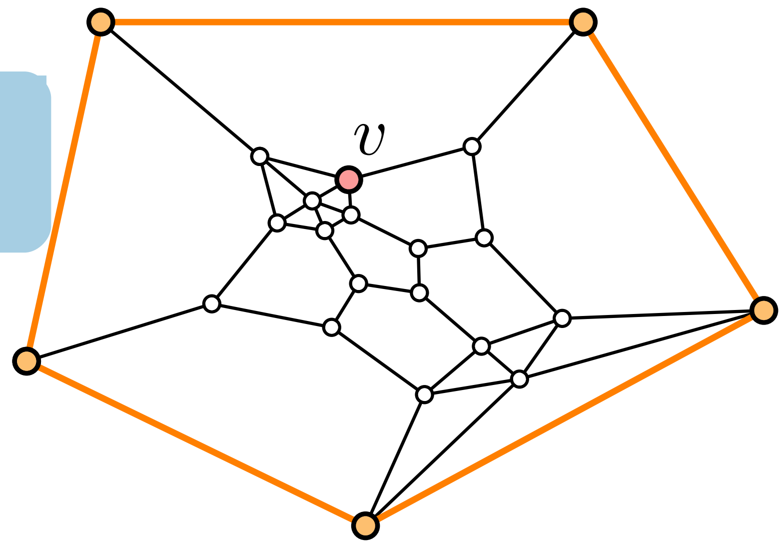
let C be a face of its unique embedding.
If we fix C on a strictly convex polygon, then the Tutte drawing of G is planar and all its faces are strictly convex.



Properties of Tutte Drawings

Property 1. Let $v \in V(G)$ be free



Properties of Tutte Drawings

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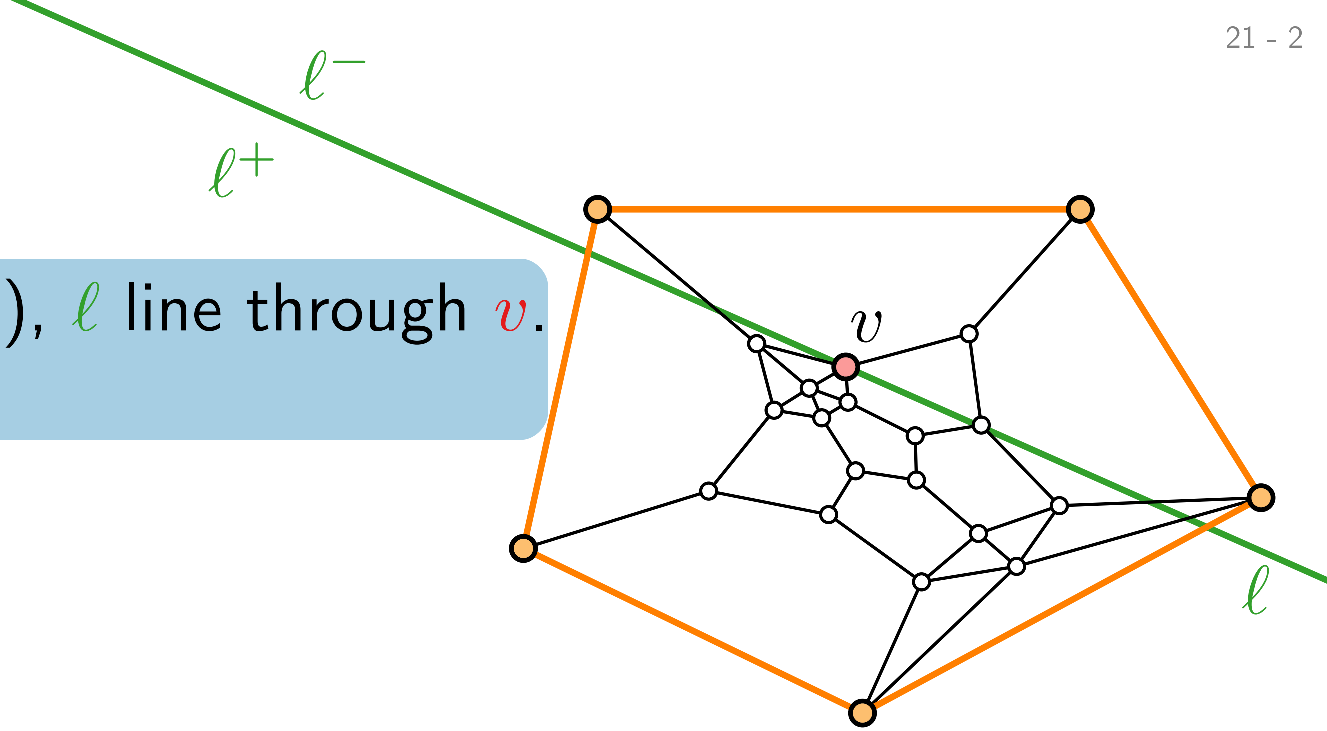

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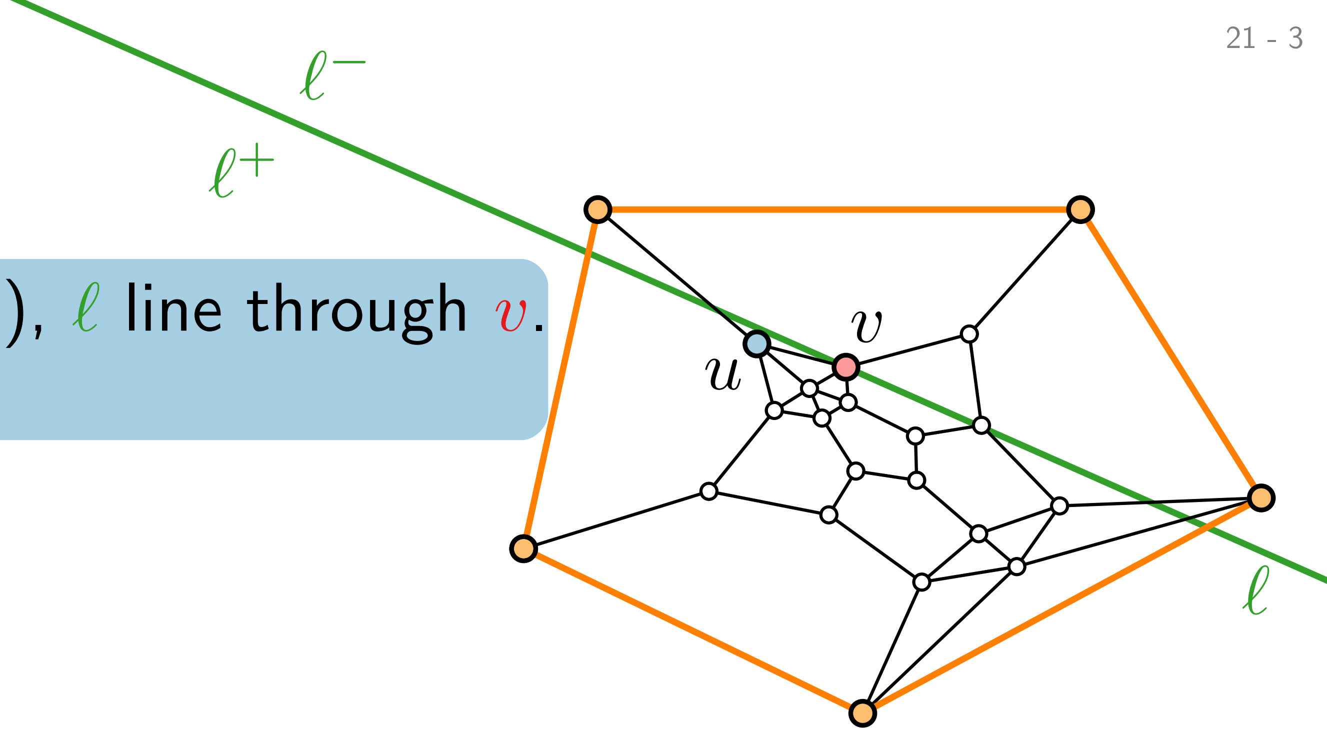
$\exists u \in \text{Adj}[v] \cap \ell^+ \Rightarrow$



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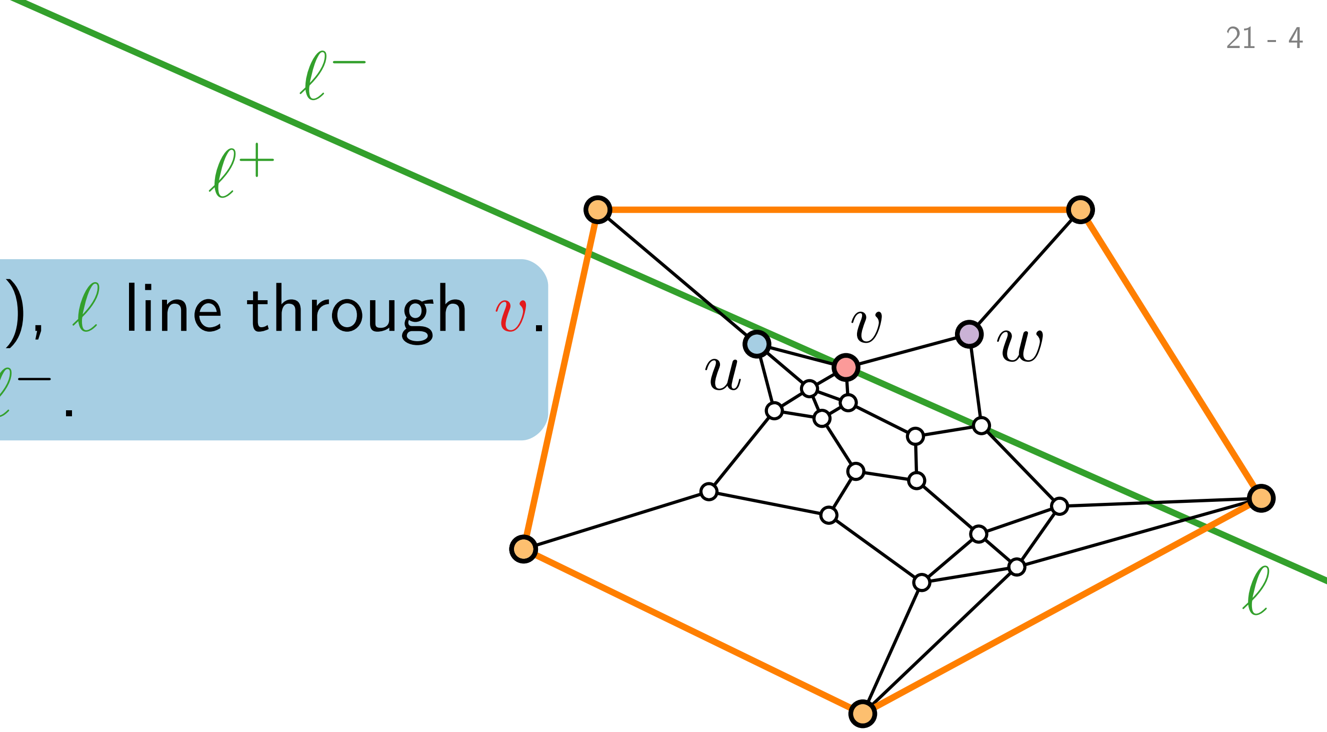

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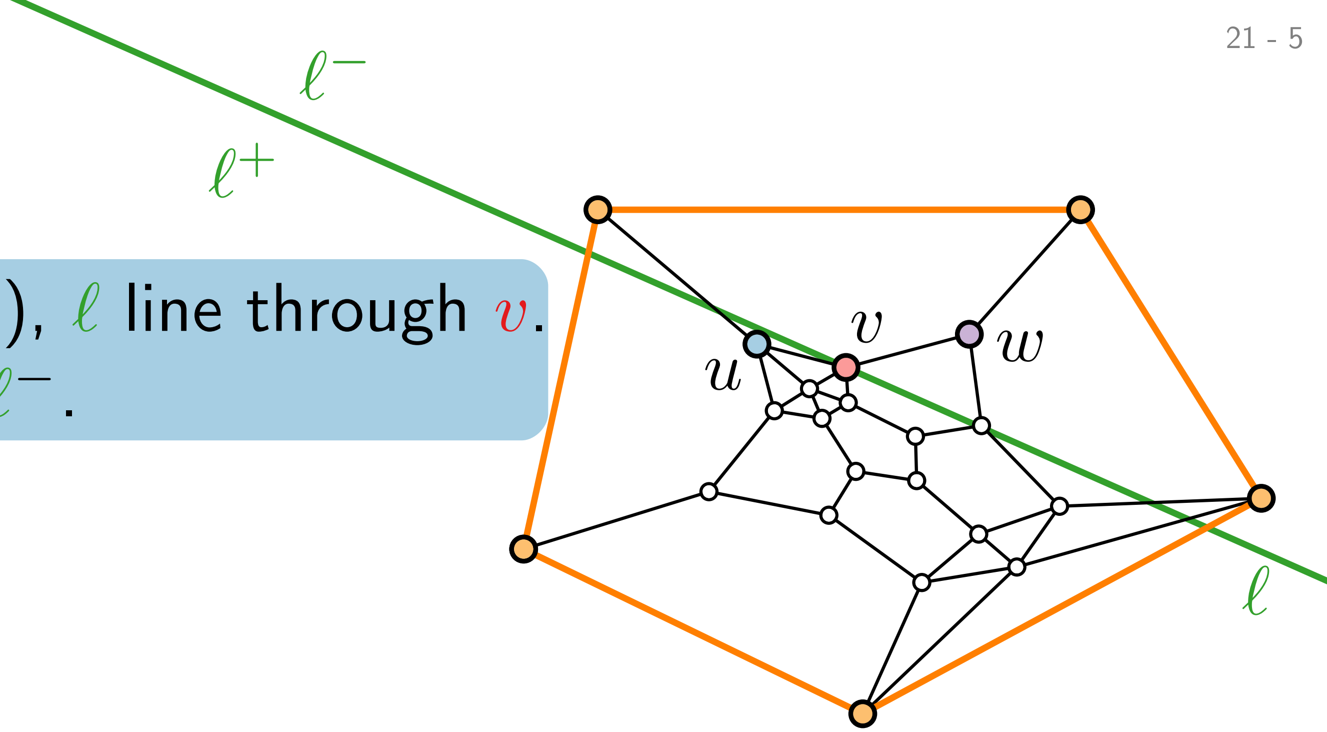
Otherwise, all forces pull v to the same side ...



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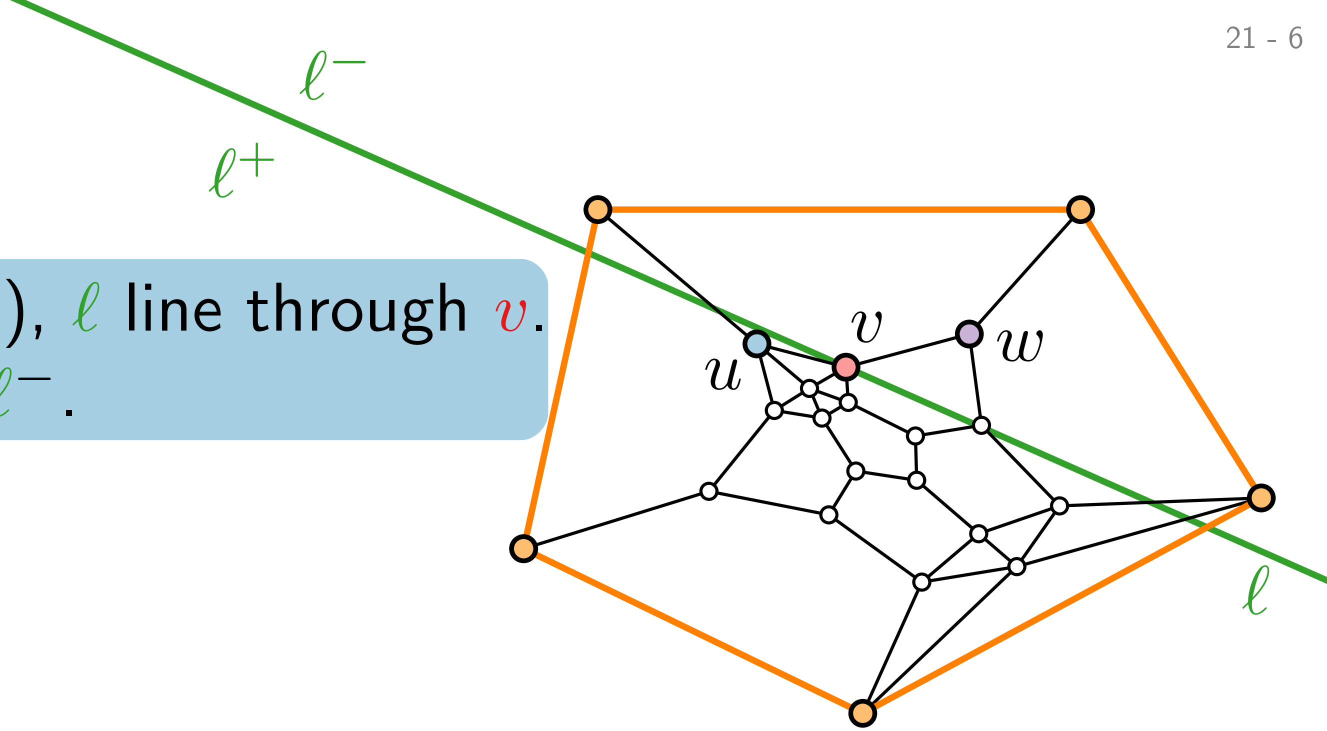

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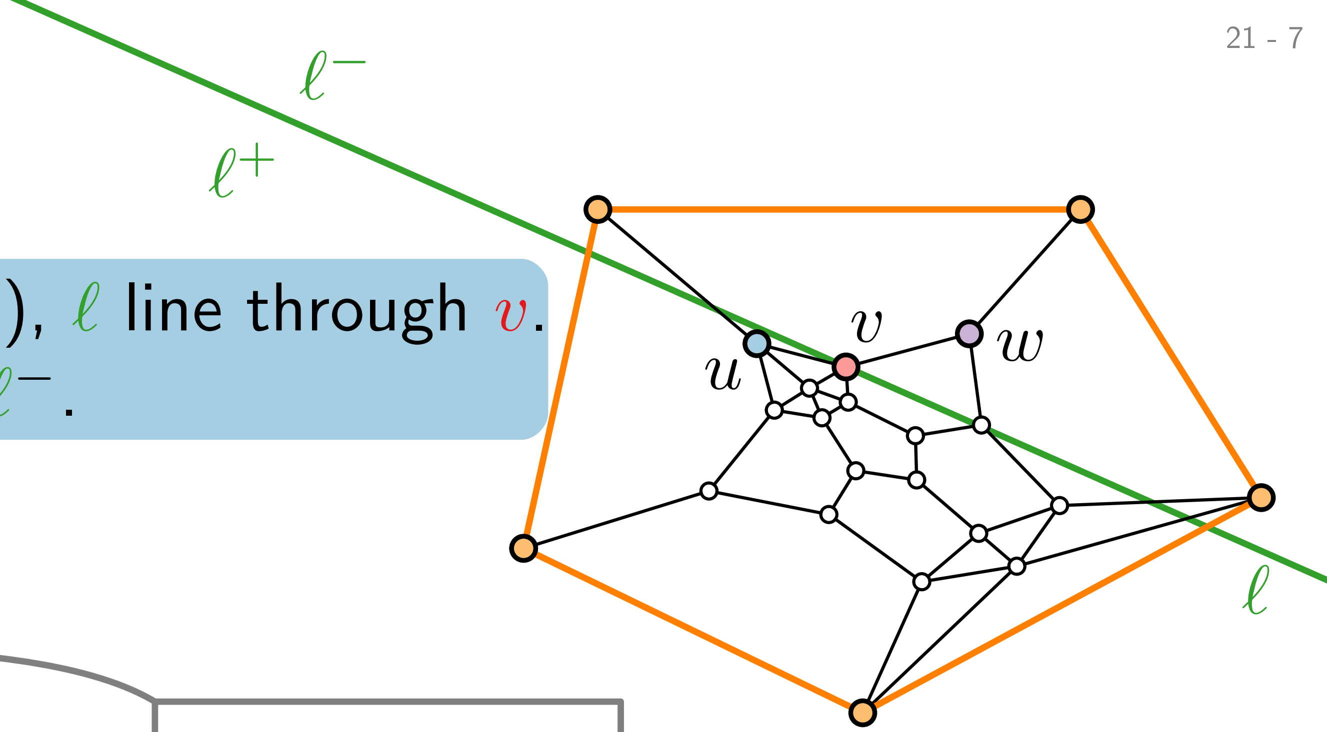

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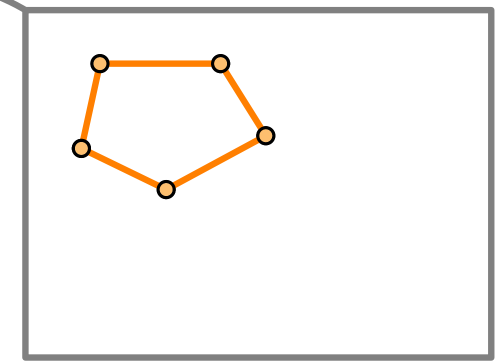

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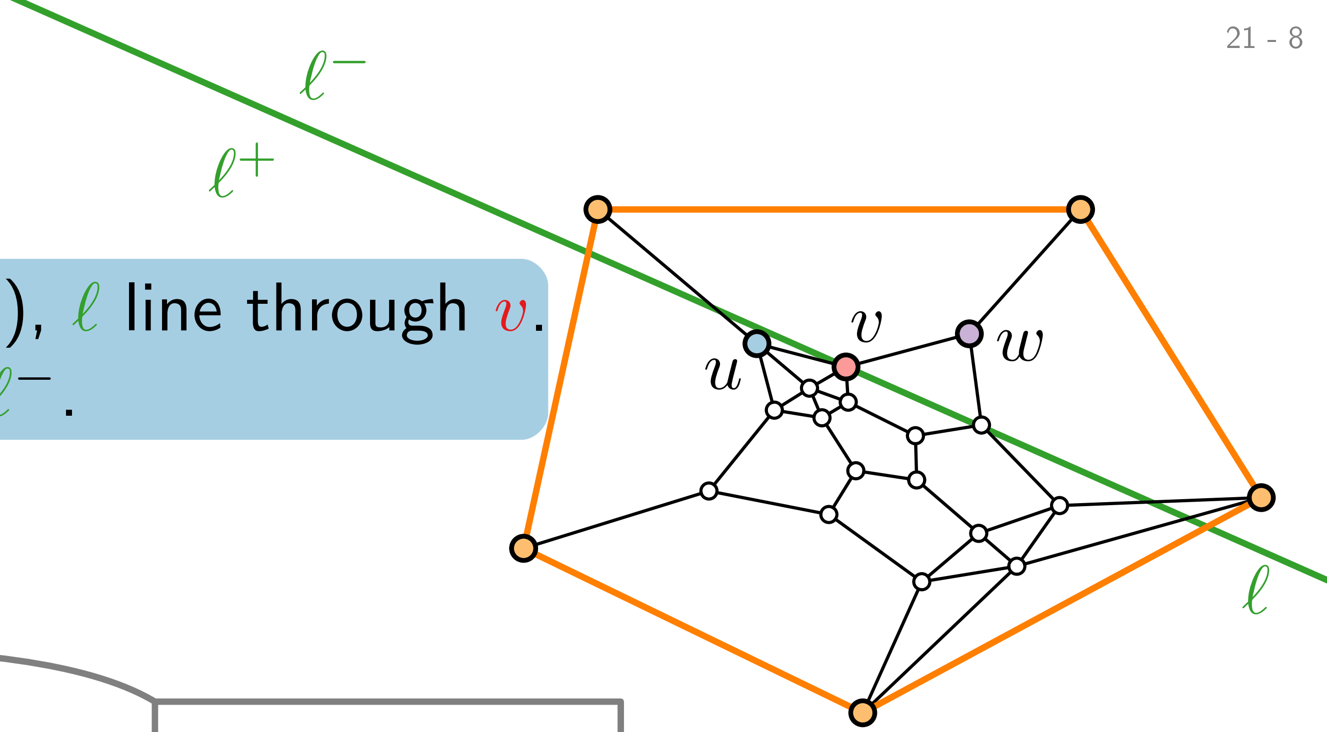

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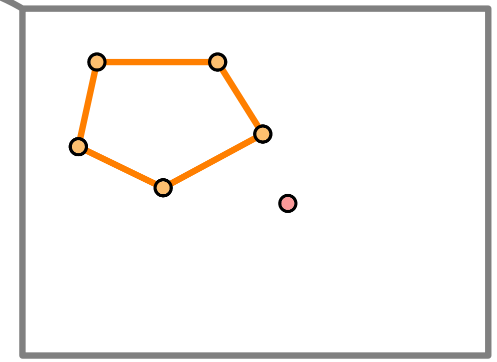

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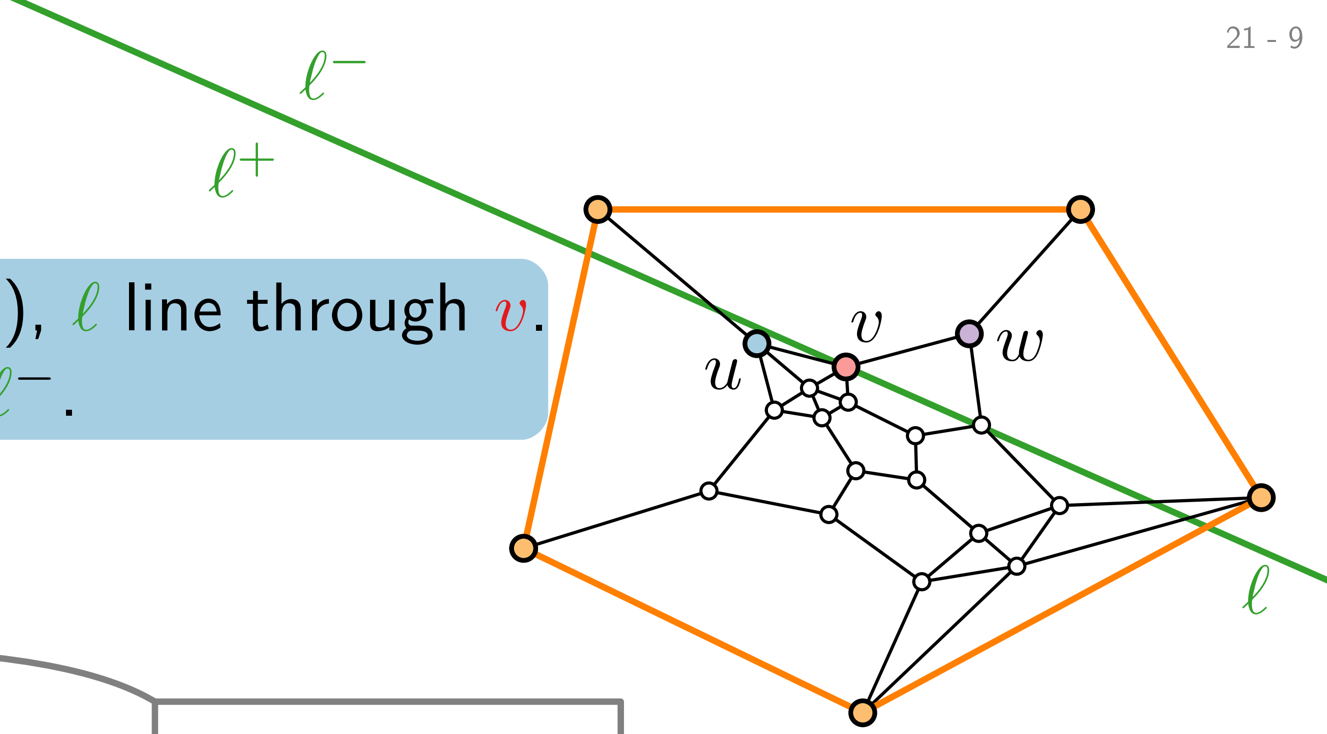

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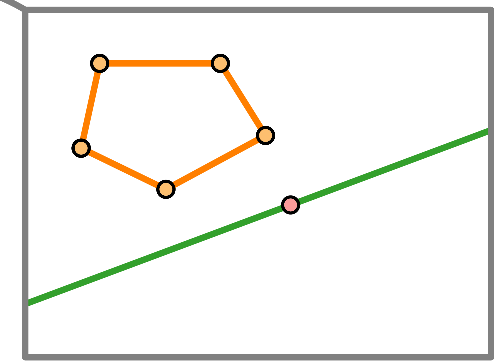

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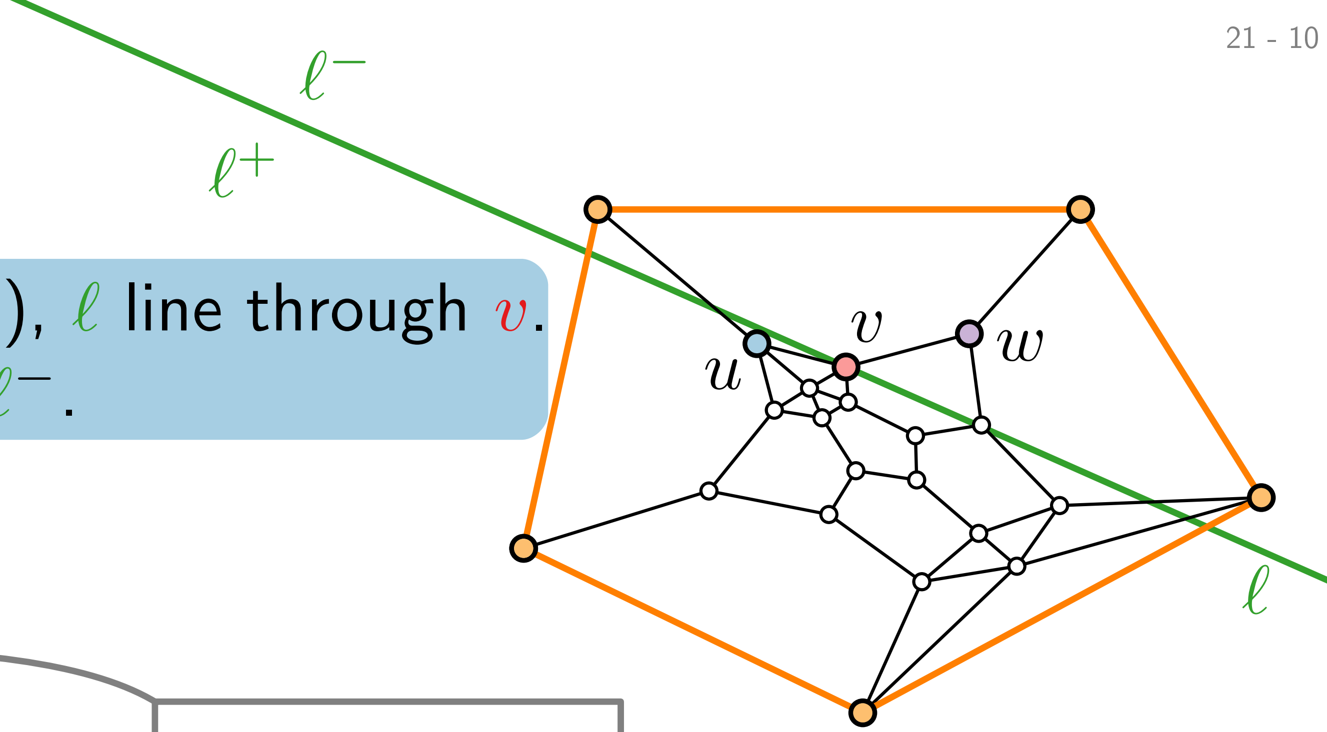

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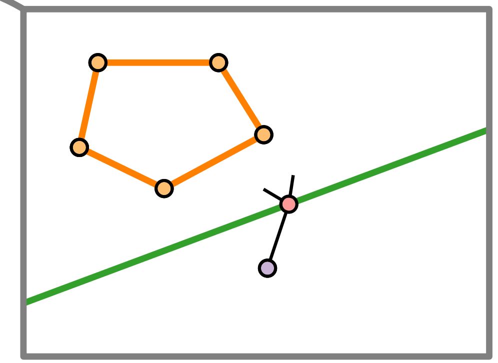

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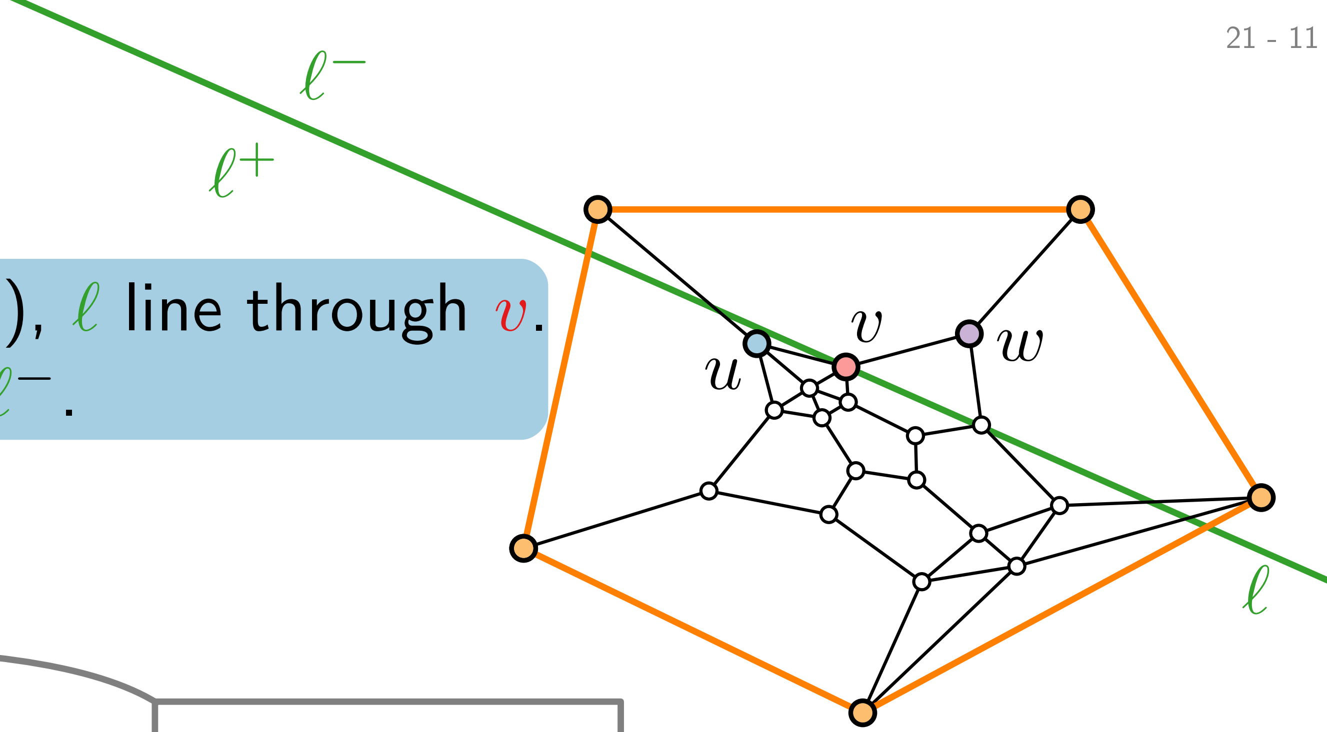

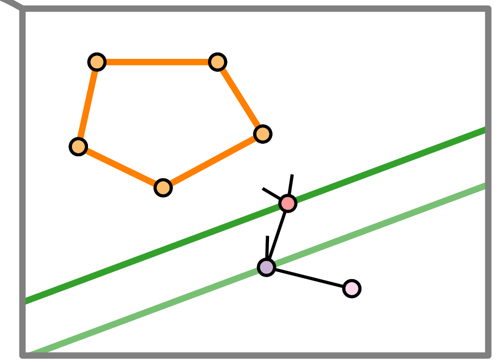

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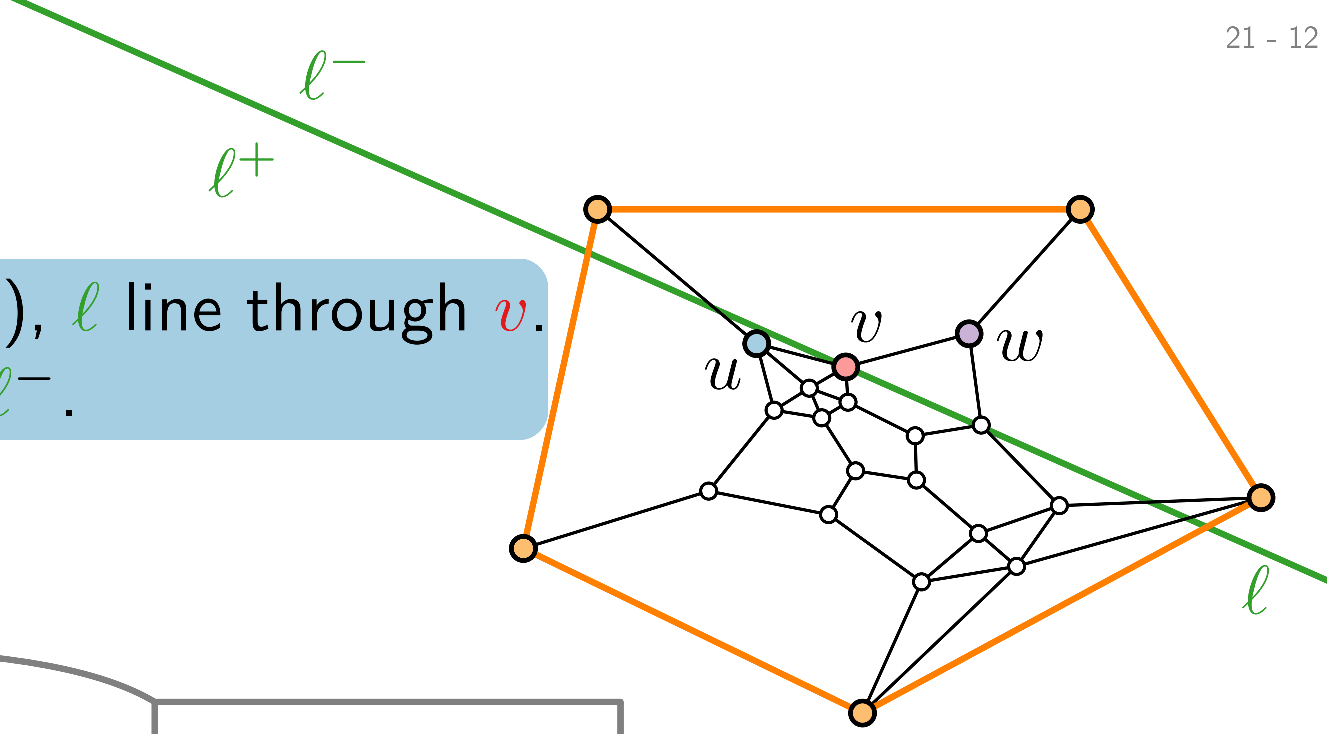

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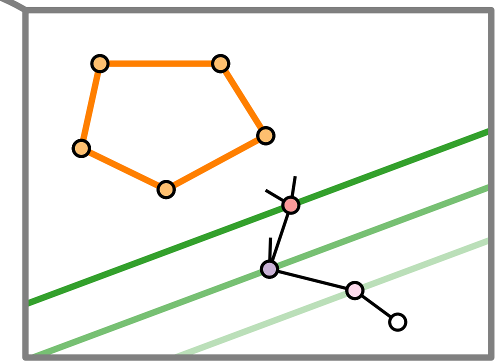

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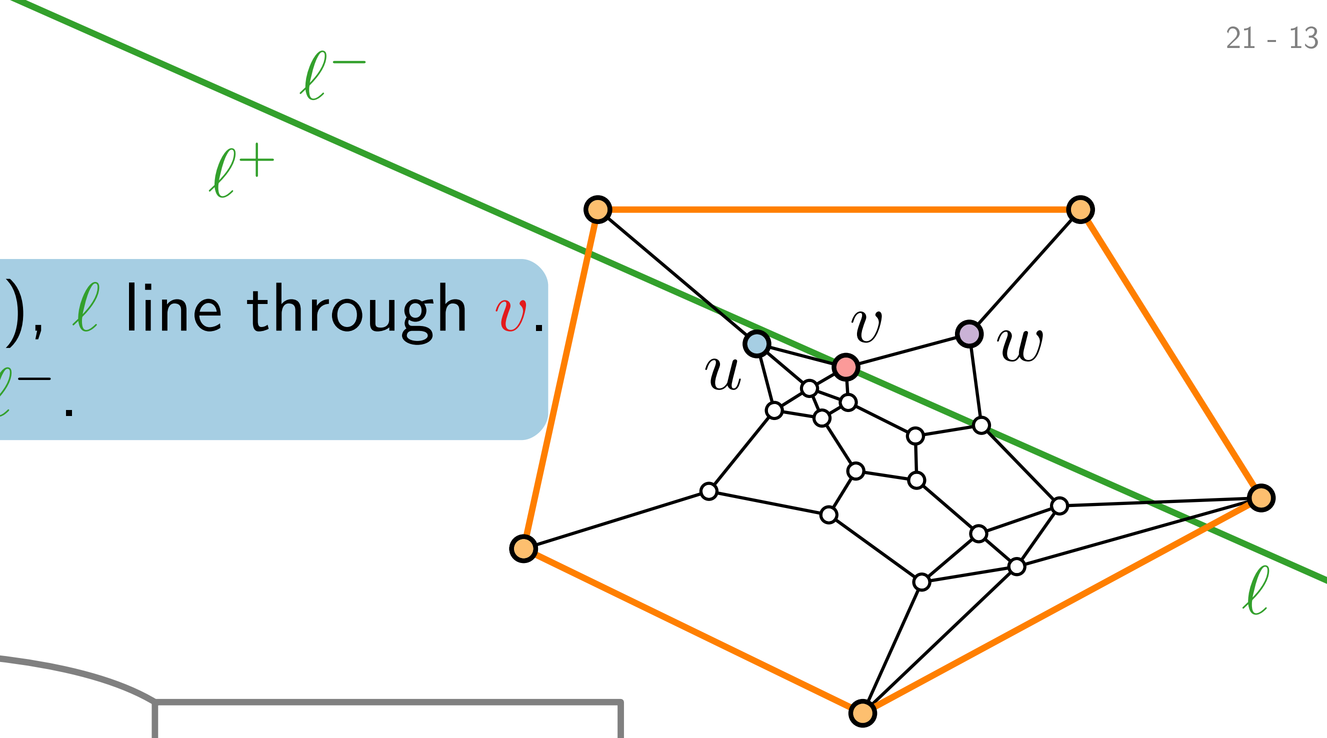

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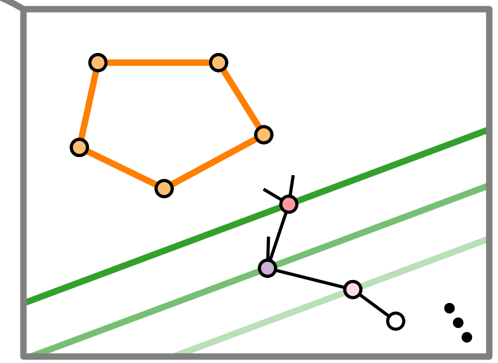

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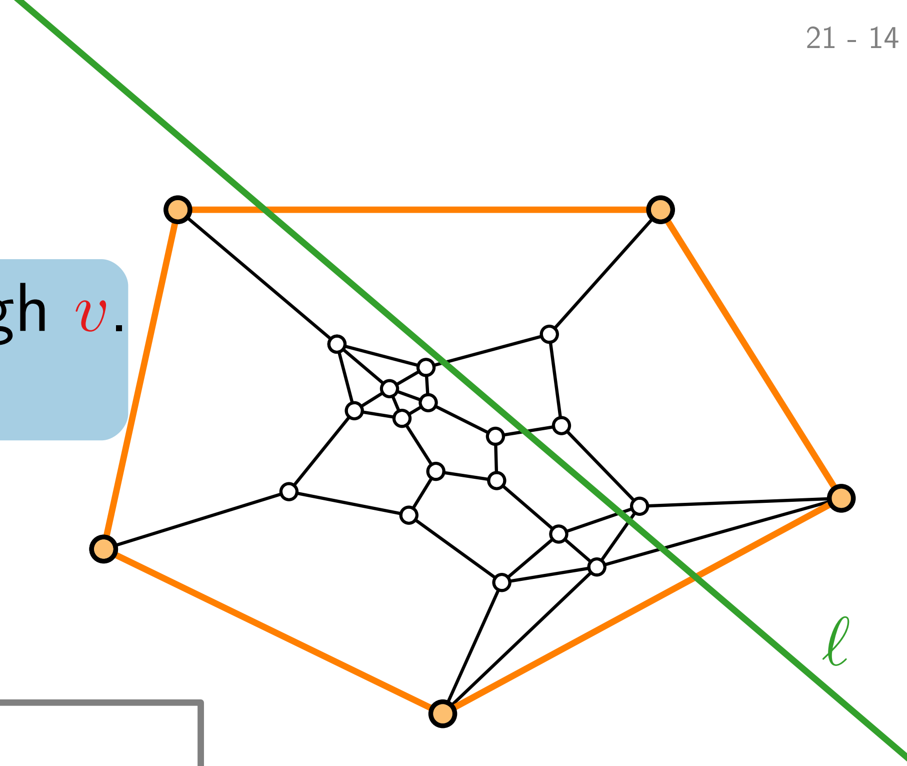

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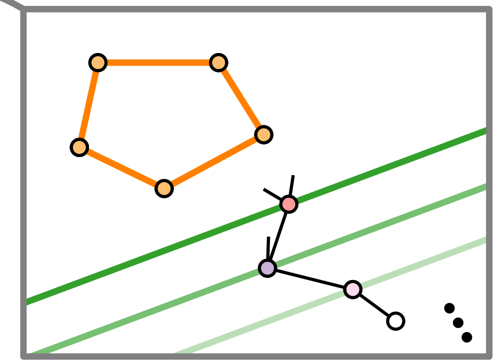

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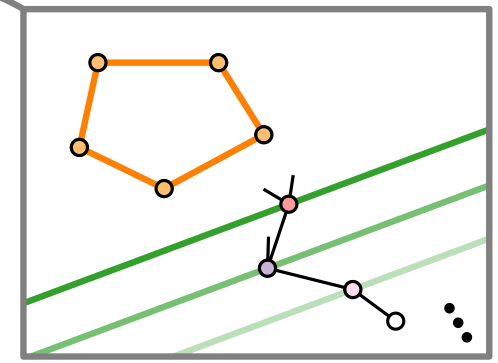

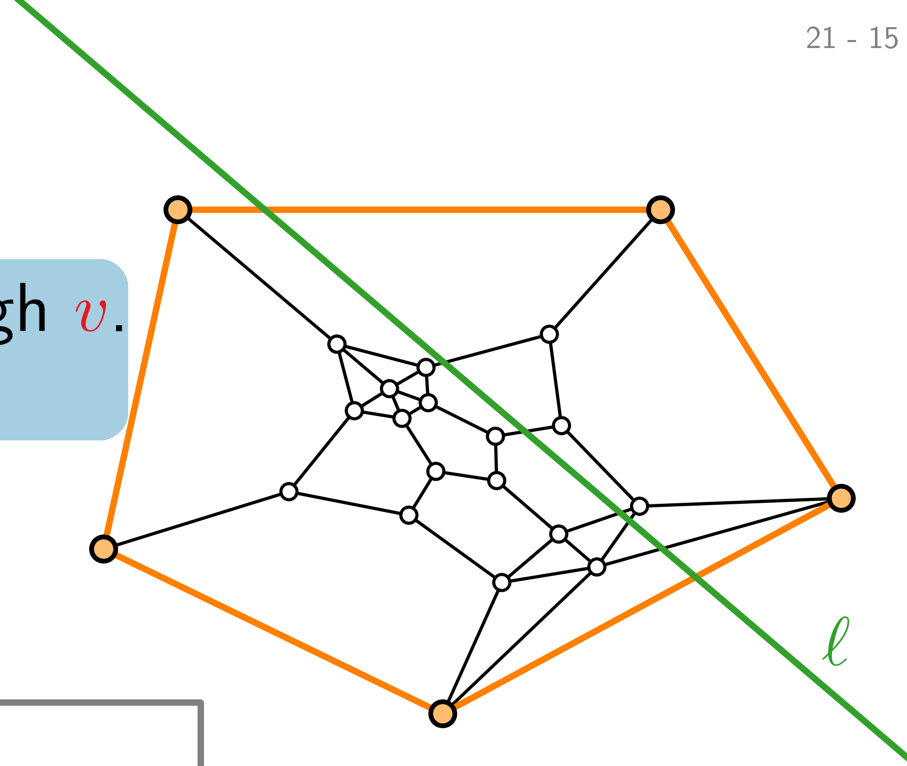

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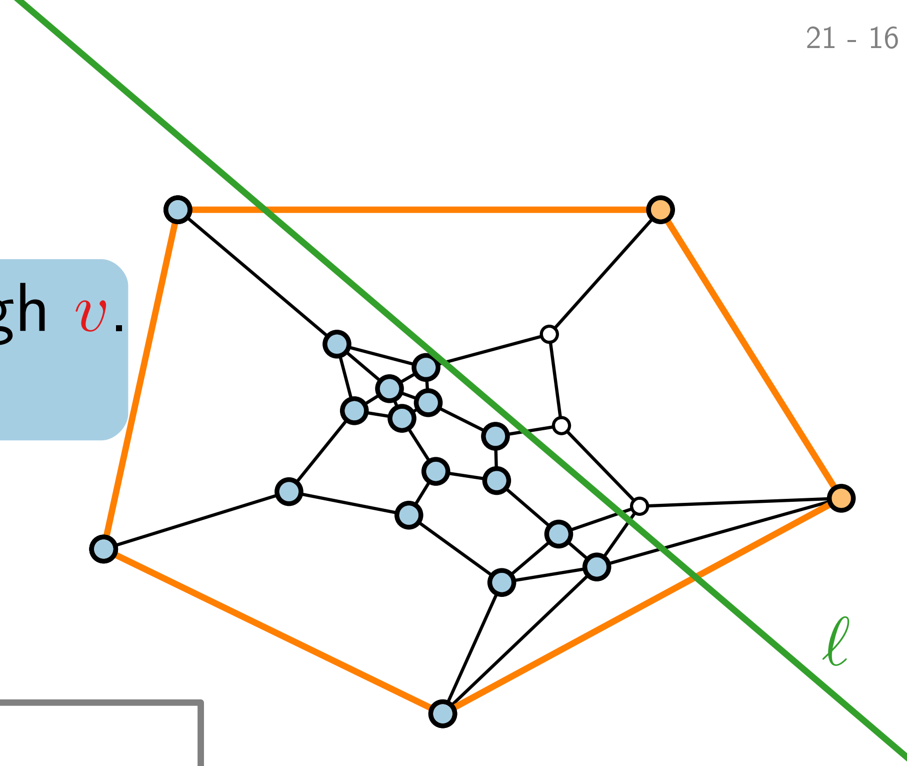

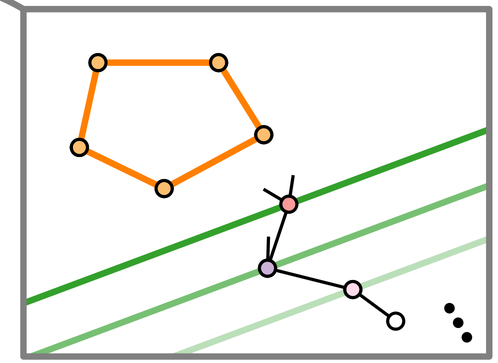

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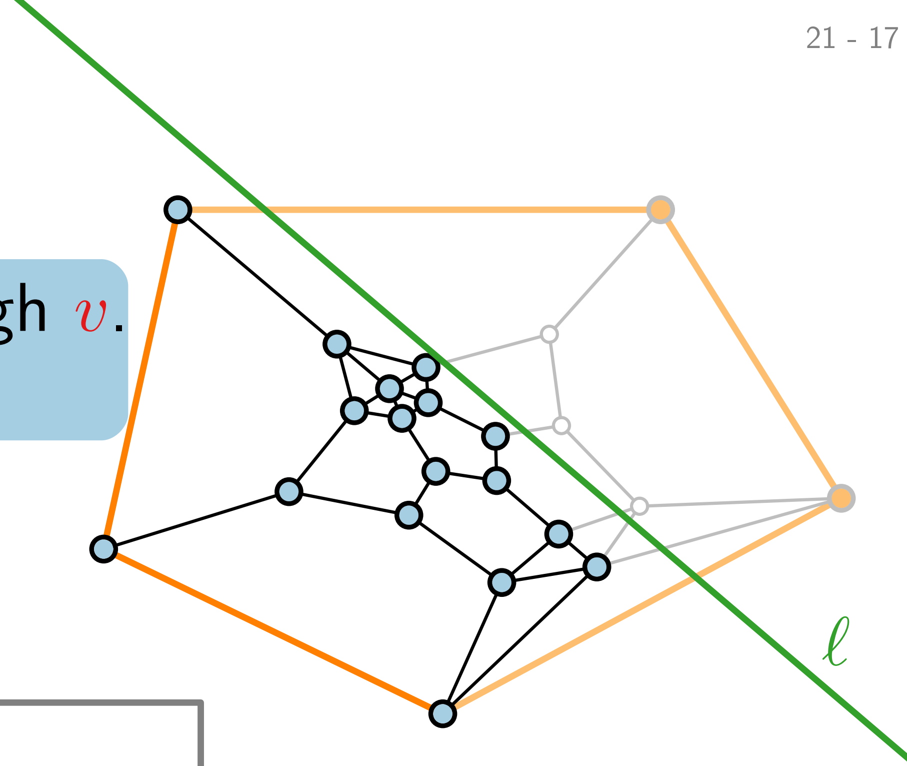

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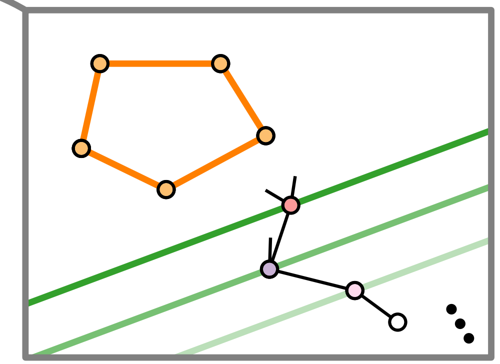

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Then $G[V_\ell]$ is connected.

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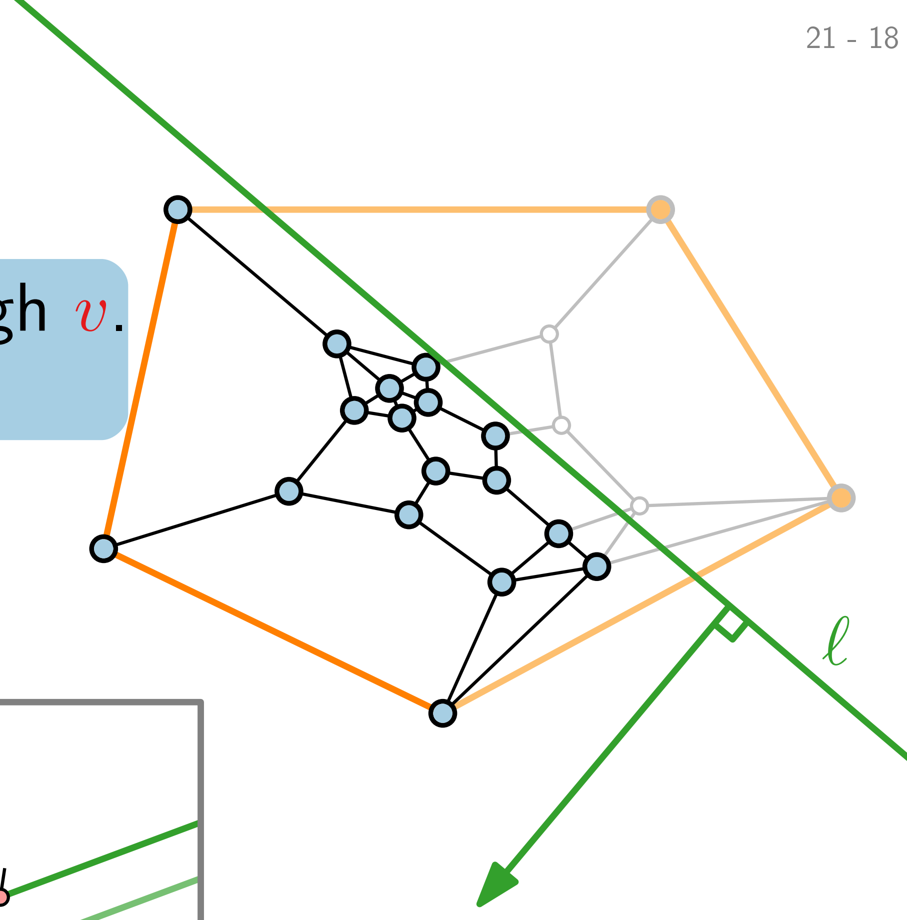

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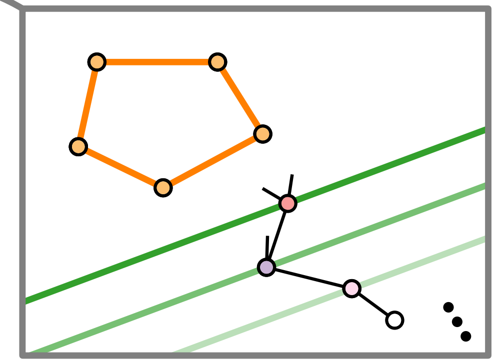

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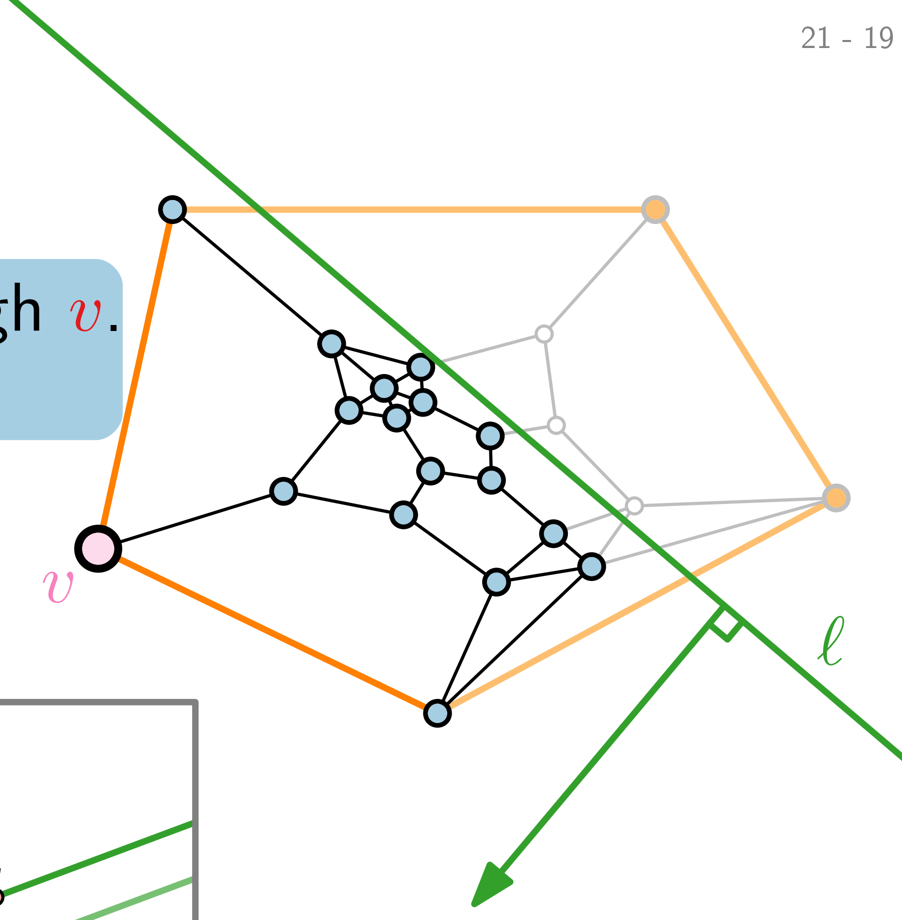

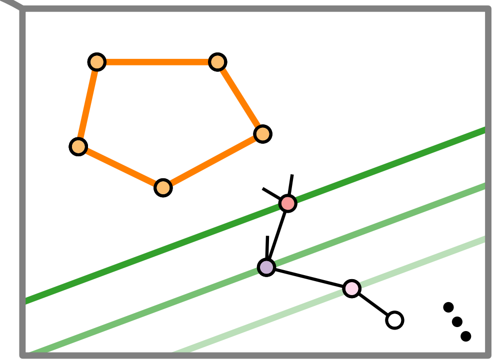

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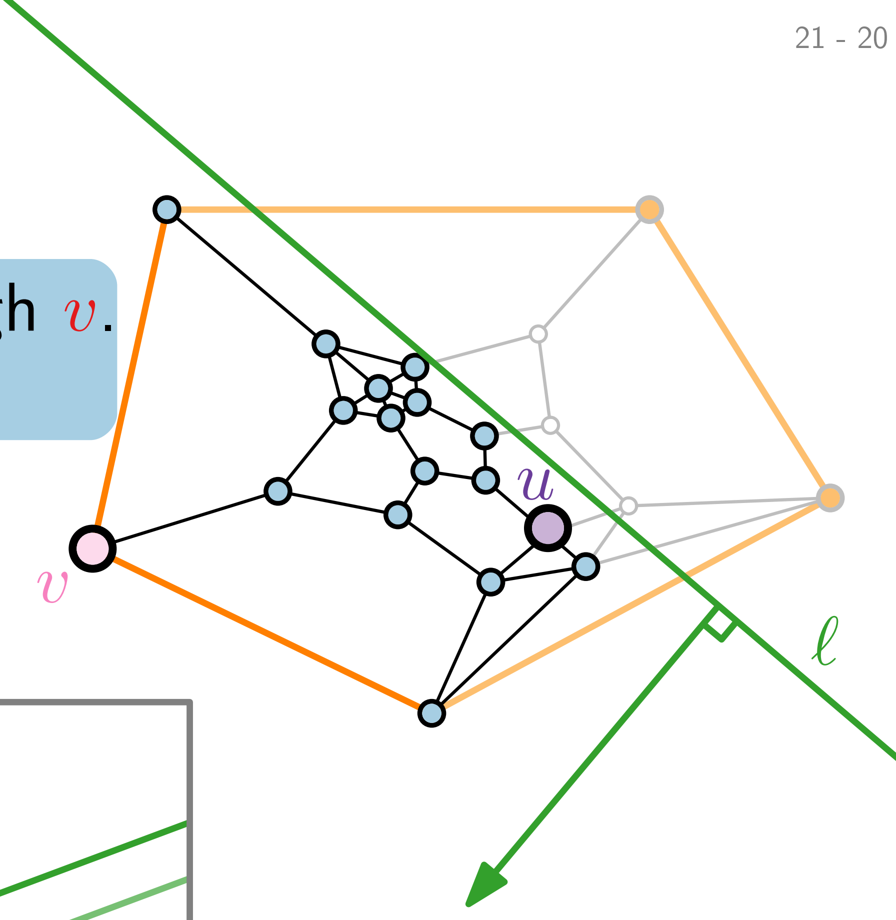

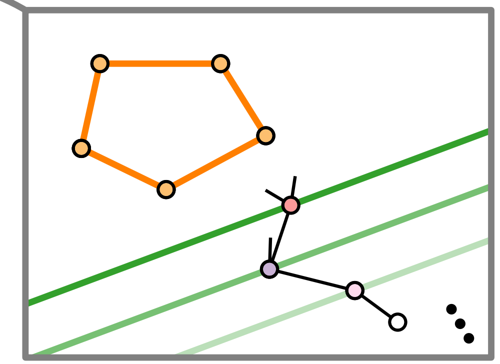

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Pick any vertex $u \in V_\ell$

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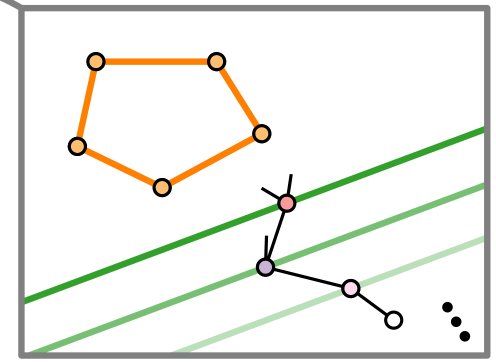

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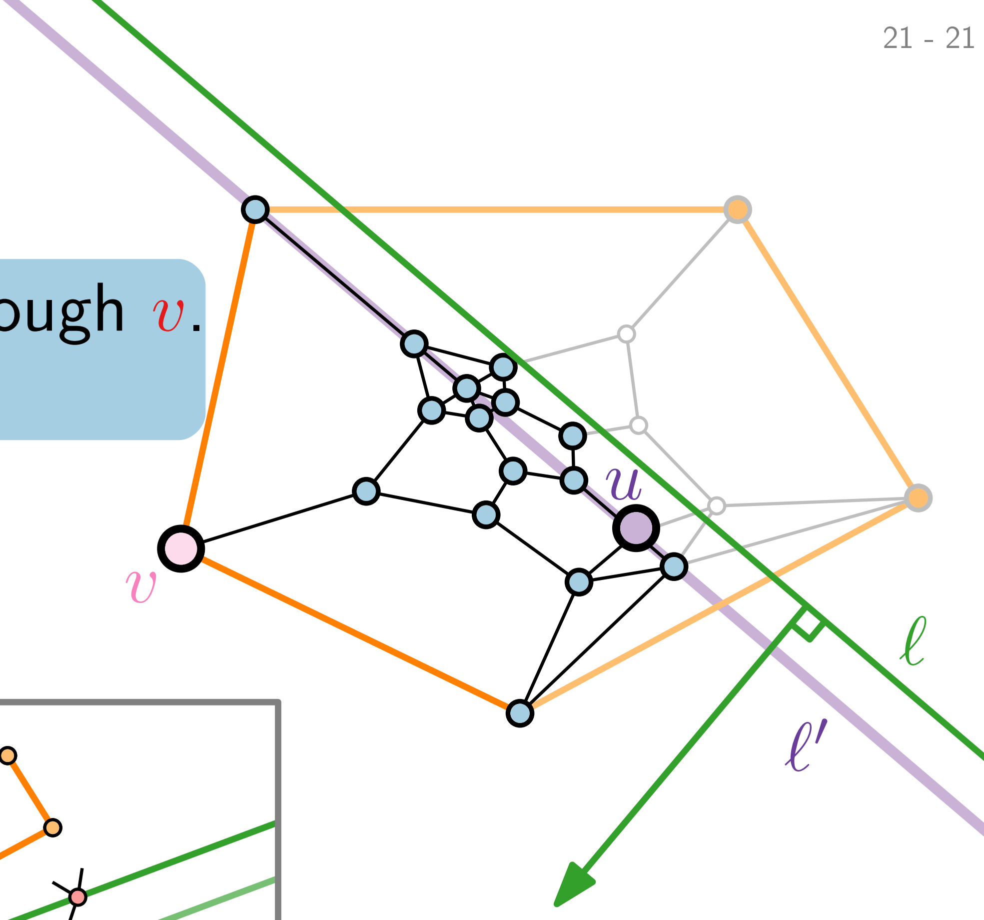

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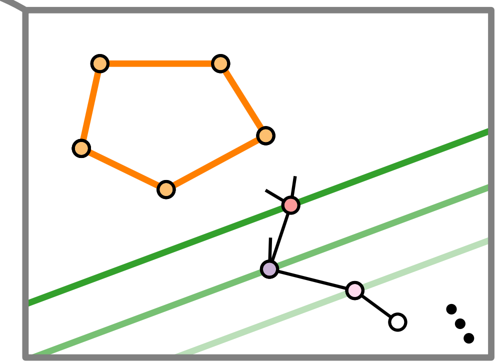

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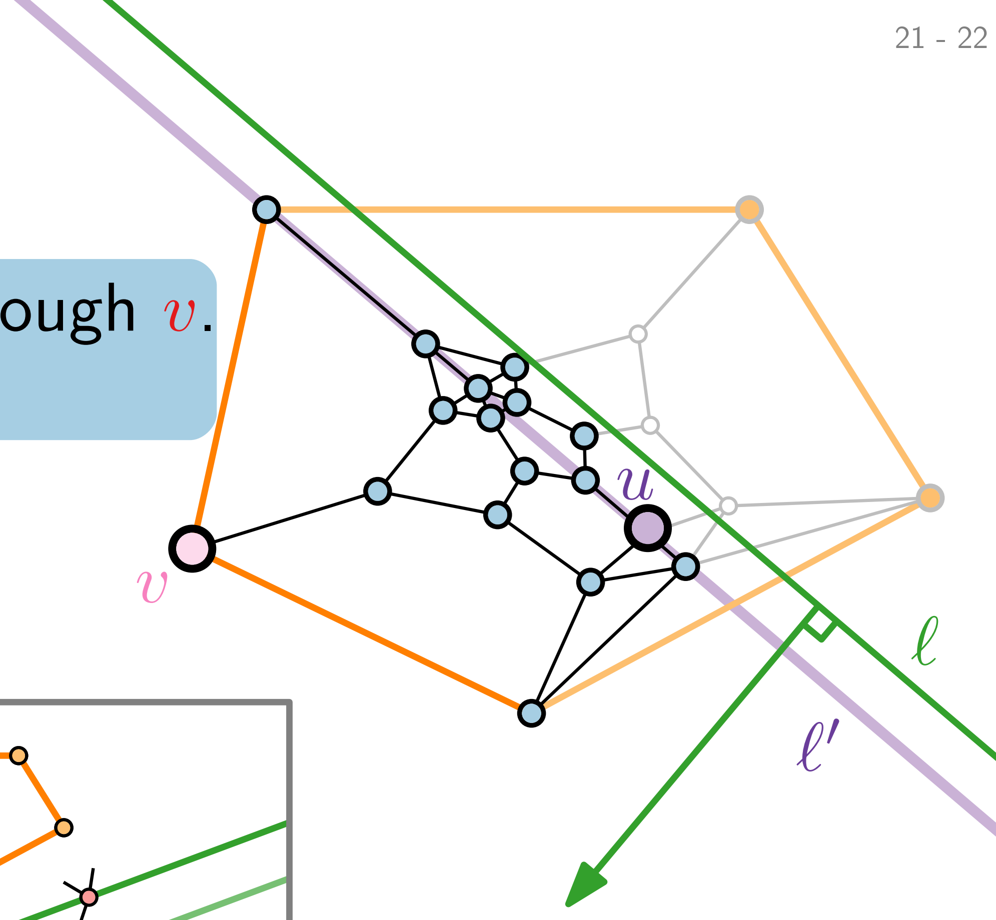

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G connected, v not on ℓ'

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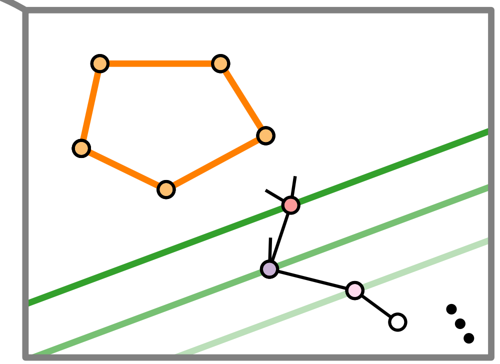

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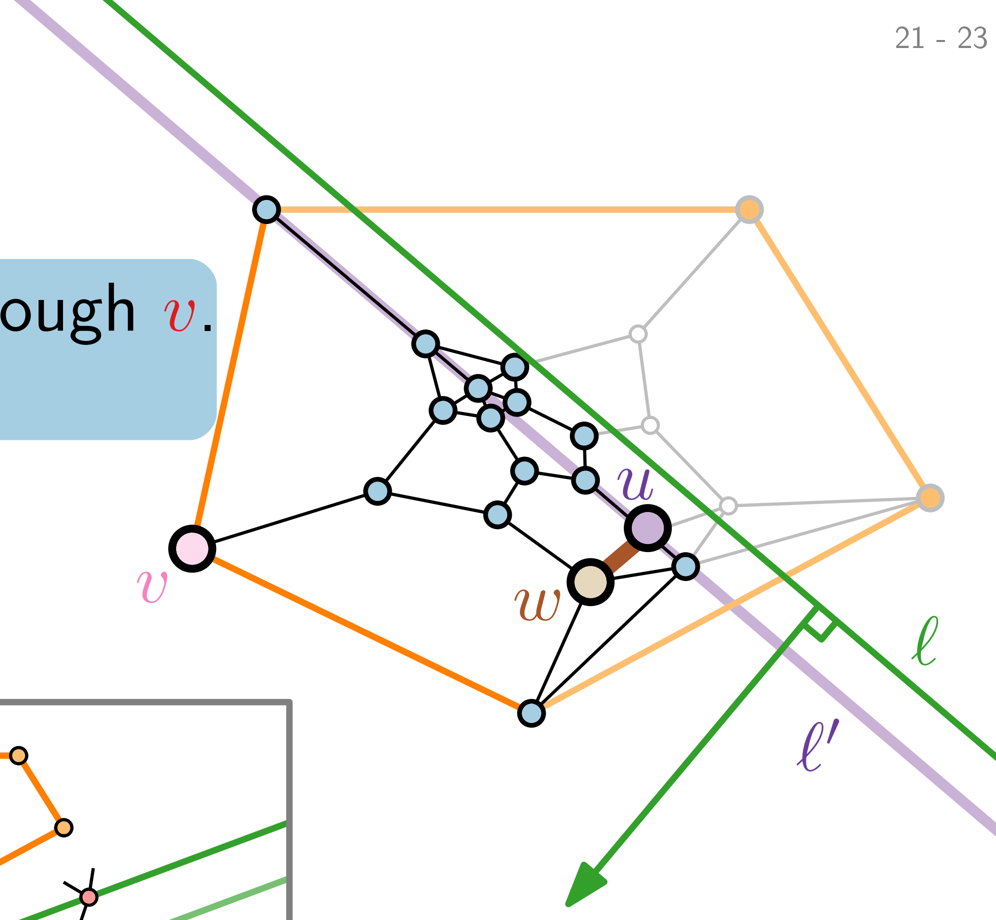

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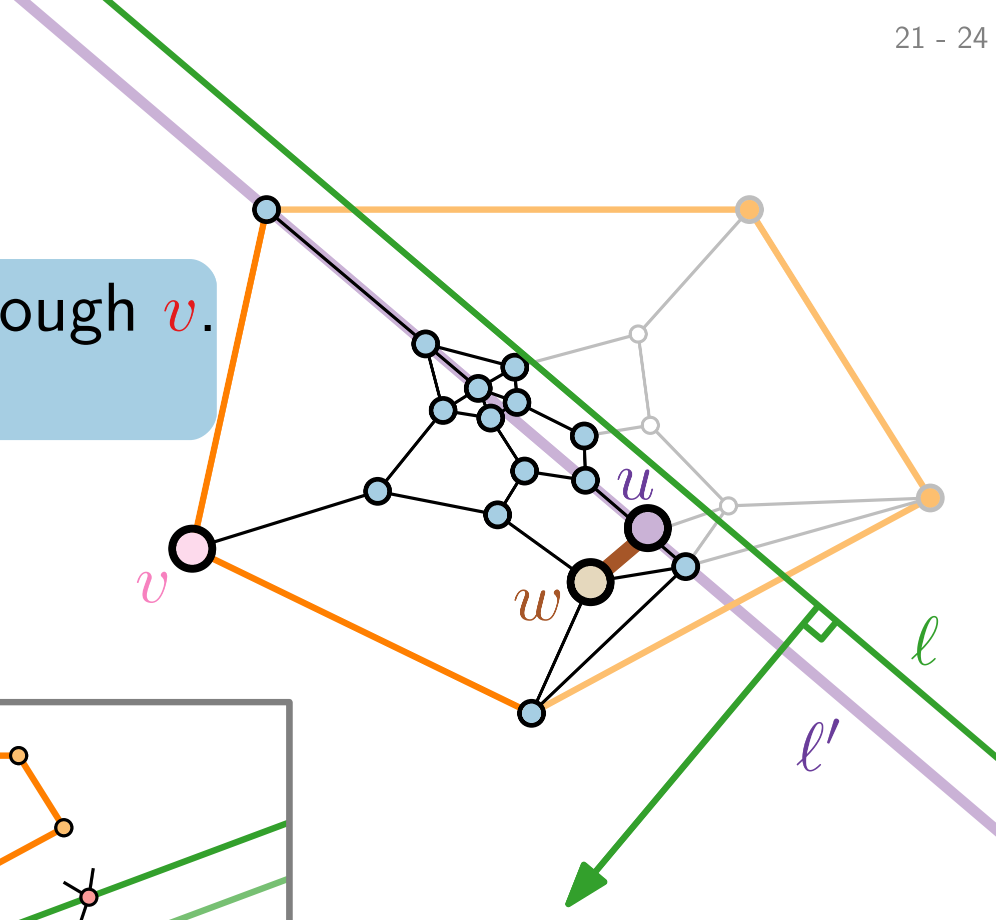

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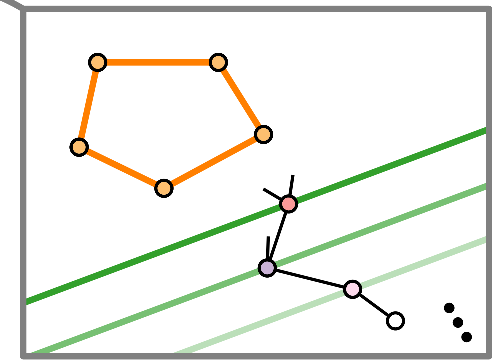

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G connected, v not on ℓ' $\Rightarrow \exists$ neighbor $w \in V_\ell$ of u on the same side of ℓ' as v.

Move ℓ' onto w and repeat $\Rightarrow \exists$ path from u to C, and along C to v.

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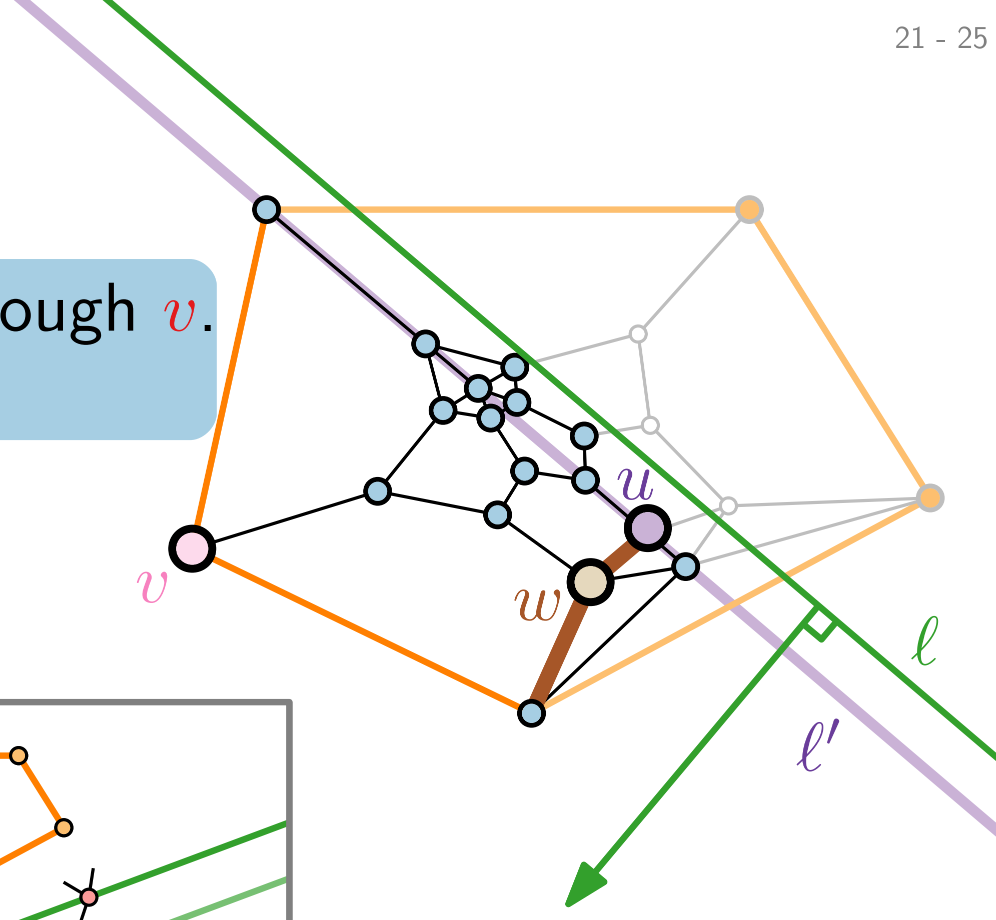

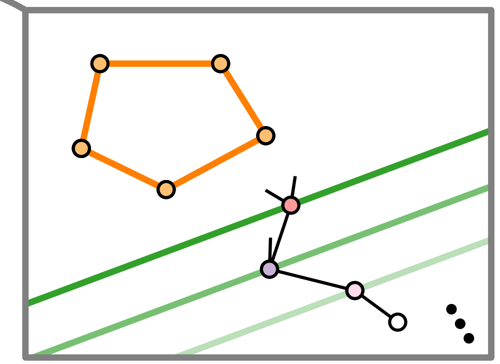

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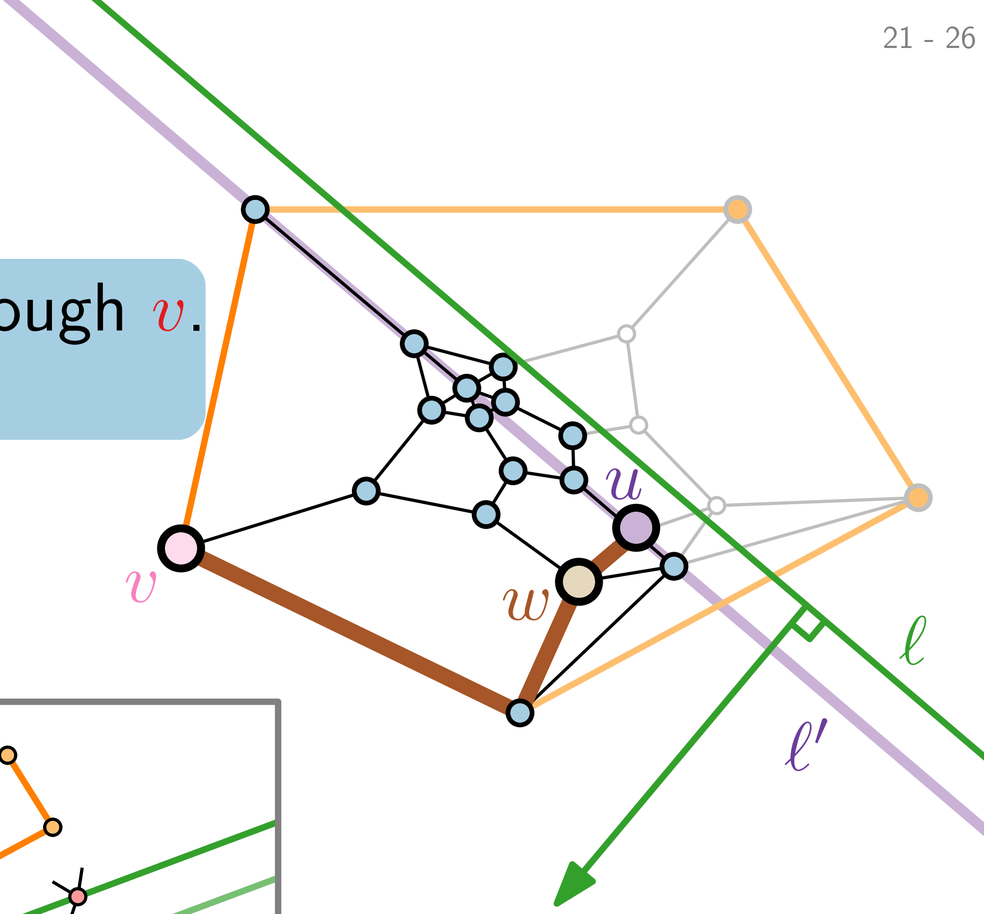
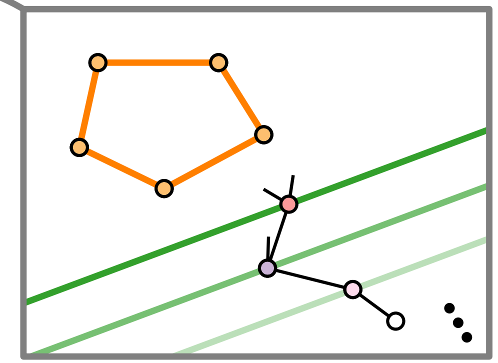


Property 2. All free vertices lie inside C.



Property 3. Let ℓ be any line.
Let $V_\ell = V(G) \cap \ell^+$.
Then $G[V_\ell]$ is connected.

Choose v farthest away from ℓ.

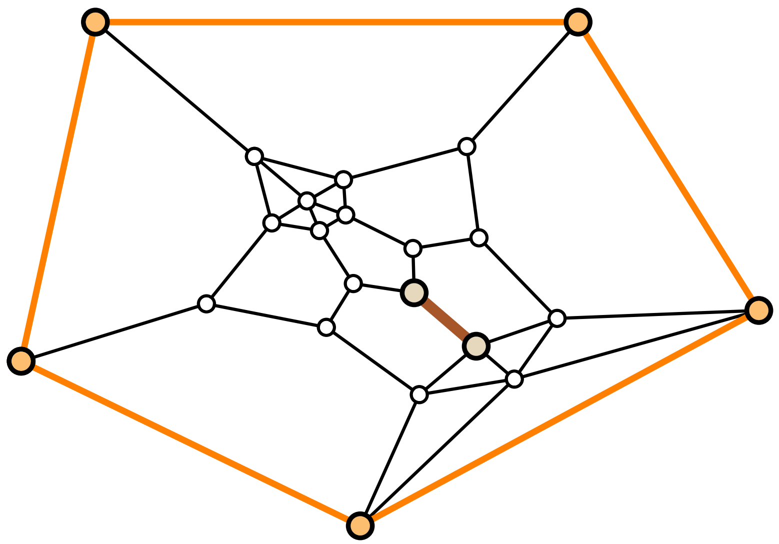
Pick any vertex $u \in V_\ell$, ℓ' parallel to ℓ through u

G connected, v not on ℓ' $\Rightarrow \exists$ neighbor $w \in V_\ell$ of u on the same side of ℓ' as v.

Move ℓ' onto w and repeat $\Rightarrow \exists$ path from u to C, and along C to v.

Proof of Tutte's Theorem

Lemma. Let uv be a non-boundary edge,



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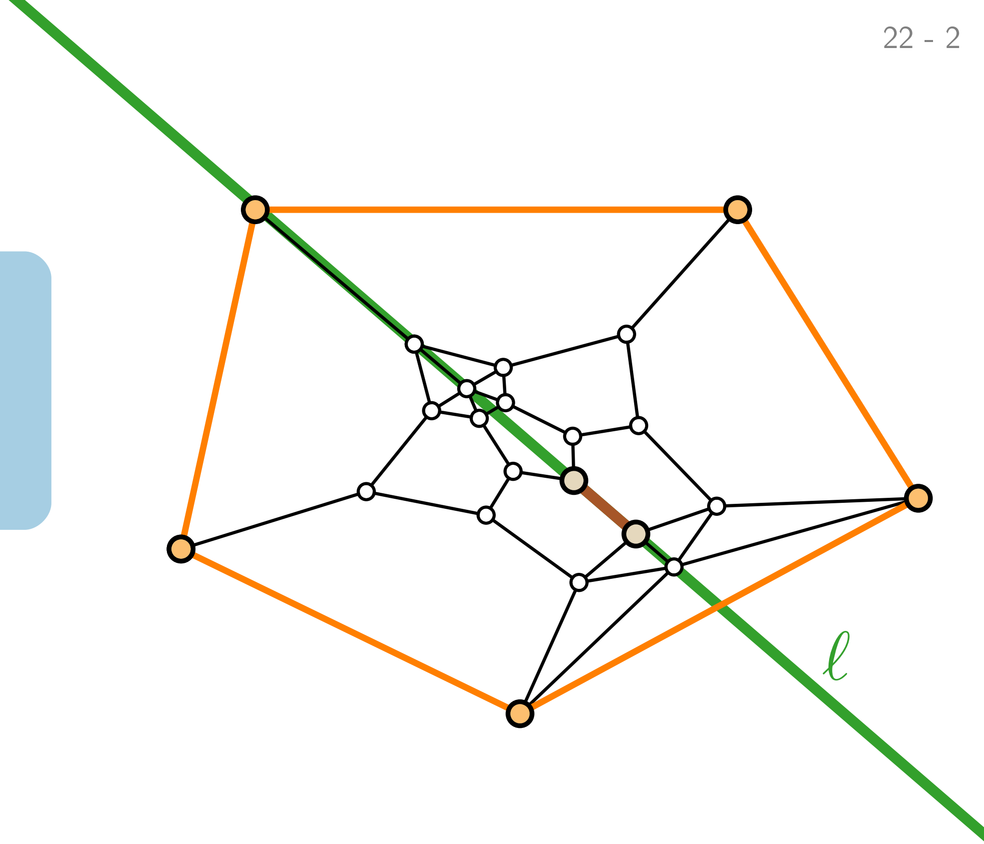

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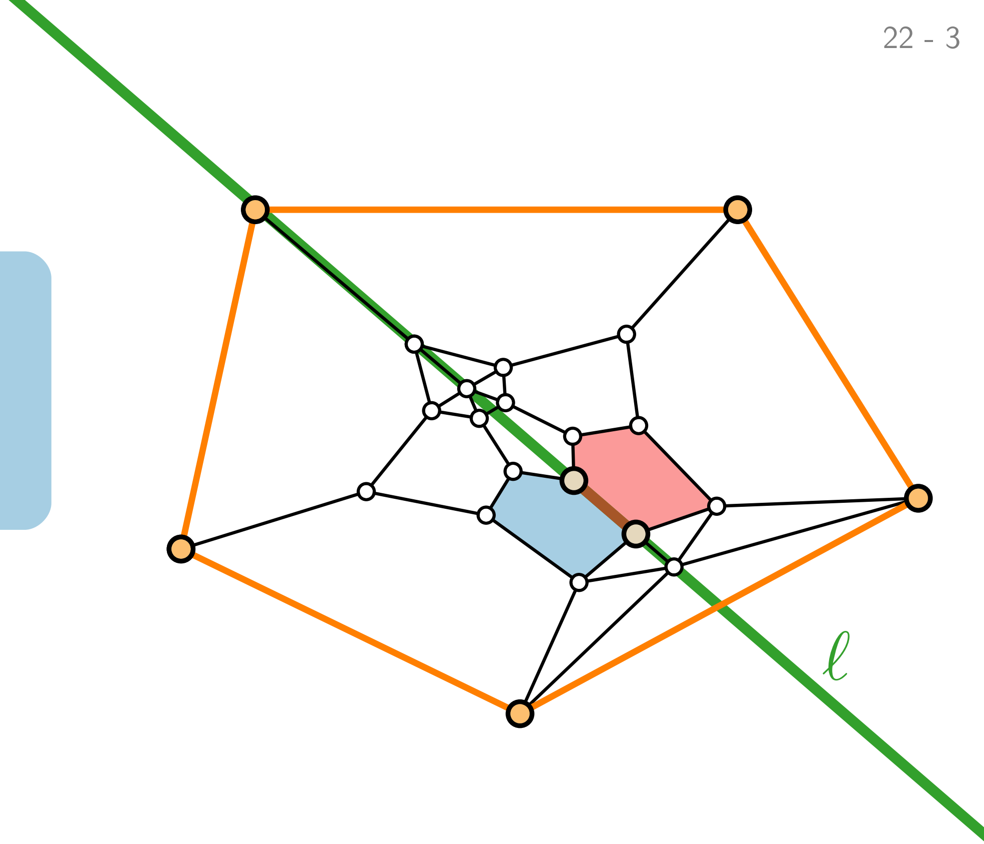

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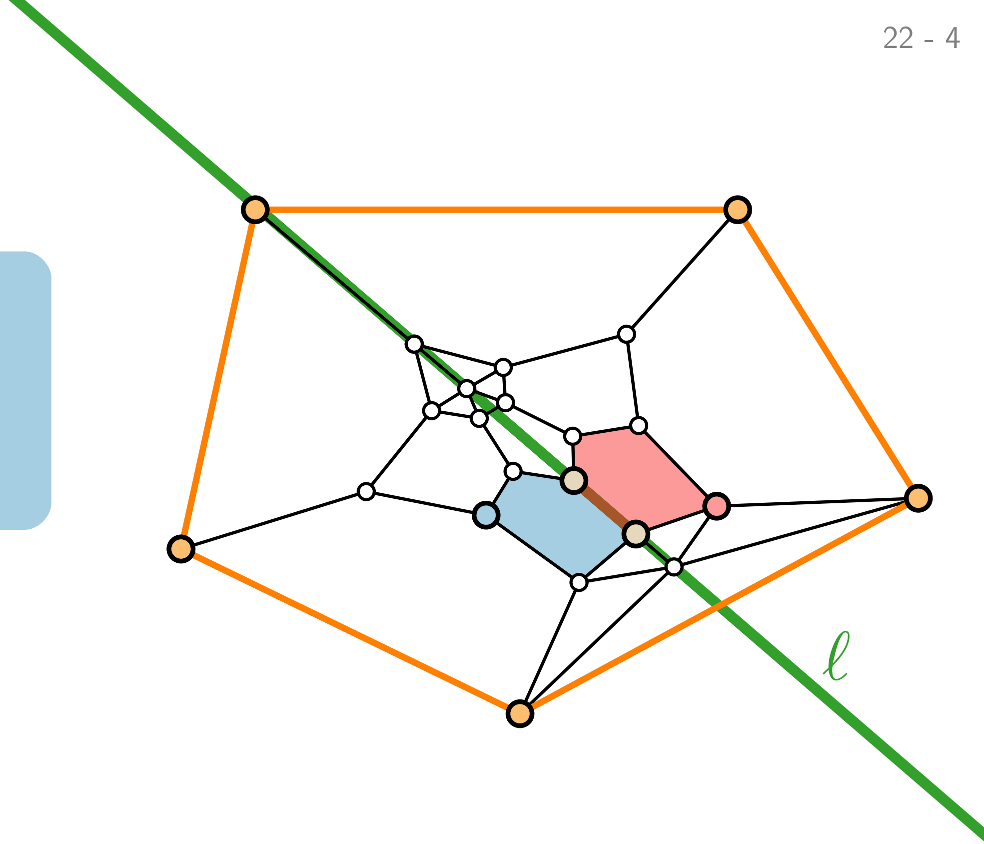

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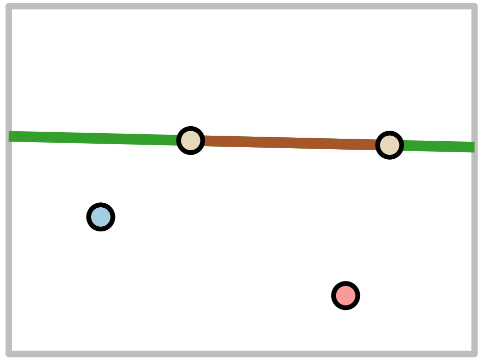

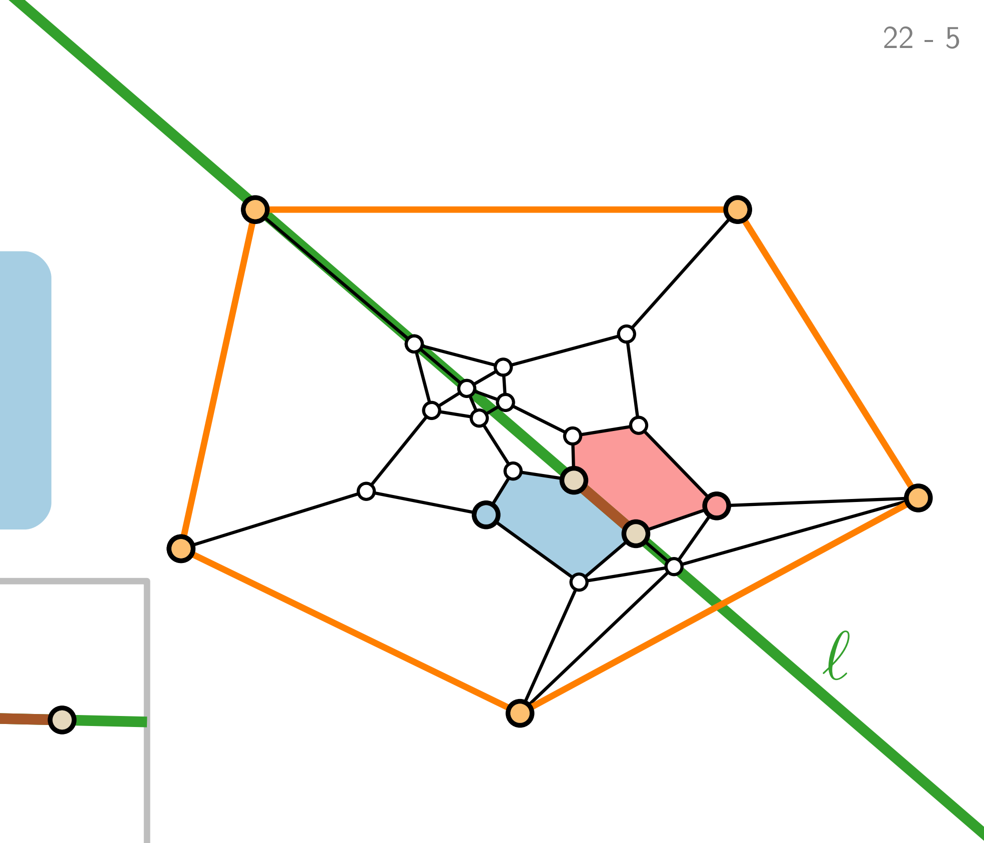

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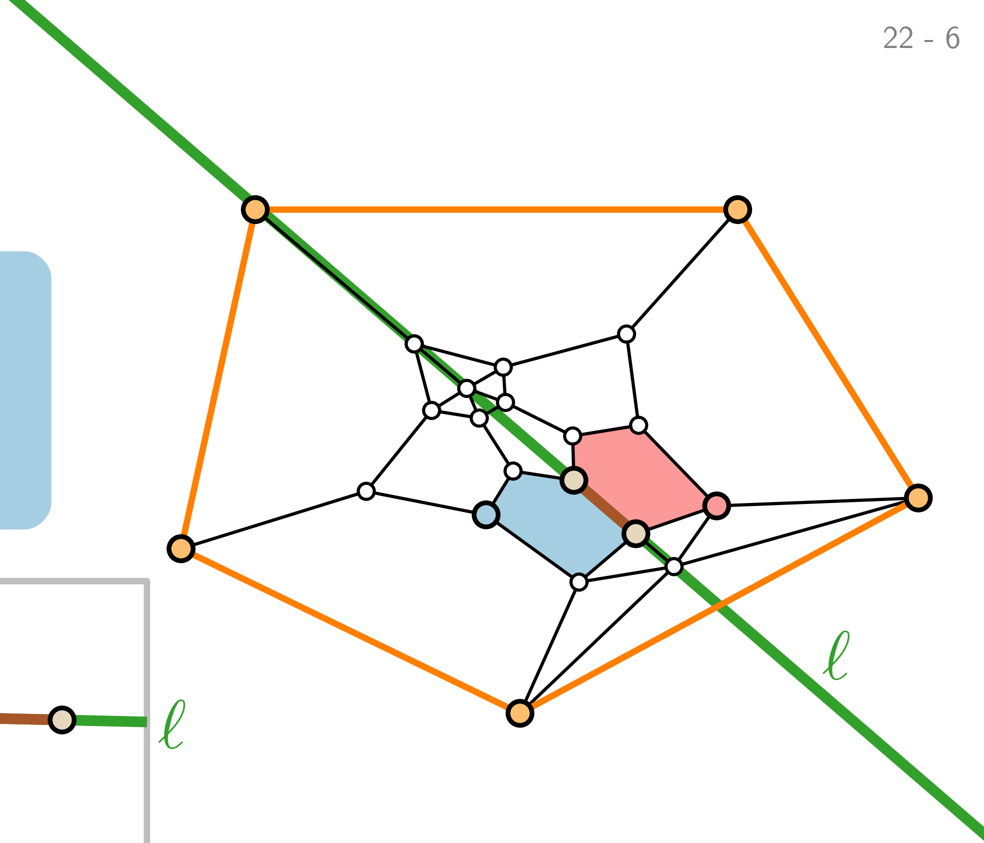

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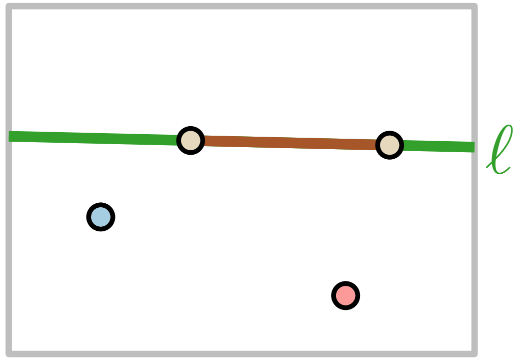

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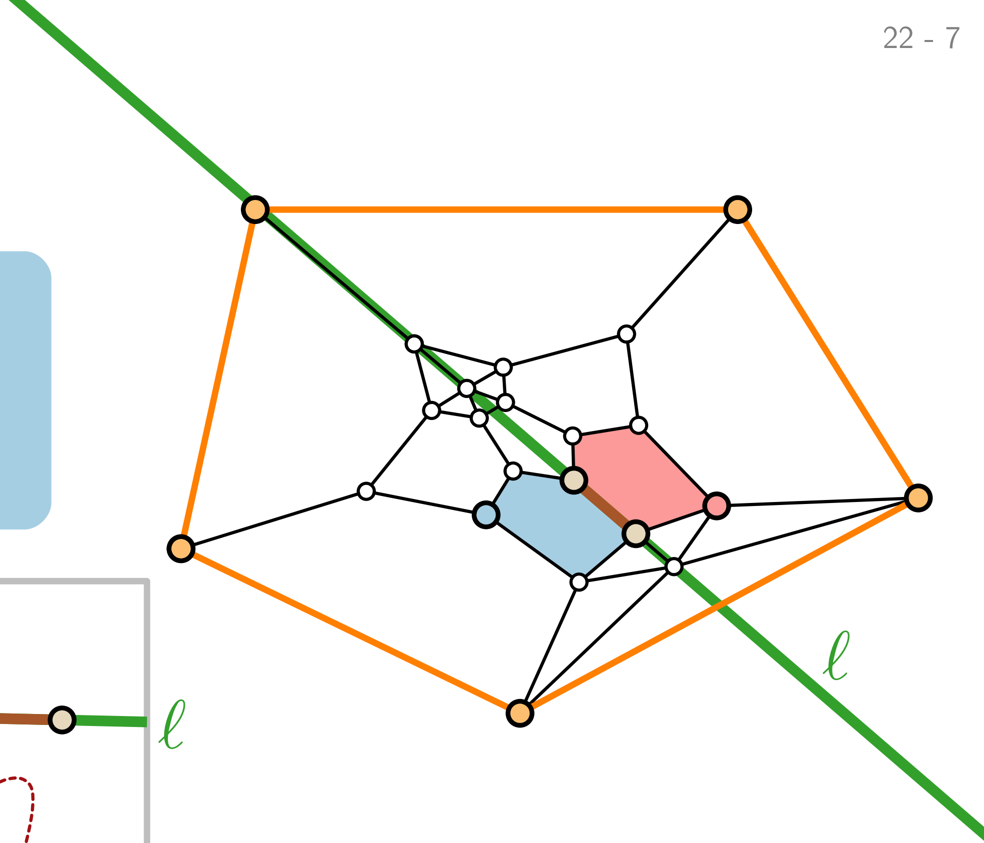

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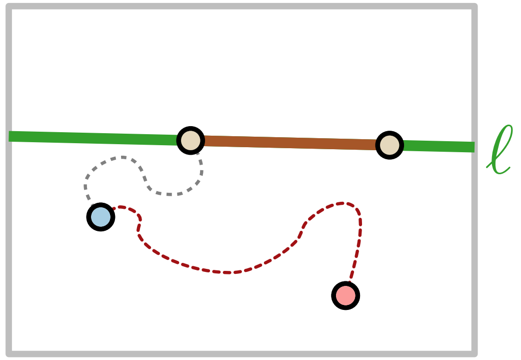

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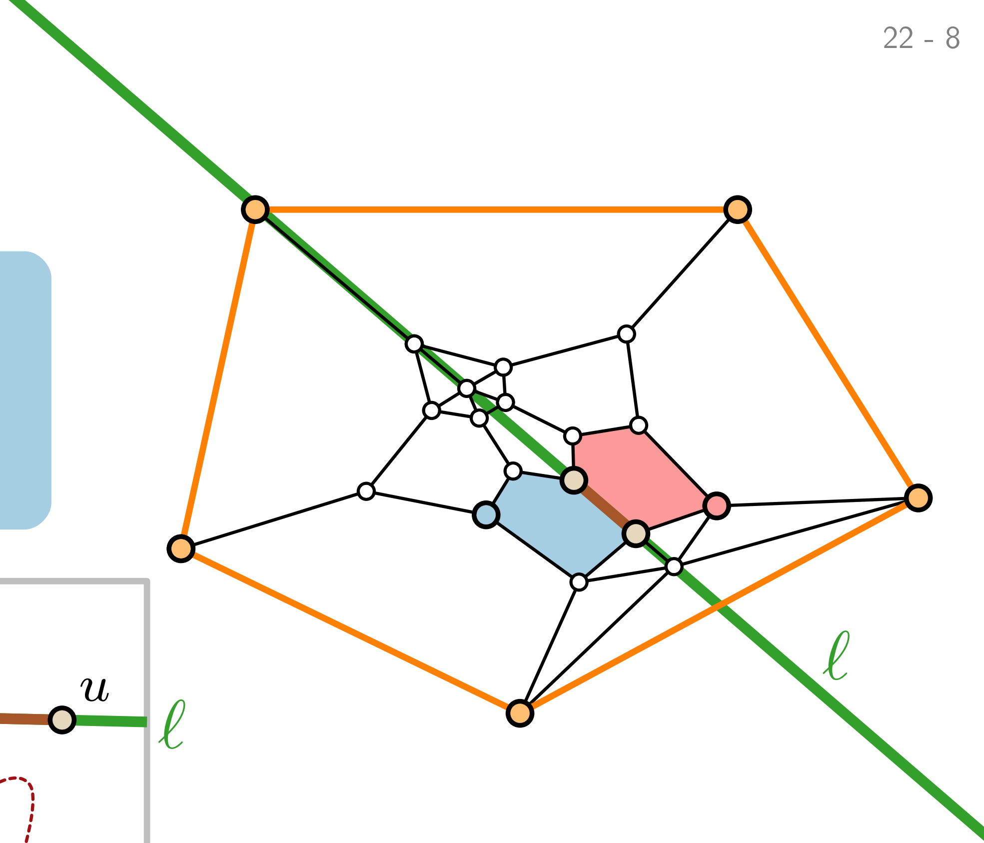

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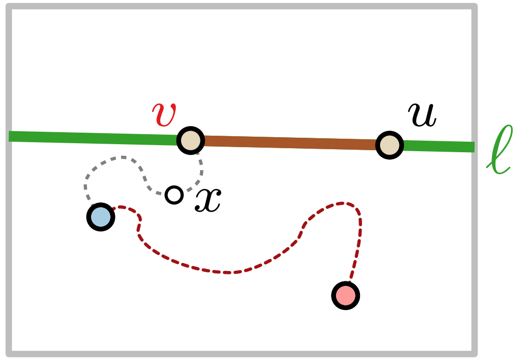

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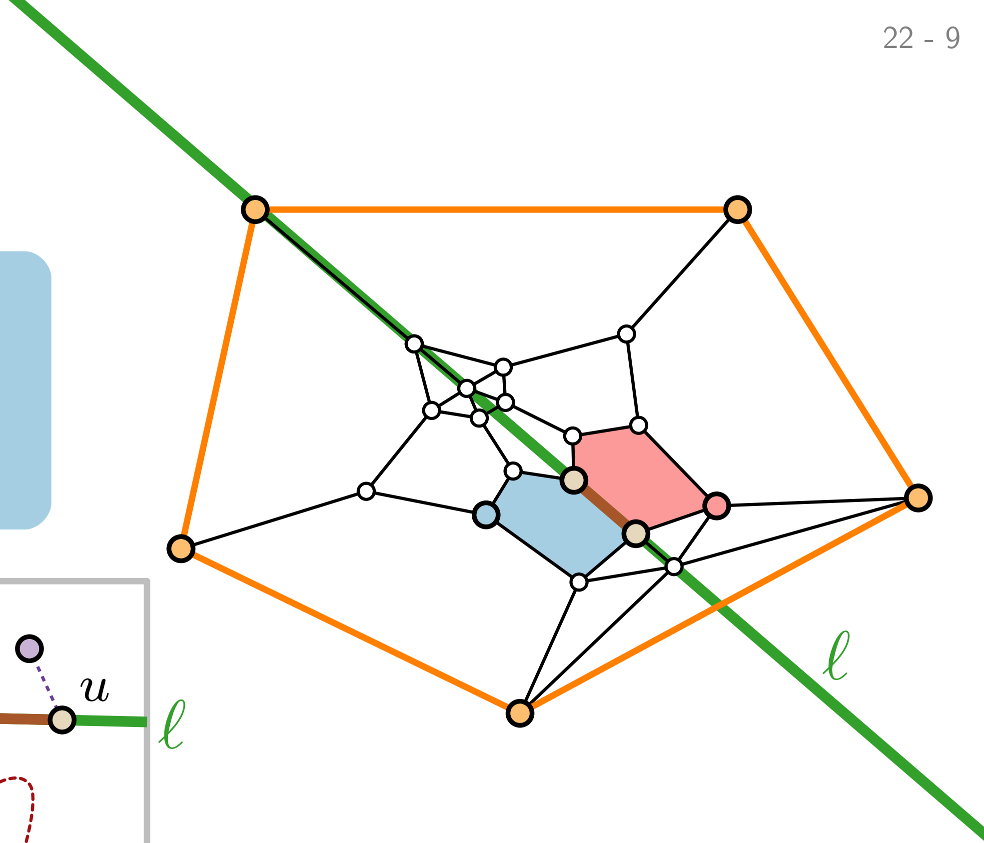

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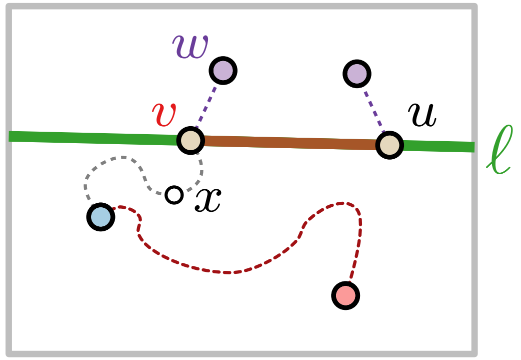

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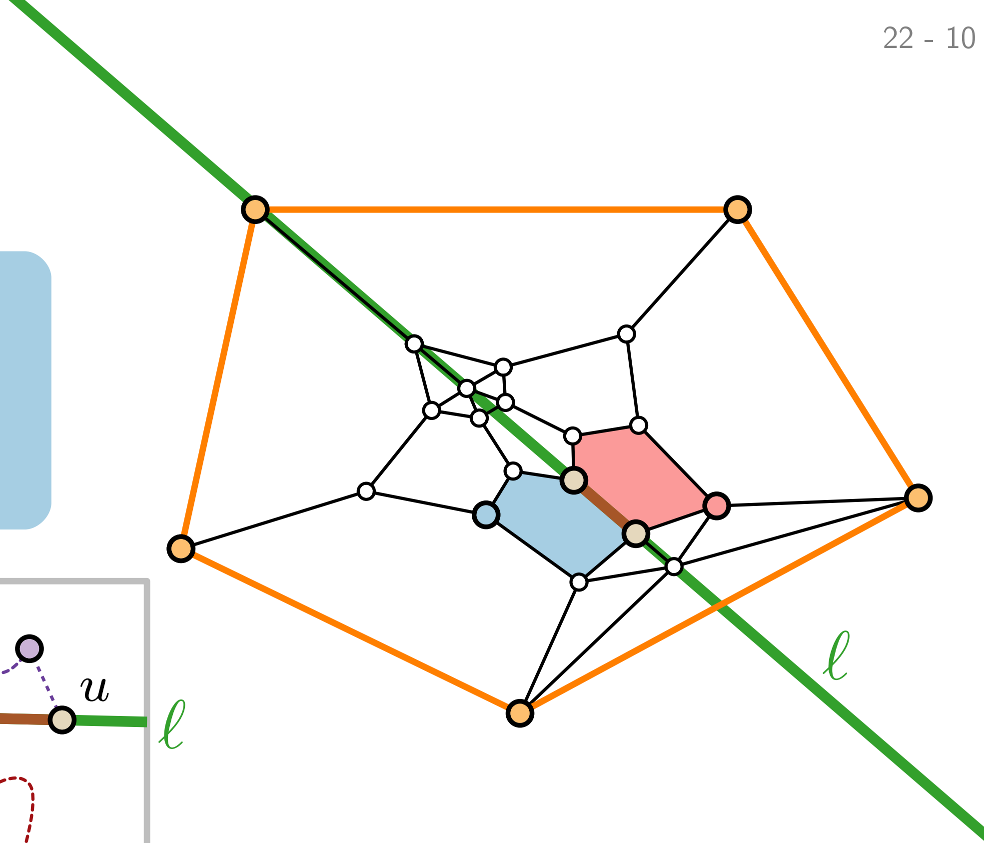

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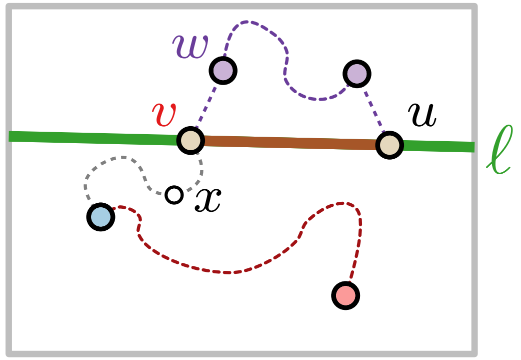

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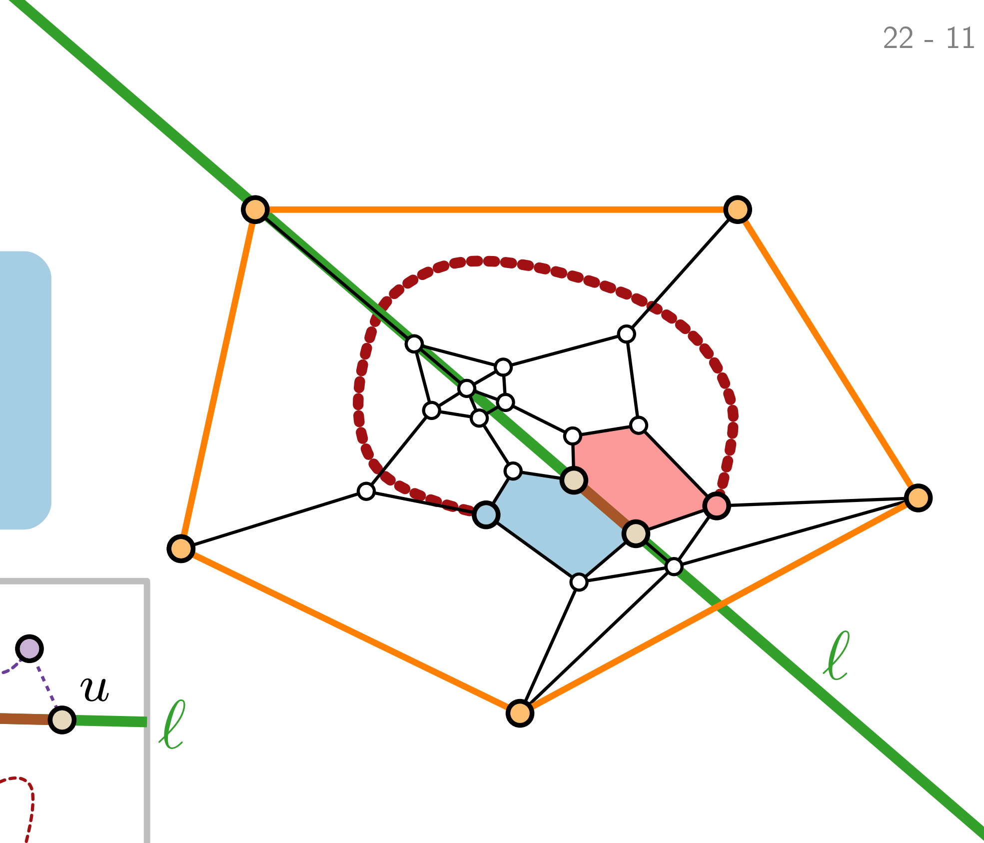

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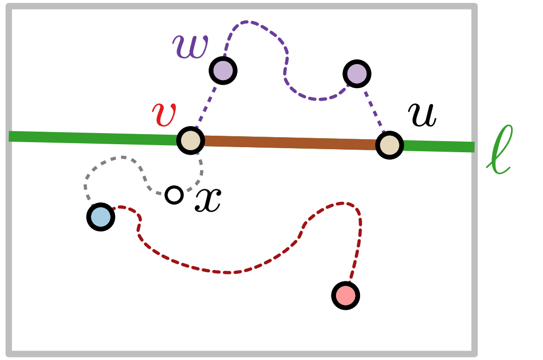

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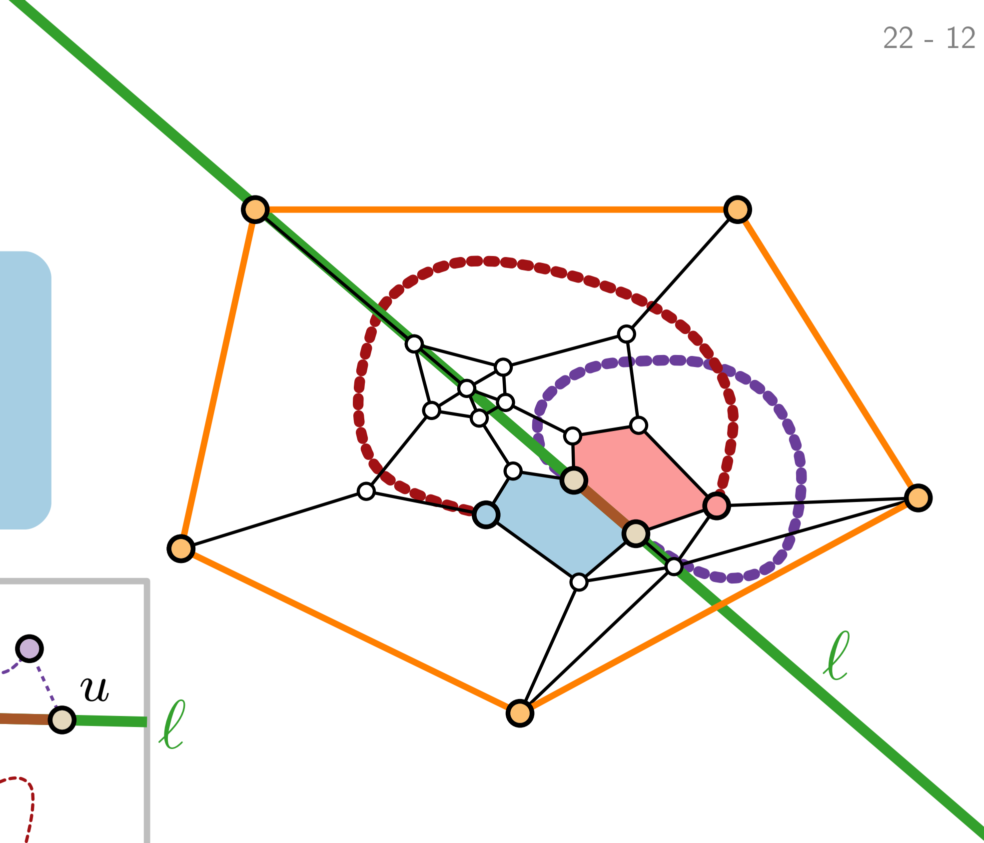

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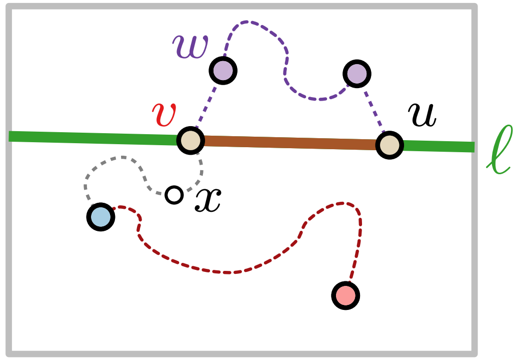

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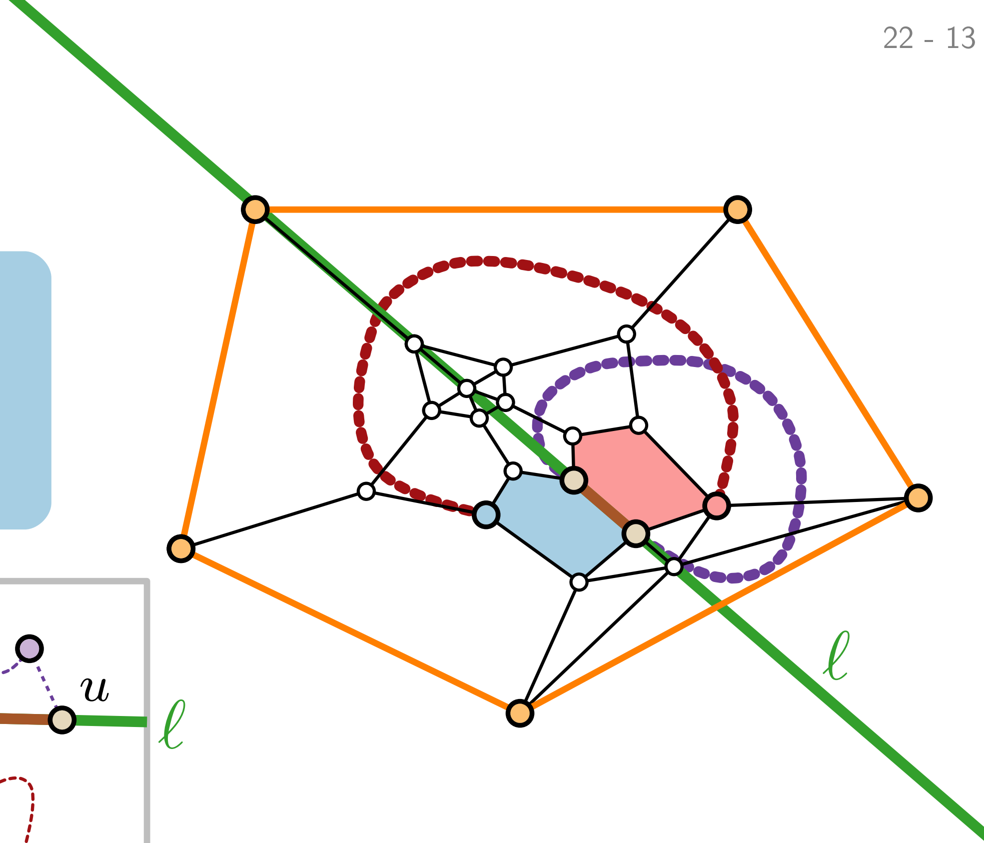

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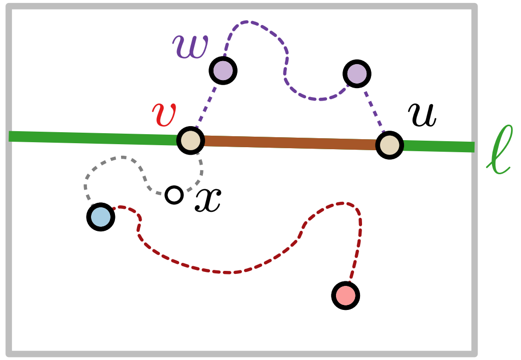

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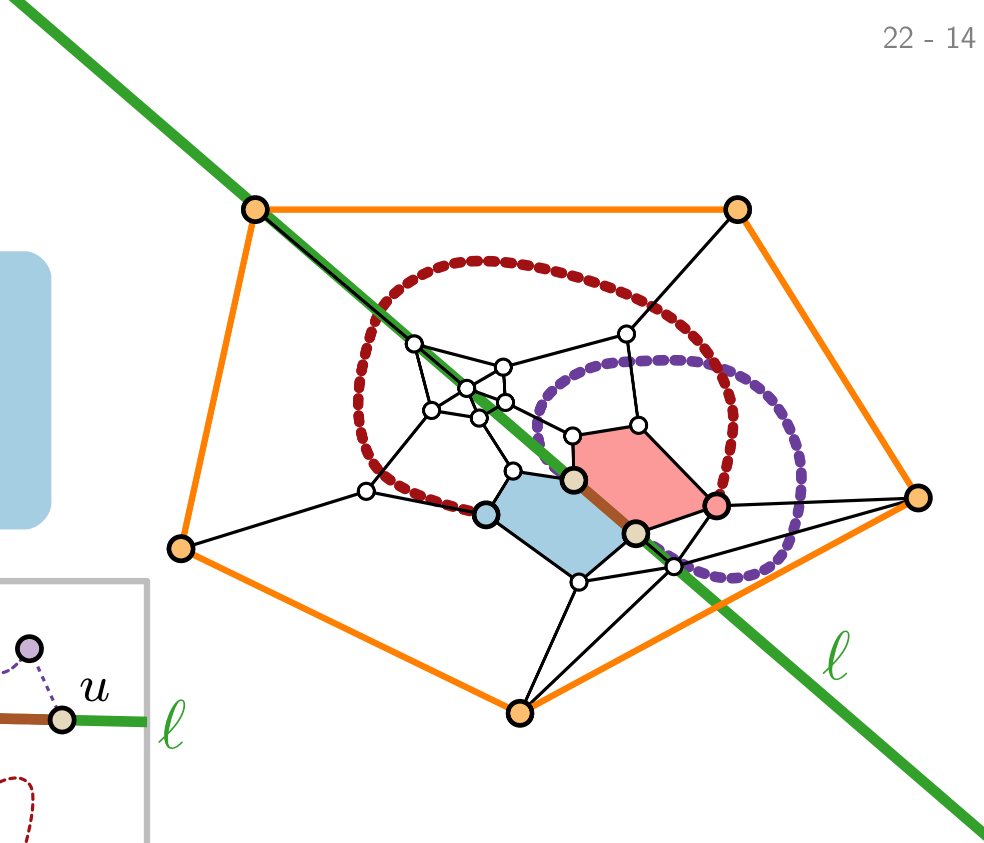

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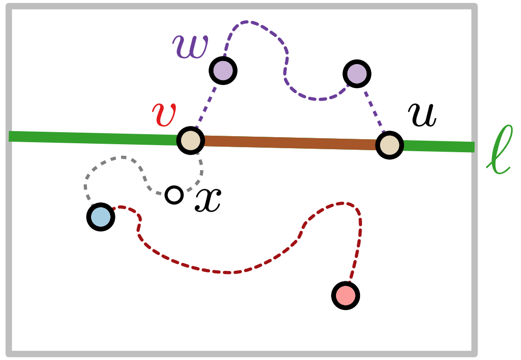

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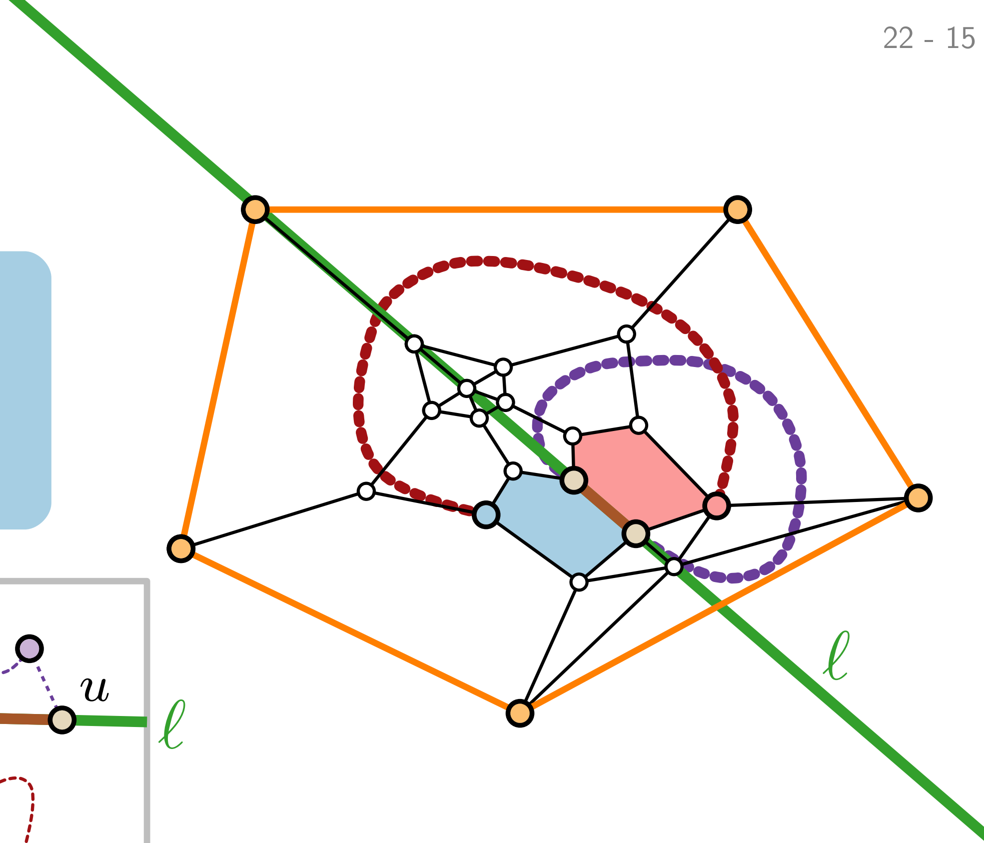
Proof of Tutte's Theorem



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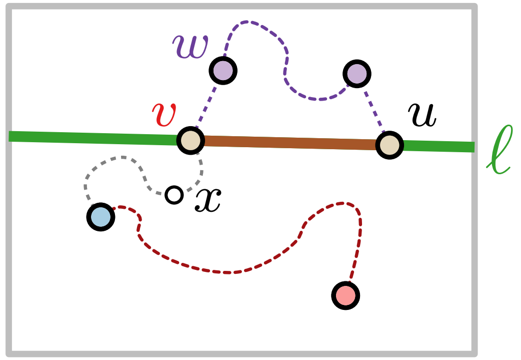

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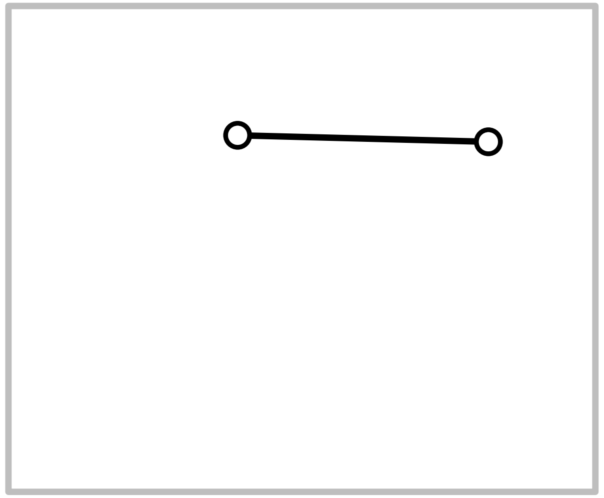

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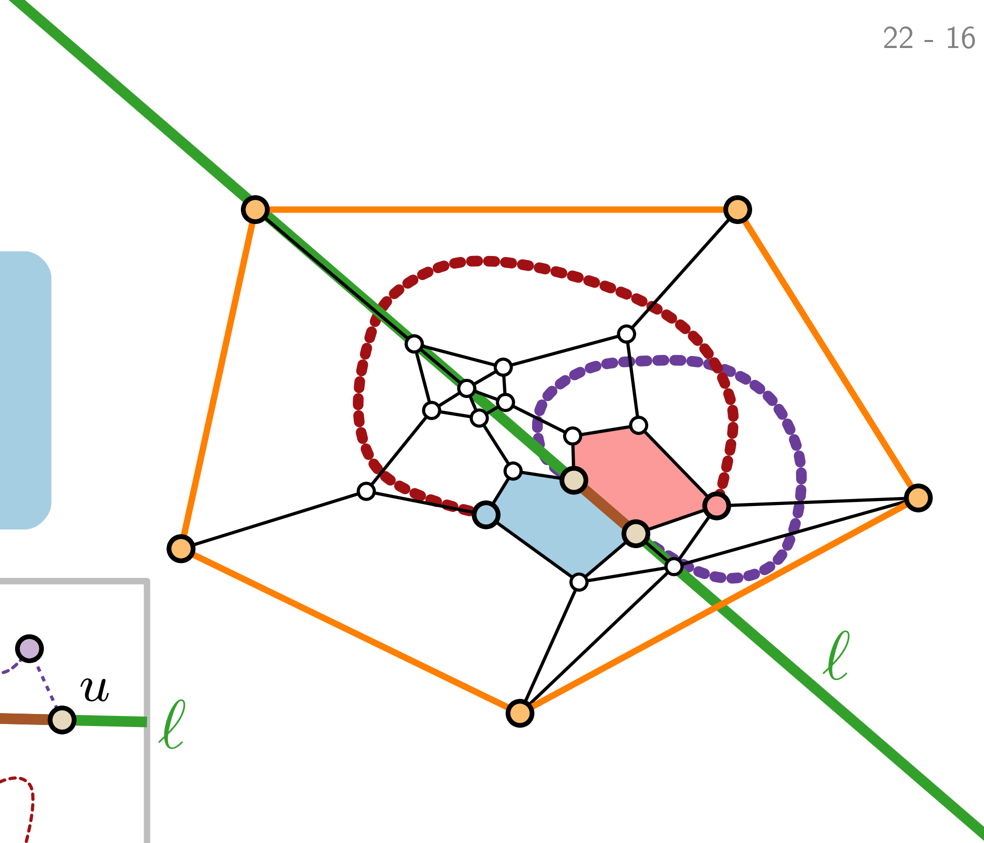

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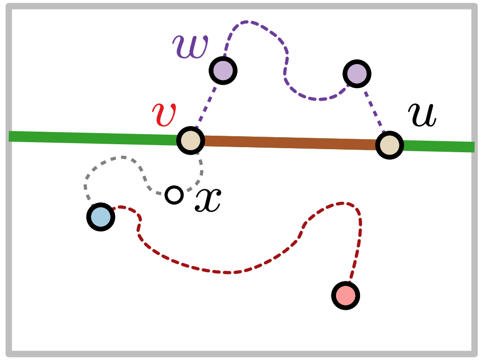

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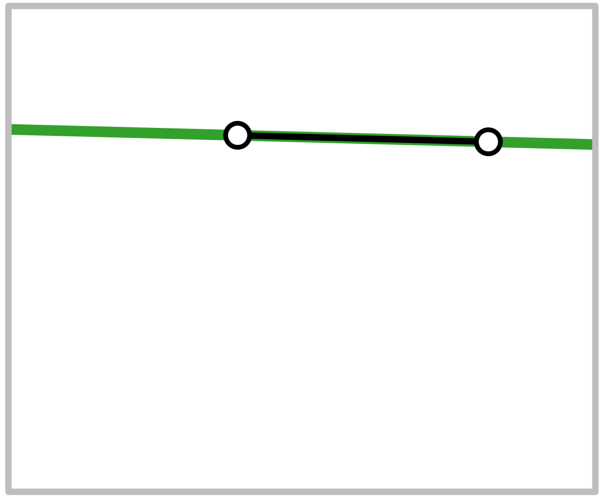

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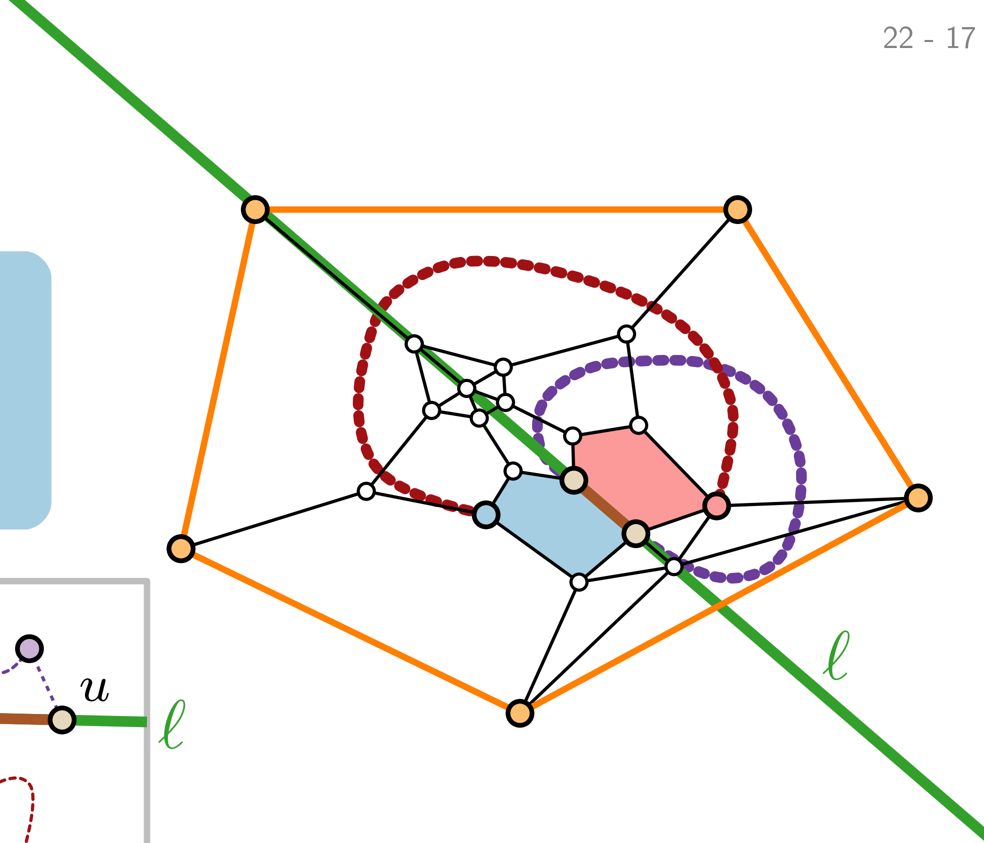

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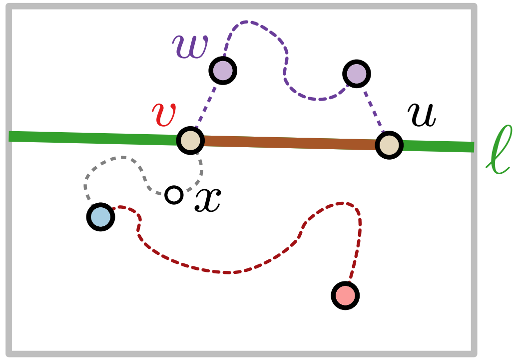

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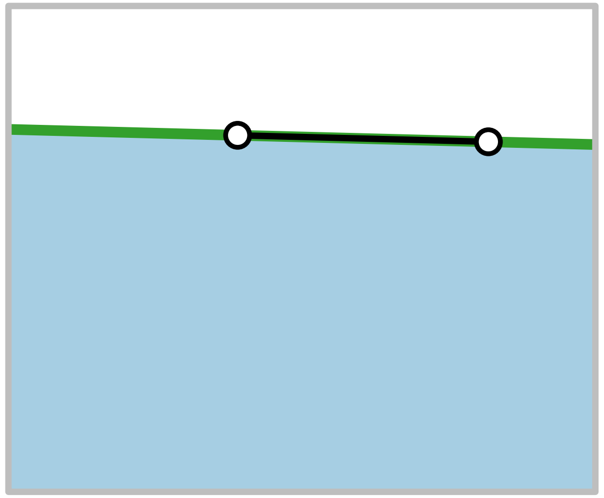

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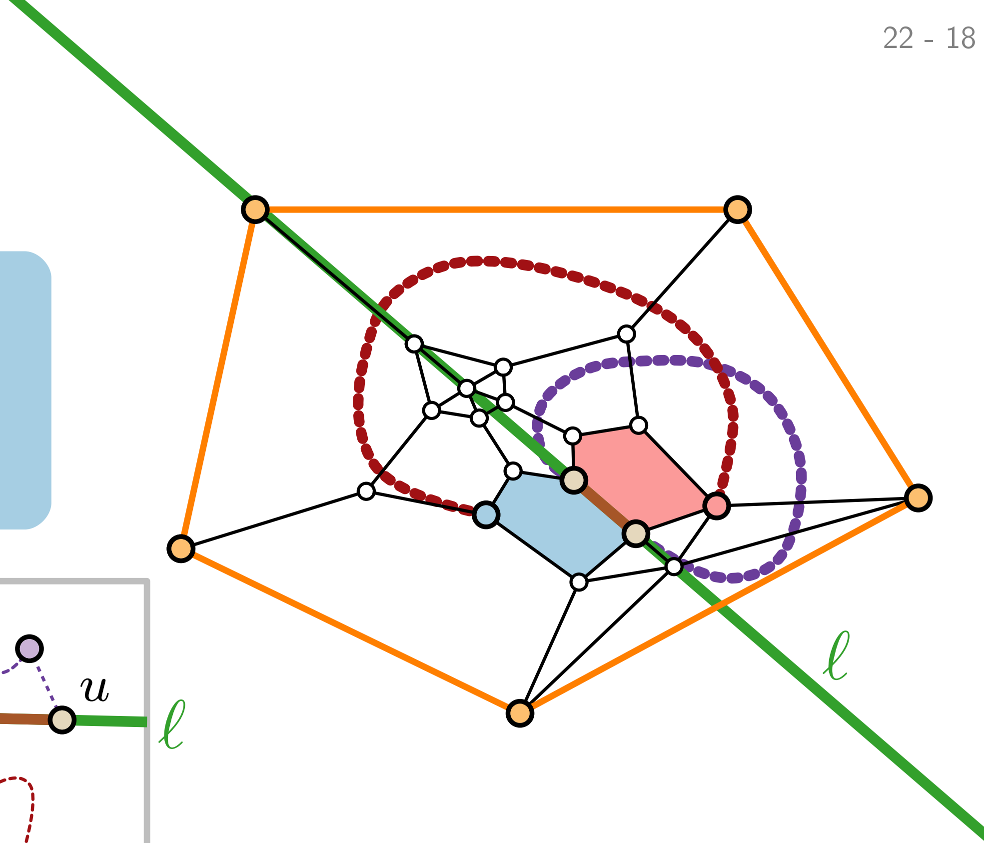

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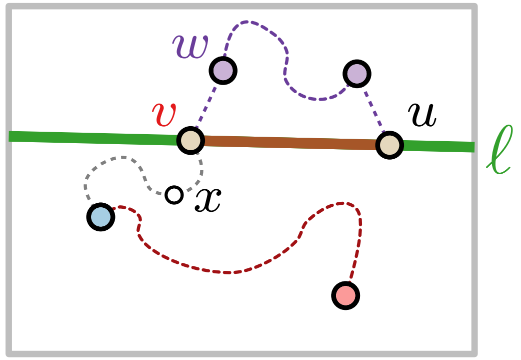

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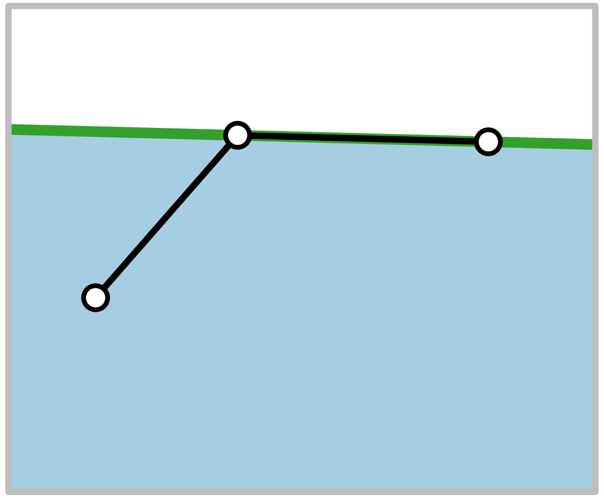

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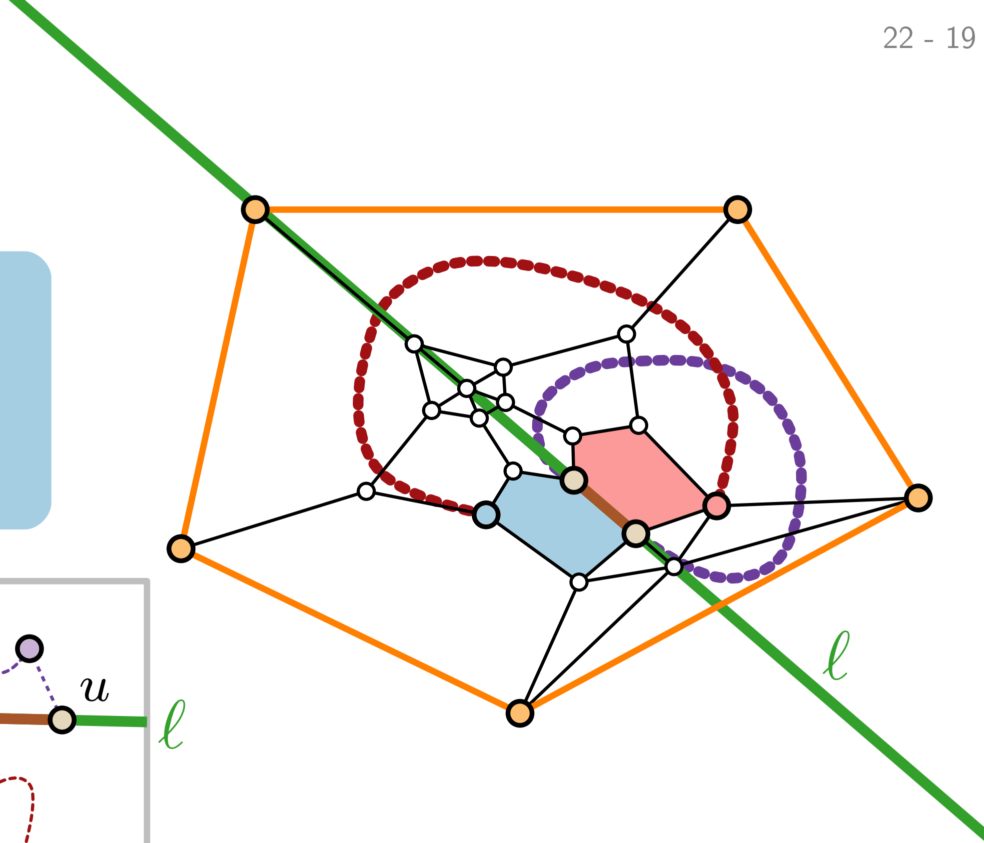

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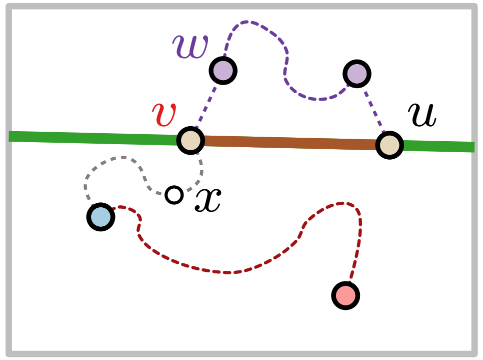

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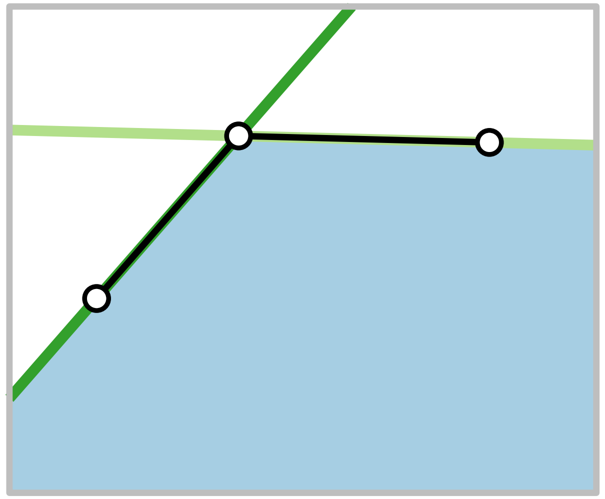

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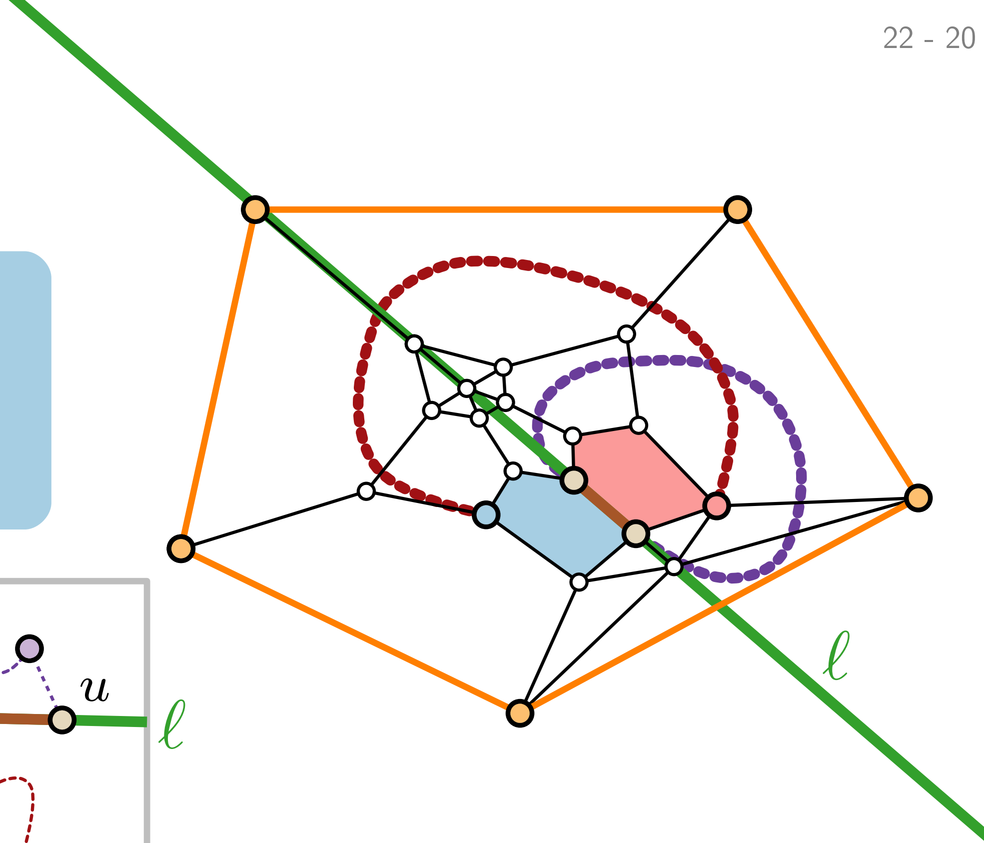

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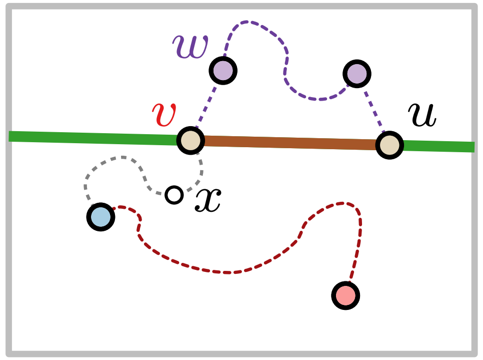

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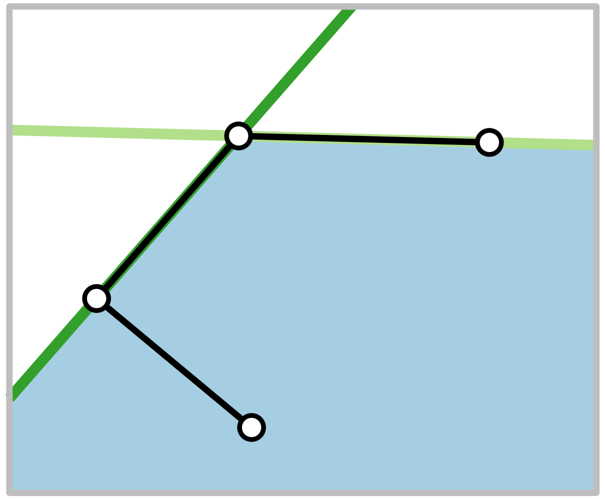

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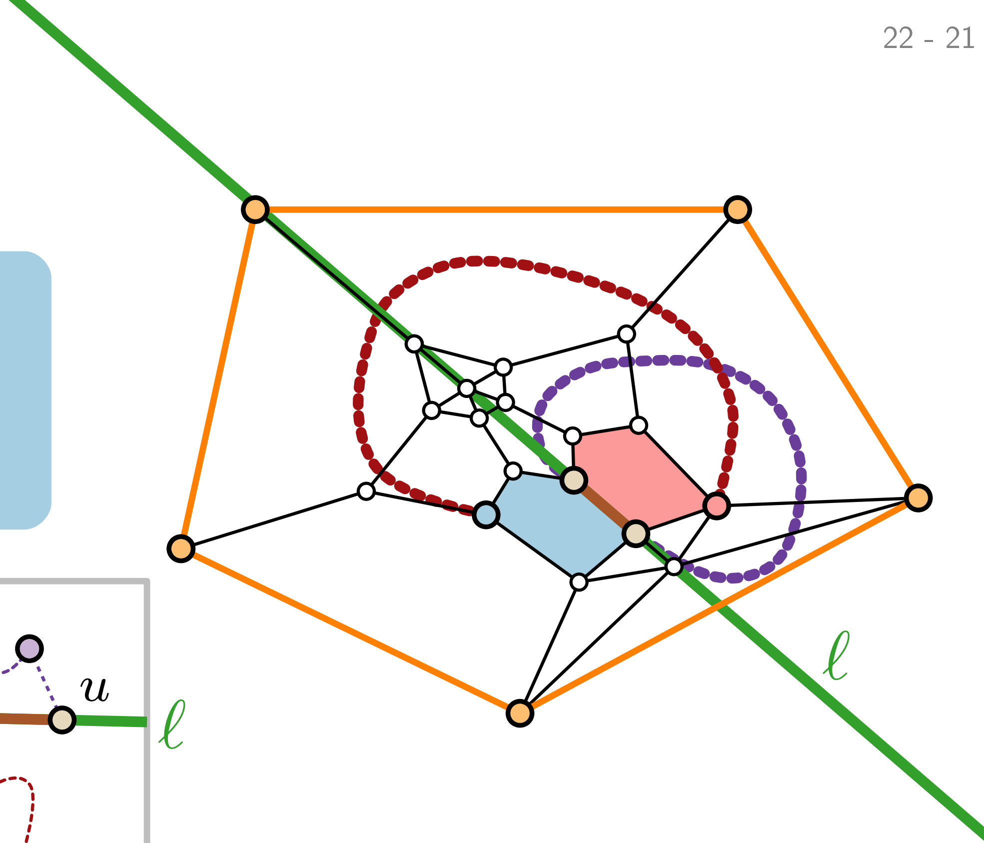

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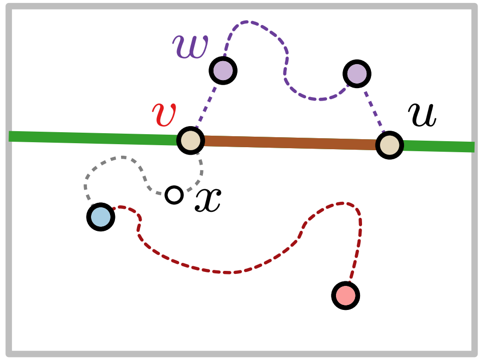

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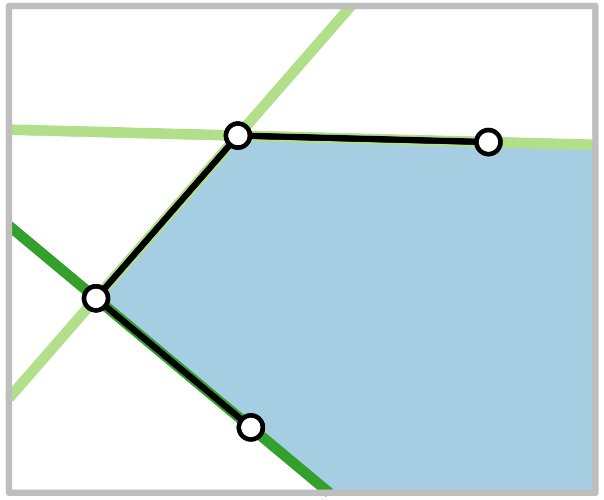

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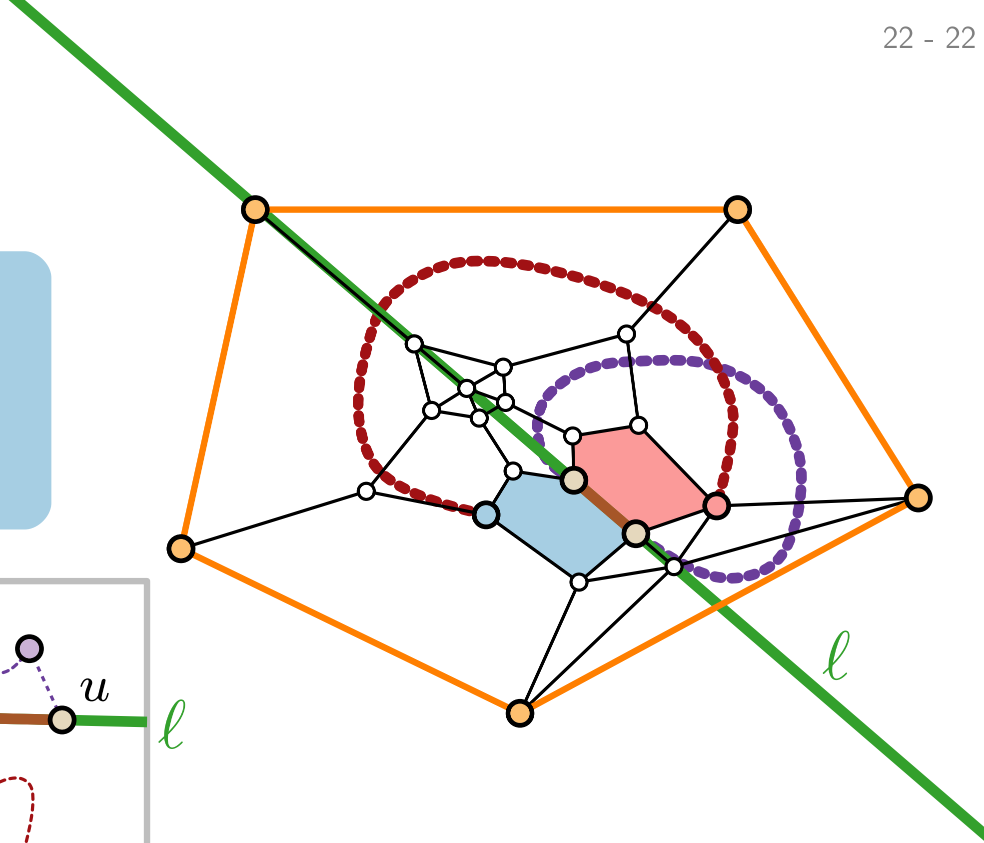

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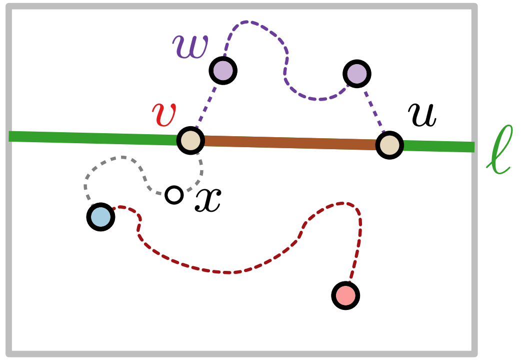

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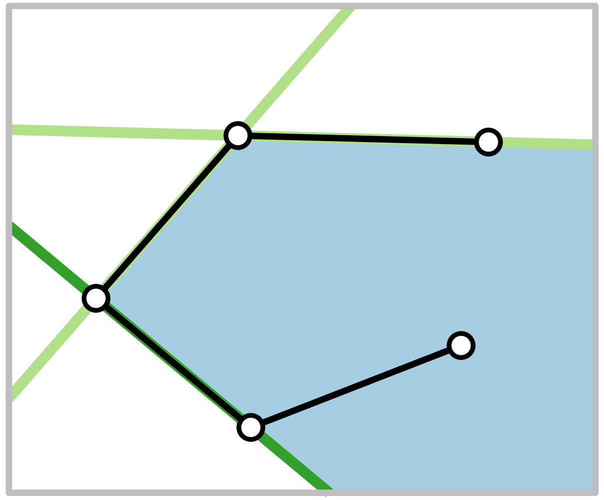

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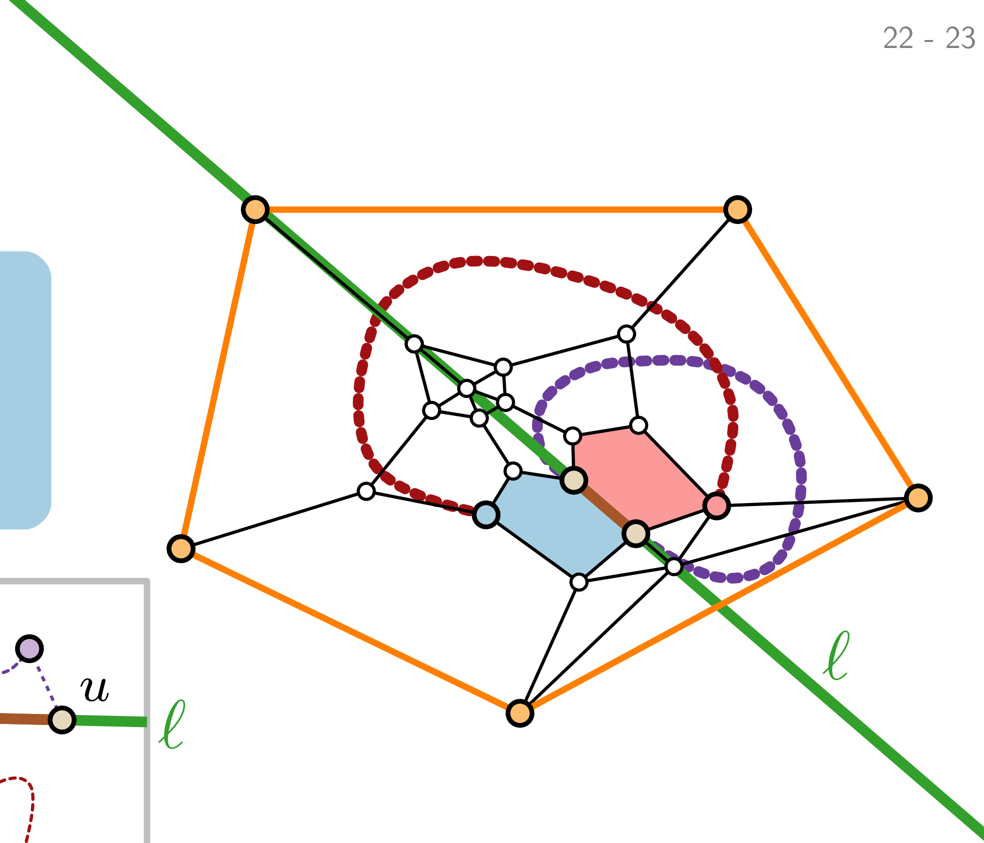

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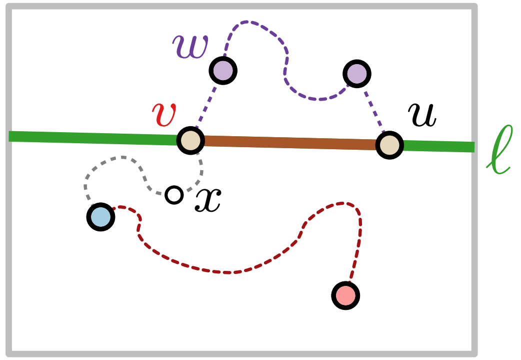

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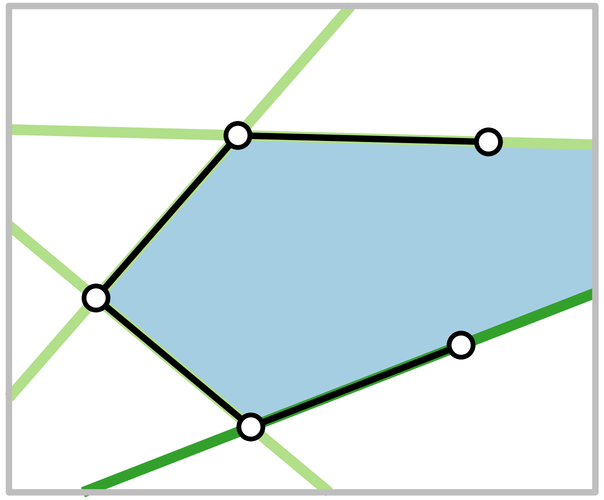

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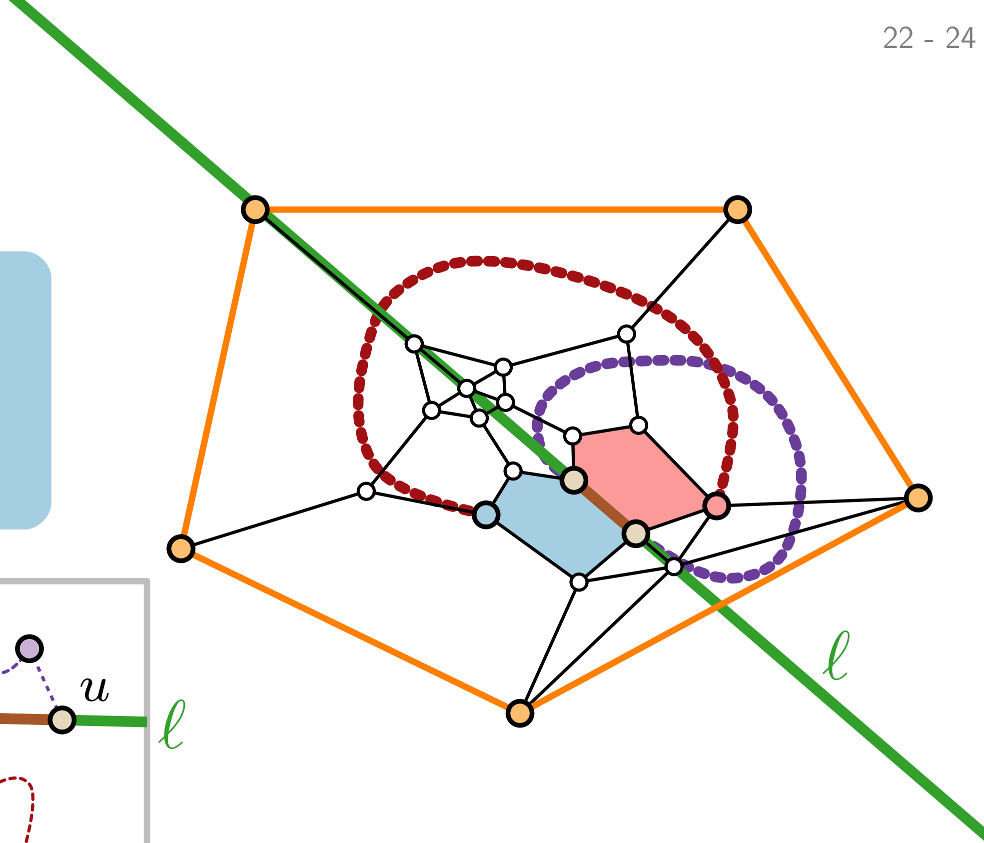

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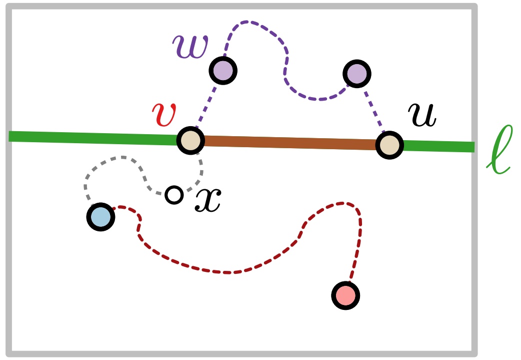

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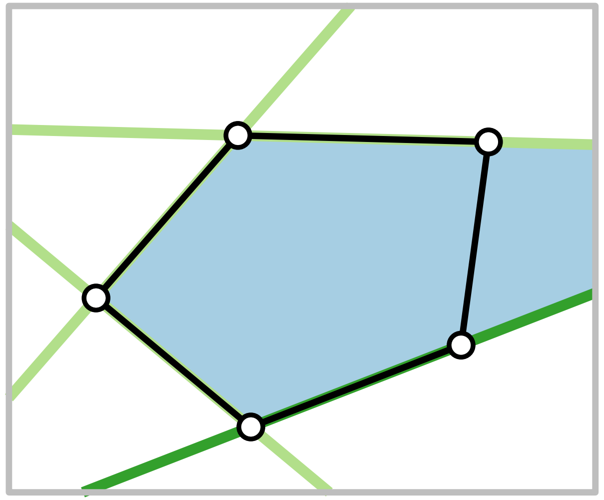

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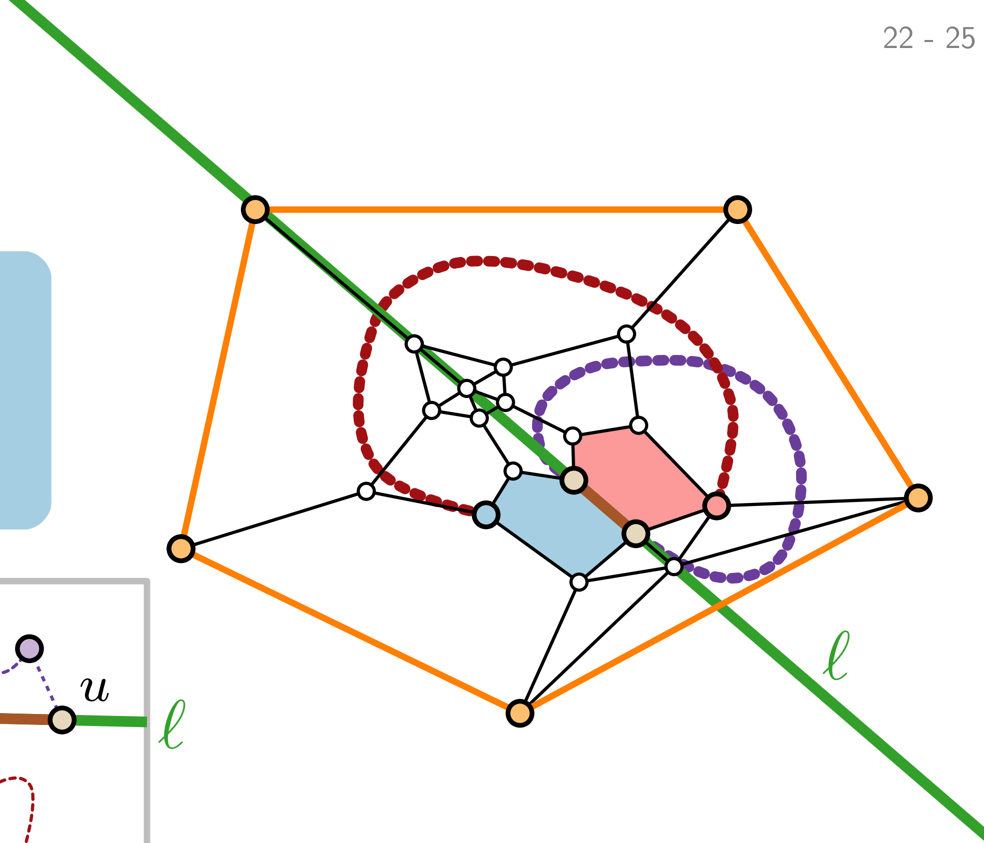

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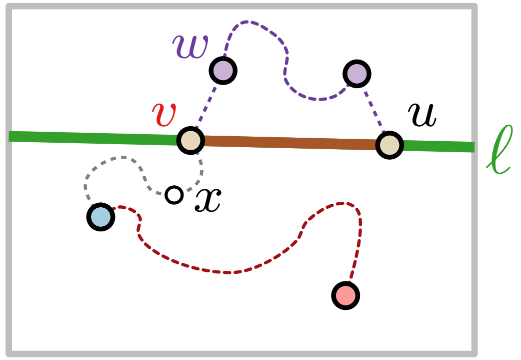

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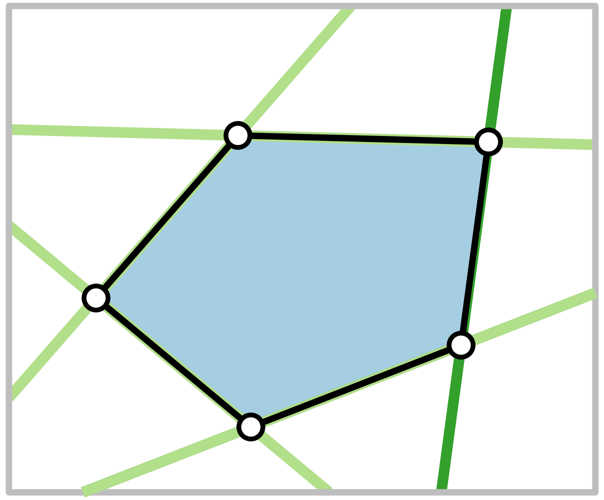

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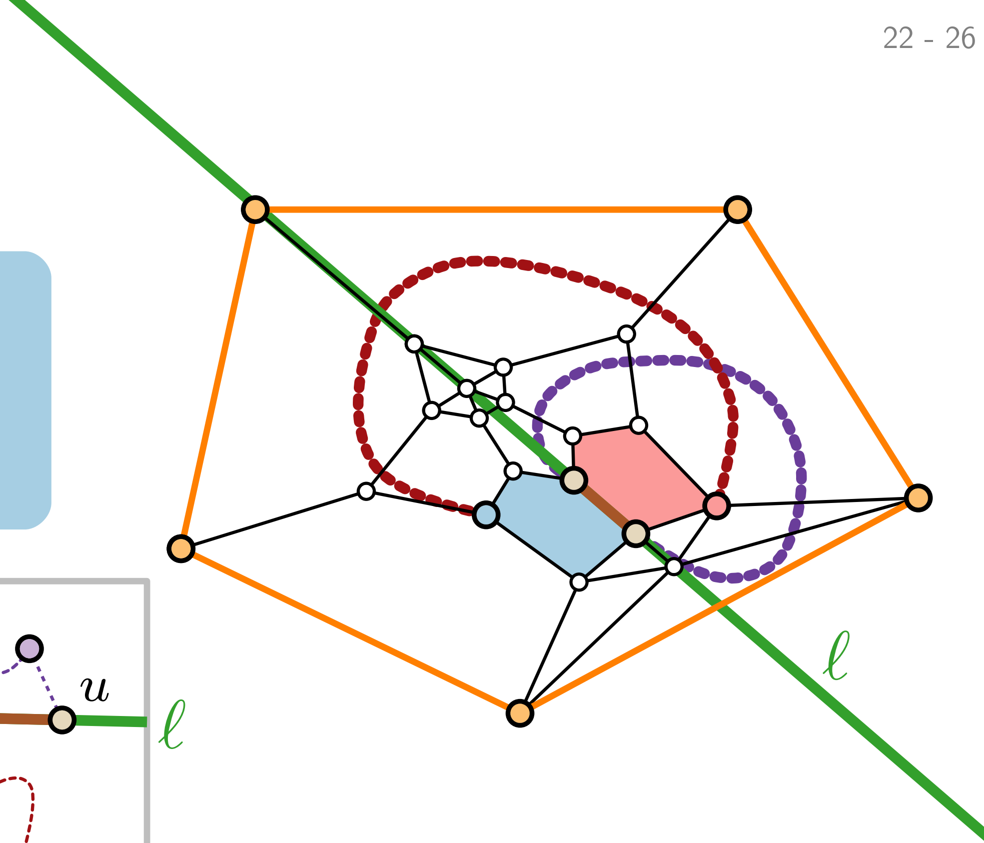

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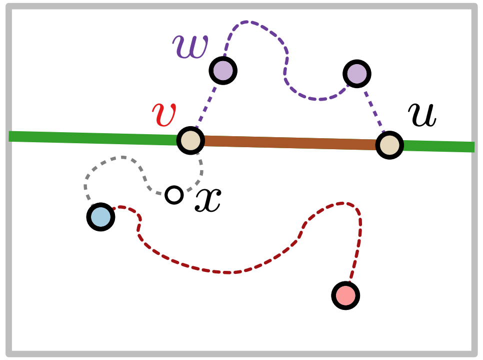

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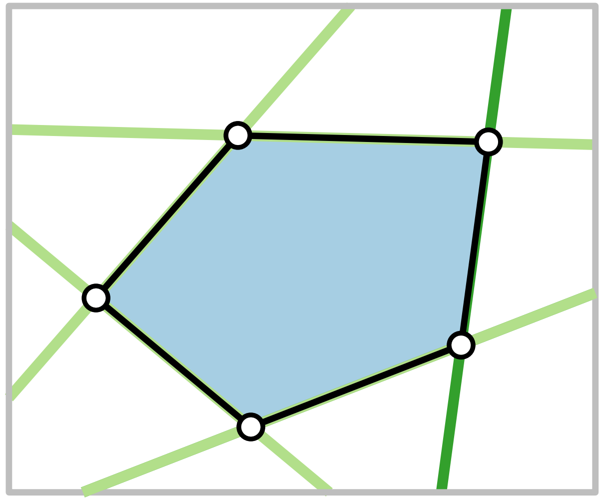

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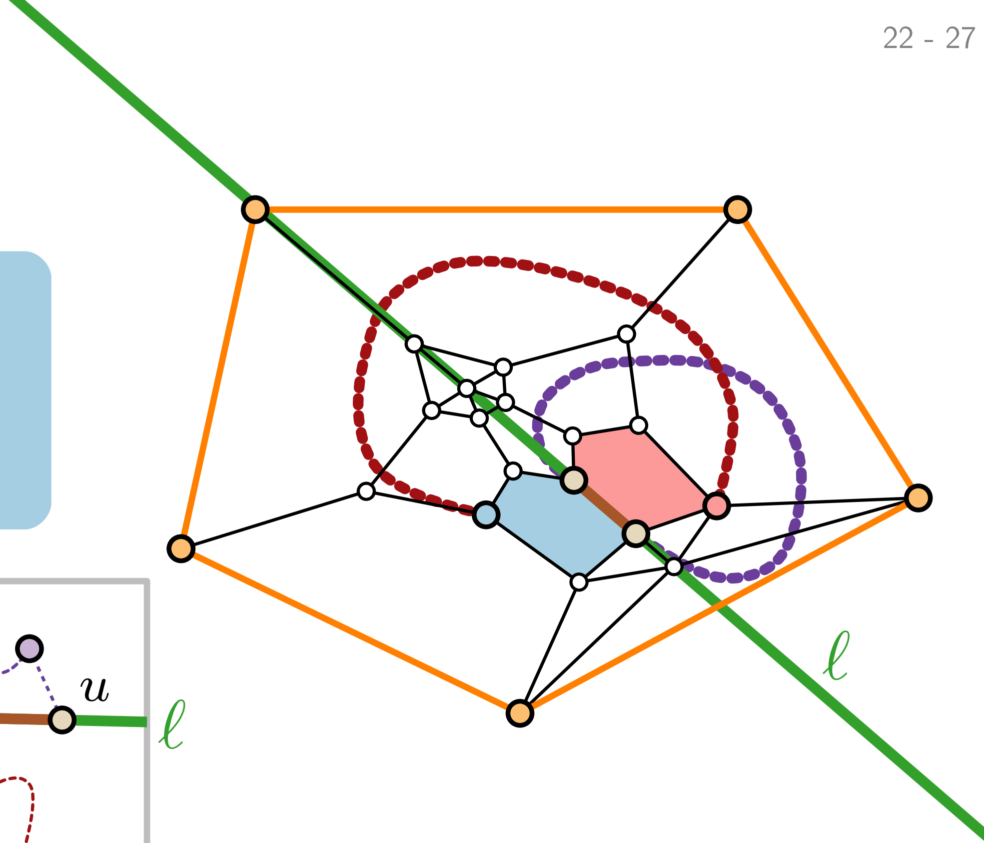

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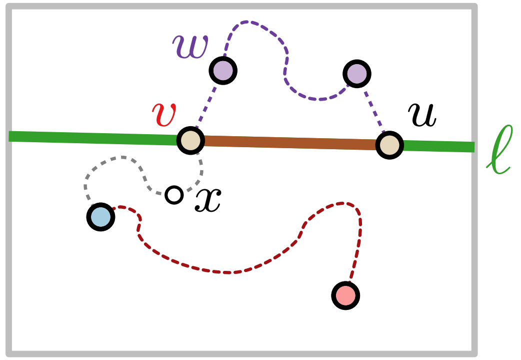

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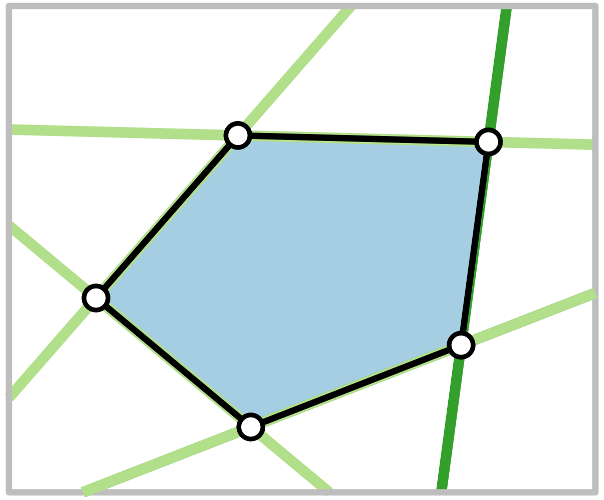

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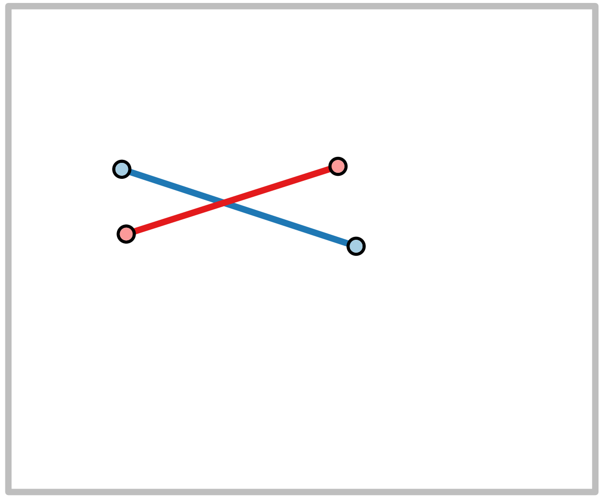

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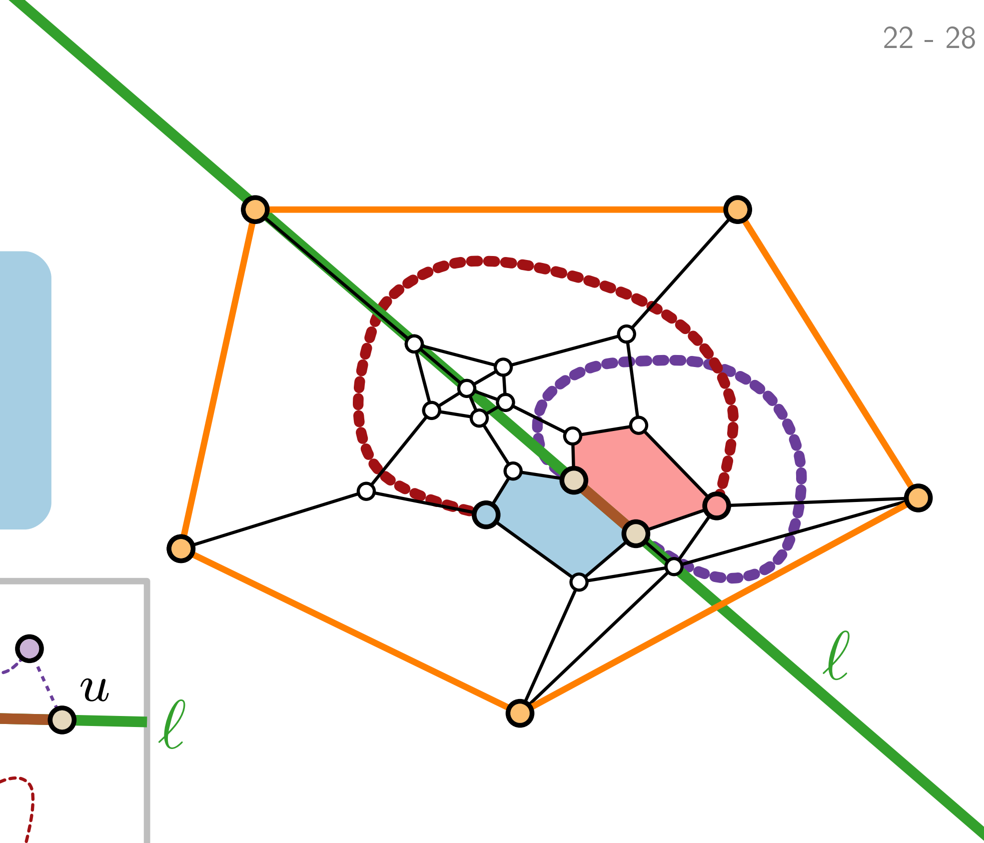

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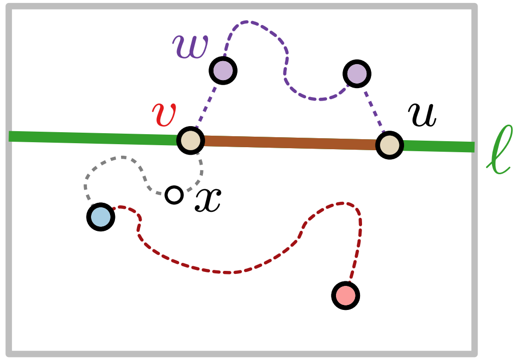

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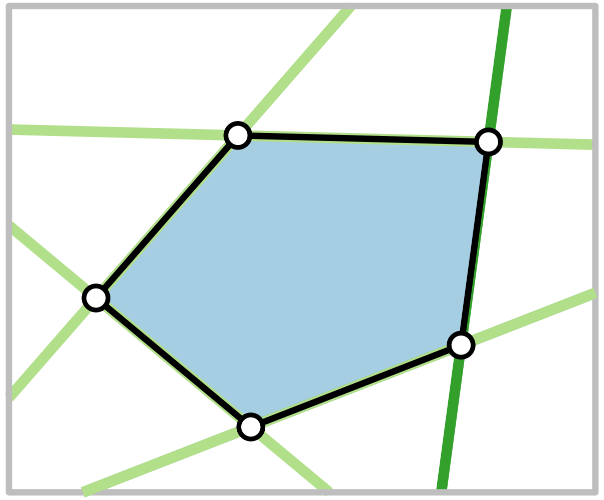

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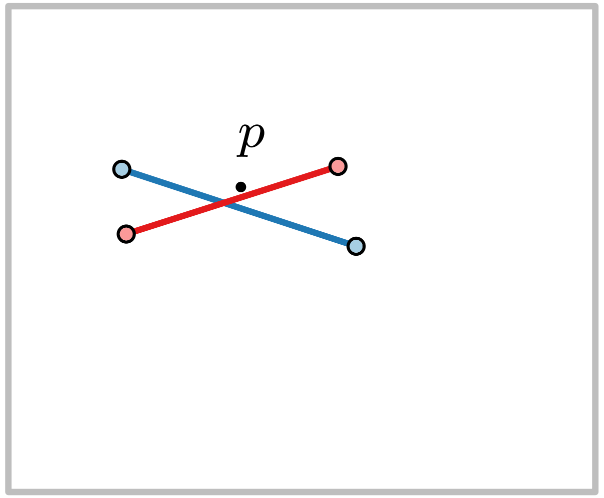

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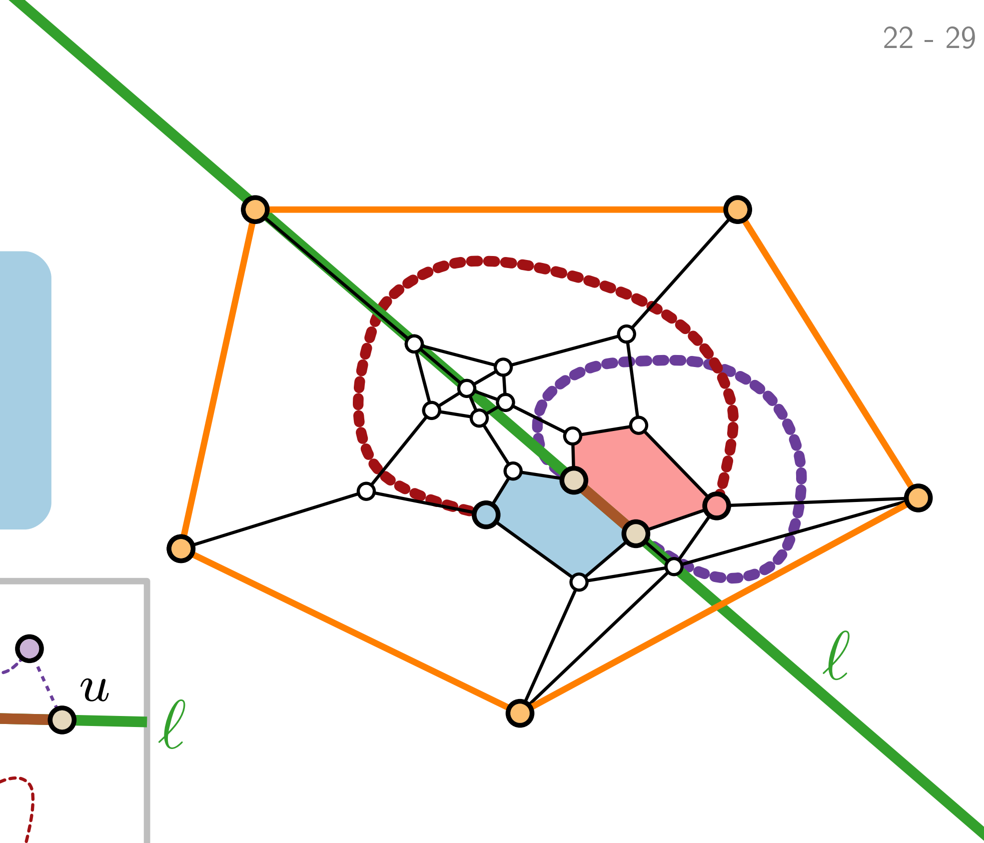

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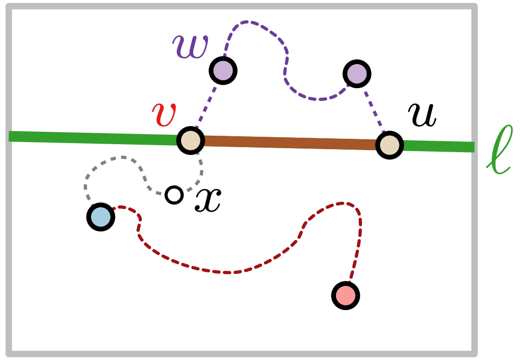

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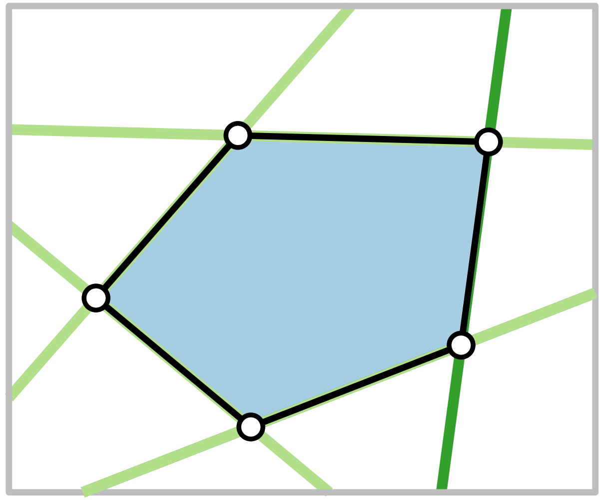

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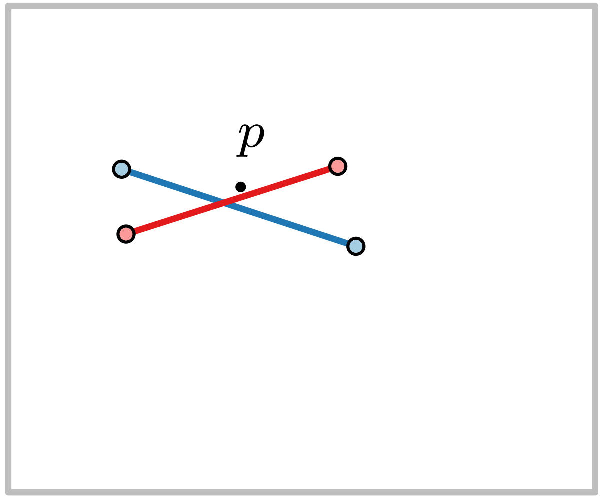

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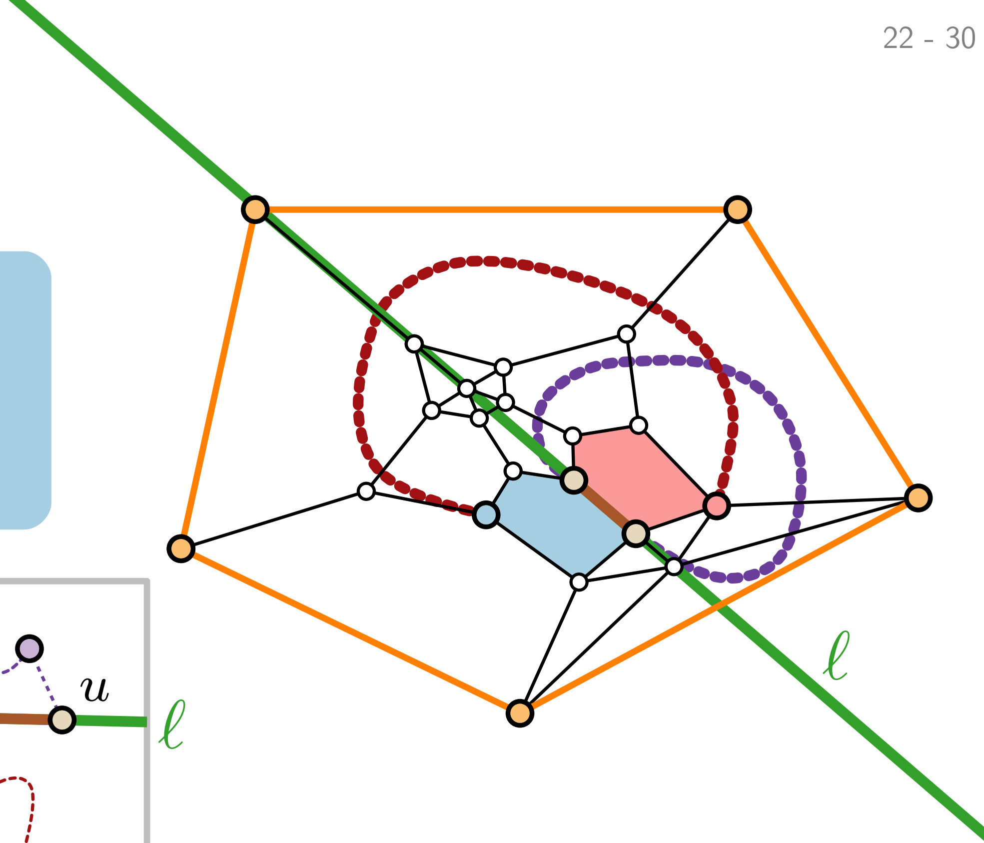

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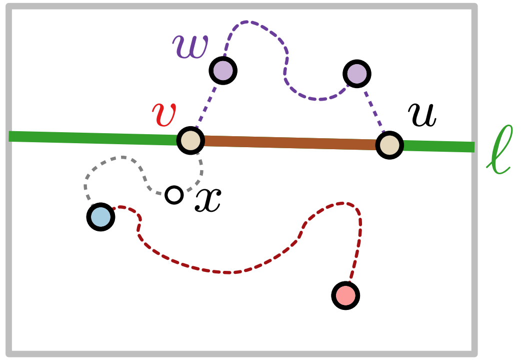

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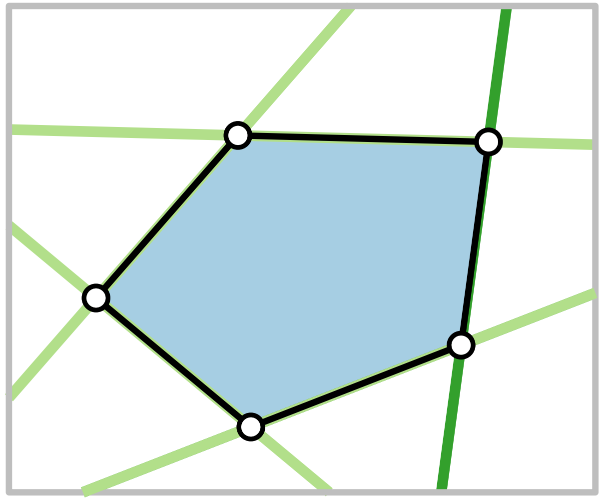

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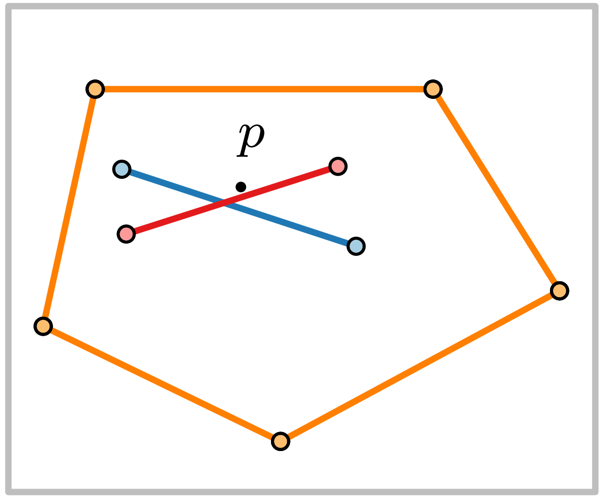

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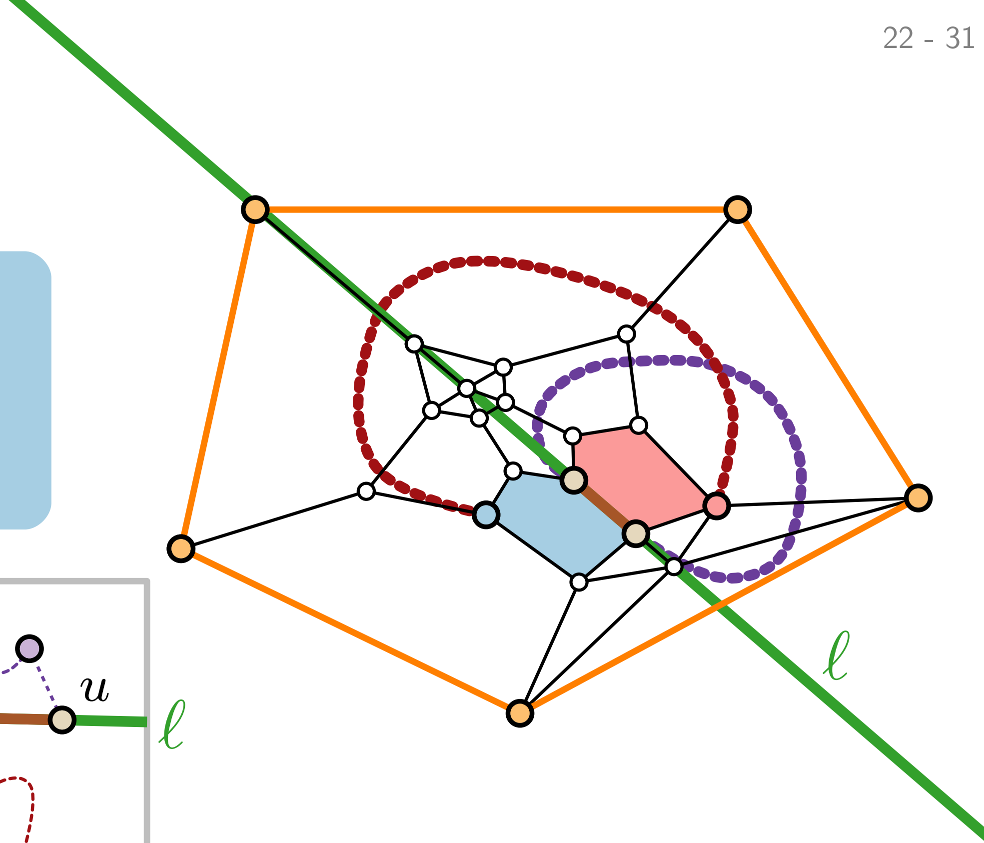

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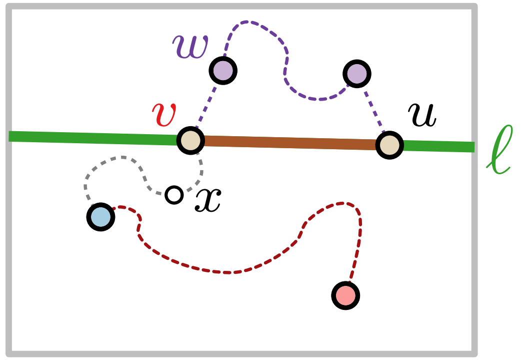

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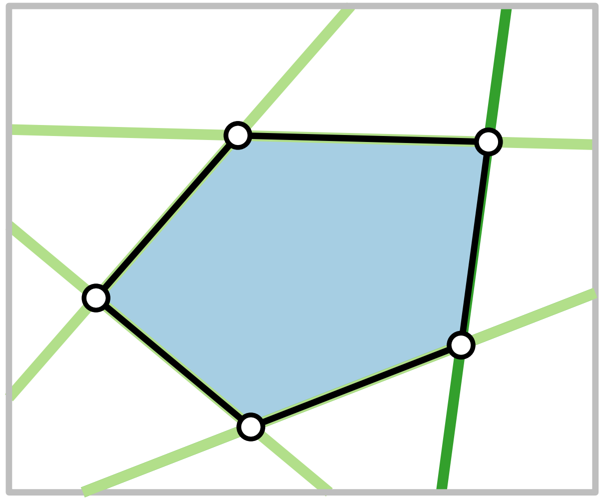

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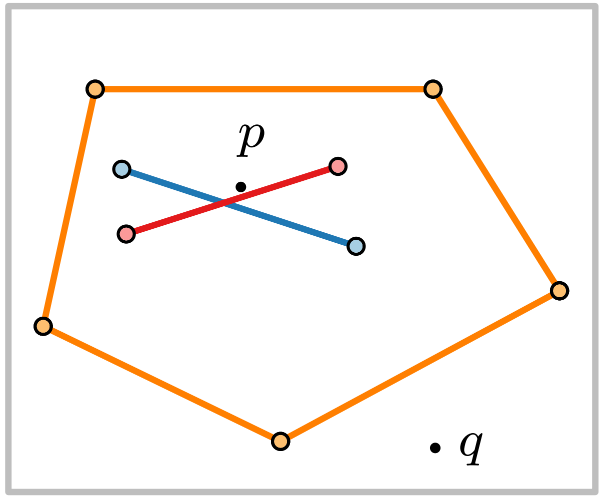

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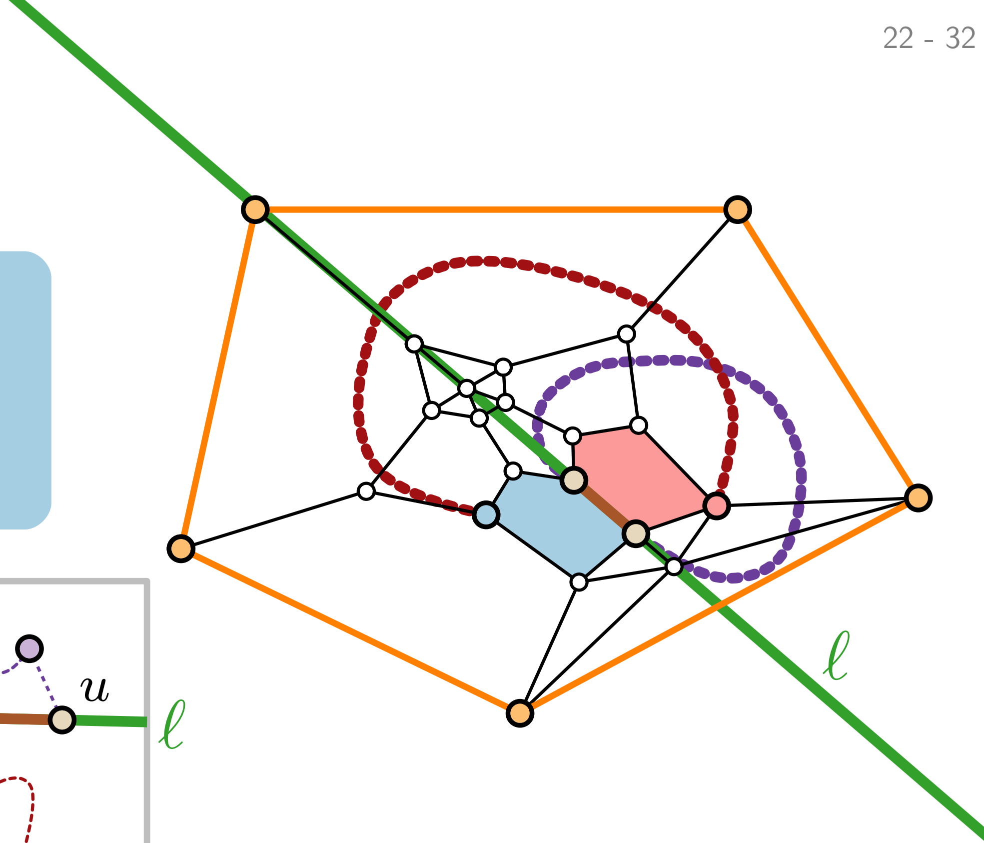

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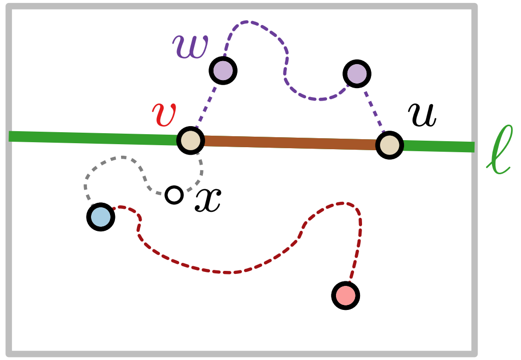

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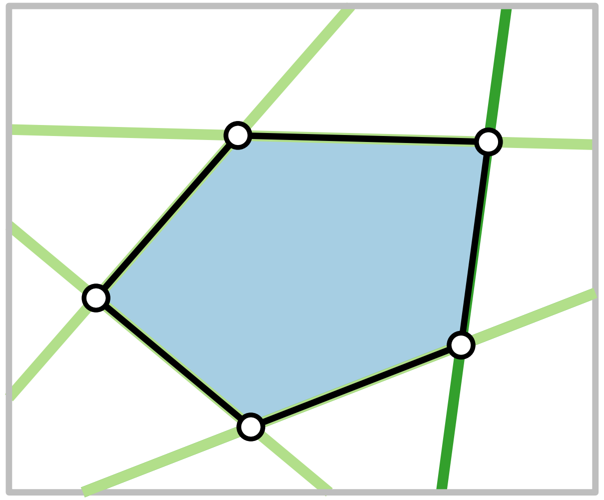

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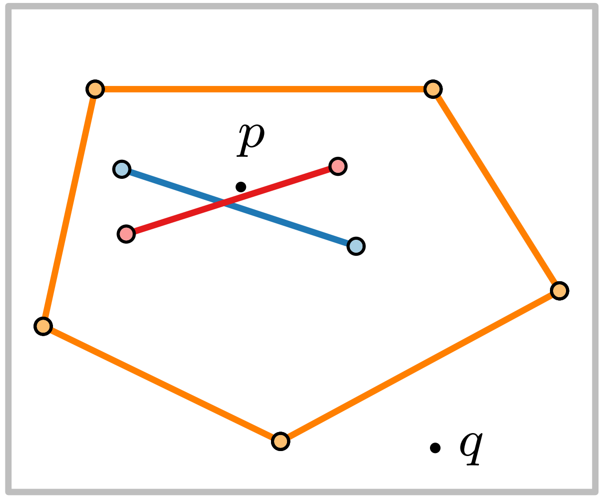

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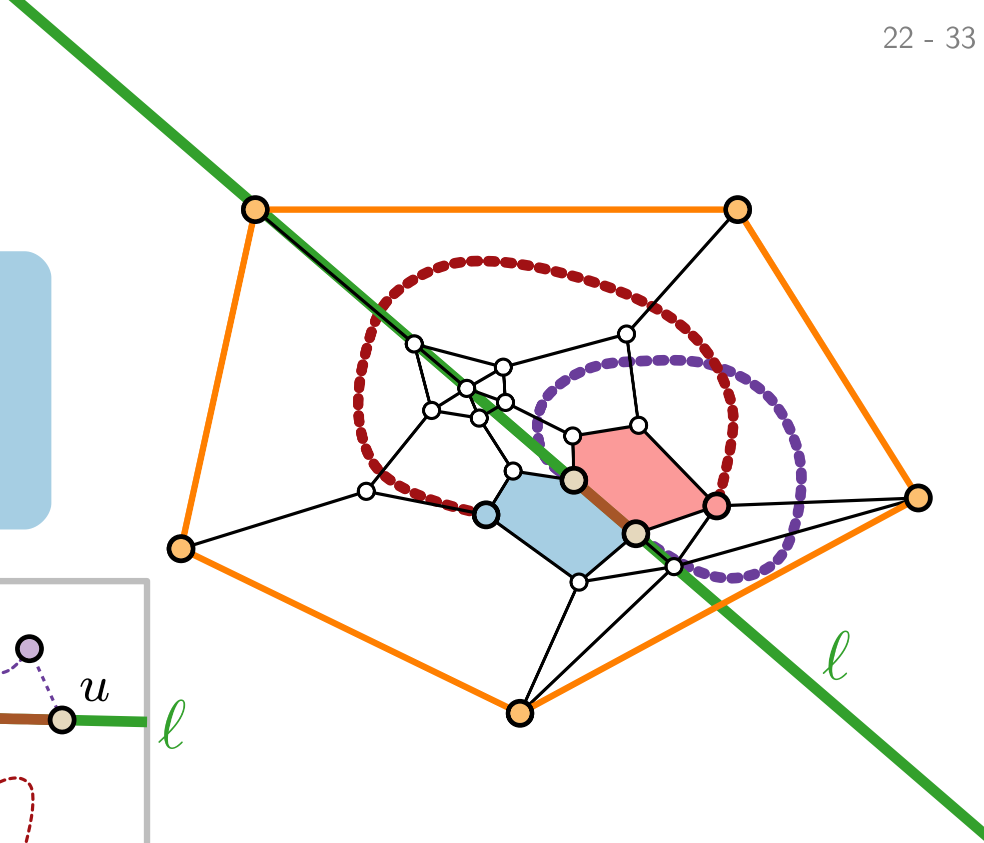

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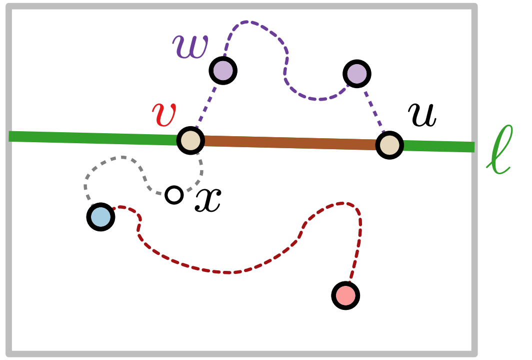

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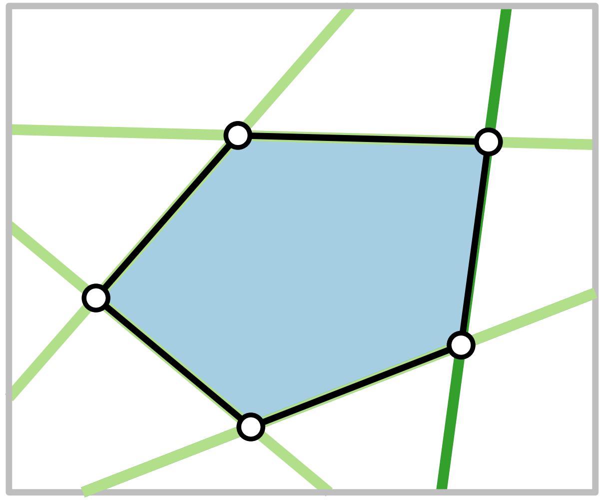

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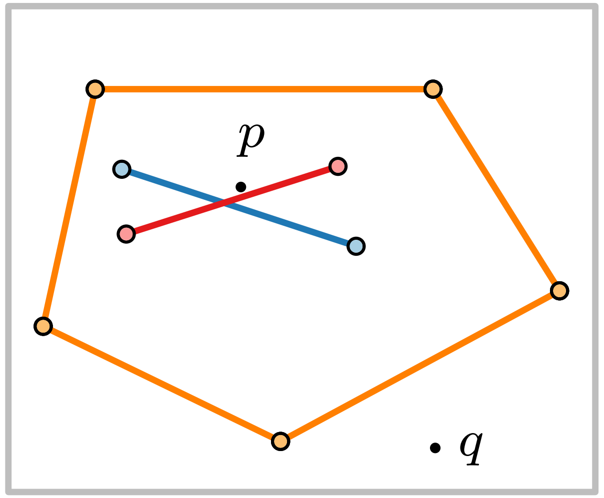

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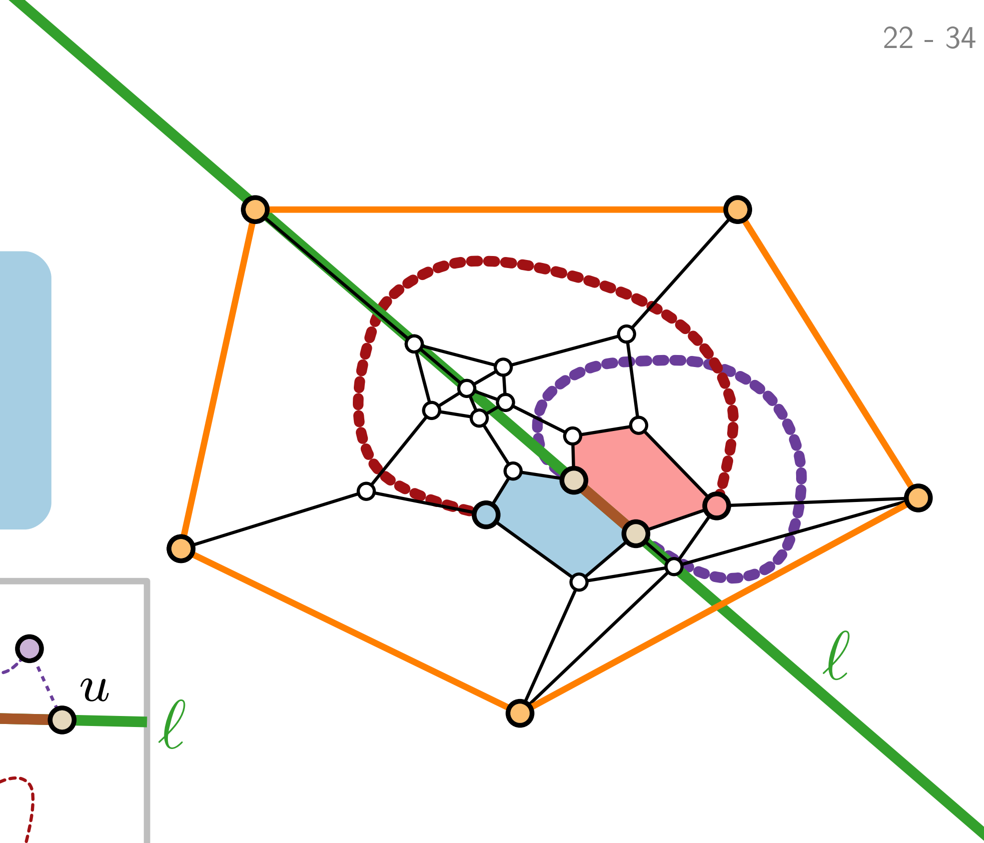

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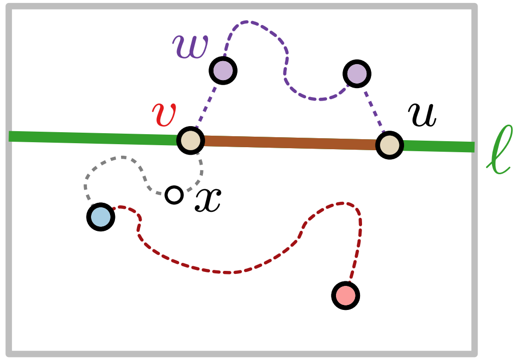

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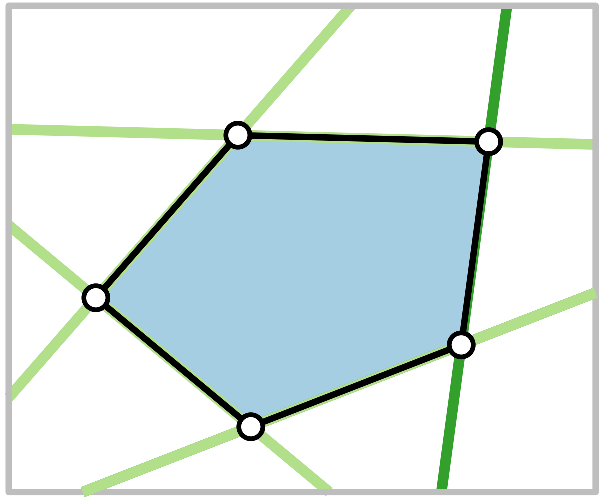

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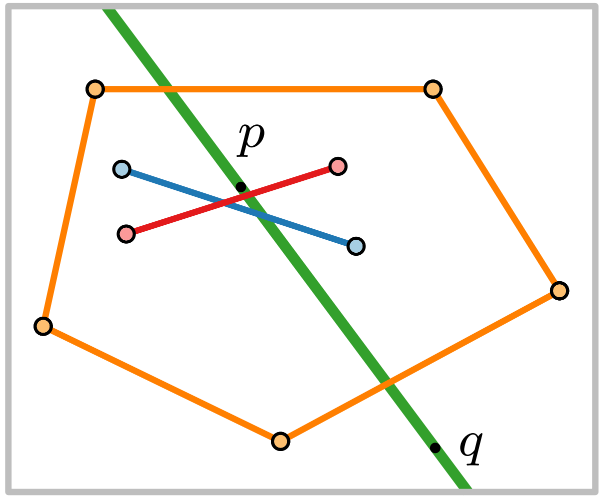

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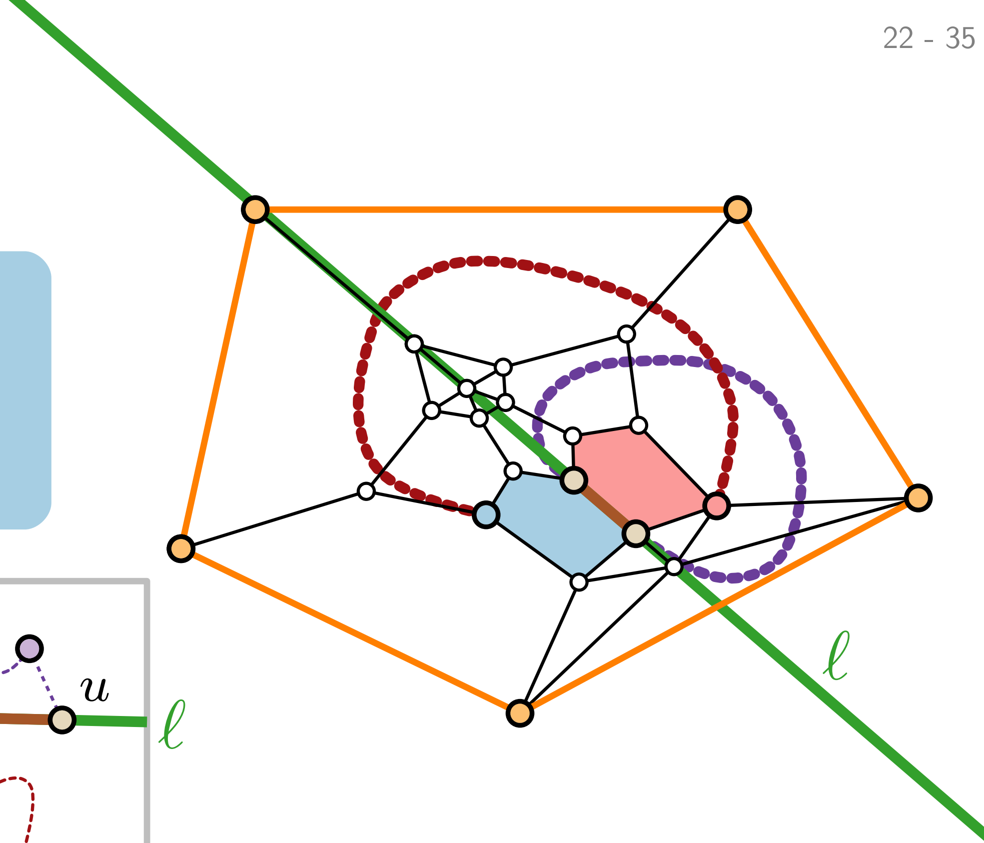

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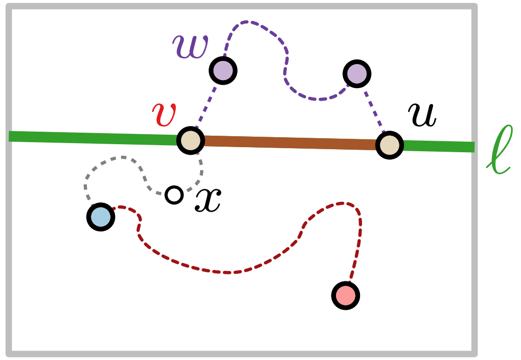

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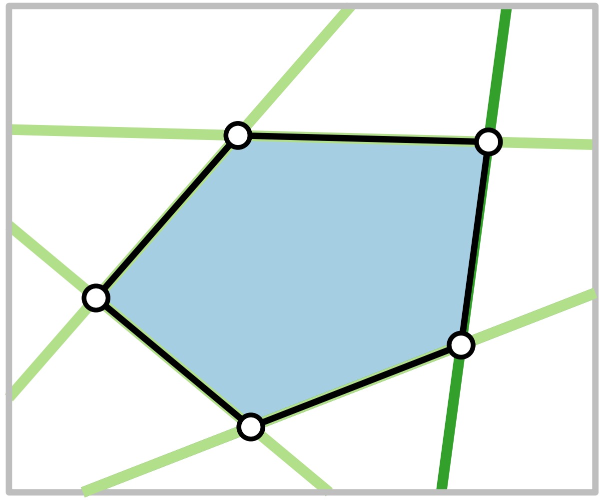

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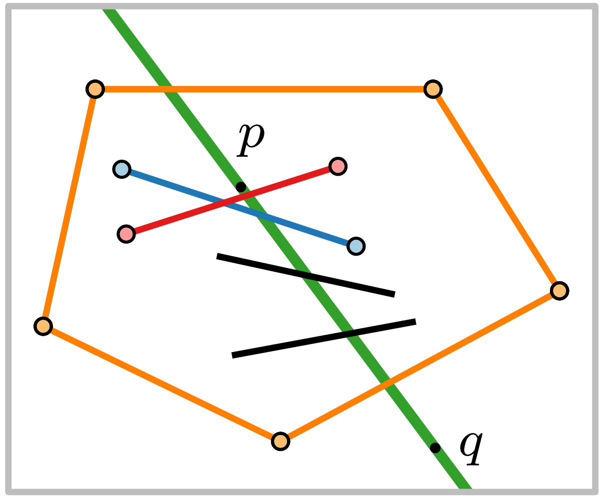

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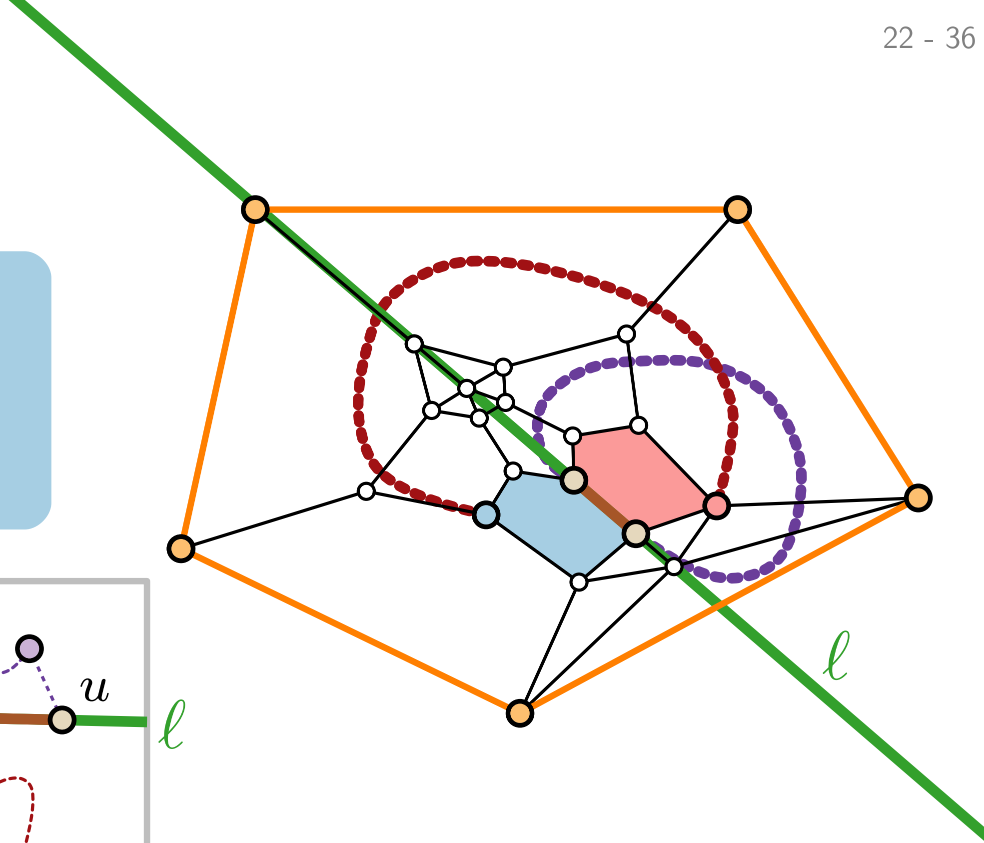

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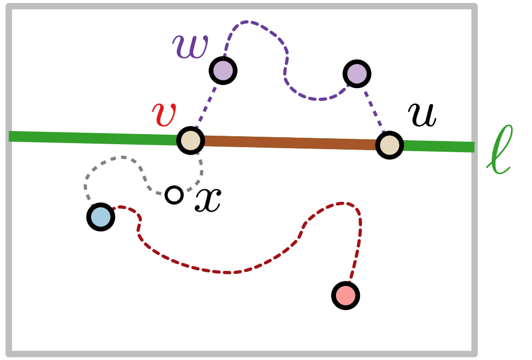

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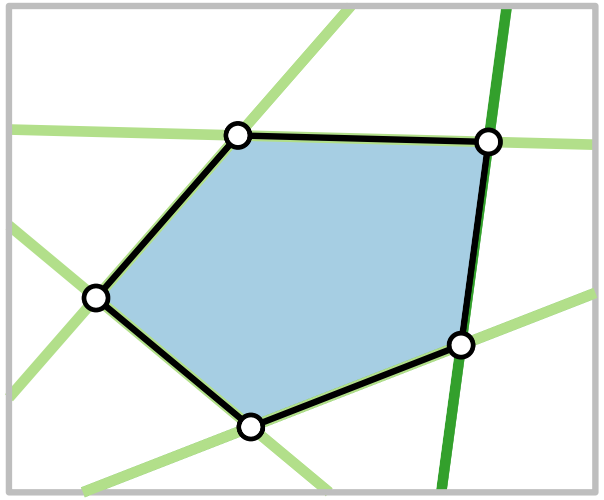

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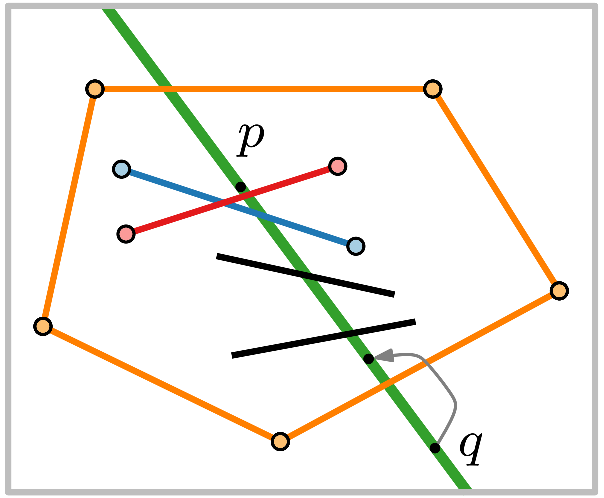

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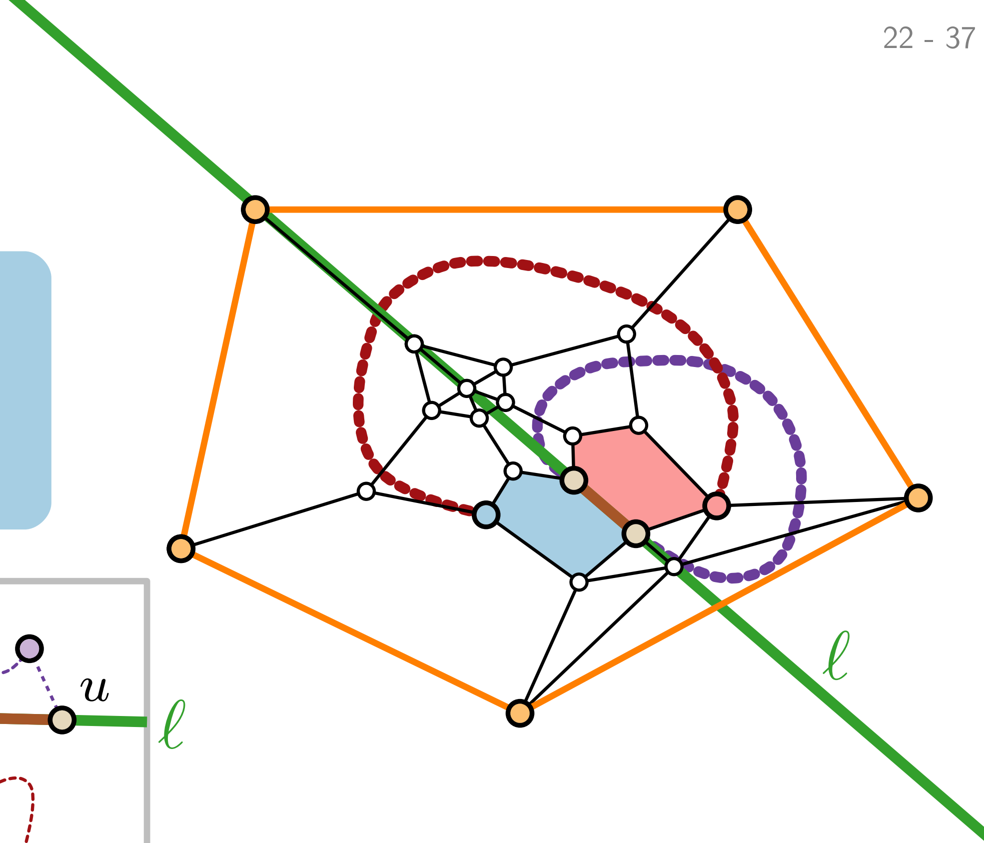

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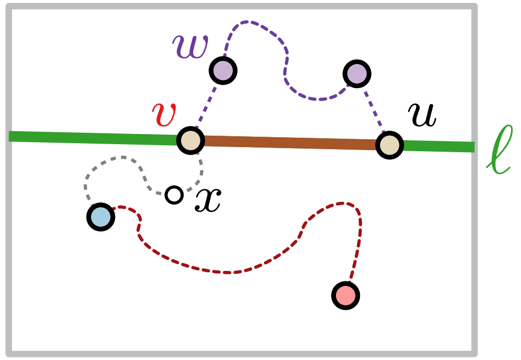

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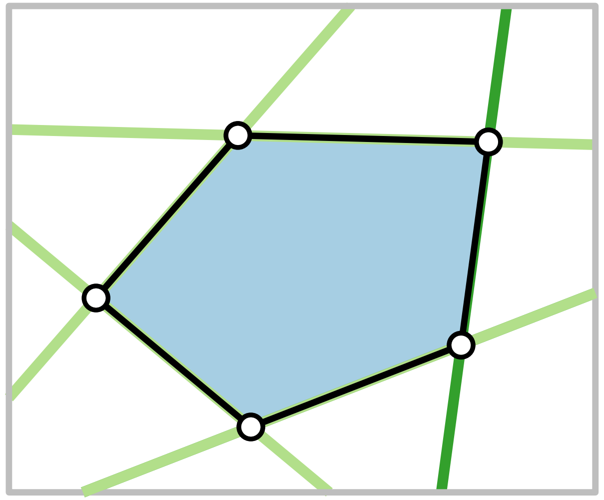

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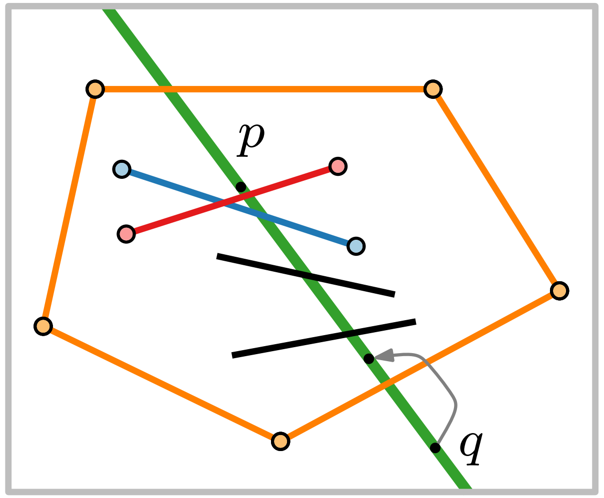

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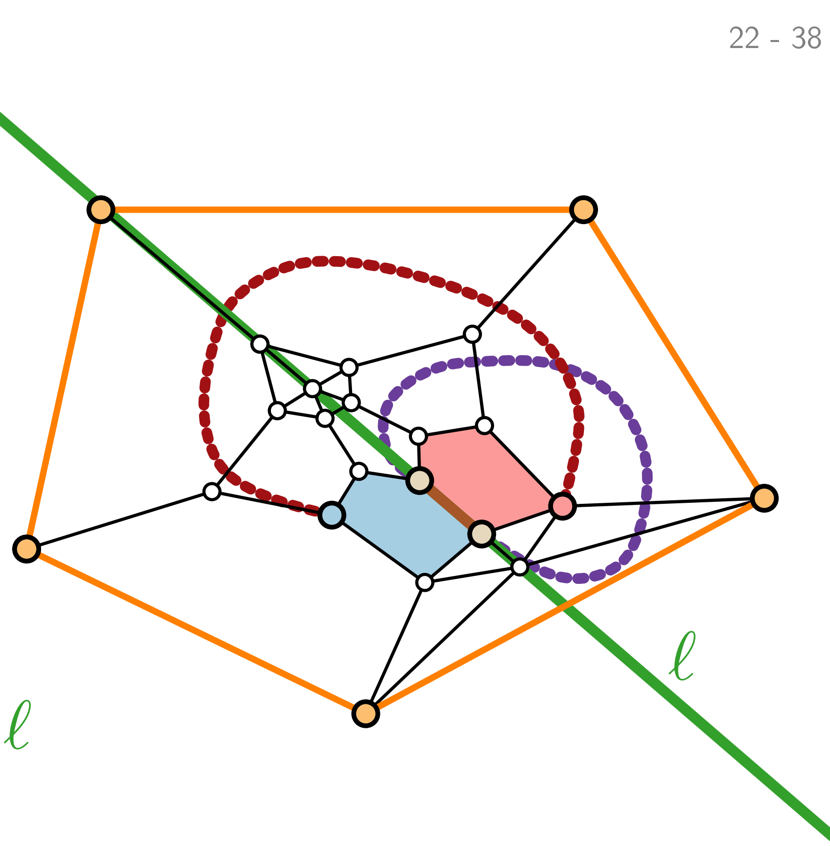

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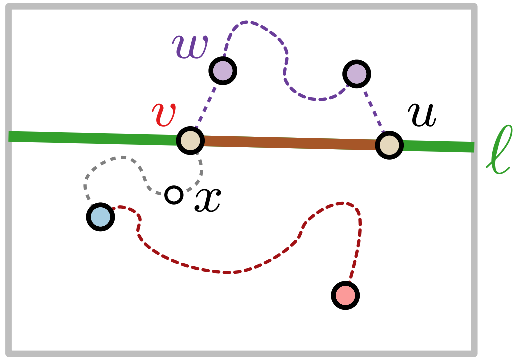

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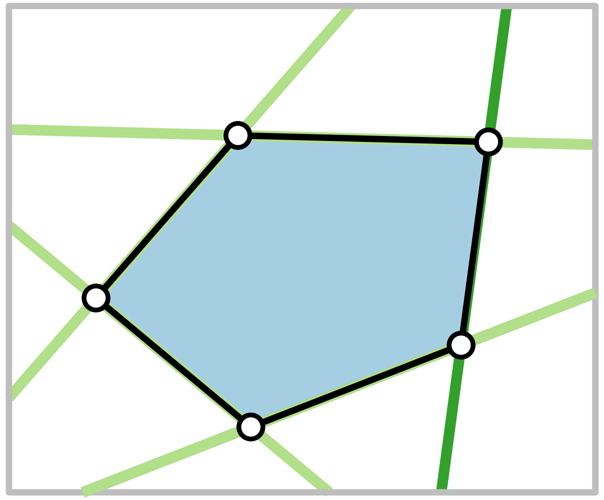

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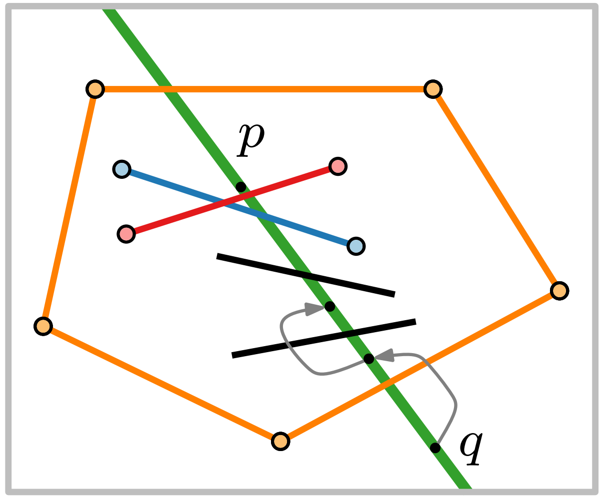

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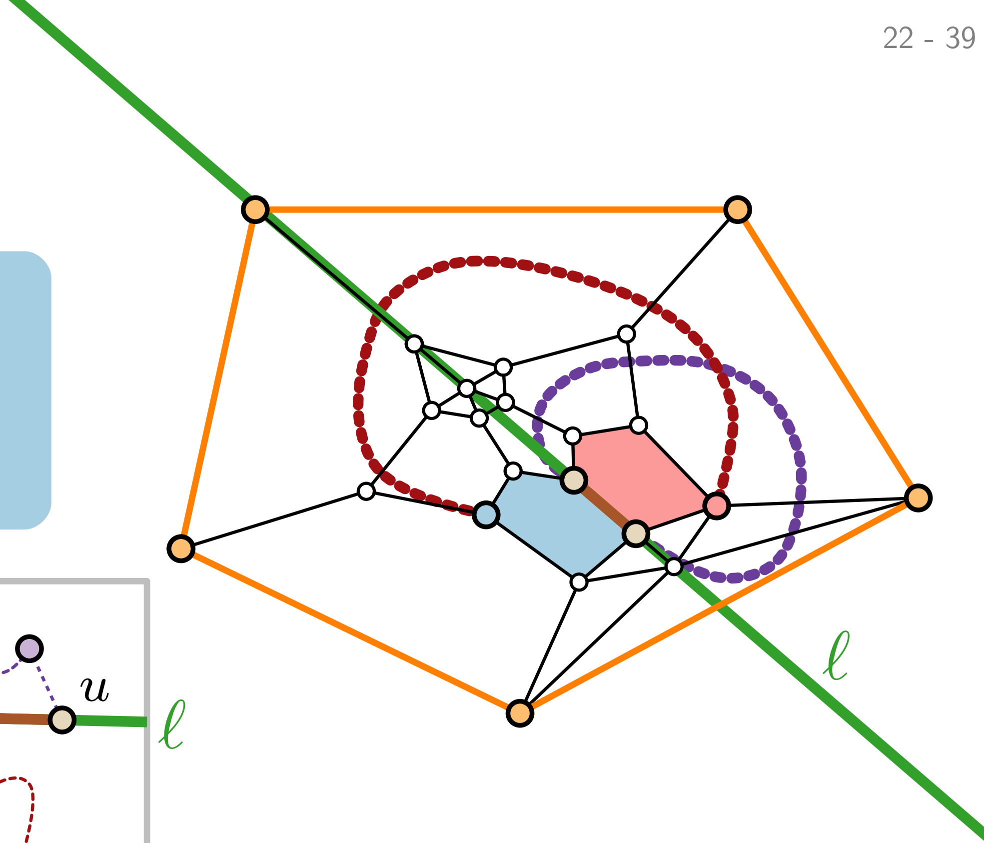

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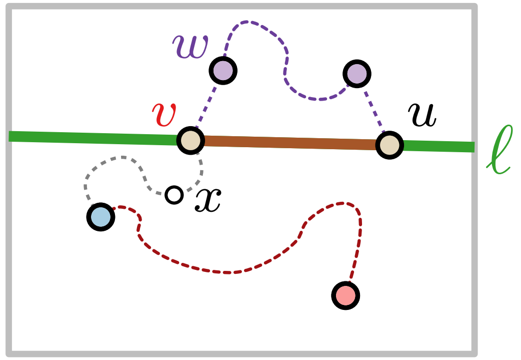

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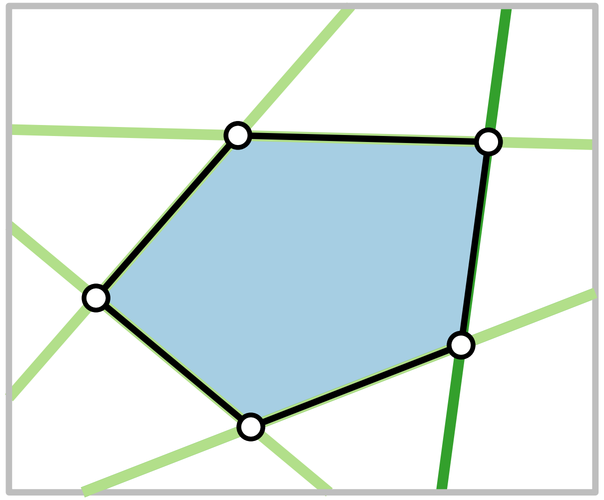

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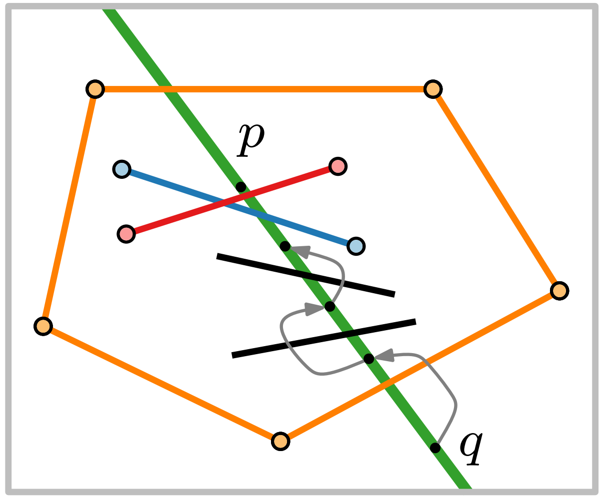

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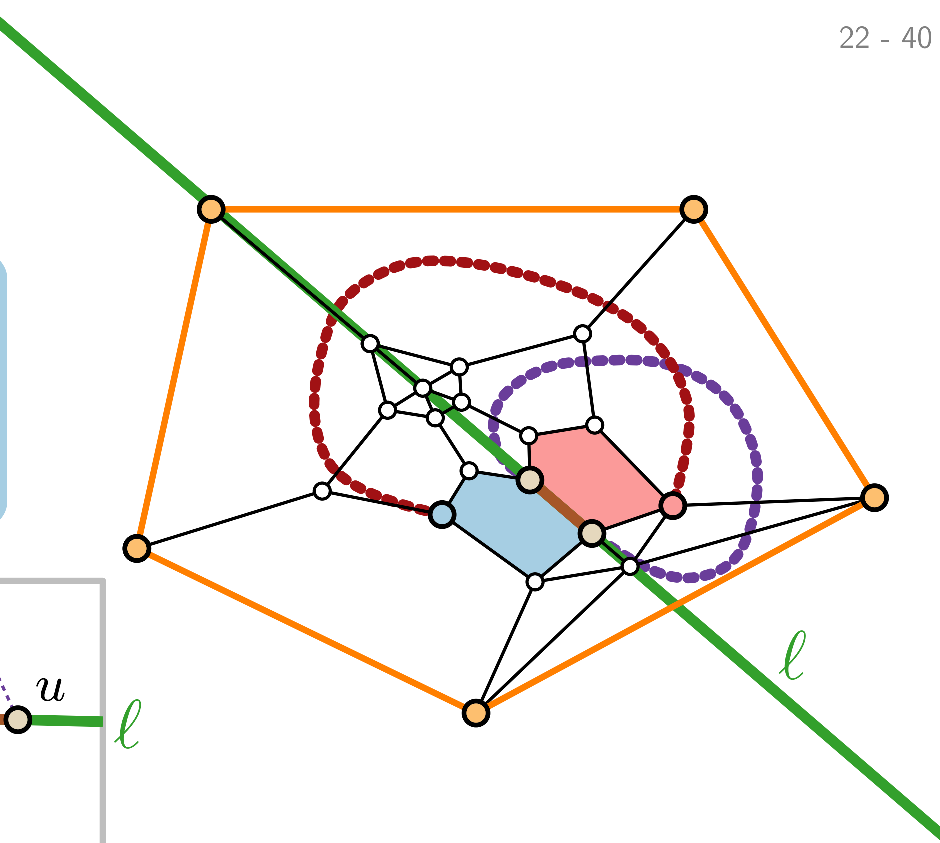

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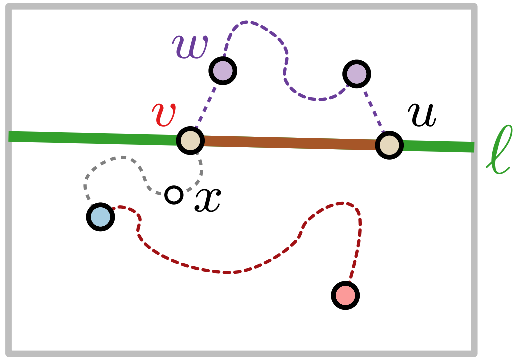

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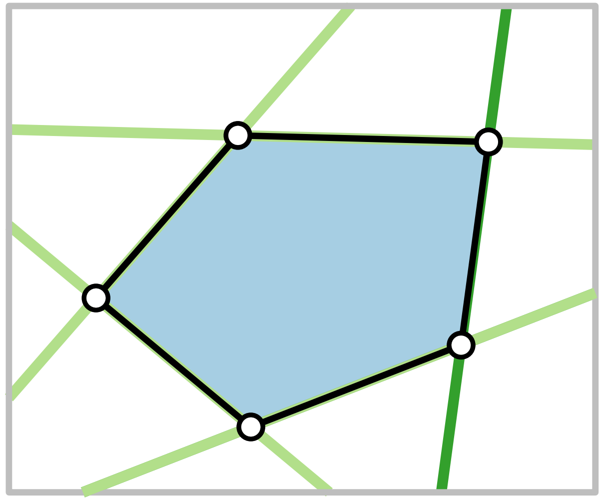

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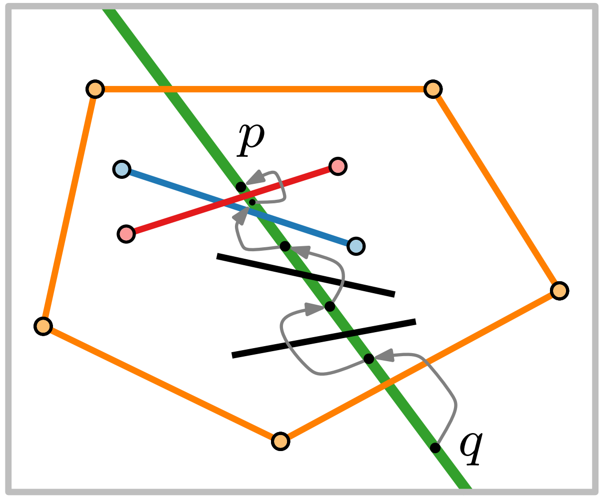

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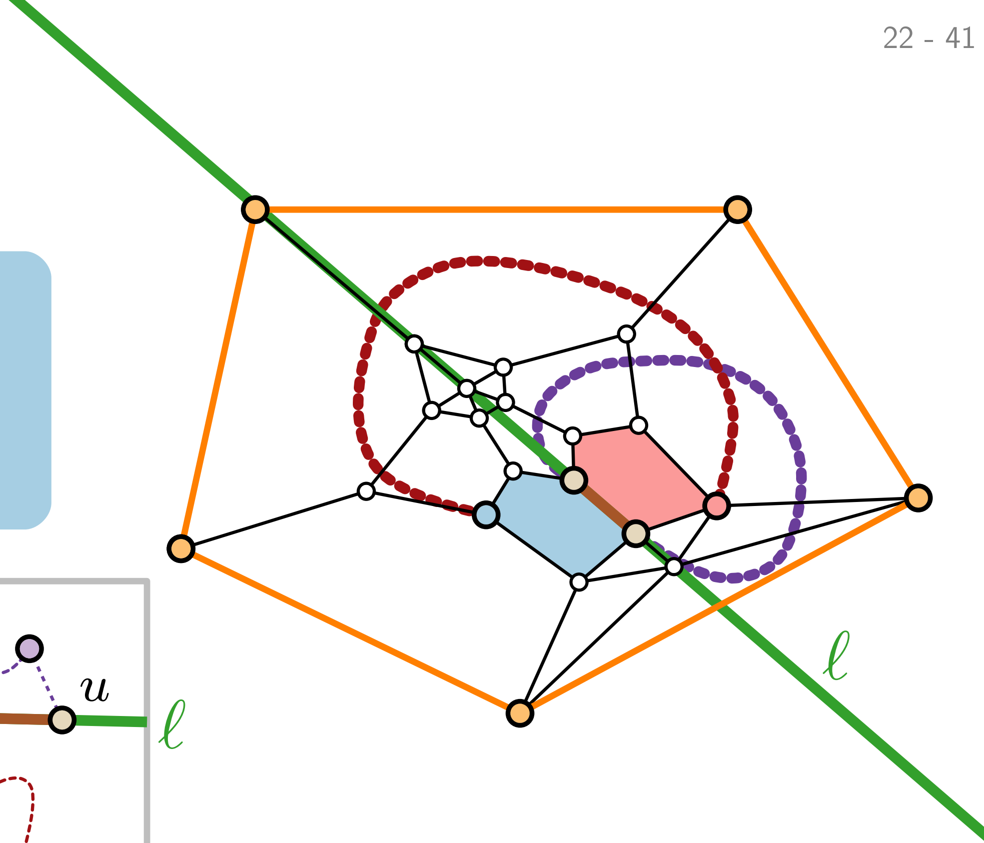

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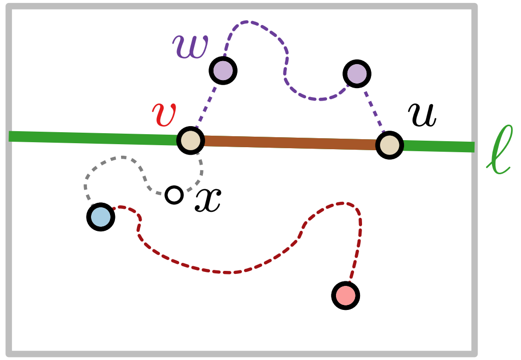

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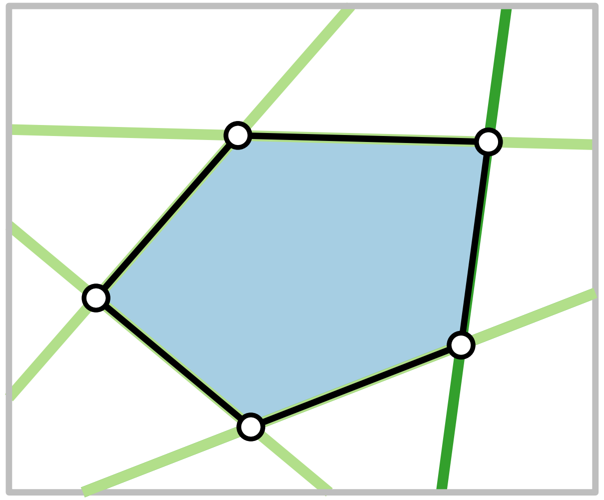

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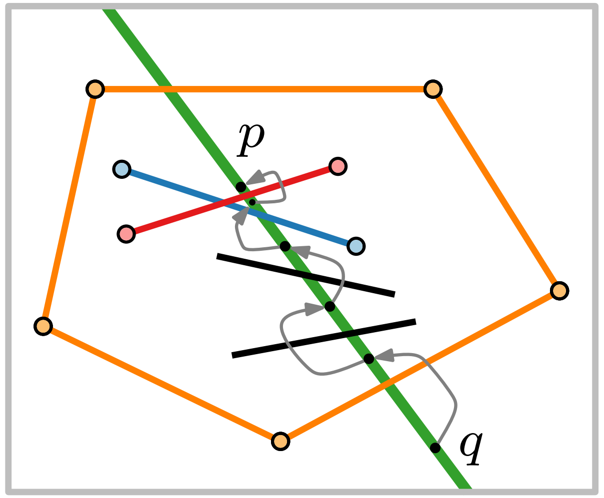

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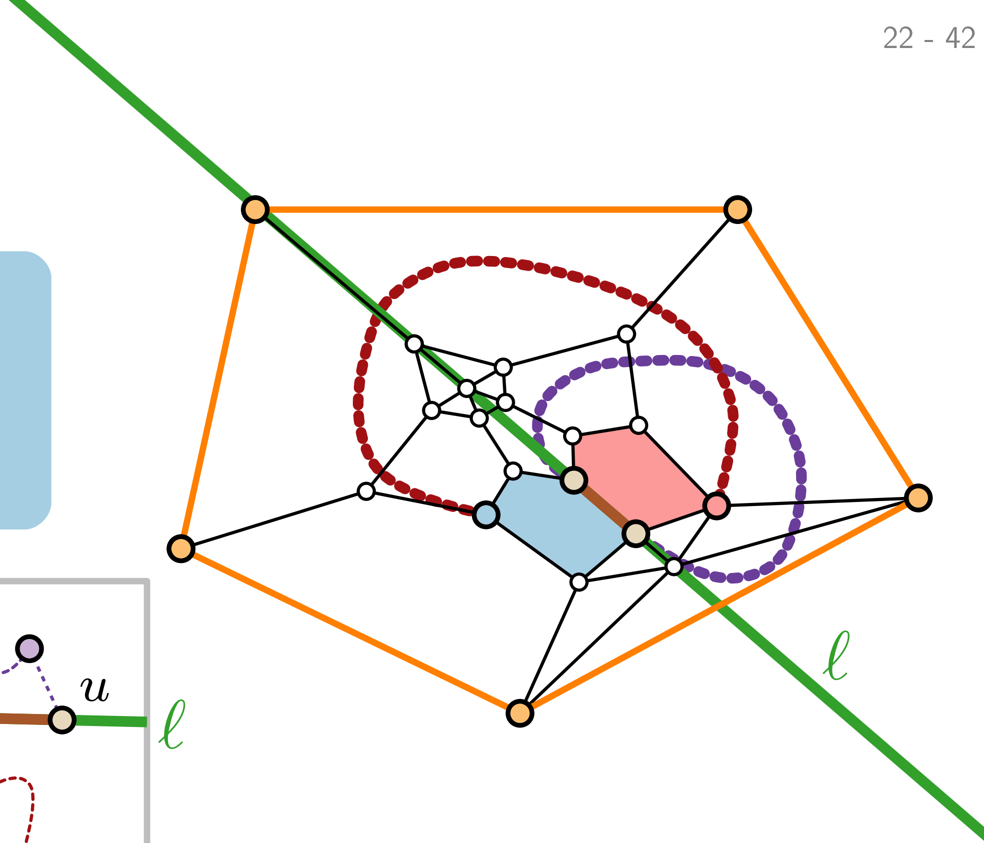

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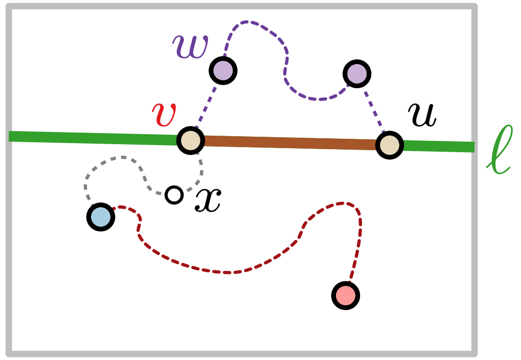

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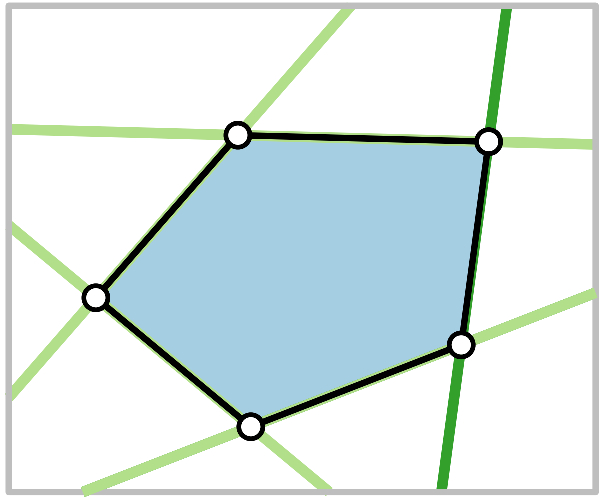

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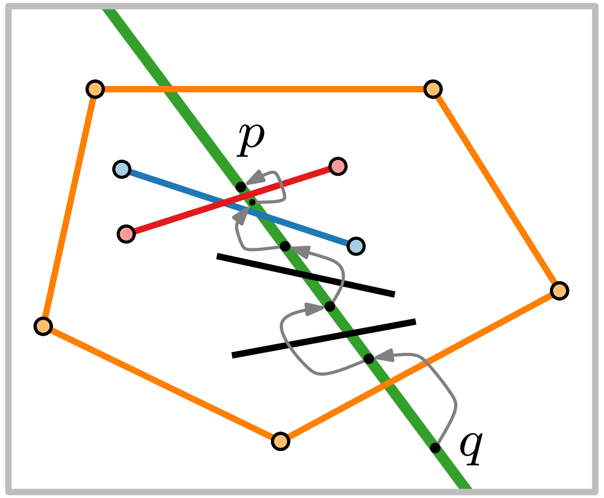

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- In practice, Tutte drawings are hardly used because the inner parts often become tiny.

Literature

Main sources:

- [GD Ch. 10] Force-Directed Methods
- [DG Ch. 4] Drawing on Physical Analogies

Original papers:

- [Eades 1984] A heuristic for graph drawing
- [Fruchterman, Reingold 1991] Graph drawing by force-directed placement
- [Tutte 1963] How to draw a graph