Approximation Algorithms

Lecture 11: MaxSat via Randomized Rounding

Part I: Maximum Satisfiability (MaxSat)

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Is a given formula in conjunctive normal form satisfiable?
E.g., $(x_1 \vee \overline{x_2} \vee x_3) \wedge (x_2 \vee \overline{x_3} \vee x_4) \wedge (x_1 \vee \overline{x_4})$.

Approximation Algorithms

Lecture 11:
MaxSat via Randomized Rounding

Part II:
A Simple Randomized Algorithm

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Thus, $\mathbf{E}[W] \geq \frac{1}{2}\sum_{j=1}^{m} w_j \geq \mathrm{OPT}/2.$ □

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Part III:
Derandomization by Conditional Expectation

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Consider a partial assignment $x_1 = b_1, \ldots, x_i = b_i$ and a clause C_j.

If C_j is already satisfied, then it contributes exactly to $\mathbf{E}[W \mid x_1 = b_1, \ldots, x_i = b_i]$.

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Consider a partial assignment $x_1 = b_1, \ldots, x_i = b_i$ and a clause C_j.

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If C_j is not yet satisfied and contains k unassigned variables, then it contributes exactly to $\mathbf{E}[W \mid x_1 = b_1, \ldots, x_i = b_i]$.

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The conditional expectation is simply the sum of the contributions from each clause. □

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Global optimization?

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Part IV:
Randomized Rounding

An ILP

maximize

subject to

where $C_j = \bigvee_{i \in P_j} x_i \vee \bigvee_{i \in N_j} \bar{x}_i$ for $j = 1, \ldots, m$.

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$$z_j \in \{0,1\}, \qquad \text{for } j = 1,\dots,m$$

where $\quad C_j = \bigvee_{i \in P_j} x_i \vee \bigvee_{i \in N_j} \bar{x}_i \quad \text{for } j = 1,\dots,m.$

An ILP

maximize $\sum_{j=1}^{m} w_j z_j$

subject to

$$y_i \in \{0,1\}, \qquad \text{for } i = 1, \ldots, n$$
$$z_j \in \{0,1\}, \qquad \text{for } j = 1, \ldots, m$$

where $C_j = \bigvee_{i \in P_j} x_i \vee \bigvee_{i \in N_j} \bar{x}_i \quad \text{for } j = 1, \ldots, m.$

An ILP

$$
\begin{aligned}
\textbf{maximize} \quad & \sum_{j=1}^{m} w_j z_j \\
\textbf{subject to} \quad & \sum_{i \in P_j} \quad + \sum_{i \in N_j} & \text{for } j = 1, \ldots, m \\
& y_i \in \{0, 1\}, & \text{for } i = 1, \ldots, n \\
& z_j \in \{0, 1\}, & \text{for } j = 1, \ldots, m
\end{aligned}
$$

where $C_j = \bigvee_{i \in P_j} x_i \vee \bigvee_{i \in N_j} \bar{x}_i$ for $j = 1, \ldots, m$.

An ILP

$$\begin{aligned} \textbf{maximize} \quad & \sum_{j=1}^{m} w_j z_j \\ \textbf{subject to} \quad & \sum_{i \in P_j} y_i + \sum_{i \in N_j} & \text{for } j = 1, \ldots, m \\ & y_i \in \{0, 1\}, & \text{for } i = 1, \ldots, n \\ & z_j \in \{0, 1\}, & \text{for } j = 1, \ldots, m \end{aligned}$$

where $C_j = \bigvee_{i \in P_j} x_i \vee \bigvee_{i \in N_j} \bar{x}_i \quad \text{for } j = 1, \ldots, m.$

An ILP

$$
\begin{aligned}
\textbf{maximize} \quad & \sum_{j=1}^{m} w_j z_j \\
\textbf{subject to} \quad & \sum_{i \in P_j} y_i + \sum_{i \in N_j} (1 - y_i) & \text{for } j = 1, \ldots, m \\
& y_i \in \{0,1\}, & \text{for } i = 1, \ldots, n \\
& z_j \in \{0,1\}, & \text{for } j = 1, \ldots, m
\end{aligned}
$$

where $C_j = \bigvee_{i \in P_j} x_i \vee \bigvee_{i \in N_j} \bar{x}_i$ for $j = 1, \ldots, m$.

An ILP

$$
\begin{aligned}
\textbf{maximize} \quad & \sum_{j=1}^{m} w_j z_j \\
\textbf{subject to} \quad & \sum_{i \in P_j} y_i + \sum_{i \in N_j} (1 - y_i) \geq \qquad \text{for } j = 1, \ldots, m \\
& y_i \in \{0, 1\}, \qquad \text{for } i = 1, \ldots, n \\
& z_j \in \{0, 1\}, \qquad \text{for } j = 1, \ldots, m
\end{aligned}
$$

where $C_j = \bigvee_{i \in P_j} x_i \vee \bigvee_{i \in N_j} \bar{x}_i$ for $j = 1, \ldots, m$.

An ILP

$$\textbf{maximize} \quad \sum_{j=1}^{m} w_j z_j$$

$$\textbf{subject to} \quad \sum_{i \in P_j} y_i + \sum_{i \in N_j} (1 - y_i) \geq z_j \quad \text{for } j = 1, \ldots, m$$

$$y_i \in \{0, 1\}, \quad \text{for } i = 1, \ldots, n$$

$$z_j \in \{0, 1\}, \quad \text{for } j = 1, \ldots, m$$

where $C_j = \bigvee_{i \in P_j} x_i \vee \bigvee_{i \in N_j} \bar{x}_i$ for $j = 1, \ldots, m$.

... and its Relaxation

$$\begin{aligned}
&\textbf{maximize} \quad \sum_{j=1}^{m} w_j z_j \\
&\textbf{subject to} \quad \sum_{i \in P_j} y_i + \sum_{i \in N_j} (1 - y_i) \geq z_j && \text{for } j = 1, \ldots, m \\
&\qquad 0 \leq y_i \leq 1, && \text{for } i = 1, \ldots, n \\
&\qquad 0 \leq z_j \leq 1, && \text{for } j = 1, \ldots, m
\end{aligned}$$

where $C_j = \bigvee_{i \in P_j} x_i \vee \bigvee_{i \in N_j} \bar{x}_i$ for $j = 1, \ldots, m$.

Randomized Rounding

Theorem. Let (y^*, z^*) be an optimal solution to the LP-relaxation.

Randomized Rounding

Theorem. Let (y^*, z^*) be an optimal solution to the LP-relaxation. Independently setting each variable x_i to 1

Randomized Rounding

Theorem. Let (y^*, z^*) be an optimal solution to the LP-relaxation. Independently setting each variable x_i to 1 with probability y_i^* provides a ( )-approximation for MaxSat.

Randomized Rounding

Theorem. Let (y^*, z^*) be an optimal solution to the LP-relaxation. Independently setting each variable x_i to 1 with probability y_i^* provides a $(1 - 1/e)$-approximation for MaxSat.

Randomized Rounding

Theorem. Let (y^*, z^*) be an optimal solution to the LP-relaxation. Independently setting each variable x_i to 1 with probability y_i^* provides a $\underbrace{(1 - 1/e)}_{\approx 0.63}$-approximation for MaxSat.

Approximation Algorithms

Lecture 11:
MaxSat via Randomized Rounding

Part V:
Randomized Rounding – Proof

Mathematical Toolkit

Let f be a function that is concave on $[0, 1]$

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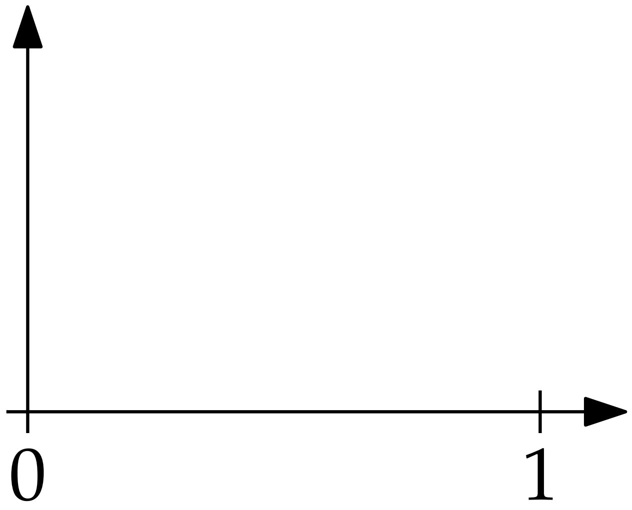

Mathematical Toolkit

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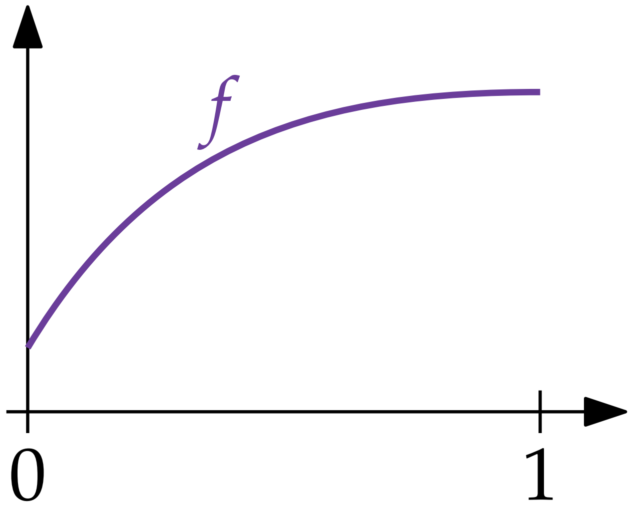

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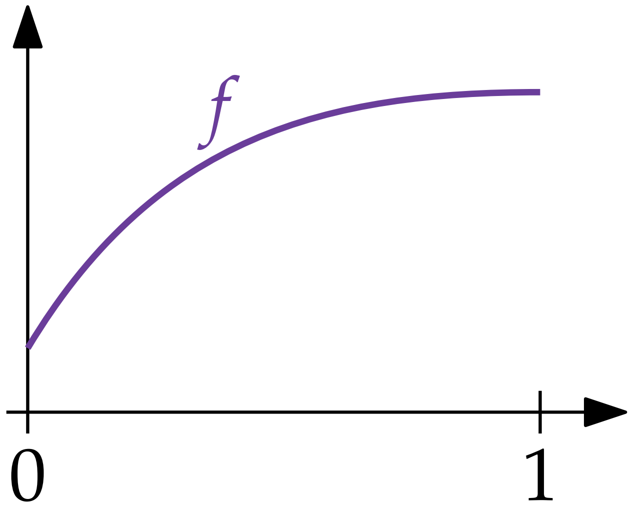
Let f be a function that is concave on $[0, 1]$ (i.e. $f''(x) \leq 0$ on $[0, 1]$) with $f(0) = a$



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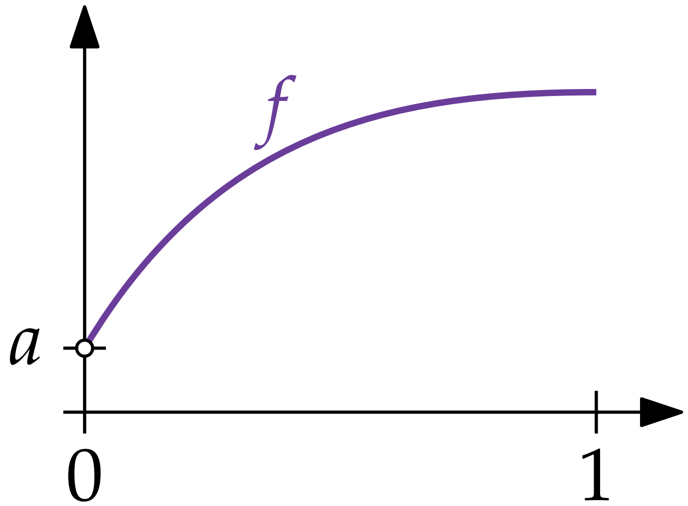
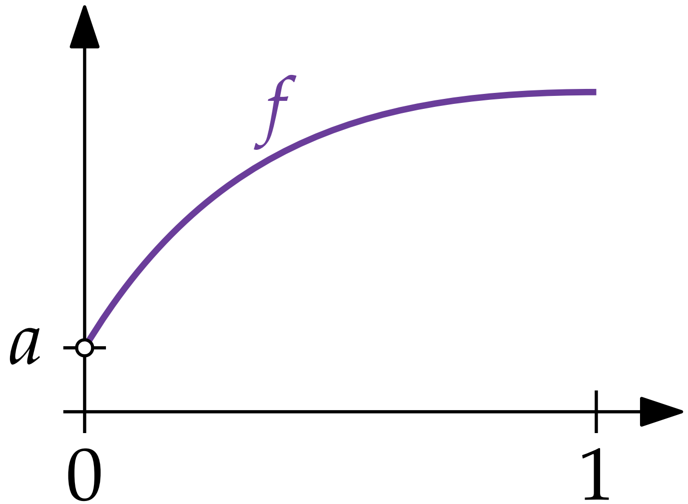


Mathematical Toolkit

Let f be a function that is concave on $[0, 1]$
(i.e. $f''(x) \leq 0$ on $[0, 1]$) with $f(0) = a$ and $f(1) = a + b$



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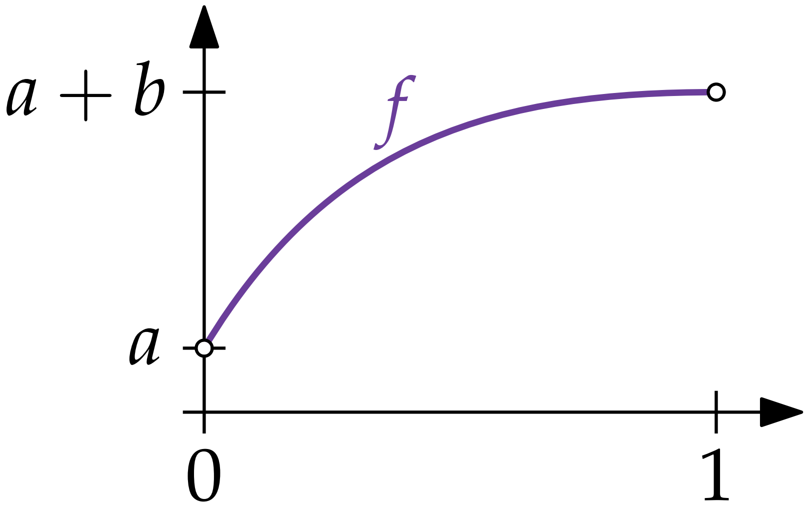
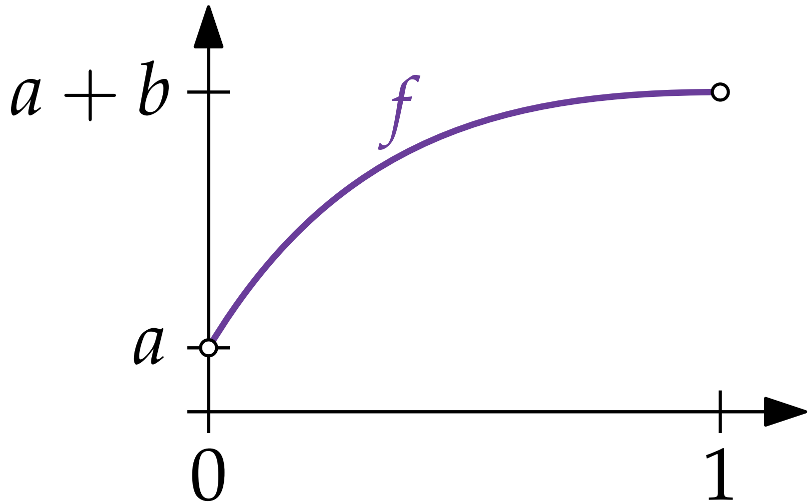


Mathematical Toolkit

Let f be a function that is concave on $[0, 1]$
(i.e. $f''(x) \leq 0$ on $[0, 1]$) with $f(0) = a$ and $f(1) = a + b$

$$\Rightarrow f(x) \geq bx + a \text{ for } x \in [0, 1].$$



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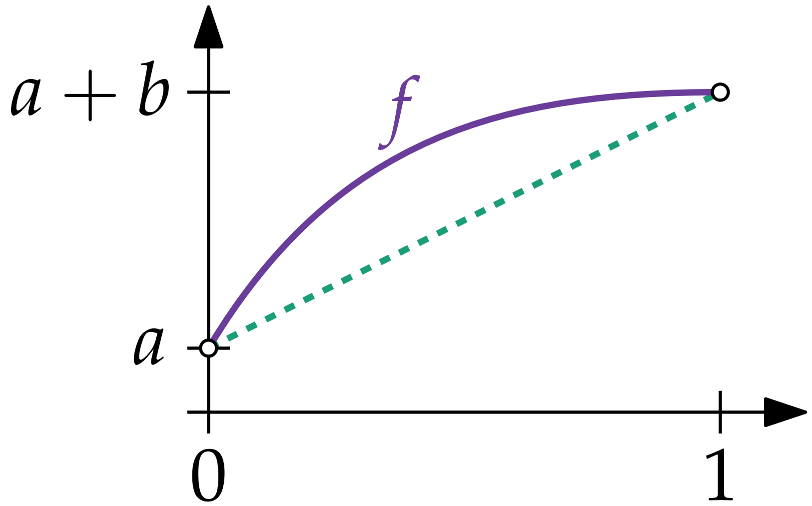

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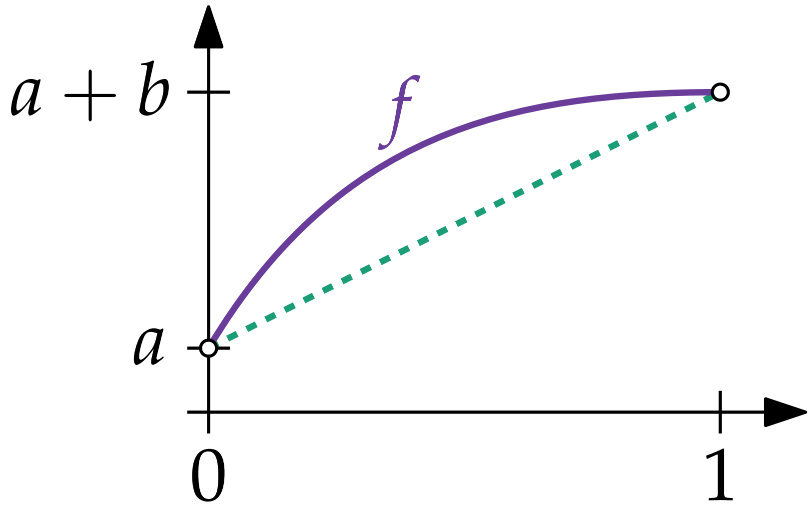

Arithmetic–Geometric Mean Inequality (AGMI):

Mathematical Toolkit

Let f be a function that is concave on $[0, 1]$
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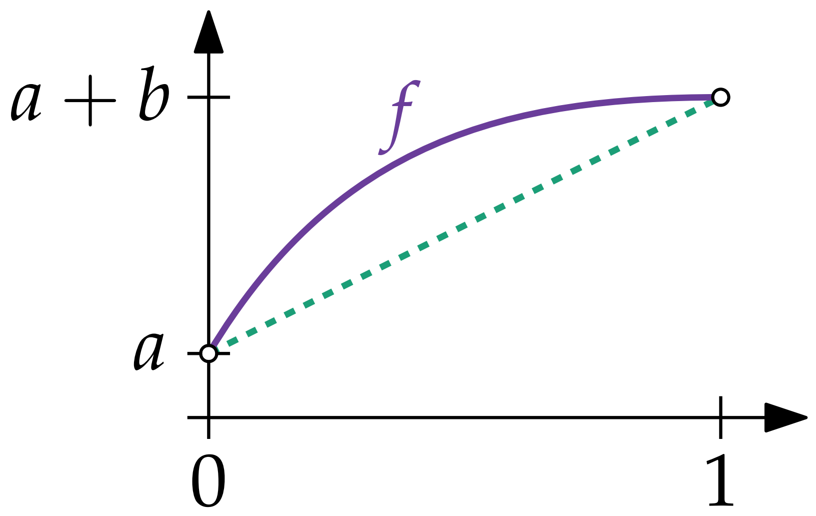
$\Rightarrow f(x) \geq bx + a$ for $x \in [0, 1]$.



Arithmetic–Geometric Mean Inequality (AGMI):

For all non-negative numbers $a_1, \ldots, a_k$:

Mathematical Toolkit

Let f be a function that is concave on $[0,1]$
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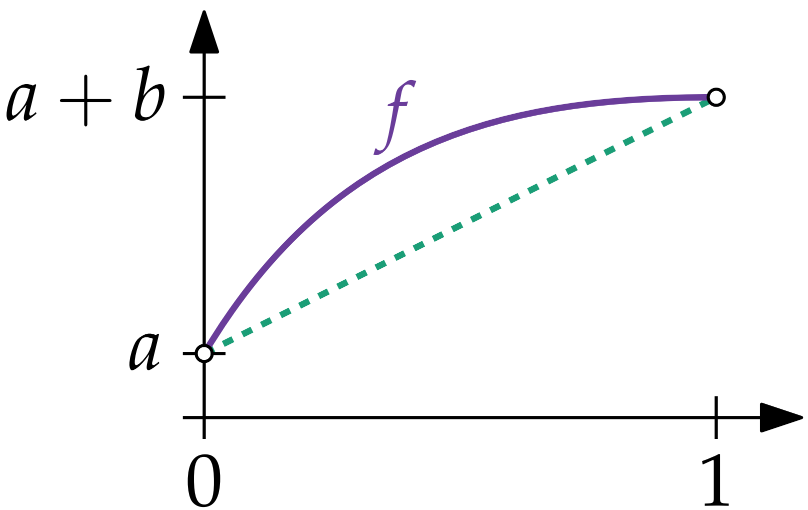

Arithmetic–Geometric Mean Inequality (AGMI):

For all non-negative numbers $a_1, \ldots, a_k$:

$$\left(\prod_{i=1}^{k} a_i\right)^{1/k} \leq$$

Mathematical Toolkit

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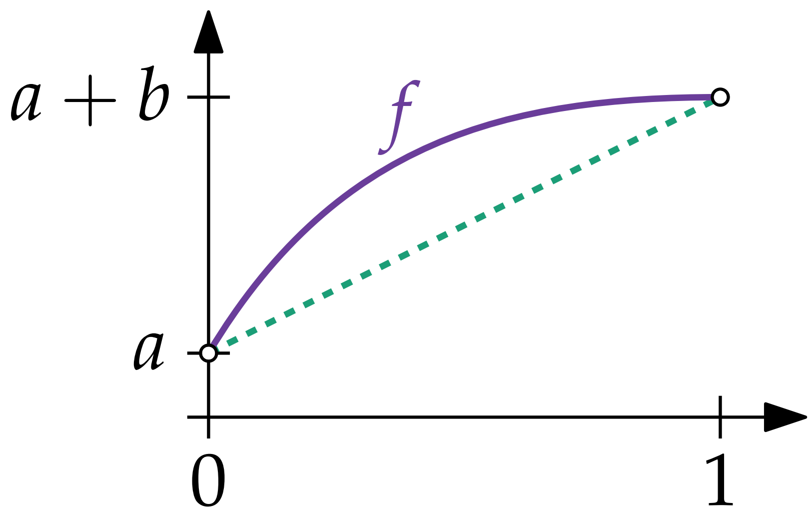

Arithmetic–Geometric Mean Inequality (AGMI):

For all non-negative numbers $a_1, \ldots, a_k$:

$$\left(\prod_{i=1}^{k} a_i\right)^{1/k} \leq \frac{1}{k}\left(\sum_{i=1}^{k} a_i\right)$$

Randomized Rounding (Proof)

Consider a fixed clause C_j of length l_j. Then we have:

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Consider a fixed clause C_j of length l_j. Then we have:

$$\mathbf{Pr}[C_j \text{ not sat.}] = \prod_{i \in P_j} (1 - y_i^*) \prod_{i \in N_j} y_i^*$$

$$\leq$$

Randomized Rounding (Proof)

Consider a fixed clause C_j of length l_j. Then we have:

$$\mathbf{Pr}[C_j \text{ not sat.}] = \prod_{i \in P_j} (1 - y_i^*) \prod_{i \in N_j} y_i^*$$

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AGMI

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AGMI

$$\leq \left(\sum_{i \in P_j} (1 - y_i^*) + \sum_{i \in N_j} y_i^* \right)$$

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AGMI

$$\leq \left[\frac{1}{l_j} \left(\sum_{i \in P_j} (1 - y_i^*) + \sum_{i \in N_j} y_i^* \right) \right]^{l_j}$$

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AGMI

$$\leq \left[\frac{1}{l_j} \left(\sum_{i \in P_j} (1 - y_i^*) + \sum_{i \in N_j} y_i^* \right) \right]^{l_j}$$

$$= \left[1 - \frac{1}{l_j} \left(\sum_{i \in P_j} y_i^* + \sum_{i \in N_j} (1 - y_i^*) \right) \right]^{l_j}$$

Randomized Rounding (Proof)

Consider a fixed clause C_j of length l_j. Then we have:

$$\mathbf{Pr}[C_j \text{ not sat.}] = \prod_{i\in P_j}(1-y_i^*)\prod_{i\in N_j} y_i^*$$

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Randomized Rounding (Proof)

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$$\leq \left[\frac{1}{l_j} \left(\sum_{i \in P_j} (1 - y_i^*) + \sum_{i \in N_j} y_i^* \right) \right]^{l_j}$$

$$= \left[1 - \frac{1}{l_j} \underbrace{\left(\sum_{i \in P_j} y_i^* + \sum_{i \in N_j} (1 - y_i^*) \right)}_{\geq \text{ by LP constraints}} \right]^{l_j}$$

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$$\left(\prod_{i=1}^{k} a_i \right)^{1/k} \leq \frac{1}{k} \left(\sum_{i=1}^{k} a_i \right)$$

AGMI

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$$\leq \left(1 - \frac{z_j^*}{l_j} \right)^{l_j}$$

Randomized Rounding (Proof)

The function $f(z_j^*) = 1 - \left(1 - \frac{z_j^*}{l_j}\right)^{l_j}$ is concave on $[0, 1]$.

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Thus

$$\mathbf{Pr}[C_j \text{ satisfied}] \geq f(z_j^*) \geq f(1) \cdot z_j^* + f(0)$$

$$\geq$$

Randomized Rounding (Proof)

The function $f(z_j^*) = 1 - \left(1 - \frac{z_j^*}{l_j}\right)^{l_j}$ is concave on $[0, 1]$.

Thus

$$\begin{aligned}\mathbf{Pr}[C_j \text{ satisfied}] &\geq f(z_j^*) \geq f(1) \cdot z_j^* + f(0) \\ &\geq \left[1 - \left(1 - \frac{1}{l_j}\right)^{l_j}\right] z_j^* \\ &\geq \end{aligned}$$

Randomized Rounding (Proof)

The function $f(z_j^*) = 1 - \left(1 - \frac{z_j^*}{l_j}\right)^{l_j}$ is concave on $[0, 1]$.

Thus

$$\mathbf{Pr}[C_j \text{ satisfied}] \geq f(z_j^*) \geq f(1) \cdot z_j^* + f(0)$$
$$\geq \left[1 - \left(1 - \frac{1}{l_j}\right)^{l_j}\right] z_j^*$$
$$\geq$$

$1 + x \leq e^x$

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The function $f(z_j^*) = 1 - \left(1 - \frac{z_j^*}{l_j}\right)^{l_j}$ is concave on $[0, 1]$.

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$$\geq$$

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$x = -\frac{1}{l_j} \Rightarrow$

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$$\geq$$

$$1 + x \leq e^x$$

$$x = -\frac{1}{l_j} \Rightarrow 1 - \frac{1}{l_j} \leq e^{-1/l_j}$$

Randomized Rounding (Proof)

The function $f(z_j^*) = 1 - \left(1 - \frac{z_j^*}{l_j}\right)^{l_j}$ is concave on $[0,1]$.

Thus

$$\begin{aligned}
\mathbf{Pr}[C_j \text{ satisfied}] &\geq f(z_j^*) \geq f(1) \cdot z_j^* + f(0) \\
&\geq \left[1 - \left(1 - \frac{1}{l_j}\right)^{l_j}\right] z_j^* \\
&\geq \left(1 - \frac{1}{e}\right) z_j^*
\end{aligned}$$

$1 + x \leq e^x$

$x = -\frac{1}{l_j} \Rightarrow 1 - \frac{1}{l_j} \leq e^{-1/l_j}$

Randomized Rounding (Proof)

Therefore

$$\mathbf{E}[W] = \sum_{j=1}^{m} \mathbf{Pr}[C_j \text{ satisfied}] \cdot w_j$$

$$\geq$$

Randomized Rounding (Proof)

Therefore

$$\mathbf{E}[W] = \sum_{j=1}^{m} \mathbf{Pr}[C_j \text{ satisfied}] \cdot w_j$$

$$\geq \left(1 - \frac{1}{e}\right) \sum_{j=1}^{m} w_j z_j^*$$

$$=$$

Randomized Rounding (Proof)

Therefore

$$\mathbf{E}[W] = \sum_{j=1}^{m} \mathbf{Pr}[C_j \text{ satisfied}] \cdot w_j$$

$$\geq \left(1 - \frac{1}{e}\right) \boxed{\sum_{j=1}^{m} w_j z_j^*}$$

LP objective function

$$=$$

Randomized Rounding (Proof)

Therefore

$$\begin{aligned}
\mathbf{E}[W] &= \sum_{j=1}^{m} \mathbf{Pr}[C_j \text{ satisfied}] \cdot w_j \\
&\geq \left(1 - \frac{1}{e}\right) \boxed{\sum_{j=1}^{m} w_j z_j^*} \\
&= \left(1 - \frac{1}{e}\right) \mathrm{OPT}_{\mathrm{LP}} \\
&\geq
\end{aligned}$$

LP objective function

Randomized Rounding (Proof)

Therefore

$$\mathbf{E}[W] = \sum_{j=1}^{m} \mathbf{Pr}[C_j \text{ satisfied}] \cdot w_j$$

$$\geq \left(1 - \frac{1}{e}\right) \boxed{\sum_{j=1}^{m} w_j z_j^*}$$ LP objective function

$$= \left(1 - \frac{1}{e}\right) \mathrm{OPT}_{\mathrm{LP}}$$

$$\geq \left(1 - \frac{1}{e}\right) \mathrm{OPT}$$

□

Randomized Rounding (Proof)

Therefore

$$\begin{aligned} \mathbf{E}[W] &= \sum_{j=1}^{m} \mathbf{Pr}[C_j \text{ satisfied}] \cdot w_j \\ &\geq \left(1 - \frac{1}{e}\right) \boxed{\sum_{j=1}^{m} w_j z_j^*} \quad \text{LP objective function} \\ &= \left(1 - \frac{1}{e}\right) \mathrm{OPT}_{\mathrm{LP}} \\ &\geq \left(1 - \frac{1}{e}\right) \mathrm{OPT} \end{aligned}$$

□

Theorem. The previous algorithm can be derandomized by the method of conditional expectation.

Approximation Algorithms

Lecture 11:
MaxSat via Randomized Rounding

Part VI:
Combining the Algorithms

Take the better of the two solutions!

Theorem. The better solution among the randomized algorithm and the randomized LP-rounding algorithm provides a -approximation for MaxSat.

Take the better of the two solutions!

Theorem. The better solution among the randomized algorithm and the randomized LP-rounding algorithm provides a $3/4$-approximation for MaxSat.

Take the better of the two solutions!

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The better solution is at least as good as the expectation of the above randomized algorithm.

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Visualization and Derandomization

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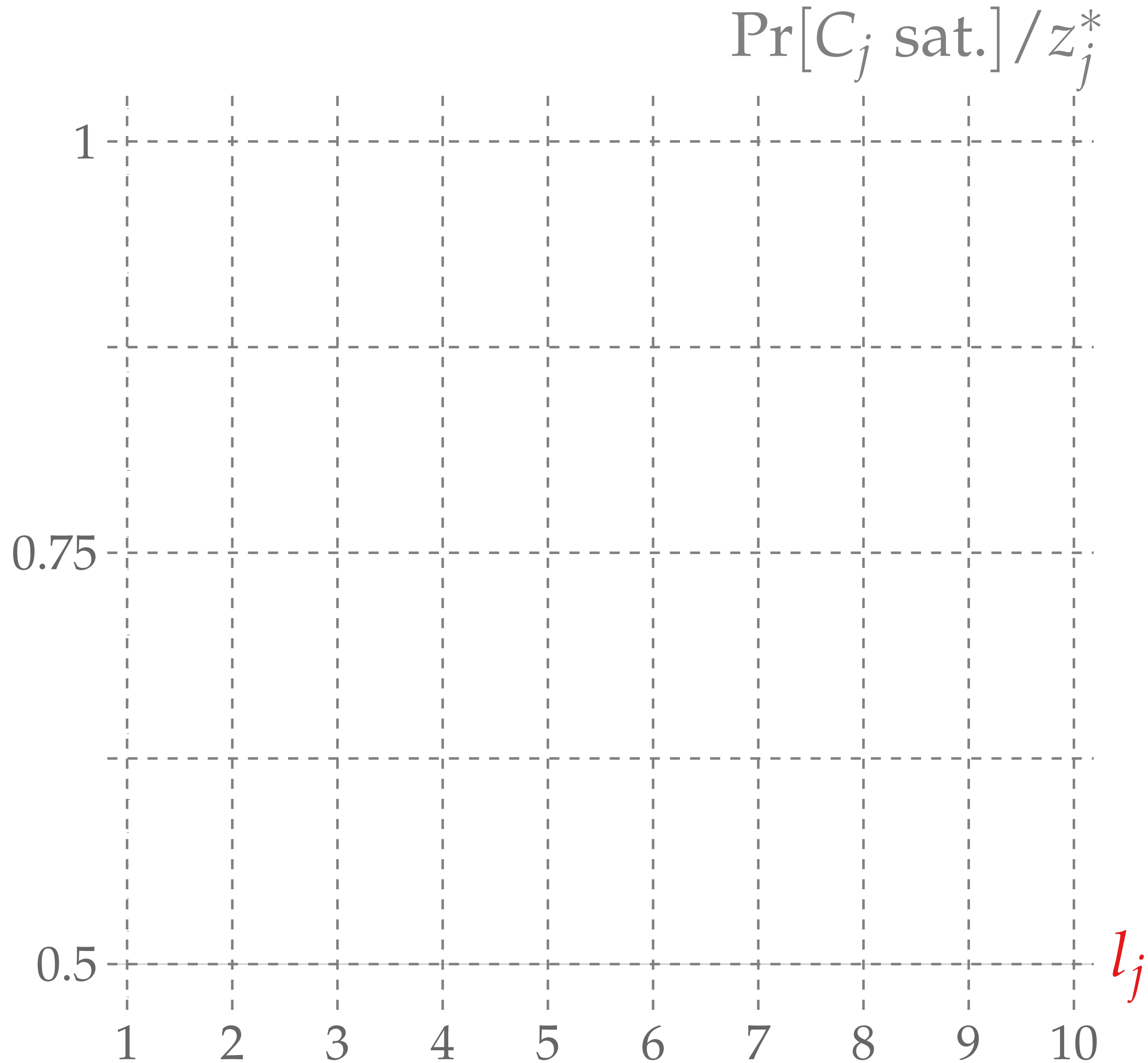

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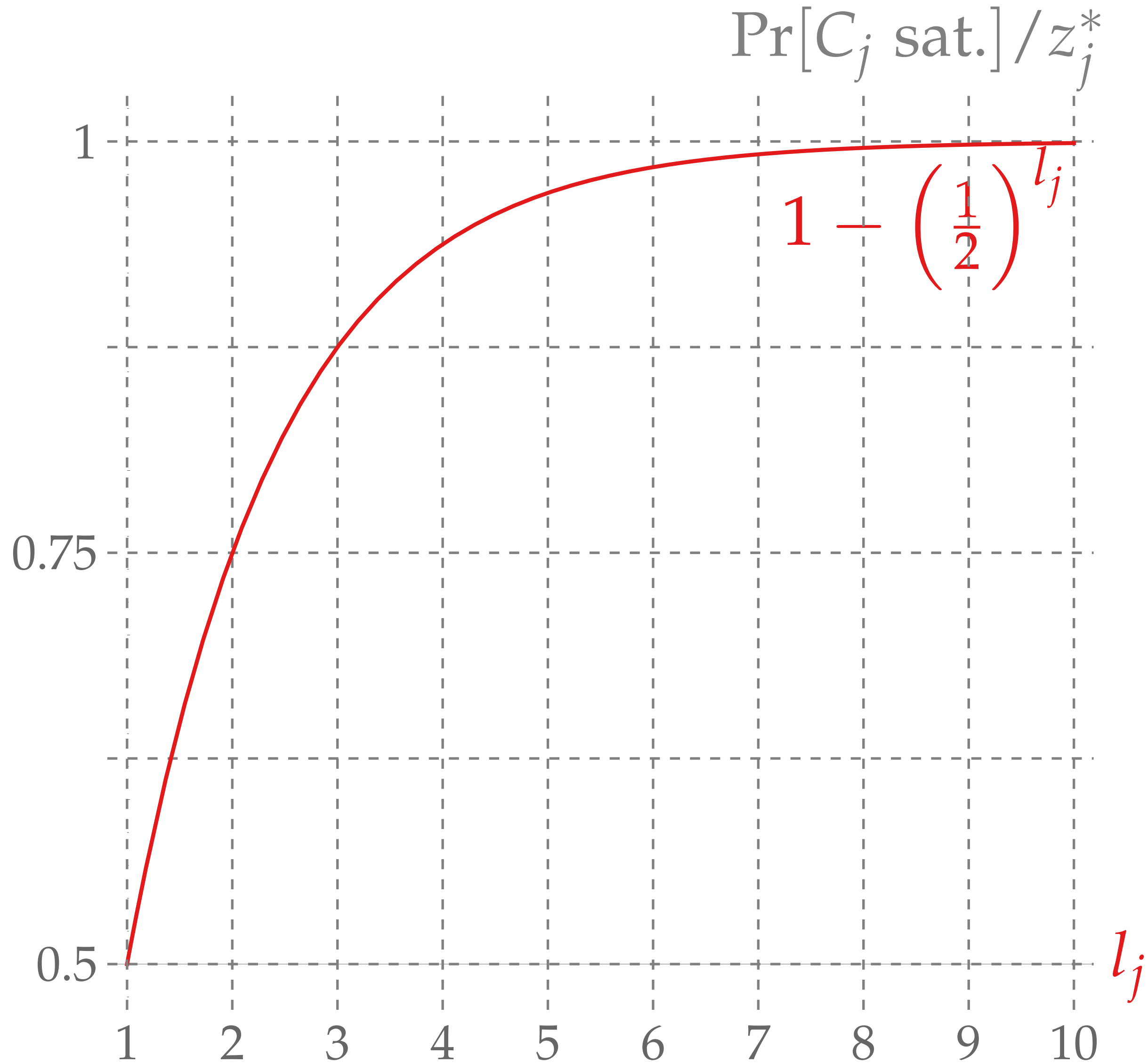

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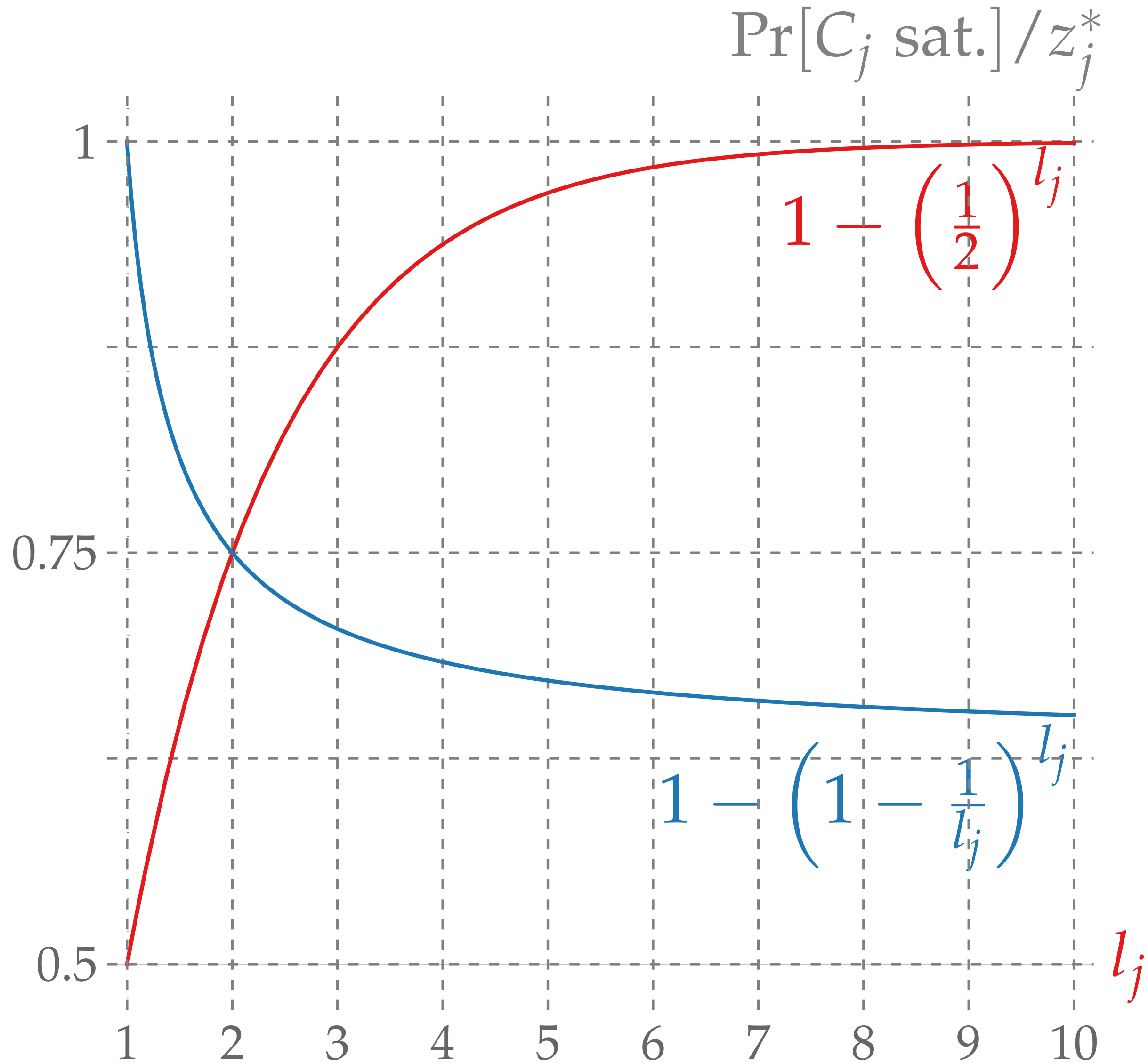

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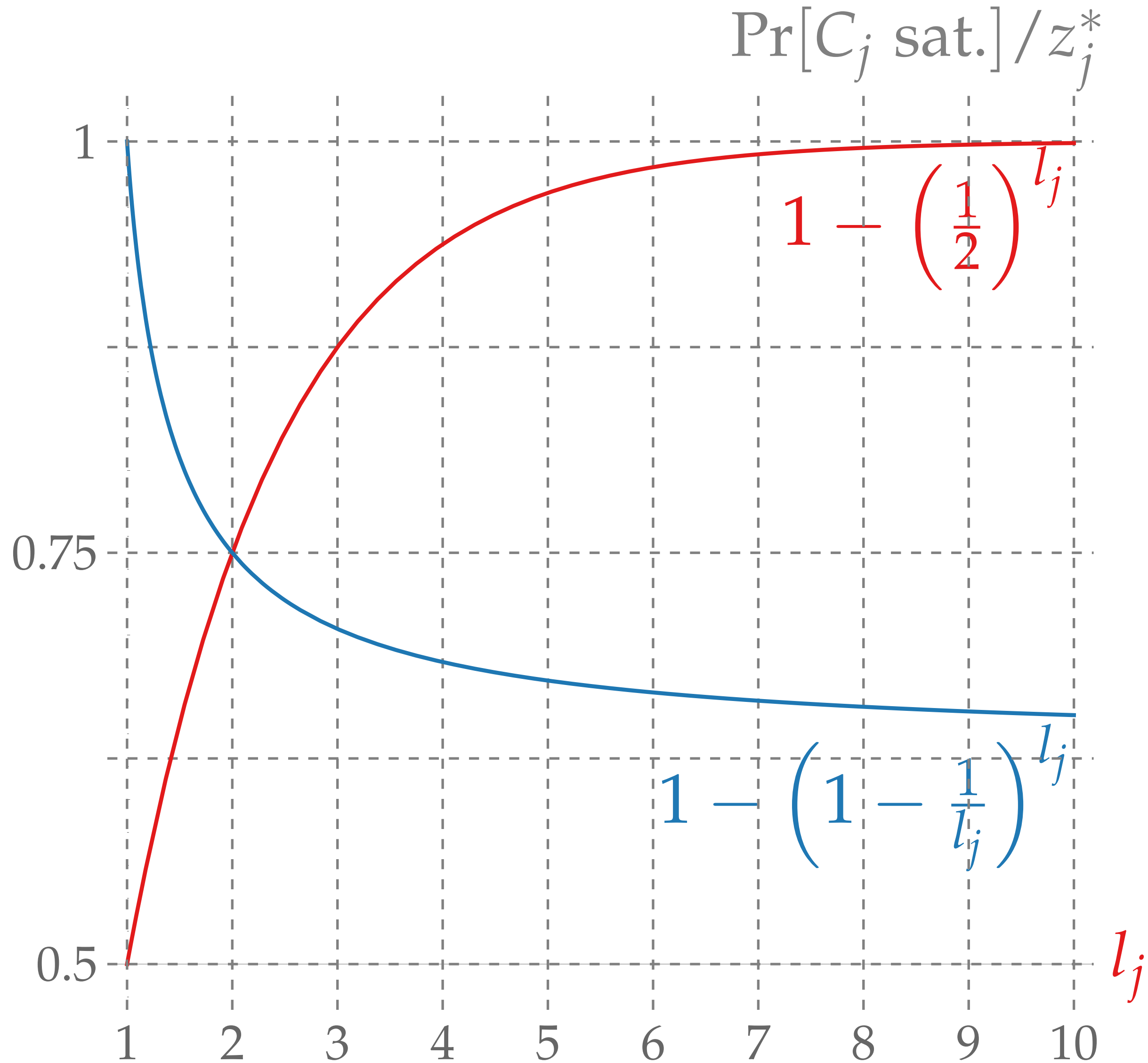

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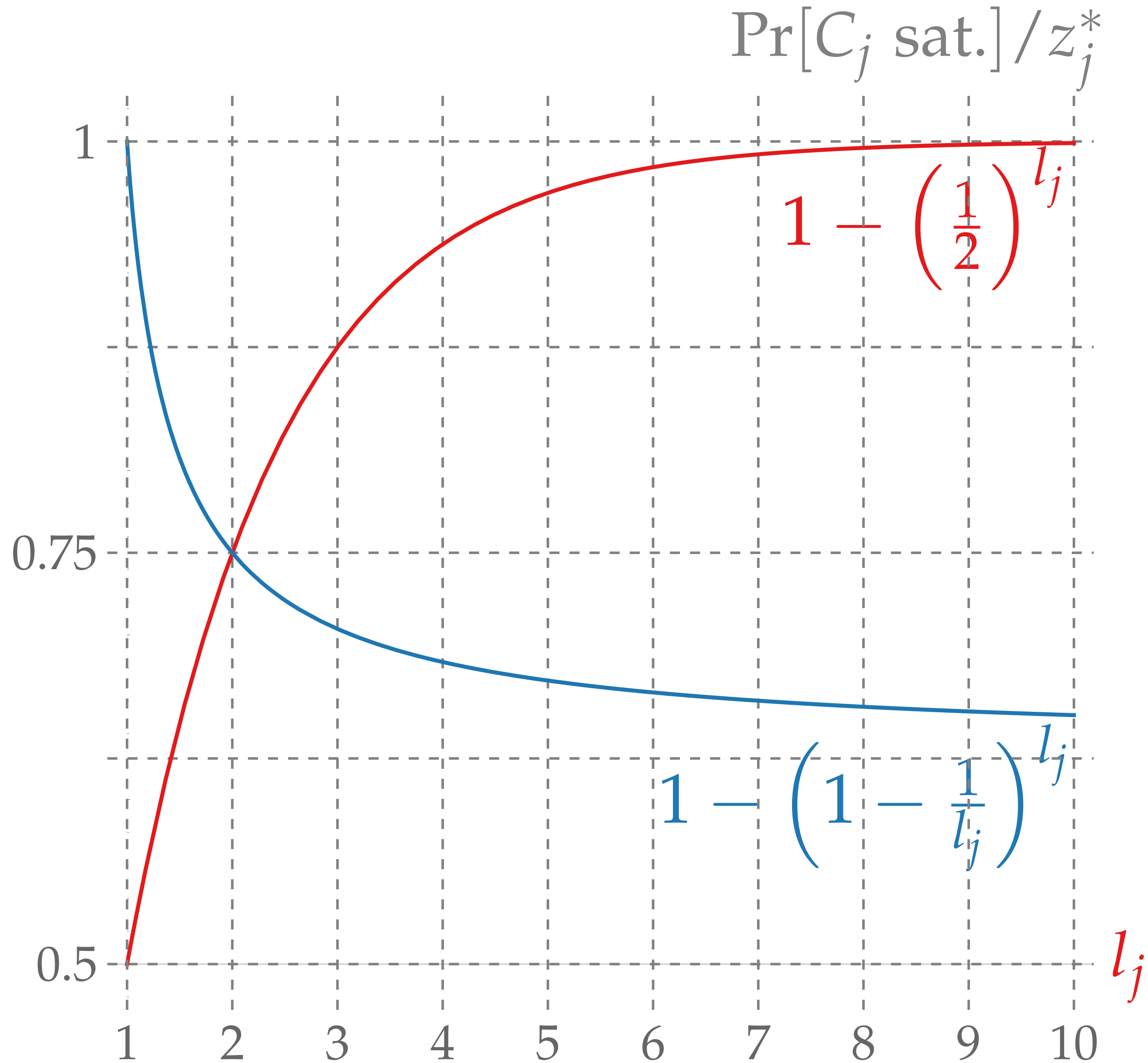

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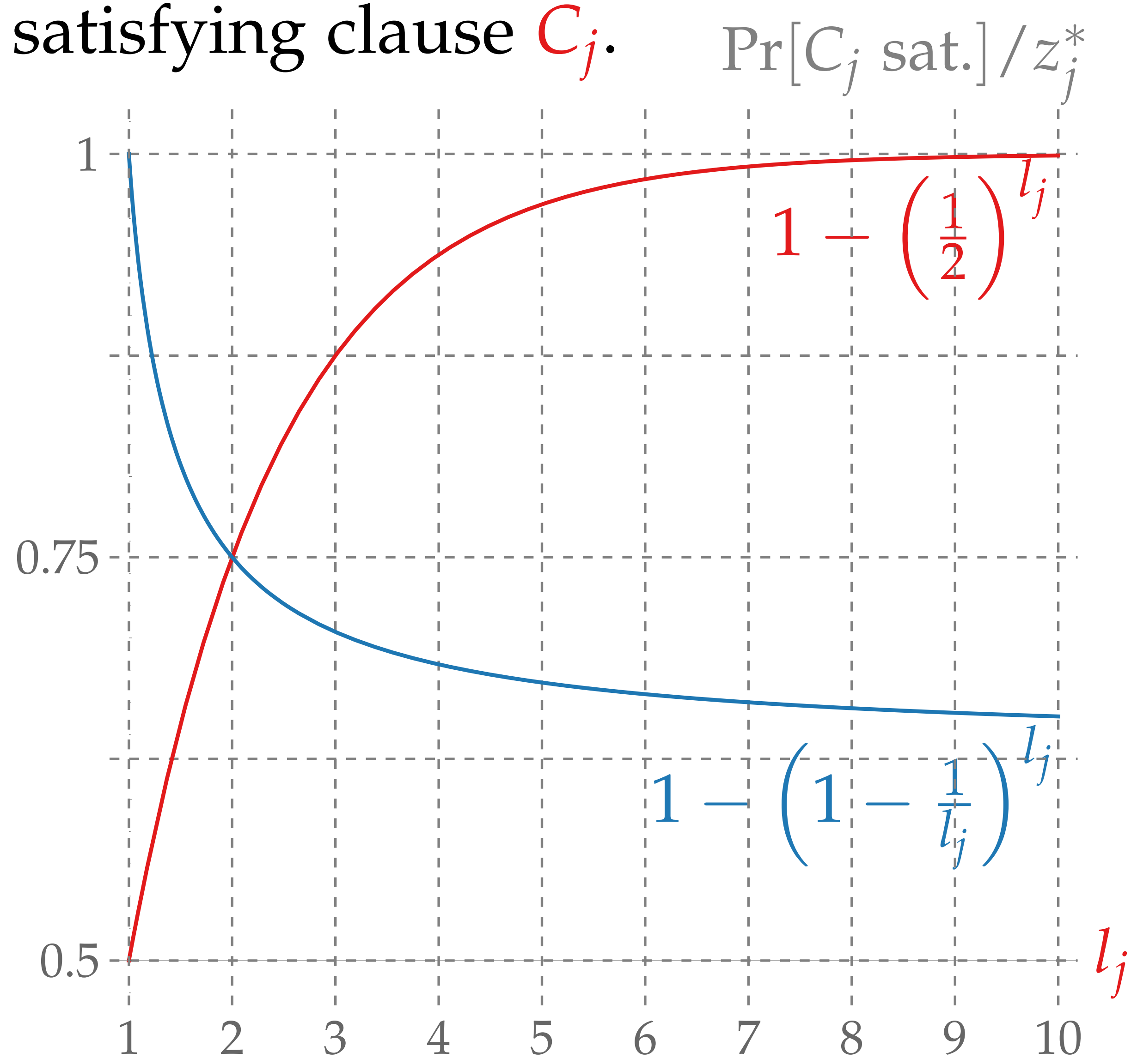

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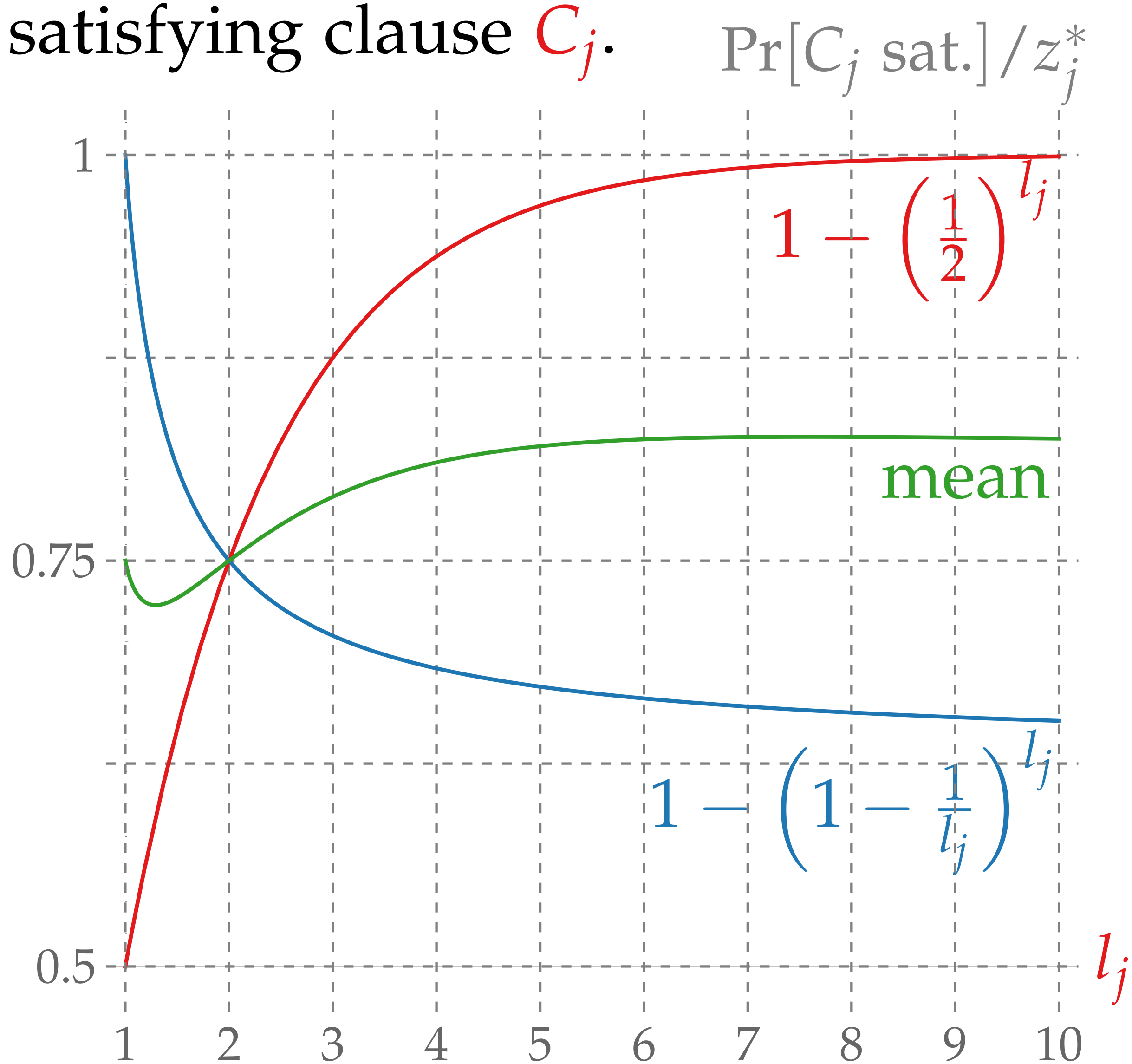

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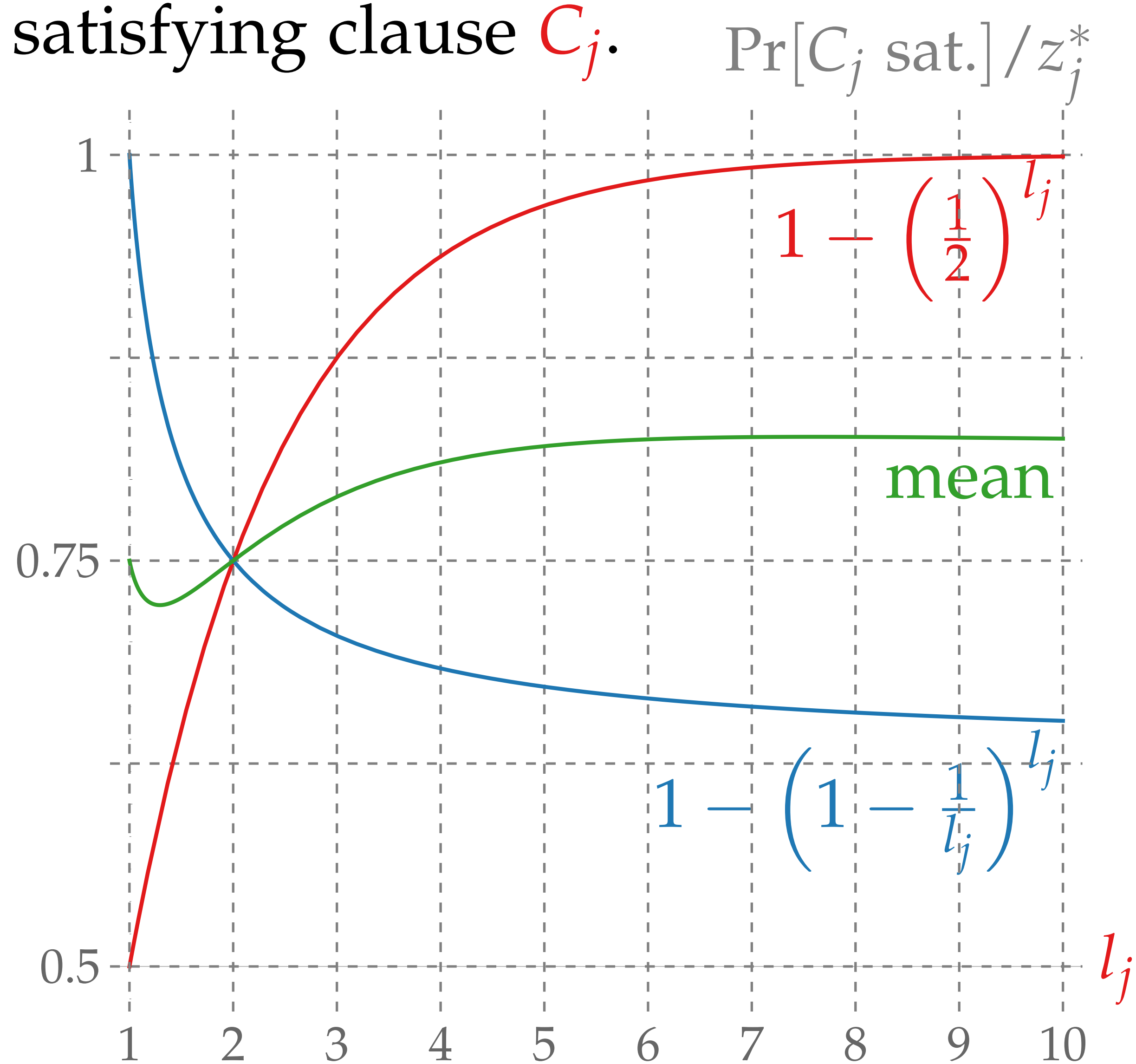

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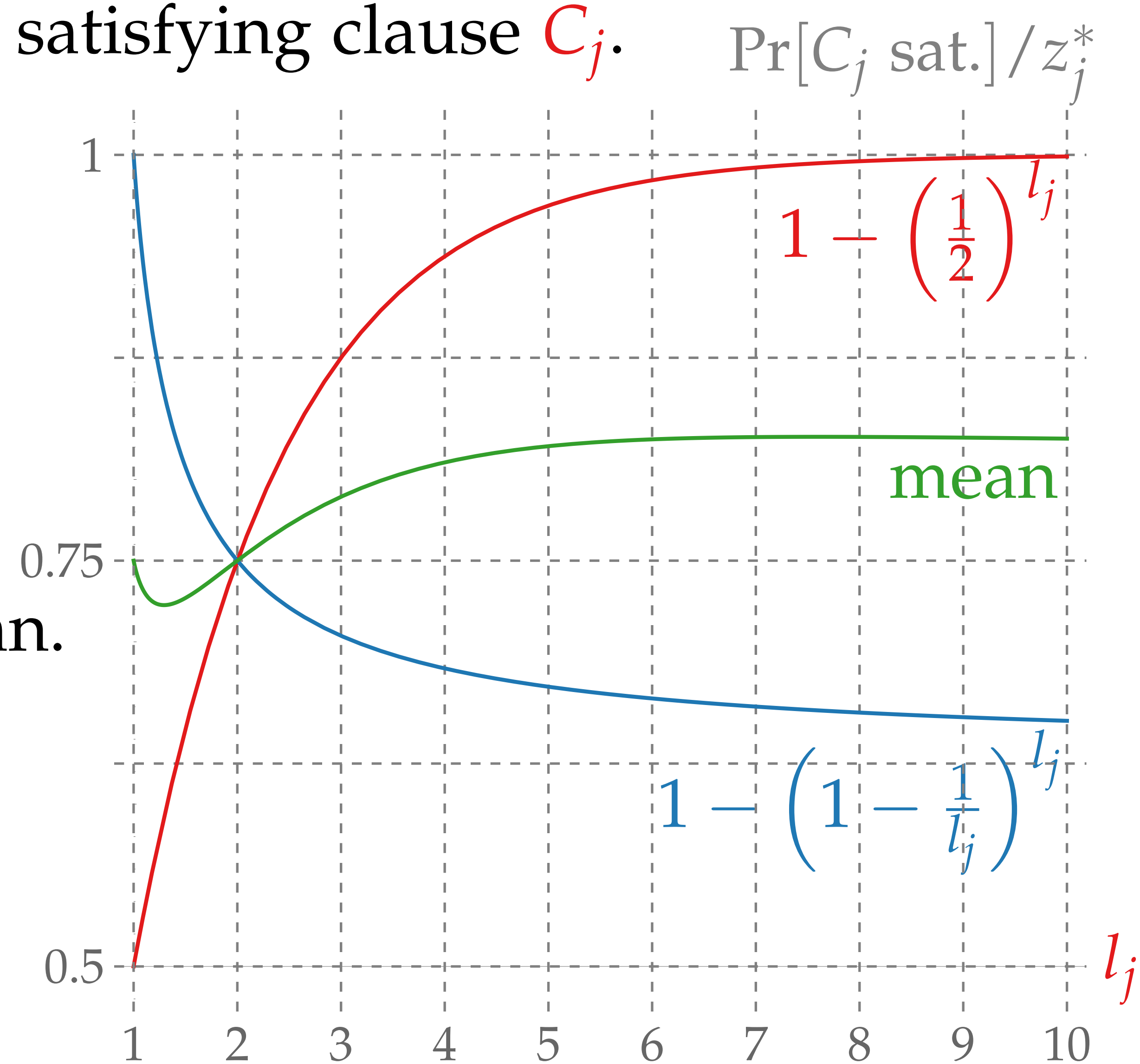

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This algorithm, too, can be derandomized by conditional expectation.