Approximation Algorithms

Lecture 4:
Linear Programming and LP-Duality

Part I:
Introduction to Linear Programming

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Maximizing Profits

You're the boss of a small company that produces two products P_1 and P_2. For the production of x_1 units of P_1 and x_2 units of P_2, your profit in € is:

$$G(x_1, x_2) = 30x_1 + 50x_2$$

Three machines M_A, M_B and M_C produce the required components A, B and C for the products. The components are used in different quantities for the products, and each machine requires some time for the production.

$$
\begin{aligned}
M_A: && 4x_1 &+ 11x_2 &\leq 880 \\
M_B: && x_1 &+ x_2 &\leq 150 \\
M_C: && &x_2 &\leq 60
\end{aligned}
$$

Which choice of (x_1, x_2) maximizes the profit?

Solution

Linear constraints:

$M_A:$	$4x_1$	$+$	$11x_2$	$\leq$	880
$M_B:$	x_1	$+$	x_2	$\leq$	150
$M_C:$			x_2	$\leq$	60
			x_1	$\geq$	0
			x_2	$\geq$	0

Linear target function:

$$G(x_1, x_2) = 30x_1 + 50x_2$$
$$= (30, 50)\binom{x_1}{x_2}$$

$G(110, 40) = 5.300$

Set of valid solutions

€5.300
€4.500
€3.000
€1.500

"profit line": orthogonal to $\binom{30}{50}$
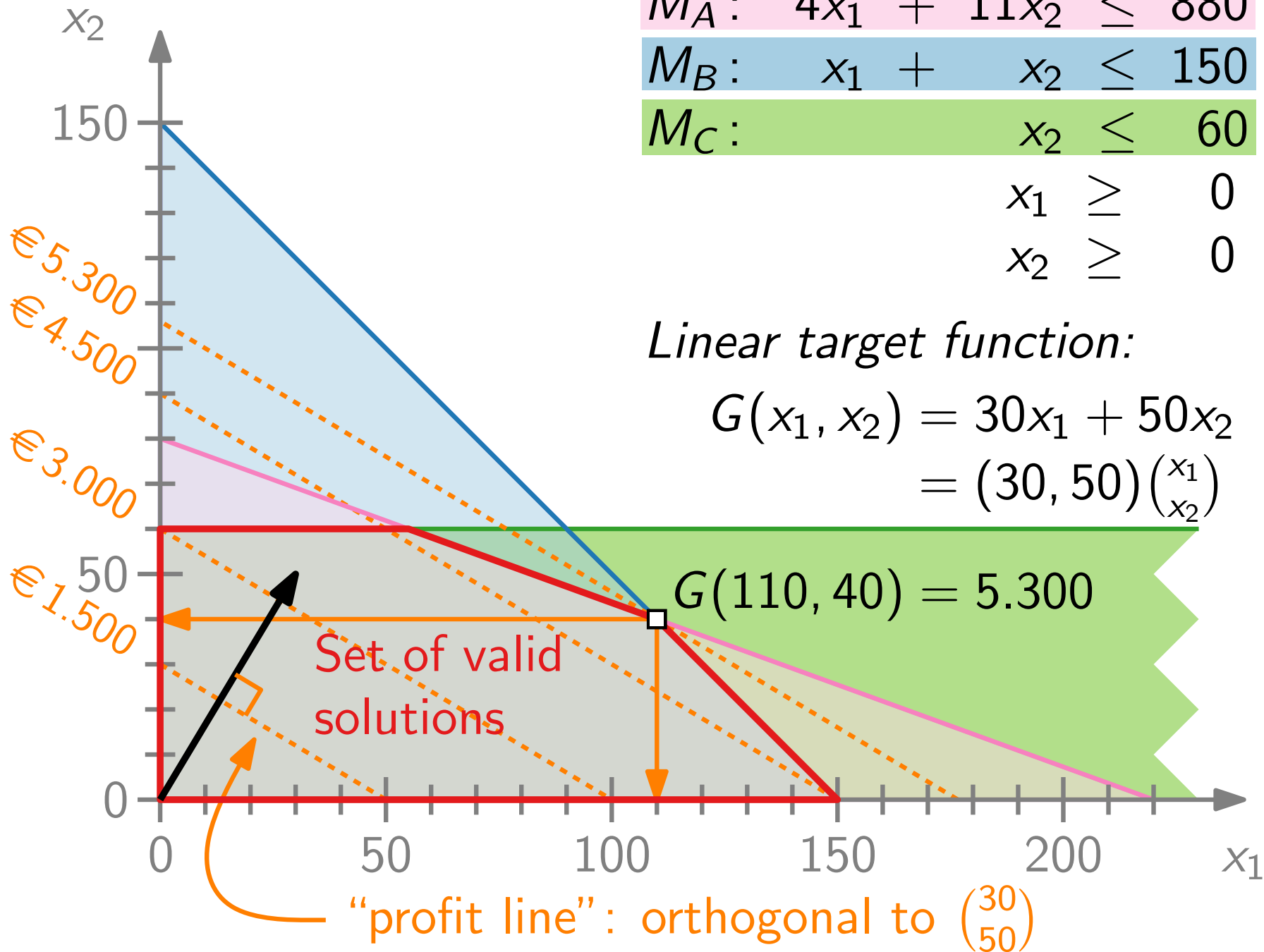
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Part II:
Upper Bounds for LPs

Motivation: Upper and Lower Bounds

Consider an NP-hard minimization problem.

Decision Problem:
Is a given U an **upper bound** on OPT?

A feasible sol. S provides efficiently verifiable "yes"-certificate.

Lower bounds / "no"-certificates?
⤳ probably not! (conjecture: NP $\neq$ coNP)

For an approximation algorithm, we need a lower bound
$L \geq \text{OPT}/\alpha$ (i.e., an approximate "no"-certificate)!

Examples:
- ■ Vertex Cover: lower bound by matchings
- ■ TSP: lower bound by MST or by cycle cover

Linear Programming

Optimize (i.e., minimize or maximize) a linear (*objective*) function subject to linear inequalities (*constraints*).

$$
\begin{aligned}
\textbf{minimize} \quad & c^\mathsf{T} x & \textit{standard form}\\
\textbf{subject to} \quad & Ax \geq b\\
& x \geq 0
\end{aligned}
$$

Example. $\quad c = \begin{pmatrix} 7 \\ 1 \\ 5 \end{pmatrix} \quad A = \begin{pmatrix} 1 & -1 & 3 \\ 5 & 2 & -1 \end{pmatrix} \quad b = \begin{pmatrix} 10 \\ 6 \end{pmatrix}$

$$
\begin{aligned}
\textbf{minimize} \quad & 7x_1 + x_2 + 5x_3\\
\textbf{subject to} \quad & x_1 - x_2 + 3x_3 \geq 10\\
& 5x_1 + 2x_2 - x_3 \geq 6\\
& x_1, x_2, x_3 \geq 0
\end{aligned}
$$

Linear Programming – Upper Bounds

Optimize (i.e., minimize or maximize) a linear (*objective*) function subject to linear inequalities (*constraints*).

$$
\begin{array}{lrrrrr}
\textbf{minimize} & 7x_1\,14\,+ & x_2\,1\,+ & 5x_3\,15 & = & 30 \\
\textbf{subject to} & x_1\,2\,- & x_2\,1\,+ & 3x_3\,9\,\geq & 10 & 10 \\
 & 5x_1\,10\,+ & 2x_2\,2\,- & x_3\,3\,\geq & 6 & 9 \\
 & & & x_1, x_2, x_3\,\geq & & 0
\end{array}
$$

Valid solution?

$x = (2, 1, 3)$

$\Rightarrow \mathrm{obj}(x) = 30$ is upper bound for OPT

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Part III:
Lower Bounds for LPs

Linear Programming – Lower Bounds

Optimize (i.e., minimize or maximize) a linear (*objective*) function subject to linear inequalities (*constraints*).

$$
\begin{array}{lrcrcrcr}
\textbf{minimize} & 7x_1 & + & x_2 & + & 5x_3 & & \\
\textbf{subject to} & 2\cdot\ x_1 & - & 2\cdot x_2 & + & 2\cdot 3x_3 & \geq & 2\cdot 10 \\
& 5x_1 & + & 2x_2 & - & x_3 & \geq & 6 \\
& & & & & x_1, x_2, x_3 & \geq & 0
\end{array}
$$

$$7x_1 + x_2 + 5x_3 \geq x_1 - x_2 + 3x_3 \quad \Rightarrow \text{OPT} \geq 10$$

$$7x_1 + x_2 + 5x_3 \geq (x_1 - x_2 + 3x_3) + (5x_1 + 2x_2 - x_3)$$
$$\geq 10 + 6 \qquad \Rightarrow \text{OPT} \geq 16$$

$$7x_1 + x_2 + 5x_3 \geq 2 \cdot (x_1 - x_2 + 3x_3) + (5x_1 + 2x_2 - x_3)$$
$$\geq 2\cdot 10 + 6 \qquad \Rightarrow \text{OPT} \geq 26$$

Linear Programming – Lower Bounds

$$
\begin{aligned}
\textbf{minimize} \quad & 7x_1 \; + \; x_2 \; + \; 5x_3 \\
\textbf{subject to} \; y_1(\;\; & x_1 \; - \; x_2 \; + \; 3x_3 \;) \;\geq\; 10 \; y_1 \\
y_2(\;\; & 5x_1 \; + \; 2x_2 \; - \; x_3 \;) \;\geq\; 6 \; y_2 \\
& x_1, x_2, x_3 \;\geq\; 0
\end{aligned}
$$

$$
\begin{aligned}
7x_1 + x_2 + 5x_3 \; &\geq\; y_1 \cdot (x_1 - x_2 + 3x_3) + y_2 \cdot (5x_1 + 2x_2 - x_3) \\
&\geq\; y_1 \cdot 10 + y_2 \cdot 6 \;\Rightarrow\; \text{OPT} \geq 10y_1 + 6y_2
\end{aligned}
$$

$10y_1 + 6y_2$ is lower bound for OPT

Linear Programming – Lower Bounds

$$
\begin{array}{rrrrrrl}
\textbf{minimize} & 7x_1 & + & x_2 & + & 5x_3 & \\
\textbf{subject to}\; y_1\,(& x_1 & - & x_2 & + & 3x_3\,) & \geq \; 10\; y_1 \\
y_2\,(& 5x_1 & + & 2x_2 & - & x_3\,) & \geq \quad 6\; y_2 \\
& & & x_1, x_2, x_3 & & & \geq \quad 0
\end{array}
$$

$$
7x_1 + x_2 + 5x_3 \;\geq\; y_1 \cdot (x_1 - x_2 + 3x_3) + y_2 \cdot (5x_1 + 2x_2 - x_3)
$$

$$
\geq y_1 \cdot 10 + y_2 \cdot 6 \;\Rightarrow\; \text{OPT} \geq 10y_1 + 6y_2
$$

maximize

Bounds for y_1, y_2:

$$
\begin{array}{rrrrl}
y_1 & + & 5y_2 & \leq & 7 \\
-y_1 & + & 2y_2 & \leq & 1 \\
3y_1 & - & y_2 & \leq & 5 \\
& & y_1, y_2 & \geq & 0
\end{array}
$$

Primal–Dual

$$
\begin{aligned}
\textbf{minimize} \quad & c^\mathsf{T} x \\
\textbf{subject to} \quad & Ax \geq b \\
& x \geq 0
\end{aligned}
$$

$$
\begin{aligned}
\textbf{maximize} \quad & b^\mathsf{T} y \\
\textbf{subject to} \quad & A^\mathsf{T} y \leq c \\
& y \geq 0
\end{aligned}
$$

$$
\begin{aligned}
\textbf{minimize} \quad & c^\mathsf{T} x \\
\textbf{subject to} \quad & Ax \geq b \\
& x \geq 0
\end{aligned}
$$

Approximation Algorithms

Lecture 4:
Linear Programming and LP-Duality

Part IV:
LP-Duality and Complementary Slackness

LP-Duality

$$
\begin{array}{lrcl}
\textbf{minimize} & c^{\mathsf{T}}x & & \\
\textbf{subject to} & Ax & \geq & b \\
& x & \geq & 0
\end{array}
$$

Primal

$$
\begin{array}{lrcl}
\textbf{maximize} & b^{\mathsf{T}}y & & \\
\textbf{subject to} & A^{\mathsf{T}}y & \leq & c \\
& y & \geq & 0
\end{array}
$$

Dual

Theorem. The primal program has a finite optimum $\Leftrightarrow$ the dual program has a finite optimum. Moreover, if $x^* = (x_1^*, \ldots, x_n^*)$ and $y^* = (y_1^*, \ldots, y_m^*)$ are *optimal* solutions for the primal and dual program, respectively, then

$$
\sum_{j=1}^{n} c_j x_j^* = \sum_{i=1}^{m} b_i y_i^* .
$$

Weak LP-Duality

minimize $\quad c^\mathsf{T} x$
subject to $\quad Ax \;\geq\; b$
$\qquad\qquad\qquad x \;\geq\; 0$

maximize $\quad b^\mathsf{T} y$
subject to $\quad A^\mathsf{T} y \;\leq\; c$
$\qquad\qquad\qquad\quad y \;\geq\; 0$

Theorem. If $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_m)$ are *valid* solutions for the primal and dual program, resp., then
$$\sum_{j=1}^{n} c_j x_j \geq \sum_{i=1}^{m} b_i y_i .$$

Proof.

$$\sum_{j=1}^{n} c_j x_j \geq \sum_{j=1}^{n} \left(\sum_{i=1}^{m} a_{ij} y_i \right) x_j = \sum_{i=1}^{m} \left(\sum_{j=1}^{n} a_{ij} x_j \right) y_i \geq \sum_{i=1}^{m} b_i y_i .$$

Complementary Slackness

$$\text{minimize} \quad c^\mathsf{T} x$$
$$\text{subject to} \quad Ax \geq b$$
$$\qquad\qquad\quad x \geq 0$$

$$\text{maximize} \quad b^\mathsf{T} y$$
$$\text{subject to} \quad A^\mathsf{T} y \leq c$$
$$\qquad\qquad\quad y \geq 0$$

Theorem. Let $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_m)$ be valid solutions for the primal and dual program, respectively. Then x and y are optimal if and only if the following conditions are met:

Primal CS:
For each $j = 1, \ldots, n$: $\quad x_j = 0 \quad$ or $\quad \sum_{i=1}^{m} a_{ij} y_i = c_j$

Dual CS:
For each $i = 1, \ldots, m$: $\quad y_i = 0 \quad$ or $\quad \sum_{j=1}^{n} a_{ij} x_j = b_i$

Proof. Follows from LP-duality:

$$\sum_{j=1}^{n} c_j x_j \geq \sum_{j=1}^{n} \left(\sum_{i=1}^{m} a_{ij} y_i \right) x_j = \sum_{i=1}^{m} \left(\sum_{j=1}^{n} a_{ij} x_j \right) y_i \geq \sum_{i=1}^{m} b_i y_i \,.$$
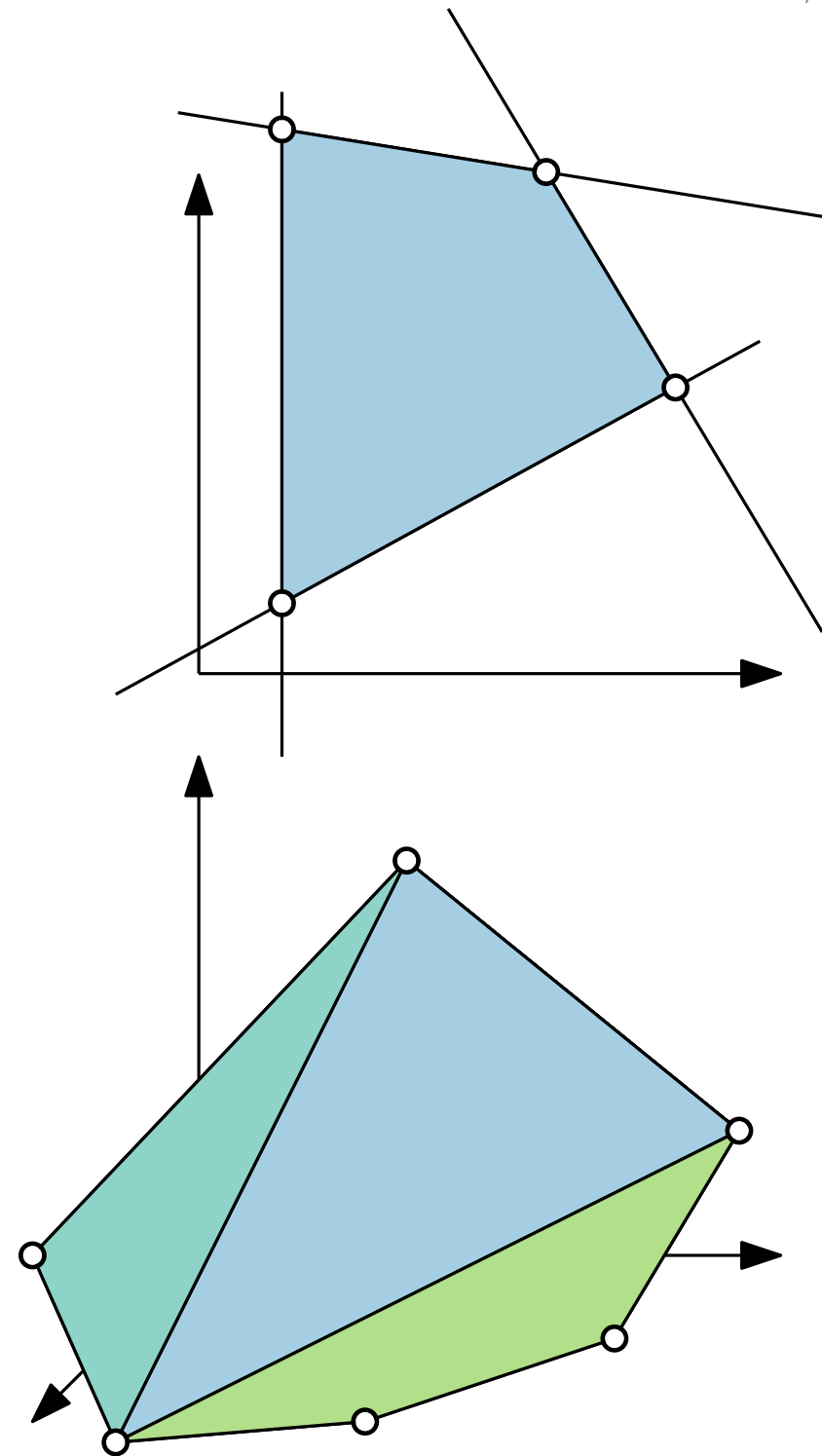
LPs and Convex Polytopes

The feasible solutions of an LP with n variables form a **convex polytope** in $\mathbb{R}^n$ (intersection of halfspaces).

Corners of the polytope are called **extreme point solutions** $\Leftrightarrow$ n linearly independent inequalities (constraints) are satisfied with equality.

If an optimal solution exists, some extreme point is also optimal.

Integer Linear Programs (ILPs)

$$
\begin{aligned}
\textbf{minimize} \quad & c^\mathsf{T}x \\
\textbf{subject to} \quad & Ax \geq b \\
& x \geq 0
\end{aligned}
$$

$$
\begin{aligned}
\textbf{minimize} \quad & c^\mathsf{T}x \\
\textbf{subject to} \quad & Ax \geq b \\
& x \in \mathbb{N}
\end{aligned}
$$

Many NP-optimization problems can be formulated as ILPs; thus ILPs are NP-hard to solve.

LP-relaxation provides a lower bound: $\mathrm{OPT_{LP}} \leq \mathrm{OPT_{ILP}}$

Approximation Algorithms

Lecture 4:
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Part V:
Min–Max Relationships

Max-Flow Problem

Given: A directed graph $G = (V, E)$ with edge capacities $c \colon E \to \mathbb{Q}_+$ and two special vertices: the source s and sink t.

Find: A maximum s–t flow (i.e., non-negative edge weights f) such that
- $f(u, v) \le c(u, v)$ for each edge $(u, v) \in E$,
- $\sum_{u \colon (u,v) \in E} f(u, v) = \sum_{z \colon (v,z) \in E} f(v, z)$ for each vertex $v \in V \setminus \{s, t\}$.

The **flow value** is the inflow to t minus the outflow from t.
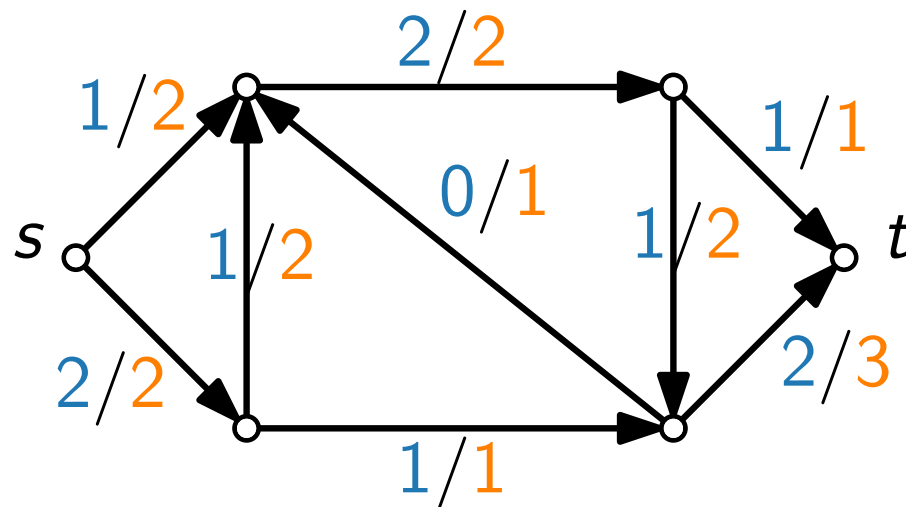
Min-Cut Problem

Given: A directed graph $G = (V, E)$ with edge capacities $c \colon E \to \mathbb{Q}_+$ and two special vertices: the source s and sink t.

Find: An $s{-}t$ cut, i.e., a vertex set X with $s \in X$ and $t \in \overline{X}$ such that the total weight $c(X, \overline{X})$ of the edges from X to $\overline{X}$ is minimum.
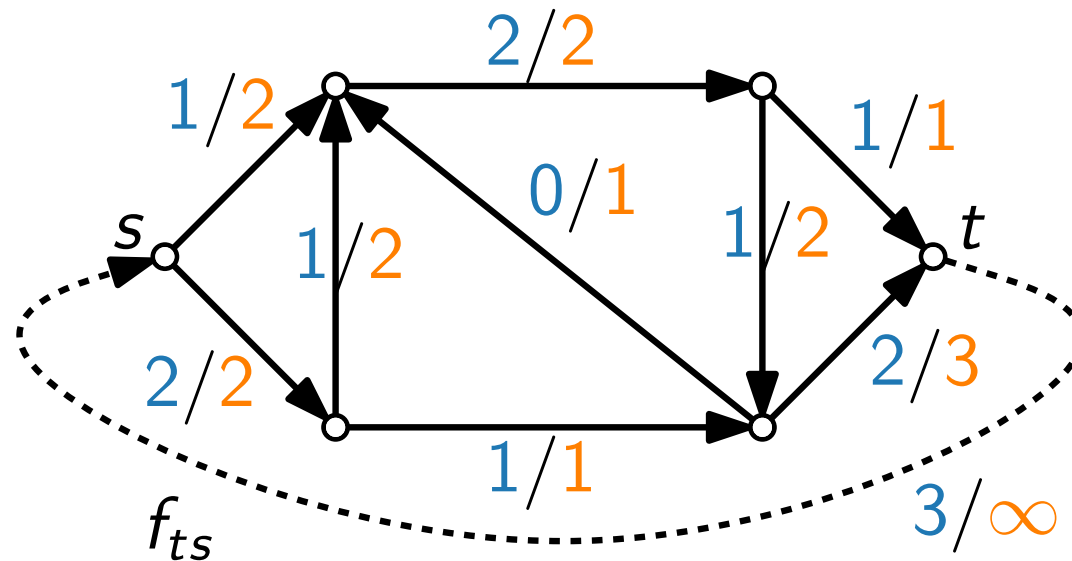
Max-Flow-Min-Cut Theorem

Theorem. The value of a maximum s–t flow and the weight of a minimum s–t cut are the same.

Proof. Special case of LP-Duality …

$$
\begin{aligned}
\textbf{maximize} \quad & f_{ts} \\
\textbf{subject to} \quad & f_{uv} \leq c_{uv} && \forall (u,v) \in E \setminus \{(t,s)\} \\
& \sum_{u:\,(u,v)\in E} f_{uv} - \sum_{z:\,(v,z)\in E} f_{vz} \leq 0 && \forall v \in V \\
& f_{uv} \geq 0 && \forall (u,v) \in E
\end{aligned}
$$

Max-Flow-Min-Cut Theorem

Theorem. The value of a maximum $s{-}t$ flow and the weight of a minimum $s{-}t$ cut are the same.

Proof.   Special case of LP-Duality ...

$$
\begin{aligned}
&\textbf{maximize} && f_{ts} \\
&\textbf{subject to} && f_{uv} \le c_{uv} && \forall (u,v) \in E \setminus \{(t,s)\} && d_{uv} \\
& && \sum_{u:\,(u,v)\in E} f_{uv} - \sum_{z:\,(v,z)\in E} f_{vz} \le 0 && \forall v \in V && p_v \\
& && f_{uv} \ge 0 && \forall (u,v) \in E
\end{aligned}
$$

maximize $c^{\mathsf{T}} x \quad = \sum_{(u,v)\in E} (0 \cdot f_{uv}) \;+\; 1 \cdot f_{ts} \quad \Rightarrow c^{\mathsf{T}} = (0, \dots, 0, 1)$

Which constraints contain f_{uv} for $(u,v) \ne (t,s)$? $d_{uv},\, p_u,\, p_v$

$$\Rightarrow d_{uv} - p_u + p_v \ge 0$$

Which constraints contain f_{ts}? $p_s,\, p_t$

$$\Rightarrow p_s - p_t \ge 1$$

Max-Flow-Min-Cut Theorem

Theorem. The value of a maximum s–t flow and the weight of a minimum s–t cut are the same.

Proof. Special case of LP-Duality ...

maximize $\quad f_{ts}$

subject to $\quad f_{uv} \leq c_{uv} \qquad\qquad\qquad \forall (u,v) \in E \setminus \{(t,s)\} \qquad\qquad d_{uv}$

$$\sum_{u:\,(u,v)\in E} f_{uv} - \sum_{z:\,(v,z)\in E} f_{vz} \leq 0 \qquad \forall v \in V \qquad\qquad p_v$$

$$f_{uv} \geq 0 \qquad\qquad\qquad\qquad \forall (u,v) \in E$$

minimize $\quad \displaystyle\sum_{(u,v)\in E\setminus\{(t,s)\}} c_{uv} \cdot d_{uv}$

subject to $\quad d_{uv} - p_u + p_v \geq 0 \qquad \forall (u,v) \in E \setminus \{(t,s)\}$

$$p_s - p_t \geq 1$$

$$d_{uv} \geq 0 \qquad\qquad\qquad\qquad \forall (u,v) \in E$$

$$p_u \geq 0 \qquad\qquad\qquad\qquad \forall u \in V$$

Approximation Algorithms

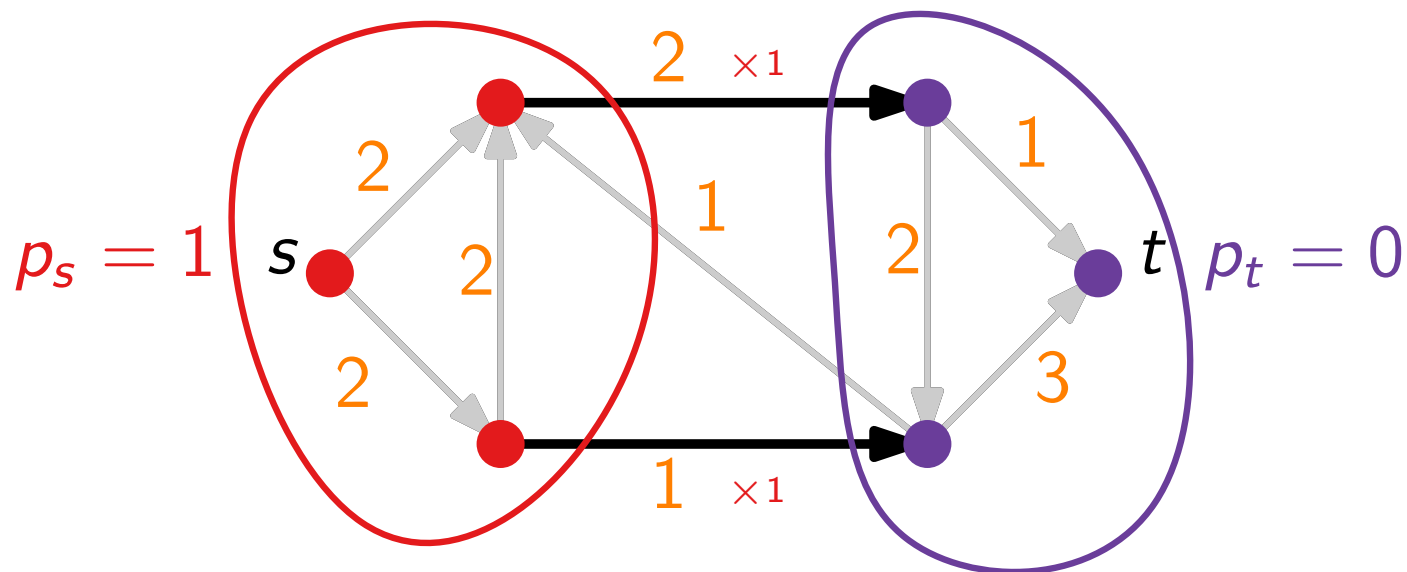## Lecture 4:
Linear Programming and LP-Duality

Part VI:
Dual LP of Max Flow

Dual LP – Interpretation as ILP

minimize $\displaystyle\sum_{(u,v)\in E\setminus\{(t,s)\}} c_{uv}\cdot d_{uv}$

subject to $d_{uv} - p_u + p_v \geq 0 \qquad \forall(u,v)\in E\setminus\{(t,s)\}$

$p_s - p_t \geq 1$

$d_{uv} \geq \cancel{0} \quad \in \{0,1\} \qquad\qquad \forall(u,v)\in E$

$p_u \geq \cancel{0} \quad \in \{0,1\} \qquad\qquad\qquad \forall u \in V$

equivalent to Min-Cut!

Dual LP – Fractional Cuts

minimize $\displaystyle\sum_{(u,v)\in E\setminus\{(t,s)\}} c_{uv}\cdot d_{uv}$ $\equiv$ LP-relaxation of the ILP

subject to $d_{uv} - p_u + p_v \geq 0 \qquad \forall(u,v)\in E\setminus\{(t,s)\}$

$p_s - p_t \geq 1$

$d_{uv} \geq 0 \qquad\qquad\qquad \forall(u,v)\in E$

$p_u \geq 0 \qquad\qquad\qquad\quad \forall u\in V$

Moreover, all extreme-point solutions all **integral**! (HW)

Note that every s–t path $s=v_0,\dots,v_k=t$ has length ≥ 1 w.r.t. d:

$$\sum_{i=0}^{k-1} d_{i,i+1} \geq \sum_{i=0}^{k-1}(p_i - p_{i+1})$$

$$= p_s - p_t = 1$$
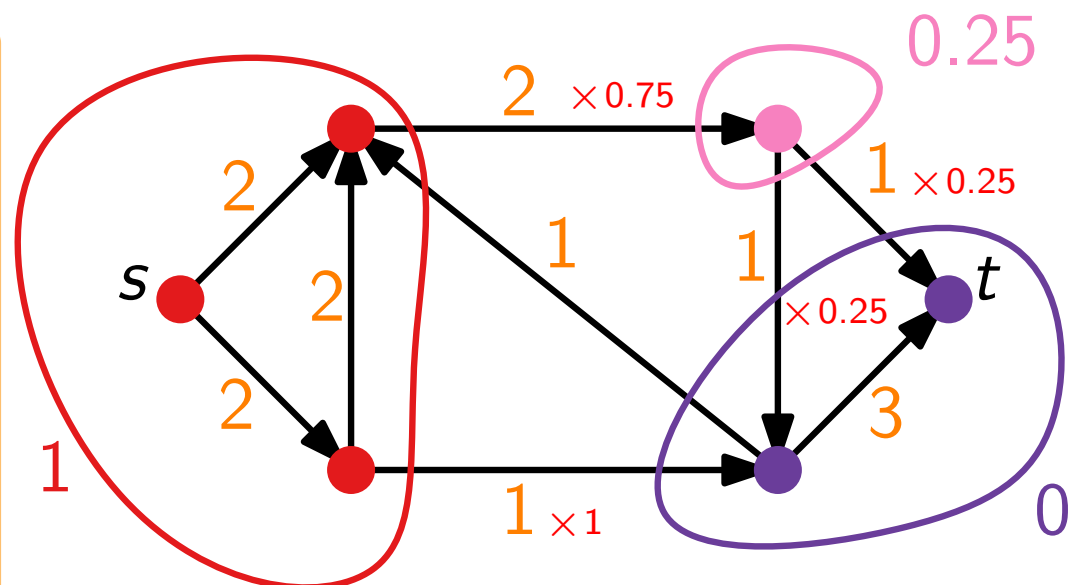
Dual LP – Complementary Slackness

maximize f_{ts}

subject to $f_{uv} \leq c_{uv}$ $\forall (u,v) \in E \setminus \{(t,s)\}$

$$\sum_{u:\,(u,v)\in E} f_{uv} - \sum_{z:\,(v,z)\in E} f_{vz} \leq 0 \qquad \forall v \in V$$

$f_{uv} \geq 0$ $\forall (u,v) \in E$

minimize $\displaystyle\sum_{(u,v)\in E \setminus \{(t,s)\}} c_{uv} \cdot d_{uv}$

subject to $d_{uv} - p_u + p_v \geq 0$

$p_s - p_t \geq 1$

$d_{uv} \geq 0$

$p_u \geq 0$

Primal CS:

$\forall j: \quad x_j = 0 \ \ \text{or} \ \ \sum_{i=1}^{m} a_{ij} y_i = c_j$

Dual CS:

$\forall i: \quad y_i = 0 \ \ \text{or} \ \ \sum_{j=1}^{n} a_{ij} x_j = b_i$

For a max flow and min cut:

- For each forward edge (u,v) of the cut: $f_{uv} = c_{uv}$.

  ($d_{uv} = 1$, so by dual CS: $f_{uv} = c_{uv}$.)

- For each backward edge (u,v) of the cut: $f_{uv} = 0$.

  (Otherwise, by primal CS: $d_{uv} - 0 + 1 = 0$.)