Approximation Algorithms

Lecture 4:
Linear Programming and LP-Duality

Part I:
Introduction to Linear Programming

Alexander Wolff

Winter 2023/24

Maximizing Profits

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Which choice of (x_1, x_2) maximizes the profit?

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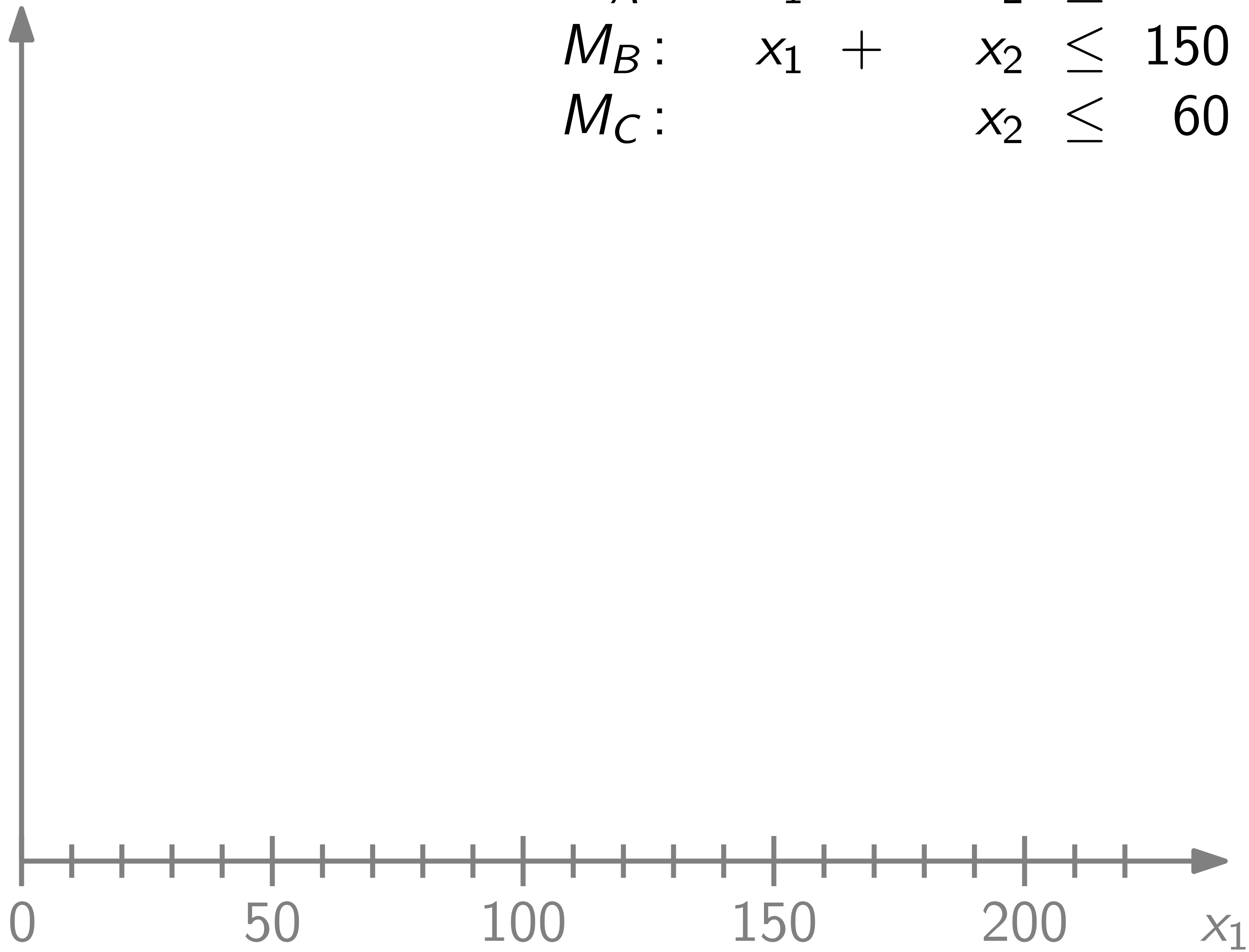

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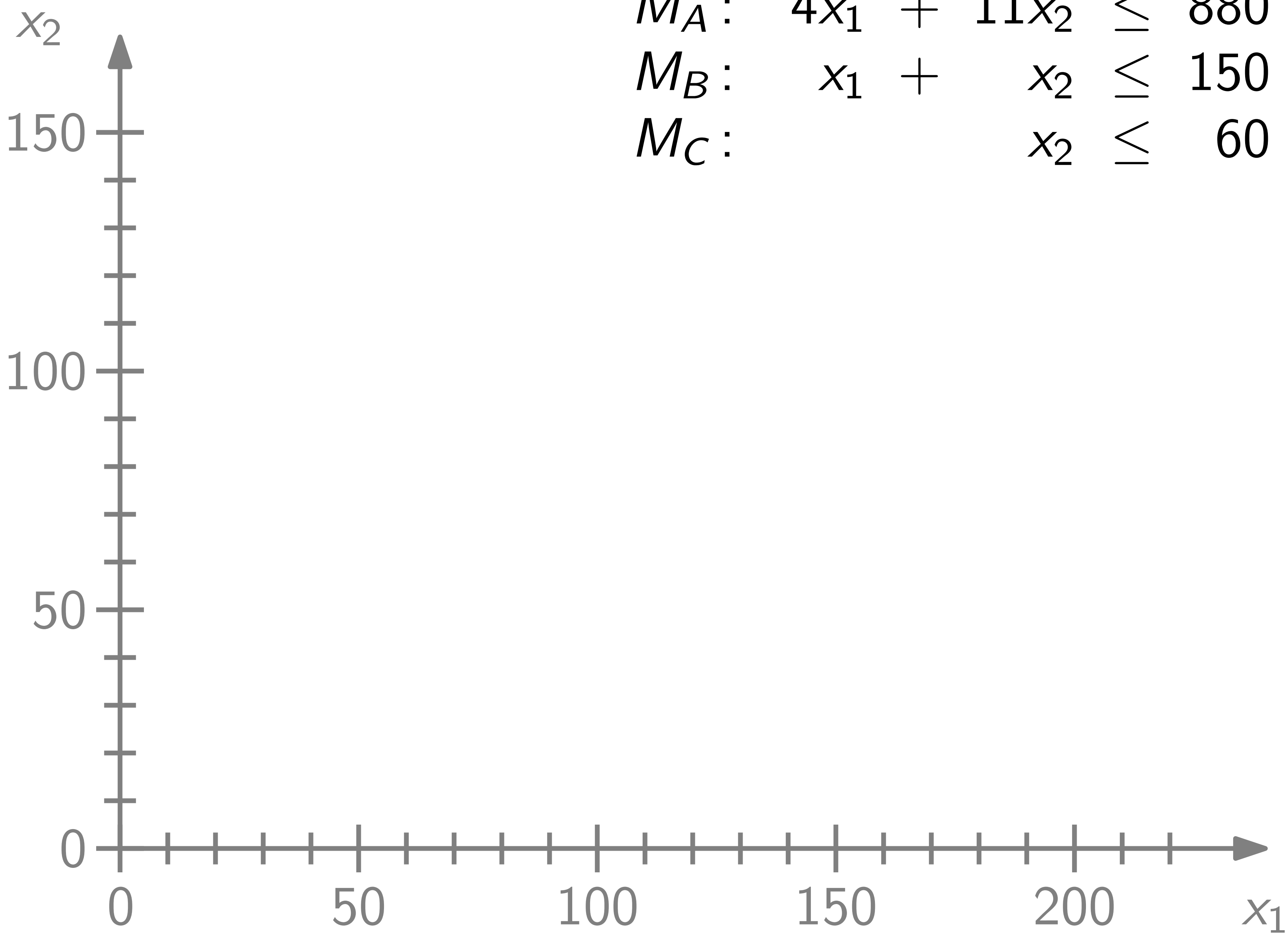

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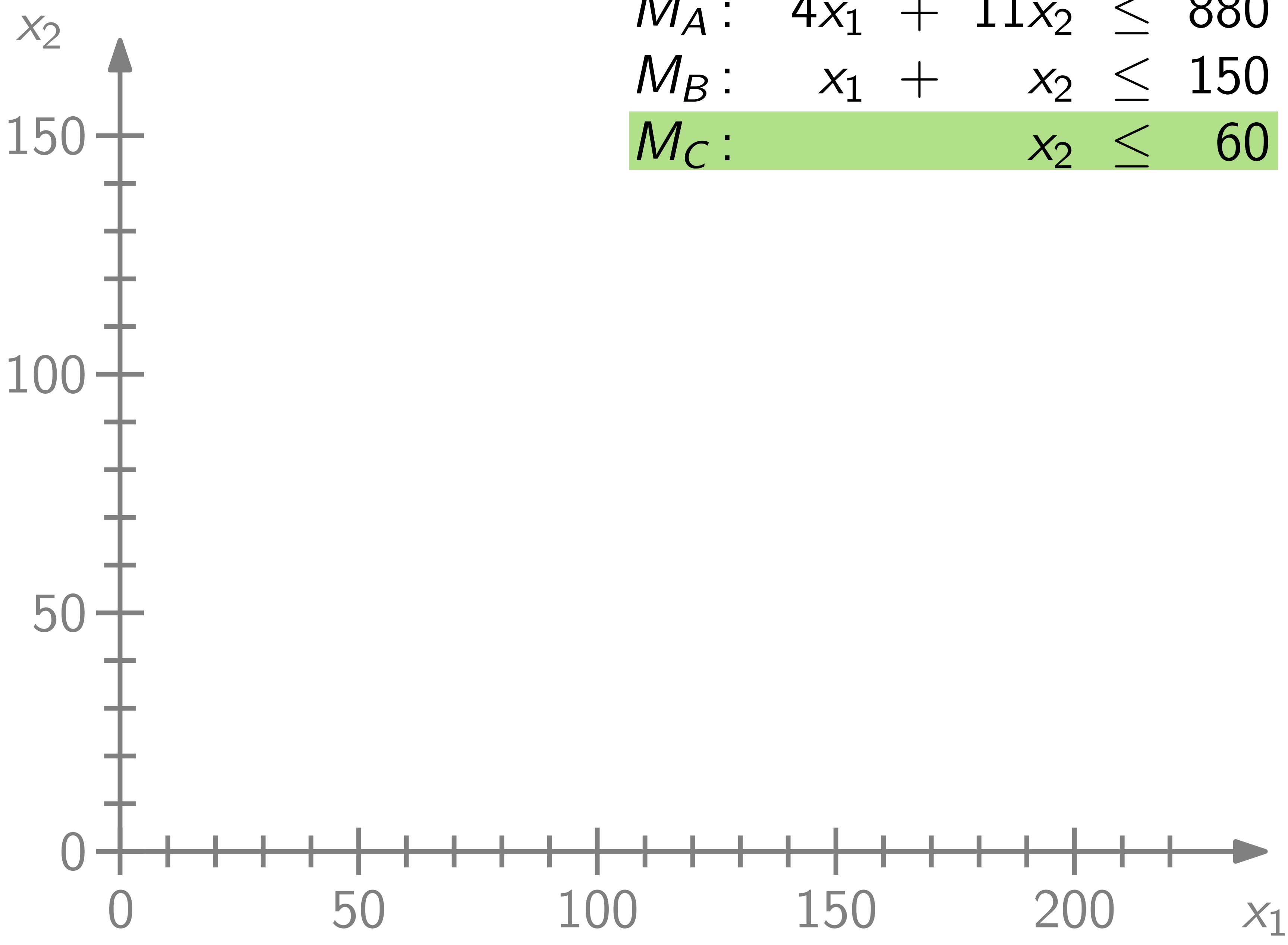

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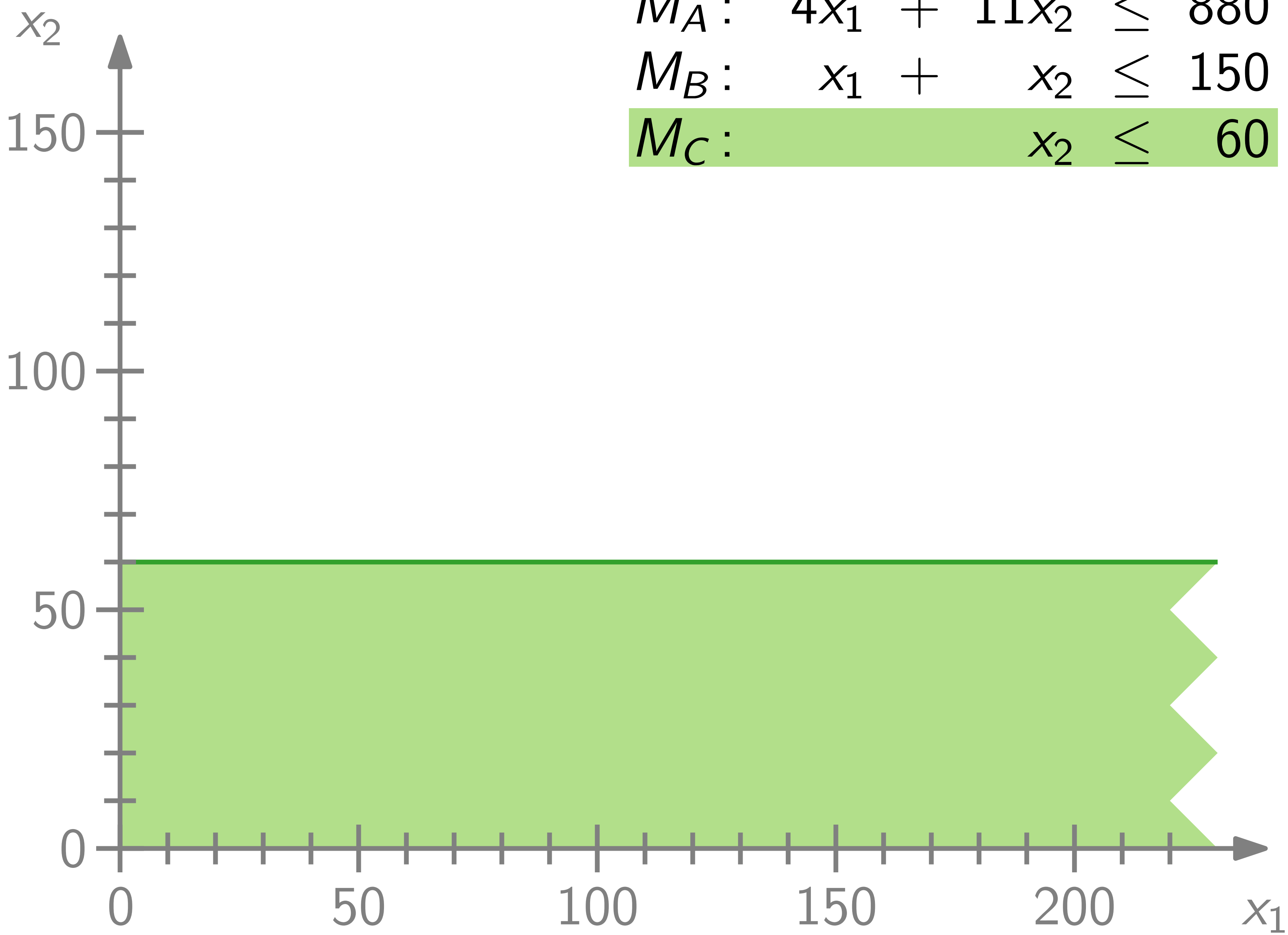

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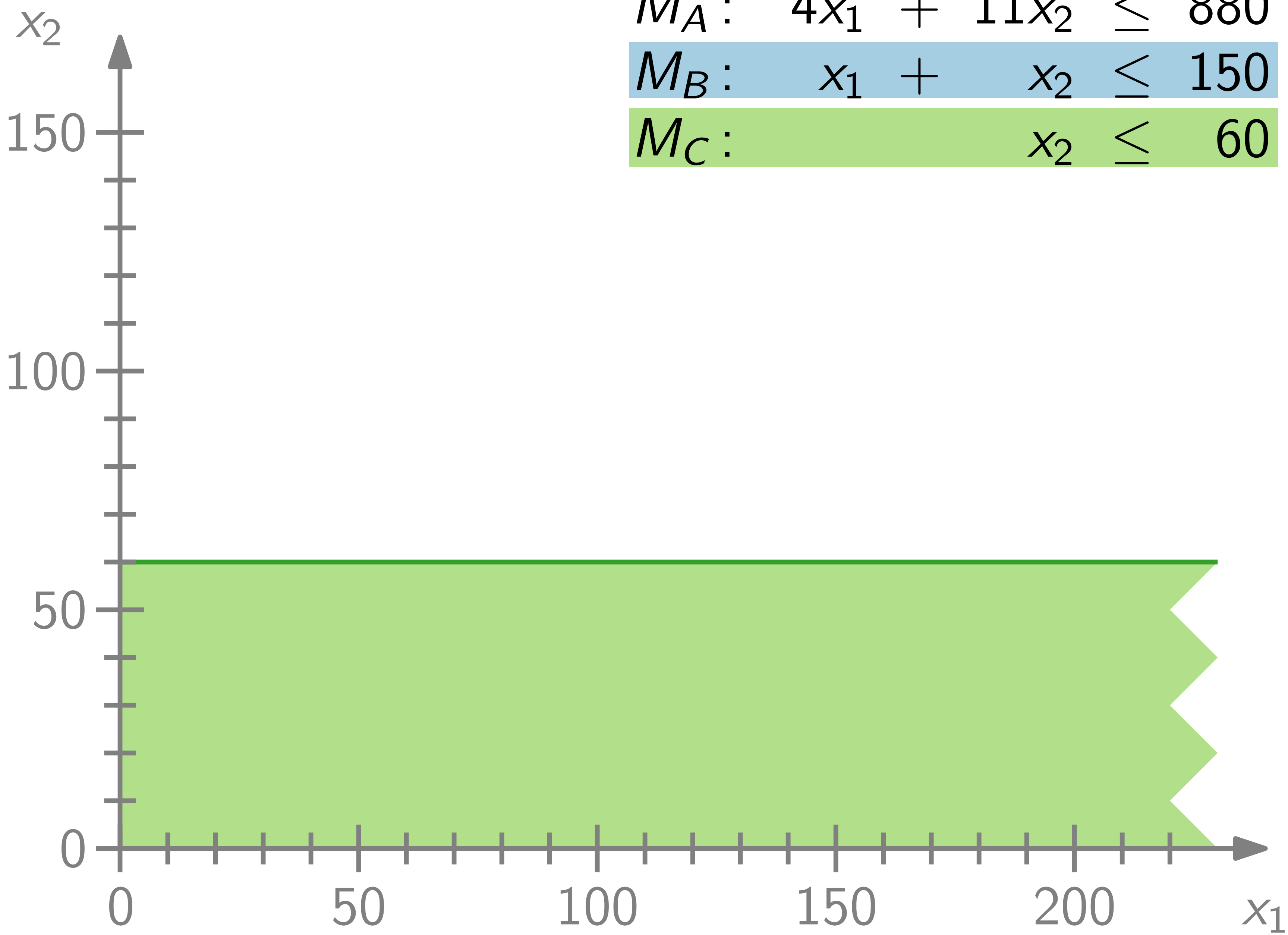

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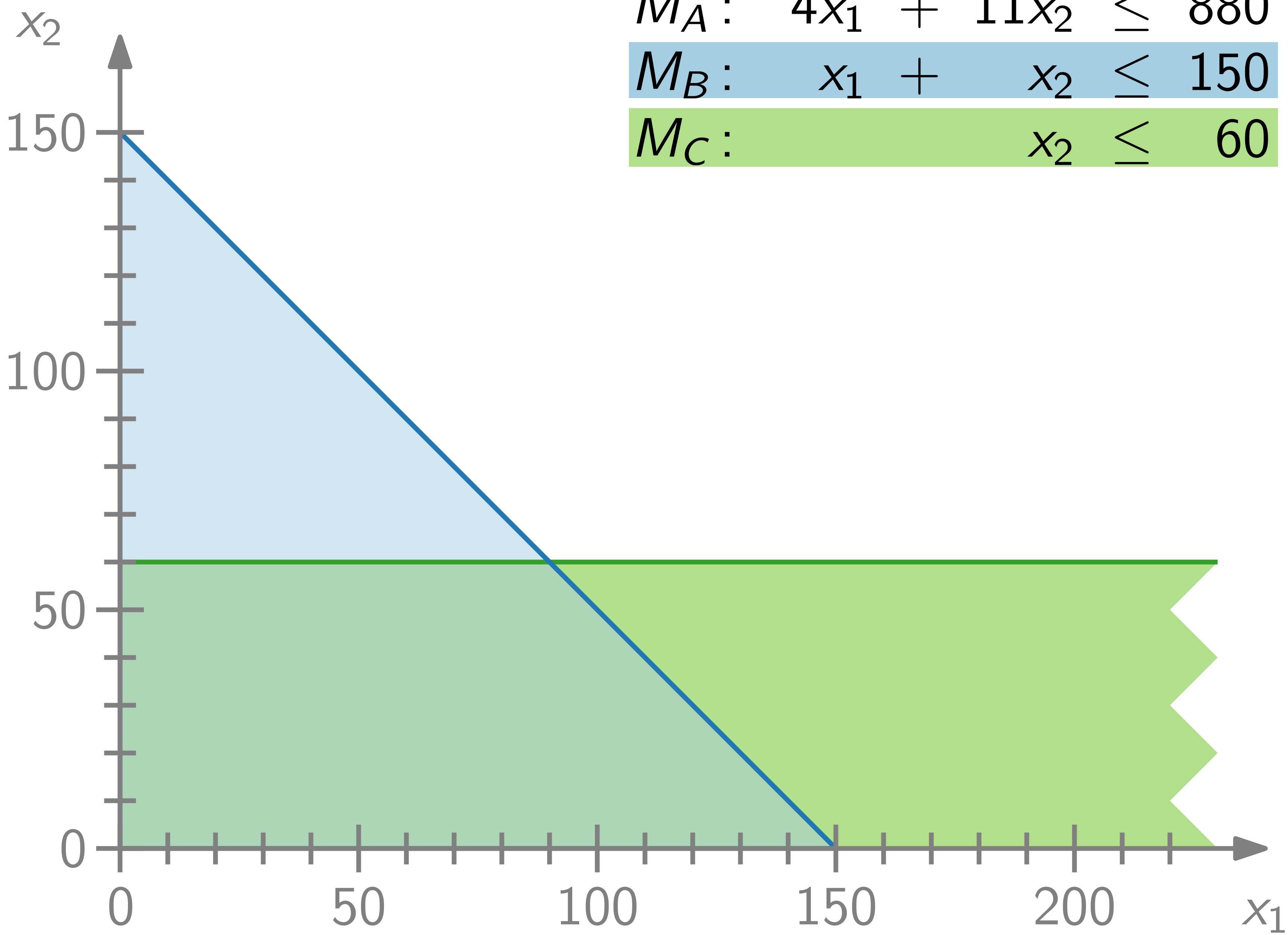

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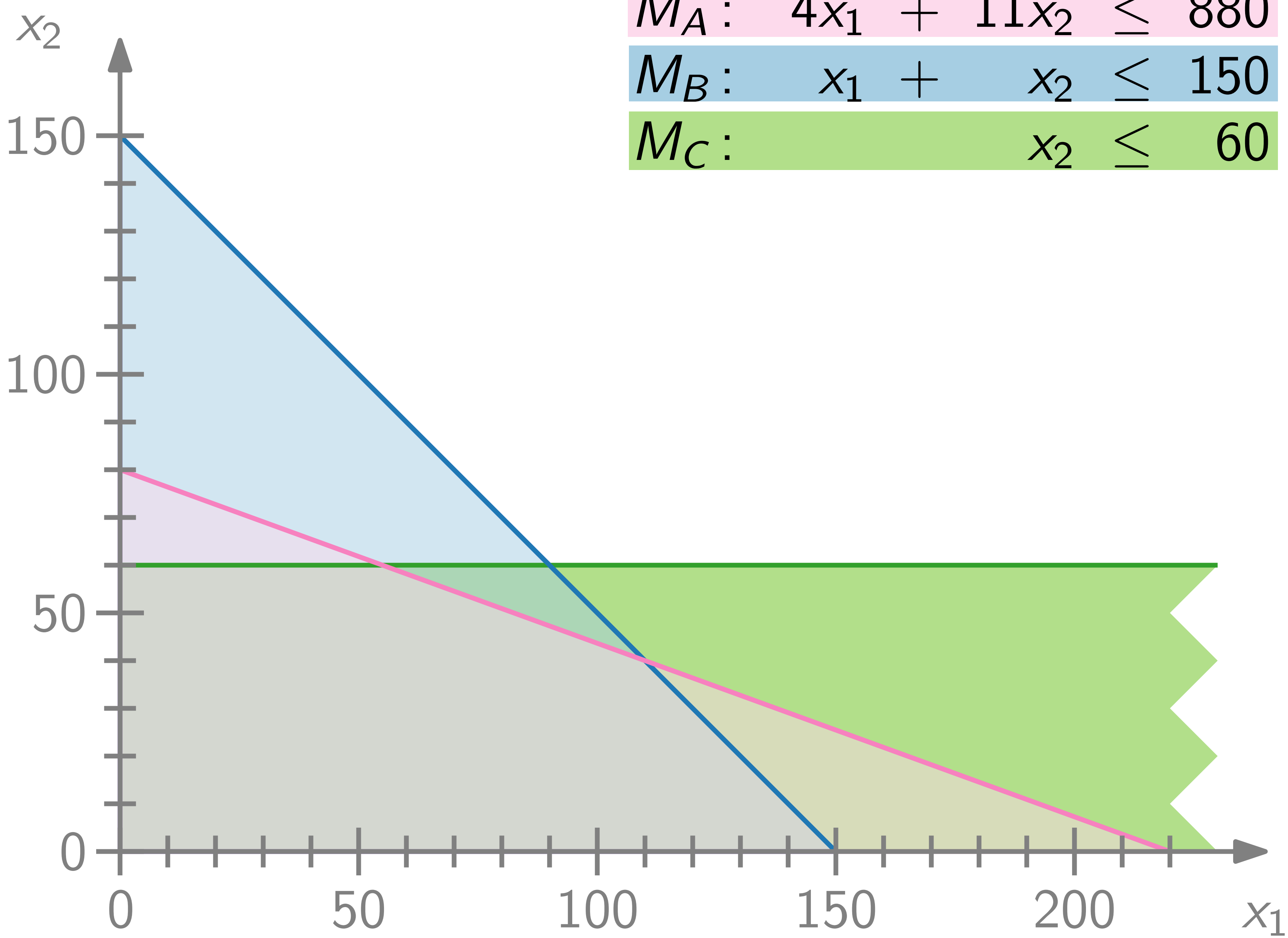

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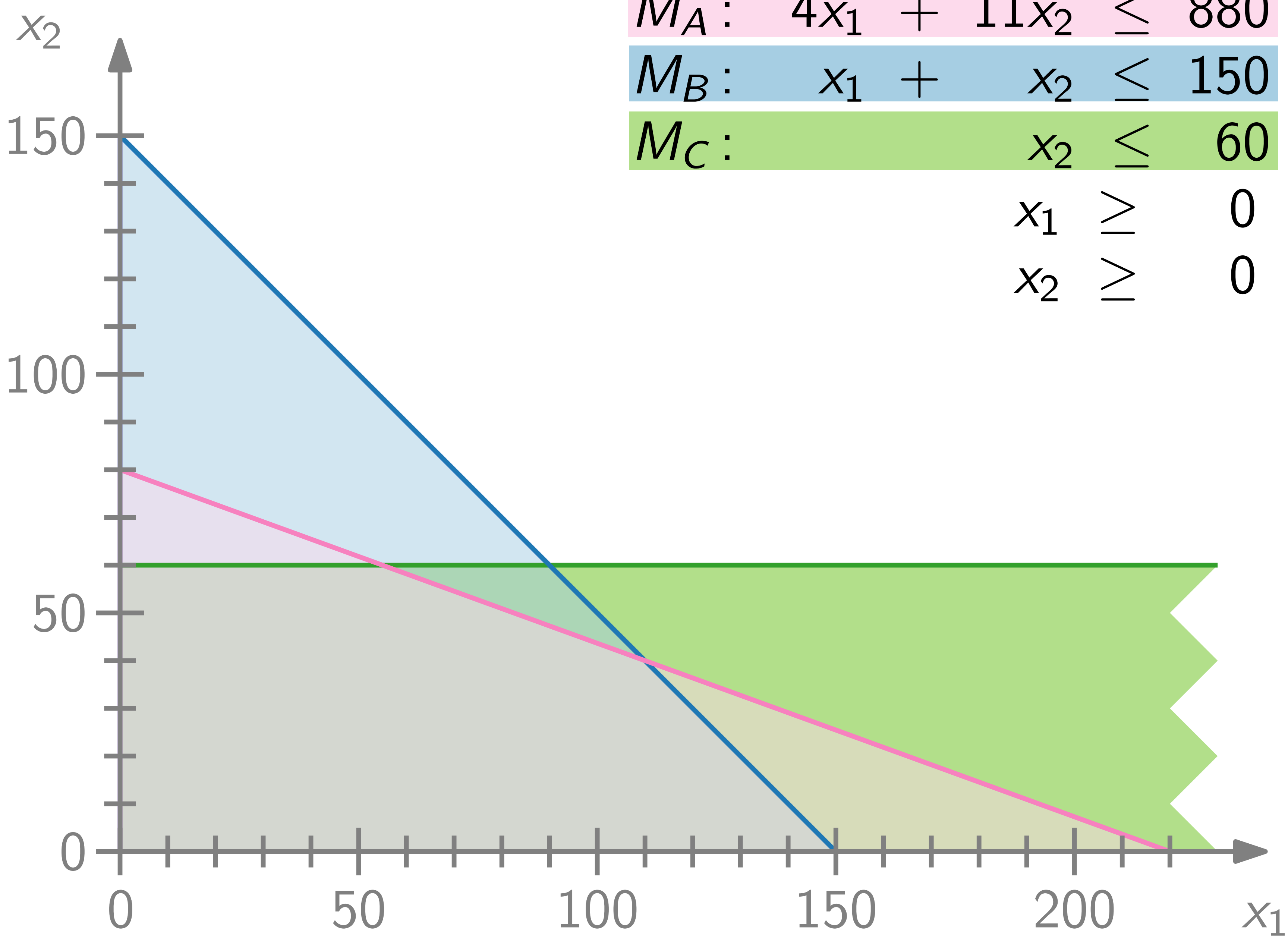

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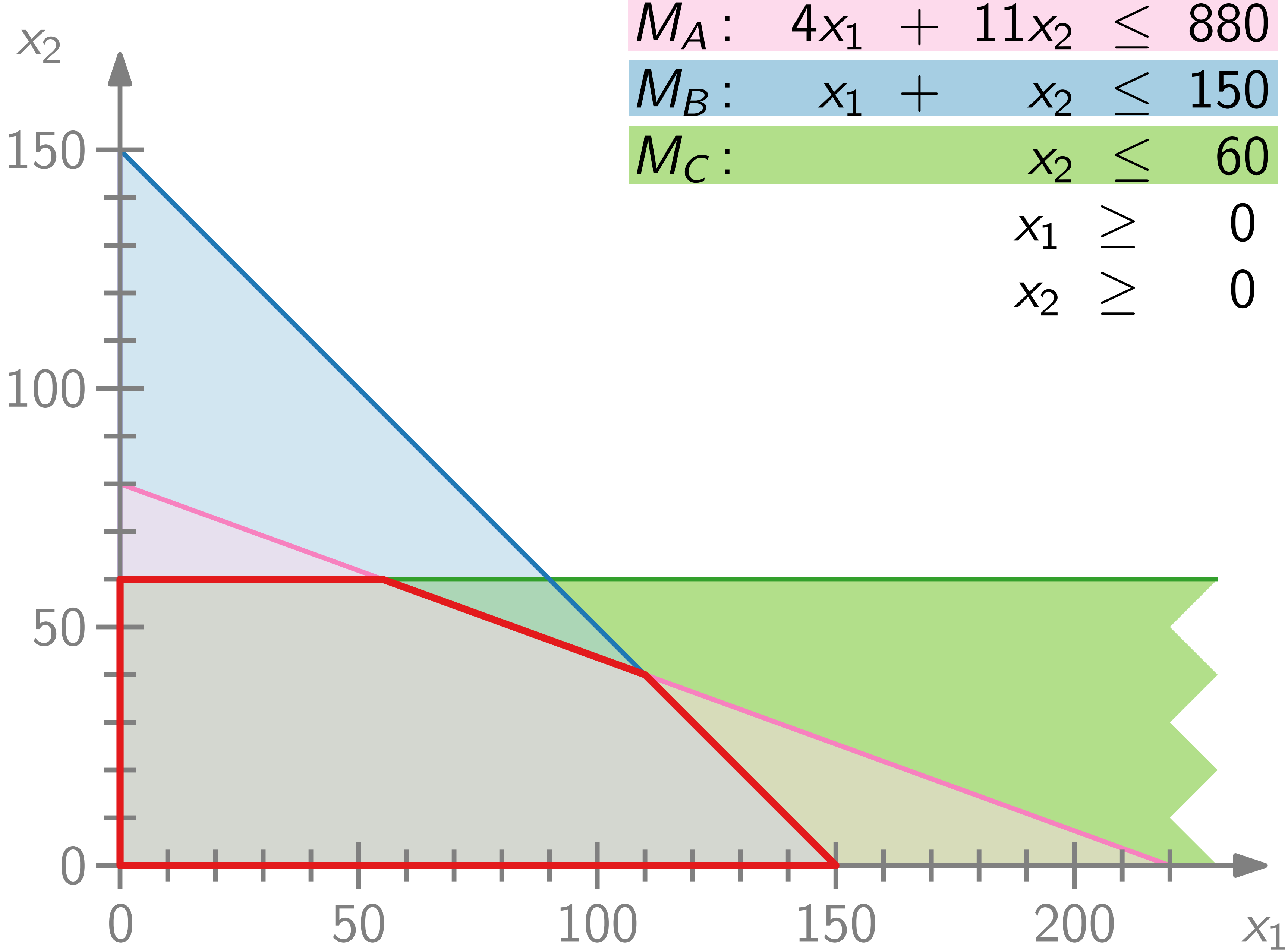

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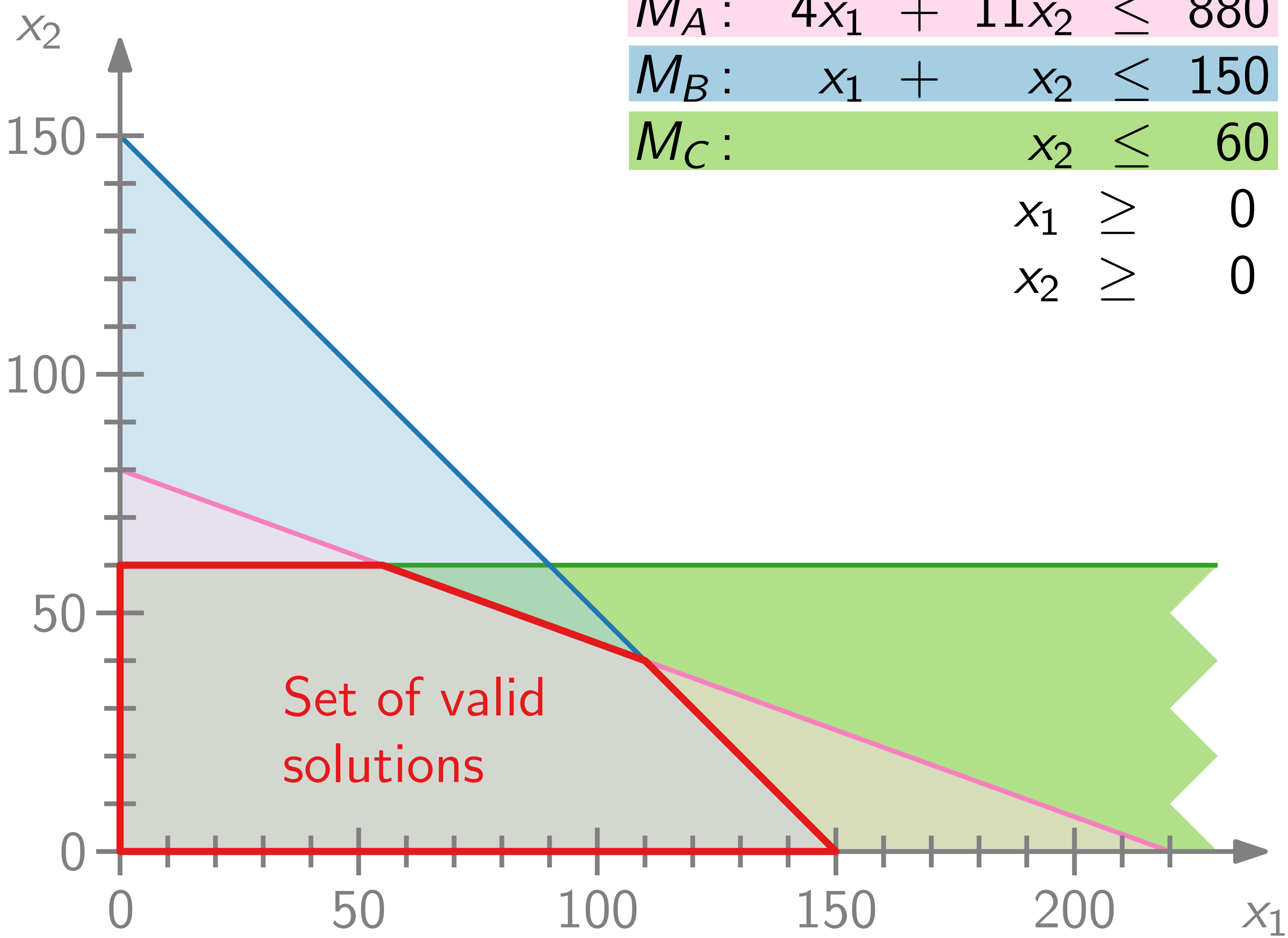

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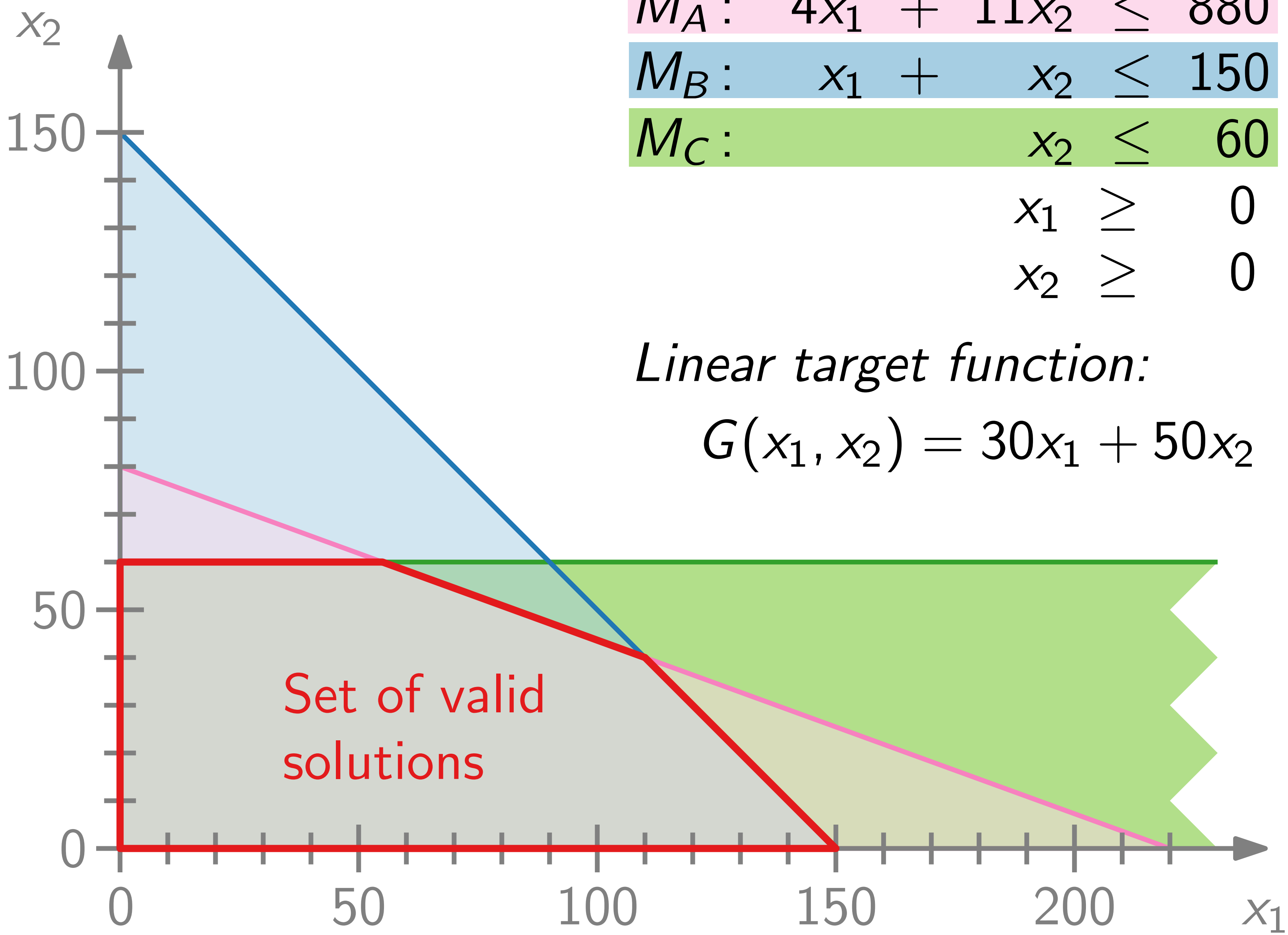

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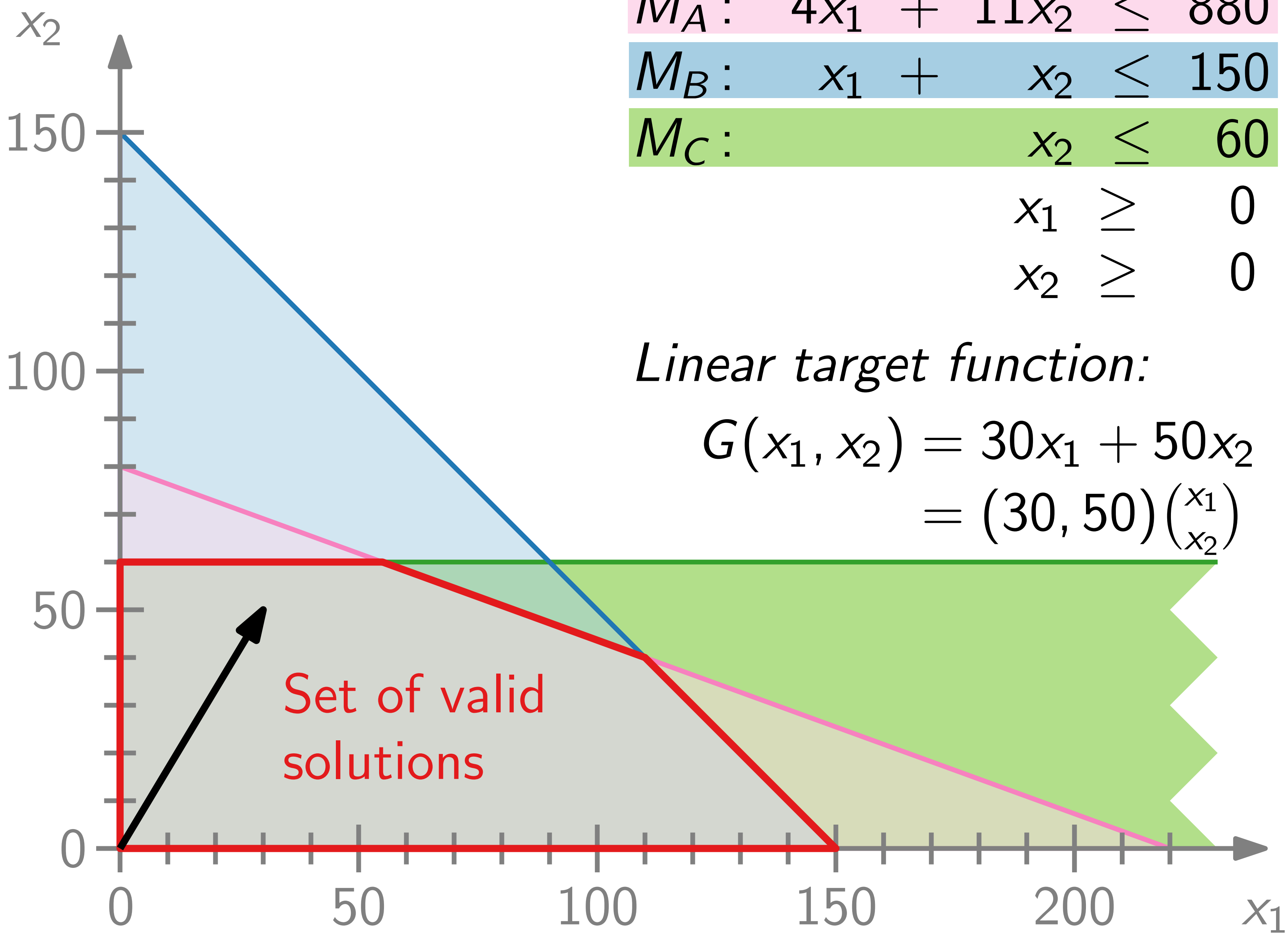

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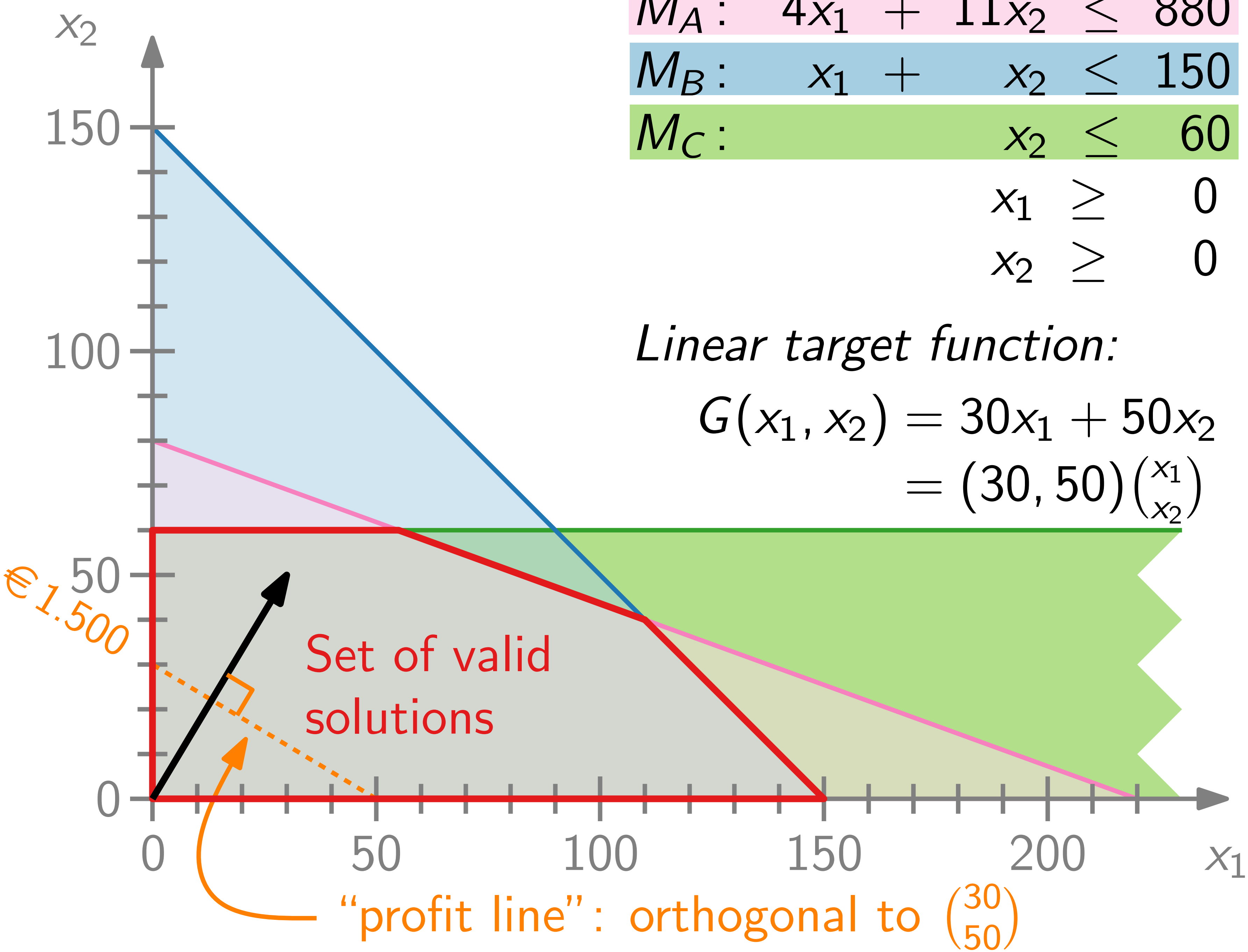

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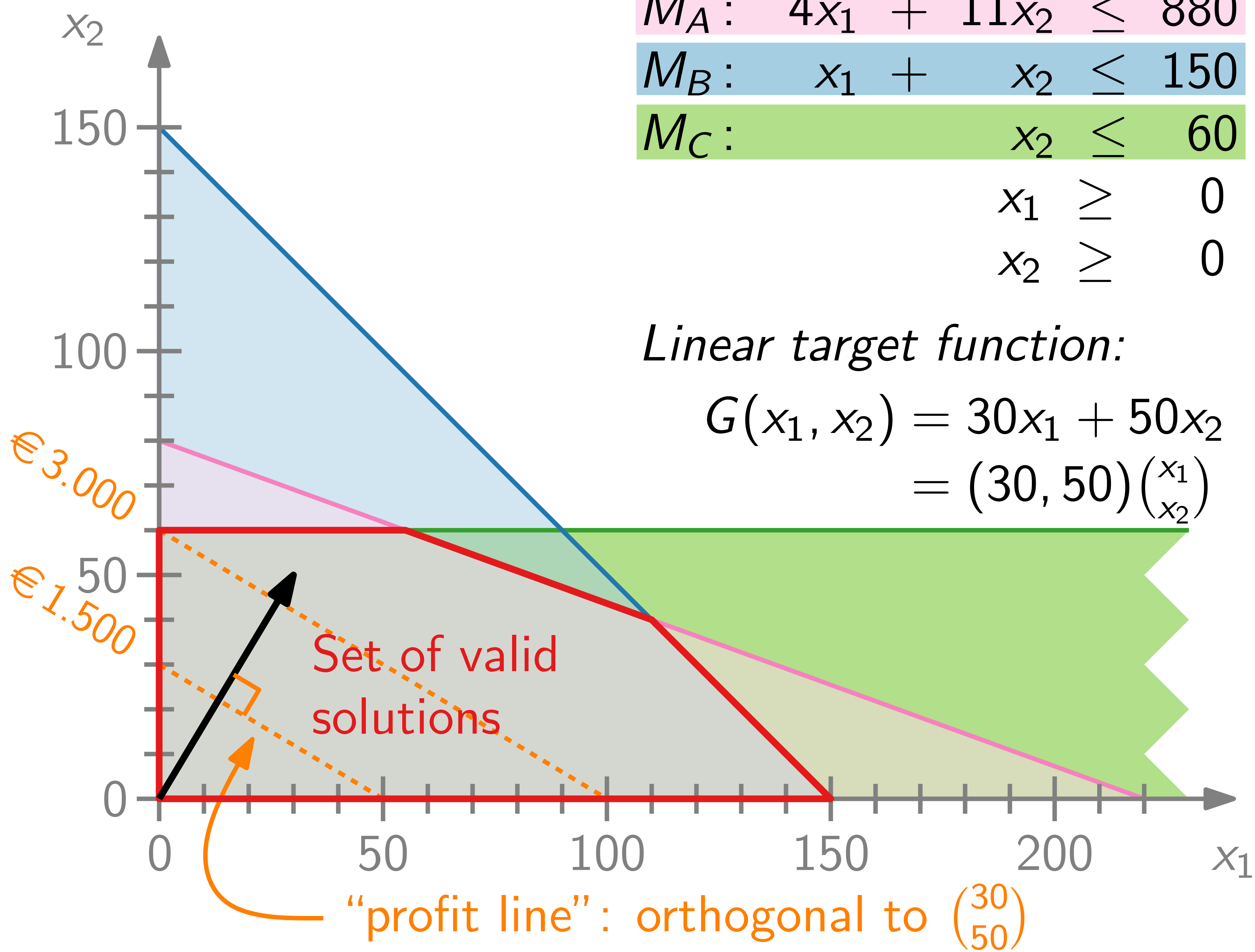

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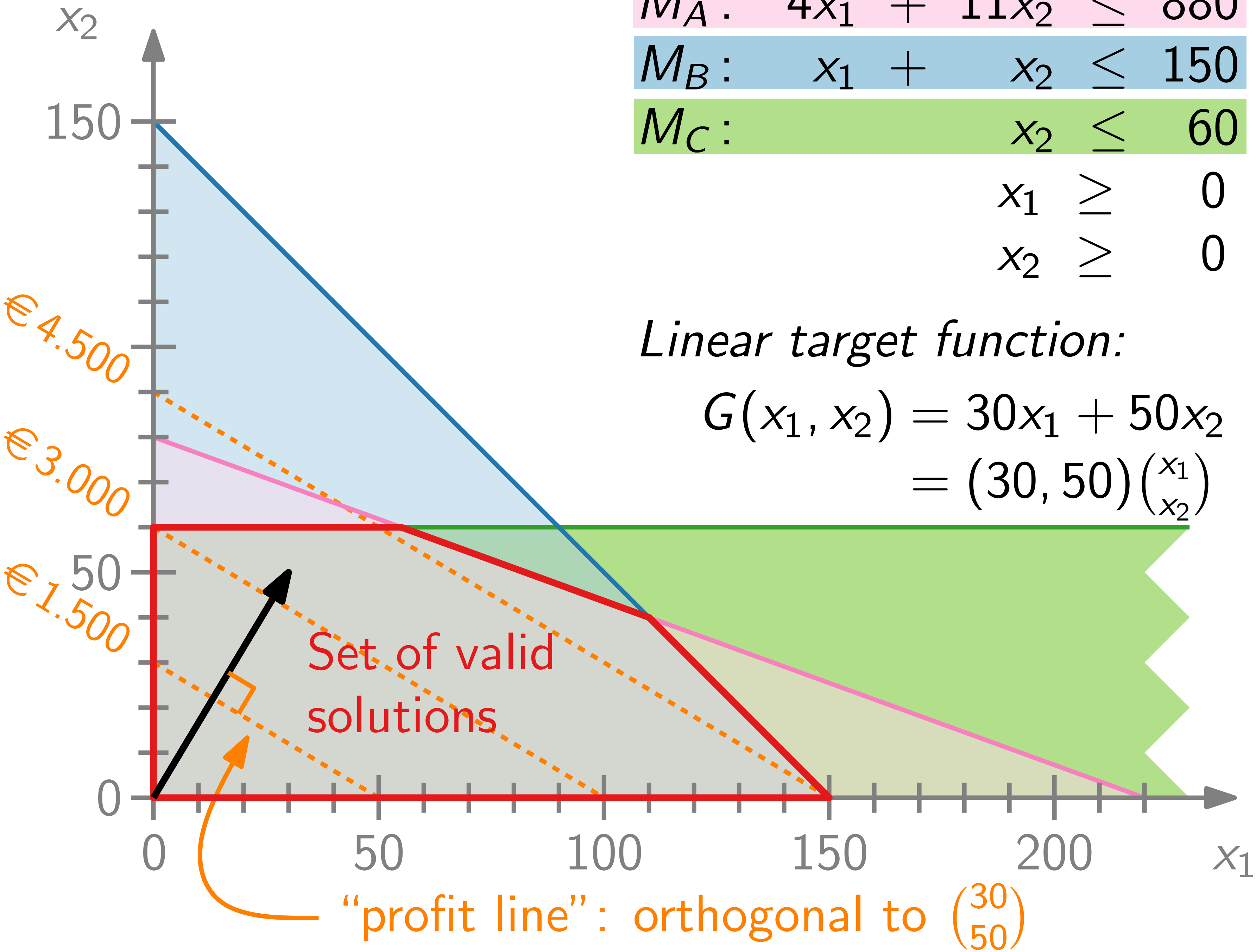

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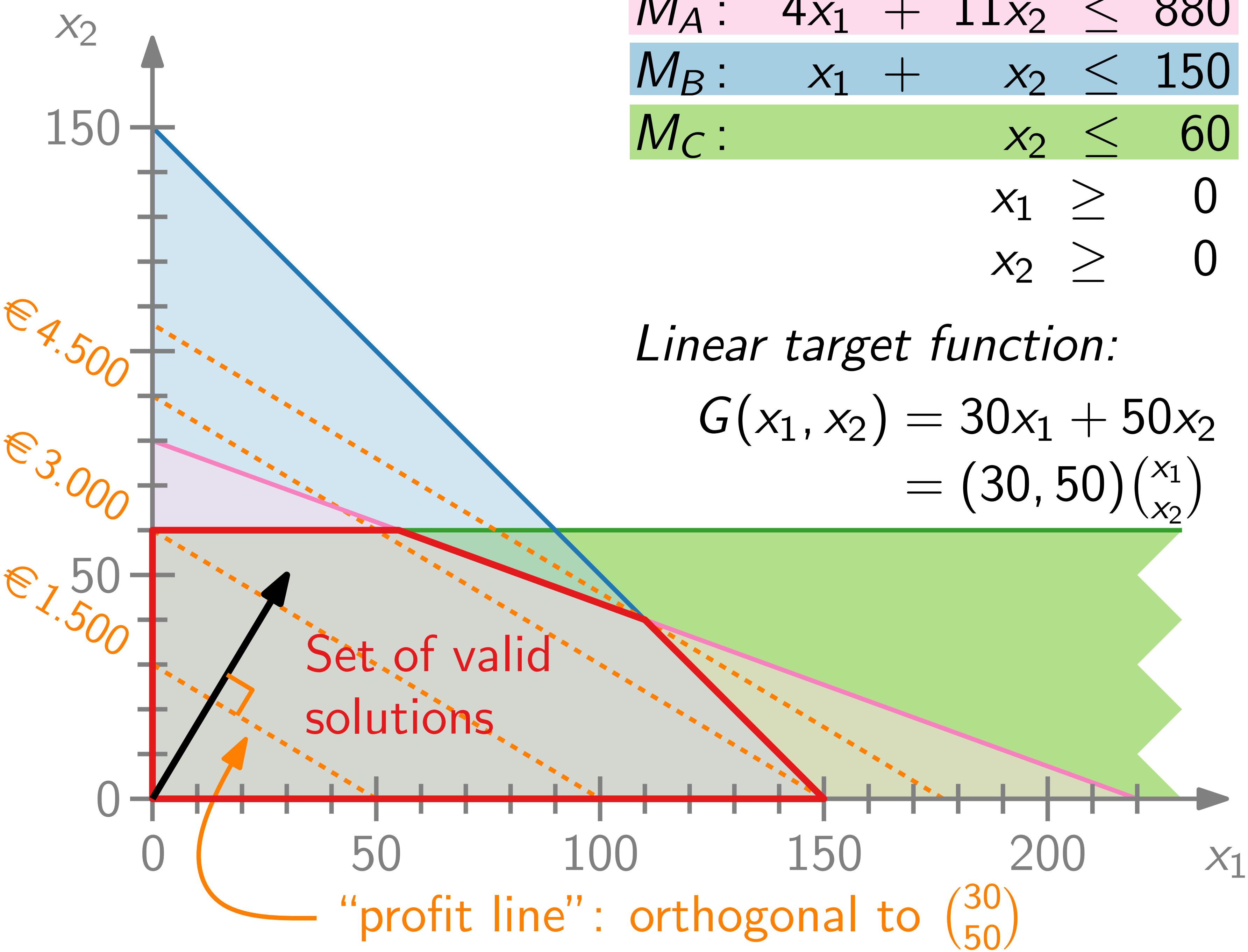

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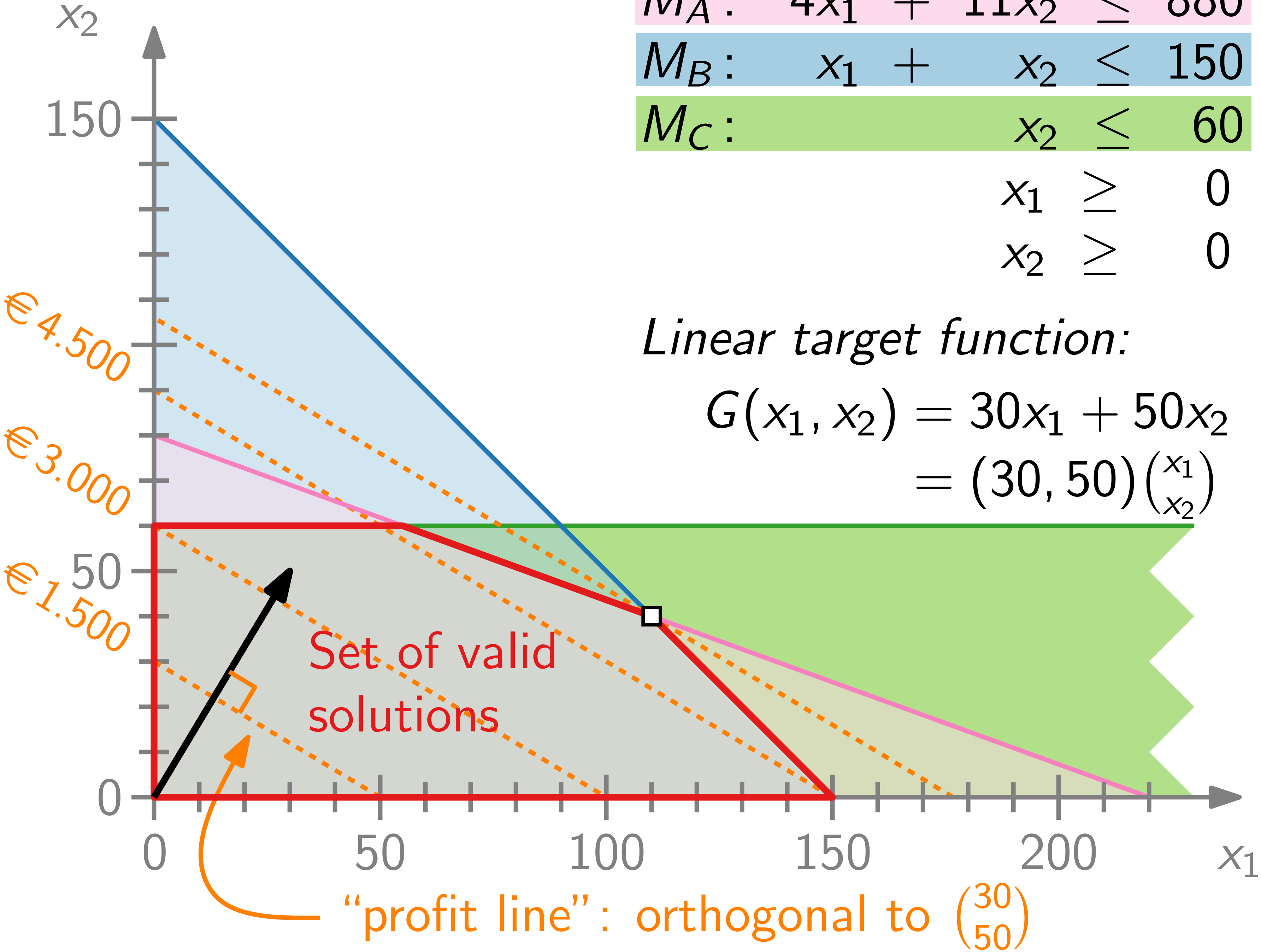

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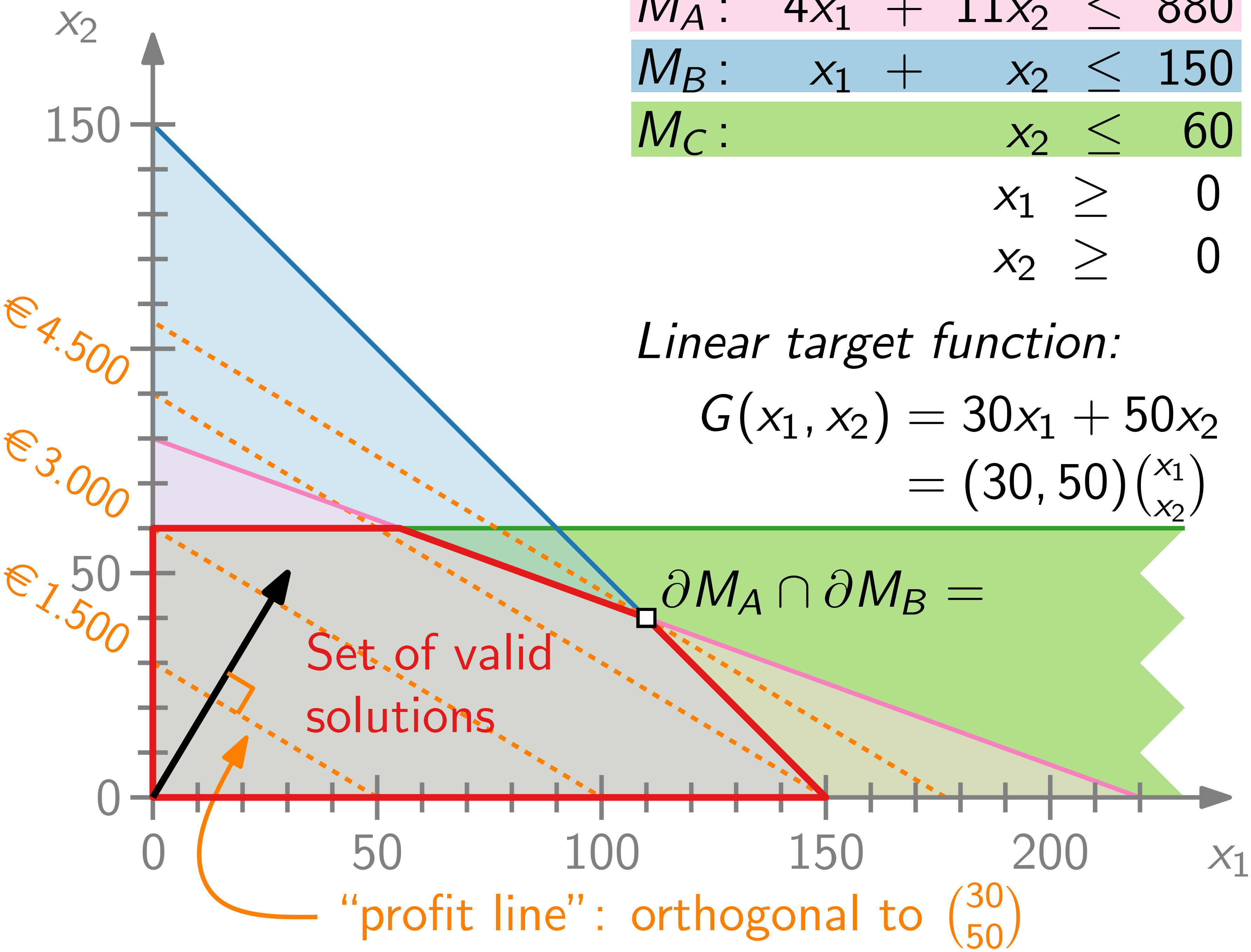

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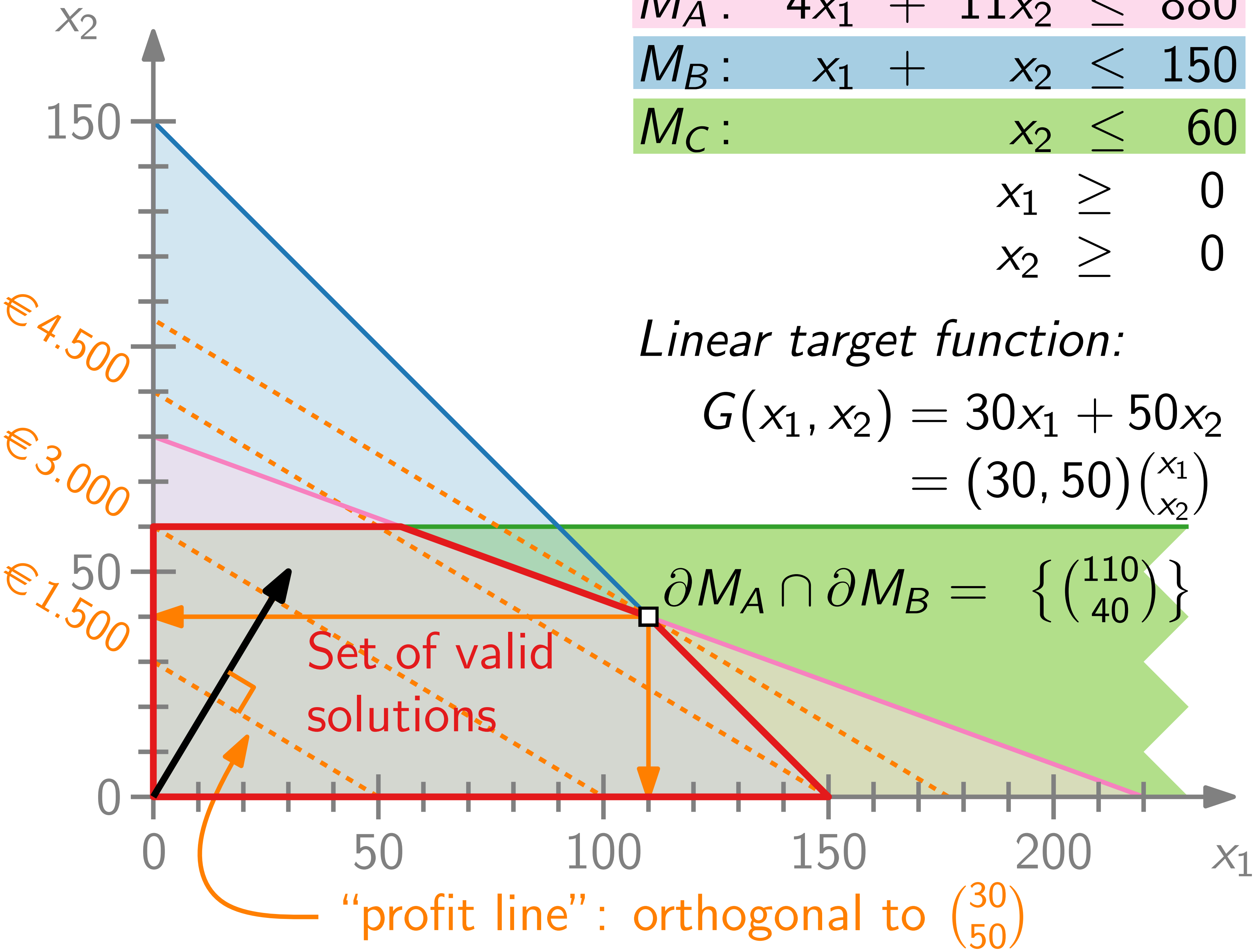

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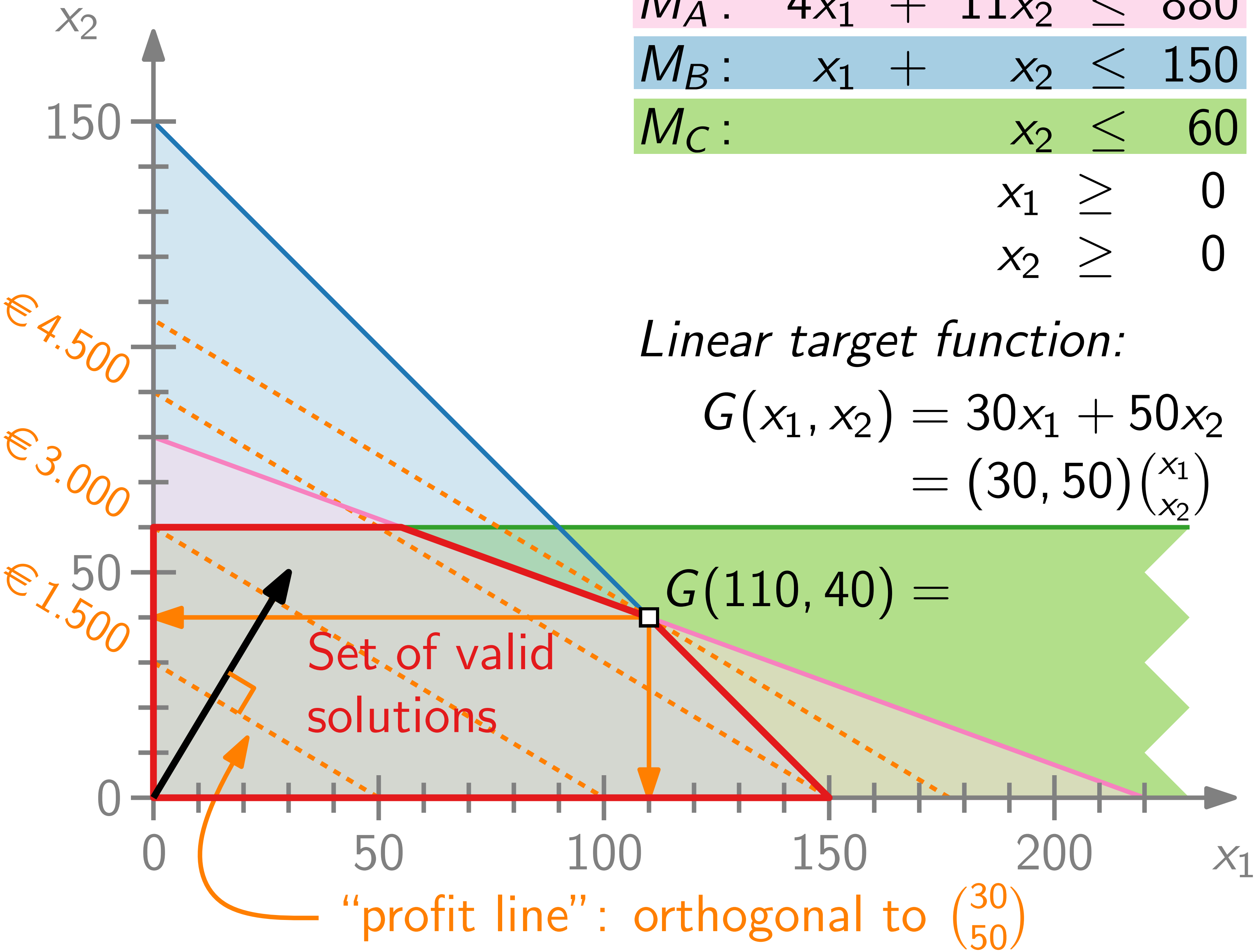

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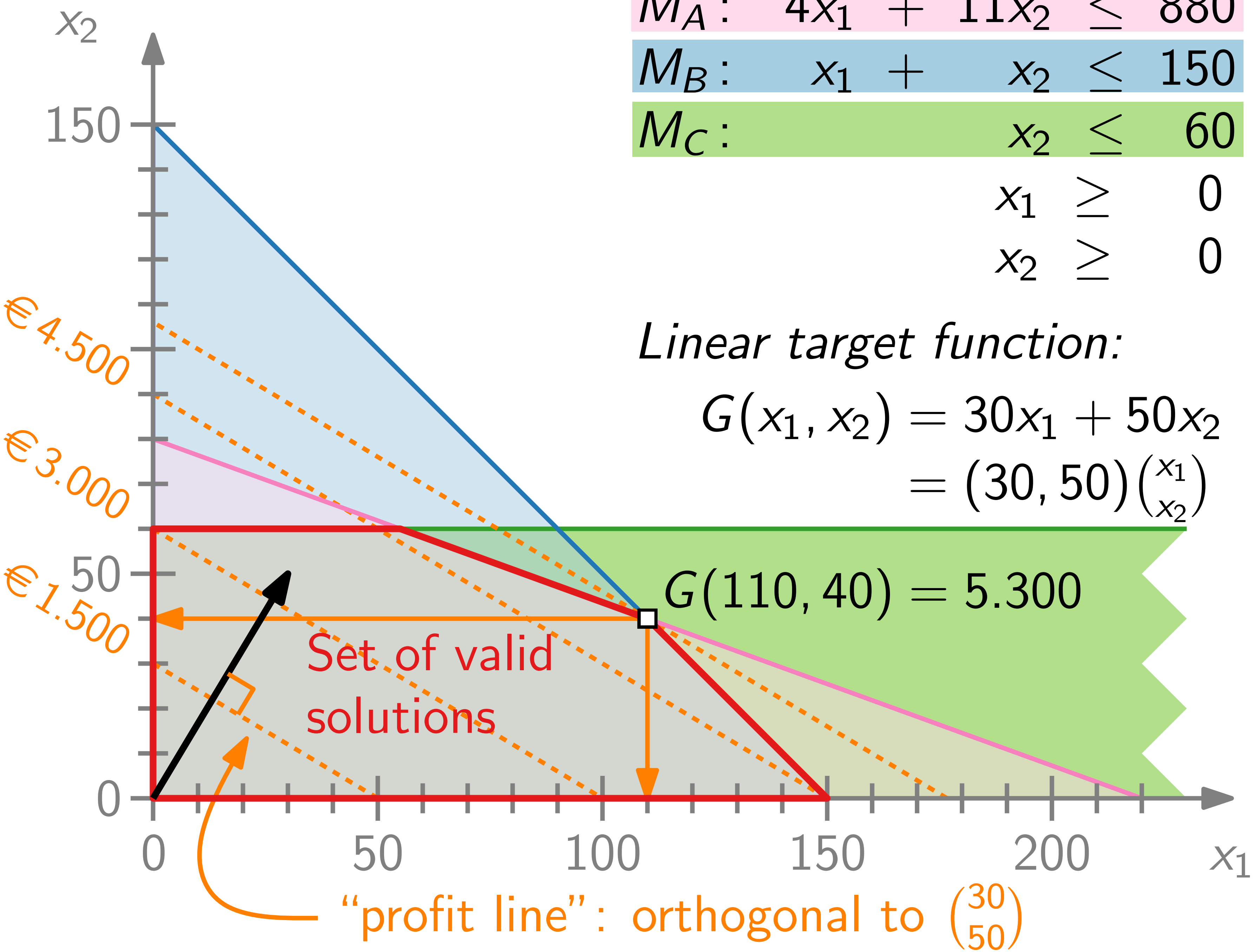

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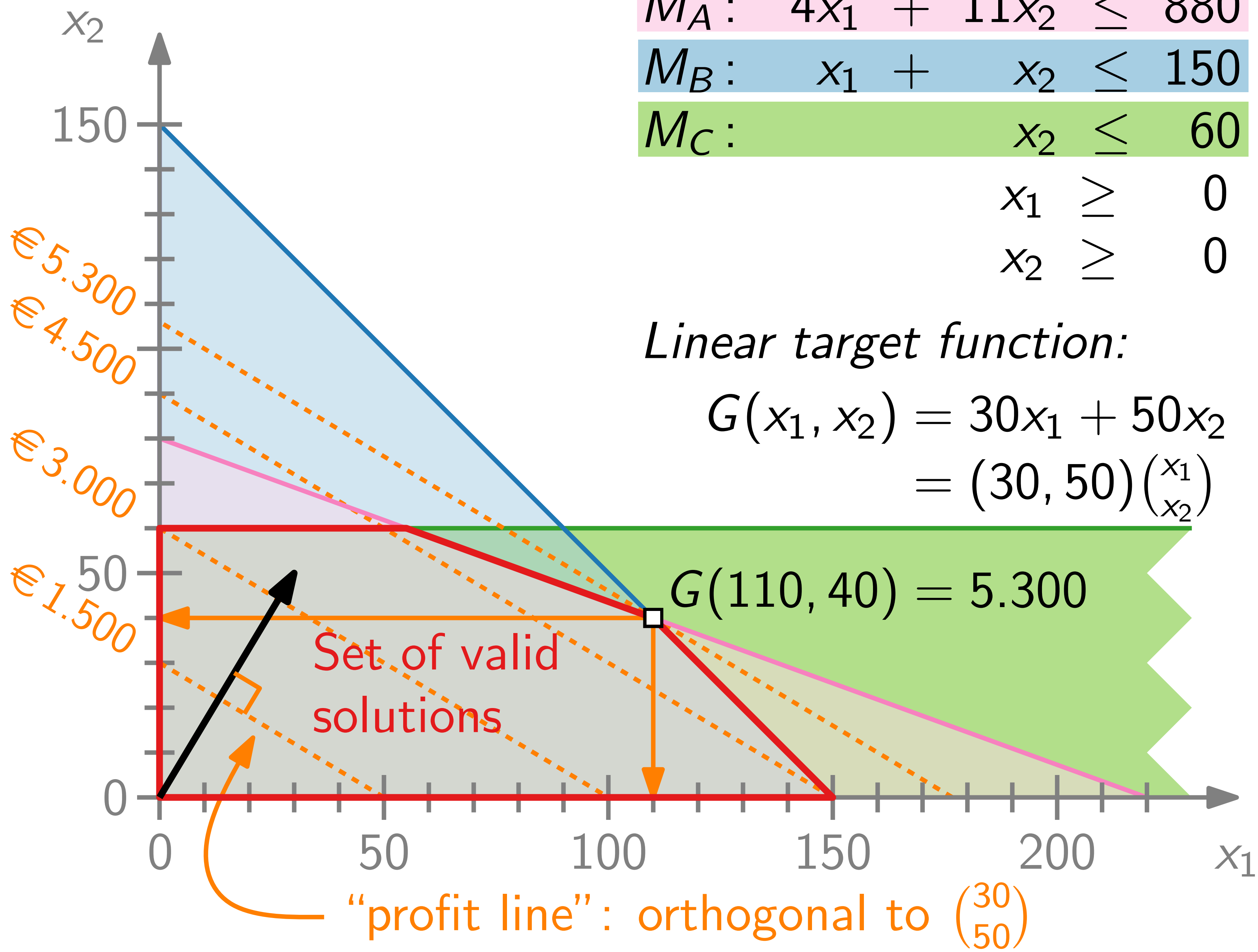

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Part II:
Upper Bounds for LPs

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Examples:

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- TSP: lower bound by MST or by cycle cover

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Optimize (i.e., minimize or maximize) a linear (*objective*) function subject to linear inequalities (*constraints*).

$$\begin{array}{lrcl} \textbf{minimize} & c^{\mathsf{T}} x & & \\ \textbf{subject to} & Ax & \geq & b \\ & x & \geq & 0 \end{array} \qquad \textit{standard form}$$

Example. $c = \begin{pmatrix} 7 \\ 1 \\ 5 \end{pmatrix}$ $A = \begin{pmatrix} 1 & -1 & 3 \\ 5 & 2 & -1 \end{pmatrix}$ $b = \begin{pmatrix} 10 \\ 6 \end{pmatrix}$

$$\begin{array}{lrcrcrcr} \textbf{minimize} & 7x_1 & + & x_2 & + & 5x_3 & & \\ \textbf{subject to} & x_1 & - & x_2 & + & 3x_3 & \geq & 10 \\ & 5x_1 & + & 2x_2 & - & x_3 & \geq & 6 \\ & & & & & x_1, x_2, x_3 & \geq & 0 \end{array}$$

Linear Programming – Upper Bounds

Optimize (i.e., minimize or maximize) a linear (*objective*) function subject to linear inequalities (*constraints*).

$$\begin{array}{llcrcrcrcr} \textbf{minimize} & 7x_1 & + & x_2 & + & 5x_3 & & \\ \textbf{subject to} & x_1 & - & x_2 & + & 3x_3 & \geq & 10 \\ & 5x_1 & + & 2x_2 & - & x_3 & \geq & 6 \\ & & & & & x_1, x_2, x_3 & \geq & 0 \end{array}$$

Linear Programming – Upper Bounds

Optimize (i.e., minimize or maximize) a linear (*objective*) function subject to linear inequalities (*constraints*).

$$\begin{array}{lrcrcrcr} \textbf{minimize} & 7x_1 & + & x_2 & + & 5x_3 & & \\ \textbf{subject to} & x_1 & - & x_2 & + & 3x_3 & \geq & 10 \\ & 5x_1 & + & 2x_2 & - & x_3 & \geq & 6 \\ & & & & & x_1, x_2, x_3 & \geq & 0 \end{array}$$

Valid solution?

Linear Programming – Upper Bounds

Optimize (i.e., minimize or maximize) a linear (*objective*) function subject to linear inequalities (*constraints*).

$$\begin{array}{lrcrcrcr} \textbf{minimize} & 7x_1 & + & x_2 & + & 5x_3 & & \\ \textbf{subject to} & x_1 & - & x_2 & + & 3x_3 & \geq & 10 \\ & 5x_1 & + & 2x_2 & - & x_3 & \geq & 6 \\ & & & & & x_1, x_2, x_3 & \geq & 0 \end{array}$$

Valid solution?

$x = (2, 1, 3)$

Linear Programming – Upper Bounds

Optimize (i.e., minimize or maximize) a linear (*objective*) function subject to linear inequalities (*constraints*).

$$
\begin{array}{lrcrcrcr}
\textbf{minimize} & 7x_1 & + & x_2 & + & 5x_3 & & \\
\textbf{subject to} & x_1 \; 2 & - & x_2 & + & 3x_3 & \geq & 10 \\
 & 5x_1 & + & 2x_2 & - & x_3 & \geq & 6 \\
 & & & & & x_1, x_2, x_3 & \geq & 0
\end{array}
$$

Valid solution?

$x = (2, 1, 3)$

Linear Programming – Upper Bounds

Optimize (i.e., minimize or maximize) a linear (*objective*) function subject to linear inequalities (*constraints*).

$$
\begin{array}{llllllll}
\textbf{minimize} & 7x_1 & + & x_2 & + & 5x_3 & & \\
\textbf{subject to} & x_1\ 2 & - & x_2\ 1 & + & 3x_3 & \geq & 10 \\
 & 5x_1 & + & 2x_2 & - & x_3 & \geq & 6 \\
 & & & & & x_1, x_2, x_3 & \geq & 0
\end{array}
$$

Valid solution?

$x = (2, 1, 3)$

Linear Programming – Upper Bounds

Optimize (i.e., minimize or maximize) a linear (*objective*) function subject to linear inequalities (*constraints*).

$$
\begin{array}{llcl}
\textbf{minimize} & 7x_1 + x_2 + 5x_3 & & \\
\textbf{subject to} & x_1 \; 2 - x_2 \; 1 + 3x_3 \; 9 & \geq & 10 \\
 & 5x_1 + 2x_2 - x_3 & \geq & 6 \\
 & x_1, x_2, x_3 & \geq & 0
\end{array}
$$

Valid solution?

$x = (2, 1, 3)$

Linear Programming – Upper Bounds

Optimize (i.e., minimize or maximize) a linear (*objective*) function subject to linear inequalities (*constraints*).

minimize	$7x_1$		$+$	x_2		$+$	$5x_3$					
subject to	x_1	2	$-$	x_2	1	$+$	$3x_3$	9	$\geq$	10	10
	$5x_1$		$+$	$2x_2$		$-$	x_3		$\geq$	6	
							x_1, x_2, x_3		$\geq$	0	

Valid solution?

$x = (2, 1, 3)$

Linear Programming – Upper Bounds

Optimize (i.e., minimize or maximize) a linear (*objective*) function subject to linear inequalities (*constraints*).

$$
\begin{array}{llllllll}
\textbf{minimize} & 7x_1 & + & x_2 & + & 5x_3 & & \\
\textbf{subject to} & x_1 \; 2 & - & x_2 \; 1 & + & 3x_3 \; 9 & \geq & 10 \; 10 \\
 & 5x_1 \; 10 & + & 2x_2 \; 2 & - & x_3 \; 3 & \geq & 6 \; 9 \\
 & & & & & x_1, x_2, x_3 & \geq & 0
\end{array}
$$

Valid solution?

$x = (2, 1, 3)$

Linear Programming – Upper Bounds

Optimize (i.e., minimize or maximize) a linear (*objective*) function subject to linear inequalities (*constraints*).

$$
\begin{array}{llllll}
\textbf{minimize} & 7x_1\ 14 + & x_2 & + & 5x_3 & \\
\textbf{subject to} & x_1\ 2 - & x_2\ 1 & + & 3x_3\ 9 \geq & 10\ 10 \\
 & 5x_1\ 10 + & 2x_2\ 2 & - & x_3\ 3 \geq & 6\ 9 \\
 & & & & x_1, x_2, x_3 \geq & 0
\end{array}
$$

Valid solution?

$x = (2, 1, 3)$

Linear Programming – Upper Bounds

Optimize (i.e., minimize or maximize) a linear (*objective*) function subject to linear inequalities (*constraints*).

$$
\begin{array}{llllllll}
\textbf{minimize} & 7x_1\ 14 & + & x_2\ 1 & + & 5x_3 & & \\
\textbf{subject to} & x_1\ 2 & - & x_2\ 1 & + & 3x_3\ 9 & \geq & 10\ 10 \\
 & 5x_1\ 10 & + & 2x_2\ 2 & - & x_3\ 3 & \geq & 6\ 9 \\
 & & & & & x_1, x_2, x_3 & \geq & 0
\end{array}
$$

Valid solution?

$x = (2, 1, 3)$

Linear Programming – Upper Bounds

Optimize (i.e., minimize or maximize) a linear (*objective*) function subject to linear inequalities (*constraints*).

$$\begin{array}{lrcrcrcr} \textbf{minimize} & 7x_1\ 14 & + & x_2\ 1 & + & 5x_3\ 15 & & \\ \textbf{subject to} & x_1\ 2 & - & x_2\ 1 & + & 3x_3\ 9 & \geq & 10\ 10 \\ & 5x_1\ 10 & + & 2x_2\ 2 & - & x_3\ 3 & \geq & 6\ 9 \\ & & & & & x_1, x_2, x_3 & \geq & 0 \end{array}$$

Valid solution?

$x = (2, 1, 3)$

Linear Programming – Upper Bounds

Optimize (i.e., minimize or maximize) a linear (*objective*) function subject to linear inequalities (*constraints*).

minimize	$7x_1$	14	$+$	x_2	1	$+$	$5x_3$	15	$=$		30
subject to	x_1	2	$-$	x_2	1	$+$	$3x_3$	9	$\geq$	10	10
	$5x_1$	10	$+$	$2x_2$	2	$-$	x_3	3	$\geq$	6	9
							x_1, x_2, x_3		$\geq$	0	

Valid solution?

$x = (2, 1, 3)$

Linear Programming – Upper Bounds

Optimize (i.e., minimize or maximize) a linear (*objective*) function subject to linear inequalities (*constraints*).

$$
\begin{array}{lrrrrrl}
\textbf{minimize} & 7x_1\, 14 + & x_2\, 1 + & 5x_3\, 15 & = & & 30 \\
\textbf{subject to} & x_1\, 2 - & x_2\, 1 + & 3x_3\, 9 & \geq & 10 & 10 \\
 & 5x_1\, 10 + & 2x_2\, 2 - & x_3\, 3 & \geq & 6 & 9 \\
 & & & x_1, x_2, x_3 & \geq & 0 &
\end{array}
$$

Valid solution?

$x = (2, 1, 3)$

$\Rightarrow \mathrm{obj}(x) = 30$ is upper bound for OPT

Approximation Algorithms

Lecture 4:
Linear Programming and LP-Duality

Part III:
Lower Bounds for LPs

Linear Programming – Lower Bounds

Optimize (i.e., minimize or maximize) a linear (*objective*) function subject to linear inequalities (*constraints*).

$$
\begin{array}{lrcrcrcr}
\textbf{minimize} & 7x_1 & + & x_2 & + & 5x_3 & & \\
\textbf{subject to} & x_1 & - & x_2 & + & 3x_3 & \geq & 10 \\
 & 5x_1 & + & 2x_2 & - & x_3 & \geq & 6 \\
 & & & & & x_1, x_2, x_3 & \geq & 0
\end{array}
$$

Linear Programming – Lower Bounds

Optimize (i.e., minimize or maximize) a linear (*objective*) function subject to linear inequalities (*constraints*).

$$
\begin{array}{lrcrcrcr}
\textbf{minimize} & 7x_1 & + & x_2 & + & 5x_3 & & \\
 & \geq & & & & & & \\
\textbf{subject to} & x_1 & - & x_2 & + & 3x_3 & \geq & 10 \\
 & 5x_1 & + & 2x_2 & - & x_3 & \geq & 6 \\
 & & & & & x_1, x_2, x_3 & \geq & 0
\end{array}
$$

Linear Programming – Lower Bounds

Optimize (i.e., minimize or maximize) a linear (*objective*) function subject to linear inequalities (*constraints*).

$$
\begin{array}{lrcrcrcr}
\textbf{minimize} & 7x_1 & + & x_2 & + & 5x_3 & & \\
 & \geq & & \geq & & & & \\
\textbf{subject to} & x_1 & - & x_2 & + & 3x_3 & \geq & 10 \\
 & 5x_1 & + & 2x_2 & - & x_3 & \geq & 6 \\
 & & & & & x_1, x_2, x_3 & \geq & 0
\end{array}
$$

Linear Programming – Lower Bounds

Optimize (i.e., minimize or maximize) a linear (*objective*) function subject to linear inequalities (*constraints*).

$$
\begin{array}{lrcrcrcr}
\textbf{minimize} & 7x_1 & + & x_2 & + & 5x_3 & & \\
 & \rotatebox{-90}{$\geq$} & & \rotatebox{-90}{$\geq$} & & \rotatebox{-90}{$\geq$} & & \\
\textbf{subject to} & x_1 & - & x_2 & + & 3x_3 & \geq & 10 \\
 & 5x_1 & + & 2x_2 & - & x_3 & \geq & 6 \\
 & & & & & x_1, x_2, x_3 & \geq & 0
\end{array}
$$

Linear Programming – Lower Bounds

Optimize (i.e., minimize or maximize) a linear (*objective*) function subject to linear inequalities (*constraints*).

$$
\begin{array}{lrcrcrcr}
\textbf{minimize} & 7x_1 & + & x_2 & + & 5x_3 & & \\
 & \rotatebox{90}{$\leq$} & & \rotatebox{90}{$\leq$} & & \rotatebox{90}{$\leq$} & & \\
\textbf{subject to} & x_1 & - & x_2 & + & 3x_3 & \geq & 10 \\
 & 5x_1 & + & 2x_2 & - & x_3 & \geq & 6 \\
 & & & & & x_1, x_2, x_3 & \geq & 0
\end{array}
$$

$$7x_1 + x_2 + 5x_3 \geq x_1 - x_2 + 3x_3$$

Linear Programming – Lower Bounds

Optimize (i.e., minimize or maximize) a linear (*objective*) function subject to linear inequalities (*constraints*).

$$
\begin{array}{lrcrcrcr}
\textbf{minimize} & 7x_1 & + & x_2 & + & 5x_3 & & \\
 & \rotatebox{90}{$\leq$} & & \rotatebox{90}{$\leq$} & & \rotatebox{90}{$\leq$} & & \\
\textbf{subject to} & x_1 & - & x_2 & + & 3x_3 & \geq & 10 \\
 & 5x_1 & + & 2x_2 & - & x_3 & \geq & 6 \\
 & & & & & x_1, x_2, x_3 & \geq & 0
\end{array}
$$

$$7x_1 + x_2 + 5x_3 \geq x_1 - x_2 + 3x_3 \Rightarrow \text{OPT} \geq 10$$

Linear Programming – Lower Bounds

Optimize (i.e., minimize or maximize) a linear (*objective*) function subject to linear inequalities (*constraints*).

$$
\begin{array}{lrcrcrcr}
\textbf{minimize} & 7x_1 & + & x_2 & + & 5x_3 & & \\
& \text{\textbackslash rotatebox}{\geq} & & \geq & & \geq & & \\
\textbf{subject to} & x_1 & - & x_2 & + & 3x_3 & \geq & 10 \\
& + & & & & & & \\
& 5x_1 & + & 2x_2 & - & x_3 & \geq & 6 \\
& & & & & x_1, x_2, x_3 & \geq & 0
\end{array}
$$

$$7x_1 + x_2 + 5x_3 \;\geq\; x_1 - x_2 + 3x_3 \;\Rightarrow\; \text{OPT} \geq 10$$

Linear Programming – Lower Bounds

Optimize (i.e., minimize or maximize) a linear (*objective*) function subject to linear inequalities (*constraints*).

$$
\begin{array}{lrcrcrcr}
\textbf{minimize} & 7x_1 & + & x_2 & + & 5x_3 & & \\
 & \geq & & \geq & & \geq & & \\
\textbf{subject to} & x_1 & - & x_2 & + & 3x_3 & \geq & 10 \\
 & + & & + & & & & \\
 & 5x_1 & + & 2x_2 & - & x_3 & \geq & 6 \\
 & & & & & x_1, x_2, x_3 & \geq & 0
\end{array}
$$

$$7x_1 + x_2 + 5x_3 \geq x_1 - x_2 + 3x_3 \Rightarrow \text{OPT} \geq 10$$

Linear Programming – Lower Bounds

Optimize (i.e., minimize or maximize) a linear (*objective*) function subject to linear inequalities (*constraints*).

$$
\begin{array}{llllllll}
\textbf{minimize} & 7x_1 & + & x_2 & + & 5x_3 & & \\
 & \rotatebox{90}{$\le$} & & \rotatebox{90}{$\le$} & & \rotatebox{90}{$\le$} & & \\
\textbf{subject to} & x_1 & - & x_2 & + & 3x_3 & \geq & 10 \\
 & + & & + & & + & & \\
 & 5x_1 & + & 2x_2 & - & x_3 & \geq & 6 \\
 & & & & & x_1, x_2, x_3 & \geq & 0
\end{array}
$$

$$7x_1 + x_2 + 5x_3 \geq x_1 - x_2 + 3x_3 \quad \Rightarrow \text{OPT} \geq 10$$

Linear Programming – Lower Bounds

Optimize (i.e., minimize or maximize) a linear (*objective*) function subject to linear inequalities (*constraints*).

$$
\begin{array}{llcrcrcl}
\textbf{minimize} & 7x_1 & + & x_2 & + & 5x_3 & & \\
 & \rotatebox{90}{$\leq$} & & \rotatebox{90}{$\leq$} & & \rotatebox{90}{$\leq$} & & \\
\textbf{subject to} & x_1 & - & x_2 & + & 3x_3 & \geq & 10 \\
 & + & & + & & + & & \\
 & 5x_1 & + & 2x_2 & - & x_3 & \geq & 6 \\
 & & & & & x_1, x_2, x_3 & \geq & 0
\end{array}
$$

$$7x_1 + x_2 + 5x_3 \geq x_1 - x_2 + 3x_3 \Rightarrow \text{OPT} \geq 10$$

$$
\begin{aligned}
7x_1 + x_2 + 5x_3 &\geq (x_1 - x_2 + 3x_3) + (5x_1 + 2x_2 - x_3) \\
&\geq 10 + 6
\end{aligned}
$$

Linear Programming – Lower Bounds

Optimize (i.e., minimize or maximize) a linear (*objective*) function subject to linear inequalities (*constraints*).

$$
\begin{array}{lrcrcrcr}
\textbf{minimize} & 7x_1 & + & x_2 & + & 5x_3 & & \\
 & \rotatebox{90}{$\le$} & & \rotatebox{90}{$\le$} & & \rotatebox{90}{$\le$} & & \\
\textbf{subject to} & x_1 & - & x_2 & + & 3x_3 & \geq & 10 \\
 & + & & + & & + & & \\
 & 5x_1 & + & 2x_2 & - & x_3 & \geq & 6 \\
 & & & & & x_1, x_2, x_3 & \geq & 0
\end{array}
$$

$$7x_1 + x_2 + 5x_3 \geq x_1 - x_2 + 3x_3 \Rightarrow \text{OPT} \geq 10$$

$$
\begin{aligned}
7x_1 + x_2 + 5x_3 &\geq (x_1 - x_2 + 3x_3) + (5x_1 + 2x_2 - x_3) \\
&\geq 10 + 6 \qquad \Rightarrow \text{OPT} \geq 16
\end{aligned}
$$

Linear Programming – Lower Bounds

Optimize (i.e., minimize or maximize) a linear (*objective*) function subject to linear inequalities (*constraints*).

$$
\begin{array}{lrcrcrcr}
\textbf{minimize} & 7x_1 & + & x_2 & + & 5x_3 & & \\
 & \rotatebox{90}{$\le$} & & \rotatebox{90}{$\le$} & & \rotatebox{90}{$\le$} & & \\
\textbf{subject to} & 2\cdot\, x_1 & - & 2\cdot\, x_2 & + & 2\cdot 3x_3 & \geq & 2\cdot 10 \\
 & + & & + & & + & & \\
 & 5x_1 & + & 2x_2 & - & x_3 & \geq & 6 \\
 & & & & & x_1, x_2, x_3 & \geq & 0
\end{array}
$$

$$7x_1 + x_2 + 5x_3 \geq x_1 - x_2 + 3x_3 \quad \Rightarrow \text{OPT} \geq 10$$

$$
\begin{aligned}
7x_1 + x_2 + 5x_3 &\geq (x_1 - x_2 + 3x_3) + (5x_1 + 2x_2 - x_3) \\
&\geq 10 + 6 \qquad \Rightarrow \text{OPT} \geq 16
\end{aligned}
$$

Linear Programming – Lower Bounds

Optimize (i.e., minimize or maximize) a linear (*objective*) function subject to linear inequalities (*constraints*).

$$\begin{array}{lrcrcrcr}
\textbf{minimize} & 7x_1 & + & x_2 & + & 5x_3 & & \\
 & \text{IV} & & \text{IV} & & \text{IV} & & \\
\textbf{subject to} & 2\cdot\ x_1 & - & 2\cdot\ x_2 & + & 2\cdot 3x_3 & \geq & 2\cdot 10 \\
 & + & & + & & + & & \\
 & 5x_1 & + & 2x_2 & - & x_3 & \geq & 6 \\
 & & & & & x_1, x_2, x_3 & \geq & 0
\end{array}$$

$$7x_1 + x_2 + 5x_3 \geq x_1 - x_2 + 3x_3 \quad \Rightarrow \text{OPT} \geq 10$$

$$\begin{aligned}
7x_1 + x_2 + 5x_3 &\geq (x_1 - x_2 + 3x_3) + (5x_1 + 2x_2 - x_3) \\
&\geq 10 + 6 \quad \Rightarrow \text{OPT} \geq 16
\end{aligned}$$

$$\begin{aligned}
7x_1 + x_2 + 5x_3 &\geq 2 \cdot (x_1 - x_2 + 3x_3) + (5x_1 + 2x_2 - x_3) \\
&\geq 2 \cdot 10 + 6
\end{aligned}$$

Linear Programming – Lower Bounds

Optimize (i.e., minimize or maximize) a linear (*objective*) function subject to linear inequalities (*constraints*).

$$
\begin{array}{lrcrcrcr}
\textbf{minimize} & 7x_1 & + & x_2 & + & 5x_3 & & \\
 & \text{IV} & & \text{IV} & & \text{IV} & & \\
\textbf{subject to} & 2\cdot\ x_1 & - & 2\cdot\ x_2 & + & 2\cdot 3x_3 & \geq & 2\cdot 10 \\
 & + & & + & & + & & \\
 & 5x_1 & + & 2x_2 & - & x_3 & \geq & 6 \\
 & & & & & x_1, x_2, x_3 & \geq & 0
\end{array}
$$

$$7x_1 + x_2 + 5x_3 \geq x_1 - x_2 + 3x_3 \Rightarrow \text{OPT} \geq 10$$

$$
\begin{aligned}
7x_1 + x_2 + 5x_3 &\geq (x_1 - x_2 + 3x_3) + (5x_1 + 2x_2 - x_3) \\
&\geq 10 + 6 \Rightarrow \text{OPT} \geq 16
\end{aligned}
$$

$$
\begin{aligned}
7x_1 + x_2 + 5x_3 &\geq 2 \cdot (x_1 - x_2 + 3x_3) + (5x_1 + 2x_2 - x_3) \\
&\geq 2\cdot 10 + 6 \Rightarrow \text{OPT} \geq 26
\end{aligned}
$$

Linear Programming – Lower Bounds

$$\begin{array}{llcccccccc}
\textbf{minimize} & 7x_1 & + & x_2 & + & 5x_3 & & \\
\textbf{subject to} & 2\cdot\ x_1 & - & 2\cdot x_2 & + & 2\cdot 3x_3 & \geq & 2\cdot 10 \\
& 5x_1 & + & 2x_2 & - & x_3 & \geq & 6 \\
& & & & & x_1, x_2, x_3 & \geq & 0
\end{array}$$

$$\begin{aligned}
7x_1 + x_2 + 5x_3 &\geq 2 \cdot (x_1 - x_2 + 3x_3) + 1 \cdot (5x_1 + 2x_2 - x_3) \\
&\geq 2 \cdot 10 + 1 \cdot 6 \quad \Rightarrow \text{OPT} \geq 2 \cdot 10 + 1 \cdot 6
\end{aligned}$$

Linear Programming – Lower Bounds

$$\begin{array}{llcccccl} \textbf{minimize} & 7x_1 & + & x_2 & + & 5x_3 & & \\ \textbf{subject to}\ y_1(& x_1 & - & x_2 & + & 3x_3\ ) & \geq & 10\ y_1 \\ & 5x_1 & + & 2x_2 & - & x_3 & \geq & 6 \\ & & & & & x_1, x_2, x_3 & \geq & 0 \end{array}$$

$$7x_1 + x_2 + 5x_3 \geq 2 \cdot (x_1 - x_2 + 3x_3) + 1 \cdot (5x_1 + 2x_2 - x_3)$$

$$\geq 2 \cdot 10 + 1 \cdot 6 \quad \Rightarrow \text{OPT} \geq 2 \cdot 10 + 1 \cdot 6$$

Linear Programming – Lower Bounds

$$
\begin{array}{lrcrcrcl}
\textbf{minimize} & 7x_1 & + & x_2 & + & 5x_3 & & \\
\textbf{subject to}\ y_1(& x_1 & - & x_2 & + & 3x_3\ ) & \geq & 10\ y_1 \\
y_2(& 5x_1 & + & 2x_2 & - & x_3\ ) & \geq & 6\ y_2 \\
 & & & & & x_1, x_2, x_3 & \geq & 0
\end{array}
$$

$$
\begin{aligned}
7x_1 + x_2 + 5x_3 &\geq 2 \cdot (x_1 - x_2 + 3x_3) + 1 \cdot (5x_1 + 2x_2 - x_3) \\
&\geq 2 \cdot 10 + 1 \cdot 6 \quad \Rightarrow \text{OPT} \geq 2 \cdot 10 + 1 \cdot 6
\end{aligned}
$$

Linear Programming – Lower Bounds

$$\begin{array}{lrcrcrcl}
\textbf{minimize} & 7x_1 & + & x_2 & + & 5x_3 & & \\
\textbf{subject to}\ y_1(& x_1 & - & x_2 & + & 3x_3\ ) & \geq & 10\ y_1 \\
y_2(& 5x_1 & + & 2x_2 & - & x_3\ ) & \geq & 6\ y_2 \\
 & & & & & x_1, x_2, x_3 & \geq & 0
\end{array}$$

$$7x_1 + x_2 + 5x_3 \geq y_1 \cdot (x_1 - x_2 + 3x_3) + y_2 \cdot (5x_1 + 2x_2 - x_3)$$

$$\geq y_1 \cdot 10 + y_2 \cdot 6 \Rightarrow \text{OPT} \geq 10y_1 + 6y_2$$

Linear Programming – Lower Bounds

$$\begin{array}{llllllllll}
\textbf{minimize} & & 7x_1 & + & x_2 & + & 5x_3 & & & \\
\textbf{subject to} & y_1(& x_1 & - & x_2 & + & 3x_3 &) & \geq & 10\ y_1 \\
 & y_2(& 5x_1 & + & 2x_2 & - & x_3 &) & \geq & 6\ y_2 \\
 & & & & & & x_1, x_2, x_3 & & \geq & 0
\end{array}$$

$$\begin{aligned}
7x_1 + x_2 + 5x_3 &\geq y_1 \cdot (x_1 - x_2 + 3x_3) + y_2 \cdot (5x_1 + 2x_2 - x_3) \\
&\geq y_1 \cdot 10 + y_2 \cdot 6 \Rightarrow \text{OPT} \geq 10y_1 + 6y_2
\end{aligned}$$

$10y_1 + 6y_2$ is lower bound for OPT

Linear Programming – Lower Bounds

$$\begin{array}{rrrrrrrcl} \textbf{minimize} & 7x_1 & + & x_2 & + & 5x_3 & & & \\ \textbf{subject to}\ y_1(& x_1 & - & x_2 & + & 3x_3 &) & \geq & 10\ y_1 \\ y_2(& 5x_1 & + & 2x_2 & - & x_3 &) & \geq & 6\ y_2 \\ & & & & & x_1, x_2, x_3 & & \geq & 0 \end{array}$$

$$\begin{aligned} 7x_1 + x_2 + 5x_3 &\geq y_1 \cdot (x_1 - x_2 + 3x_3) + y_2 \cdot (5x_1 + 2x_2 - x_3) \\ &\geq y_1 \cdot 10 + y_2 \cdot 6 \Rightarrow \text{OPT} \geq 10y_1 + 6y_2 \end{aligned}$$

Bounds for y_1, y_2:

Linear Programming – Lower Bounds

$$\begin{array}{lrcrcrcrl}
\textbf{minimize} & & 7x_1 & + & x_2 & + & 5x_3 & & \\
& & \text{IV} & & \text{IV} & & \text{IV} & & \\
\textbf{subject to} & y_1(& x_1 & - & x_2 & + & 3x_3 &) \geq & 10 \; y_1 \\
& & + & & + & & + & & \\
& y_2(& 5x_1 & + & 2x_2 & - & x_3 &) \geq & 6 \; y_2 \\
& & & & & & x_1, x_2, x_3 & \geq & 0
\end{array}$$

$$\begin{aligned}
7x_1 + x_2 + 5x_3 &\geq y_1 \cdot (x_1 - x_2 + 3x_3) + y_2 \cdot (5x_1 + 2x_2 - x_3) \\
&\geq y_1 \cdot 10 + y_2 \cdot 6 \Rightarrow \text{OPT} \geq 10y_1 + 6y_2
\end{aligned}$$

Bounds for y_1, y_2:

Linear Programming – Lower Bounds

$$\begin{array}{lrcrcrcrl}
\textbf{minimize} & 7x_1 & + & x_2 & + & 5x_3 & & & \\
 & \text{IV} & & \text{IV} & & \text{IV} & & & \\
\textbf{subject to}\ y_1(& x_1 & - & x_2 & + & 3x_3 &) & \geq & 10\ y_1 \\
 & + & & + & & + & & & \\
y_2(& 5x_1 & + & 2x_2 & - & x_3 &) & \geq & 6\ y_2 \\
 & & & & x_1, & x_2, & x_3 & \geq & 0
\end{array}$$

$$7x_1 + x_2 + 5x_3 \geq y_1 \cdot (x_1 - x_2 + 3x_3) + y_2 \cdot (5x_1 + 2x_2 - x_3)$$

$$\geq y_1 \cdot 10 + y_2 \cdot 6 \Rightarrow \text{OPT} \geq 10y_1 + 6y_2$$

$$y_1 + 5y_2 \leq 7$$

Bounds for y_1, y_2:

Linear Programming – Lower Bounds

$$
\begin{array}{llcccccccc}
\textbf{minimize} & & 7x_1 & + & x_2 & + & 5x_3 & & & \\
& & \text{IV} & & \text{IV} & & \text{IV} & & & \\
\textbf{subject to} & y_1(& x_1 & - & x_2 & + & 3x_3 &) & \geq & 10\ y_1 \\
& & + & & + & & + & & & \\
& y_2(& 5x_1 & + & 2x_2 & - & x_3 &) & \geq & 6\ y_2 \\
& & & & & & x_1, x_2, x_3 & & \geq & 0
\end{array}
$$

$$
\begin{aligned}
7x_1 + x_2 + 5x_3 &\geq y_1 \cdot (x_1 - x_2 + 3x_3) + y_2 \cdot (5x_1 + 2x_2 - x_3) \\
&\geq y_1 \cdot 10 + y_2 \cdot 6 \Rightarrow \text{OPT} \geq 10y_1 + 6y_2
\end{aligned}
$$

Bounds for y_1, y_2:

$$
\begin{array}{rcrcl}
y_1 & + & 5y_2 & \leq & 7 \\
-y_1 & + & 2y_2 & \leq & 1
\end{array}
$$

Linear Programming – Lower Bounds

$$\begin{array}{rrcrcrcl}
\textbf{minimize} & 7x_1 & + & x_2 & + & 5x_3 & & \\
 & \text{IV} & & \text{IV} & & \text{IV} & & \\
\textbf{subject to} \; y_1(& x_1 & - & x_2 & + & 3x_3 \,) & \geq & 10 \; y_1 \\
 & + & & + & & + & & \\
y_2(& 5x_1 & + & 2x_2 & - & x_3 \,) & \geq & 6 \; y_2 \\
 & & & & & x_1, x_2, x_3 & \geq & 0
\end{array}$$

$$\begin{aligned}
7x_1 + x_2 + 5x_3 &\geq y_1 \cdot (x_1 - x_2 + 3x_3) + y_2 \cdot (5x_1 + 2x_2 - x_3) \\
&\geq y_1 \cdot 10 + y_2 \cdot 6 \Rightarrow \text{OPT} \geq 10y_1 + 6y_2
\end{aligned}$$

Bounds for y_1, y_2:

$$\begin{array}{rcrcl}
y_1 & + & 5y_2 & \leq & 7 \\
-y_1 & + & 2y_2 & \leq & 1 \\
3y_1 & - & y_2 & \leq & 5
\end{array}$$

Linear Programming – Lower Bounds

$$
\begin{array}{llrcrcrcl}
\textbf{minimize} & & 7x_1 & + & x_2 & + & 5x_3 & & \\
 & & \text{IV} & & \text{IV} & & \text{IV} & & \\
\textbf{subject to} & y_1(& x_1 & - & x_2 & + & 3x_3 &) \geq & 10\ y_1 \\
 & & + & & + & & + & & \\
 & y_2(& 5x_1 & + & 2x_2 & - & x_3 &) \geq & 6\ y_2 \\
 & & & & & & x_1, x_2, x_3 & \geq & 0
\end{array}
$$

$$
\begin{aligned}
7x_1 + x_2 + 5x_3 &\geq y_1 \cdot (x_1 - x_2 + 3x_3) + y_2 \cdot (5x_1 + 2x_2 - x_3) \\
&\geq y_1 \cdot 10 + y_2 \cdot 6 \Rightarrow \text{OPT} \geq 10y_1 + 6y_2
\end{aligned}
$$

maximize

Bounds for y_1, y_2:

$$
\begin{array}{rcrcl}
y_1 & + & 5y_2 & \leq & 7 \\
-y_1 & + & 2y_2 & \leq & 1 \\
3y_1 & - & y_2 & \leq & 5 \\
 & & y_1, y_2 & \geq & 0
\end{array}
$$

Linear Programming – Lower Bounds

$$
\begin{array}{llcccccccl}
\textbf{minimize} & & 7x_1 & + & x_2 & + & 5x_3 & & & \\
& & \text{IV} & & \text{IV} & & \text{IV} & & & \\
\textbf{subject to} & y_1(& x_1 & - & x_2 & + & 3x_3 &) & \geq & 10\ y_1 \\
& & + & & + & & + & & & \\
& y_2(& 5x_1 & + & 2x_2 & - & x_3 &) & \geq & 6\ y_2 \\
& & & & & & x_1, x_2, x_3 & & \geq & 0
\end{array}
$$

$$7x_1 + x_2 + 5x_3 \geq y_1 \cdot (x_1 - x_2 + 3x_3) + y_2 \cdot (5x_1 + 2x_2 - x_3)$$

$$\geq y_1 \cdot 10 + y_2 \cdot 6 \Rightarrow \text{OPT} \geq 10y_1 + 6y_2$$

$$
\begin{array}{lccccl}
\textbf{maximize} & 10y_1 & + & 6y_2 & & \\
\textbf{subject to} & y_1 & + & 5y_2 & \leq & 7 \\
& -y_1 & + & 2y_2 & \leq & 1 \\
& 3y_1 & - & y_2 & \leq & 5 \\
& & & y_1, y_2 & \geq & 0
\end{array}
$$

Linear Programming – Lower Bounds

Primal

$$\begin{array}{lrcrcrcr} \textbf{minimize} & 7x_1 & + & x_2 & + & 5x_3 & & \\ \textbf{subject to} & x_1 & - & x_2 & + & 3x_3 & \geq & 10 \\ & 5x_1 & + & 2x_2 & - & x_3 & \geq & 6 \\ & & & & & x_1, x_2, x_3 & \geq & 0 \end{array}$$

$$7x_1 + x_2 + 5x_3 \geq y_1 \cdot (x_1 - x_2 + 3x_3) + y_2 \cdot (5x_1 + 2x_2 - x_3)$$

$$\geq y_1 \cdot 10 + y_2 \cdot 6 \Rightarrow \text{OPT} \geq 10y_1 + 6y_2$$

$$\begin{array}{lrcrcr} \textbf{maximize} & 10y_1 & + & 6y_2 & & \\ \textbf{subject to} & y_1 & + & 5y_2 & \leq & 7 \\ & -y_1 & + & 2y_2 & \leq & 1 \\ & 3y_1 & - & y_2 & \leq & 5 \\ & & & y_1, y_2 & \geq & 0 \end{array}$$

Linear Programming – Lower Bounds

Primal

$$\begin{array}{llcrcrcrcr}
\textbf{minimize} & 7x_1 & + & x_2 & + & 5x_3 & & \\
\textbf{subject to} & x_1 & - & x_2 & + & 3x_3 & \geq & 10 \\
 & 5x_1 & + & 2x_2 & - & x_3 & \geq & 6 \\
 & & & & & x_1, x_2, x_3 & \geq & 0
\end{array}$$

$$7x_1 + x_2 + 5x_3 \geq y_1 \cdot (x_1 - x_2 + 3x_3) + y_2 \cdot (5x_1 + 2x_2 - x_3)$$

$$\geq y_1 \cdot 10 + y_2 \cdot 6 \Rightarrow \text{OPT} \geq 10y_1 + 6y_2$$

Dual

$$\begin{array}{llcrcr}
\textbf{maximize} & 10y_1 & + & 6y_2 & & \\
\textbf{subject to} & y_1 & + & 5y_2 & \leq & 7 \\
 & -y_1 & + & 2y_2 & \leq & 1 \\
 & 3y_1 & - & y_2 & \leq & 5 \\
 & & & y_1, y_2 & \geq & 0
\end{array}$$

Linear Programming – Lower Bounds

$$\begin{array}{lrcrcrcl} \textbf{minimize} & 7x_1 & + & x_2 & + & 5x_3 & & \\ \textbf{subject to} & x_1 & - & x_2 & + & 3x_3 & \geq & 10 \\ & 5x_1 & + & 2x_2 & - & x_3 & \geq & 6 \\ & & & & & x_1, x_2, x_3 & \geq & 0 \end{array}$$

Primal

$$7x_1 + x_2 + 5x_3 \geq y_1 \cdot (x_1 - x_2 + 3x_3) + y_2 \cdot (5x_1 + 2x_2 - x_3)$$

$$\geq y_1 \cdot 10 + y_2 \cdot 6 \Rightarrow \text{OPT} \geq 10y_1 + 6y_2$$

$$\begin{array}{lrcrcl} \textbf{maximize} & 10y_1 & + & 6y_2 & & \\ \textbf{subject to} & y_1 & + & 5y_2 & \leq & 7 \\ & -y_1 & + & 2y_2 & \leq & 1 \\ & 3y_1 & - & y_2 & \leq & 5 \\ & & & y_1, y_2 & \geq & 0 \end{array}$$

Dual

Any feasible solution to the **dual** program provides a lower bound for the optimum of the **primal** program.

Linear Programming – Lower Bounds

Primal

$$\begin{array}{llcrcrcll}
\textbf{minimize} & 7x_1 & + & x_2 & + & 5x_3 & & \\
\textbf{subject to} & x_1 & - & x_2 & + & 3x_3 & \geq & 10 \\
 & 5x_1 & + & 2x_2 & - & x_3 & \geq & 6 \\
 & & & & & x_1, x_2, x_3 & \geq & 0
\end{array}$$

$$\begin{aligned}
7x_1 + x_2 + 5x_3 &\geq y_1 \cdot (x_1 - x_2 + 3x_3) + y_2 \cdot (5x_1 + 2x_2 - x_3) \\
&\geq y_1 \cdot 10 + y_2 \cdot 6 \Rightarrow \text{OPT} \geq 10y_1 + 6y_2
\end{aligned}$$

Dual

$$\begin{array}{llcrcl}
\textbf{maximize} & 10y_1 & + & 6y_2 & & \\
\textbf{subject to} & y_1 & + & 5y_2 & \leq & 7 \\
 & -y_1 & + & 2y_2 & \leq & 1 \\
 & 3y_1 & - & y_2 & \leq & 5 \\
 & & & y_1, y_2 & \geq & 0
\end{array}$$

Any feasible solution to the **dual** program provides a lower bound for the optimum of the **primal** program.

Both $x = (\frac{7}{4}, 0, \frac{11}{4})$ and $y = (2, 1)$ provide objective value 26.

Linear Programming – Lower Bounds

Primal

$$\begin{array}{llcr} \textbf{minimize} & 7x_1 + x_2 + 5x_3 & & \\ \textbf{subject to} & x_1 - x_2 + 3x_3 & \geq & 10 \\ & 5x_1 + 2x_2 - x_3 & \geq & 6 \\ & x_1, x_2, x_3 & \geq & 0 \end{array}$$

$$7x_1 + x_2 + 5x_3 \geq y_1 \cdot (x_1 - x_2 + 3x_3) + y_2 \cdot (5x_1 + 2x_2 - x_3)$$

$$\geq y_1 \cdot 10 + y_2 \cdot 6 \Rightarrow \text{OPT} \geq 10y_1 + 6y_2$$

Dual

$$\begin{array}{llcr} \textbf{maximize} & 10y_1 + 6y_2 & & \\ \textbf{subject to} & y_1 + 5y_2 & \leq & 7 \\ & -y_1 + 2y_2 & \leq & 1 \\ & 3y_1 - y_2 & \leq & 5 \\ & y_1, y_2 & \geq & 0 \end{array}$$

Any feasible solution to the **dual** program provides a lower bound for the optimum of the **primal** program.

Both $x = (\frac{7}{4}, 0, \frac{11}{4})$ and $y = (2, 1)$ provide objective value 26. = OPT

Primal–Dual

primal program

$$
\begin{array}{lrcl}
\textbf{minimize} & c^{\mathsf{T}} x & & \\
\textbf{subject to} & Ax & \geq & b \\
 & x & \geq & 0
\end{array}
$$

Primal–Dual

primal program

$$\begin{array}{lrcl} \textbf{minimize} & c^{\mathsf{T}}x & & \\ \textbf{subject to} & Ax & \geq & b \\ & x & \geq & 0 \end{array}$$

dual program

$$\begin{array}{lrcl} \textbf{maximize} & b^{\mathsf{T}}y & & \\ \textbf{subject to} & A^{\mathsf{T}}y & \leq & c \\ & y & \geq & 0 \end{array}$$

Primal–Dual

primal program

$$\begin{array}{lrcl} \textbf{minimize} & c^\intercal x & & \\ \textbf{subject to} & Ax & \geq & b \\ & x & \geq & 0 \end{array}$$

dual program

$$\begin{array}{lrcl} \textbf{maximize} & b^\intercal y & & \\ \textbf{subject to} & A^\intercal y & \leq & c \\ & y & \geq & 0 \end{array}$$

dual of the dual program

Primal–Dual

primal program

$$\begin{array}{lrcl} \textbf{minimize} & c^{\mathsf{T}} x & & \\ \textbf{subject to} & Ax & \geq & b \\ & x & \geq & 0 \end{array}$$

dual program

$$\begin{array}{lrcl} \textbf{maximize} & b^{\mathsf{T}} y & & \\ \textbf{subject to} & A^{\mathsf{T}} y & \leq & c \\ & y & \geq & 0 \end{array}$$

dual of the dual program

$$\begin{array}{lrcl} \textbf{minimize} & c^{\mathsf{T}} x & & \\ \textbf{subject to} & Ax & \geq & b \\ & x & \geq & 0 \end{array}$$

Approximation Algorithms

Lecture 4:
Linear Programming and LP-Duality

Part IV:
LP-Duality and Complementary Slackness

LP-Duality

Primal

$$\begin{array}{llll} \textbf{minimize} & c^\intercal x & & \\ \textbf{subject to} & Ax & \geq & b \\ & x & \geq & 0 \end{array}$$

Dual

$$\begin{array}{llll} \textbf{maximize} & b^\intercal y & & \\ \textbf{subject to} & A^\intercal y & \leq & c \\ & y & \geq & 0 \end{array}$$

LP-Duality

Primal

$$\begin{array}{lrcl} \textbf{minimize} & c^\intercal x & & \\ \textbf{subject to} & Ax & \geq & b \\ & x & \geq & 0 \end{array}$$

Dual

$$\begin{array}{lrcl} \textbf{maximize} & b^\intercal y & & \\ \textbf{subject to} & A^\intercal y & \leq & c \\ & y & \geq & 0 \end{array}$$

Theorem. The primal program has a finite optimum
$\Leftrightarrow$ the dual program has a finite optimum.

LP-Duality

$$\begin{array}{llll} \textbf{minimize} & c^\intercal x & & \\ \textbf{subject to} & Ax & \geq & b \\ & x & \geq & 0 \end{array}$$

Primal

$$\begin{array}{llll} \textbf{maximize} & b^\intercal y & & \\ \textbf{subject to} & A^\intercal y & \leq & c \\ & y & \geq & 0 \end{array}$$

Dual

Theorem. The primal program has a finite optimum $\Leftrightarrow$ the dual program has a finite optimum. Moreover, if $x^* = (x_1^*, \ldots, x_n^*)$ and $y^* = (y_1^*, \ldots, y_m^*)$ are *optimal* solutions for the primal and dual program, respectively, then

LP-Duality

$$\begin{array}{llll} \textbf{minimize} & c^\intercal x & & \\ \textbf{subject to} & Ax & \geq & b \\ & x & \geq & 0 \end{array}$$

Primal

$$\begin{array}{llll} \textbf{maximize} & b^\intercal y & & \\ \textbf{subject to} & A^\intercal y & \leq & c \\ & y & \geq & 0 \end{array}$$

Dual

Theorem. The primal program has a finite optimum $\Leftrightarrow$ the dual program has a finite optimum. Moreover, if $x^* = (x_1^*, \ldots, x_n^*)$ and $y^* = (y_1^*, \ldots, y_m^*)$ are *optimal* solutions for the primal and dual program, respectively, then

$$\sum_{j=1}^{n} c_j x_j^* = \sum_{i=1}^{m} b_i y_i^* .$$

Weak LP-Duality

minimize	$c^{\mathsf{T}} x$		
subject to	Ax	$\geq$	b
	x	$\geq$	0

maximize	$b^{\mathsf{T}} y$		
subject to	$A^{\mathsf{T}} y$	$\leq$	c
	y	$\geq$	0

Theorem. If $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_m)$ are *valid* solutions for the primal and dual program, resp., then

Weak LP-Duality

minimize	$c^\intercal x$		
subject to	Ax	$\geq$	b
	x	$\geq$	0

maximize	$b^\intercal y$		
subject to	$A^\intercal y$	$\leq$	c
	y	$\geq$	0

Theorem. If $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_m)$ are *valid* solutions for the primal and dual program, resp., then

$$\sum_{j=1}^{n} c_j x_j \geq \sum_{i=1}^{m} b_i y_i \,.$$

Weak LP-Duality

minimize	$c^\intercal x$		
subject to	Ax	$\geq$	b
	x	$\geq$	0

maximize	$b^\intercal y$		
subject to	$A^\intercal y$	$\leq$	c
	y	$\geq$	0

Theorem. If $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_m)$ are *valid* solutions for the primal and dual program, resp., then

$$\sum_{j=1}^{n} c_j x_j \geq \sum_{i=1}^{m} b_i y_i \,.$$

Proof.

Weak LP-Duality

minimize	$c^\intercal x$		
subject to	Ax	$\geq$	b
	x	$\geq$	0

maximize	$b^\intercal y$		
subject to	$A^\intercal y$	$\leq$	c
	y	$\geq$	0

Theorem. If $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_m)$ are *valid* solutions for the primal and dual program, resp., then

$$\sum_{j=1}^{n} c_j x_j \geq \sum_{i=1}^{m} b_i y_i \,.$$

Proof.

$$\sum_{j=1}^{n} c_j x_j \geq \qquad \sum_{i=1}^{m} b_i y_i \,.$$

Weak LP-Duality

minimize	$c^\intercal x$		
subject to	Ax	$\geq$	b
	x	$\geq$	0

maximize	$b^\intercal y$		
subject to	$A^\intercal y$	$\leq$	c
	y	$\geq$	0

Theorem. If $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_m)$ are *valid* solutions for the primal and dual program, resp., then

$$\sum_{j=1}^{n} c_j x_j \geq \sum_{i=1}^{m} b_i y_i \, .$$

Proof.

$$\sum_{j=1}^{n} c_j x_j \geq \qquad \sum_{i=1}^{m} b_i y_i \, .$$

Weak LP-Duality

minimize	$c^\intercal x$		
subject to	Ax	$\geq$	b
	x	$\geq$	0

maximize	$b^\intercal y$		
subject to	$A^\intercal y$	$\leq$	c
	y	$\geq$	0

Theorem. If $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_m)$ are *valid* solutions for the primal and dual program, resp., then

$$\sum_{j=1}^{n} c_j x_j \geq \sum_{i=1}^{m} b_i y_i \, .$$

Proof.

$$\sum_{j=1}^{n} c_j x_j \geq \qquad \sum_{i=1}^{m} b_i y_i \, .$$

Weak LP-Duality

minimize	$c^\intercal x$		
subject to	Ax	$\geq$	b
	x	$\geq$	0

maximize	$b^\intercal y$		
subject to	$A^\intercal y$	$\leq$	c
	y	$\geq$	0

Theorem. If $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_m)$ are *valid* solutions for the primal and dual program, resp., then

$$\sum_{j=1}^{n} c_j x_j \geq \sum_{i=1}^{m} b_i y_i \, .$$

Proof.

$$\sum_{j=1}^{n} c_j x_j \geq \qquad \sum_{i=1}^{m} b_i y_i \, .$$

Weak LP-Duality

minimize	$c^\mathsf{T} x$		
subject to	Ax	$\geq$	b
	x	$\geq$	0

maximize	$b^\mathsf{T} y$		
subject to	$A^\mathsf{T} y$	$\leq$	c
	y	$\geq$	0

Theorem. If $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_m)$ are *valid* solutions for the primal and dual program, resp., then

$$\sum_{j=1}^{n} c_j x_j \geq \sum_{i=1}^{m} b_i y_i \,.$$

Proof.

$$\sum_{j=1}^{n} c_j x_j \overset{\geq}{\geq} \sum_{j=1}^{n} \left(\sum_{i=1}^{m} a_{ij} y_i \right) x_j = \qquad \geq \sum_{i=1}^{m} b_i y_i \,.$$

Weak LP-Duality

minimize	$c^\top x$		
subject to	Ax	$\geq$	b
	x	$\geq$	0

maximize	$b^\top y$		
subject to	$A^\top y$	$\leq$	c
	y	$\geq$	0

Theorem. If $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_m)$ are *valid* solutions for the primal and dual program, resp., then

$$\sum_{j=1}^{n} c_j x_j \geq \sum_{i=1}^{m} b_i y_i \,.$$

Proof.

$$\sum_{j=1}^{n} c_j x_j \geq \sum_{j=1}^{n} \left(\sum_{i=1}^{m} a_{ij} y_i \right) x_j = \sum_{j=1}^{n} \left(\sum_{i=1}^{m} a_{ij} y_i x_j \right) \geq \sum_{i=1}^{m} b_i y_i \,.$$

Weak LP-Duality

minimize	$c^\intercal x$		
subject to	Ax	$\geq$	b
	x	$\geq$	0

maximize	$b^\intercal y$		
subject to	$A^\intercal y$	$\leq$	c
	y	$\geq$	0

Theorem. If $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_m)$ are *valid* solutions for the primal and dual program, resp., then

$$\sum_{j=1}^{n} c_j x_j \geq \sum_{i=1}^{m} b_i y_i \,.$$

Proof.

$$\sum_{j=1}^{n} c_j x_j \overset{\geq}{\geq} \sum_{j=1}^{n} \left(\sum_{i=1}^{m} a_{ij} y_i \right) x_j = \sum_{i=1}^{m} \left(\sum_{j=1}^{n} a_{ij} y_i x_j \right) \geq \sum_{i=1}^{m} b_i y_i \,.$$

Weak LP-Duality

minimize	$c^\intercal x$		
subject to	Ax	$\geq$	b
	x	$\geq$	0

maximize	$b^\intercal y$		
subject to	$A^\intercal y$	$\leq$	c
	y	$\geq$	0

Theorem. If $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_m)$ are *valid* solutions for the primal and dual program, resp., then

$$\sum_{j=1}^{n} c_j x_j \geq \sum_{i=1}^{m} b_i y_i \,.$$

Proof.

$$\sum_{j=1}^{n} c_j x_j \geq \sum_{j=1}^{n} \left(\sum_{i=1}^{m} a_{ij} y_i \right) x_j = \sum_{i=1}^{m} \left(\sum_{j=1}^{n} a_{ij} x_j \right) y_i \geq \sum_{i=1}^{m} b_i y_i \,.$$

Weak LP-Duality

minimize	$c^\intercal x$		
subject to	Ax	$\geq$	b
	x	$\geq$	0

maximize	$b^\intercal y$		
subject to	$A^\intercal y$	$\leq$	c
	y	$\geq$	0

Theorem. If $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_m)$ are *valid* solutions for the primal and dual program, resp., then

$$\sum_{j=1}^{n} c_j x_j \geq \sum_{i=1}^{m} b_i y_i \, .$$

Proof.

$$\sum_{j=1}^{n} c_j x_j \geq \sum_{j=1}^{n} \left(\sum_{i=1}^{m} a_{ij} y_i \right) x_j = \sum_{i=1}^{m} \left(\sum_{j=1}^{n} a_{ij} x_j \right) y_i \geq \sum_{i=1}^{m} b_i y_i \, .$$

Weak LP-Duality

minimize	$c^\top x$		
subject to	Ax	$\geq$	b
	x	$\geq$	0

maximize	$b^\top y$		
subject to	$A^\top y$	$\leq$	c
	y	$\geq$	0

Theorem. If $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_m)$ are *valid* solutions for the primal and dual program, resp., then

$$\sum_{j=1}^{n} c_j x_j \geq \sum_{i=1}^{m} b_i y_i \,.$$

Proof.

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Weak LP-Duality

minimize	$c^\mathsf{T} x$
subject to	$Ax \geq b$
	$x \geq 0$

maximize	$b^\mathsf{T} y$
subject to	$A^\mathsf{T} y \leq c$
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Complementary Slackness

$$\begin{array}{llll} \textbf{minimize} & c^\intercal x & & \\ \textbf{subject to} & Ax & \geq & b \\ & x & \geq & 0 \end{array}$$

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The feasible solutions of an LP with n variables form a **convex polytope** in $\mathbb{R}^n$ (intersection of halfspaces).

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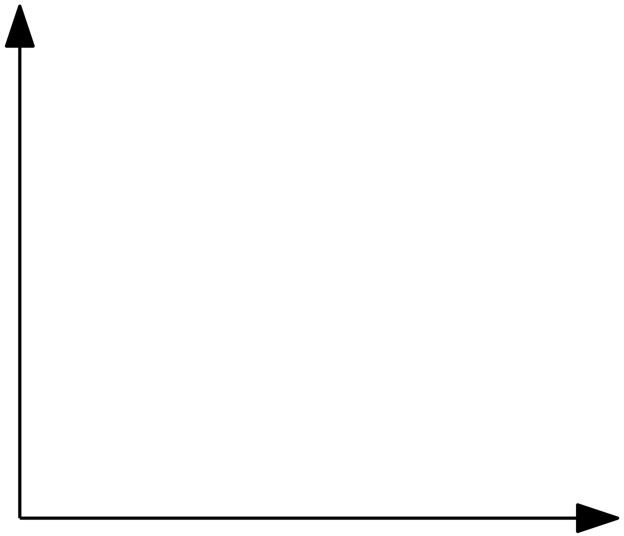

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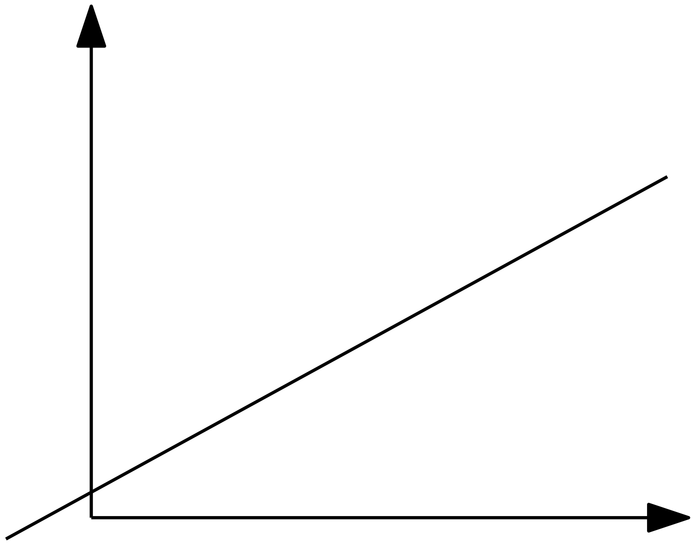

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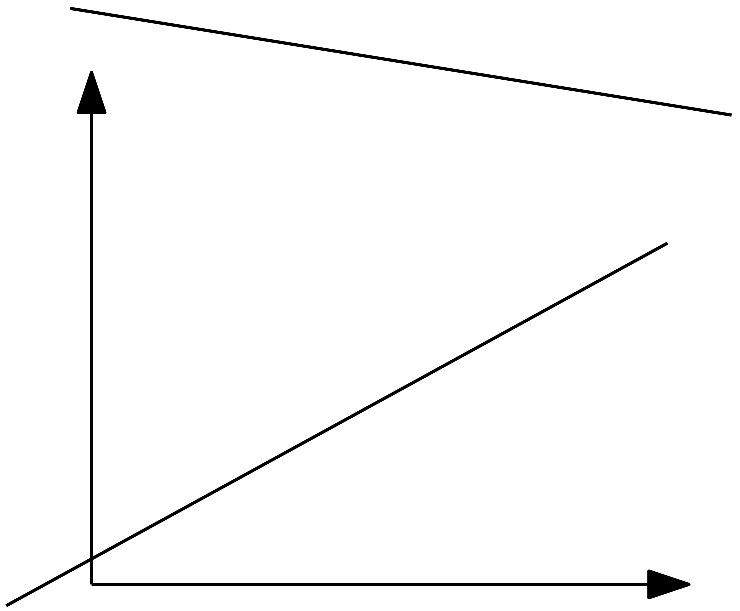

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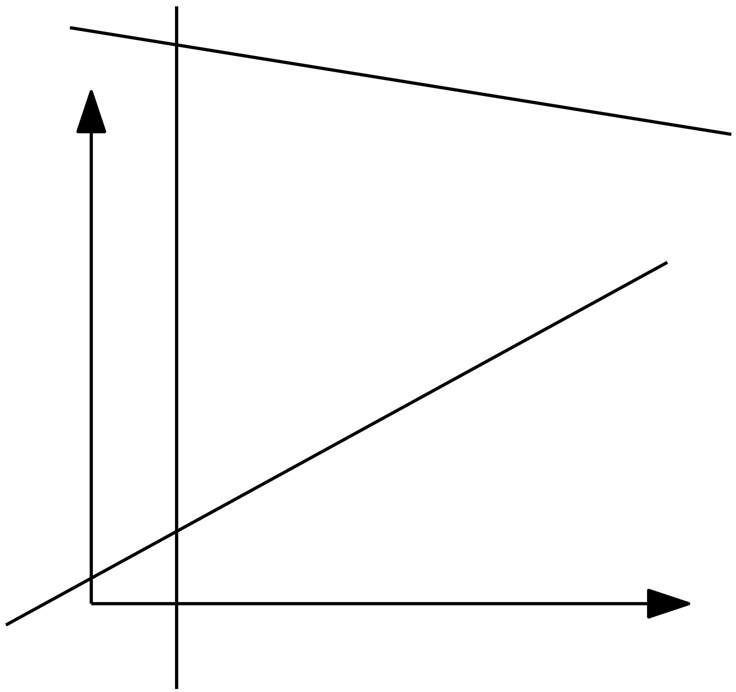

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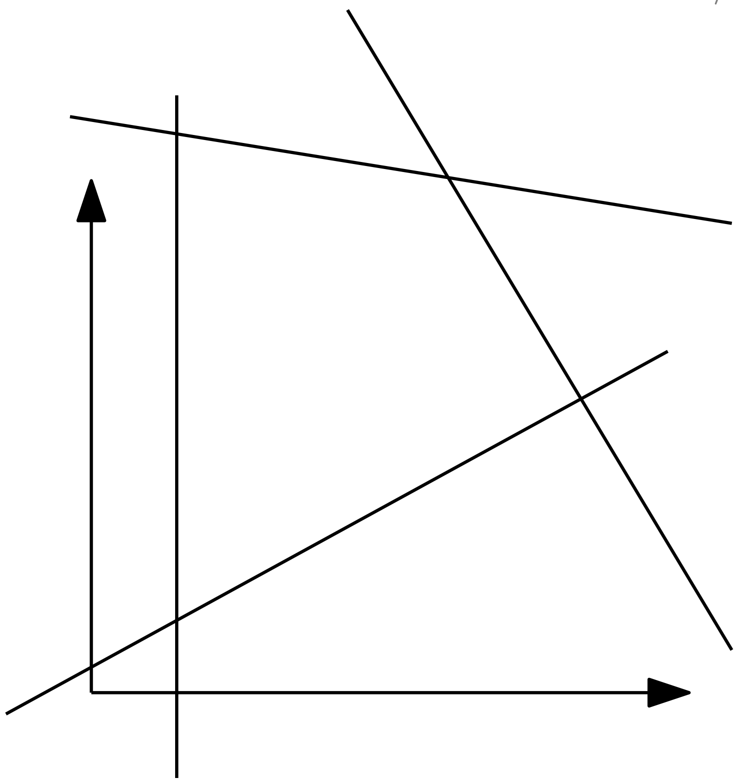

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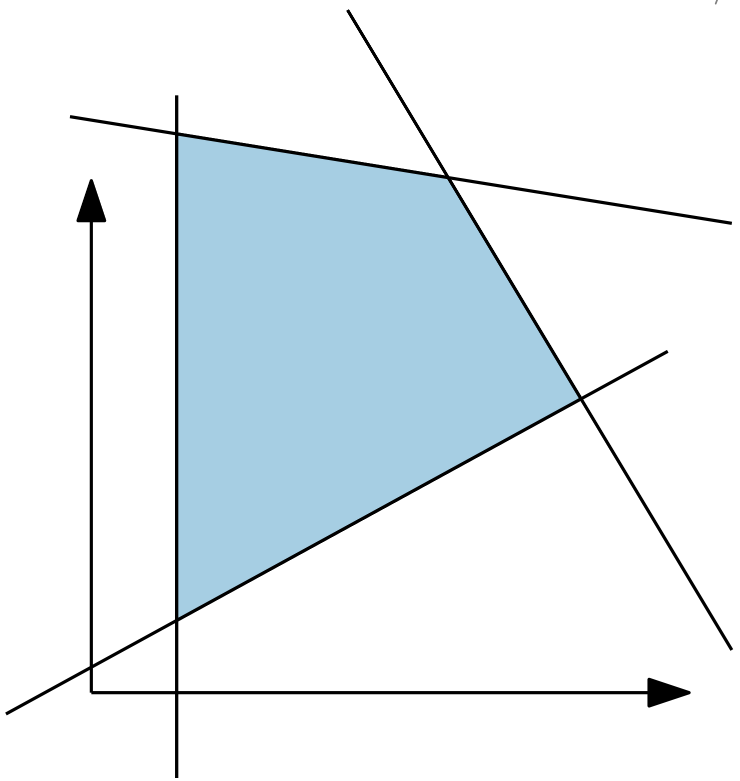

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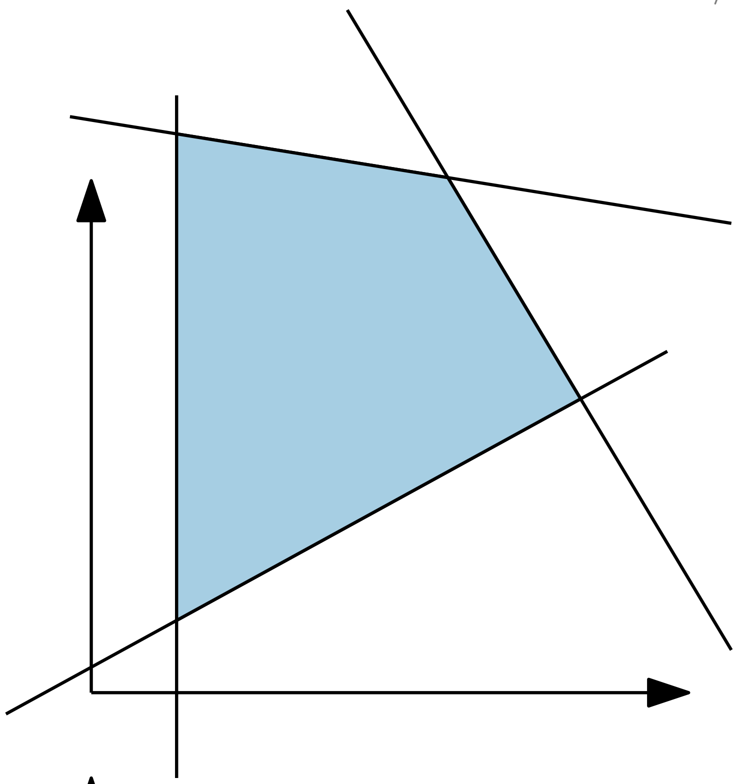

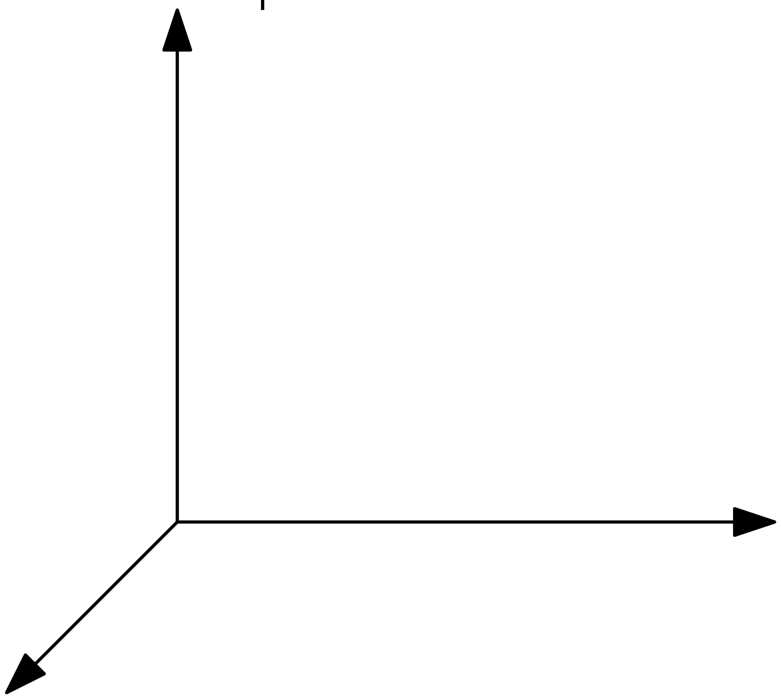

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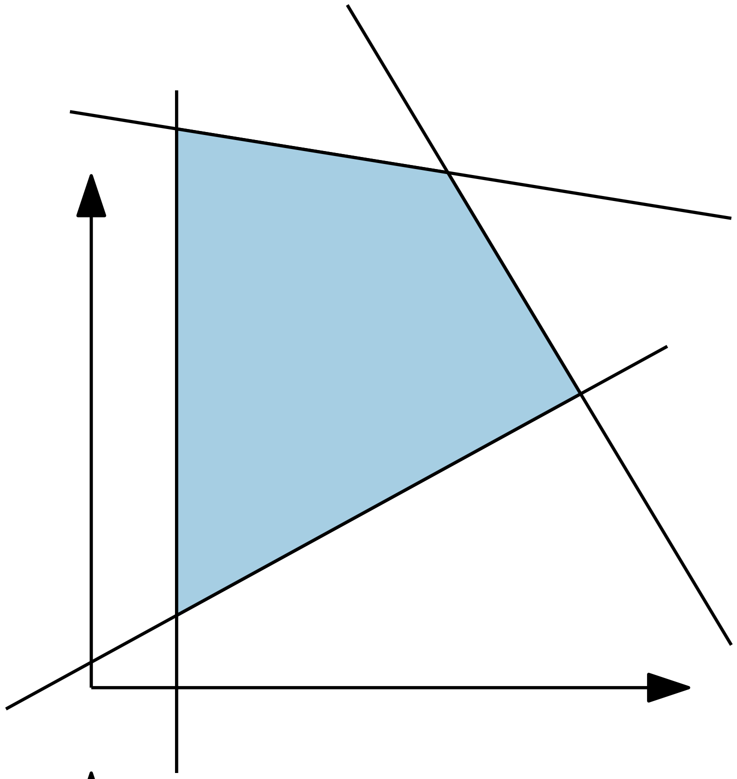

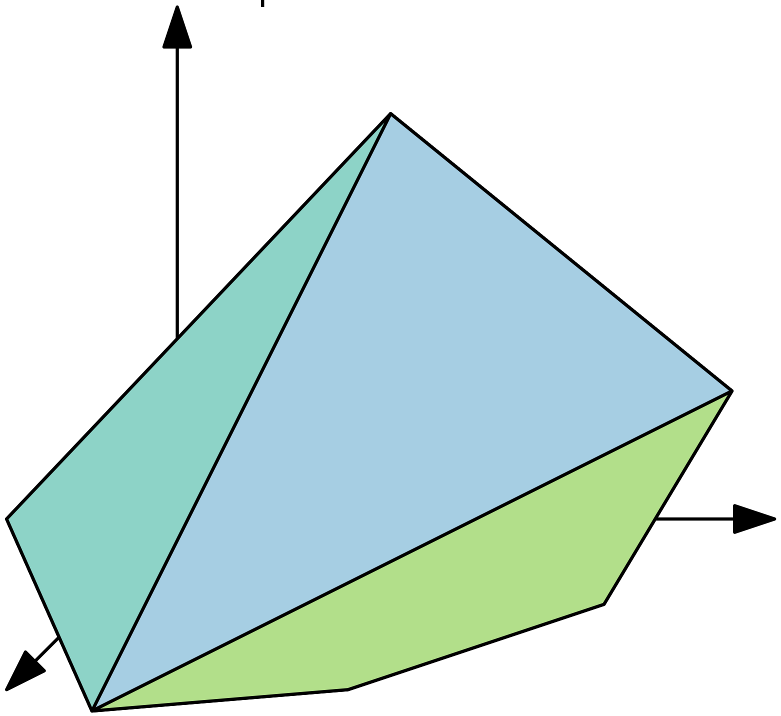

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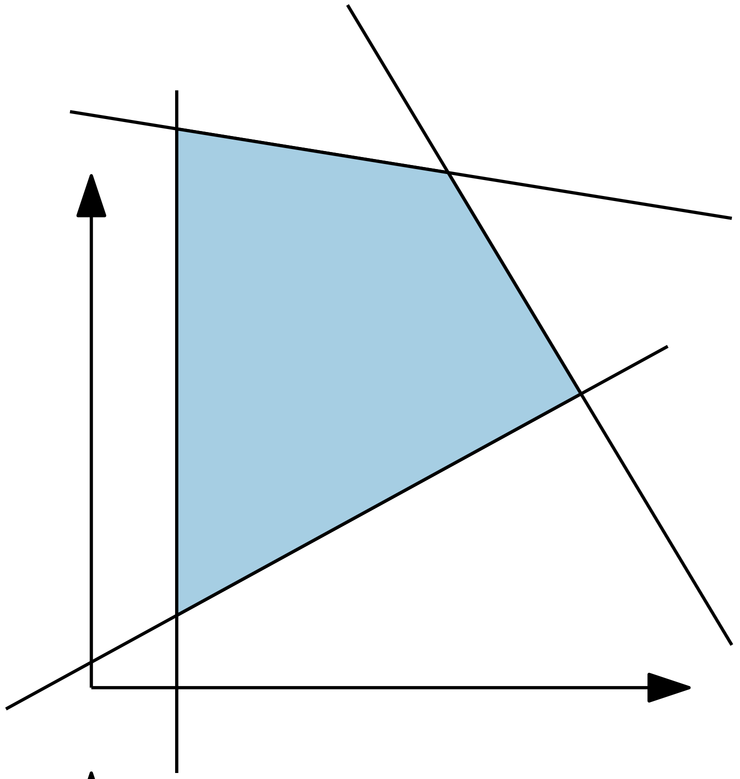
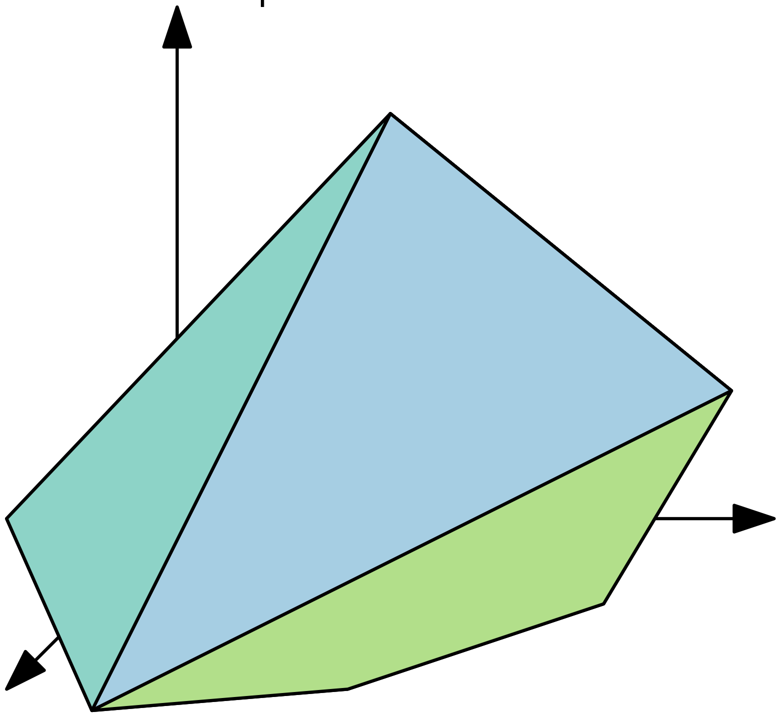
The feasible solutions of an LP with n variables form a **convex polytope** in $\mathbb{R}^n$ (intersection of halfspaces).

Corners of the polytope are called **extreme point solutions** $\Leftrightarrow$ n linearly independent inequalities (constraints) are satisfied with equality.





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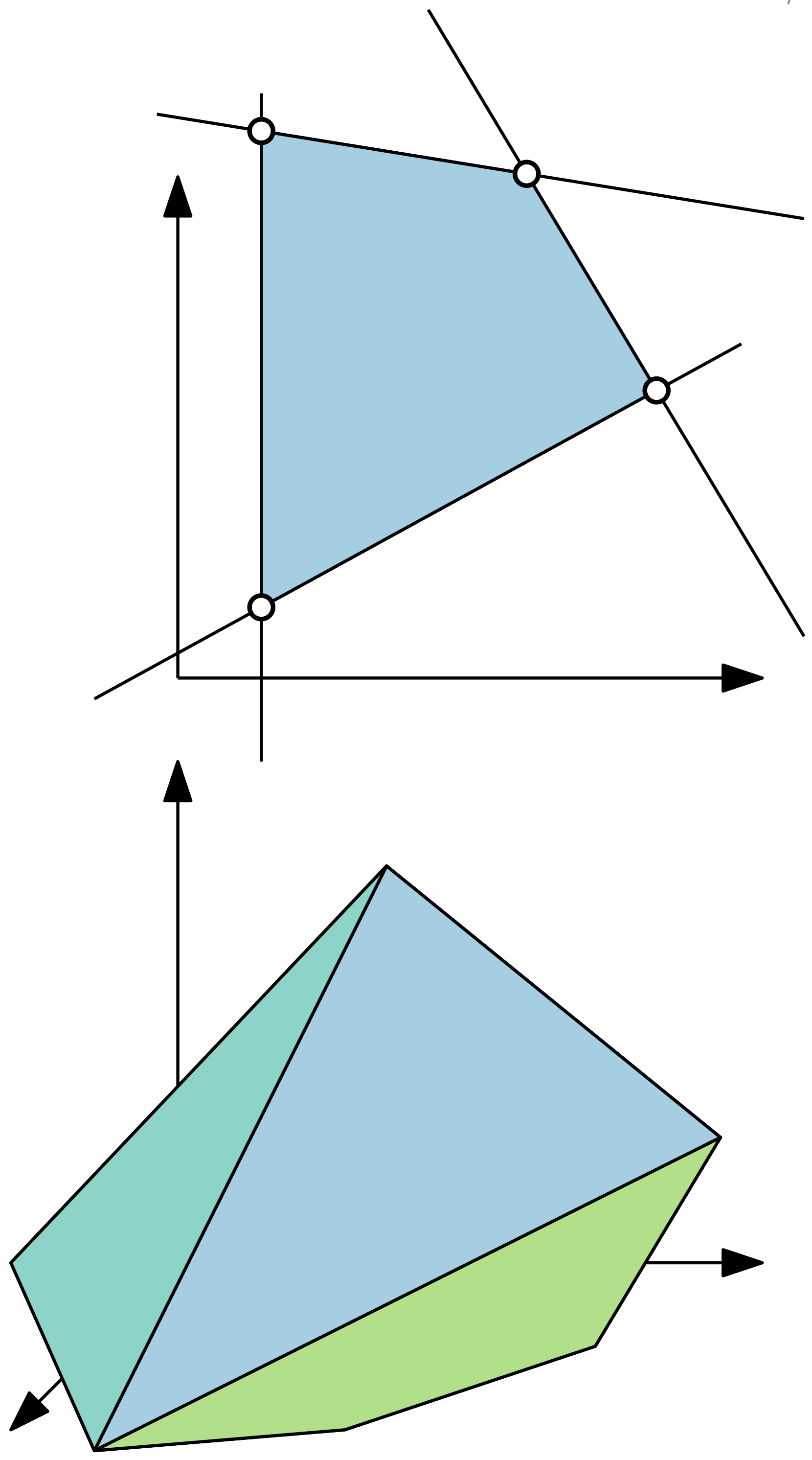

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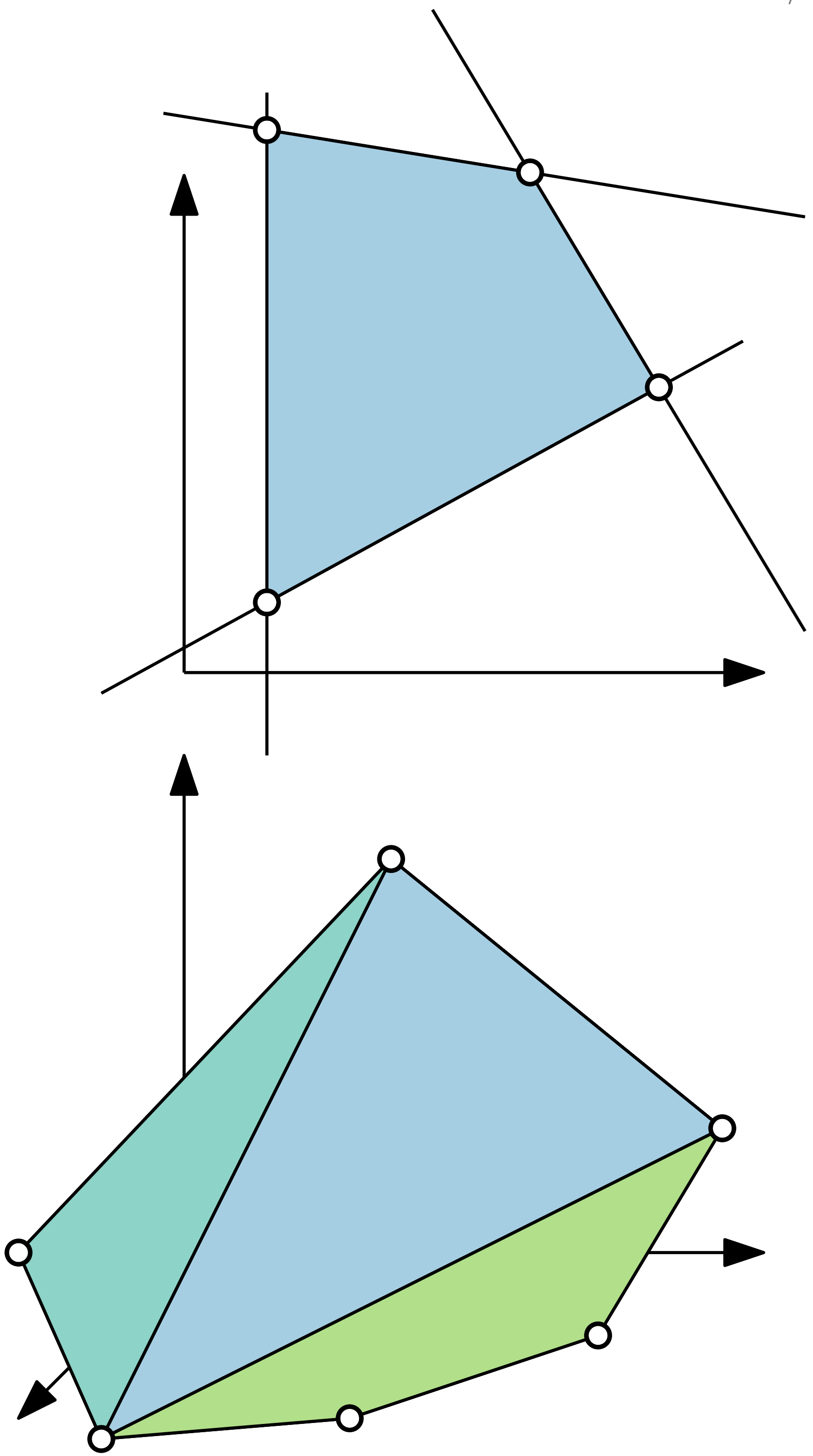

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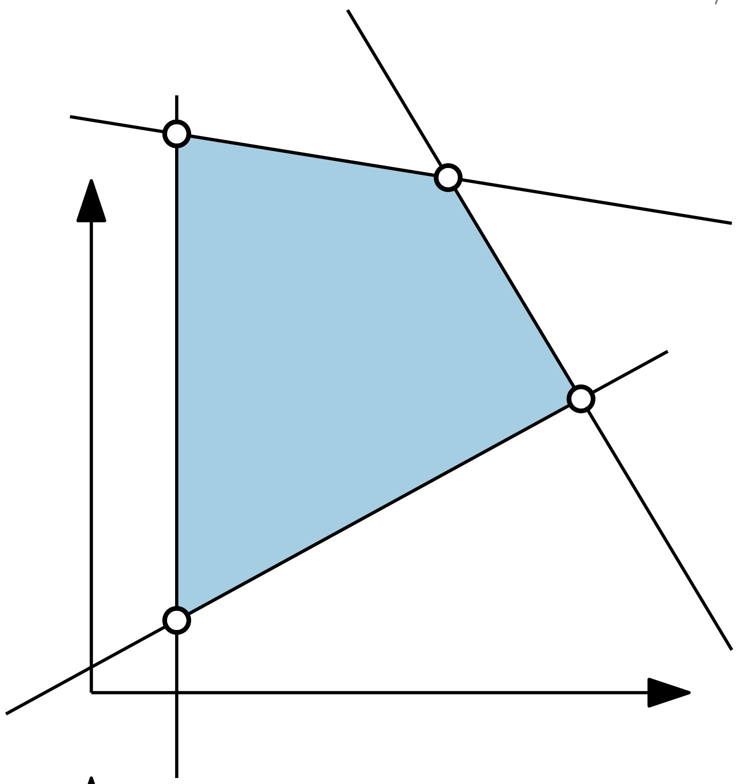

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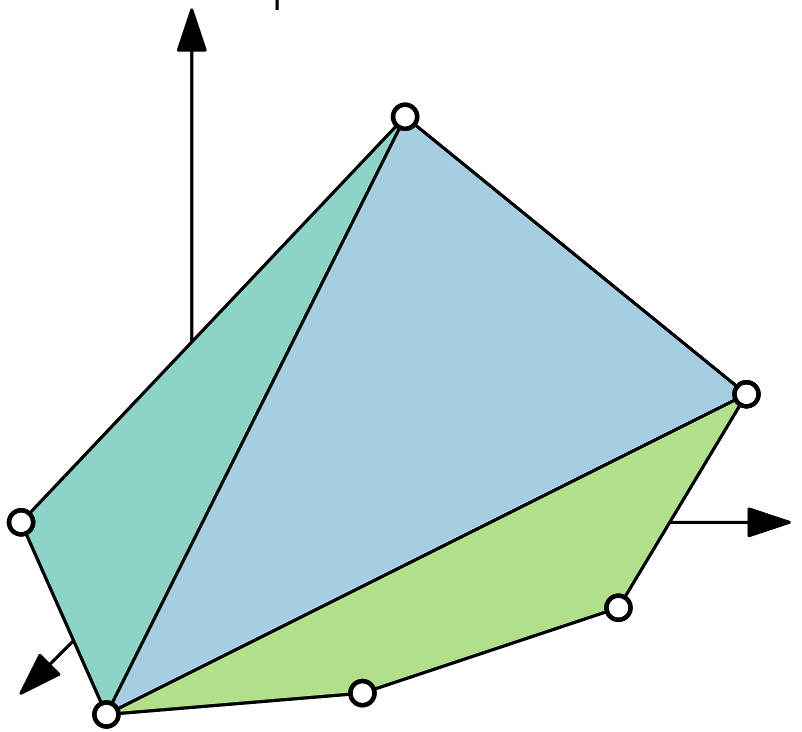

If an optimal solution exists, some extreme point is also optimal.

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LP-relaxation provides a lower bound: $\mathrm{OPT}_{\mathrm{LP}} \leq \mathrm{OPT}_{\mathrm{ILP}}$

Approximation Algorithms

Lecture 4:
Linear Programming and LP-Duality

Part V:
Min–Max Relationships

Max-Flow Problem

Given: A directed graph $G = (V, E)$ with edge capacities $c\colon E \to \mathbb{Q}_+$ and two special vertices: the source s and sink t.

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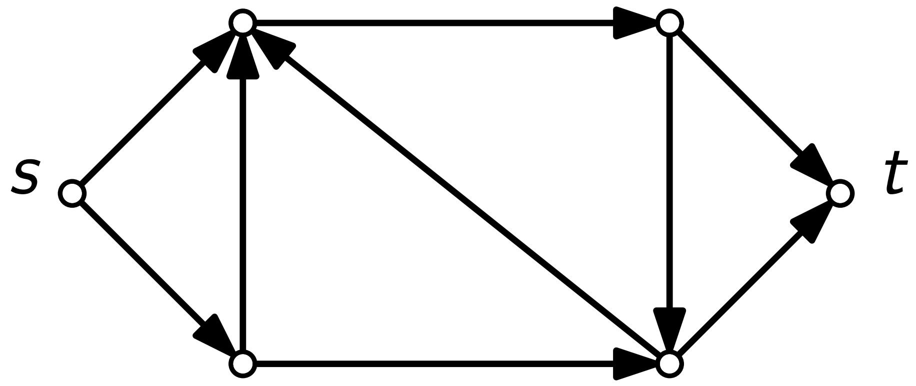

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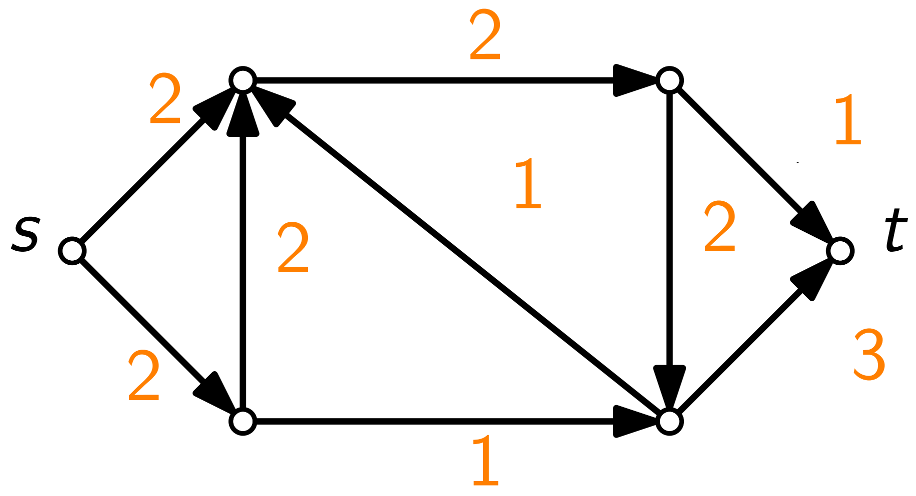

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Given: A directed graph $G = (V, E)$ with edge capacities $c\colon E \to \mathbb{Q}_+$ and two special vertices: the source s and sink t.

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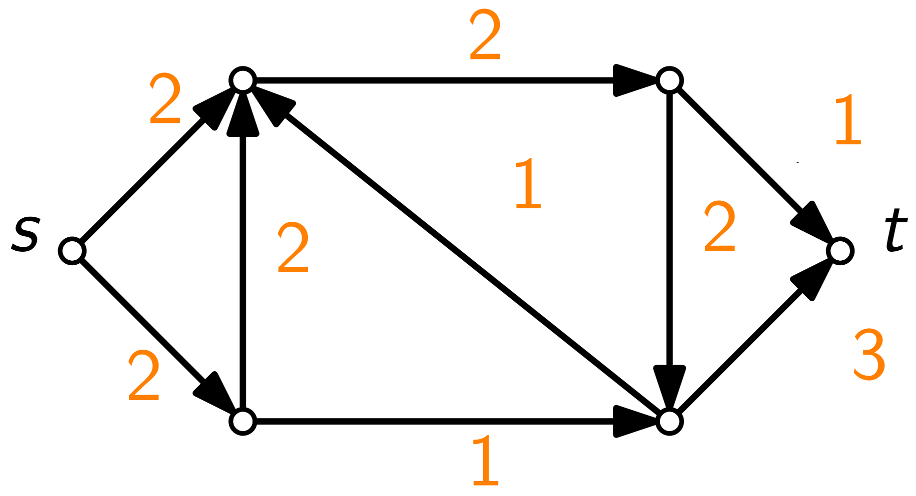

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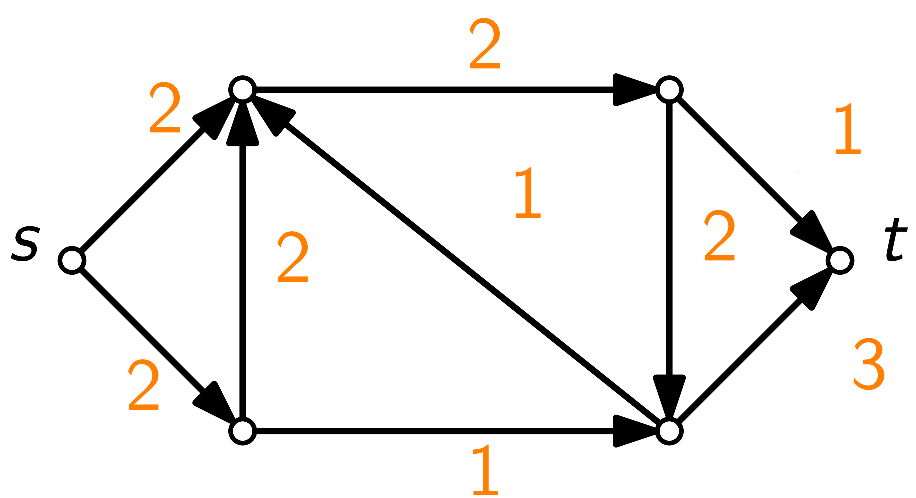

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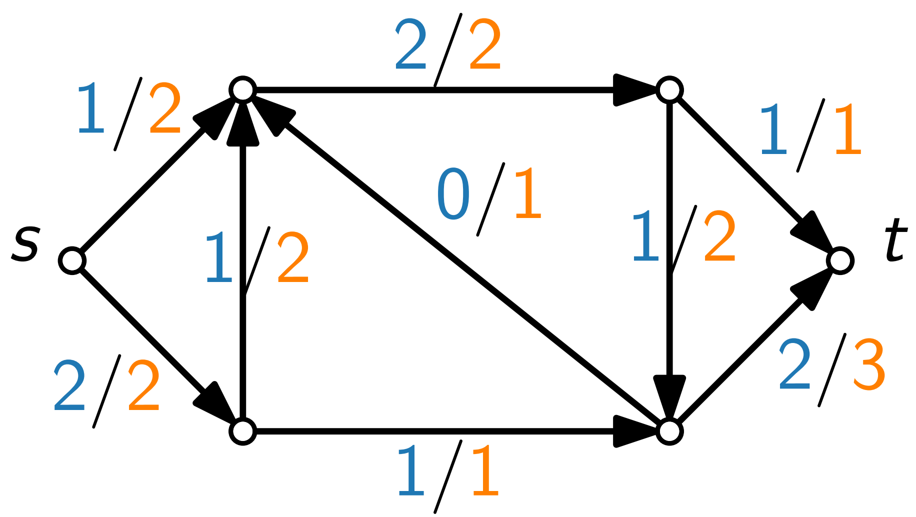

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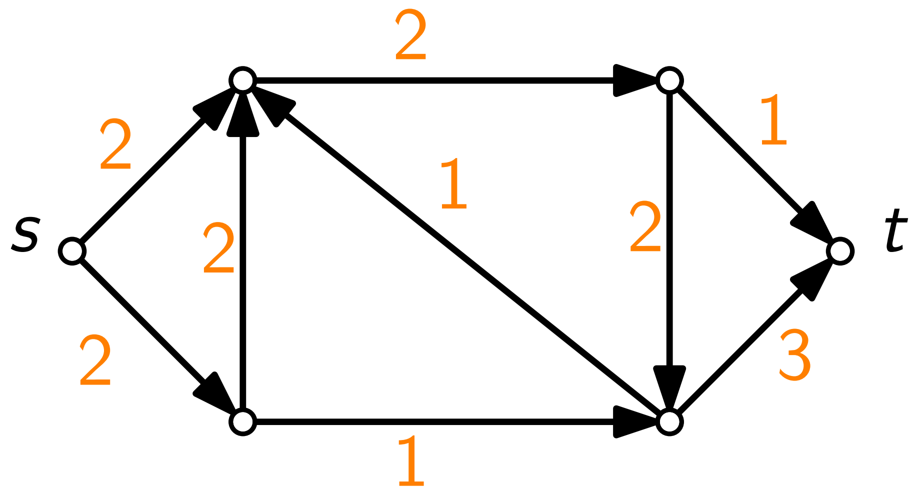

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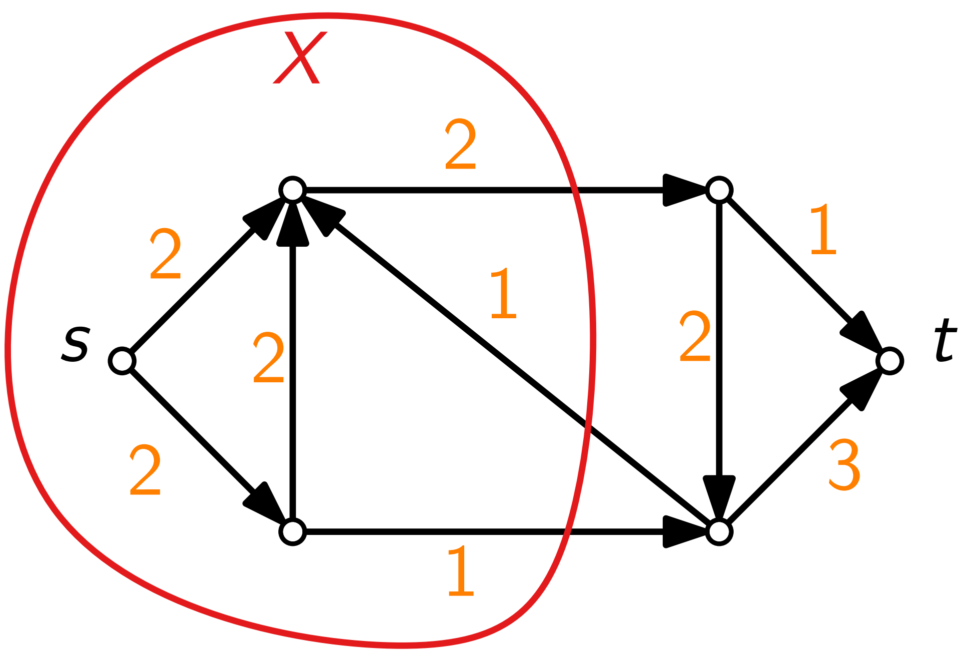

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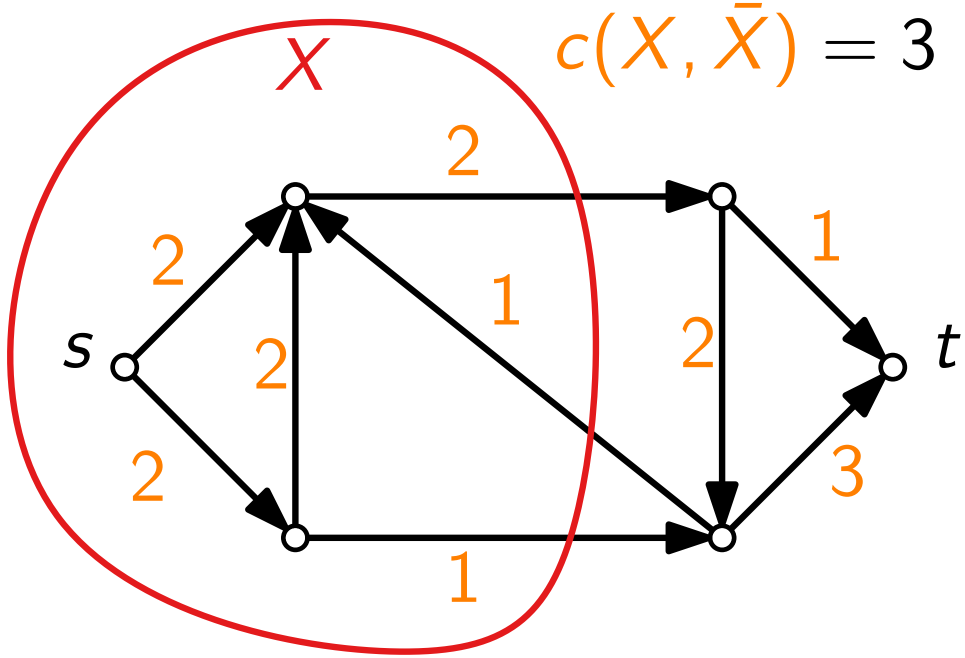

Max-Flow-Min-Cut Theorem

Theorem. The value of a maximum s–t flow and the weight of a minimum s–t cut are the same.

Max-Flow-Min-Cut Theorem

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Proof. Special case of LP-Duality . . .

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maximize
subject to

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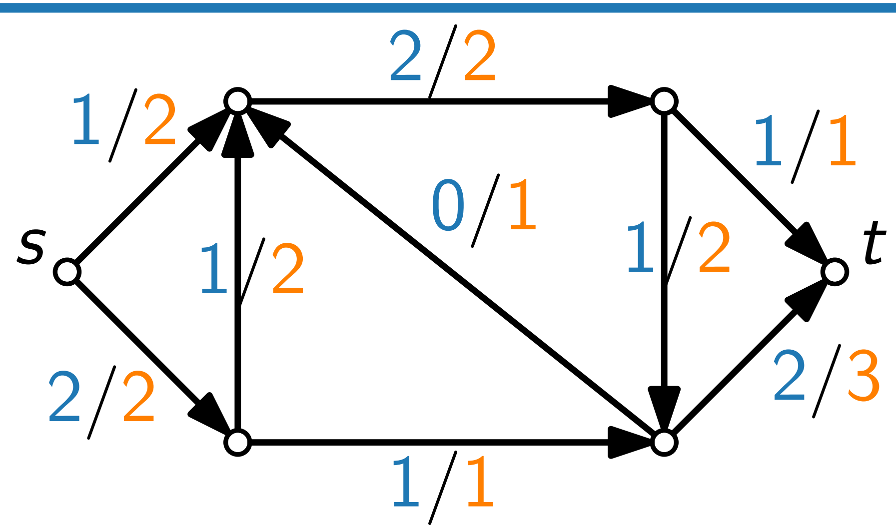

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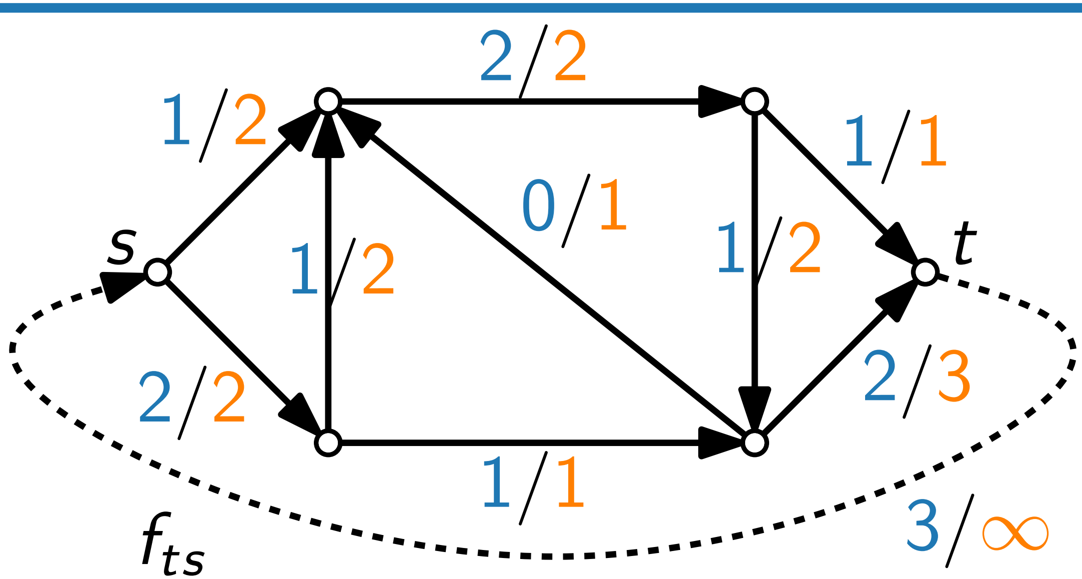
subject to



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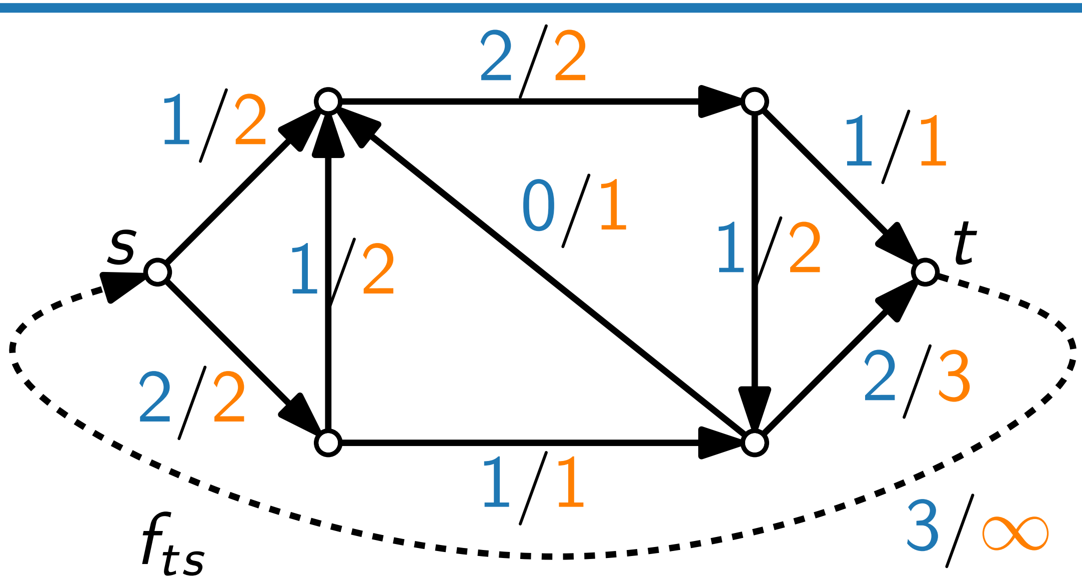
maximize f_{ts}
subject to $f_{uv} \leq c_{uv}$ $\qquad \forall (u,v) \in E \setminus \{(t,s)\}$



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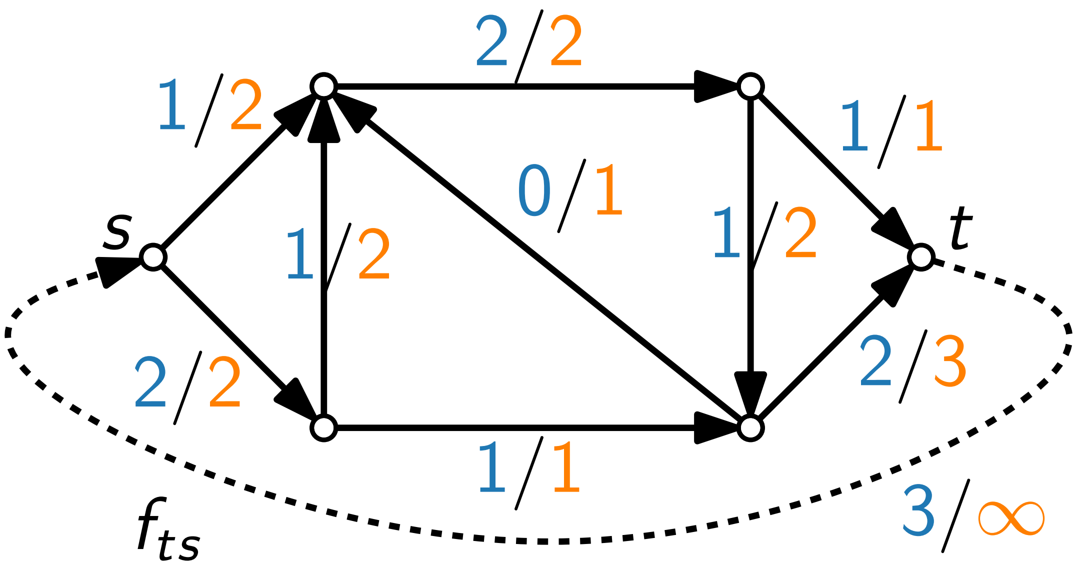
$$\sum_{u\colon (u,v)\in E} f_{uv} - \sum_{z\colon (v,z)\in E} f_{vz} \leq 0 \quad \forall v \in V$$



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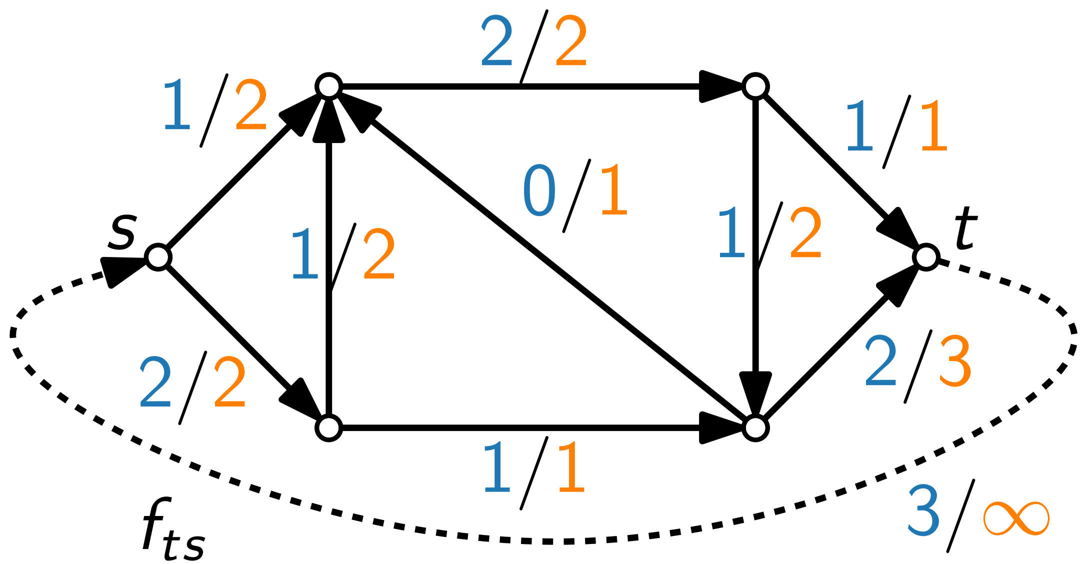
$$\begin{array}{lll} \textbf{maximize} & f_{ts} & \\ \textbf{subject to} & f_{uv} \leq c_{uv} & \forall (u,v) \in E \setminus \{(t,s)\} \\ & \sum\limits_{u\colon (u,v)\in E} f_{uv} - \sum\limits_{z\colon (v,z)\in E} f_{vz} \leq 0 & \forall v \in V \\ & f_{uv} \geq 0 & \forall (u,v) \in E \end{array}$$



Max-Flow-Min-Cut Theorem

Theorem. The value of a maximum s–t flow and the weight of a minimum s–t cut are the same.

Proof. Special case of LP-Duality . . .

maximize	f_{ts}		
subject to	$f_{uv} \leq c_{uv}$	$\forall (u,v) \in E \setminus \{(t,s)\}$	d_{uv}
	$\sum_{u\colon (u,v)\in E} f_{uv} - \sum_{z\colon (v,z)\in E} f_{vz} \leq 0$	$\forall v \in V$	
	$f_{uv} \geq 0$	$\forall (u,v) \in E$	

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\end{array}
$$

maximize $c^\intercal x$

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Proof. Special case of LP-Duality ...

maximize	f_{ts}		
subject to	$f_{uv} \leq c_{uv}$	$\forall (u,v) \in E \setminus \{(t,s)\}$	d_{uv}
	$\displaystyle\sum_{u\colon (u,v)\in E} f_{uv} - \sum_{z\colon (v,z)\in E} f_{vz} \leq 0$	$\forall v \in V$	p_v
	$f_{uv} \geq 0$	$\forall (u,v) \in E$	

maximize $c^{\mathsf{T}} x \quad = \sum_{(u,v)\in E} (0 \cdot f_{uv})$

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& f_{uv} \ge 0 & \forall (u,v) \in E &
\end{array}$$

maximize $c^\intercal x \quad = \sum_{(u,v)\in E} (0 \cdot f_{uv}) + 1 \cdot f_{ts}$

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maximize $c^\mathsf{T} x \quad = \sum_{(u,v)\in E} (0 \cdot f_{uv}) + 1 \cdot f_{ts} \quad \Rightarrow c^\mathsf{T} = (0, \ldots, 0, 1)$

Which constraints contain f_{uv} for $(u,v) \neq (t,s)$?

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$$\Rightarrow d_{uv} - p_u + p_v \ge 0$$

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Proof. Special case of LP-Duality ...

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	$f_{uv} \geq 0$	$\forall (u,v) \in E$	

maximize $c^\mathsf{T} x \quad = \sum_{(u,v)\in E} (0 \cdot f_{uv}) \ + 1 \cdot f_{ts} \quad \Rightarrow c^\mathsf{T} = (0, \ldots, 0, 1)$

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Which constraints contain f_{uv} for $(u,v) \neq (t,s)$? $d_{uv},\ p_u,\ p_v$

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Which constraints contain f_{ts}? $p_s,\ p_t$

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Which constraints contain f_{uv} for $(u,v) \neq (t,s)$? d_{uv}, p_u, p_v

$$\Rightarrow d_{uv} - p_u + p_v \geq 0$$

Which constraints contain f_{ts}? p_s, p_t

$$\Rightarrow p_s - p_t \geq 1$$

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maximize f_{ts}

subject to

$$f_{uv} \leq c_{uv} \quad \forall (u,v) \in E \setminus \{(t,s)\} \qquad d_{uv}$$

$$\sum_{u\colon (u,v)\in E} f_{uv} - \sum_{z\colon (v,z)\in E} f_{vz} \leq 0 \quad \forall v \in V \qquad p_v$$

$$f_{uv} \geq 0 \quad \forall (u,v) \in E$$

subject to

$$d_{uv} - p_u + p_v \geq 0 \quad \forall (u,v) \in E \setminus \{(t,s)\}$$

$$p_s - p_t \geq 1$$

$$d_{uv} \geq 0 \quad \forall (u,v) \in E$$

$$p_u \geq 0 \quad \forall u \in V$$

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\textbf{subject to} & f_{uv} \leq c_{uv} & \forall (u,v) \in E \setminus \{(t,s)\} & d_{uv} \\
& \displaystyle\sum_{u\colon (u,v)\in E} f_{uv} - \sum_{z\colon (v,z)\in E} f_{vz} \leq 0 & \forall v \in V & p_v \\
& f_{uv} \geq 0 & \forall (u,v) \in E &
\end{array}
$$

$$
\begin{array}{lll}
\textbf{minimize} & \displaystyle\sum_{(u,v)\in E\setminus\{(t,s)\}} c_{uv} \cdot d_{uv} & \\
\textbf{subject to} & d_{uv} - p_u + p_v \geq 0 & \forall (u,v) \in E \setminus \{(t,s)\} \\
& p_s - p_t \geq 1 & \\
& d_{uv} \geq 0 & \forall (u,v) \in E \\
& p_u \geq 0 & \forall u \in V
\end{array}
$$

Approximation Algorithms

Lecture 4:
Linear Programming and LP-Duality

Part VI:
Dual LP of Max Flow

Dual LP – Interpretation as ILP

$$\begin{aligned}
&\textbf{minimize} && \sum_{(u,v)\in E\setminus\{(t,s)\}} c_{uv}\cdot d_{uv} && \\
&\textbf{subject to} && d_{uv}-p_u+p_v\geq 0 && \forall (u,v)\in E\setminus\{(t,s)\}\\
& && p_s-p_t\geq 1 && \\
& && d_{uv}\geq 0 && \forall (u,v)\in E\\
& && p_u\geq 0 && \forall u\in V
\end{aligned}$$

Dual LP – Interpretation as ILP

$$\begin{array}{lll}
\textbf{minimize} & \displaystyle\sum_{(u,v)\in E\setminus\{(t,s)\}} c_{uv} \cdot d_{uv} & \\
\textbf{subject to} & d_{uv} - p_u + p_v \geq 0 & \forall (u,v) \in E \setminus \{(t,s)\} \\
& p_s - p_t \geq 1 & \\
& d_{uv} \geq \cancel{0} \; \in \{0,1\} & \forall (u,v) \in E \\
& p_u \geq \cancel{0} \; \in \{0,1\} & \forall u \in V
\end{array}$$

Dual LP – Interpretation as ILP

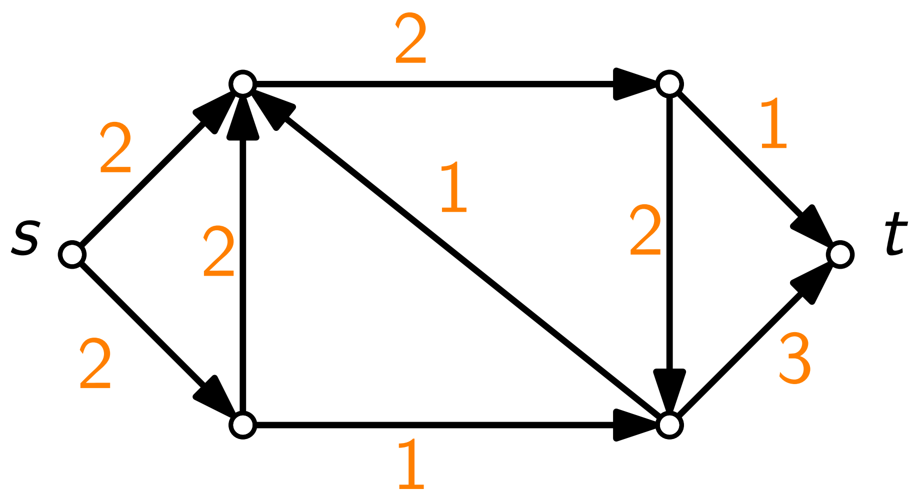
$$\textbf{minimize} \quad \sum_{(u,v)\in E\setminus\{(t,s)\}} c_{uv}\cdot d_{uv}$$

$$\textbf{subject to} \quad d_{uv} - p_u + p_v \geq 0 \qquad \forall (u,v)\in E\setminus\{(t,s)\}$$

$$p_s - p_t \geq 1$$

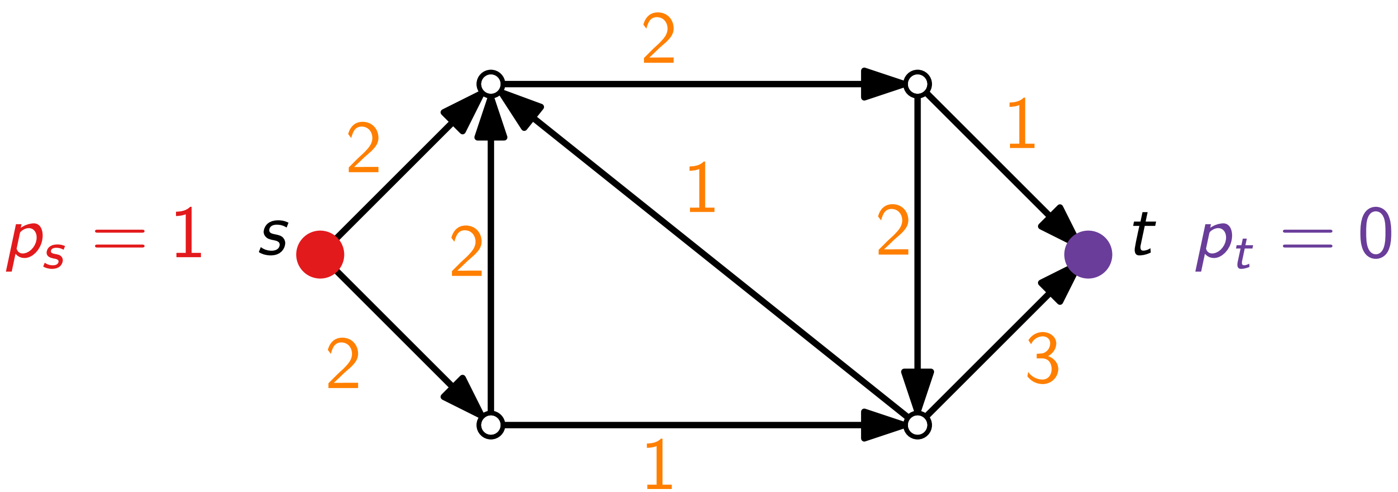
$$d_{uv} \cancel{\geq 0} \in \{0,1\} \qquad \forall (u,v)\in E$$

$$p_u \cancel{\geq 0} \in \{0,1\} \qquad \forall u \in V$$



Dual LP – Interpretation as ILP

$$
\begin{aligned}
\textbf{minimize} \quad & \sum_{(u,v)\in E\setminus\{(t,s)\}} c_{uv} \cdot d_{uv} \\
\textbf{subject to} \quad & d_{uv} - p_u + p_v \geq 0 && \forall (u,v) \in E \setminus \{(t,s)\} \\
& p_s - p_t \geq 1 \\
& d_{uv} \geq \cancel{0} \quad \in \{0,1\} && \forall (u,v) \in E \\
& p_u \geq \cancel{0} \quad \in \{0,1\} && \forall u \in V
\end{aligned}
$$



Dual LP – Interpretation as ILP

minimize $\displaystyle\sum_{(u,v)\in E\setminus\{(t,s)\}} c_{uv} \cdot d_{uv}$

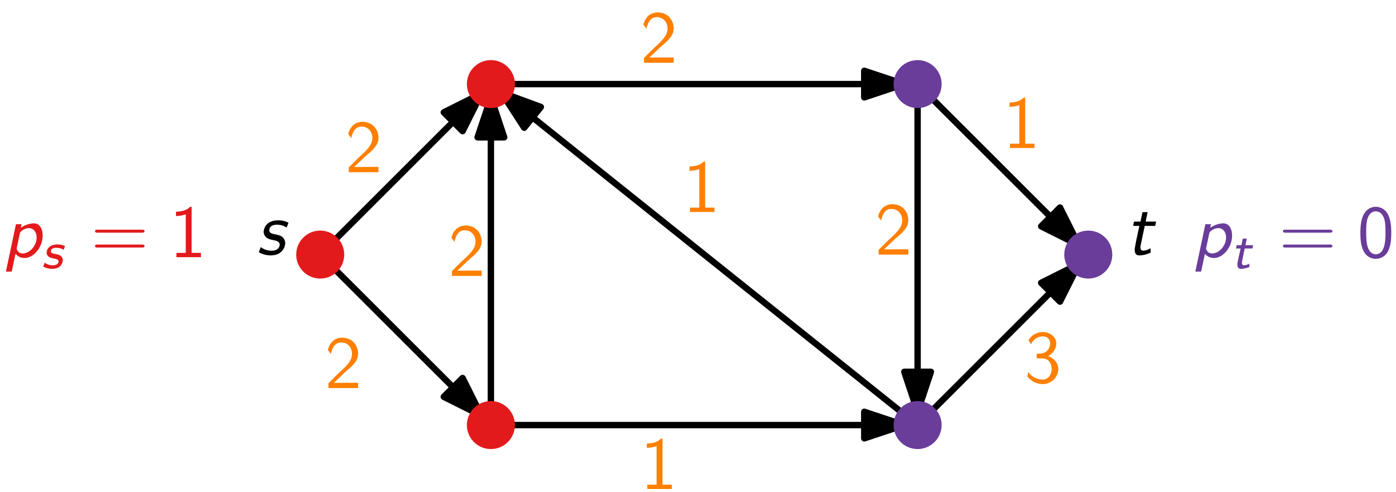
subject to

$d_{uv} - p_u + p_v \geq 0 \qquad \forall (u,v) \in E \setminus \{(t,s)\}$

$p_s - p_t \geq 1$

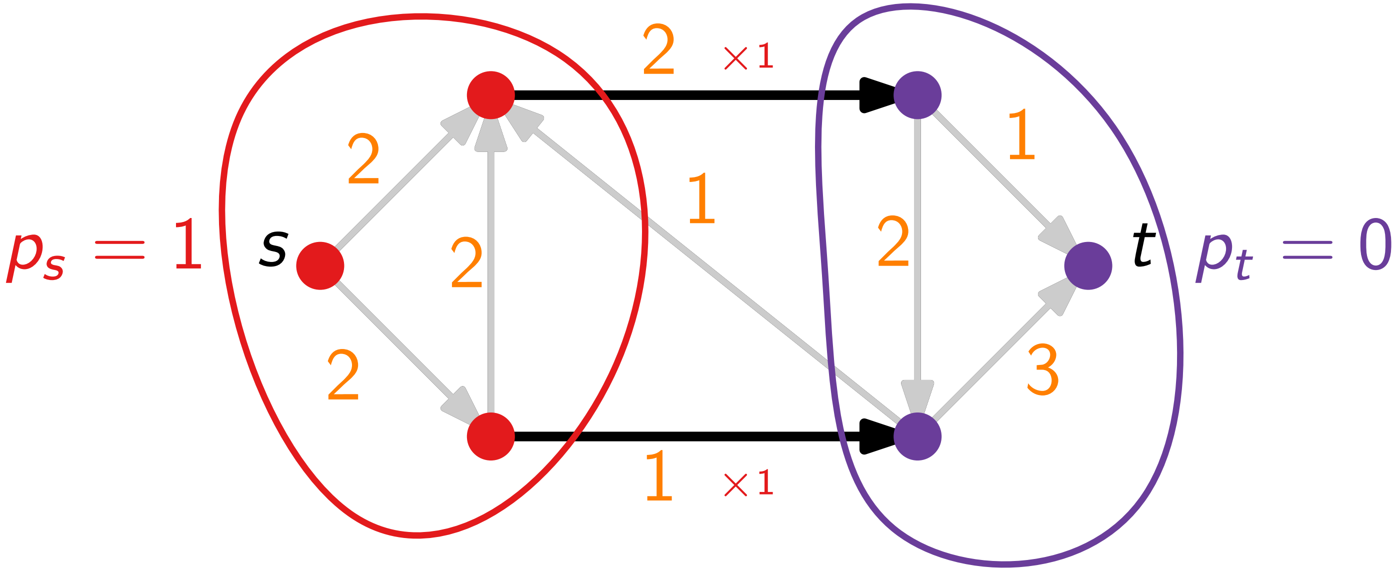
d_{uv} ~~≥ 0~~ $\in \{0,1\} \qquad \forall (u,v) \in E$

p_u ~~≥ 0~~ $\in \{0,1\} \qquad \forall u \in V$



Dual LP – Interpretation as ILP

$$\begin{array}{lll} \textbf{minimize} & \displaystyle\sum_{(u,v)\in E\setminus\{(t,s)\}} c_{uv}\cdot d_{uv} & \\ \textbf{subject to} & d_{uv} - p_u + p_v \geq 0 & \forall (u,v)\in E\setminus\{(t,s)\} \\ & p_s - p_t \geq 1 & \\ & d_{uv} \geq 0 \quad \in\{0,1\} & \forall (u,v)\in E \\ & p_u \geq 0 \quad \in\{0,1\} & \forall u\in V \end{array}$$



Dual LP – Interpretation as ILP

minimize $\displaystyle\sum_{(u,v)\in E\setminus\{(t,s)\}} c_{uv} \cdot d_{uv}$

subject to

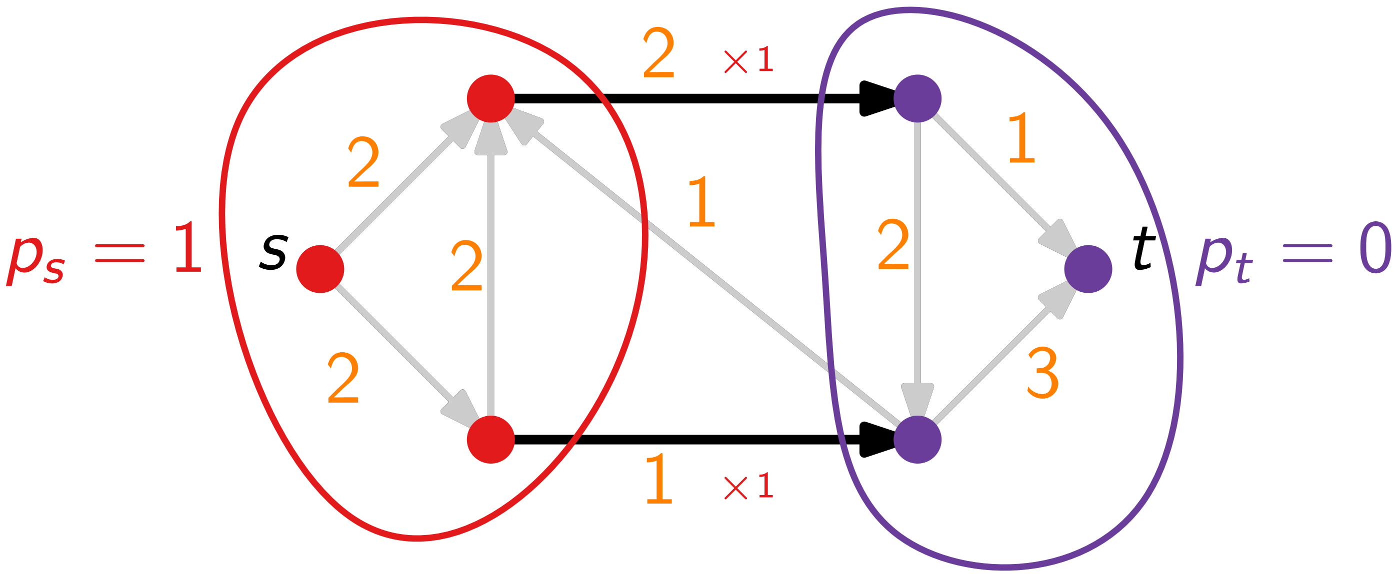
$d_{uv} - p_u + p_v \geq 0 \qquad \forall (u,v) \in E \setminus \{(t,s)\}$

$p_s - p_t \geq 1$

d_{uv} ~~≥ 0~~ $\in \{0,1\} \qquad \forall (u,v) \in E$

p_u ~~≥ 0~~ $\in \{0,1\} \qquad \forall u \in V$

equivalent to Min-Cut!

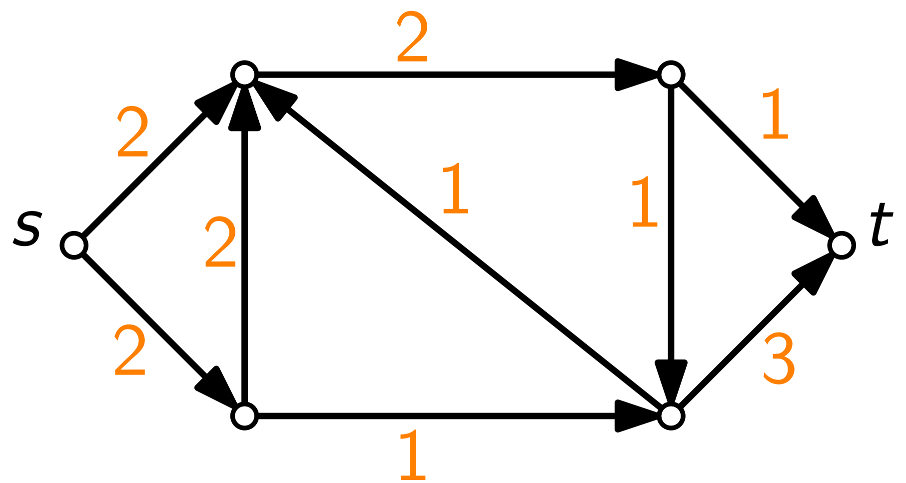


Dual LP – Fractional Cuts

$$\begin{aligned}
&\textbf{minimize} && \sum_{(u,v)\in E\setminus\{(t,s)\}} c_{uv}\cdot d_{uv} && \equiv \text{LP-relaxation of the ILP}\\
&\textbf{subject to} && d_{uv} - p_u + p_v \geq 0 && \forall (u,v)\in E\setminus\{(t,s)\}\\
& && p_s - p_t \geq 1 && \\
& && d_{uv} \geq 0 && \forall (u,v)\in E\\
& && p_u \geq 0 && \forall u\in V
\end{aligned}$$

Dual LP – Fractional Cuts

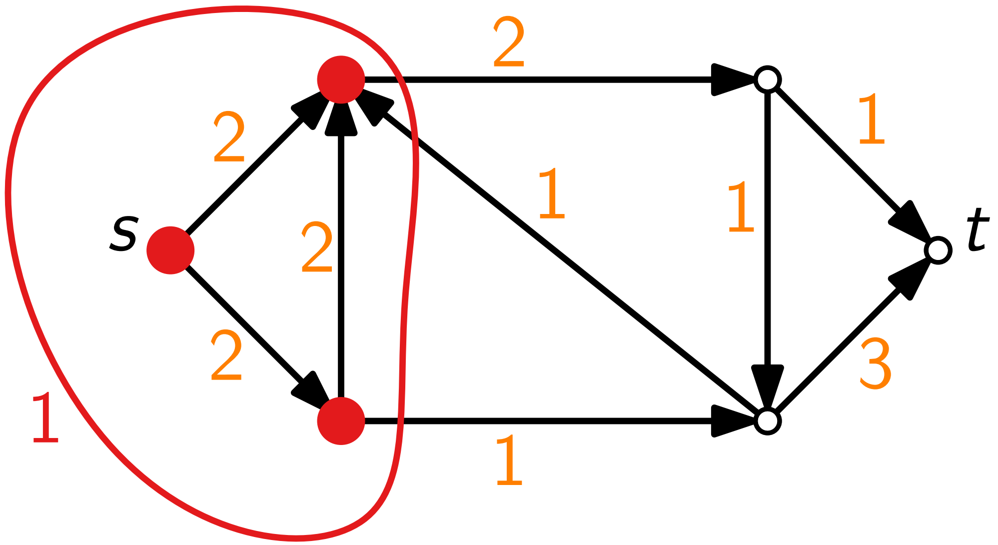
$$\begin{array}{lll} \textbf{minimize} & \displaystyle\sum_{(u,v)\in E\setminus\{(t,s)\}} c_{uv}\cdot d_{uv} & \equiv \text{LP-relaxation of the ILP} \\ \textbf{subject to} & d_{uv} - p_u + p_v \geq 0 & \forall (u,v)\in E\setminus\{(t,s)\} \\ & p_s - p_t \geq 1 & \\ & d_{uv} \geq 0 & \forall (u,v)\in E \\ & p_u \geq 0 & \forall u\in V \end{array}$$



Dual LP – Fractional Cuts

$$\text{minimize} \sum_{(u,v)\in E\setminus\{(t,s)\}} c_{uv} \cdot d_{uv} \qquad \equiv \text{LP-relaxation of the ILP}$$

$$\begin{aligned}
\text{subject to} \quad & d_{uv} - p_u + p_v \geq 0 && \forall (u,v) \in E \setminus \{(t,s)\} \\
& p_s - p_t \geq 1 && \\
& d_{uv} \geq 0 && \forall (u,v) \in E \\
& p_u \geq 0 && \forall u \in V
\end{aligned}$$



Dual LP – Fractional Cuts

$$
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\textbf{minimize} \quad & \sum_{(u,v)\in E\setminus\{(t,s)\}} c_{uv} \cdot d_{uv} & \equiv \text{LP-relaxation of the ILP} \\
\textbf{subject to} \quad & d_{uv} - p_u + p_v \geq 0 & \forall (u,v) \in E \setminus \{(t,s)\} \\
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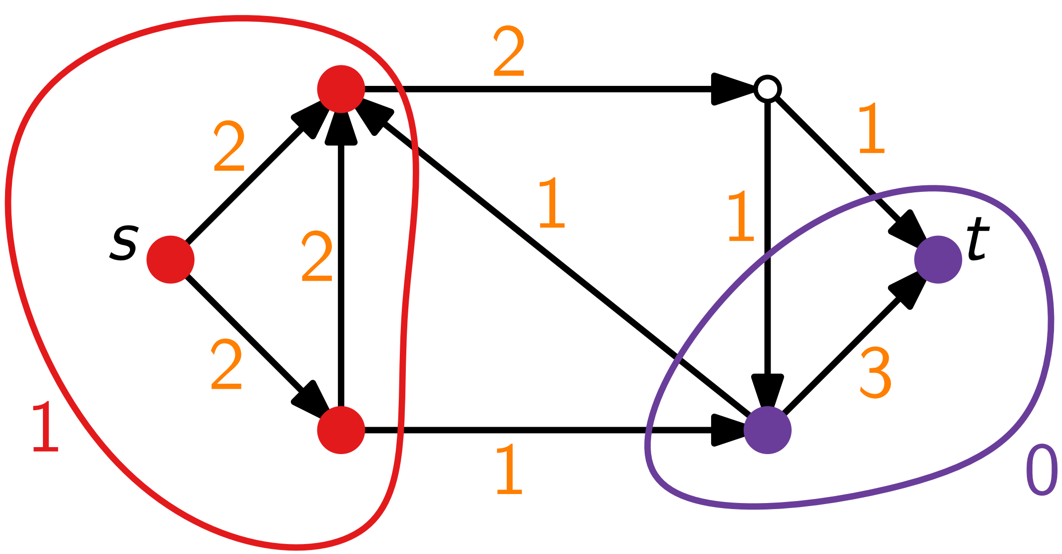
& p_u \geq 0 & \forall u \in V
\end{aligned}
$$



Dual LP – Fractional Cuts

$$\begin{aligned}
\textbf{minimize} \quad & \sum_{(u,v)\in E\setminus\{(t,s)\}} c_{uv} \cdot d_{uv} \\
\textbf{subject to} \quad & d_{uv} - p_u + p_v \geq 0 && \forall (u,v) \in E \setminus \{(t,s)\} \\
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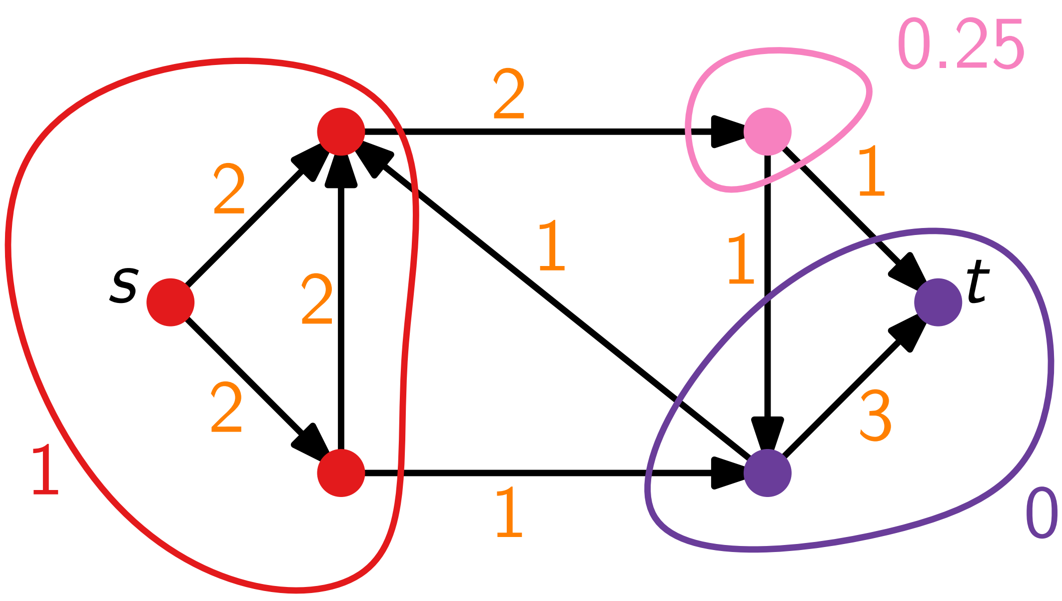
\end{aligned}$$

$\equiv$ LP-relaxation of the ILP



Dual LP – Fractional Cuts

$$\begin{array}{lll}
\textbf{minimize} & \displaystyle\sum_{(u,v)\in E\setminus\{(t,s)\}} c_{uv} \cdot d_{uv} & \\
\textbf{subject to} & d_{uv} - p_u + p_v \geq 0 & \forall (u,v) \in E \setminus \{(t,s)\} \\
& p_s - p_t \geq 1 & \\
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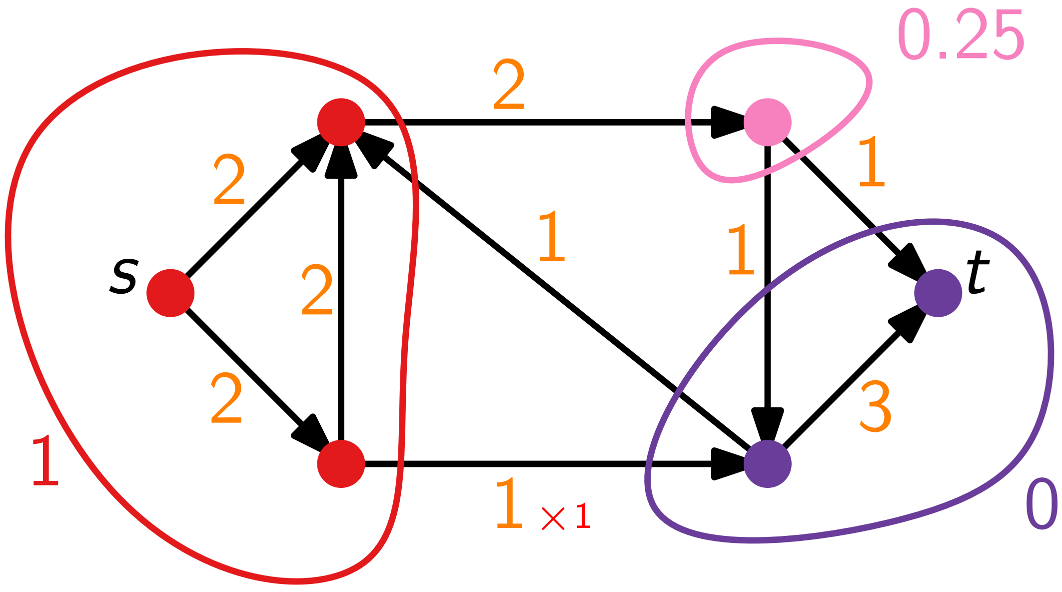
\end{array}$$

≡ LP-relaxation of the ILP



Dual LP – Fractional Cuts

$$\begin{aligned} &\textbf{minimize} && \sum_{(u,v)\in E\setminus\{(t,s)\}} c_{uv}\cdot d_{uv} \\ &\textbf{subject to} && d_{uv} - p_u + p_v \geq 0 && \forall (u,v)\in E\setminus\{(t,s)\} \\ &&& p_s - p_t \geq 1 \\ &&& d_{uv} \geq 0 && \forall (u,v)\in E \\ &&& p_u \geq 0 && \forall u \in V \end{aligned}$$

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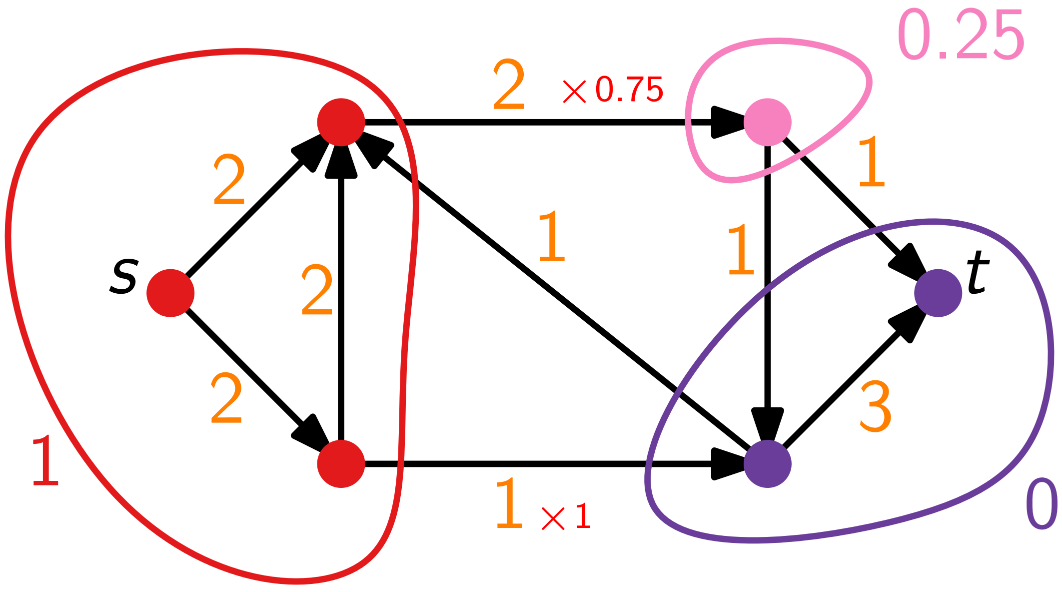

Dual LP – Fractional Cuts

$\equiv$ LP-relaxation of the ILP

$$\begin{aligned}
\textbf{minimize} \quad & \sum_{(u,v)\in E\setminus\{(t,s)\}} c_{uv} \cdot d_{uv} \\
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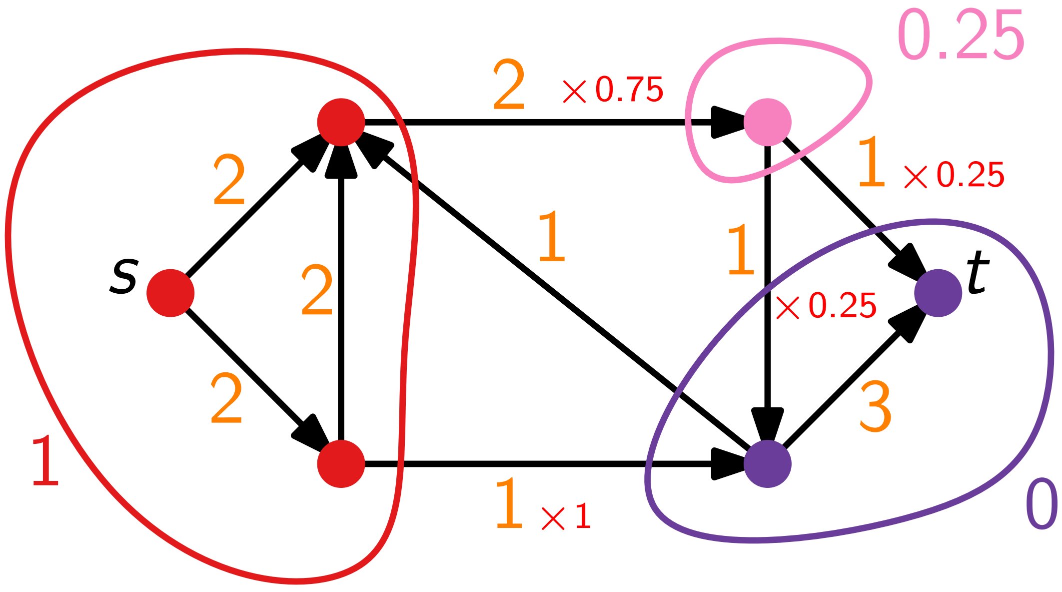

Dual LP – Fractional Cuts

$$
\begin{array}{lll}
\textbf{minimize} & \displaystyle\sum_{(u,v)\in E\setminus\{(t,s)\}} c_{uv}\cdot d_{uv} & \\
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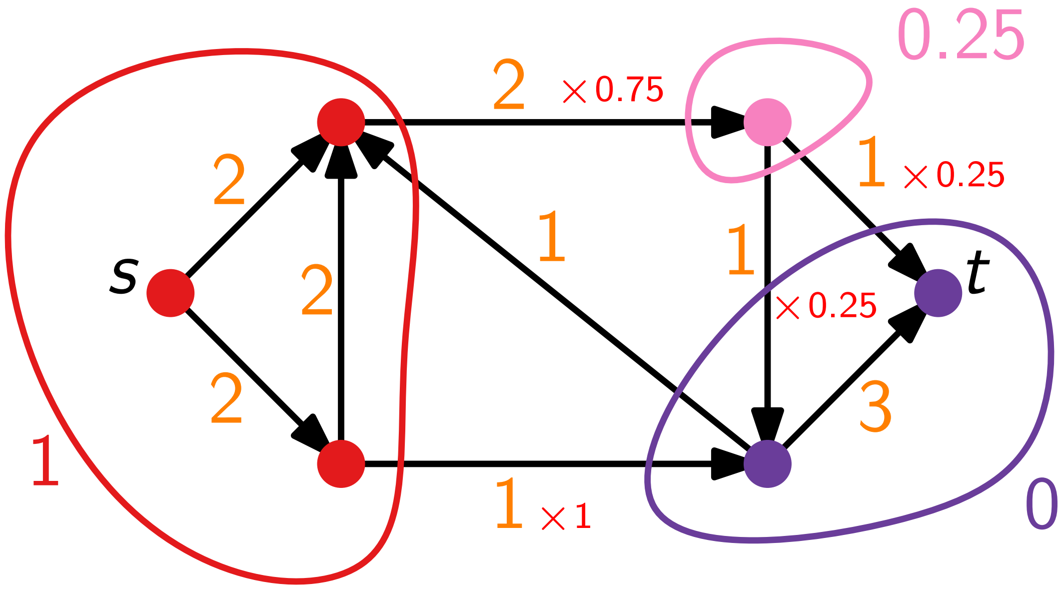
$$

$\equiv$ LP-relaxation of the ILP

Note that every s–t path $s = v_0, \ldots, v_k = t$ has length ≥ 1 w.r.t. d:



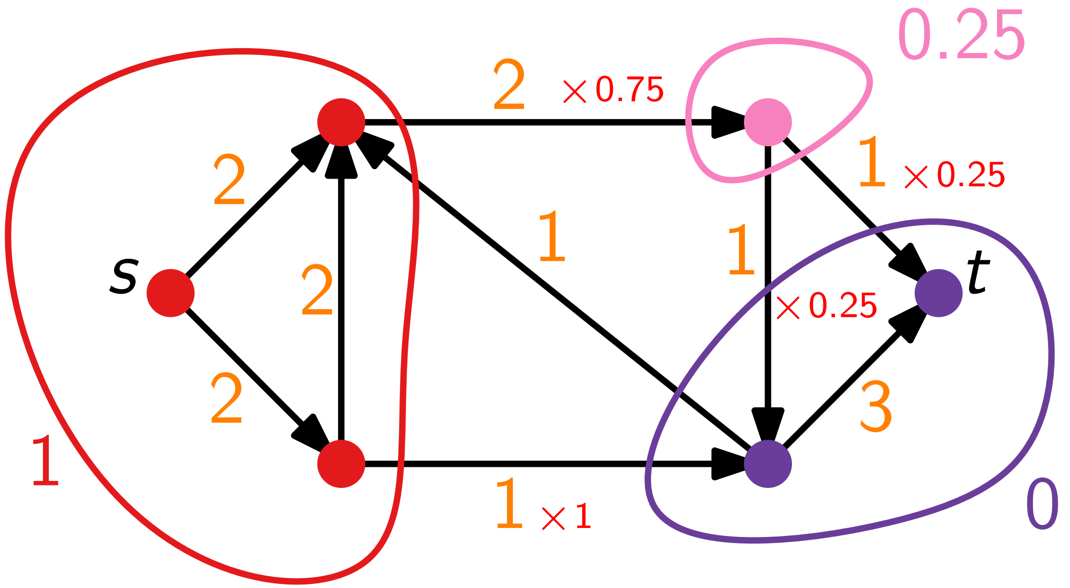
Dual LP – Fractional Cuts

$$\begin{array}{lll} \textbf{minimize} & \displaystyle\sum_{(u,v)\in E\setminus\{(t,s)\}} c_{uv}\cdot d_{uv} & \\ \textbf{subject to} & d_{uv} - p_u + p_v \geq 0 & \forall (u,v)\in E\setminus\{(t,s)\} \\ & p_s - p_t \geq 1 & \\ & d_{uv}\geq 0 & \forall (u,v)\in E \\ & p_u \geq 0 & \forall u\in V \end{array}$$

≡ LP-relaxation of the ILP

Note that every s–t path $s = v_0, \ldots, v_k = t$ has length ≥ 1 w.r.t. d:

$$\sum_{i=0}^{k-1} d_{i,i+1} \geq$$



Dual LP – Fractional Cuts

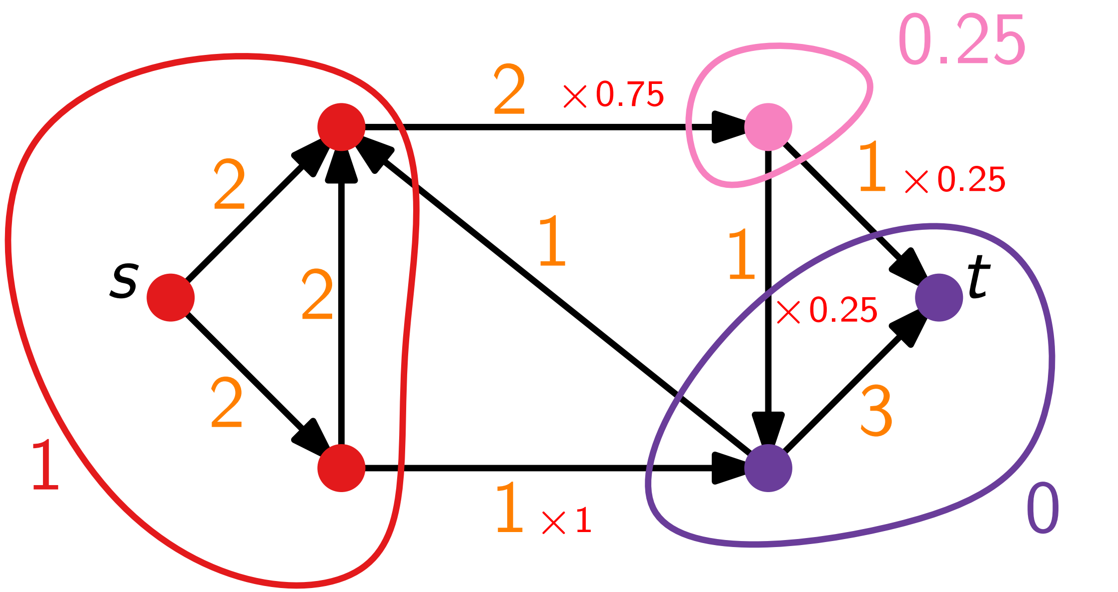
minimize $\sum_{(u,v)\in E\setminus\{(t,s)\}} c_{uv} \cdot d_{uv}$ $\equiv$ LP-relaxation of the ILP

subject to
$d_{uv} - p_u + p_v \geq 0 \quad \forall (u,v) \in E \setminus \{(t,s)\}$
$p_s - p_t \geq 1$
$d_{uv} \geq 0 \quad \forall (u,v) \in E$
$p_u \geq 0 \quad \forall u \in V$

Note that every s–t path $s = v_0, \ldots, v_k = t$ has length ≥ 1 w.r.t. d:

$$\sum_{i=0}^{k-1} d_{i,i+1} \geq \sum_{i=0}^{k-1} (p_i - p_{i+1})$$

$$=$$



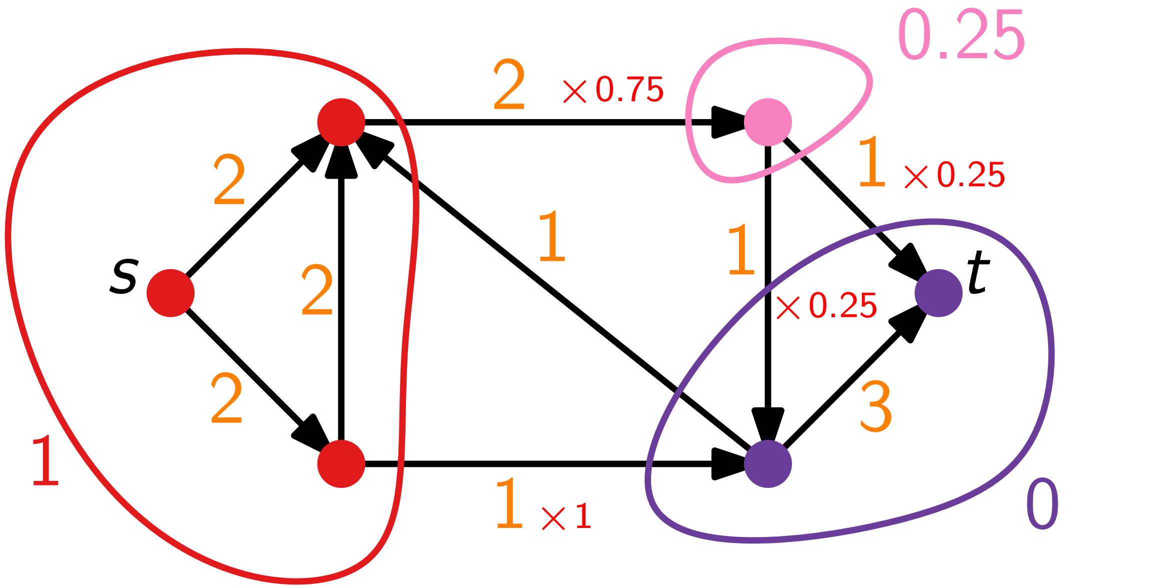
Dual LP – Fractional Cuts

$$\begin{array}{lll} \textbf{minimize} & \sum_{(u,v)\in E\setminus\{(t,s)\}} c_{uv}\cdot d_{uv} & \equiv \text{LP-relaxation of the ILP} \\ \textbf{subject to} & d_{uv} - p_u + p_v \geq 0 & \forall (u,v)\in E\setminus\{(t,s)\} \\ & p_s - p_t \geq 1 & \\ & d_{uv} \geq 0 & \forall (u,v)\in E \\ & p_u \geq 0 & \forall u \in V \end{array}$$

Note that every s–t path $s = v_0, \dots, v_k = t$ has length ≥ 1 w.r.t. d:

$$\sum_{i=0}^{k-1} d_{i,i+1} \geq \sum_{i=0}^{k-1}(p_i - p_{i+1})$$

$$= p_s - p_t = 1$$



Dual LP – Fractional Cuts

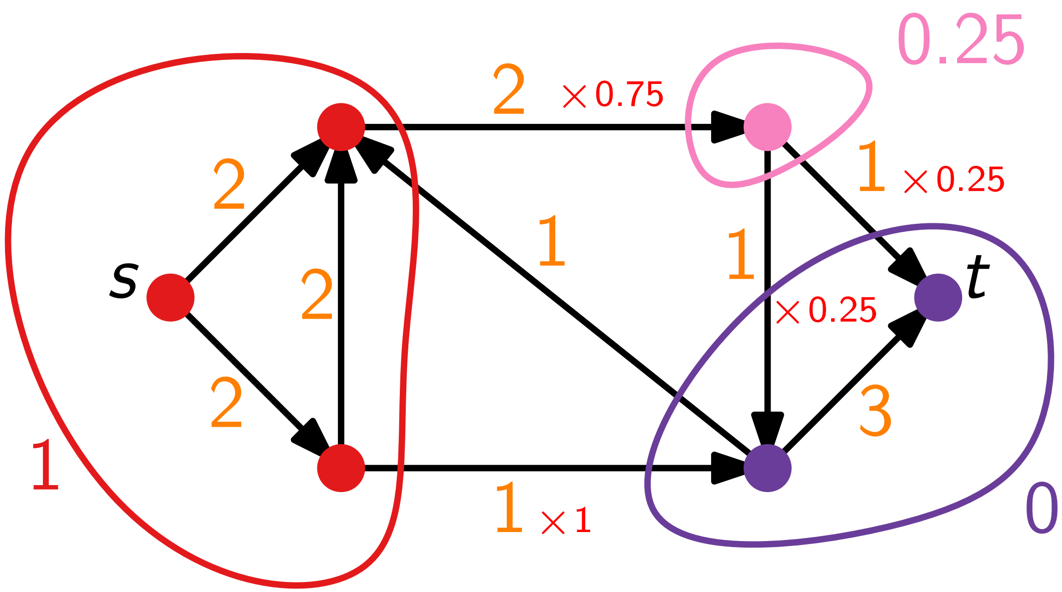
minimize $\sum_{(u,v)\in E\setminus\{(t,s)\}} c_{uv} \cdot d_{uv}$ $\equiv$ LP-relaxation of the ILP

subject to
$d_{uv} - p_u + p_v \geq 0 \quad \forall (u,v) \in E \setminus \{(t,s)\}$
$p_s - p_t \geq 1$
$d_{uv} \geq 0 \quad \forall (u,v) \in E$
$p_u \geq 0 \quad \forall u \in V$

Moreover, all extreme-point solutions all **integral**! (HW)

Note that every s–t path $s = v_0, \ldots, v_k = t$ has length ≥ 1 w.r.t. d:

$$\sum_{i=0}^{k-1} d_{i,i+1} \geq \sum_{i=0}^{k-1} (p_i - p_{i+1})$$
$$= p_s - p_t = 1$$



Dual LP – Complementary Slackness

maximize f_{ts}

subject to

$$f_{uv} \le c_{uv} \qquad \forall (u,v) \in E \setminus \{(t,s)\}$$

$$\sum_{u\colon (u,v)\in E} f_{uv} - \sum_{z\colon (v,z)\in E} f_{vz} \le 0 \qquad \forall v \in V$$

$$f_{uv} \ge 0 \qquad \forall (u,v) \in E$$

minimize $\sum_{(u,v)\in E\setminus\{(t,s)\}} c_{uv} \cdot d_{uv}$

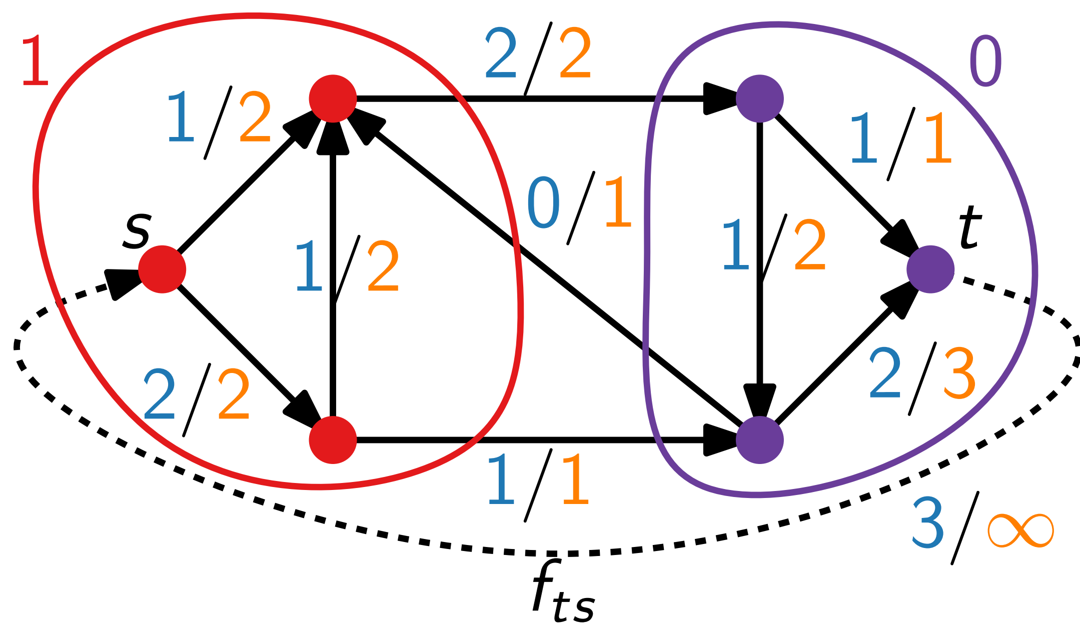
subject to

$$d_{uv} - p_u + p_v \ge 0 \qquad \forall (u,v) \in E \setminus \{(t,s)\}$$

$$p_s - p_t \ge 1$$

$$d_{uv} \ge 0 \qquad \forall (u,v) \in E$$

$$p_u \ge 0 \qquad \forall u \in V$$



Dual LP – Complementary Slackness

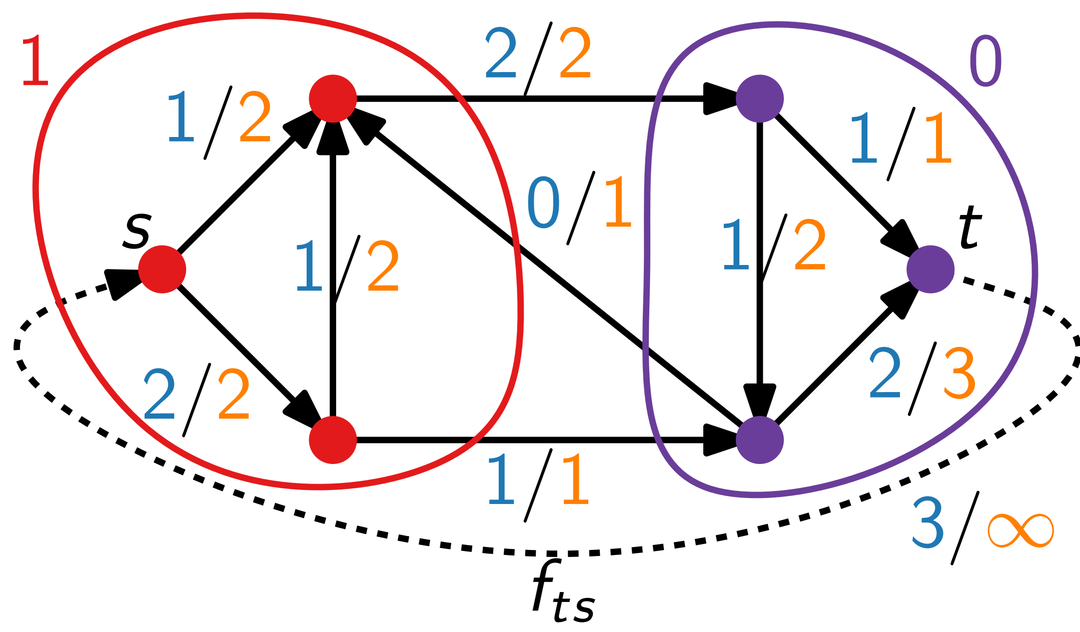
maximize f_{ts}

subject to
$$f_{uv} \leq c_{uv} \quad \forall (u,v) \in E \setminus \{(t,s)\}$$
$$\sum_{u:\,(u,v)\in E} f_{uv} - \sum_{z:\,(v,z)\in E} f_{vz} \leq 0 \quad \forall v \in V$$
$$f_{uv} \geq 0 \quad \forall (u,v) \in E$$

minimize $$\sum_{(u,v)\in E\setminus\{(t,s)\}} c_{uv} \cdot d_{uv}$$

subject to
$$d_{uv} - p_u + p_v \geq 0 \quad \forall (u,v) \in E \setminus \{(t,s)\}$$
$$p_s - p_t \geq 1$$
$$d_{uv} \geq 0 \quad \forall (u,v) \in E$$
$$p_u \geq 0 \quad \forall u \in V$$

For a max flow and min cut:



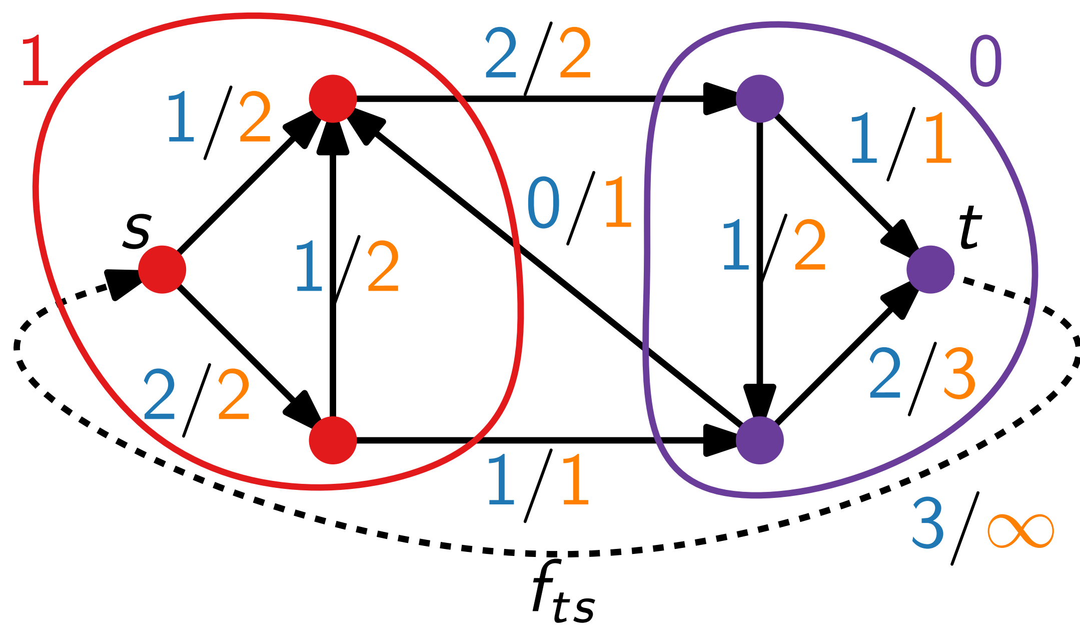
Dual LP – Complementary Slackness

maximize	f_{ts}	
subject to	$f_{uv} \leq c_{uv}$	$\forall (u,v) \in E \setminus \{(t,s)\}$
	$\sum_{u:\,(u,v)\in E} f_{uv} - \sum_{z:\,(v,z)\in E} f_{vz} \leq 0$	$\forall v \in V$
	$f_{uv} \geq 0$	$\forall (u,v) \in E$

minimize	$\sum_{(u,v)\in E\setminus\{(t,s)\}} c_{uv} \cdot d_{uv}$	
subject to	$d_{uv} - p_u + p_v \geq 0$	$\forall (u,v) \in E \setminus \{(t,s)\}$
	$p_s - p_t \geq 1$	
	$d_{uv} \geq 0$	$\forall (u,v) \in E$
	$p_u \geq 0$	$\forall u \in V$

For a max flow and min cut:

- For each forward edge (u,v) of the cut:



Dual LP – Complementary Slackness

maximize f_{ts}
subject to $f_{uv} \leq c_{uv} \quad \forall (u,v) \in E \setminus \{(t,s)\}$

$$\sum_{u:\ (u,v)\in E} f_{uv} - \sum_{z:\ (v,z)\in E} f_{vz} \leq 0 \quad \forall v \in V$$

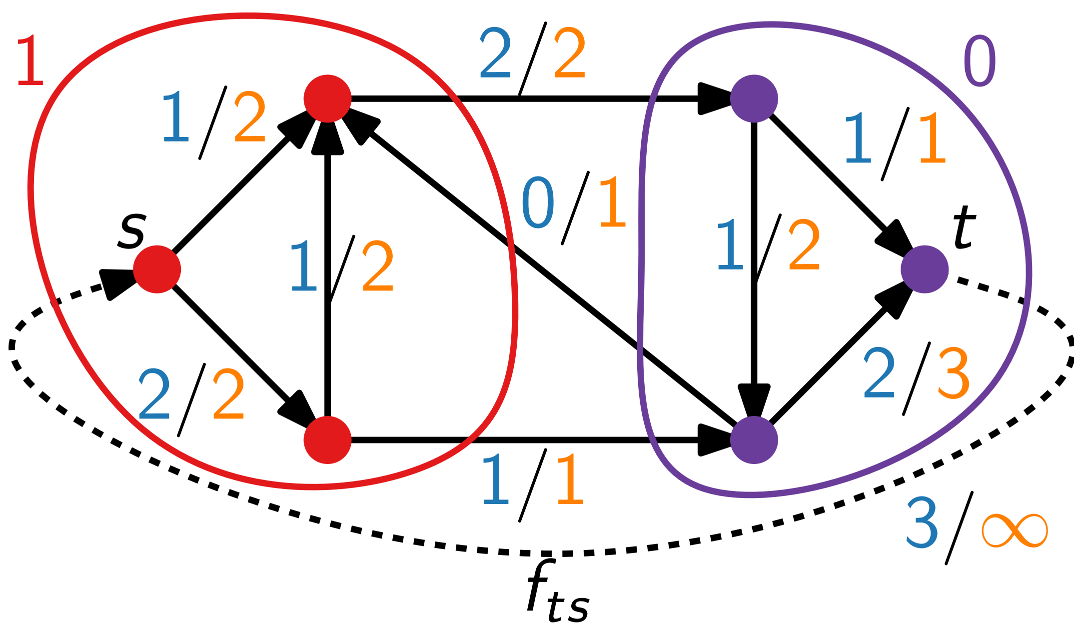
$f_{uv} \geq 0 \quad \forall (u,v) \in E$

minimize $\sum_{(u,v)\in E\setminus\{(t,s)\}} c_{uv} \cdot d_{uv}$
subject to $d_{uv} - p_u + p_v \geq 0$
$p_s - p_t \geq 1$
$d_{uv} \geq 0$
$p_u \geq 0$

Primal CS:
$\forall j$: $x_j = 0$ or $\sum_{i=1}^{m} a_{ij} y_i = c_j$

Dual CS:
$\forall i$: $y_i = 0$ or $\sum_{j=1}^{n} a_{ij} x_j = b_i$

For a max flow and min cut:
- For each forward edge (u,v) of the cut:



Dual LP – Complementary Slackness

maximize	f_{ts}
subject to	$f_{uv} \leq c_{uv} \qquad \forall (u,v) \in E \setminus \{(t,s)\}$
	$\sum_{u:\,(u,v)\in E} f_{uv} - \sum_{z:\,(v,z)\in E} f_{vz} \leq 0 \qquad \forall v \in V$
	$f_{uv} \geq 0 \qquad \forall (u,v) \in E$

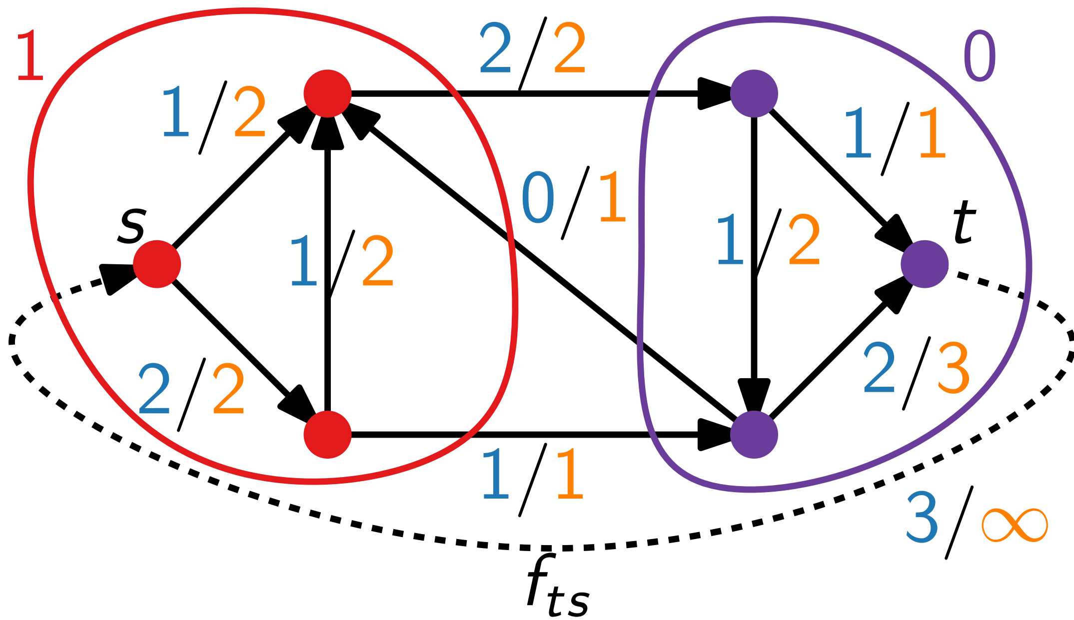
minimize	$\sum_{(u,v)\in E\setminus\{(t,s)\}} c_{uv} \cdot d_{uv}$
subject to	$d_{uv} - p_u + p_v \geq 0$
	$p_s - p_t \geq 1$
	$d_{uv} \geq 0$
	$p_u \geq 0$

Primal CS:
$\forall j$: $x_j = 0$ or $\sum_{i=1}^{m} a_{ij} y_i = c_j$

Dual CS:
$\forall i$: $y_i = 0$ or $\sum_{j=1}^{n} a_{ij} x_j = b_i$

For a max flow and min cut:

- For each forward edge (u,v) of the cut: $f_{uv} = c_{uv}$.



Dual LP – Complementary Slackness

$$\begin{array}{lll} \textbf{maximize} & f_{ts} & \\ \textbf{subject to} & f_{uv} \leq c_{uv} & \forall (u,v) \in E \setminus \{(t,s)\} \\ & \sum_{u:\,(u,v)\in E} f_{uv} - \sum_{z:\,(v,z)\in E} f_{vz} \leq 0 & \forall v \in V \\ & f_{uv} \geq 0 & \forall (u,v) \in E \end{array}$$

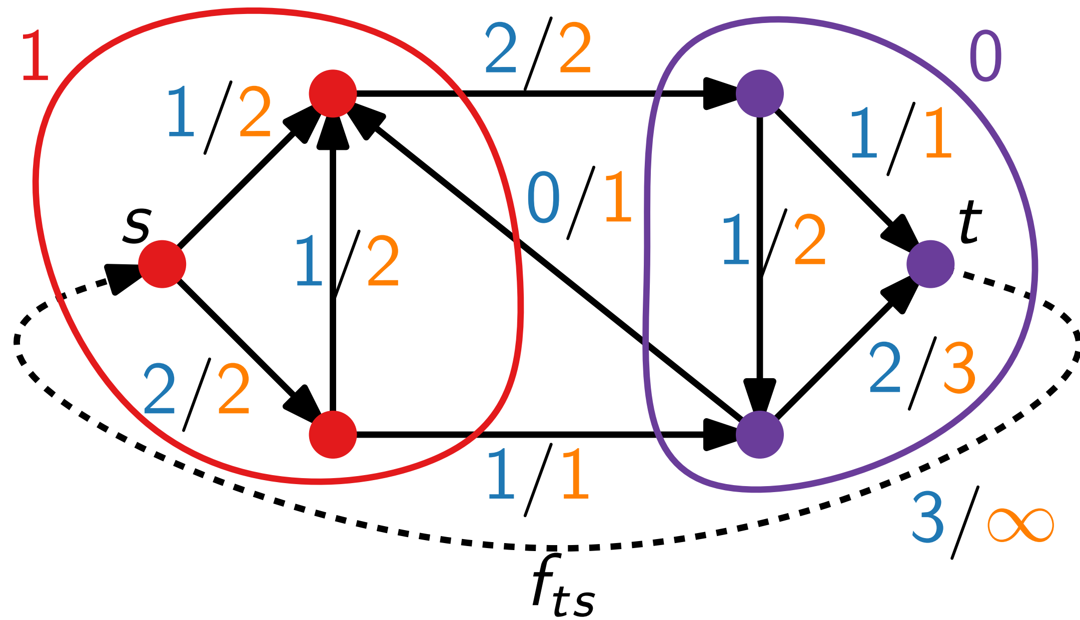
$$\begin{array}{ll} \textbf{minimize} & \sum_{(u,v)\in E\setminus\{(t,s)\}} c_{uv} \cdot d_{uv} \\ \textbf{subject to} & d_{uv} - p_u + p_v \geq 0 \\ & p_s - p_t \geq 1 \\ & d_{uv} \geq 0 \\ & p_u \geq 0 \end{array}$$

Primal CS:
$\forall j$: $x_j = 0$ or $\sum_{i=1}^{m} a_{ij} y_i = c_j$

Dual CS:
$\forall i$: $y_i = 0$ or $\sum_{j=1}^{n} a_{ij} x_j = b_i$

For a max flow and min cut:

- For each forward edge (u,v) of the cut: $f_{uv} = c_{uv}$.
  ($d_{uv} = 1$, so by dual CS: $f_{uv} = c_{uv}$.)



Dual LP – Complementary Slackness

maximize f_{ts}

subject to $f_{uv} \le c_{uv}$ $\quad \forall (u,v) \in E \setminus \{(t,s)\}$

$$\sum_{u\colon (u,v)\in E} f_{uv} - \sum_{z\colon (v,z)\in E} f_{vz} \le 0 \qquad \forall v \in V$$

$f_{uv} \ge 0$ $\quad \forall (u,v) \in E$

minimize $\sum_{(u,v)\in E\setminus\{(t,s)\}} c_{uv} \cdot d_{uv}$

subject to $d_{uv} - p_u + p_v \ge 0$

$p_s - p_t \ge 1$

$d_{uv} \ge 0$

$p_u \ge 0$

Primal CS:

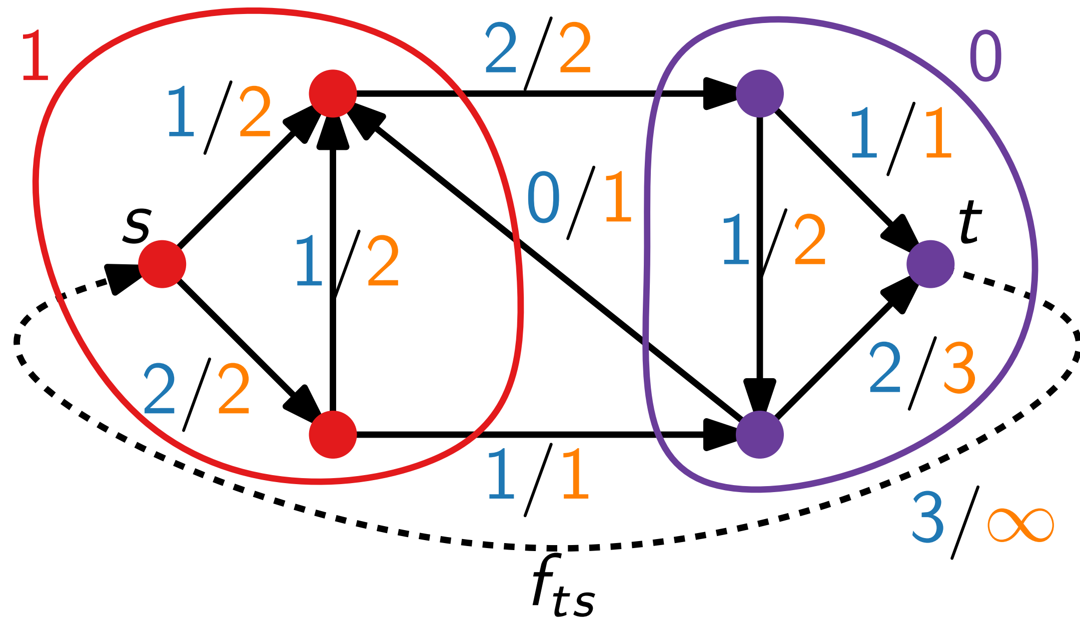
$\forall j$: $\quad x_j = 0$ or $\sum_{i=1}^{m} a_{ij} y_i = c_j$

Dual CS:

$\forall i$: $\quad y_i = 0$ or $\sum_{j=1}^{n} a_{ij} x_j = b_i$

For a max flow and min cut:

- For each forward edge (u,v) of the cut: $f_{uv} = c_{uv}$.
  ($d_{uv} = 1$, so by dual CS: $f_{uv} = c_{uv}$.)
- For each backward edge (u,v) of the cut:



Dual LP – Complementary Slackness

maximize	f_{ts}	
subject to	$f_{uv} \leq c_{uv}$	$\forall (u,v) \in E \setminus \{(t,s)\}$
	$\displaystyle\sum_{u:\,(u,v)\in E} f_{uv} - \sum_{z:\,(v,z)\in E} f_{vz} \leq 0$	$\forall v \in V$
	$f_{uv} \geq 0$	$\forall (u,v) \in E$

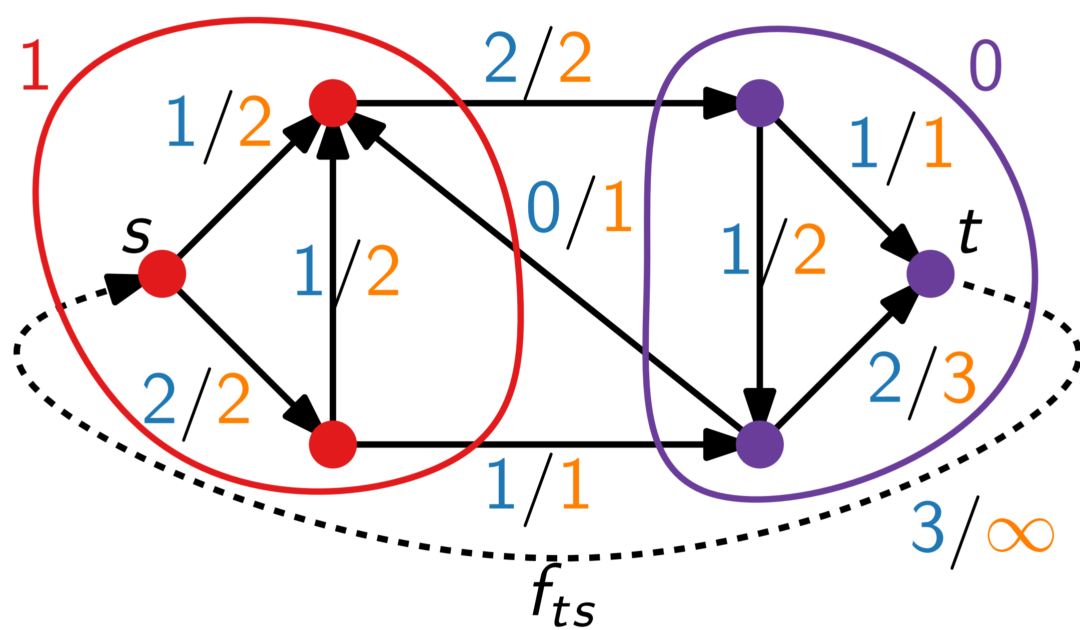
minimize	$\displaystyle\sum_{(u,v)\in E\setminus\{(t,s)\}} c_{uv} \cdot d_{uv}$
subject to	$d_{uv} - p_u + p_v \geq 0$
	$p_s - p_t \geq 1$
	$d_{uv} \geq 0$
	$p_u \geq 0$

Primal CS:
$\forall j$: $x_j = 0$ or $\sum_{i=1}^{m} a_{ij} y_i = c_j$

Dual CS:
$\forall i$: $y_i = 0$ or $\sum_{j=1}^{n} a_{ij} x_j = b_i$

For a max flow and min cut:

- For each forward edge (u,v) of the cut: $f_{uv} = c_{uv}$.
  ($d_{uv} = 1$, so by dual CS: $f_{uv} = c_{uv}$.)
- For each backward edge (u,v) of the cut: $f_{uv} = 0$.



Dual LP – Complementary Slackness

maximize f_{ts}

subject to

$$f_{uv} \le c_{uv} \quad \forall (u,v) \in E \setminus \{(t,s)\}$$

$$\sum_{u:\,(u,v)\in E} f_{uv} - \sum_{z:\,(v,z)\in E} f_{vz} \le 0 \quad \forall v \in V$$

$$f_{uv} \ge 0 \quad \forall (u,v) \in E$$

minimize $\sum_{(u,v)\in E\setminus\{(t,s)\}} c_{uv} \cdot d_{uv}$

subject to

$$d_{uv} - p_u + p_v \ge 0$$

$$p_s - p_t \ge 1$$

$$d_{uv} \ge 0$$

$$p_u \ge 0$$

Primal CS:
$\forall j$: $x_j = 0$ or $\sum_{i=1}^m a_{ij} y_i = c_j$

Dual CS:
$\forall i$: $y_i = 0$ or $\sum_{j=1}^n a_{ij} x_j = b_i$

For a max flow and min cut:

- For each forward edge (u,v) of the cut: $f_{uv} = c_{uv}$.
  ($d_{uv} = 1$, so by dual CS: $f_{uv} = c_{uv}$.)
- For each backward edge (u,v) of the cut: $f_{uv} = 0$.
  (Otherwise, by primal CS: $d_{uv} - 0 + 1 = 0$.)