Approximation Algorithms

Lecture 3: SteinerTree and MultiwayCut

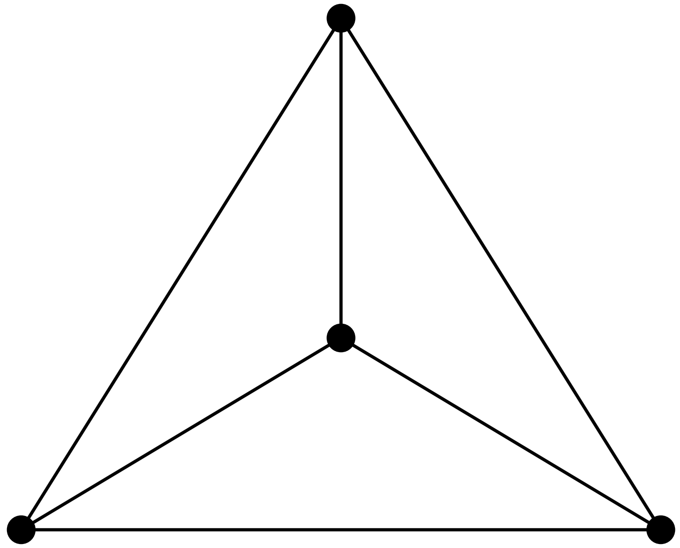
Part I: SteinerTree

Alexander Wolff

Winter 2023/24

SteinerTree

Given: A graph G



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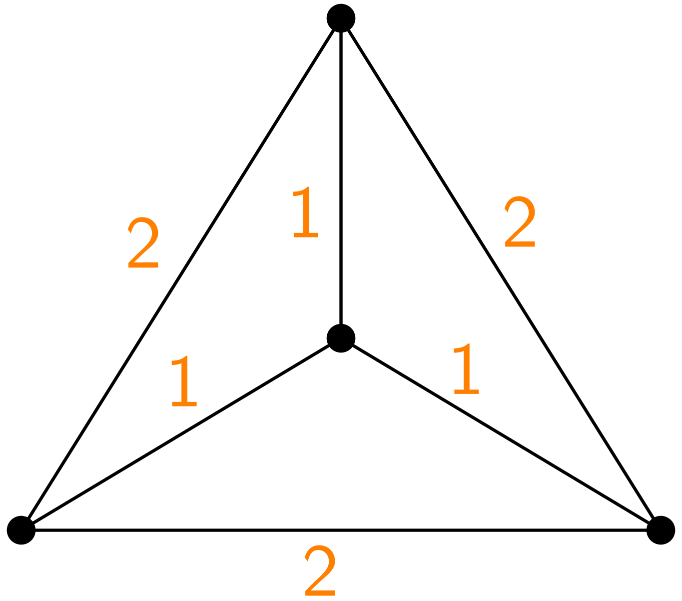
Given: A graph G with edge weights $c\colon E(G) \to \mathbb{Q}^+$



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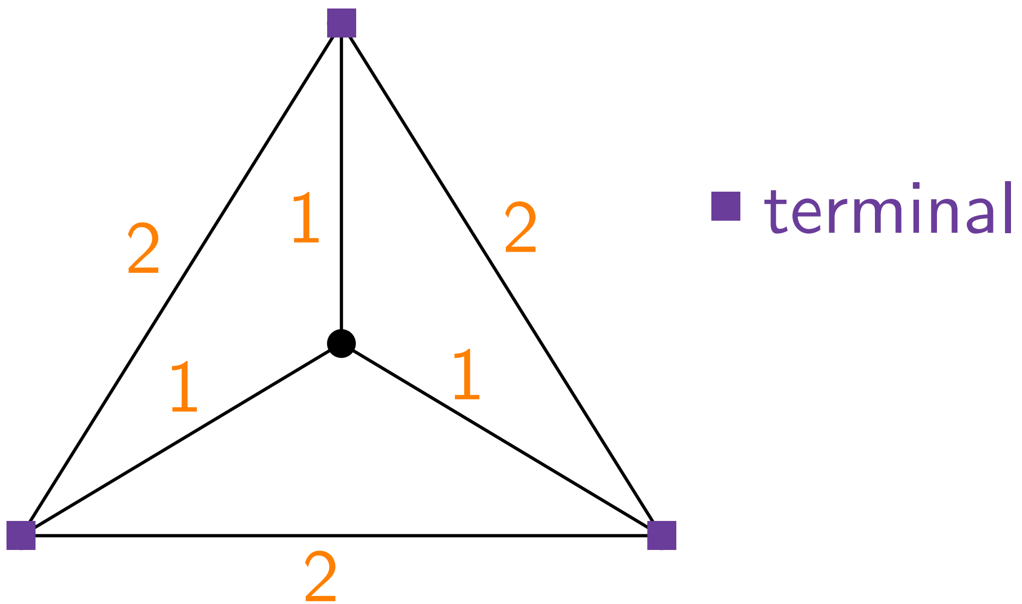

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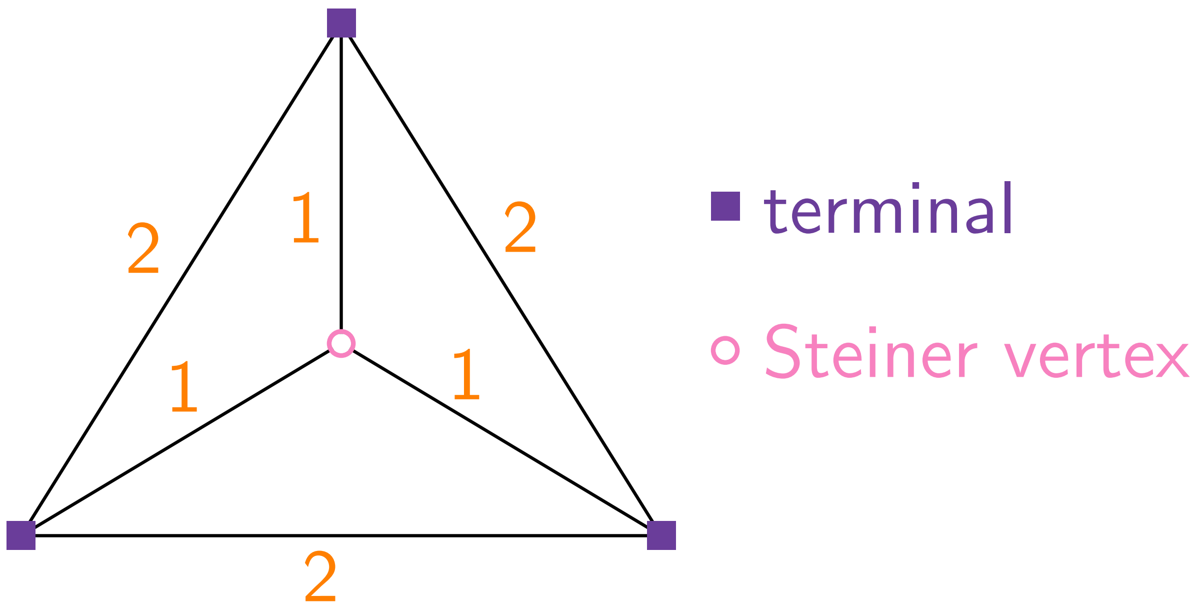

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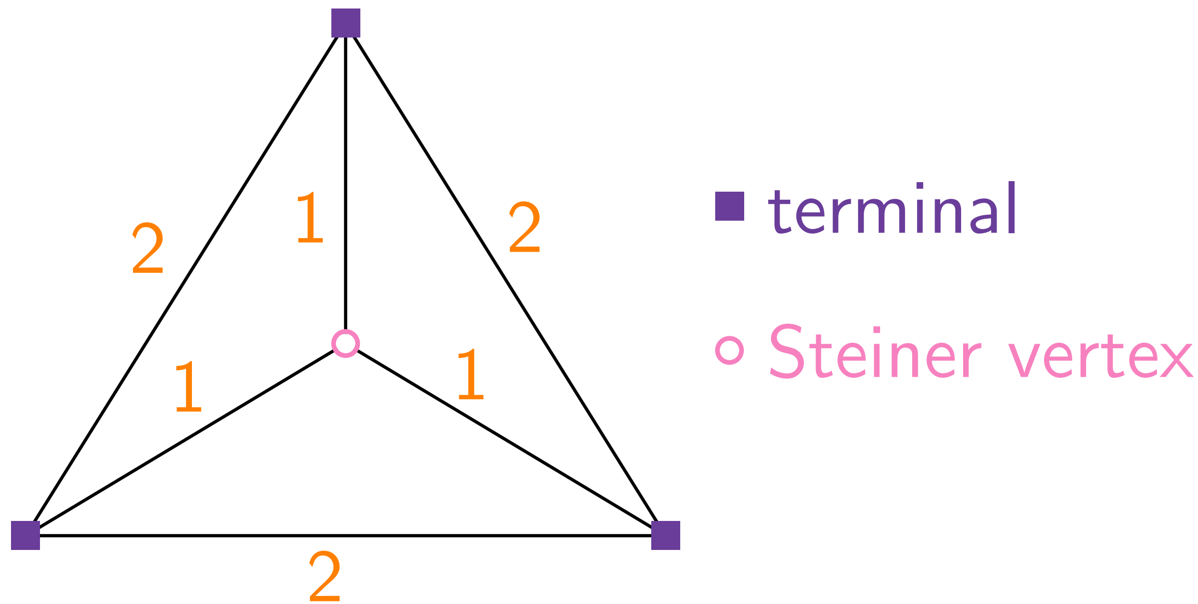

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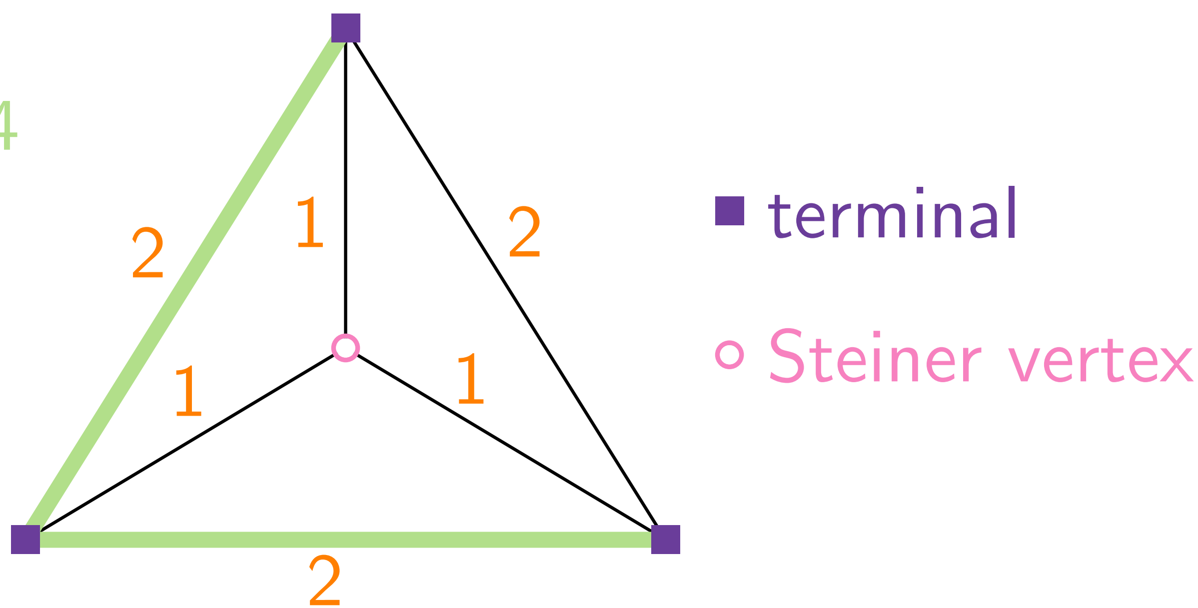

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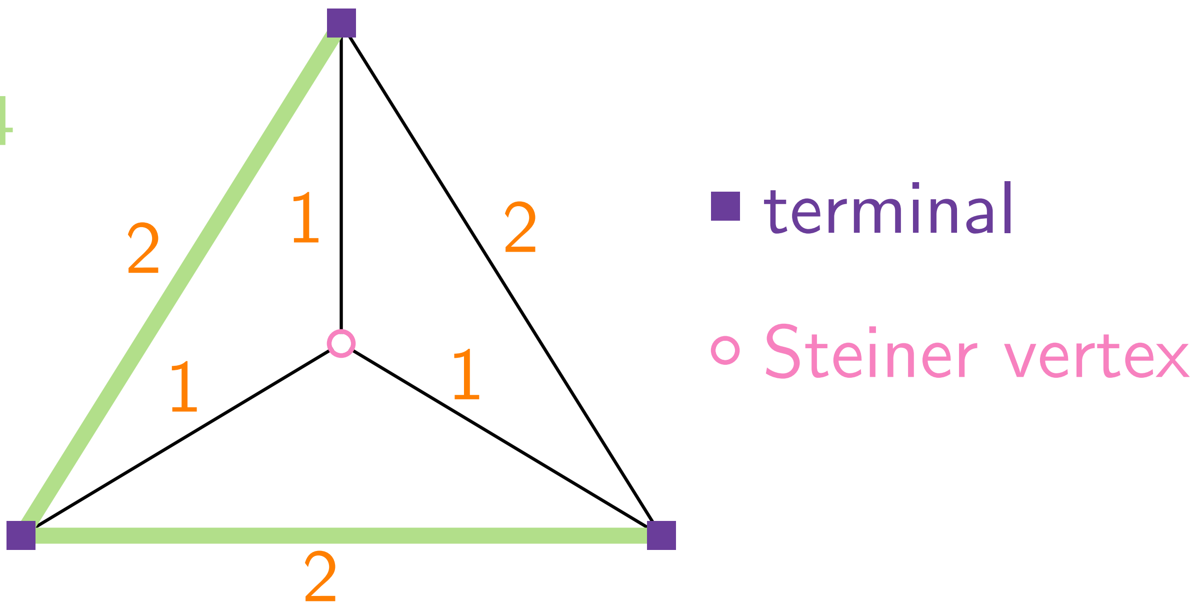

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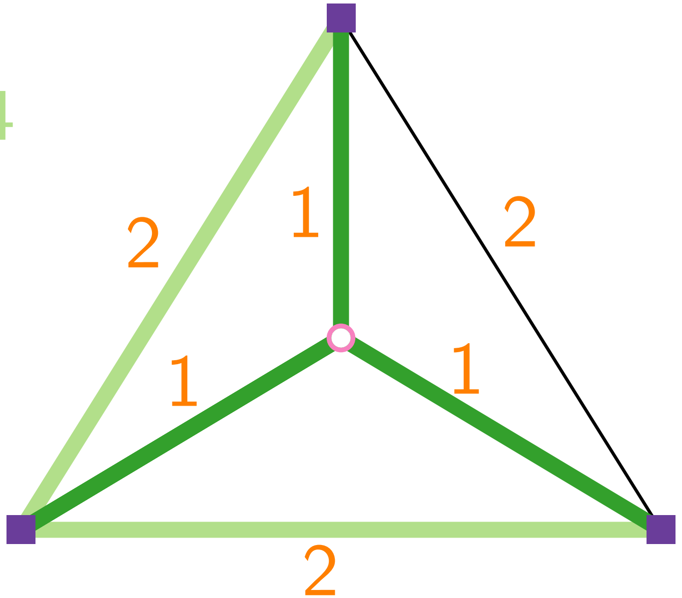

■ terminal

○ Steiner vertex

MetricSteinerTree

Restriction of SteinerTree where the graph G is complete and the cost function is **metric**

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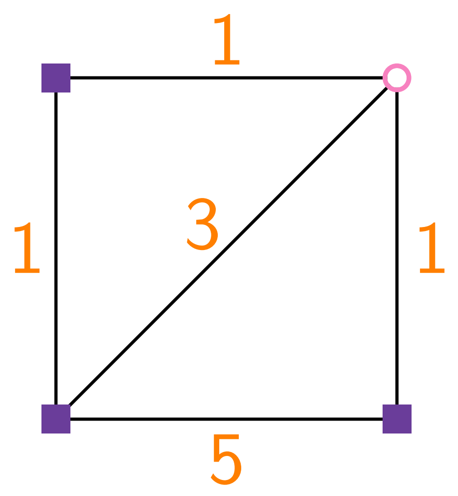

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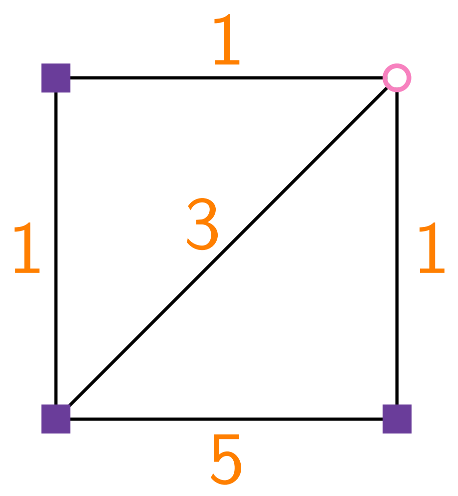

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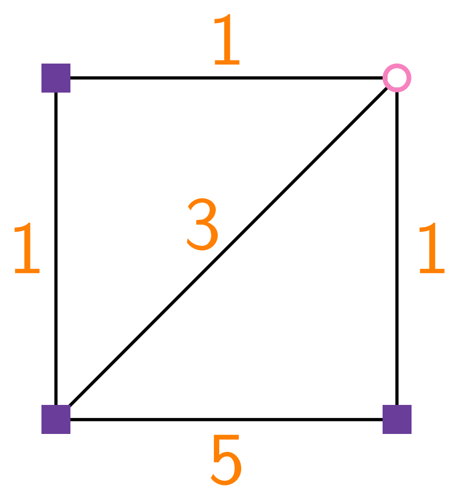

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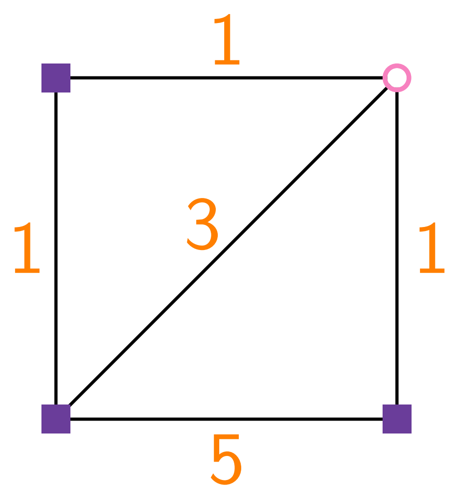

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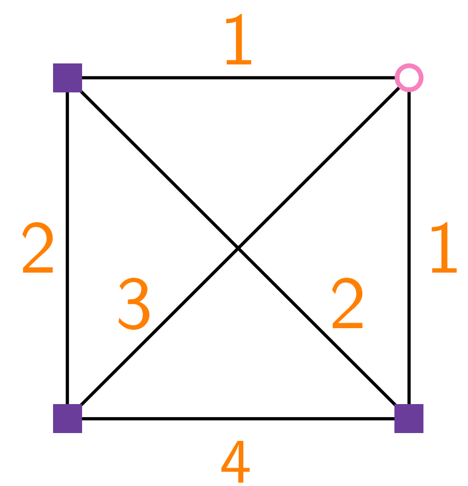

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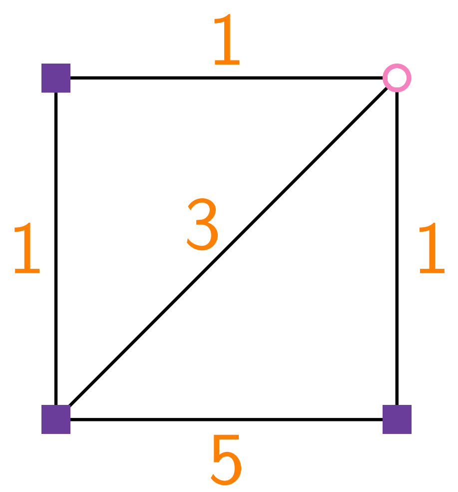

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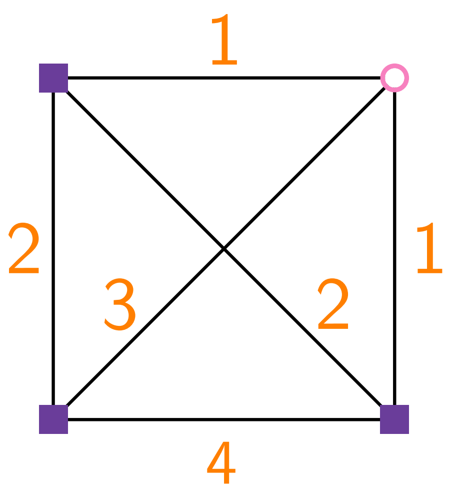

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Approximation Algorithms

Lecture 3:
SteinerTree and MultiwayCut

Part II:
Approximation Preserving Reduction

Approximation-Preserving Reduction

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problems Π_1 Π_2

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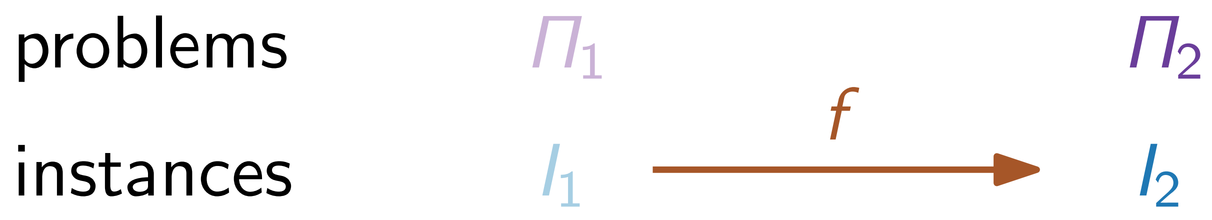

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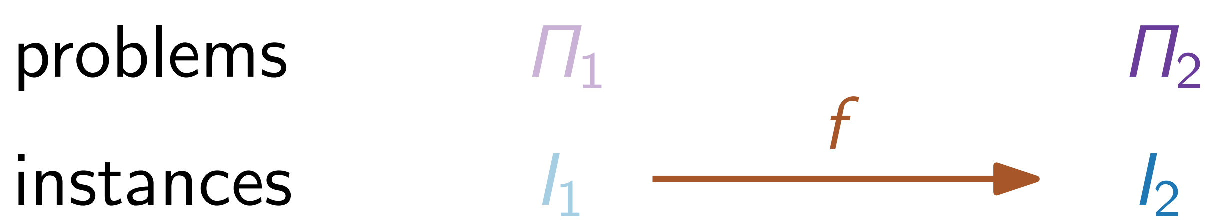

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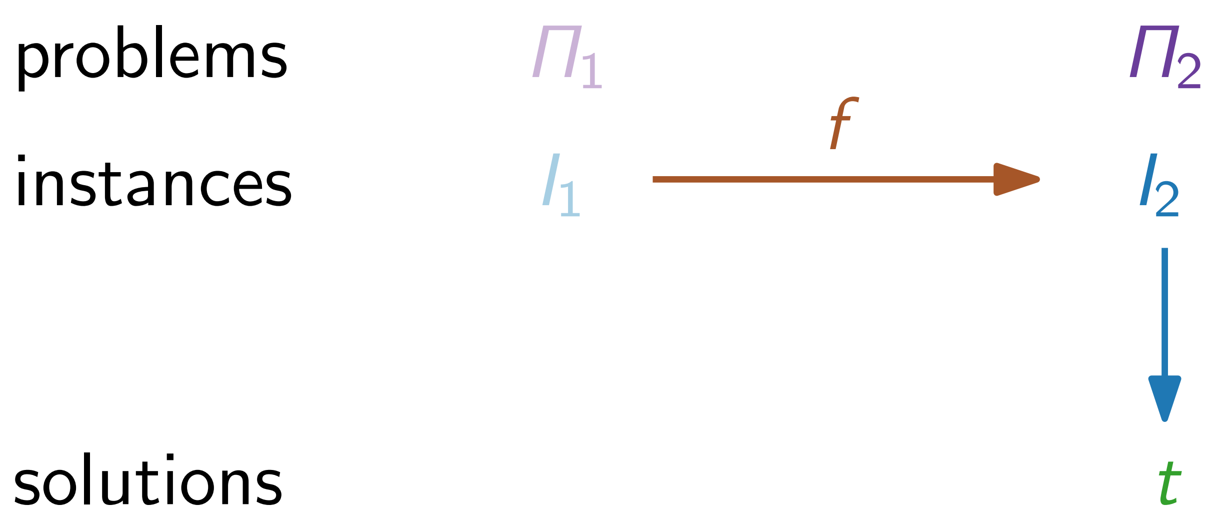

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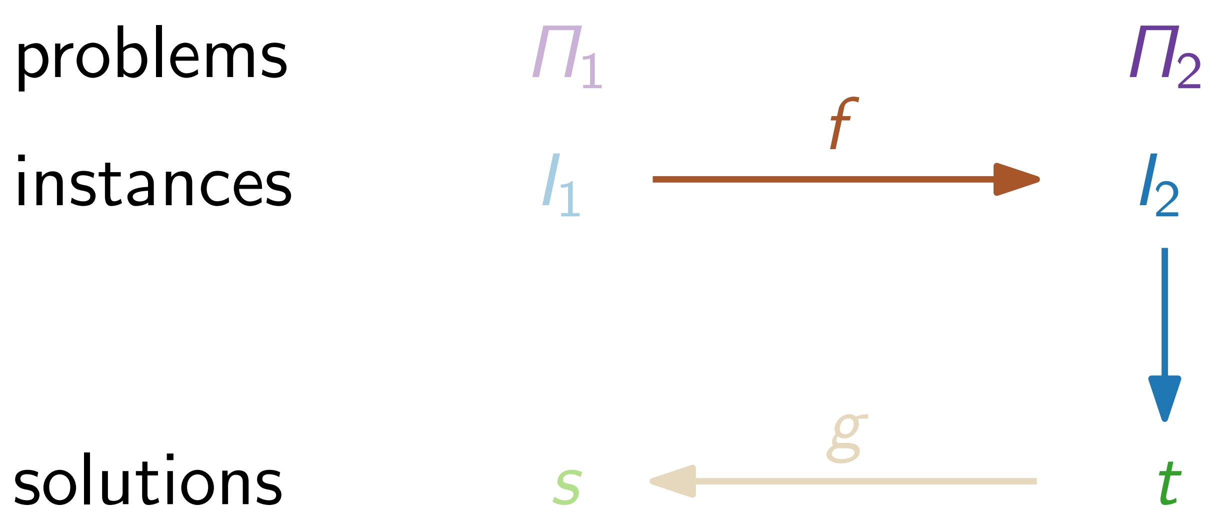

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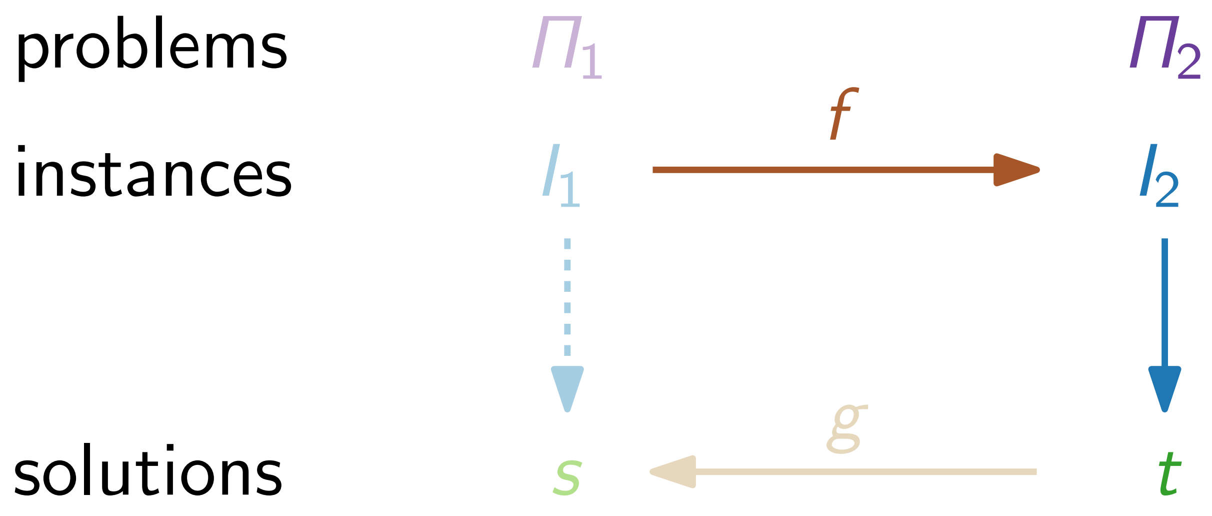

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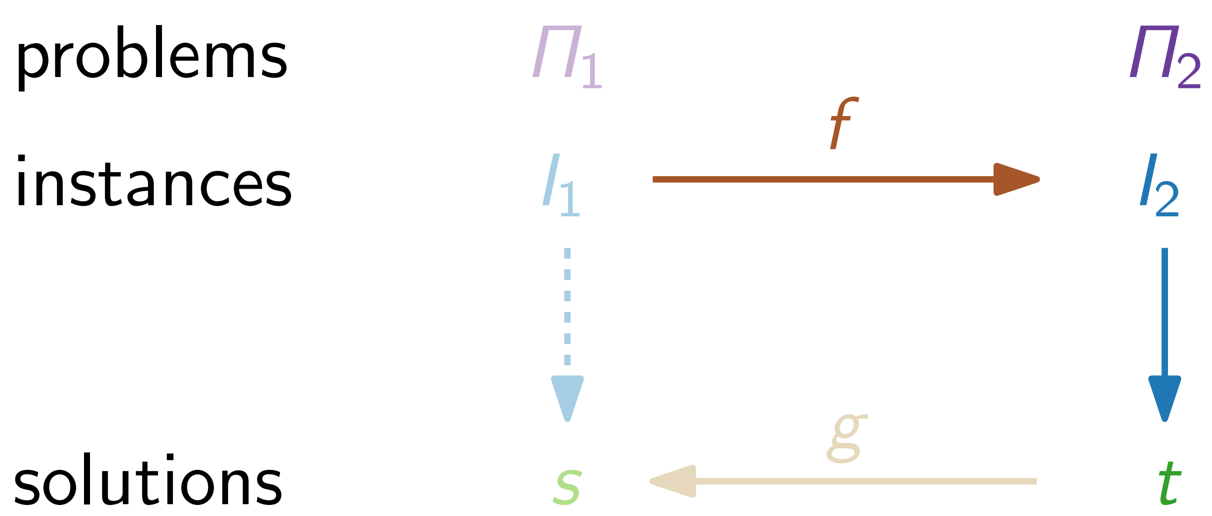

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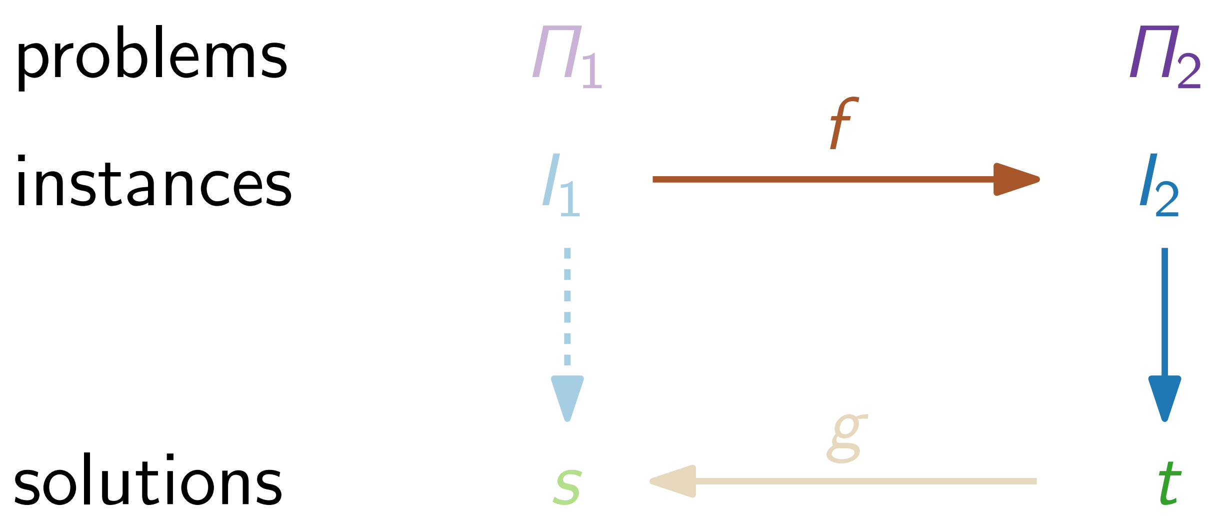

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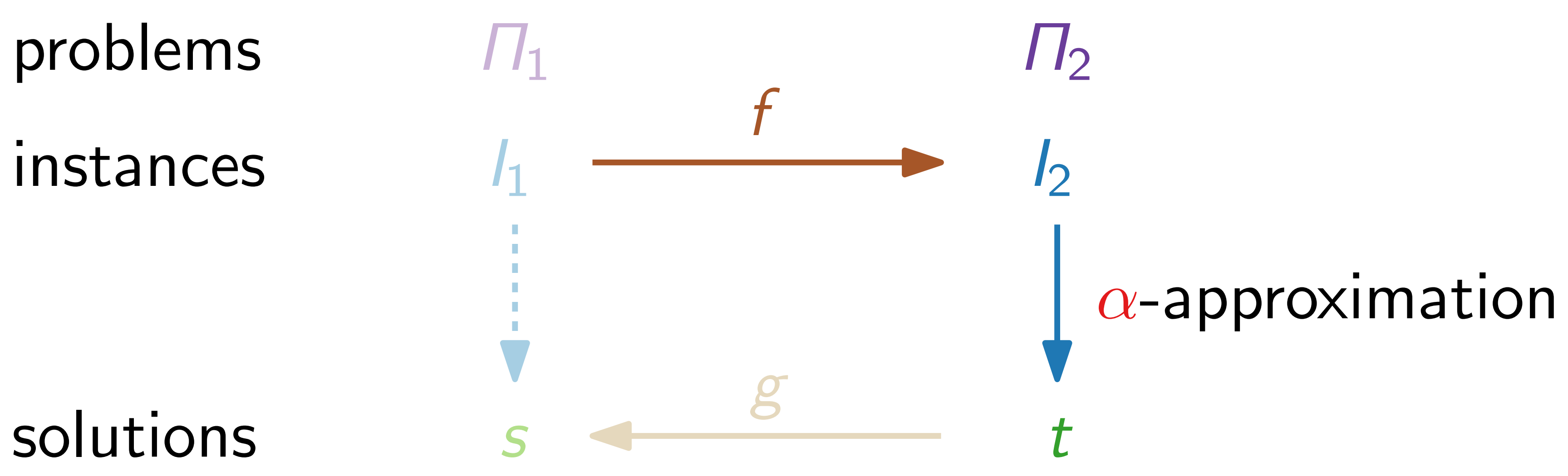

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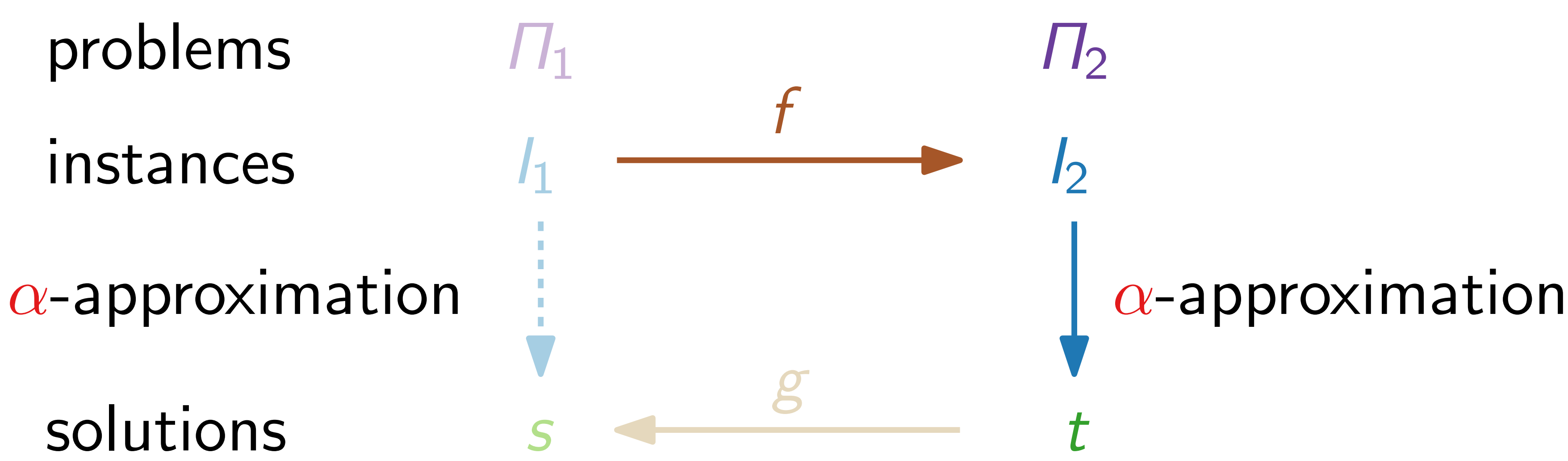

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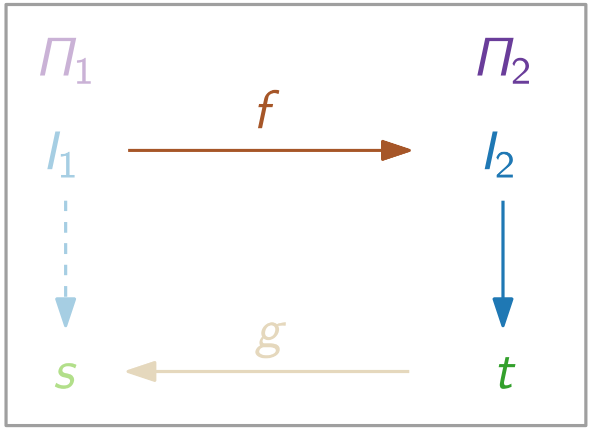

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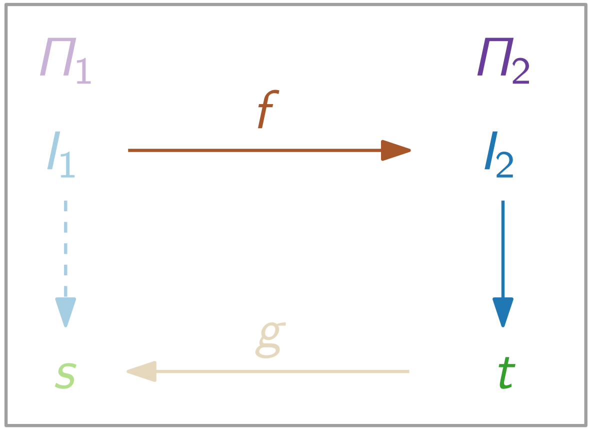

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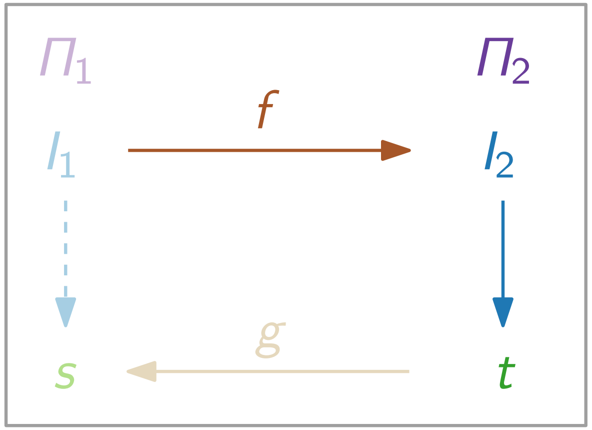

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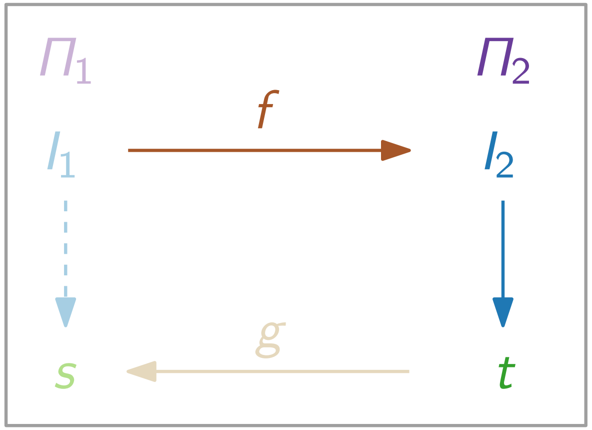

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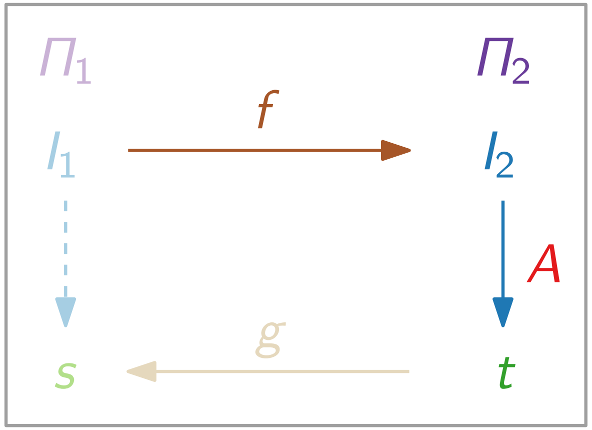

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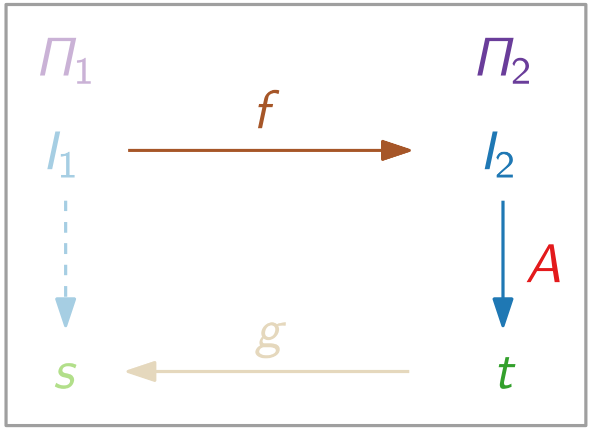

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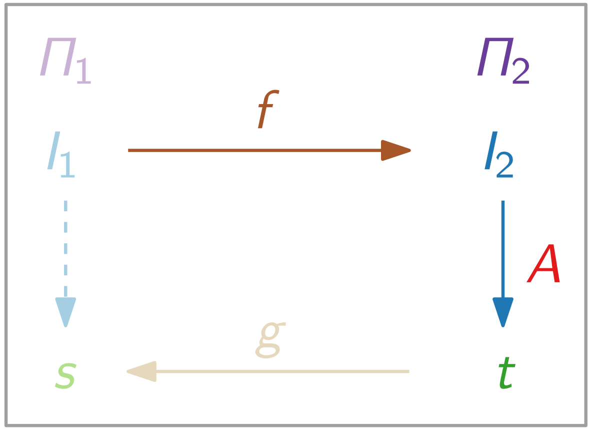

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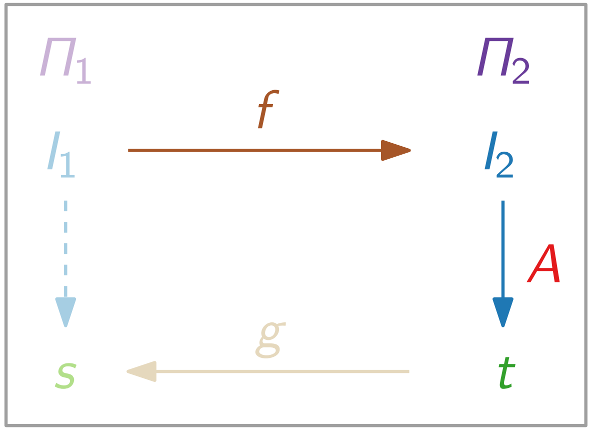

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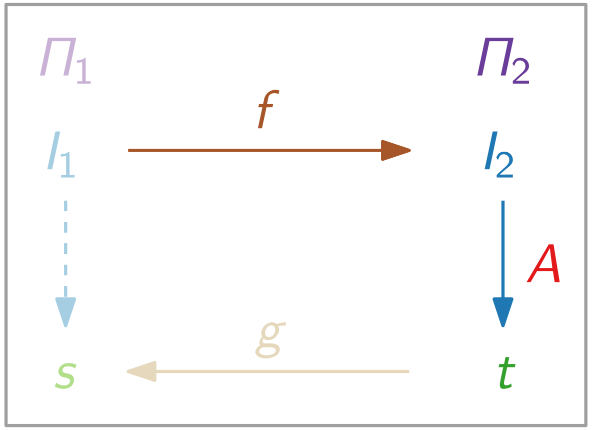

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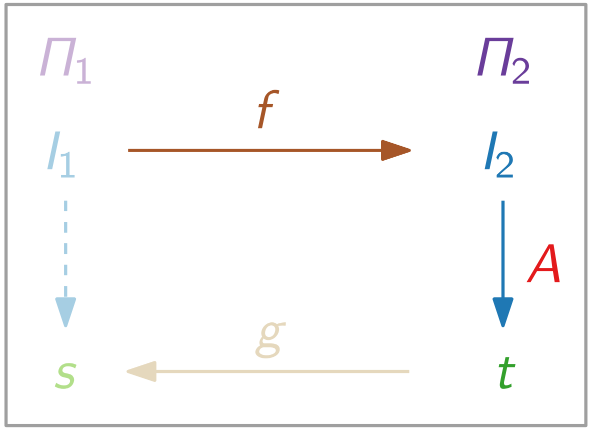

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Approximation Algorithms

Lecture 3:
SteinerTree and MultiwayCut

Part III:
Reduction to MetricSteinerTree

MetricSteinerTree

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Proof. (1) Mapping f $\quad I_1 \xrightarrow{f} I_2$

MetricSteinerTree

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Proof. (1) Mapping f

$$I_1 \xrightarrow{f} I_2$$

Instance I_1 of SteinerTree:

Graph $G_1 = (V, E_1)$, edge weights c_1, partition $V = T \mathbin{\dot\cup} S$

METRICSTEINERTREE

Theorem. There is an approximation-preserving reduction from STEINERTREE to METRICSTEINERTREE.

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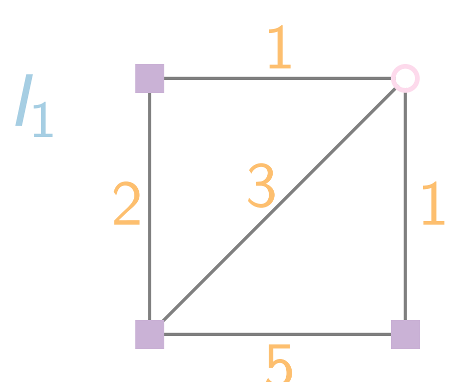

MetricSteinerTree

Theorem. There is an approximation-preserving reduction from SteinerTree to MetricSteinerTree.

Proof. (1) Mapping f

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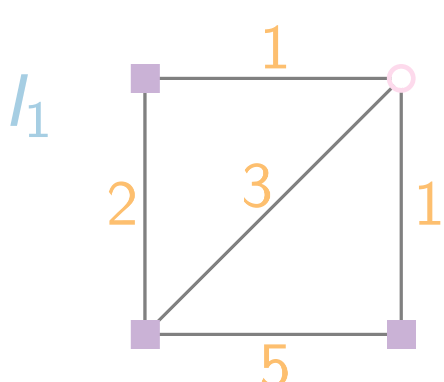
Complete graph $G_2 = (V, E_2)$, partition T, S as in I_1



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Proof. (1) Mapping f $\quad I_1 \xrightarrow{f} I_2$

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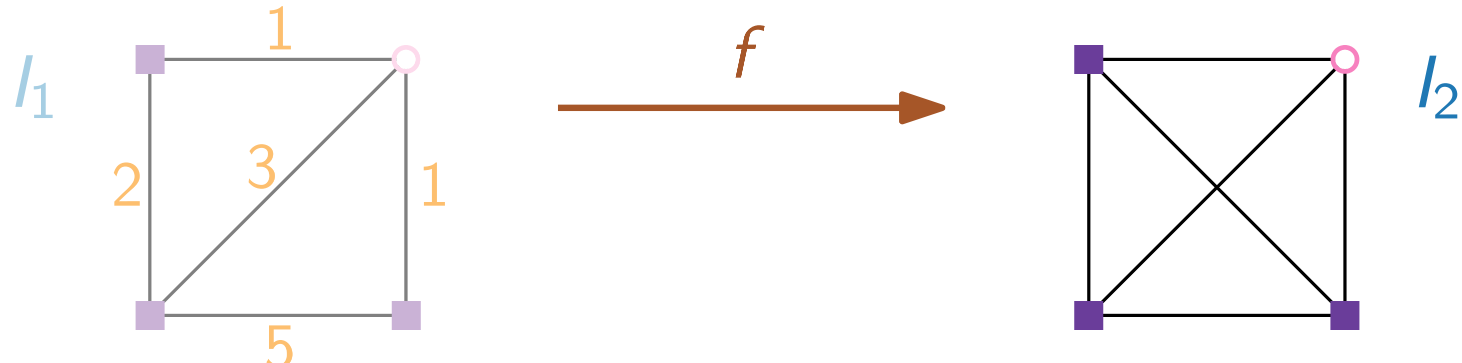

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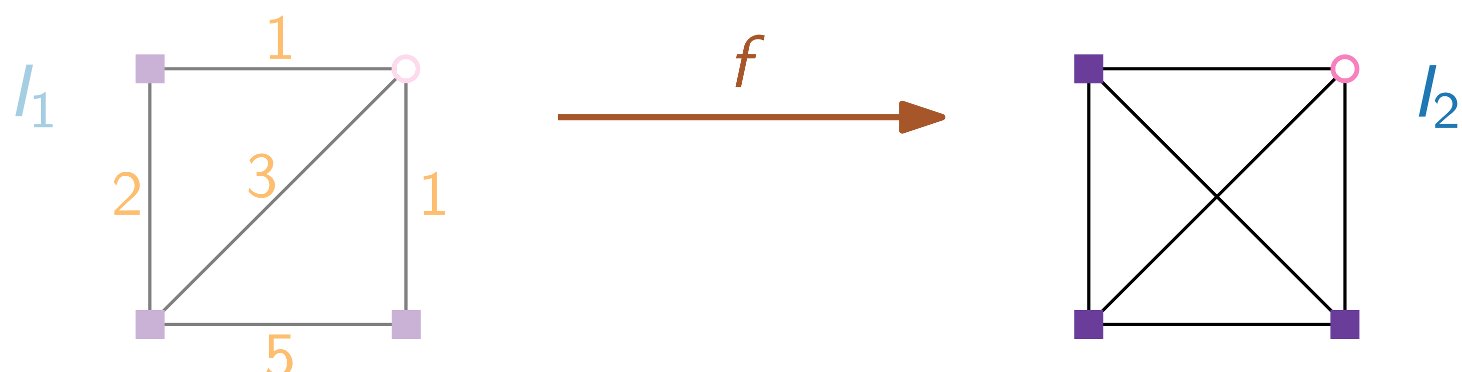

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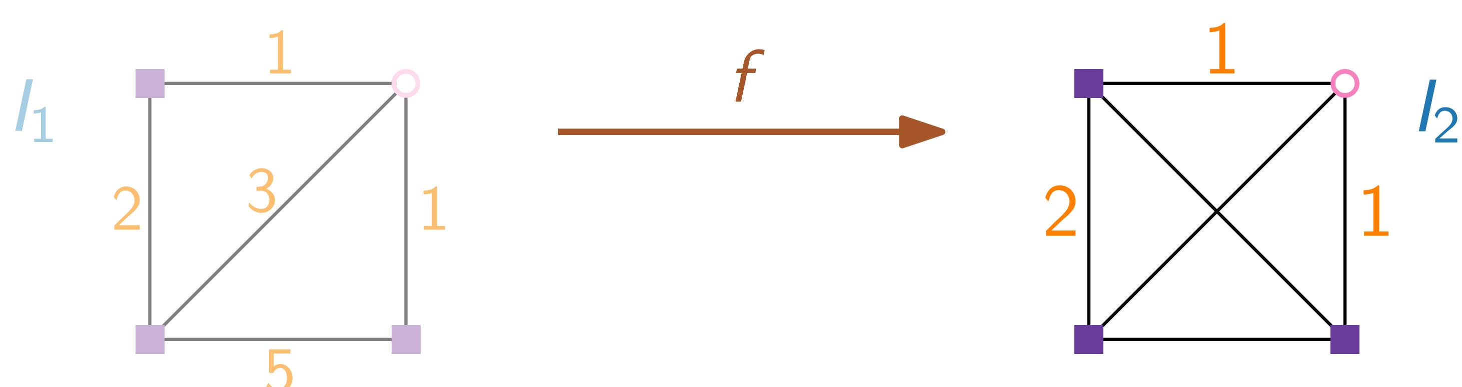

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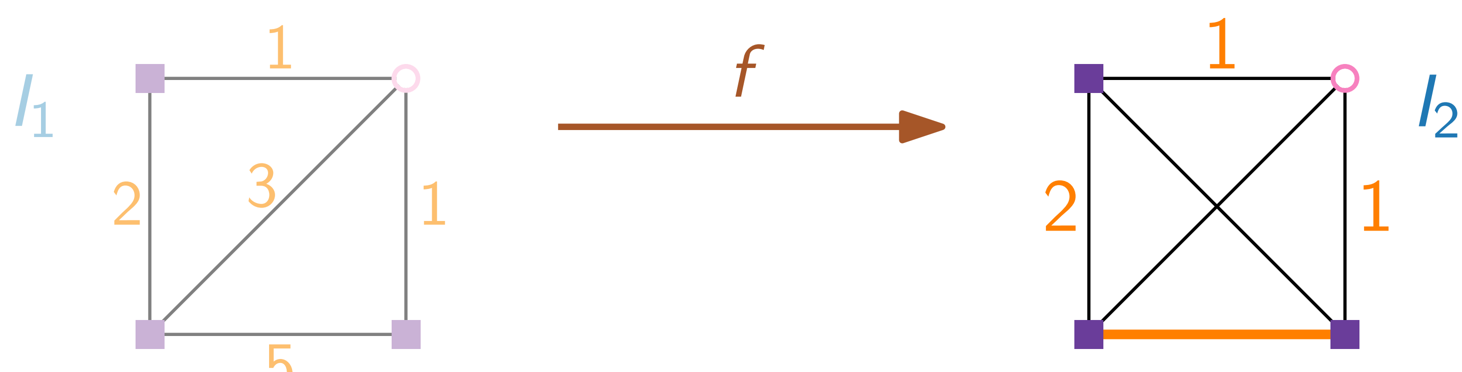

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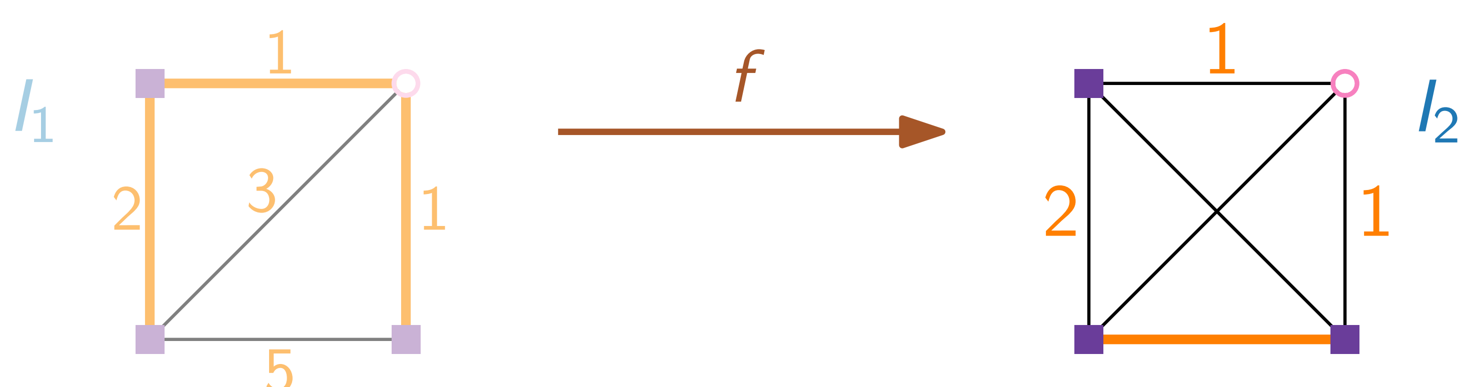

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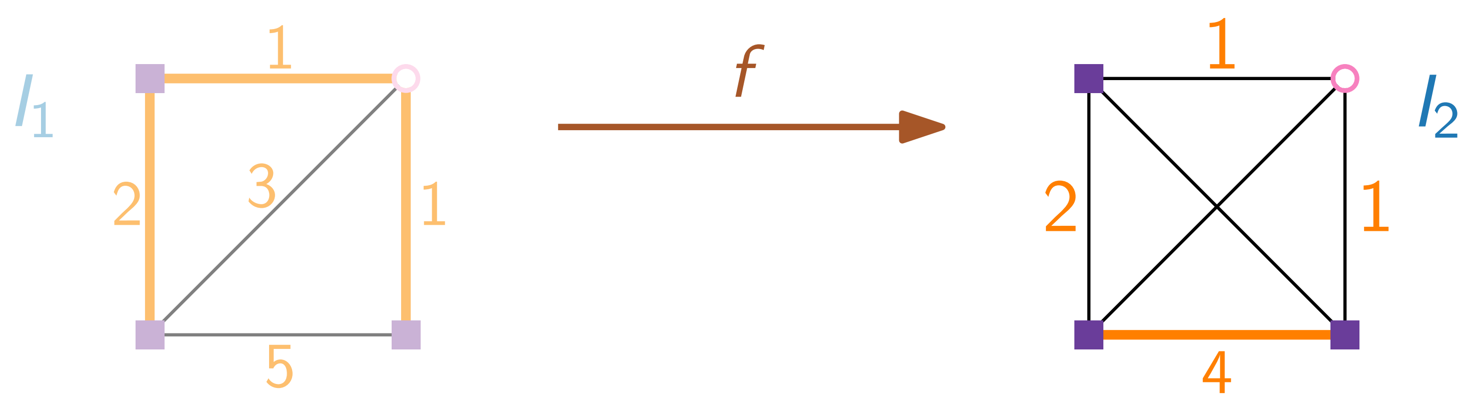

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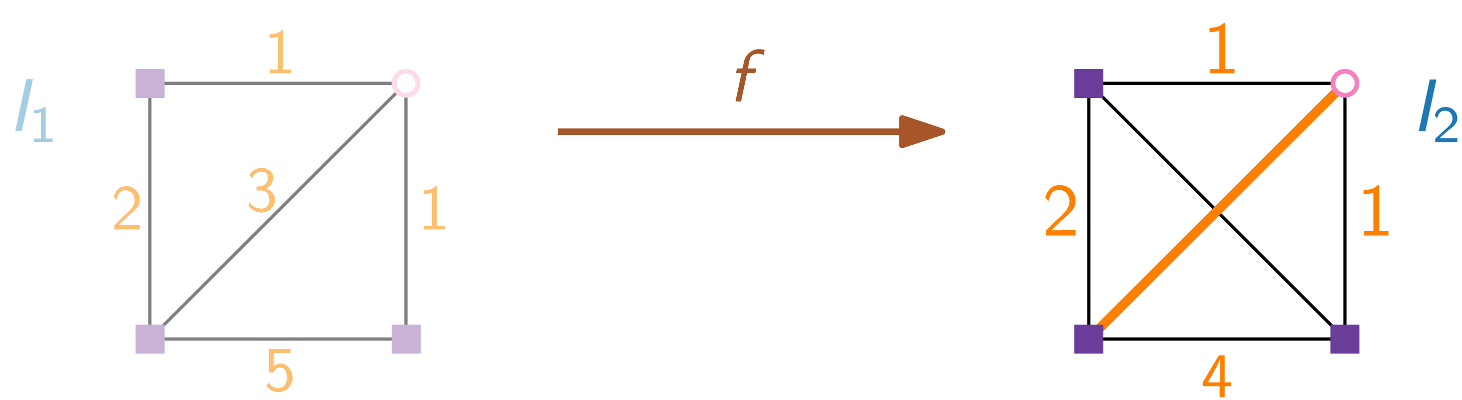

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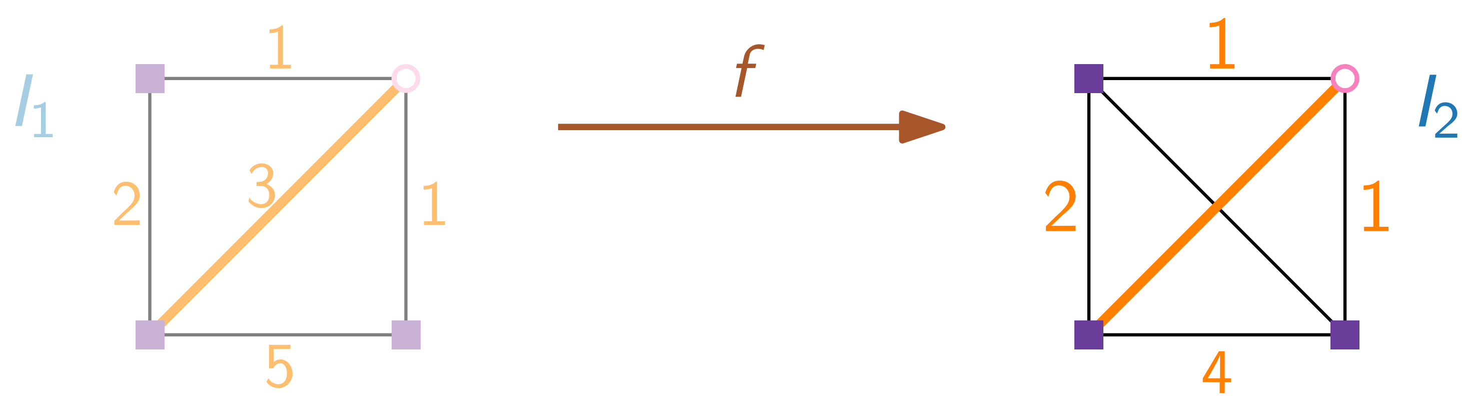

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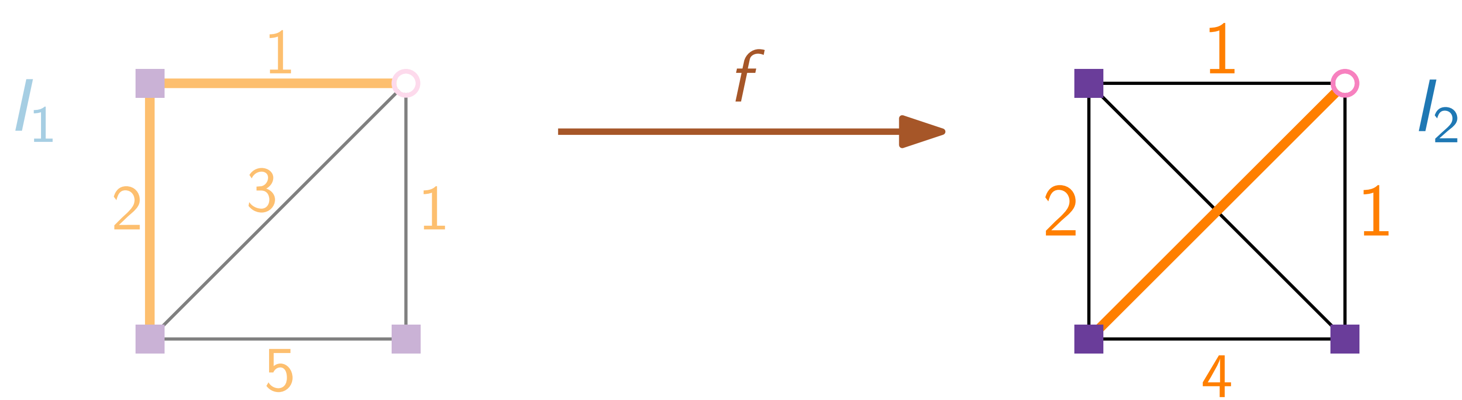

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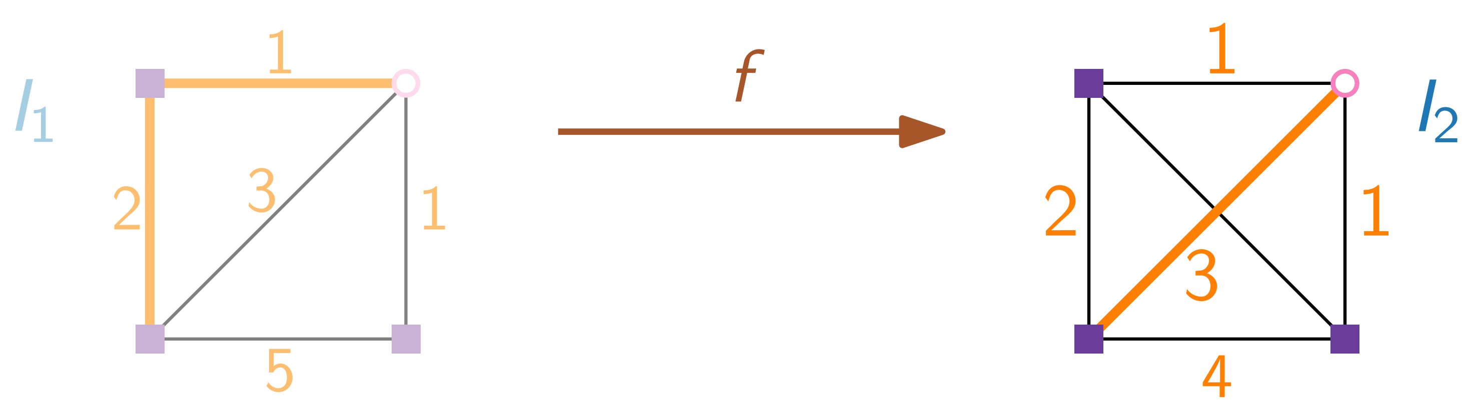

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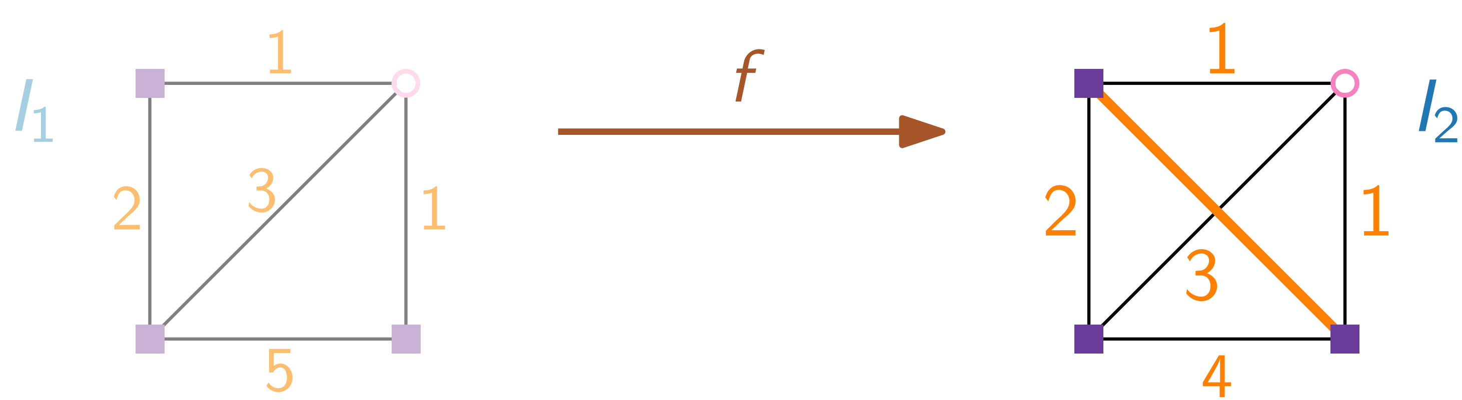

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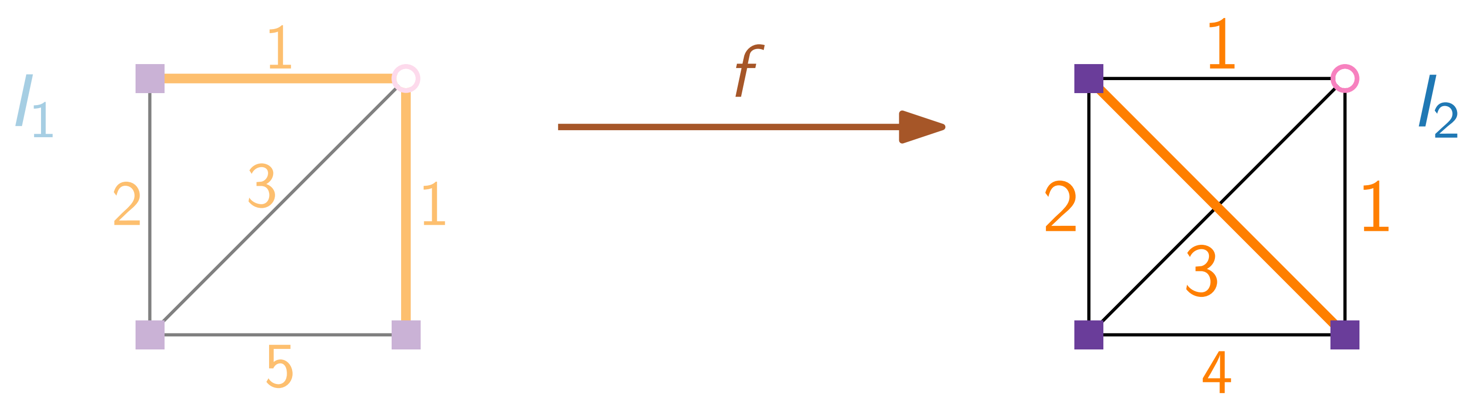

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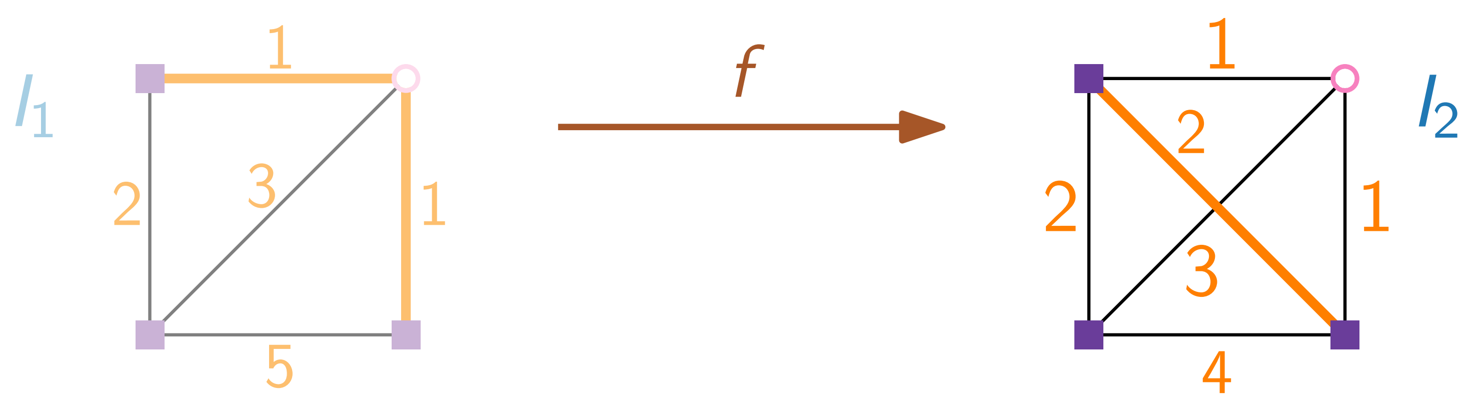

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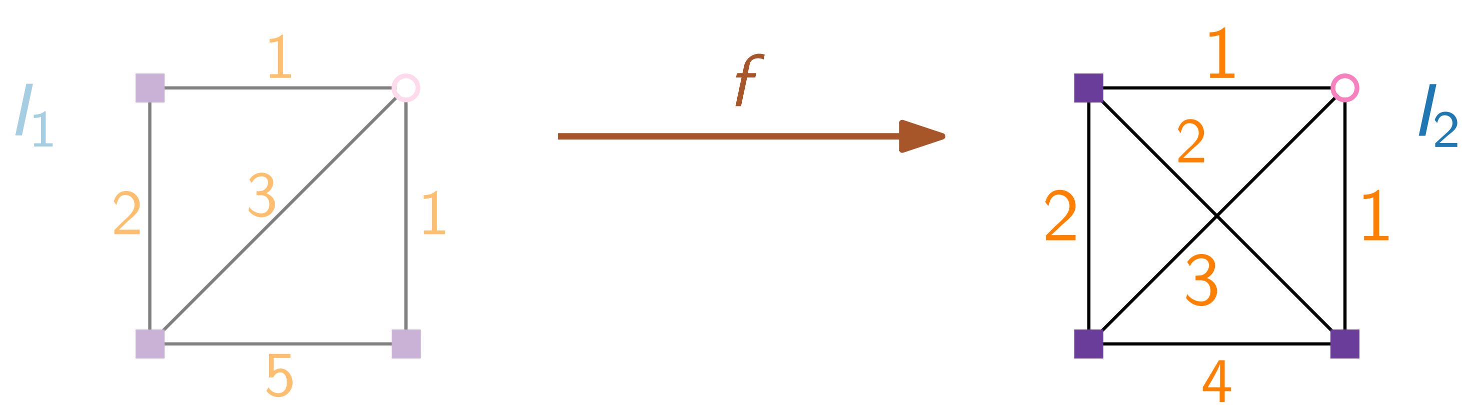

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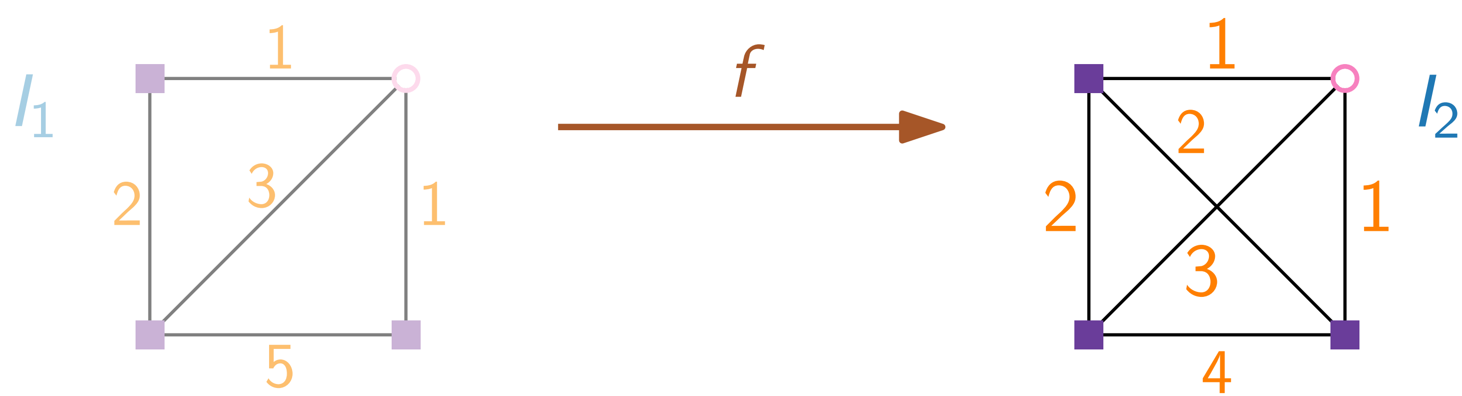
$c_2(u, v) \leq c_1(u, v)$ for every edge $(u, v) \in E_1$.



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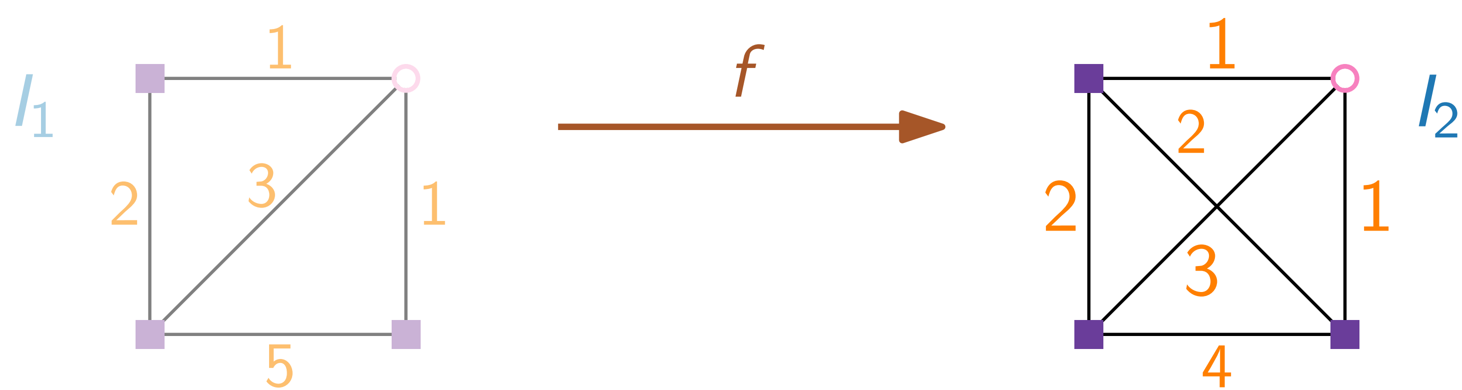
Theorem. There is an approximation-preserving reduction from SteinerTree to MetricSteinerTree.

Proof. (2) $\mathrm{OPT}(I_2) \leq \mathrm{OPT}(I_1)$



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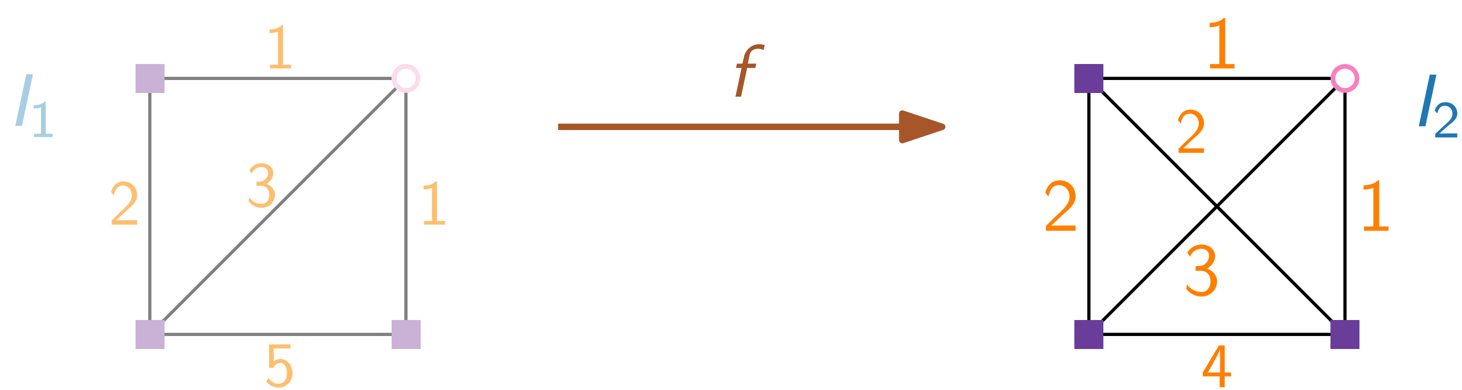
Theorem. There is an approximation-preserving reduction from SteinerTree to MetricSteinerTree.

Proof. (2) $\mathrm{OPT}(I_2) \leq \mathrm{OPT}(I_1)$

Let B^* be an optimal Steiner tree for I_1.



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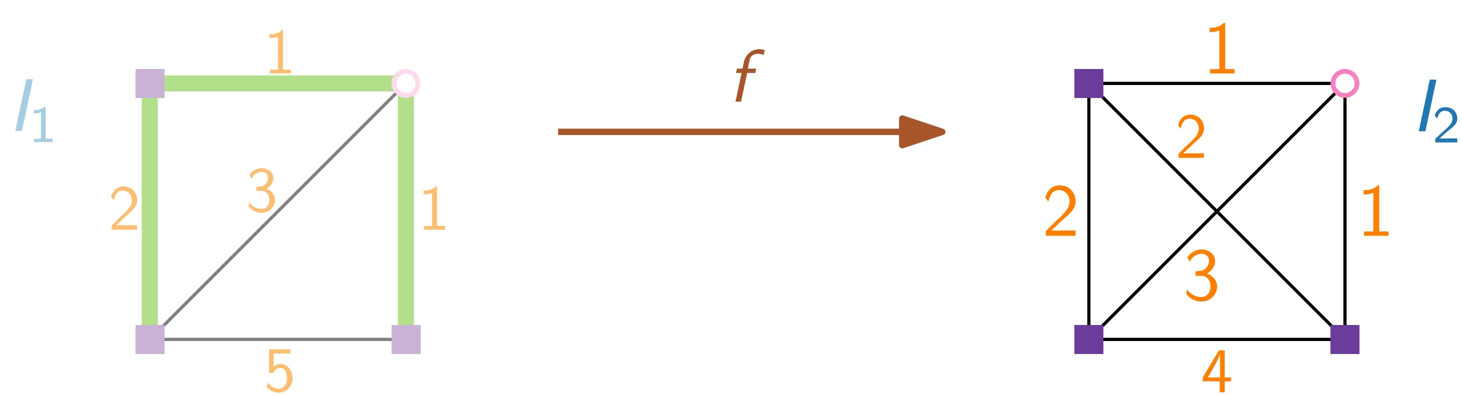

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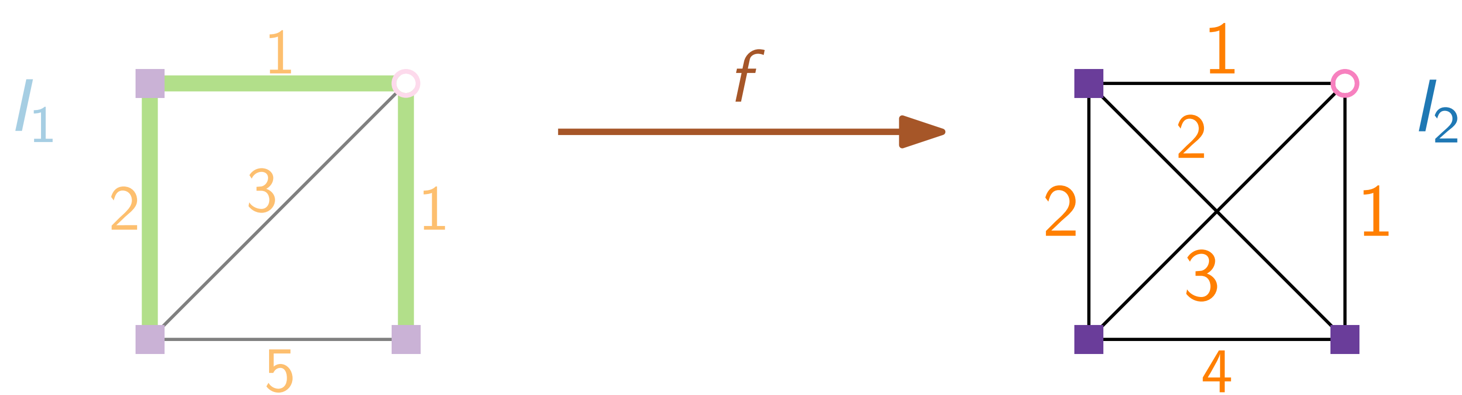

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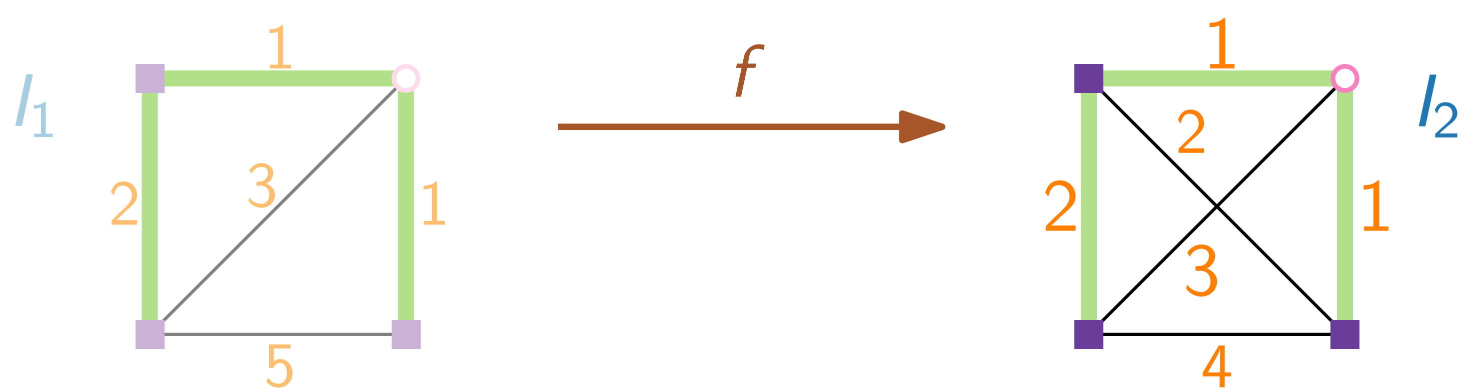

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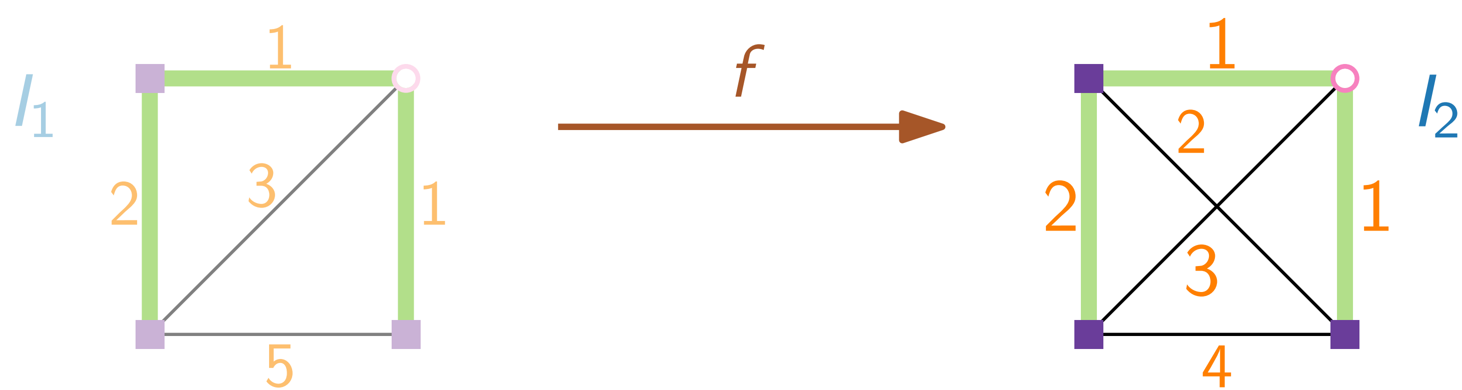
$\mathrm{OPT}(I_2)$



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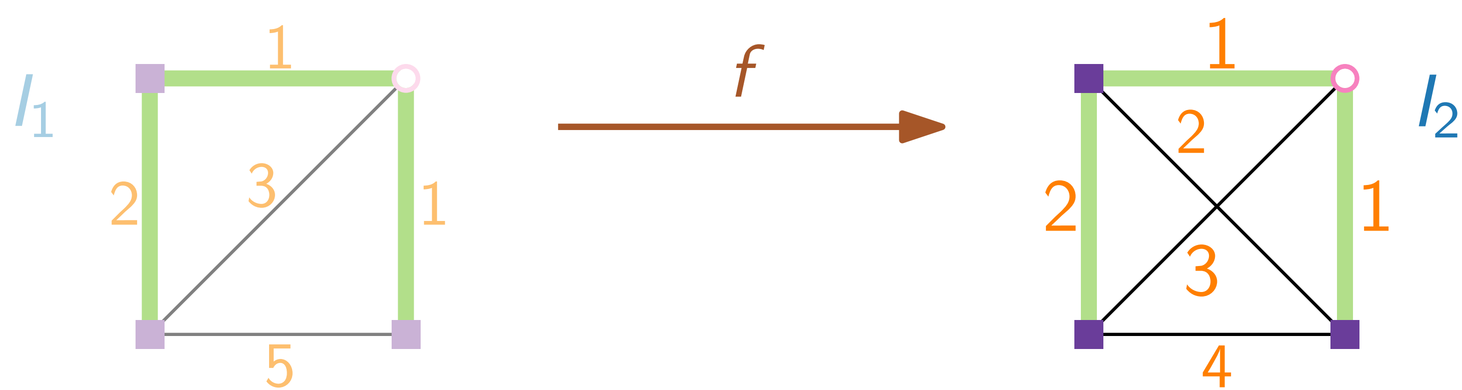

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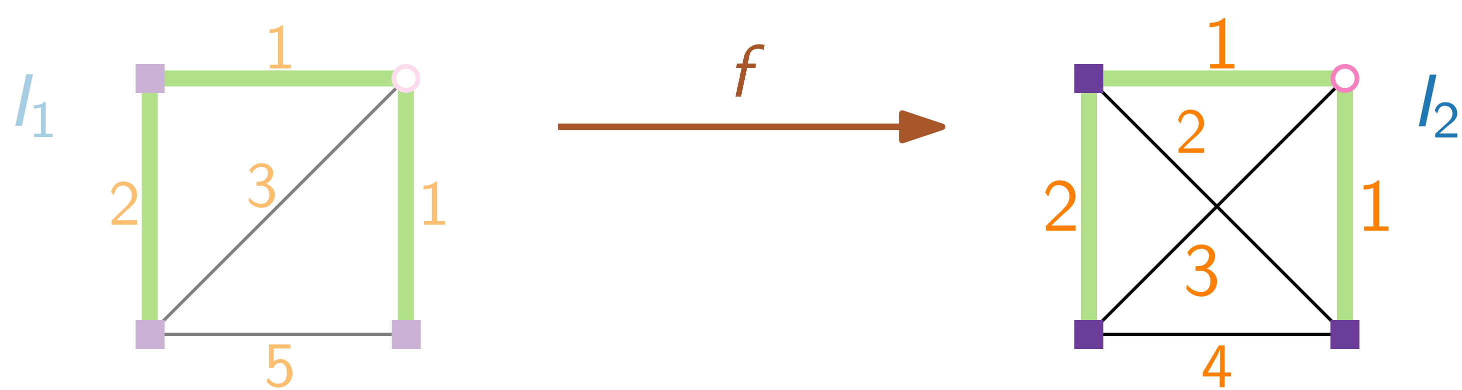

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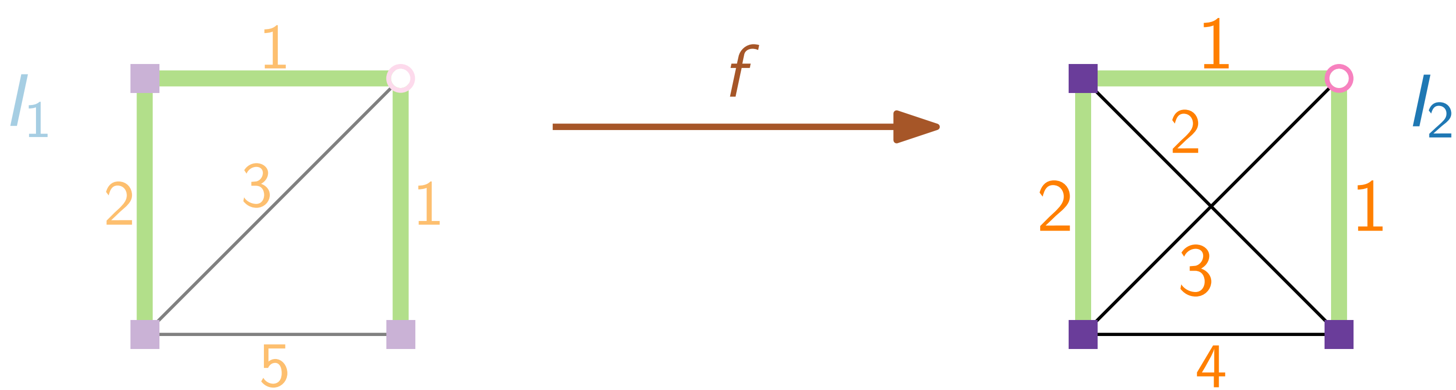

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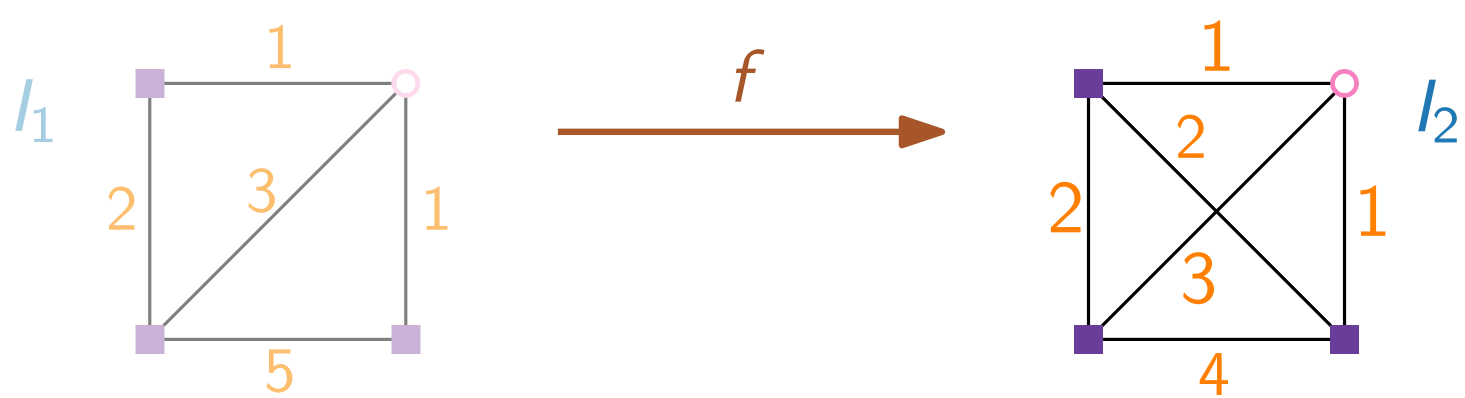
Proof. (3) Mapping g



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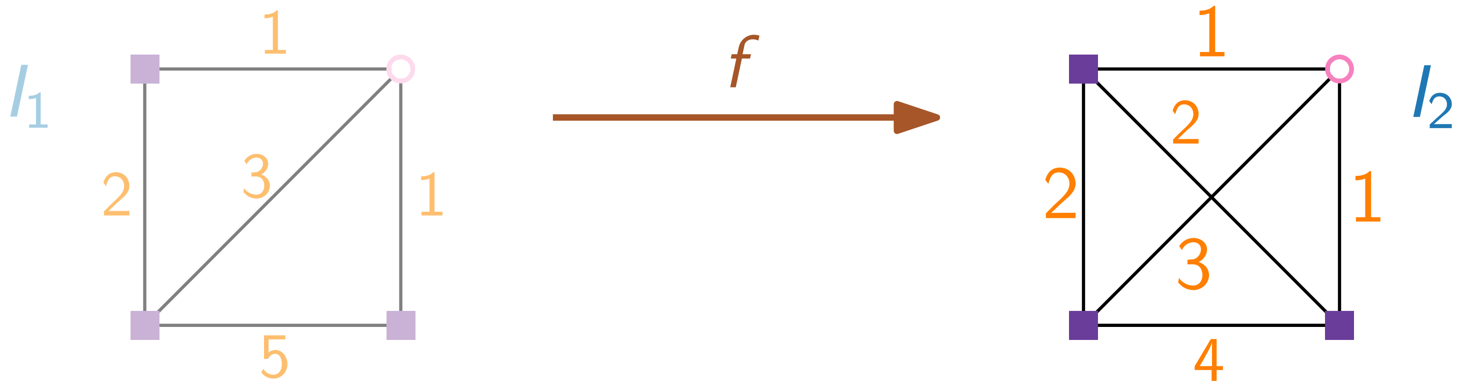

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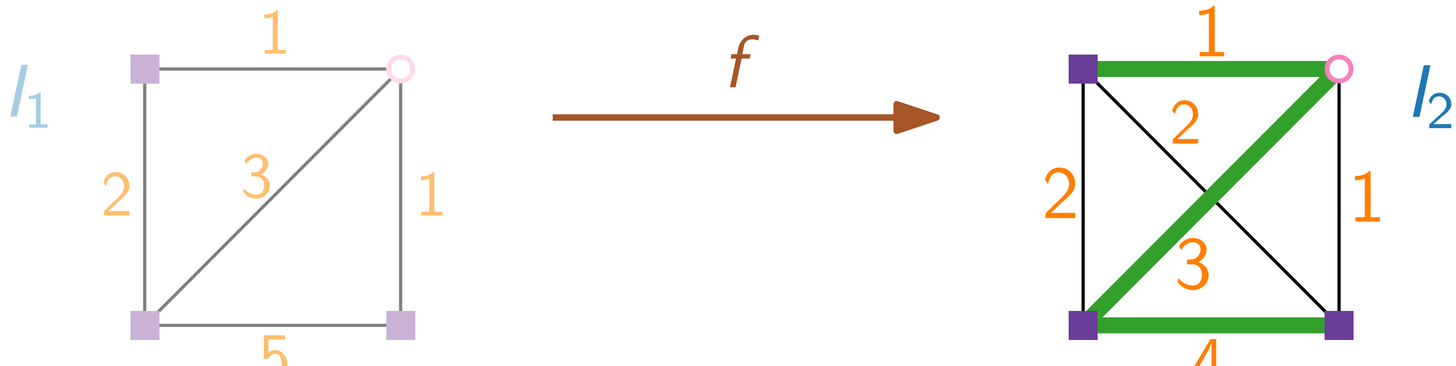

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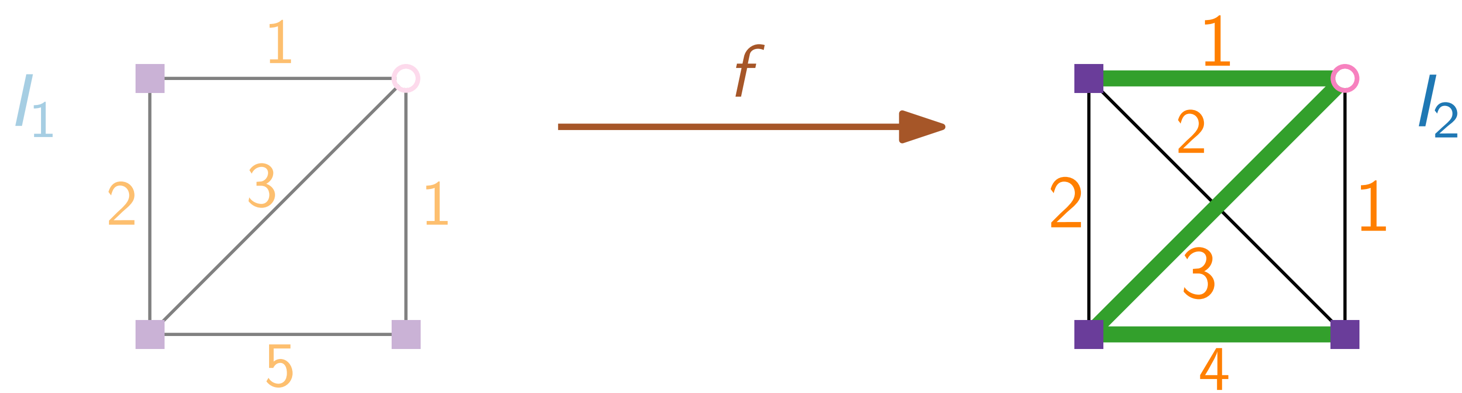

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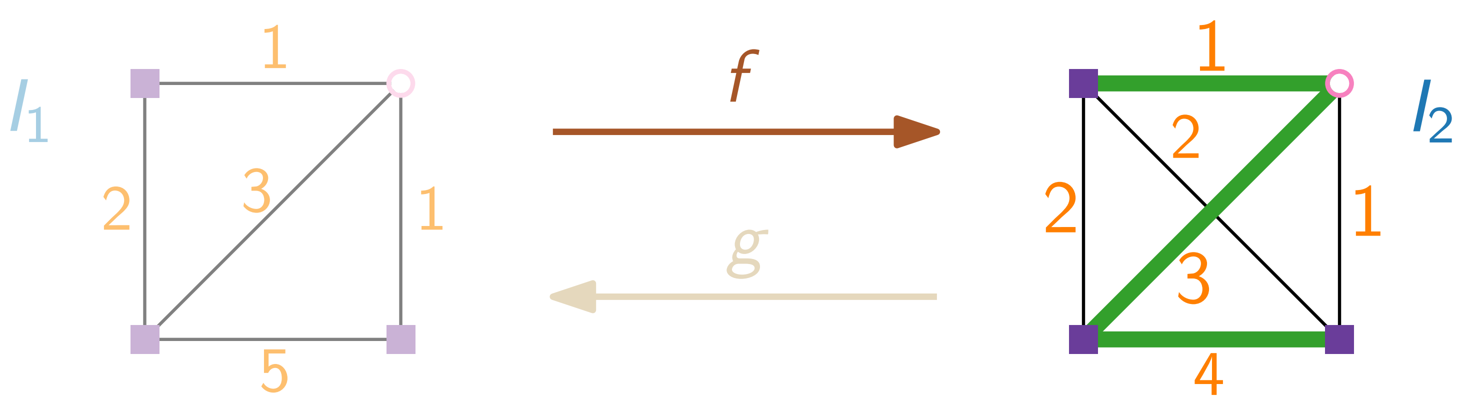

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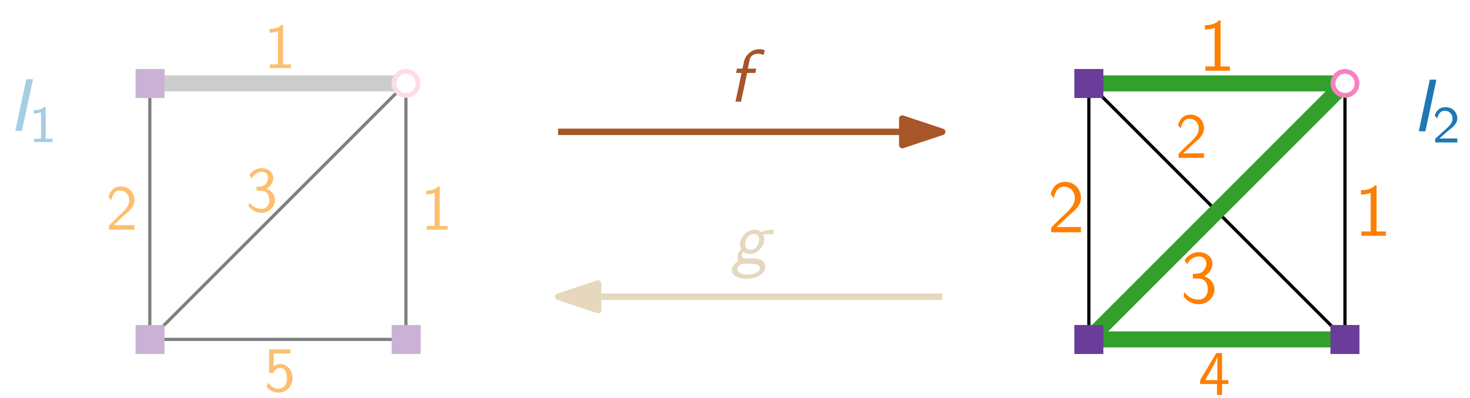

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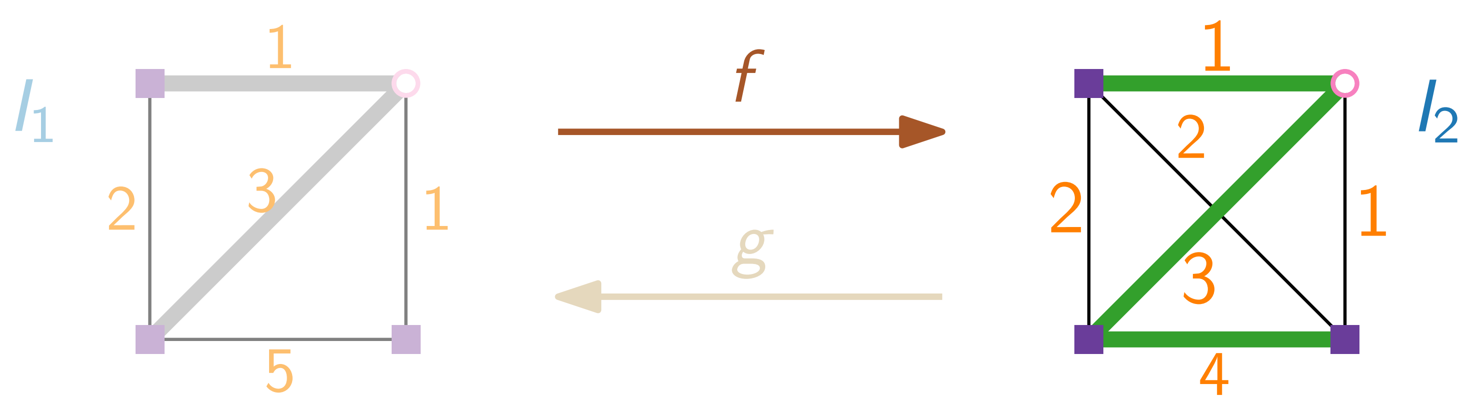

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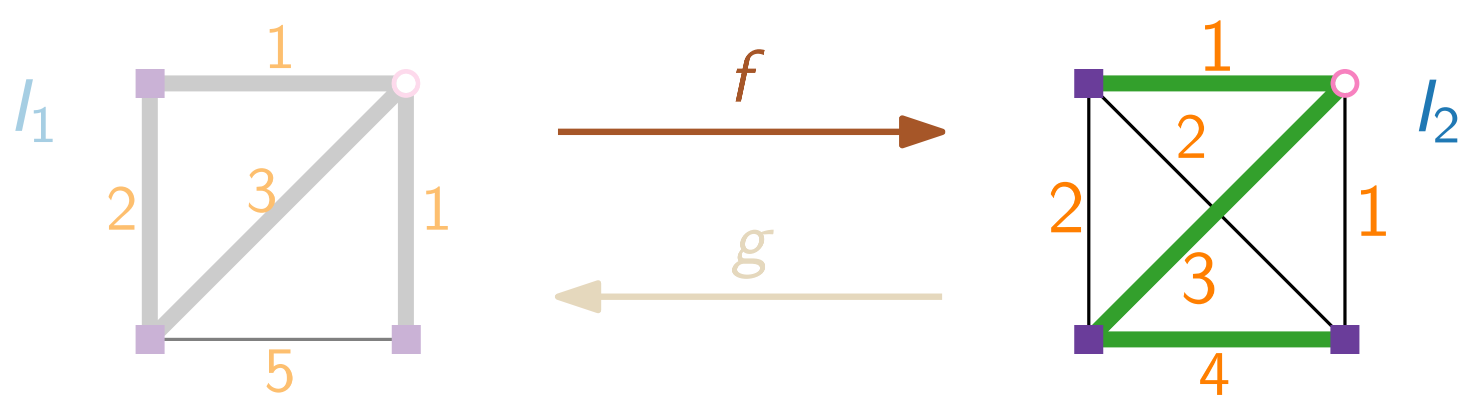

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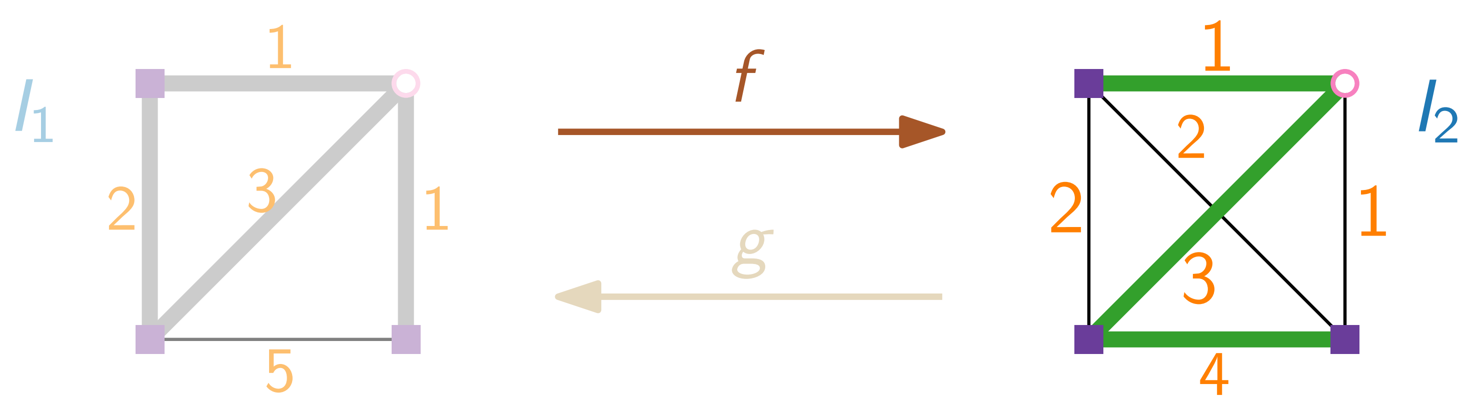

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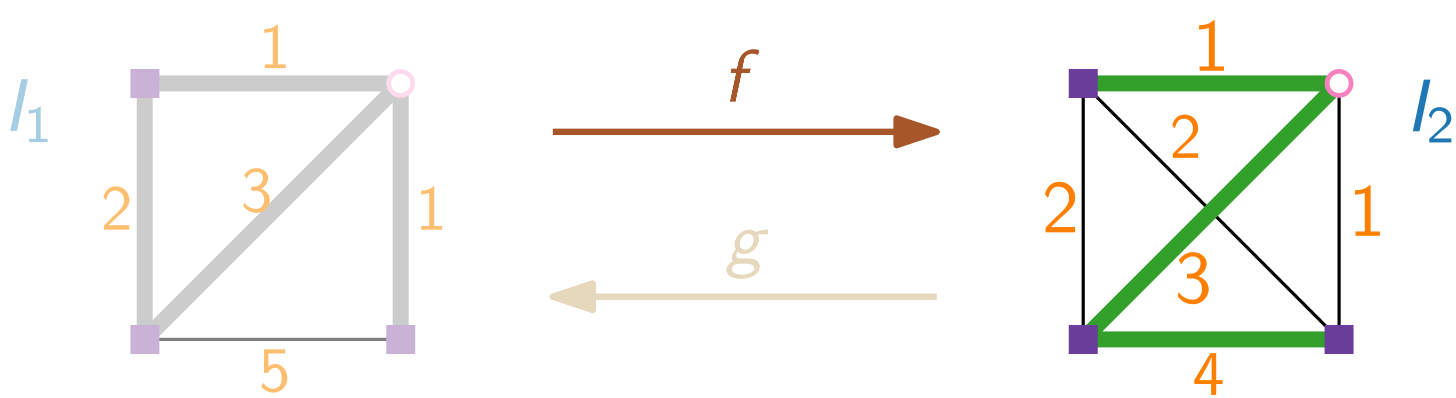

MetricSteinerTree

Theorem. There is an approximation-preserving reduction from SteinerTree to MetricSteinerTree.

Proof. (3) Mapping g $s \xleftarrow{g} t$

Let B_2 be a Steiner tree of G_2.

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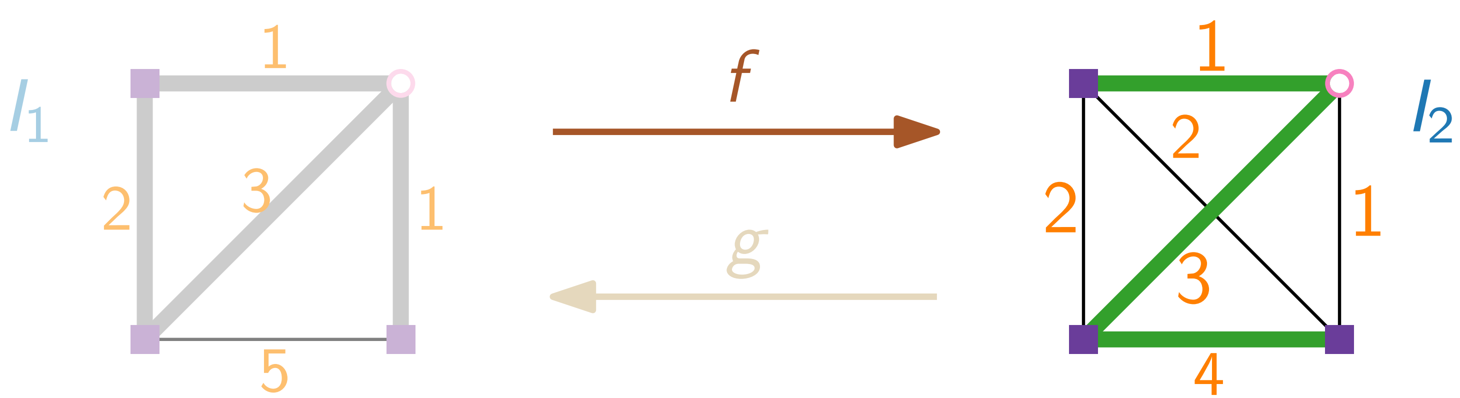

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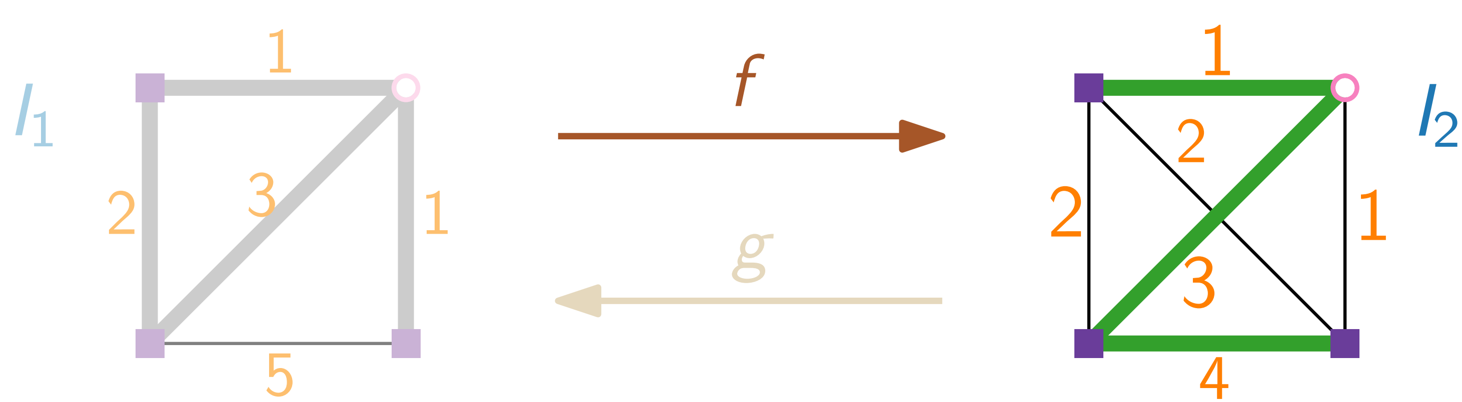

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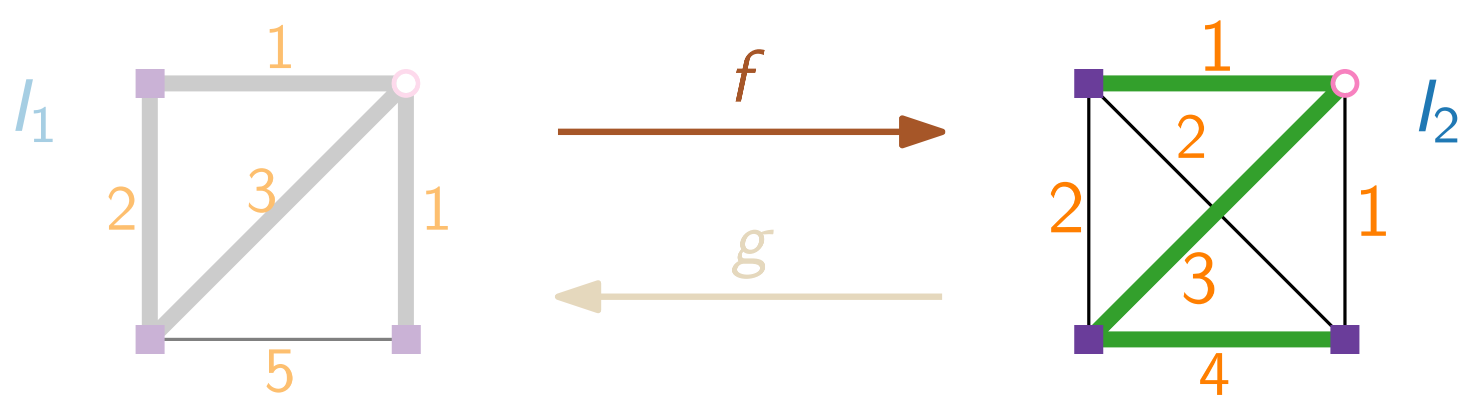

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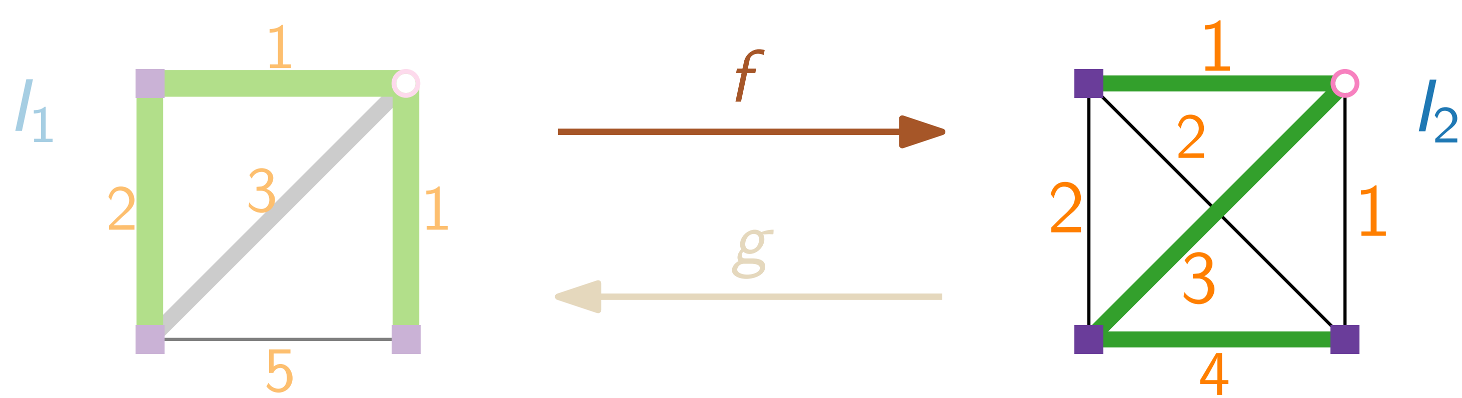

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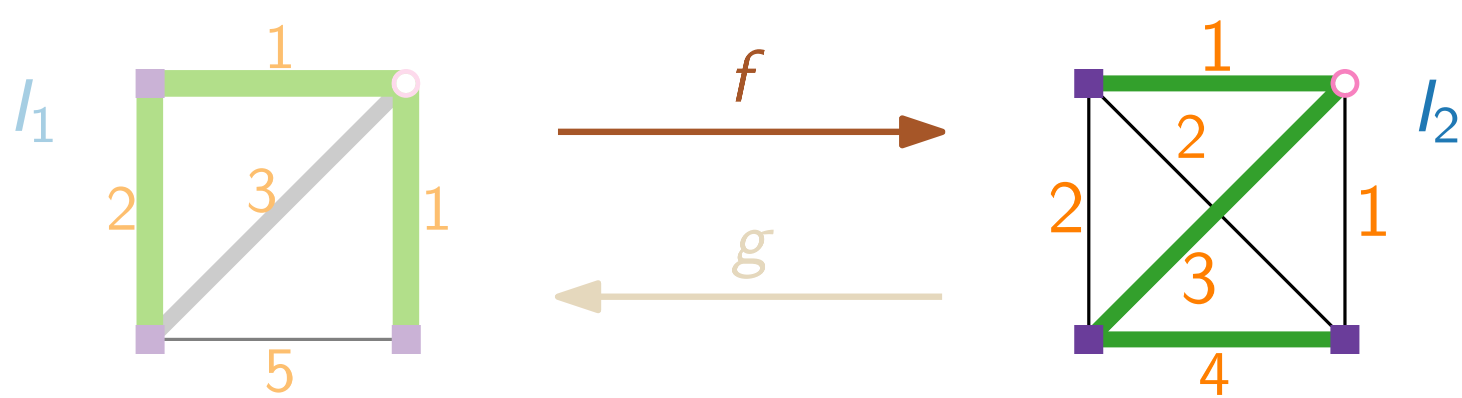

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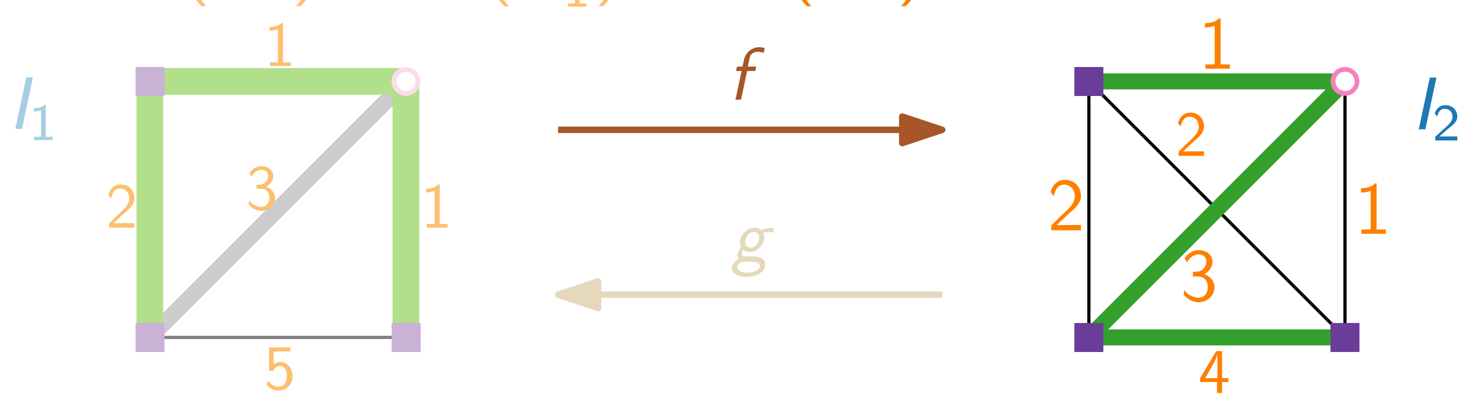
Note that $c_1(B_1) \leq c_1(G_1') \leq c_2(B_2)$.



Approximation Algorithms

Lecture 3:
SteinerTree and MultiwayCut

Part IV:
2-Approximation for SteinerTree

2-Approximation for SteinerTree

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Theorem. For an instance of MetricSteinerTree, let B be a minimum spanning tree (MST) of the subgraph $G[T]$ induced by the terminal set T. Then $c(B) \leq 2 \cdot \mathrm{OPT}$.

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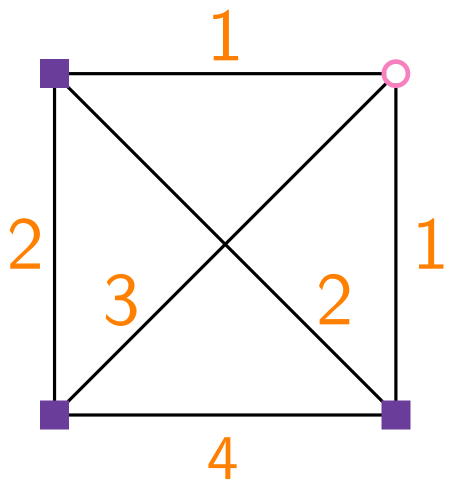

G

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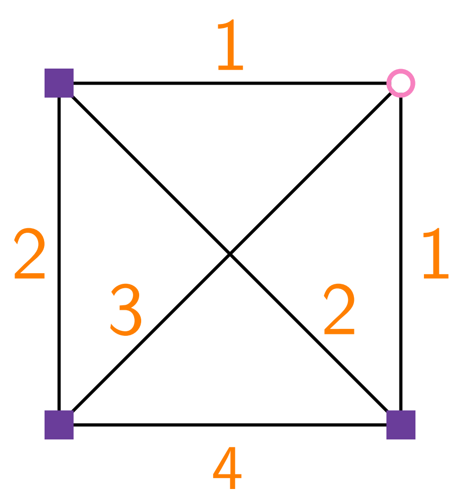

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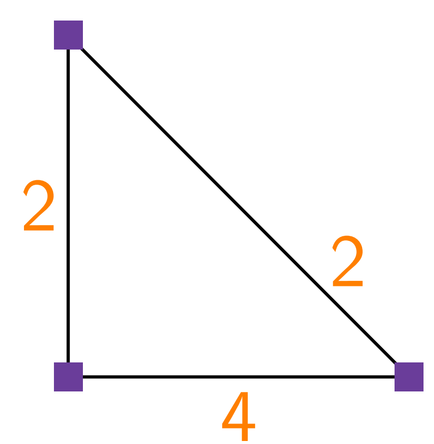

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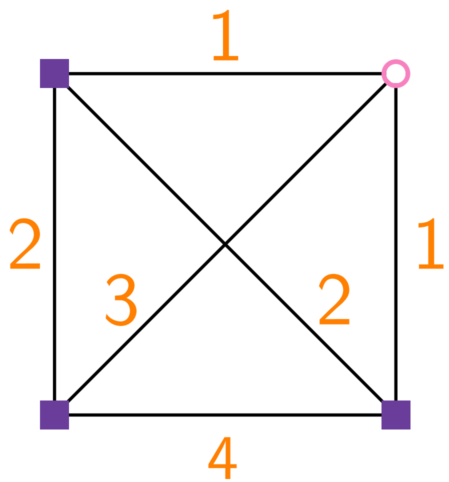

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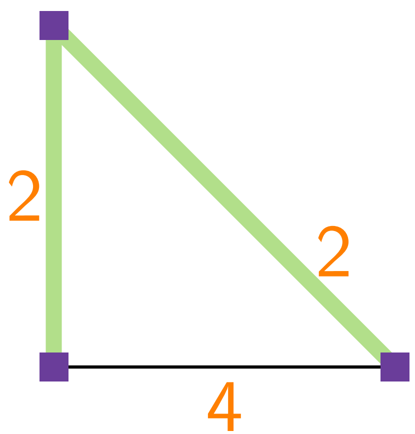

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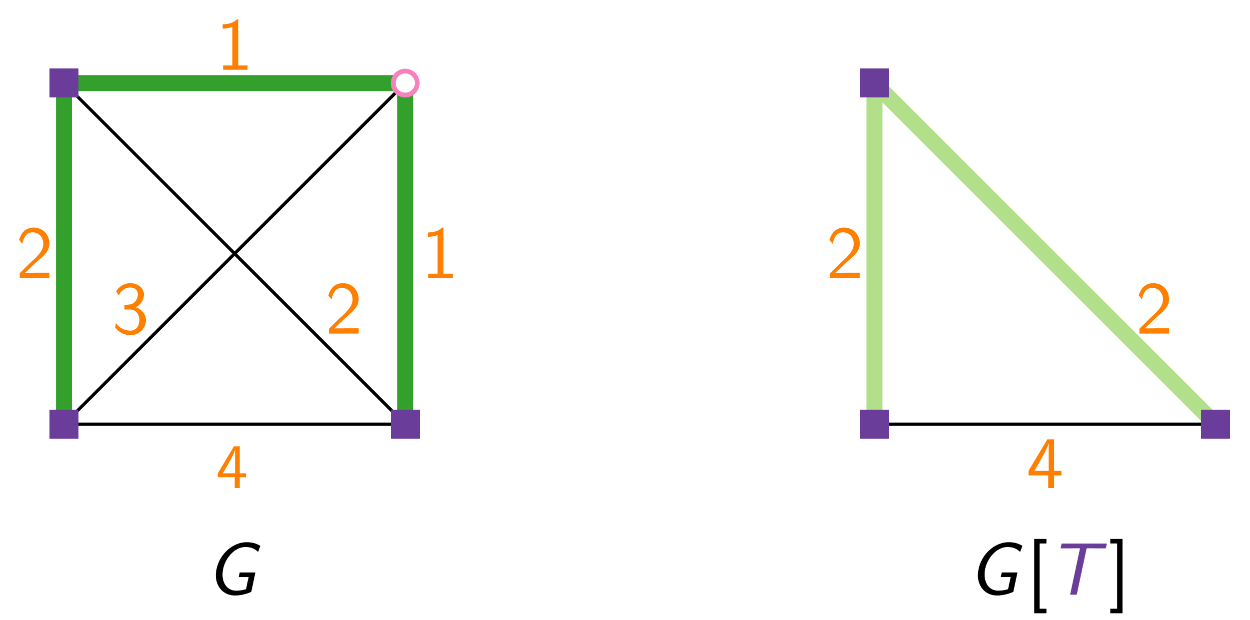

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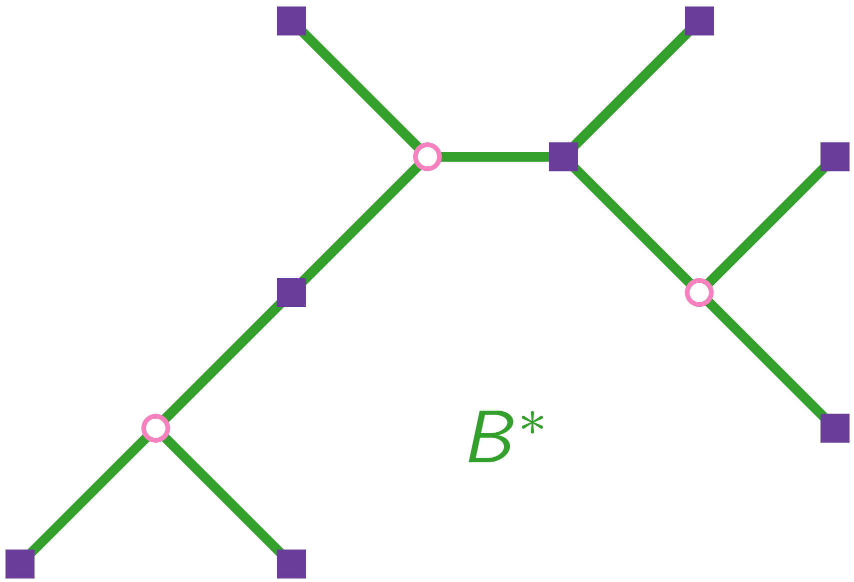

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Duplicate all edges of B^*.

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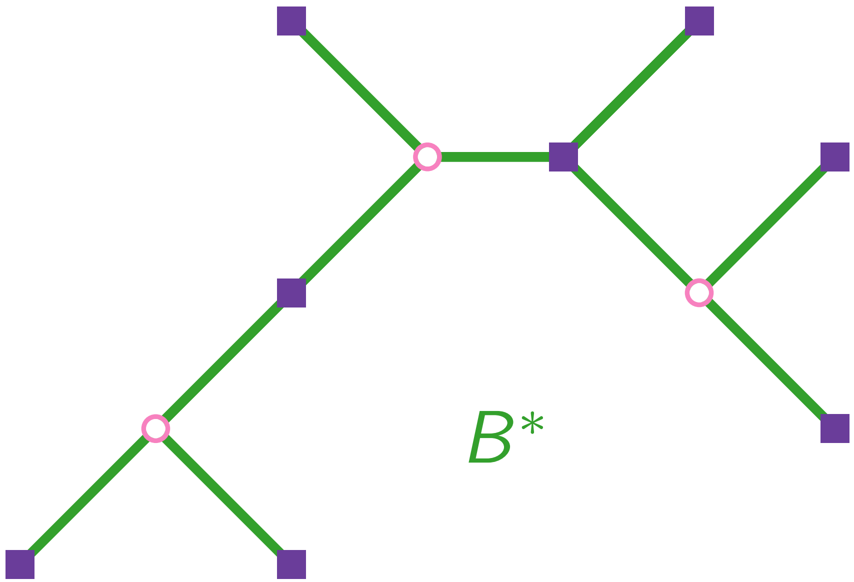

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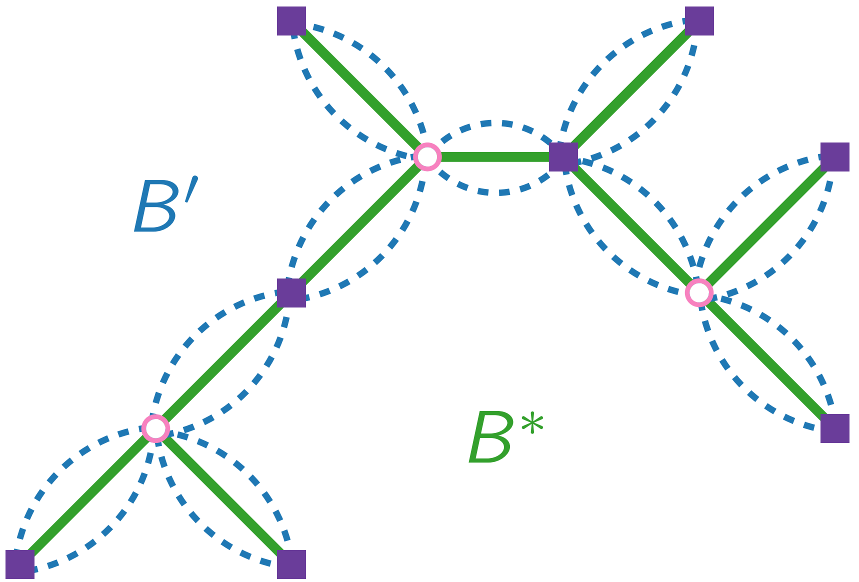

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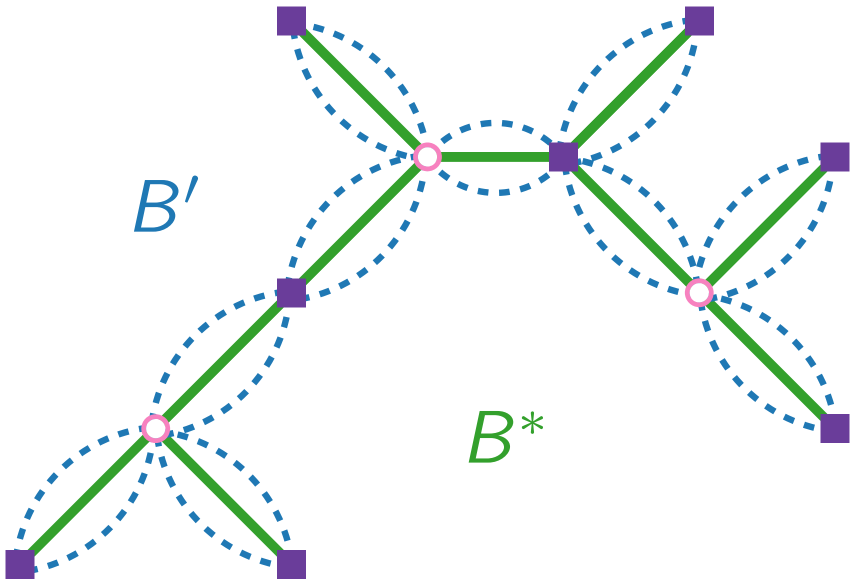

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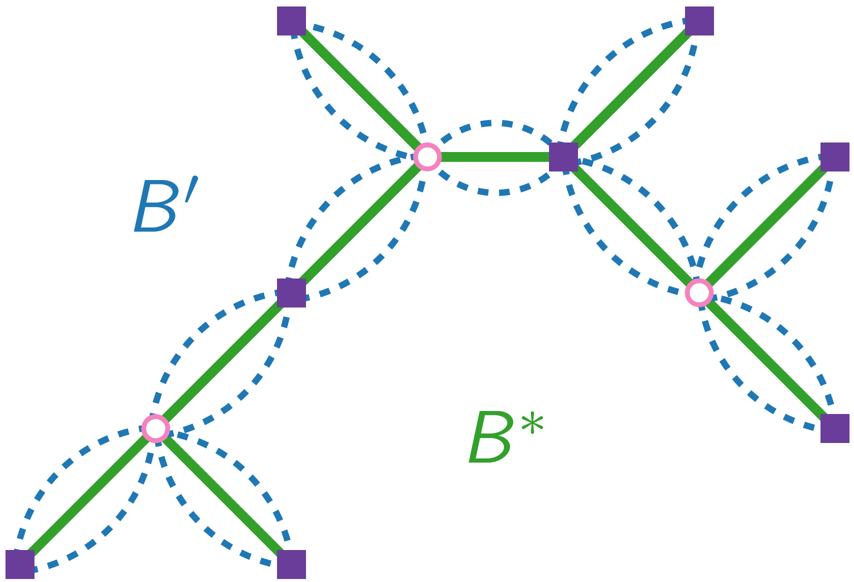

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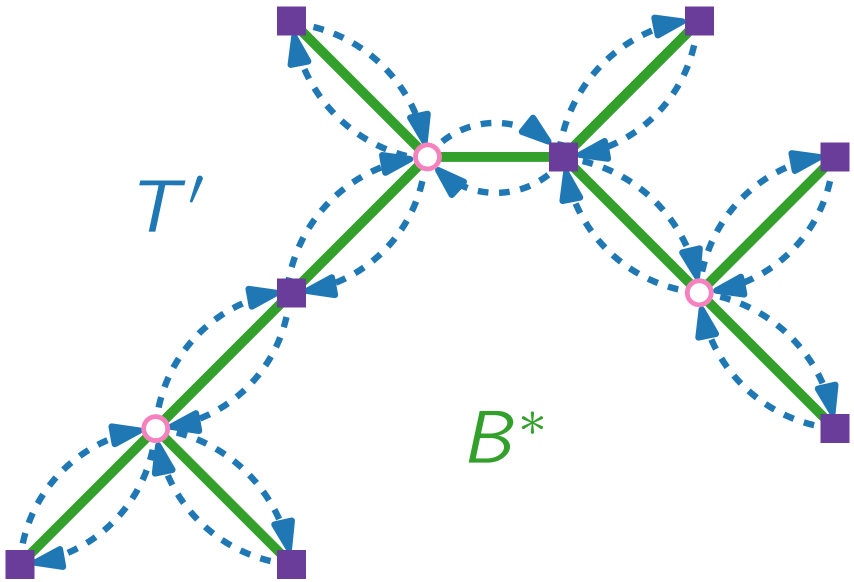

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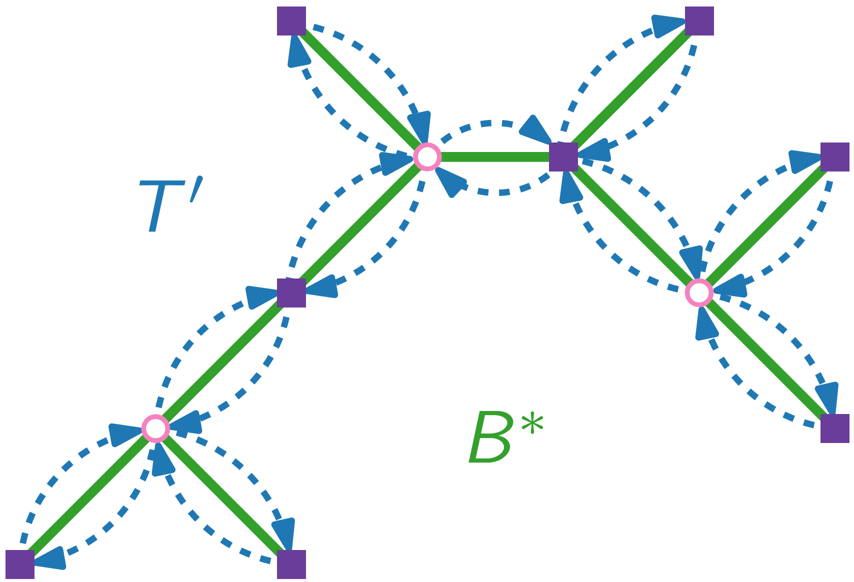

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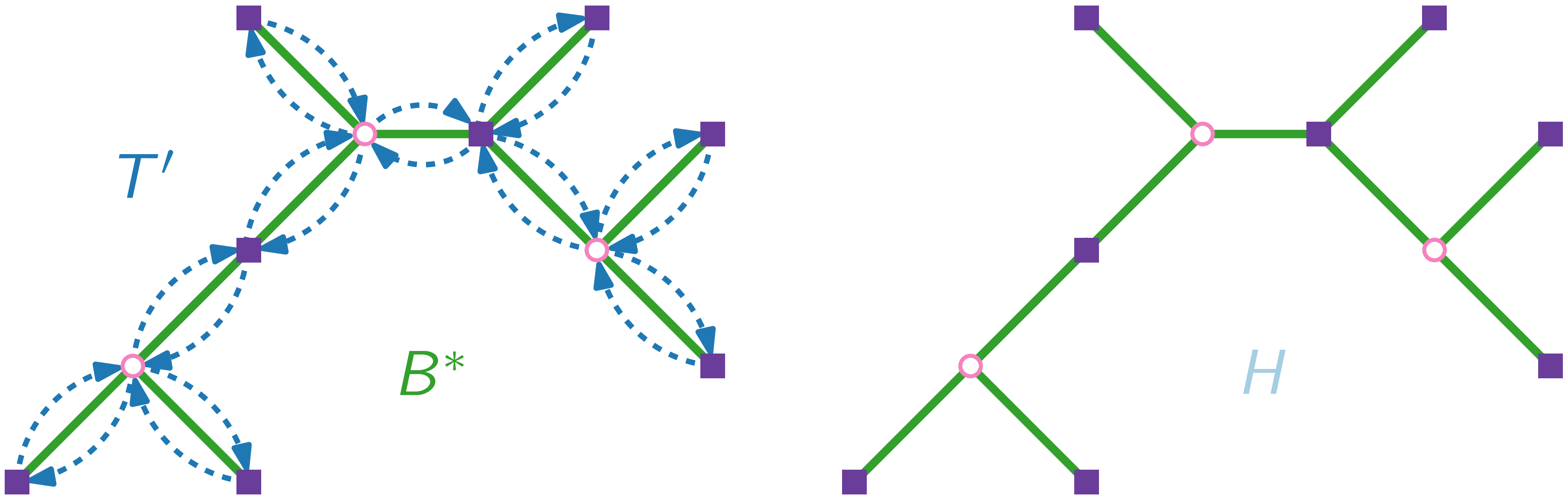

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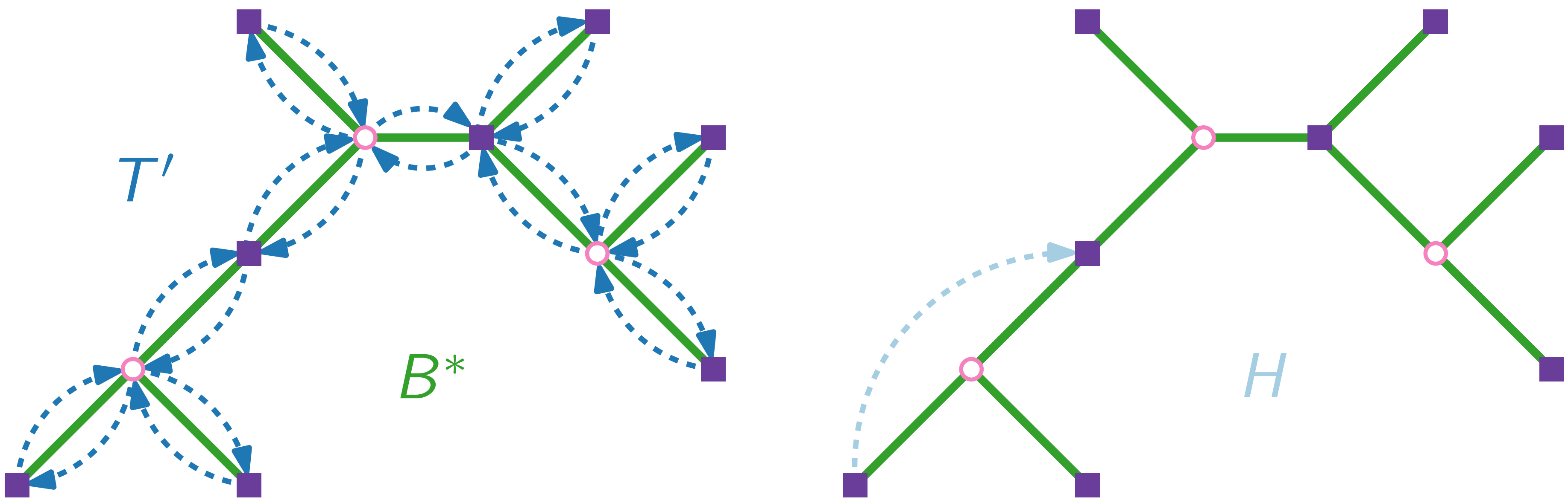

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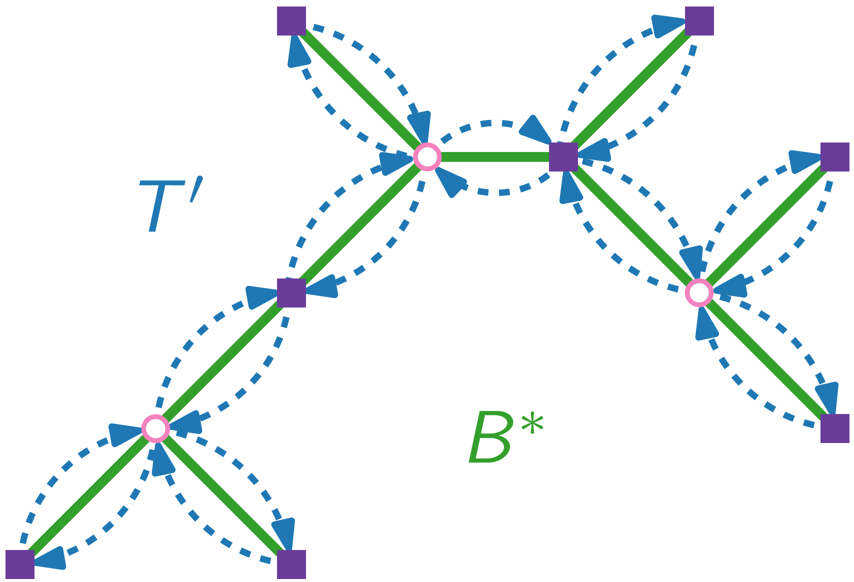

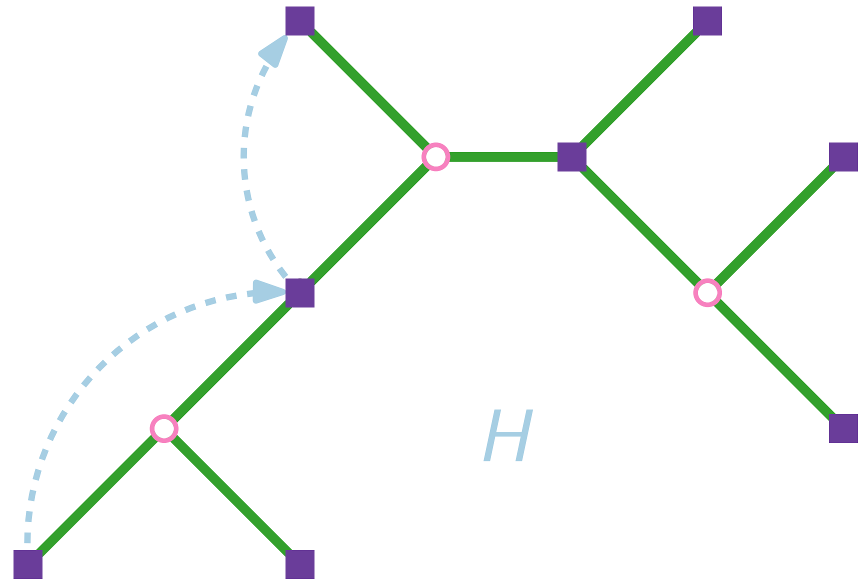

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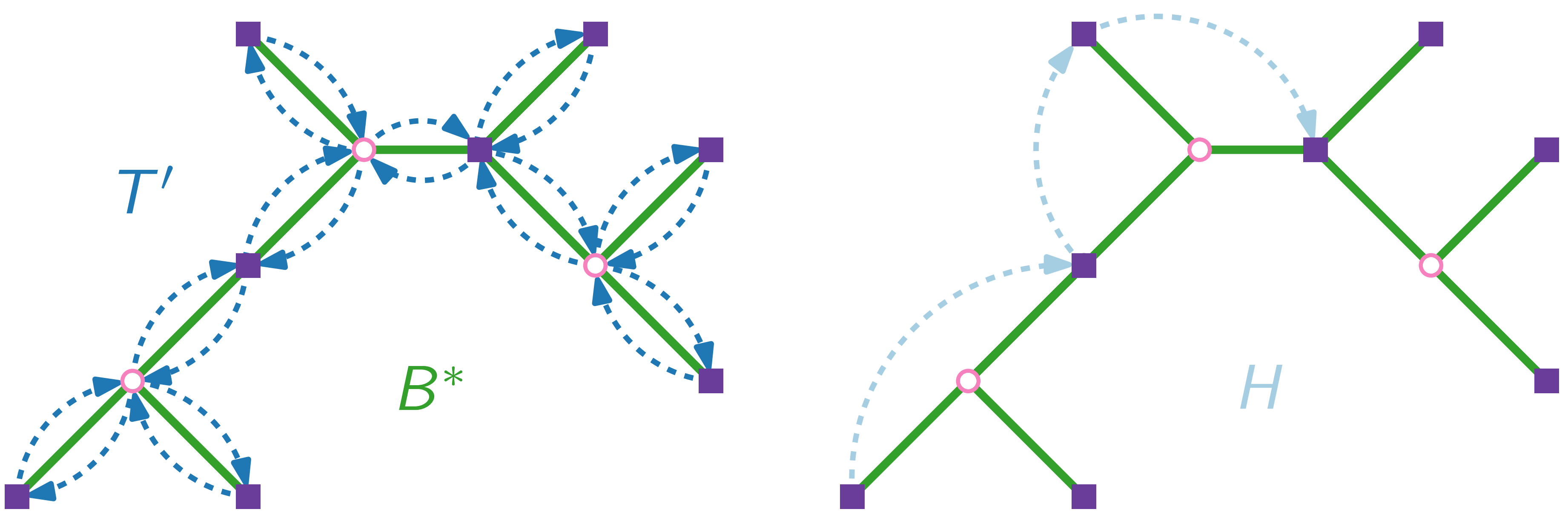

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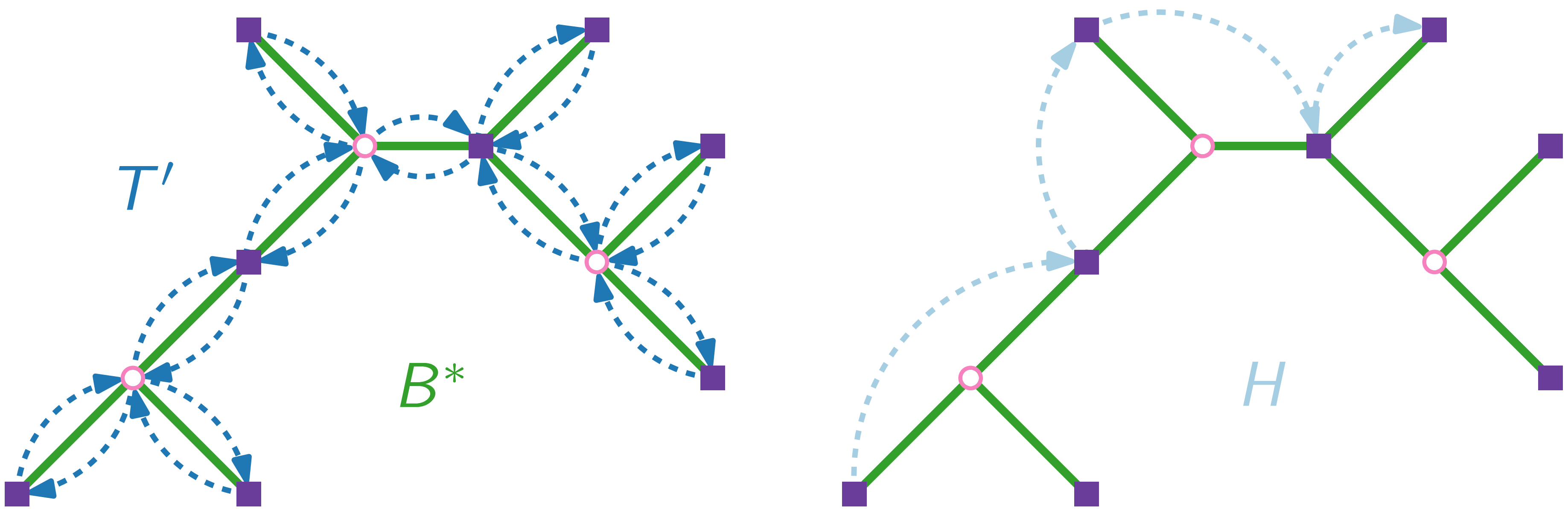

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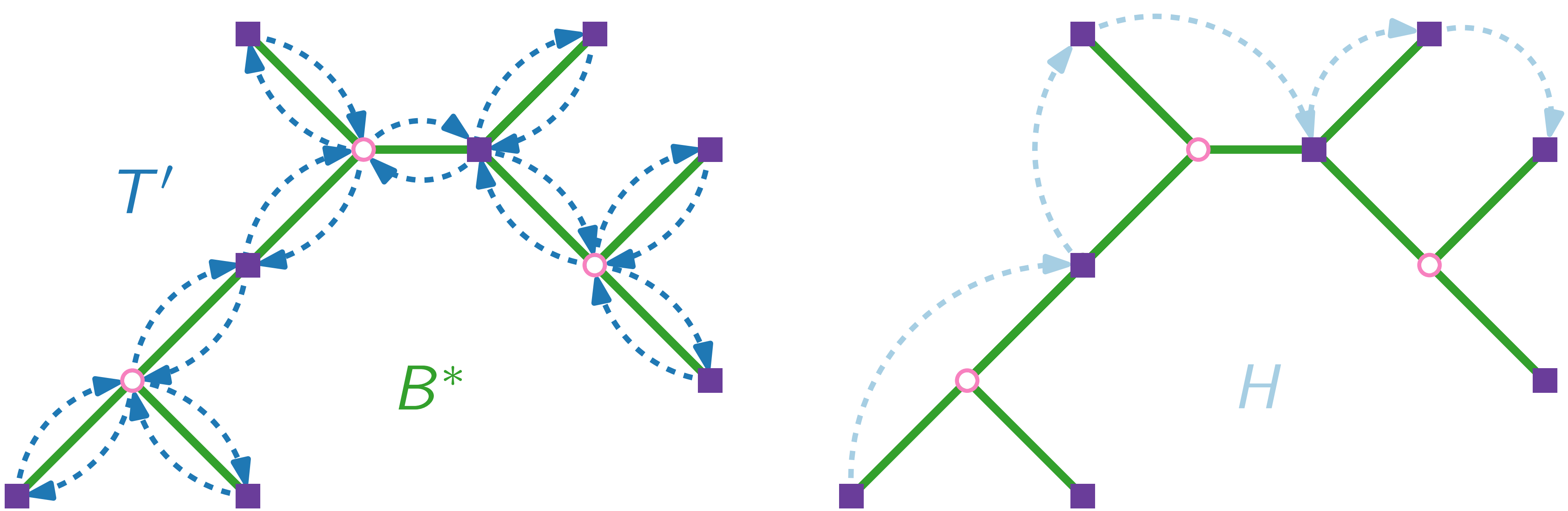

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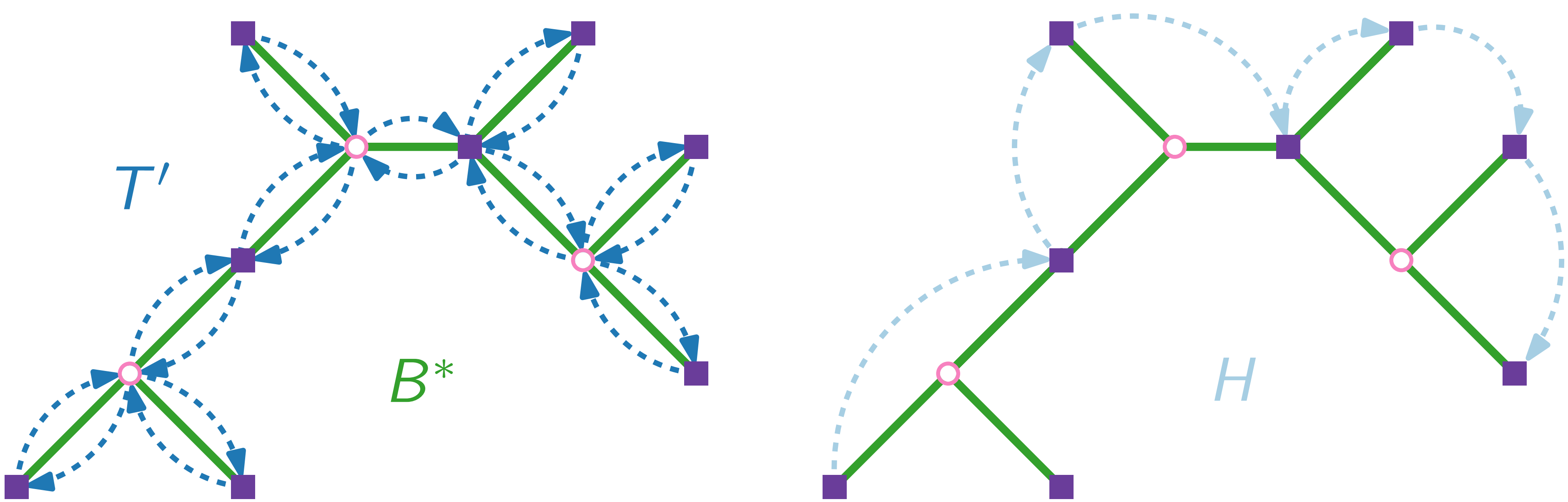

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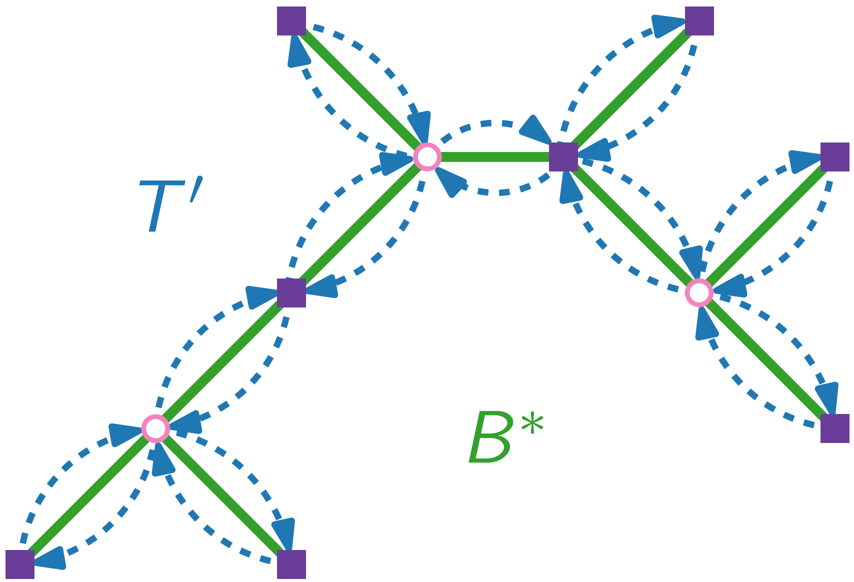

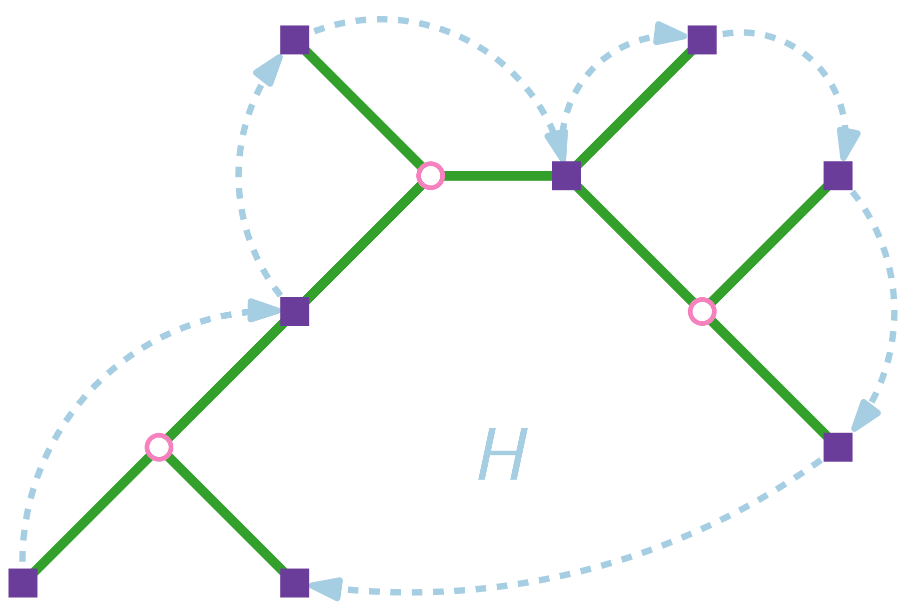

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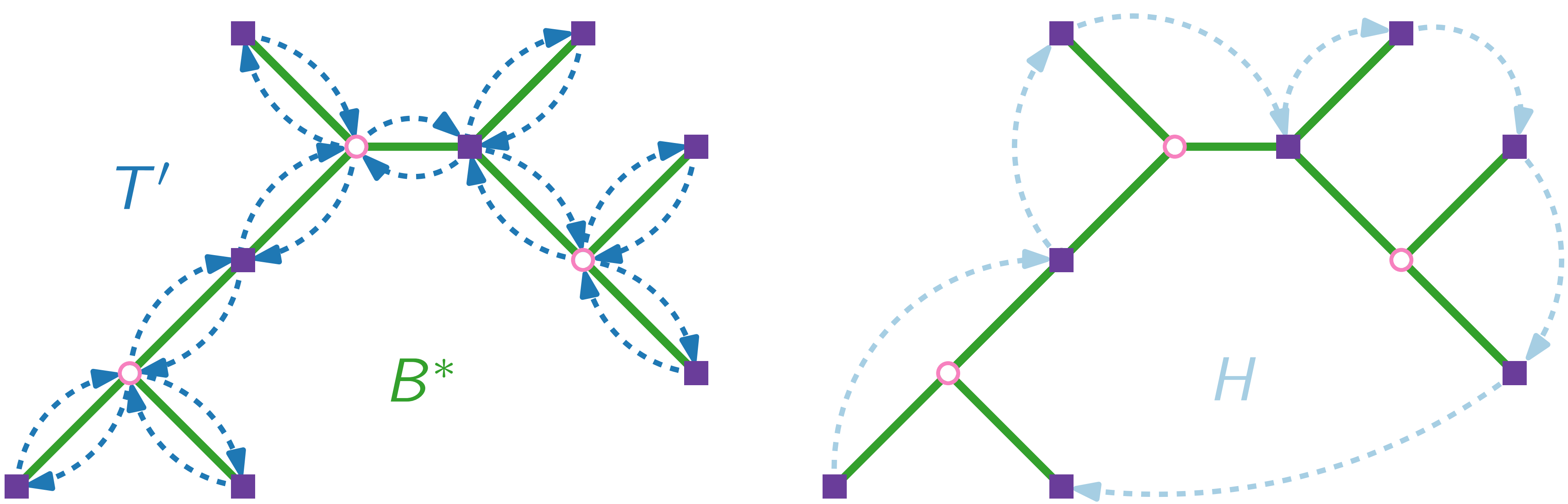
$\Rightarrow c(H) \leq c(T') = 2 \cdot \mathrm{OPT}$ since G is metric.



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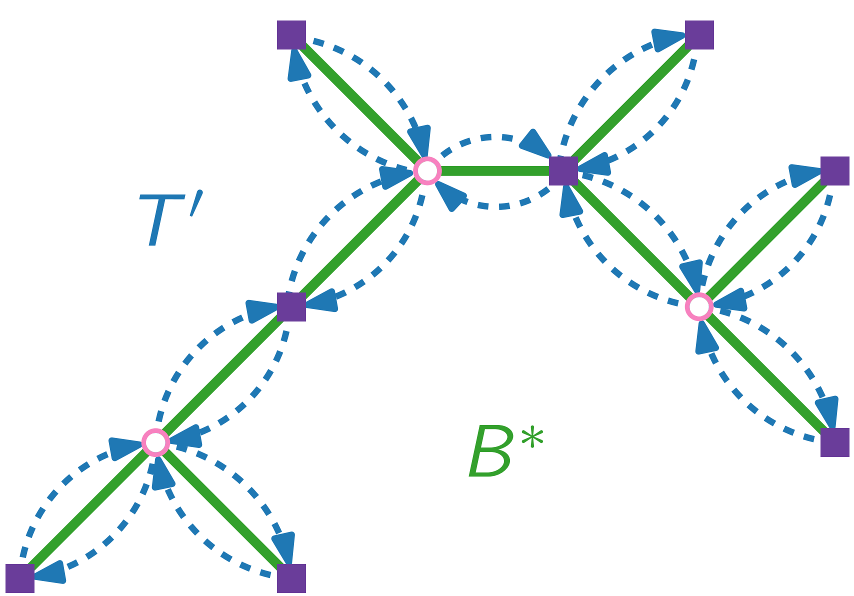

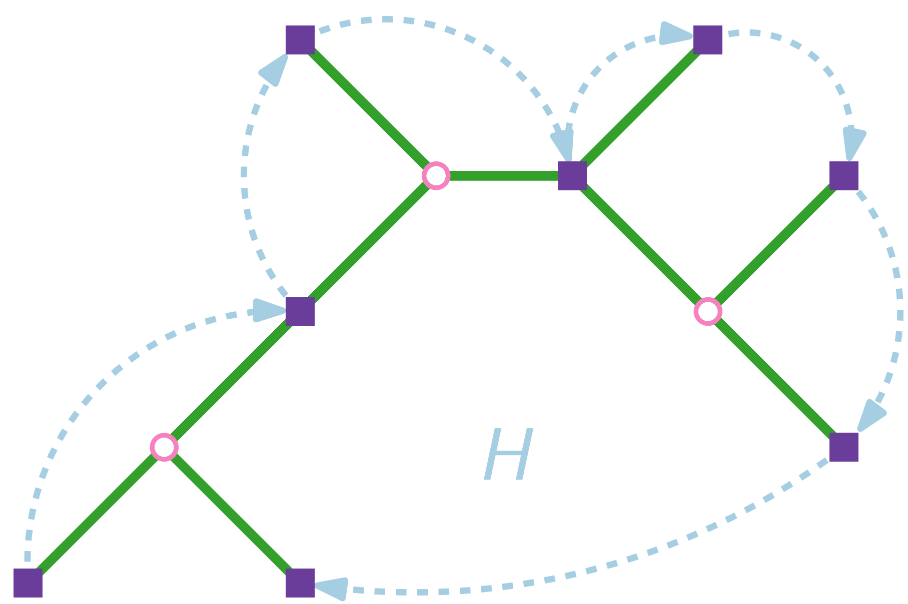

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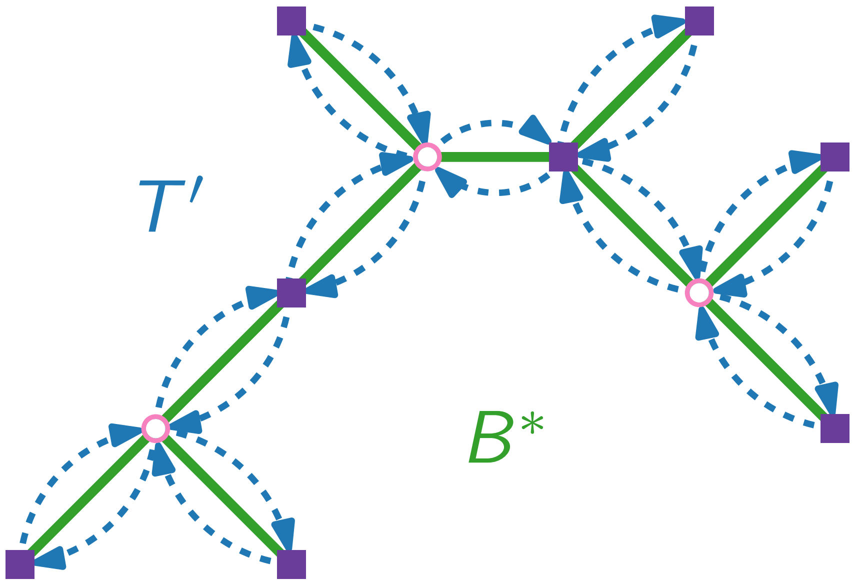

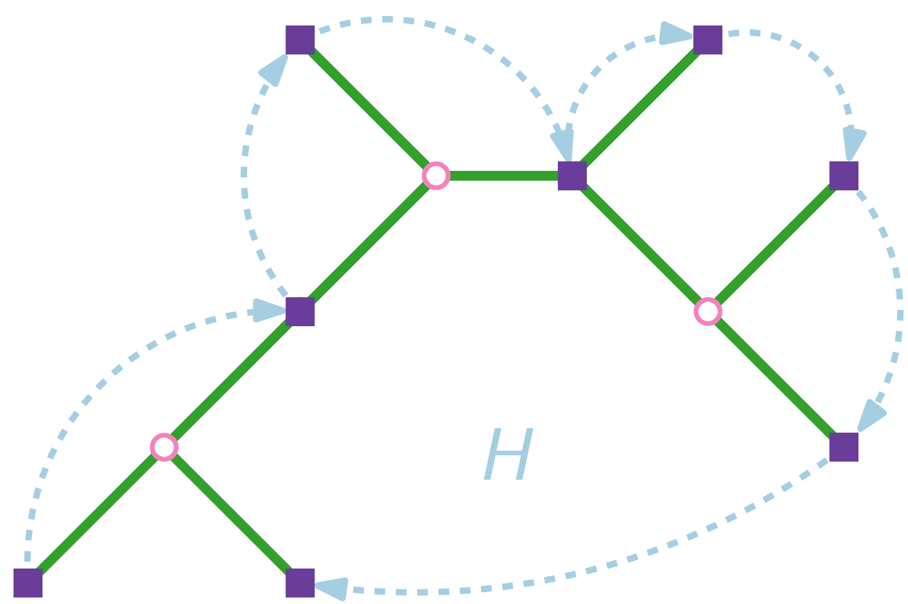

Analysis Tight?

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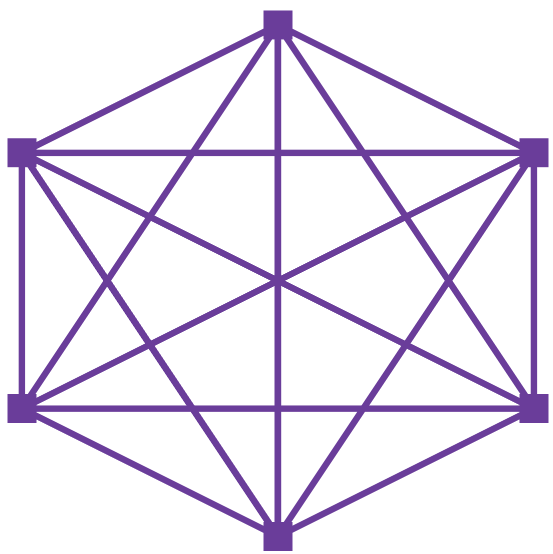
■ terminal

K_n



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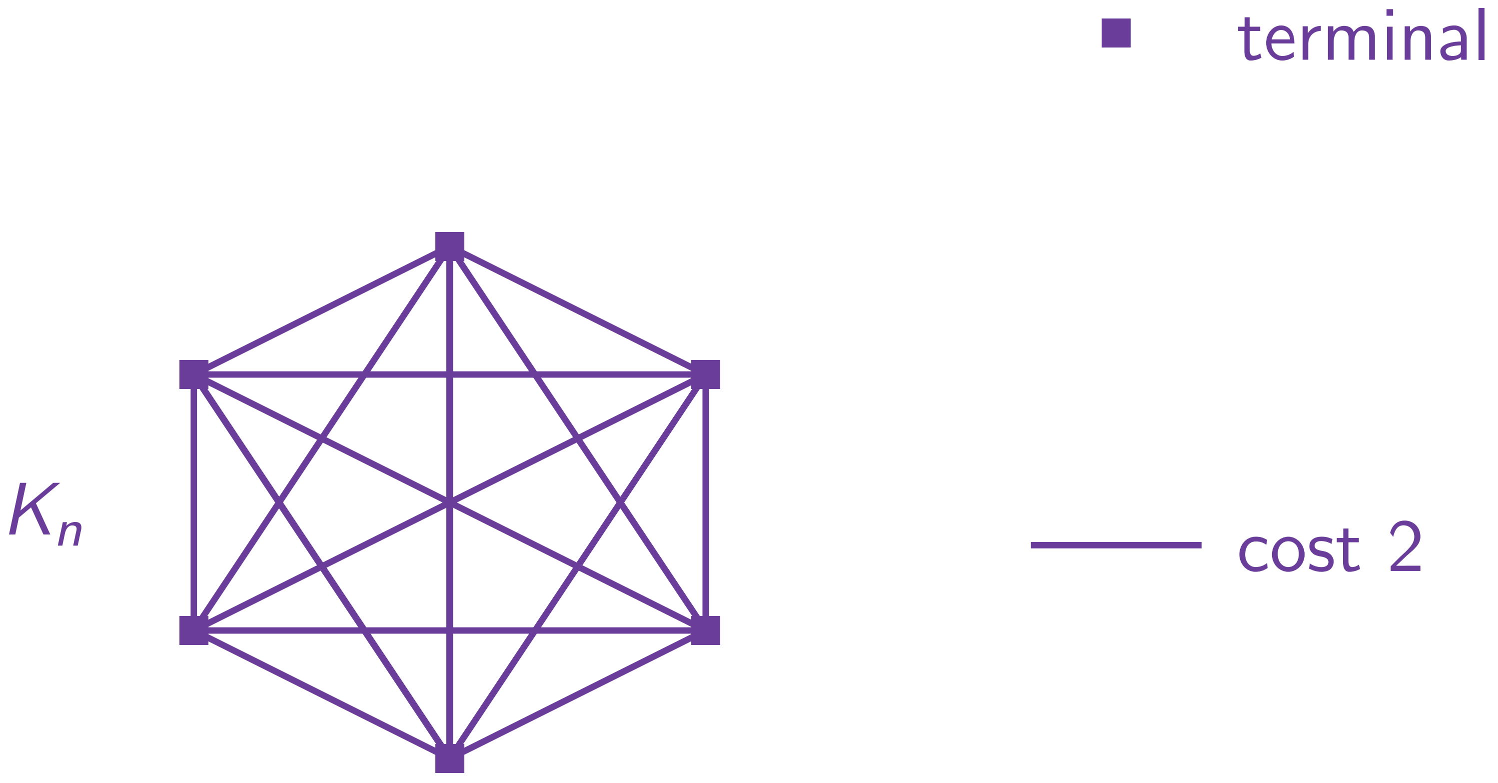

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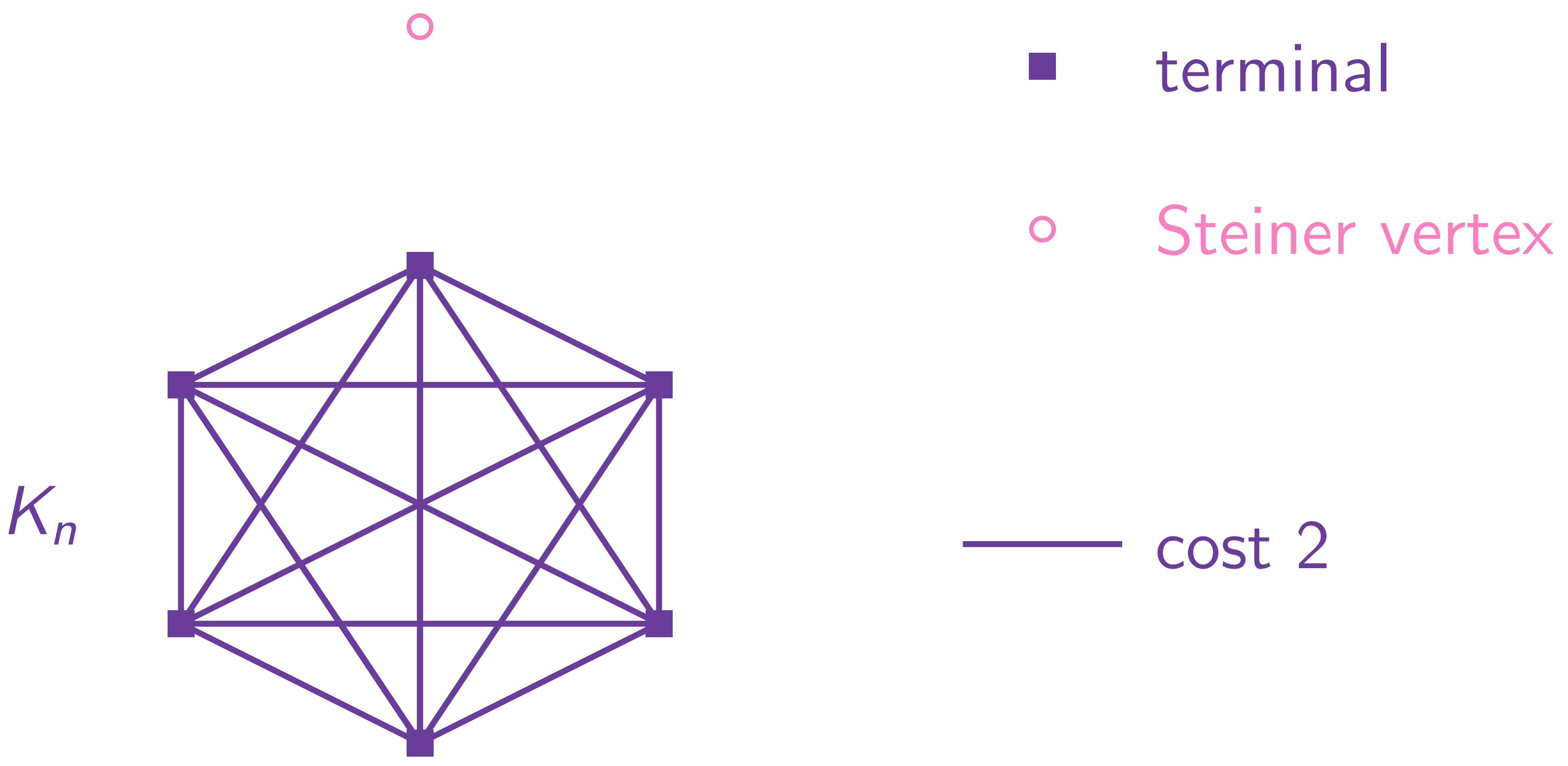

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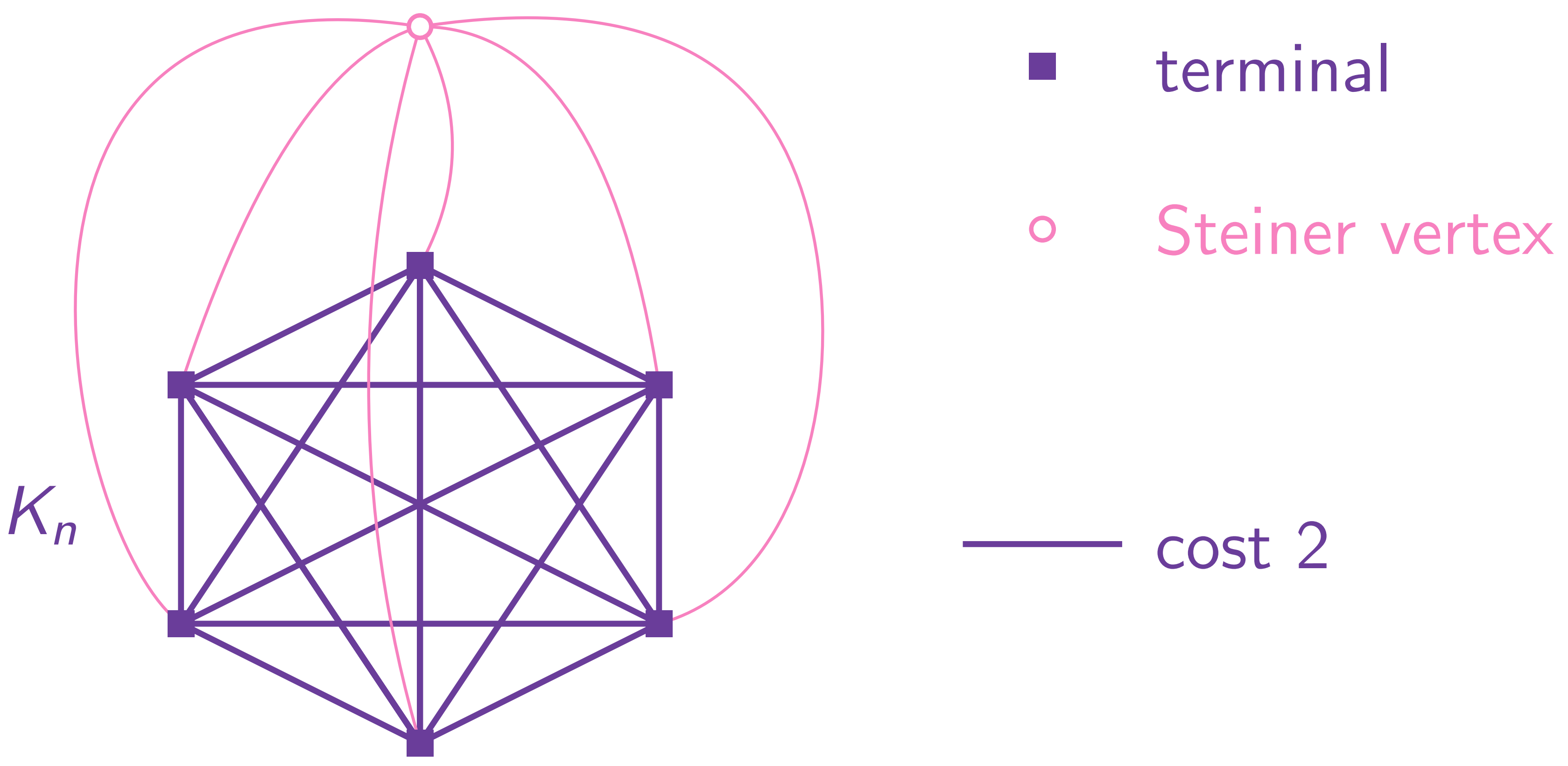

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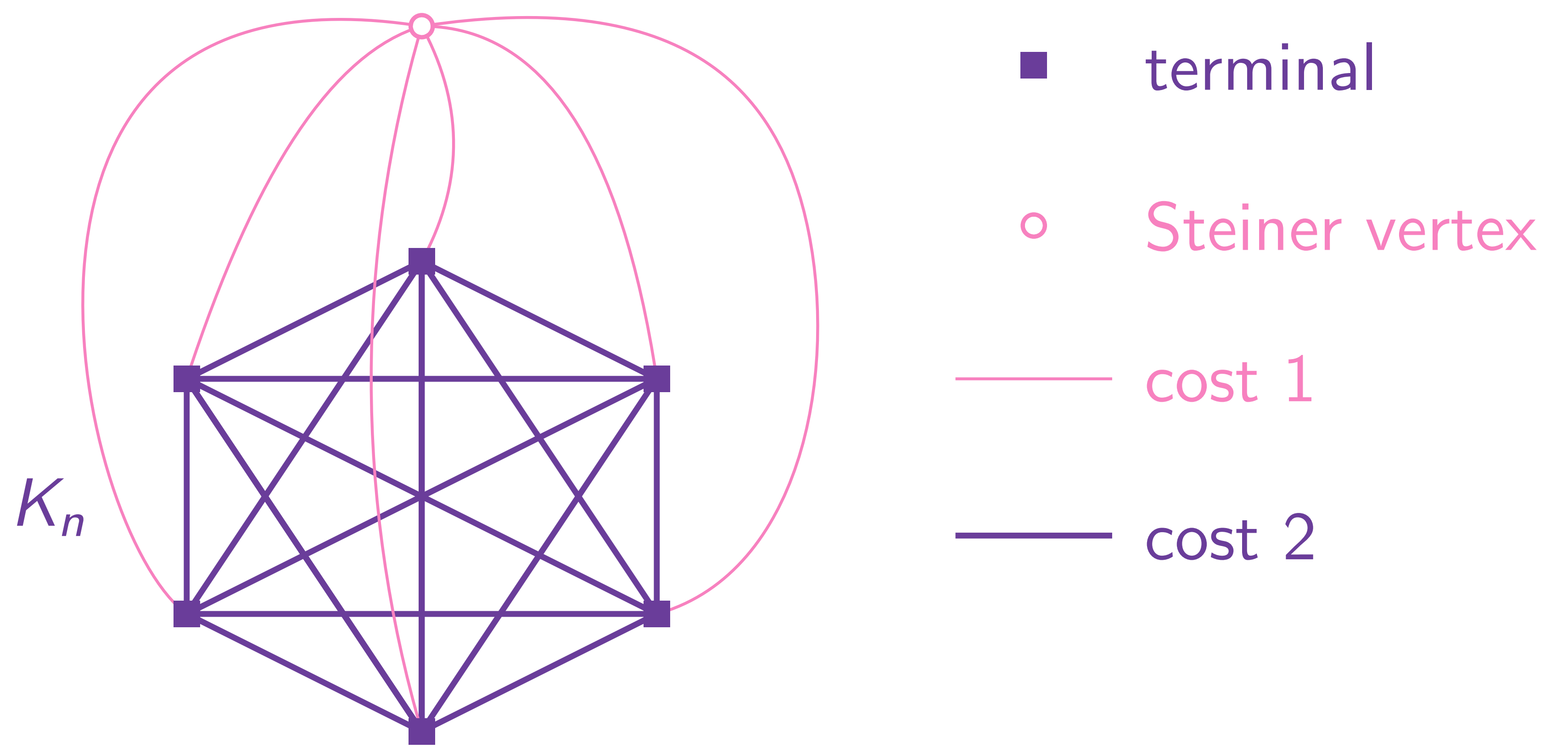

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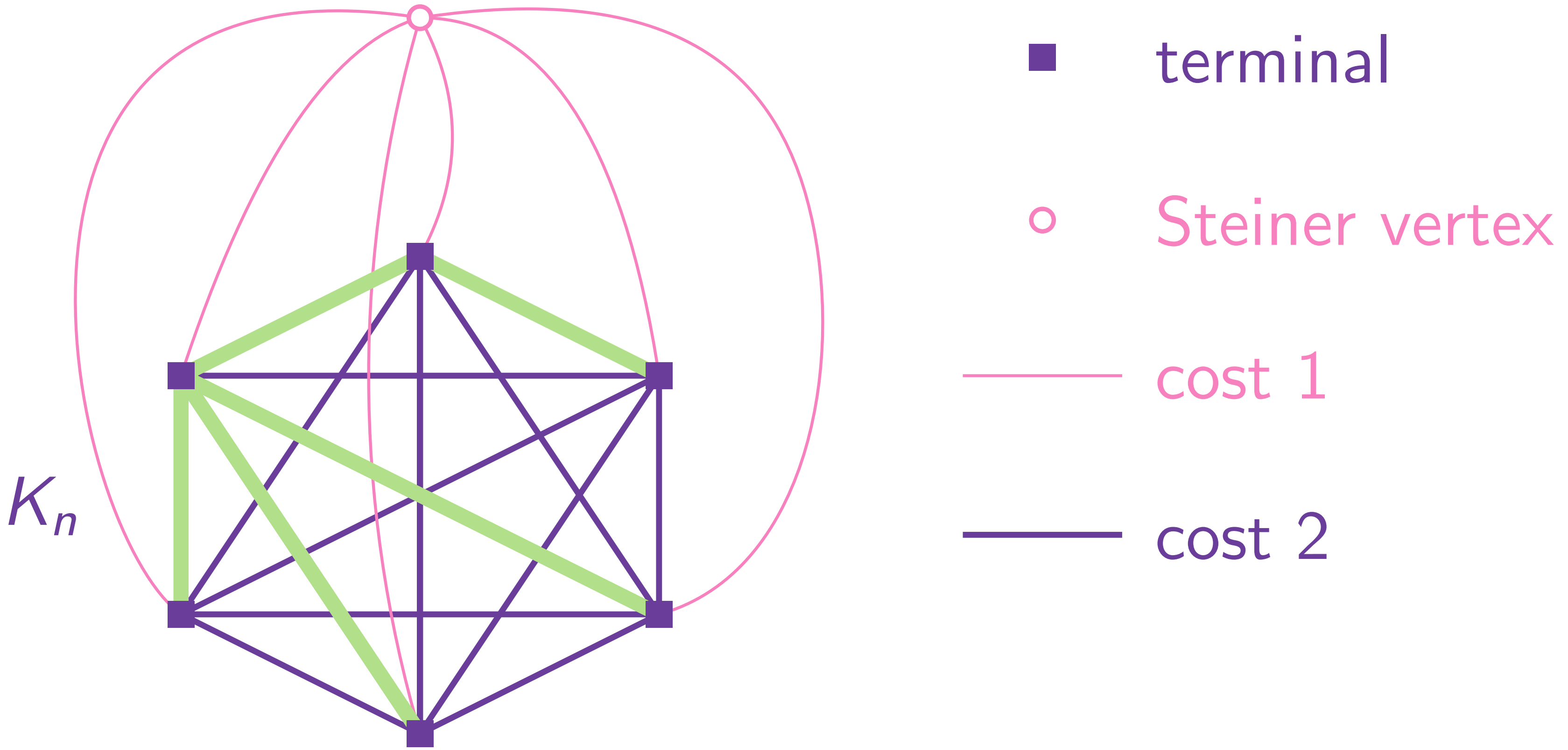
MST of $G[T]$ with cost $2(n-1)$



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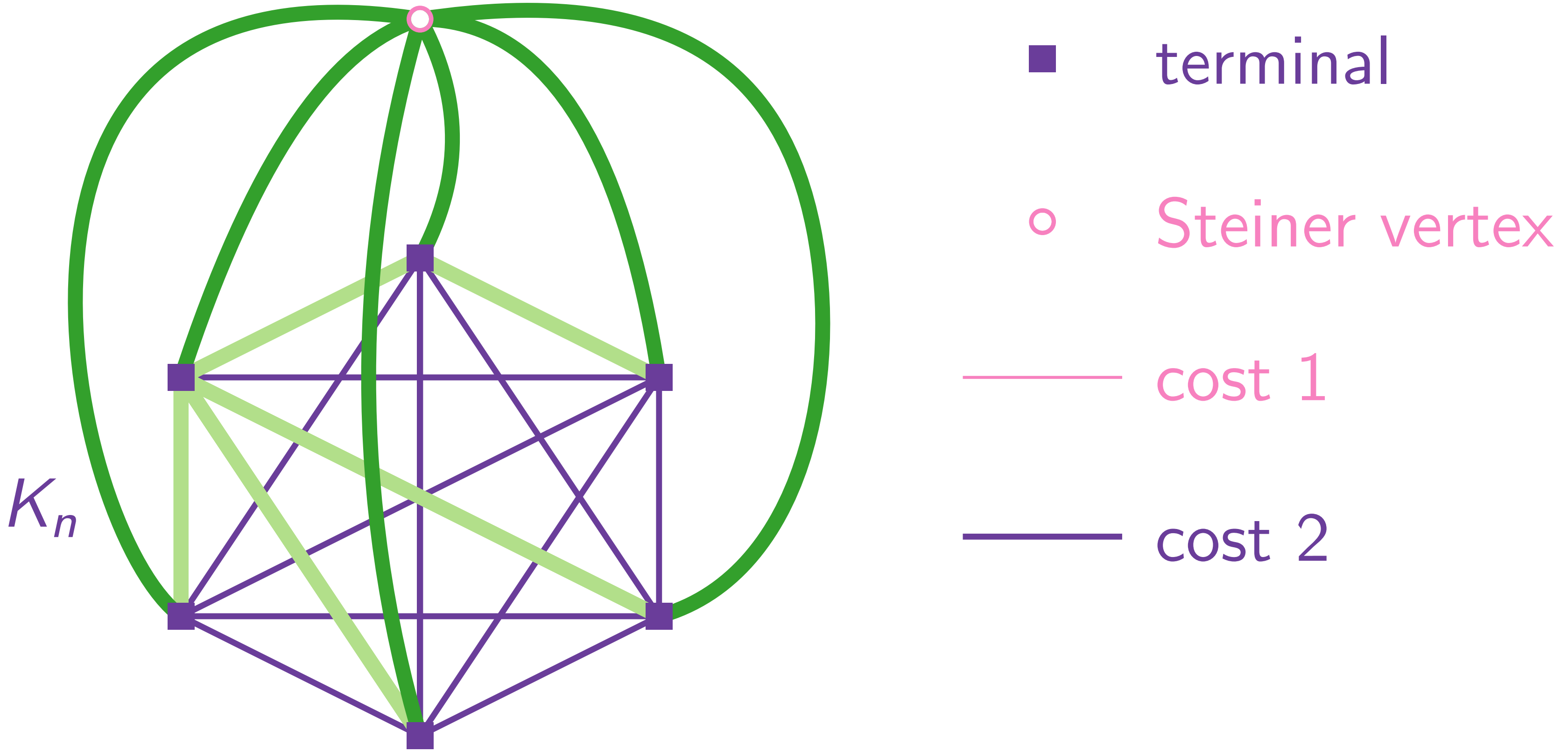
MST of $G[T]$ with cost $2(n-1)$
Optimal solution with cost n



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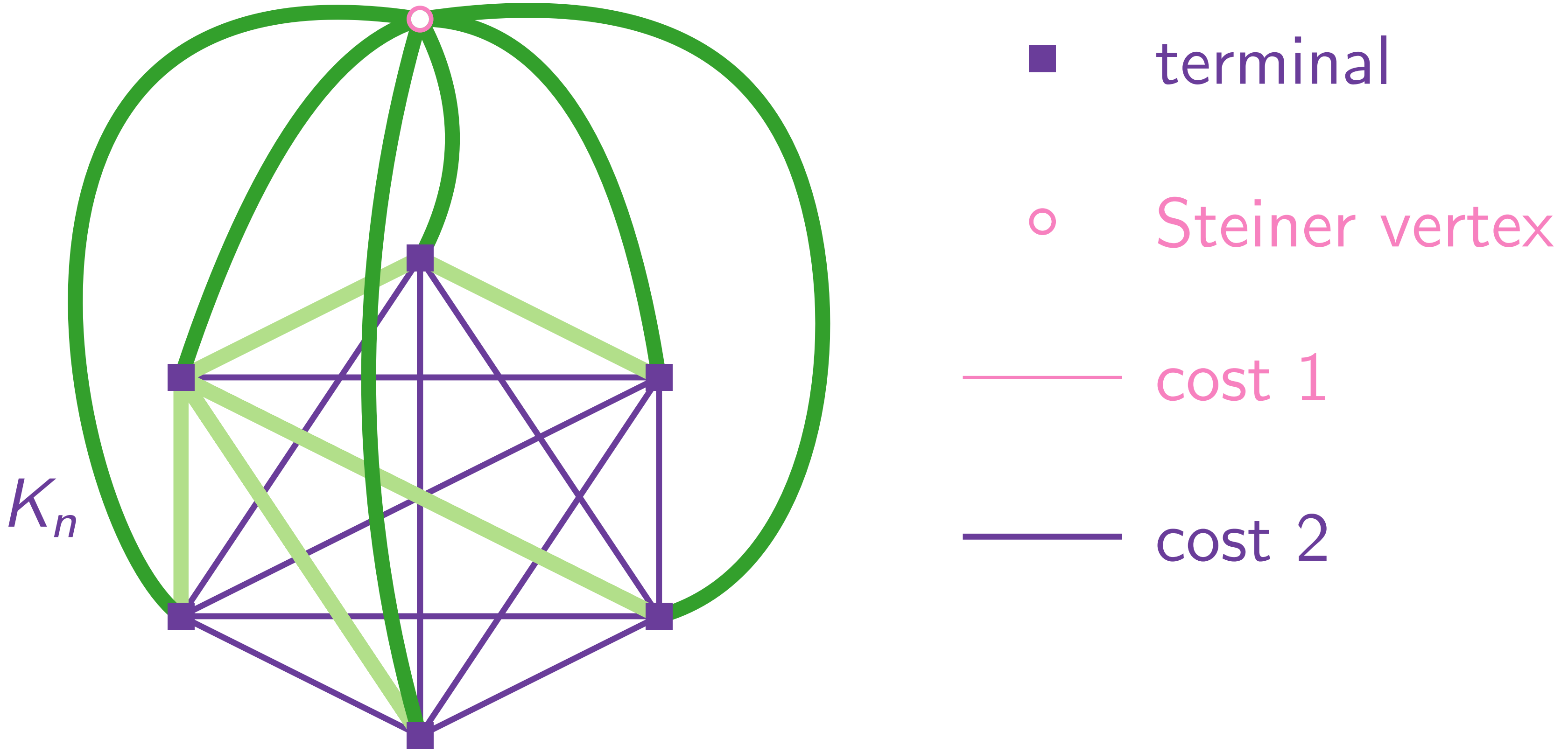
$$\frac{2(n-1)}{n} \to 2$$



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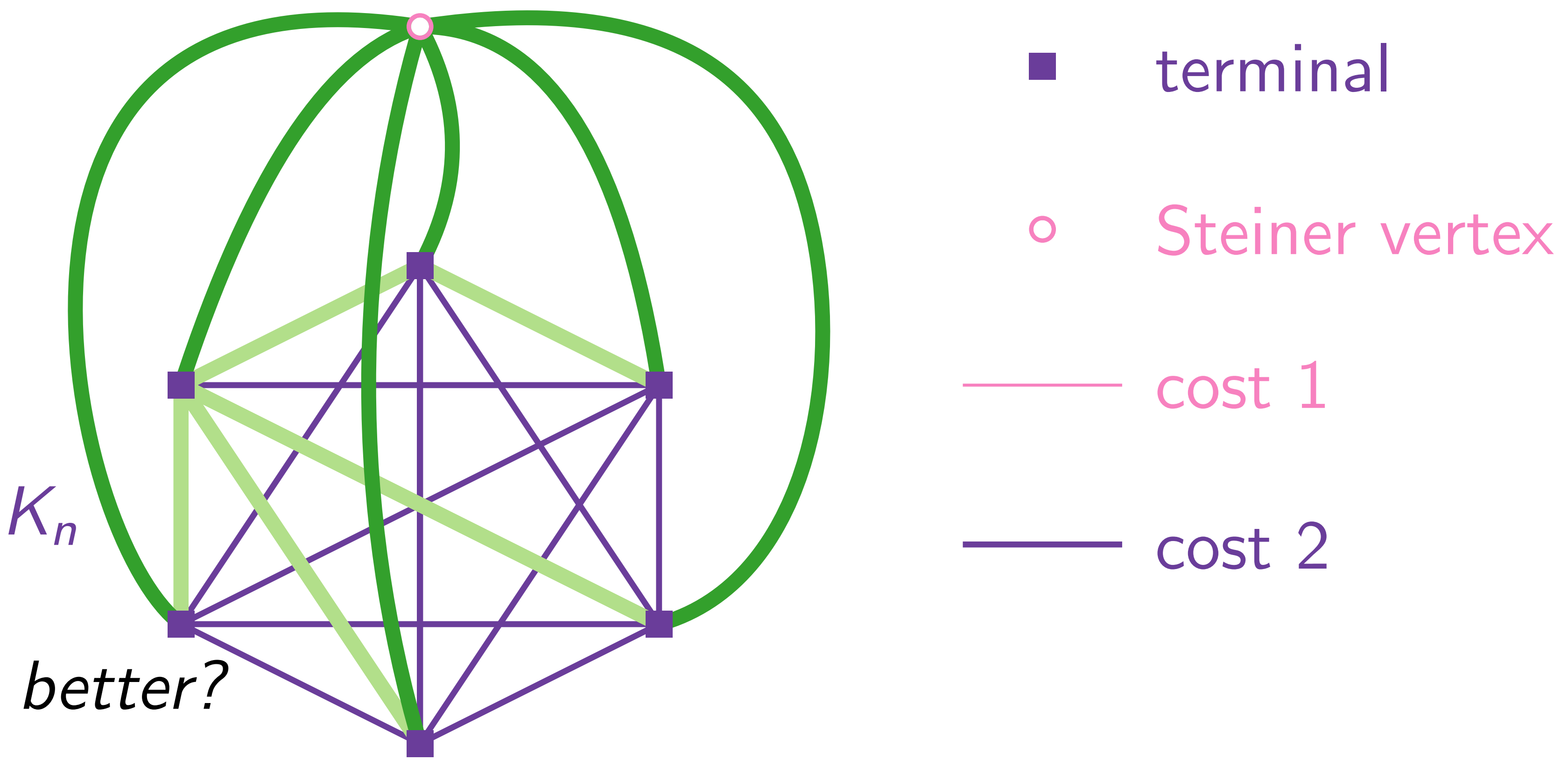

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MST of $G[T]$ with cost $2(n-1)$
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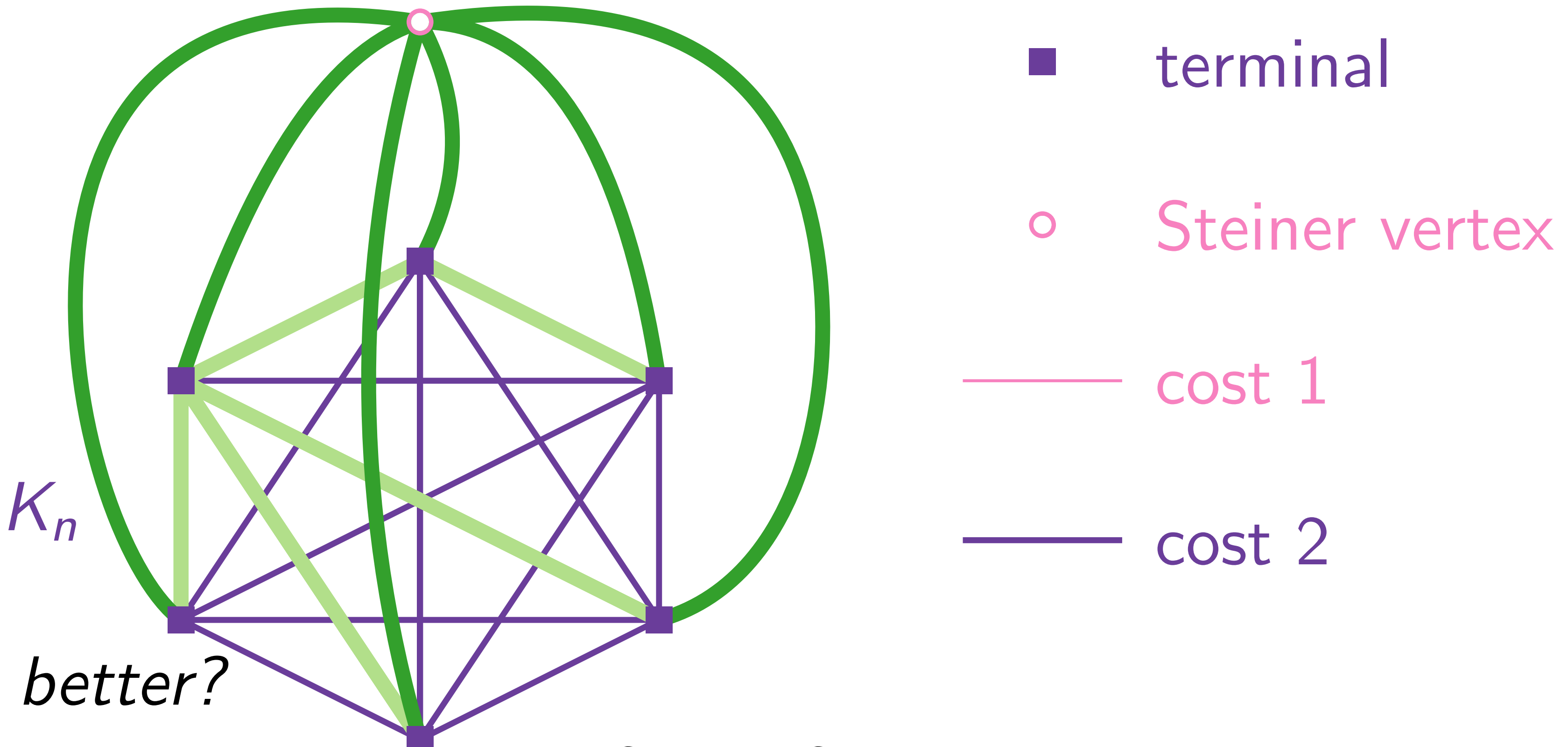

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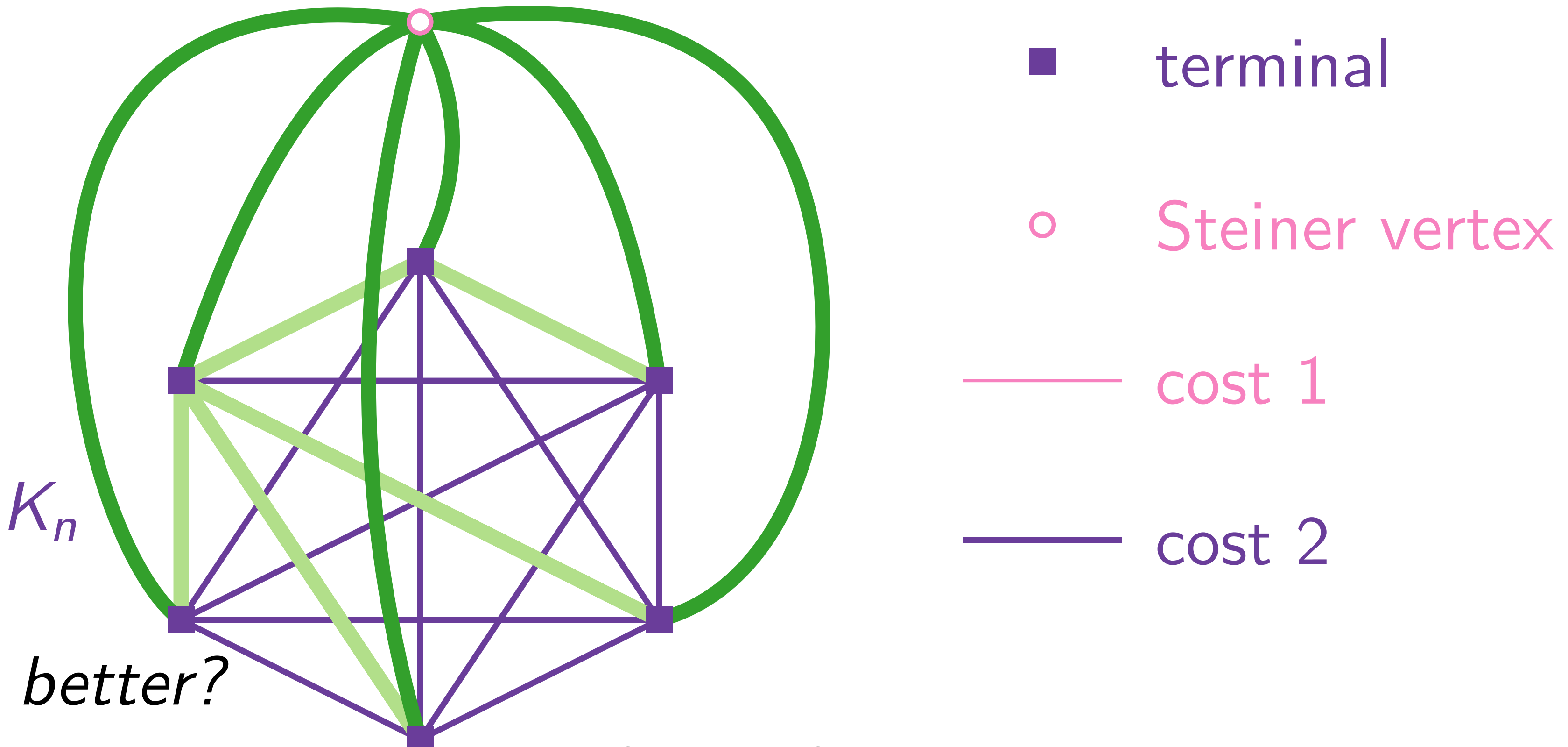

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SteinerTree cannot be approximated within factor $\frac{96}{95} \approx 1.0105$ (unless P=NP) [Chlebík & Chlebíková, TCS'08]

Approximation Algorithms

Lecture 3:
SteinerTree and MultiwayCut

Part V:
MultiwayCut

MultiwayCut

Given: A connected graph G

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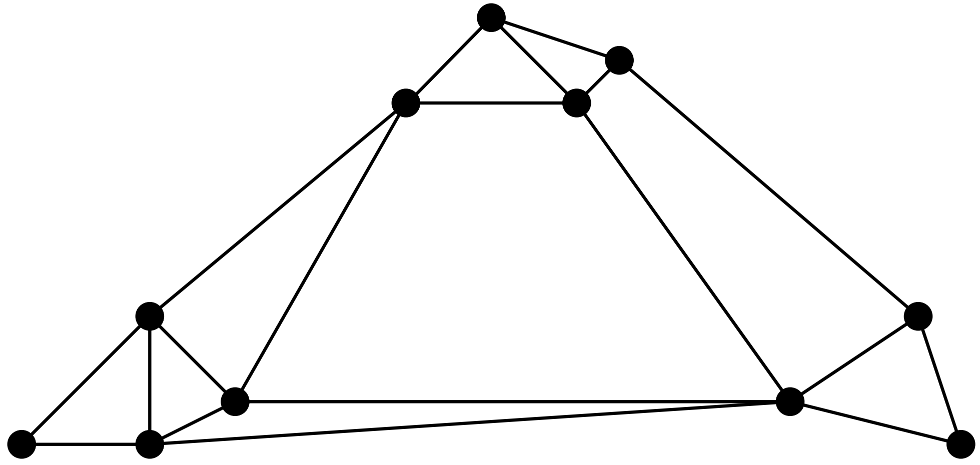

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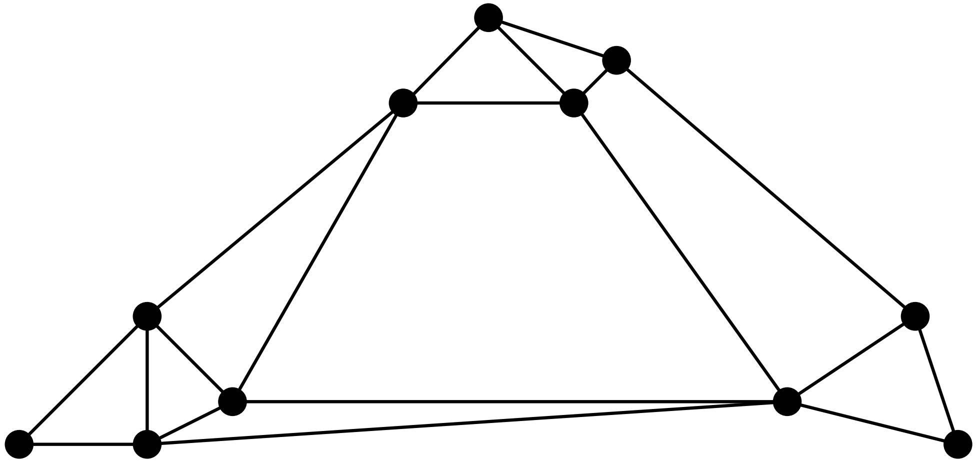

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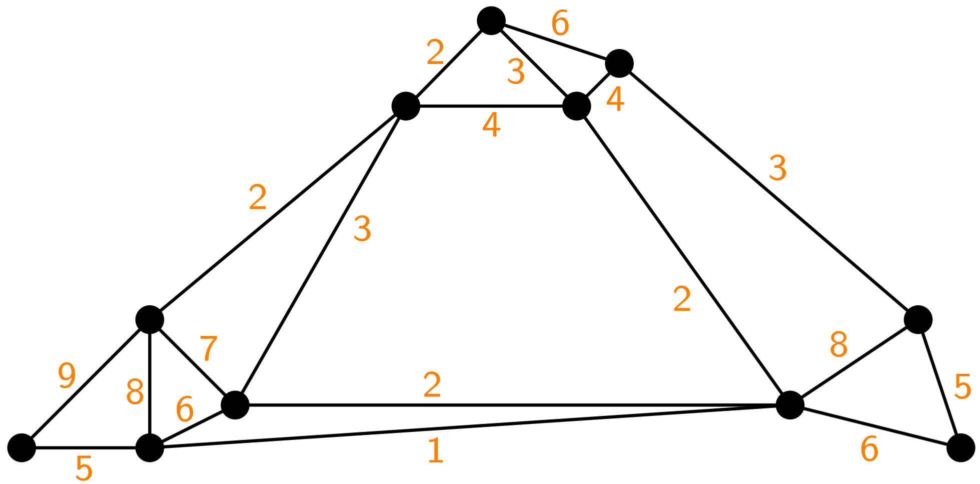

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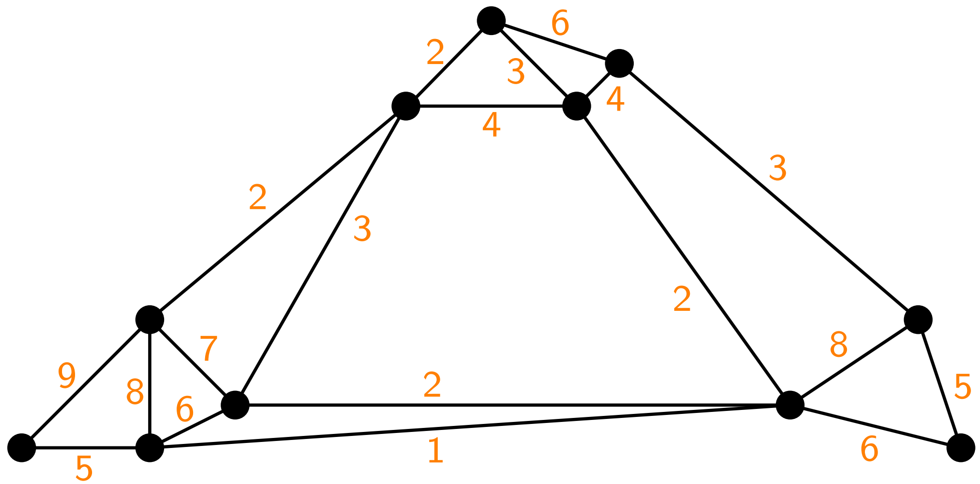

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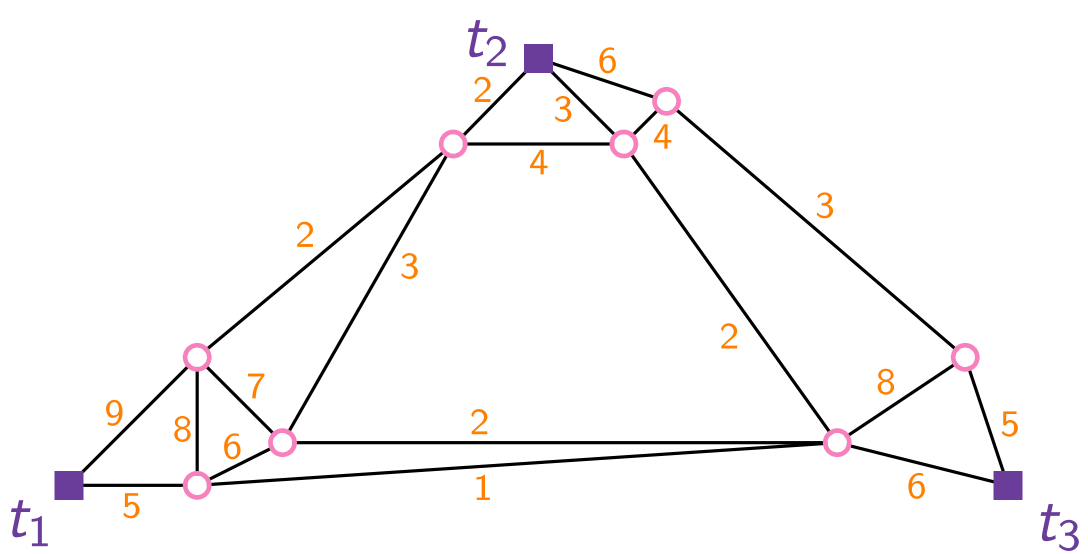

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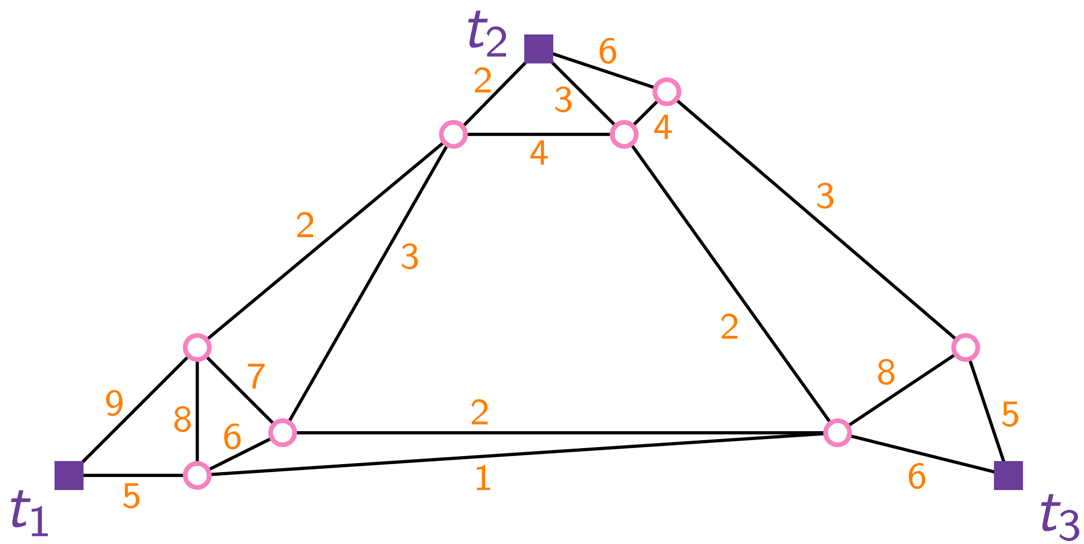

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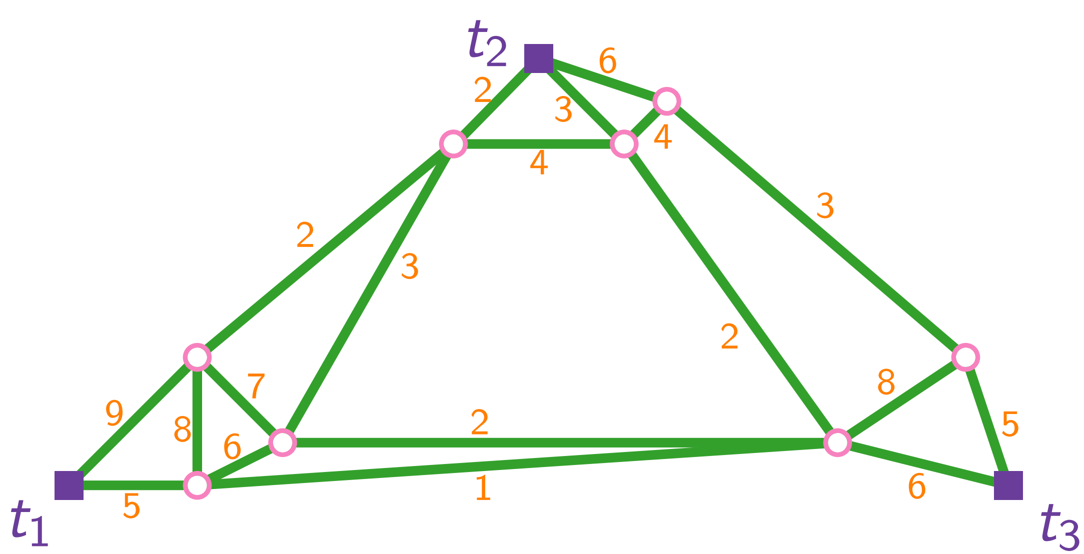

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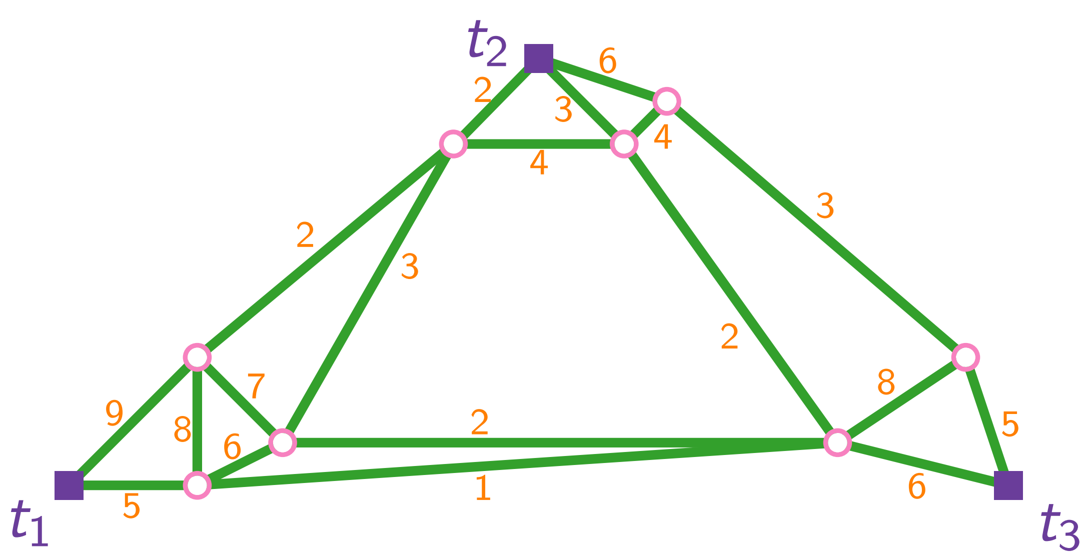

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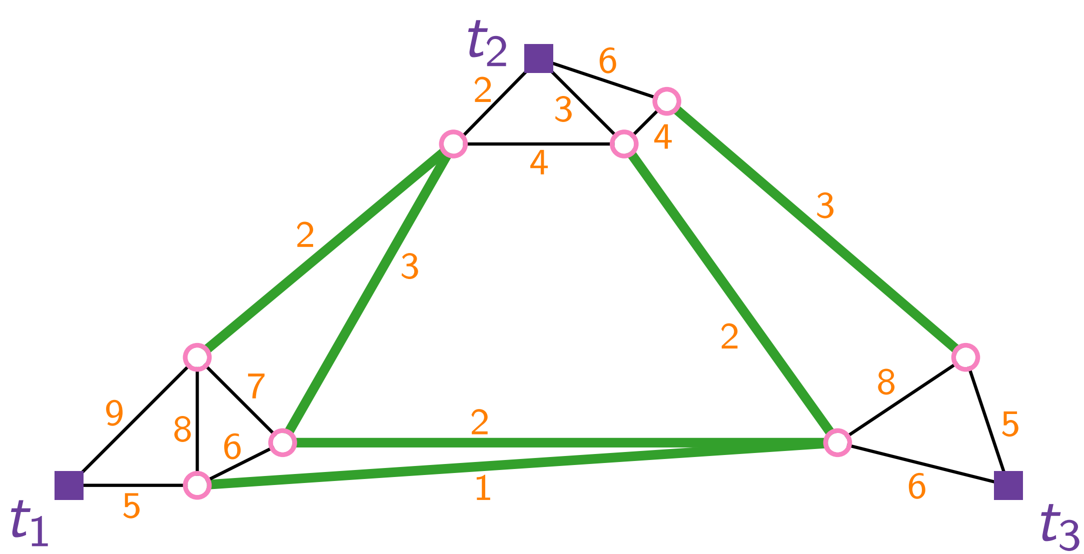

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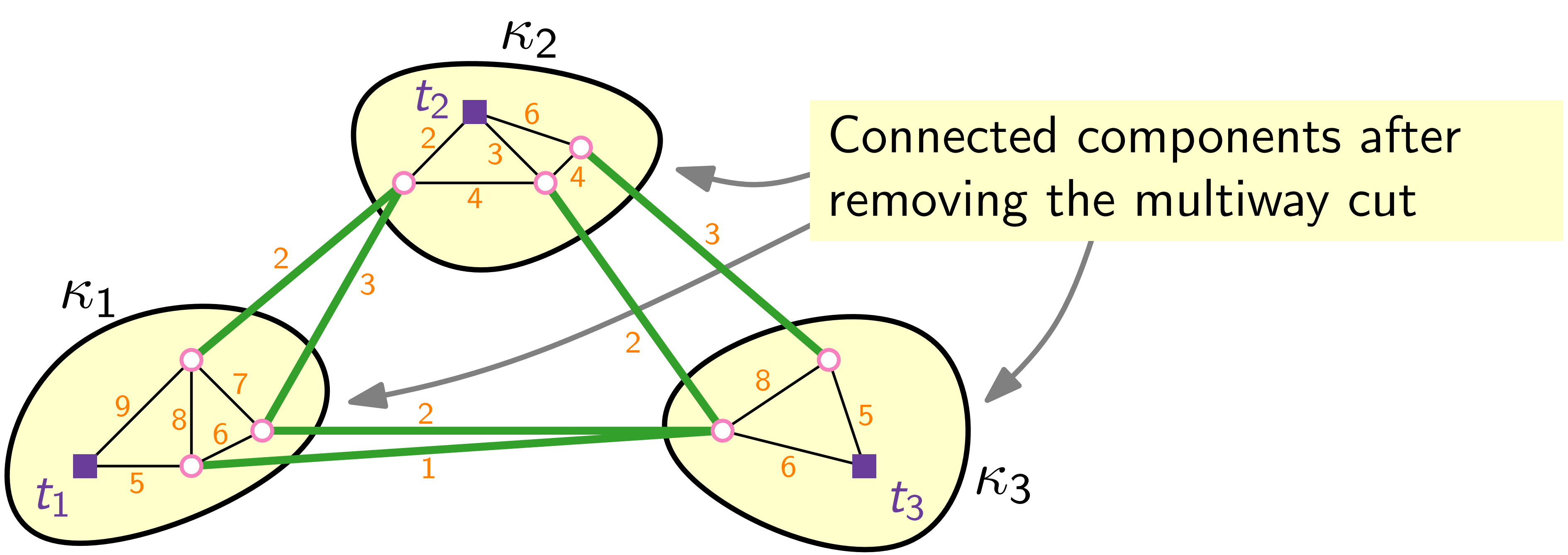

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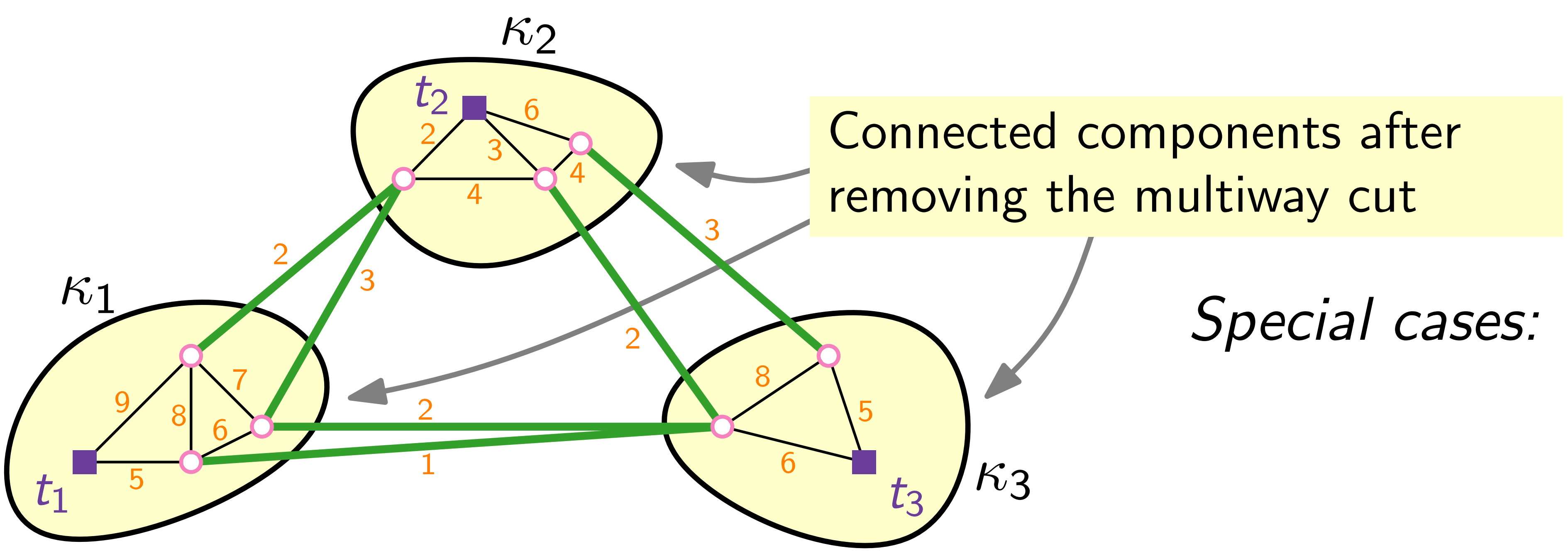

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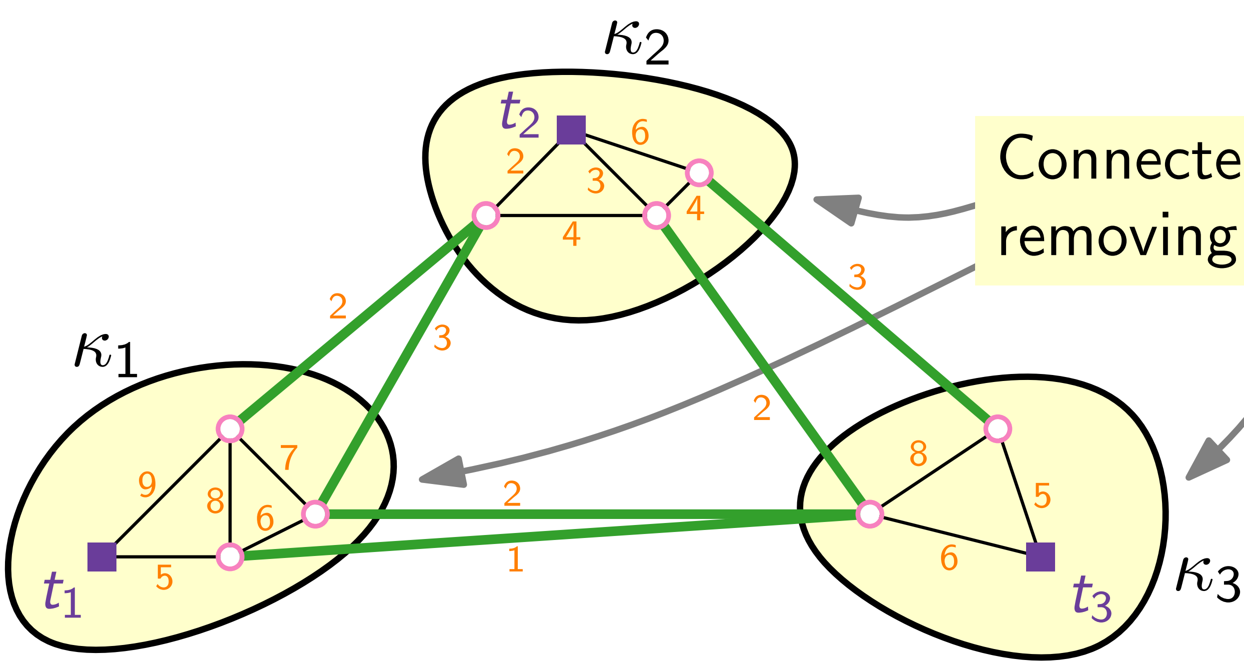

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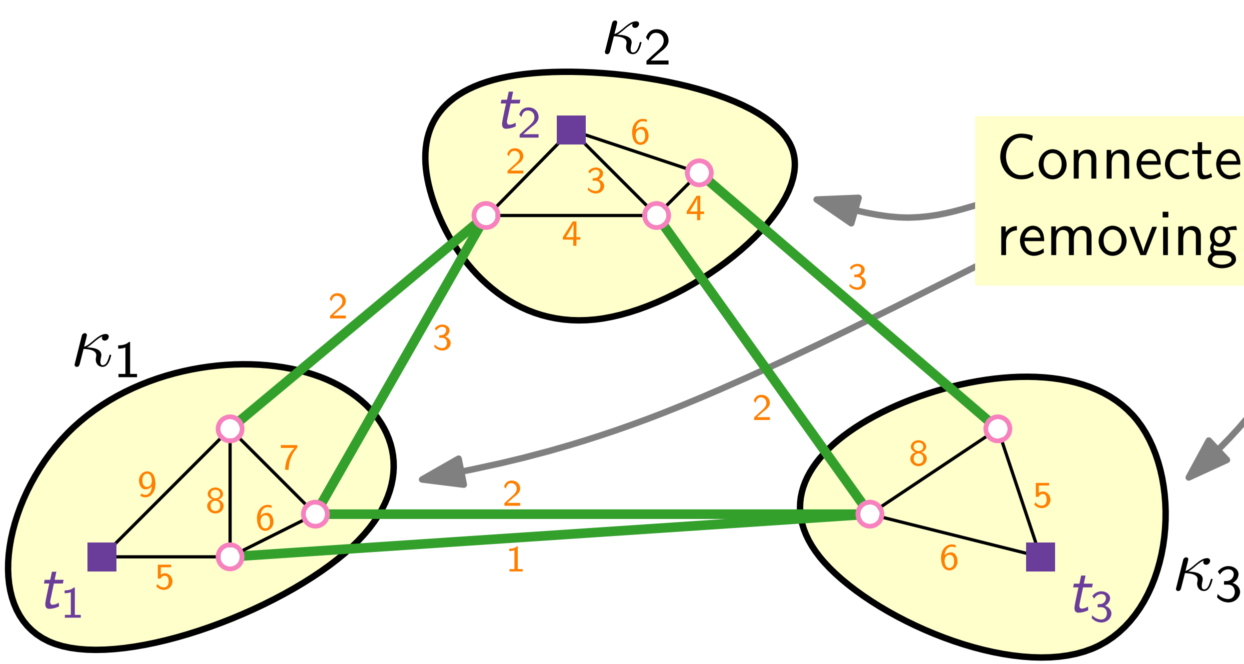

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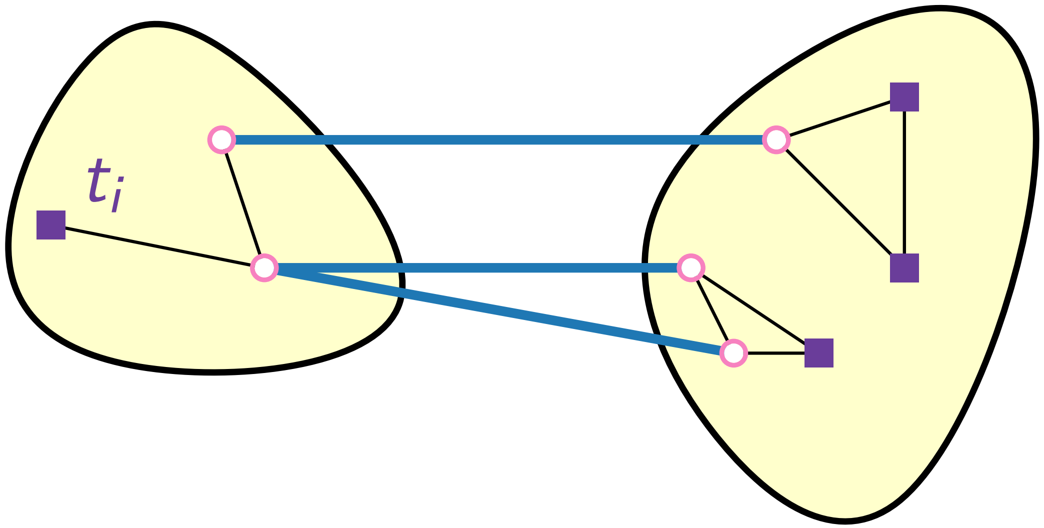

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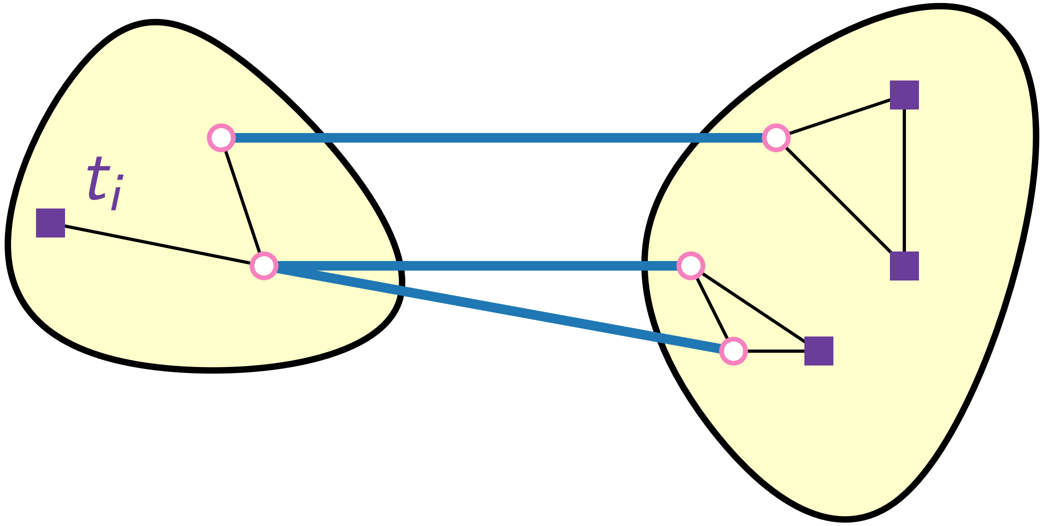

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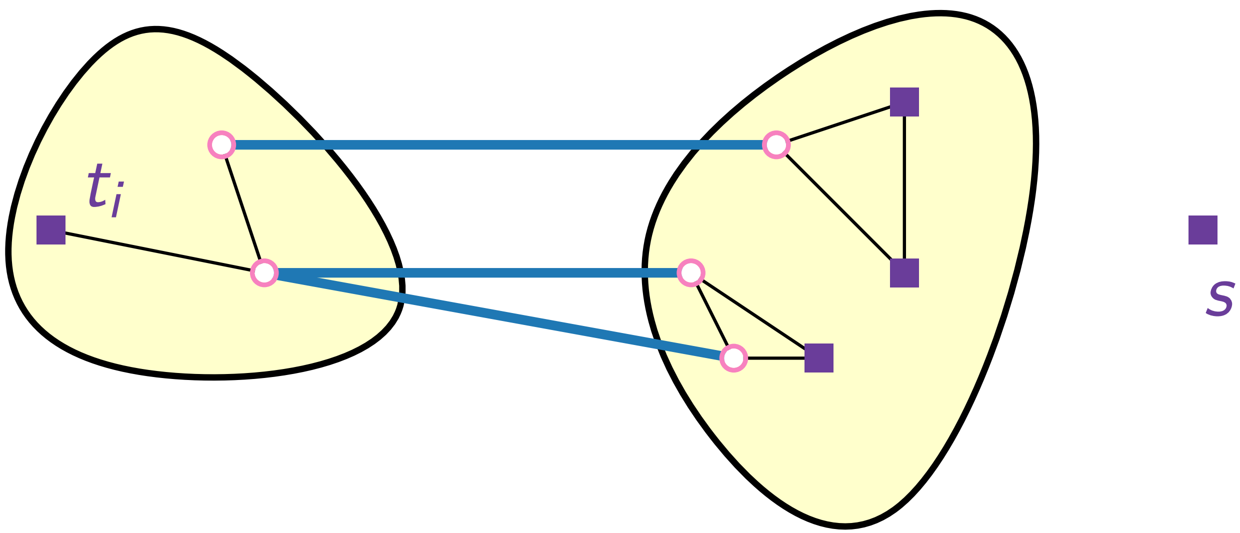

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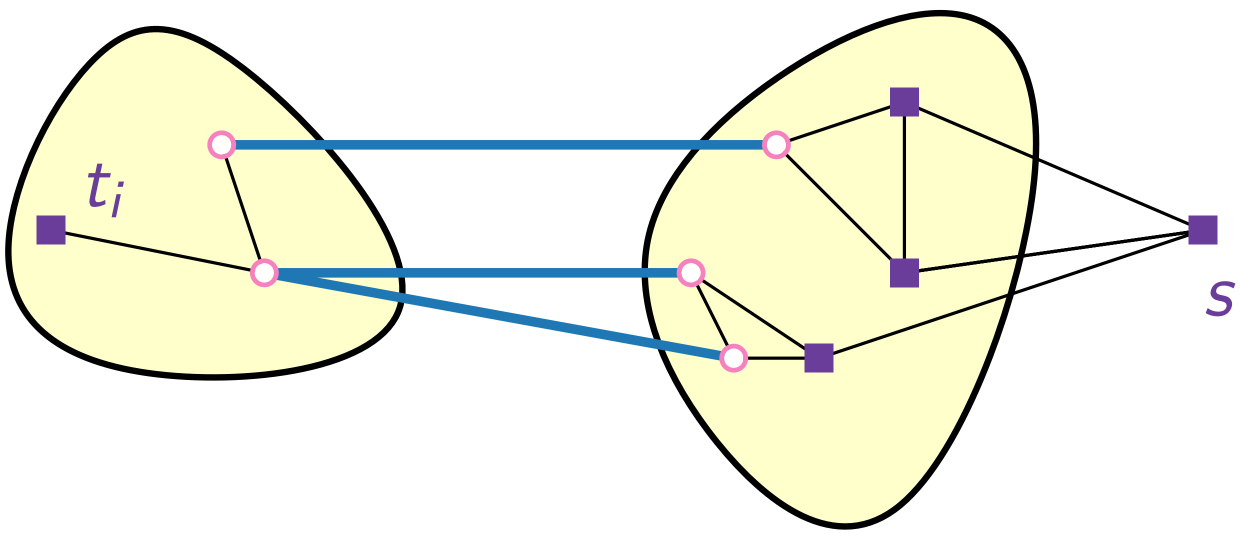

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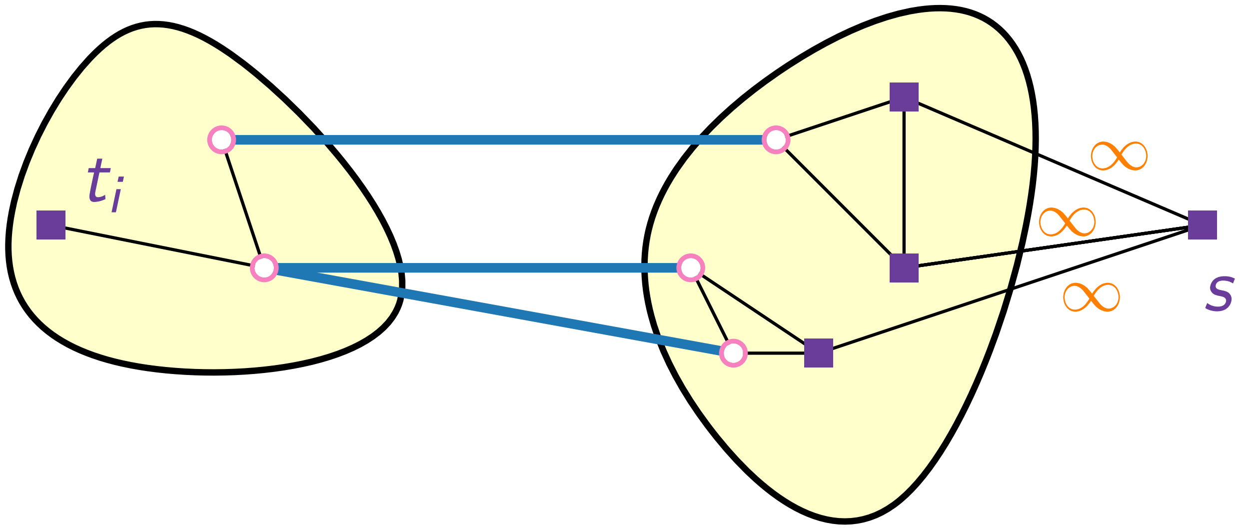

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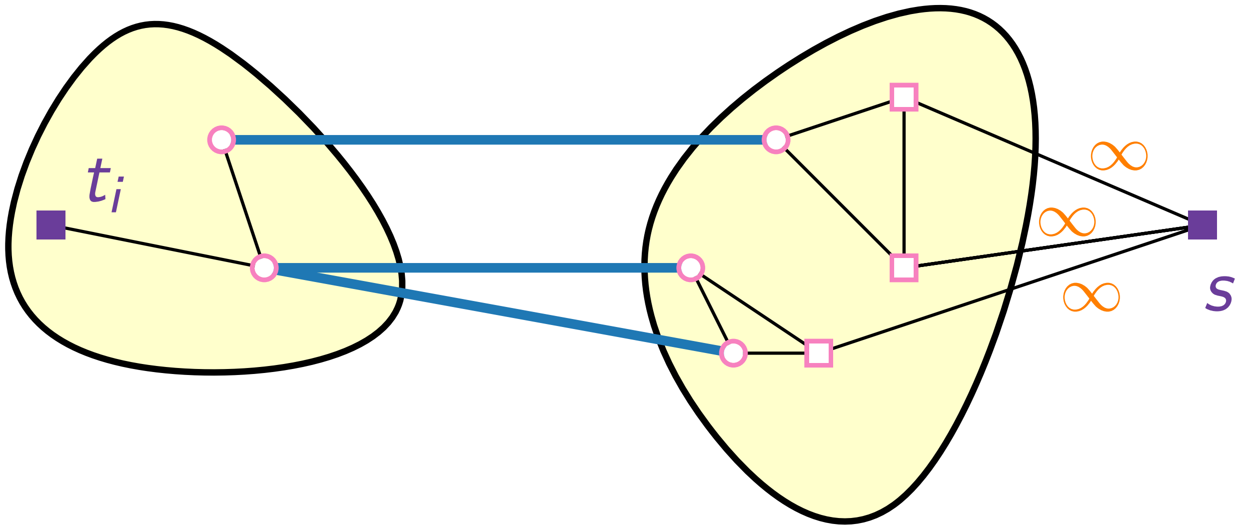

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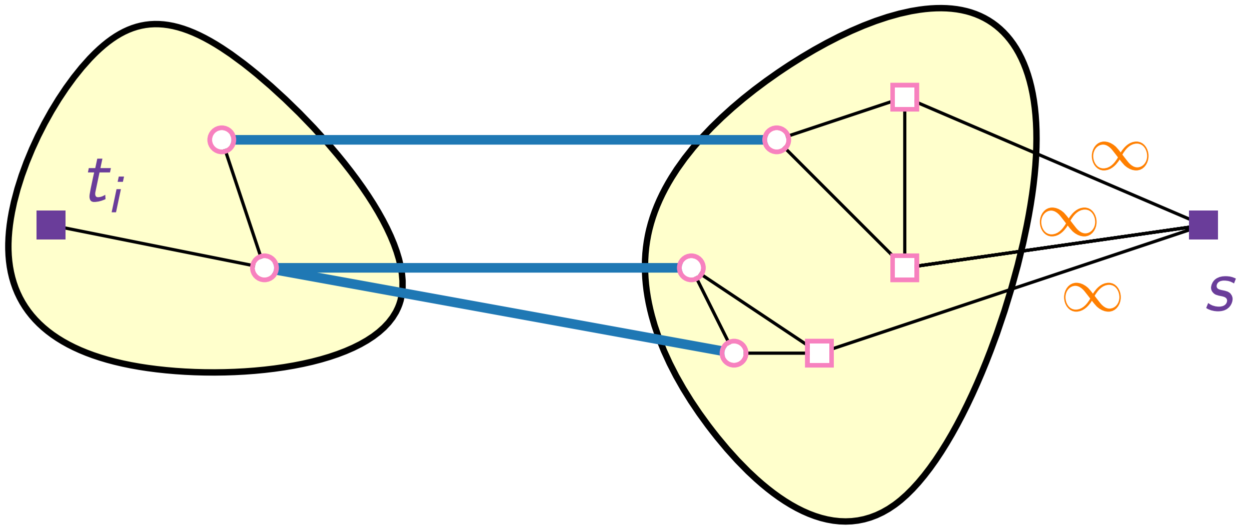

Add dummy terminal s and find a minimum-cost s–t_i cut.

Approximation Algorithms

Lecture 3: SteinerTree and MultiwayCut

Part VI: Algorithm for MultiwayCut

Algorithm MultiwayCut

For $i = 1, \ldots, k$:

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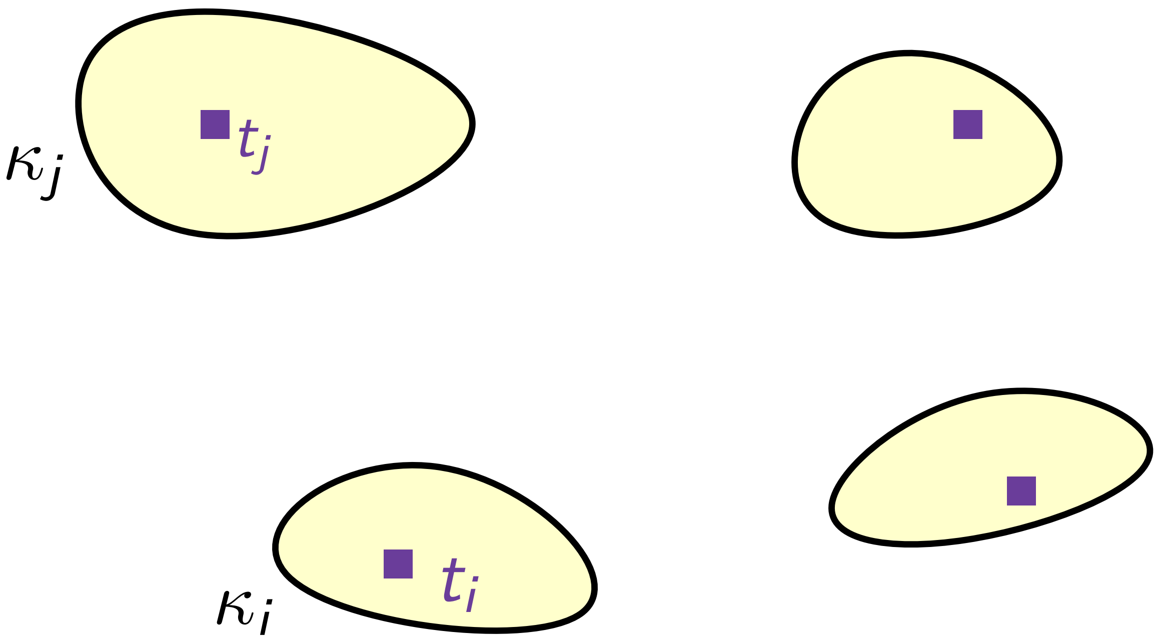

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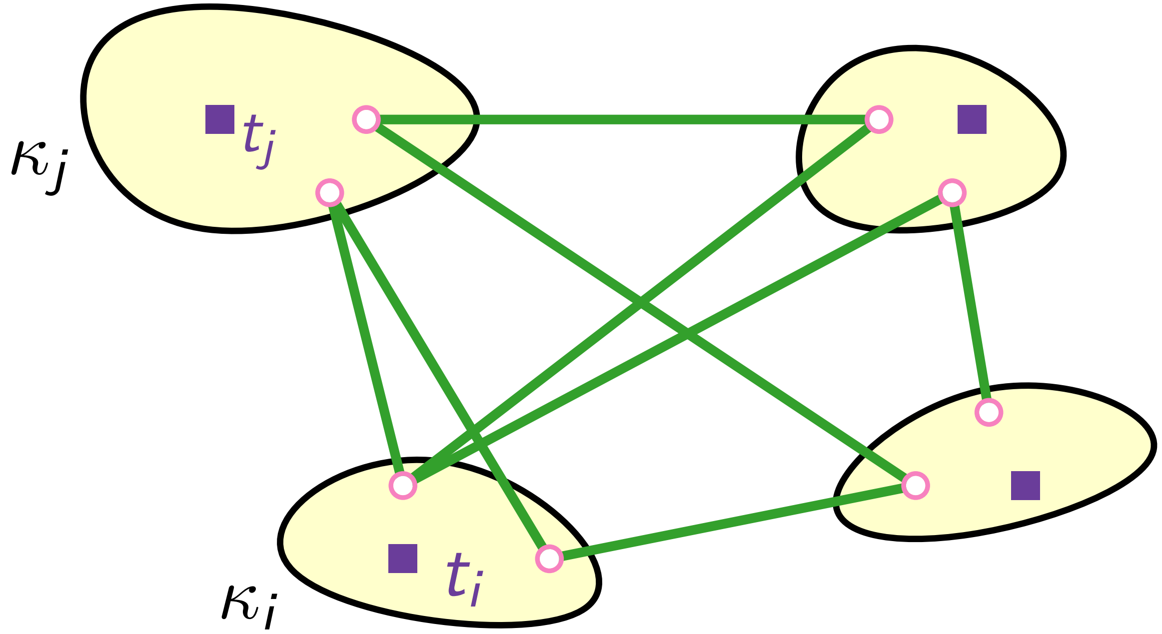

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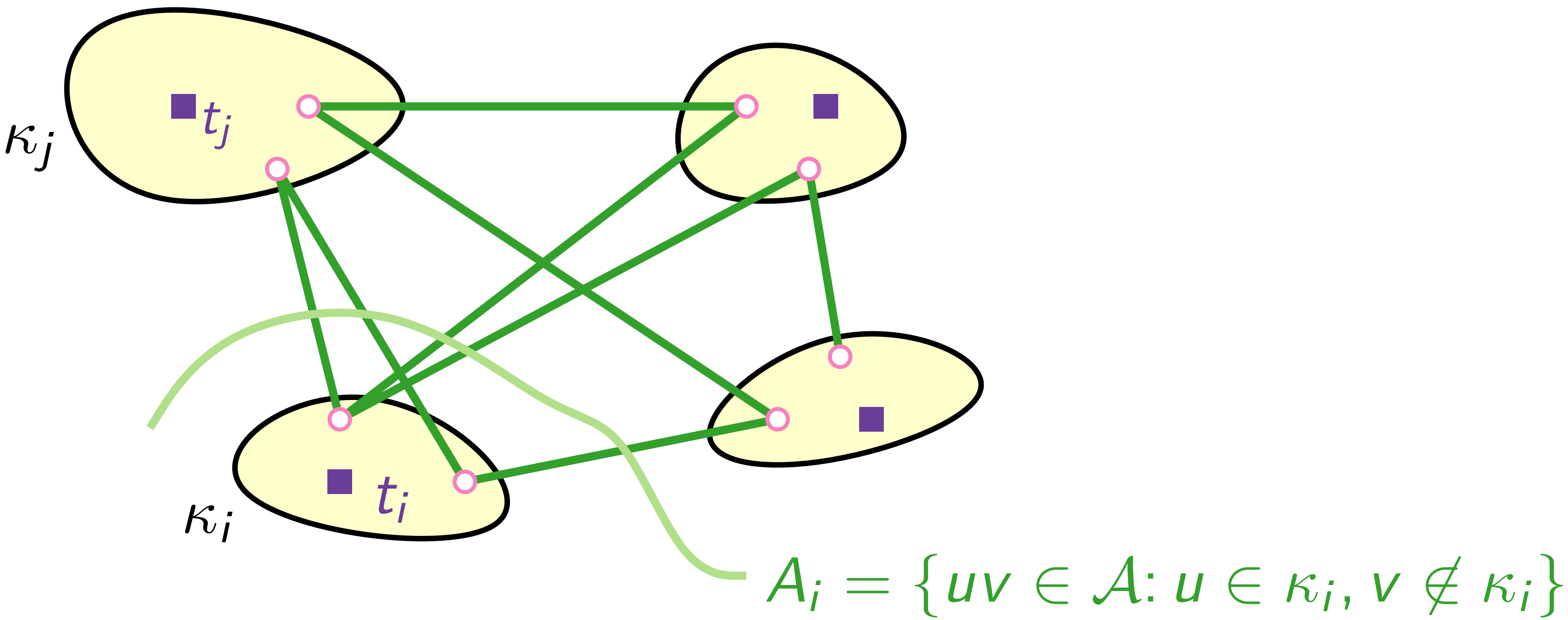

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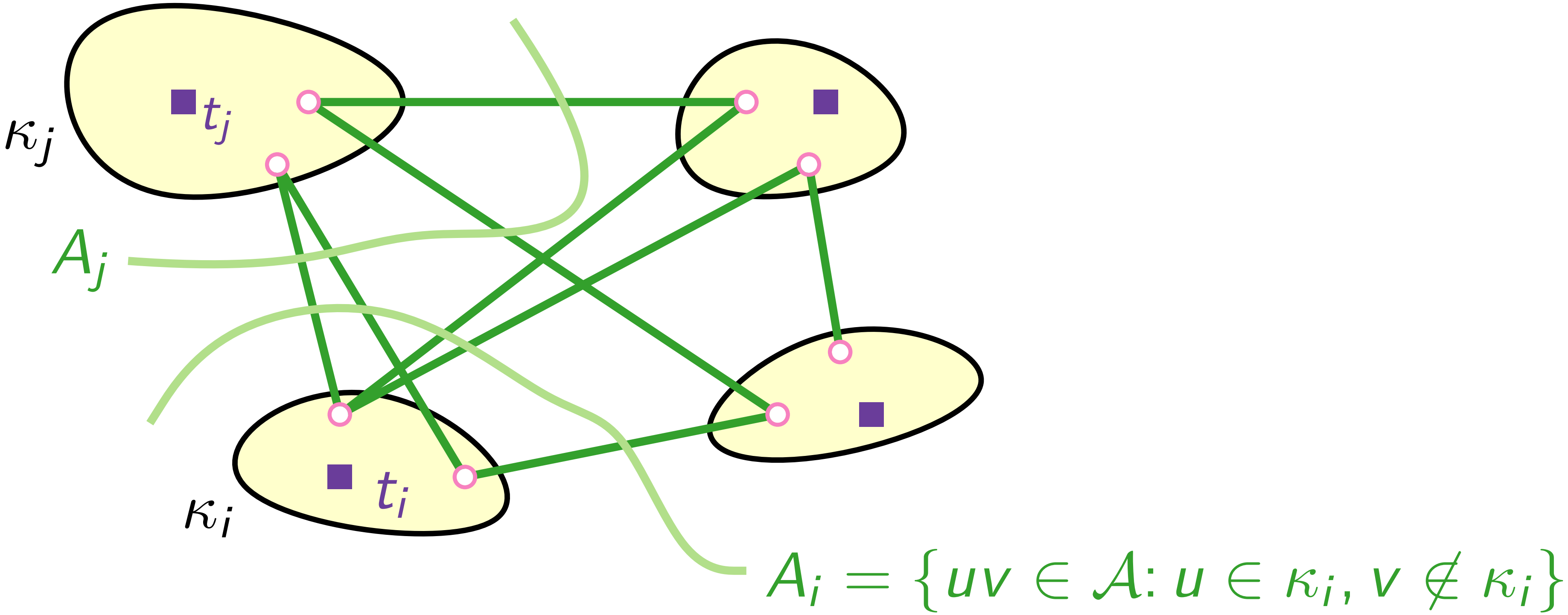

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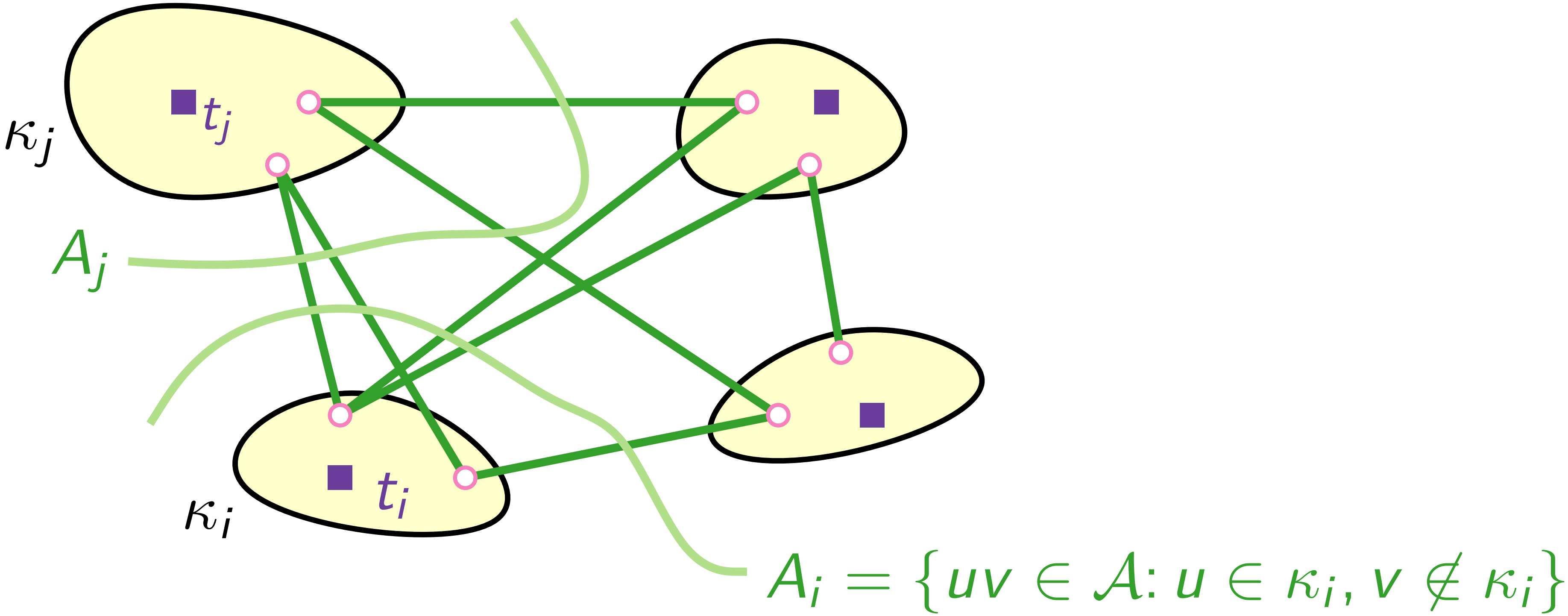

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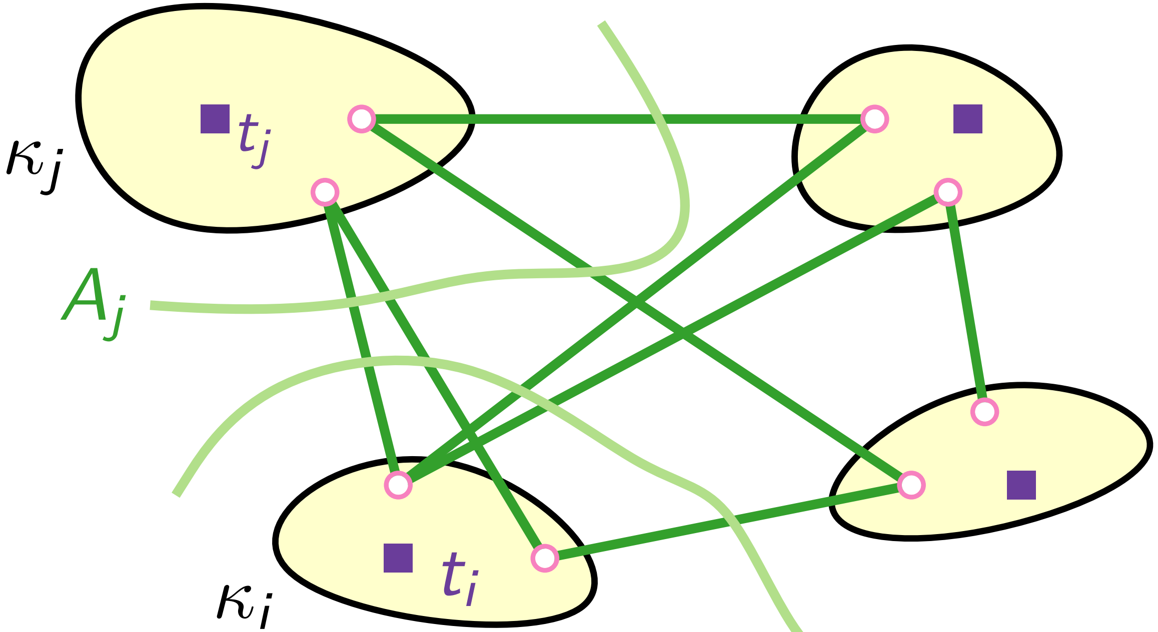

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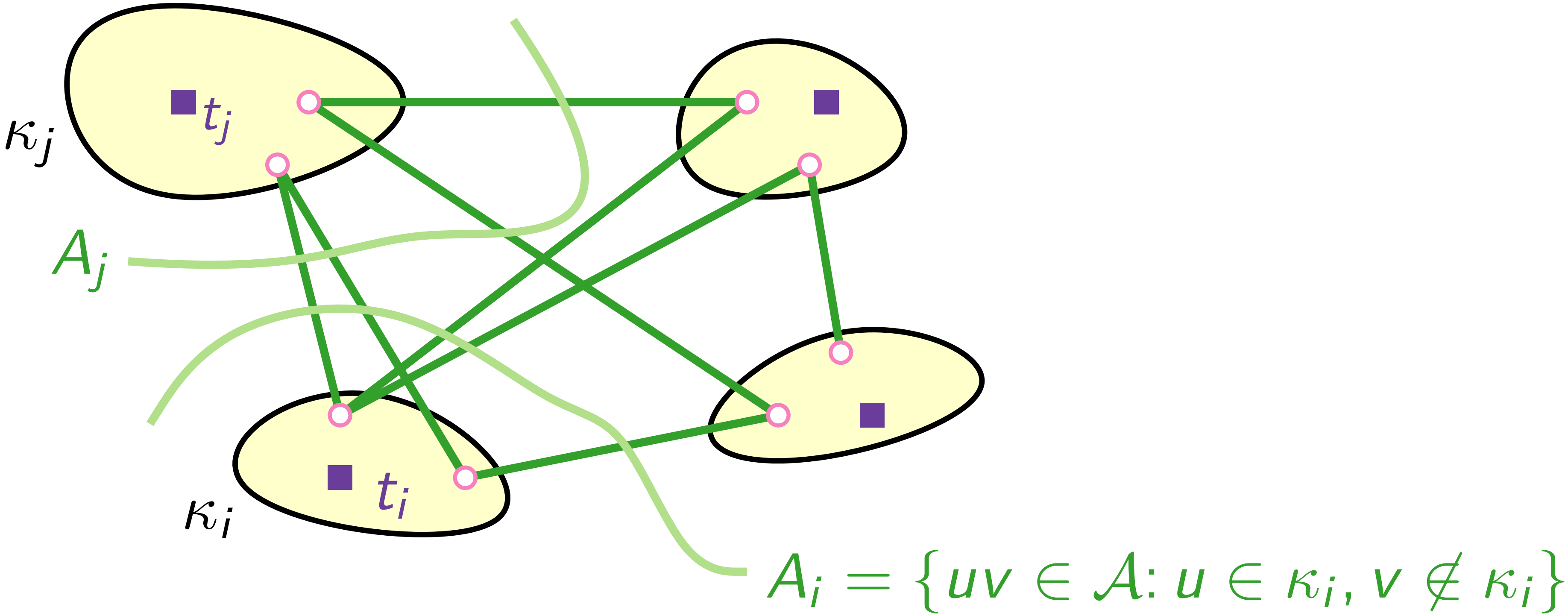

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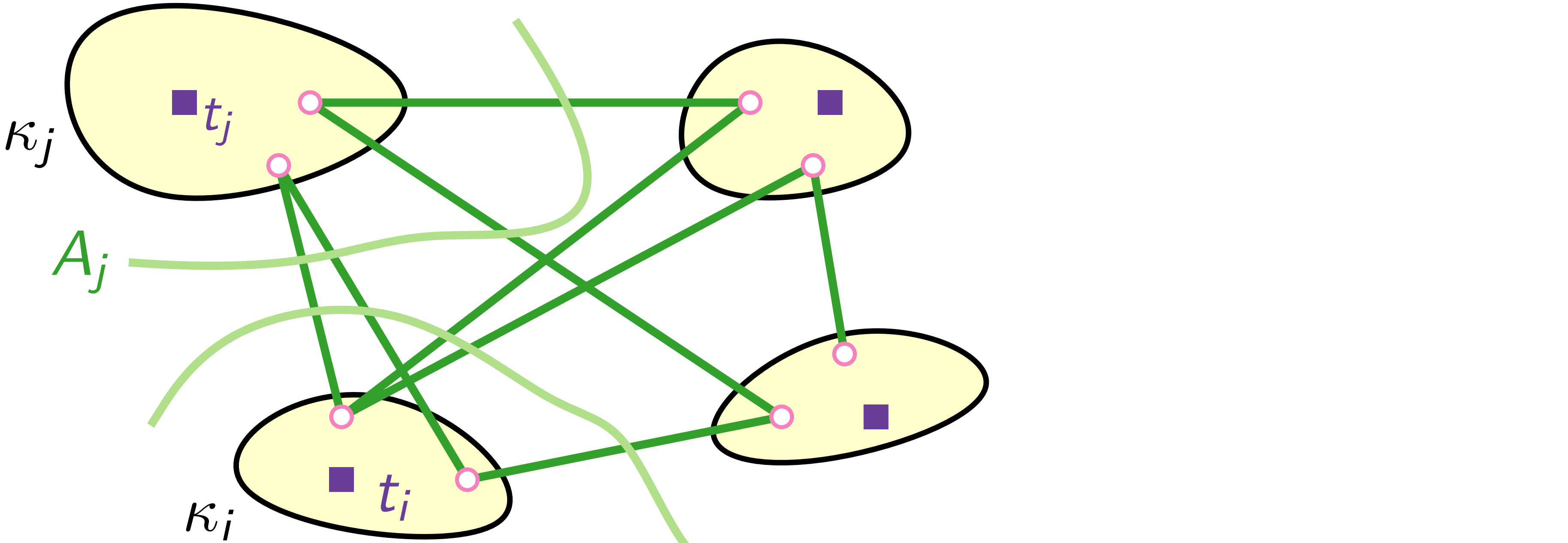

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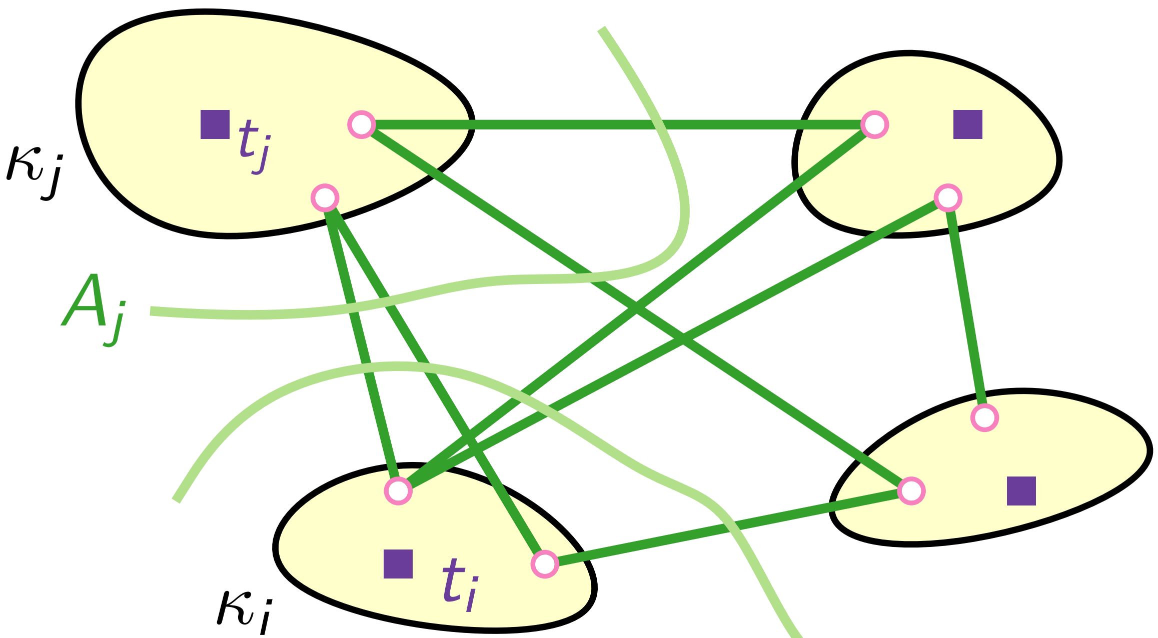

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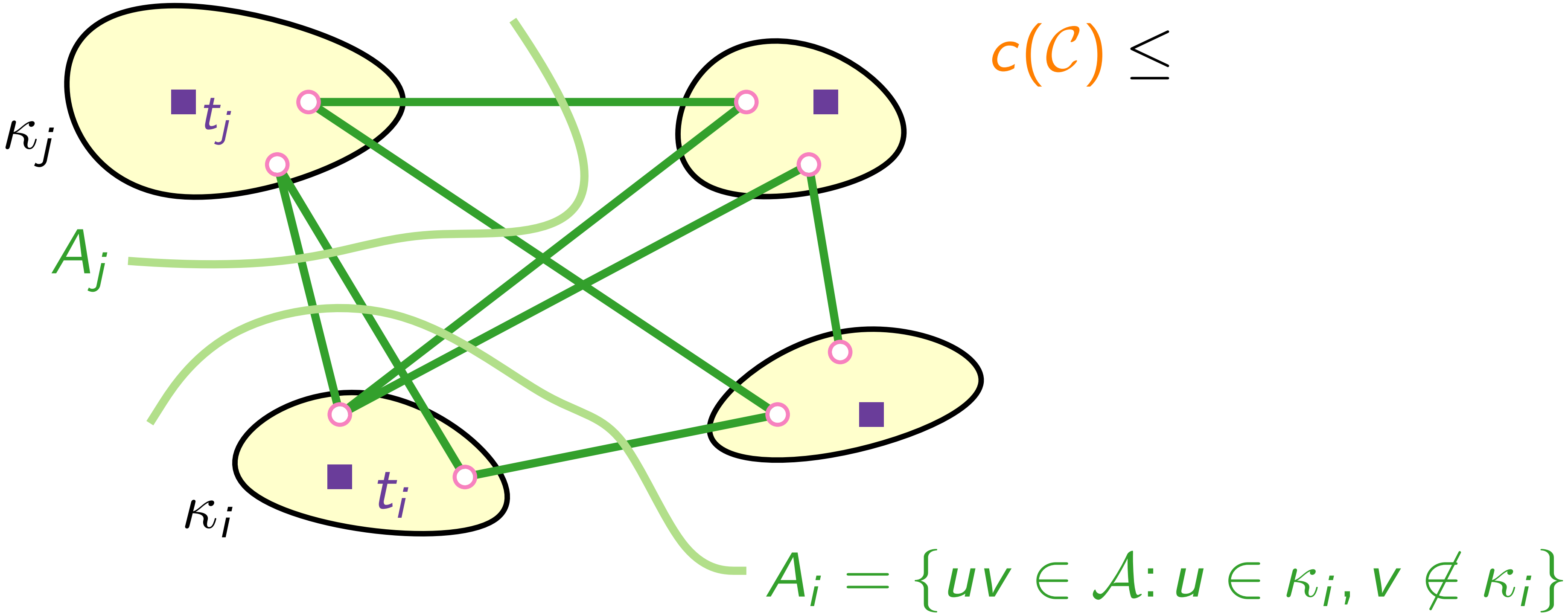

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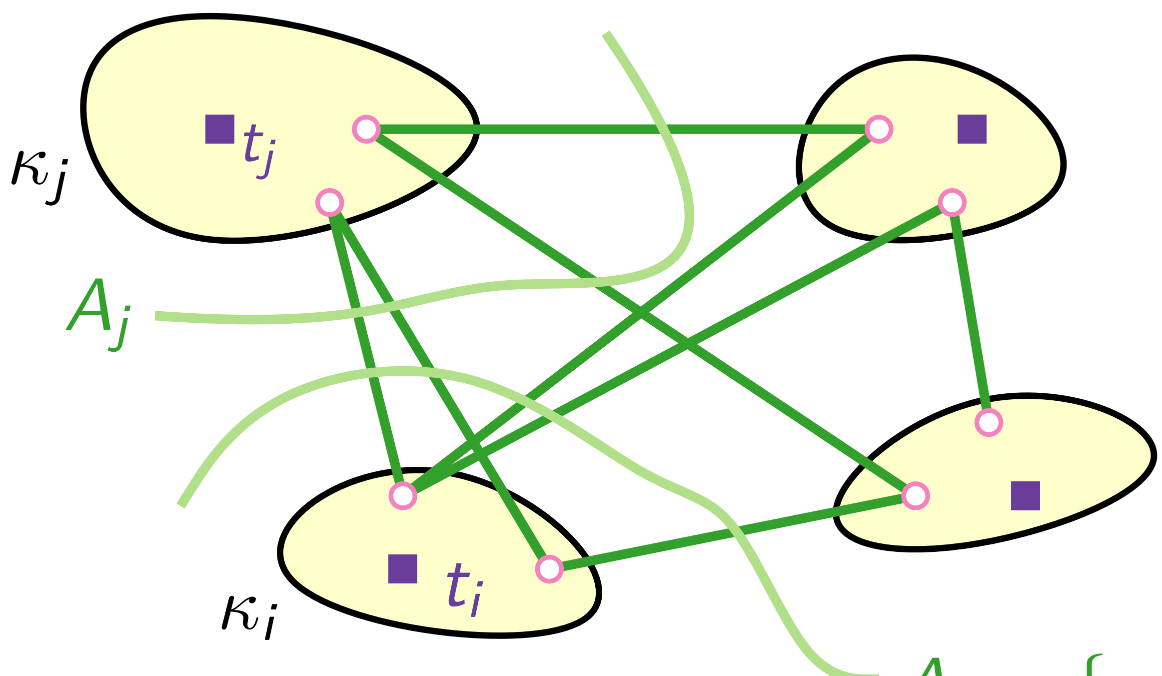

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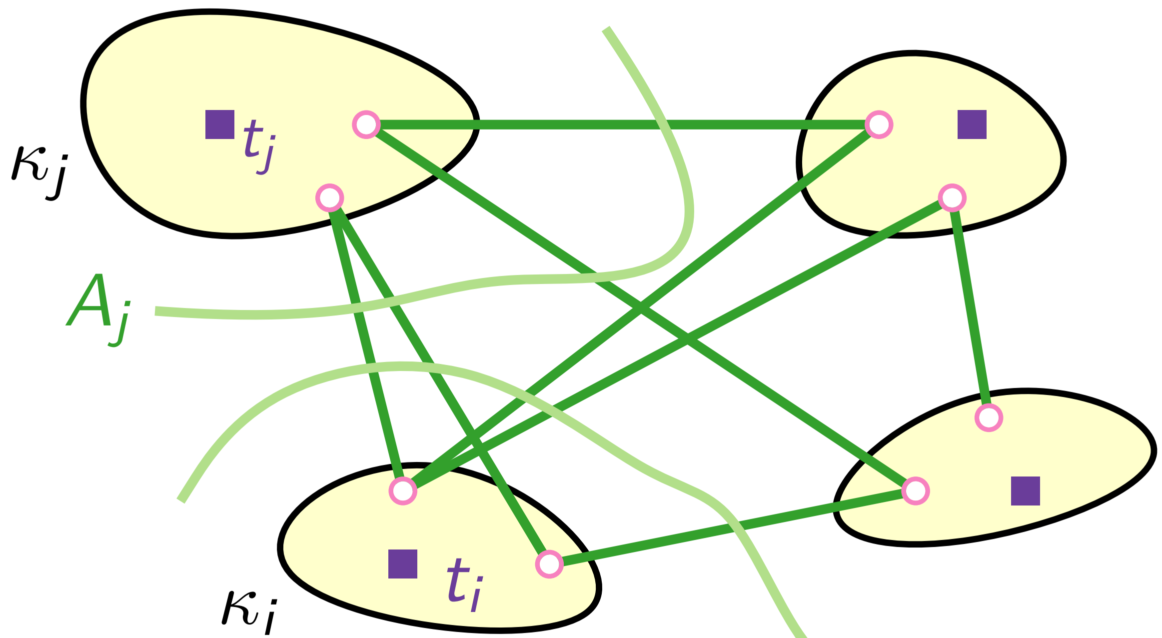

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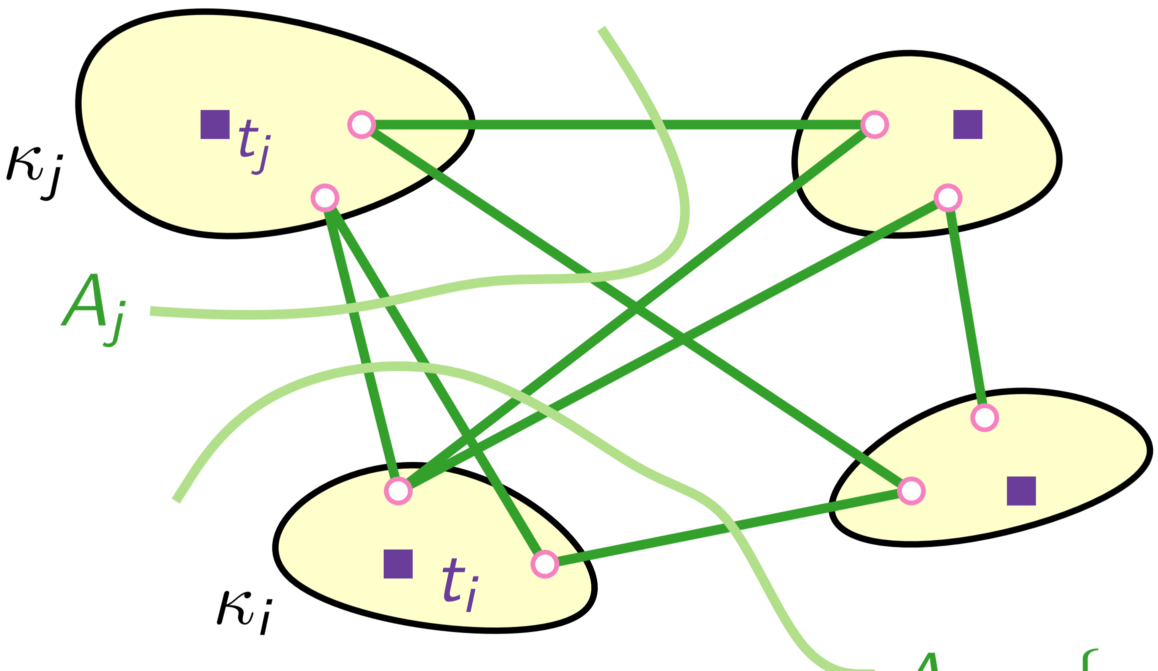

$$c(\mathcal{C}) \le \left(1 - \frac{1}{k}\right) \sum_{i=1}^{k} c(C_i)$$
$$\le \left(1 - \frac{1}{k}\right) \sum_{i=1}^{k} c(A_i)$$
$$\le \left(1 - \frac{1}{k}\right) \cdot 2 \cdot c(\mathcal{A})$$

$A_i = \{uv \in \mathcal{A} : u \in \kappa_i, v \notin \kappa_i\}$

Observation. $\mathcal{A} = \bigcup_{i=1}^{k} A_i$ and $\sum_{i=1}^{k} c(A_i) \le 2 \cdot c(\mathcal{A}) = 2 \cdot \mathrm{OPT}$.

Approximation Factor

Theorem. This algorithm is a factor-$(2 - 2/k)$ approximation algorithm for MULTIWAYCUT.

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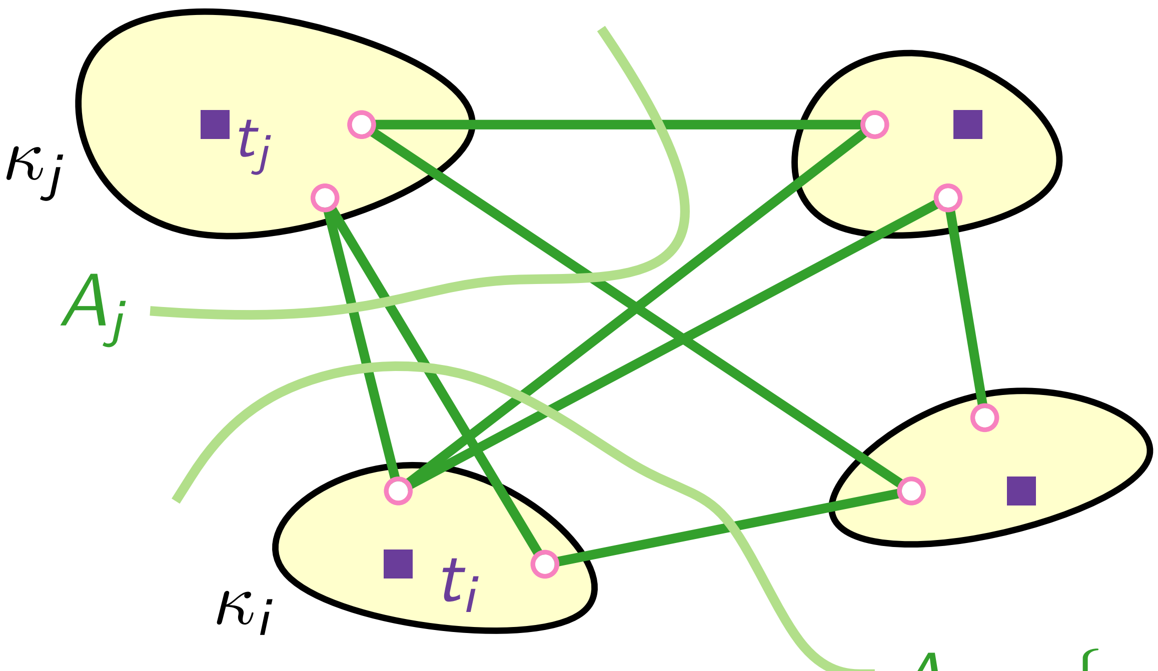

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$$\leq \left(1 - \frac{1}{k}\right) \cdot 2 \cdot c(\mathcal{A})$$
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Analysis Tight?

K_k

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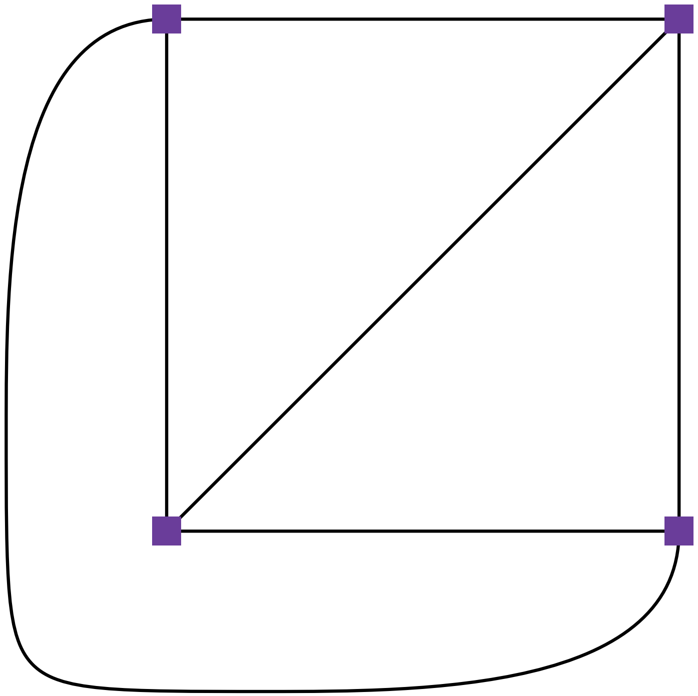

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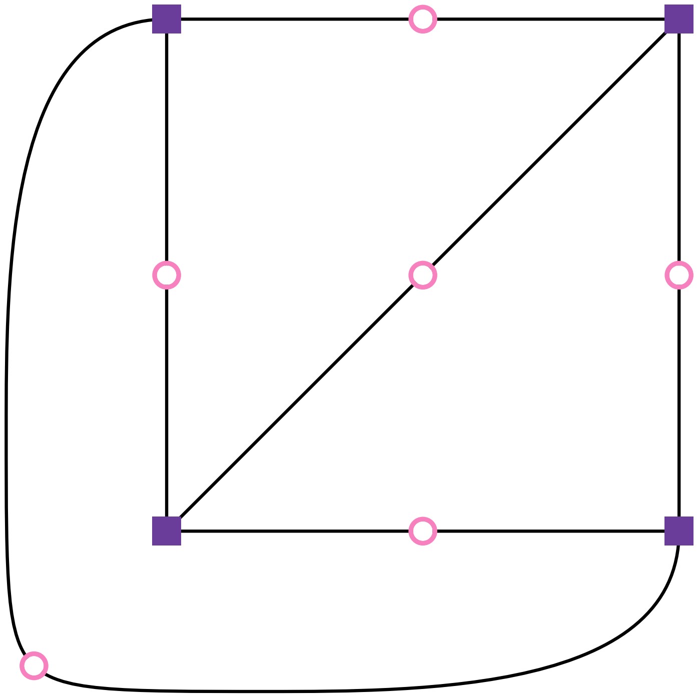

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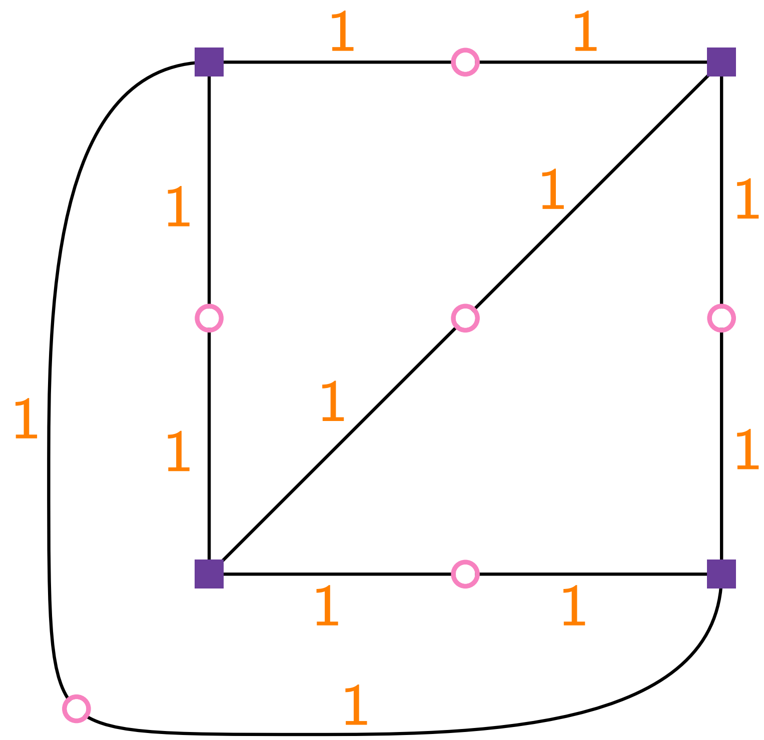

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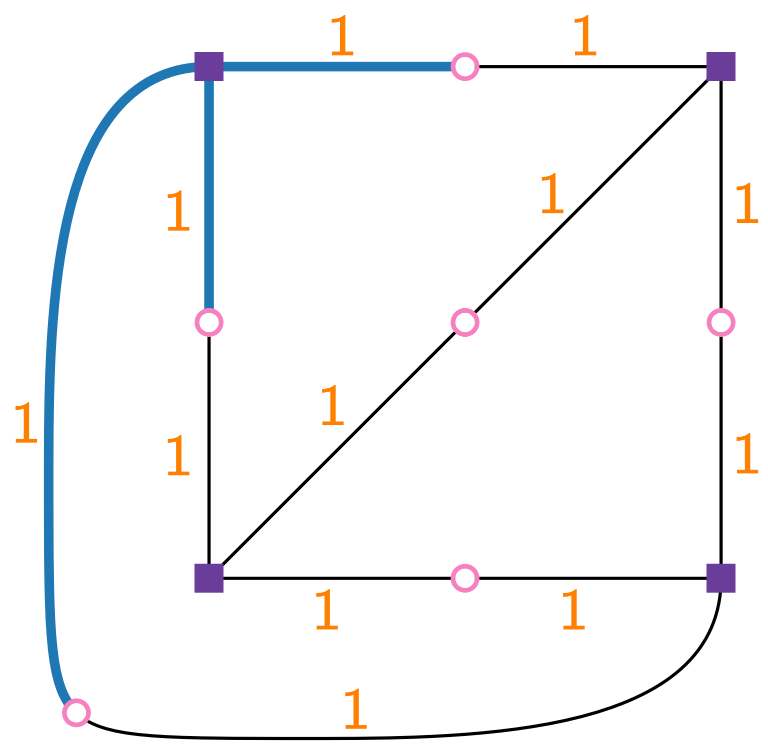

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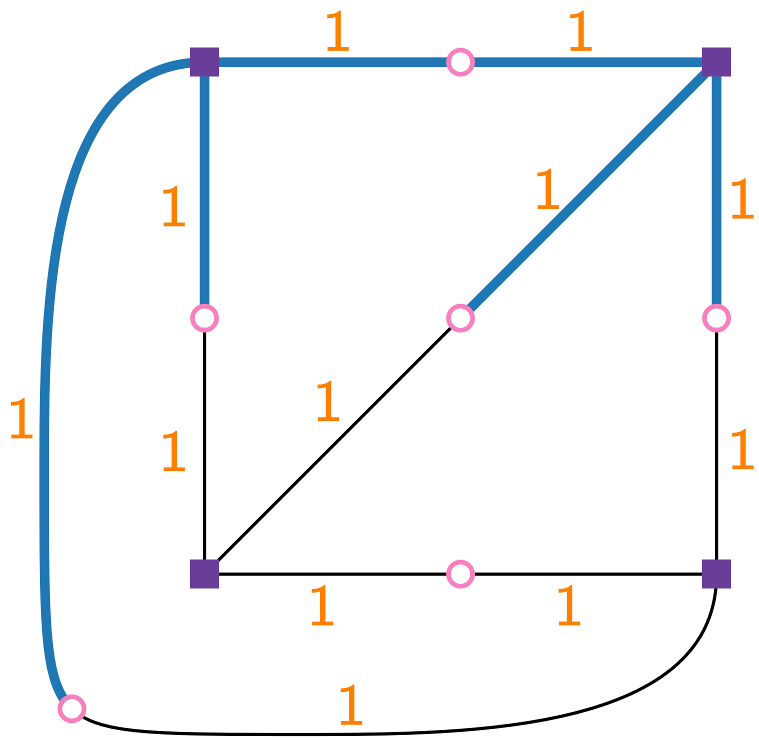

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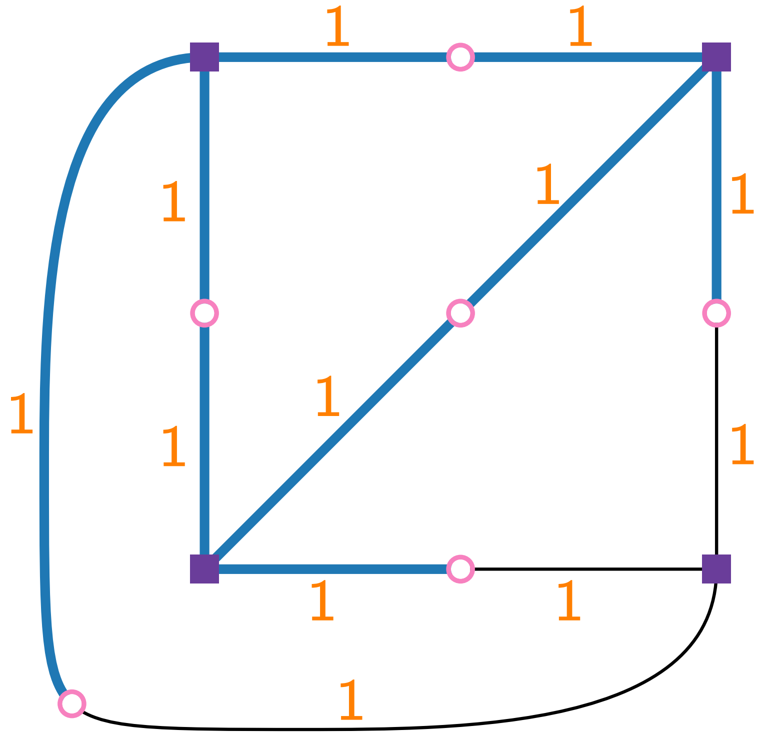

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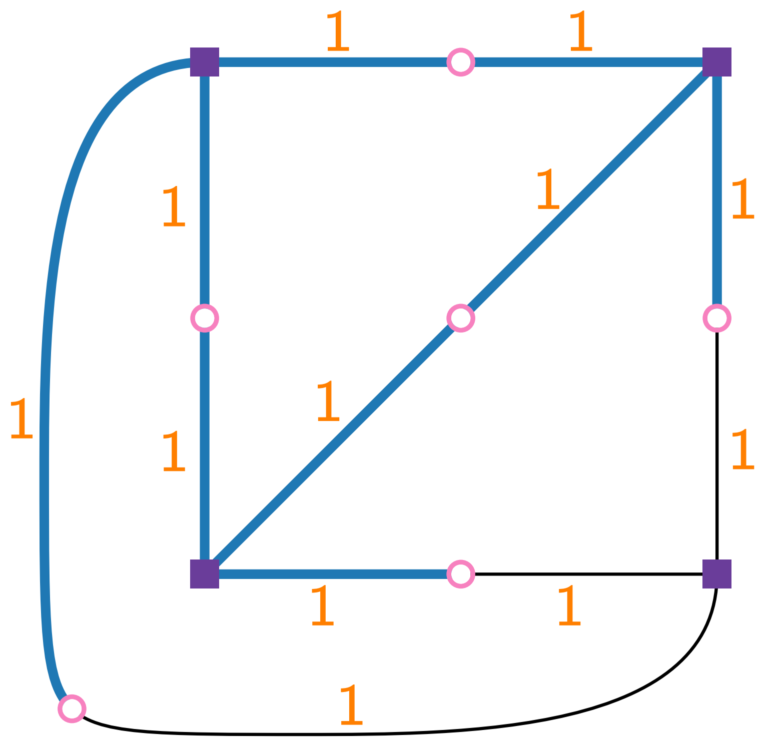
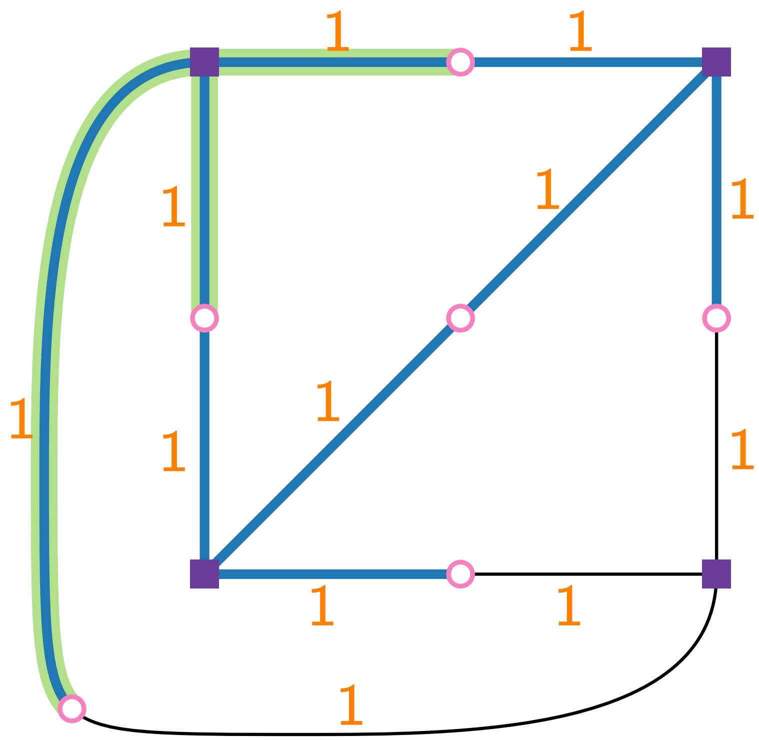


$\text{ALG} = (k-1)(k-1)$

Analysis Tight?

K_k



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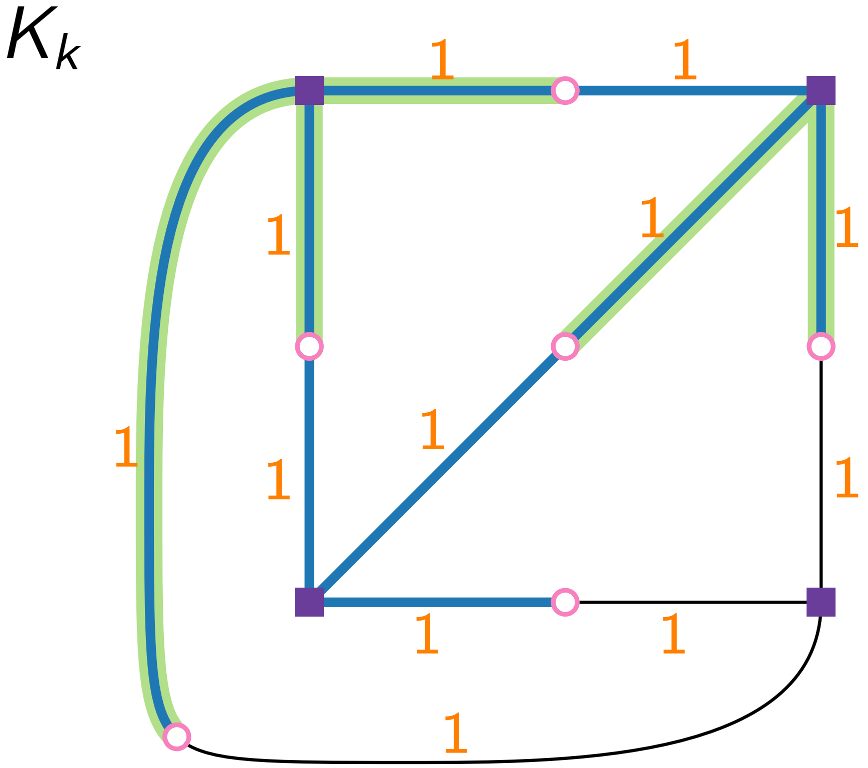

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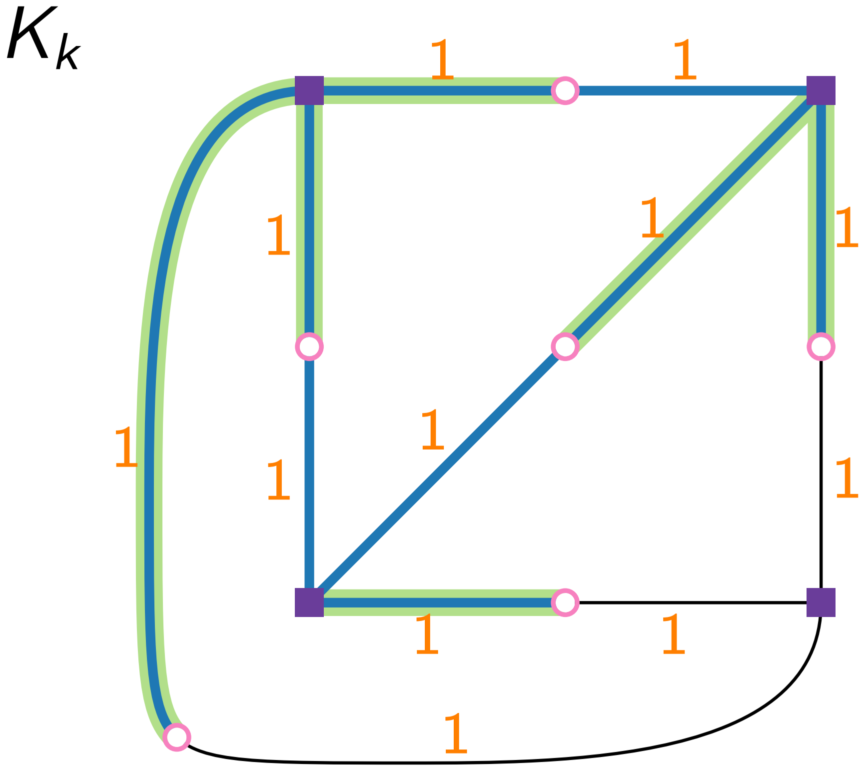

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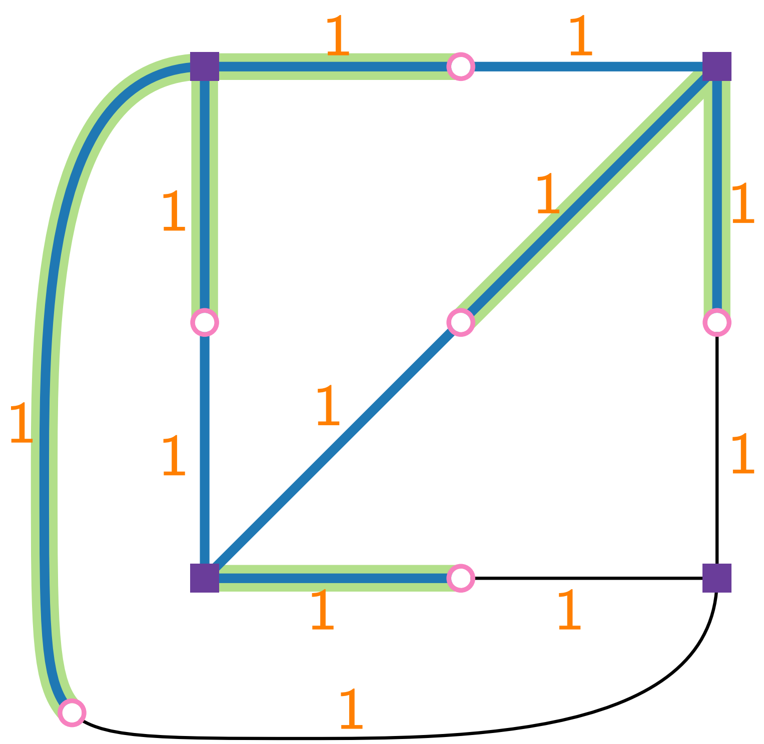

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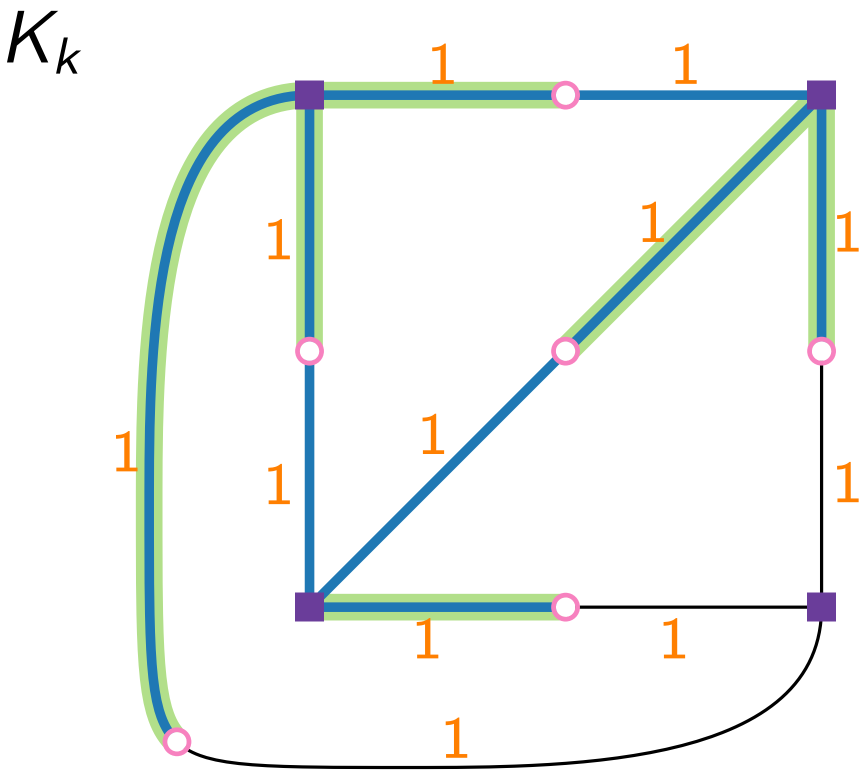
Analysis Tight?



$\text{ALG} = (k-1)(k-1)$

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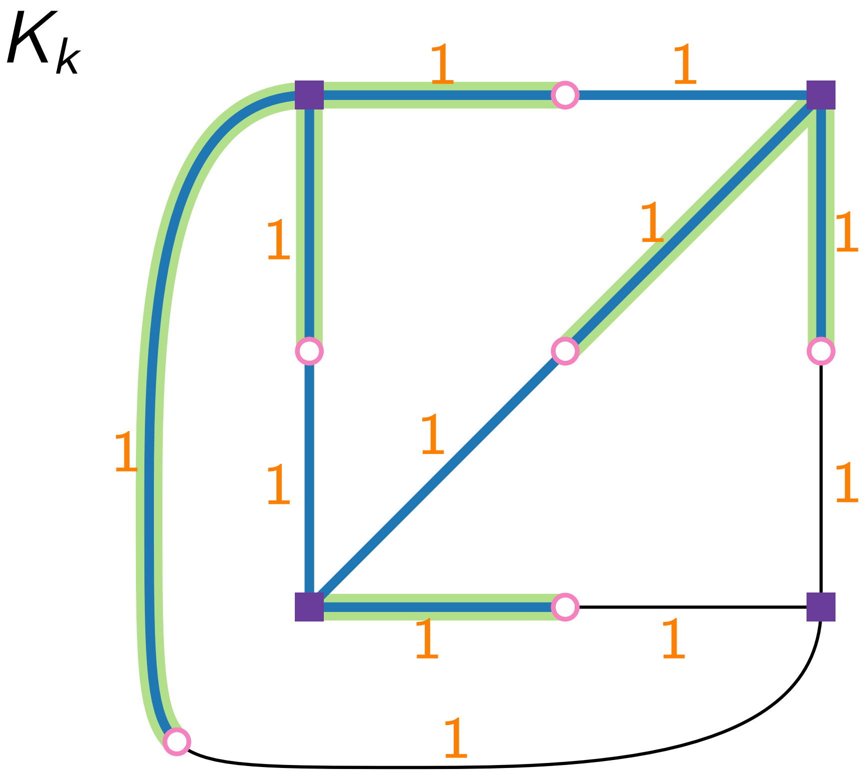
Analysis Tight?



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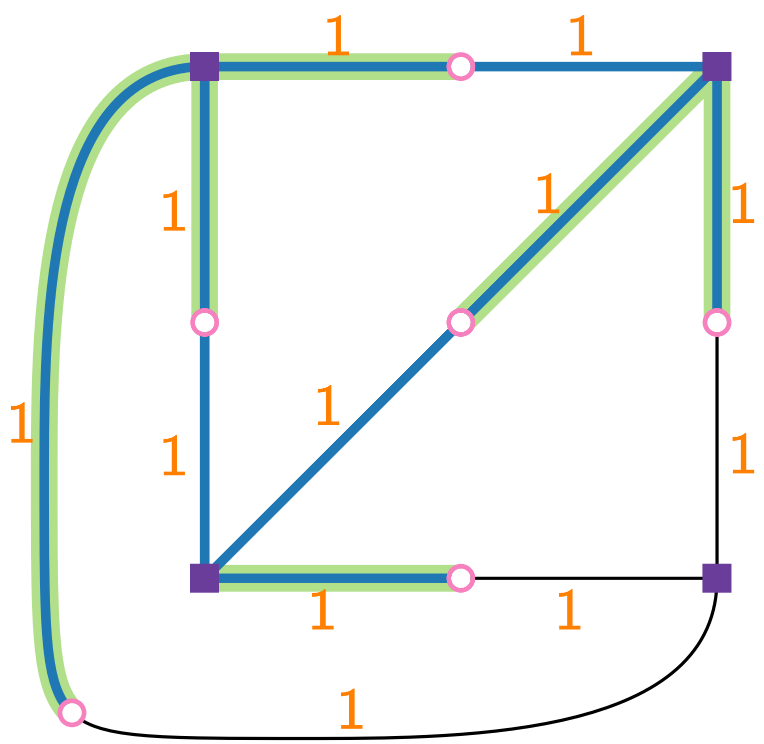
K_k



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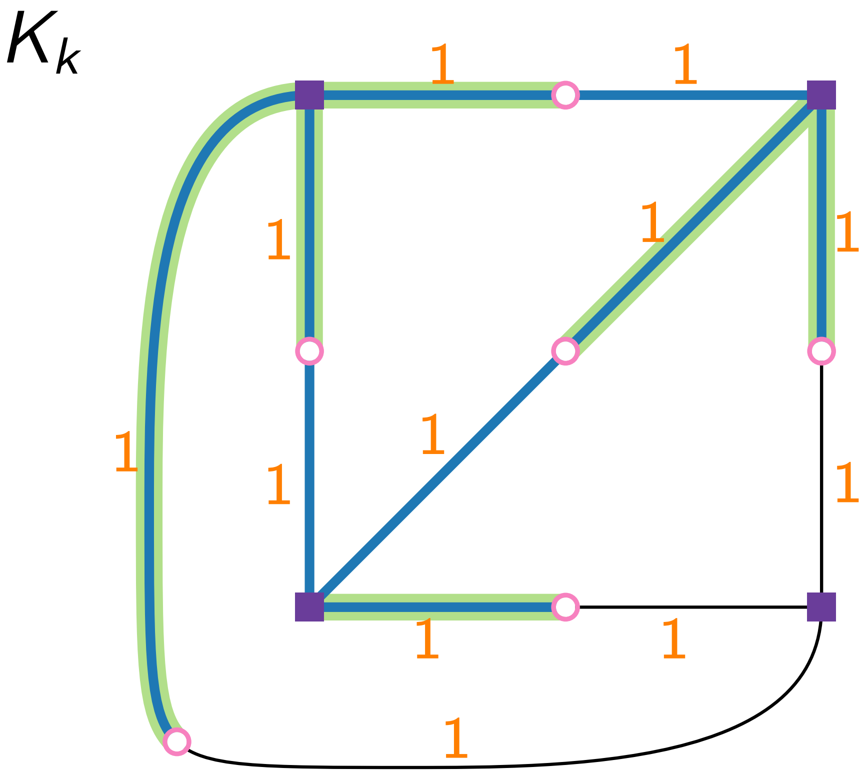

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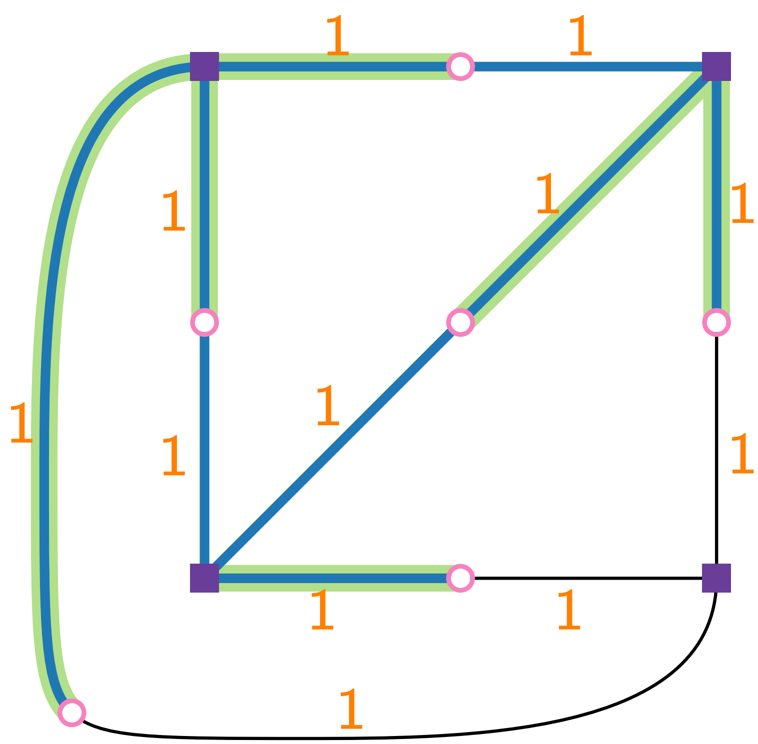
K_k



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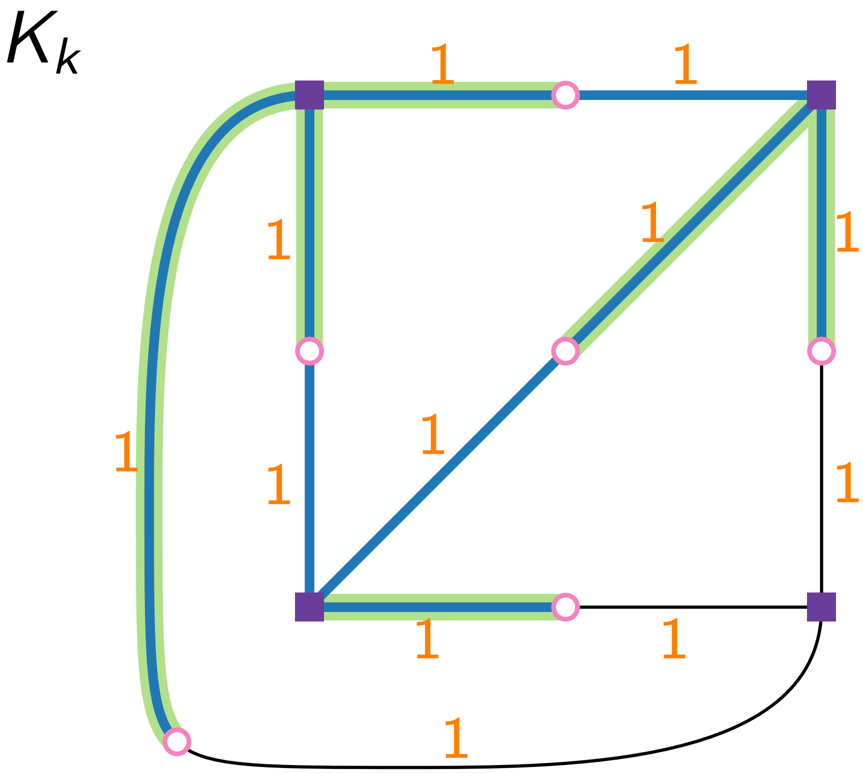

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The best known approximation factor for MultiwayCut is $1.2965 - \frac{1}{k}$.

[Sharma & Vondrák, STOC'14]

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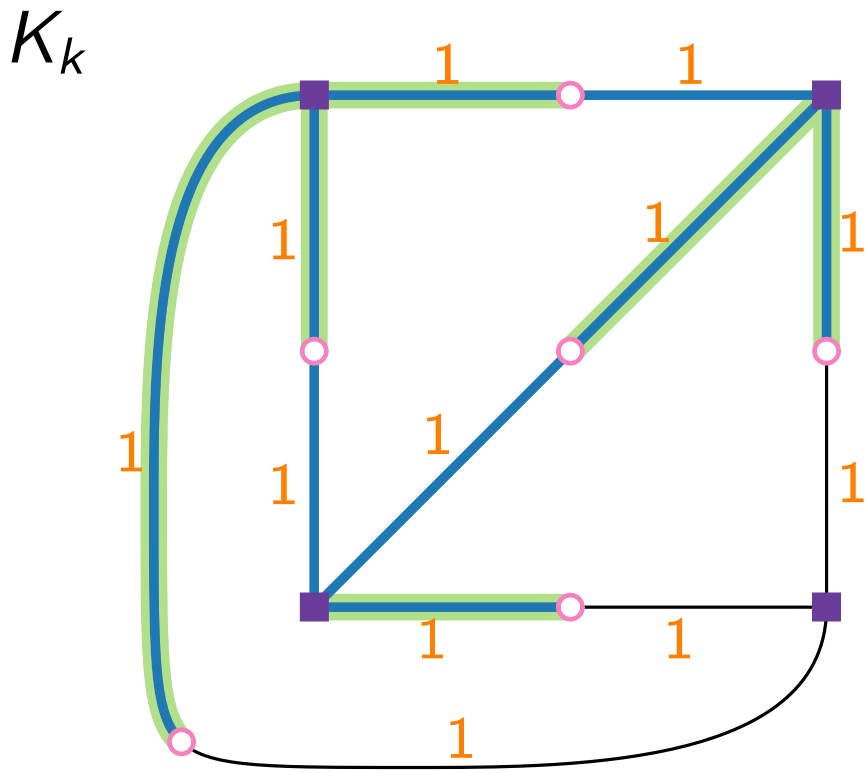

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MultiwayCut cannot be approximated within factor $1.20016 - O(1/k)$ (unless $P = NP$).
[Bérczi, Chandrasekaran, Király & Madan, MP'18]