Approximation Algorithms

Lecture 2: SetCover and ShortestSuperString

Part I: SetCover

Alexander Wolff

Winter 2023/24

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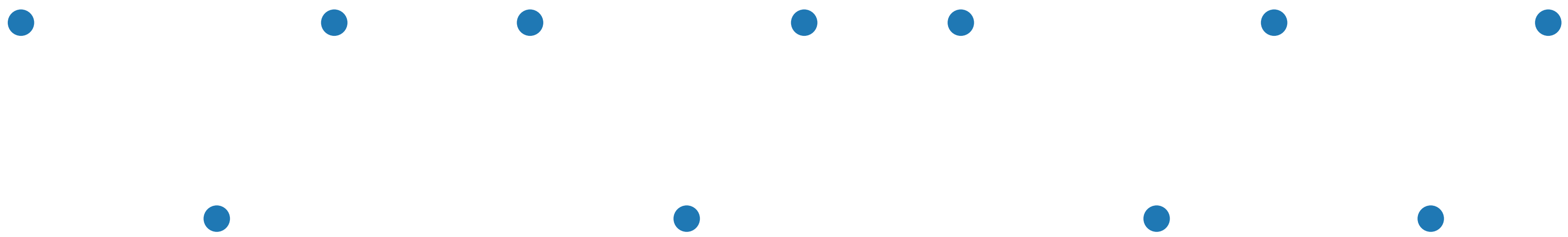

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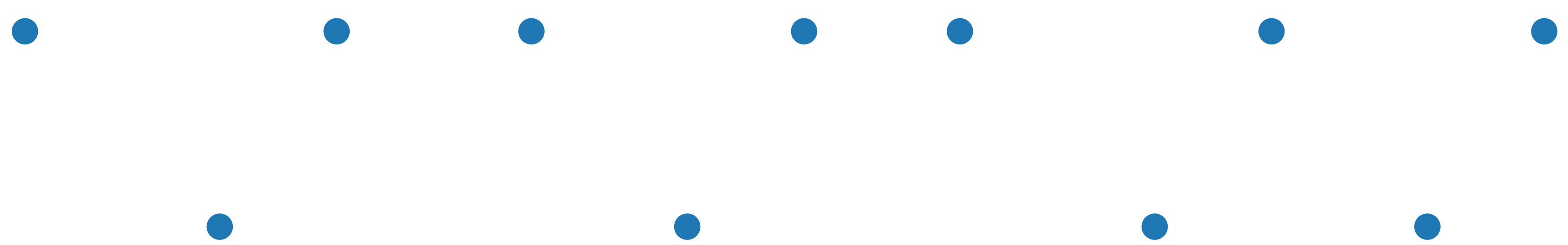

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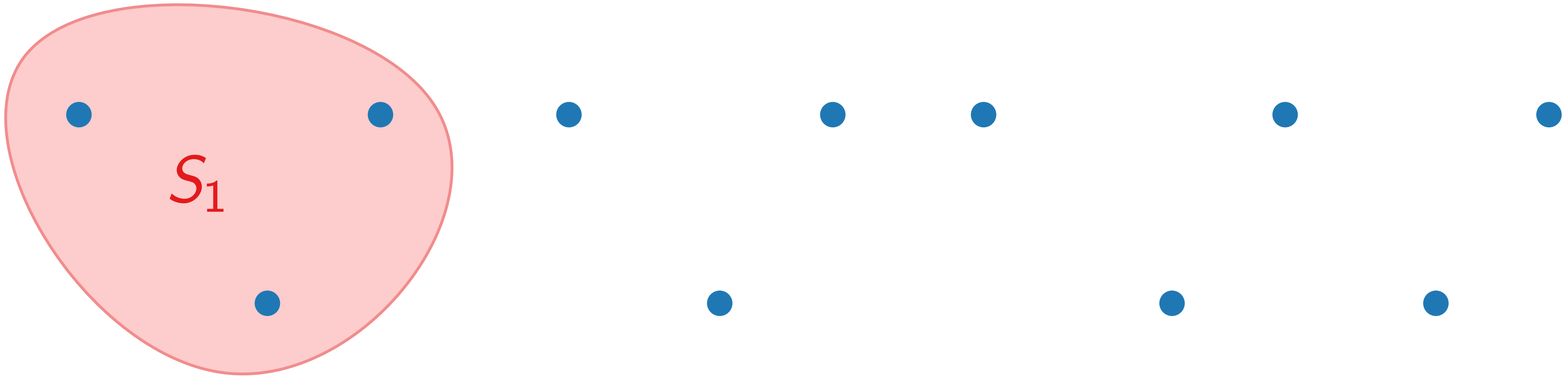

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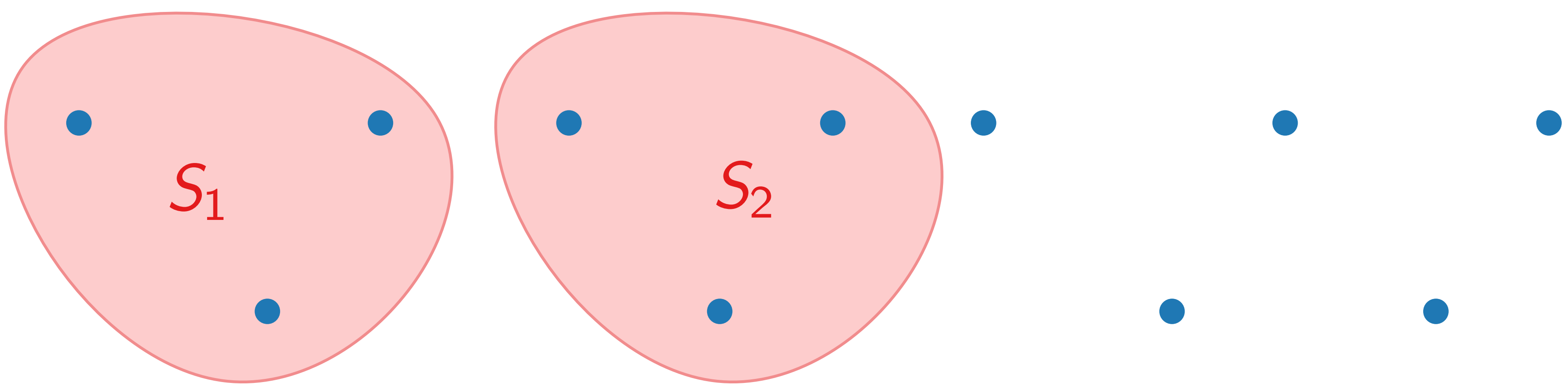

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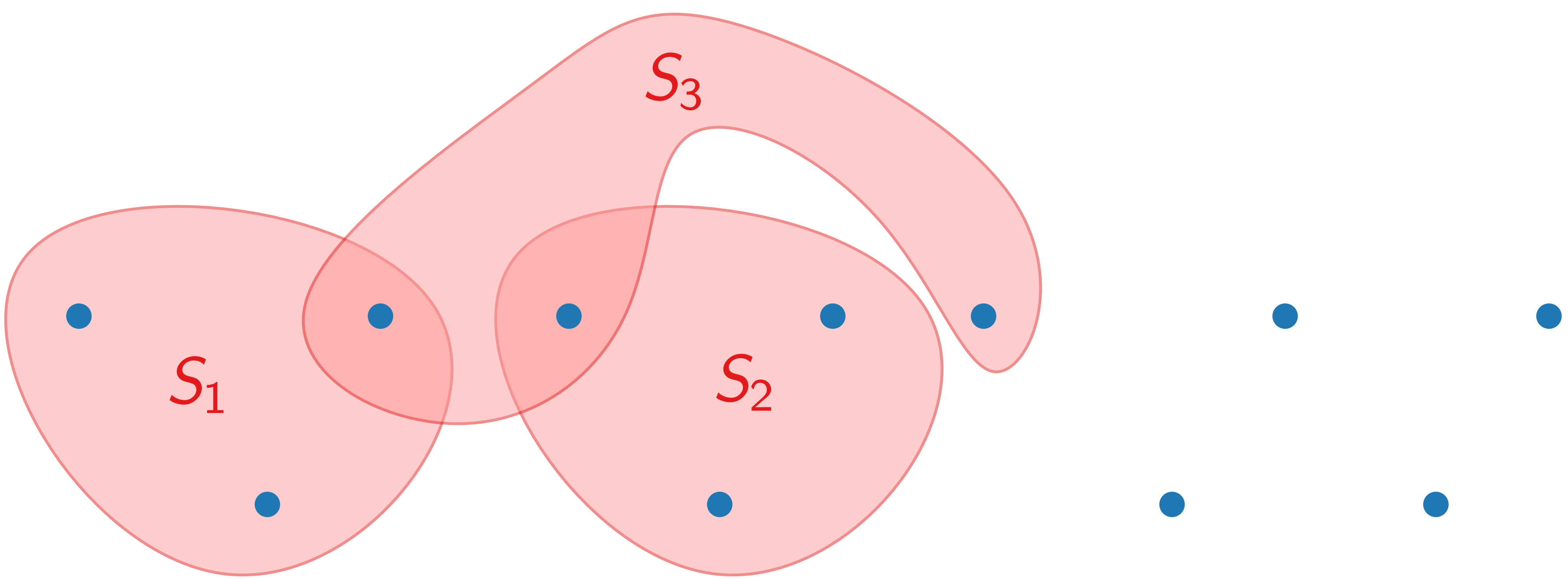

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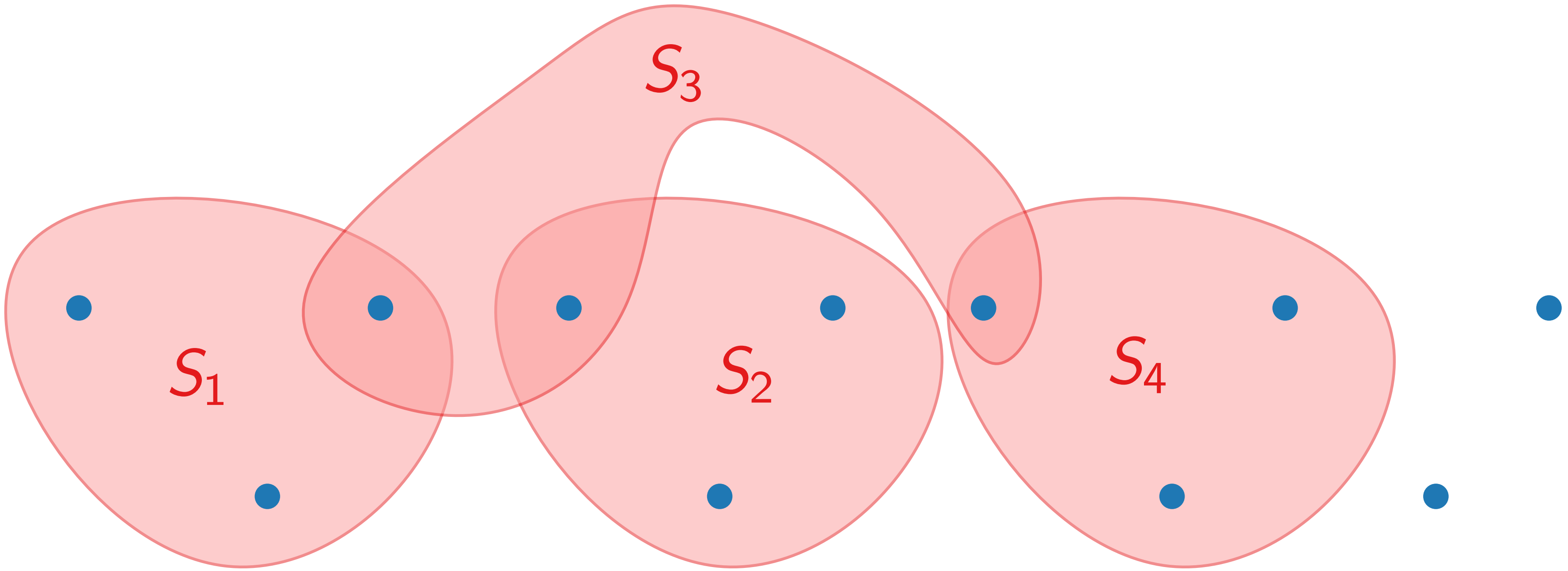

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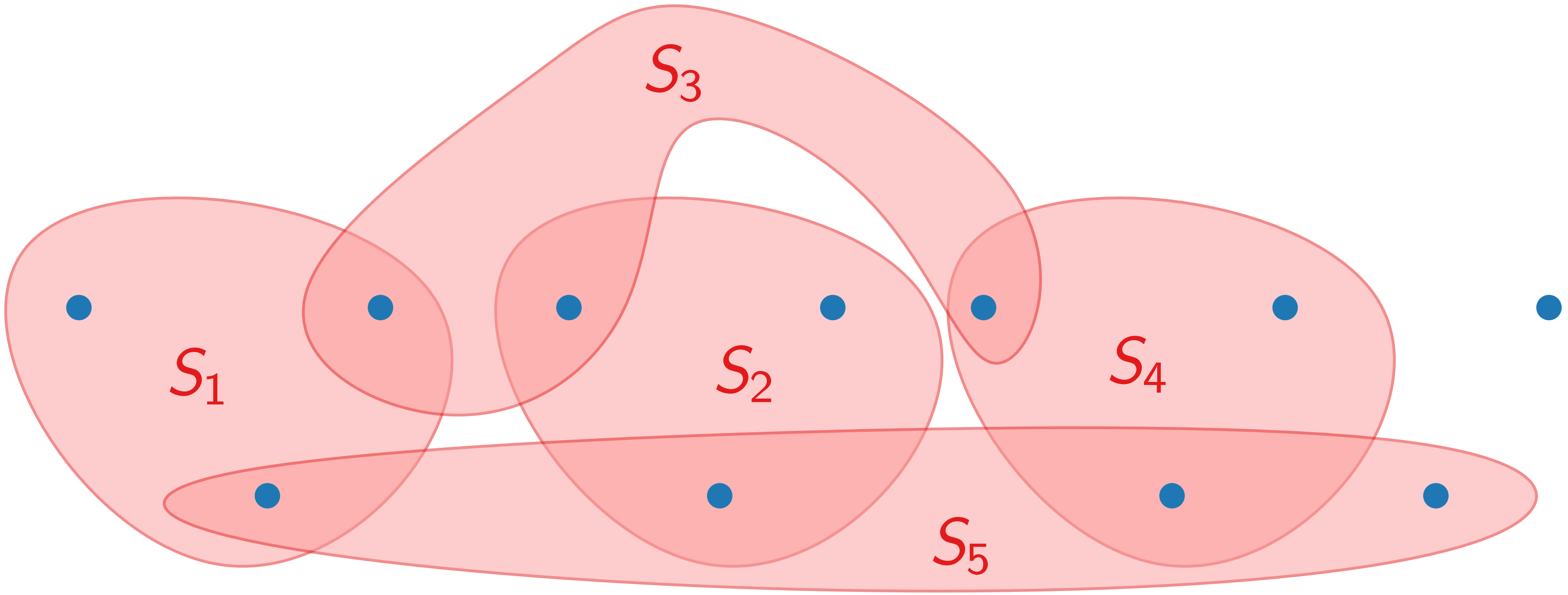

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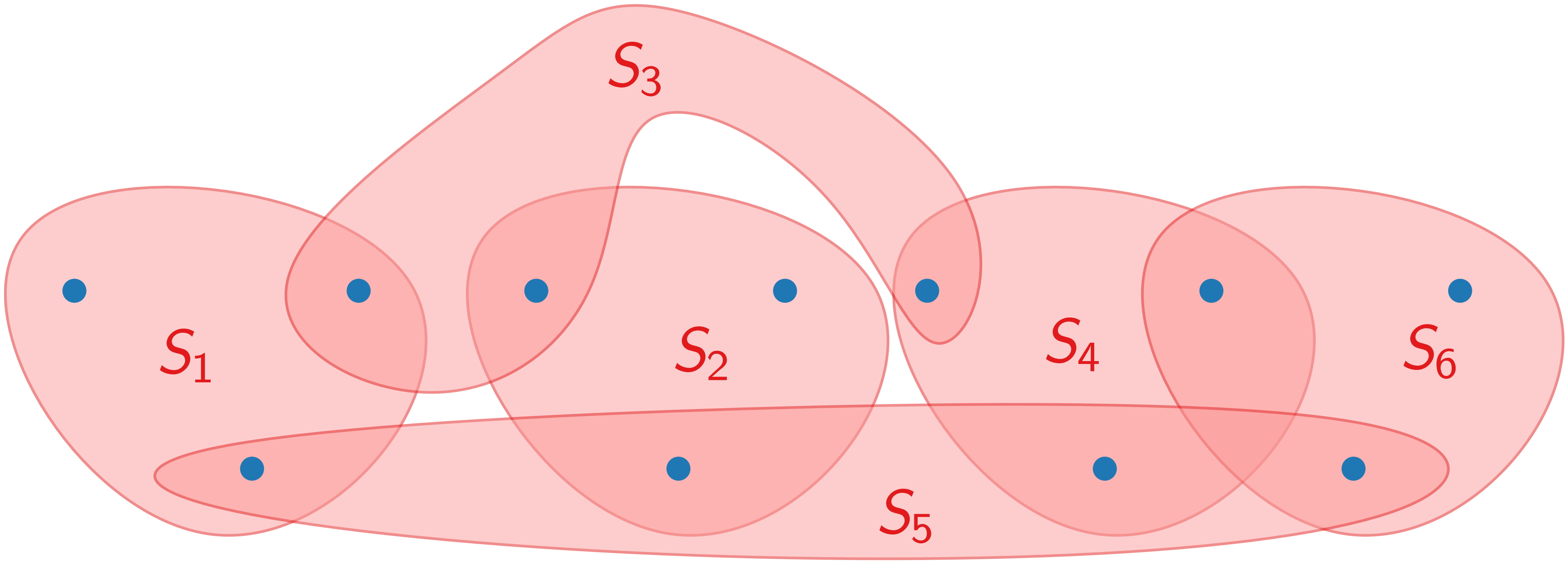

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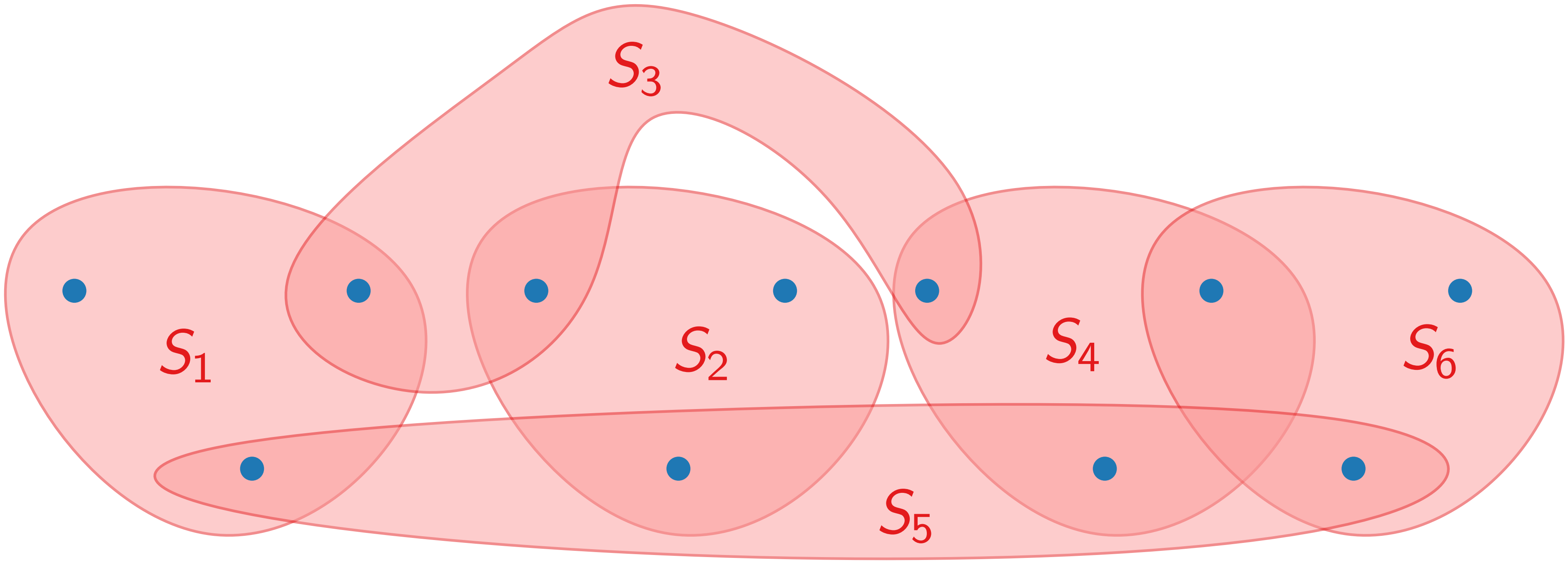

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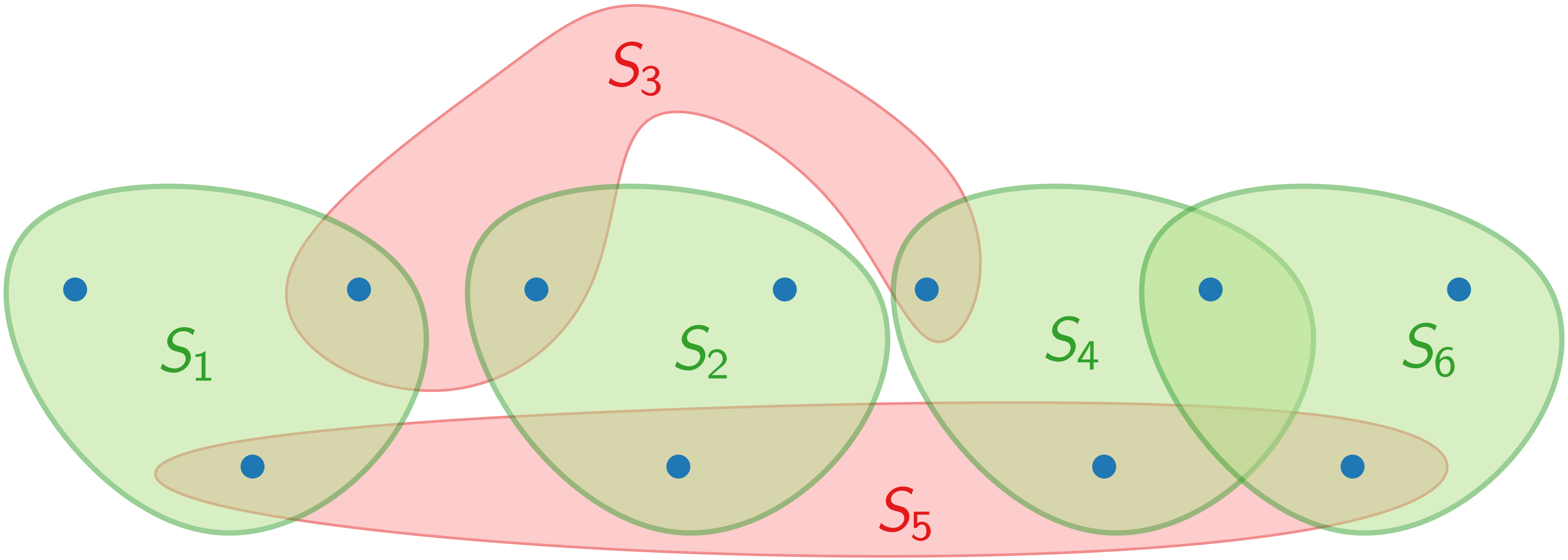

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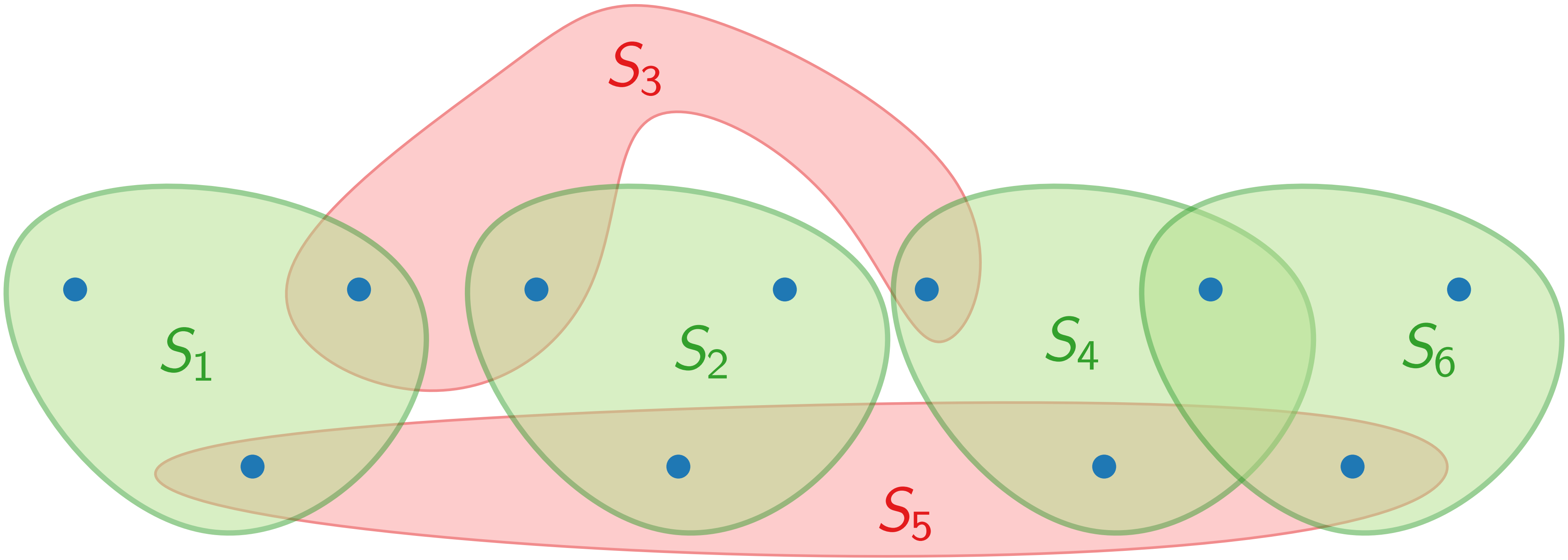

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Each $S \in \mathcal{S}$ has cost $c(S) > 0$.

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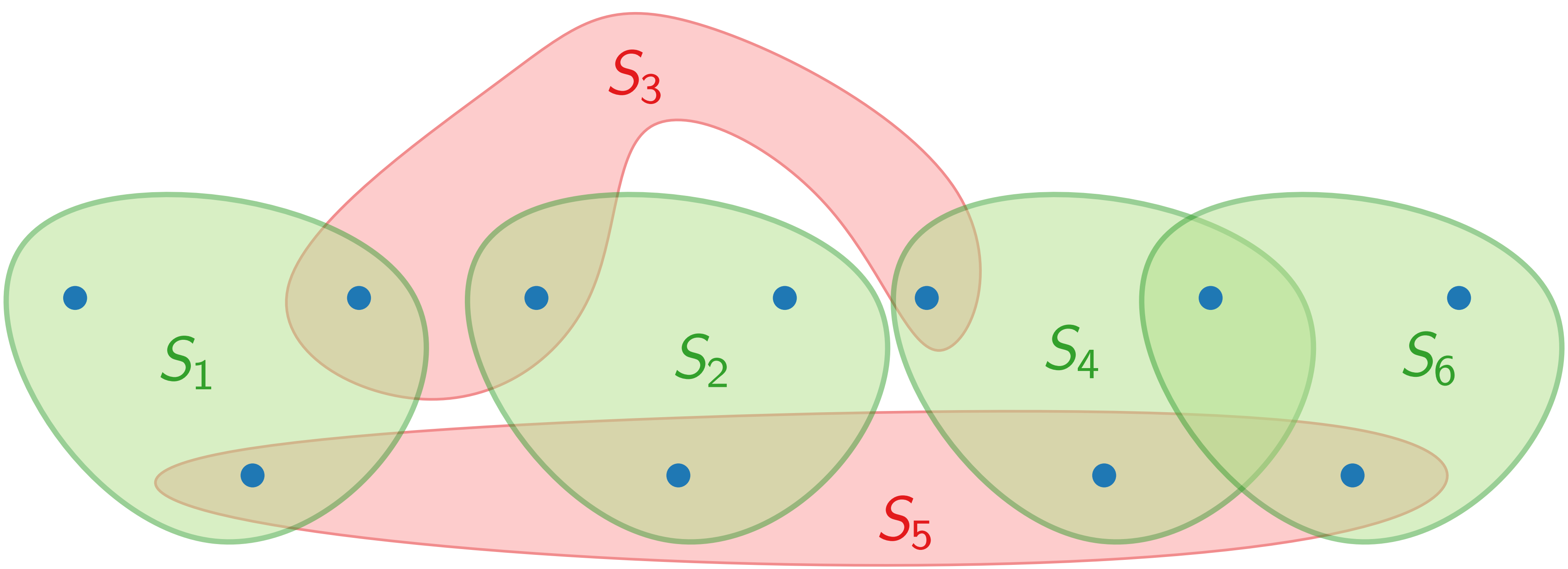

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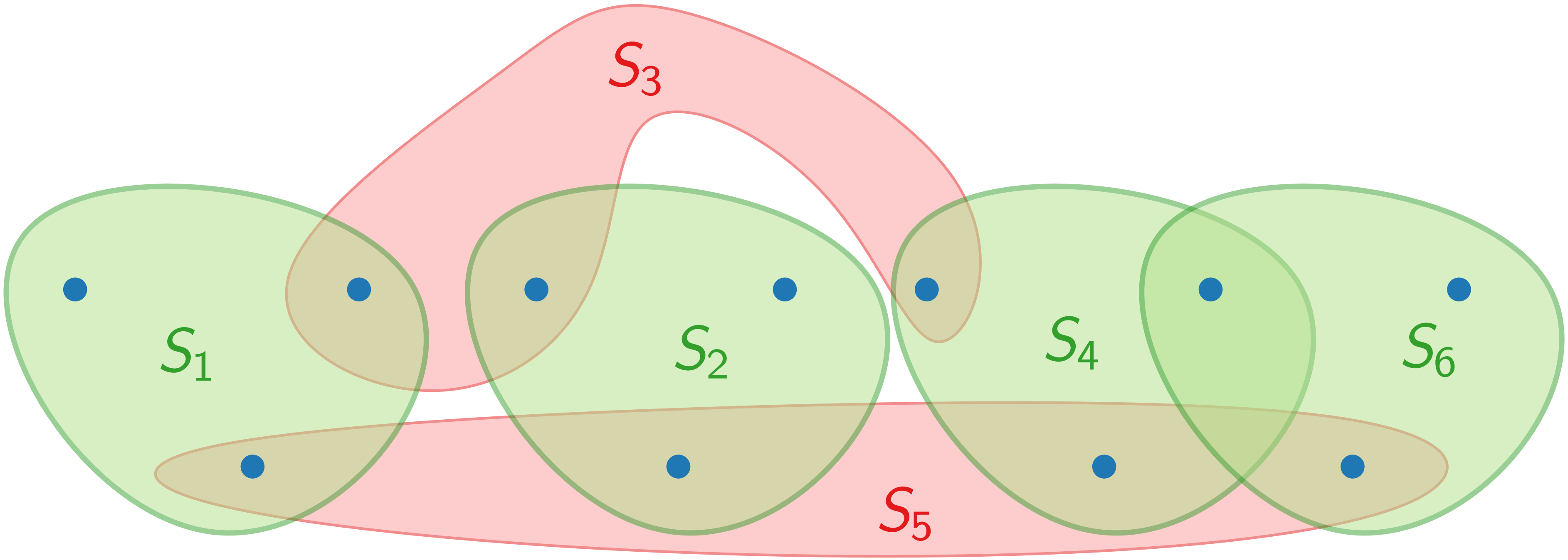

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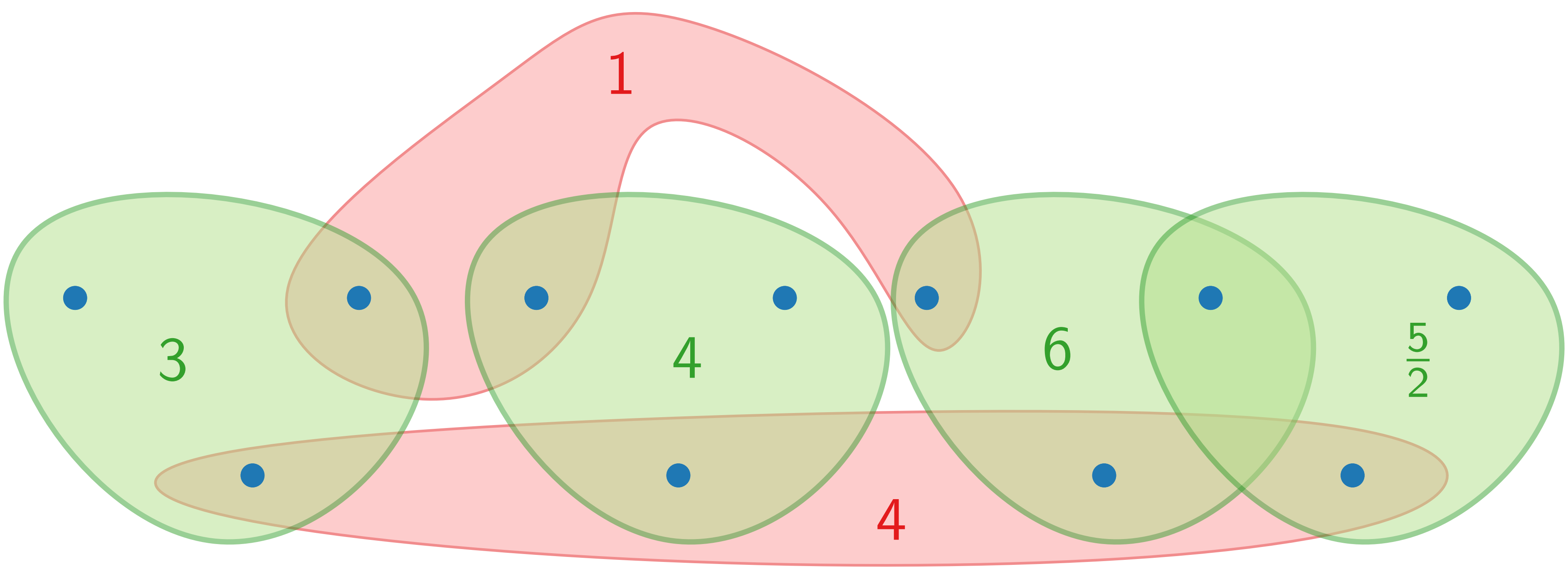

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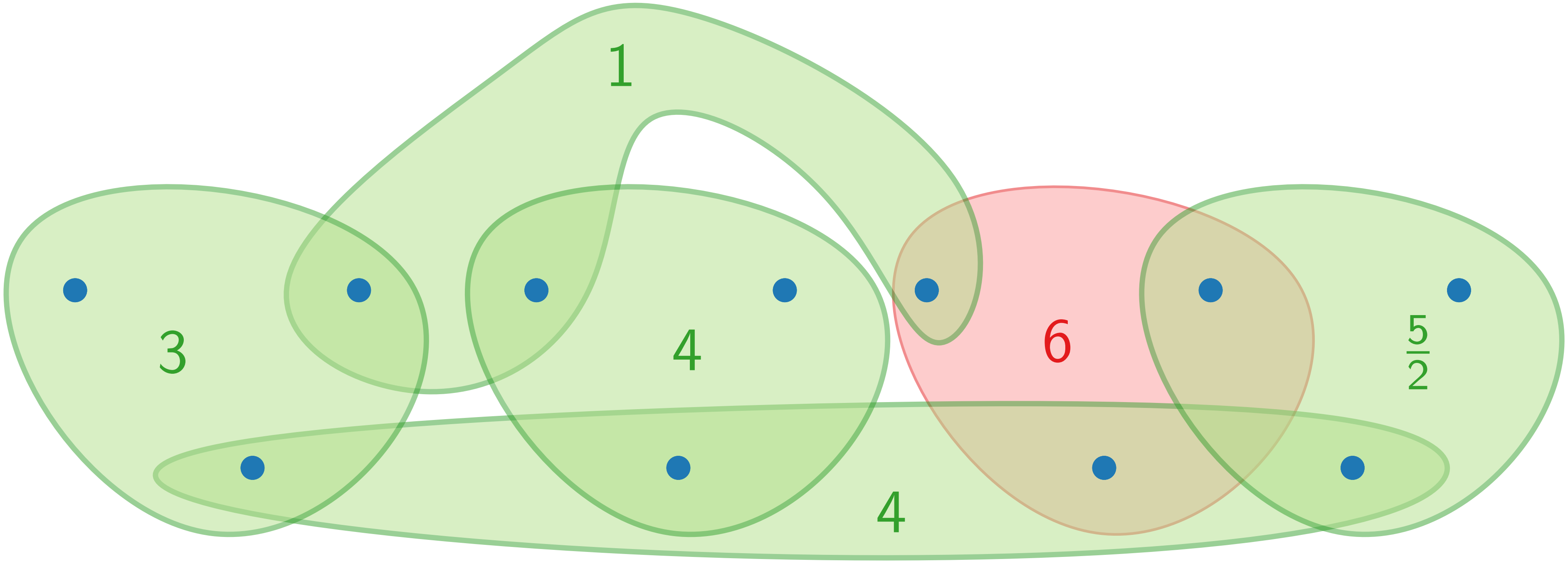

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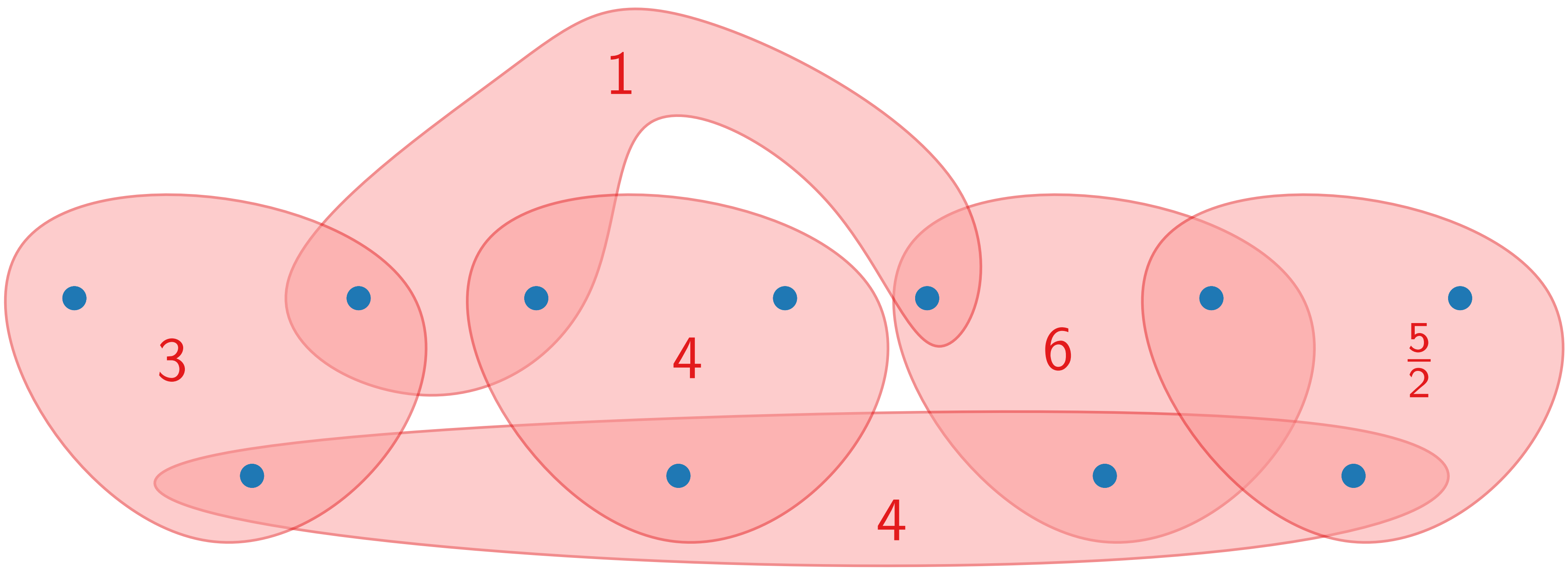
Part II:
Greedy for SetCover

Iterative "Buying" of Elements

What is the real cost of picking a set?



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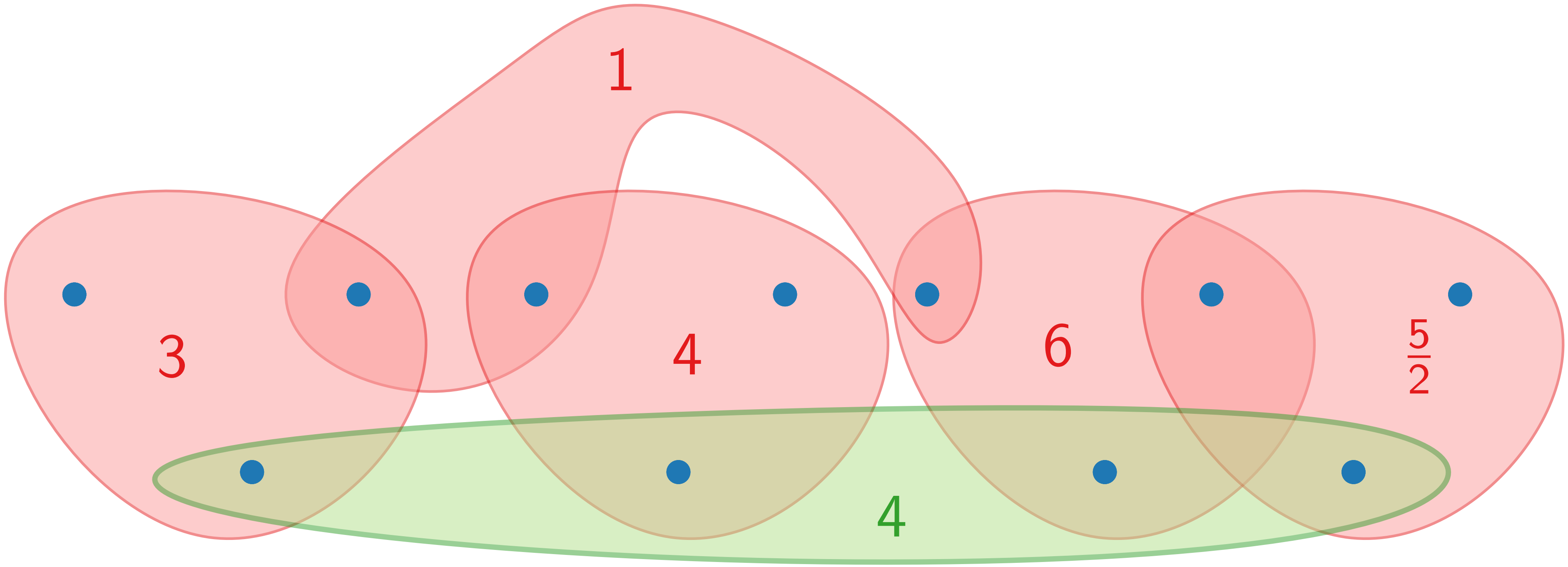

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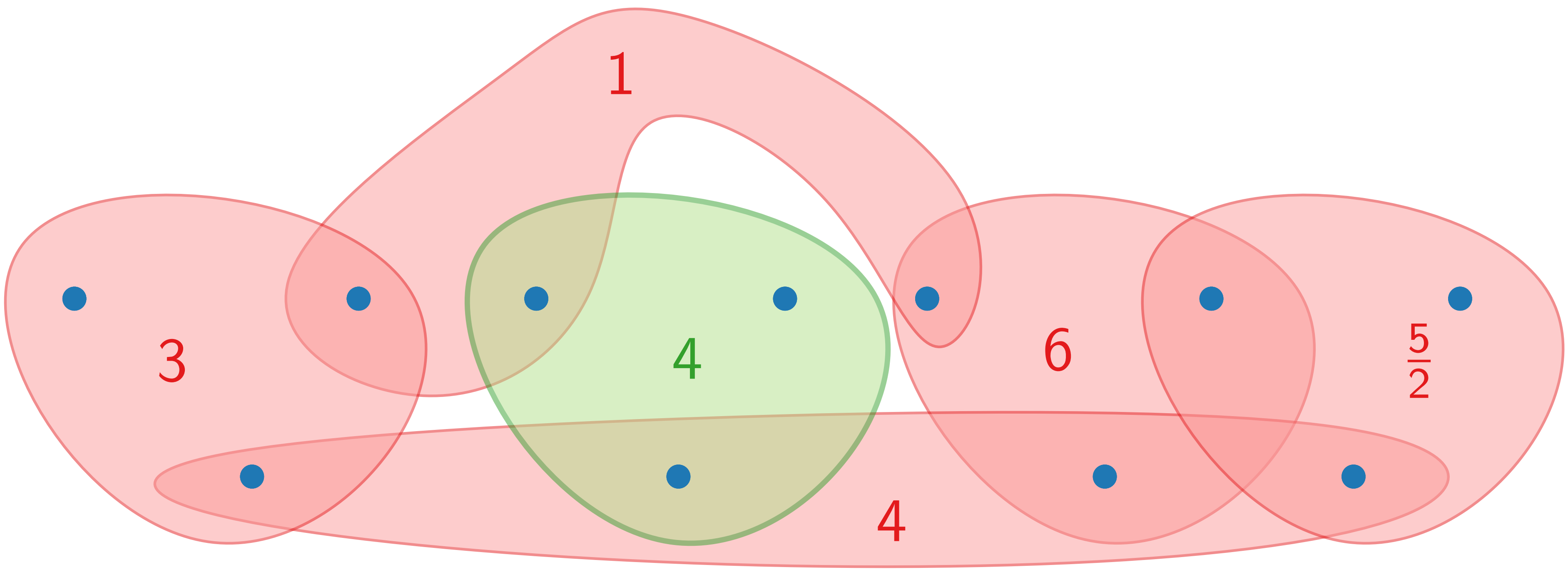

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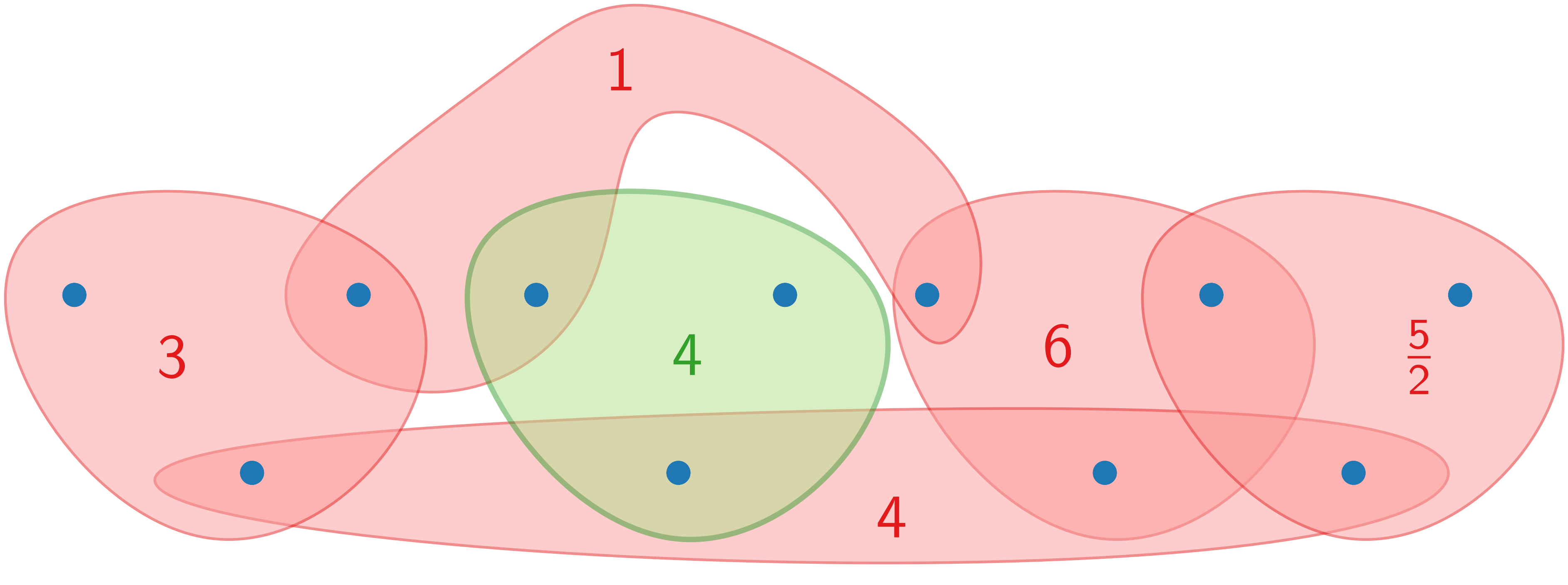

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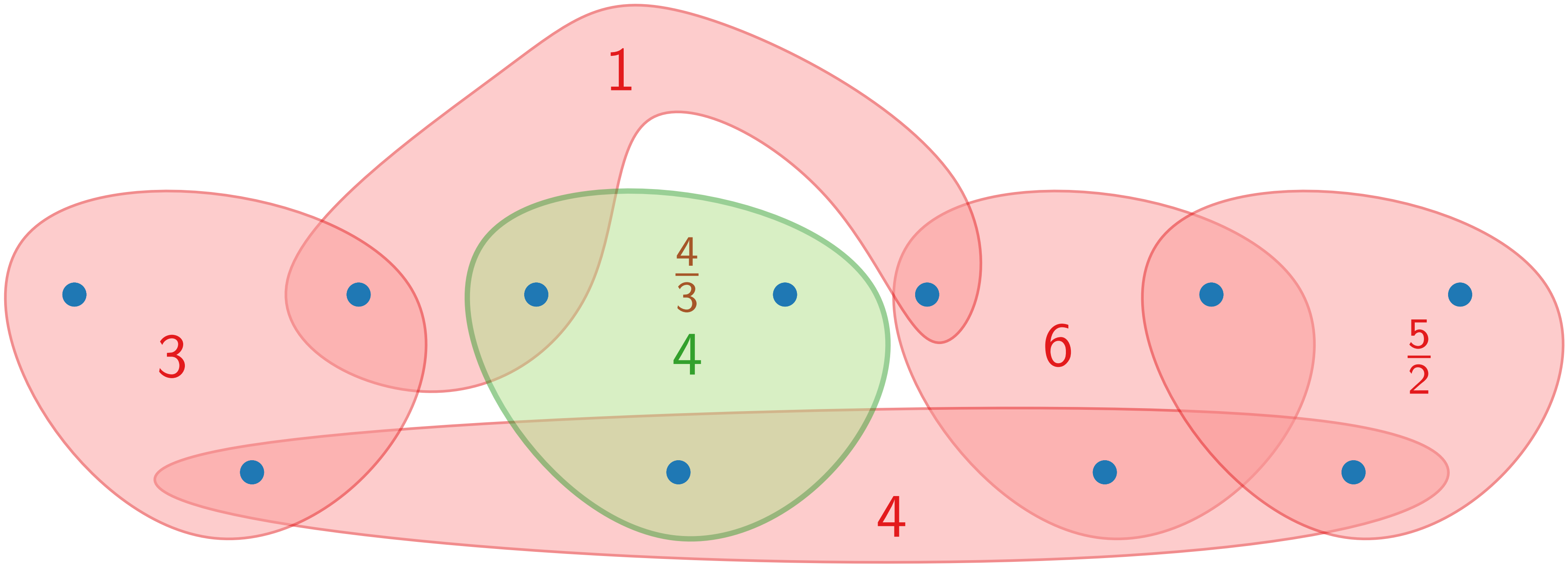

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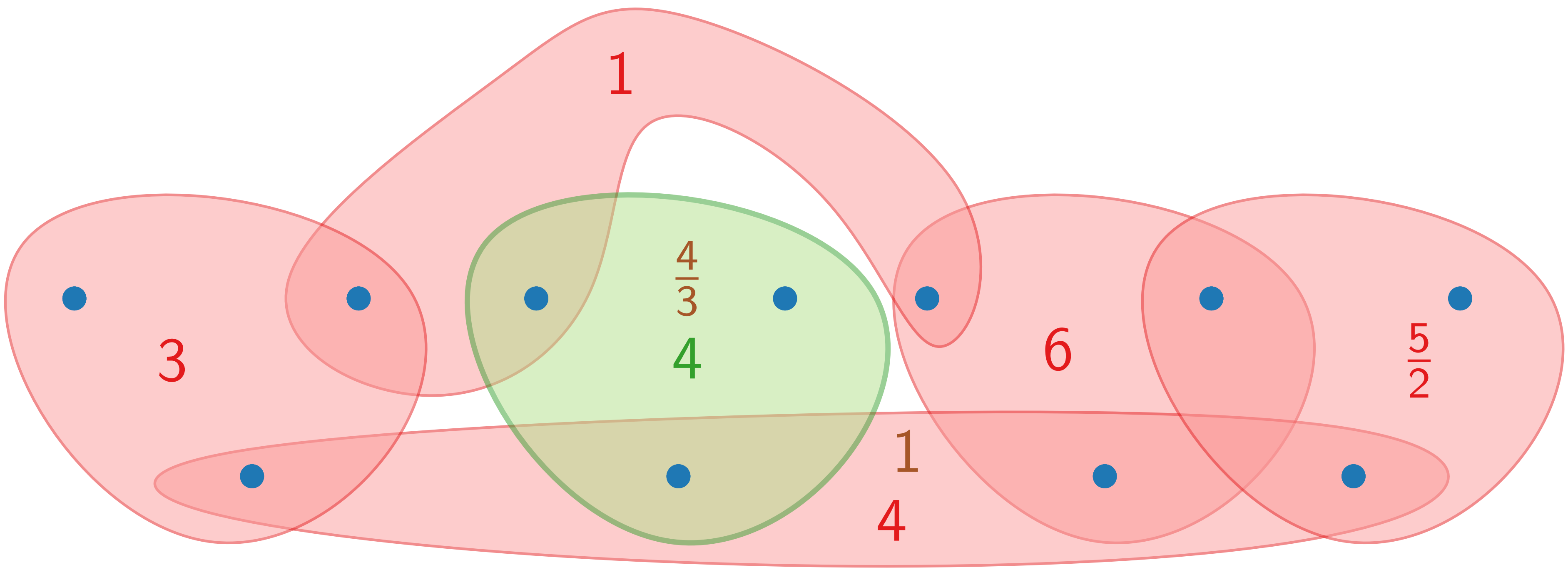

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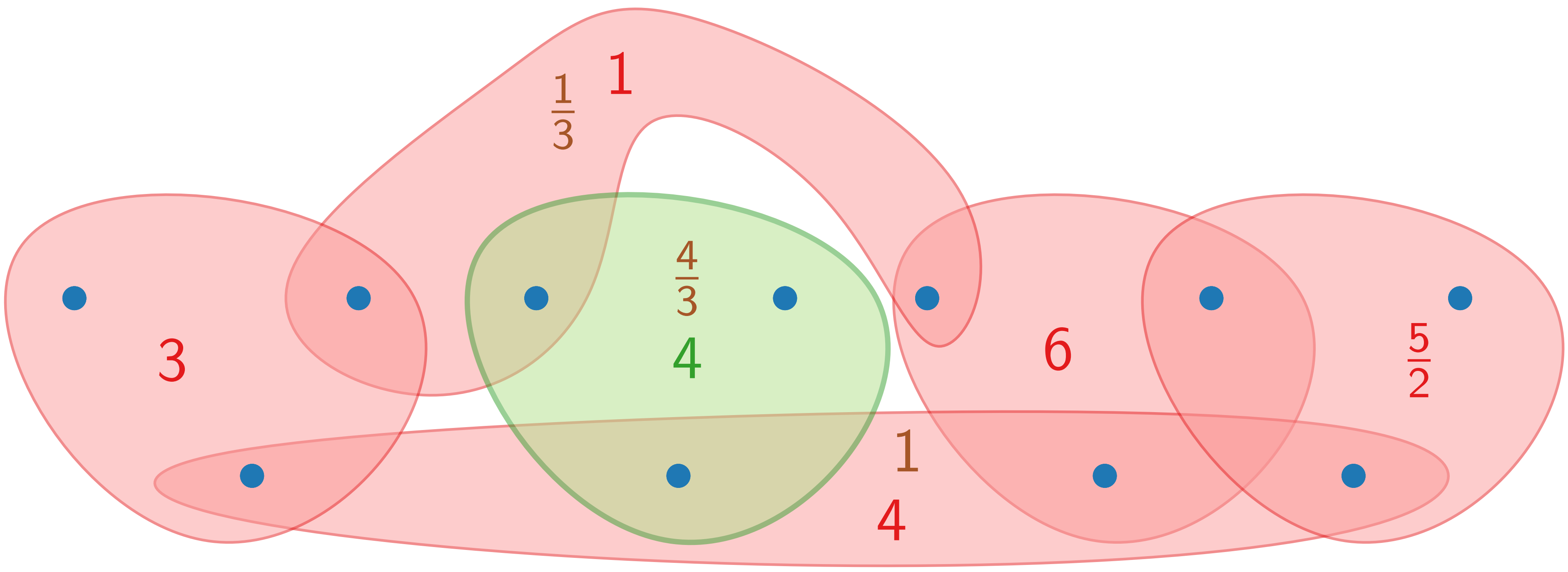

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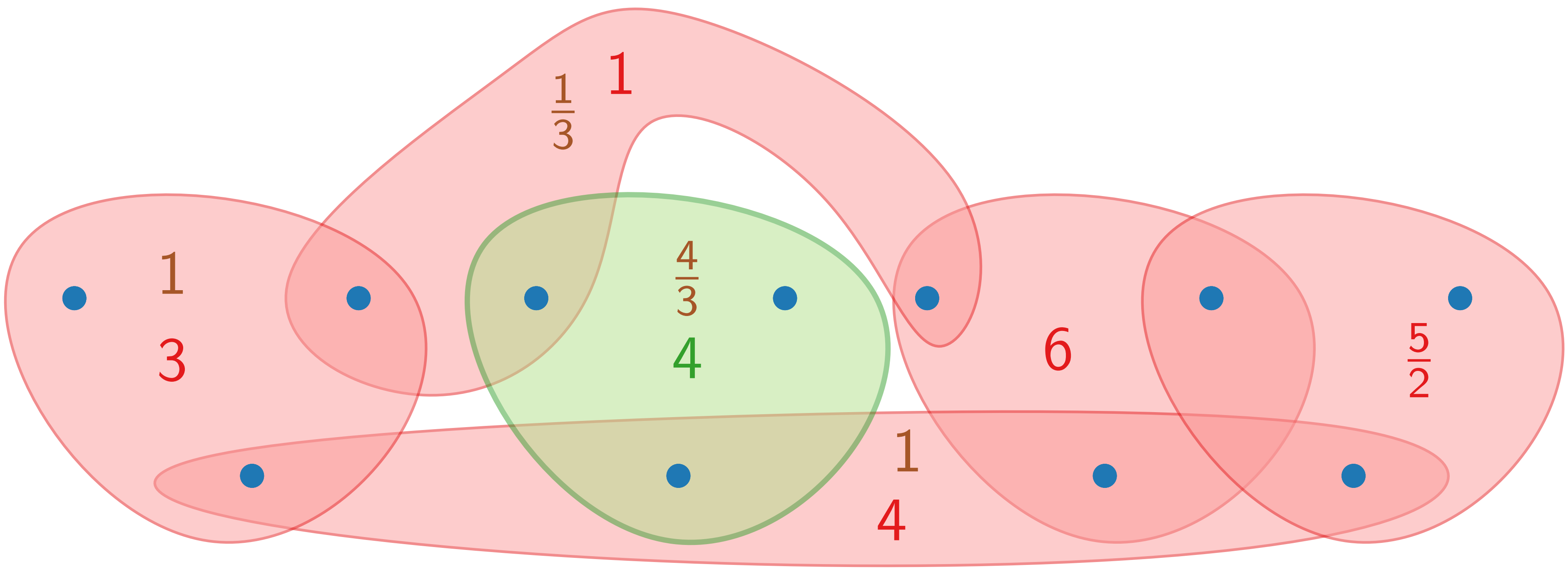

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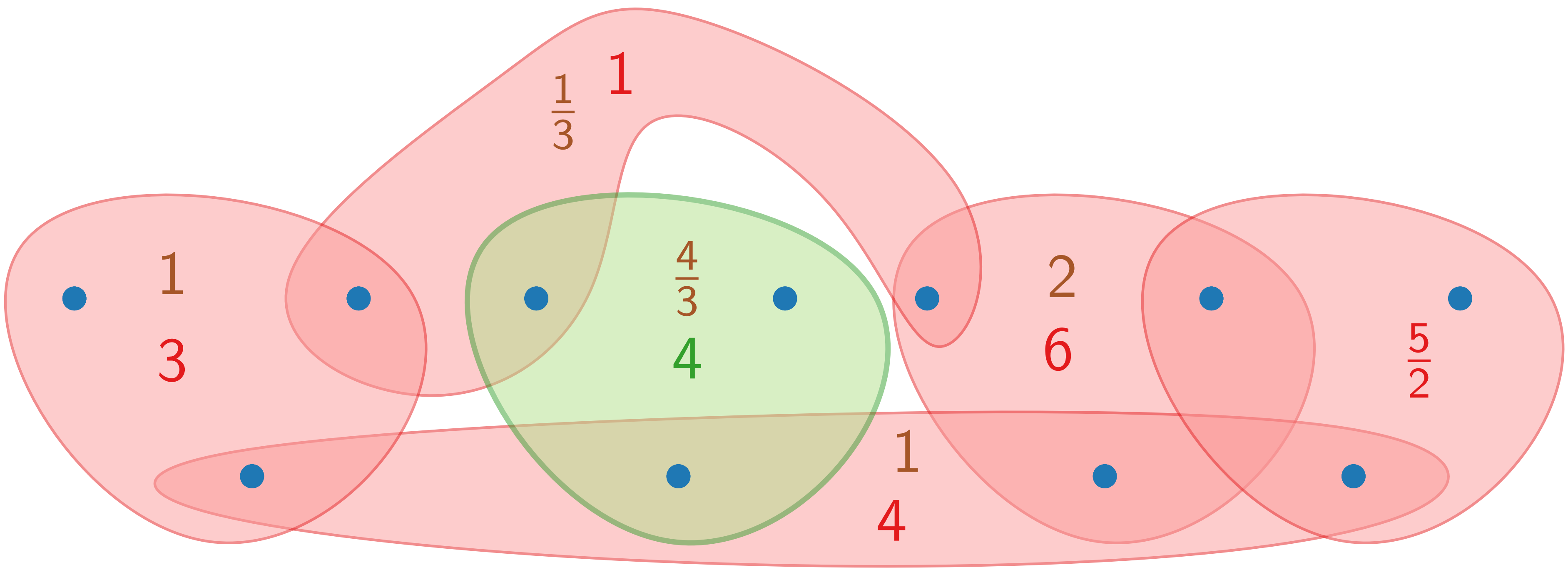

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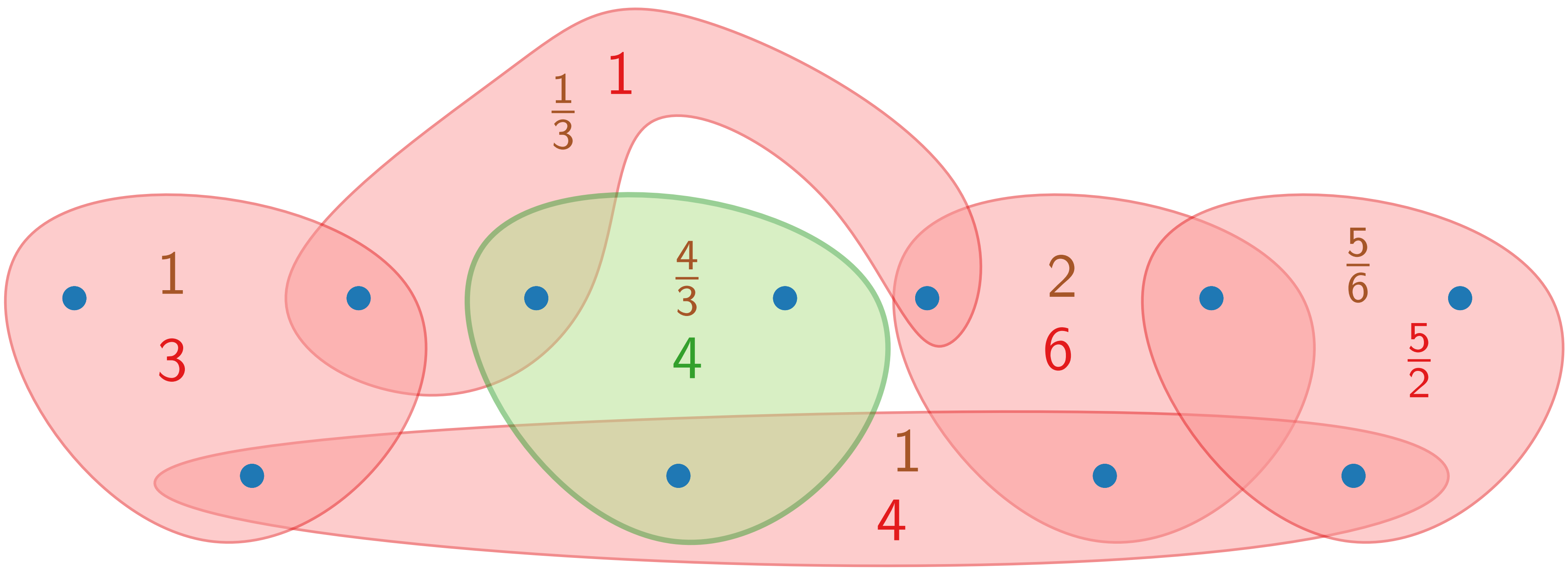

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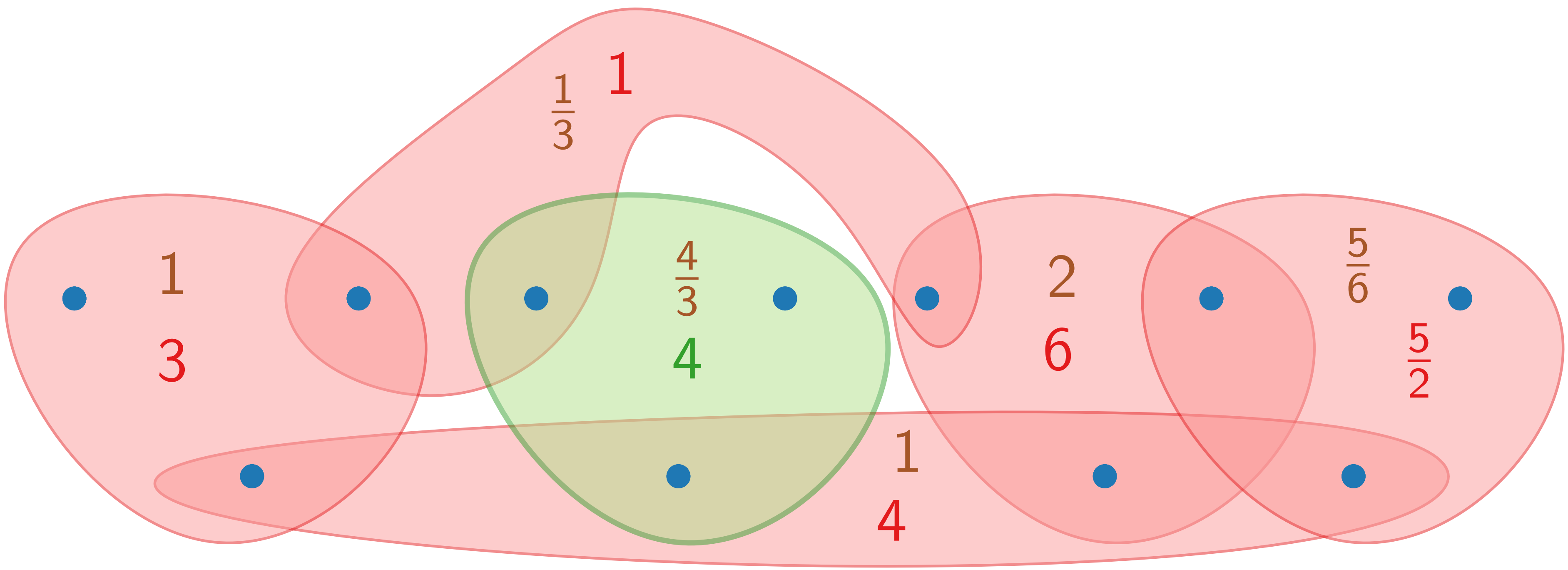

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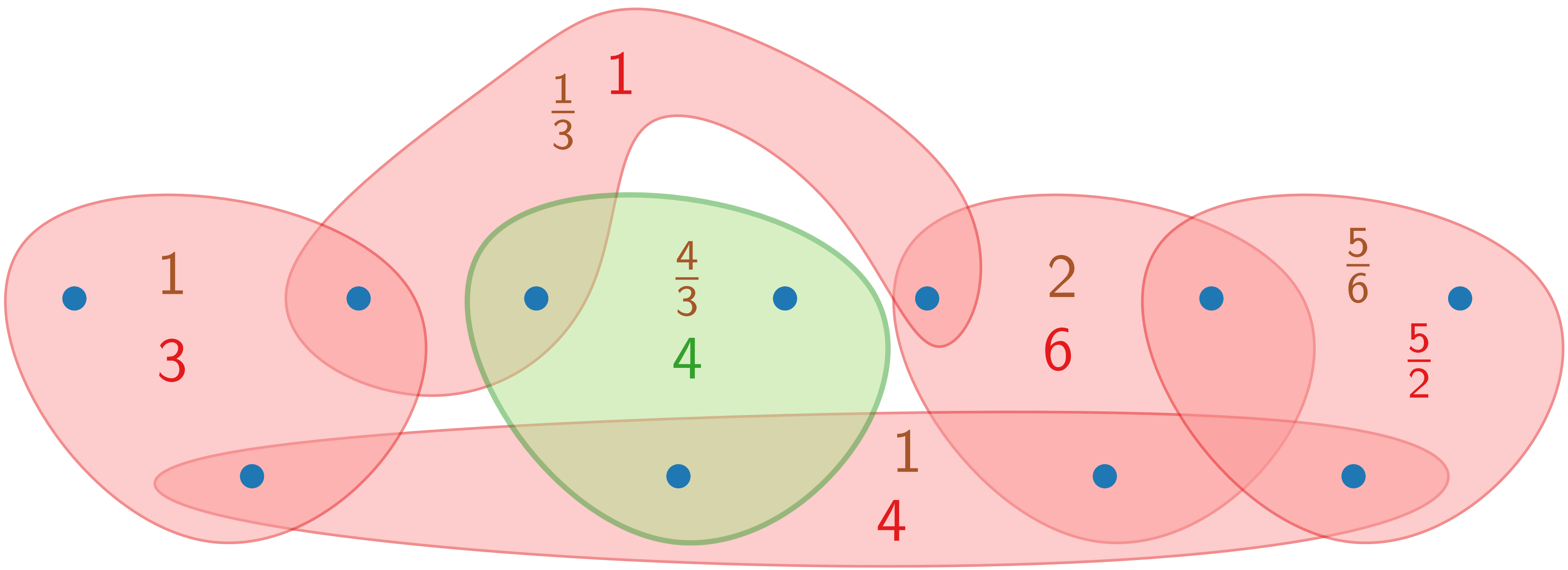

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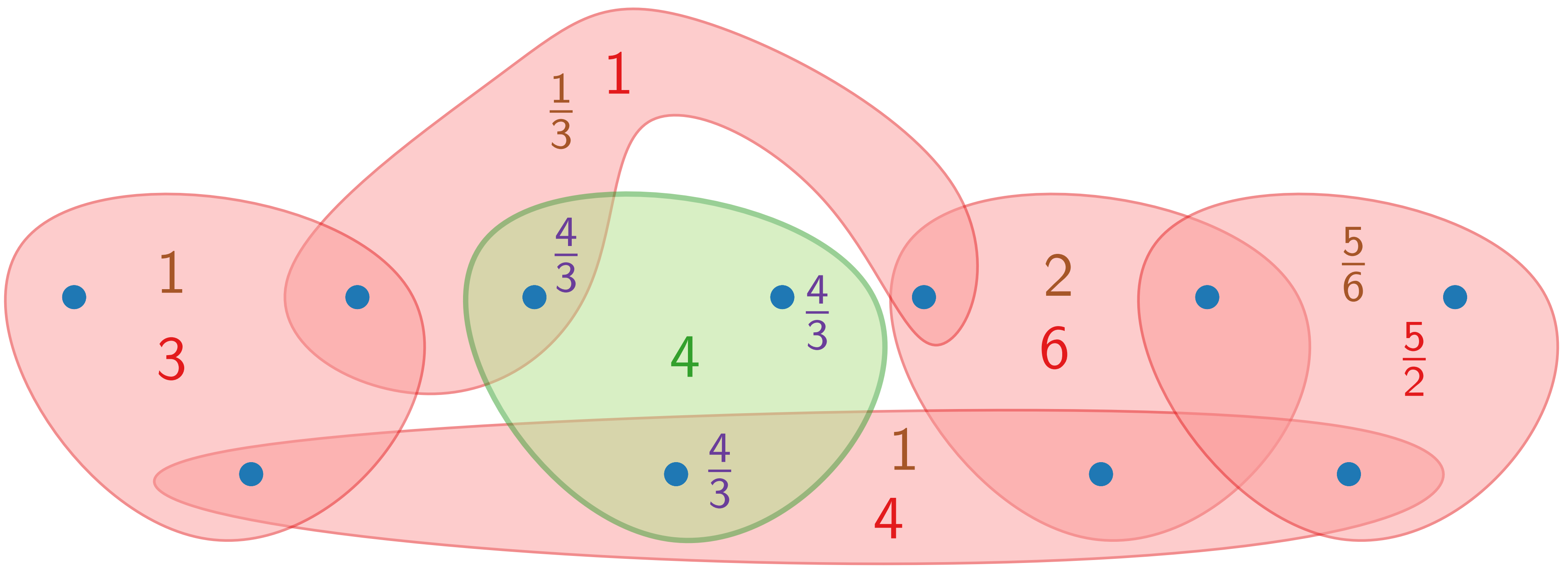

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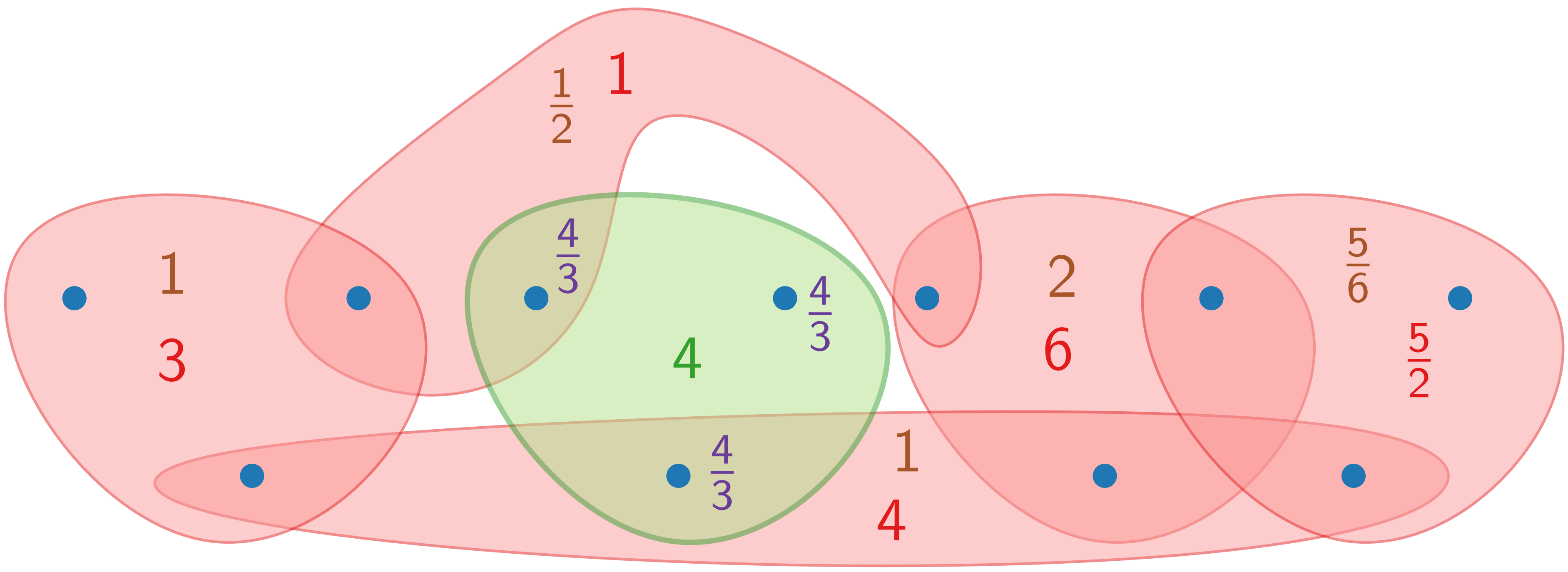

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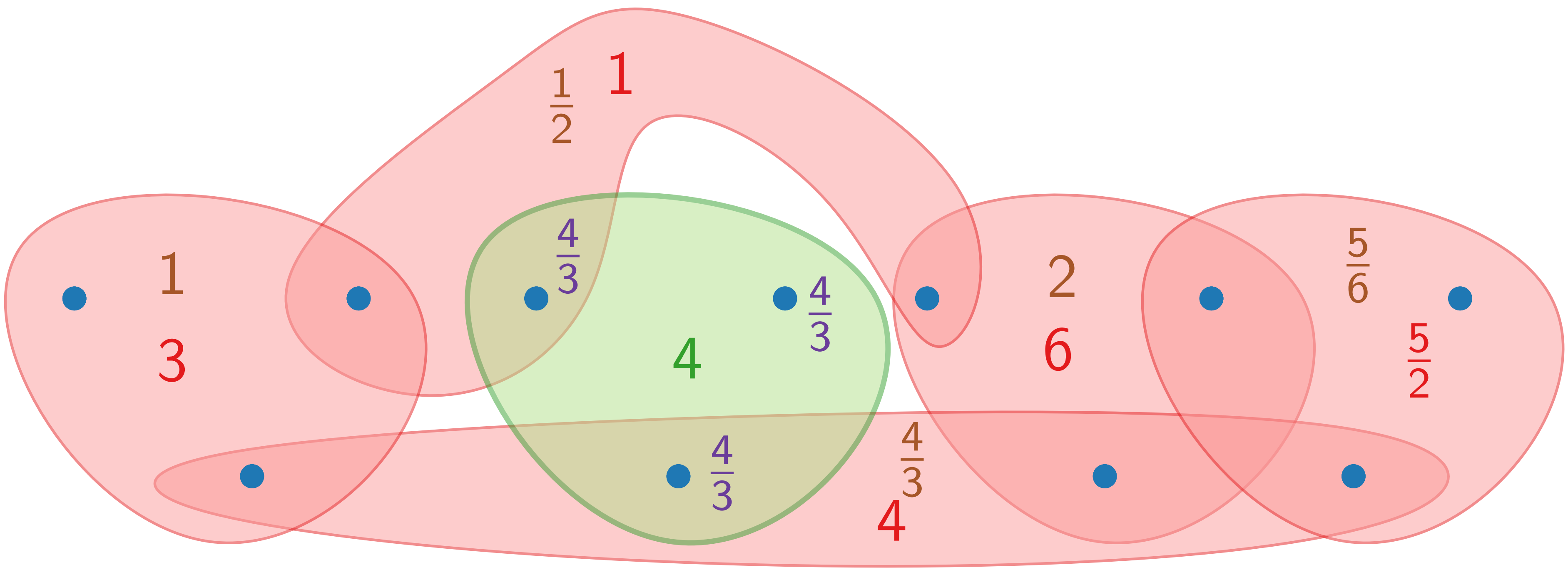

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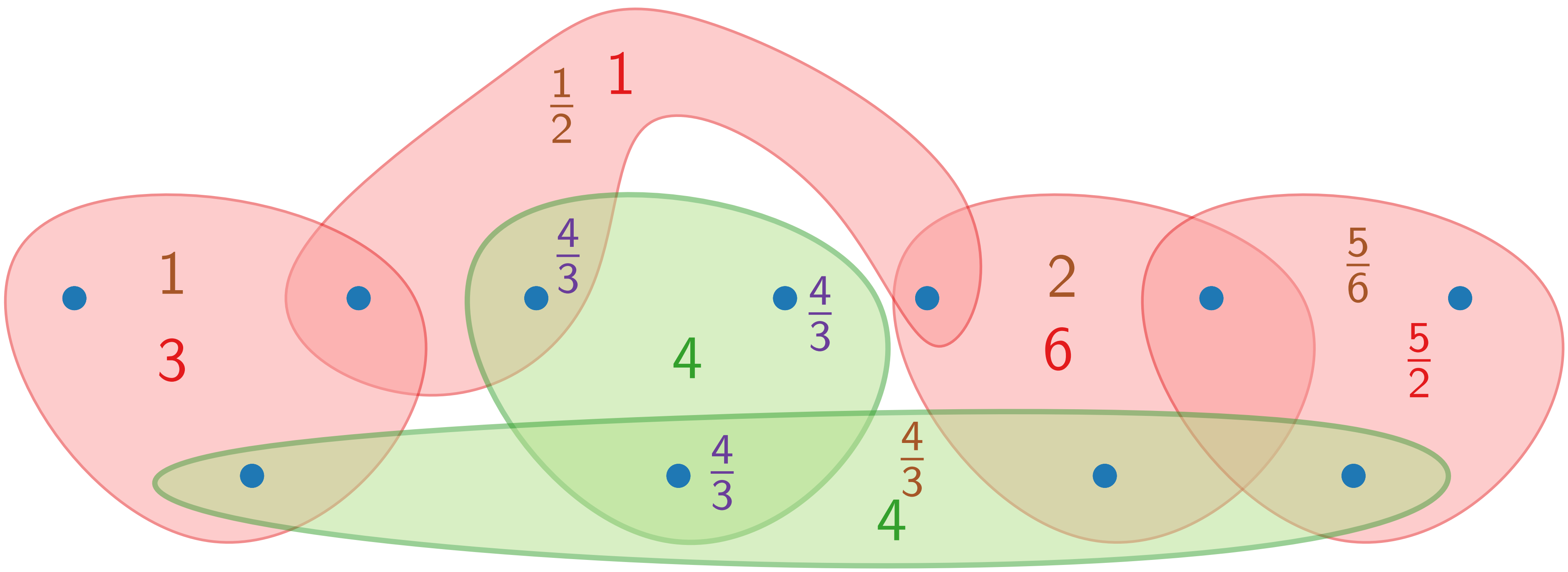

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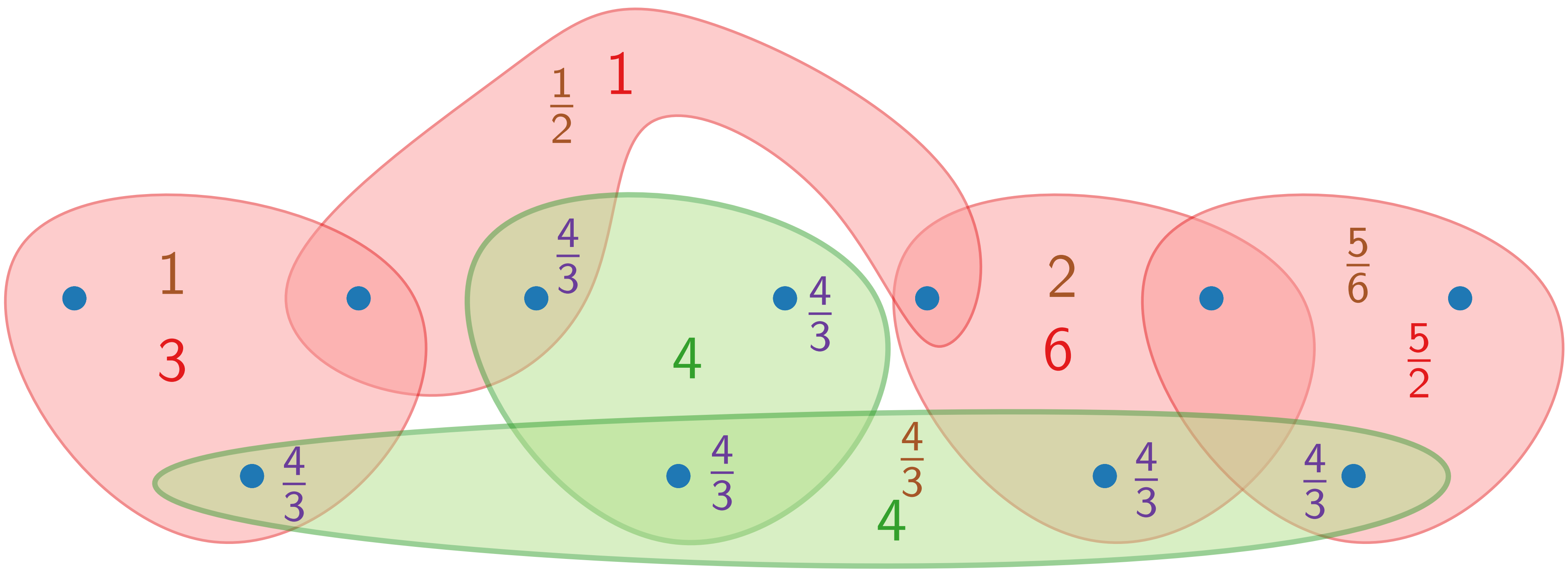

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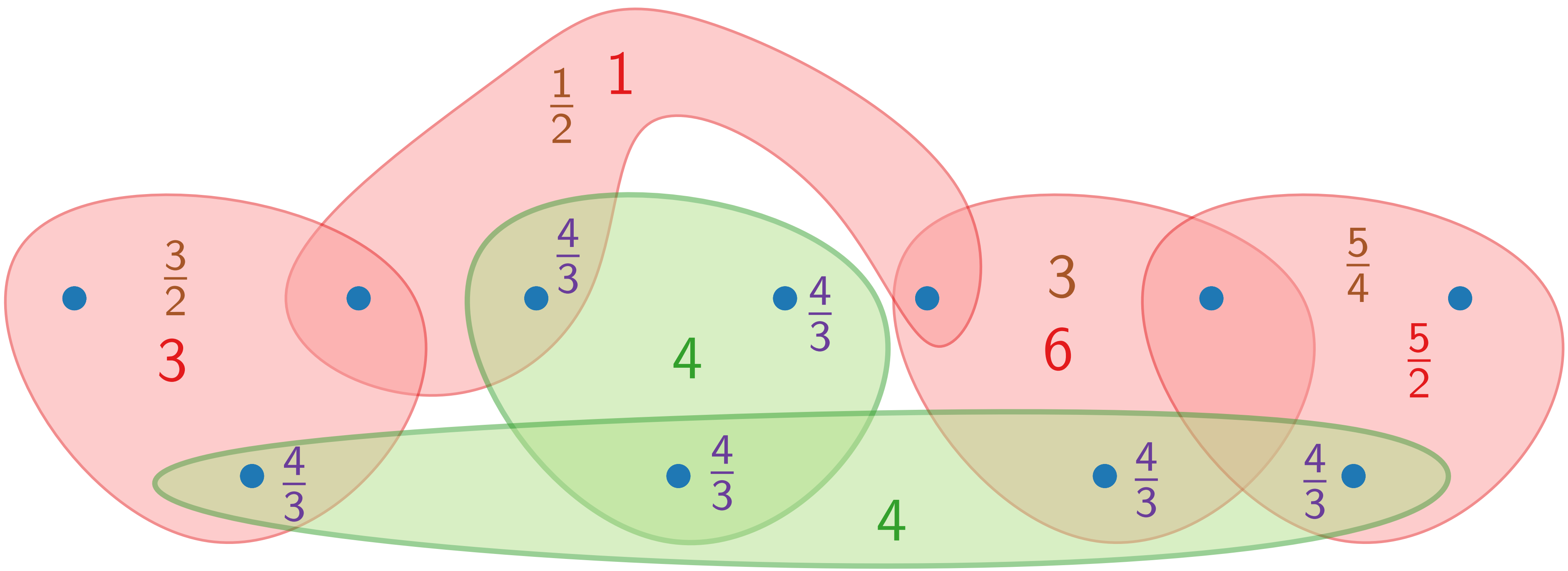

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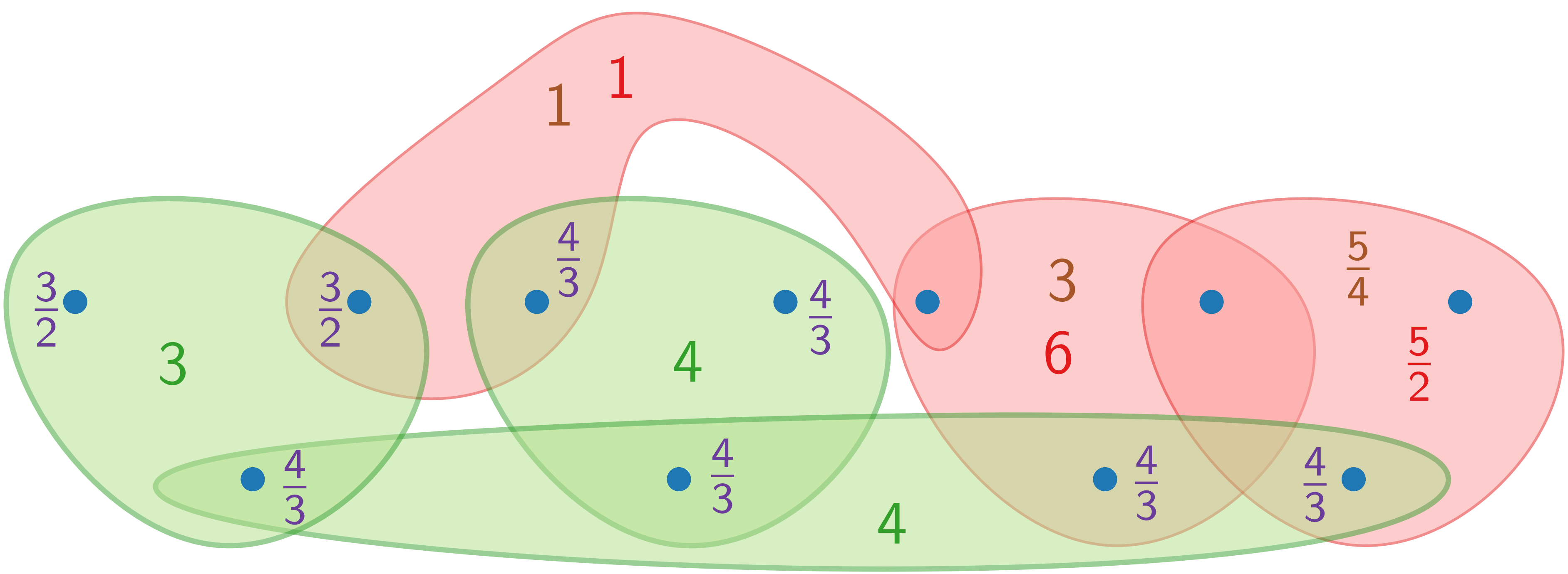

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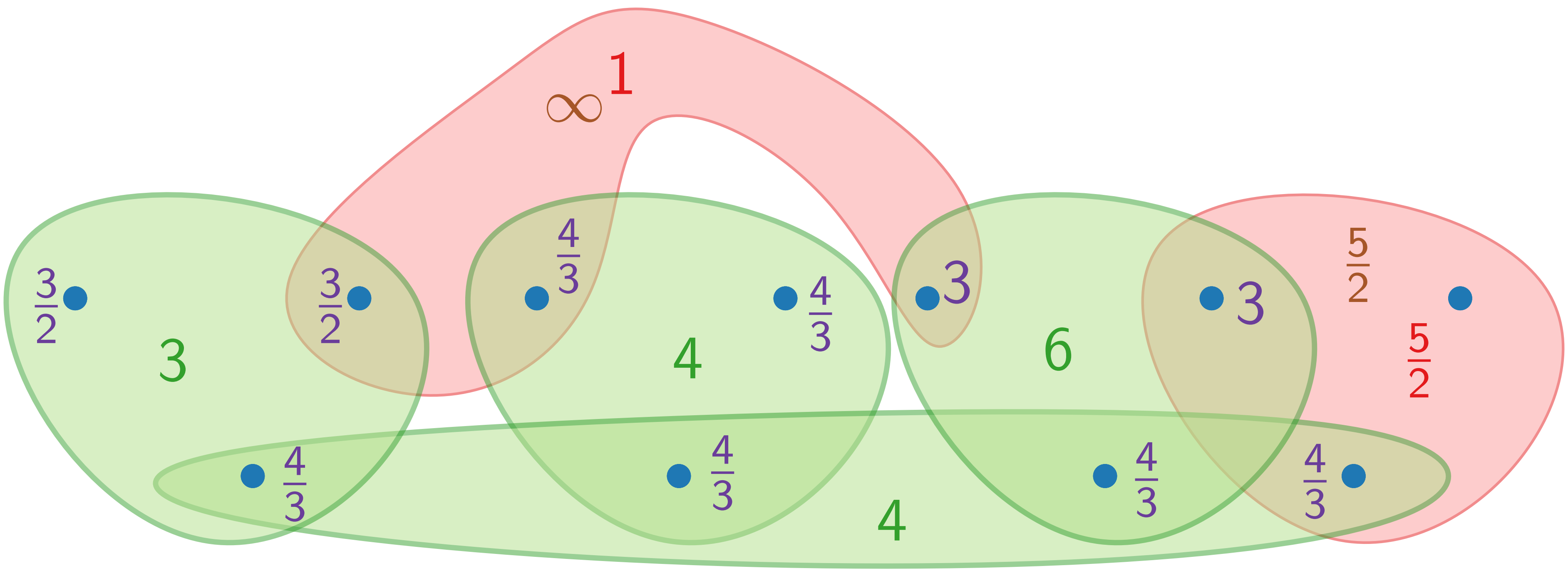

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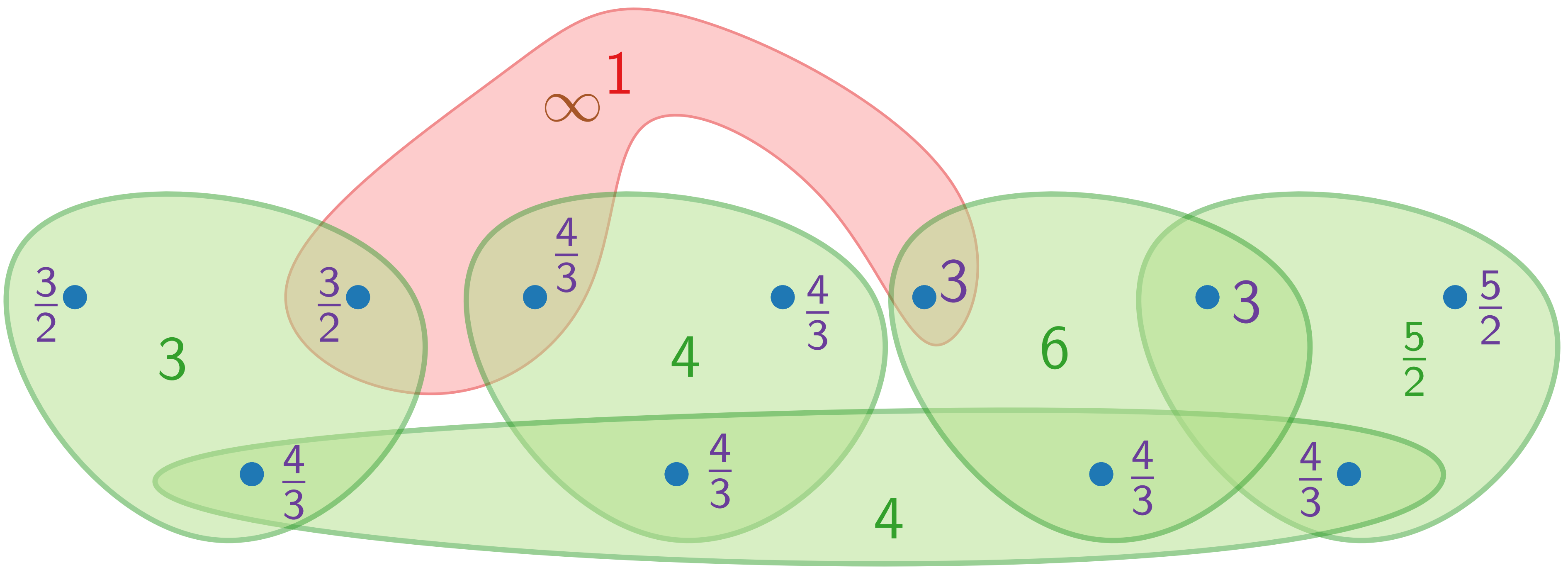

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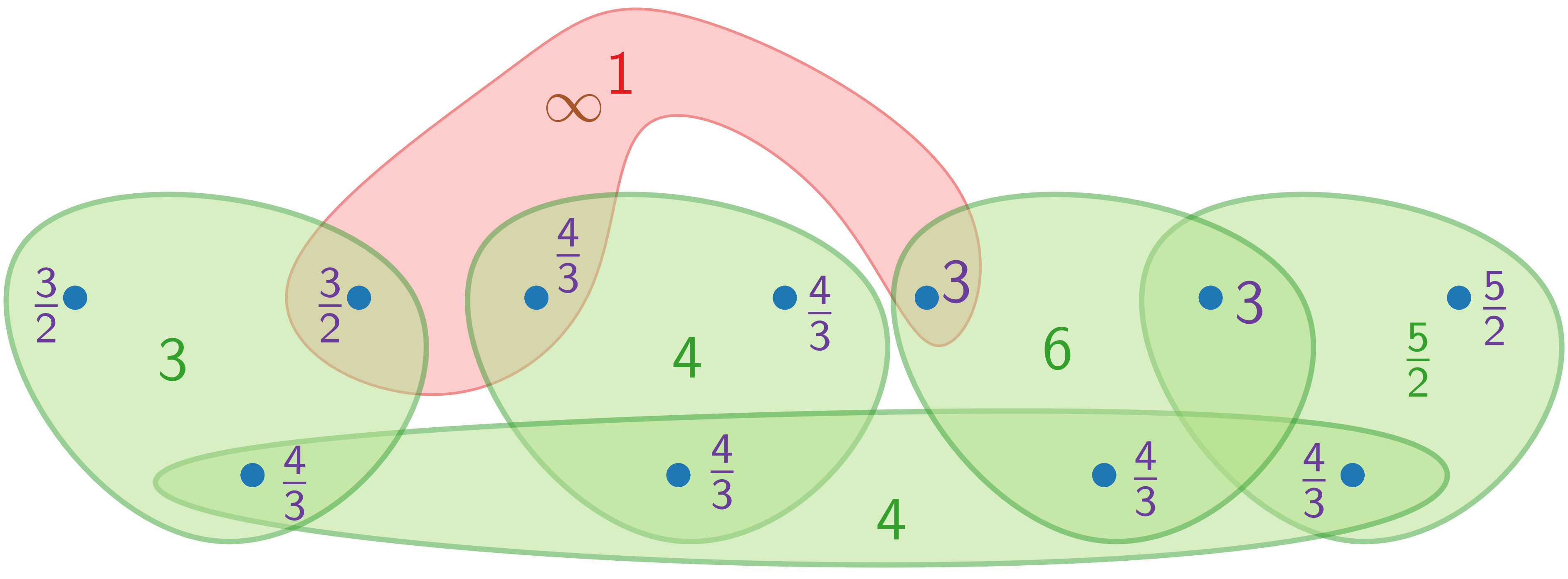
total cost: $\sum_{u \in U} \text{price}(u)$



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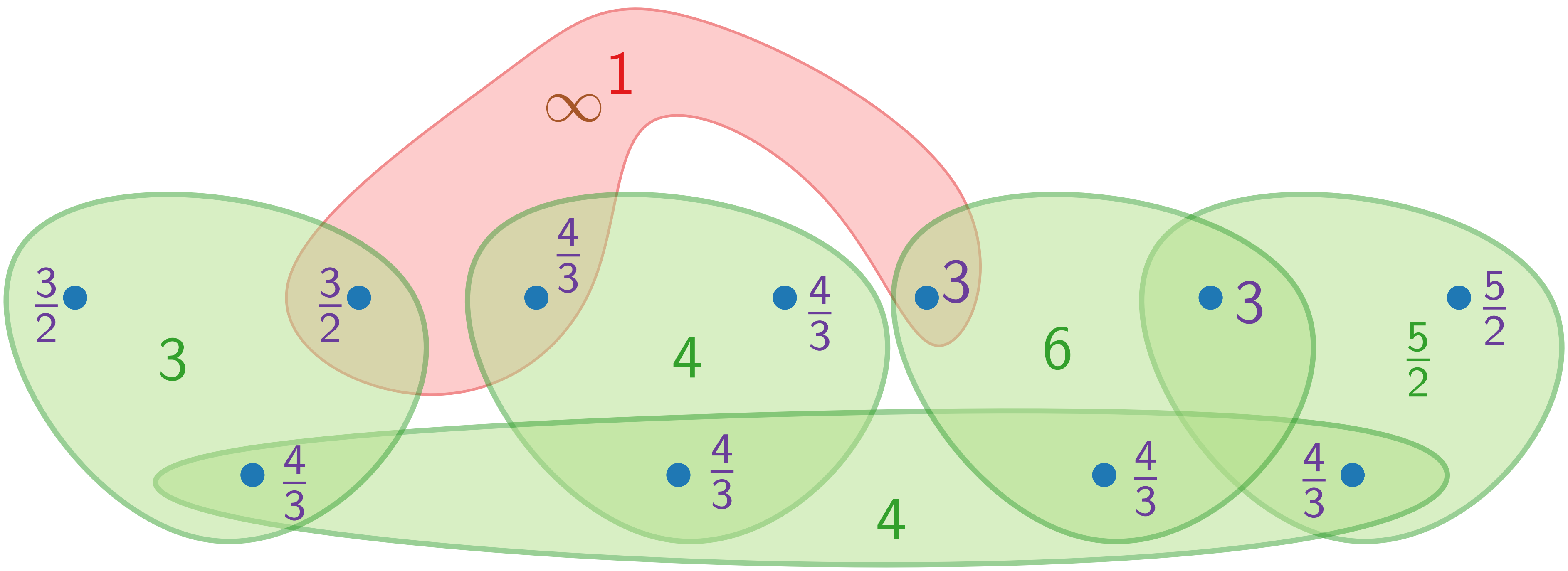
Greedy: Always choose the set with minimum per-element cost.



Greedy for SetCover

GreedySetCover(U, $\mathcal{S}$, c)

$C \leftarrow \emptyset$

$\mathcal{S}' \leftarrow \emptyset$

return $\mathcal{S}'$ // Cover of U

Greedy for SetCover

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while $C \neq U$ **do**

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Greedy for SetCover

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$C \leftarrow \emptyset$
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while $C \neq U$ **do**
    $S \leftarrow$ set in $\mathcal{S}$ that minimizes $\frac{c(S)}{|S \setminus C|}$

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Greedy for SetCover

```
GreedySetCover(U, 𝒮, c)
  C ← ∅
  𝒮' ← ∅
  while C ≠ U do
    S ← set in 𝒮 that minimizes c(S)/|S \ C|
    foreach u ∈ S \ C do
      price(u) ← c(S)/|S \ C|

  return 𝒮'                                  // Cover of U
```

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  foreach $u \in S \setminus C$ **do**

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Approximation Algorithms

Lecture 2:
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Part III:
Analysis

Analysis

Theorem. GreedySetCover is a factor-$\mathcal{H}_k$ approximation algorithm for $\textsc{SetCover}$, where k is the cardinality of the largest set in $\mathcal{S}$ and $\mathcal{H}_k := 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{k} \to 0.5 + \ln k$.

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Lemma. Let $S \in \mathcal{S}$, and let $u_1, \ldots, u_\ell$ be the elements of S in the order in which they are covered ("bought") by GreedySetCover. Then

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Proof.

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Lemma. Let $S \in \mathcal{S}$, and let $u_1, \ldots, u_\ell$ be the elements of S in the order in which they are covered ("bought") by GreedySetCover. Then $\text{price}(u_j) \le c(S)/(\ell - j + 1)$.

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- Price by alg. no larger due to greedy choice.

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$$\leq \sum_{i=1}^{m} c(S_i) \cdot \mathcal{H}_k = \text{OPT} \cdot \mathcal{H}_k \qquad \square$$

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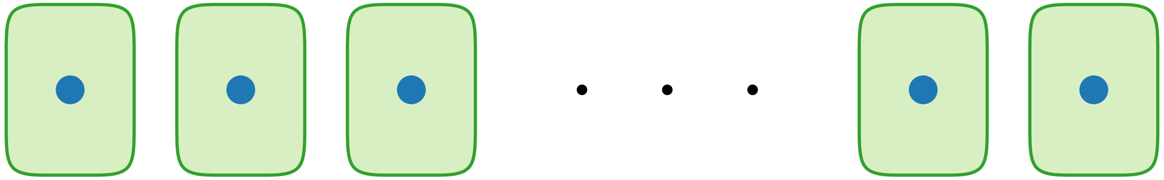

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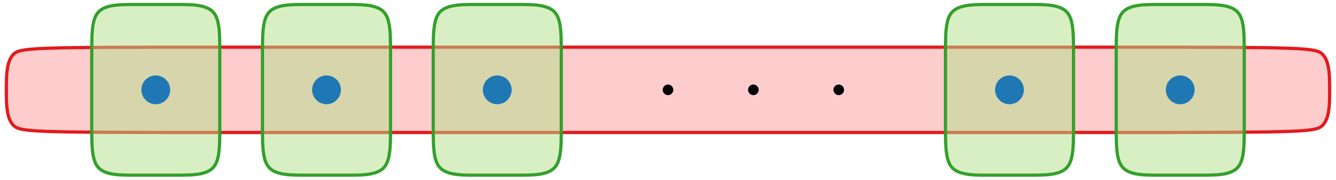

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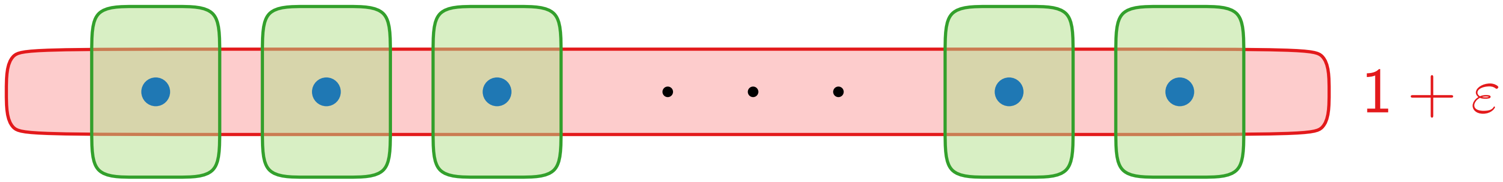

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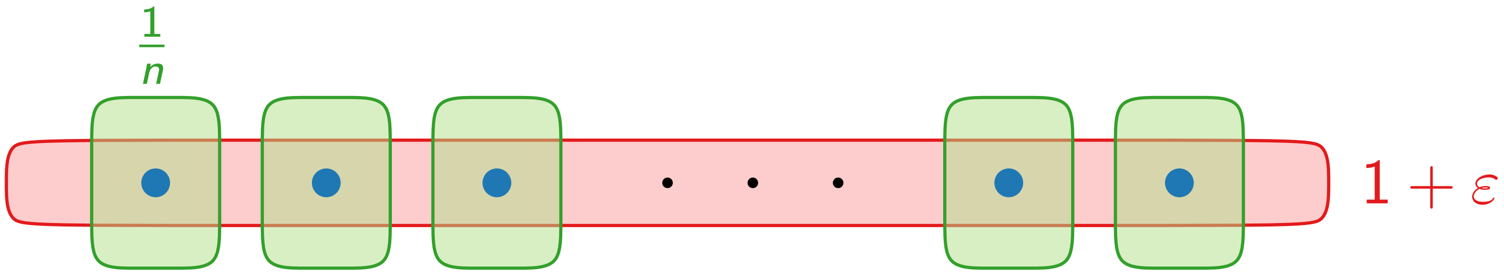

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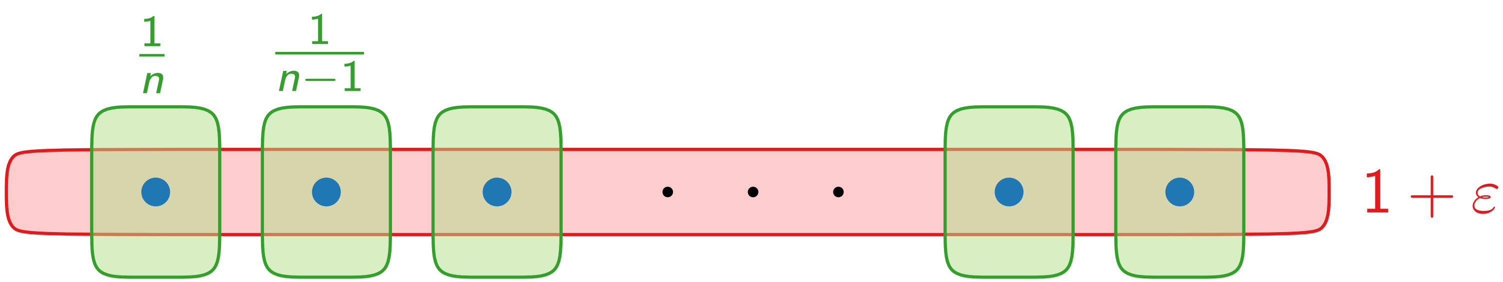

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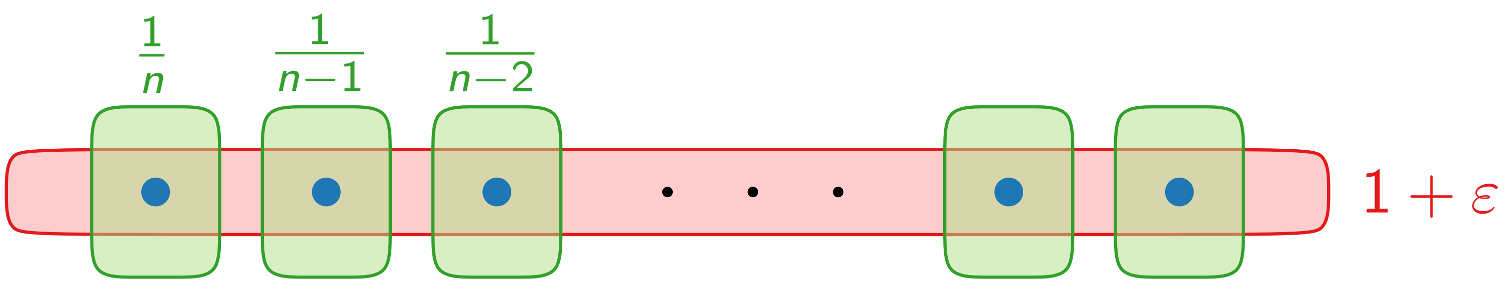

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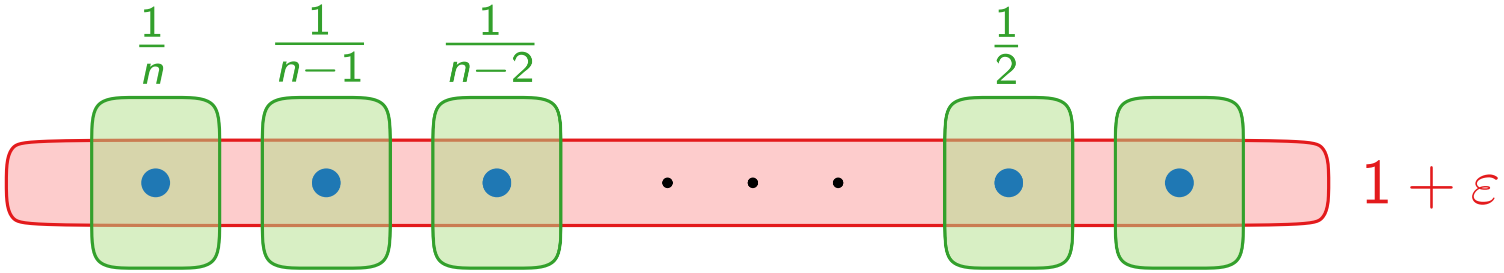

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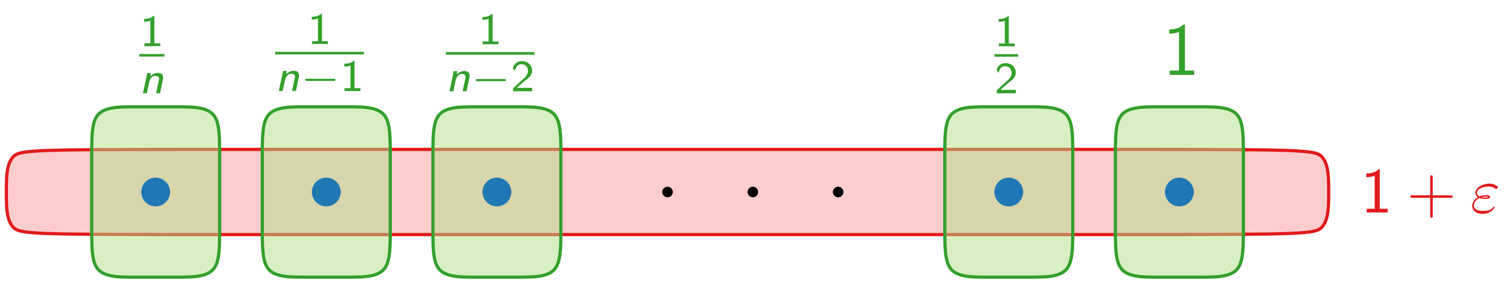

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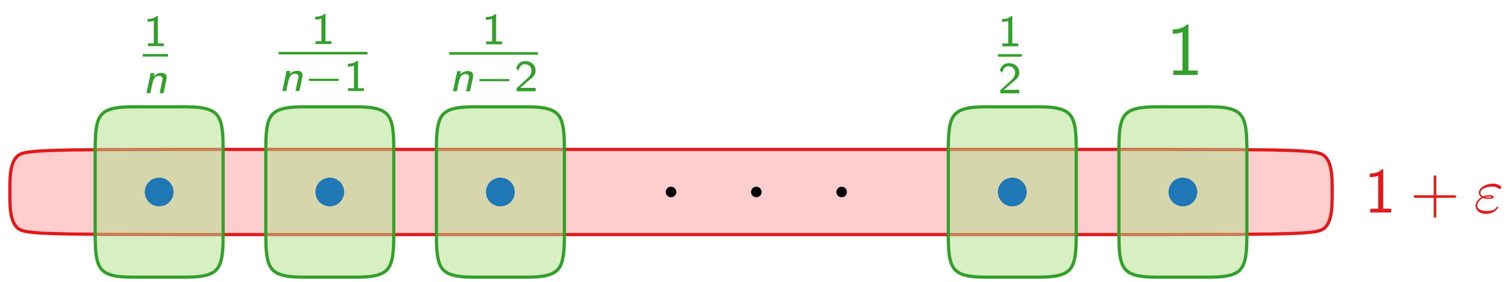

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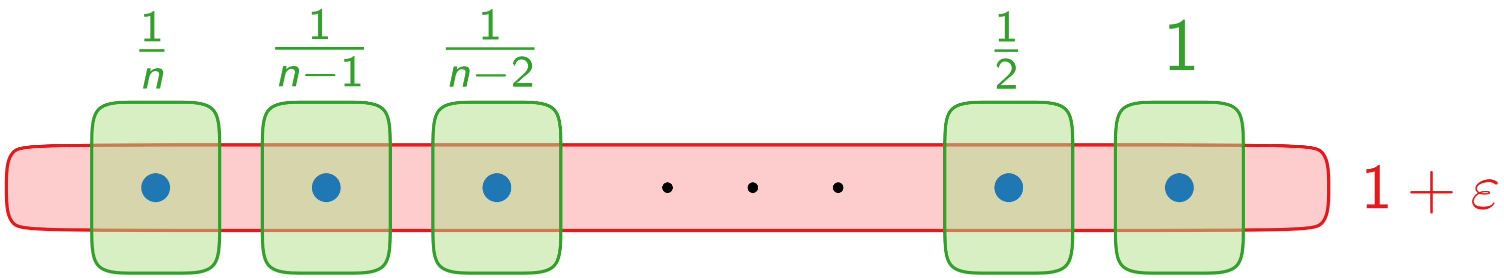

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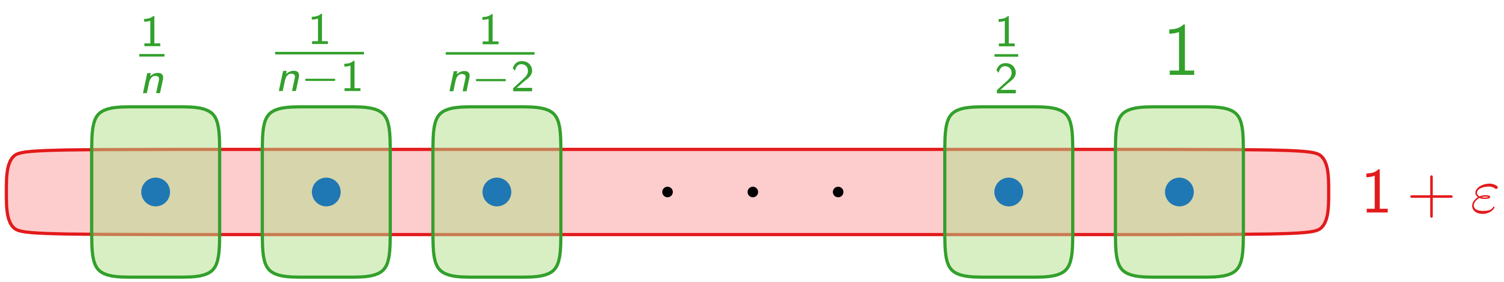

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Can we do better?

No – for any $\varepsilon > 0$, it is NP-hard to approximate SetCover with factor $(1 - \varepsilon) \cdot \ln n$

[Feige, JACM 1998]
[Dinur, Steurer, STOC 2014]

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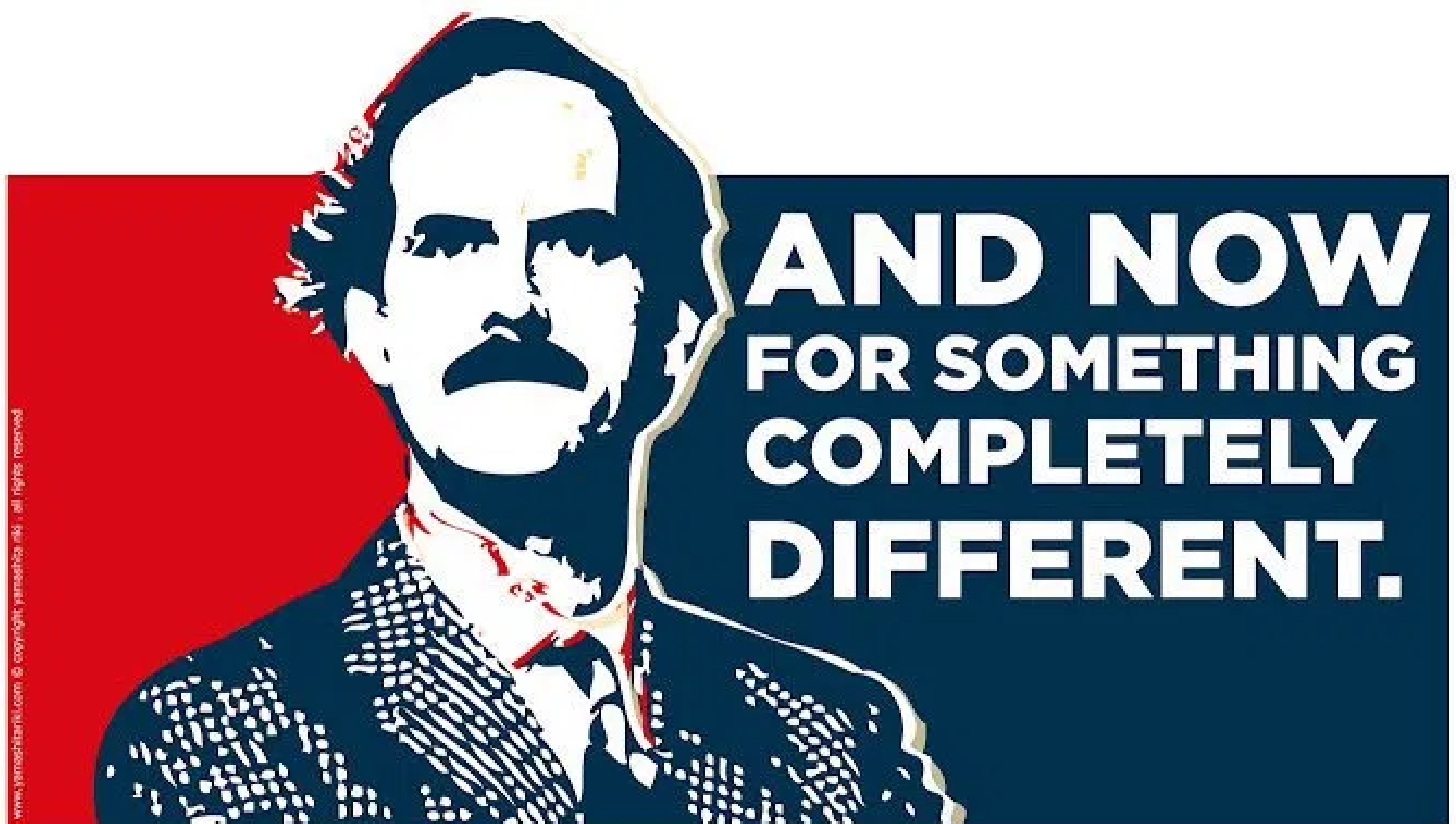

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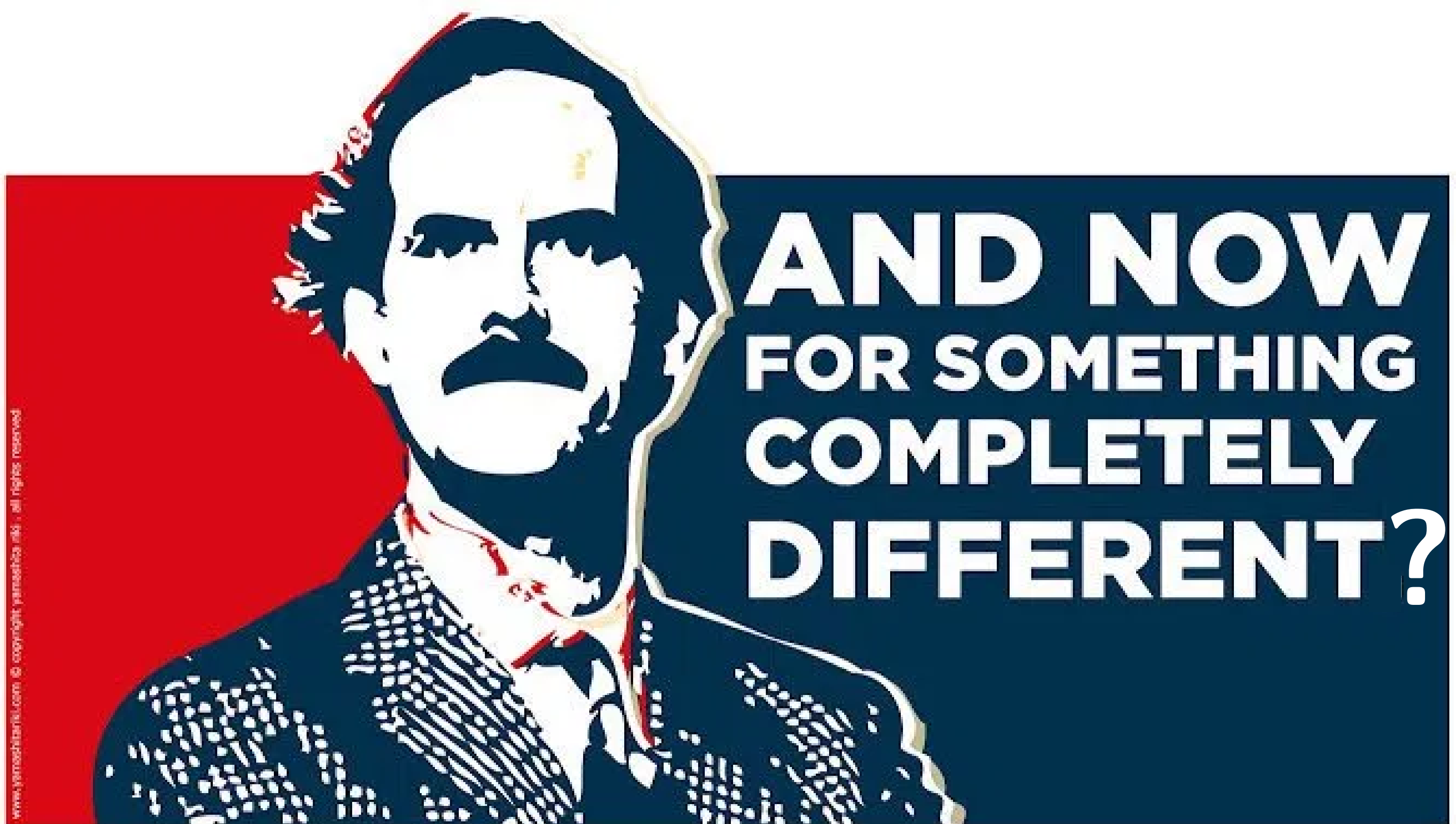

Approximation Algorithms

Lecture 2:
SetCover and ShortestSuperString

Part IV:
ShortestSuperString

ShortestSuperString (SSS)

Given a set $\{s_1, \ldots, s_n\} \subseteq \Sigma^+$ of strings over a finite alphabet Σ.

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Find a **shortest string** s (*superstring*) such that,
for each $i \in \{1, \ldots, n\}$, the string s_i is a *substring* of s.

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$$\begin{array}{l} abc \\ bcb \\ cbaa \end{array}$$

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$abcbaa$ "covers" all strings in U

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$\ \ \ \ cbaa$

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W.l.o.g.: No string s_i is a substring of any other string s_j.

SSS as a SetCover Problem

SetCover Instance: ground set U, set family $\mathcal{S}$, costs c.

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Let be σ_{ijk} be the unique string with prefix s_i and suffix s_j where s_i and s_j *overlap* on k characters (for suitable i, j, k)

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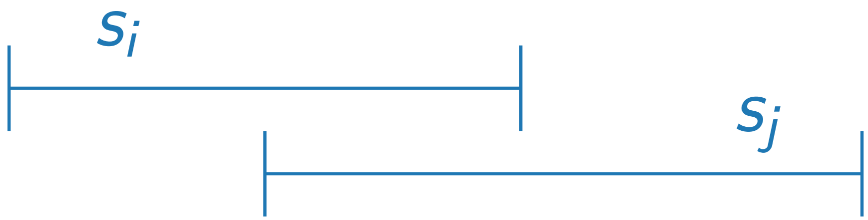

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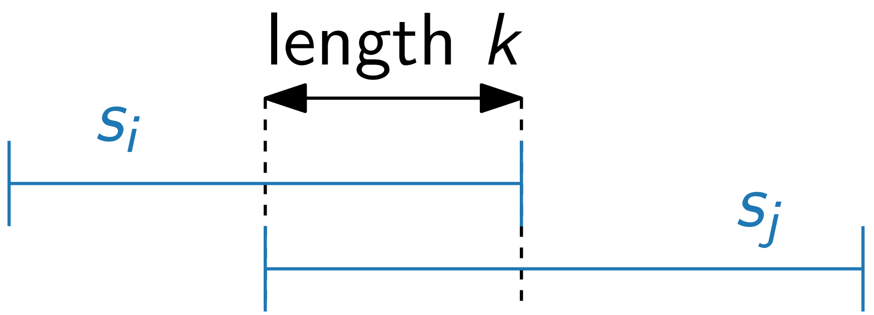

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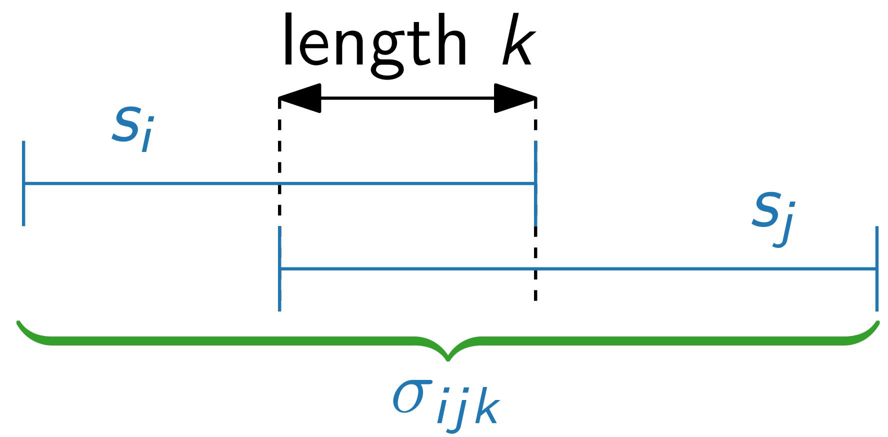

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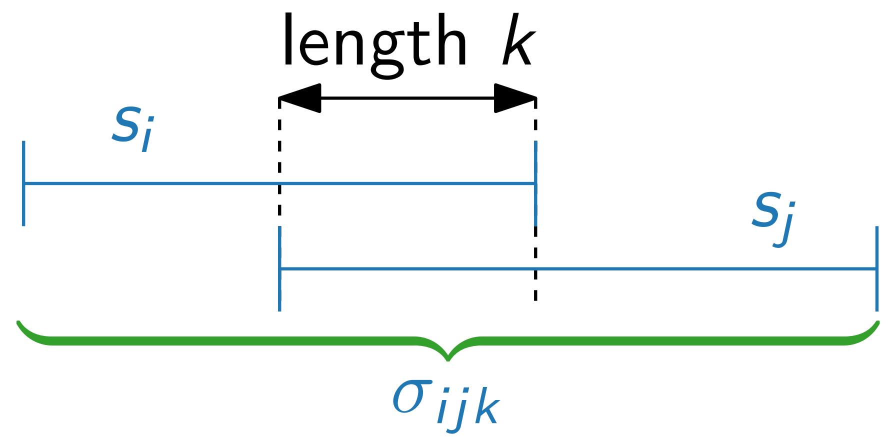

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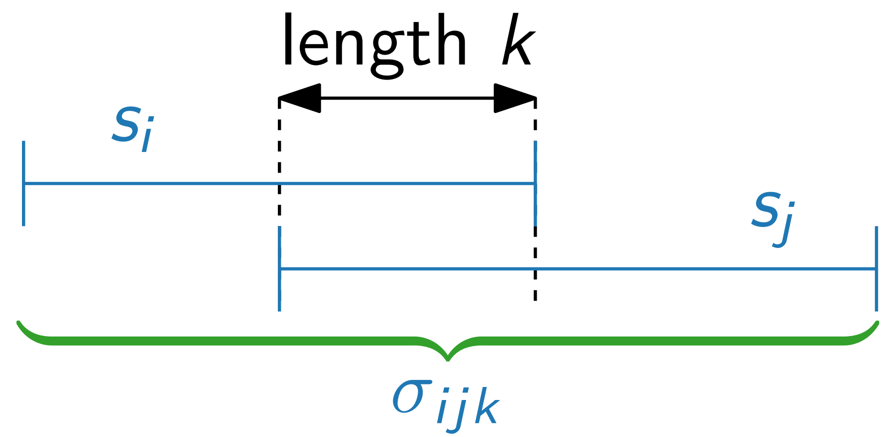

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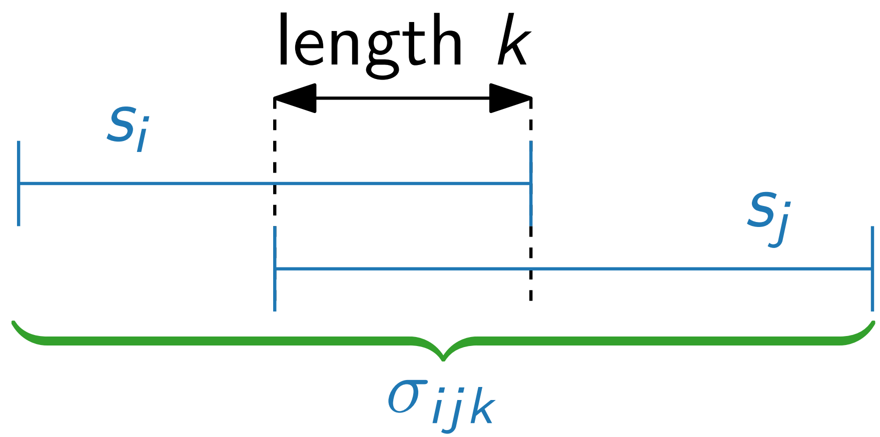

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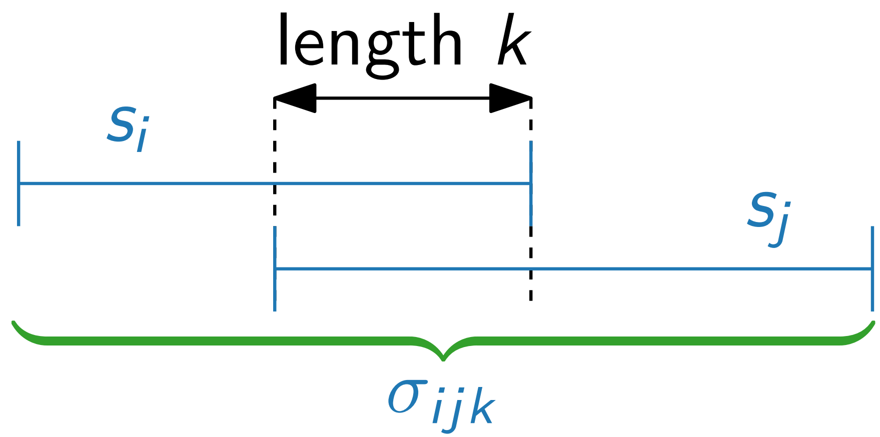

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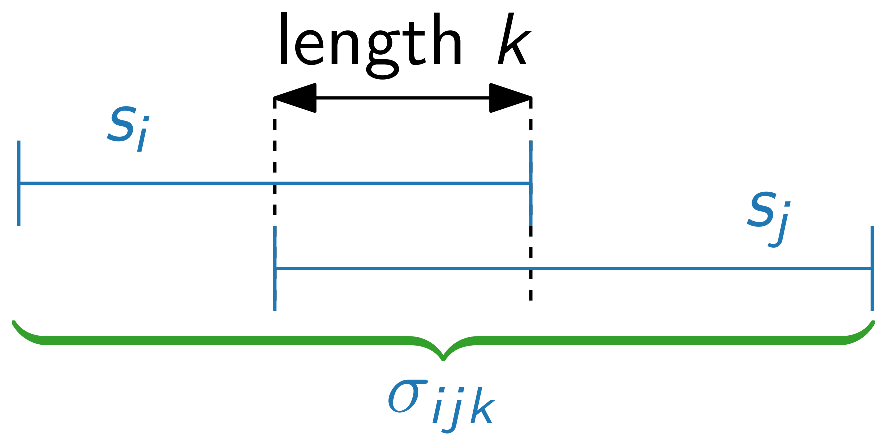

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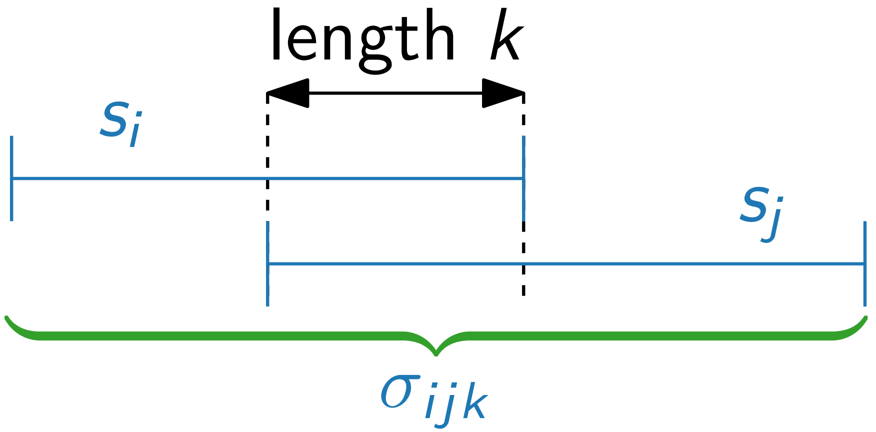

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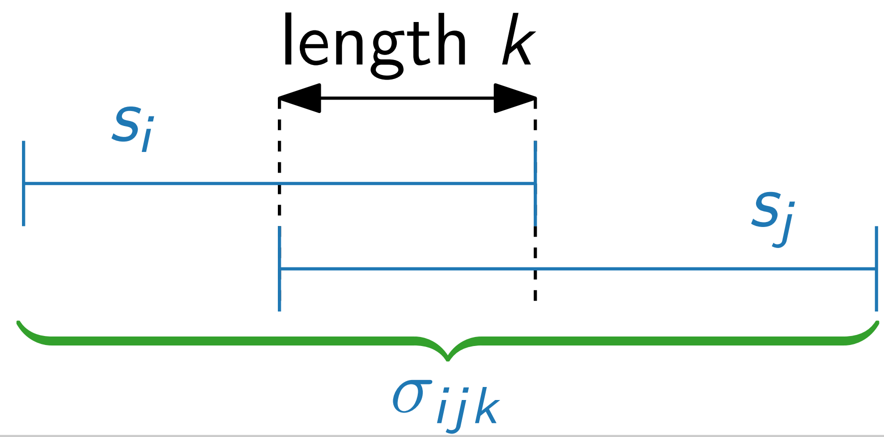

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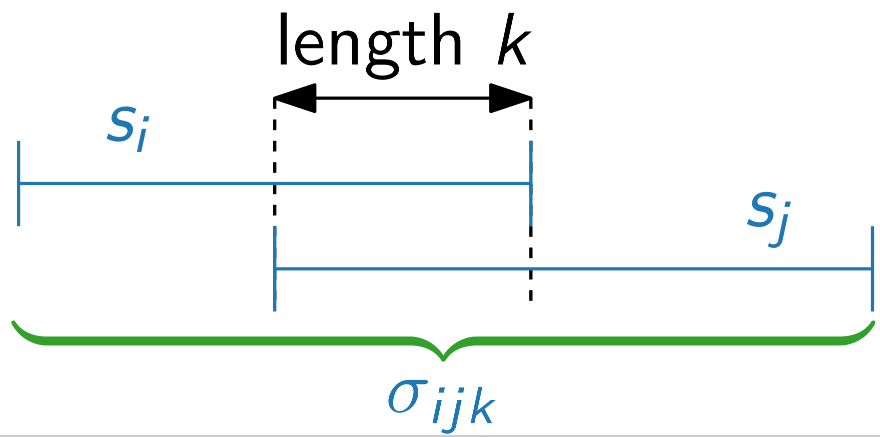

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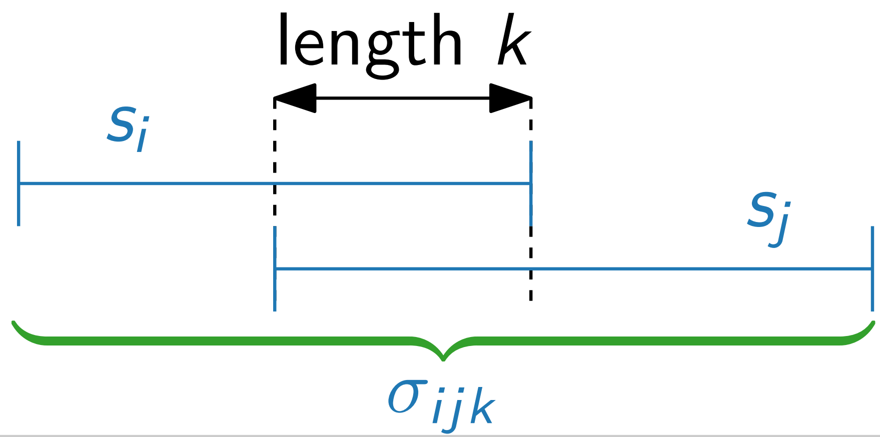

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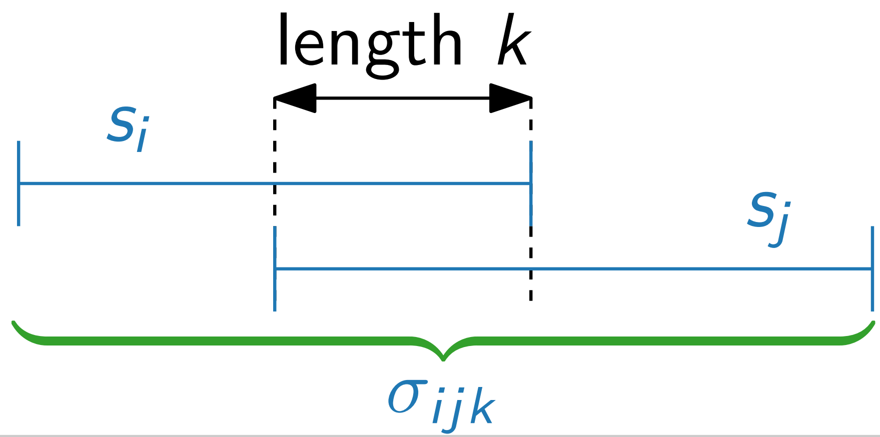

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$\mathcal{S} = \{S(\sigma_{ijk}) \mid 1 \leq i, j \leq n, k > 0\}$

Approximation Algorithms

Lecture 2:
SetCover and ShortestSuperString

Part V:
Solving ShortestSuperString via SetCover

Relating SSS and SetCover

Lemma. Let $\mathsf{OPT}_{\mathsf{SSS}}$ be the length of a shortest superstring of U, and let $\mathsf{OPT}_{\mathsf{SC}}$ be the minimum cost of the corresponding SetCover instance. Then

$$\mathsf{OPT}_{\mathsf{SSS}} \leq \mathsf{OPT}_{\mathsf{SC}}.$$

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Thus, $\mathrm{OPT}_{\mathrm{SSS}} \leq |s| = \mathrm{OPT}_{\mathrm{SC}}$.

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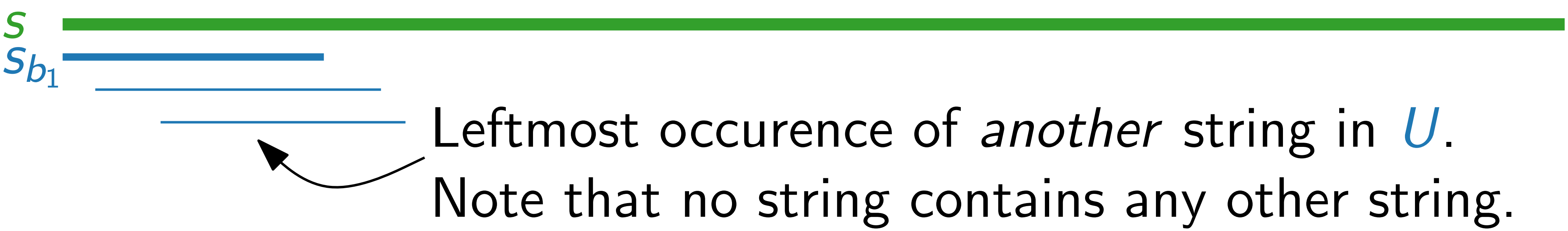

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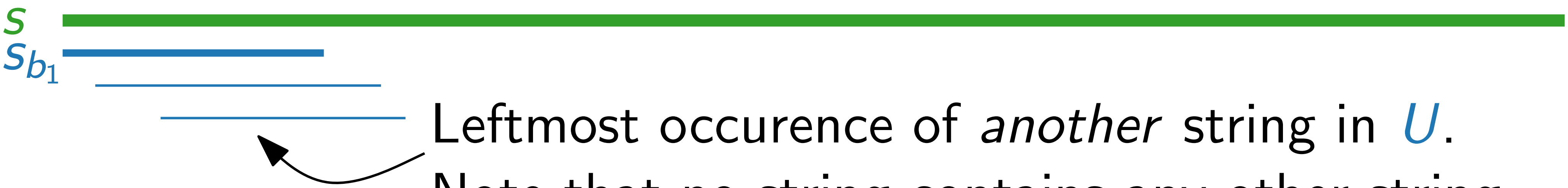

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Note that no string contains any other string.
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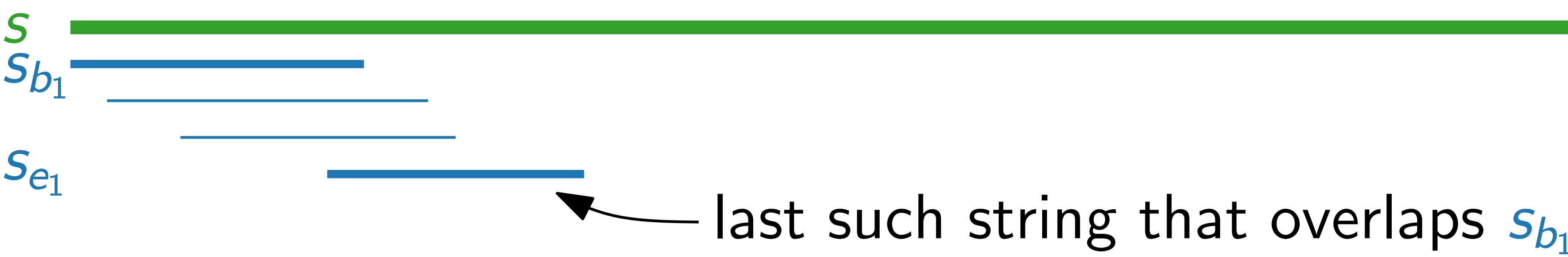

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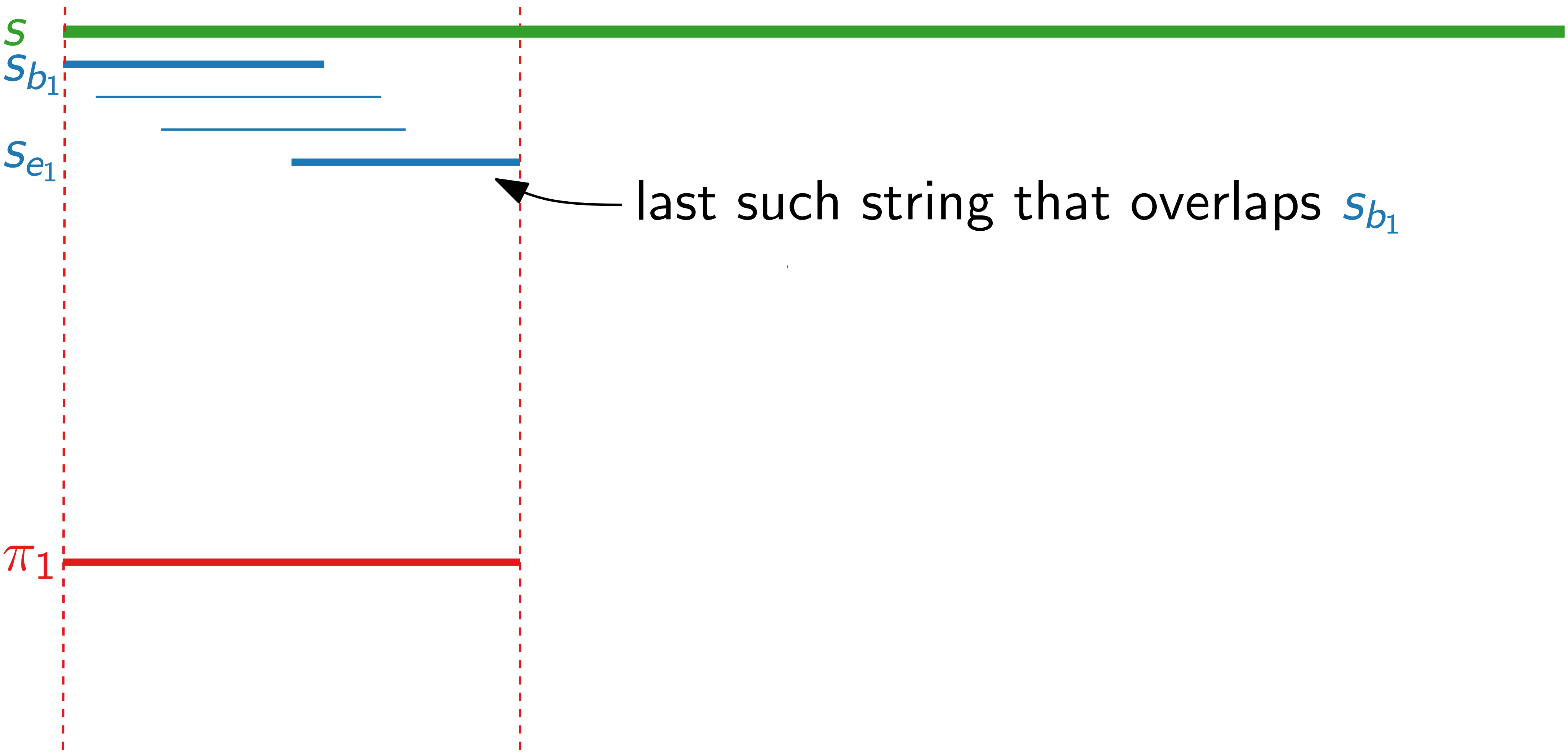

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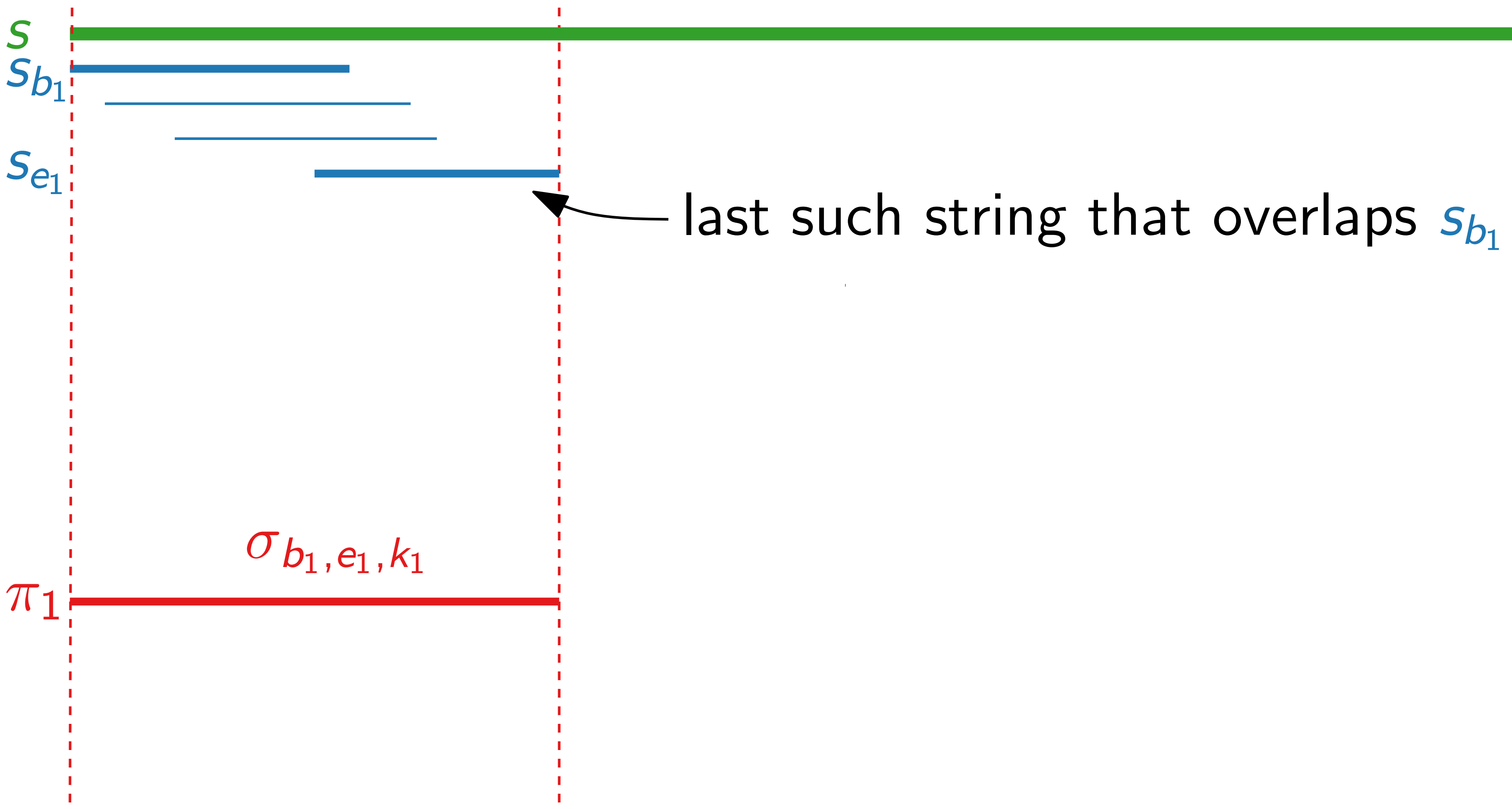

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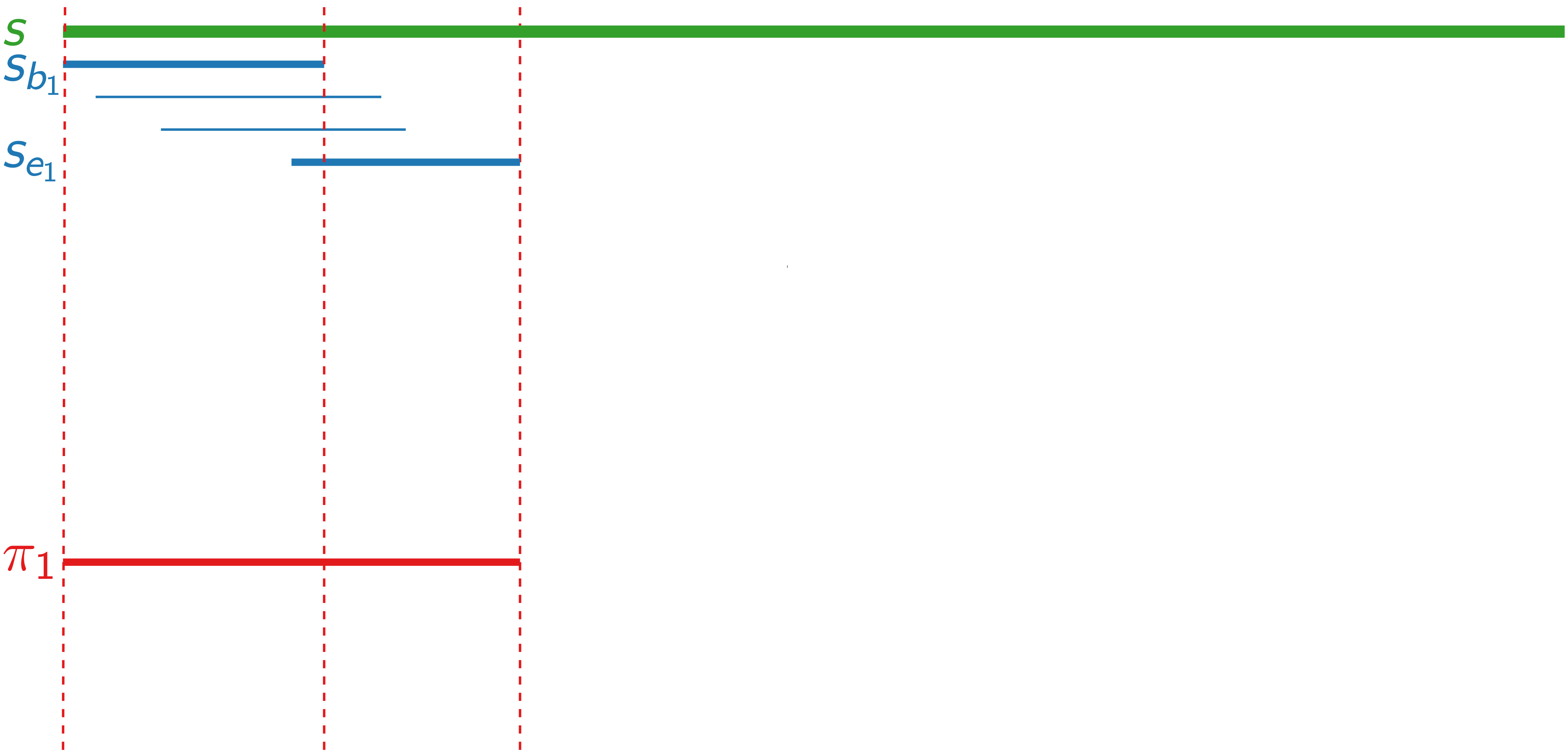

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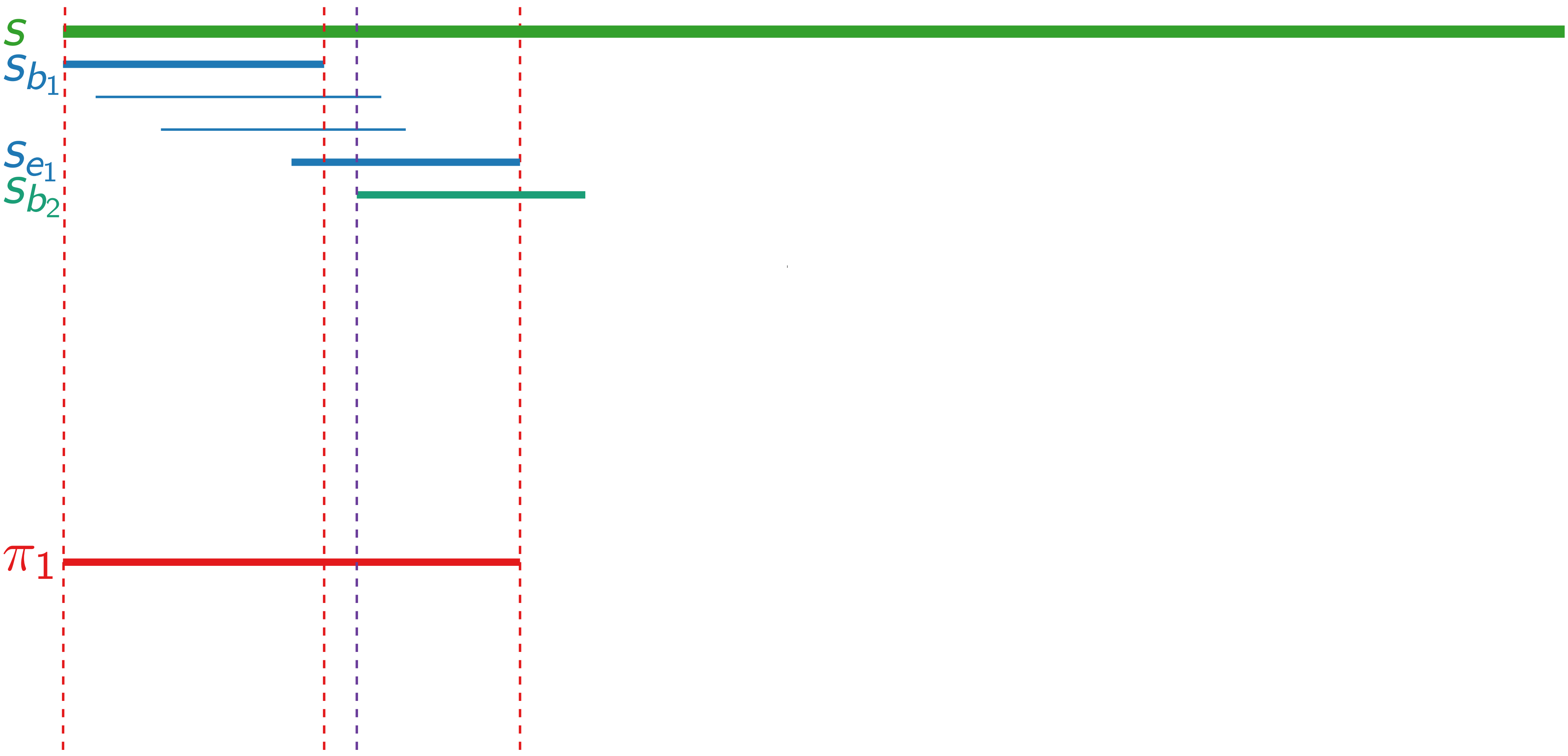

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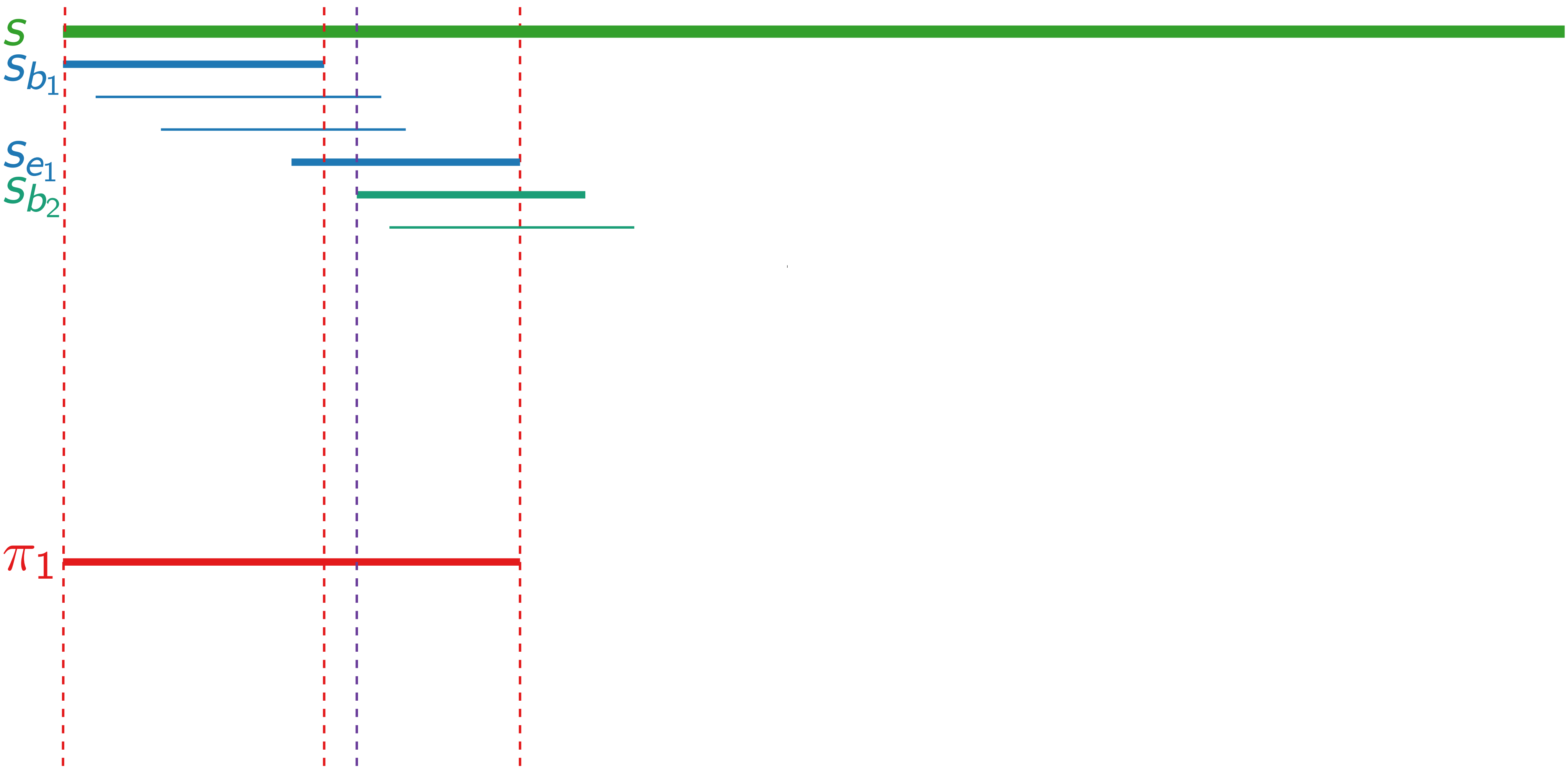

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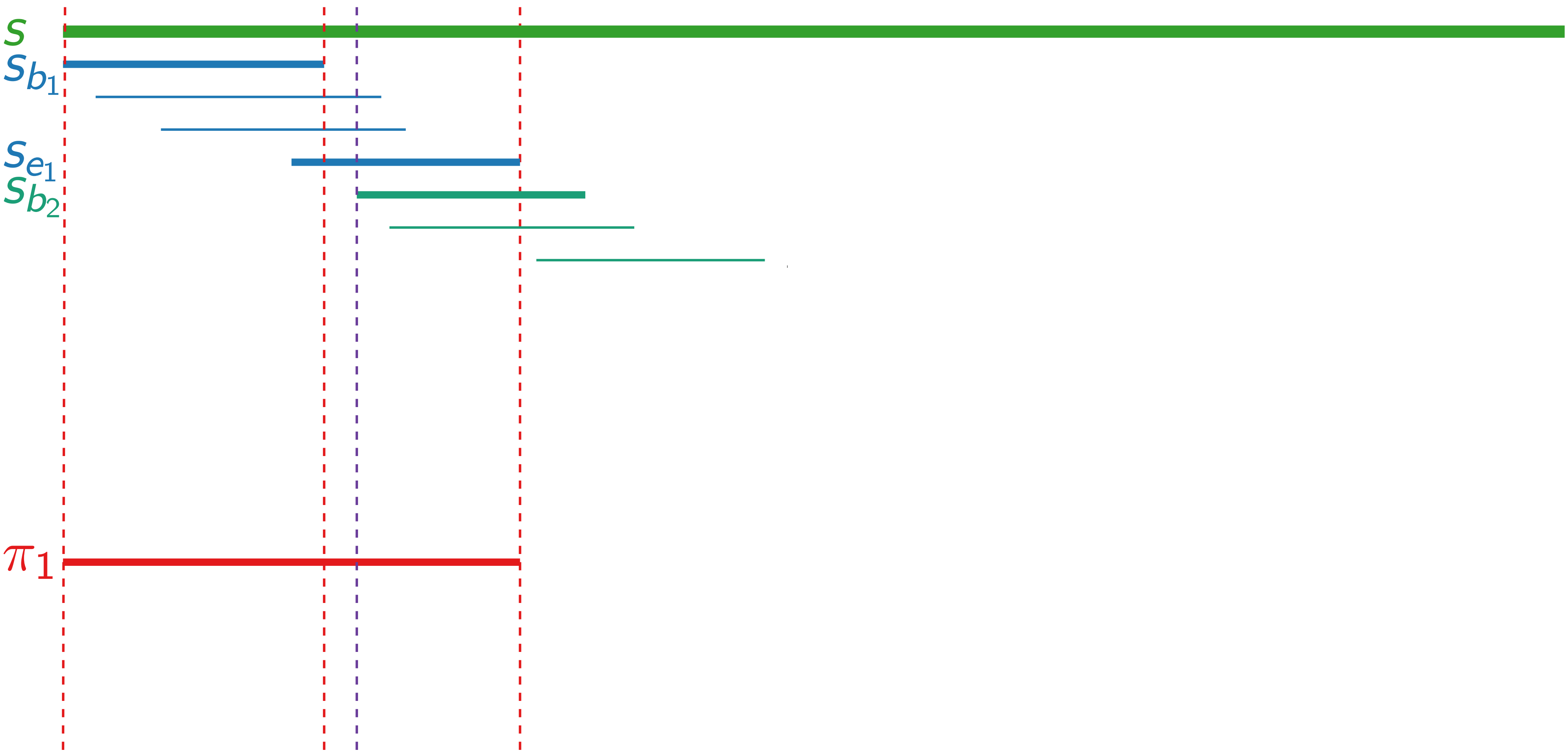

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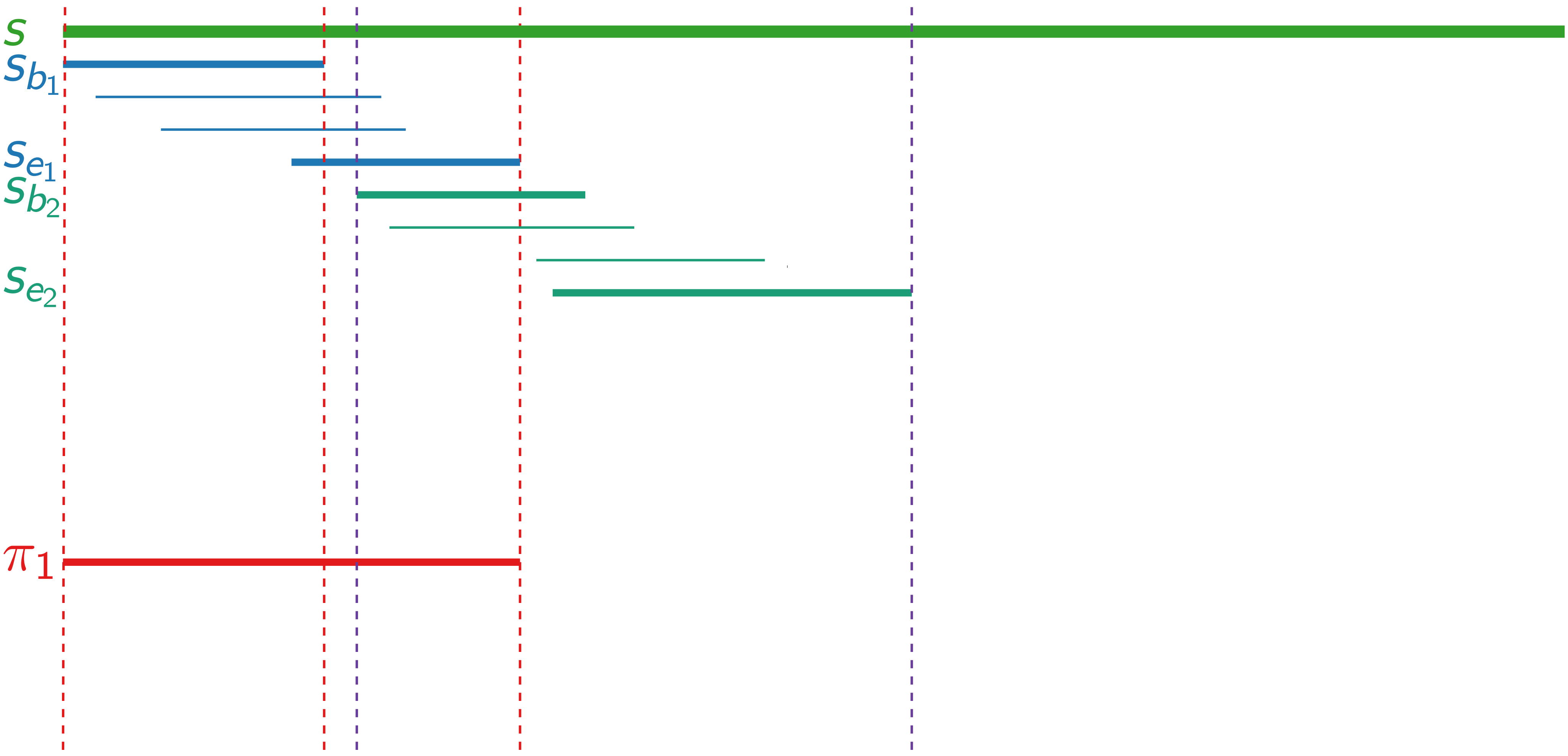

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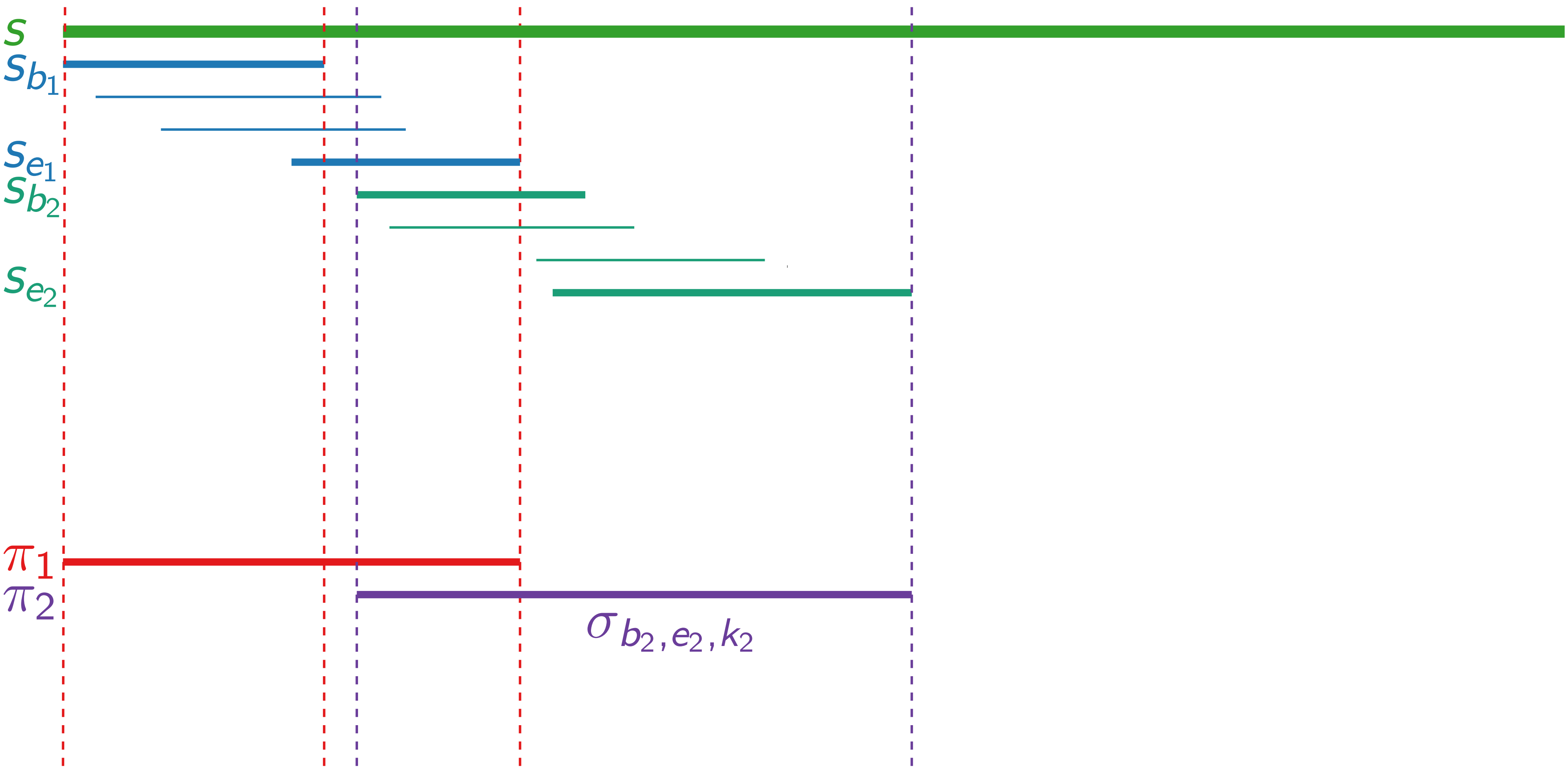

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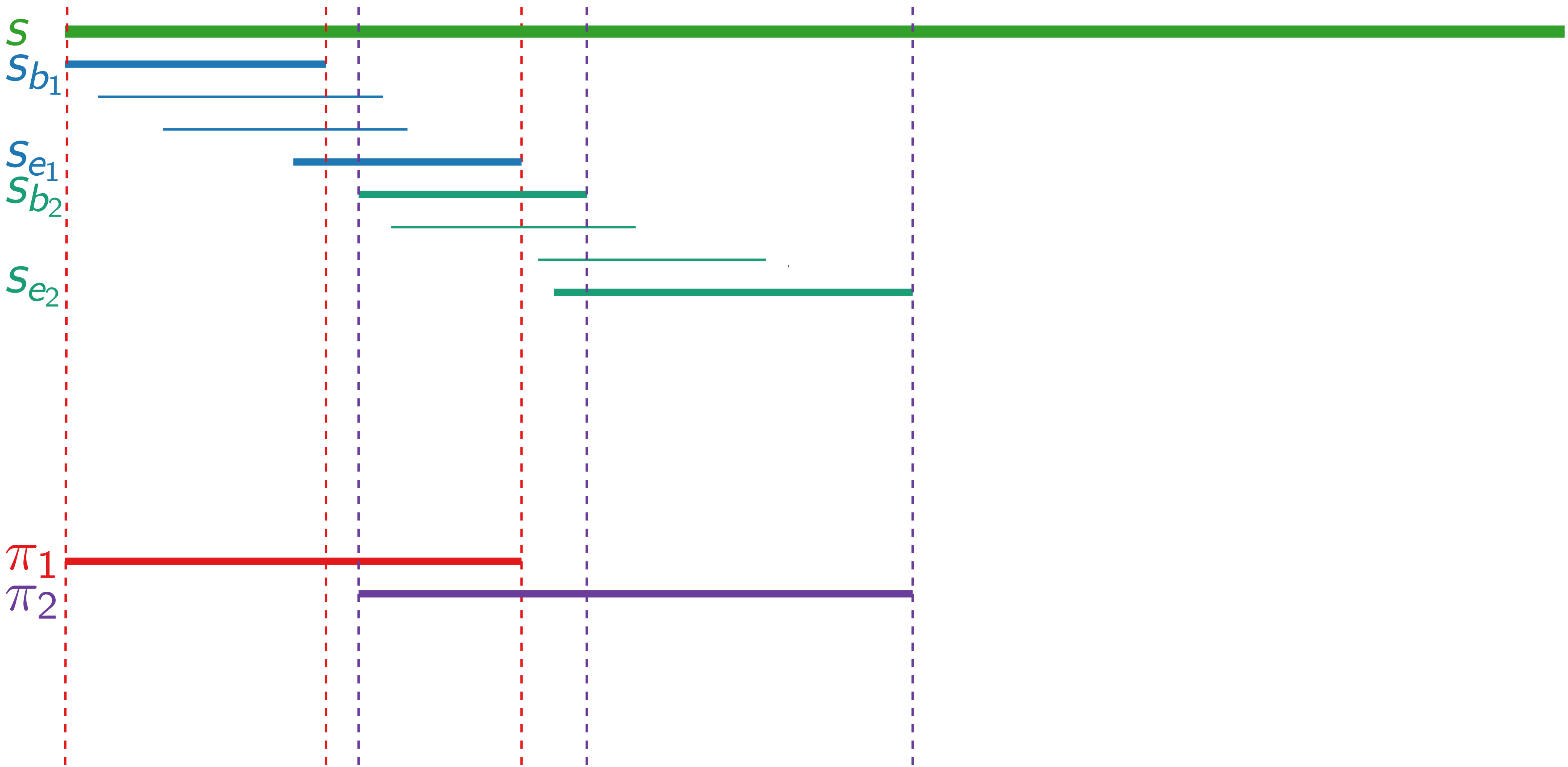

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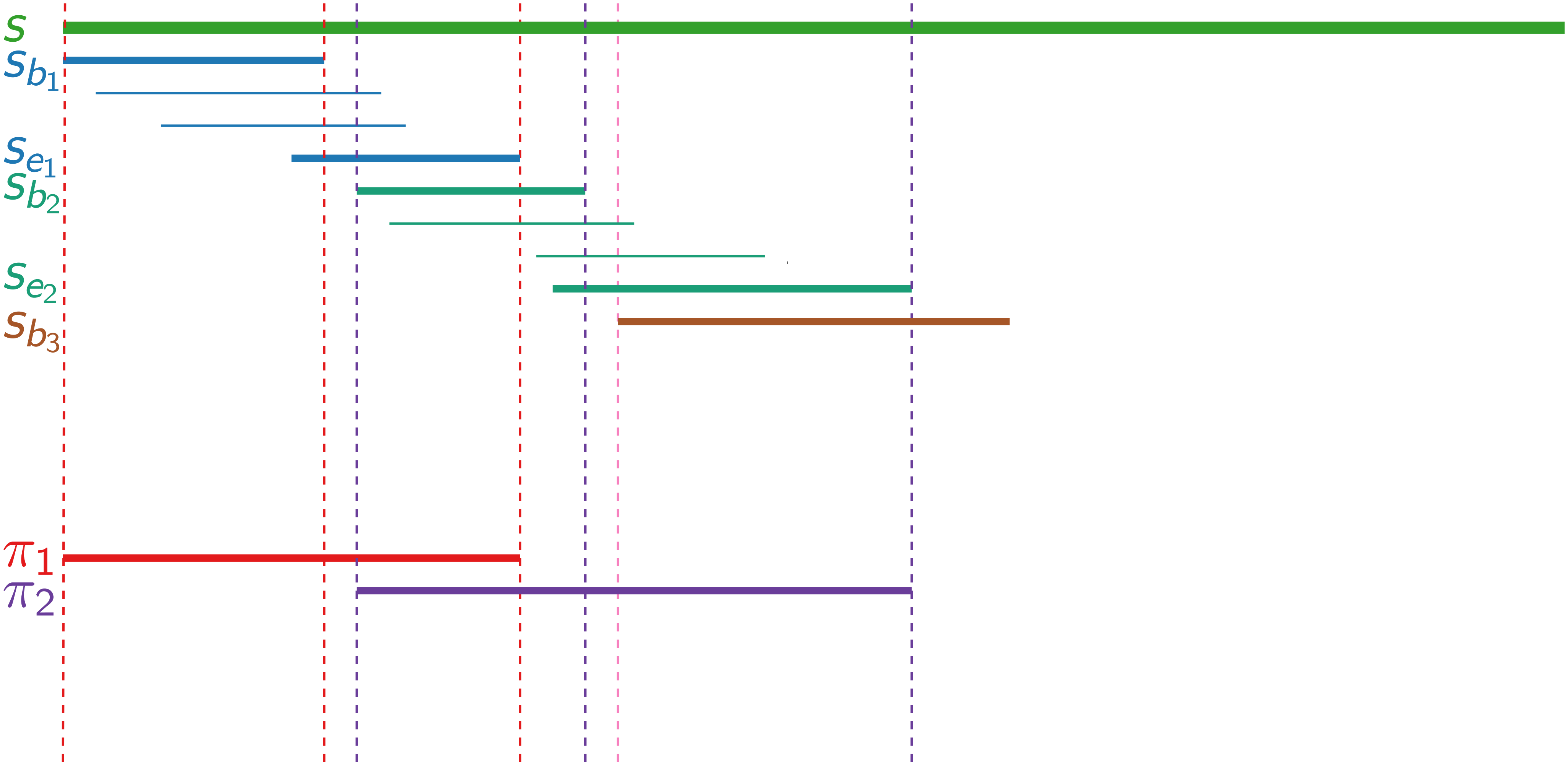

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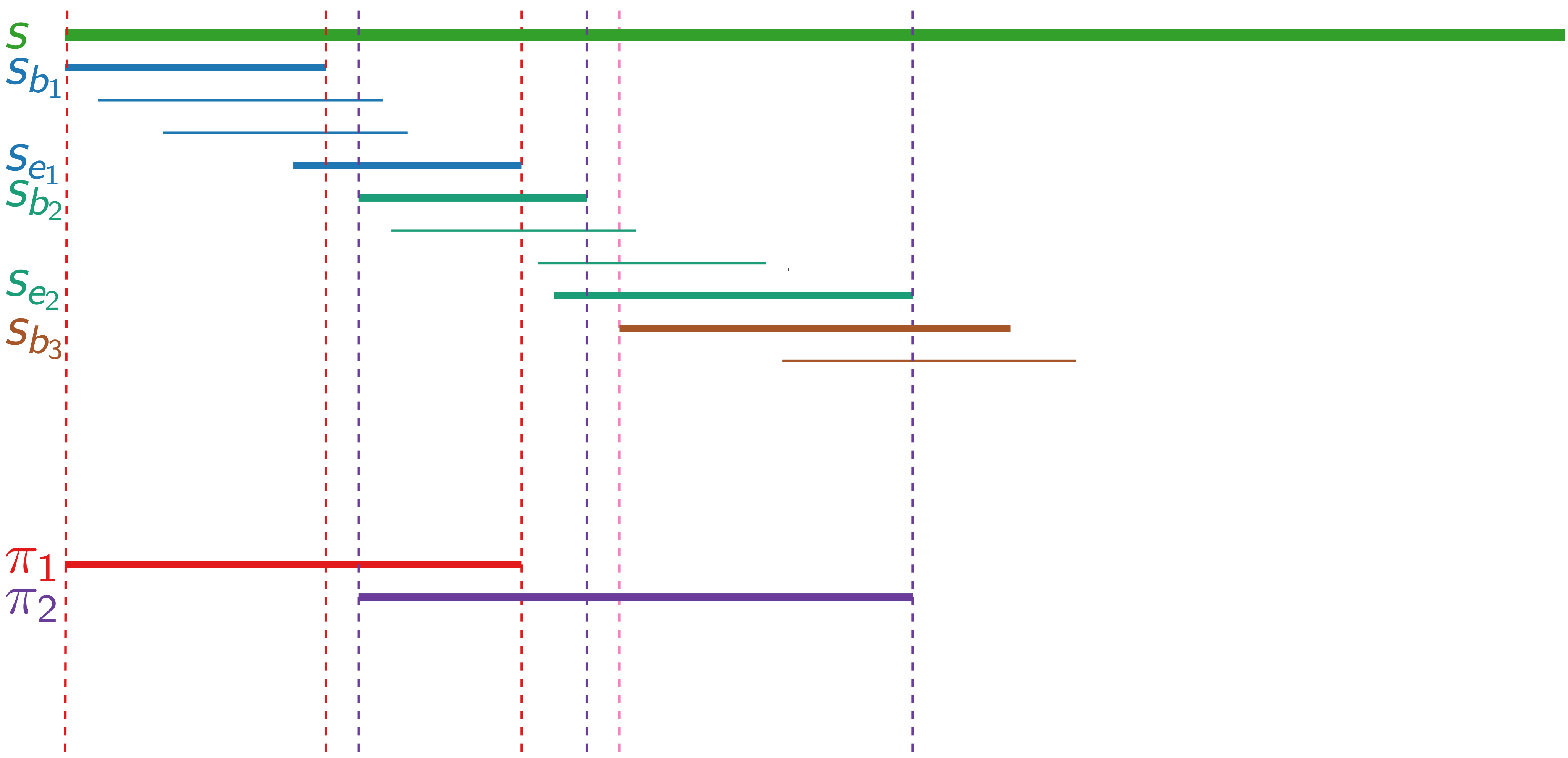

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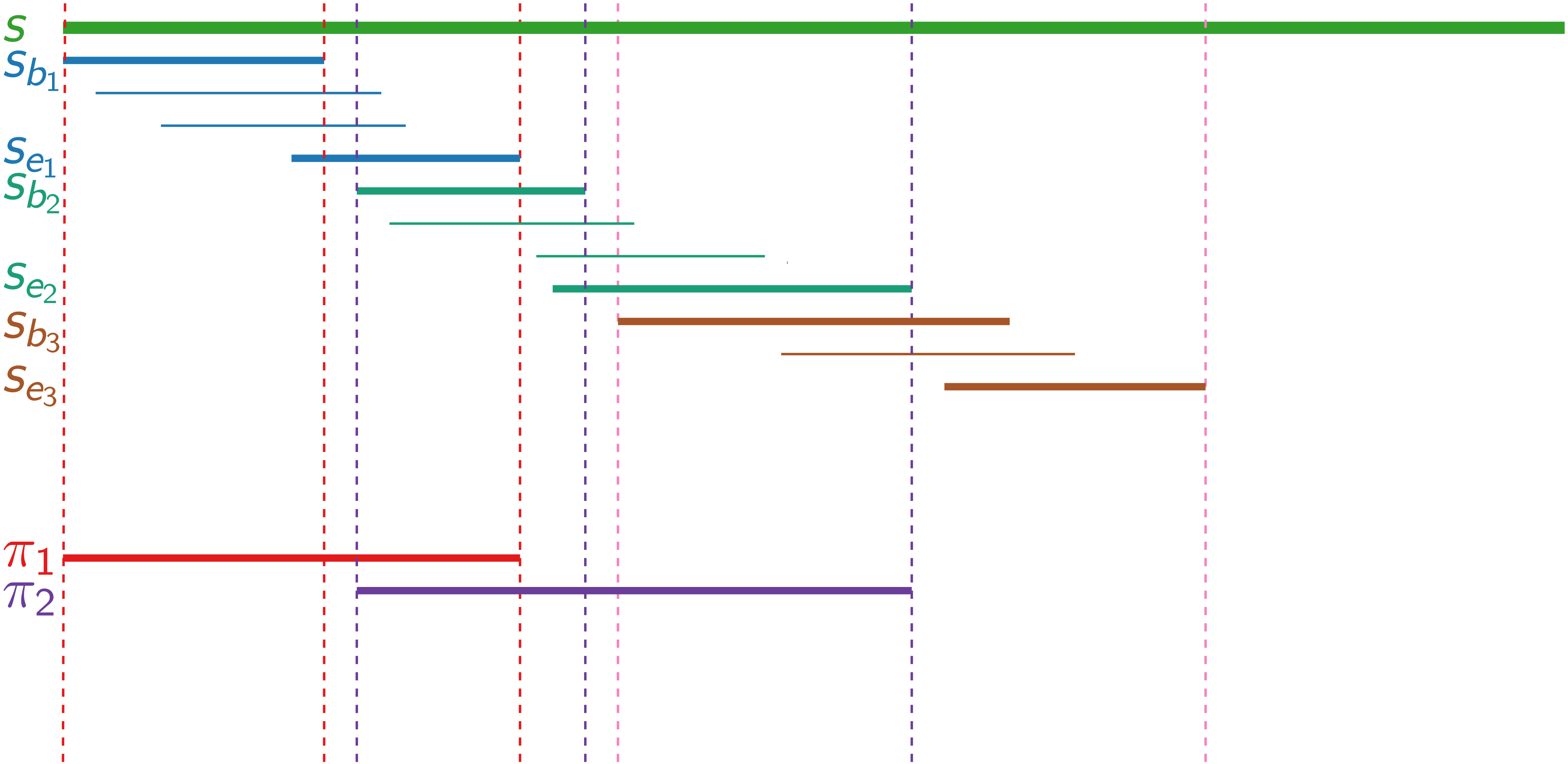

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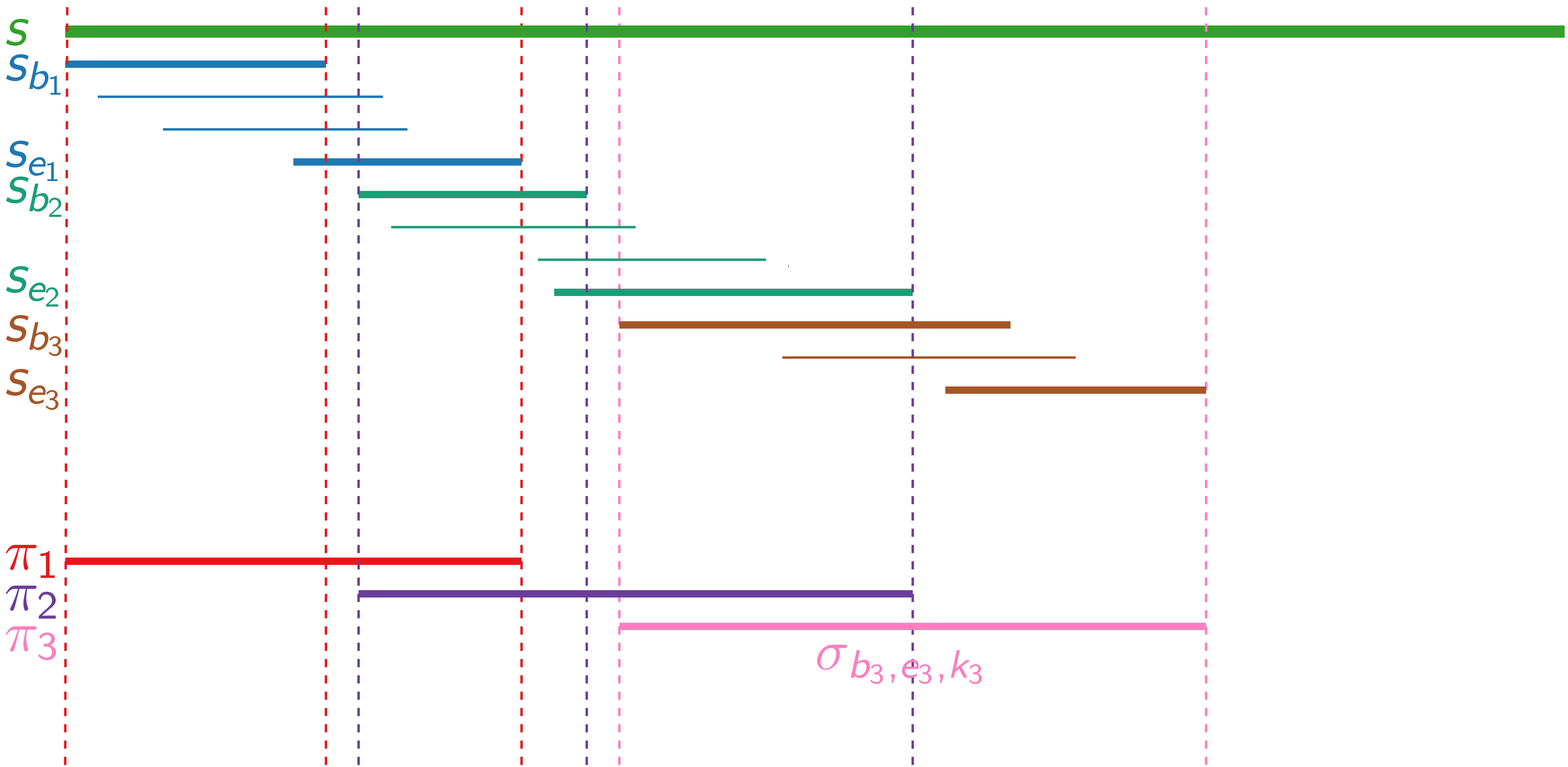

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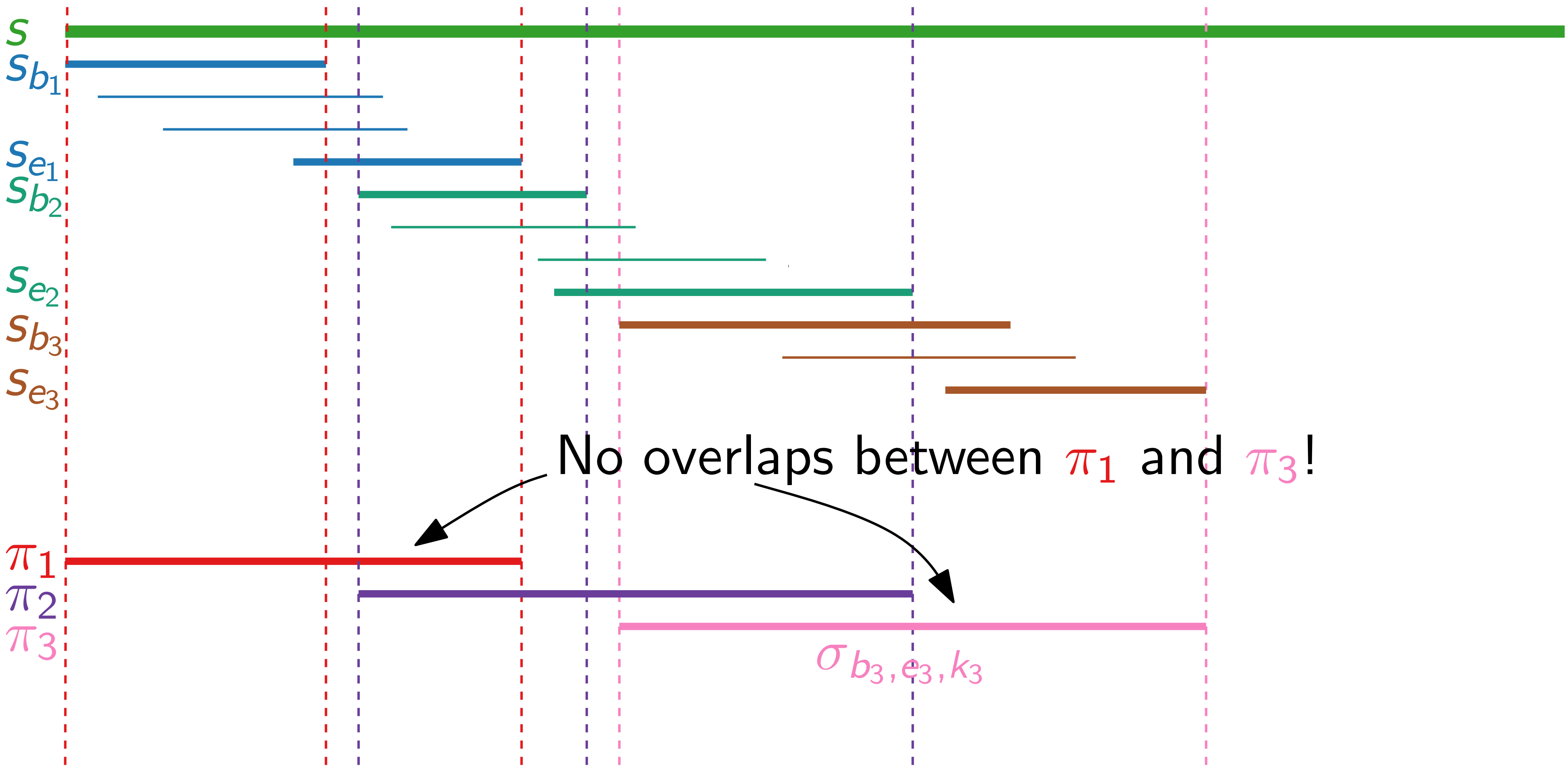

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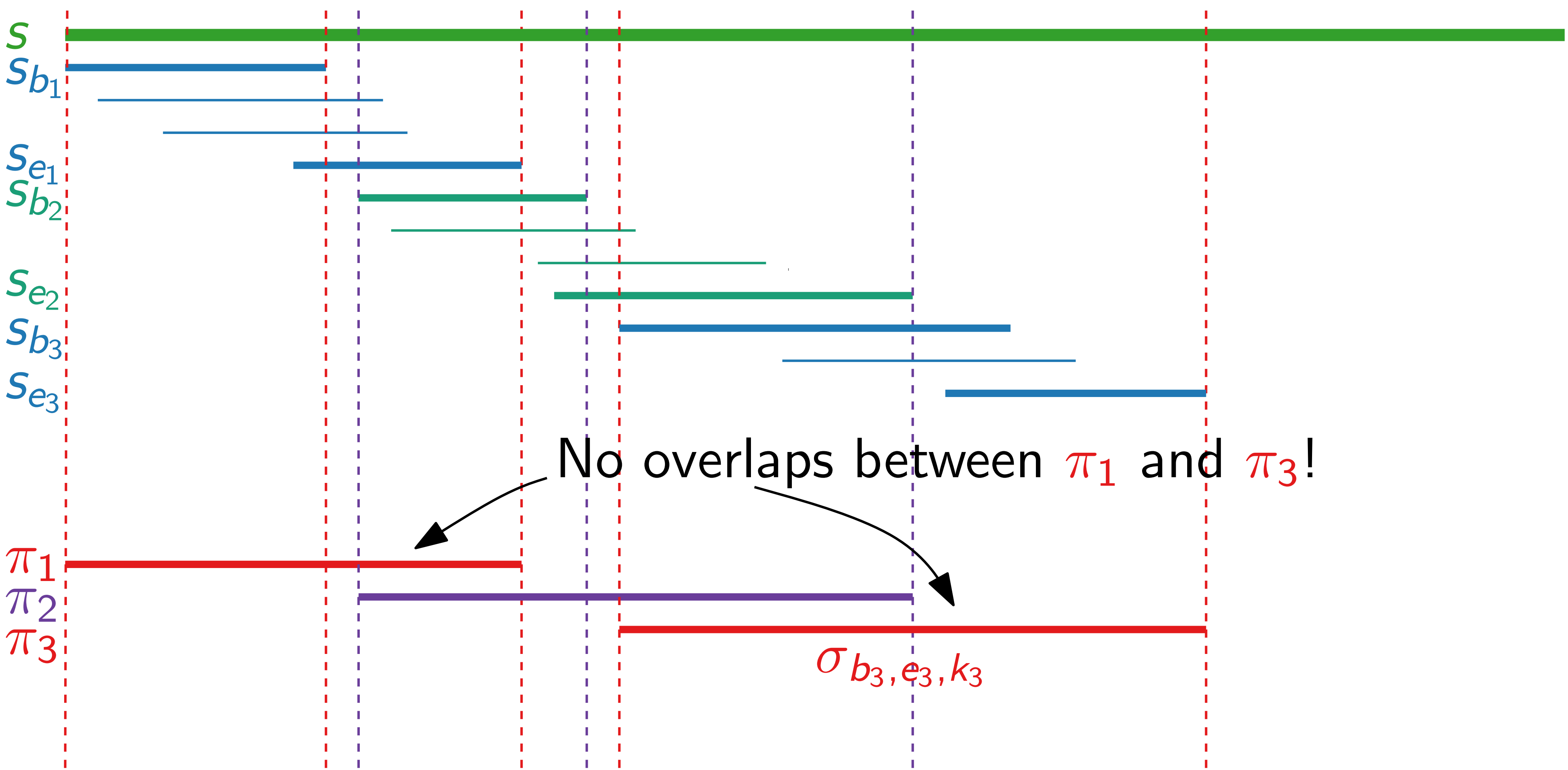

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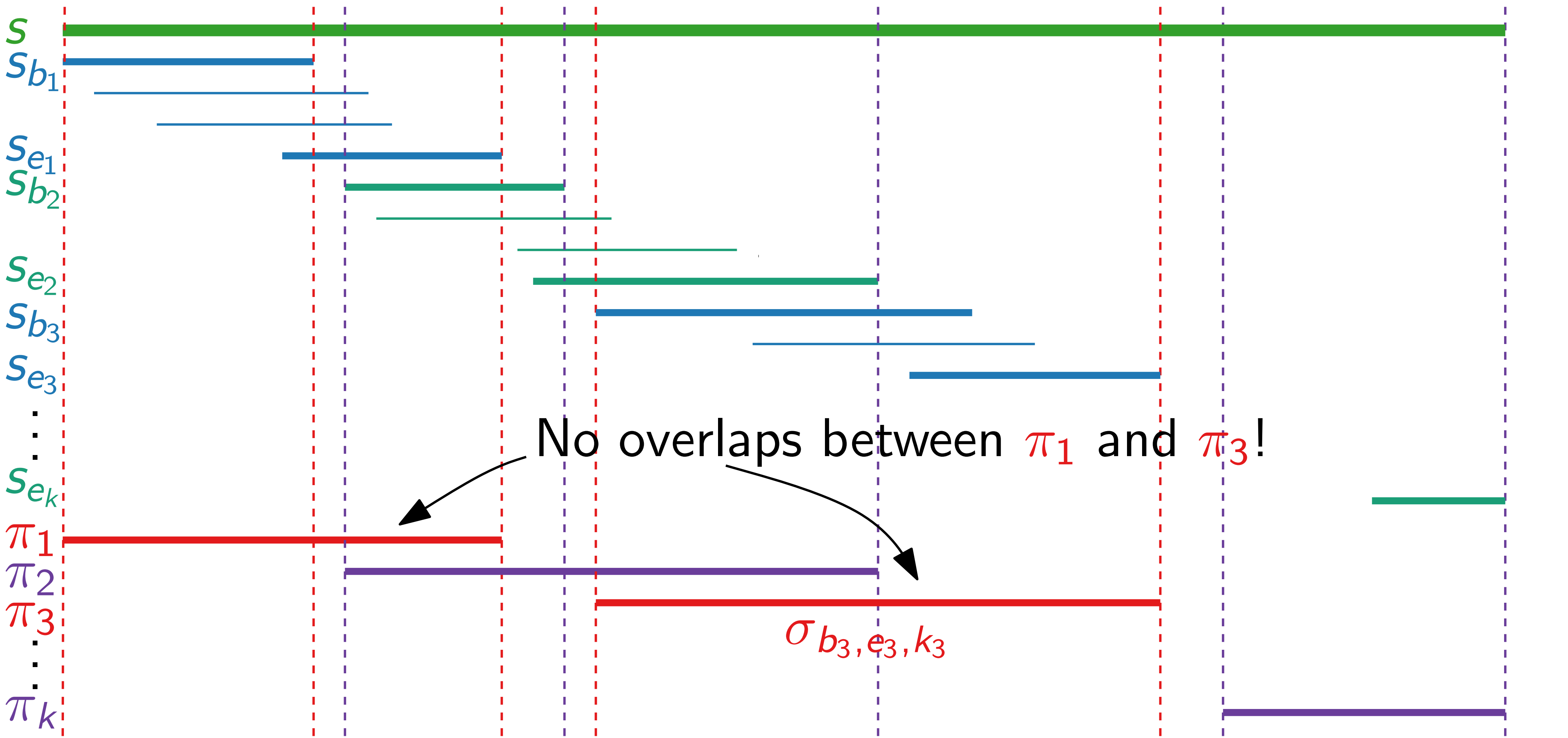

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Theorem. GreedySetCover is a factor-$\mathcal{H}_k$ approximation algorithm for $\textsc{SetCover}$, where k is the cardinality of the largest set in $\mathcal{S}$ and $\mathcal{H}_k := 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{k} \leq 1 + \ln k$.

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- $\textsc{ShortestSuperString}$ cannot be approximated within factor $\frac{333}{332} \approx 1.003$ (unless $\mathrm{P} = \mathrm{NP}$).
  [Karpinski & Schmied: CATS 2013]