

Approximation Algorithms

Lecture 1: Introduction and Vertex Cover

Part I: Organizational

Organizational

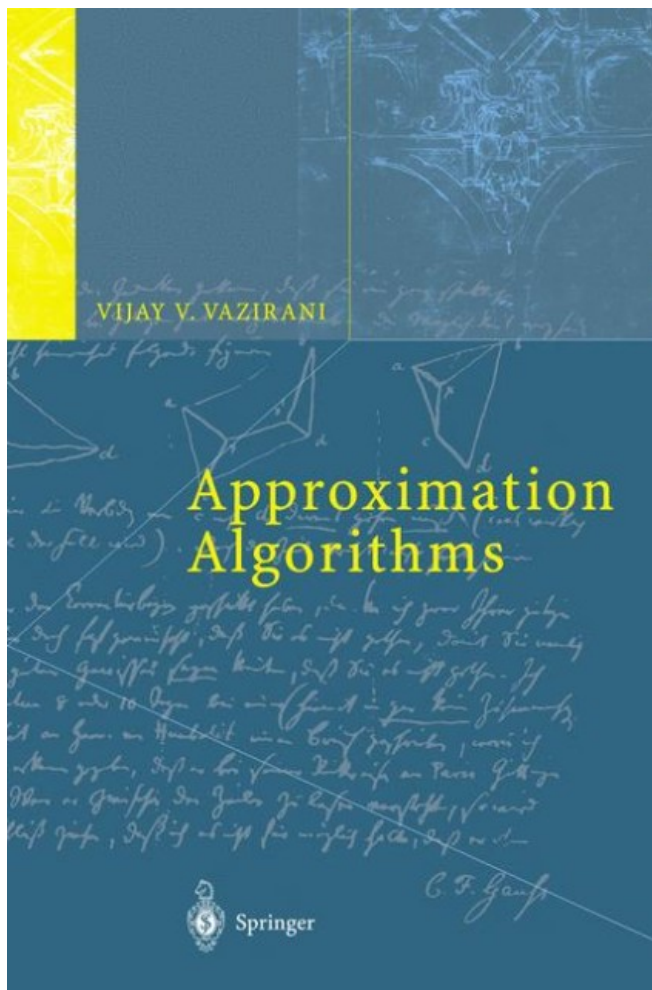
Lectures: on site (English/German, depending on audience)
Fri, 10:15–11:45 (ÜR I)

Tutorials: roughly one exercise sheet per lecture
discussing old solutions and solving new tasks
Tue, 10:15–11:45 (SE I), starting Oct. 24.
Bonus (+0.3 on final grade) for $\geq 50\%$ points

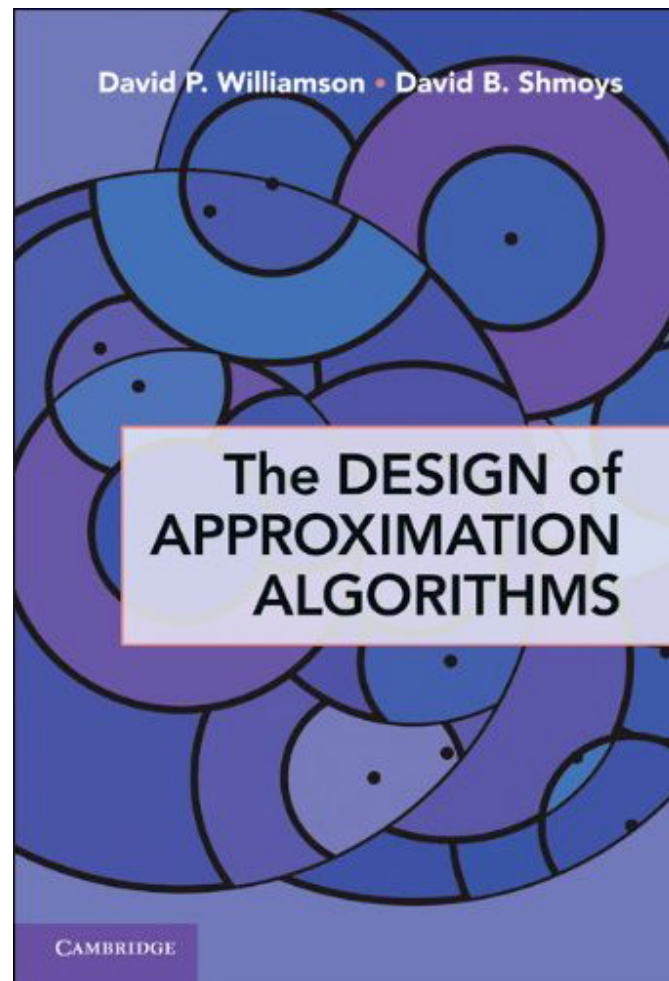
Questions/Tasks during the lecture

Most slides are due to Joachim Spoerhase,
polishing & colors are due to Philipp Kindermann – thanks!

Textbooks



Vijay V. Vazirani:
Approximation Algorithms
Springer-Verlag, 2003.



D. P. Williamson & D. B. Shmoys:
The Design of Approximation Algorithms
Cambridge-Verlag, 2011.

<http://www.designofapproxalgs.com/> ←

Approximation Algorithms

„All exact science is dominated by the idea of approximation.“

– Bertrand Russell
(1872 – 1970)



Approximation Algorithms

- Many optimization problems are NP-hard!
(For example, the traveling salesperson problem.)
- \rightsquigarrow an optimal solution cannot be efficiently computed unless $P=NP$.
- However, good approximate solutions can often be found efficiently!
- **Techniques** for the design and analysis of approximation algorithms arise from studying specific optimization problems.

Overview

Combinatorial algorithms

- Introduction (Vertex Cover)
- Set Cover via Greedy
- Shortest Superstring
via reduction to SC
- Steiner Tree via MST
- Multiway Cut via Greedy
- k -Center via Parametrized Pruning
- Min-Degree Spanning Tree
and local search
- Knapsack via DP and Scaling
- Euclidean TSP via Quadtrees

LP-based algorithms

- introduction to LP-Duality
- Set Cover via LP Rounding
- Set Cover via Primal–Dual
Schema
- Maximum Satisfiability
- Scheduling und Extreme Point
Solutions
- Steiner Forest via Primal–Dual

Approximation Algorithms

Lecture 1:

Introduction and Vertex Cover

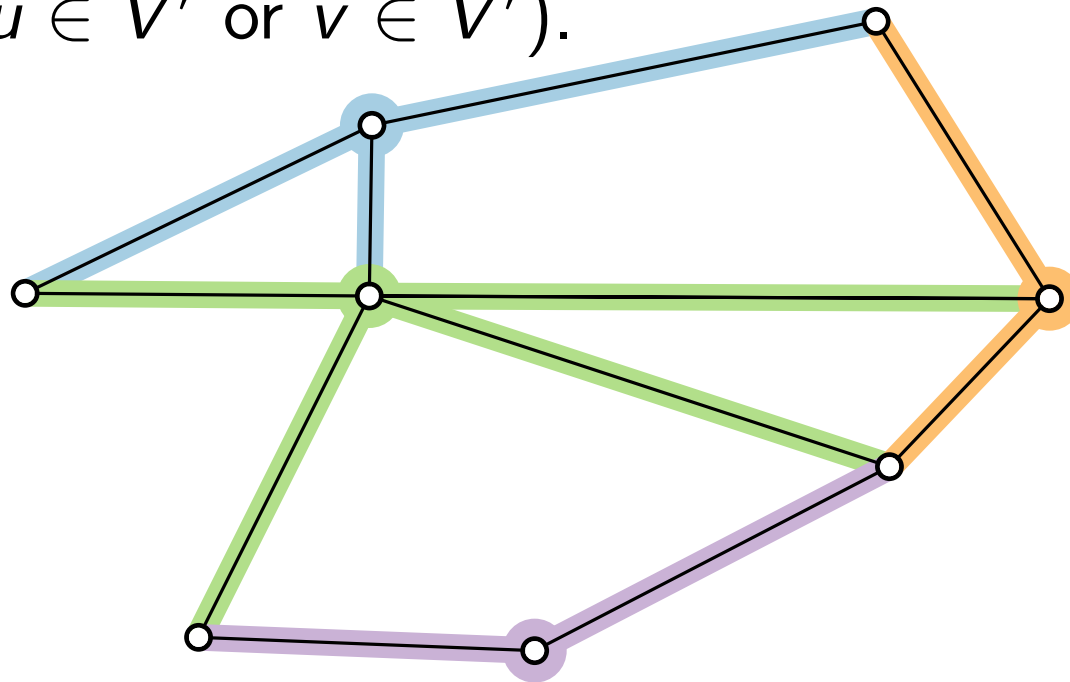
Part II:

(Cardinality) Vertex Cover

VERTEXCOVER (card.)

Input: Graph $G = (V, E)$

Output: a minimum **vertex cover**, that is, a minimum-cardinality vertex set $V' \subseteq V$ such that every edge is **covered** (i.e., for every $uv \in E$, it holds that $u \in V'$ or $v \in V'$).



Optimum ($\text{OPT} = 4$) – but in general NP-hard to find :-)

Approximation Algorithms

Lecture 1:

Introduction and Vertex Cover

Part III:

NP-Optimization Problem

NP-Optimization Problem

An **NP-optimization problem** Π is given by:

- A set D_Π of **instances**.

We denote the size of an instance $I \in D_\Pi$ by $|I|$.

- For each instance $I \in D_\Pi$,

a set $S_\Pi(I) \neq \emptyset$ of **feasible solutions** for I such that:

- for each solution $s \in S_\Pi(I)$,

its size $|s|$ is polynomially bounded in $|I|$, and

- there is a polynomial-time algorithm that decides, for each pair (s, I) , whether $s \in S_\Pi(I)$.

- A polynomial time computable **objective function** obj_Π which assigns a positive objective value $\text{obj}_\Pi(I, s) \geq 0$ to any given pair (s, I) with $s \in S_\Pi(I)$.

- Π is either a minimization or maximization problem.

VERTEXCOVER: NP-Optimization Problem

Task: Fill in the gaps for $\Pi = \text{VERTEX COVER}$.

$D_\Pi =$ Set of all graphs

For $I \in D_\Pi$: $|I| =$ Number of vertices $|V|$

$G=(V, E)$ $S_\Pi(I) =$ Set of all vertex covers of G

■ Why is $|s| \in \text{poly}(|I|)$ for every $s \in S_\Pi(I)$?

$$s \subseteq V \Rightarrow |s| \leq |V| = |I|$$

■ For a given pair (s, I) , how can we efficiently decide whether $s \in S_\Pi(I)$? Test whether all edges are covered.

$$\text{obj}_\Pi(I, s) = |s|$$

Π is a minimization problem.

Optimum and Optimal Objective Value

Let Π be a **maximization problem** and $I \in D_\Pi$ an instance of Π .

A feasible solution $s^* \in S_\Pi(I)$ is **optimal** if $\text{obj}_\Pi(I, s^*)$ is **maximal** among the objective values attained by the feasible solutions of I .

The optimal value $\text{obj}_\Pi(I, s^*)$ of the objective function is denoted by $\text{OPT}_\Pi(I)$ or simply by OPT in context.

Approximation Algorithms

maximization problem $\alpha: \mathbb{N} \rightarrow \mathbb{Q}$

Let Π be a minimization problem and ~~$\alpha \in \mathbb{Q}^+$~~ .

A factor- α approximation algorithm for Π is an efficient algorithm that provides, for **any** instance $I \in D_\Pi$, a feasible solution $s \in S_\Pi(I)$ such that

$$\frac{\text{obj}_\Pi(I, s)}{\text{OPT}_\Pi(I)} \stackrel{\geq}{\leq} \alpha \cdot \alpha(|I|).$$

Approximation Algorithms

Lecture 1:

Introduction and Vertex Cover

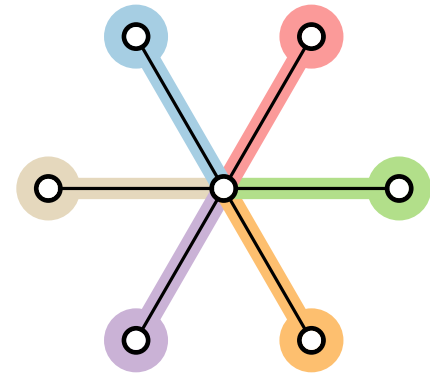
Part IV:

Approximation Algorithm for VERTEXCOVER

Approximation Alg. for VERTEXCOVER

Ideas?

- Edge-Greedy
- Vertex-Greedy



Quality?

Problem: How can we estimate $\text{obj}_\Pi(I, s)/\text{OPT}$, when it is hard to compute OPT ?

Idea: Find a “good” lower bound $L \leq \text{OPT}$ for OPT and compare it to our approximate solution.

$$\frac{\text{obj}_\Pi(I, s)}{\text{OPT}} \leq \frac{\text{obj}_\Pi(I, s)}{L}$$

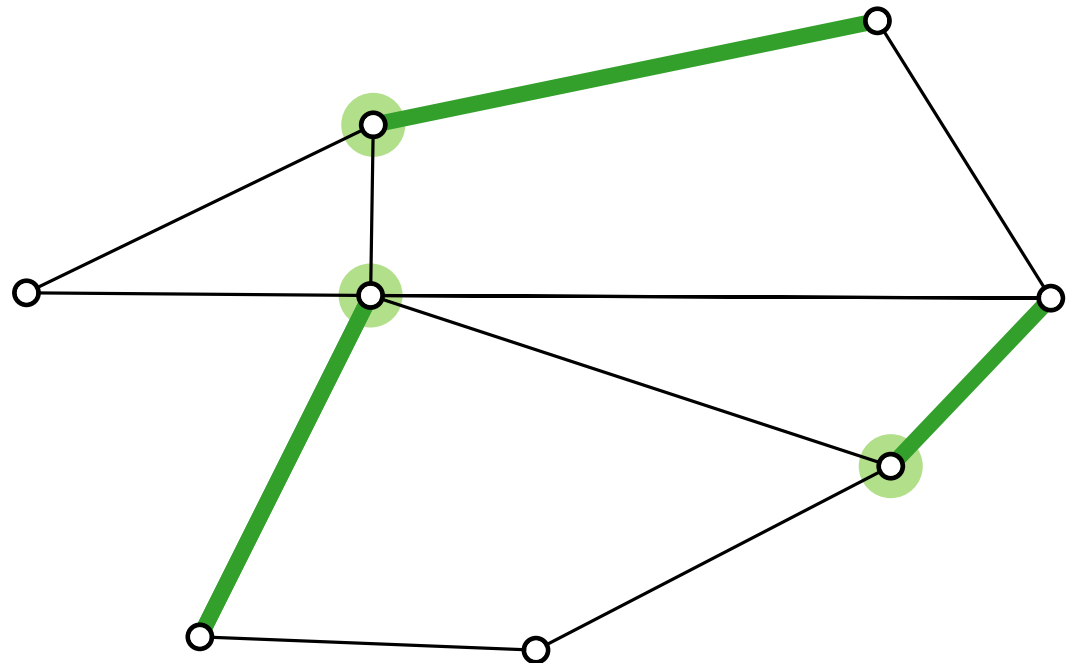
Lower Bound by Matchings

Given a graph G , a set M of edges of G is a **matching** if no two edges of M are adjacent (i.e., share an end vertex).

M is **maximal** if there is no matching M' with $M' \supsetneq M$.

$$\text{OPT} \geq |M|$$

Vertex cover of M



Lower Bound by Matchings

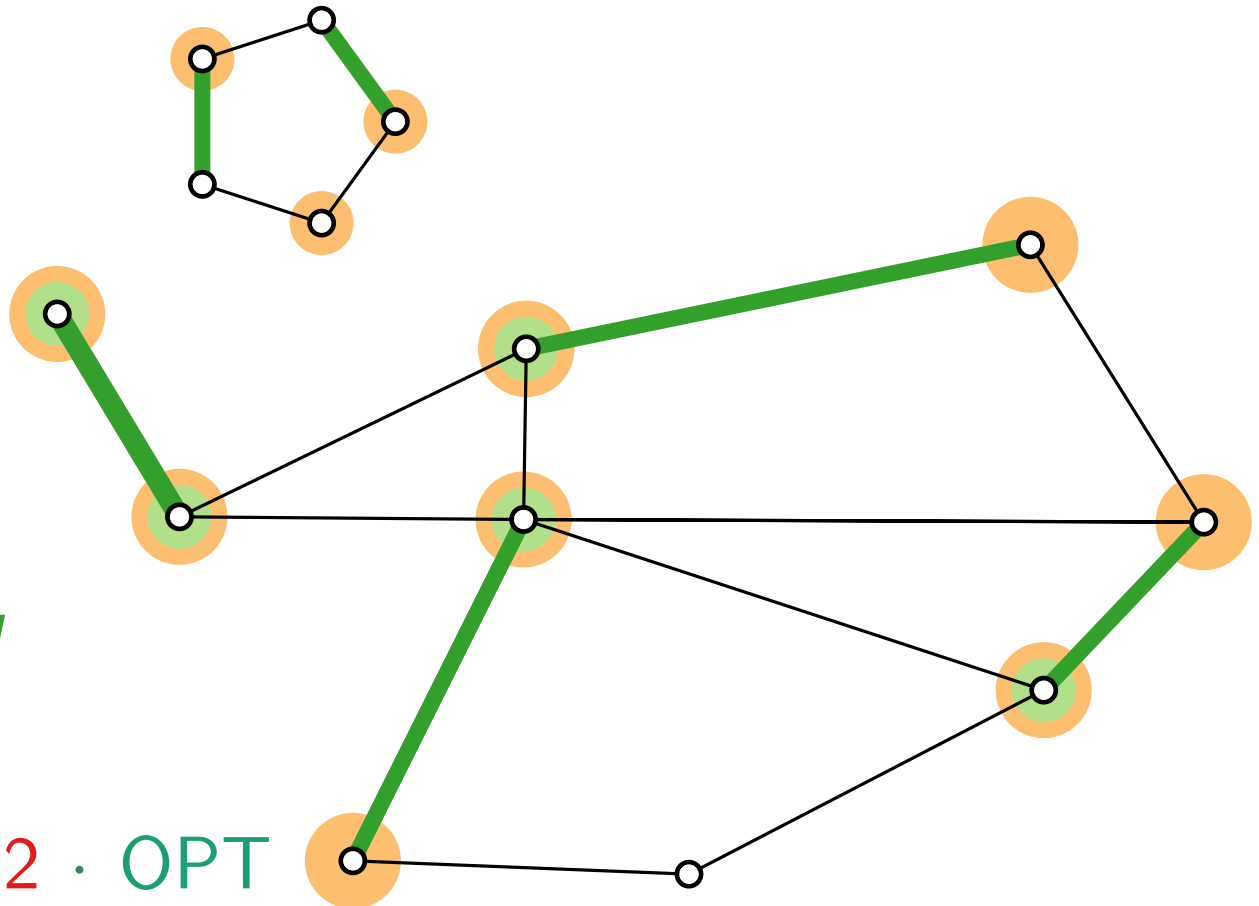
Given a graph G , a set M of edges of G is a **matching** if no two edges of M are adjacent (i.e., share an end vertex).

M is **maximal** if there is no matching M' with $M' \supsetneq M$.

$$\text{OPT} \geq |M|$$
~~$$\text{OPT} = |M| ?$$~~

Vertex cover of M
Vertex cover of E

$$\text{ALG} = 2 \cdot |M| \leq 2 \cdot \text{OPT}$$



Approximation Alg. for VERTEXCOVER

Algorithm VertexCover(G)

$M \leftarrow \emptyset$

foreach $e \in E(G)$ **do**

if e is not adjacent to any edge in M **then**
 $M \leftarrow M \cup \{e\}$

return $\{u, v \mid uv \in M\}$

Theorem. The above algorithm is a factor-2 approximation algorithm for VERTEXCOVER.

Proof. $ALG = 2 \cdot |M| \leq 2 \cdot OPT$



Approximability of VERTEX COVER

The best known approximation factor for VERTEXCOVER is $2 - \Theta(1/\sqrt{\log n})$.

If $P \neq NP$, VERTEXCOVER cannot be approximated within a factor of 1.3606.

VERTEXCOVER cannot be approximated within a factor of $2 - \Theta(1)$ – if the *Unique Games Conjecture* holds.

Approximation Algorithms

Lecture 1:

Introduction and Vertex Cover

Part V:

An LP-based Algorithm for VERTEXCOVER

Task

Write an integer linear program (ILP) for VERTEXCOVER:

Using integer (and/or real) variables, express the problem using

- linear constraints and
- a linear objective function.

You can iterate over the vertices / edges of the given graph G .

Variables: for each vertex v of G , we introduce $x_v \in \{0, 1\}$.

Objective: minimize $\sum_{v \in V(G)} x_v$

v not in the solution
 v in the solution

Constraints: for each edge uv of G , we require that

$$x_u + x_v \geq 1.$$

Standard ILP Format

LP relaxation

$$\begin{array}{ll} \text{minimize} & \sum_{v \in V(G)} x_v \\ \text{subject to} & x_u + x_v \geq 1 \quad \text{for each } uv \in E(G) \\ & x_v \geq 0 \quad \text{for each } v \in V(G) \\ & \del{x_v \in \{0, 1\}} \end{array}$$

Problem: It's NP-hard to solve ILPs in general.

But: LPs can be solved efficiently (in $O(L \cdot n^{3.5})$ time),
where $n = \#$ variables and $L =$ total bit complexity of coefficients.

Problem': Now we can get fractional solutions, i.e., in $(0, 1)$.

Task: Find a graph G with $\text{OPT}_{\text{LP}} \neq \text{OPT}_{\text{ILP}}$!

Solution? Round the LP solution to get an integral solution!

Rounding the LP Solution

$$\begin{array}{ll} \text{minimize} & \sum_{v \in V(G)} x_v \\ \text{subject to} & x_u + x_v \geq 1 \quad \text{for each } uv \in E(G) \\ & x_v \geq 0 \quad \text{for each } v \in V(G) \end{array}$$

For each $v \in V(G)$: Set $x'_v = \begin{cases} 1 & \text{if } x_v \geq 0.5, \\ 0 & \text{otherwise.} \end{cases}$

Need to check: Is $(x'_v)_{v \in V(G)}$ a feasible solution?

In other words: Is $\{v \in V(G) : x'_v = 1\}$ a vertex cover of G ?

Need to make sure that every edge uv of G is covered.

Is $x'_u = 0 = x'_v$ possible? But then $x_u < 0.5$ and $x_v < 0.5$.

This contradicts $x_u + x_v \geq 1 \Rightarrow x'_u = 1$ or $x'_v = 1 \Rightarrow (x'_v)$
feasible!

Cost of the Solution

$$\begin{array}{ll} \text{minimize} & \sum_{v \in V(G)} x_v \\ \text{subject to} & x_u + x_v \geq 1 \quad \text{for each } uv \in E(G) \\ & x_v \geq 0 \quad \text{for each } v \in V(G) \end{array}$$

For each $v \in V(G)$: Set $x'_v = \begin{cases} 1 & \text{if } x_v \geq 0.5, \\ 0 & \text{otherwise.} \end{cases}$

$$\text{ALG} = \sum_{v \in V(G)} x'_v \leq 2 \cdot \sum_{v \in V(G)} x_v = 2 \cdot \text{OPT}_{\text{LP}} \leq 2 \cdot \text{OPT}_{\text{ILP}}$$

Theorem. The LP rounding algorithm is a factor-2 approximation algorithm for **VERTEXCOVER**.

Cost of the Solution

$$\begin{array}{ll} \text{minimize} & \sum_{v \in V(G)} x_v \cdot w(v) \\ \text{subject to} & x_u + x_v \geq 1 \quad \text{for each } uv \in E(G) \\ & x_v \geq 0 \quad \text{for each } v \in V(G) \end{array}$$

For each $v \in V(G)$: Set $x'_v = \begin{cases} 1 & \text{if } x_v \geq 0.5, \\ 0 & \text{otherwise.} \end{cases}$

$$\text{ALG} = \sum_{v \in V(G)} x'_v \cdot w(v) \leq 2 \cdot \sum_{v \in V(G)} x_v \cdot w(v) = 2 \cdot \text{OPT}_{\text{LP}} \leq 2 \cdot \text{OPT}_{\text{ILP}}$$

Theorem. The LP rounding algorithm is a factor-2 approximation algorithm for **WEIGHTED VERTEX COVER**.