Approximation Algorithms

Lecture 1: Introduction and Vertex Cover

Part I: Organizational

Alexander Wolff

Winter 2023/24

Organizational

Lectures: on site (English/German, depending on audience)

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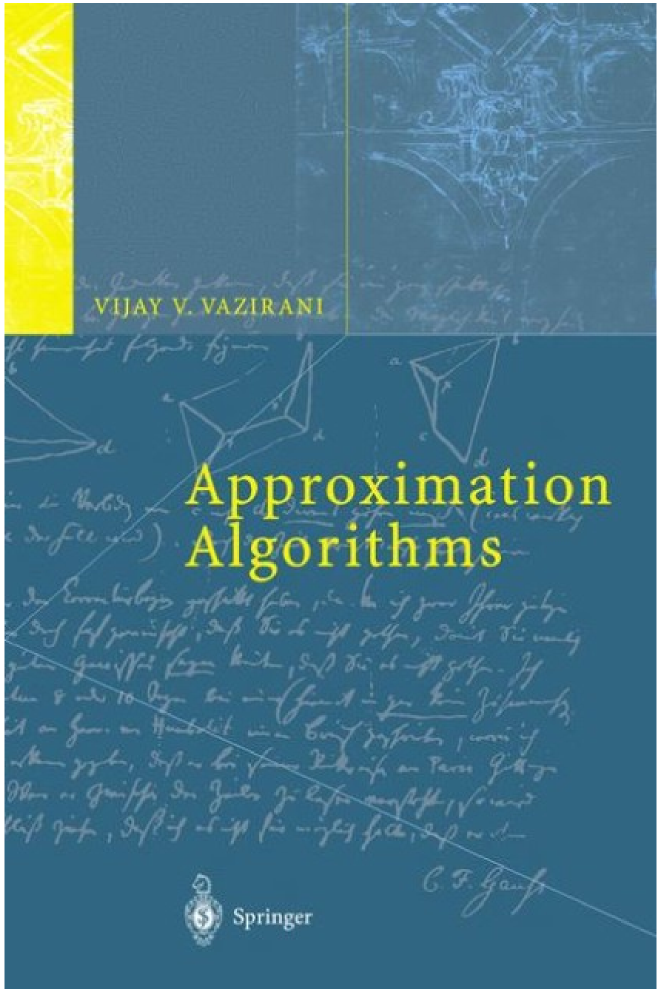
Most slides are due to Joachim Spoerhase,
polishing & colors are due to Philipp Kindermann – thanks!

Textbooks



Vijay V. Vazirani:
Approximation Algorithms
Springer-Verlag, 2003.

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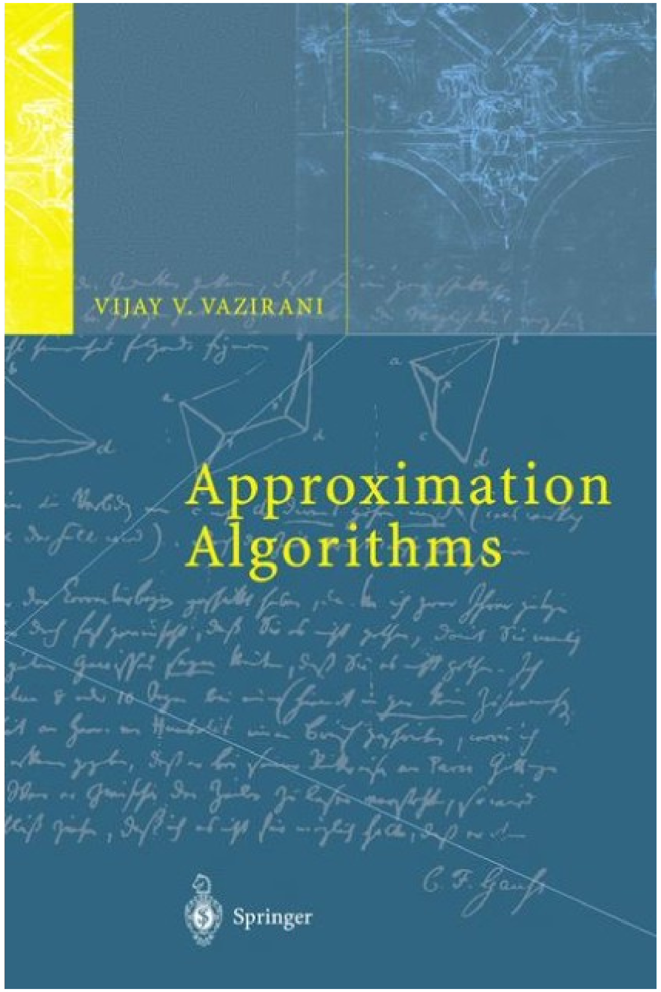

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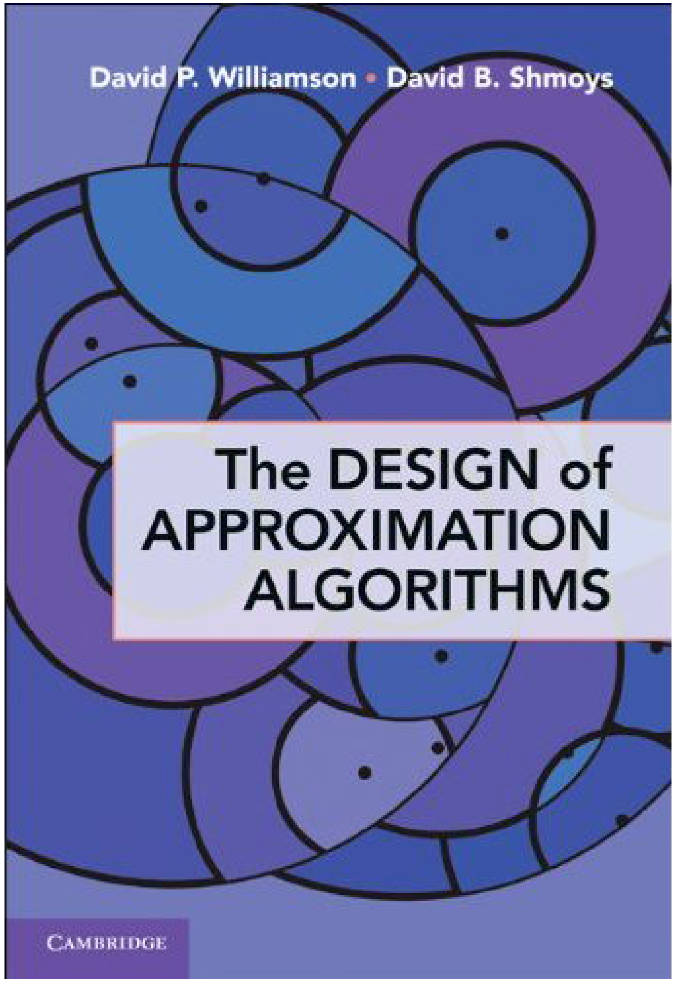

D. P. Williamson & D. B. Shmoys:
The Design of Approximation Algorithms
Cambridge-Verlag, 2011.

http://www.designofapproxalgs.com/

Approximation Algorithms

„All exact science is dominated by the idea of approximation.“

– Bertrand Russell (1872 – 1970)

Approximation Algorithms

- Many optimization problems are NP-hard!
  (For example, the traveling salesperson problem.)

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- Many optimization problems are NP-hard!
  (For example, the traveling salesperson problem.)
- $\leadsto$ an optimal solution cannot be efficiently computed unless P=NP.
- However, good approximate solutions can often be found efficiently!
- **Techniques** for the design and analysis of approximation algorithms arise from studying specific optimization problems.

Overview

Combinatorial algorithms

- Introduction (Vertex Cover)
- Set Cover via Greedy
- Shortest Superstring
  via reduction to SC
- Steiner Tree via MST
- Multiway Cut via Greedy
- k-Center via Parametrized Pruning
- Min-Degree Spanning Tree
  and local search
- Knapsack via DP and Scaling
- Euclidean TSP via Quadtrees

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LP-based algorithms

- introduction to LP-Duality
- Set Cover via LP Rounding
- Set Cover via Primal–Dual Schema
- Maximum Satisfiability
- Scheduling und Extreme Point Solutions
- Steiner Forest via Primal–Dual

Approximation Algorithms

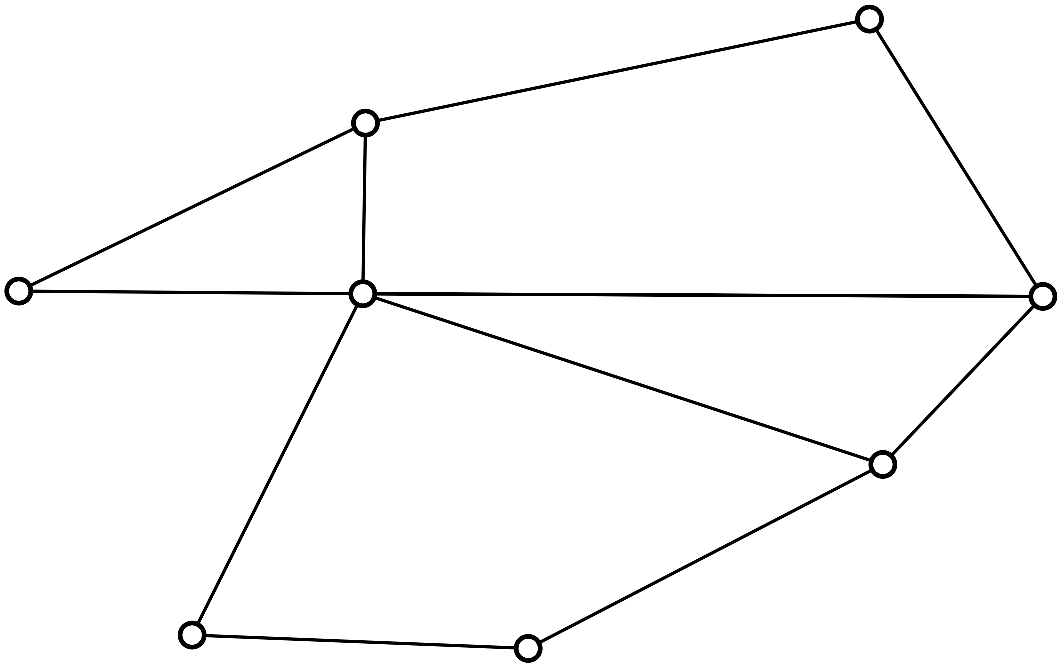
Lecture 1:
Introduction and Vertex Cover

Part II:
(Cardinality) Vertex Cover

VertexCover (card.)

Input: Graph $G = (V, E)$

Output:



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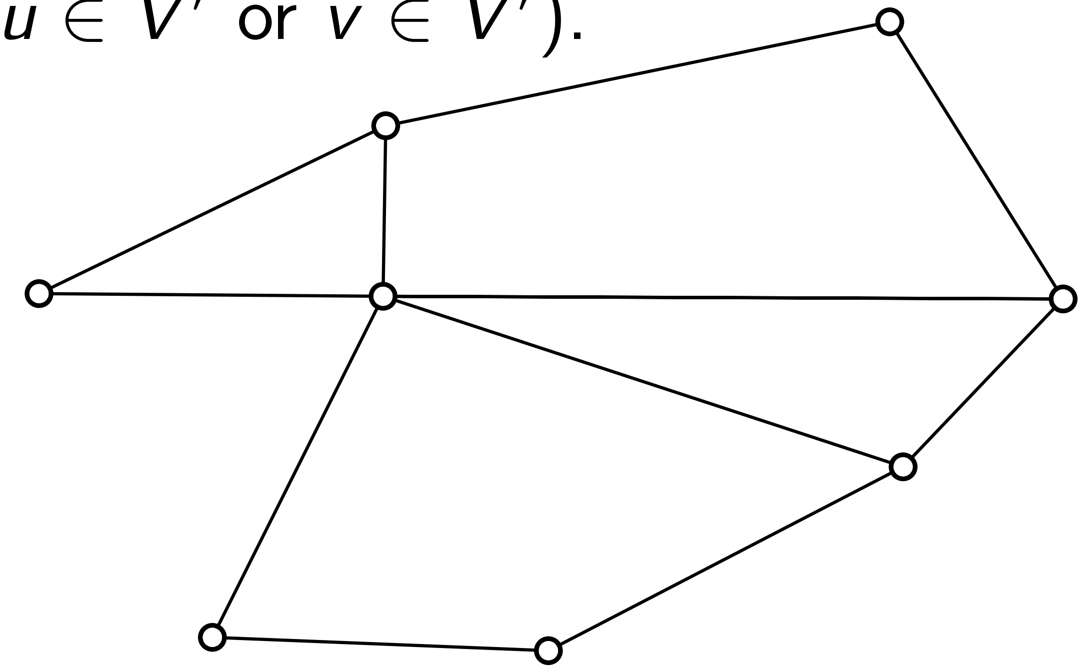

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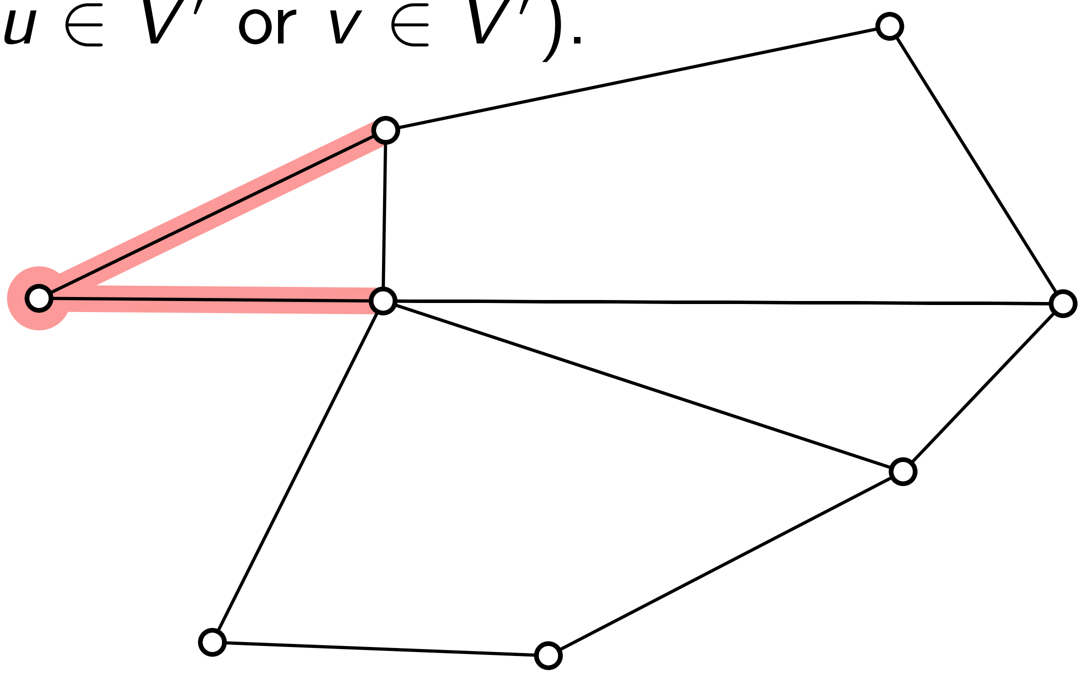

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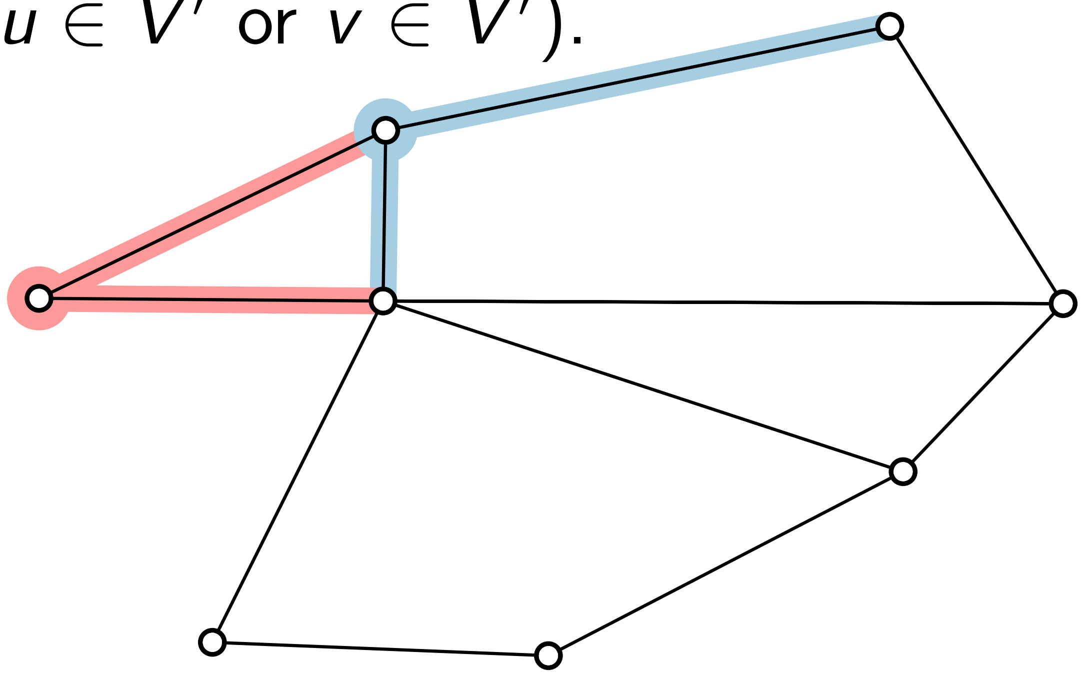

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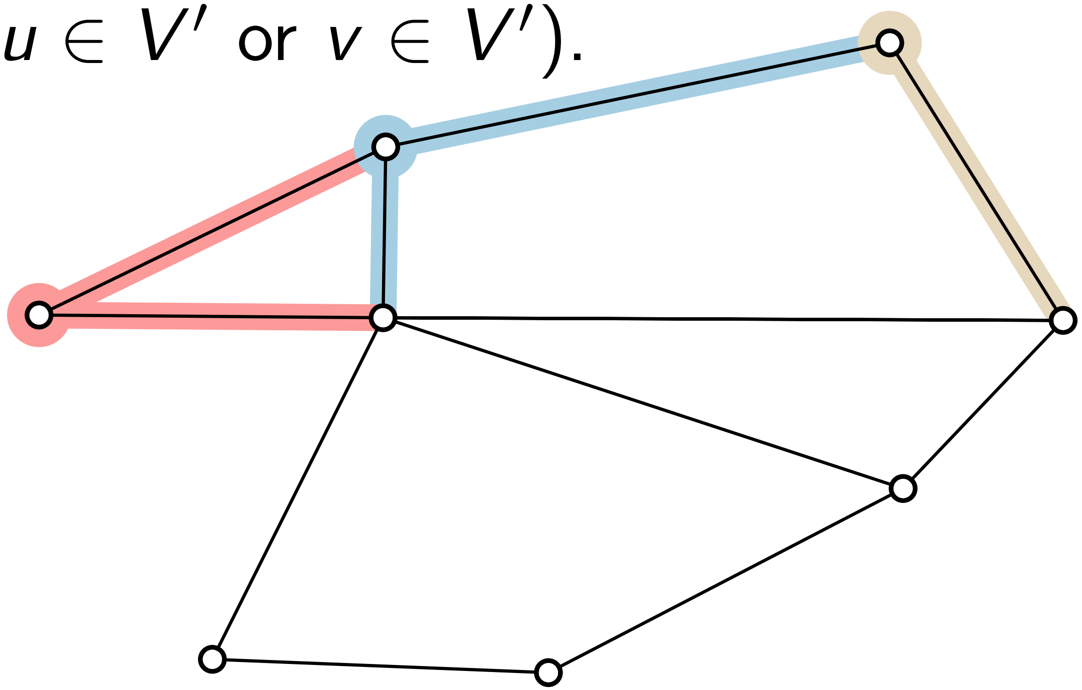

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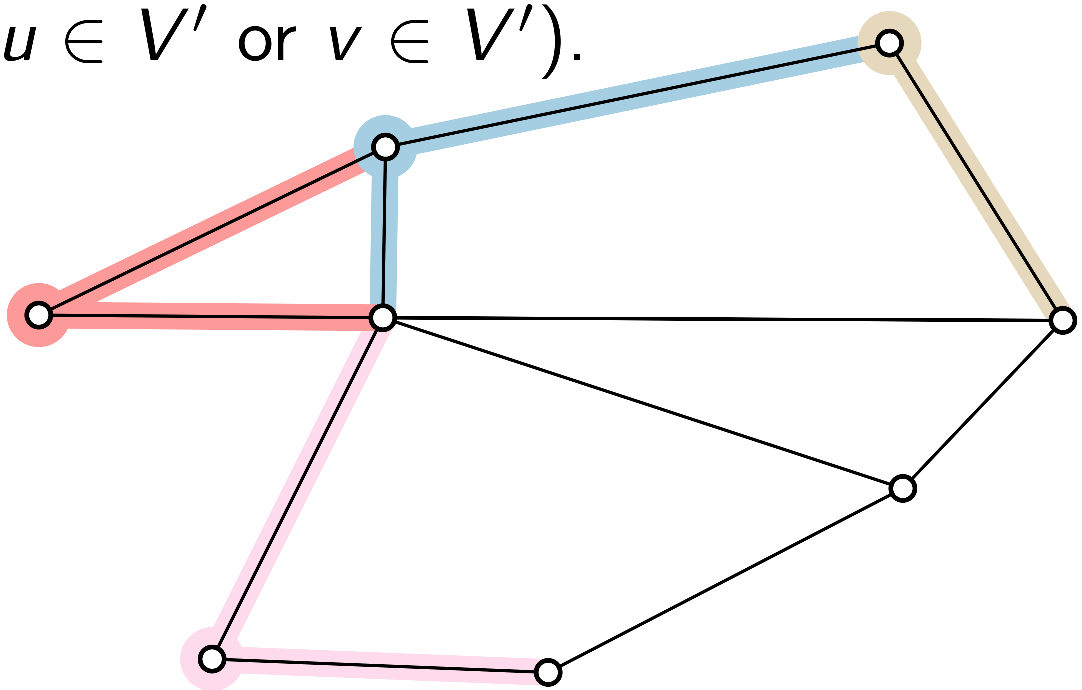

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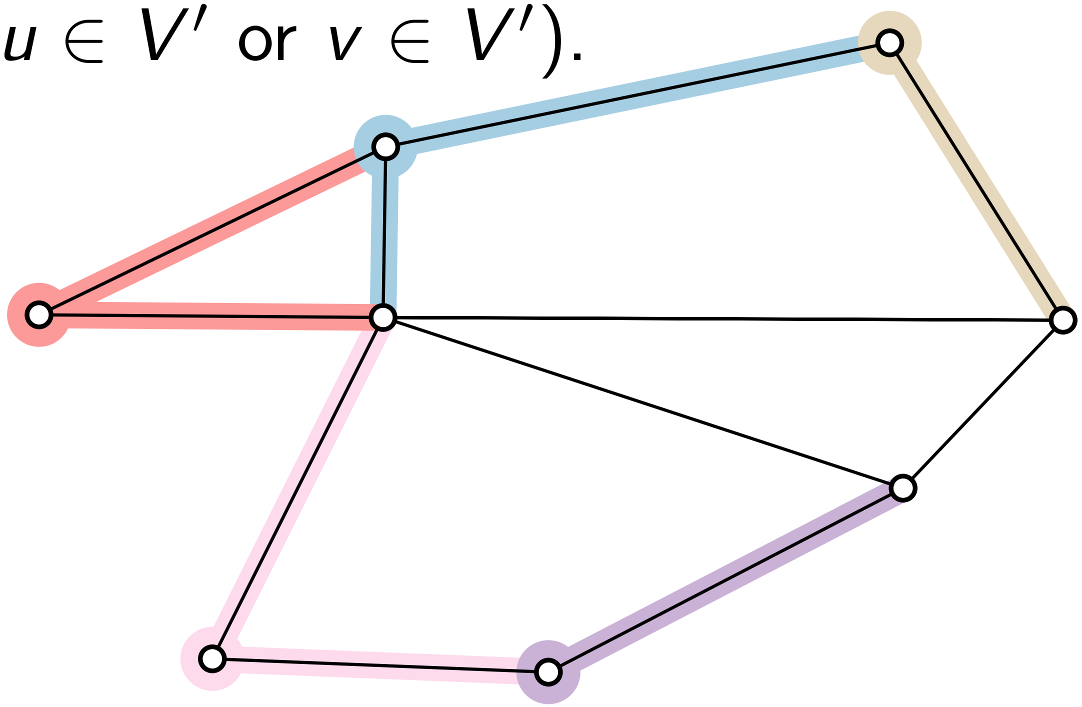

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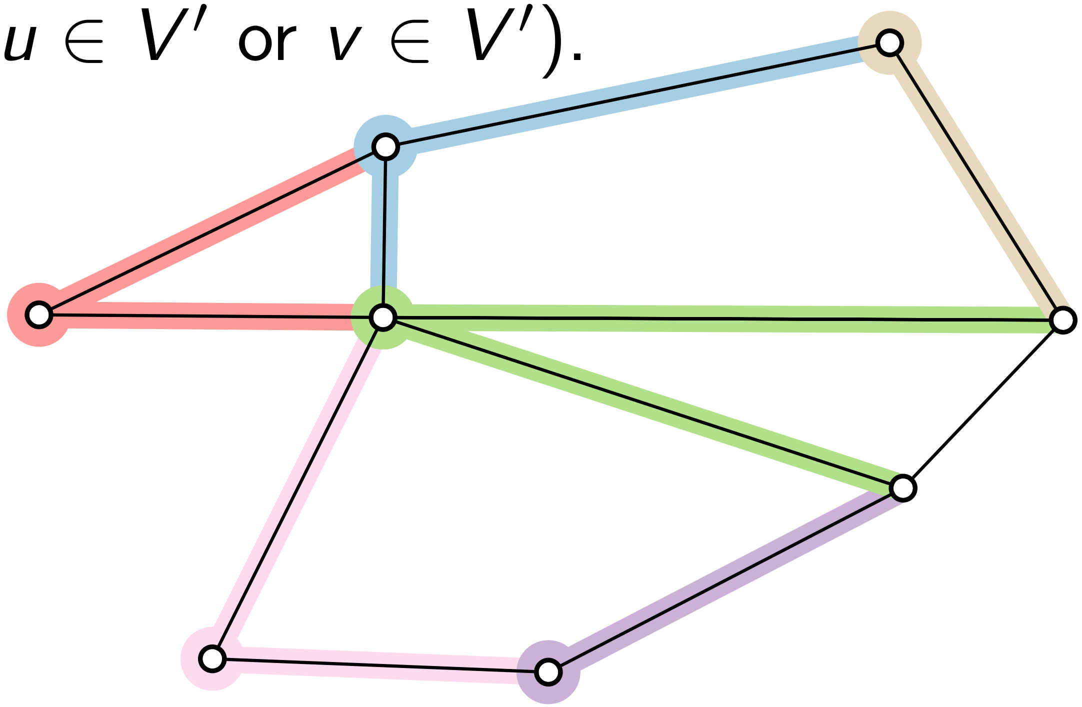

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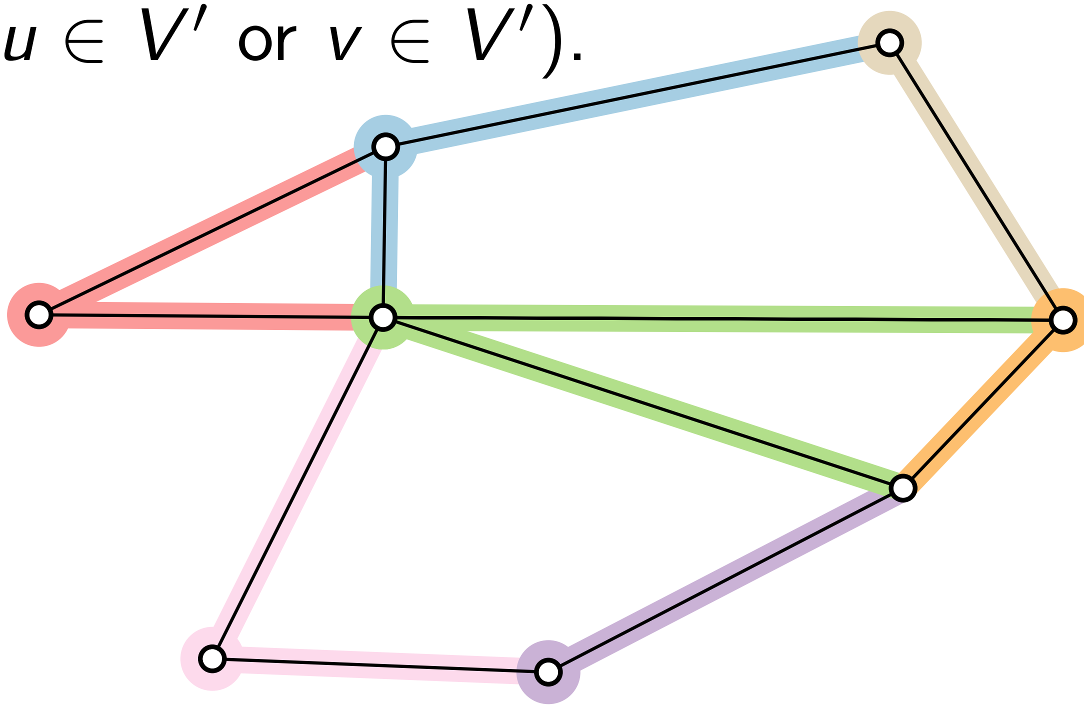

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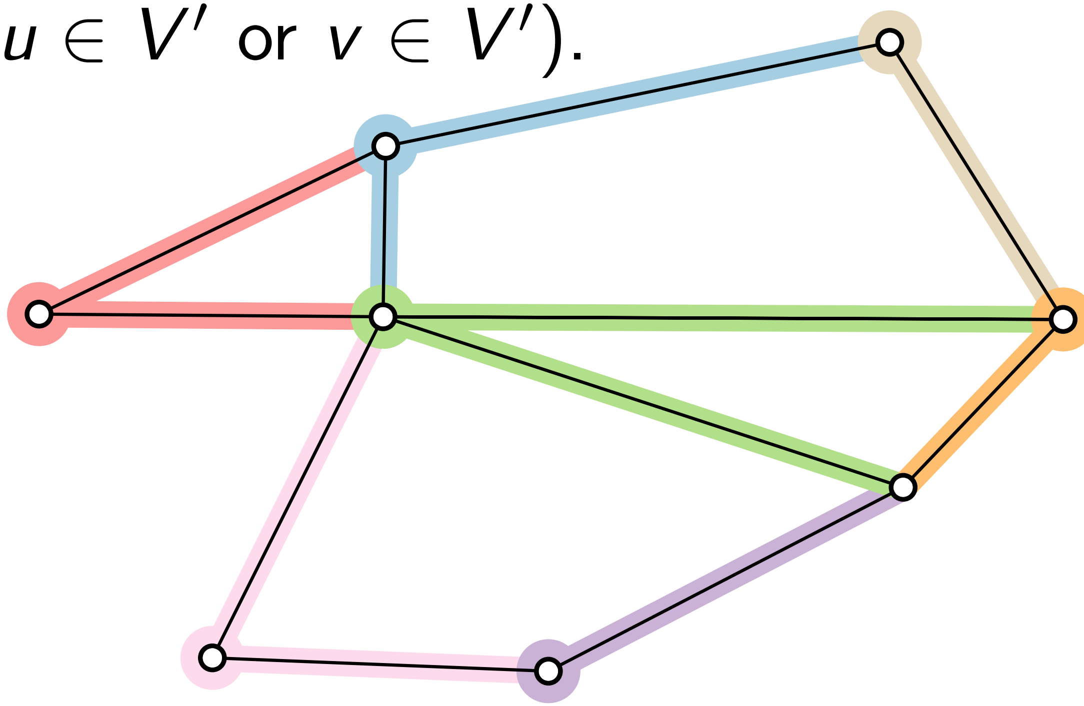

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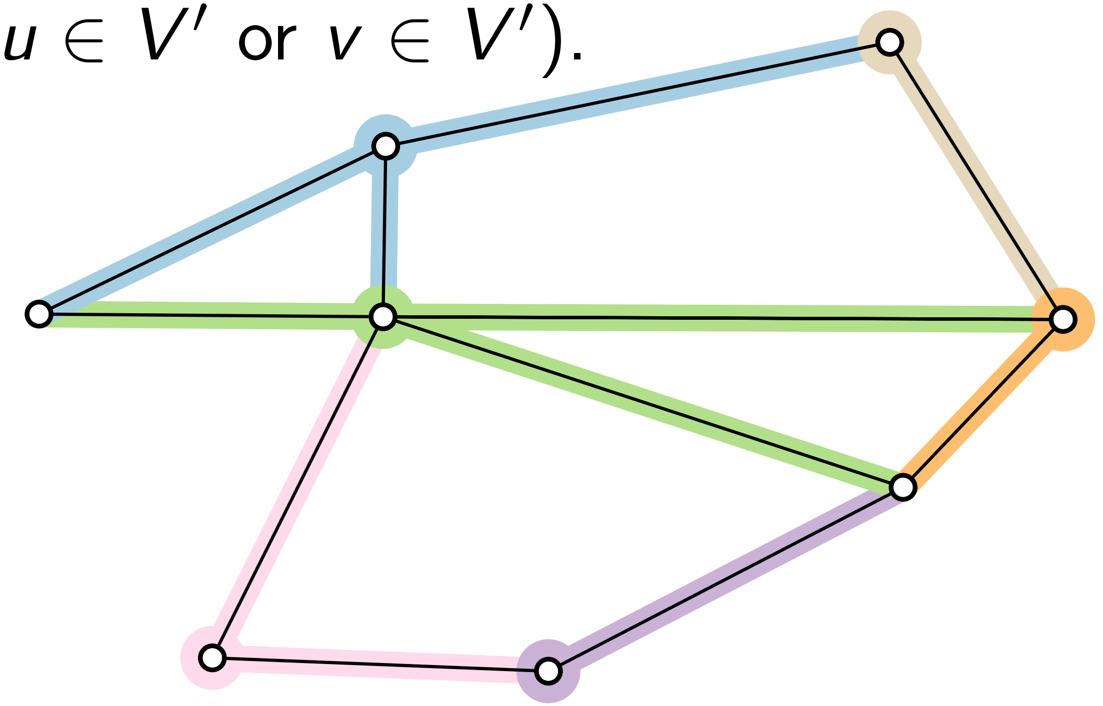

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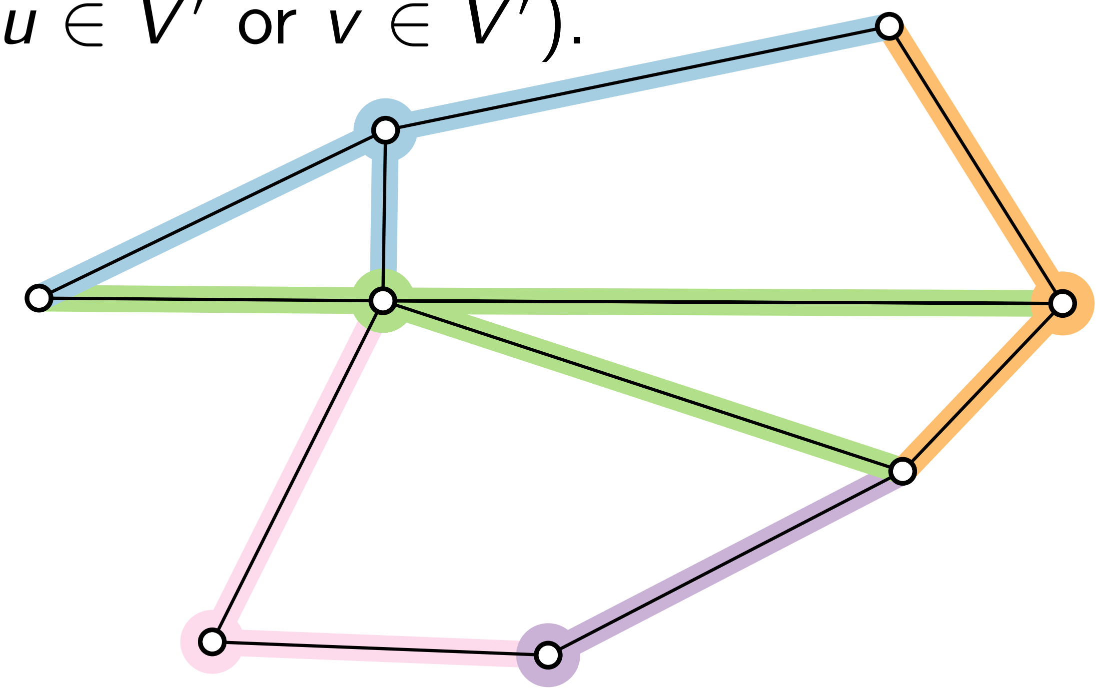

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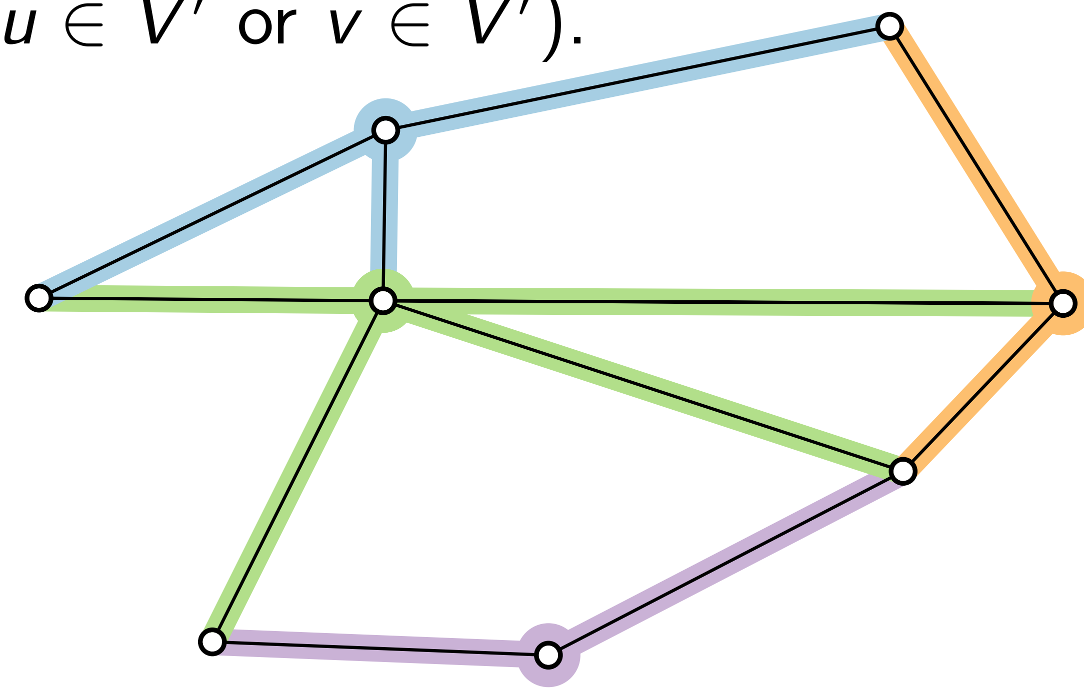

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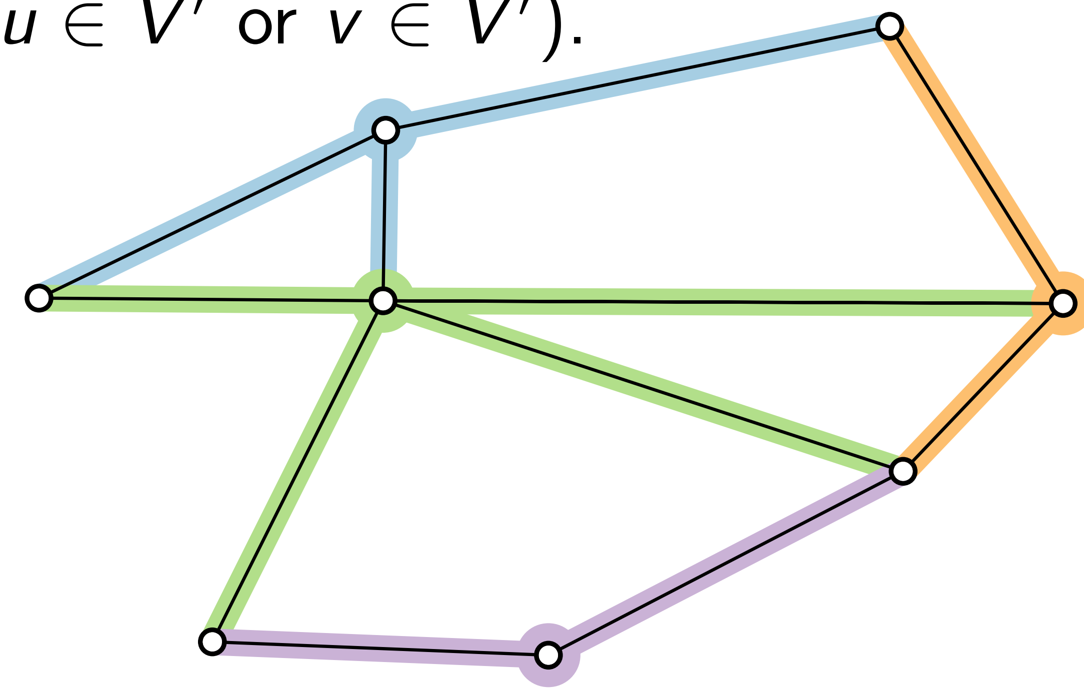

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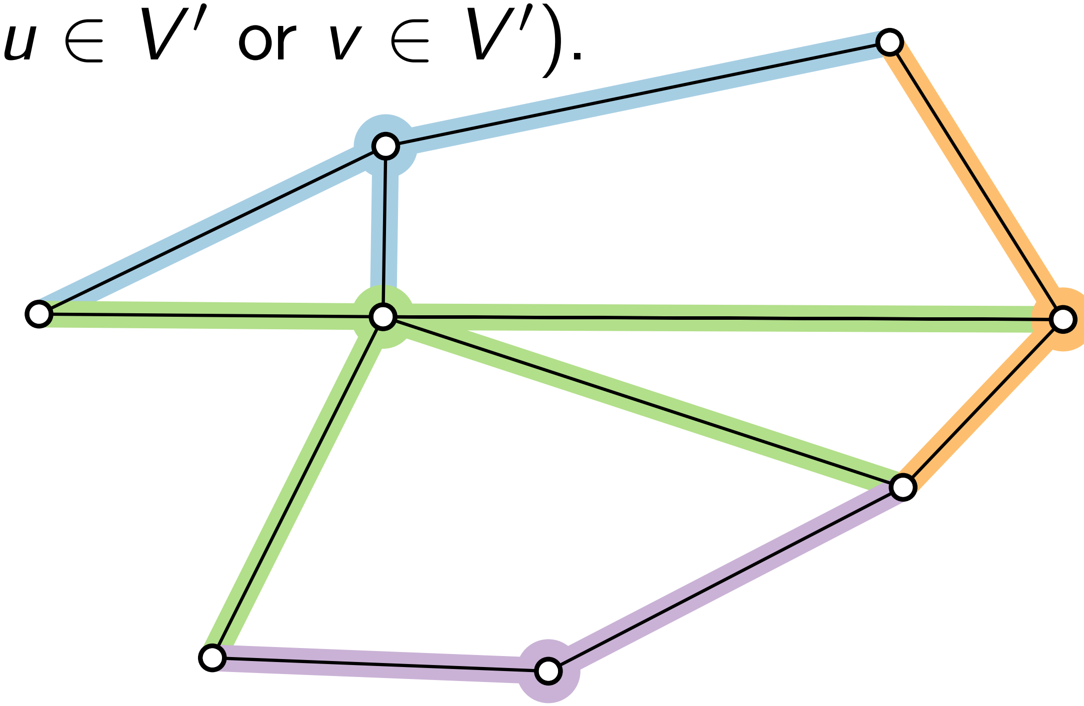

Optimum (OPT $= 4$) – but in general NP-hard to find :-(

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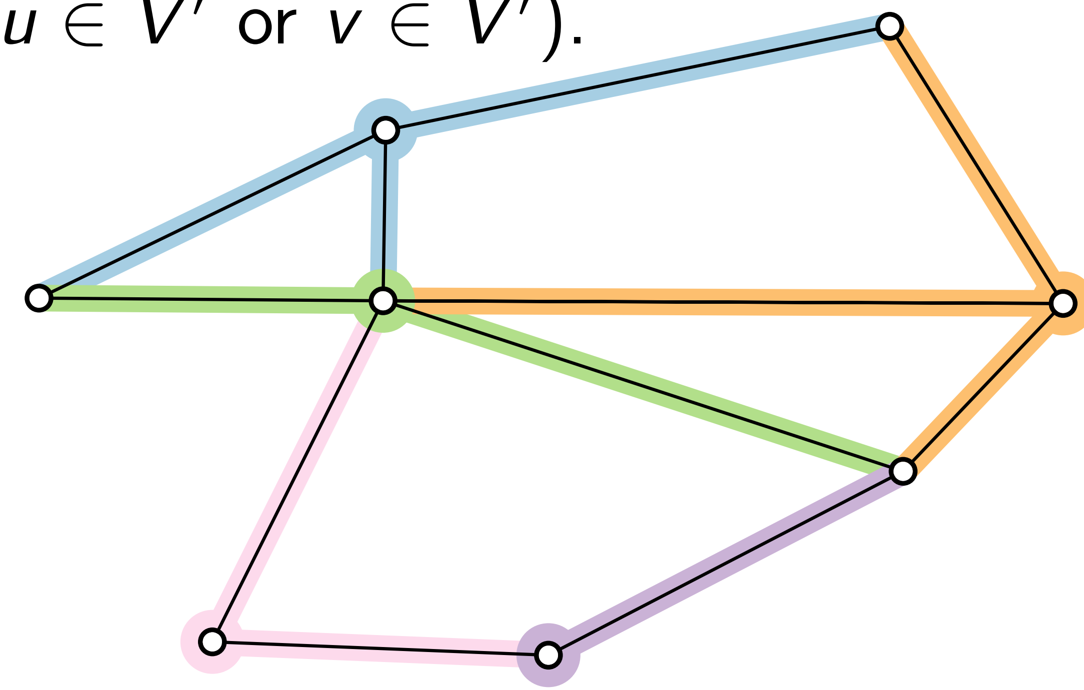

"good" (5/4-) approximate solution

Approximation Algorithms

Lecture 1:
Introduction and Vertex Cover

Part III:
NP-Optimization Problem

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  which assigns a positive objective value $\text{obj}_\Pi(I, s) \geq 0$ to
  any given pair (s, I) with $s \in S_\Pi(I)$.
- Π is either a minimization or maximization problem.

VertexCover: NP-Optimization Problem

Task: Fill in the gaps for $\Pi = \textsc{Vertex Cover}$.

$D_\Pi =$

For $I \in D_\Pi$: $|I| =$

$S_\Pi(I) =$

- Why is $|s| \in \text{poly}(|I|)$ for every $s \in S_\Pi(I)$?

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Π is a minimization problem.

Optimum and Optimal Objective Value

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A feasible solution $s^* \in S_\Pi(I)$ is **optimal** if $\mathrm{obj}_\Pi(I, s^*)$ is minimal among the objective values attained by the feasible solutions of I.

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The optimal value $\mathrm{obj}_\Pi(I, s^*)$ of the objective function is denoted by $\mathrm{OPT}_\Pi(I)$ or simply by OPT in context.

Approximation Algorithms

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$$\frac{\mathrm{obj}_\Pi(I, s)}{\mathrm{OPT}_\Pi(I)} \;\overset{\geq}{\leq}\; \cancel{\alpha}\,\alpha(|I|).$$

Approximation Algorithms

Lecture 1:
Introduction and Vertex Cover

Part IV:
Approximation Algorithm for VertexCover

Approximation Alg. for VertexCover

Ideas?

Approximation Alg. for VertexCover

Ideas?

- Edge-Greedy

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- Vertex-Greedy

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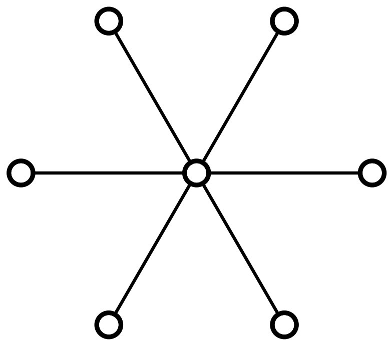

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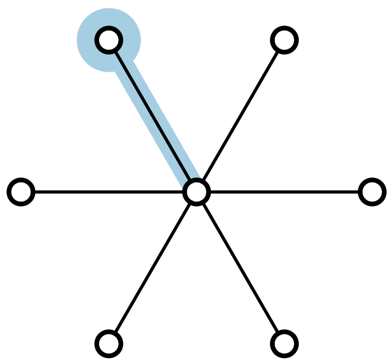

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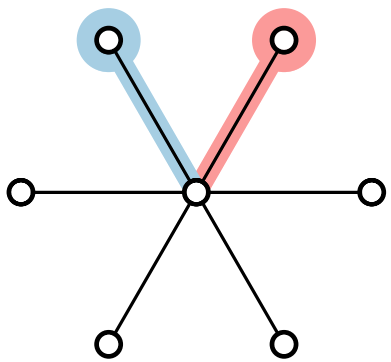

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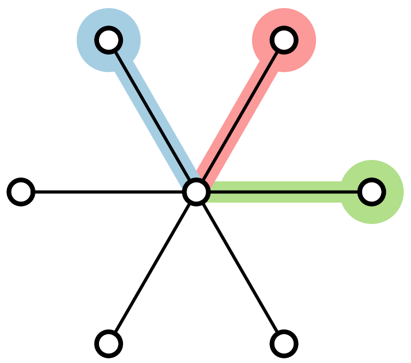

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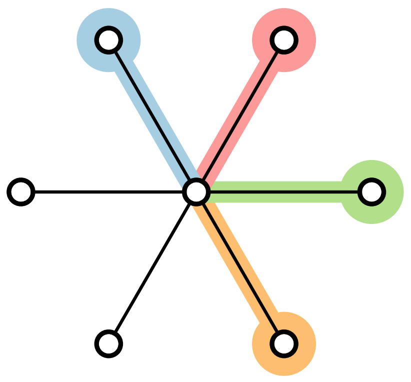

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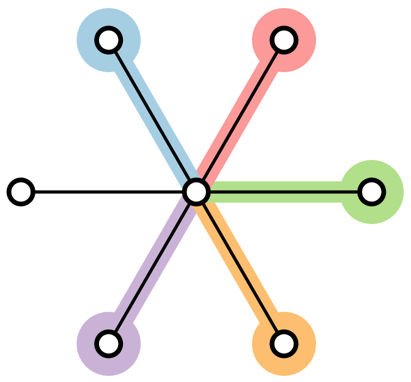

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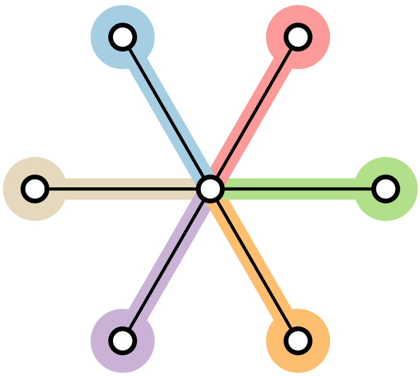

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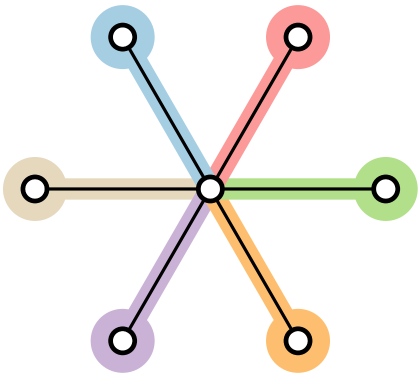

Quality?

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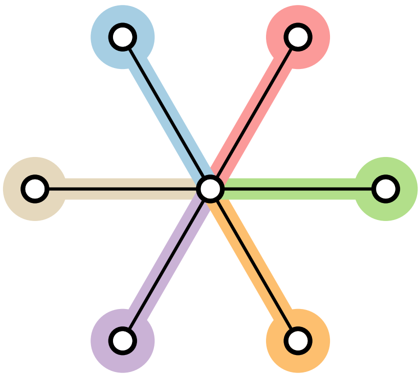

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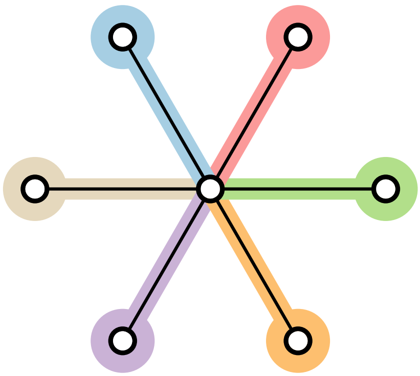

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Idea: Find a "good" lower bound $L \leq \text{OPT}$ for OPT and compare it to our approximate solution.

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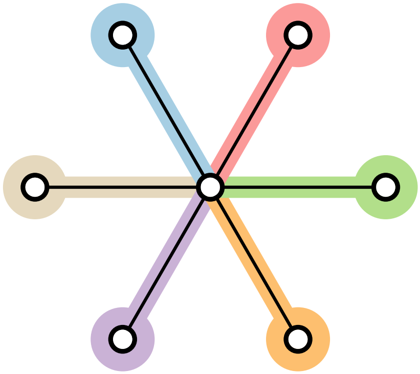

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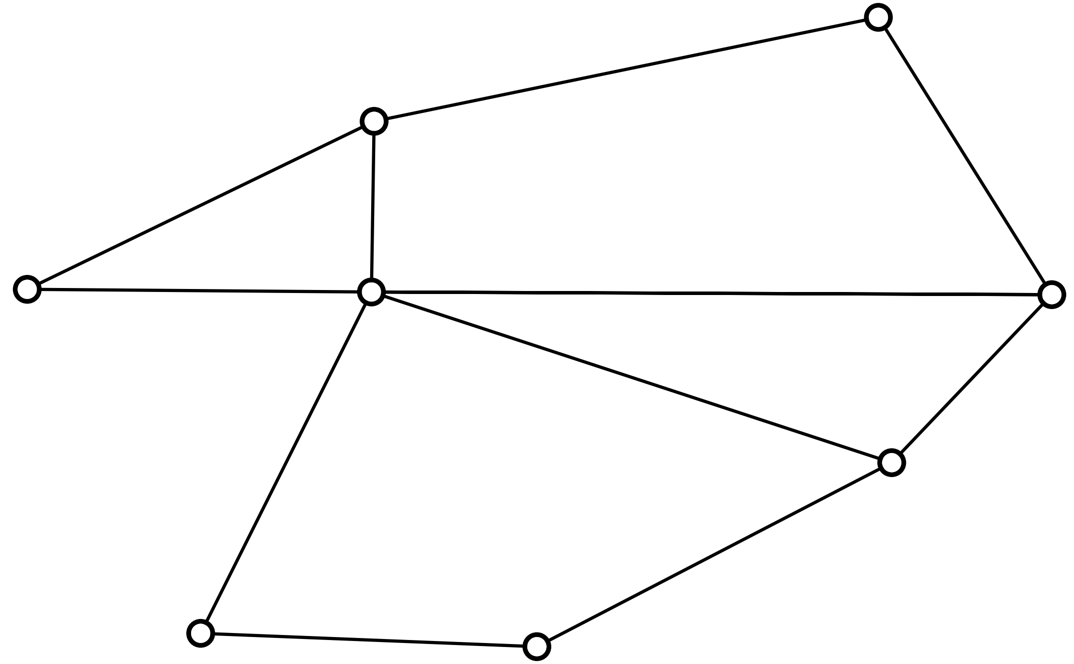
$$\frac{\text{obj}_\Pi(I, s)}{\text{OPT}} \leq \frac{\text{obj}_\Pi(I, s)}{L}$$

Lower Bound by Matchings



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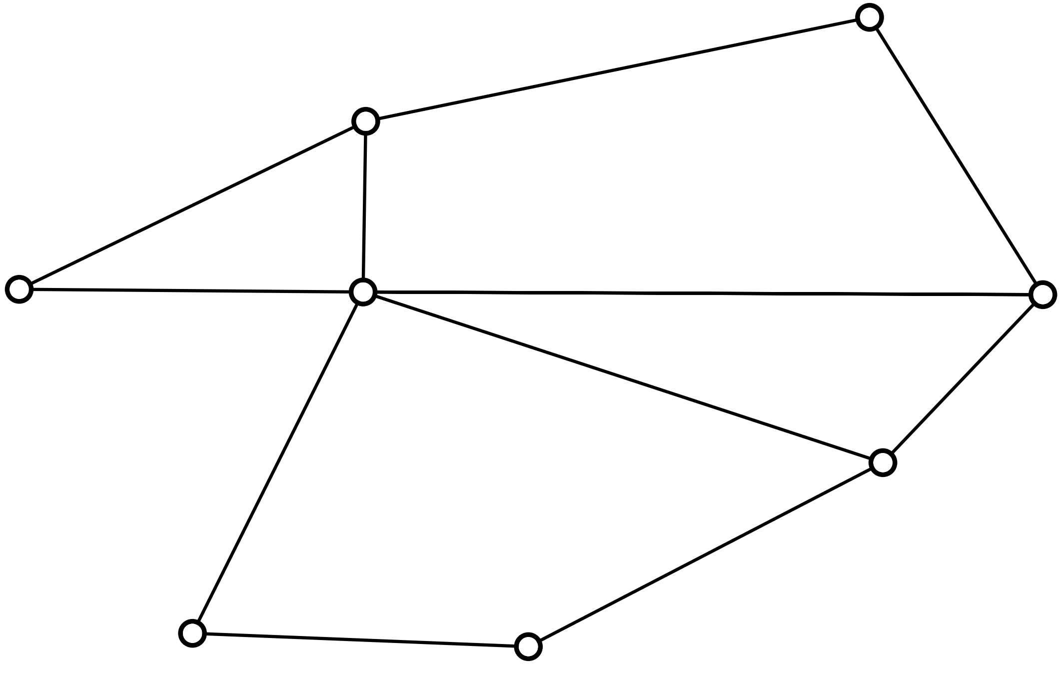

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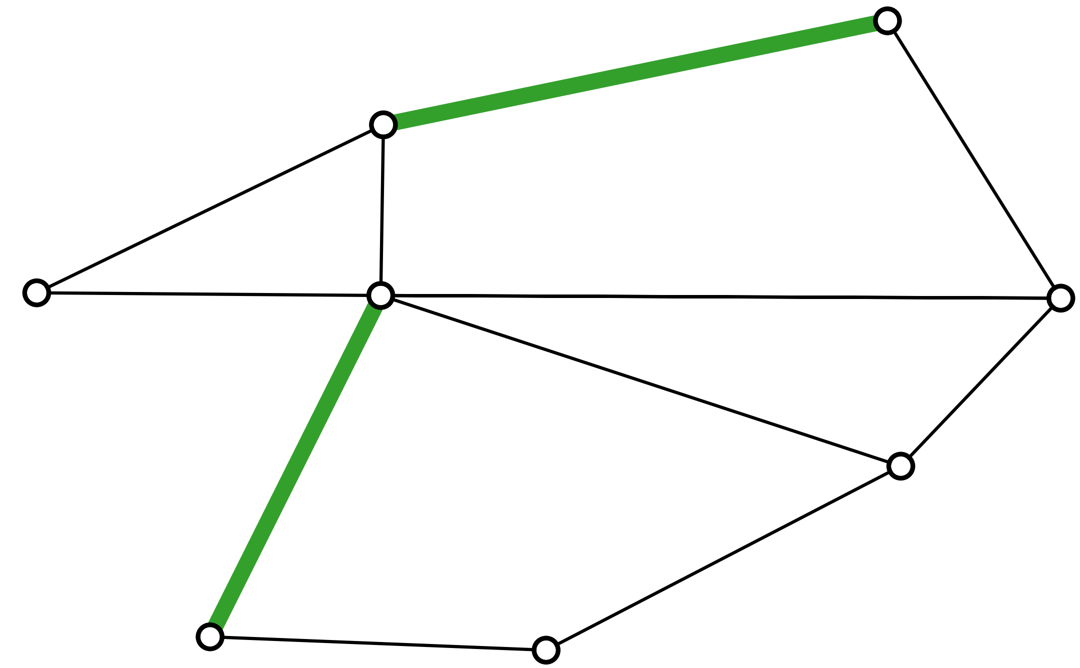

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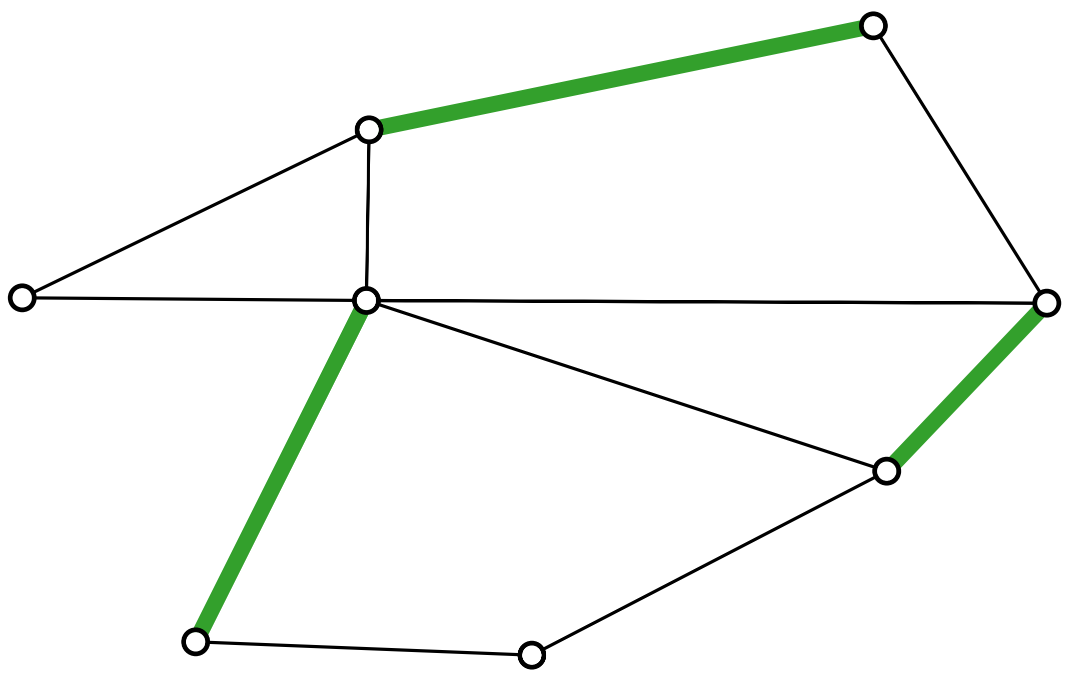

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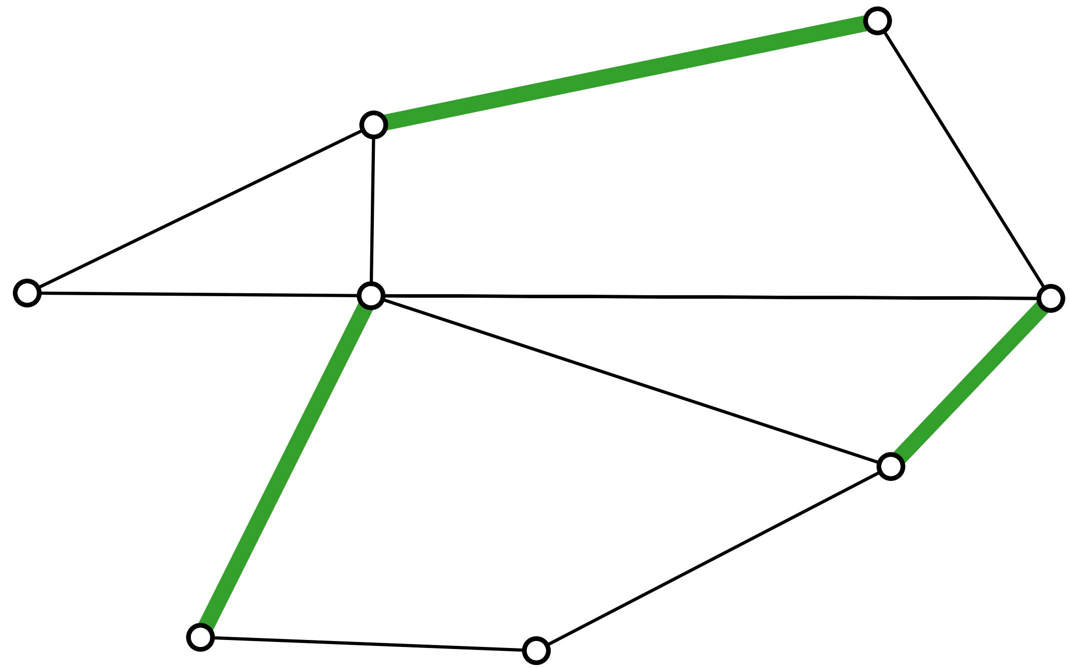

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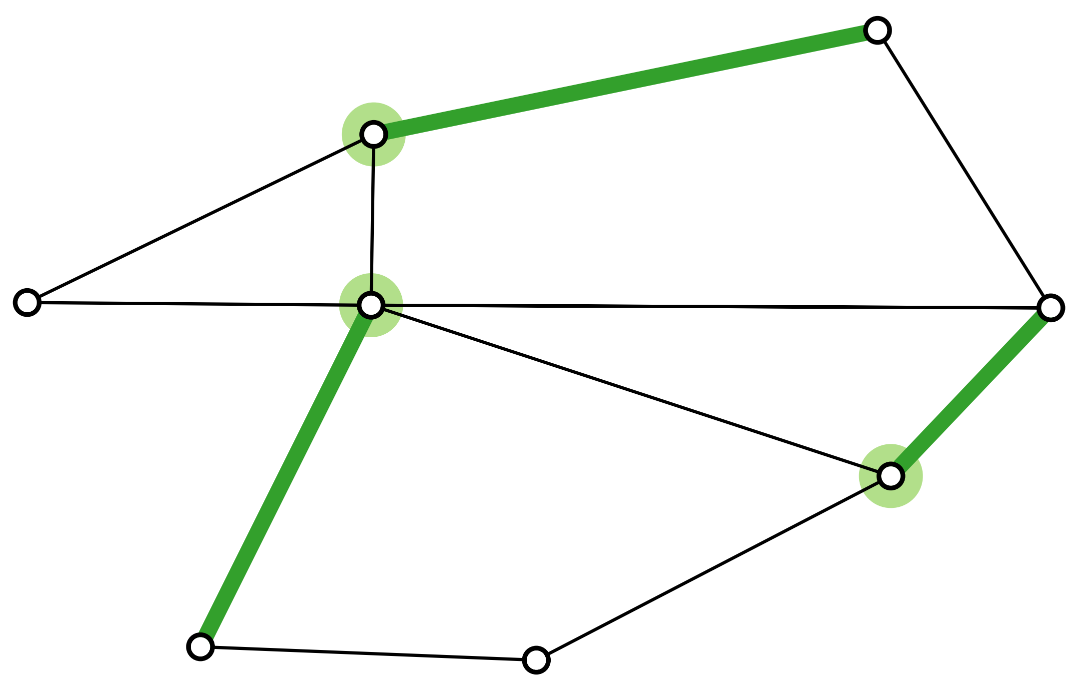
Vertex cover of M



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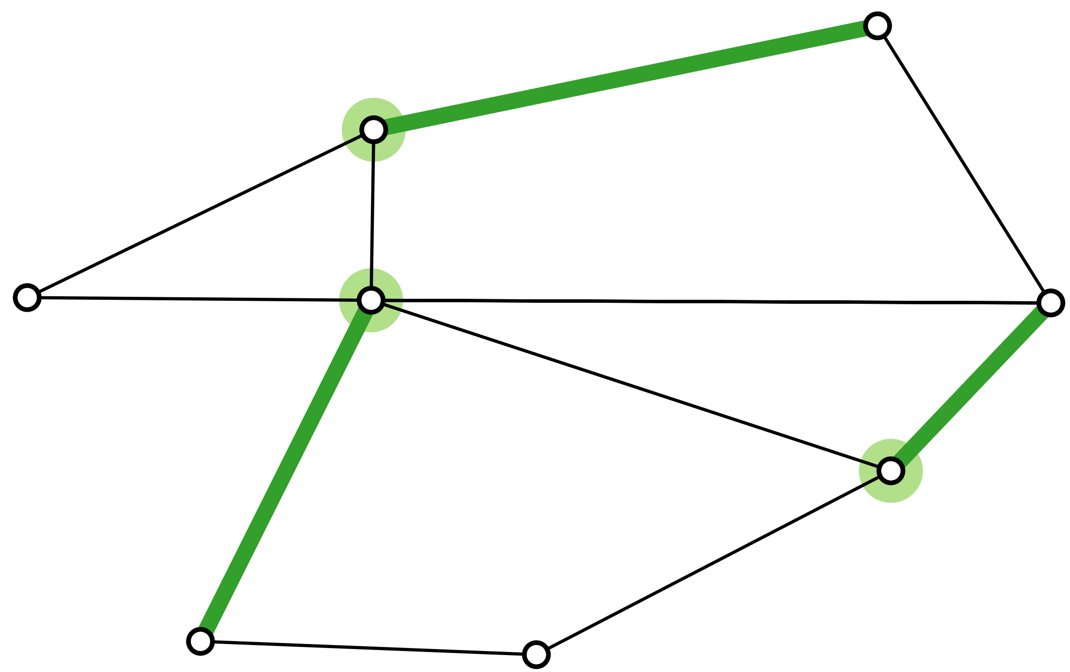

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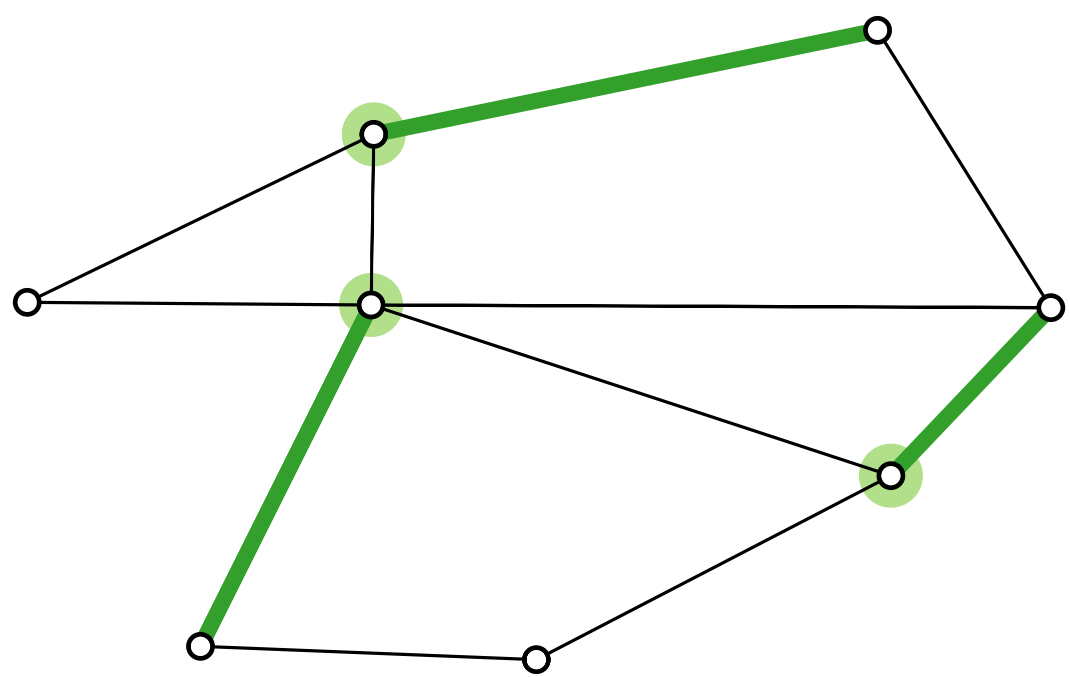

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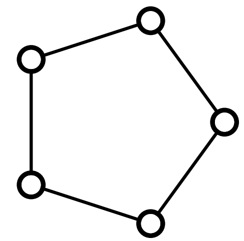

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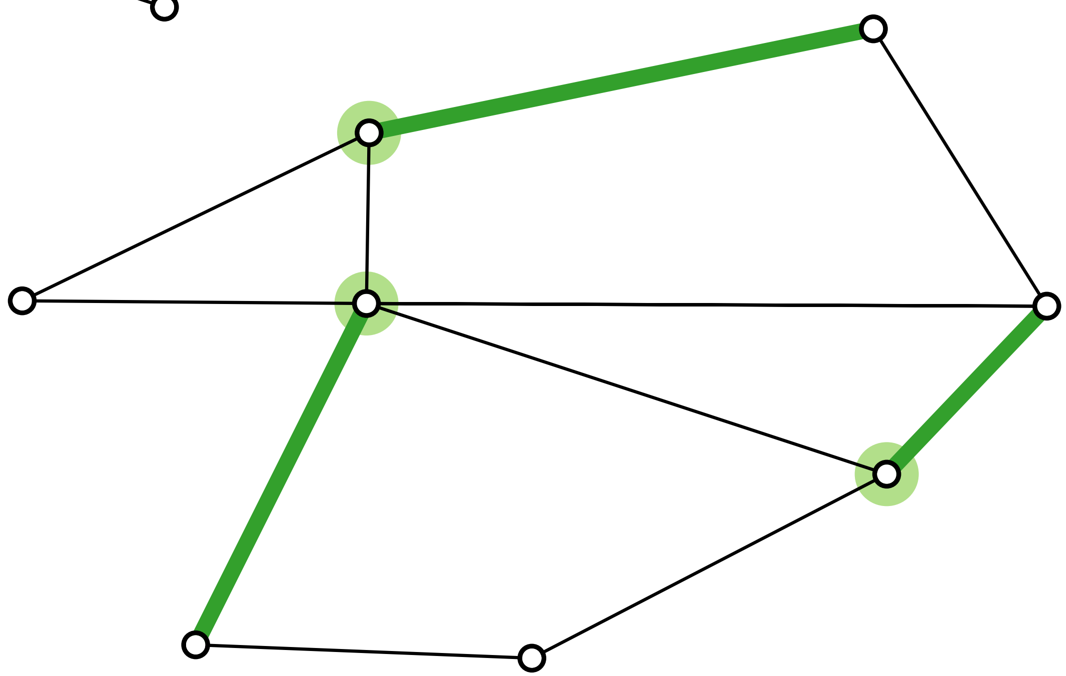

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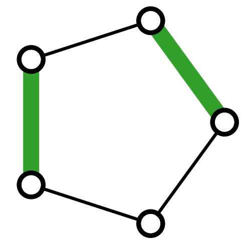

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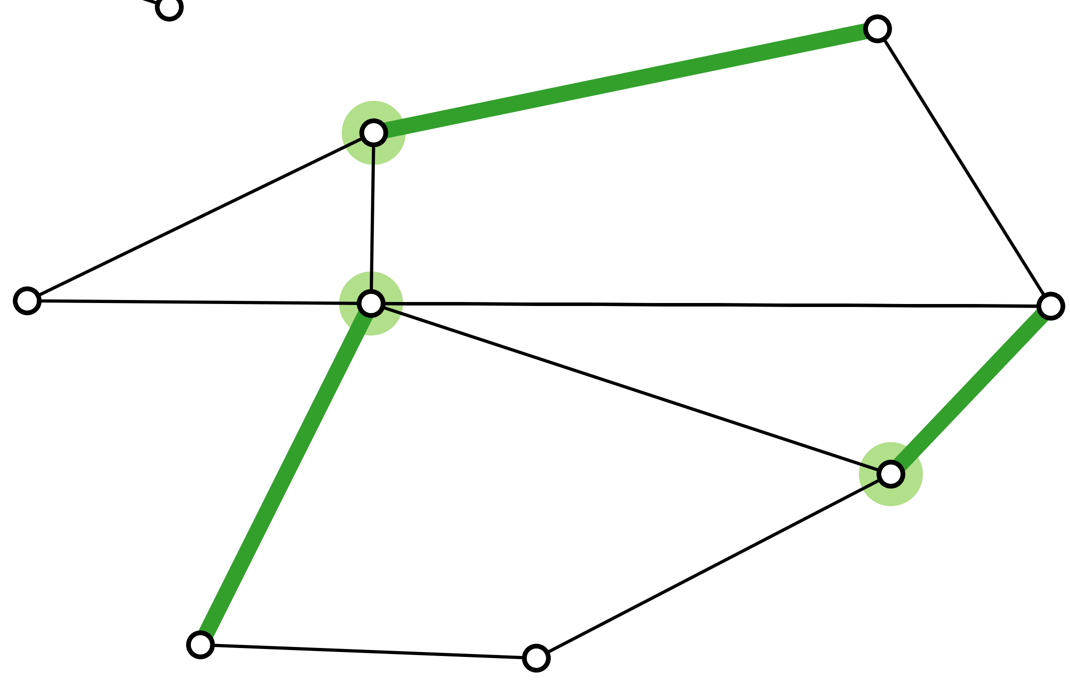

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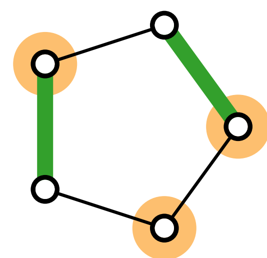

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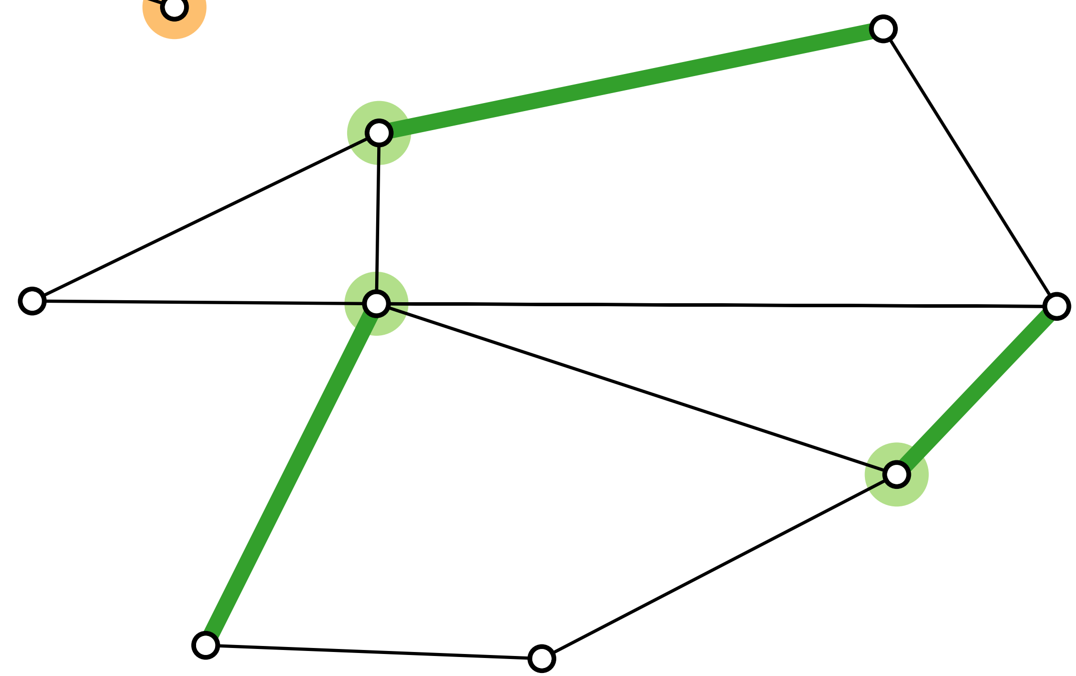

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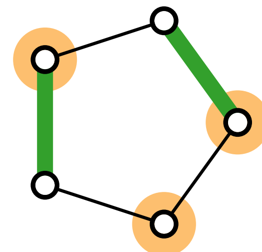

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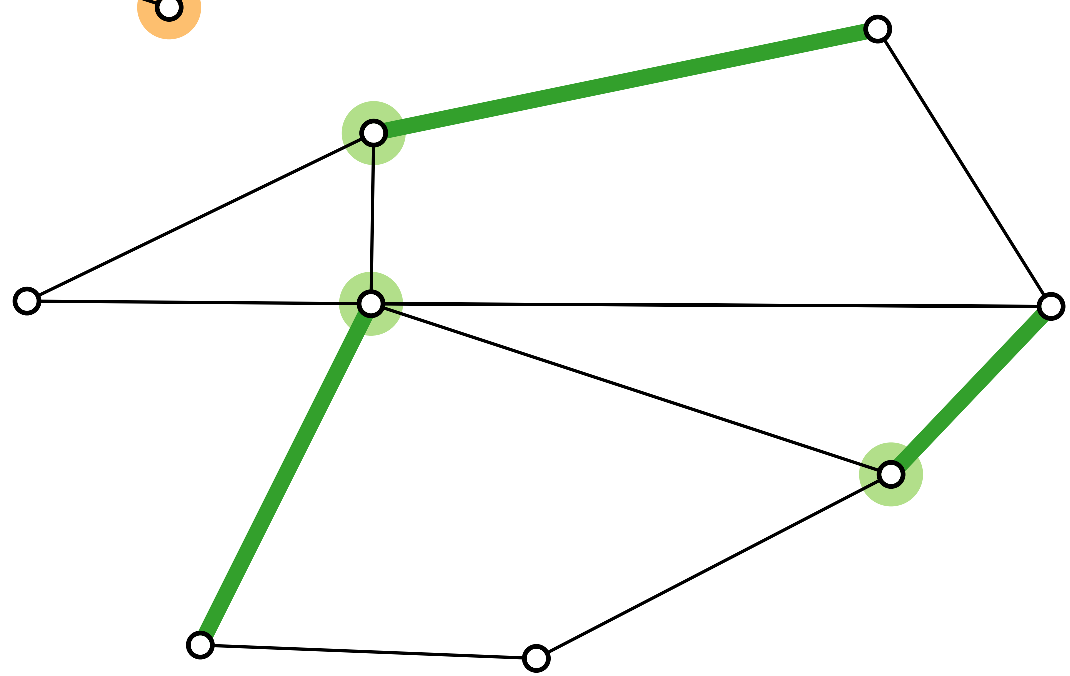

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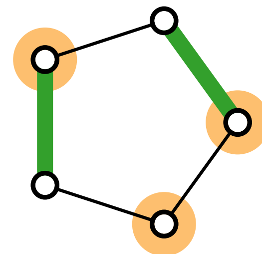

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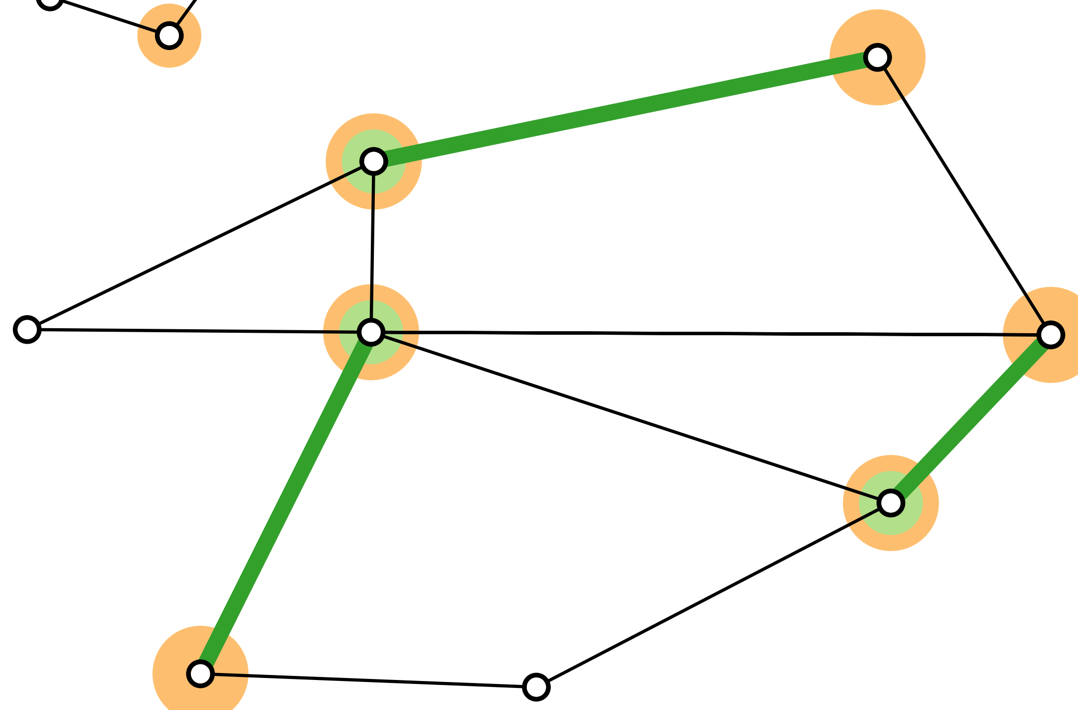
Vertex cover of M

Vertex cover of E



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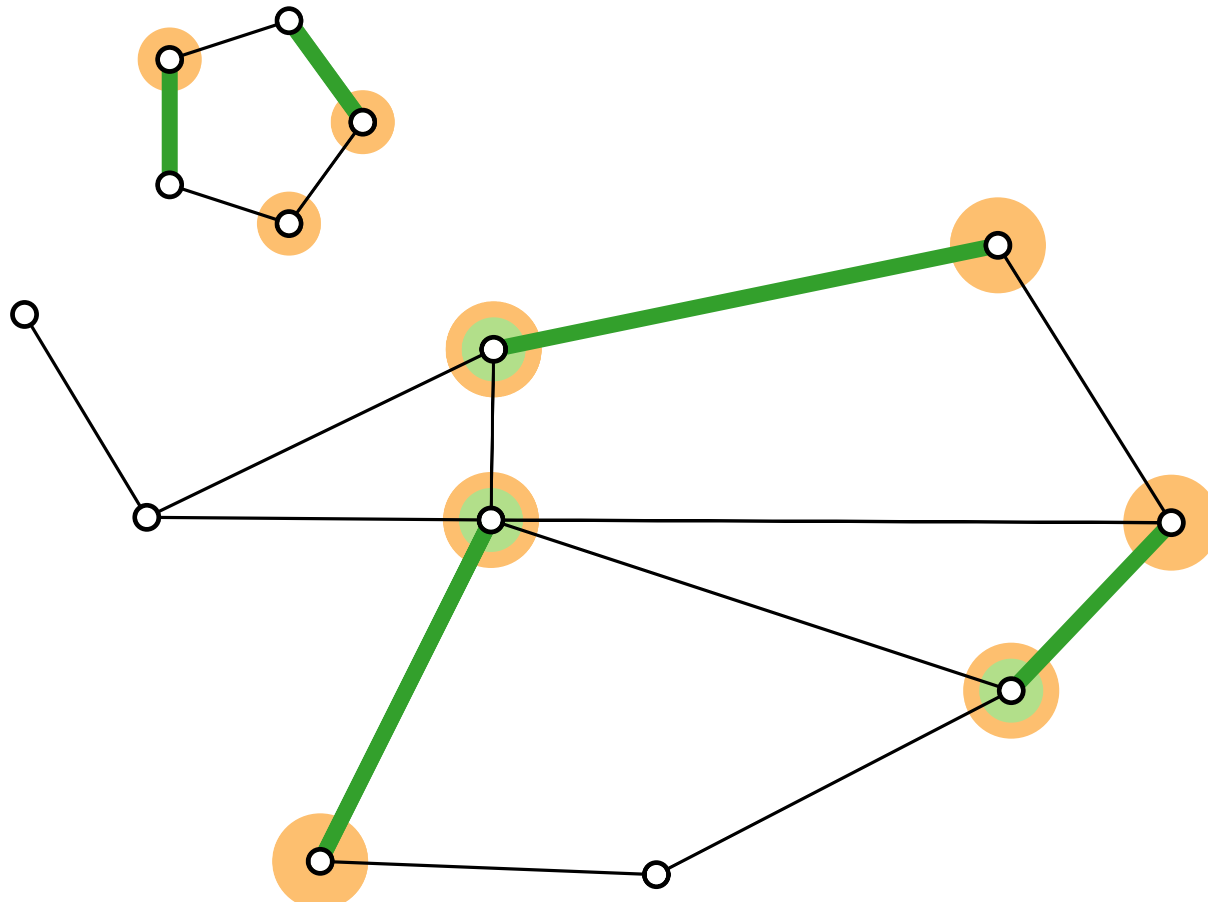

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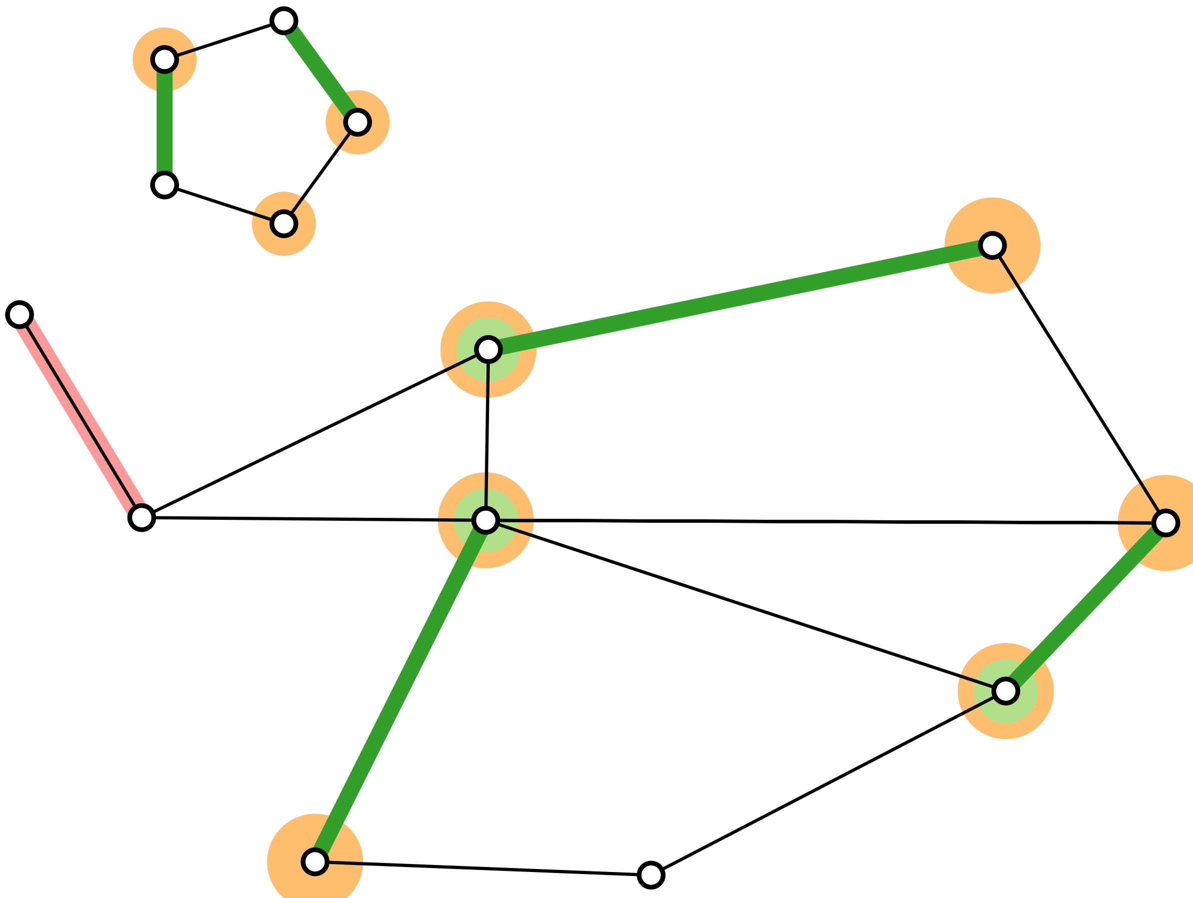

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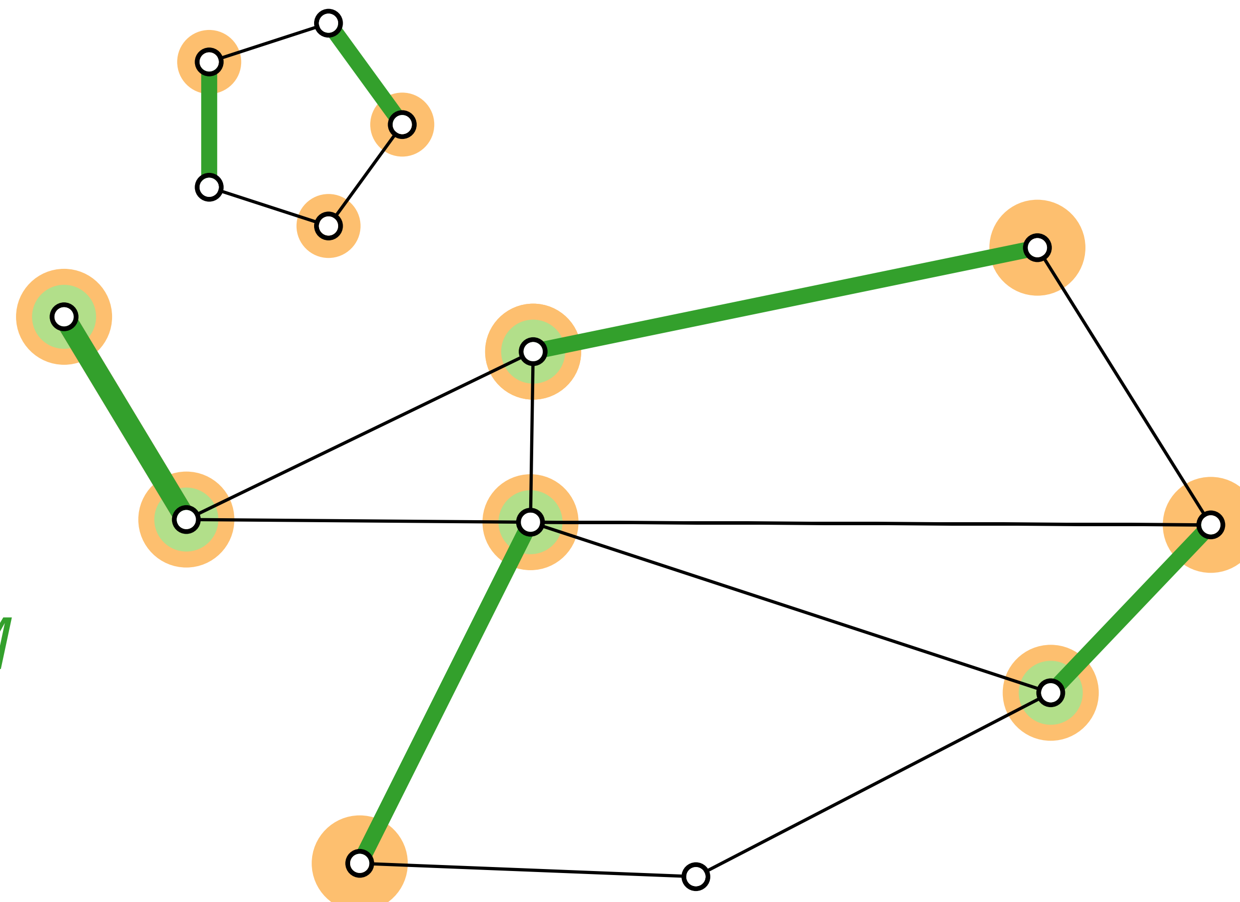

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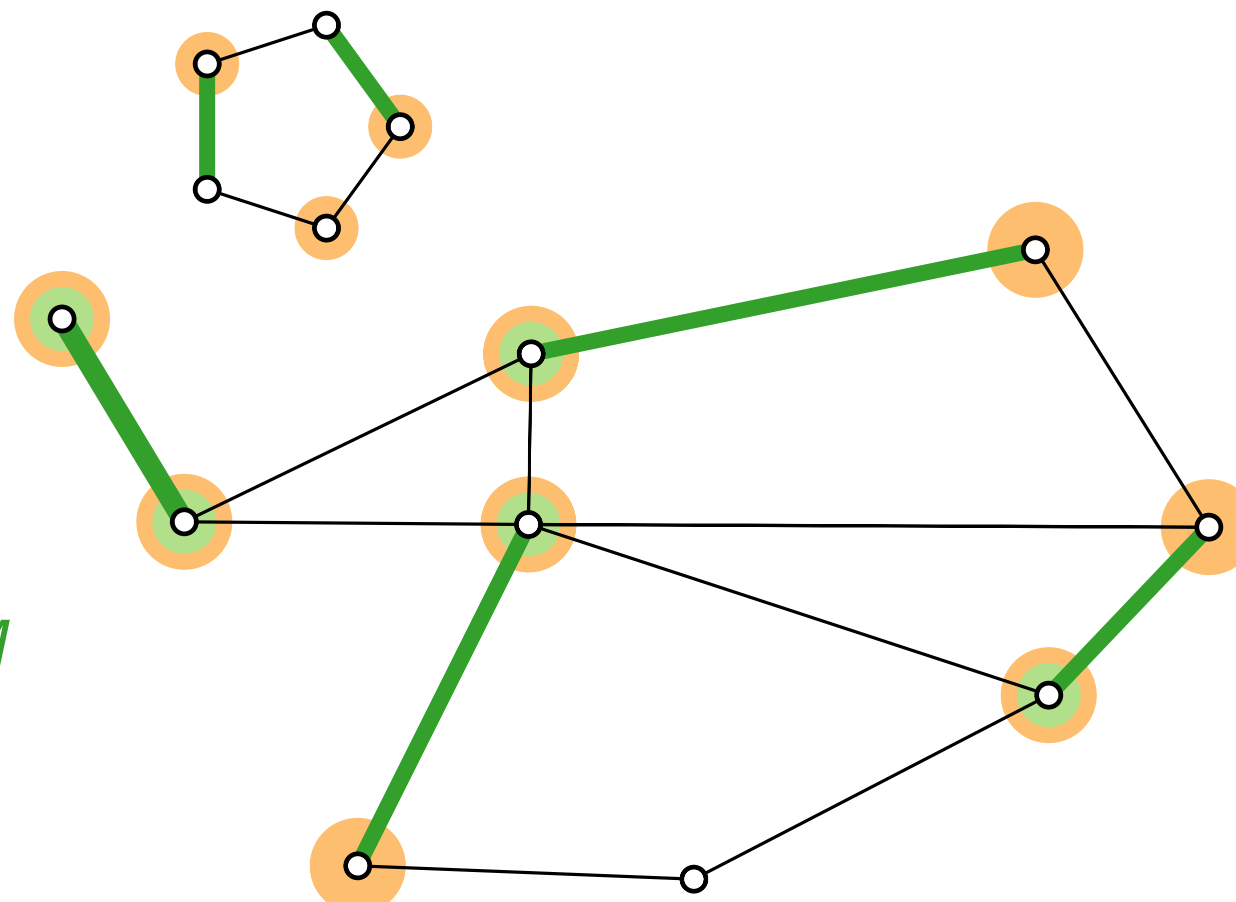

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$\mathrm{ALG} = 2 \cdot |M| \leq$

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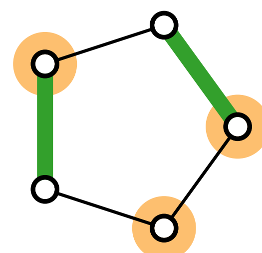

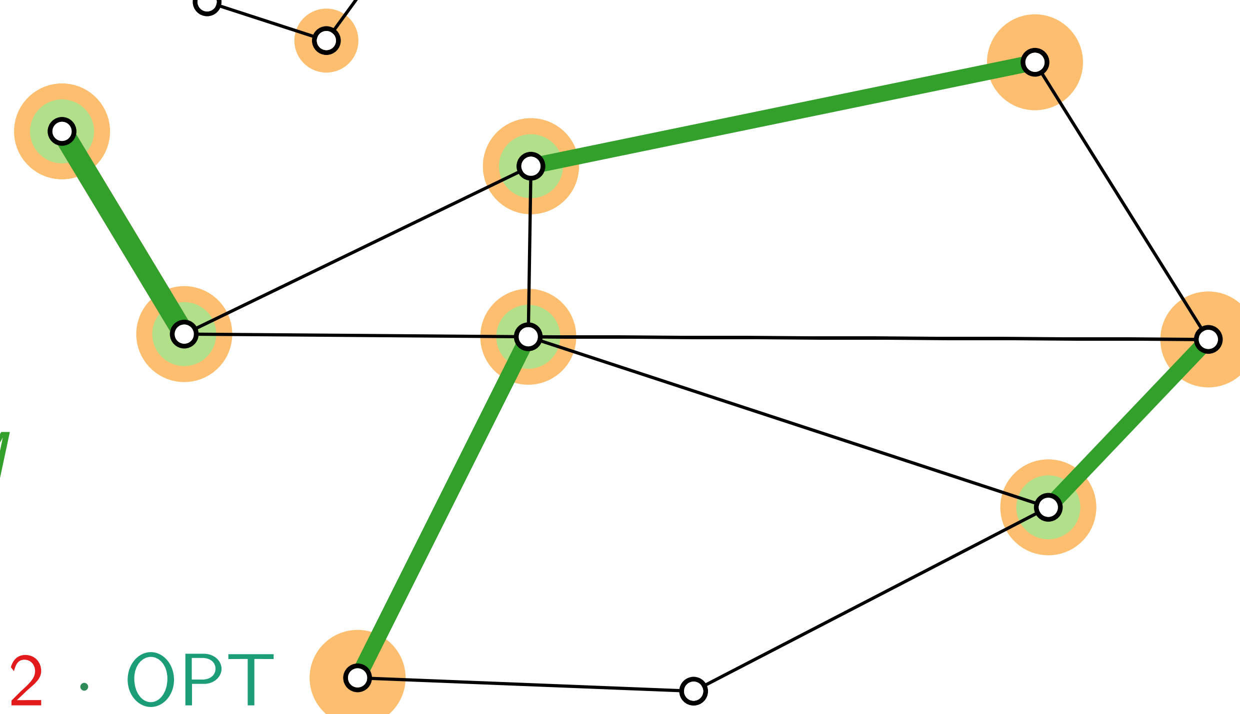

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Approximation Alg. for VertexCover

Algorithm VertexCover(G)

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Theorem. The above algorithm is a factor-2 approximation algorithm for VertexCover.

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Approximability of Vertex Cover

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VertexCover cannot be approximated within a factor of $2 - \Theta(1)$ – if the *Unique Games Conjecture* holds.

Approximation Algorithms

Lecture 1:
Introduction and Vertex Cover

Part V:
An LP-based Algorithm for VertexCover

Task

Write an integer linear program (ILP) for VertexCover:

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Write an integer linear program (ILP) for VERTEXCOVER:

Using integer (and/or real) variables, express the problem using

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Constraints:

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$$x_u + x_v \geq 1.$$

Standard ILP Format

$$\begin{array}{lll} \text{minimize} & \sum_{v \in V(G)} x_v & \\ \text{subject to} & x_u + x_v \geq 1 & \text{for each } uv \in E(G) \\ & x_v \in \{0, 1\} & \text{for each } v \in V(G) \end{array}$$

Standard ILP Format

$$\begin{aligned} &\text{minimize} \quad \textstyle\sum_{v \in V(G)} x_v \\ &\text{subject to} \quad x_u + x_v \geq 1 \qquad \text{for each } uv \in E(G) \\ &\qquad\qquad\quad x_v \in \{0, 1\} \qquad \text{for each } v \in V(G) \end{aligned}$$

Problem:

Standard ILP Format

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Problem: It's NP-hard to solve ILPs in general.

Standard ILP Format

minimize	$\sum_{v \in V(G)} x_v$	
subject to	$x_u + x_v \geq 1$	for each $uv \in E(G)$
	$x_v \in \{0, 1\}$	for each $v \in V(G)$

Problem: It's NP-hard to solve ILPs in general.

But: LPs can be solved efficiently (in $O(L \cdot n^{3.5})$ time),

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Problem: It's NP-hard to solve ILPs in general.

But: LPs can be solved efficiently (in $O(L \cdot n^{3.5})$ time),
where $n = \#$ variables and $L =$ total bit complexity of coefficients.

Standard ILP Format

LP relaxation

minimize $\sum_{v \in V(G)} x_v$

subject to $x_u + x_v \geq 1$ for each $uv \in E(G)$

$x_v \geq 0$ ~~$x_v \in \{0, 1\}$~~ for each $v \in V(G)$

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Problem′: Now we can get fractional solutions, i.e., in $(0, 1)$.

Standard ILP Format

LP relaxation

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	$x_v \geq 0$ ~~$x_v \in \{0, 1\}$~~	for each $v \in V(G)$

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Task: Find a graph G with $\mathrm{OPT}_{\mathrm{LP}} \neq \mathrm{OPT}_{\mathrm{ILP}}$!

Standard ILP Format

LP relaxation

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Problem: It's NP-hard to solve ILPs in general.

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Task: Find a graph G with $\mathrm{OPT}_{\mathrm{LP}} \neq \mathrm{OPT}_{\mathrm{ILP}}$!

Solution?

Standard ILP Format

LP relaxation

minimize $\sum_{v \in V(G)} x_v$

subject to $x_u + x_v \geq 1$ for each $uv \in E(G)$

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where $n = \#$ variables and $L =$ total bit complexity of coefficients.

Problem′: Now we can get fractional solutions, i.e., in $(0, 1)$.

Task: Find a graph G with $\mathrm{OPT}_{\mathrm{LP}} \neq \mathrm{OPT}_{\mathrm{ILP}}$!

Solution? Round the LP solution to get an integral solution!

Rounding the LP Solution

minimize	$\sum_{v \in V(G)} x_v$	
subject to	$x_u + x_v \geq 1$	for each $uv \in E(G)$
	$x_v \geq 0$	for each $v \in V(G)$

For each $v \in V(G)$:

Rounding the LP Solution

$$\begin{array}{lll} \text{minimize} & \sum_{v \in V(G)} x_v & \\ \text{subject to} & x_u + x_v \geq 1 & \text{for each } uv \in E(G) \\ & x_v \geq 0 & \text{for each } v \in V(G) \end{array}$$

For each $v \in V(G)$: Set $x'_v = \begin{cases} 1 & \text{if } x_v \geq 0.5, \\ 0 & \text{otherwise.} \end{cases}$

Rounding the LP Solution

$$
\begin{array}{lll}
\text{minimize} & \sum_{v \in V(G)} x_v & \\
\text{subject to} & x_u + x_v \geq 1 & \text{for each } uv \in E(G) \\
& x_v \geq 0 & \text{for each } v \in V(G)
\end{array}
$$

For each $v \in V(G)$: Set $x'_v = \begin{cases} 1 & \text{if } x_v \geq 0.5, \\ 0 & \text{otherwise.} \end{cases}$

Need to check:

Rounding the LP Solution

$$\begin{aligned} &\text{minimize} && \textstyle\sum_{v \in V(G)} x_v && \\ &\text{subject to} && x_u + x_v \geq 1 && \text{for each } uv \in E(G) \\ & && x_v \geq 0 && \text{for each } v \in V(G) \end{aligned}$$

For each $v \in V(G)$: Set $x'_v = \begin{cases} 1 & \text{if } x_v \geq 0.5, \\ 0 & \text{otherwise.} \end{cases}$

Need to check: Is $(x'_v)_{v \in V(G)}$ a feasible solution?

Rounding the LP Solution

minimize	$\sum_{v \in V(G)} x_v$	
subject to	$x_u + x_v \geq 1$	for each $uv \in E(G)$
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In other words: Is $\{v \in V(G) \colon x'_v = 1\}$ a vertex cover of G?

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Is $x'_u = 0 = x'_v$ possible?

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Is $x'_u = 0 = x'_v$ possible? But then $x_u < 0.5$ and $x_v < 0.5$.

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Is $x'_u = 0 = x'_v$ possible? But then $x_u < 0.5$ and $x_v < 0.5$.

This contradicts $x_u + x_v \geq 1$.

Rounding the LP Solution

$$\begin{array}{lll} \text{minimize} & \sum_{v \in V(G)} x_v & \\ \text{subject to} & x_u + x_v \geq 1 & \text{for each } uv \in E(G) \\ & x_v \geq 0 & \text{for each } v \in V(G) \end{array}$$

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Need to make sure that every edge uv of G is covered.

Is $x'_u = 0 = x'_v$ possible? But then $x_u < 0.5$ and $x_v < 0.5$.

This contradicts $x_u + x_v \geq 1. \Rightarrow x'_u = 1$ or $x'_v = 1$

Rounding the LP Solution

$$\begin{array}{lll} \text{minimize} & \sum_{v \in V(G)} x_v & \\ \text{subject to} & x_u + x_v \geq 1 & \text{for each } uv \in E(G) \\ & x_v \geq 0 & \text{for each } v \in V(G) \end{array}$$

For each $v \in V(G)$: Set $x'_v = \begin{cases} 1 & \text{if } x_v \geq 0.5, \\ 0 & \text{otherwise.} \end{cases}$

Need to check: Is $(x'_v)_{v \in V(G)}$ a feasible solution?

In other words: Is $\{v \in V(G) \colon x'_v = 1\}$ a vertex cover of G?

Need to make sure that every edge uv of G is covered.

Is $x'_u = 0 = x'_v$ possible? But then $x_u < 0.5$ and $x_v < 0.5$.

This contradicts $x_u + x_v \geq 1$. $\Rightarrow$ $x'_u = 1$ or $x'_v = 1$ $\Rightarrow$ (x'_v) feasible!

Cost of the Solution

$$\begin{array}{lll} \text{minimize} & \sum_{v \in V(G)} x_v & \\ \text{subject to} & x_u + x_v \geq 1 & \text{for each } uv \in E(G) \\ & x_v \geq 0 & \text{for each } v \in V(G) \end{array}$$

For each $v \in V(G)$: Set $x'_v = \begin{cases} 1 & \text{if } x_v \geq 0.5, \\ 0 & \text{otherwise.} \end{cases}$

ALG $=$

Cost of the Solution

$$\begin{aligned} \text{minimize} \quad & \textstyle\sum_{v \in V(G)} x_v \\ \text{subject to} \quad & x_u + x_v \geq 1 && \text{for each } uv \in E(G) \\ & x_v \geq 0 && \text{for each } v \in V(G) \end{aligned}$$

For each $v \in V(G)$: Set $x'_v = \begin{cases} 1 & \text{if } x_v \geq 0.5, \\ 0 & \text{otherwise.} \end{cases}$

$\text{ALG} = \sum_{v \in V(G)} x'_v \leq$

Cost of the Solution

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For each $v \in V(G)$: Set $x'_v = \begin{cases} 1 & \text{if } x_v \geq 0.5, \\ 0 & \text{otherwise.} \end{cases}$

$$\text{ALG} = \sum_{v \in V(G)} x'_v \leq 2 \cdot \sum_{v \in V(G)} x_v$$

Cost of the Solution

$$\begin{array}{lll} \text{minimize} & \sum_{v \in V(G)} x_v & \\ \text{subject to} & x_u + x_v \geq 1 & \text{for each } uv \in E(G) \\ & x_v \geq 0 & \text{for each } v \in V(G) \end{array}$$

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$$\text{ALG} = \sum_{v \in V(G)} x'_v \leq 2 \cdot \sum_{v \in V(G)} x_v = 2 \cdot \text{OPT}_{\text{LP}}$$

Cost of the Solution

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subject to	$x_u + x_v \geq 1$	for each $uv \in E(G)$
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Cost of the Solution

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Theorem. The LP rounding algorithm is a factor-2 approximation algorithm for VertexCover.

Cost of the Solution

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Theorem. The LP rounding algorithm is a factor-2 approximation algorithm for WeightedVertexCover.

Cost of the Solution

minimize	$\sum_{v \in V(G)} x_v \cdot w(v)$	
subject to	$x_u + x_v \geq 1$	for each $uv \in E(G)$
	$x_v \geq 0$	for each $v \in V(G)$

For each $v \in V(G)$: Set $x'_v = \begin{cases} 1 & \text{if } x_v \geq 0.5, \\ 0 & \text{otherwise.} \end{cases}$

$$\text{ALG} = \sum_{v \in V(G)} x'_v \leq 2 \cdot \sum_{v \in V(G)} x_v = 2 \cdot \text{OPT}_{\text{LP}} \leq 2 \cdot \text{OPT}_{\text{ILP}}$$

Theorem. The LP rounding algorithm is a factor-2 approximation algorithm for WeightedVertexCover.

Cost of the Solution

minimize	$\sum_{v \in V(G)} x_v \cdot w(v)$	
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	$x_v \geq 0$	for each $v \in V(G)$

For each $v \in V(G)$: Set $x'_v = \begin{cases} 1 & \text{if } x_v \geq 0.5, \\ 0 & \text{otherwise.} \end{cases}$

$$\text{ALG} = \sum_{v \in V(G)} x'_v \cdot w(v) \leq 2 \cdot \sum_{v \in V(G)} x_v \cdot w(v) = 2 \cdot \text{OPT}_{\text{LP}} \leq 2 \cdot \text{OPT}_{\text{ILP}}$$

Theorem. The LP rounding algorithm is a factor-2 approximation algorithm for WeightedVertexCover.