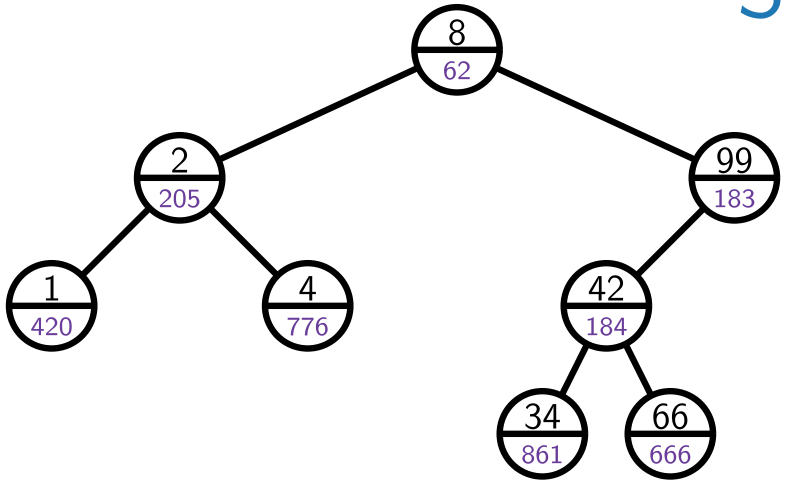
Julius-Maximilians-
UNIVERSITÄT
WÜRZBURG

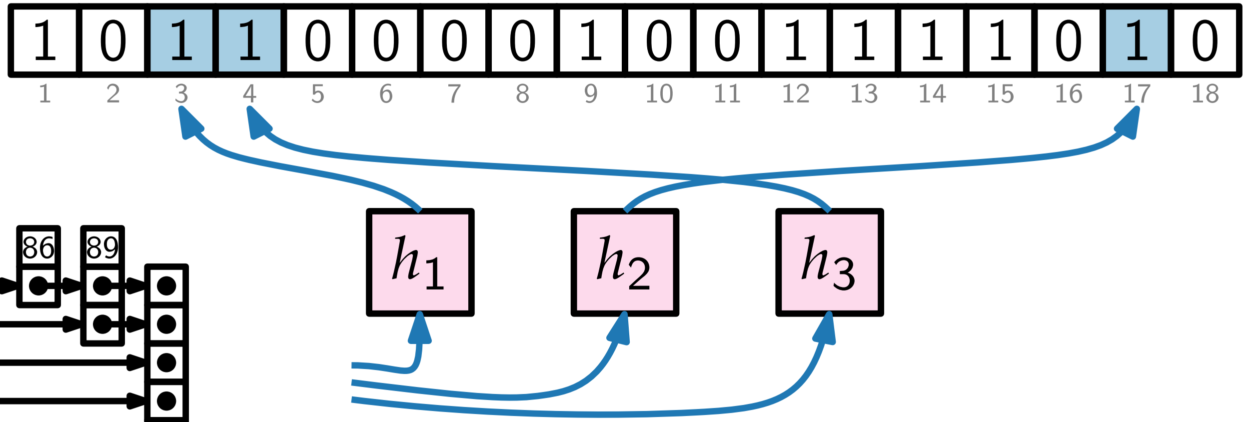
# Advanced Algorithms

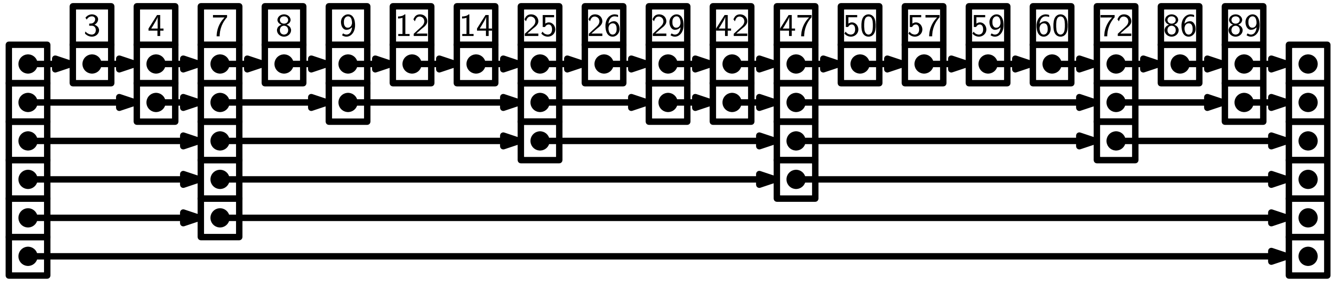
## Randomized and Probabilistic Data Structures

### Skip Lists, Treaps & Bloom Filters




Johannes Zink · WS23/24



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  - Count–min sketch (estimates the frequency of different events in a data stream)

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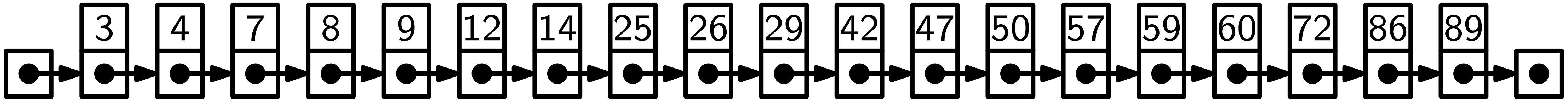


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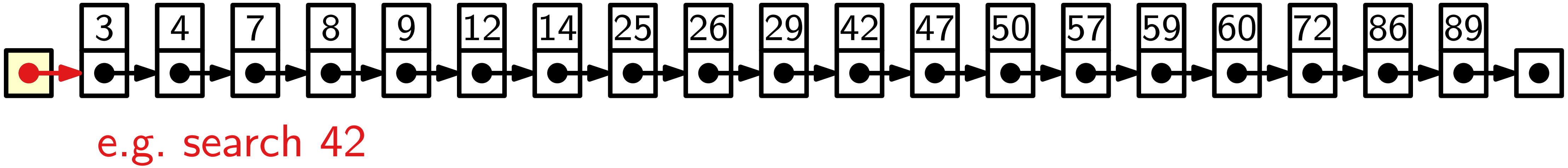

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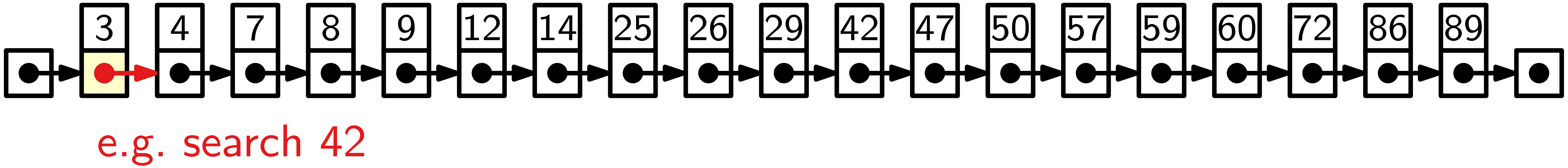

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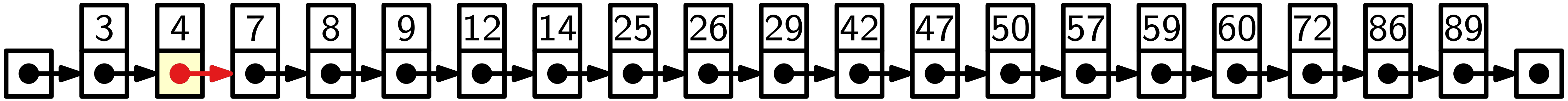


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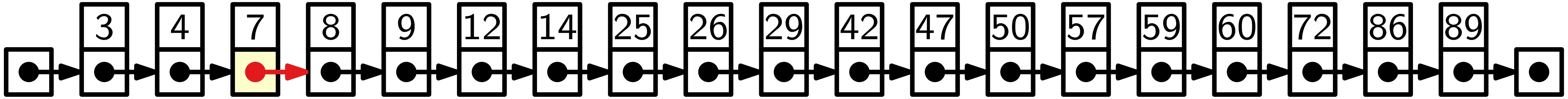


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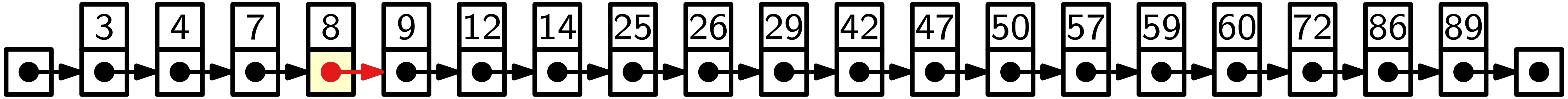


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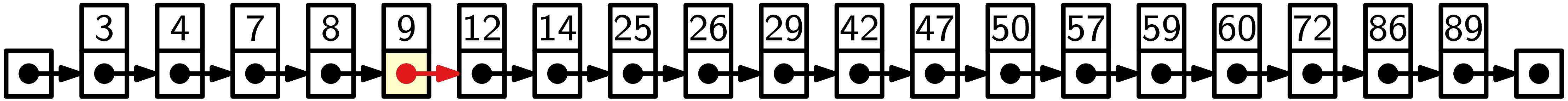


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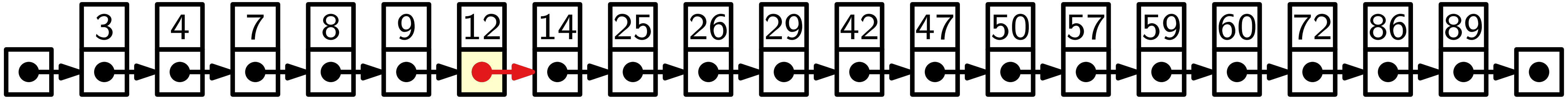


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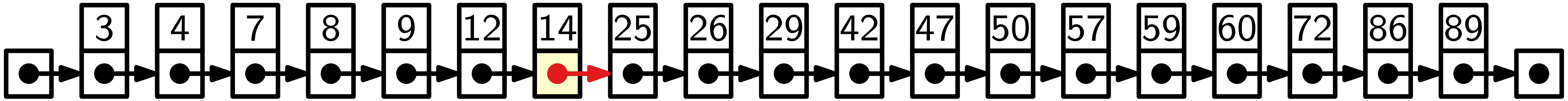


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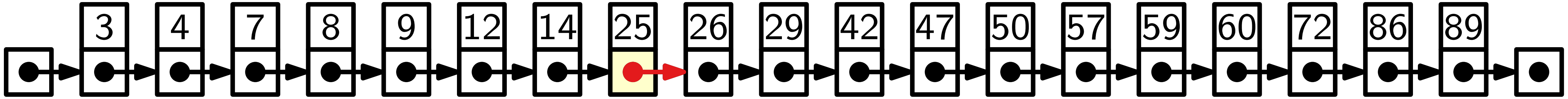


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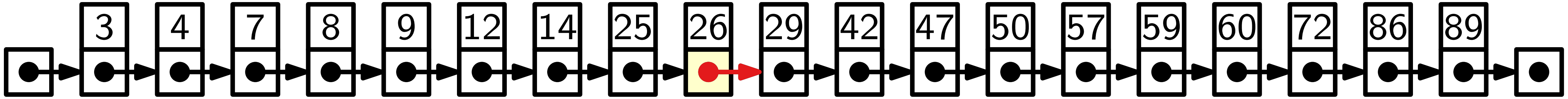


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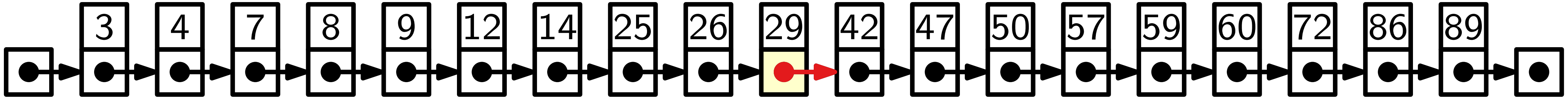


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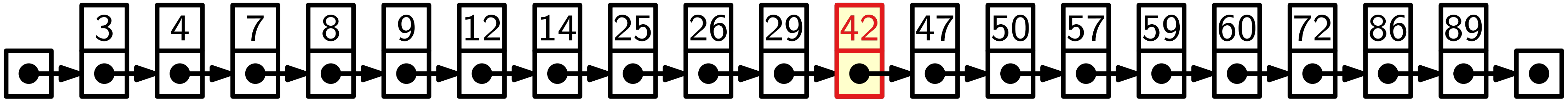


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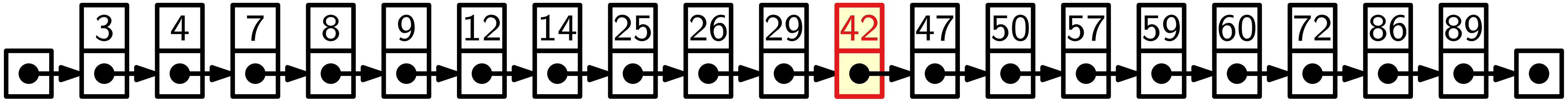


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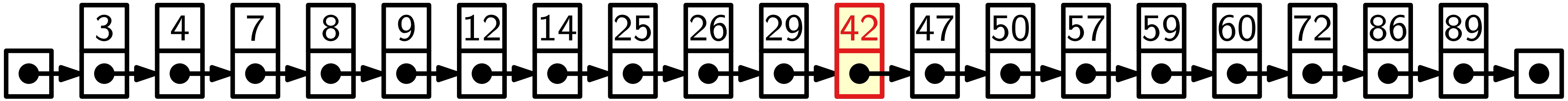


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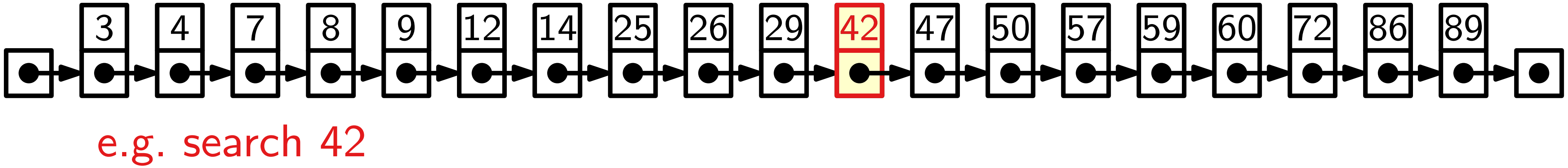


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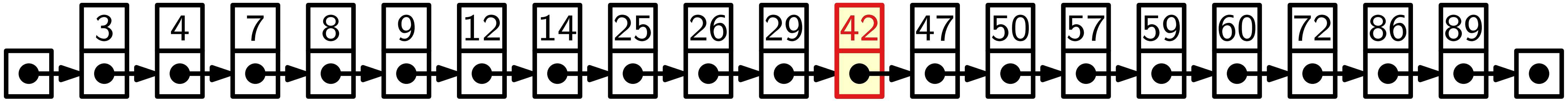


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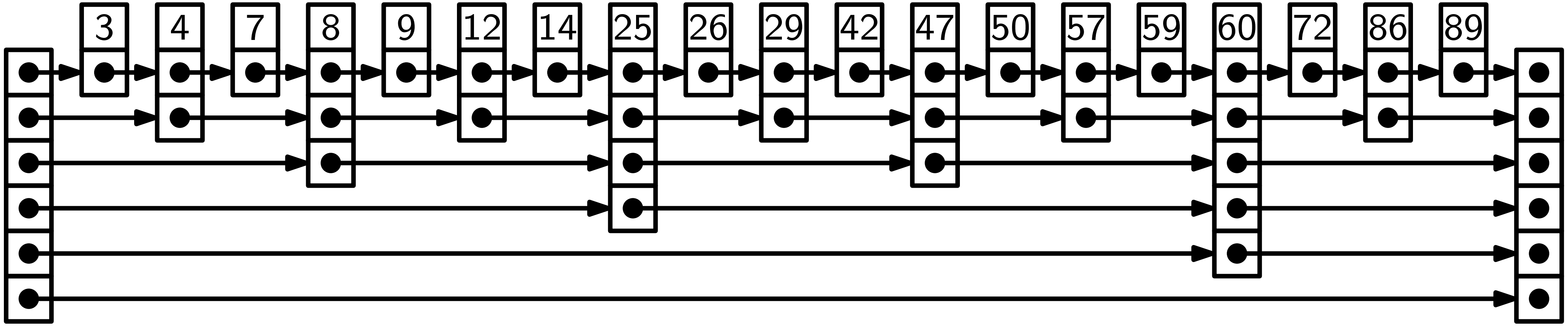


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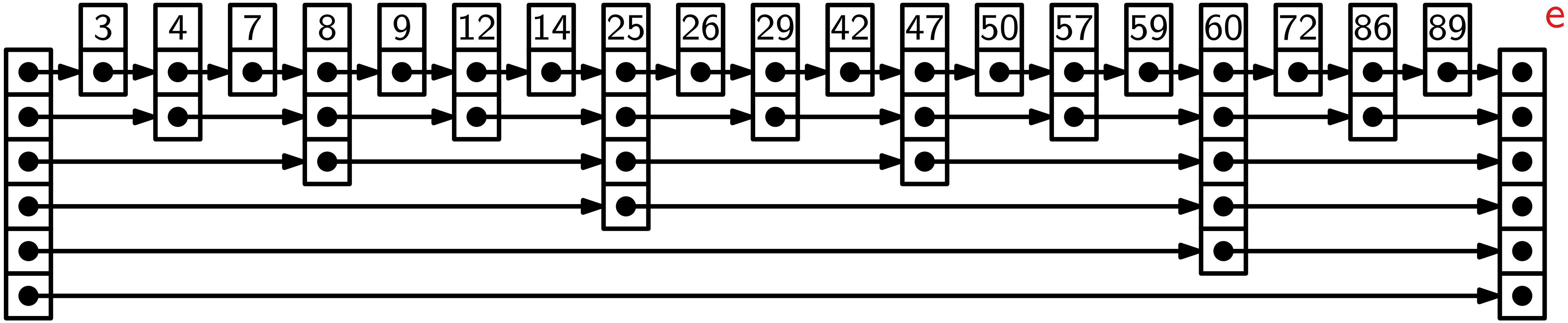

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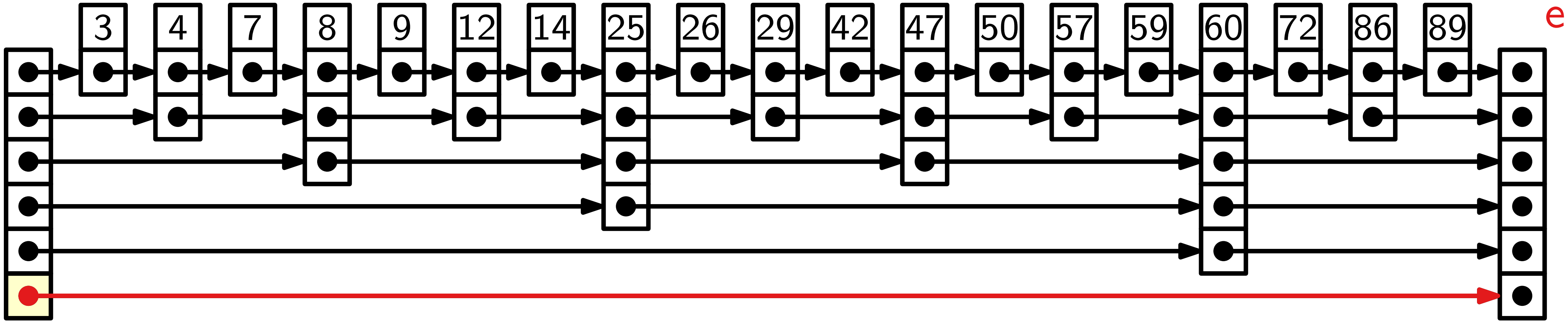


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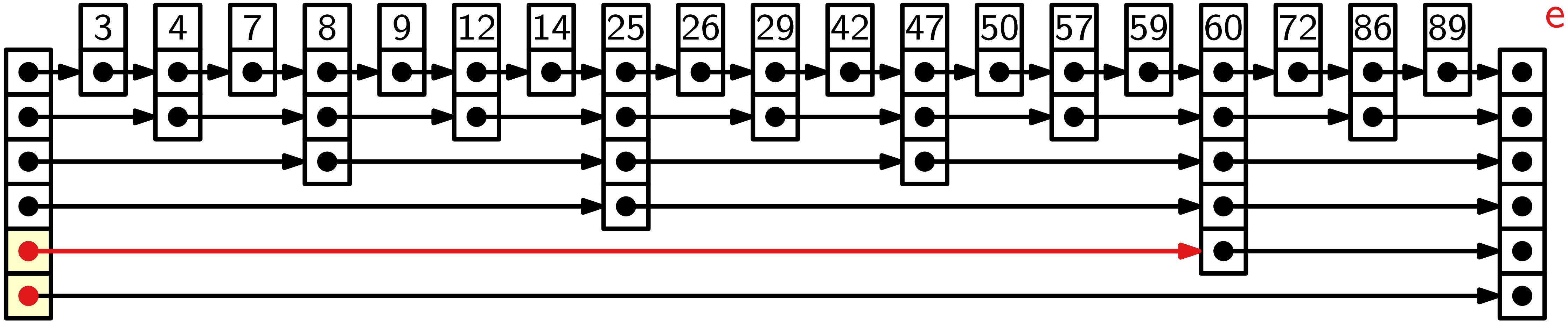


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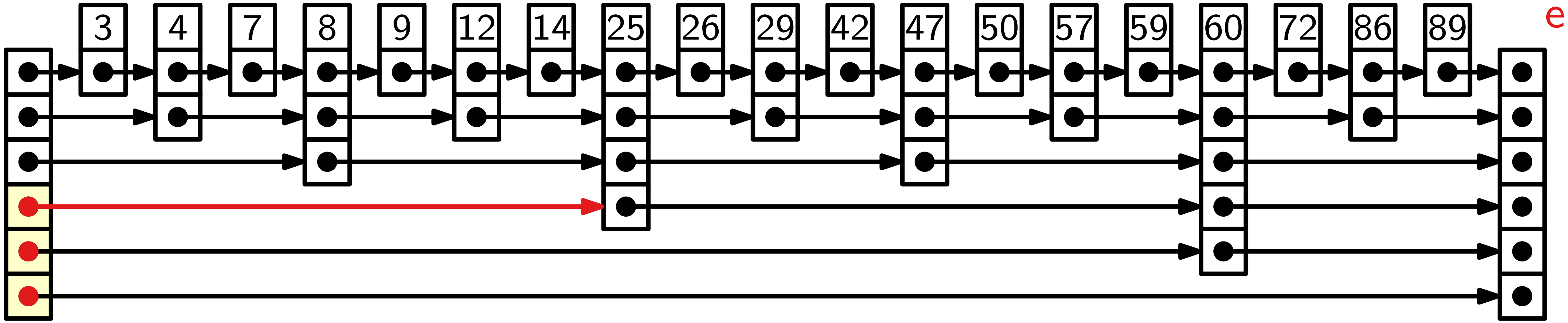


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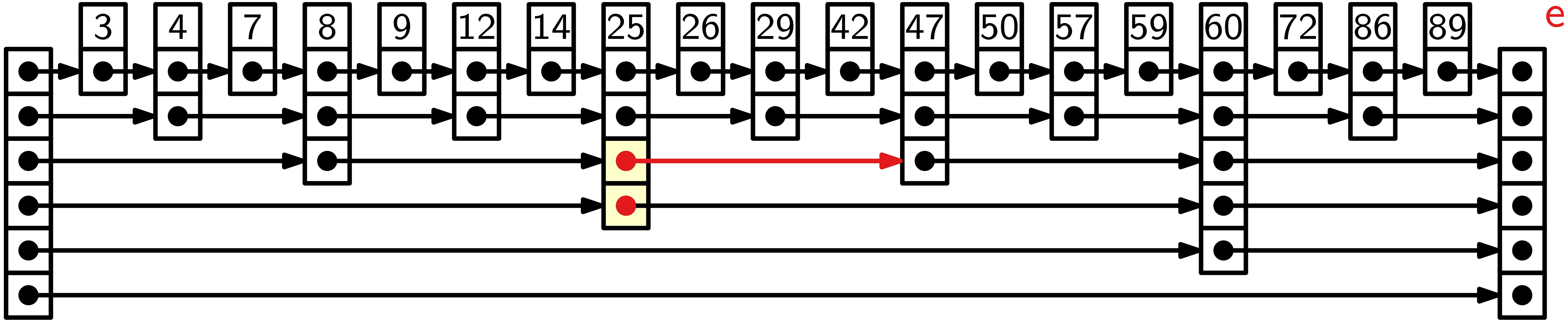


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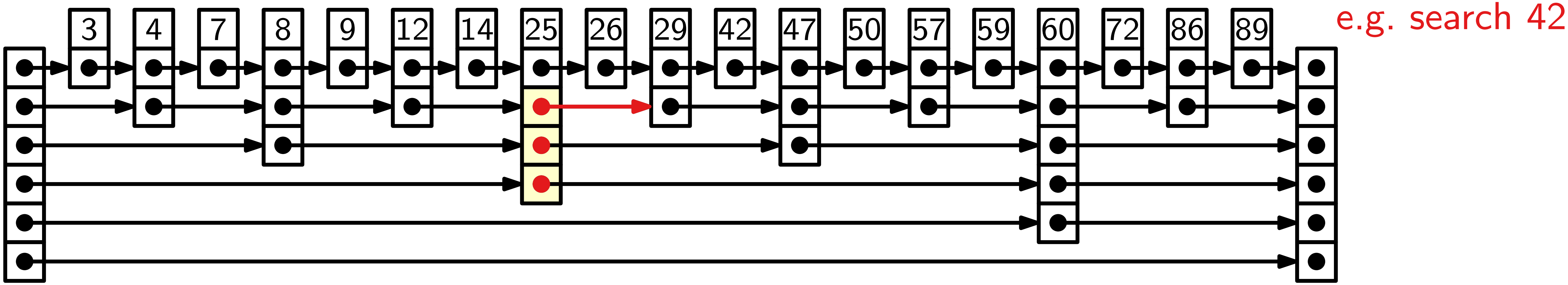

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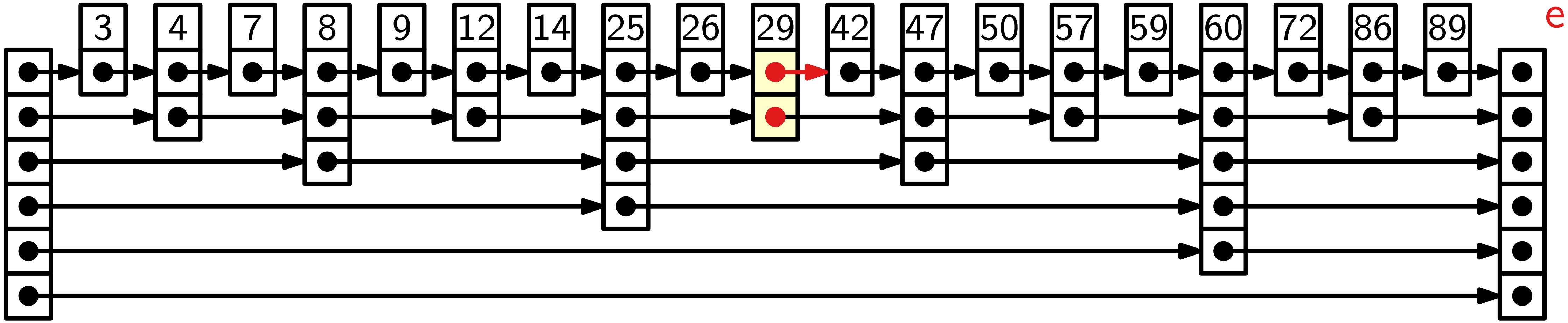

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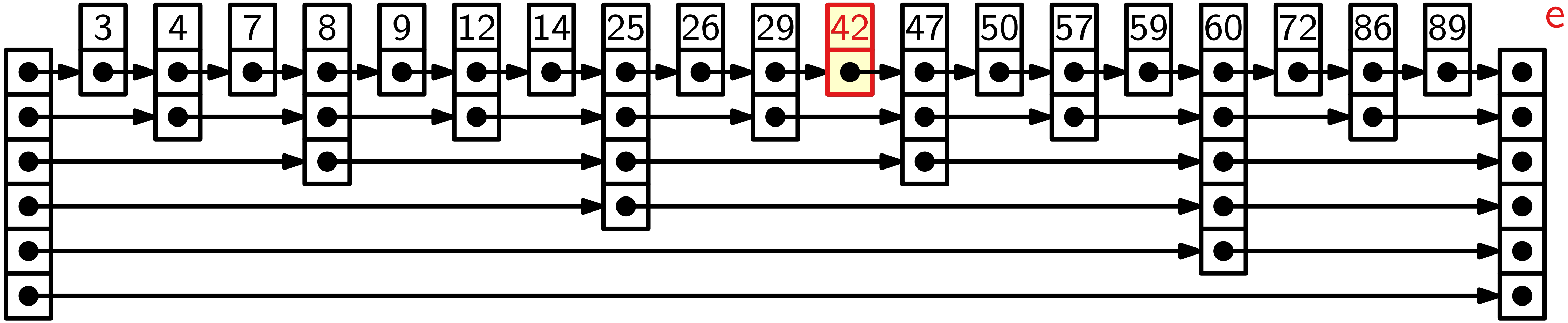

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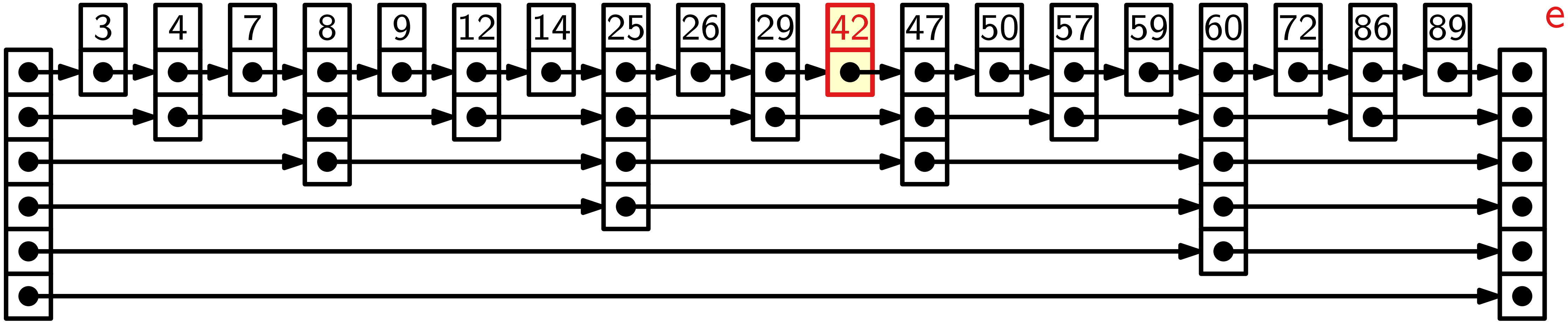


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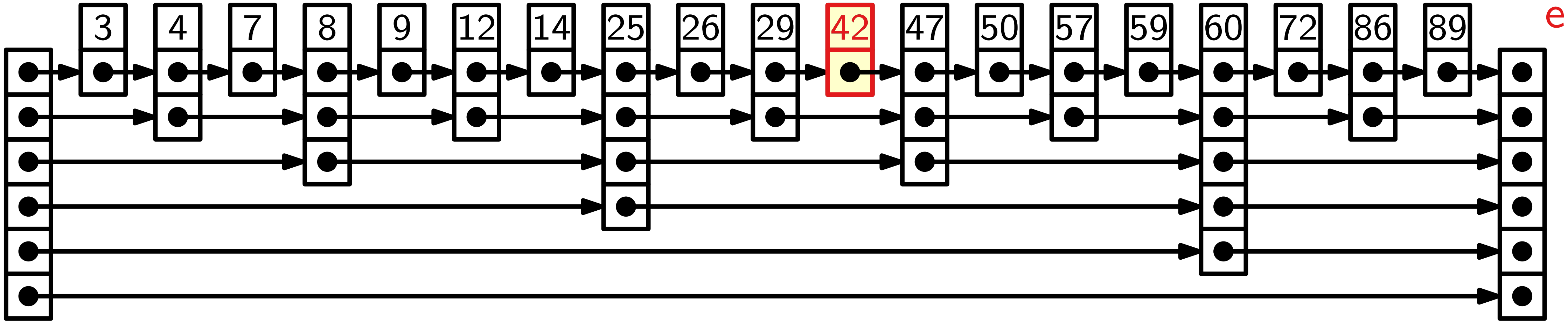


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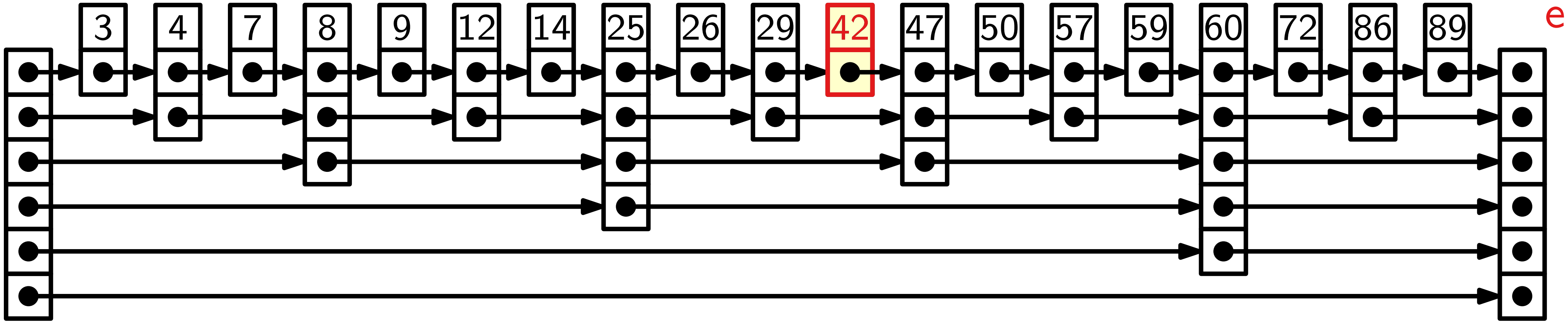


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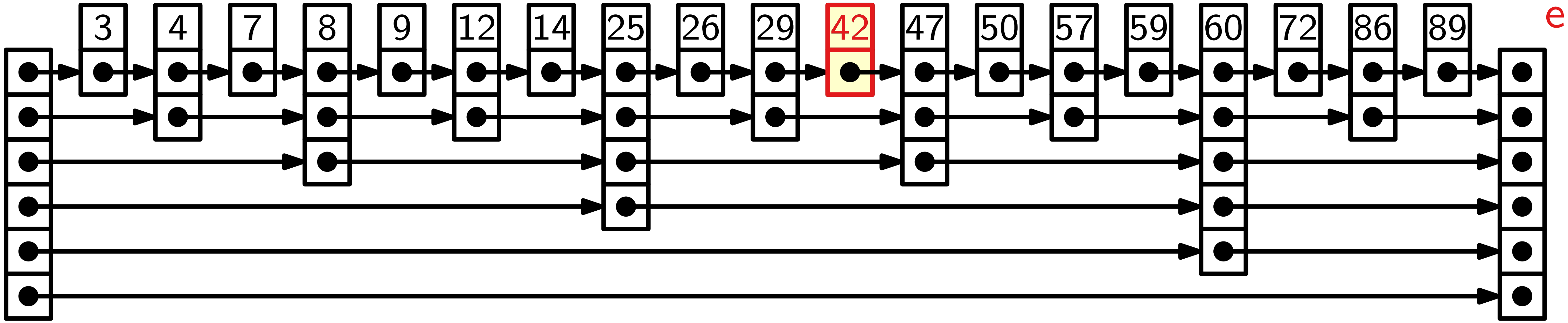


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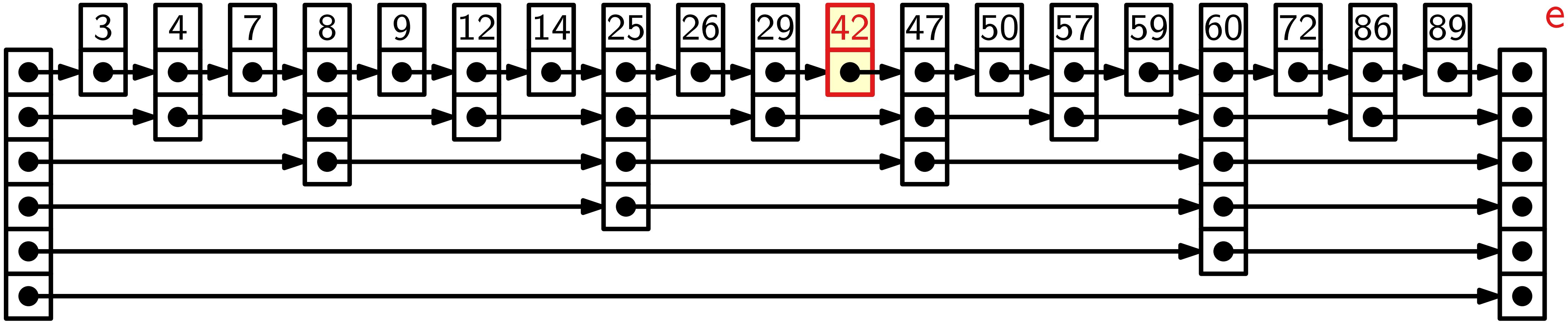


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**Idea:** Keep a skip list, but assign each entry a random height (number of lists it occurs in) s.t. lower heights are more likely to occur.

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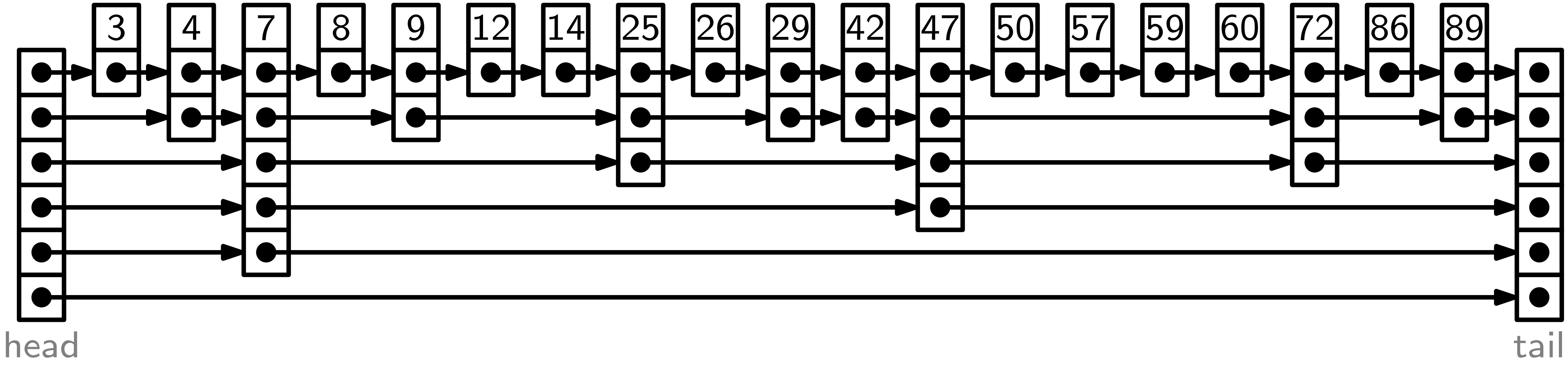


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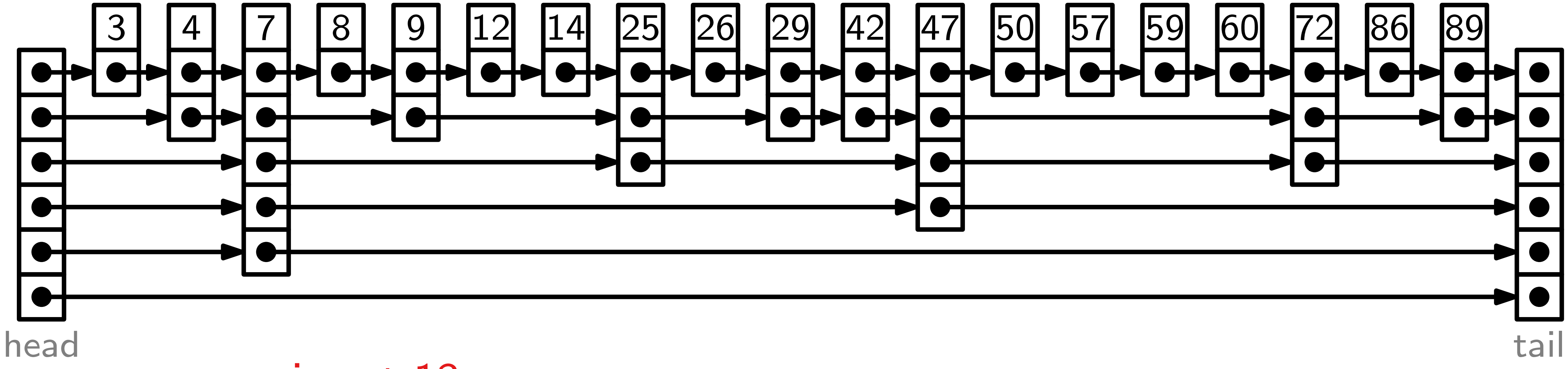


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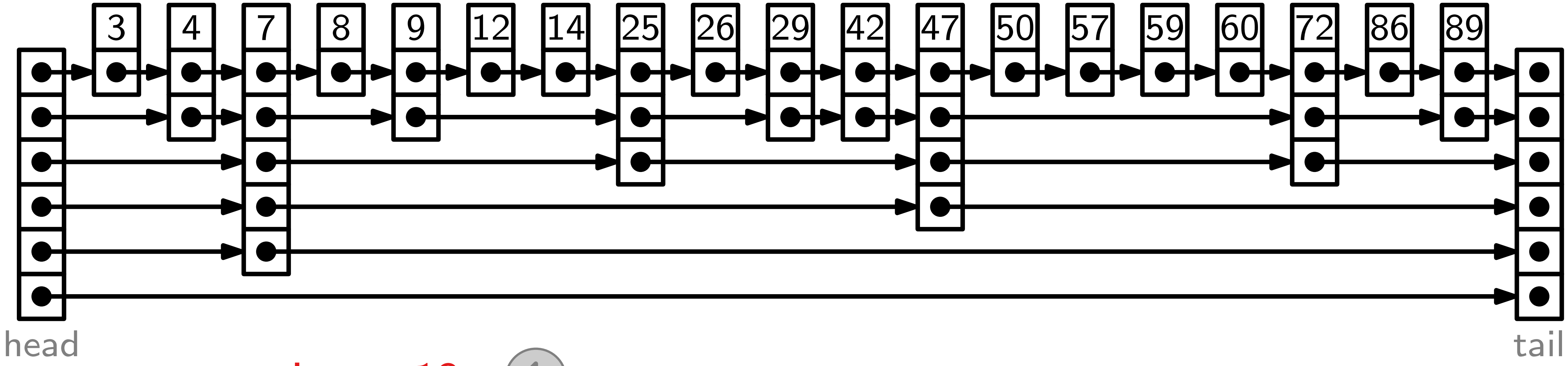


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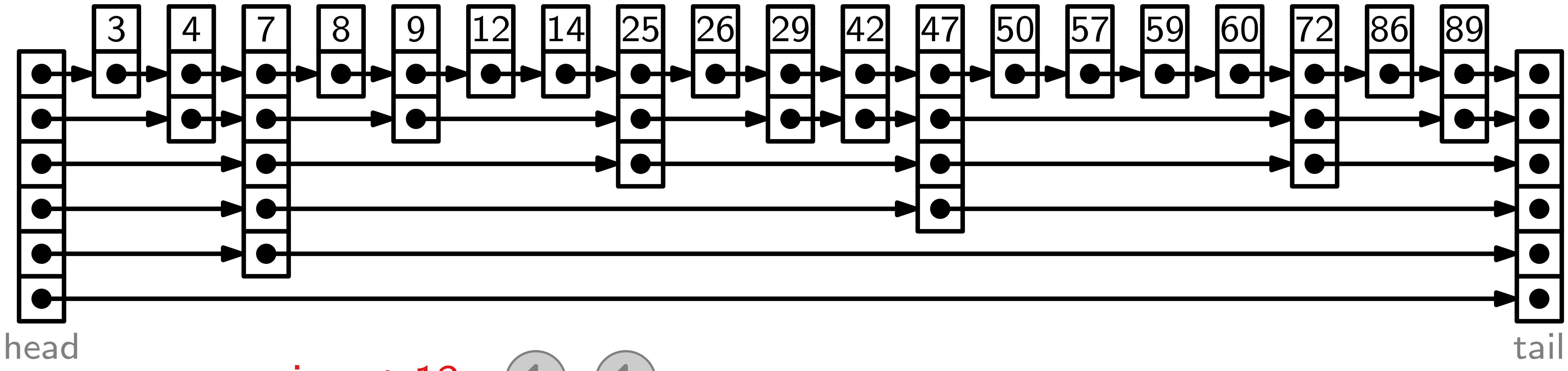


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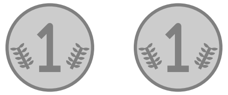

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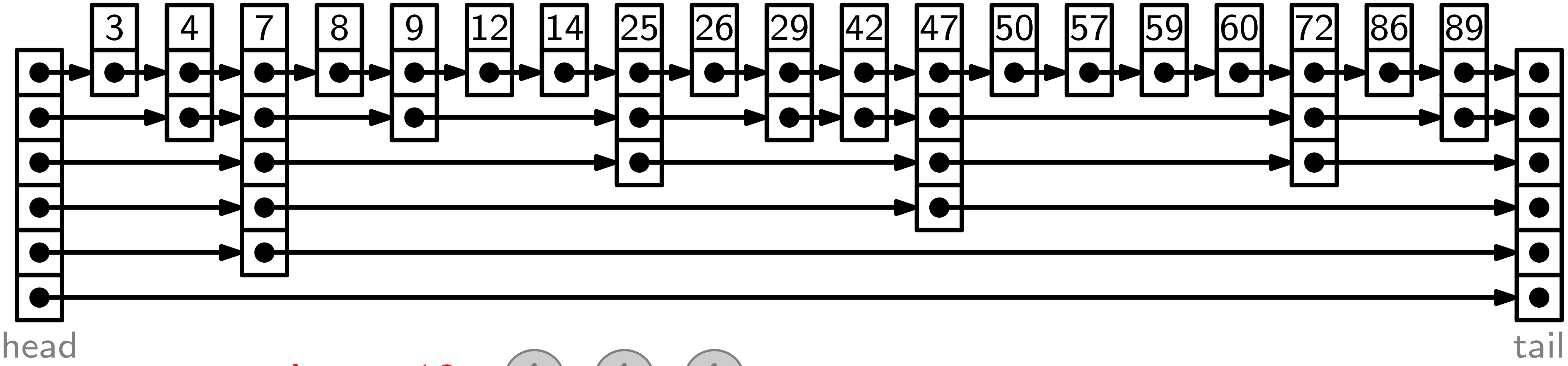


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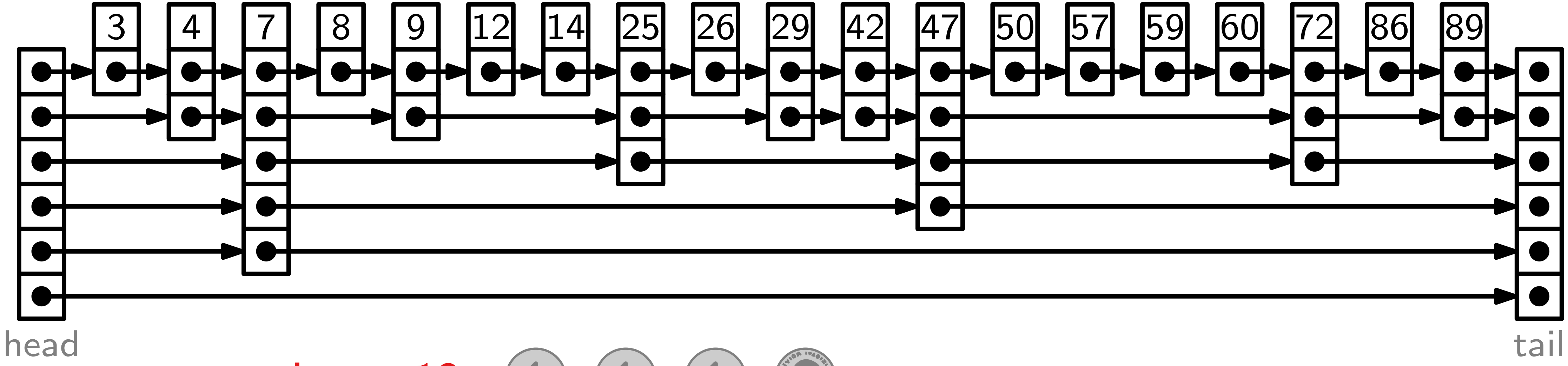


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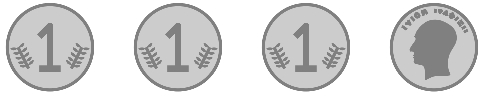

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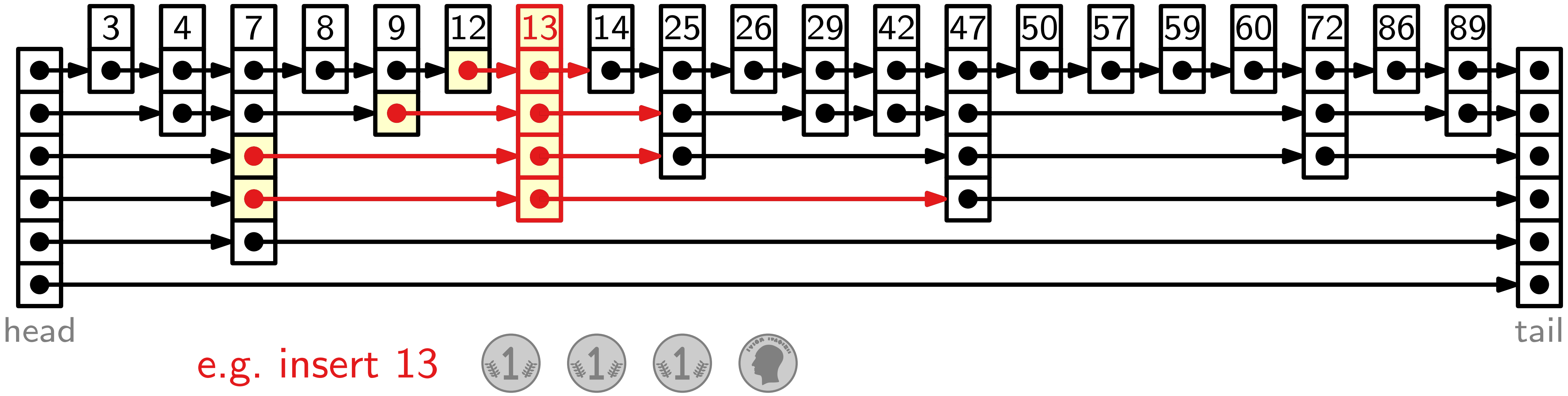


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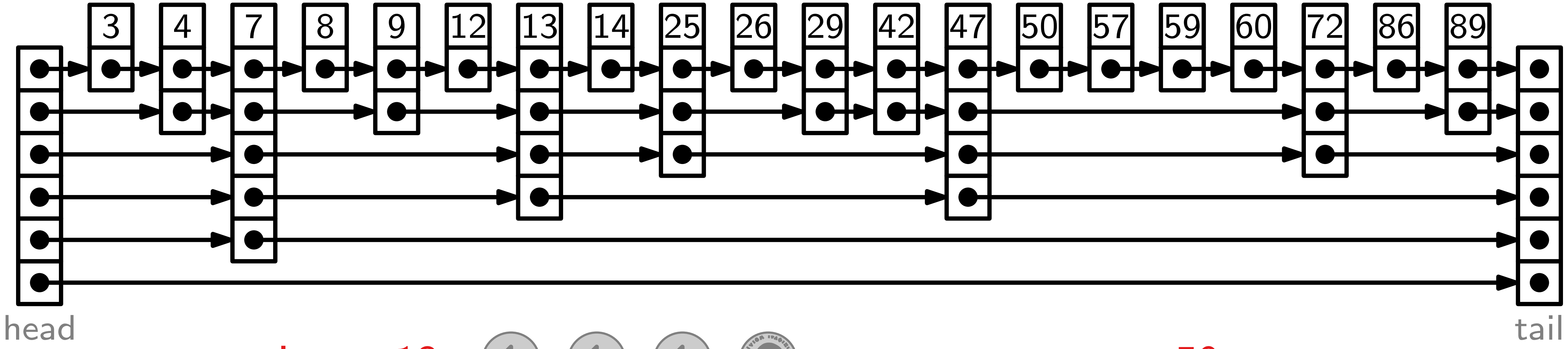


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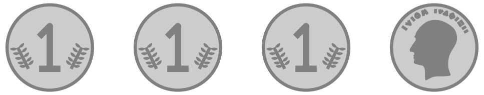
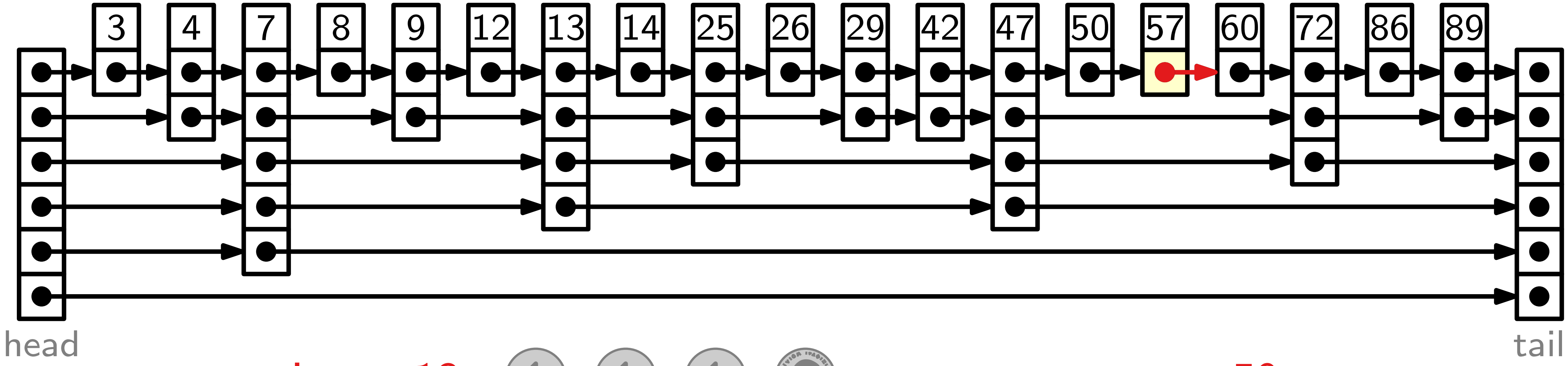


e.g. remove 59

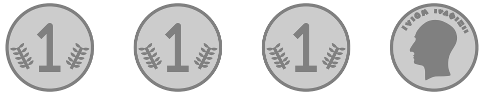
# Randomized Skip Lists

For a new entry, flip a coin until it shows HEADS. The number of flips will be its height.




(We let the head/tail of the list have a height of $\lfloor \log_2 n \rfloor + 1$.)

e.g. insert 13



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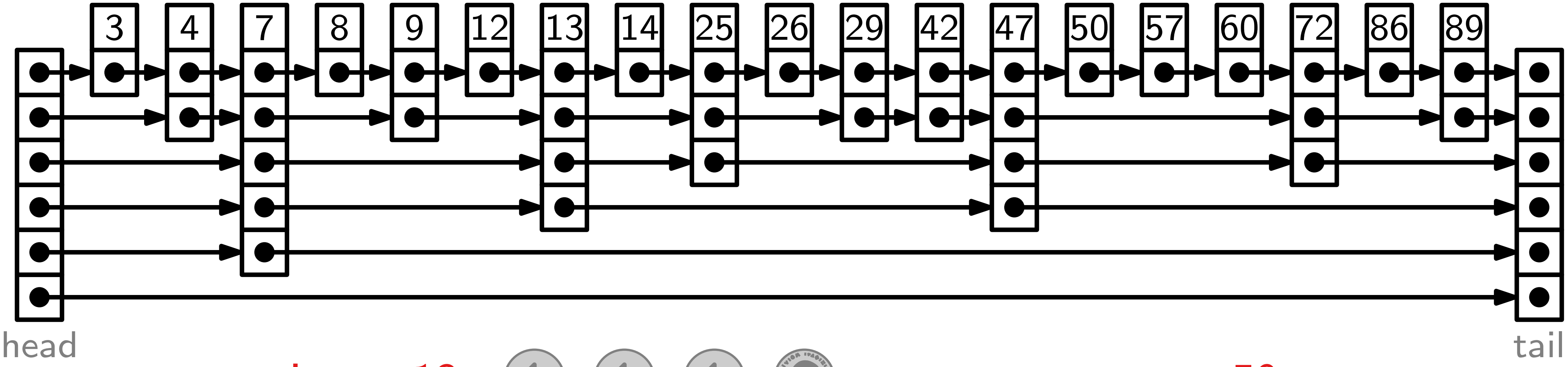


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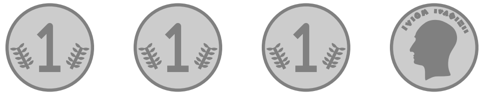


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Insertion and deletion works in $O(\log n)$ time + the time to search a key.
(We store the $O(\log n)$ pointers that need to be updated while searching.
Searching works in the same way as for deterministic skip lists.)

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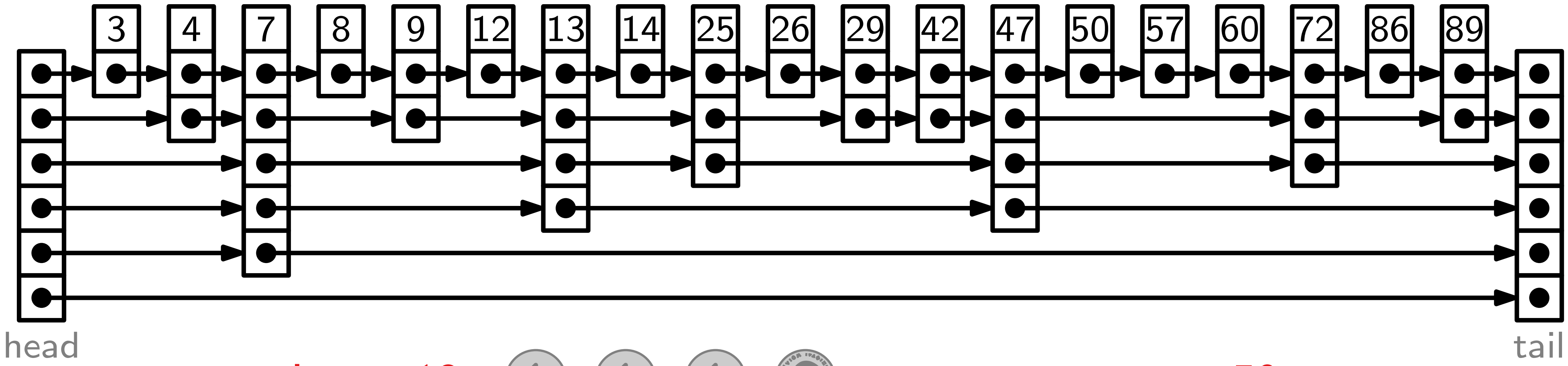


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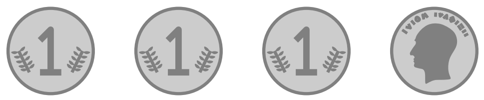

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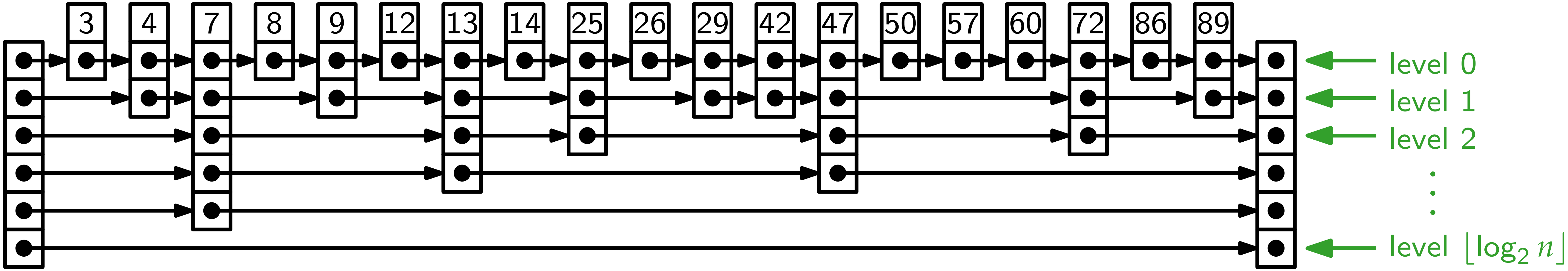
**Theorem 1.** Searching in a rand. skip list can be done in expected $O(\log n)$ time.

# Randomized Skip Lists

## Proof of Theorem 1.

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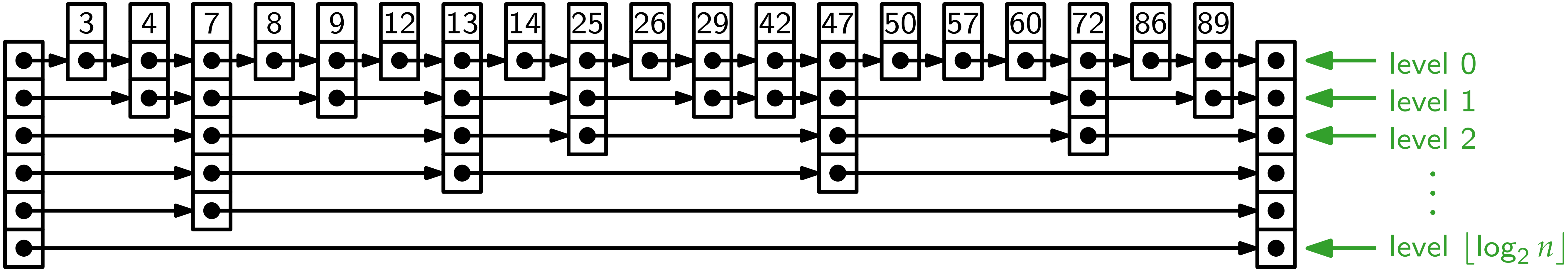


How long is the search path to reach the key we search for?

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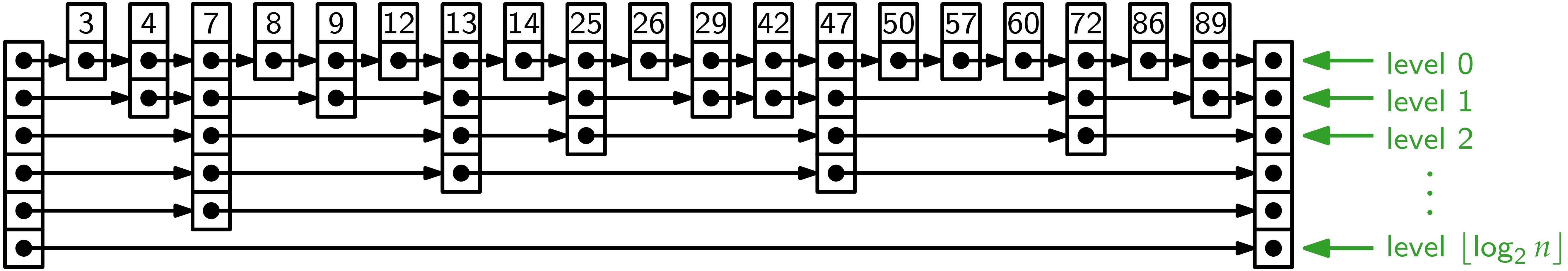


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We do backwards analysis ($\rightarrow$ see lecture on rand. algorithms) on the search path.

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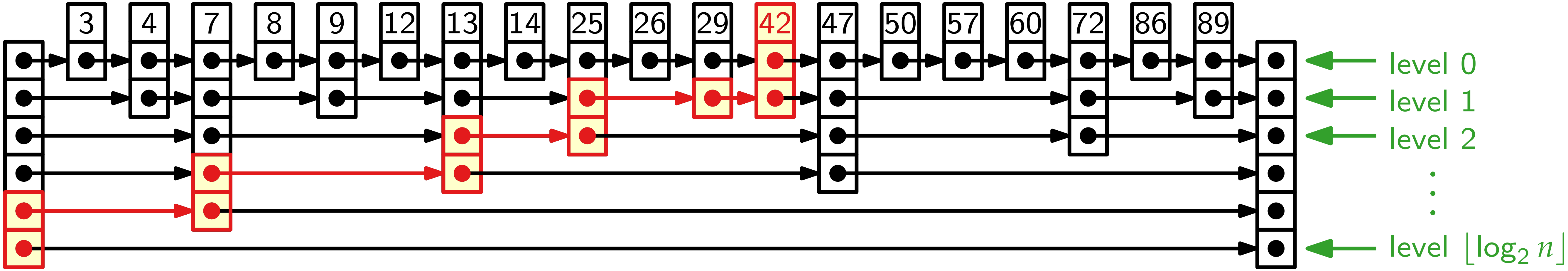


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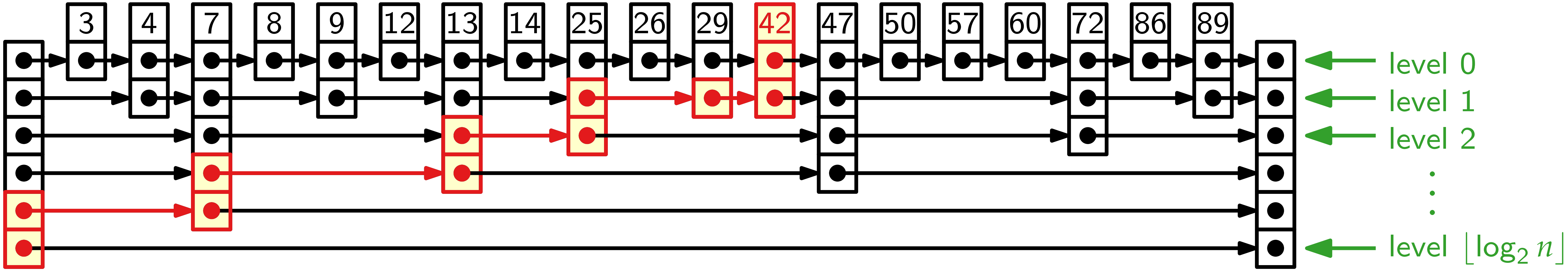


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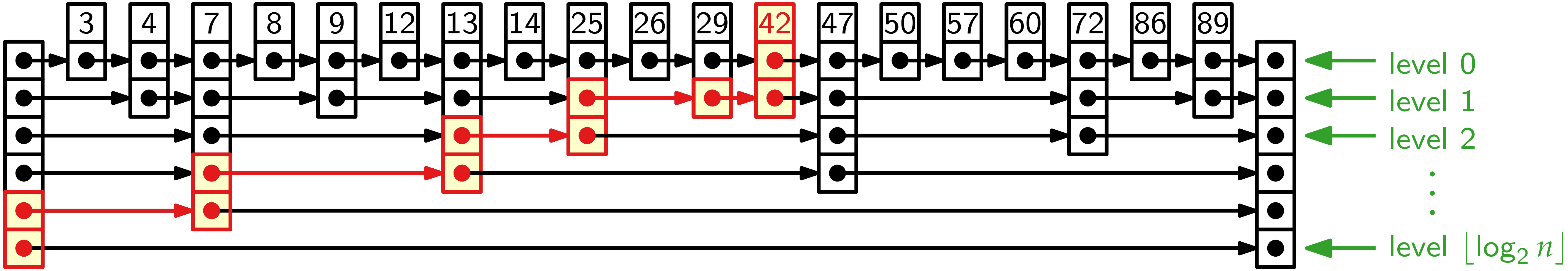


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In the reverse search path, we always go to the next greater level if possible, otherwise we follow the (reverse) pointer to the left.

If we are at level $i$, the probability that we can go a level up is $1/2$ by construction.

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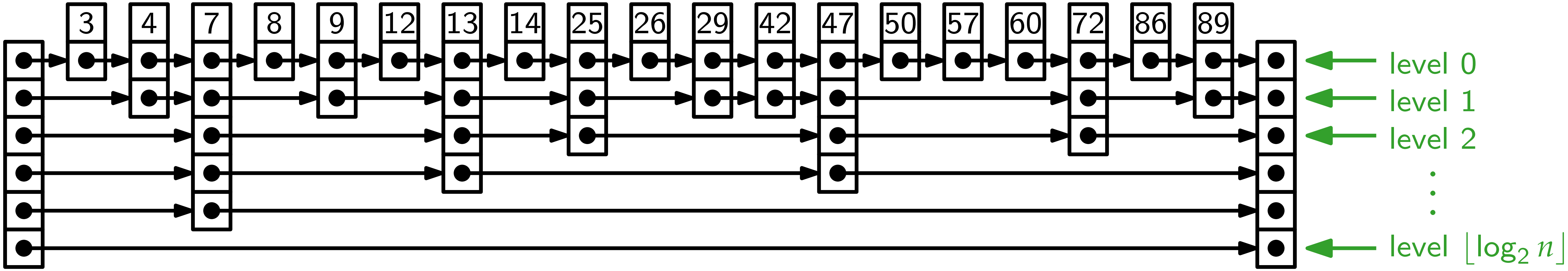
# Randomized Skip Lists

## Proof of Theorem 1.

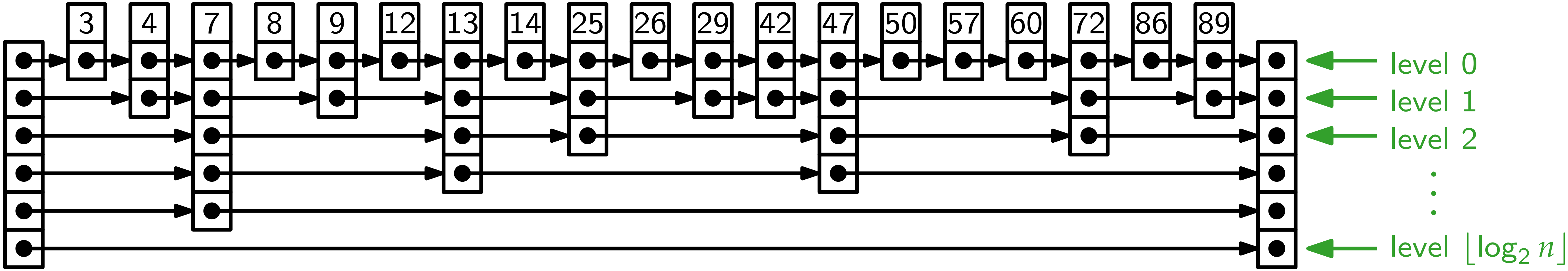



Let $X_i$ be a random variable denoting the number of steps we take on level $i$ or lower.

**Theorem 1.** Searching in a rand. skip list can be done in expected $O(\log n)$ time.

# Randomized Skip Lists

## Proof of Theorem 1.




Let $X_i$ be a random variable denoting the number of steps we take on level $i$ or lower.

$$E[X_i] = 1 + \tfrac{1}{2}E[X_{i-1}] + \tfrac{1}{2}E[X_i]$$ (for $i < 0$: $E[X_i] = 0$)

in the previous step we used a (reverse) pointer to the left

in the previous step we went a level up

current step we take on level $i$ (start with the last step we take on level $i$; we don't skip levels)

**Theorem 1.** Searching in a rand. skip list can be done in expected $O(\log n)$ time.

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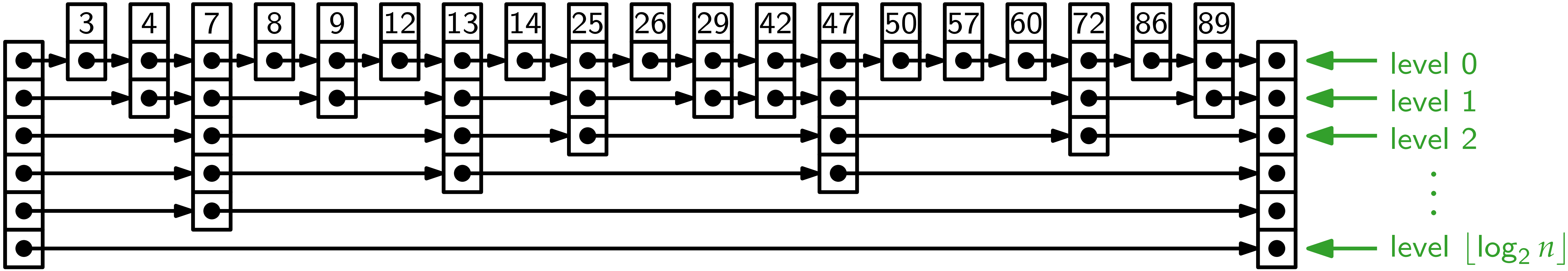


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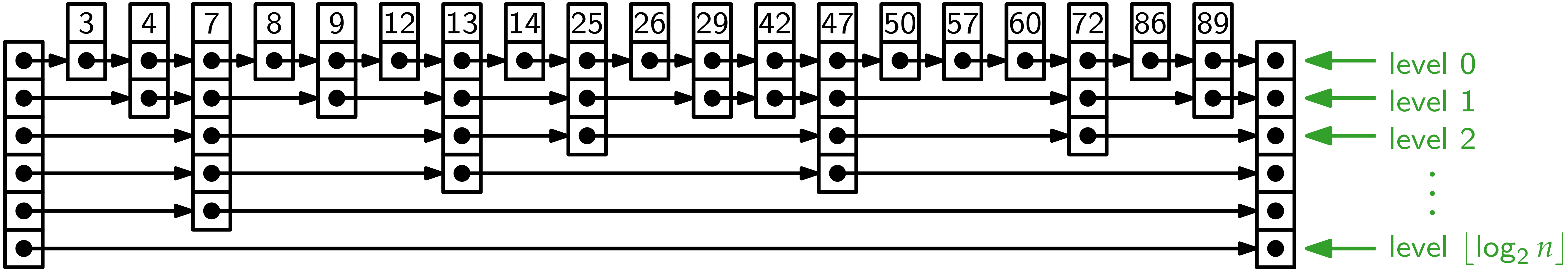


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$E[X_i] = 1 + \frac{1}{2}E[X_{i-1}] + \frac{1}{2}E[X_i] \iff E[X_i] = 2 + E[X_{i-1}]$ (for $i < 0$: $E[X_i] = 0$)

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# Randomized Skip Lists

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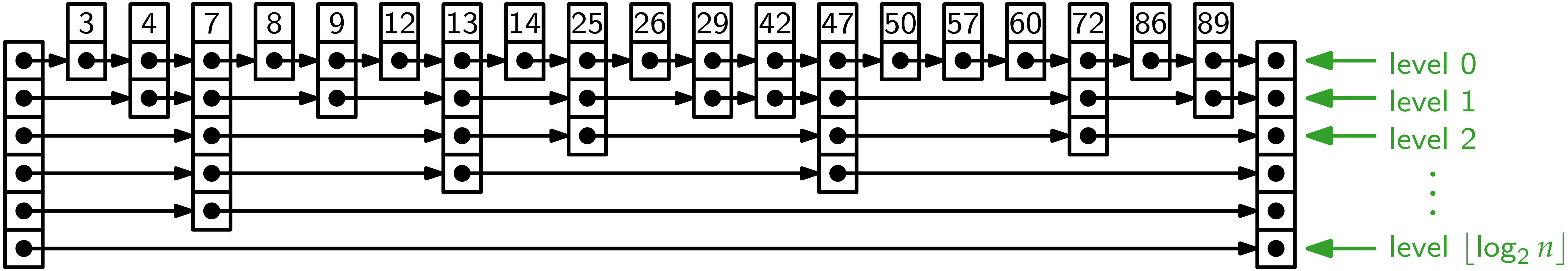


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□

**Theorem 1.** Searching in a rand. skip list can be done in expected $O(\log n)$ time.

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For every key $y$ in the left (right) subtree of $x$ it holds that $y \leq x$ ($y \geq x$).

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- The priorities in the tree fulfill the heap property.

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We build a treap for the key set $S = \{34, 8, 99, 1, 4, 2, 42, 66\}$ by inserting each key.

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we start with the empty tree

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create a new node to insert key 34

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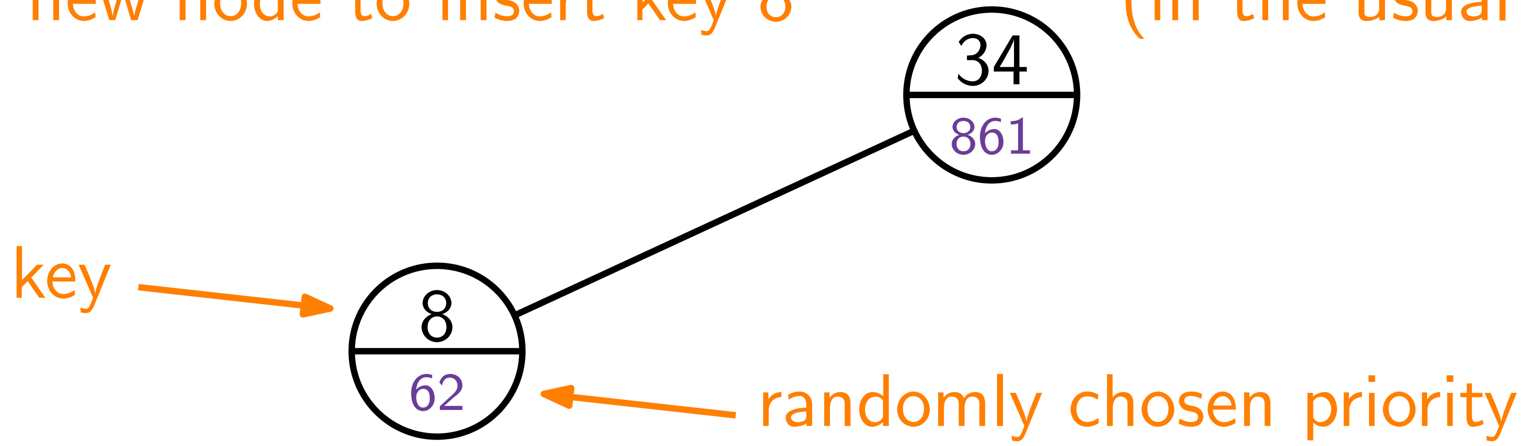
We build a treap for the key set $S = \{34, 8, 99, 1, 4, 2, 42, 66\}$ by inserting each key.

create a new node to insert key 8 (in the usual way for a binary search tree)



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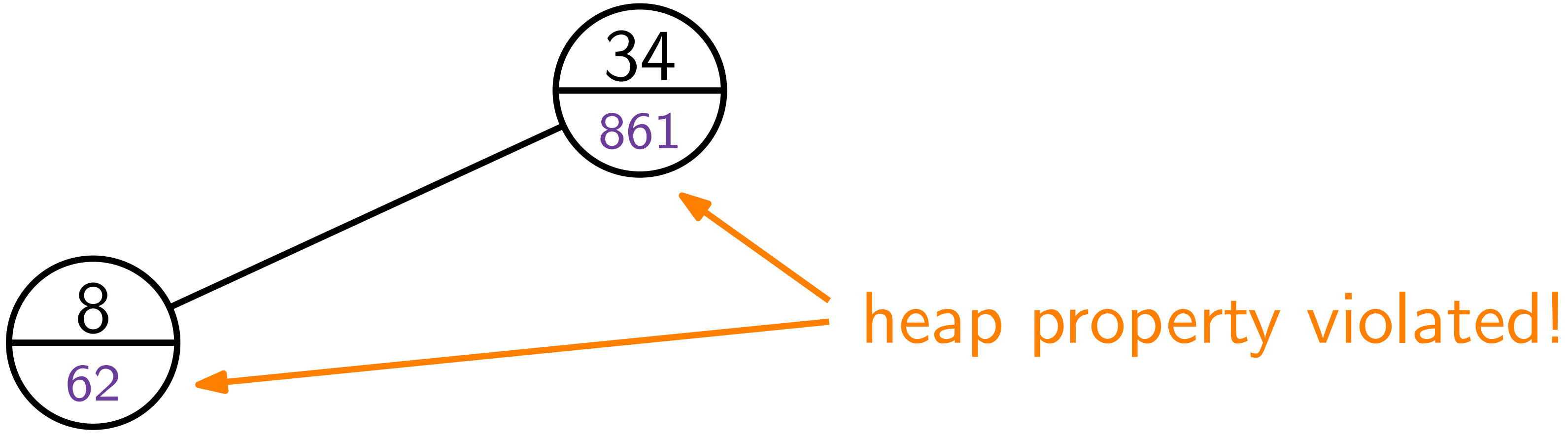

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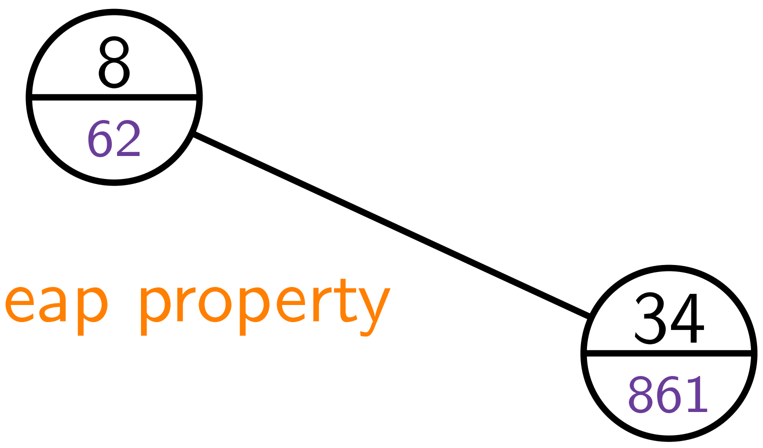


do a right rotation to fix heap property

# Building Treaps

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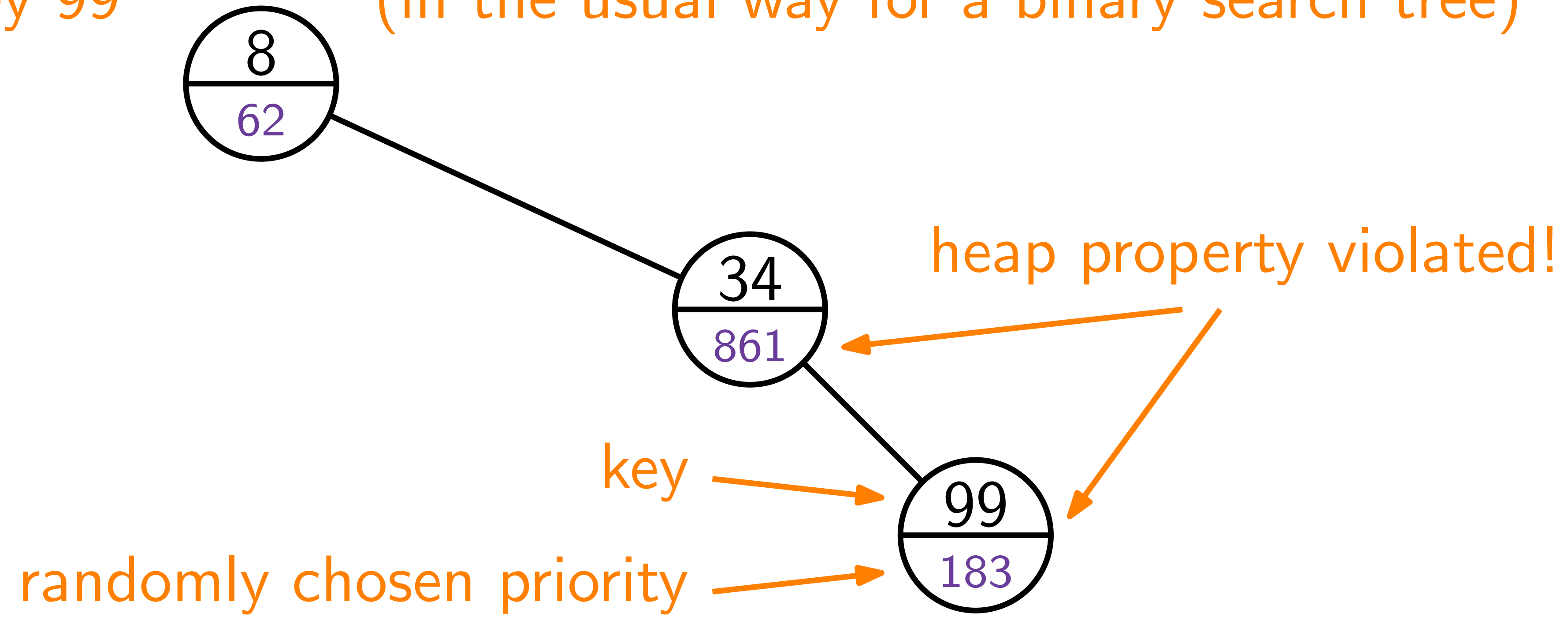
create a new node to insert key 99 (in the usual way for a binary search tree)



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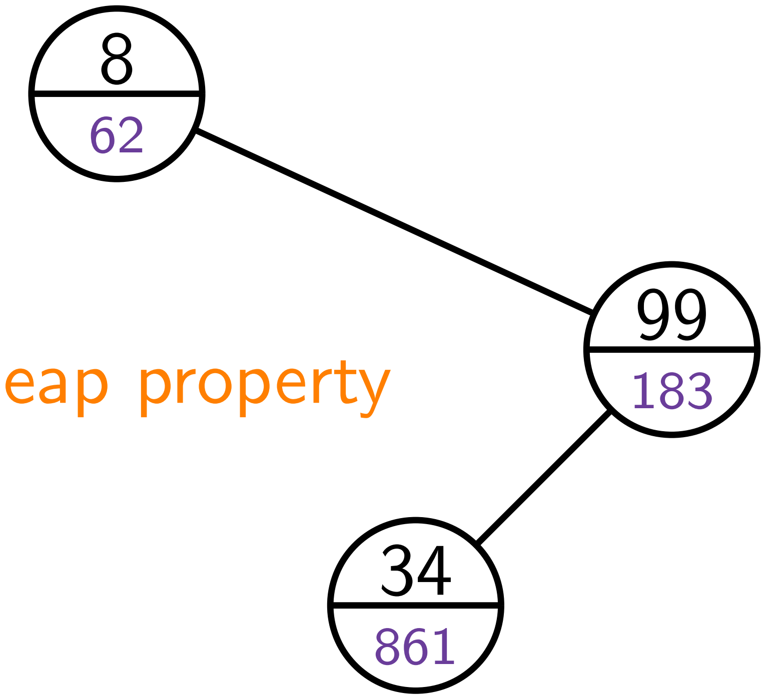
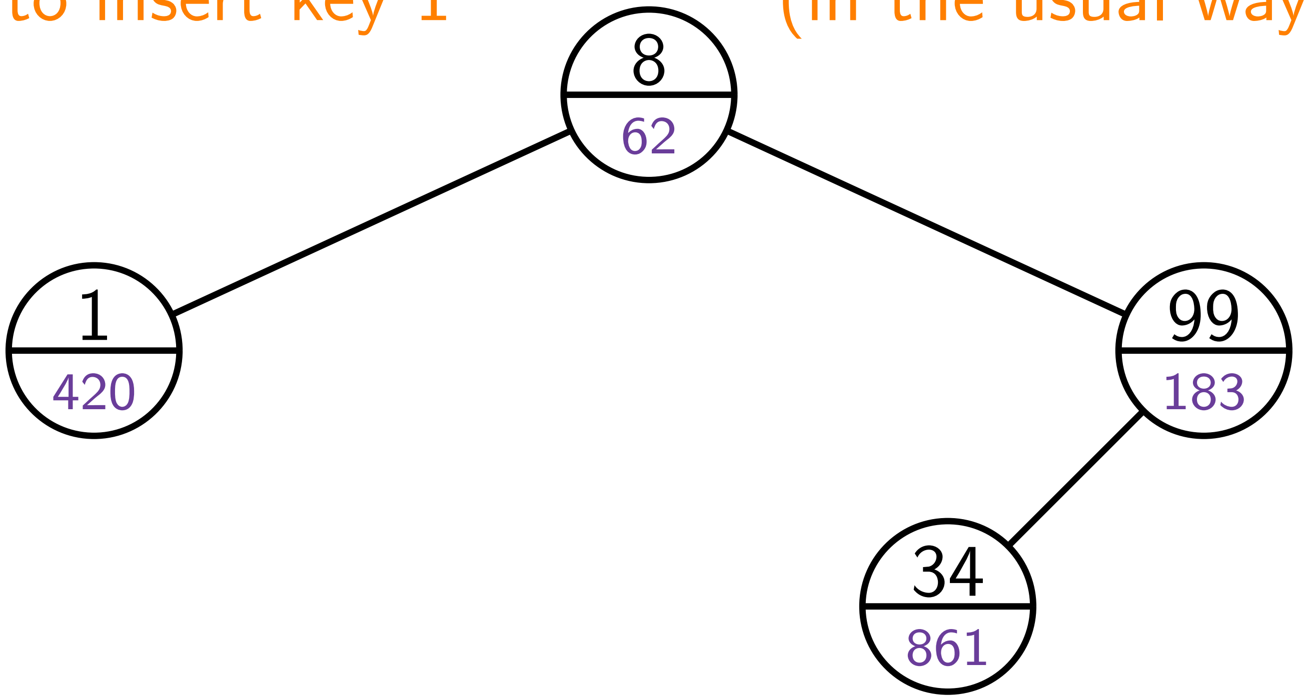
do a left rotation to fix heap property



# Building Treaps

We build a treap for the key set $S = \{34, 8, 99, 1, 4, 2, 42, 66\}$ by inserting each key.

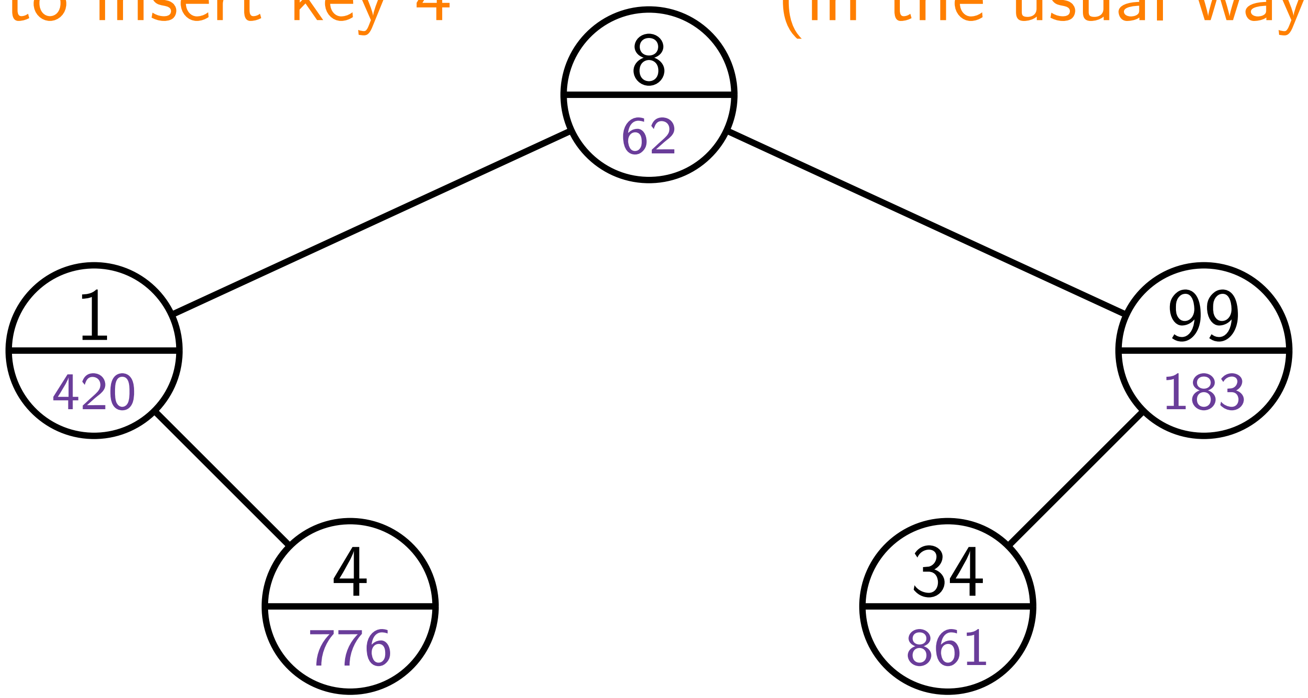
create a new node to insert key 1 (in the usual way for a binary search tree)



# Building Treaps

We build a treap for the key set $S = \{34, 8, 99, 1, 4, 2, 42, 66\}$ by inserting each key.

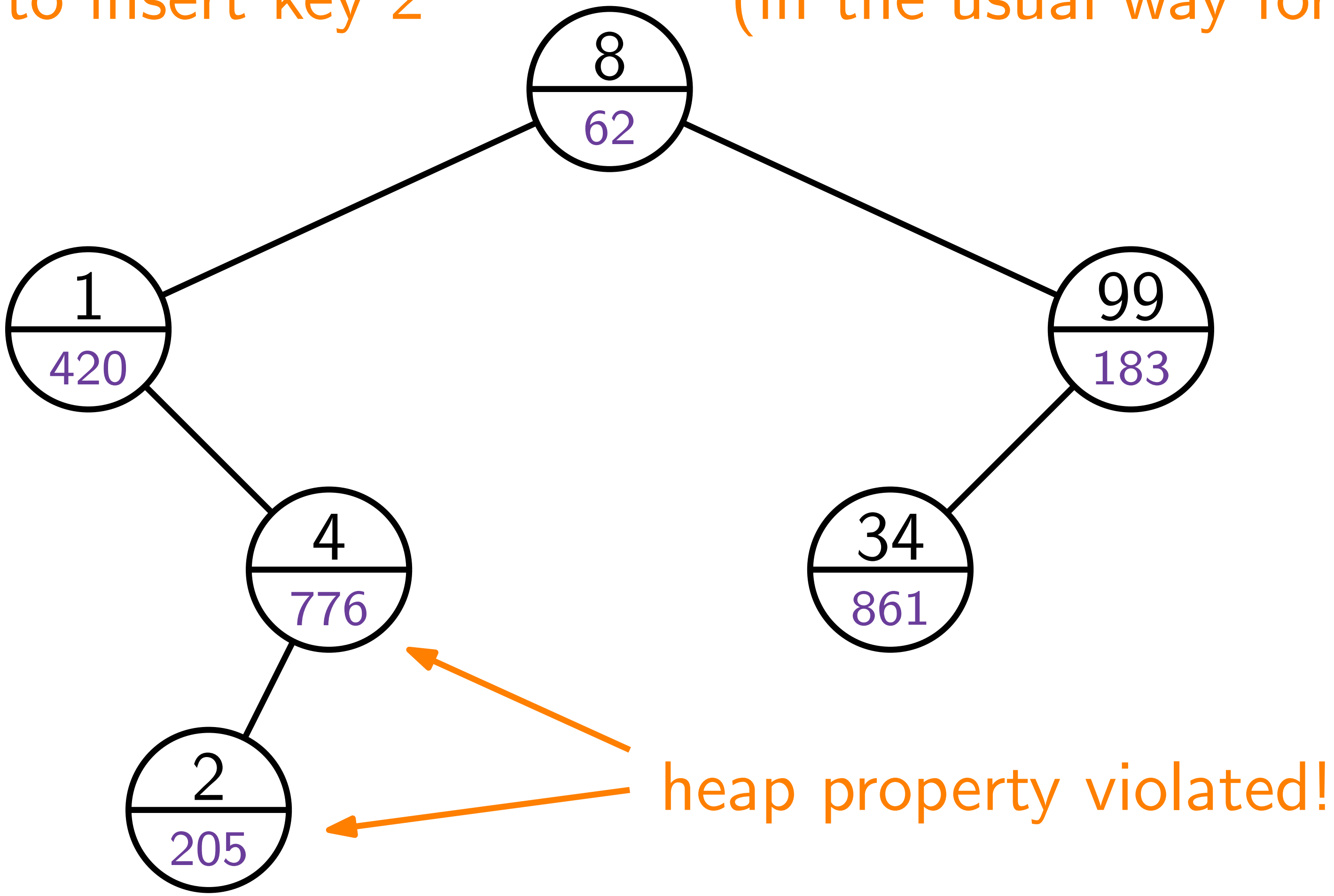
create a new node to insert key 4 (in the usual way for a binary search tree)



# Building Treaps

We build a treap for the key set $S = \{34, 8, 99, 1, 4, 2, 42, 66\}$ by inserting each key.

create a new node to insert key 2 (in the usual way for a binary search tree)



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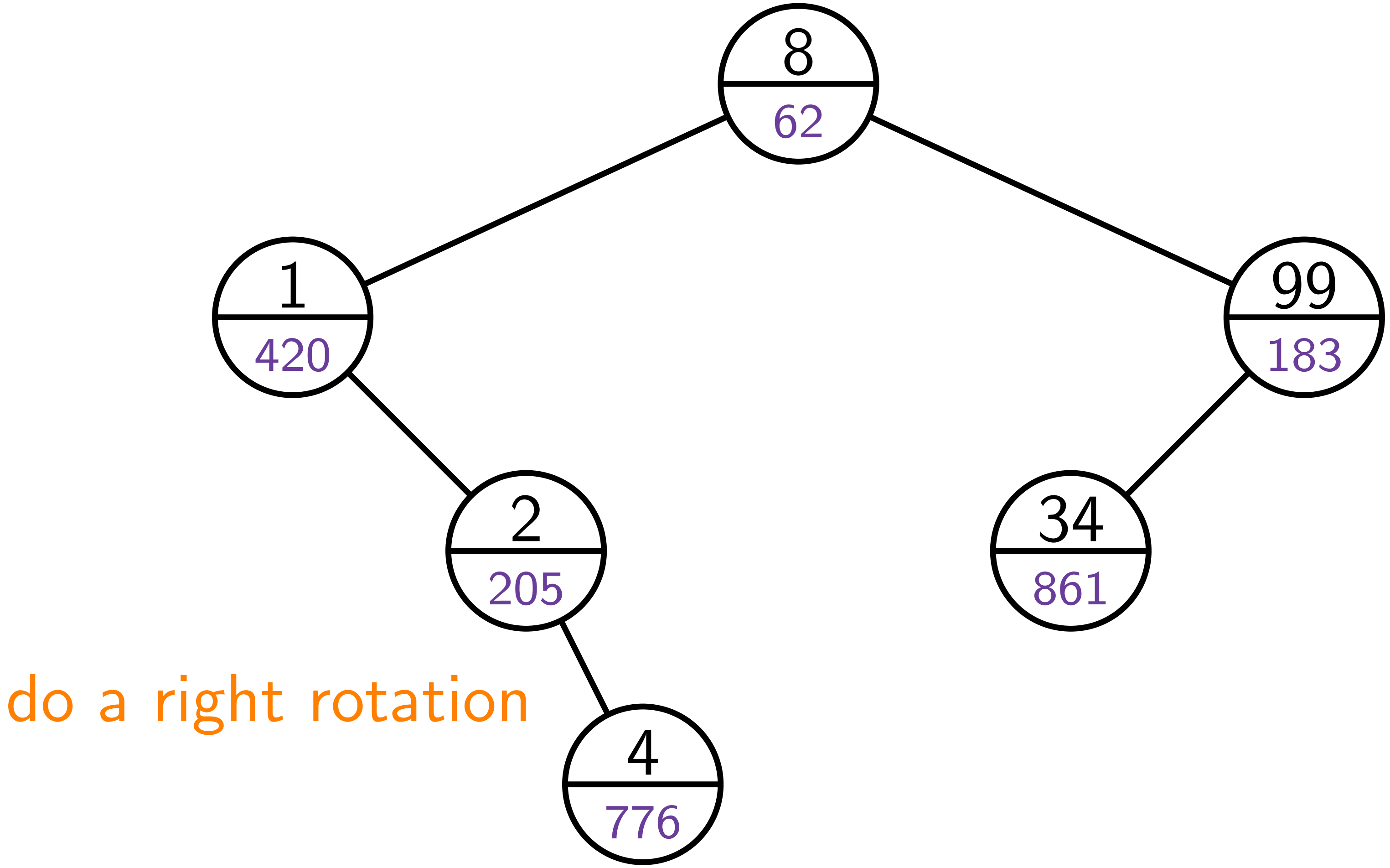

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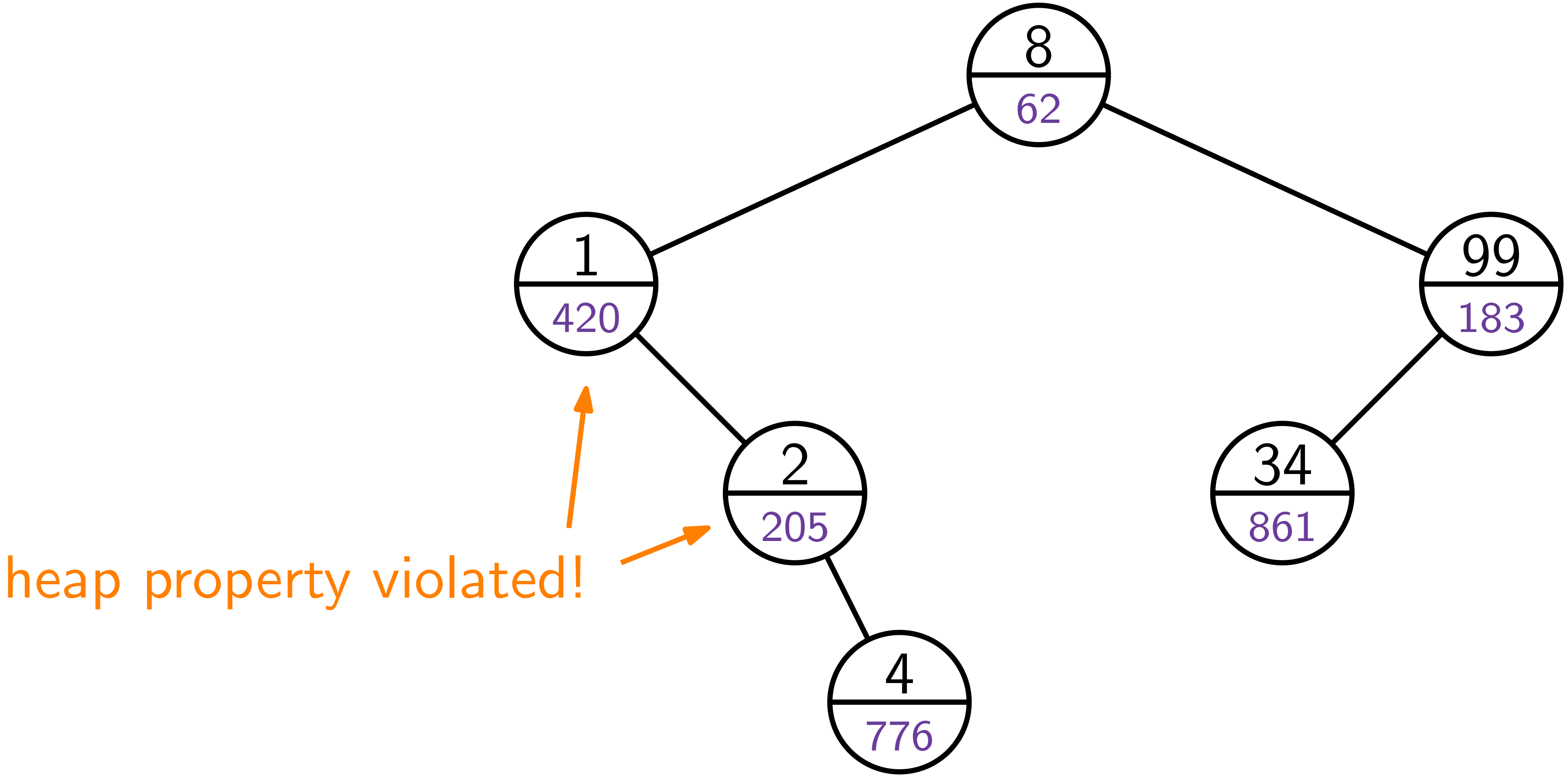

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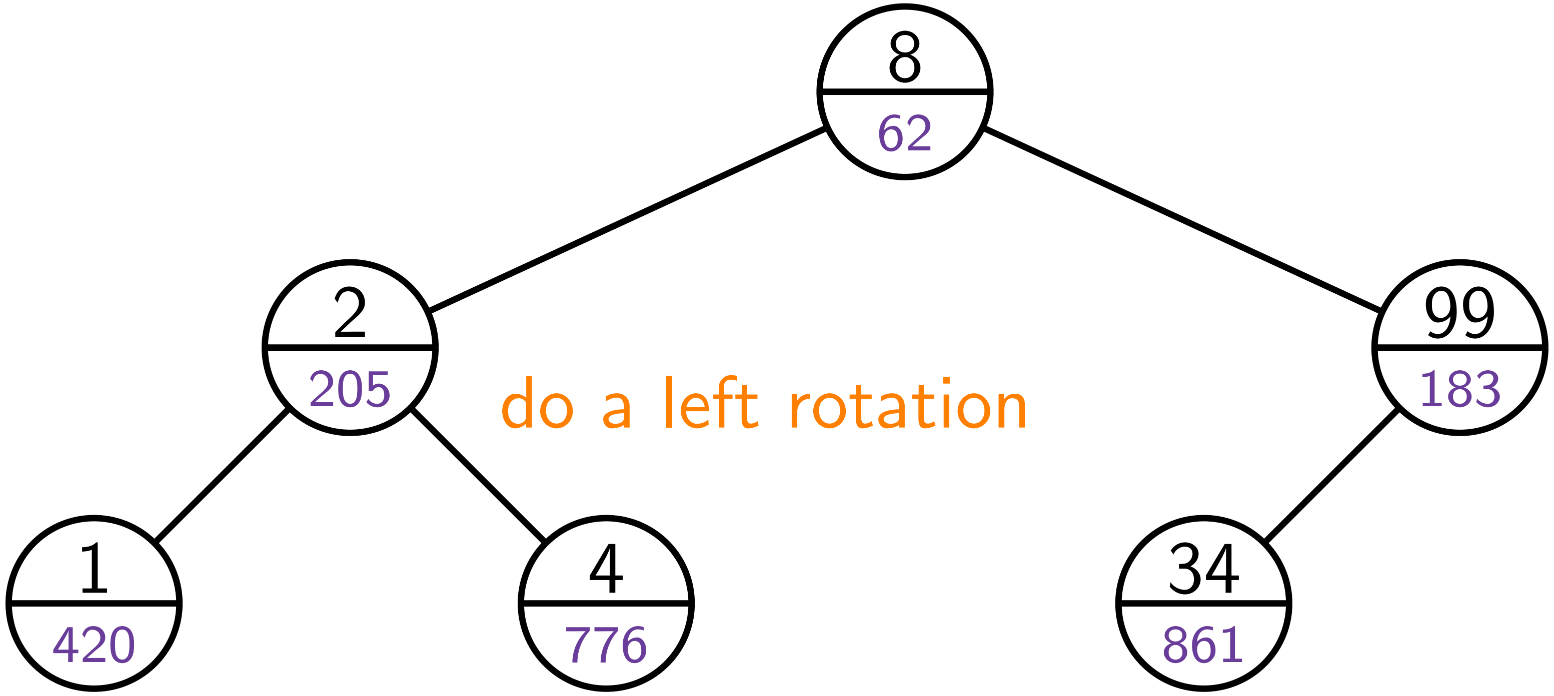
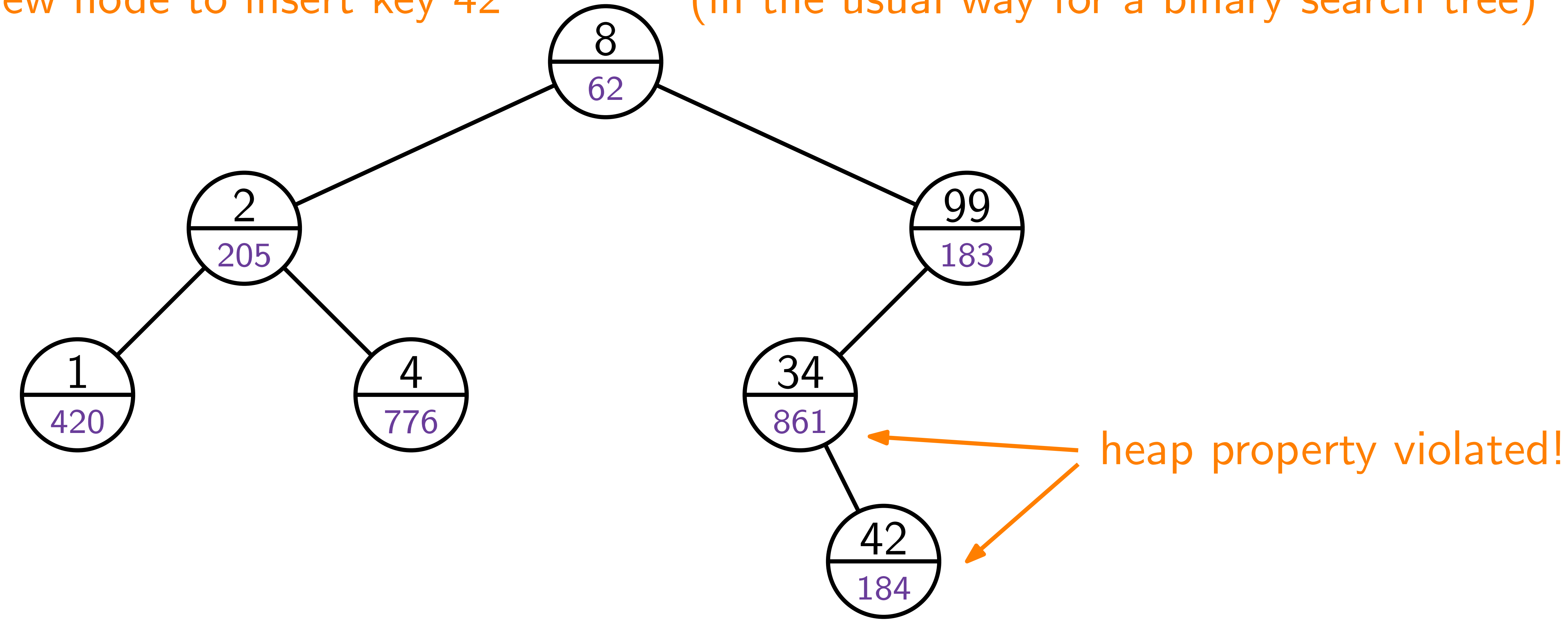


# Building Treaps

We build a treap for the key set $S = \{34, 8, 99, 1, 4, 2, 42, 66\}$ by inserting each key.

create a new node to insert key 42 (in the usual way for a binary search tree)



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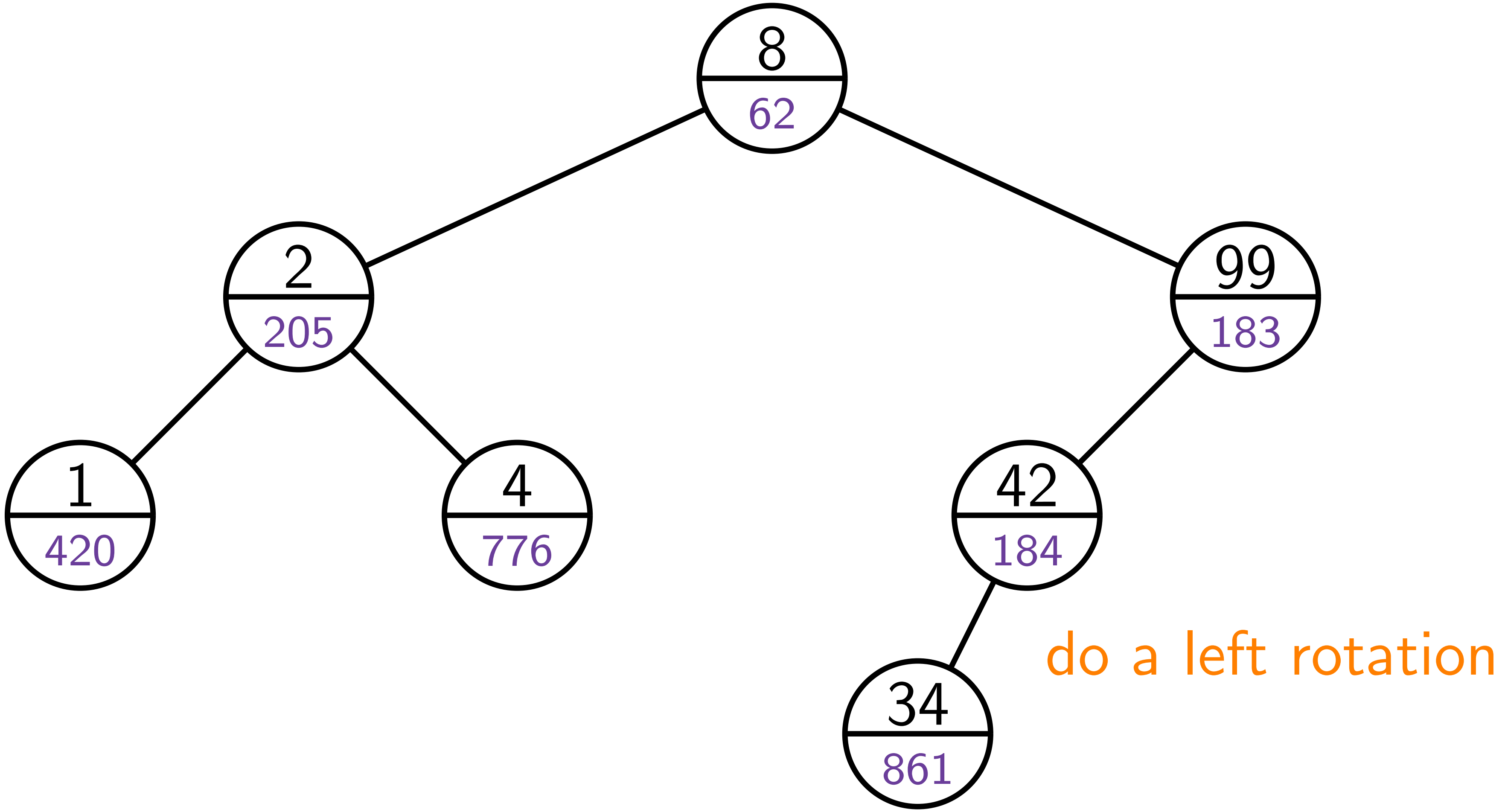
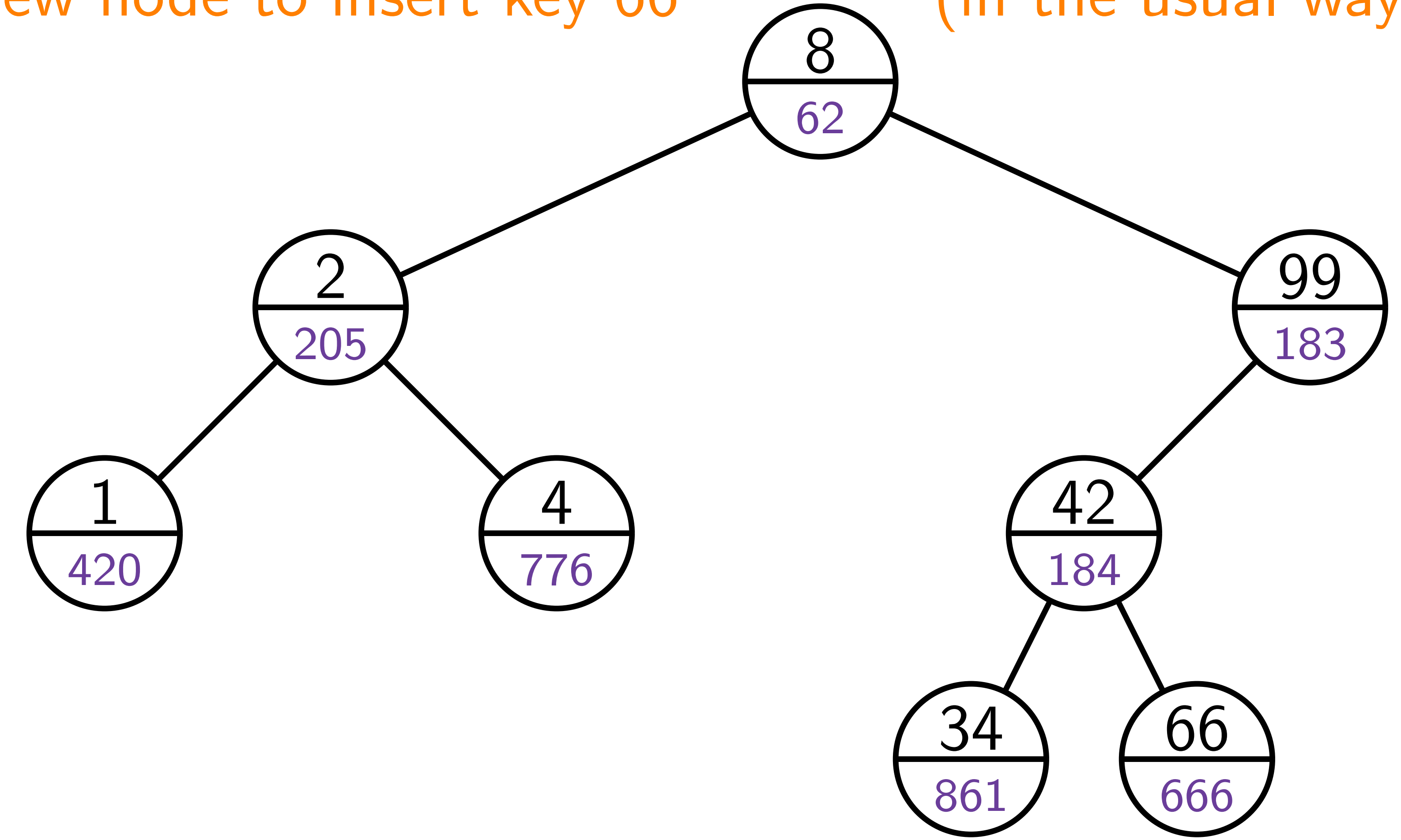


# Building Treaps

We build a treap for the key set $S = \{34, 8, 99, 1, 4, 2, 42, 66\}$ by inserting each key.

create a new node to insert key 66 (in the usual way for a binary search tree)



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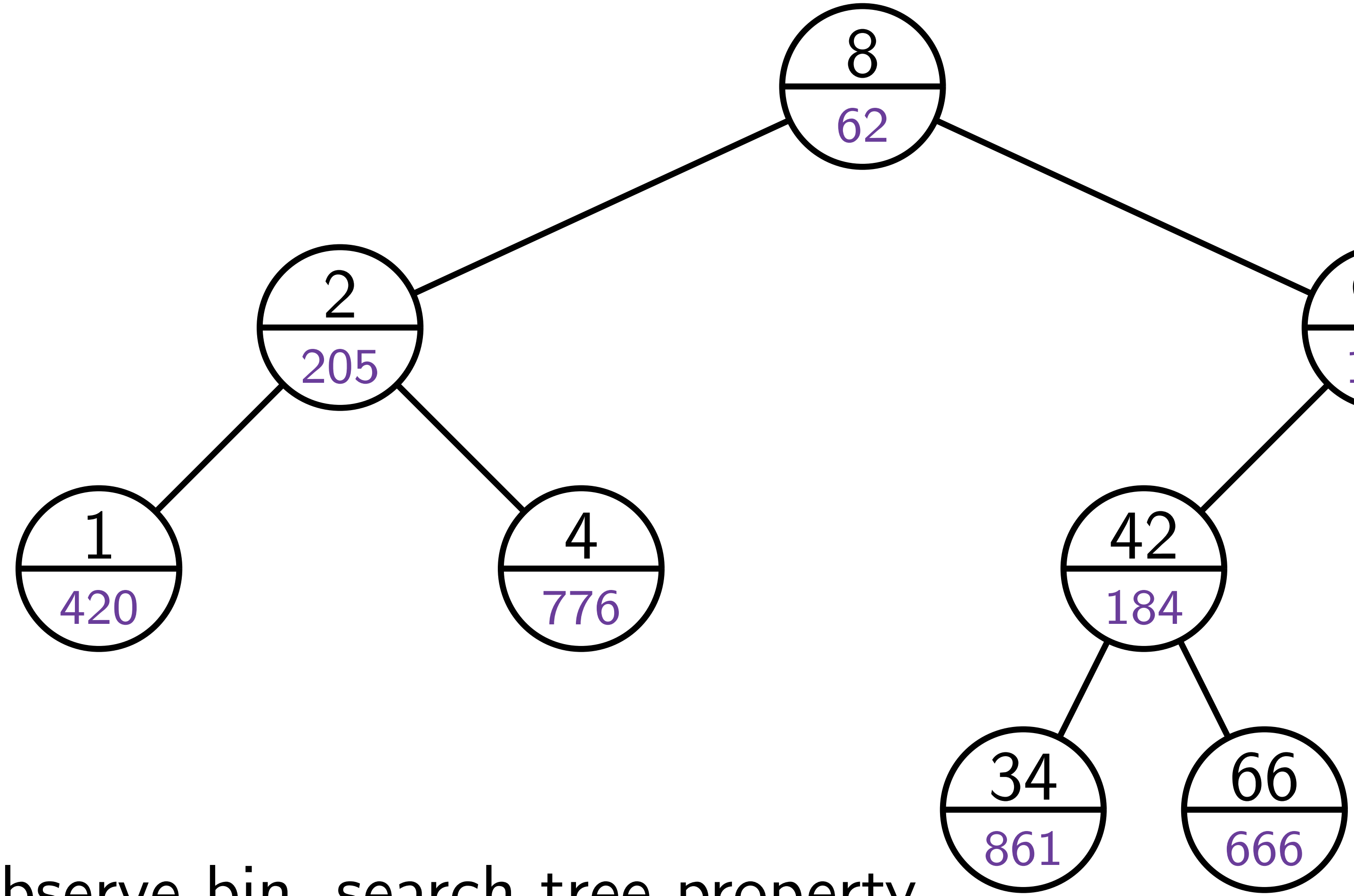


$\Rightarrow$ Keys observe bin.-search-tree property.

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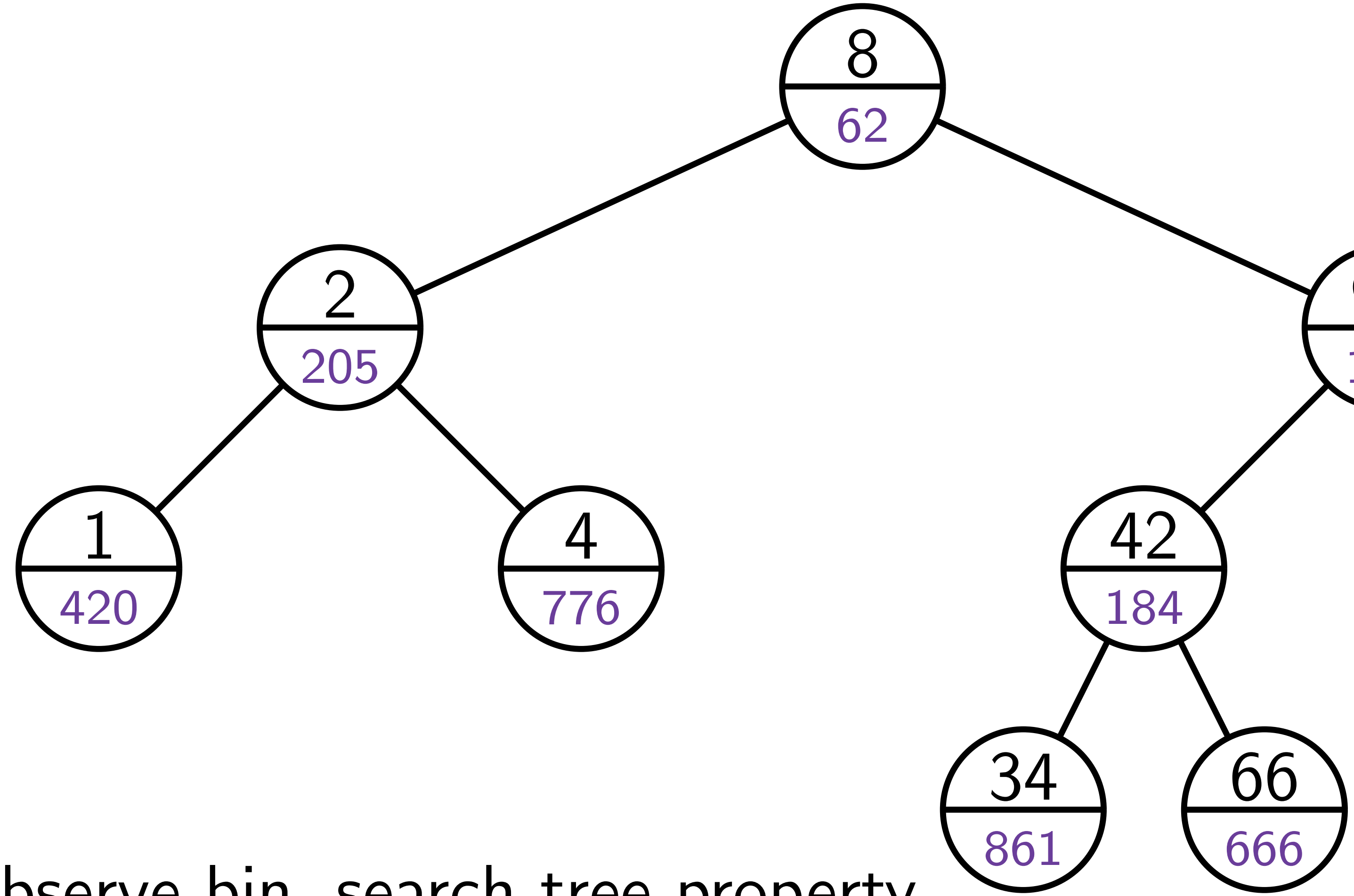


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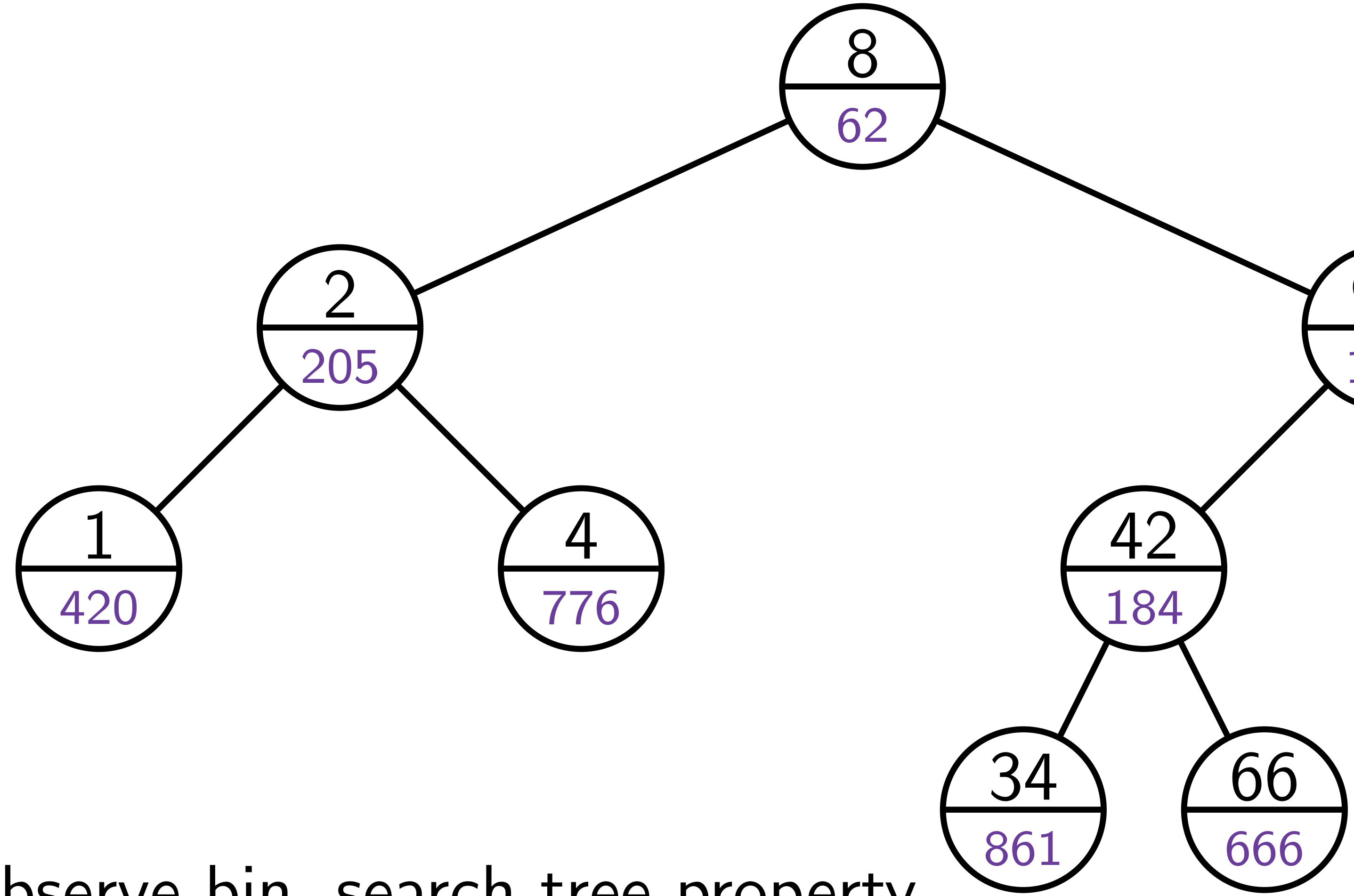


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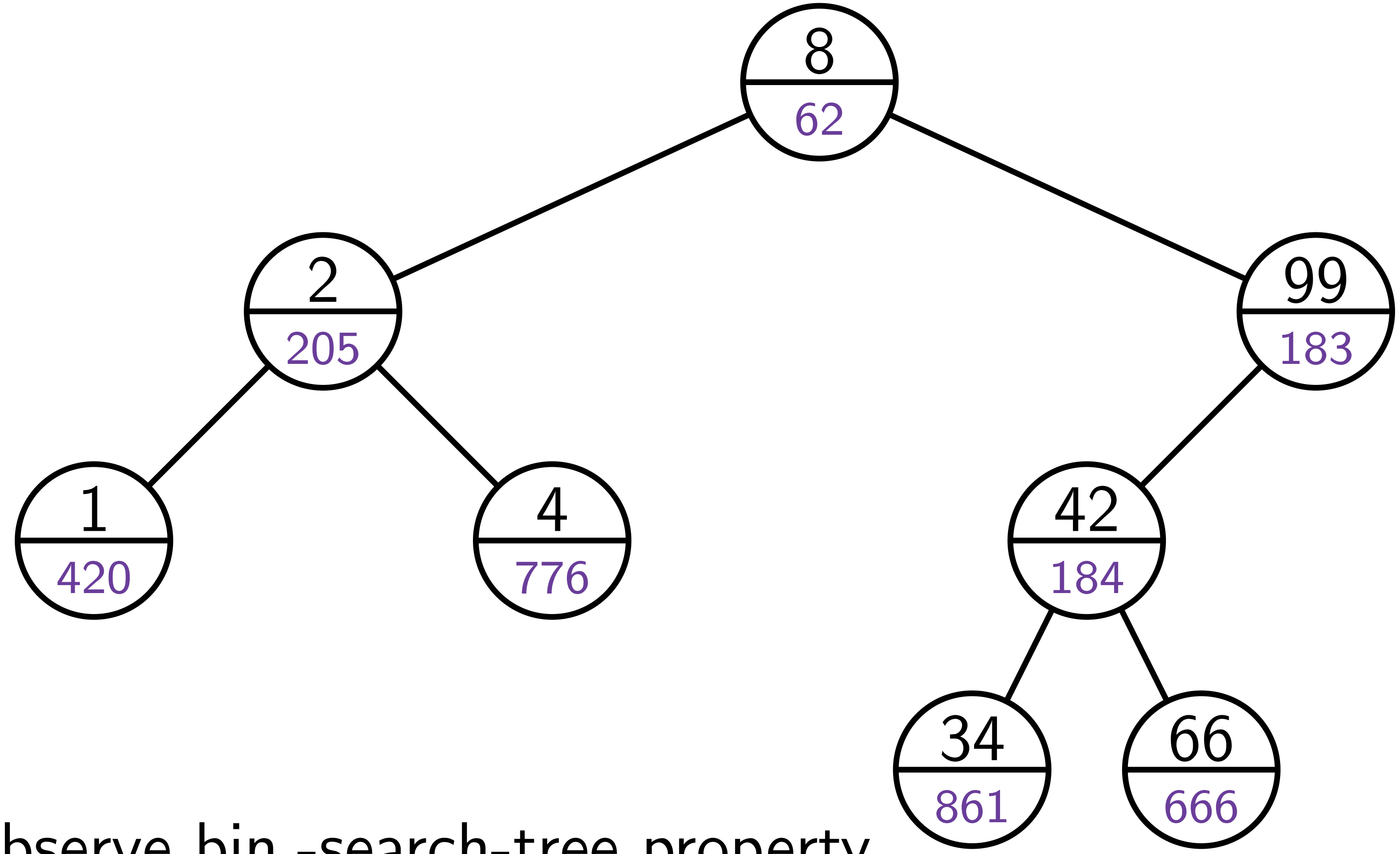


- We assume that all priorities are distinct.
- Note that the resulting tree is not necessarily balanced (in w.c. it has linear depth, but we'll see later that in expectation it is better).
- Deletion of an element works similar to insertion. $\rightarrow$ **Exercise**

$\Rightarrow$ Keys observe bin.-search-tree property.
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# Properties of Treaps

**Theorem 2.** Given the pairs of keys and priorities, the structure of a treap is unique.

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- The node with lowest priority is in the root.
- Due to the binary-search-tree property, all other nodes are uniquely assigned to the left or the right subtree of the root.
- Recursively apply this argument to the left and to the right subtree.

$\square$

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Let $x_1, x_2, \ldots, x_n$ be the keys in a treap in increasing order.

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- Hence, $Y_{ij} = 1$ if and only if $x_j$ has the lowest priority among $\{x_i, x_{i+1}, \ldots, x_j\}$. Since all priorities are chosen uniformly at random, this has prob. $1/(j-i+1)$. $\square$

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**Theorem 5.** Searching, inserting, and deleting a key in a treap can be done in expected $O(\log n)$ time.

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- Bloom filter:
  - $(+)$ $\Theta(1)$ time for containment check
  - $(-)$ may produce false positives
  - $(+)$ very low space consumption that does not depend on the lengths of the keys
  - $(-)$ allows adding keys (in $\Theta(1)$ time), but not removing keys

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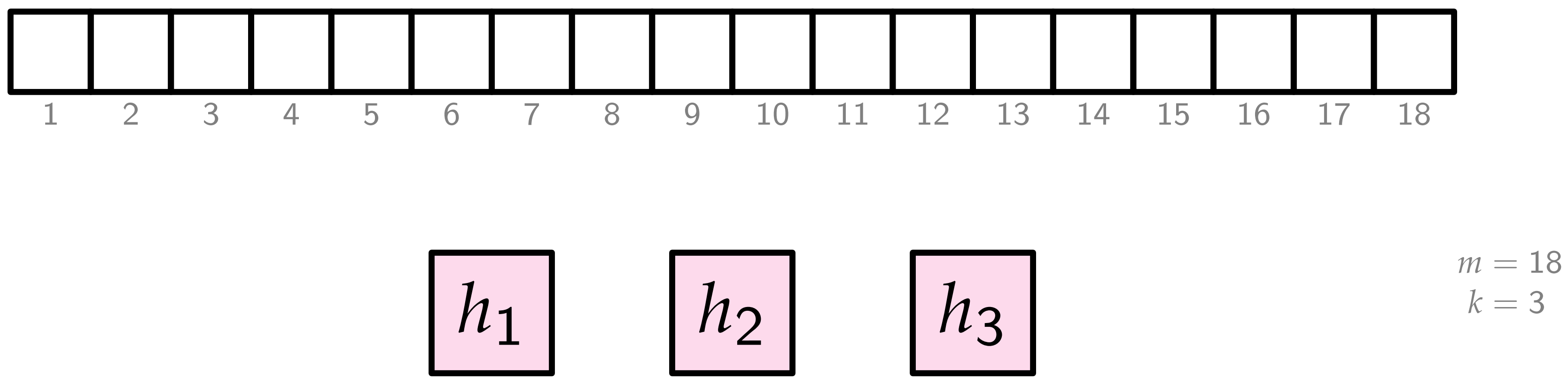

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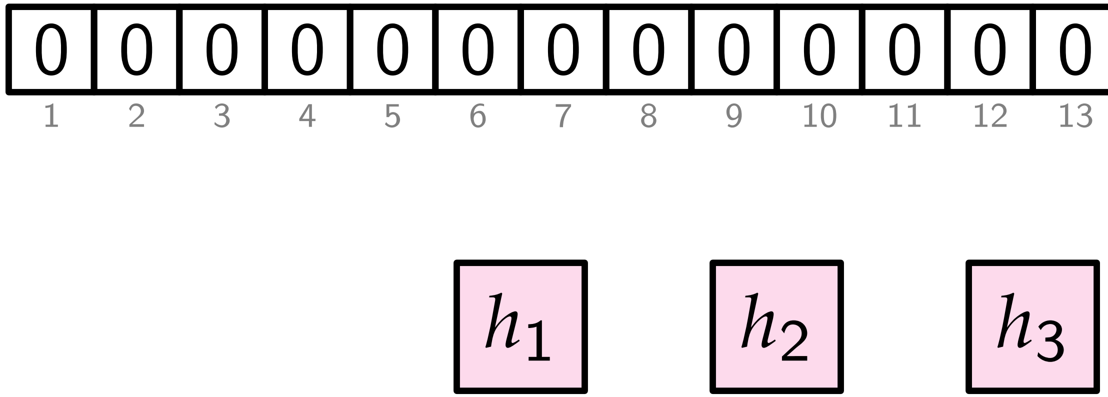


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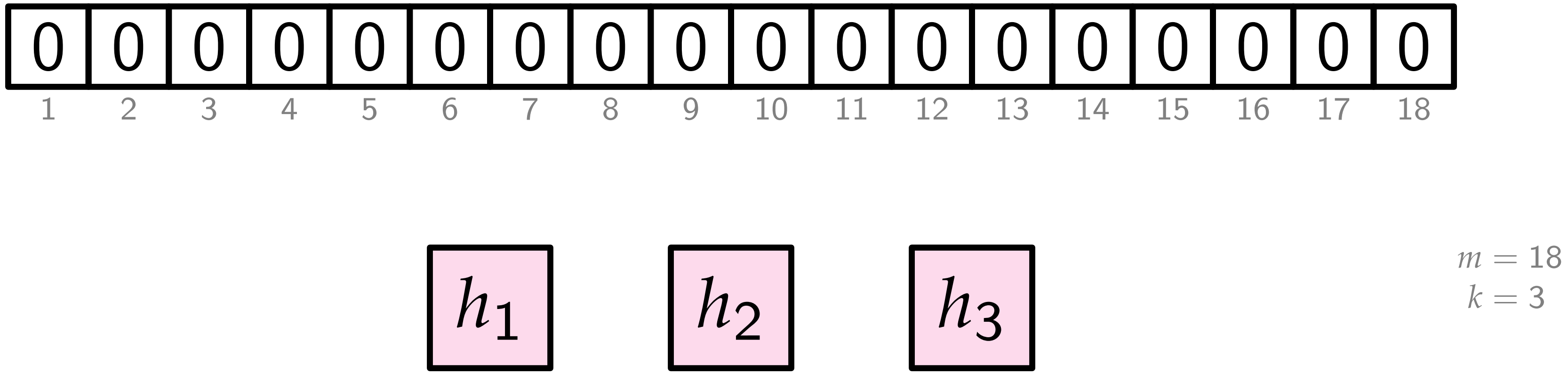

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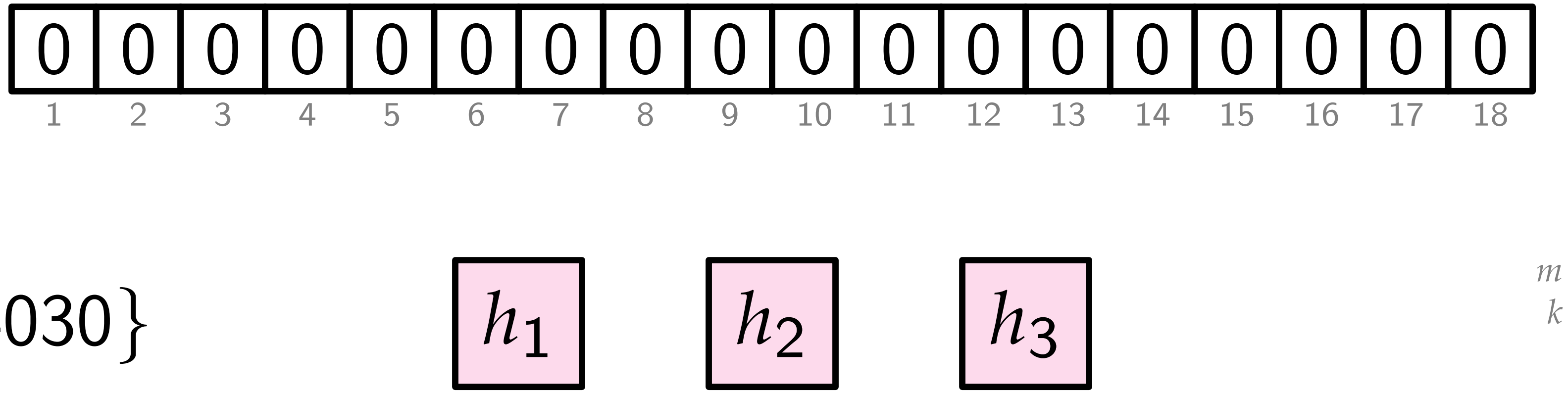


$S = \{2345, 8234, 12492, 34030\}$

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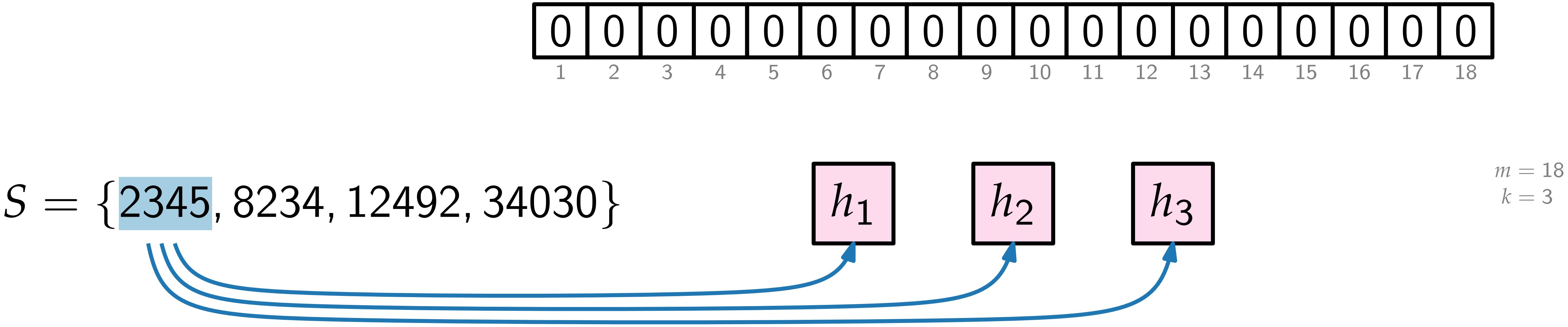

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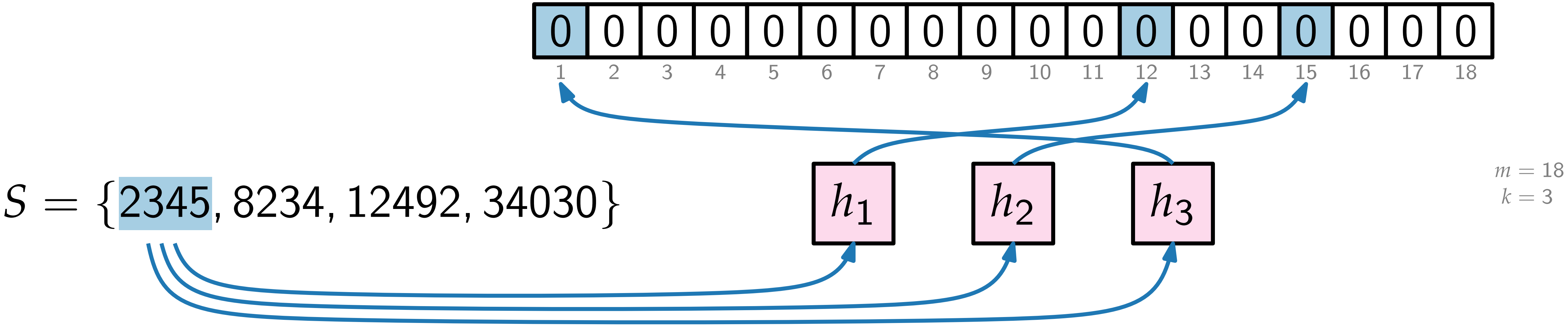

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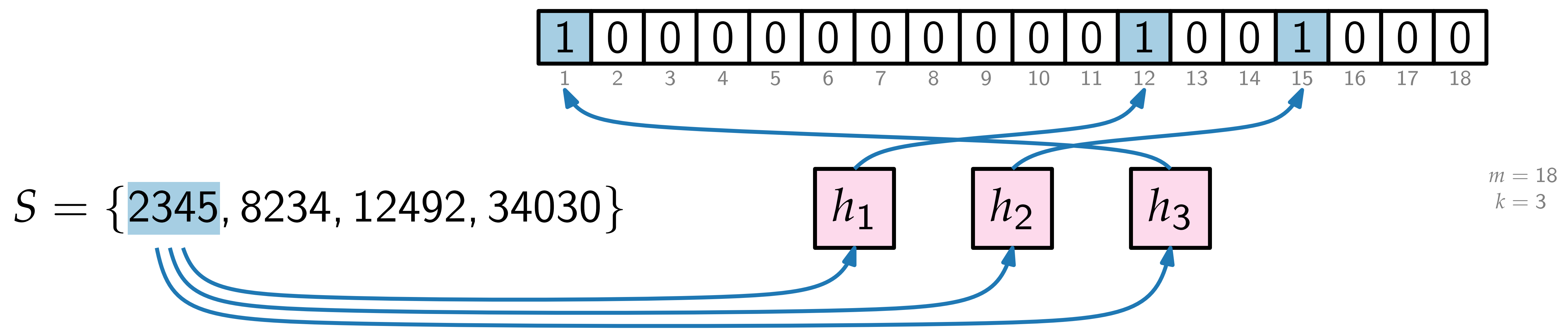

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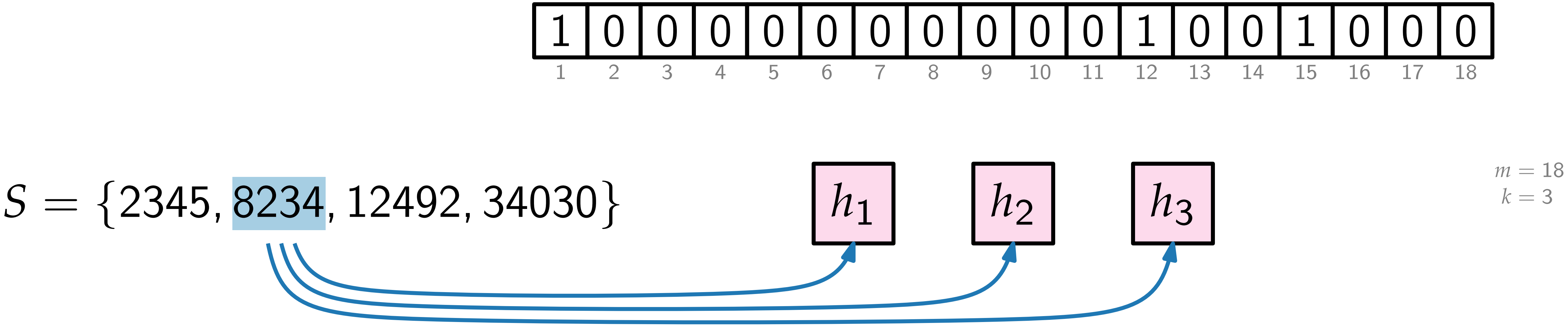

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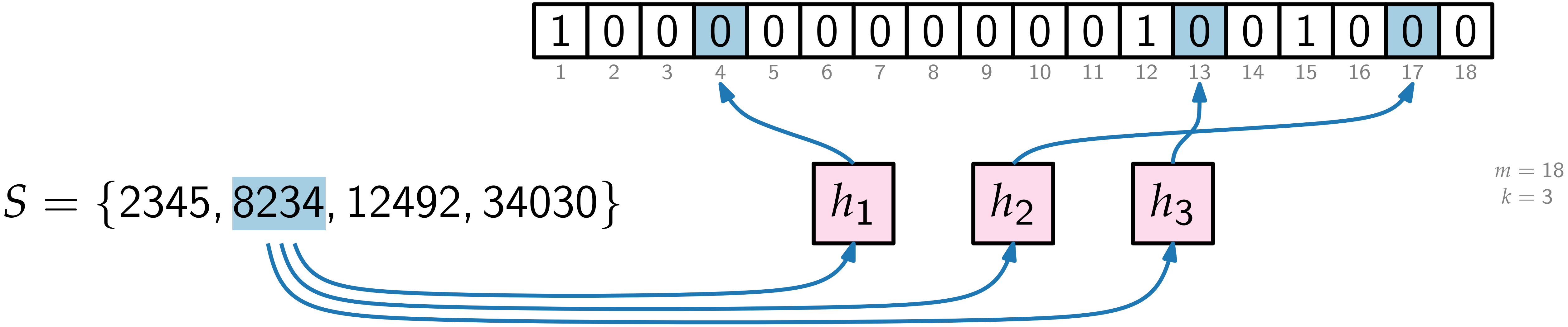

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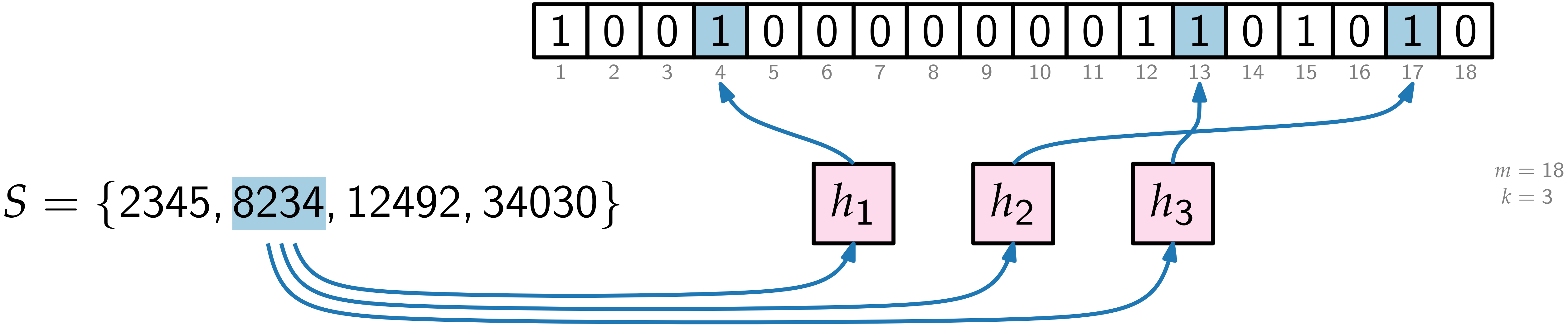

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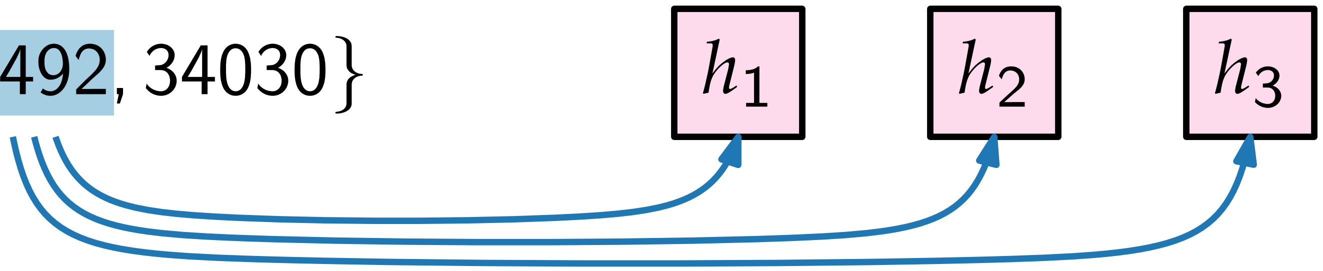


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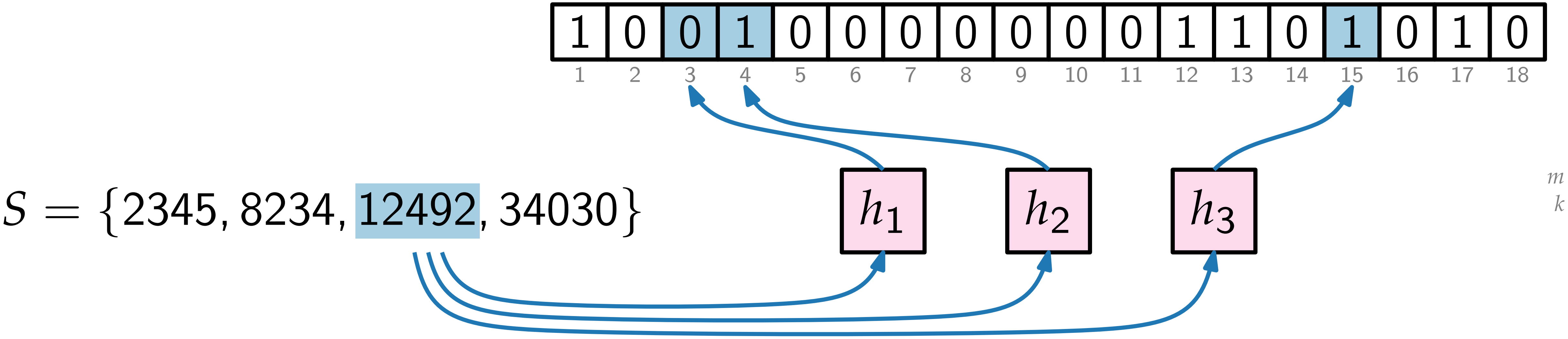


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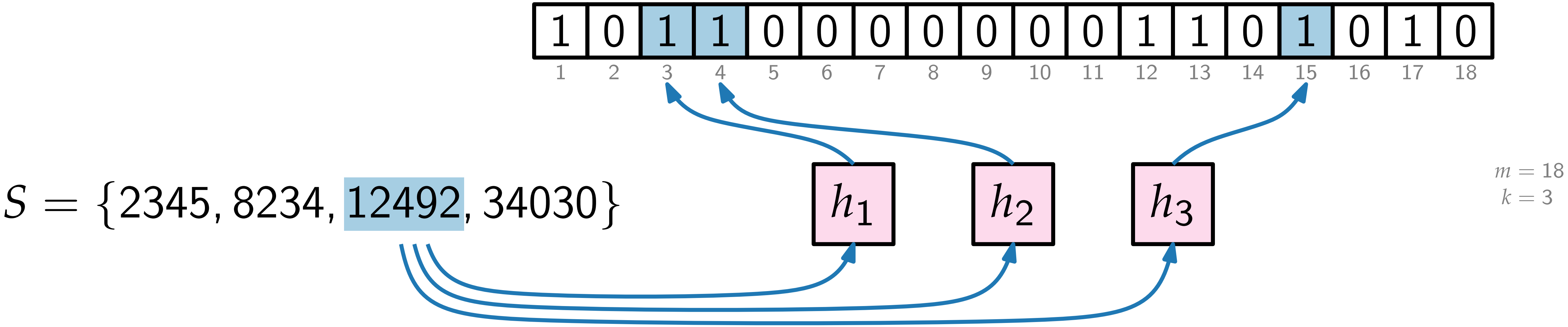

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$h_1$ $h_2$ $h_3$

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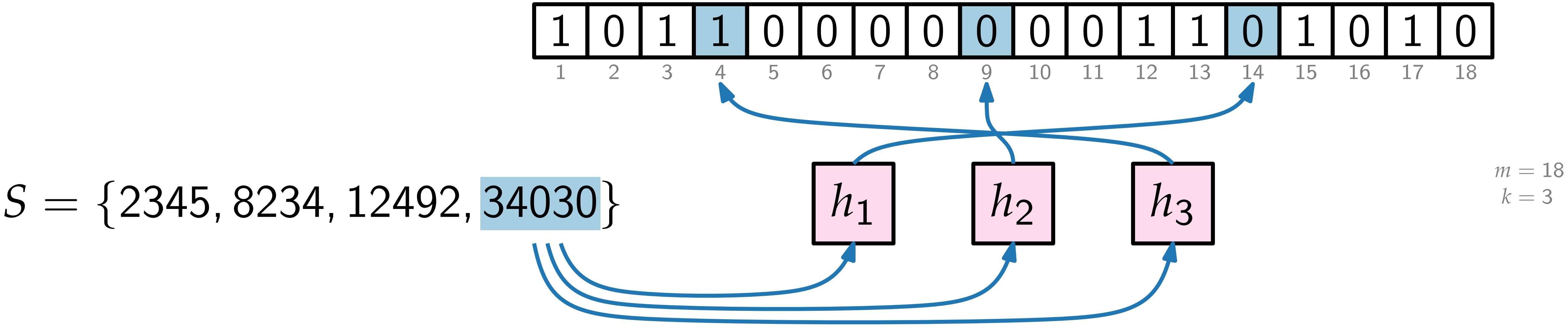

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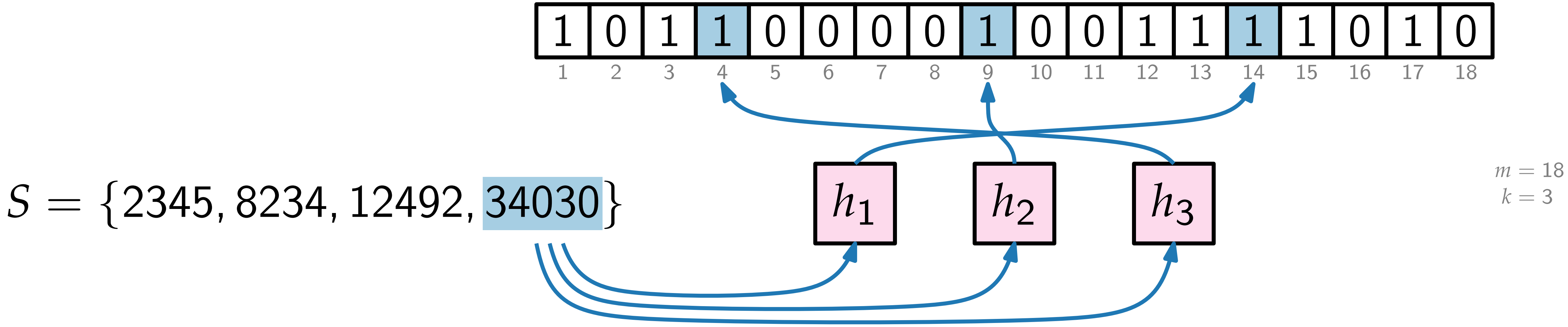

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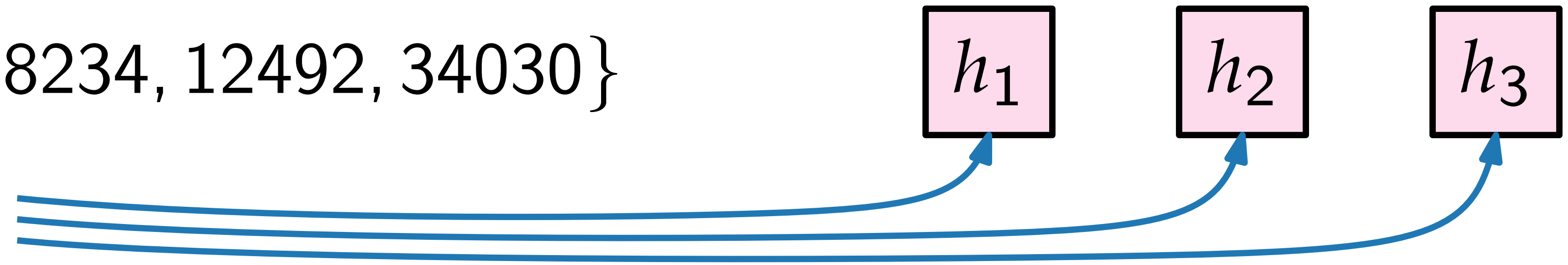
$a = 2346$



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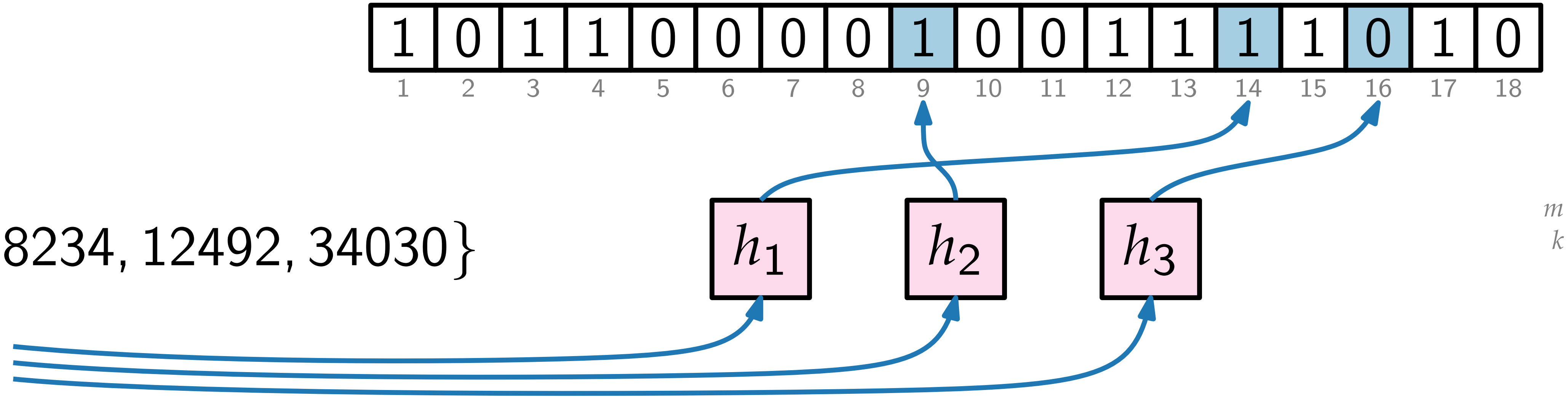


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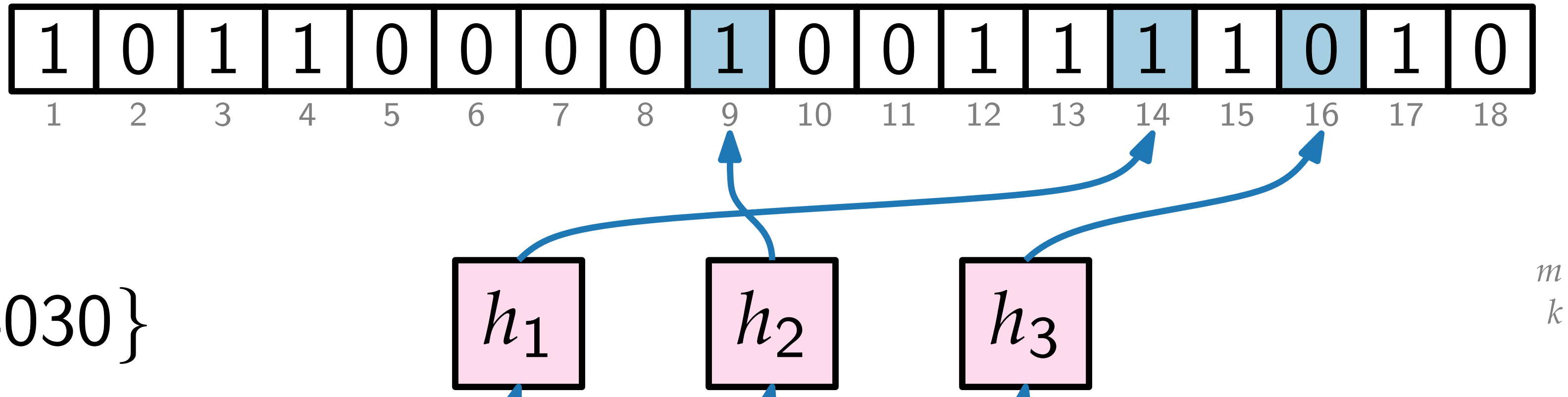


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$a = 2346$ (correct) ✗

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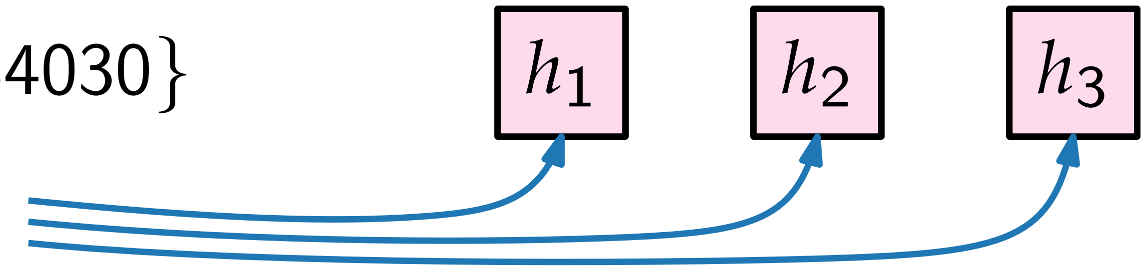

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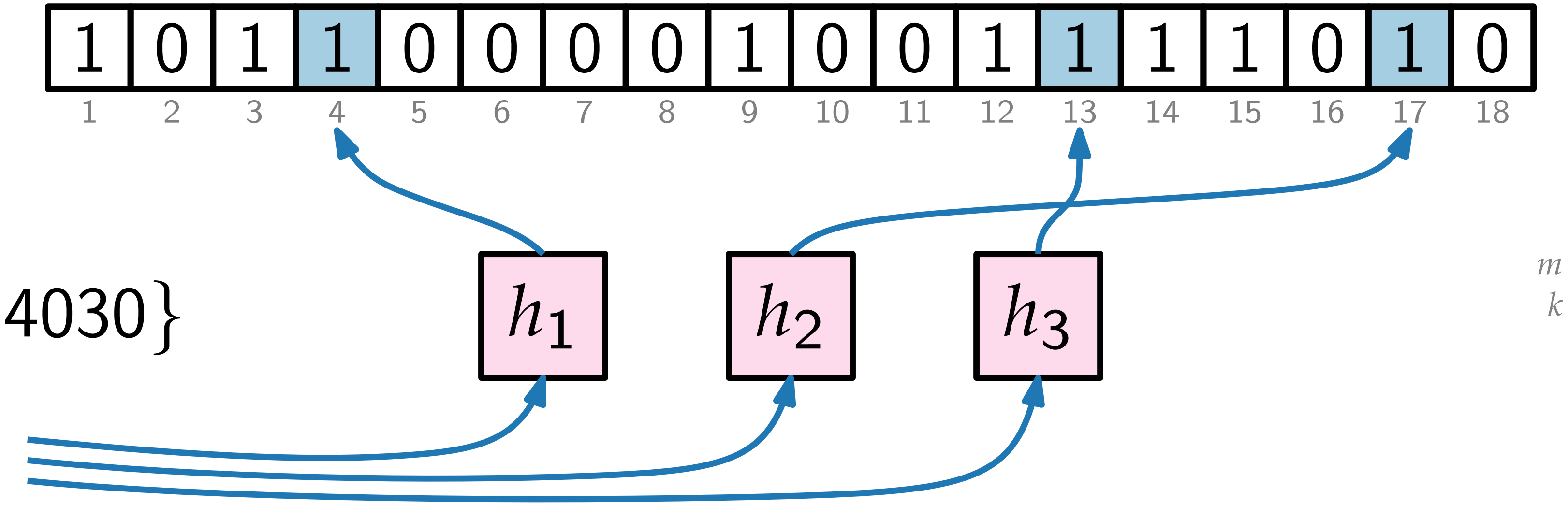


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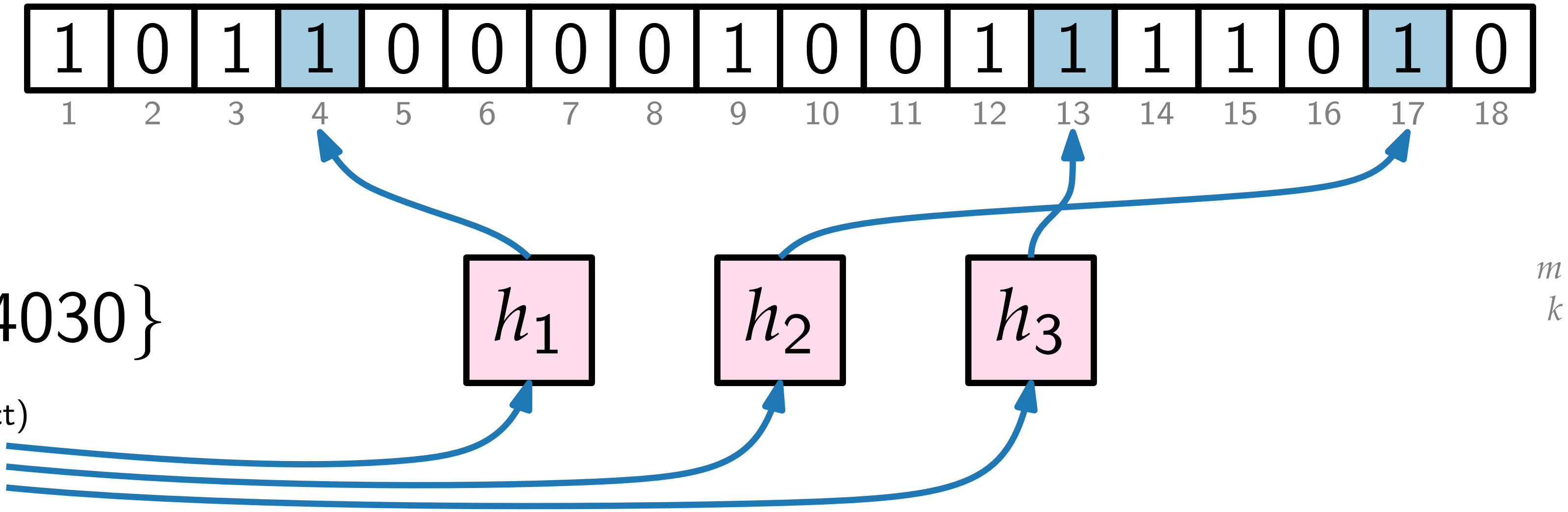


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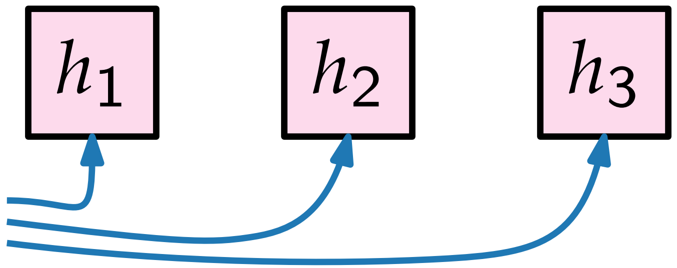

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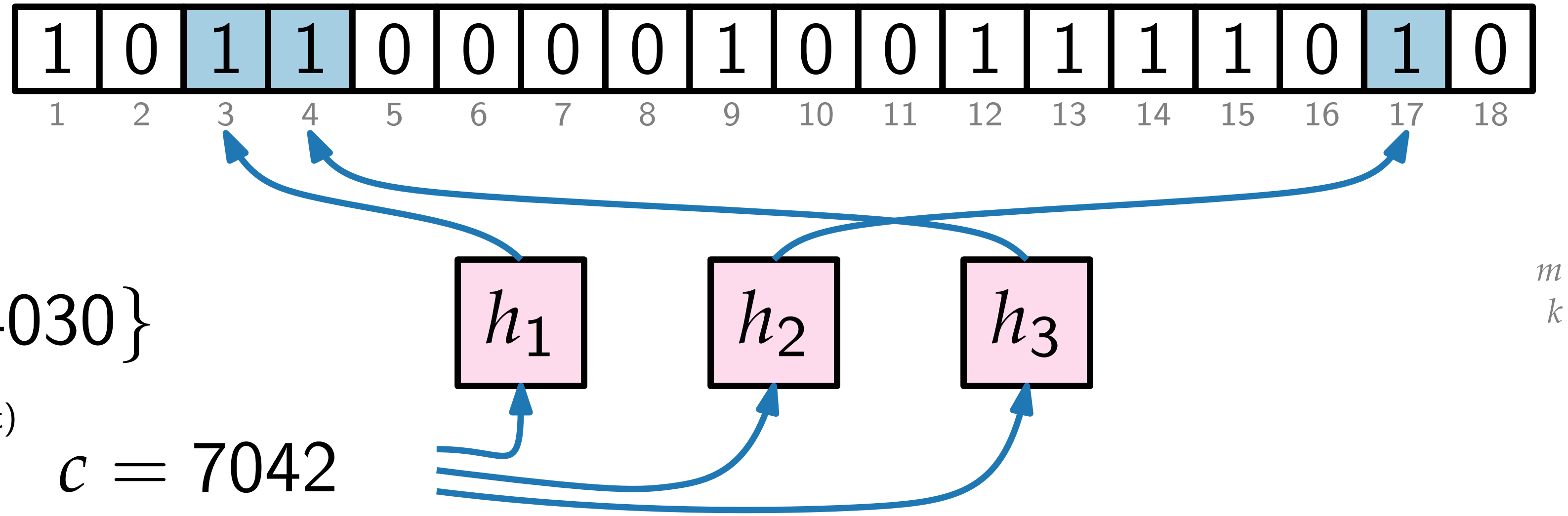


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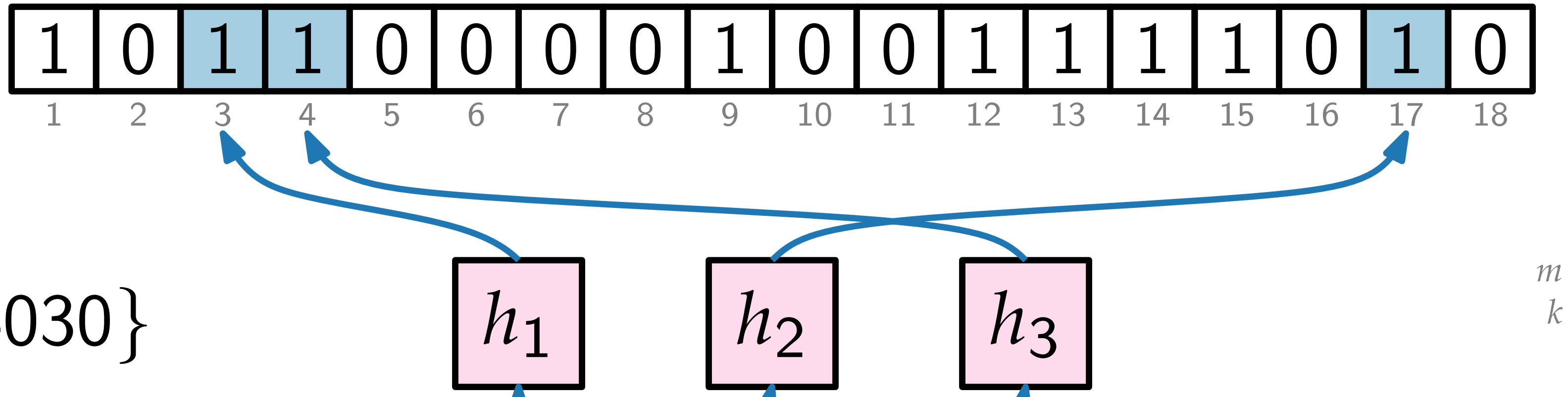


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The probabilities that all bits at positions $h_1(a), \ldots, h_k(a)$ are set to 1 are not independent. However, one can still show that the error probability $\varepsilon$ for a false positive is relatively close to $\varepsilon \approx \left(1 - \left(1 - \frac{1}{m}\right)^{kn}\right)^k \approx \left(1 - e^{-kn/m}\right)^k$.

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So, the number of bits in our array depends on the desired error probabilty,

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If we only use the optimal $k$, the error probibilty $\varepsilon \approx \left(\frac{1}{2}\right)^{m \ln 2 / n}$.

Thus, the optimal number of bits per key in our set is $\frac{m}{n} \approx -\frac{\log_2 \varepsilon}{\ln 2} \approx -1.44 \log_2 \varepsilon$.

So, the number of bits in our array depends on the desired error probabilty,

| |
|---|
| error probabiltiy $\varepsilon$ |
| bits per key $\frac{m}{n}$ |
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10 %

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|---|---|
| error probabiltiy $\varepsilon$ | 0.1 |
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| error probabiltiy $\varepsilon$ | 0.1 | 0.01 (1 %) |
|---|---|---|
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1 ‰

| error probabiltiy $\varepsilon$ | 0.1 | 0.01 | 0.001 |
|---|---|---|---|
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1 in a million ($10^6$)

| error probabiltiy $\varepsilon$ | 0.1 | 0.01 | 0.001 | 0.000001 |
|---|---|---|---|---|
| bits per key $\frac{m}{n}$ | 12 | 23 | 34 | 67 |
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So, the number of bits in our array depends on the desired error probabilty,

1 in a billion ($10^9$)

| error probabiltiy $\varepsilon$ | 0.1 | 0.01 | 0.001 | 0.000001 | 0.000000001 |
|---|---|---|---|---|---|
| bits per key $\frac{m}{n}$ | 12 | 23 | 34 | 67 | 100 |
| # hash functions $k$ | 4 | 7 | 10 | 20 | 30 |

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1 in a trillion ($10^{12}$)

| error probabiltiy $\varepsilon$ | 0.1 | 0.01 | 0.001 | 0.000001 | 0.000000001 | 0.000000000001 |
|---|---|---|---|---|---|---|
| bits per key $\frac{m}{n}$ | 12 | 23 | 34 | 67 | 100 | 133 |
| # hash functions $k$ | 4 | 7 | 10 | 20 | 30 | 40 |

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. . . but not on the lengths of the keys.
(We could check for whole documents whether they are there or not.)

# Discussion of Skip Lists and Treaps

- (Randomized) skip list provide a simpler alternative to balanced binary search trees with (in expectation) the same asymptotic time complexities for the basic operations.

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- Beside treaps, there is a random. data structure called *randomized binary search tree*.
- A randomized binary search tree does not store priorities, instead, when a key is inserted, it replaces with probability $1/(n+1)$ the current root; if it does not replace the root, this process is repeated in the corresponding subtree.
- Hence, a randomized binary search tree stores less information (the size of the subtree instead of a priority) and there is no risk of a collision between priorities. However, there are more requests to the random number generator.

# Discussion of Bloom Filters

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- There are some refinements of classical Bloom filters to overcome some disadvantages, e.g., *counting Bloom filters*:
  - Instead of $m$ bits, $m$ counters are stored.
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  - To insert a key, the counters determined by the hash functions are incremented.
  - To remove a key, the counters determined by the hash functions are decremented.
- Bloom filters are used for
  - Internet search engines
  - caching objects in Internet applications (is an image or a digest in the cache?)
  - databases (Google Bigtable, Apache HBase, Apache Cassandra, PostgreSQL)
  - web browsers (Google Chrome used one to identify malicious URLs)
  - crypto currencies (finding logs in Ethereum)
  - hiding real data, while indicating if an object is in a set (e.g., database of criminals)

# Literature

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