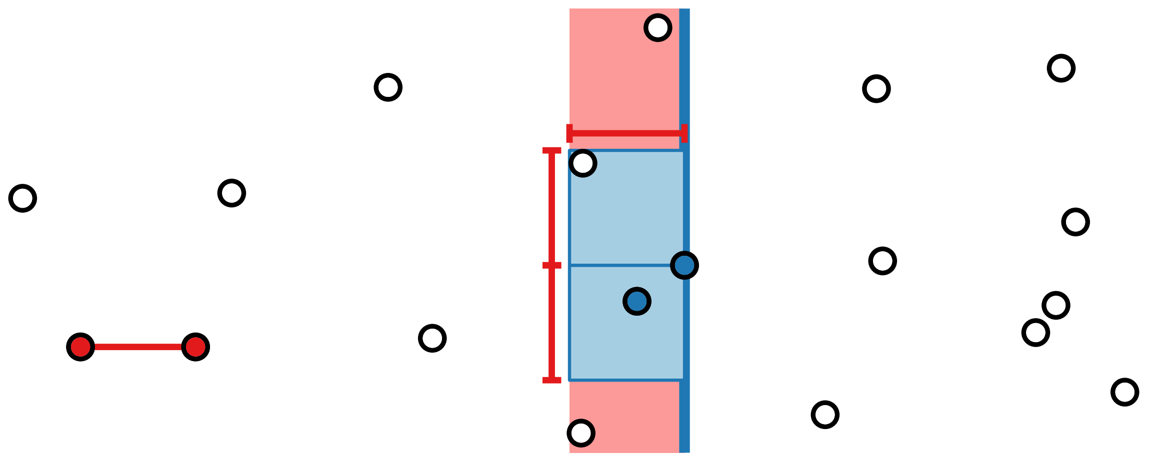
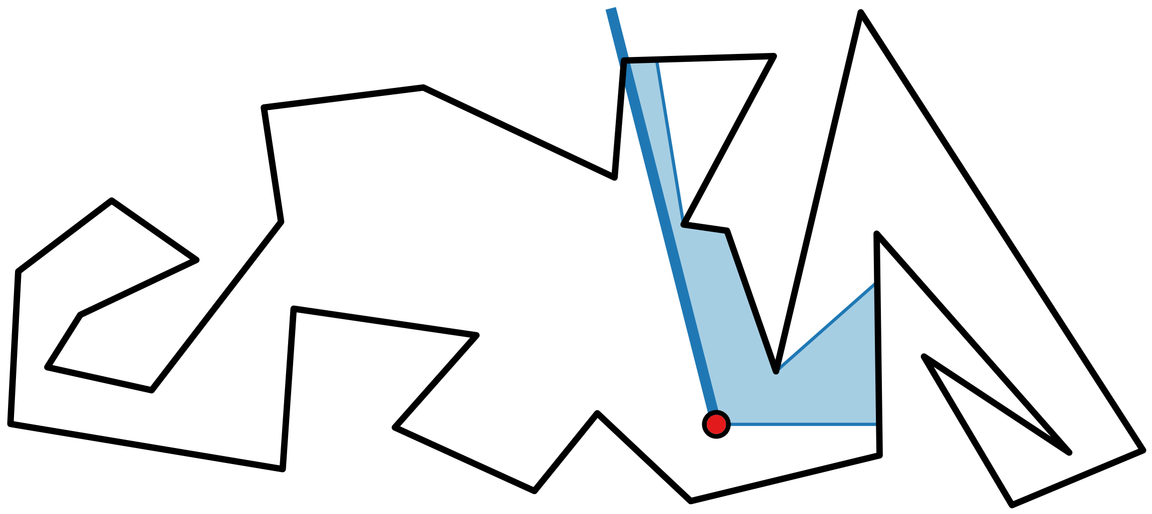
Advanced Algorithms

Computational Geometry

Sweep-Line Algorithms

Johannes Zink · WS23/24



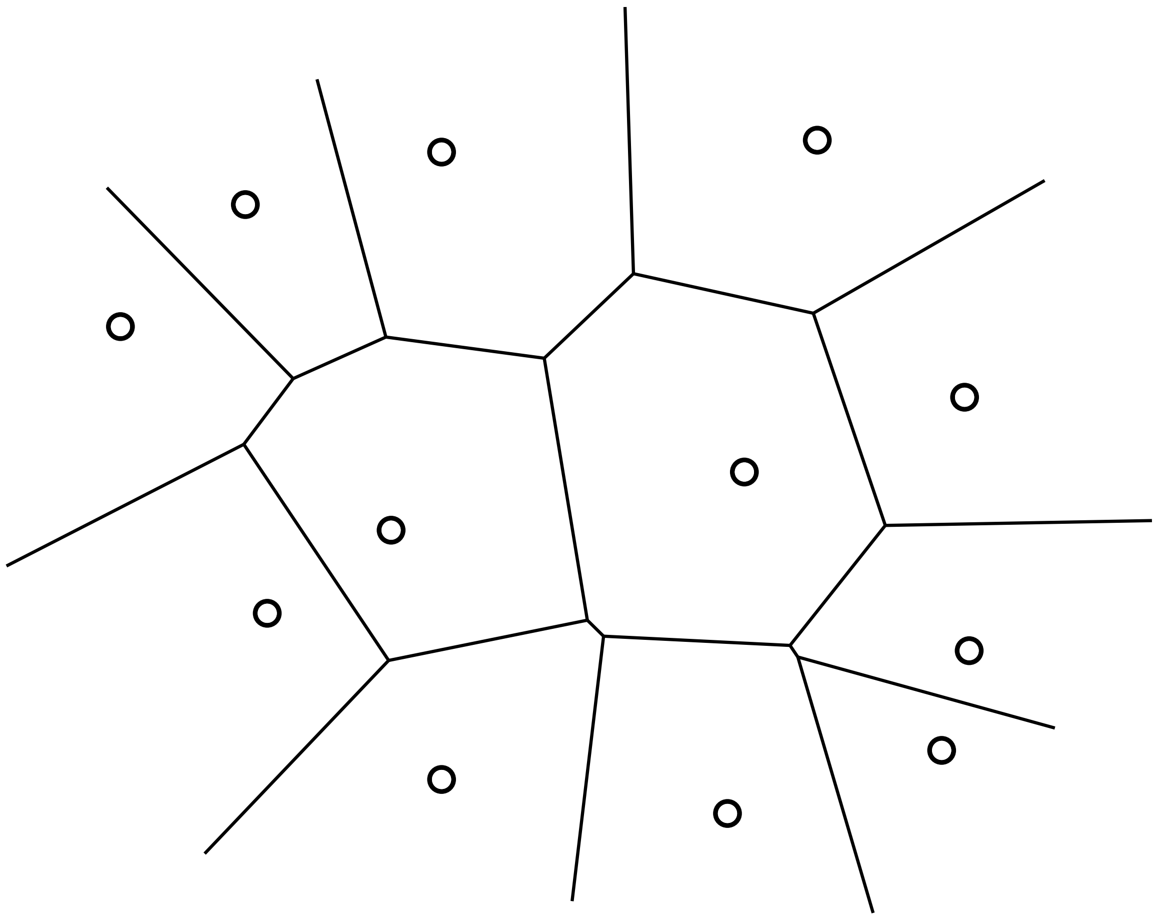


Introduction

Computational geometry is about algorithmic problems that involve geometric objects such as points, line segments, lines, polygons, circles, planes, polyhedra, . . .

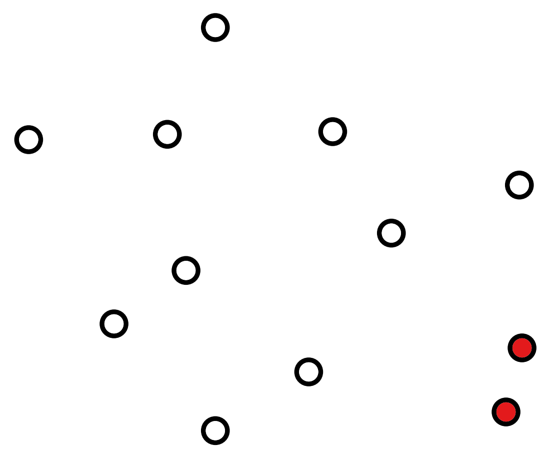
Some problems:

- Closest Pair
- Line segment intersection
- Determining visibility
- Guarding an art gallery
- Triangulating a polygon
- Motion planning
- Finding the closest post office
- and many more.



We offer an entire course on computational geometry in the winter term!

Closest Pair



Given: (multi-)set of points $P \subseteq \mathbb{R}^2$.

Task: Find a pair of distinct elements $p_a, p_b \in P$ such that the Euclidean distance $||p_a - p_b||$ is minimum.

Deterministic algorithms:

Brute-force	$\mathcal{O}(n^2)$		
Divide and conquer (recall from ADS)	$\mathcal{O}(n \log n)$	(optimal)	
Sweep line	$\mathcal{O}(n \log n)$	(optimal)	**now!**

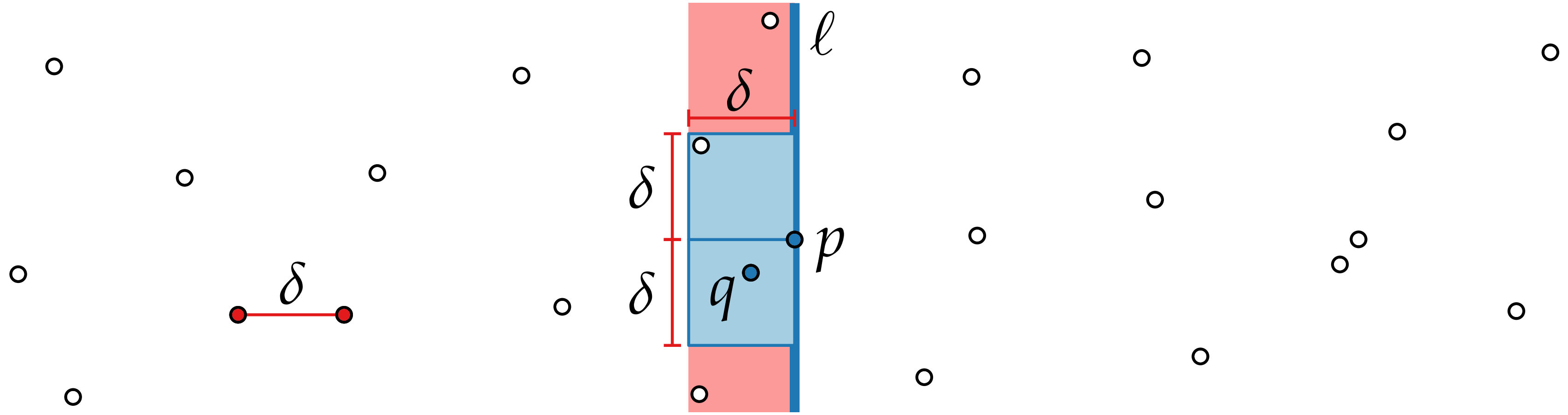
Randomized algorithm:

Randomized incremental construction	$\mathcal{O}(n)$	(expected runtime)
	later in this course!	

A Sweep Line Approach for CLOSEST PAIR

Assumption: The points in P have pairwise distinct x-coordinates.

Idea: Sweep the plane from left to right with a vertical line ℓ (the **sweep line**).



Invariant: a closest pair of the points to the left of ℓ and its distance δ is already known.

Observations:

- This partial solution can only change when ℓ sweeps a point p of P.
- Each new closest pair consists of p and a point q with distance $< \delta$ to ℓ.
- q needs to be located in a $\delta \times 2\delta$ rectangle R to the left of p.
- R contains $\mathcal{O}(1)$ points of $P \setminus \{p\}$ since their pairwise distance is $\geq \delta$. (packing argument)

Computing the Points in R Efficiently

Let S denote the vertical slab of width δ to the left of ℓ.

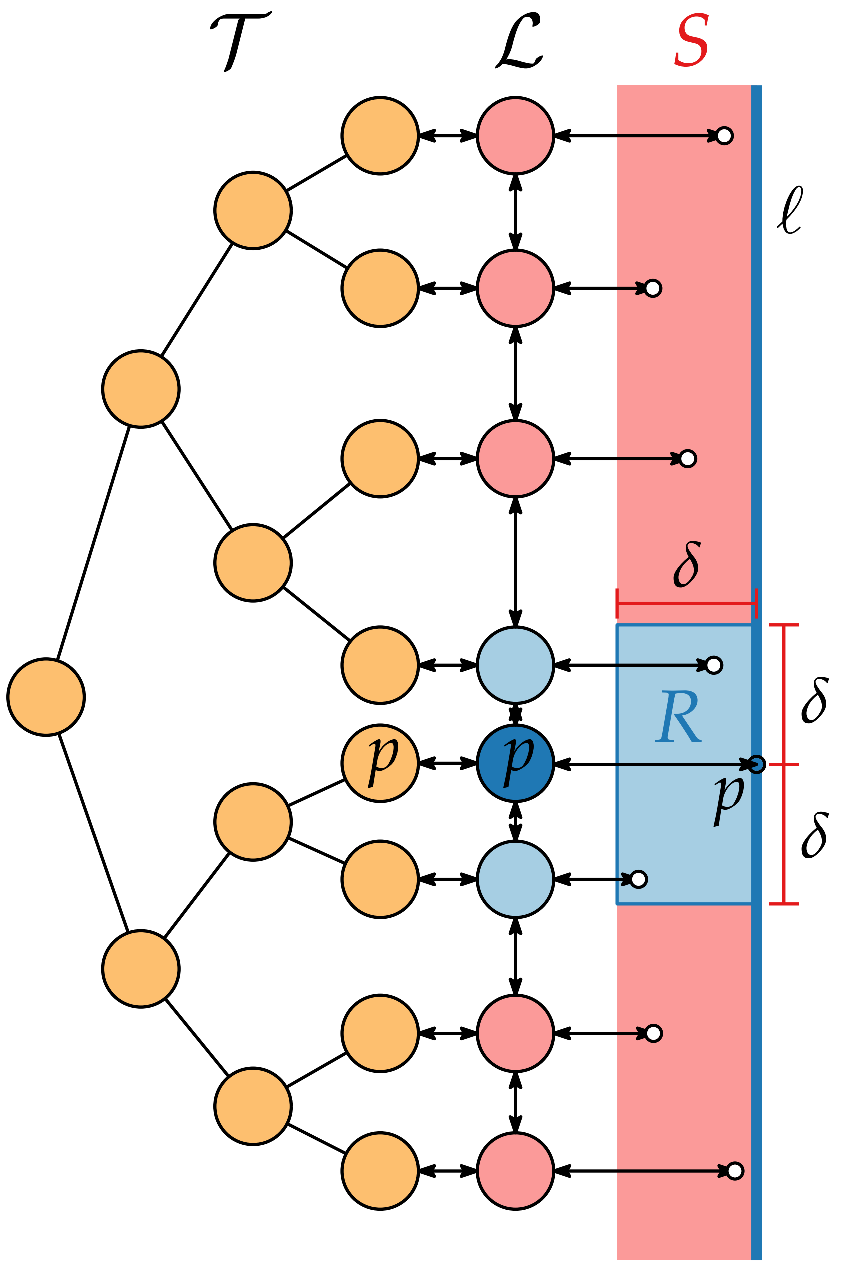
Assume that the points $P \cap S$ are stored in a **linked list** $\mathcal{L}$ sorted according to their y-coordinates.

$\Rightarrow$ Given a pointer to p, we can determine the points in R by searching the interval $[\mathrm{y}(p) - \delta,\ \mathrm{y}(p) + \delta]$. This takes $\mathcal{O}(1)$ time since R contains $\mathcal{O}(1)$ points.

To ensure that $\mathcal{L}$ can be updated efficiently, we additionally store the points $P \cap S$ in a **balanced binary search tree** $\mathcal{T}$ using the y-coordinates as keys.

The corresponding elements in $\mathcal{L}$ and $\mathcal{T}$ are linked.

$\Rightarrow$ when a point is inserted in $\mathcal{T}$ in $\mathcal{O}(\log n)$ time, its according position in $\mathcal{L}$ can be determined in $\mathcal{O}(1)$ time.



Invariant 2: when we reach a point p, $\mathcal{T}$ and $\mathcal{L}$ contain exactly the points in $P \cap S$.

Algorithm

$p_1, p_2, \ldots, p_n =$ points of P sorted according to their x-coordinates
$P_{\mathrm{min}} =$ nil // current closest pair
$\delta = \infty$ // distance of current closest pair
$k = 1$ // index of the left-most point in $\mathcal{L}$ and $\mathcal{T}$
initialize $\mathcal{L}$ and $\mathcal{T}$ with p_1
for $i = 2, 3, \ldots, n$ **do**
  insert p_i into $\mathcal{L}$ and $\mathcal{T}$
  for $p_j \in [\mathrm{y}(p_i) - \delta,\ \mathrm{y}(p_i) + \delta] \setminus \{p_i\}$ **do**
    if $||p_j - p_i|| < \delta$ **do**
      $P_{\mathrm{min}} = \{p_j, p_i\};\quad \delta = ||p_j - p_i||$
  while $\mathrm{x}(p_k) < \mathrm{x}(p_{i+1}) - \delta$ **do**
    delete p_k from $\mathcal{L}$ and $\mathcal{T}$
    $k = k + 1$
return P_{min}

Algorithm

$p_1, p_2, \dots, p_n$ = points of P sorted according to their x-coordinates $\mathcal{O}(n \log n)$
$P_{\text{min}} = \text{nil}$ // current closest pair
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$k = 1$ // index of the left-most point in $\mathcal{L}$ and $\mathcal{T}$
initialize $\mathcal{L}$ and $\mathcal{T}$ with p_1
for $i = 2, 3, \dots, n$ **do** $\mathcal{O}(n)$
  insert p_i into $\mathcal{L}$ and $\mathcal{T}$ $\mathcal{O}(\log n)$
  for $p_j \in [\mathrm{y}(p_i) - \delta,\ \mathrm{y}(p_i) + \delta] \setminus \{p_i\}$ **do** $\mathcal{O}(1)$
    if $||p_j - p_i|| < \delta$ **do**
      $P_{\text{min}} = \{p_j, p_i\}$; $\delta = ||p_j - p_i||$
  while $\mathrm{x}(p_k) < \mathrm{x}(p_{i+1}) - \delta$ **do** $\mathcal{O}(n)$ in total
    delete p_k from $\mathcal{L}$ and $\mathcal{T}$ $\mathcal{O}(\log n)$
    $k = k + 1$
return P_{min}

$\mathcal{O}(1)$

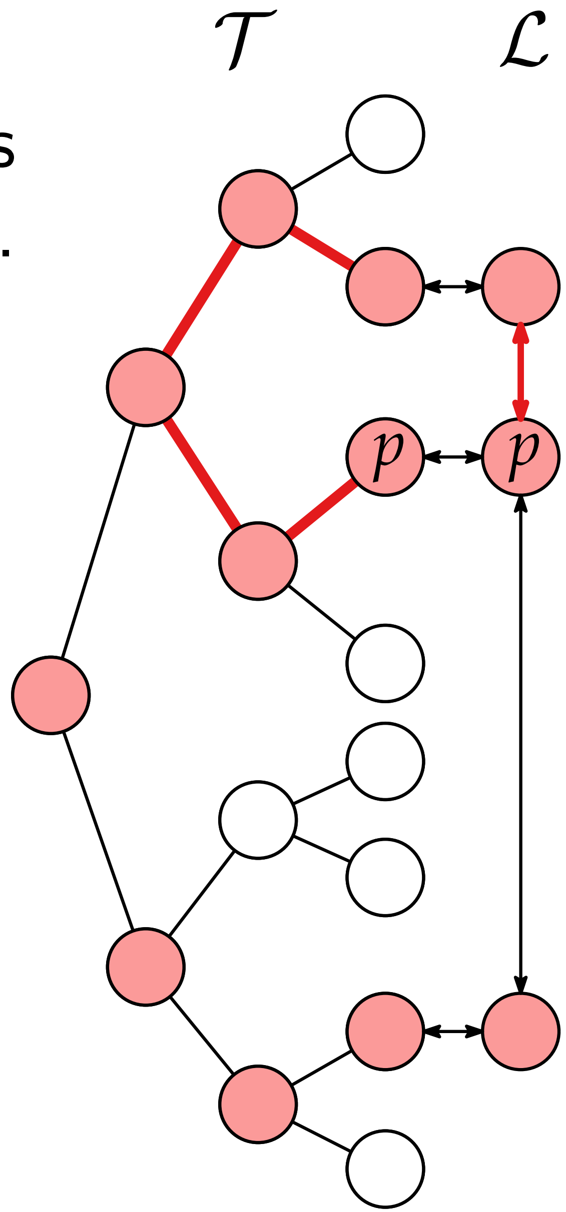
$\mathcal{O}(n \log n)$

$\mathcal{O}(n \log n)$ in total

$\Rightarrow$ **Total runtime:** $\mathcal{O}(n \log n)$

Remarks on the Implementation

- The list $\mathcal{L}$ is actually not necessary: given a point p in $\mathcal{T}$, its neighbors in the ordering can be determined in $\mathcal{O}(\log n)$ time.
- The tree $\mathcal{T}$ does not need to be dynamic! A static tree on all points suffices if each point currently in S and all its ancestors are marked. $\rightarrow$ simple and space efficient (1 bit of extra information / node).
- We assumed that the points in P have pairwise distinct x-coordinates. This situation can be established by rotating P or tilting ℓ slightly.
  Simply, visit the points in lexicographical order!



Summary and Discussion

The **sweep line approach** is an important design paradigm (like divide and conquer, prune and search, dynamic programming, greedy, ...) in computational geometry.

Main idea: Sweep the plane with a line ℓ while maintaining two invariants:

- A **partial solution** for the input to the left of ℓ is known.
- The part of the input to the left of ℓ that is still relevant for updating the partial solution is encoded in a suitable data structure (**sweep line status**).

The partial solution and the sweep line status only change at specific positions (**events**) that may be part of the input or arise during the execution of the algorithm.

The sweep line paradigm is **powerful** and leads to **simple** algorithms for many problems:
computing Voronoi diagrams, crossings in an arrangement of line segments,
intersection/union of two polygons, decompositions of polygons,
certain triangulations, visibility polygons, ...

Outlook: Computing Visibility Polygons

The sweep "line" does not always have to move from left to right!

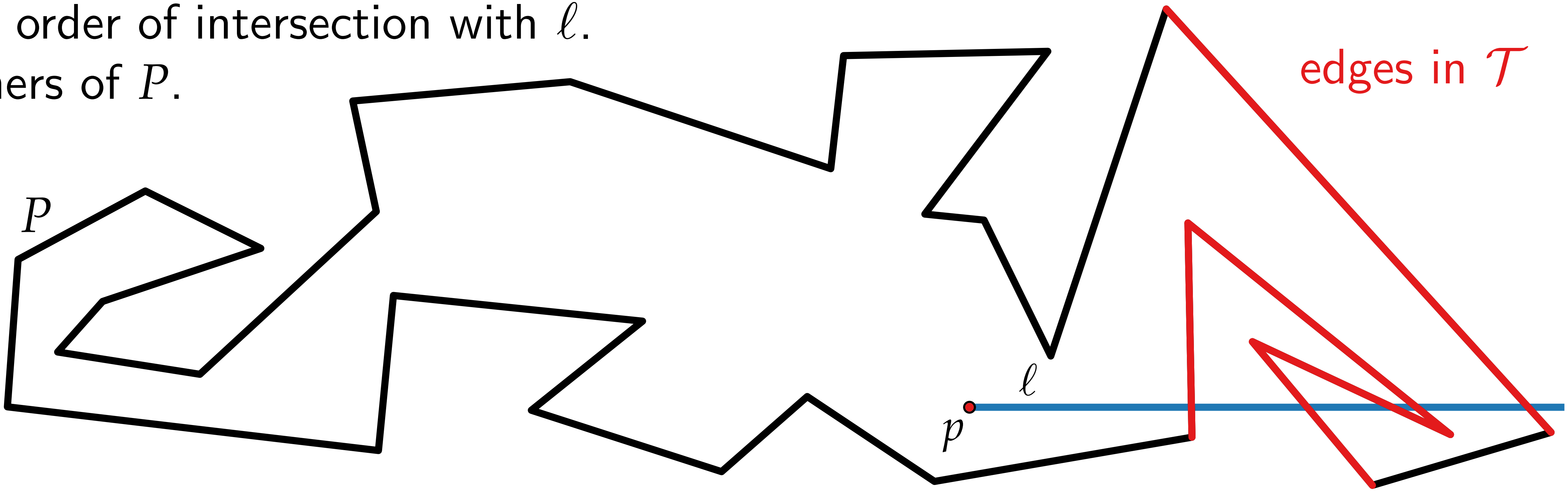
Given: A polygon P with n corners and a point p in its interior.

Task: Compute the visibility polygon of p with respect to P.

Idea: Sweep a **ray** ℓ radially around p.

Sweep line status: Edges of P intersected by ℓ are stored in a balanced binary search tree $\mathcal{T}$ in the order of intersection with ℓ.

Events: Corners of P.



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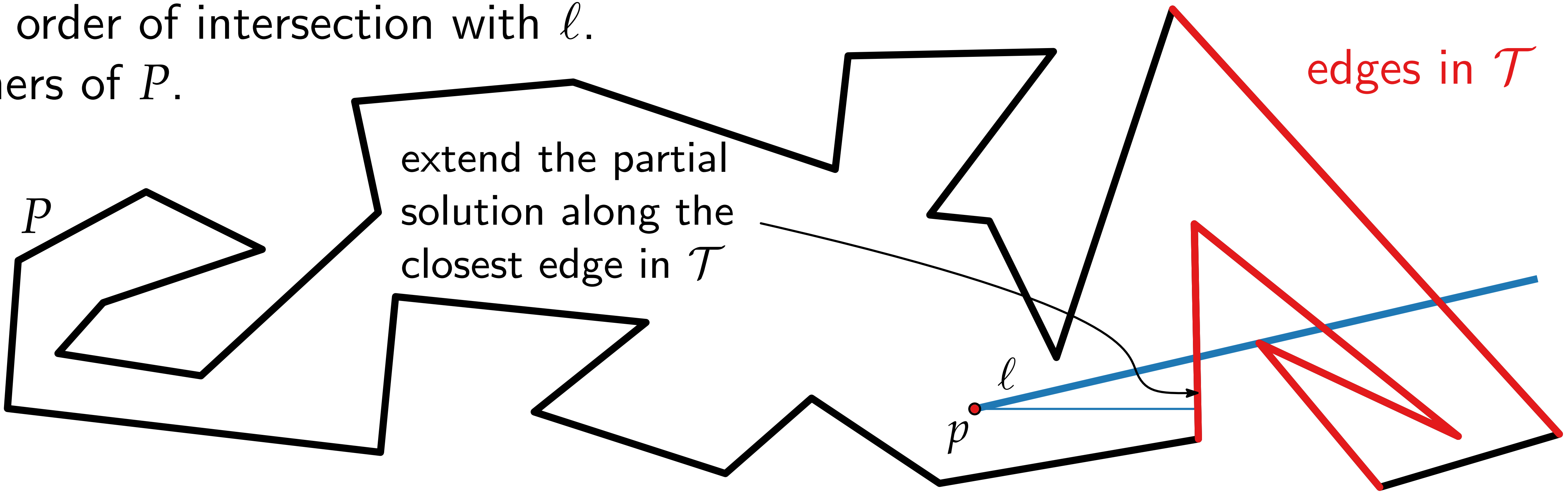

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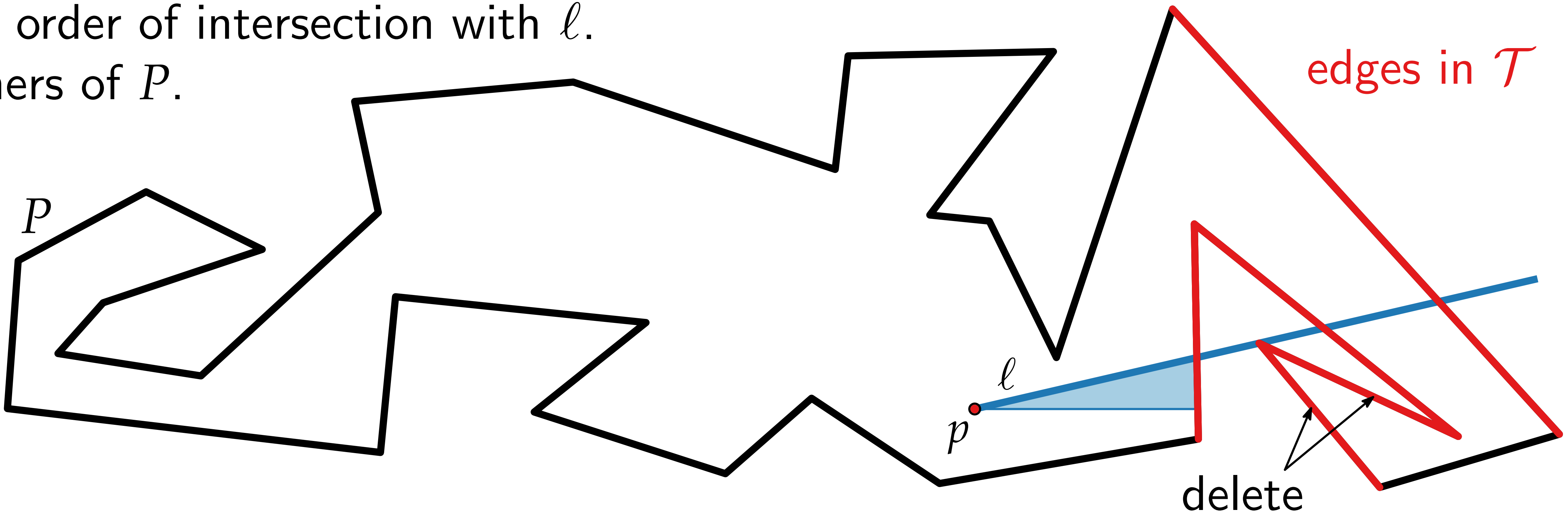

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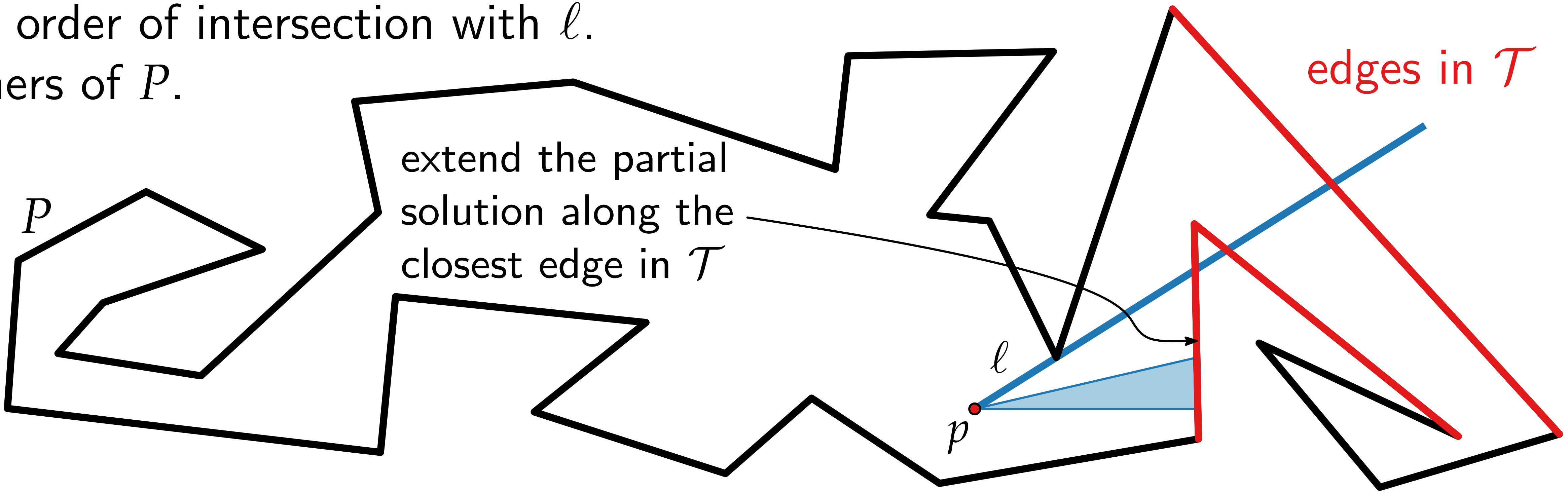

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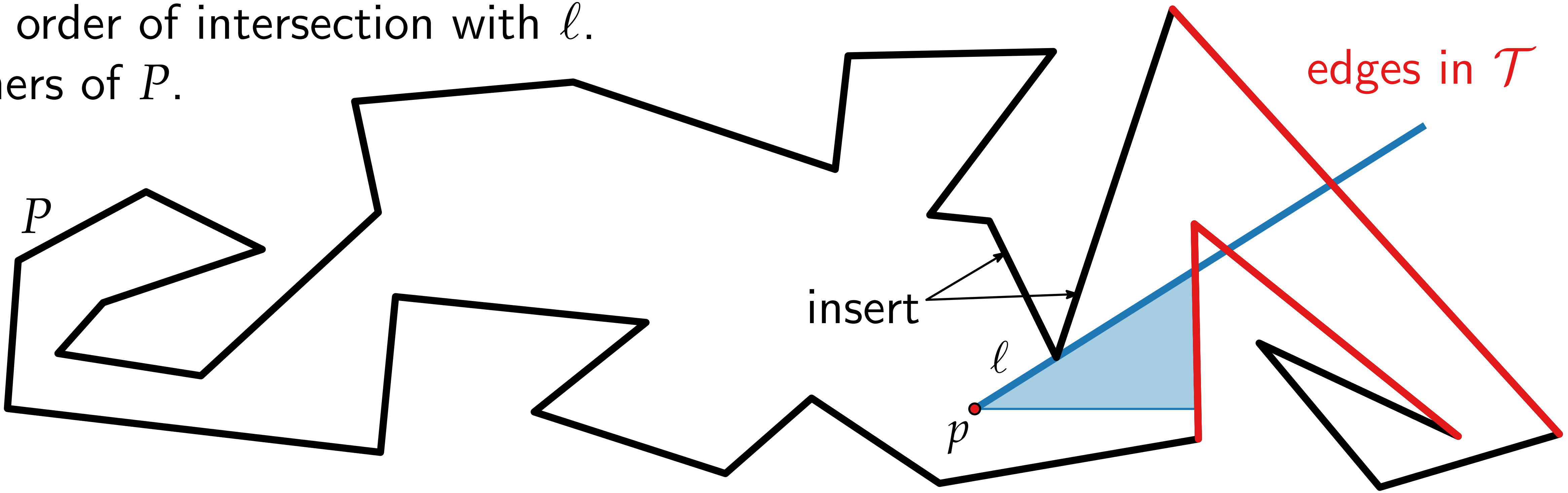

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Total runtime: $\mathcal{O}(n \log n)$

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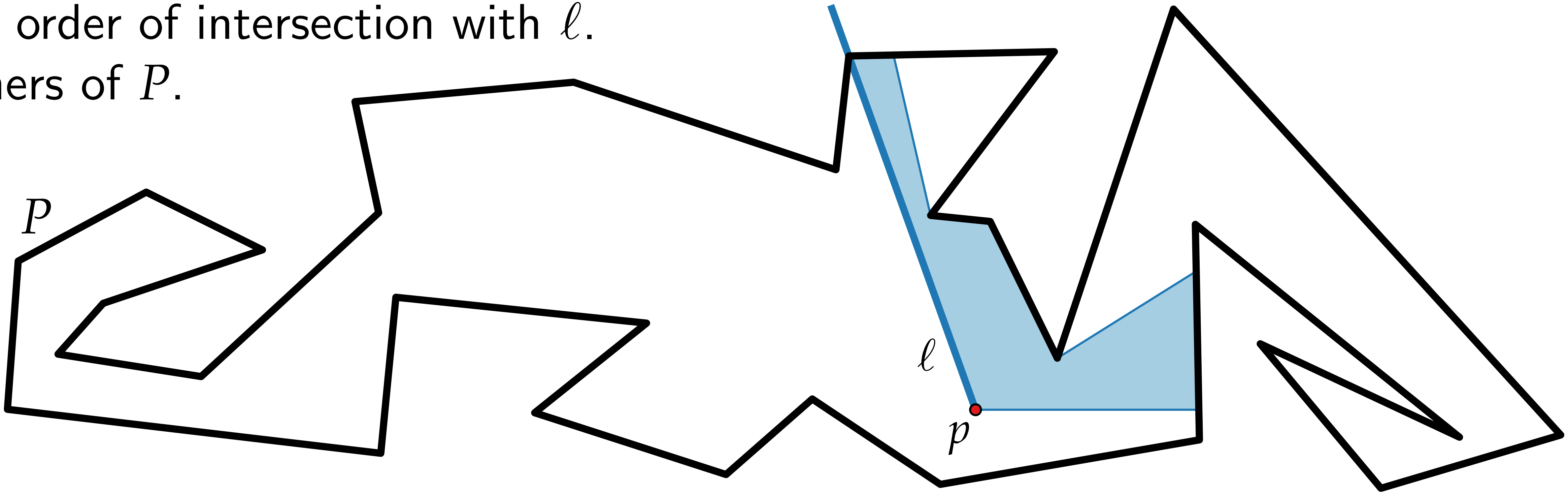

Literature

Rolf Klein. Algorithmische Geometrie: Grundlagen, Methoden, Anwendungen. Springer Verlag 2005.