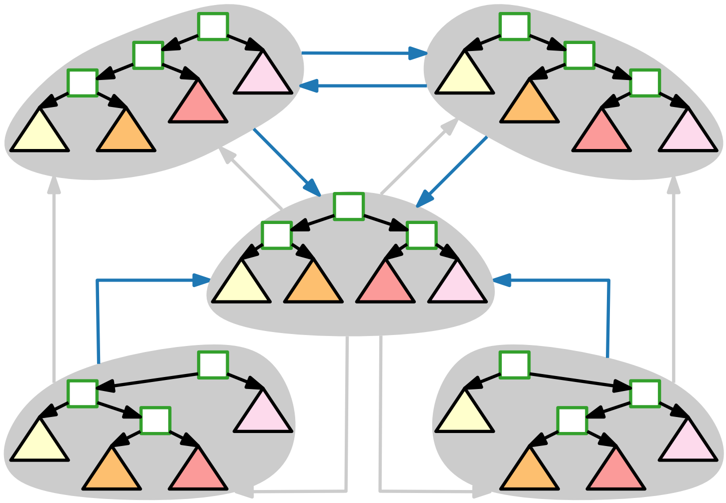
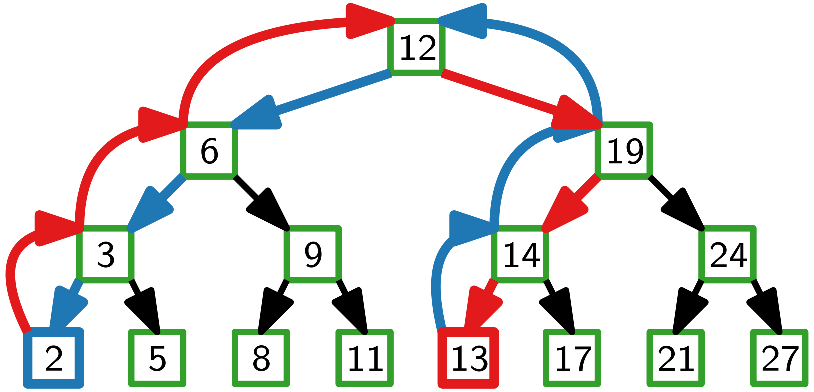
Advanced Algorithms

Optimal Binary Search Trees

Splay Trees

Johannes Zink · WS23/24





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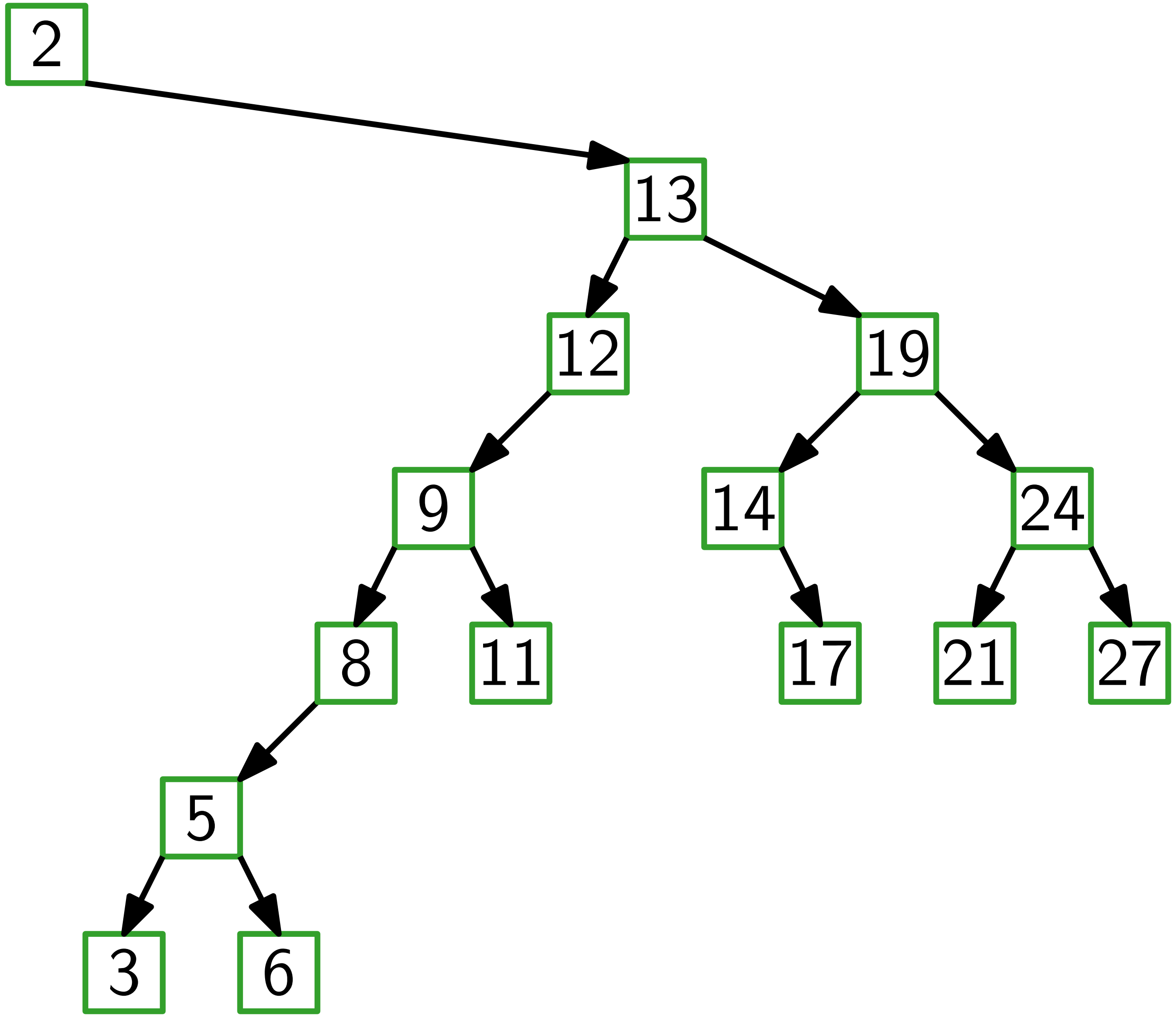
Binary search tree (BST):



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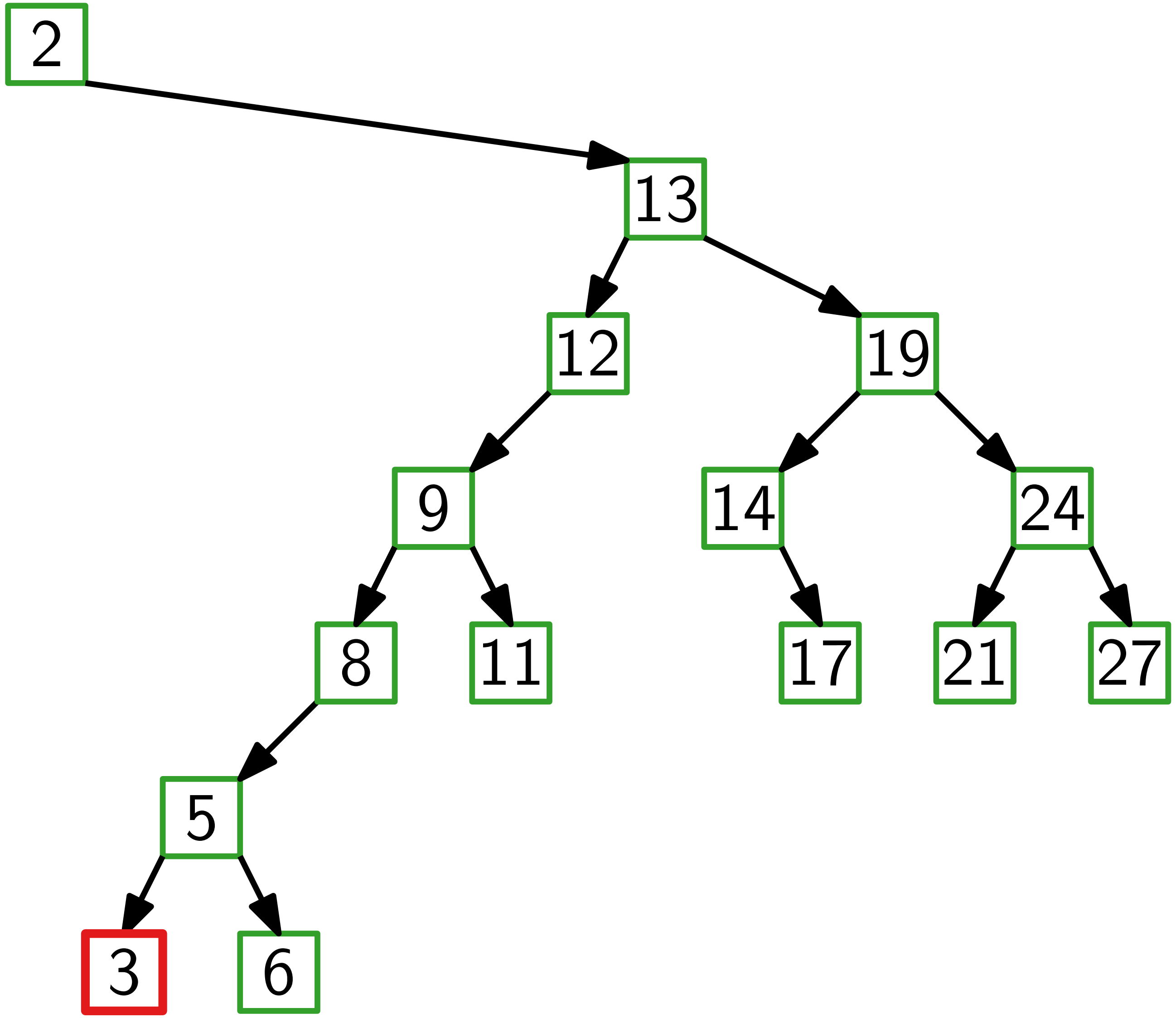

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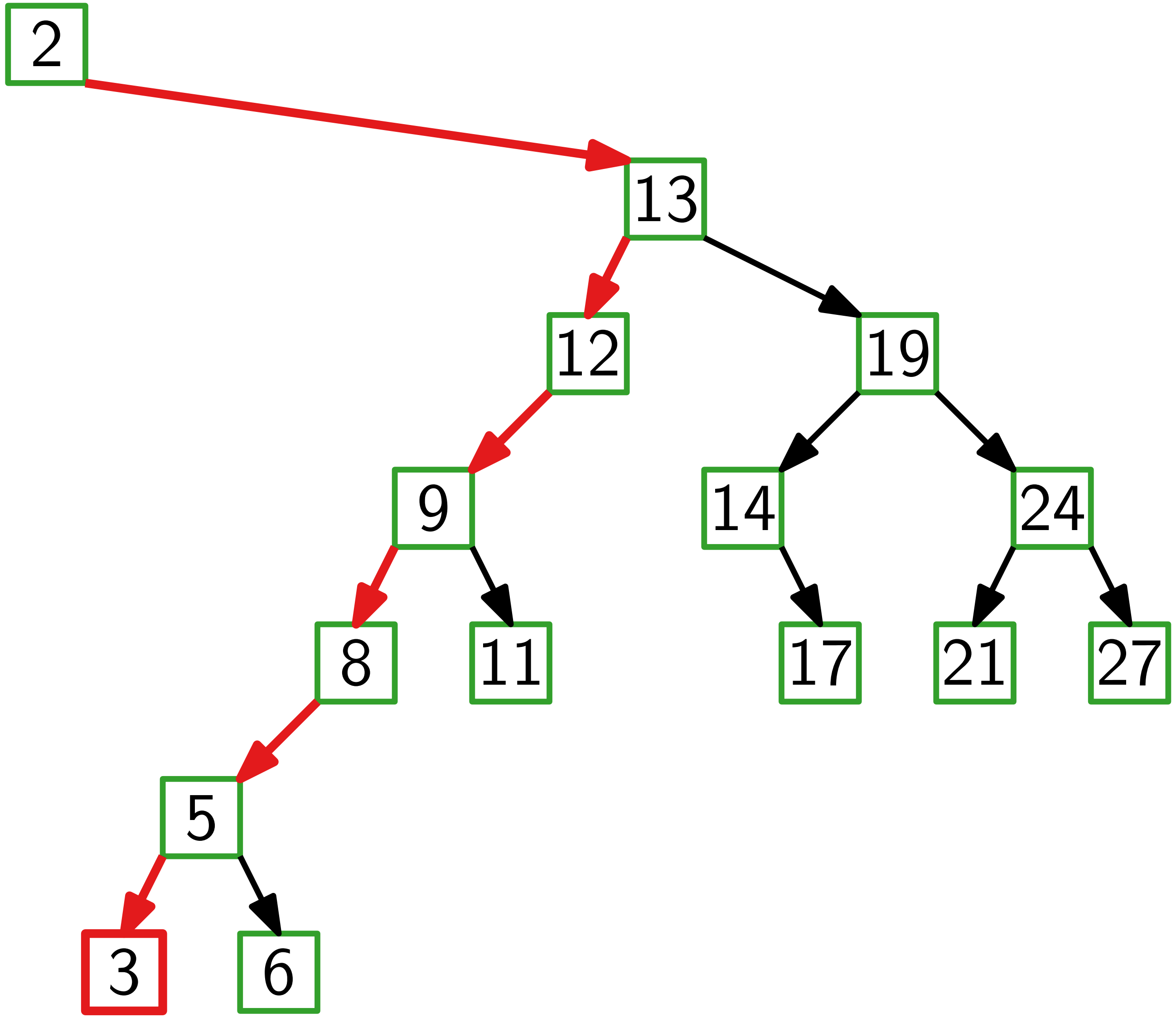

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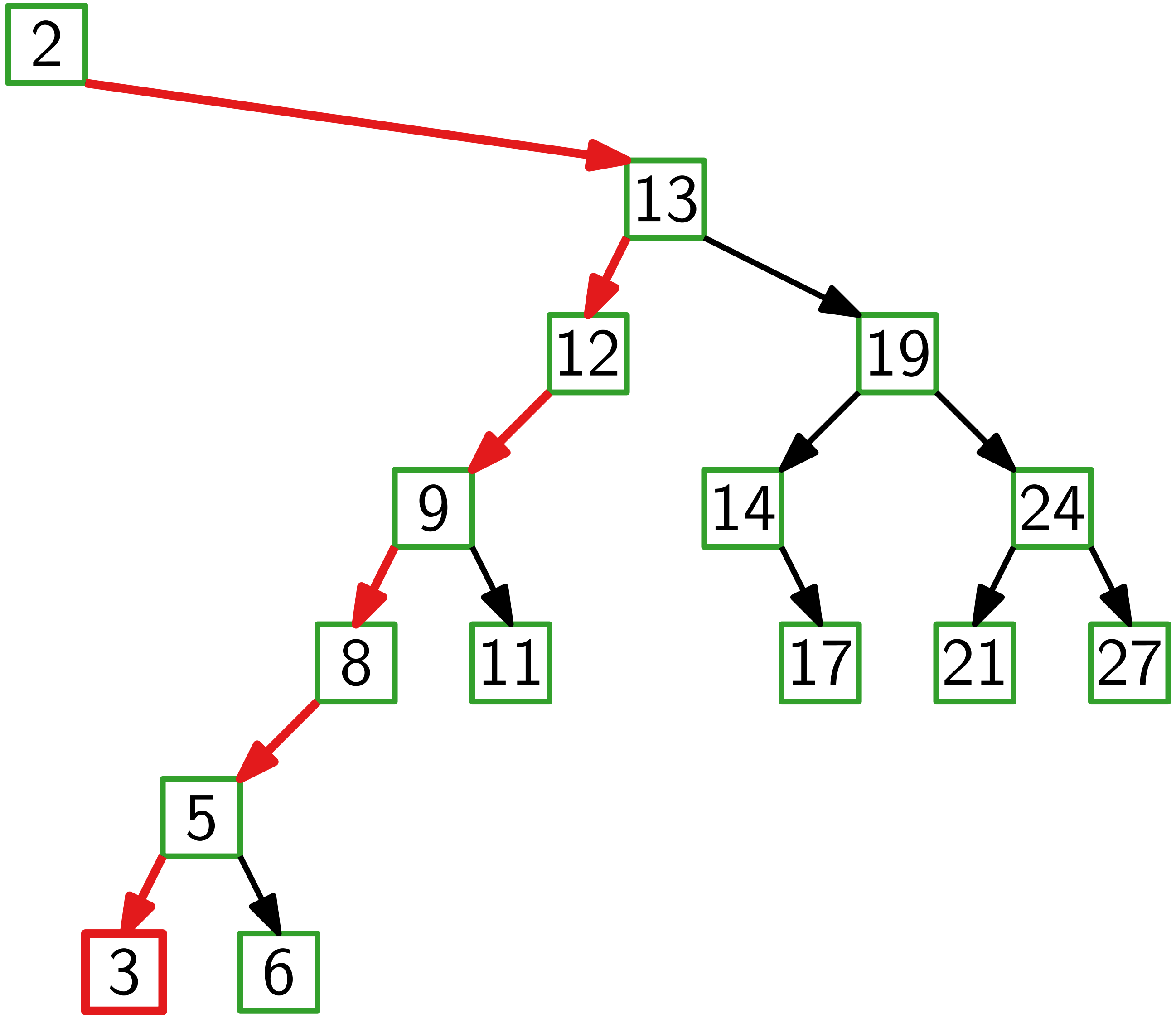

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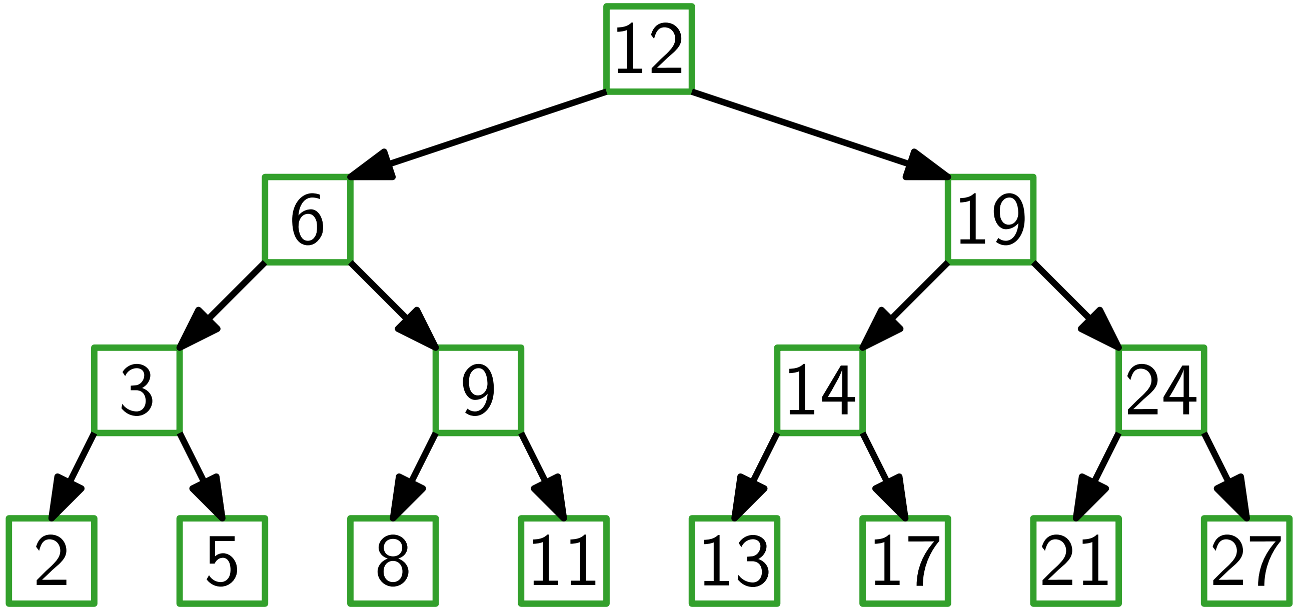

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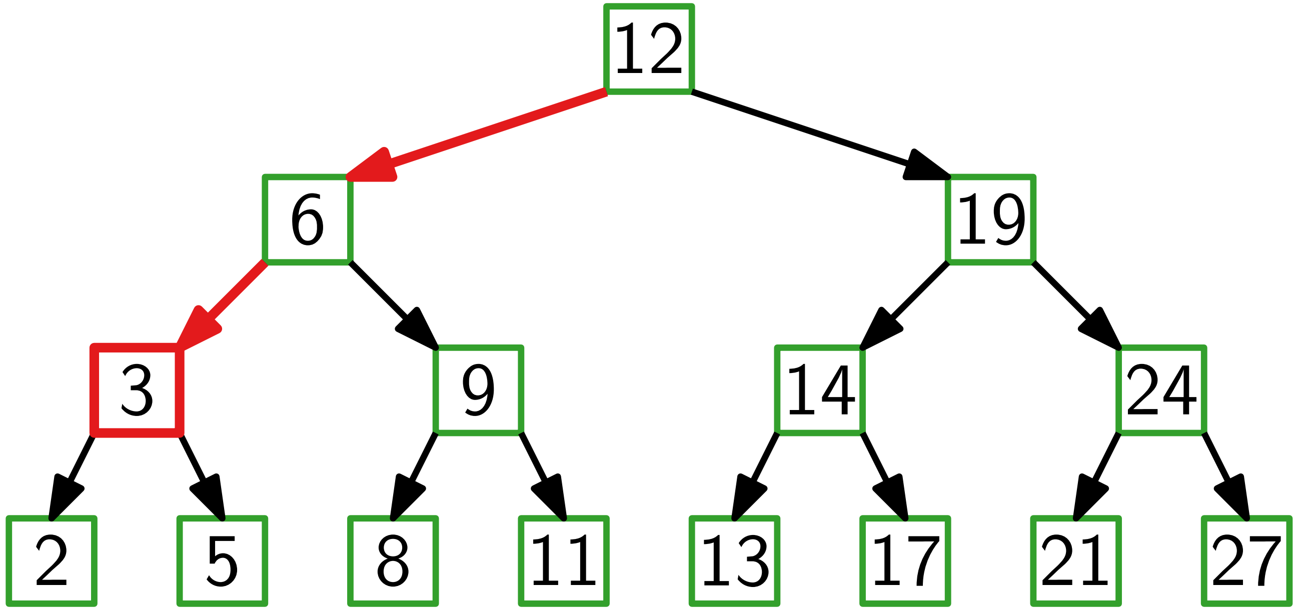

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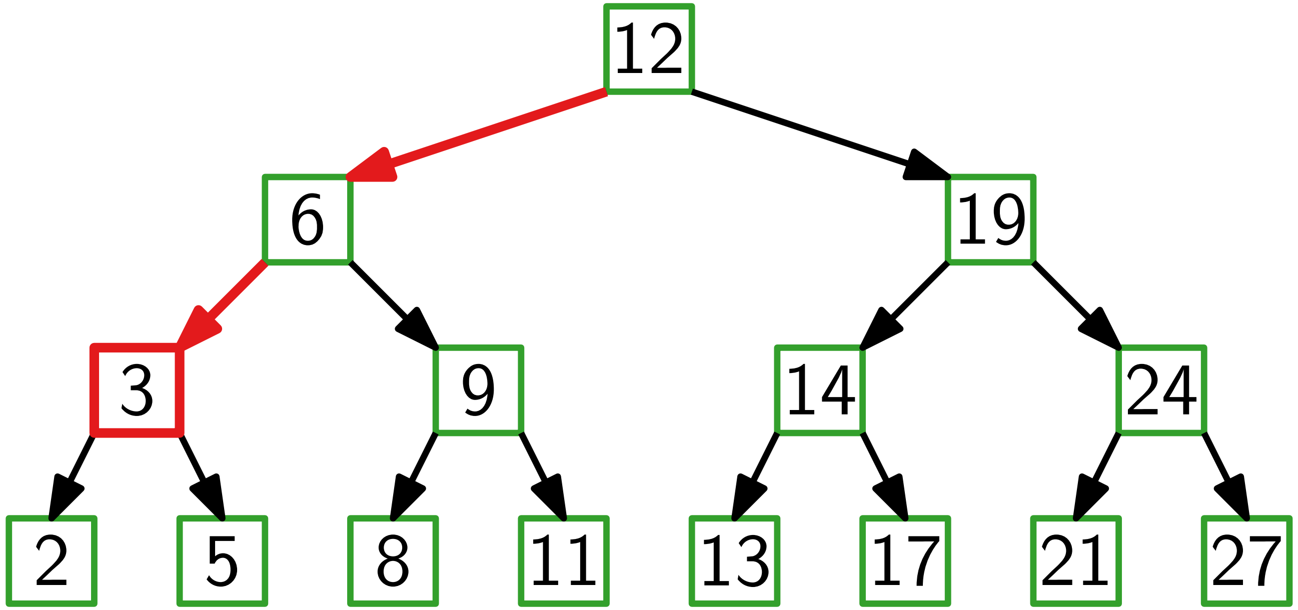

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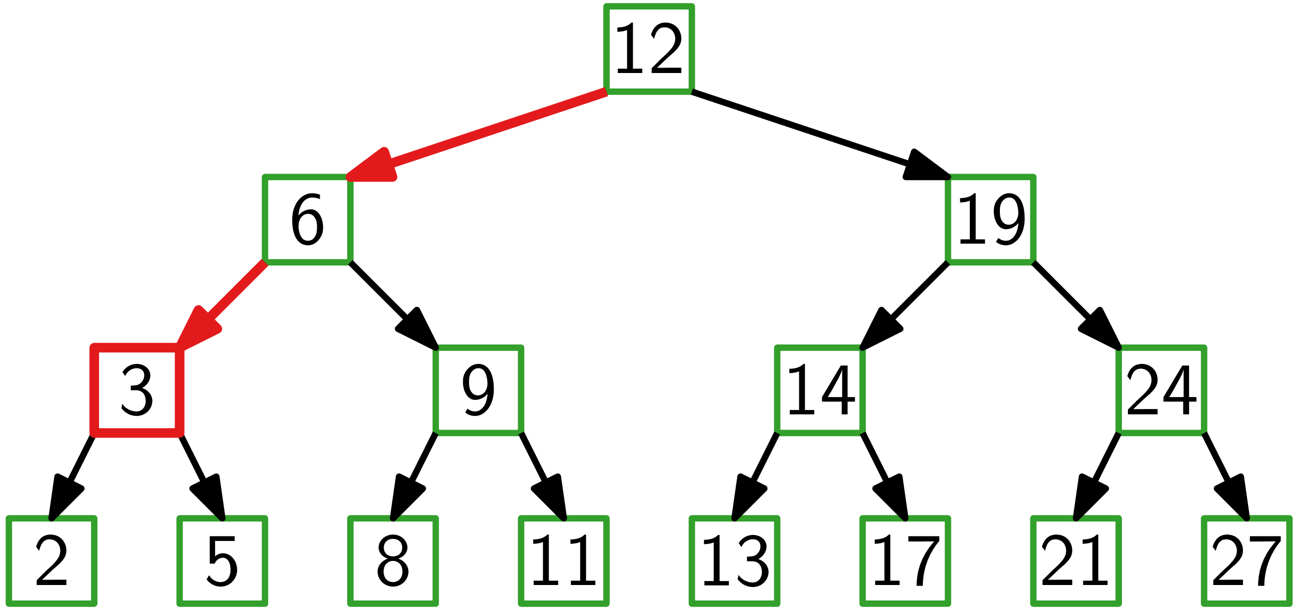

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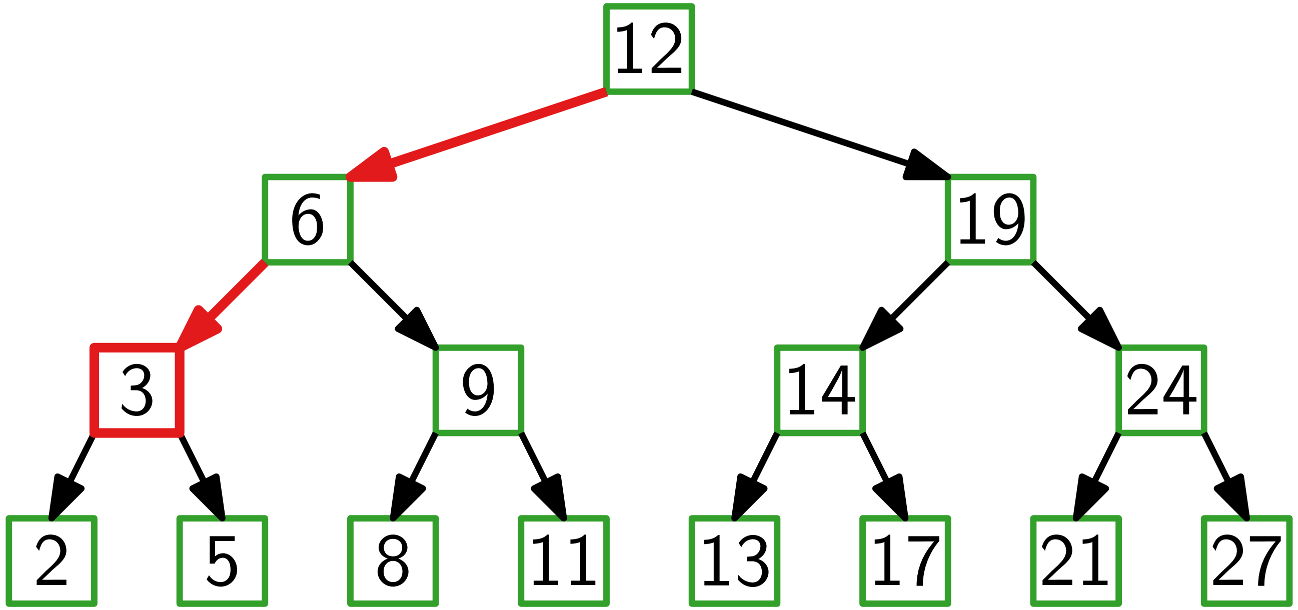

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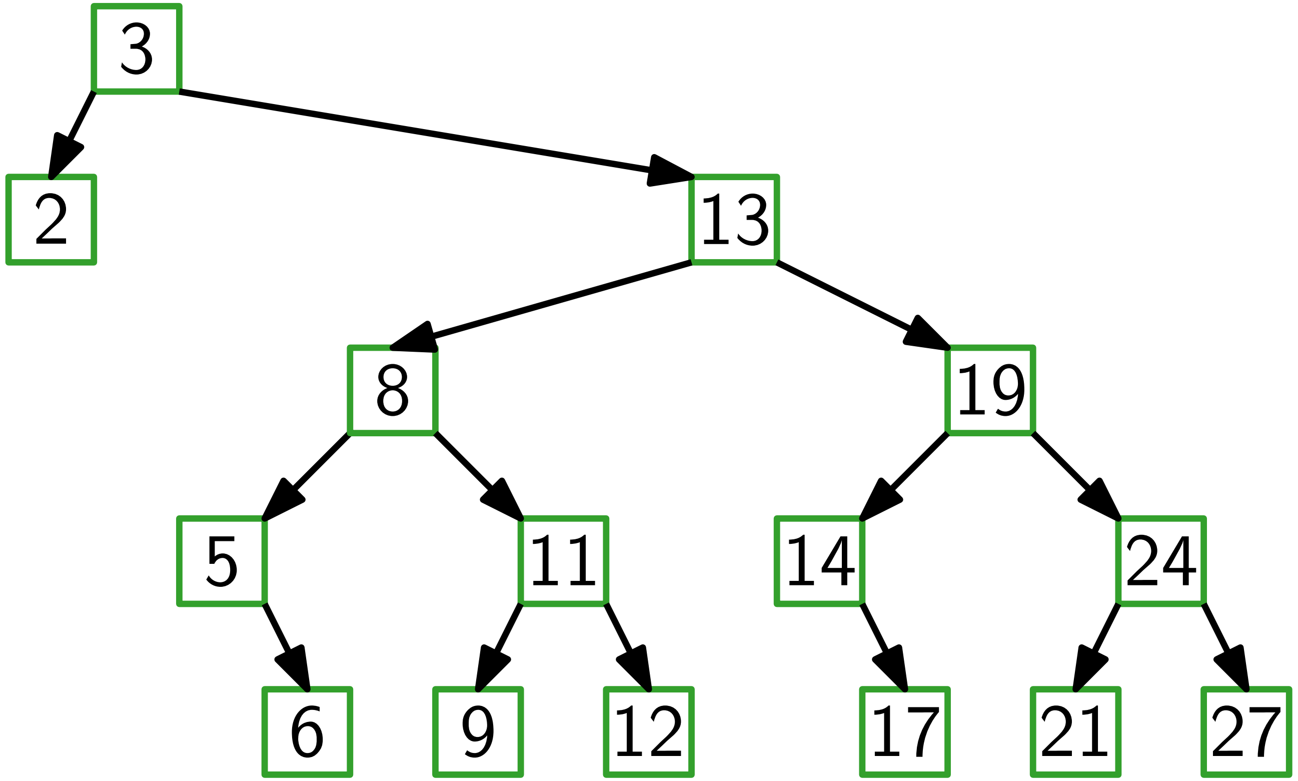

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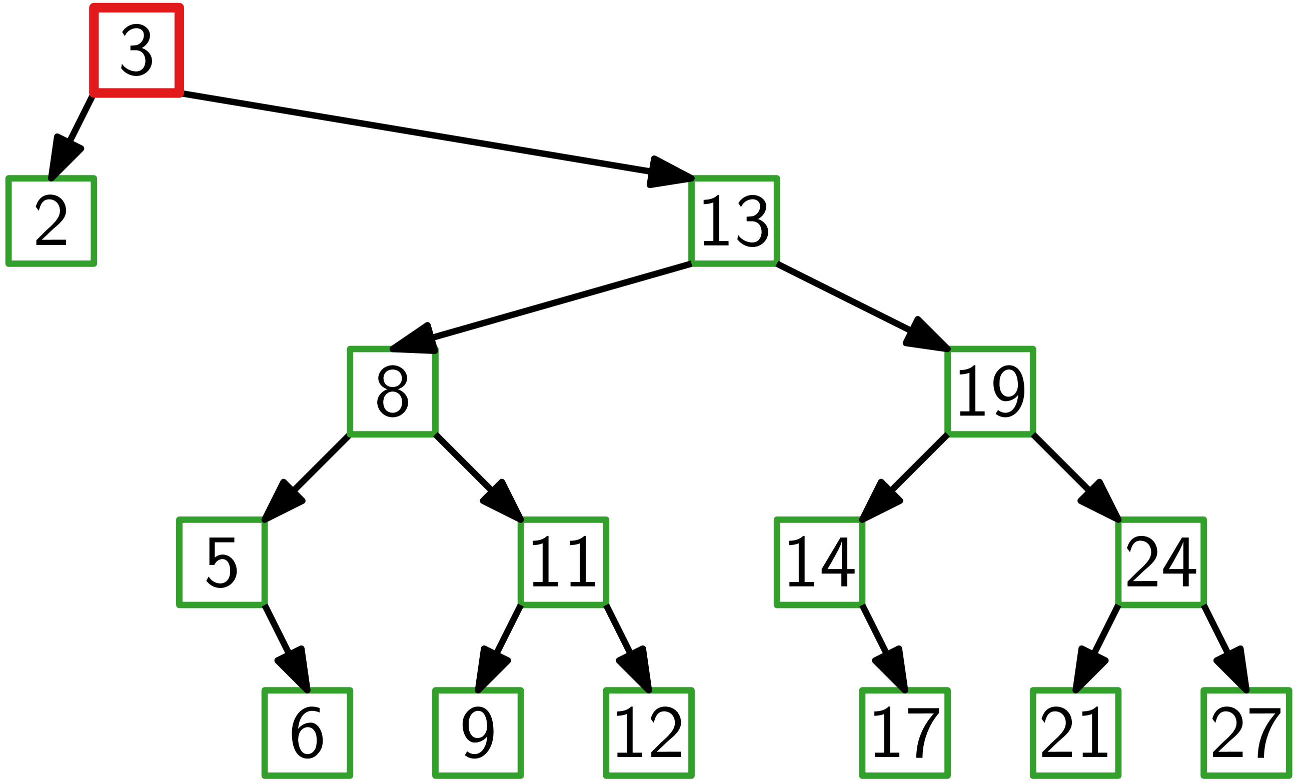

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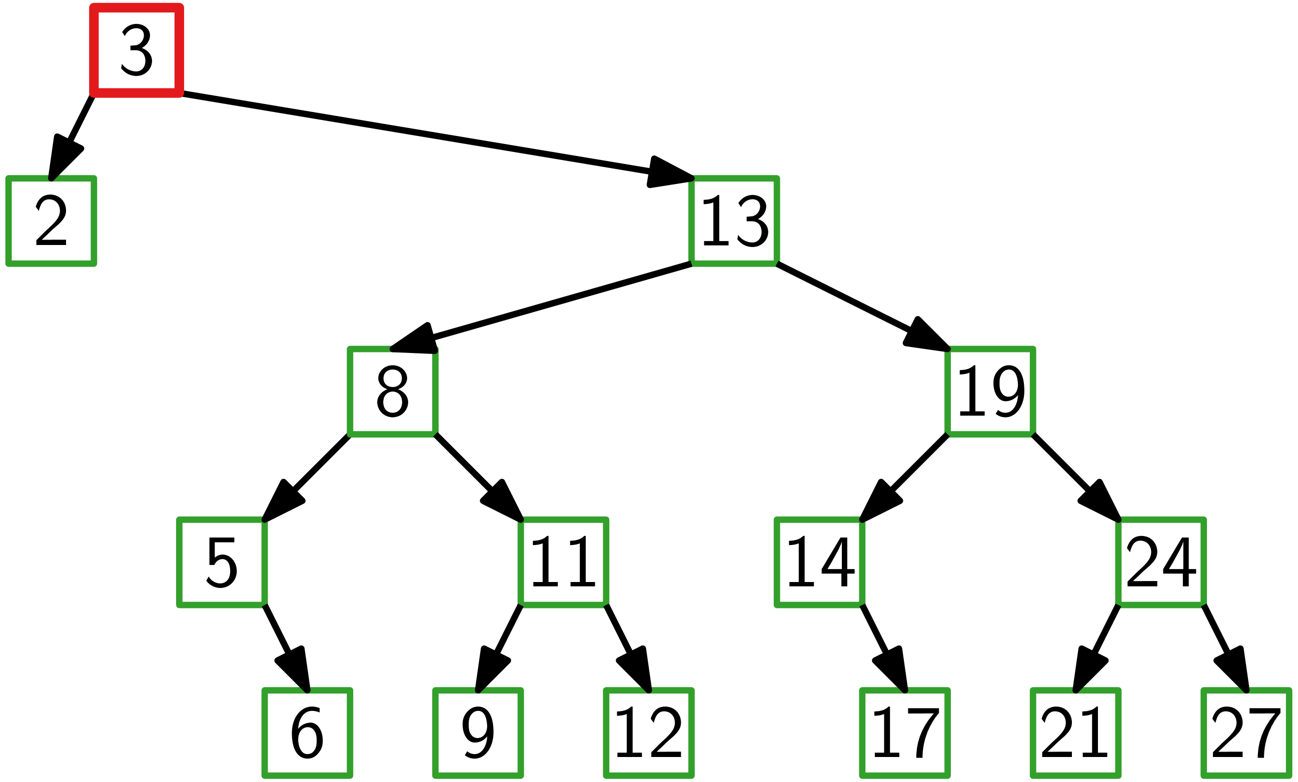

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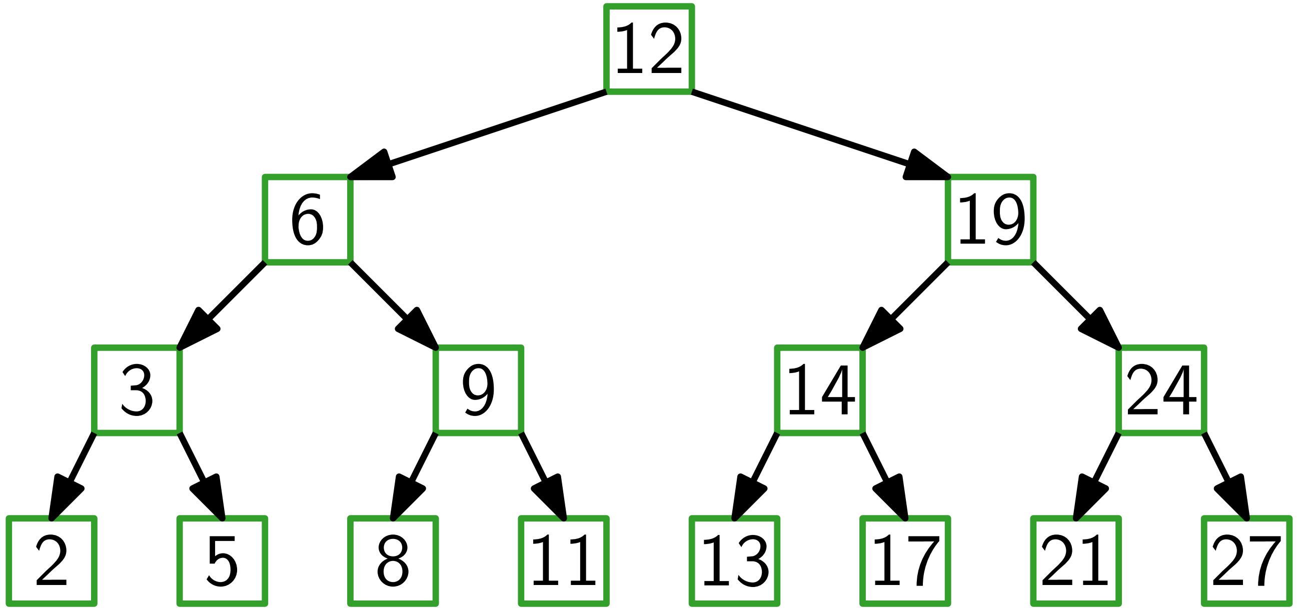

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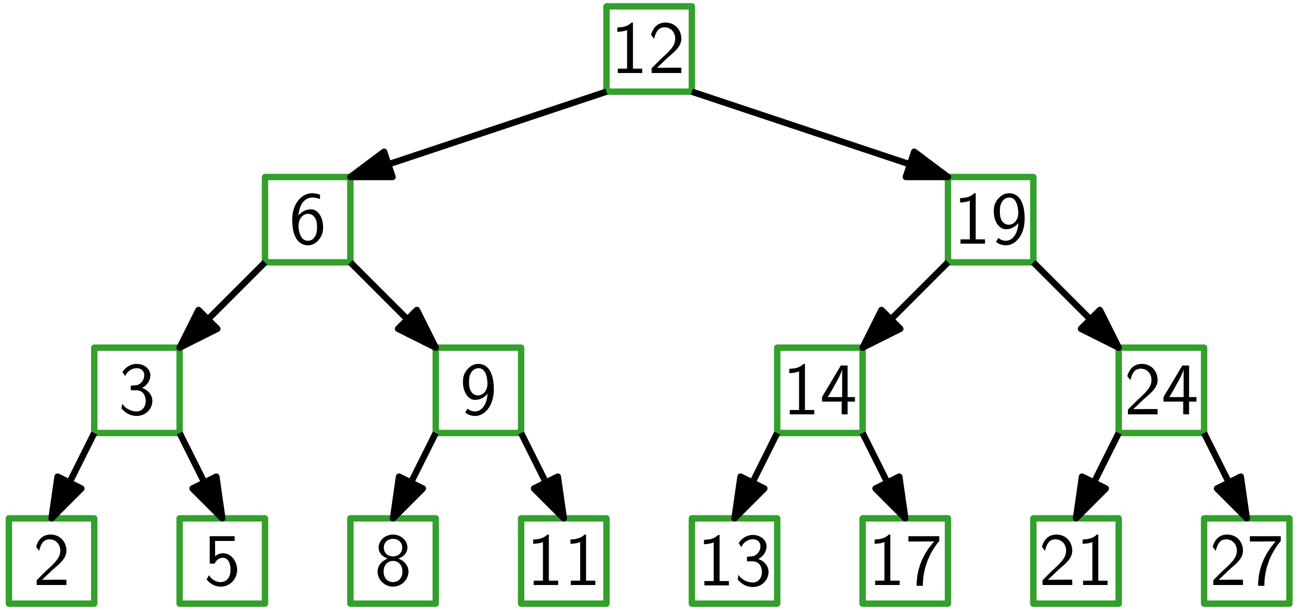

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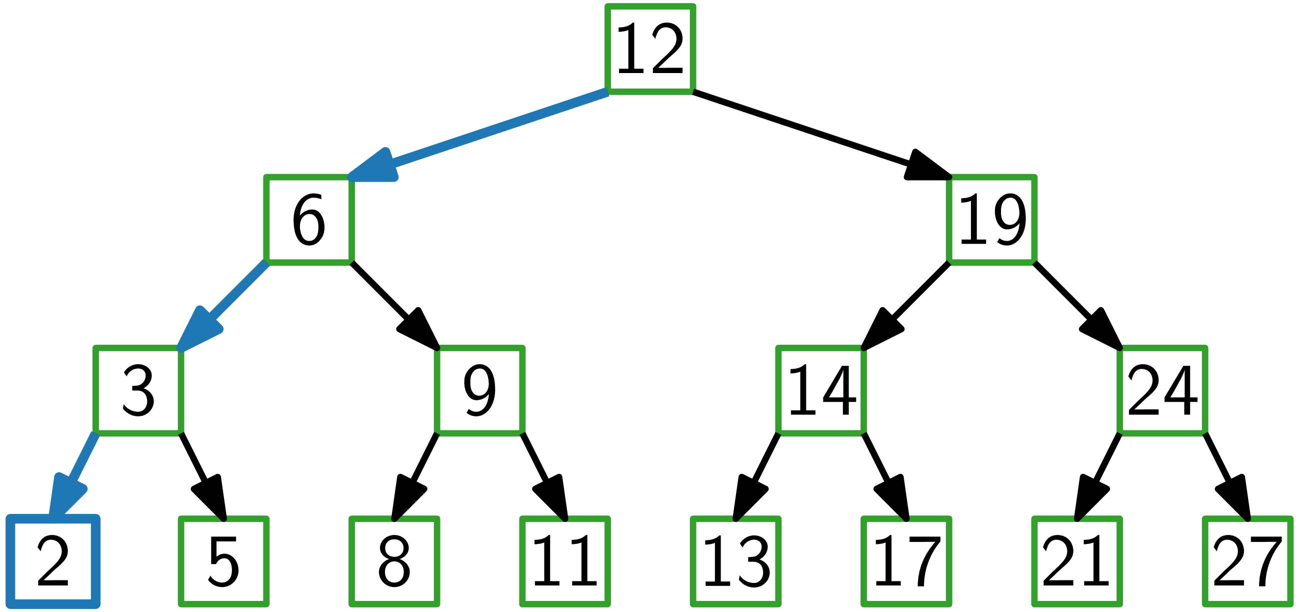

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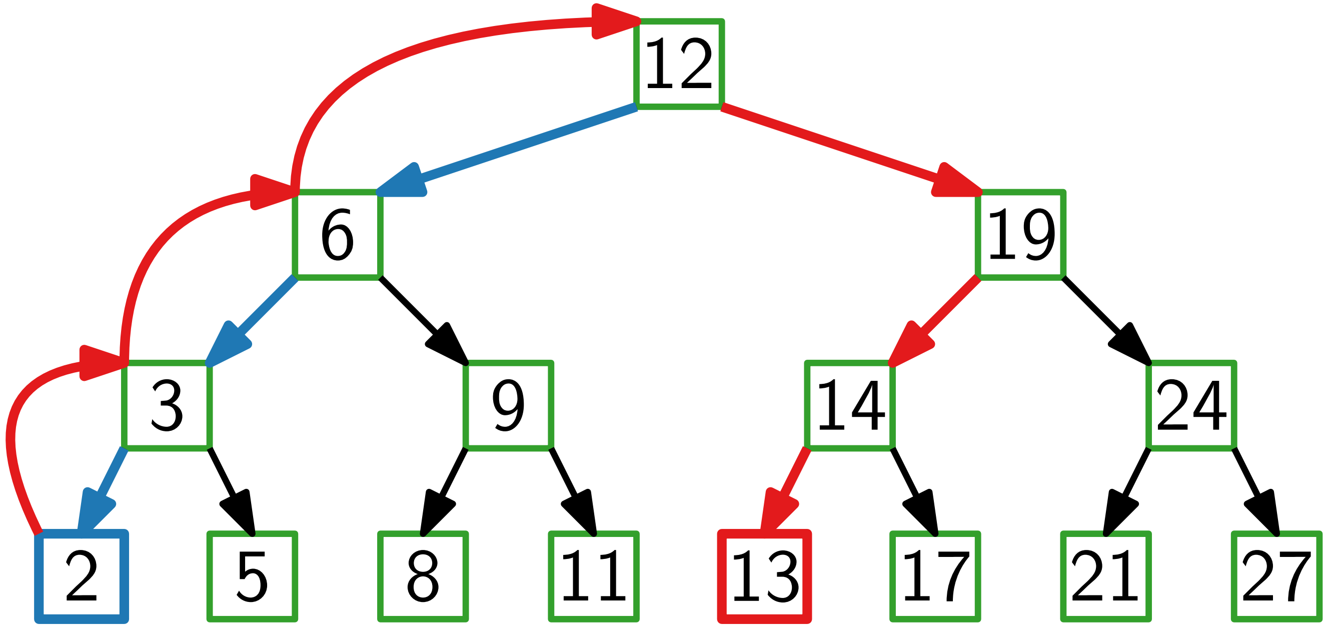

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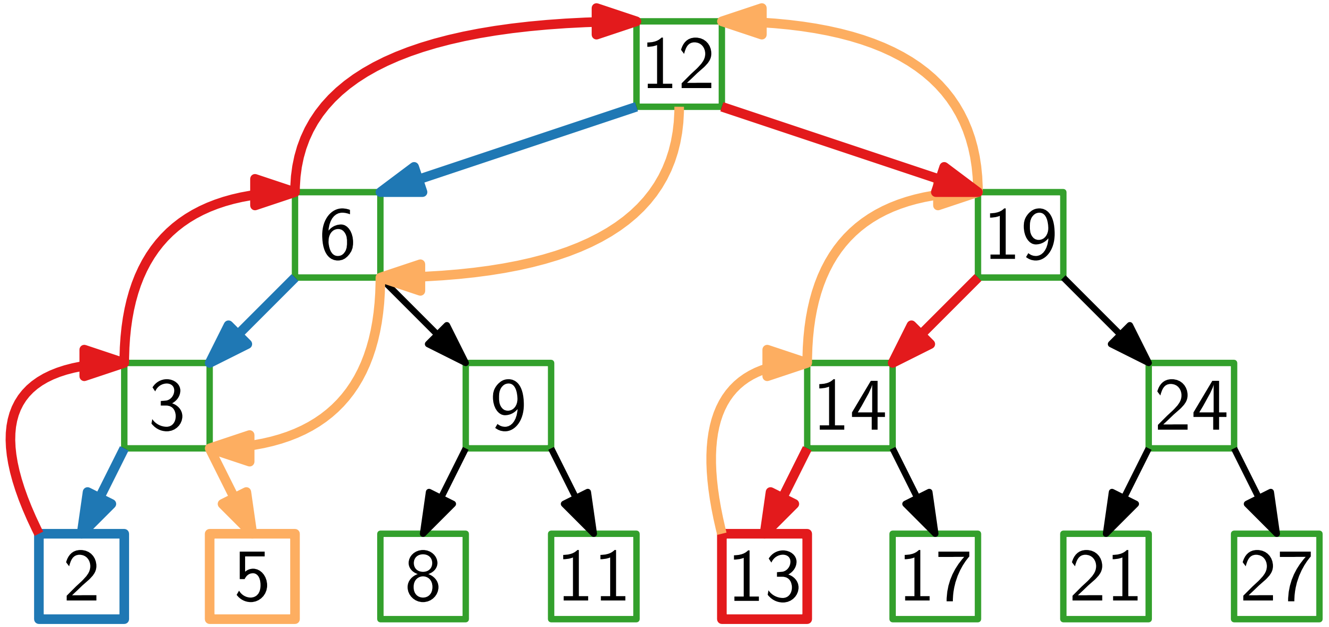

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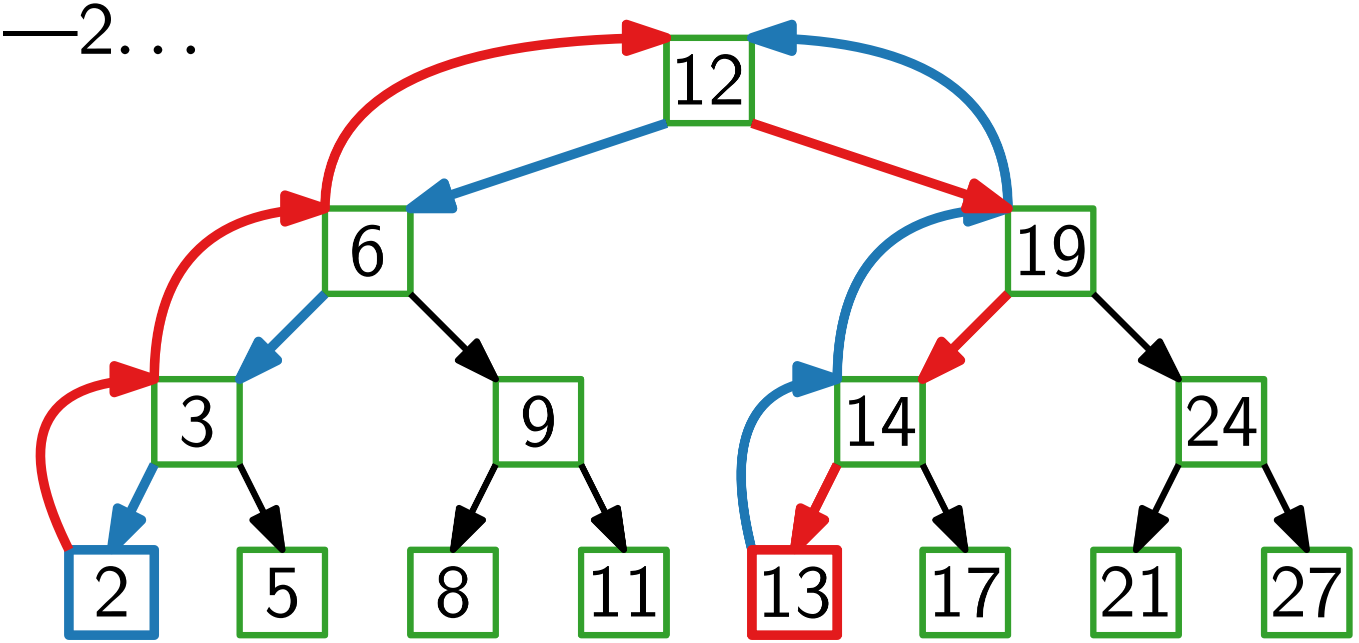

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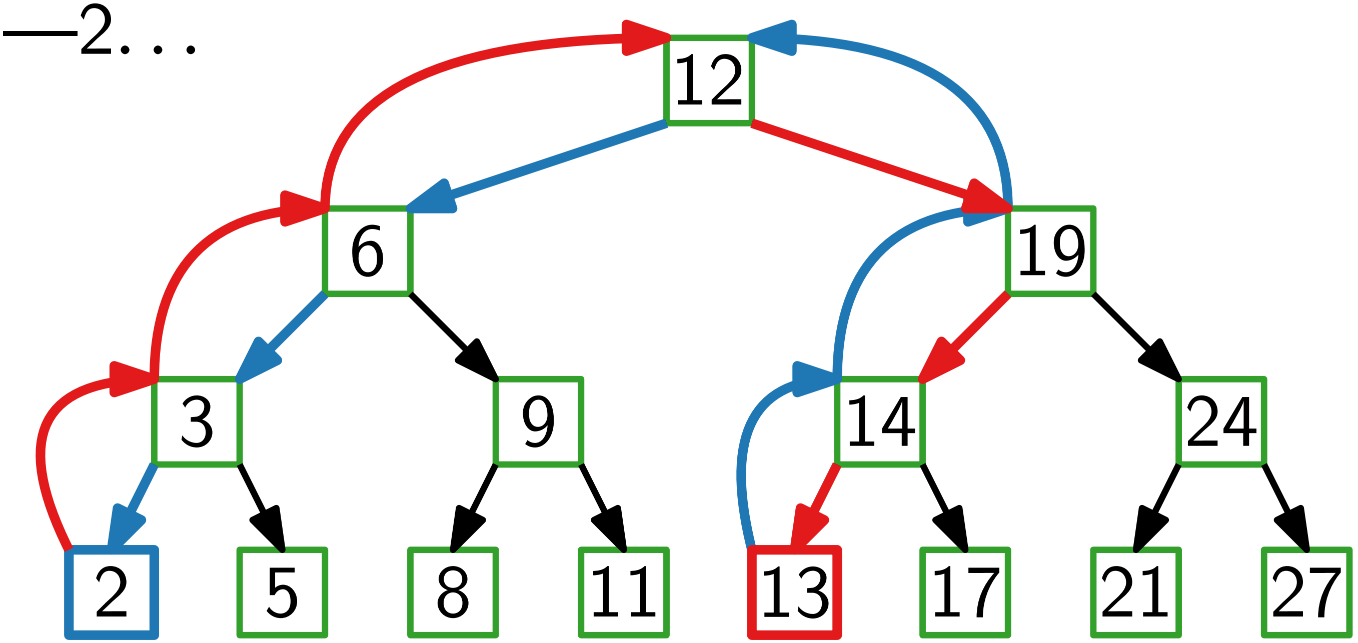

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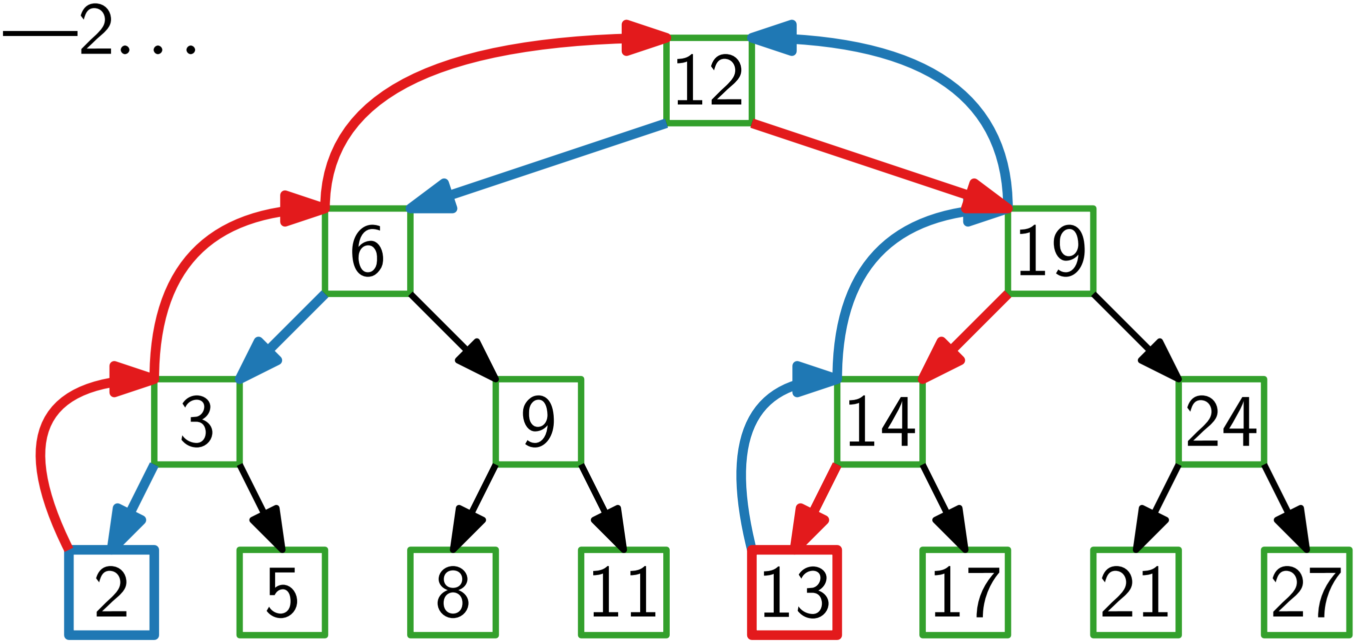

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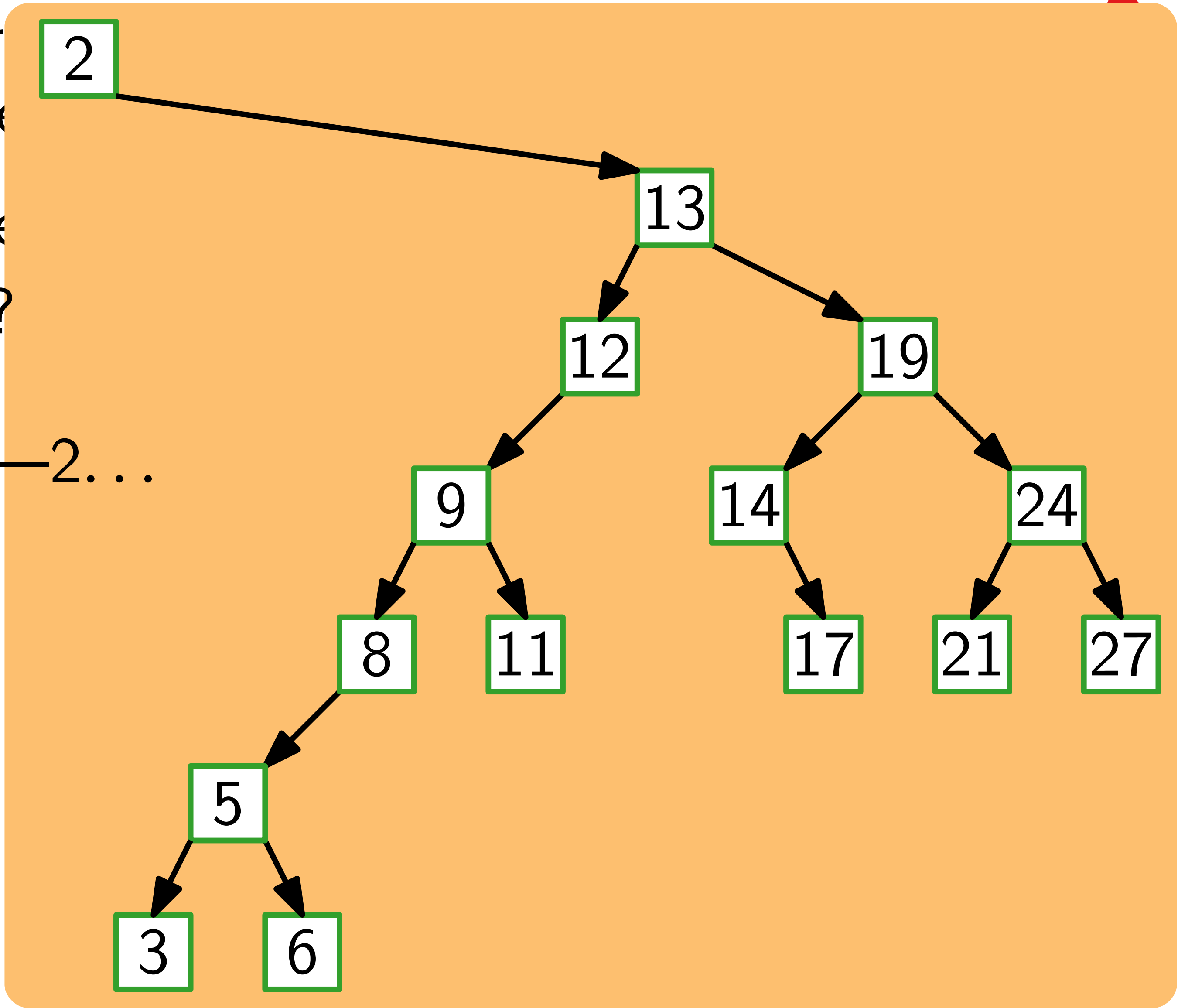

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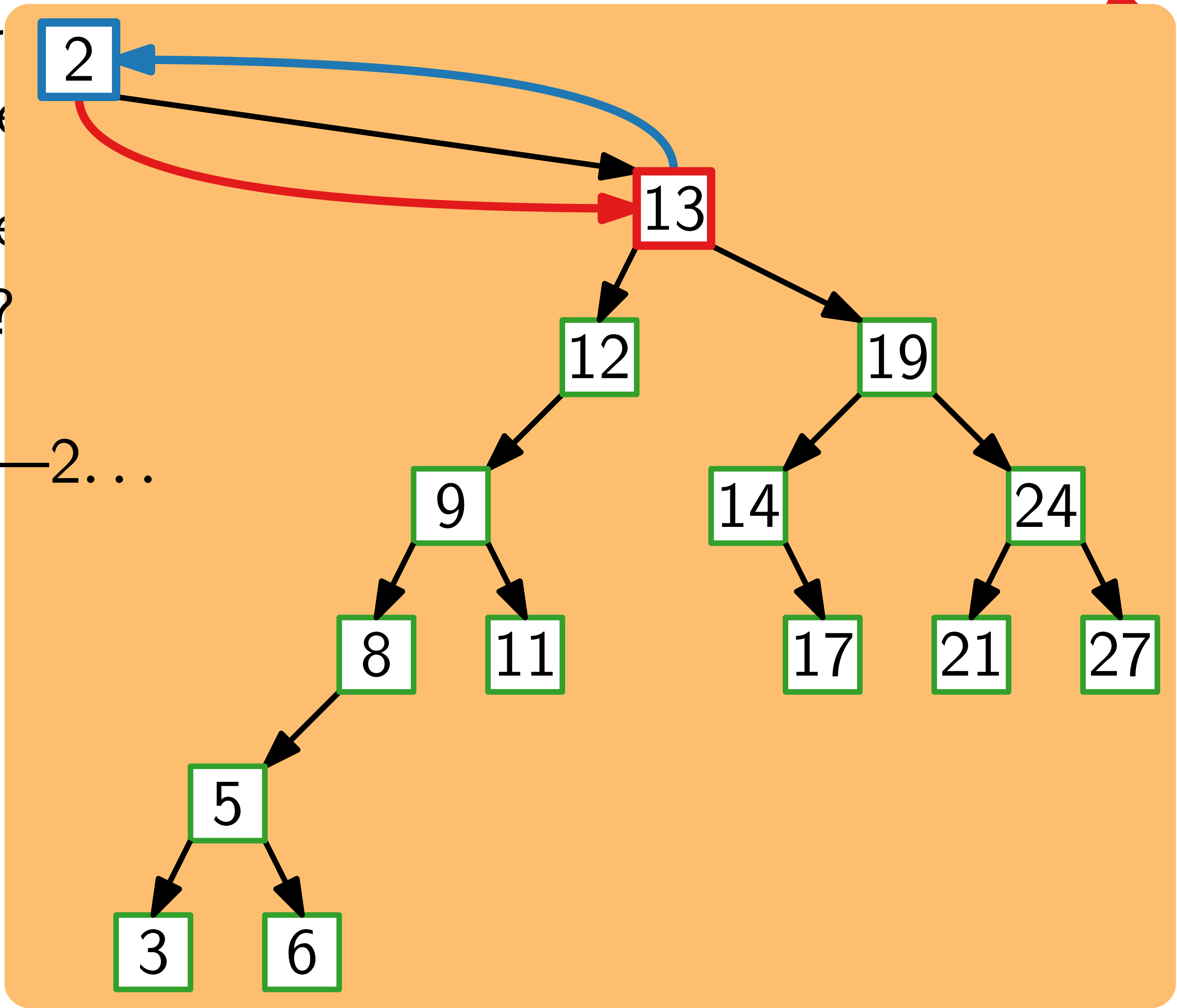

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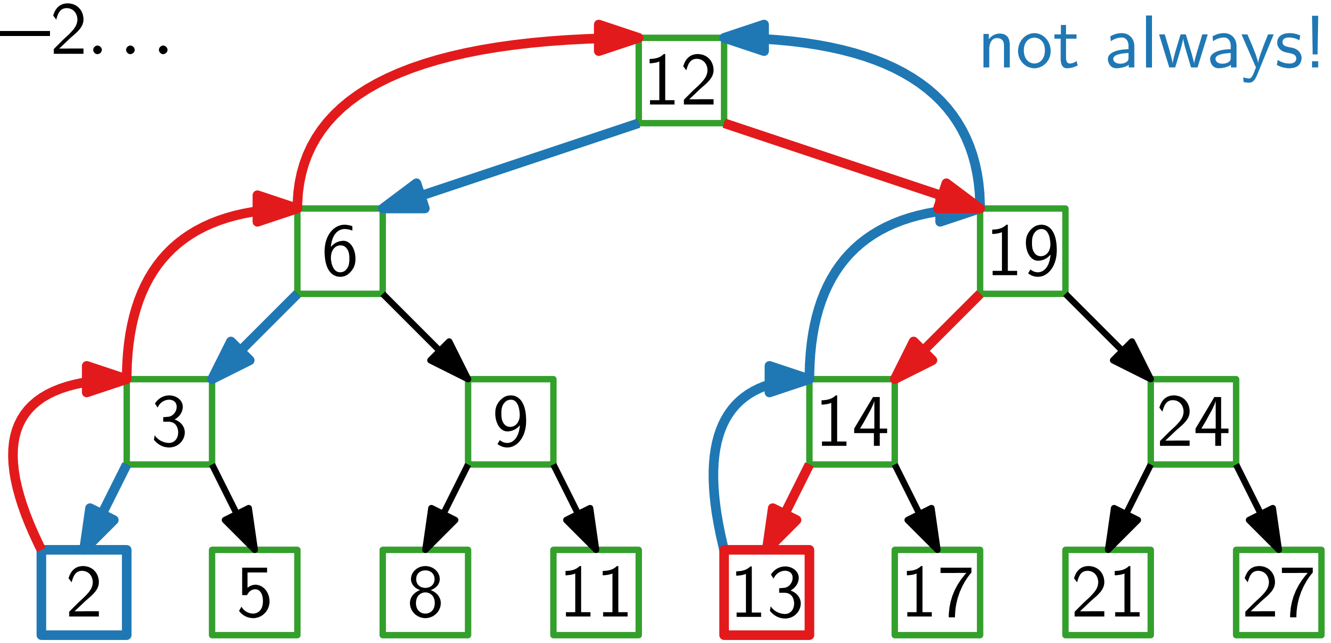
optimal?
not always!



The performance of a BST depends on the model!

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Given a BST, what is the worst sequence of queries?

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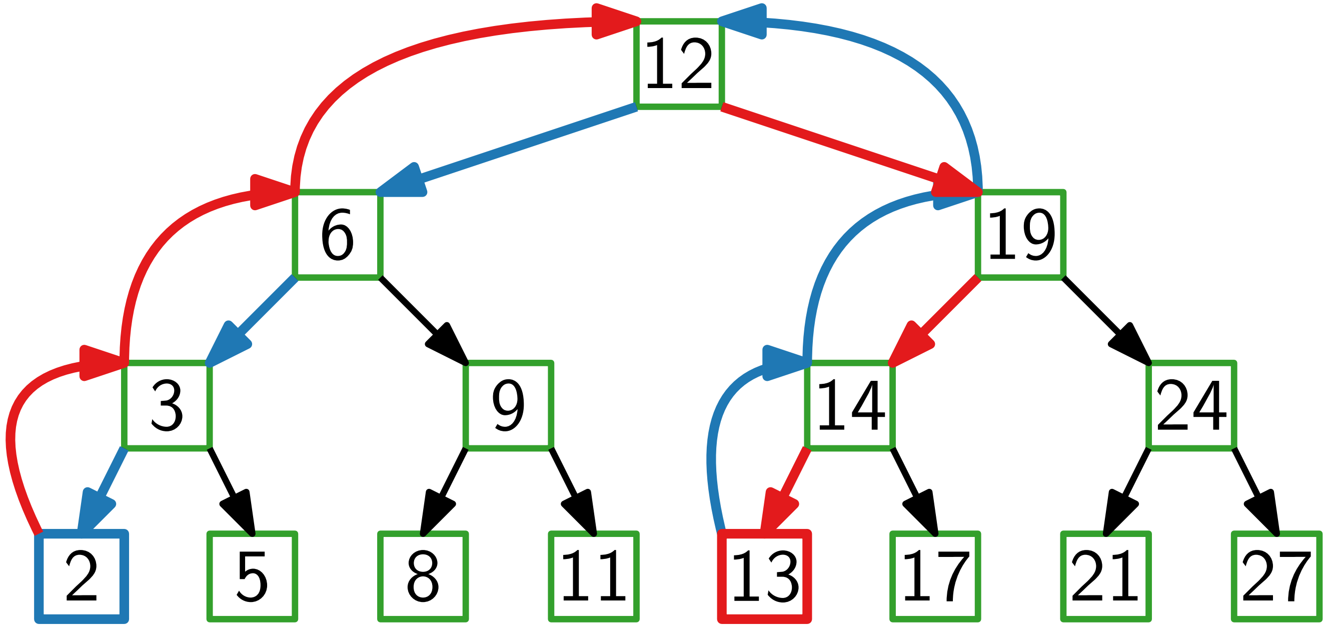

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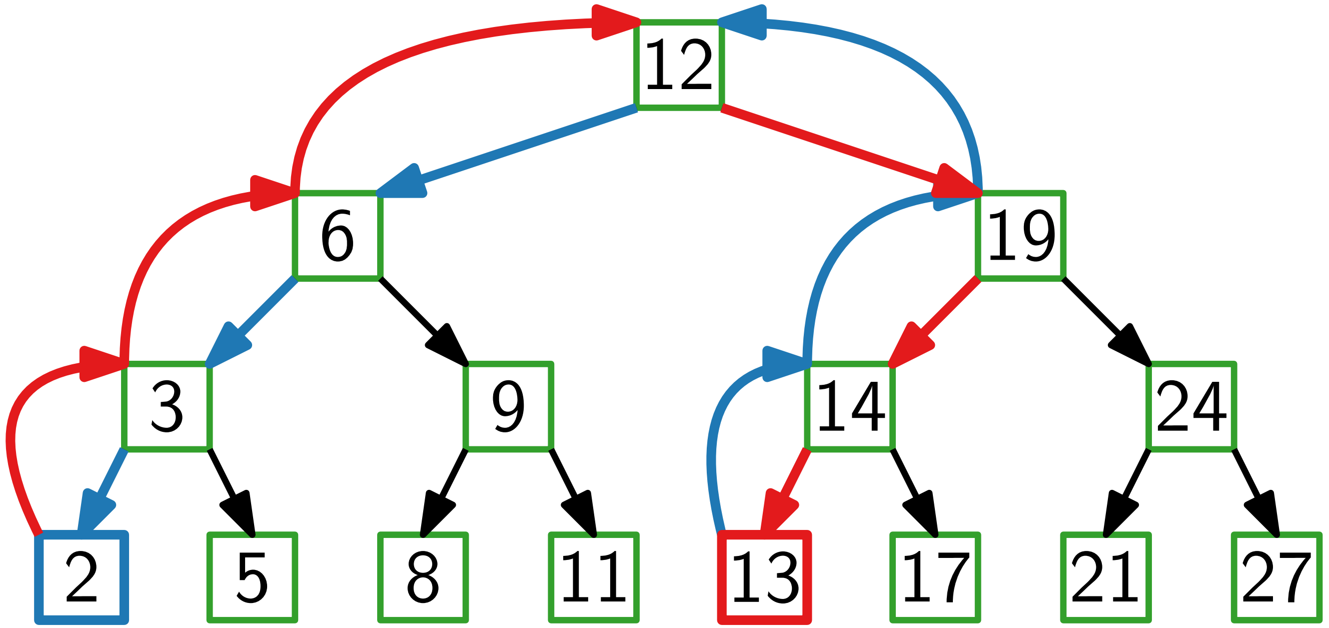
Lemma. The worst-case malicious query cost in any BST with n nodes is at least $\Omega(\log n)$ per query.



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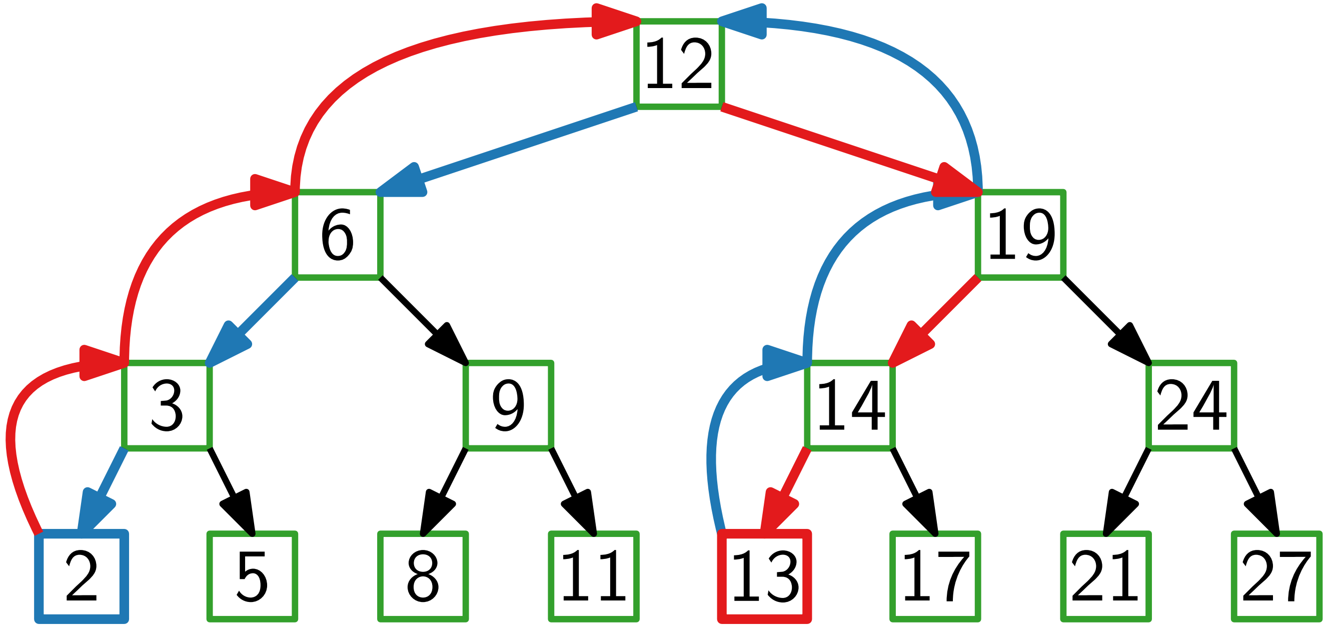

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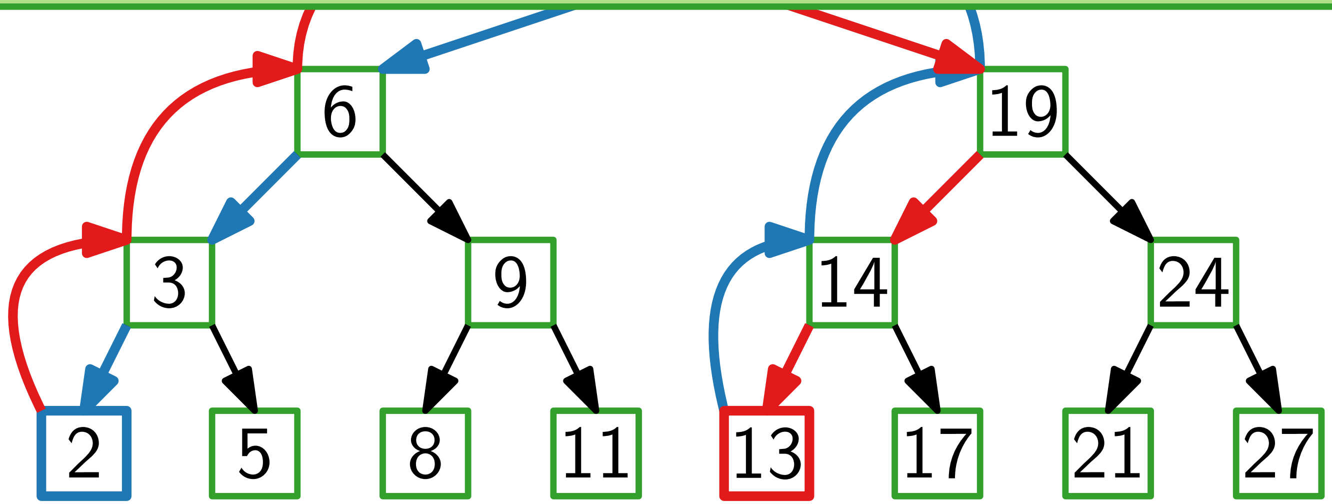
$\Rightarrow$ the (amortized) cost of each query is $O(\log n)$ (for at least n queries)



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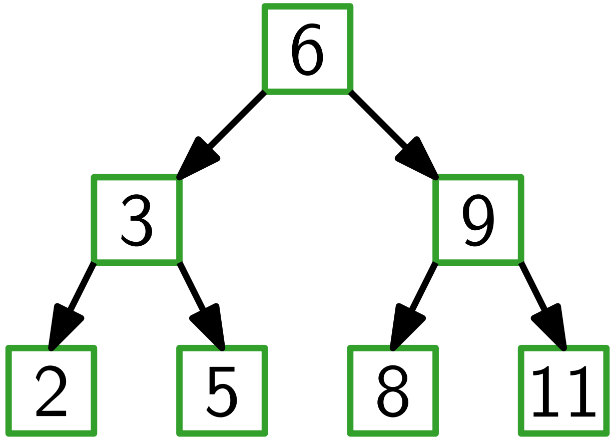
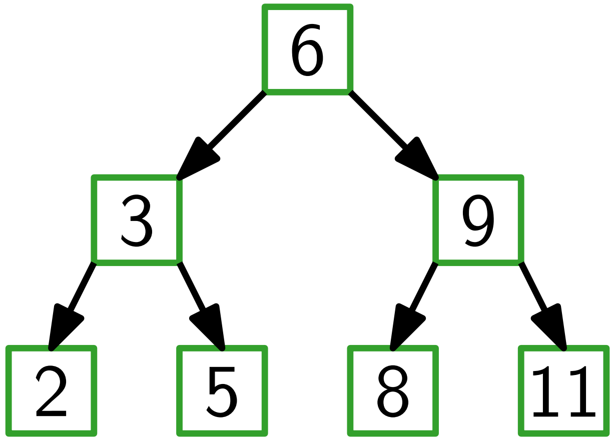


Model 2: Known Probability Distribution

Access Probabilities:

2	3	5	6	8	9	11
2%	20%	30%	8%	20%	15%	5%



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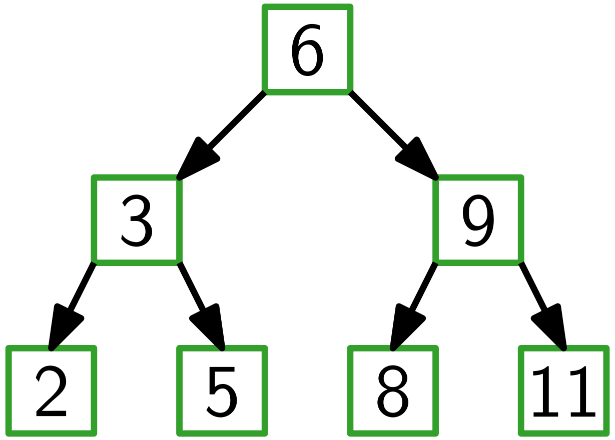

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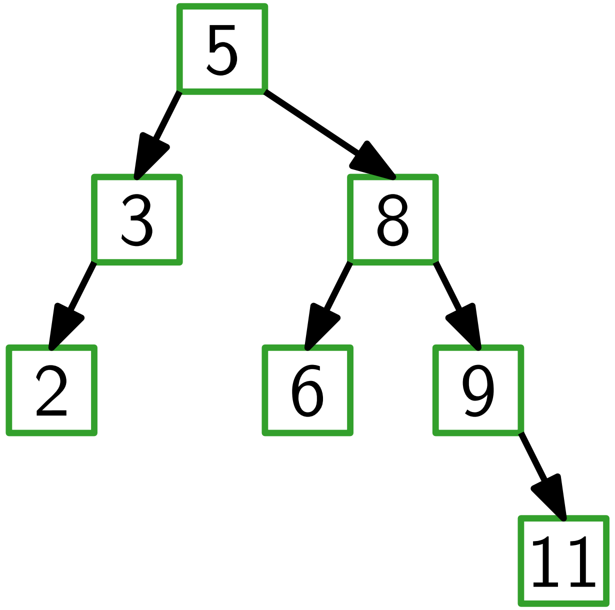

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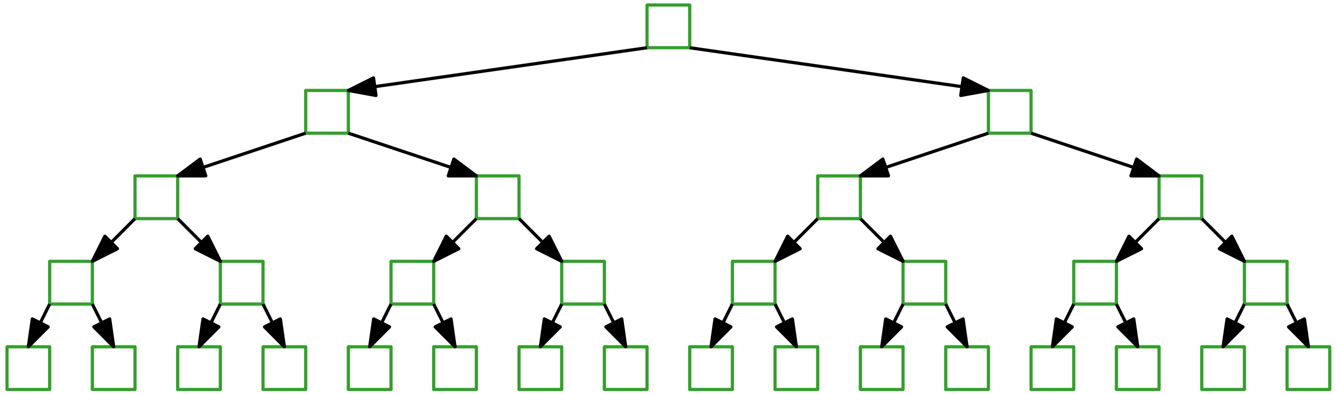

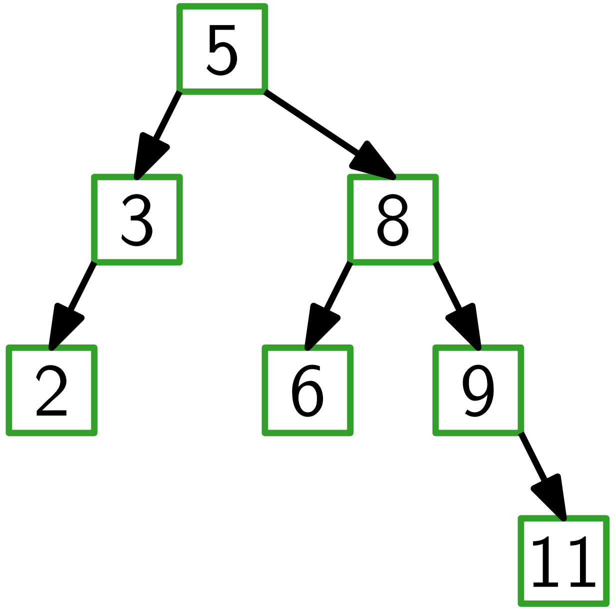

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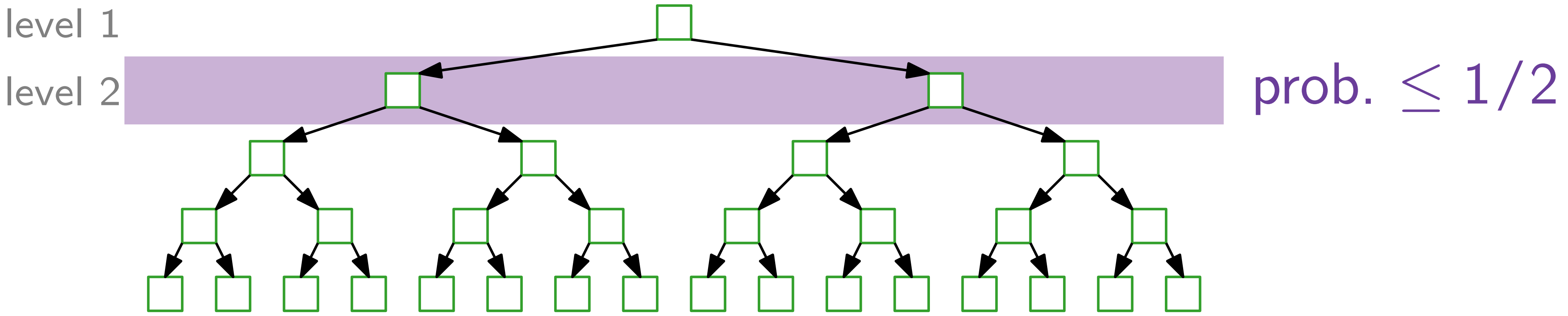

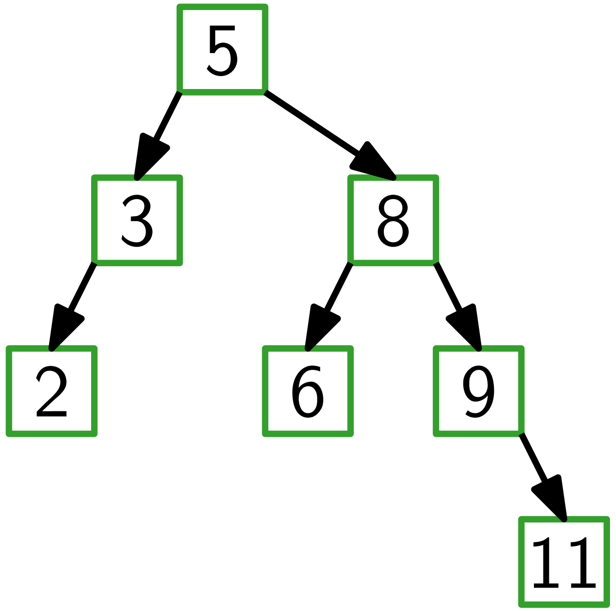

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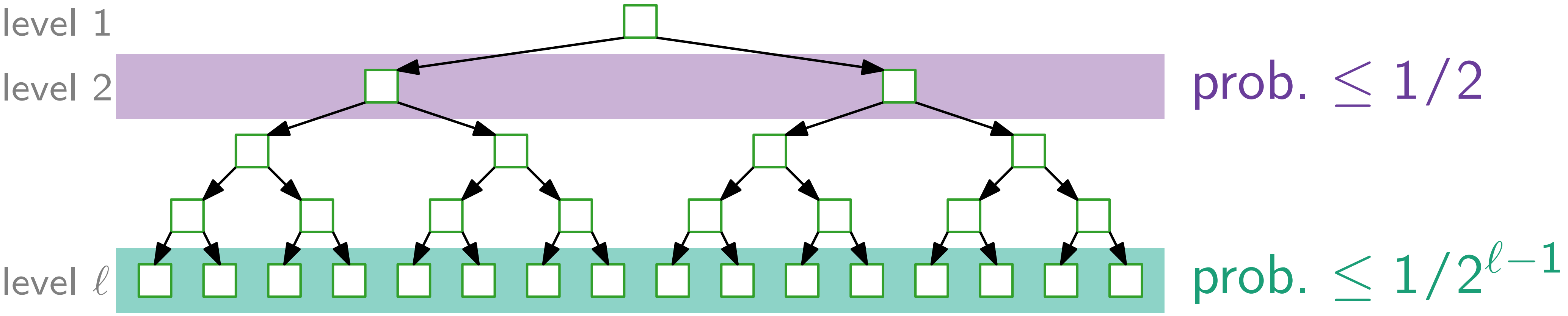

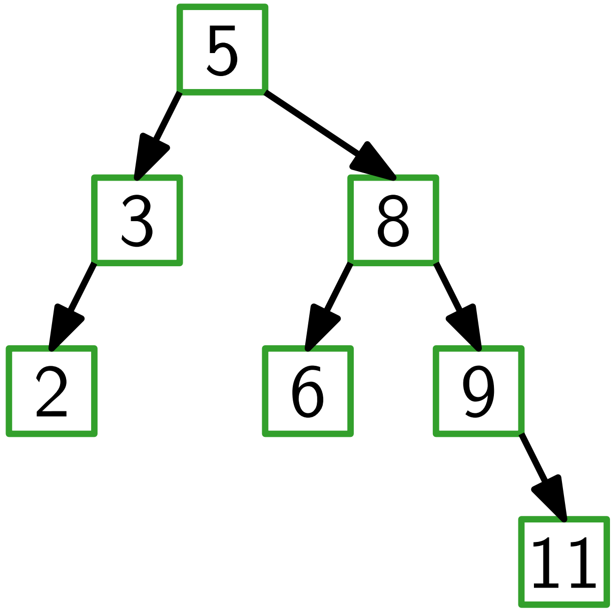

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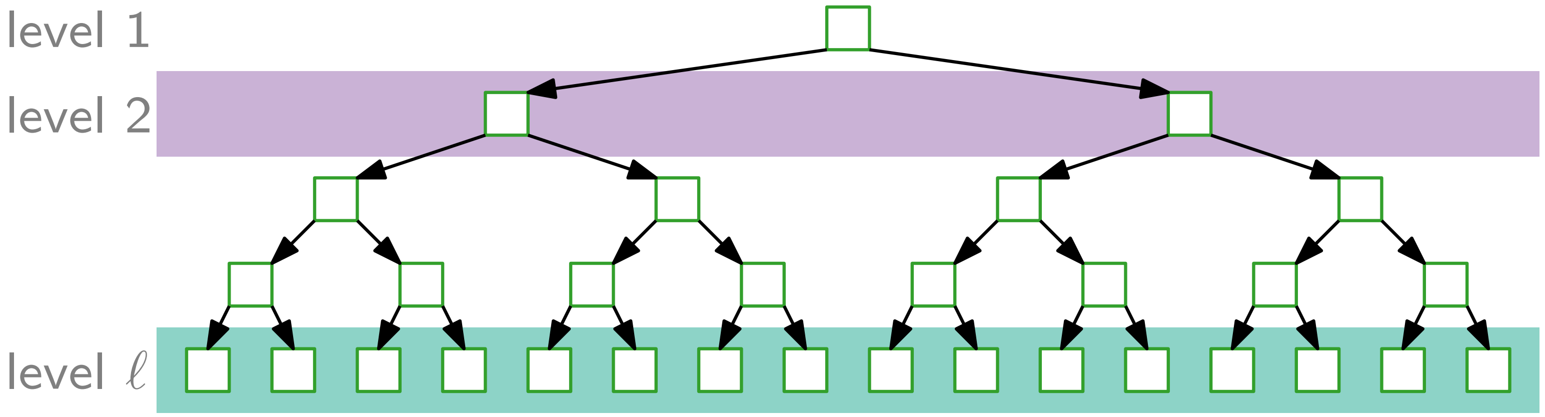

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OPT: prob. $p \Rightarrow$ level

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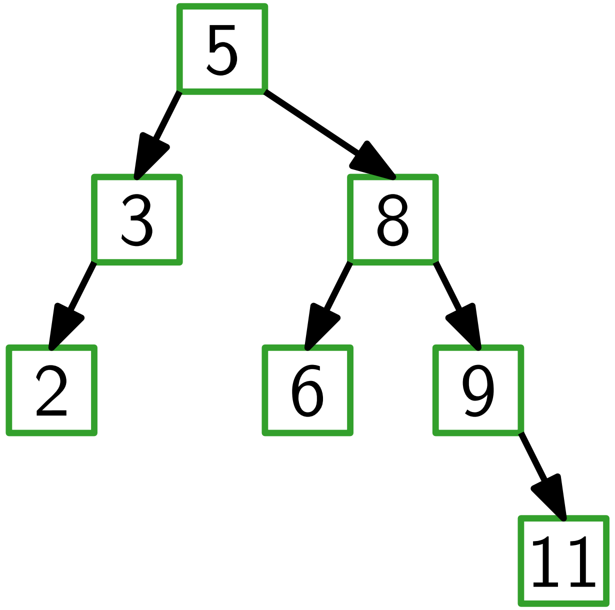

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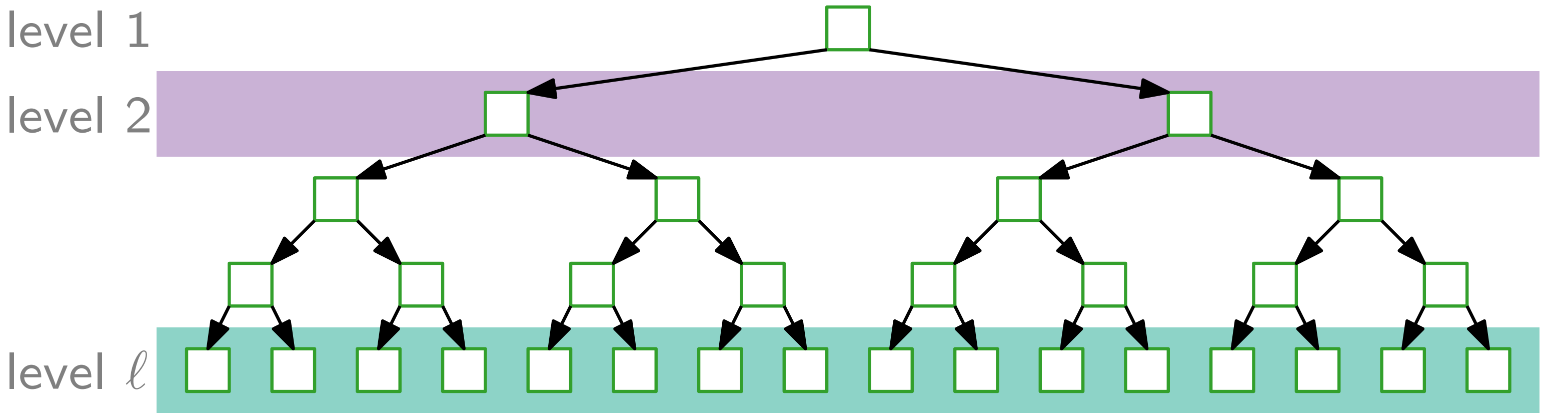

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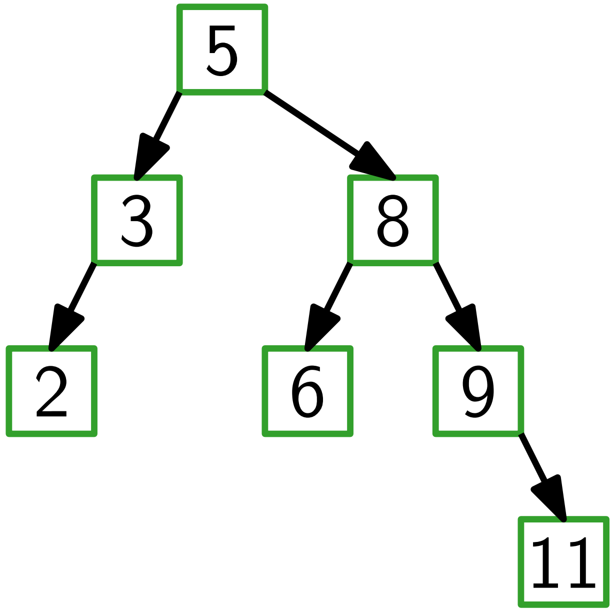
$$p \leq \frac{1}{2^{i-1}}$$



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$$p \le \frac{1}{2^{i-1}} \quad \Leftrightarrow$$

$$\log_2 p \le \log_2 \frac{1}{2^{i-1}} \quad \Leftrightarrow$$

$$\log_2 p \le 1 - i \quad \Leftrightarrow$$

$$i \le 1 - \log_2 p$$

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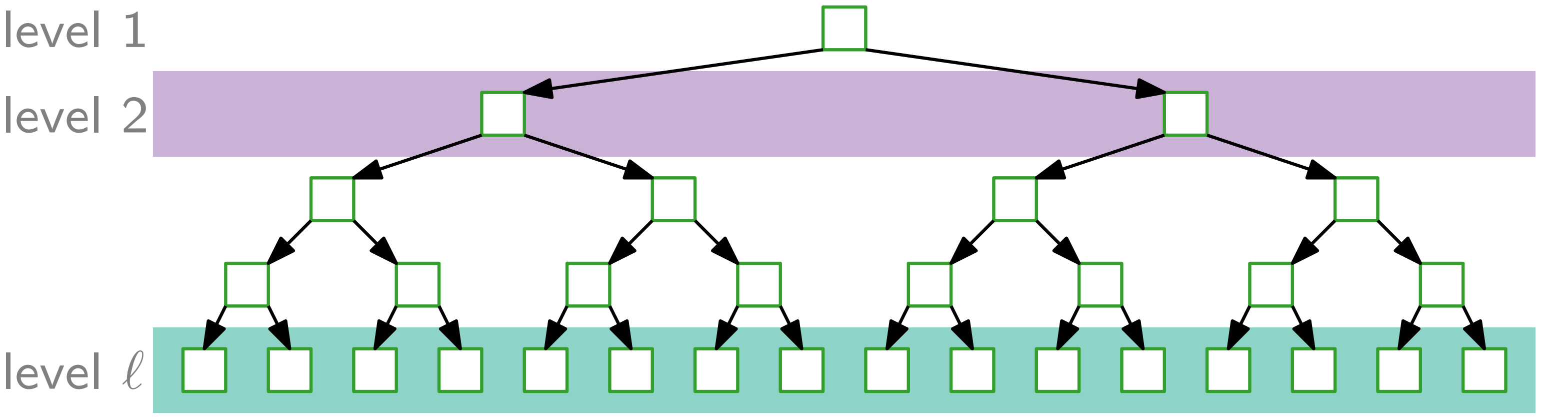

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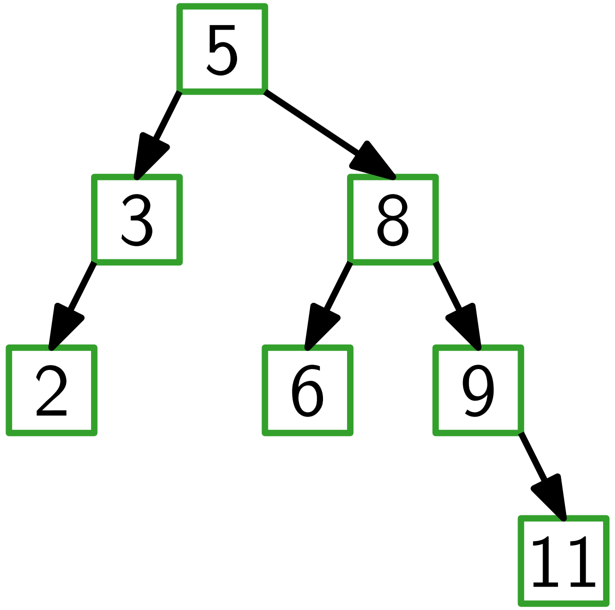

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$$
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\log_2 p &\le \log_2 \frac{1}{2^{i-1}} &&\Leftrightarrow\\
\log_2 p &\le 1 - i &&\Leftrightarrow\\
i &\le 1 - \log_2 p
\end{aligned}
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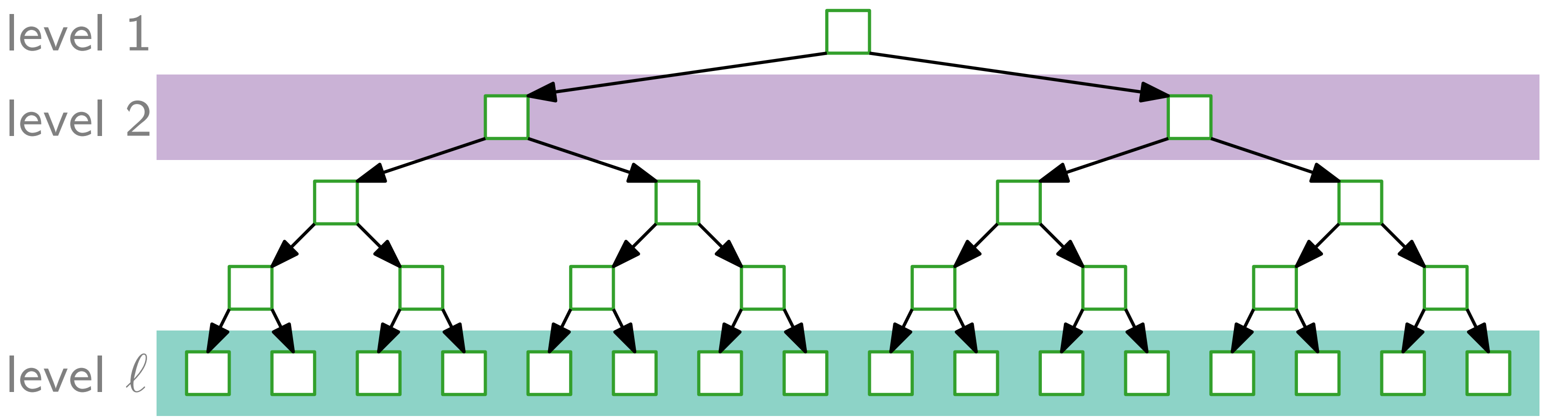

prob. $\le 1/2$

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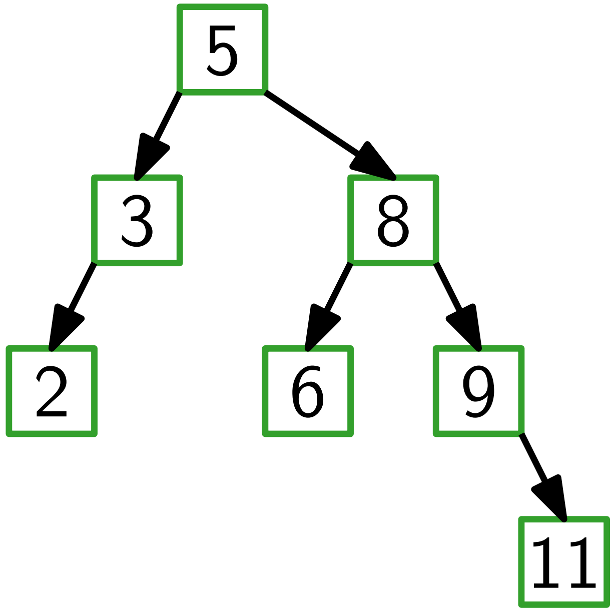

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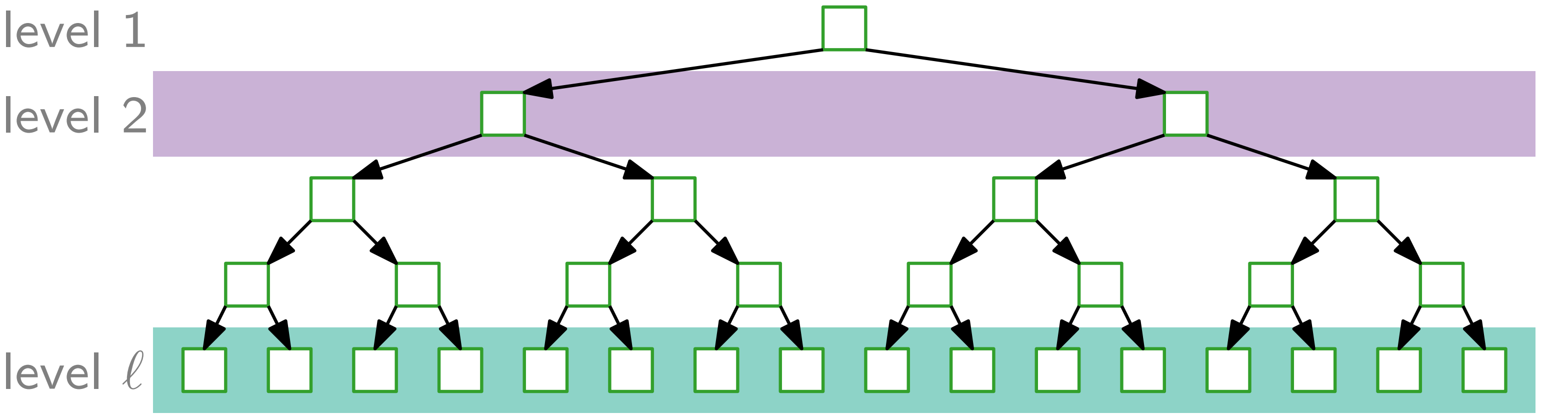
2%	20%	30%	8%	20%	15%	5%

Idea: Place nodes with higher probability higher in the tree.



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Lemma. The expected query cost in any BST is at least $\Omega(1 + H)$ per query with $H = \sum_{i=1}^{n} -p_i \log p_i$.

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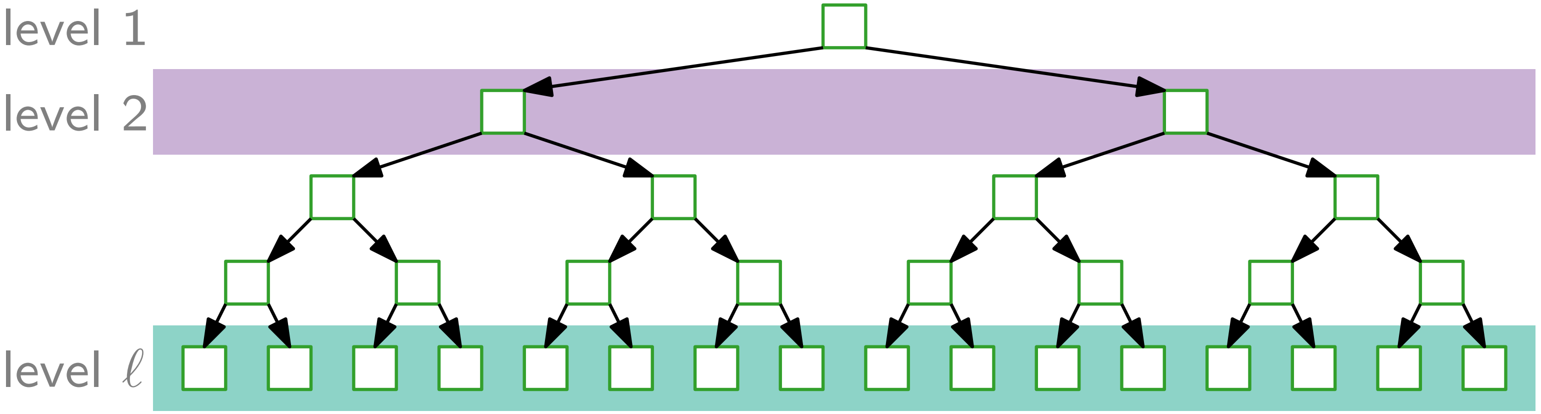

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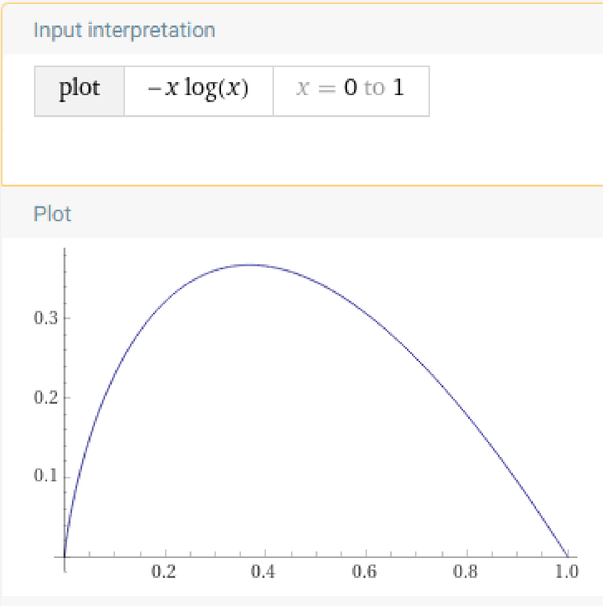
Input interpretation

plot | $-x \log(x)$ | $x = 0$ to 1

Plot

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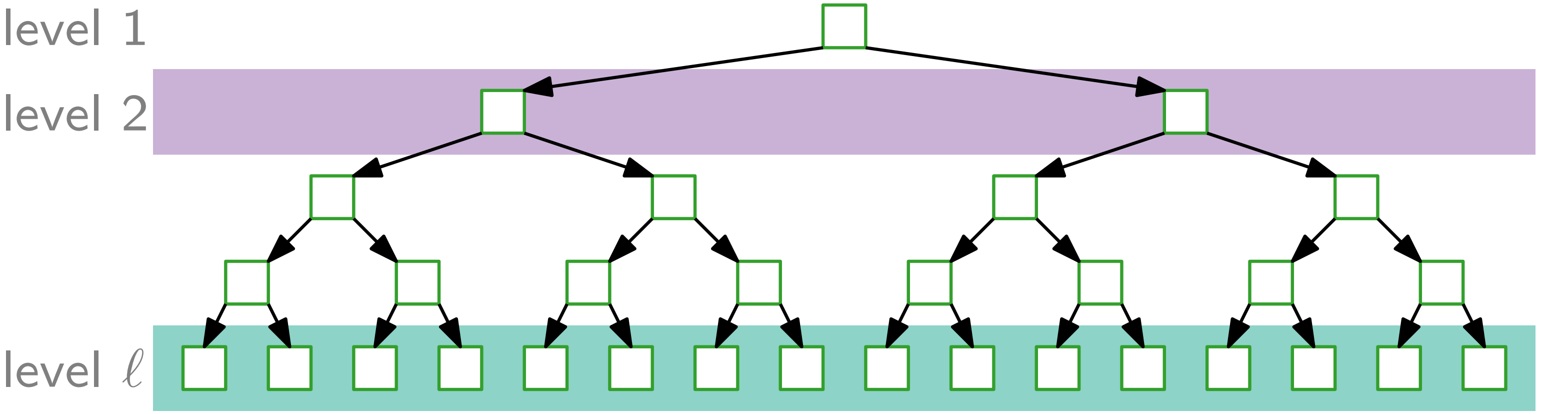

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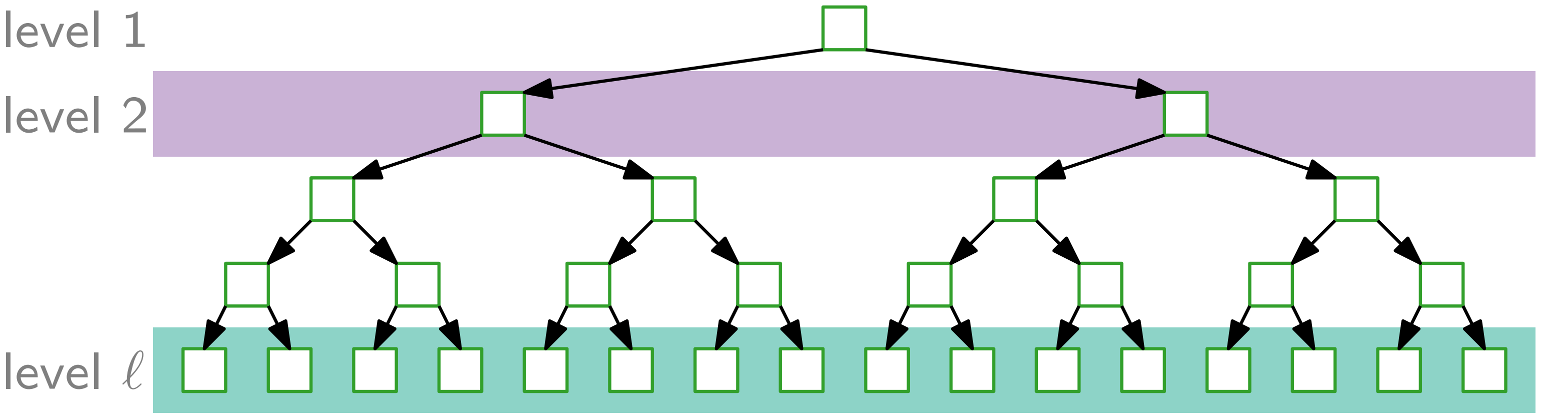

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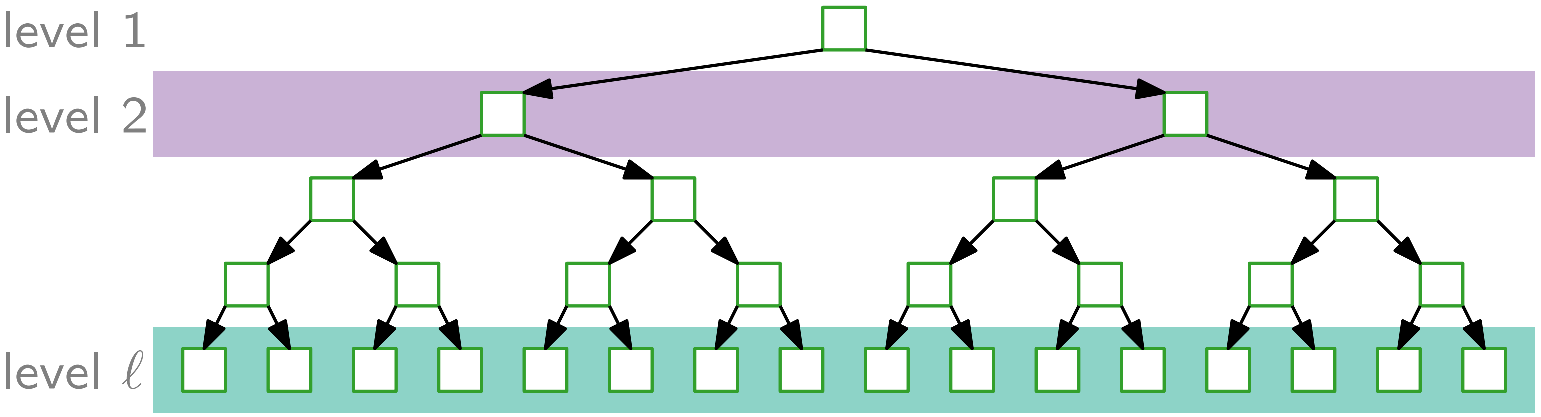

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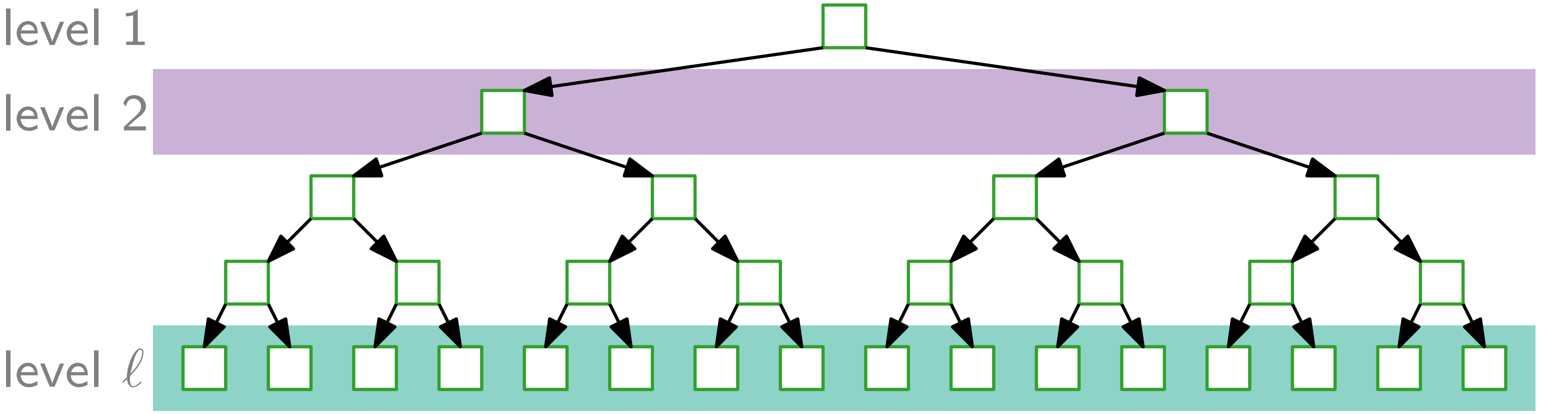

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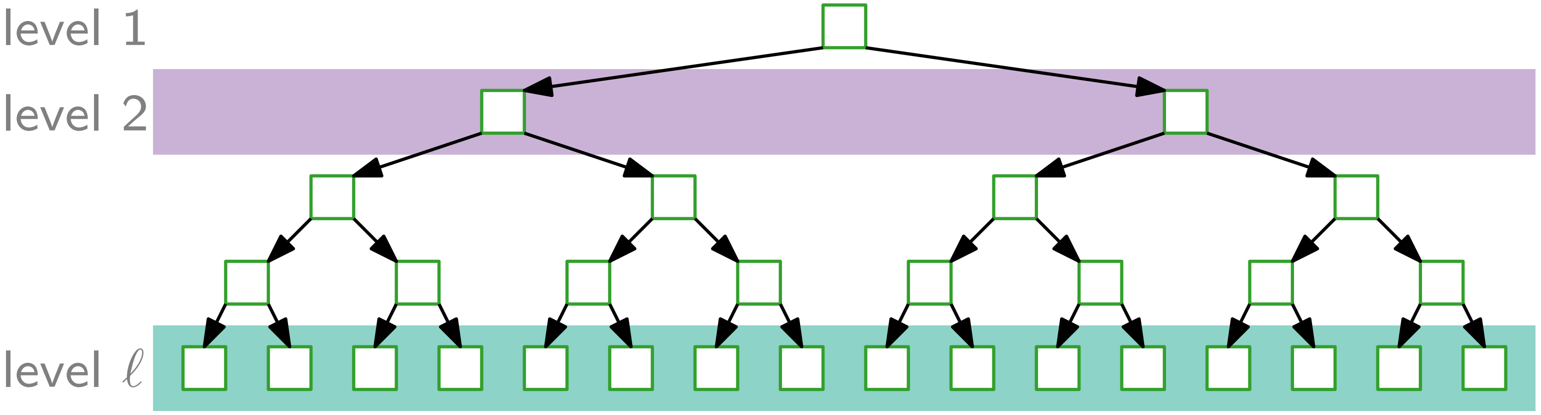

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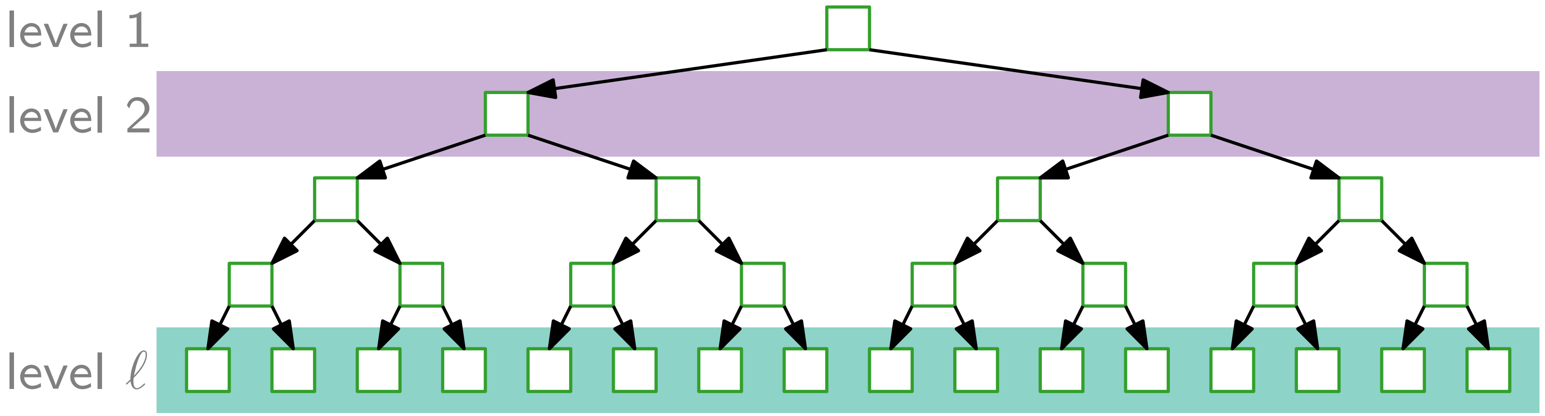

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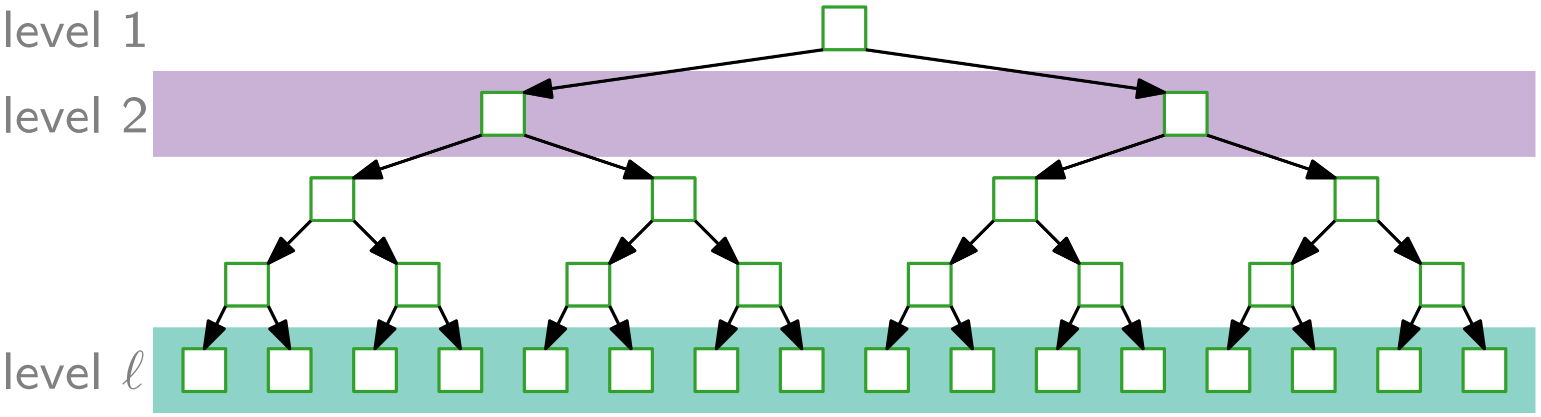

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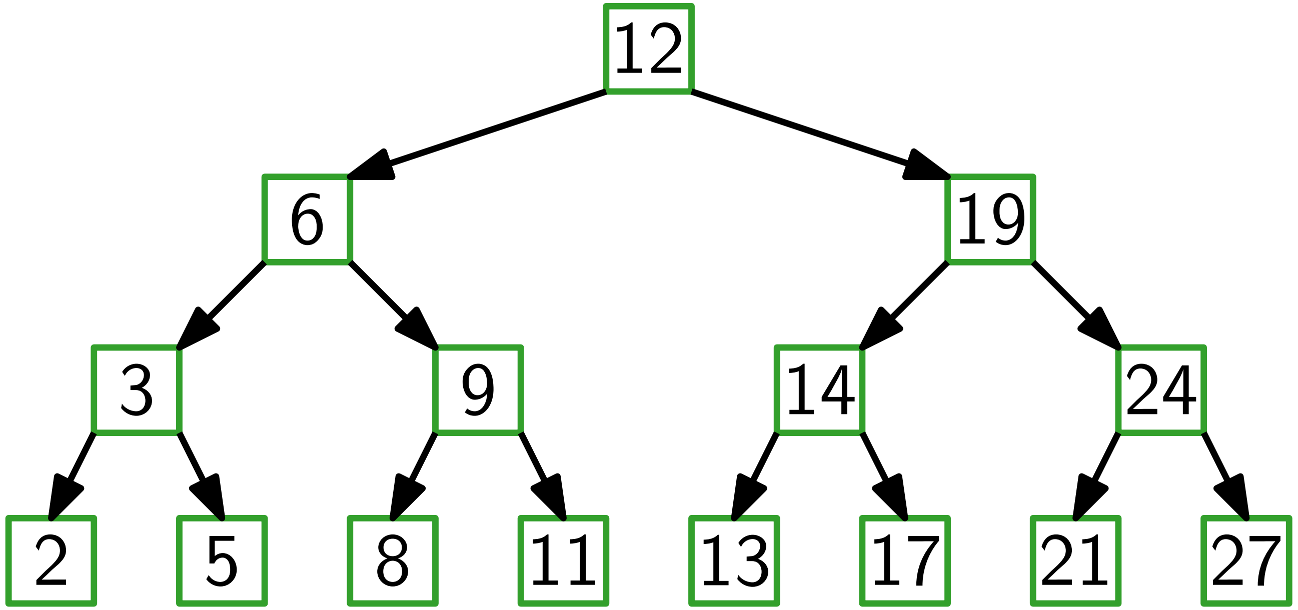

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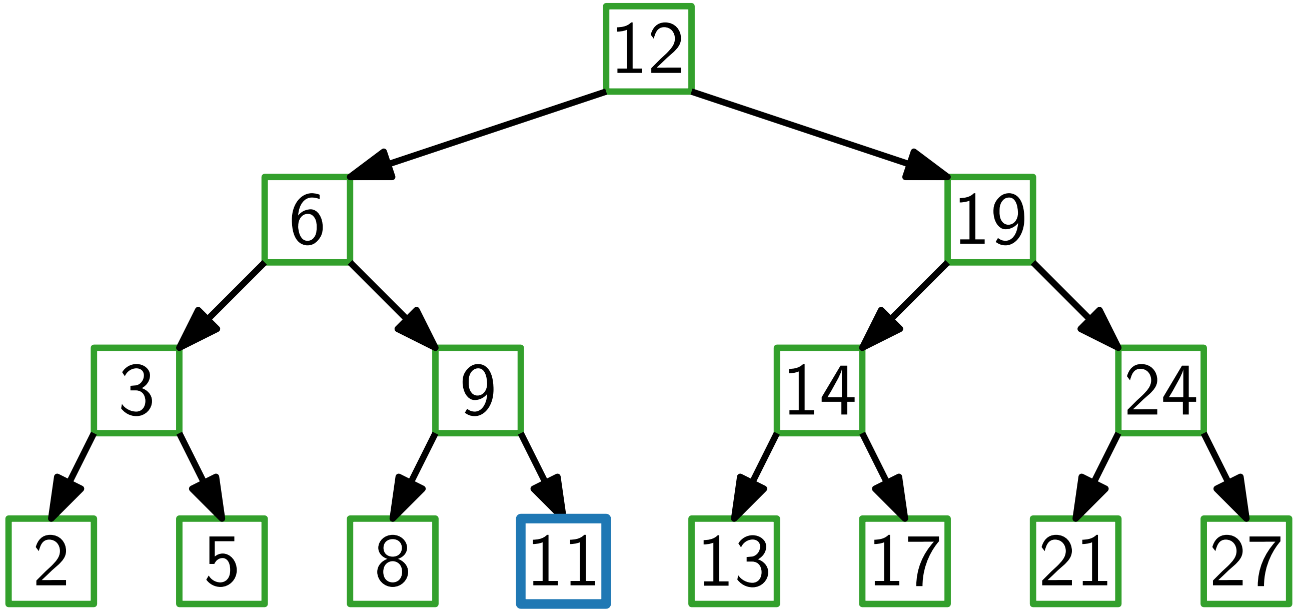

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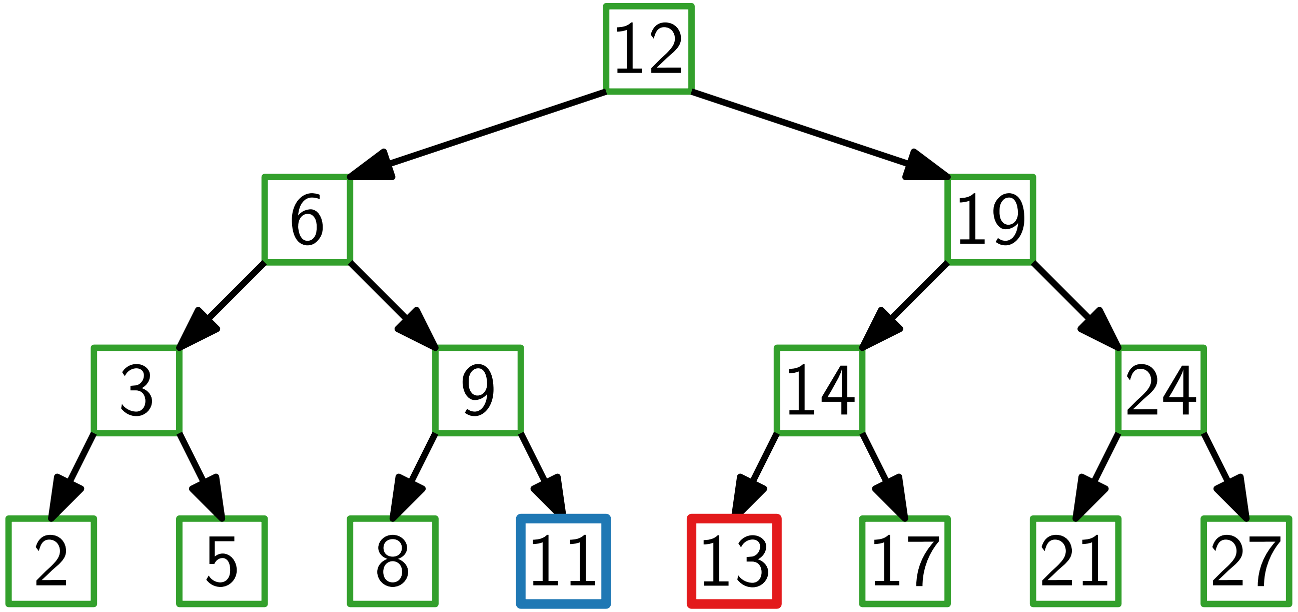

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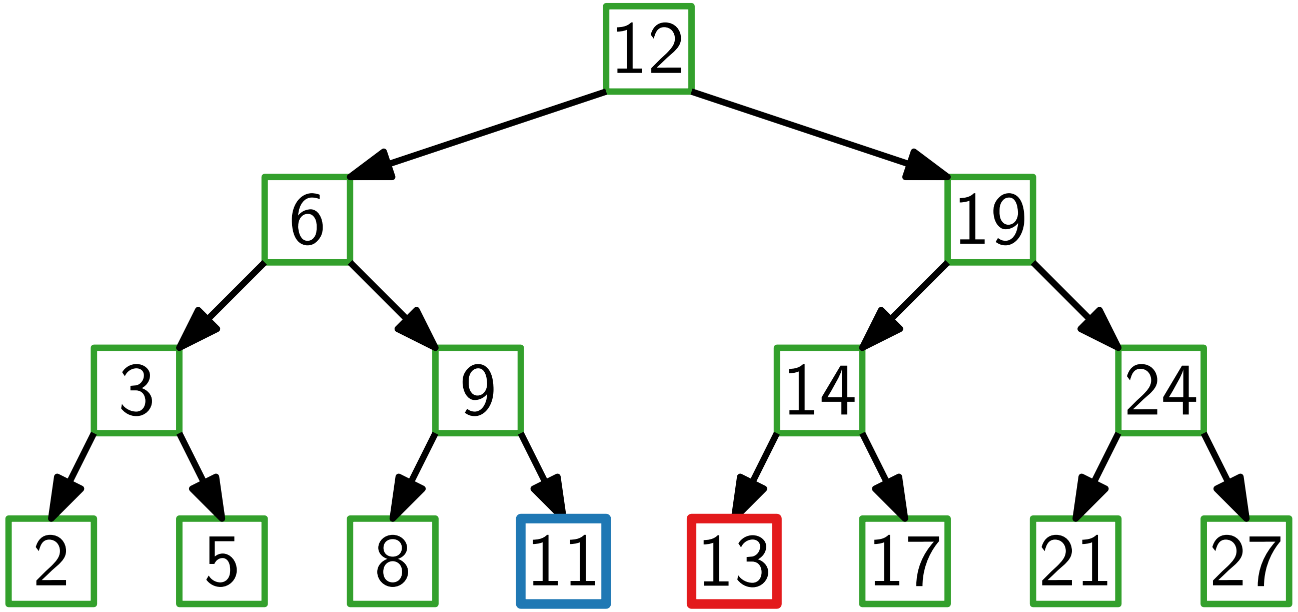

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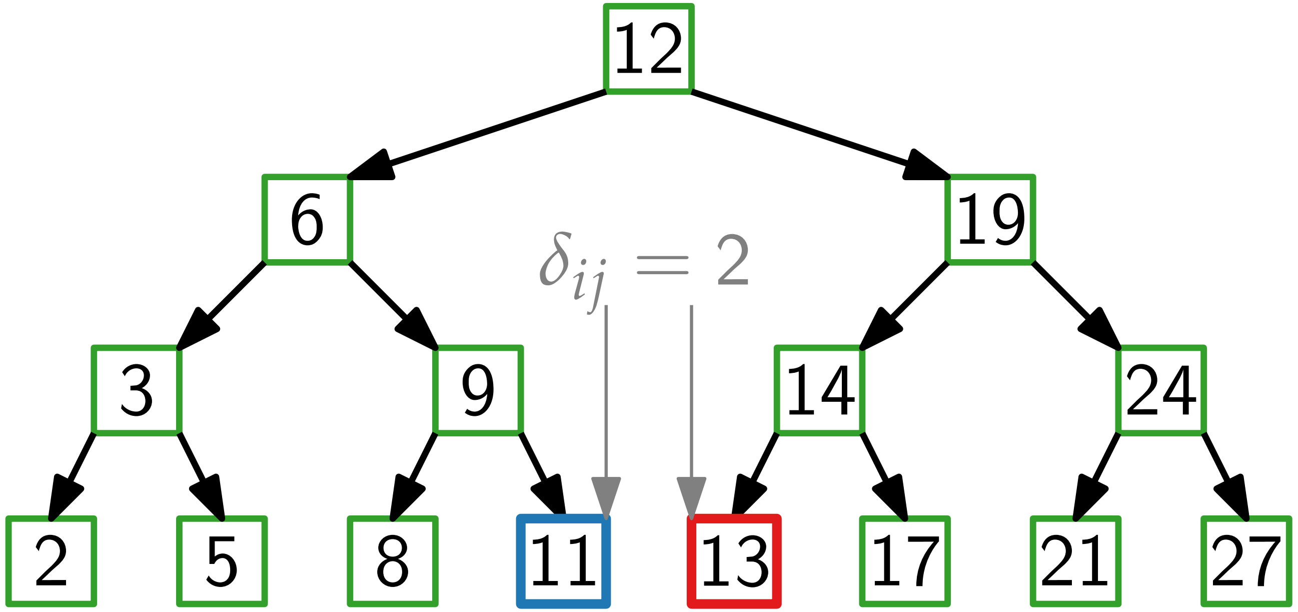

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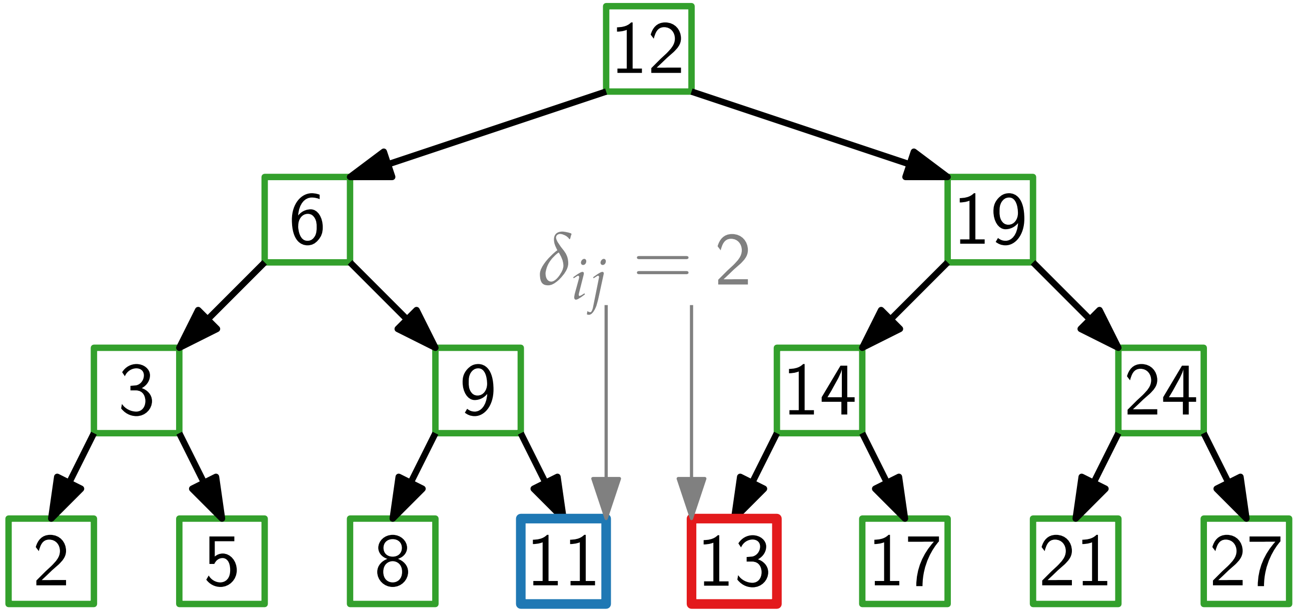

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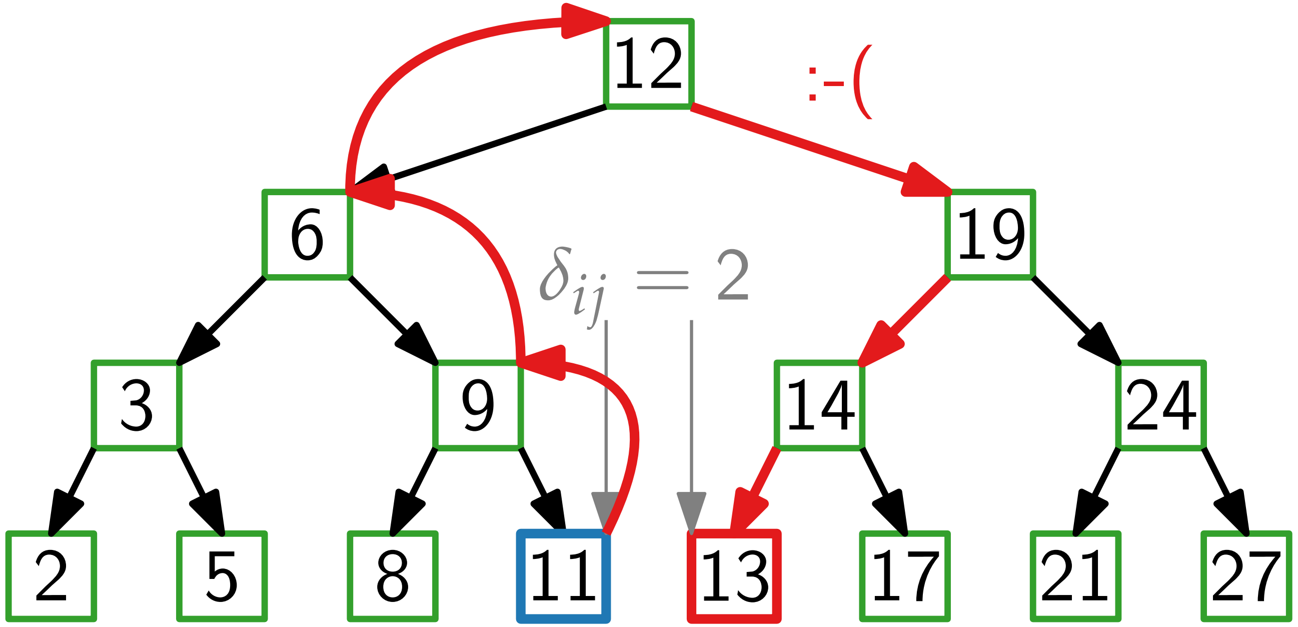

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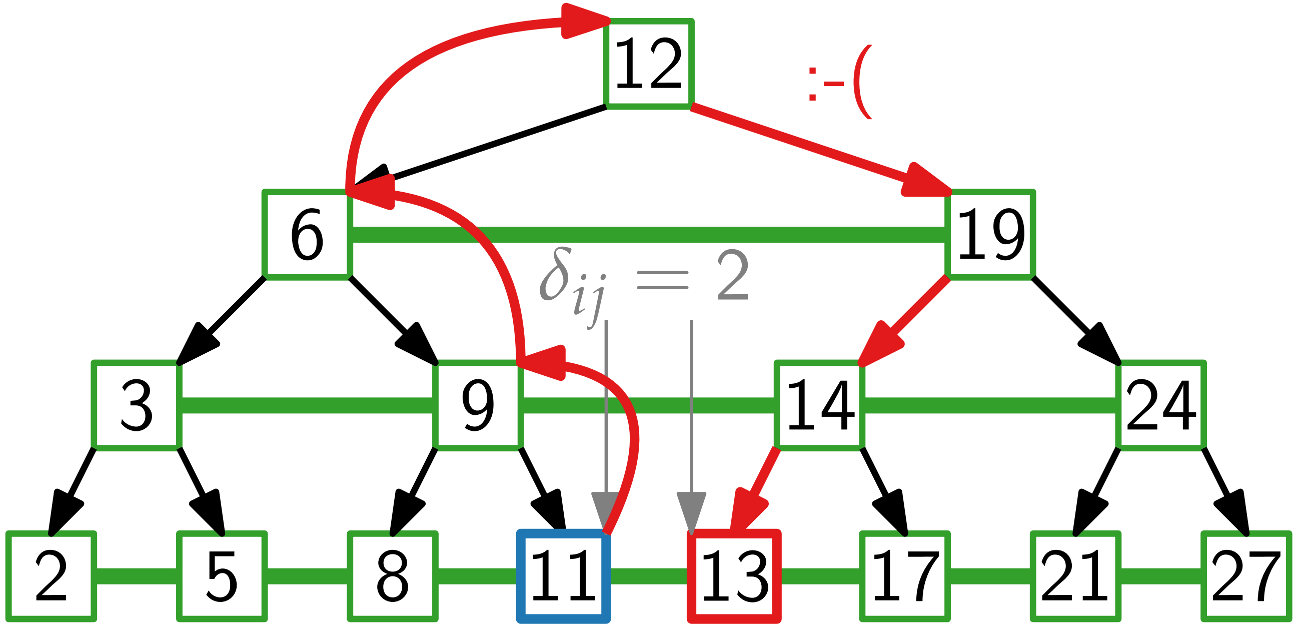

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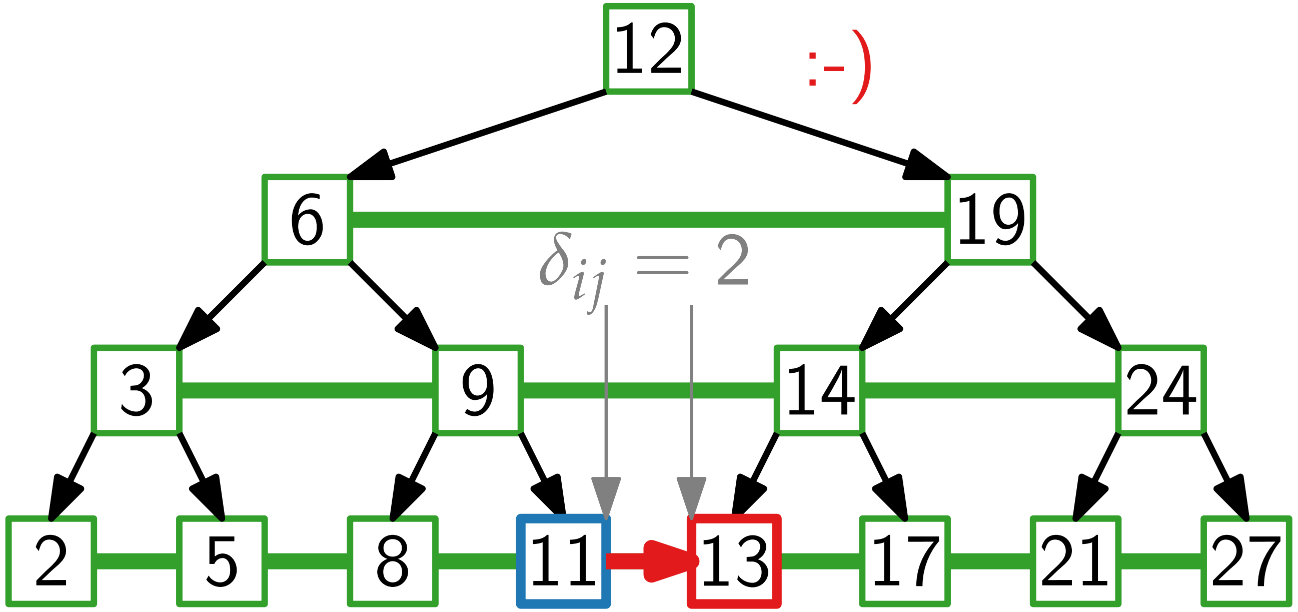

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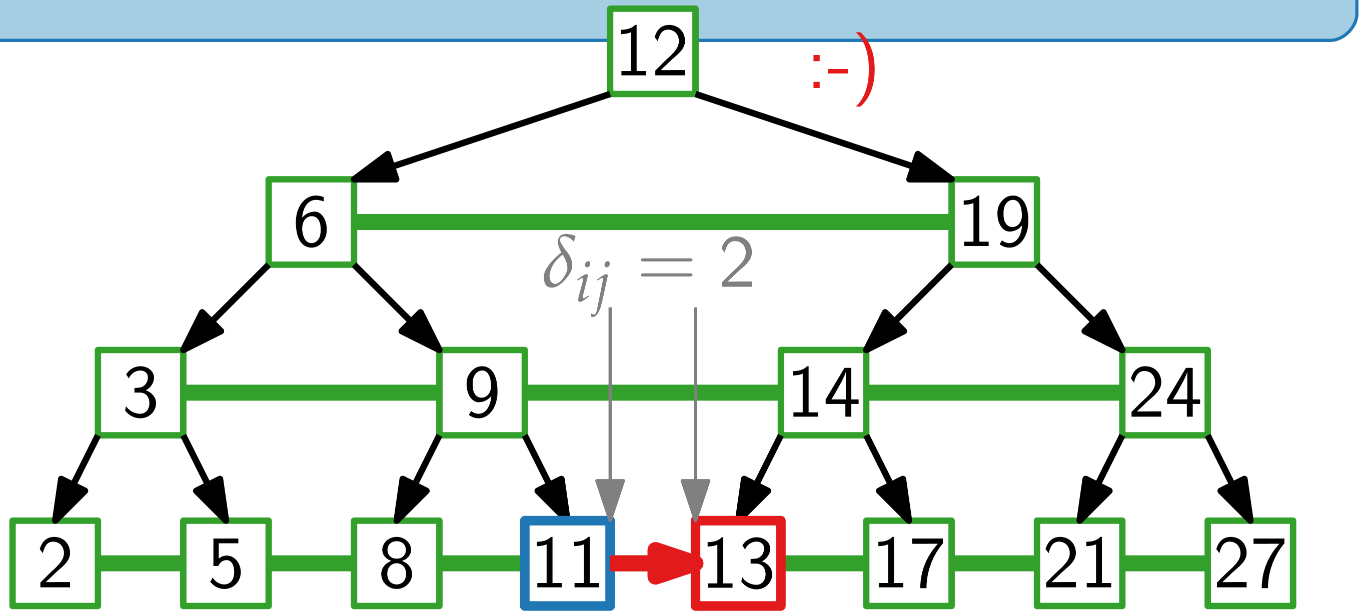
Lemma. A level-linked Red-Black-Tree has the dynamic finger property.



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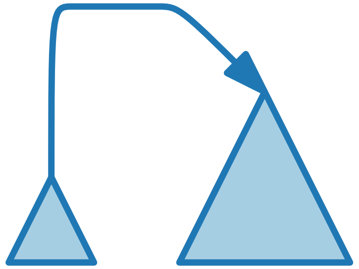

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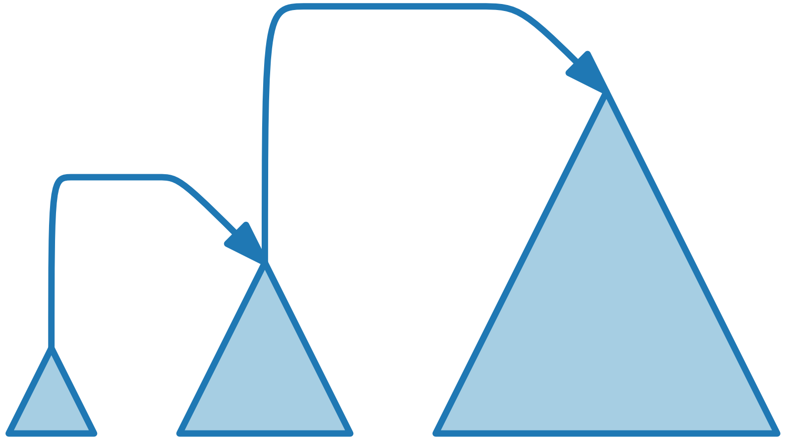

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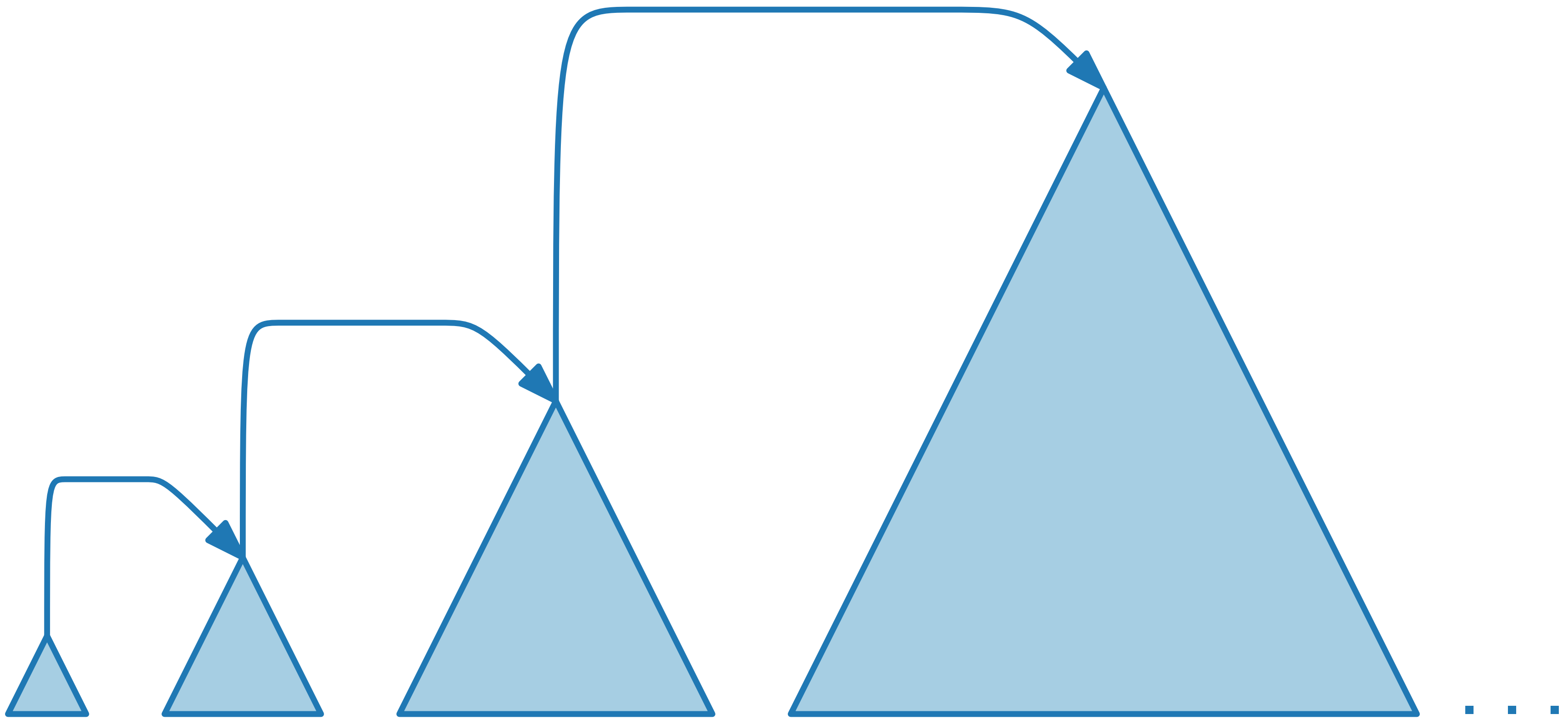

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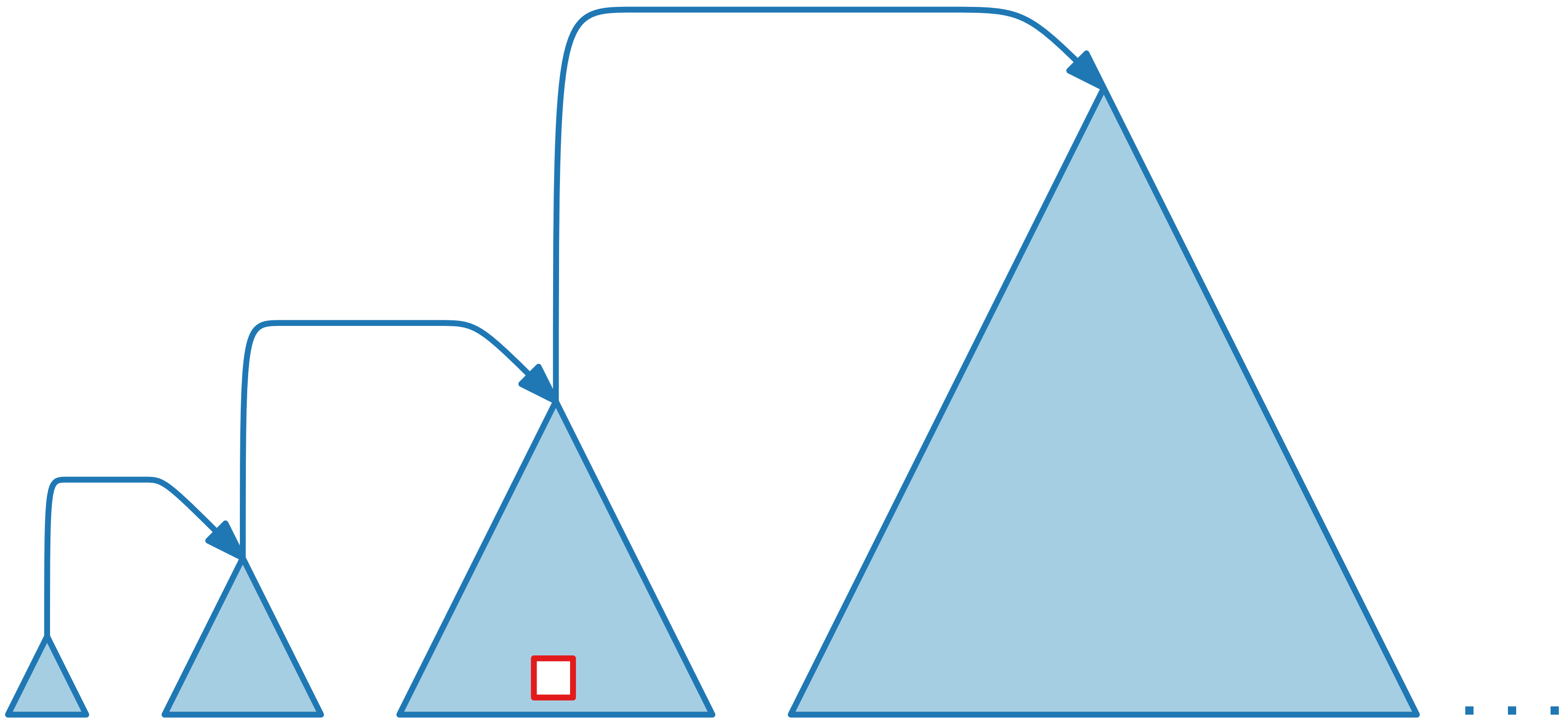

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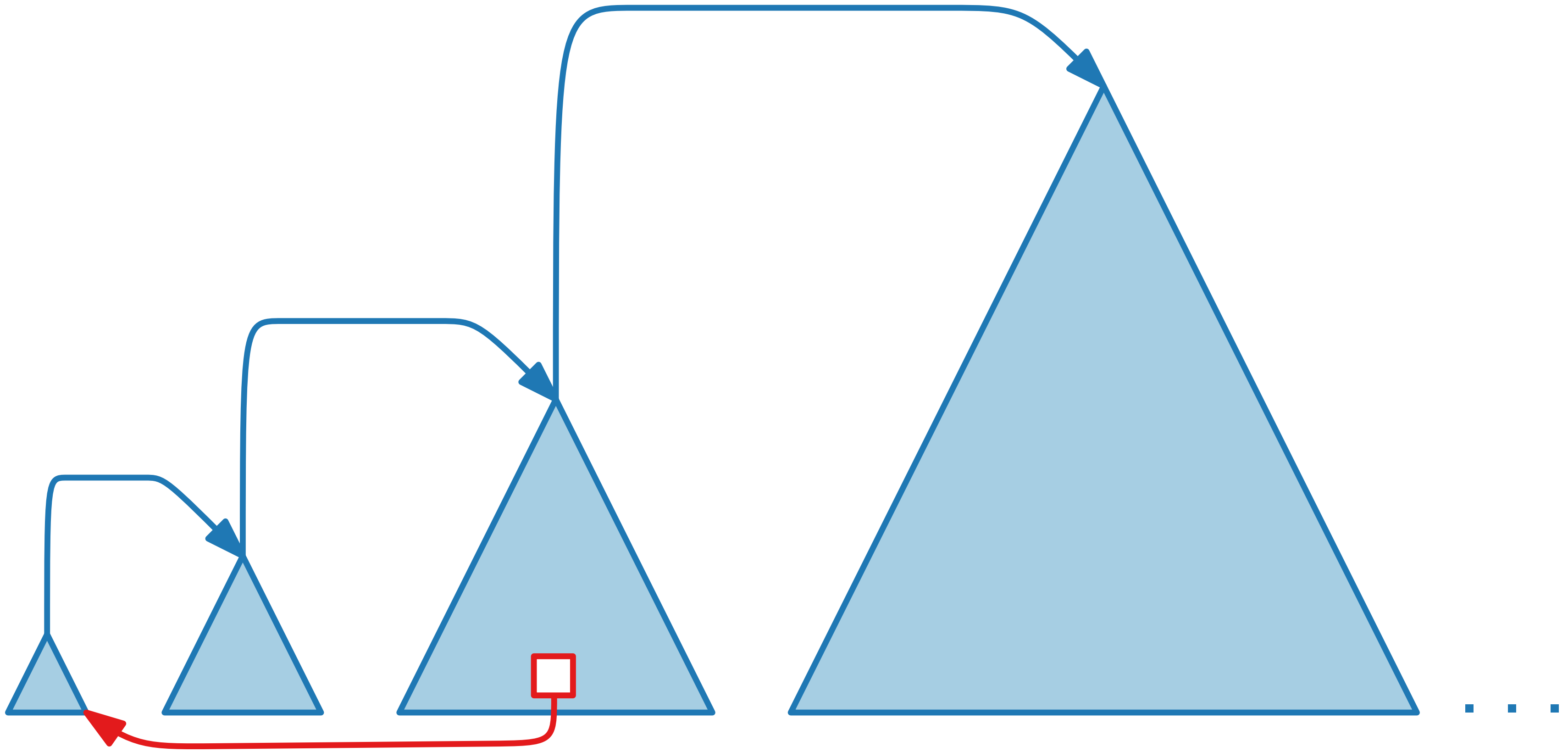

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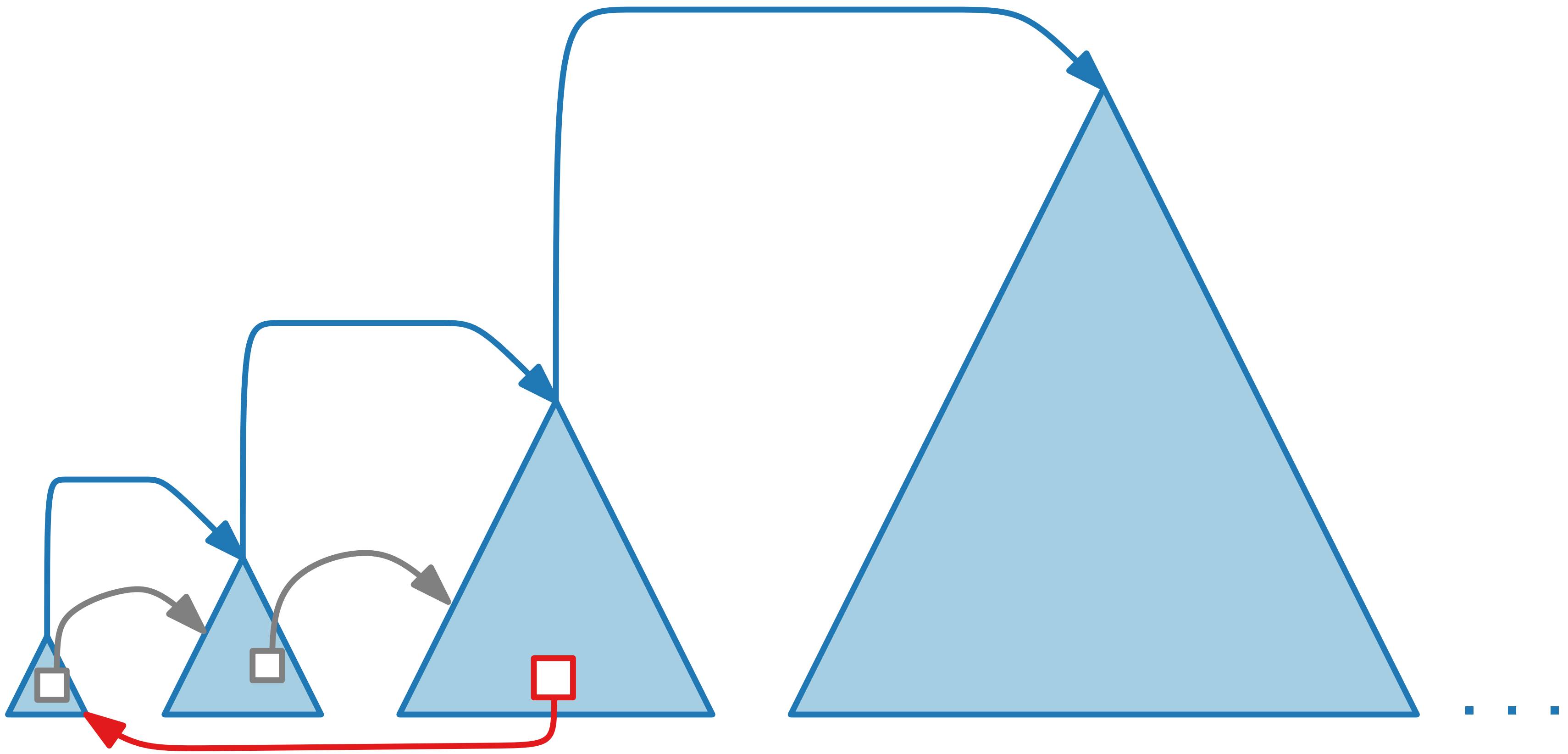

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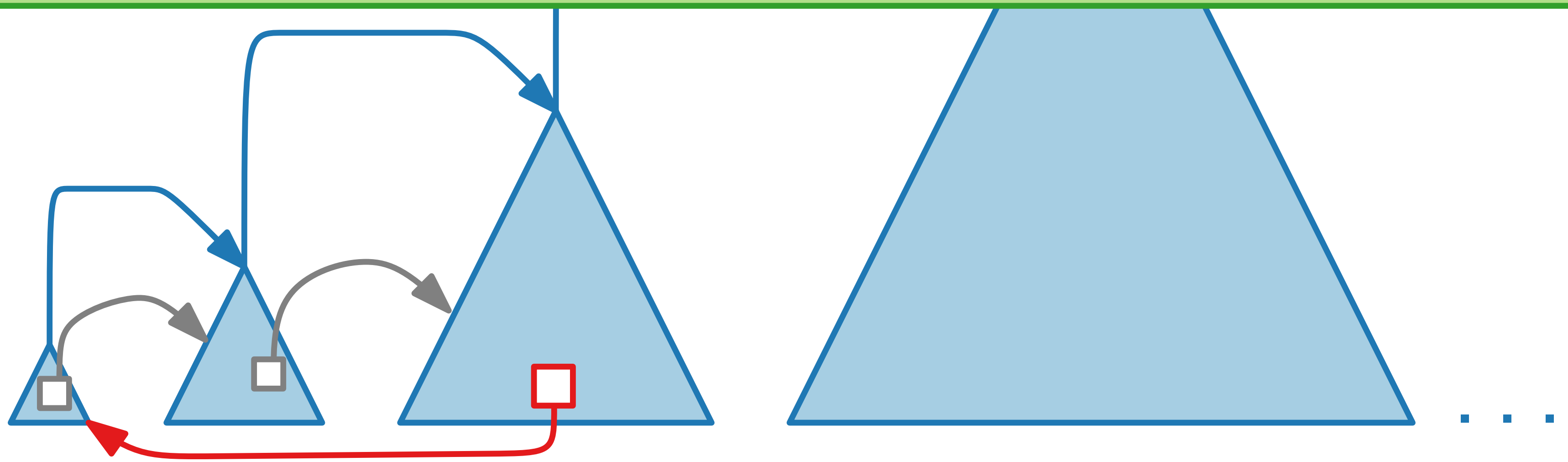

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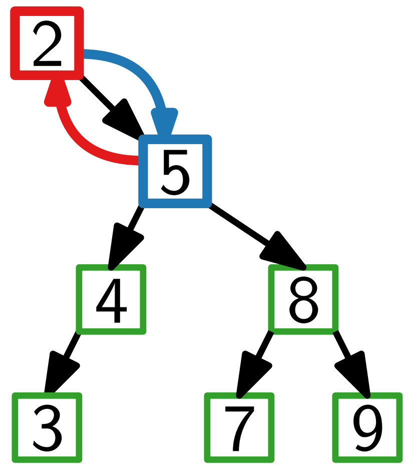

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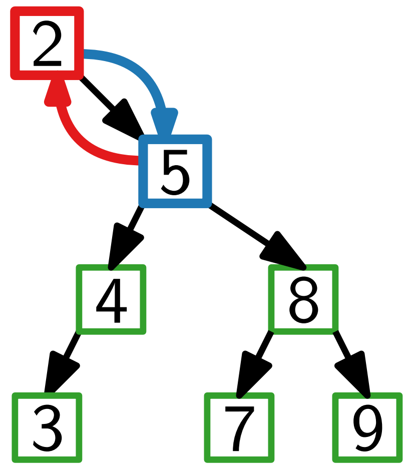

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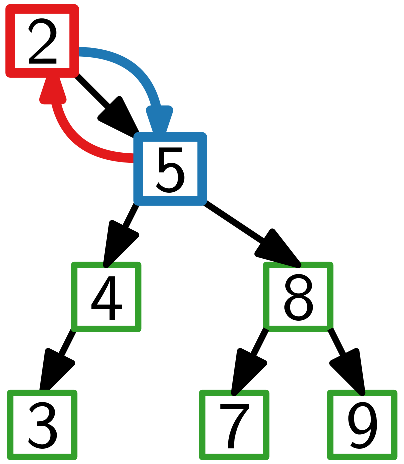

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Definition. A BST is **statically optimal** if queries take (amortized) $O(\mathrm{OPT}_S)$ time for every S.

All These Models . . .

Balanced: Queries take (amortized) $O(\log n)$ time

Entropy: Queries take expected $O(1 + H)$ time

Dynamic Finger: Queries take $O(\log \delta_i)$ time (δ_i: rank diff.)

Working Set: Queries take $O(\log t)$ time (t: recency)

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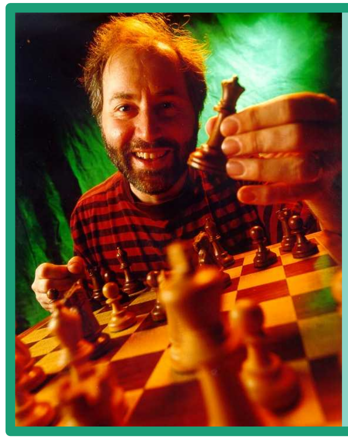
Yes!

Splay Trees



Daniel D. Sleator

Robert E. Tarjan

J. ACM 1985

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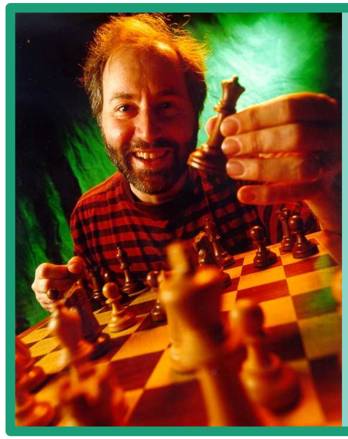

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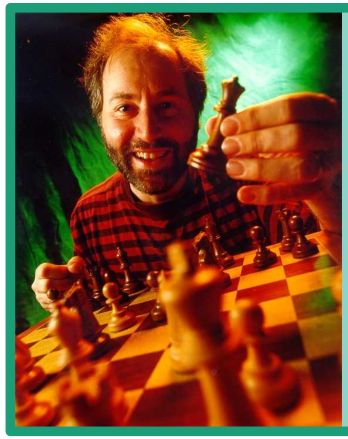

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Known from the lecture *algorithms and data structures* (ADS):

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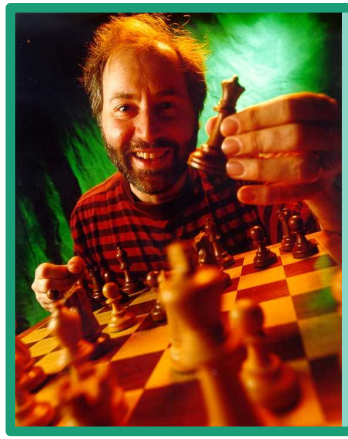

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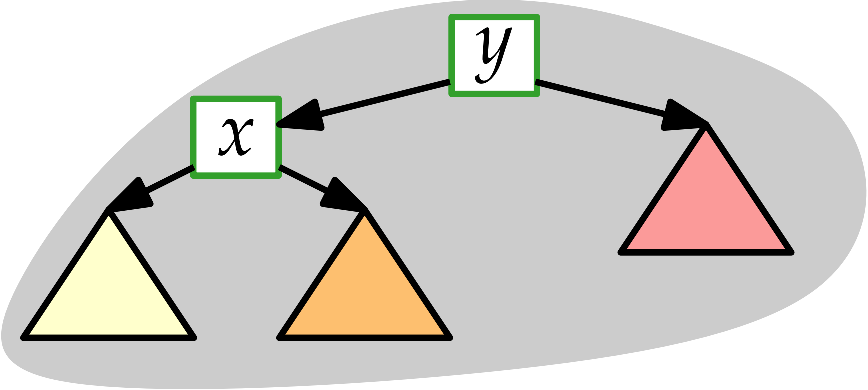

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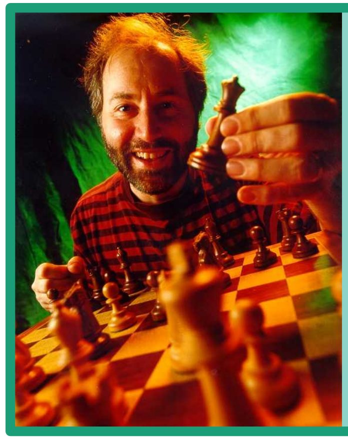

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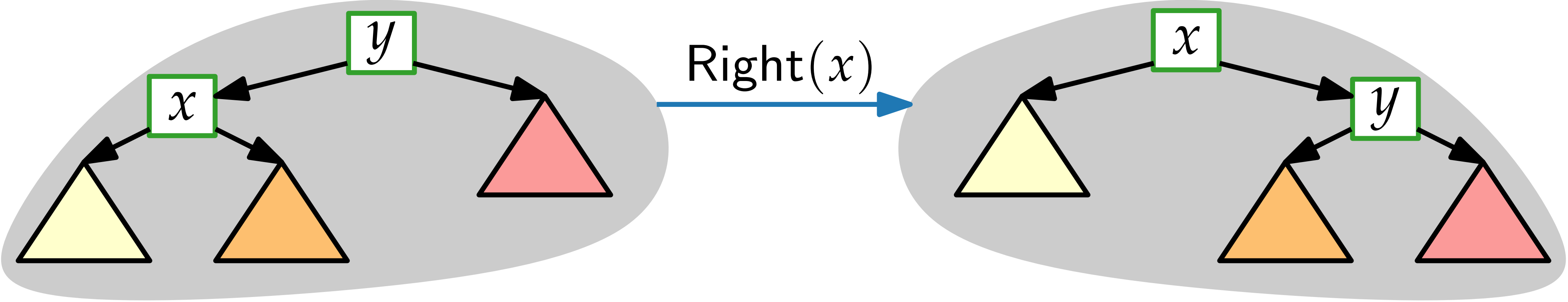

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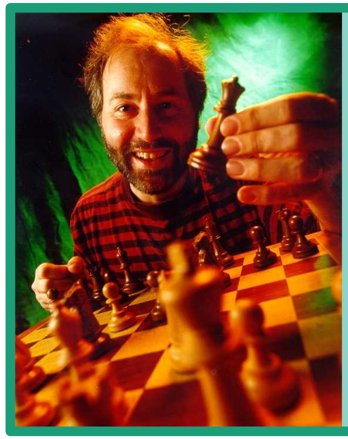

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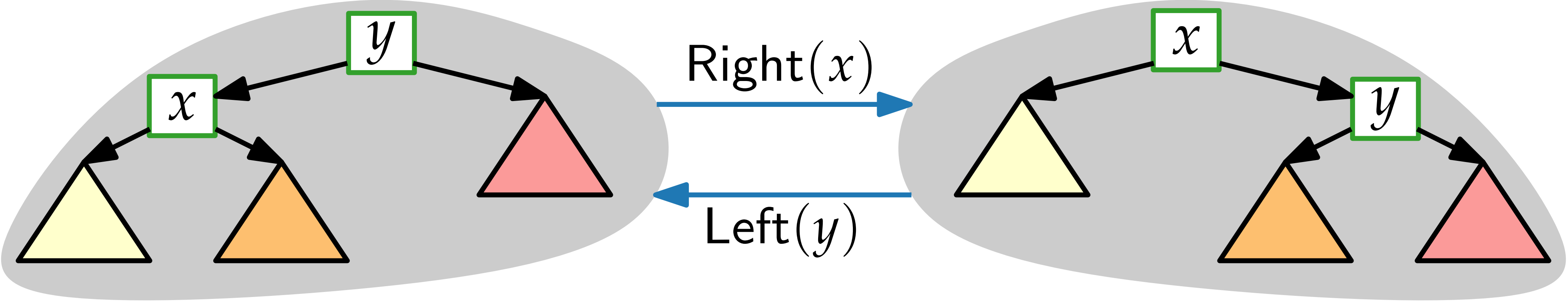

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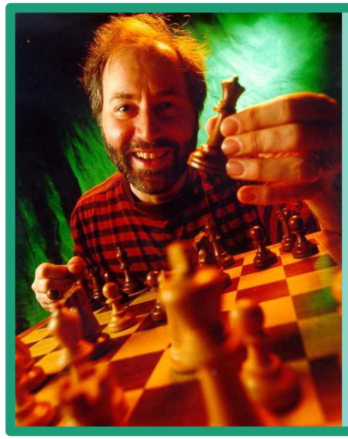

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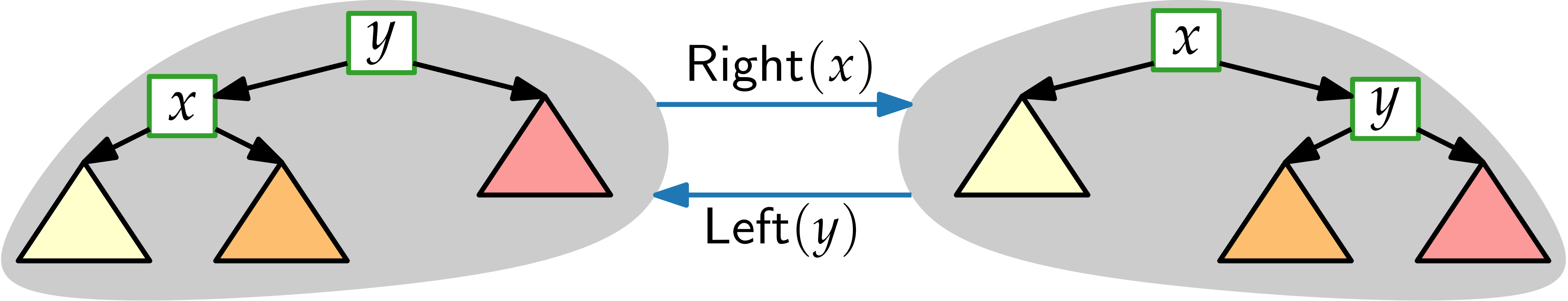

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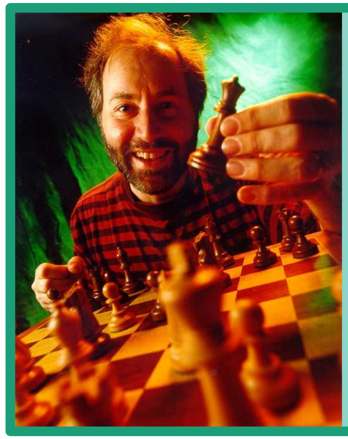

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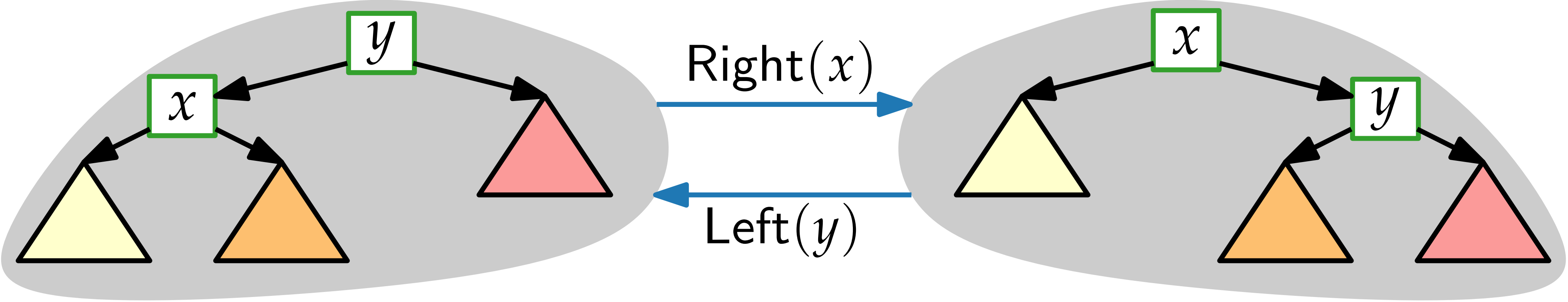

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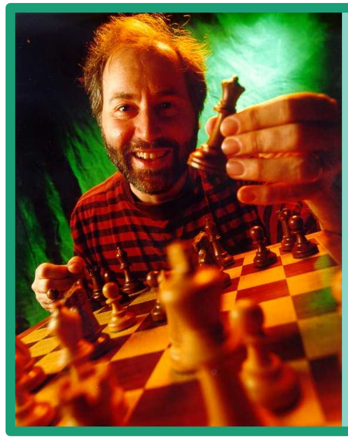

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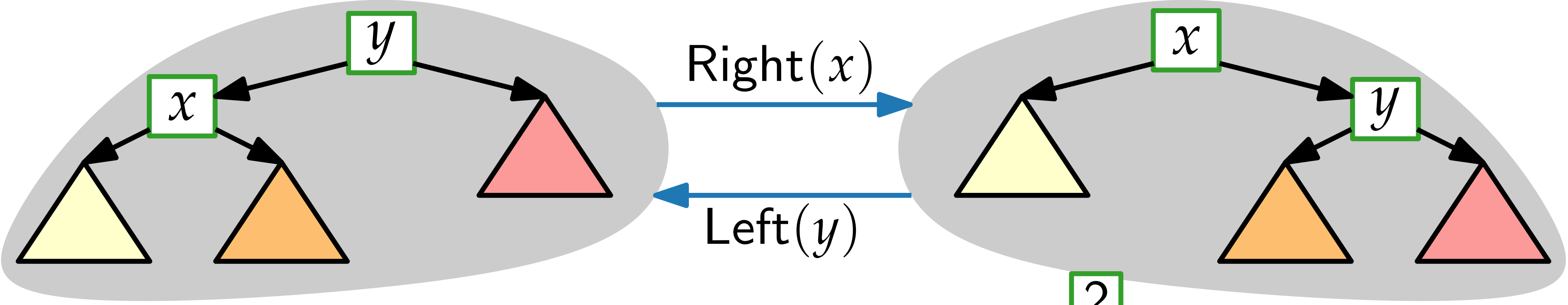

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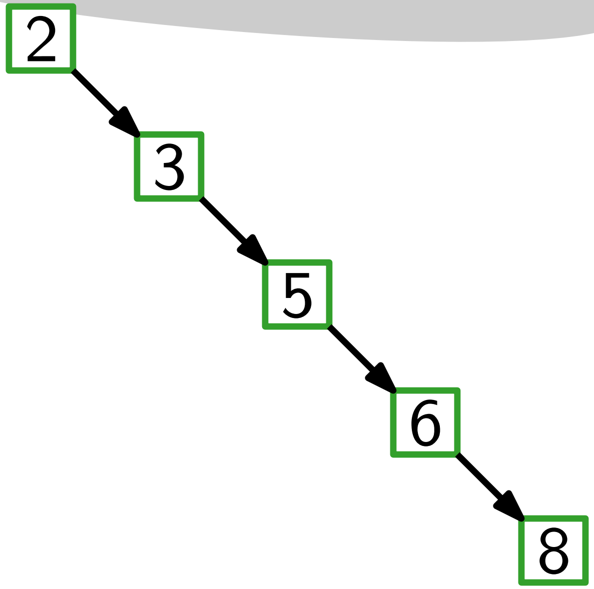

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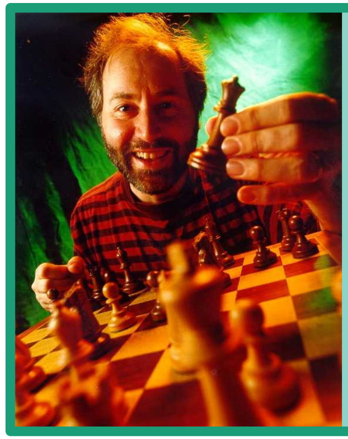

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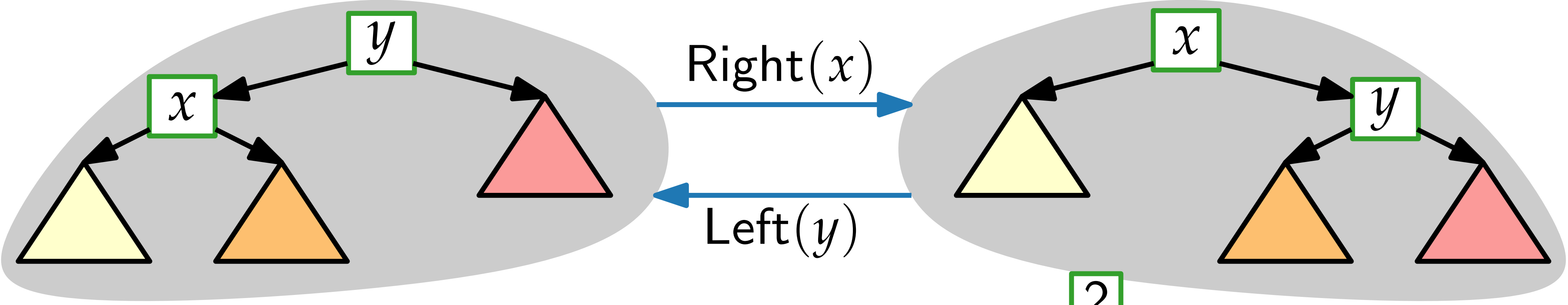

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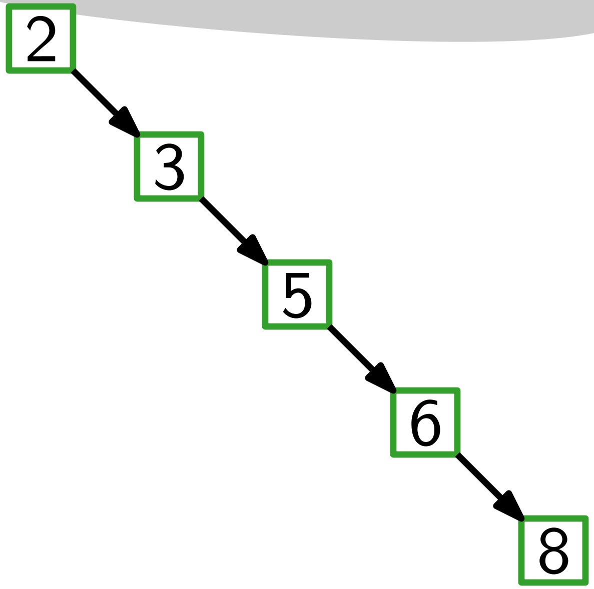

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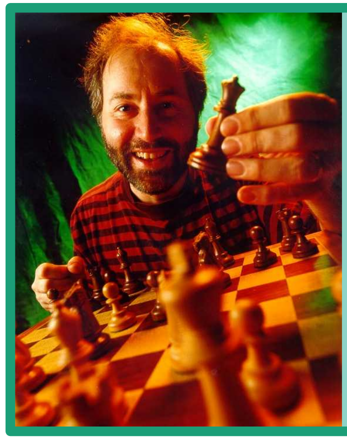

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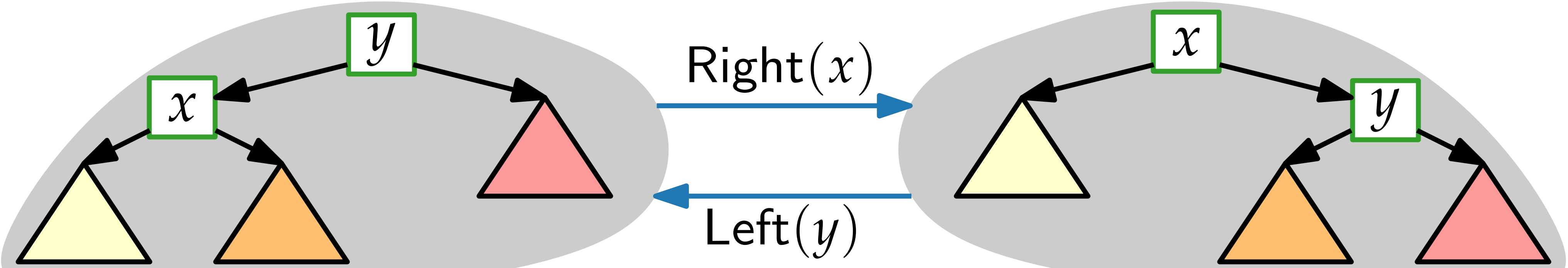

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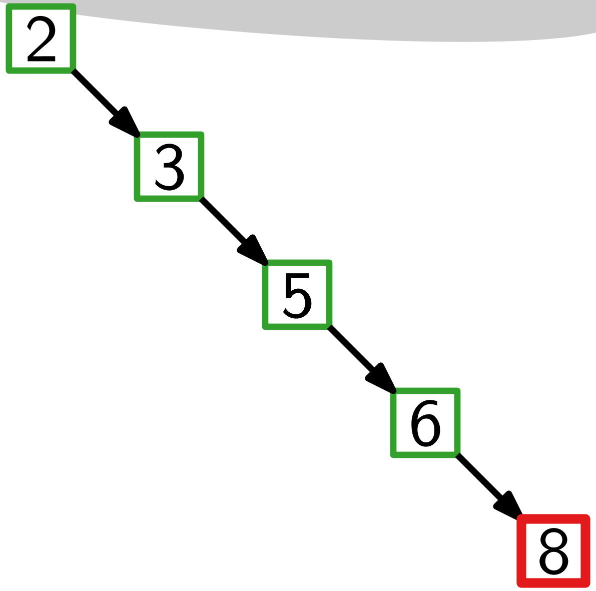

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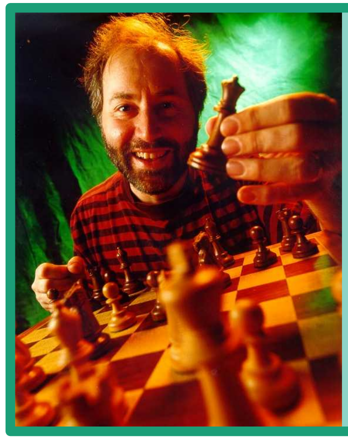

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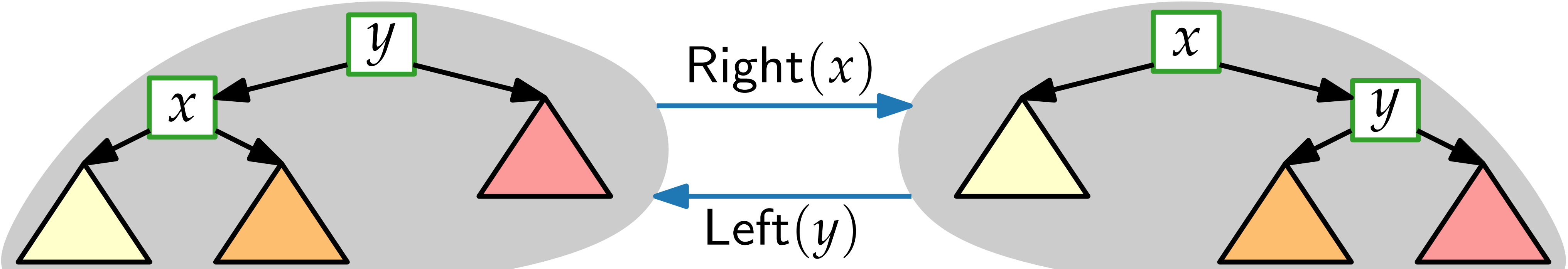

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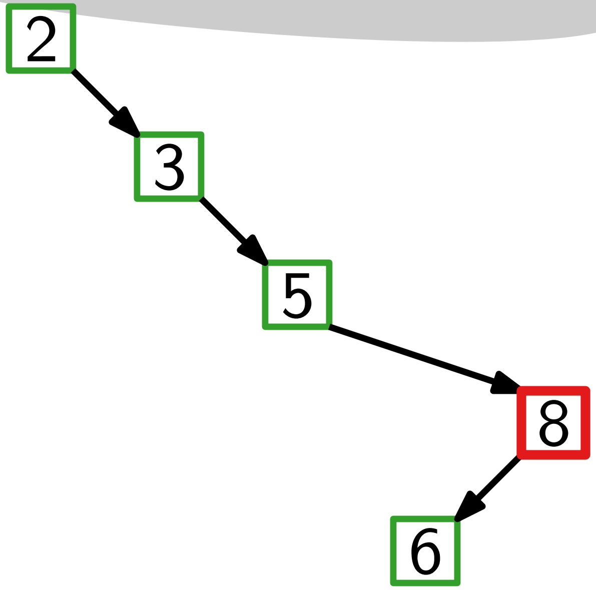

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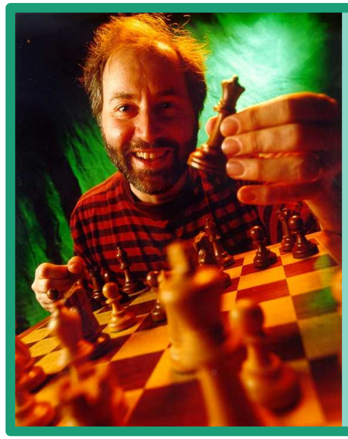

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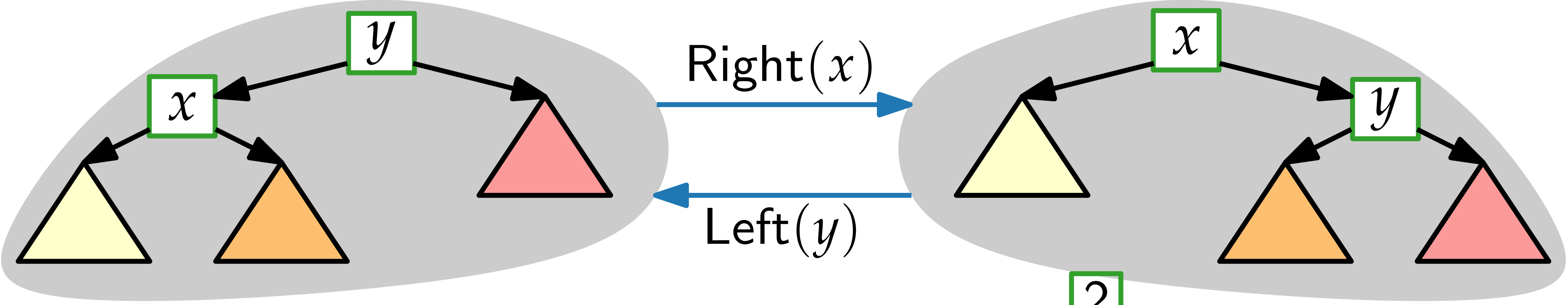

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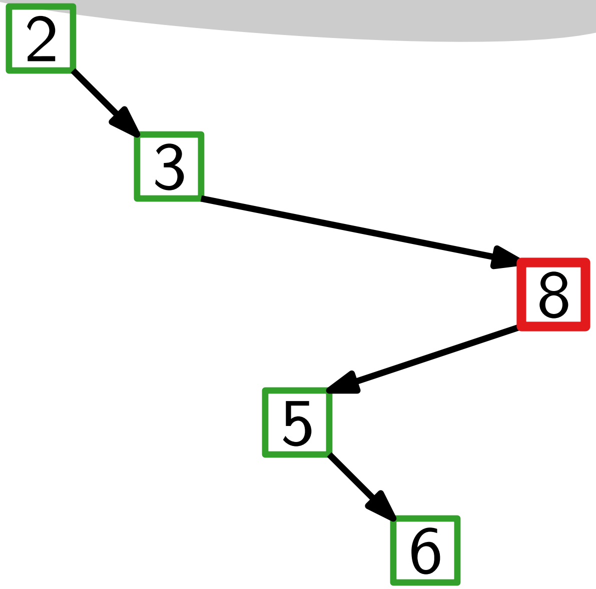

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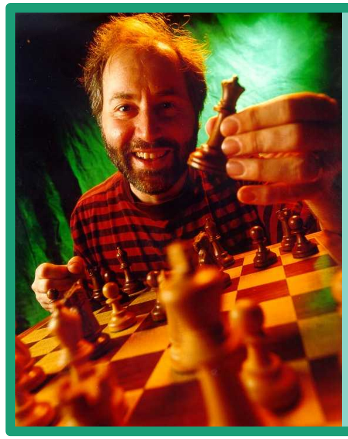

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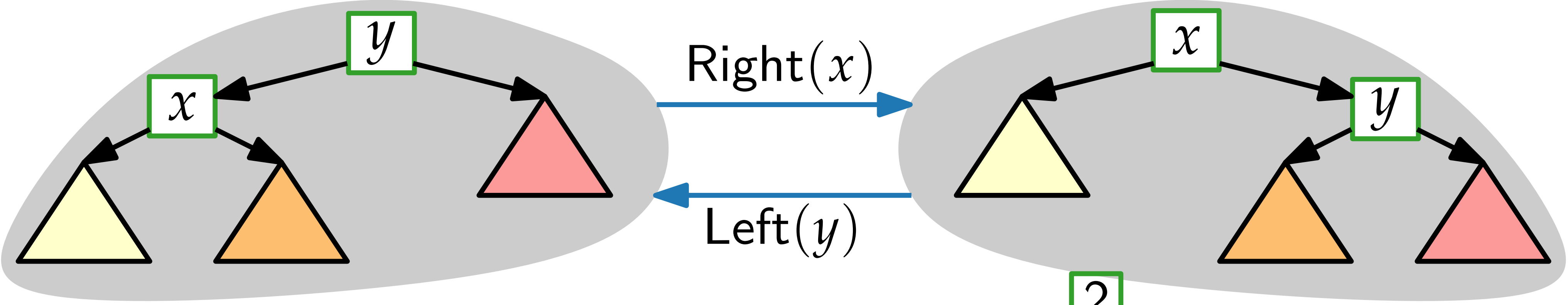

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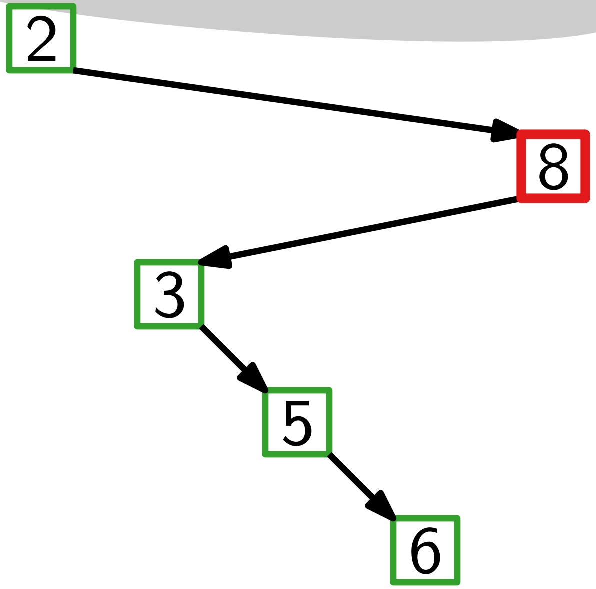

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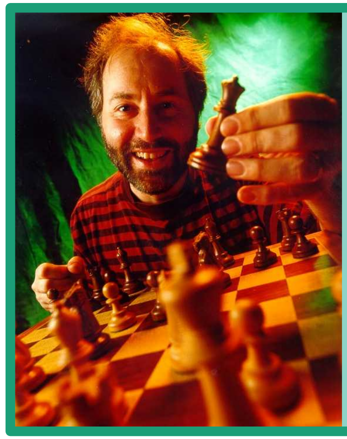

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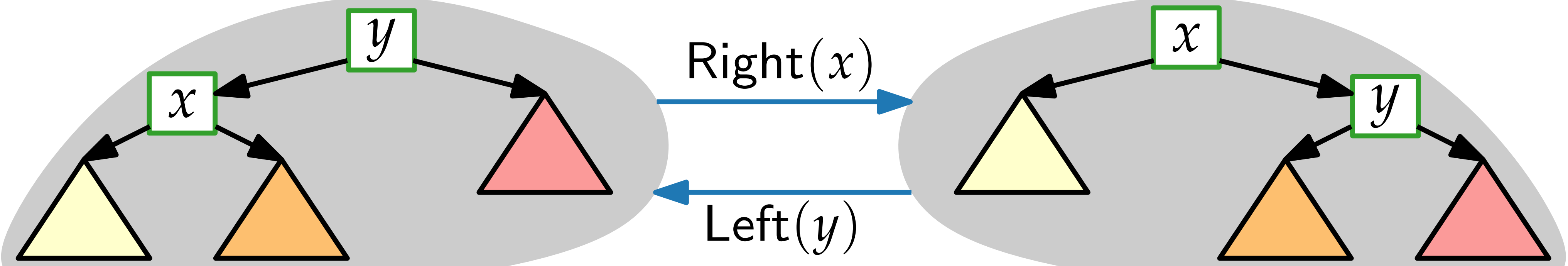

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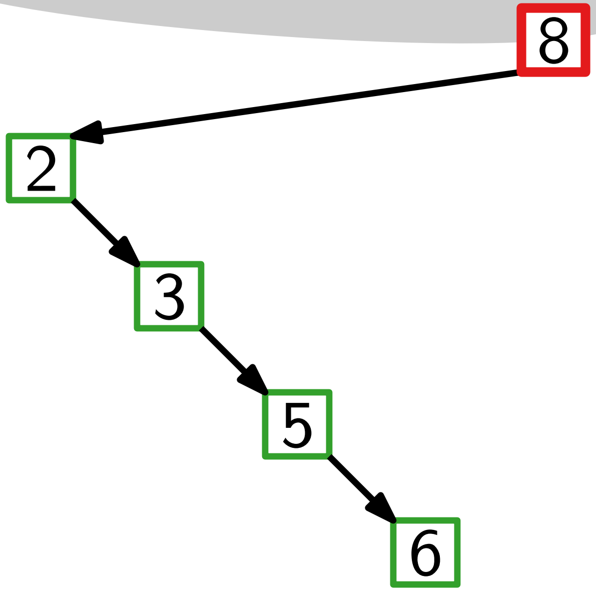

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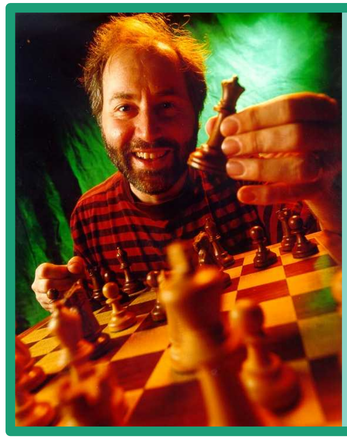

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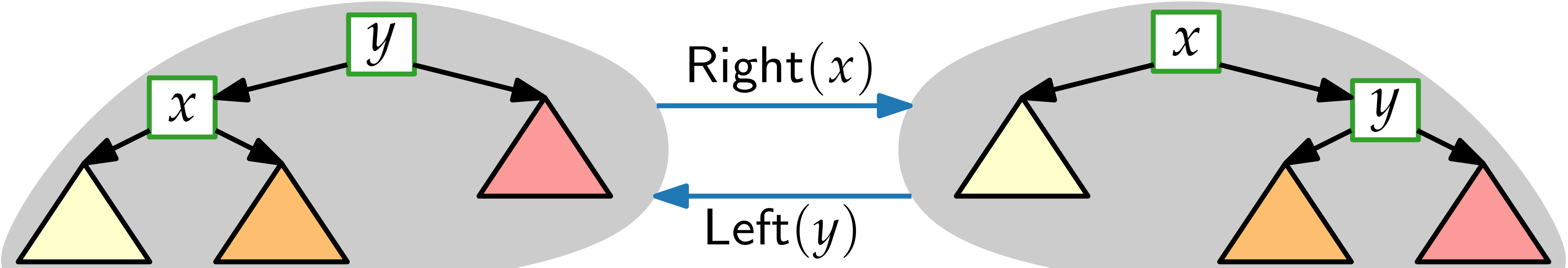

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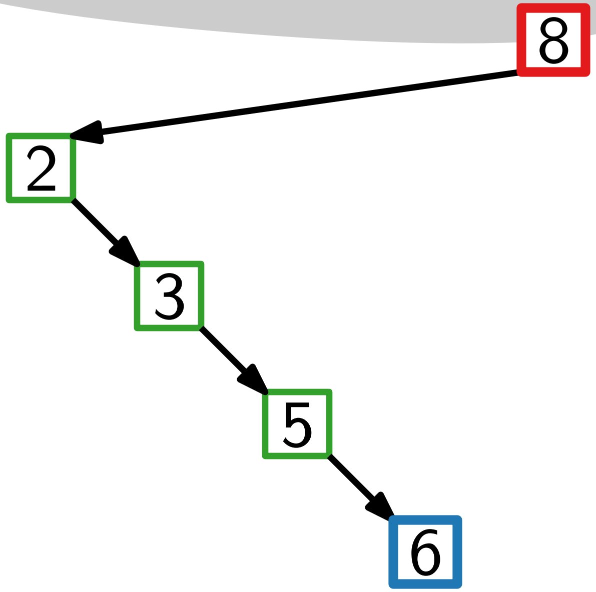

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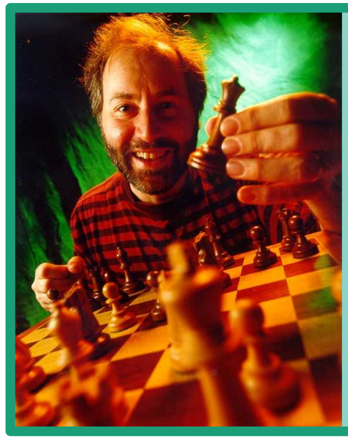

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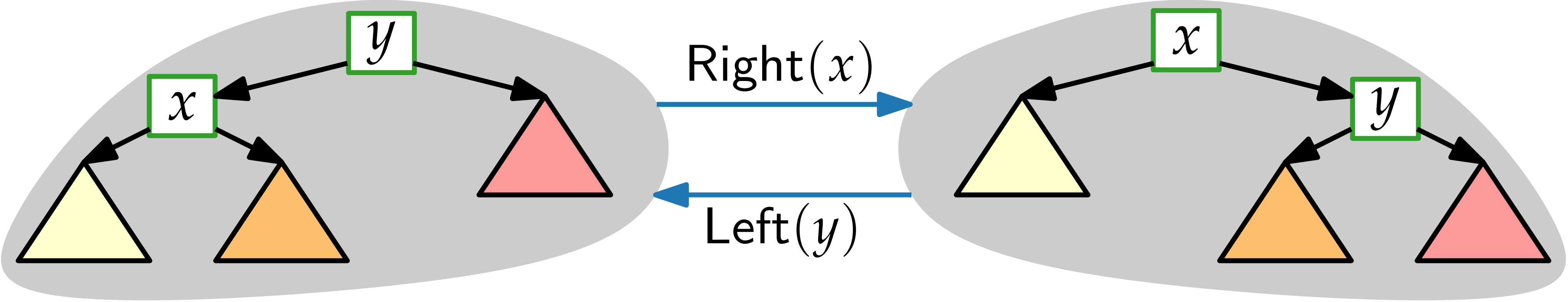

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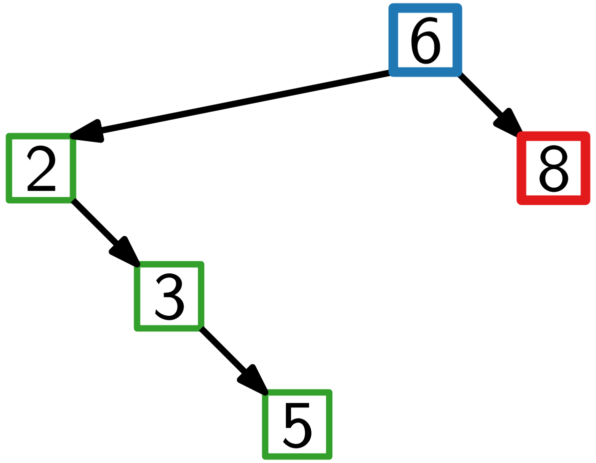

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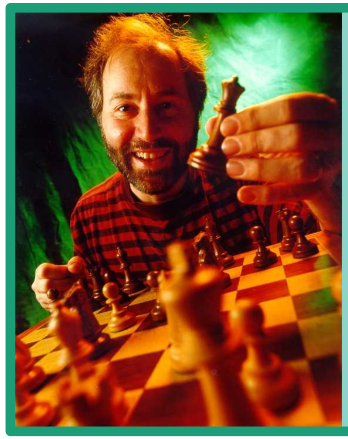

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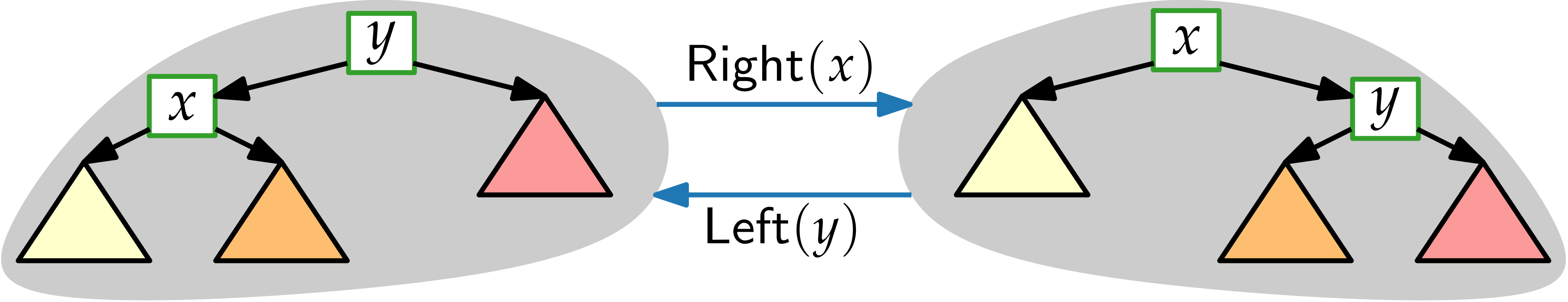

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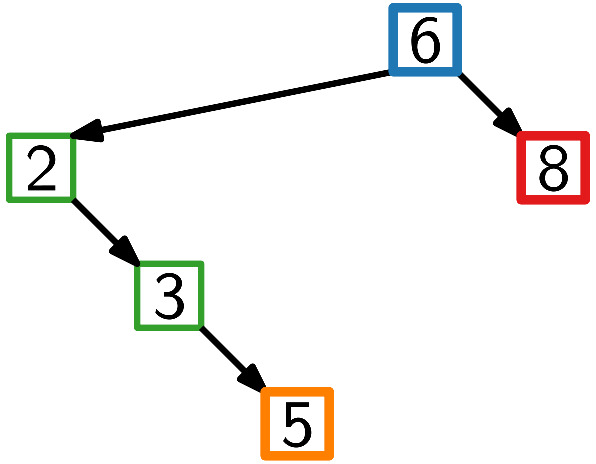

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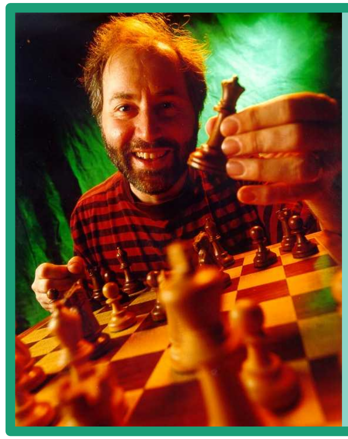

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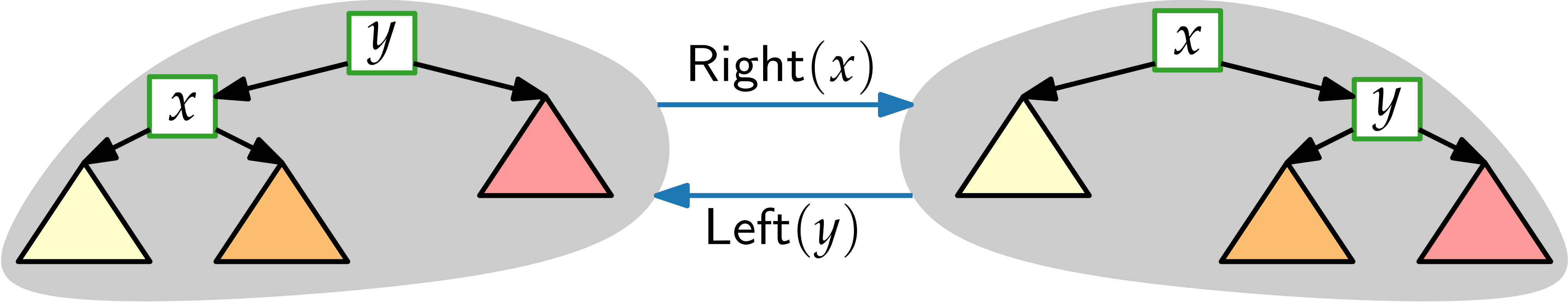

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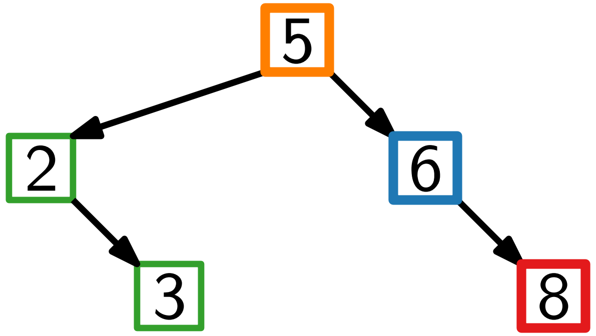

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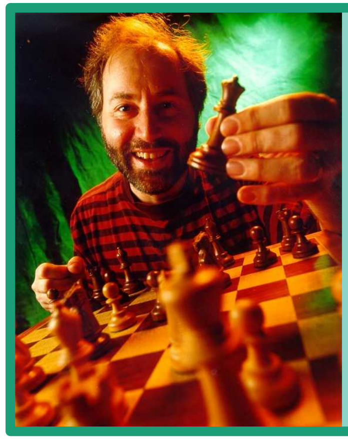

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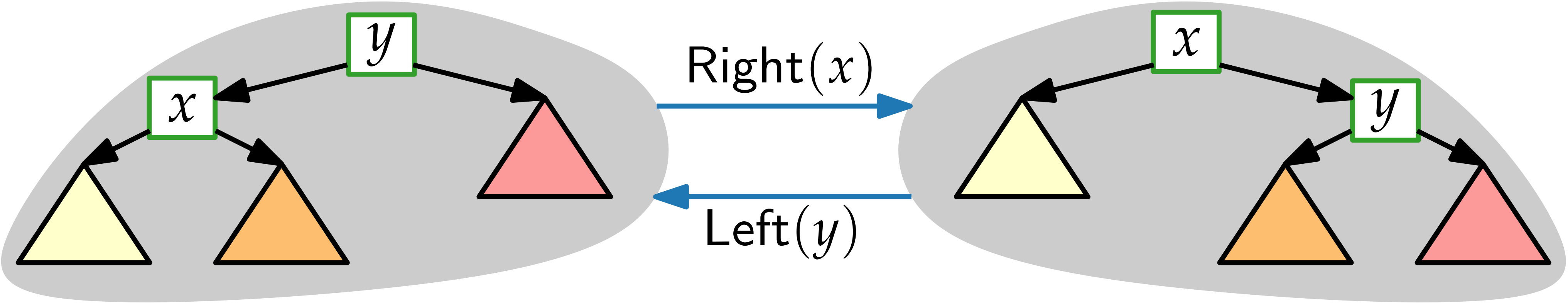

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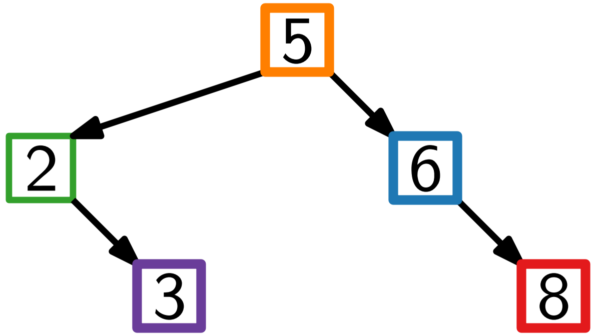

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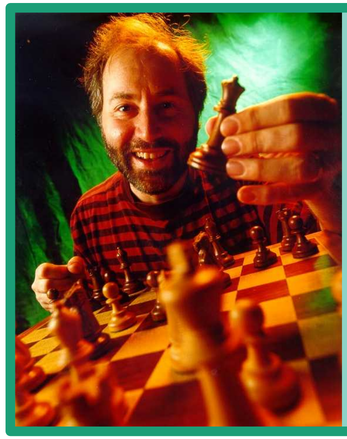

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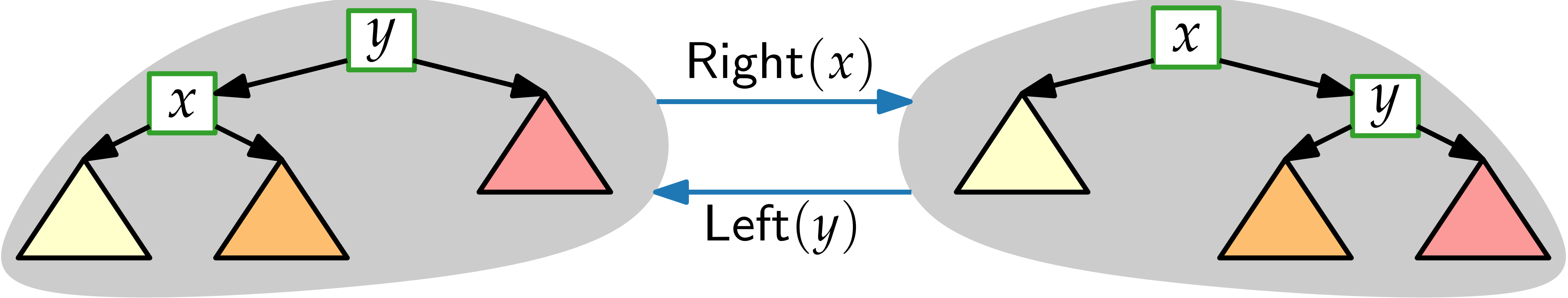

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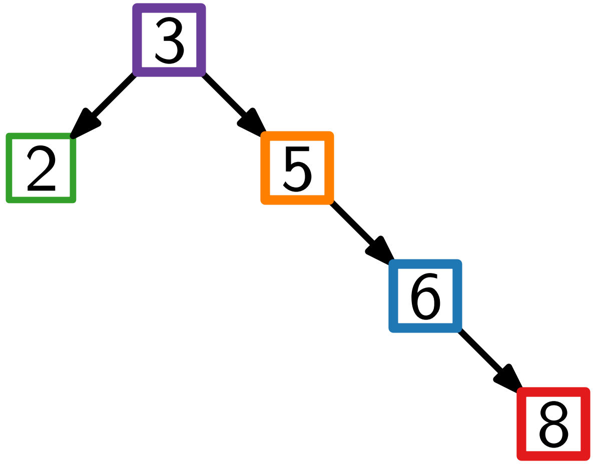

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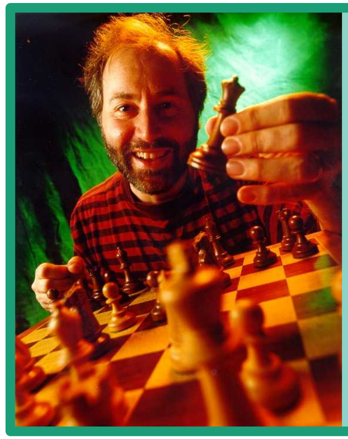

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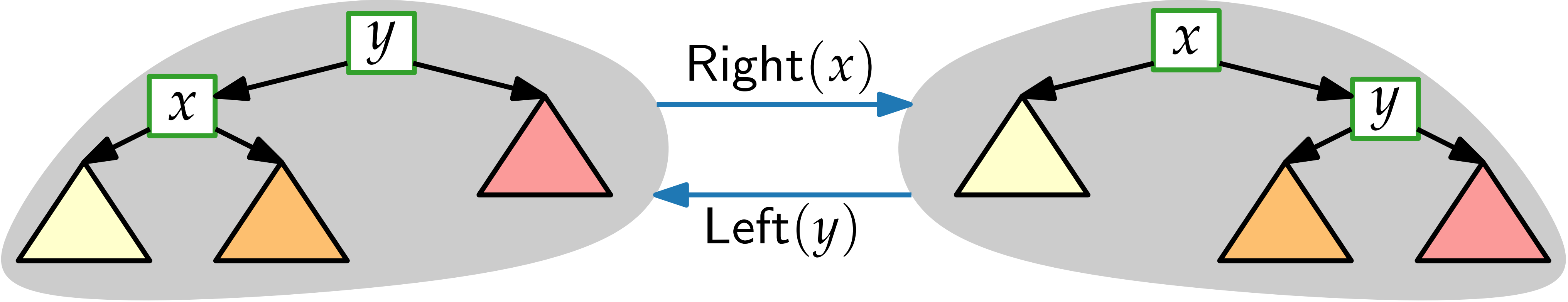

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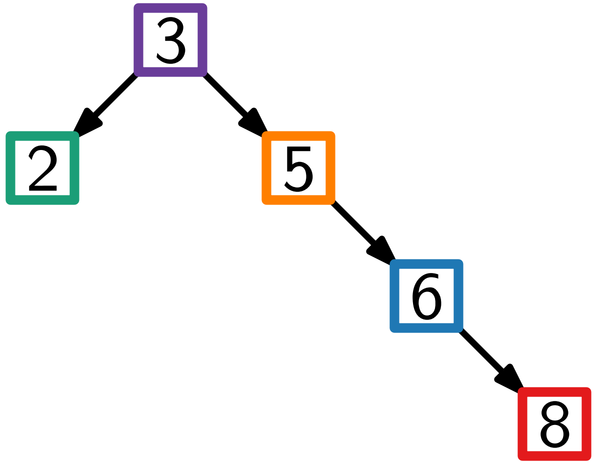

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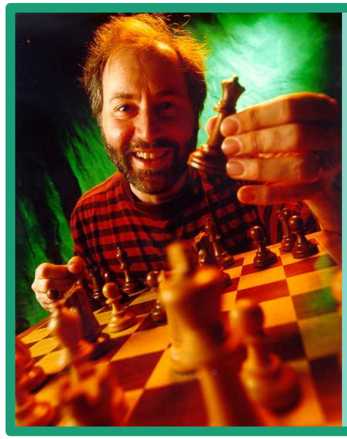

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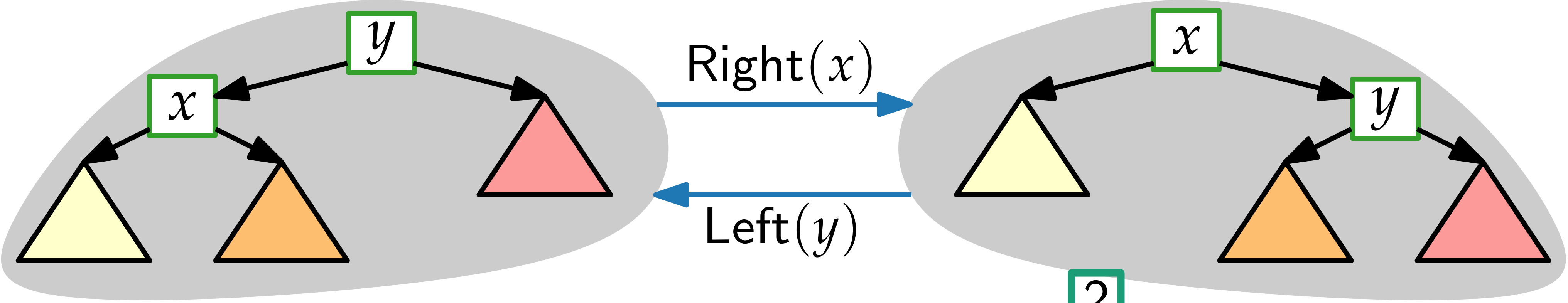

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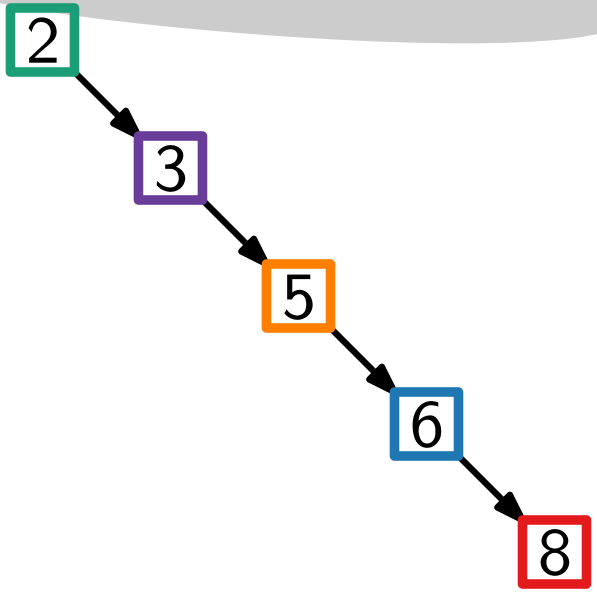

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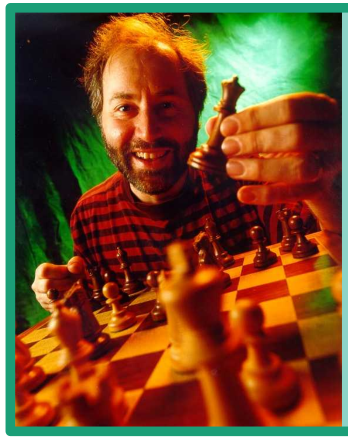

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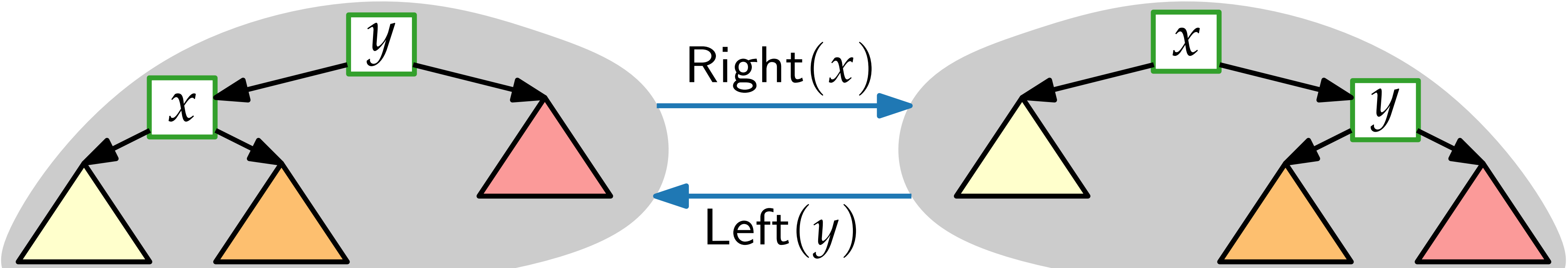

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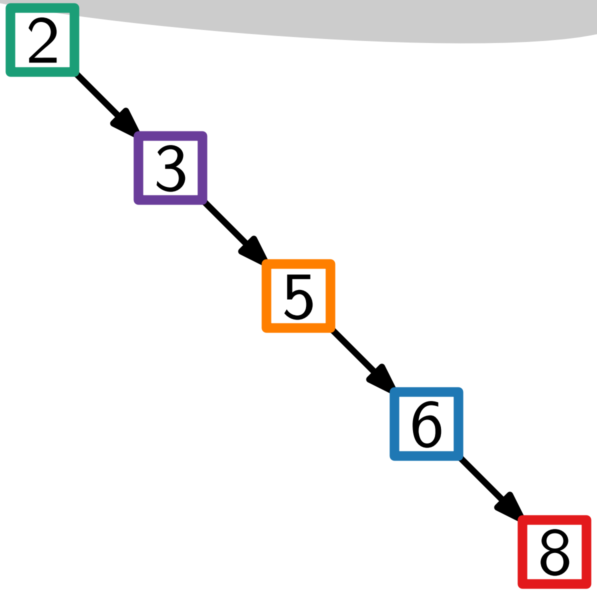
We're back at the start...



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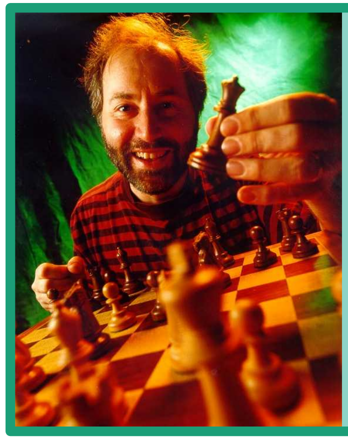

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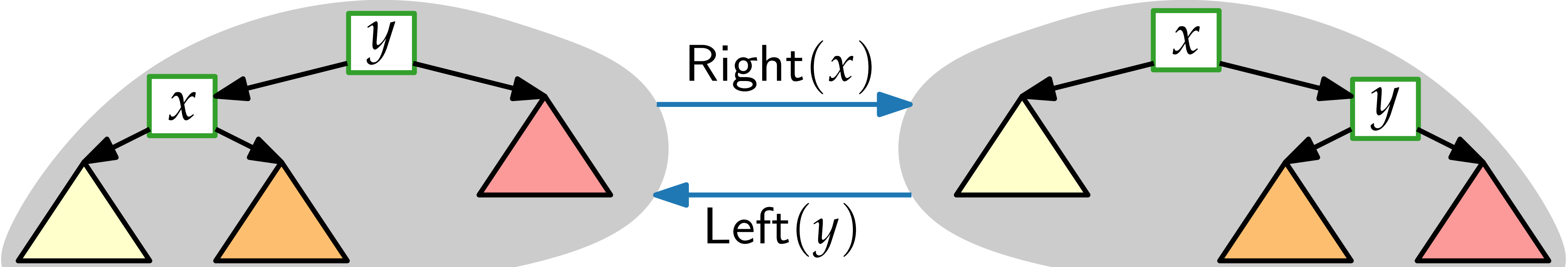
Idea: Whenever we query a key, rotate it to the root.

Known from the lecture *algorithms and data structures* (ADS):



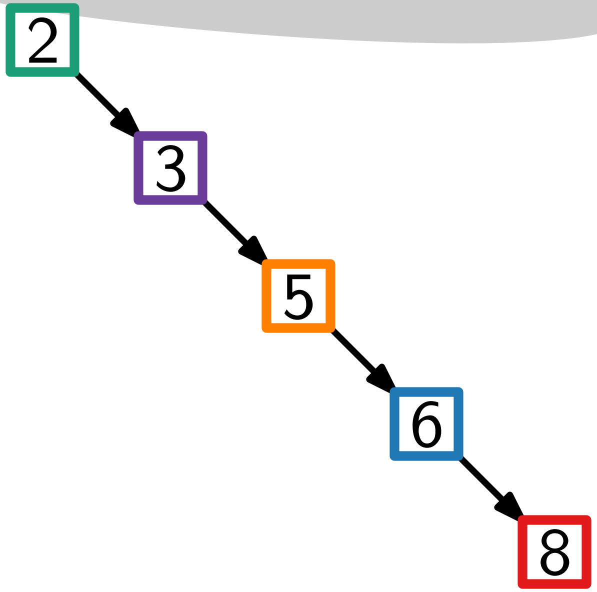
New:

Splay(x): Rotate x to the root

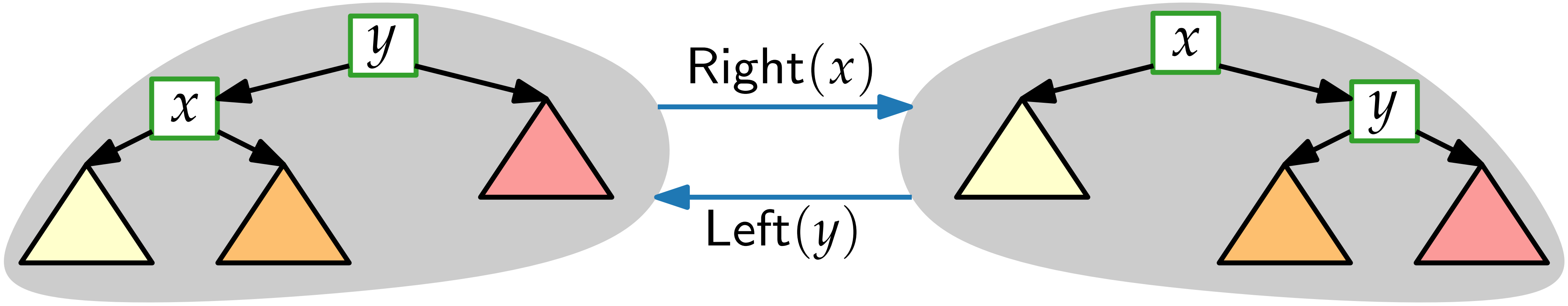
Query(x): Splay(x), then return root

Query(8) Query(6) Query(5)
Query(3)
Query(2)

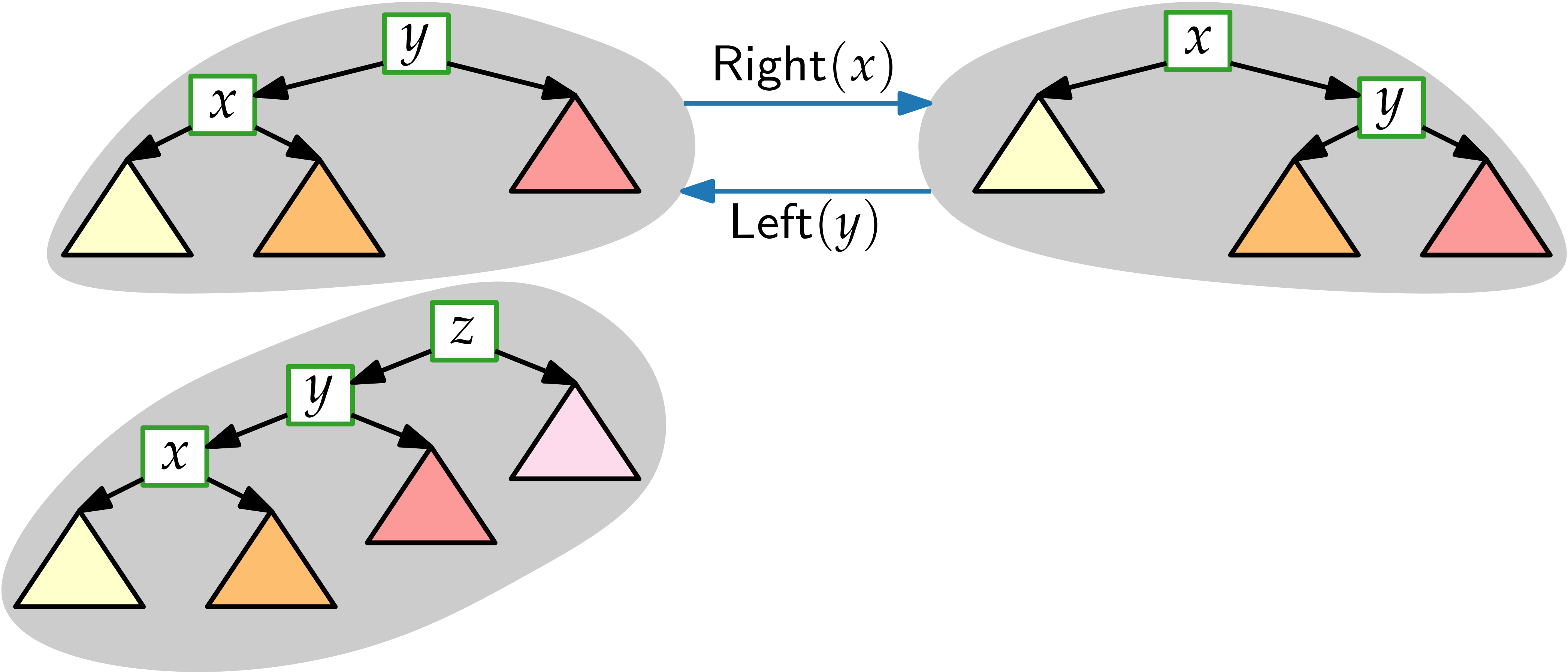
We're back at the start...
and we did $\Theta(n^2)$ rotations



Rotations II



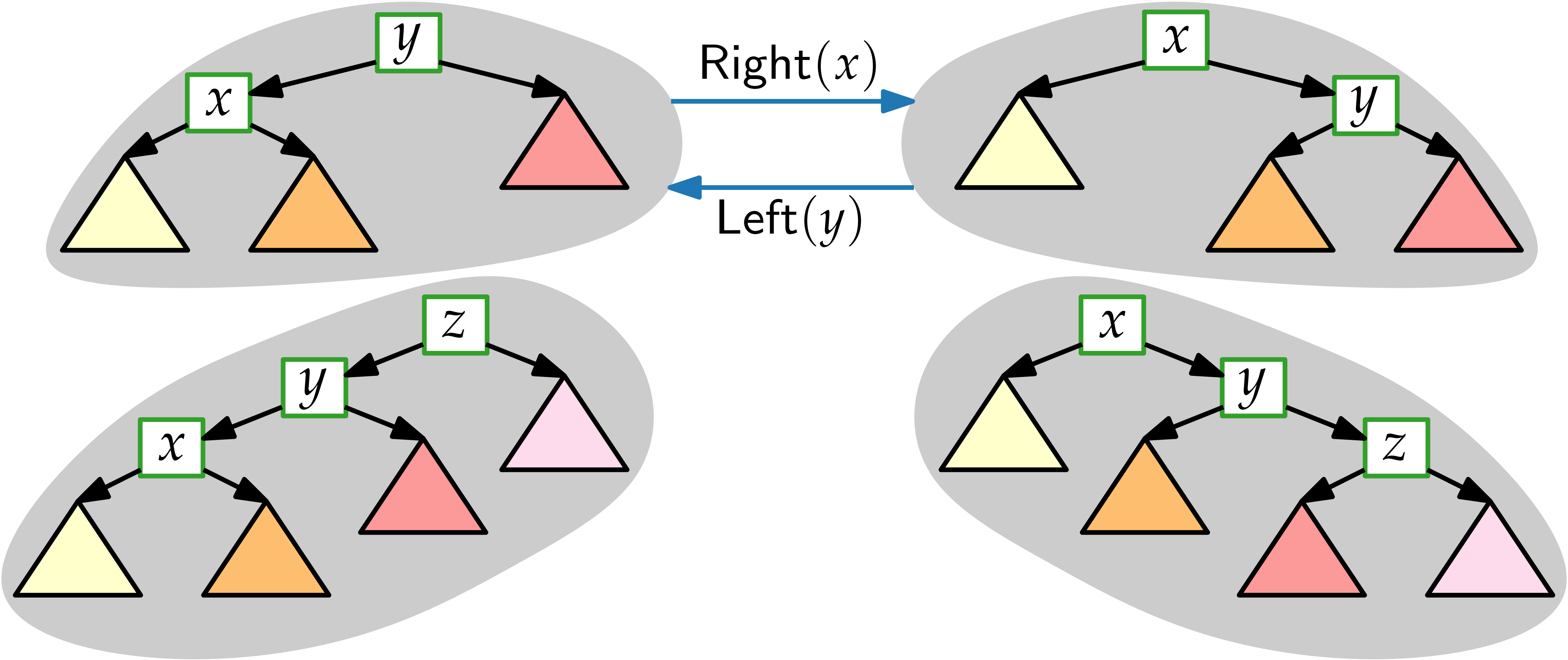
Rotations II



Right(x)

Left(y)

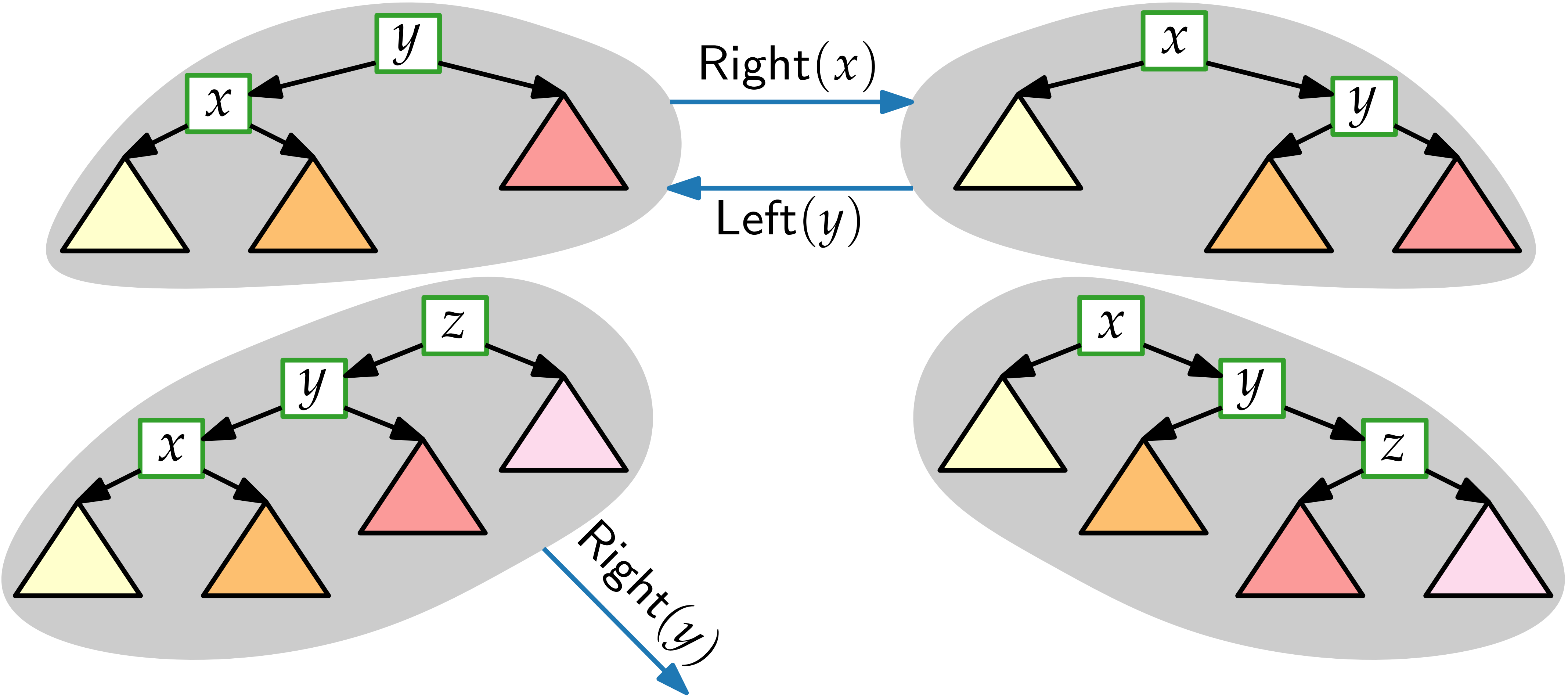
Rotations II



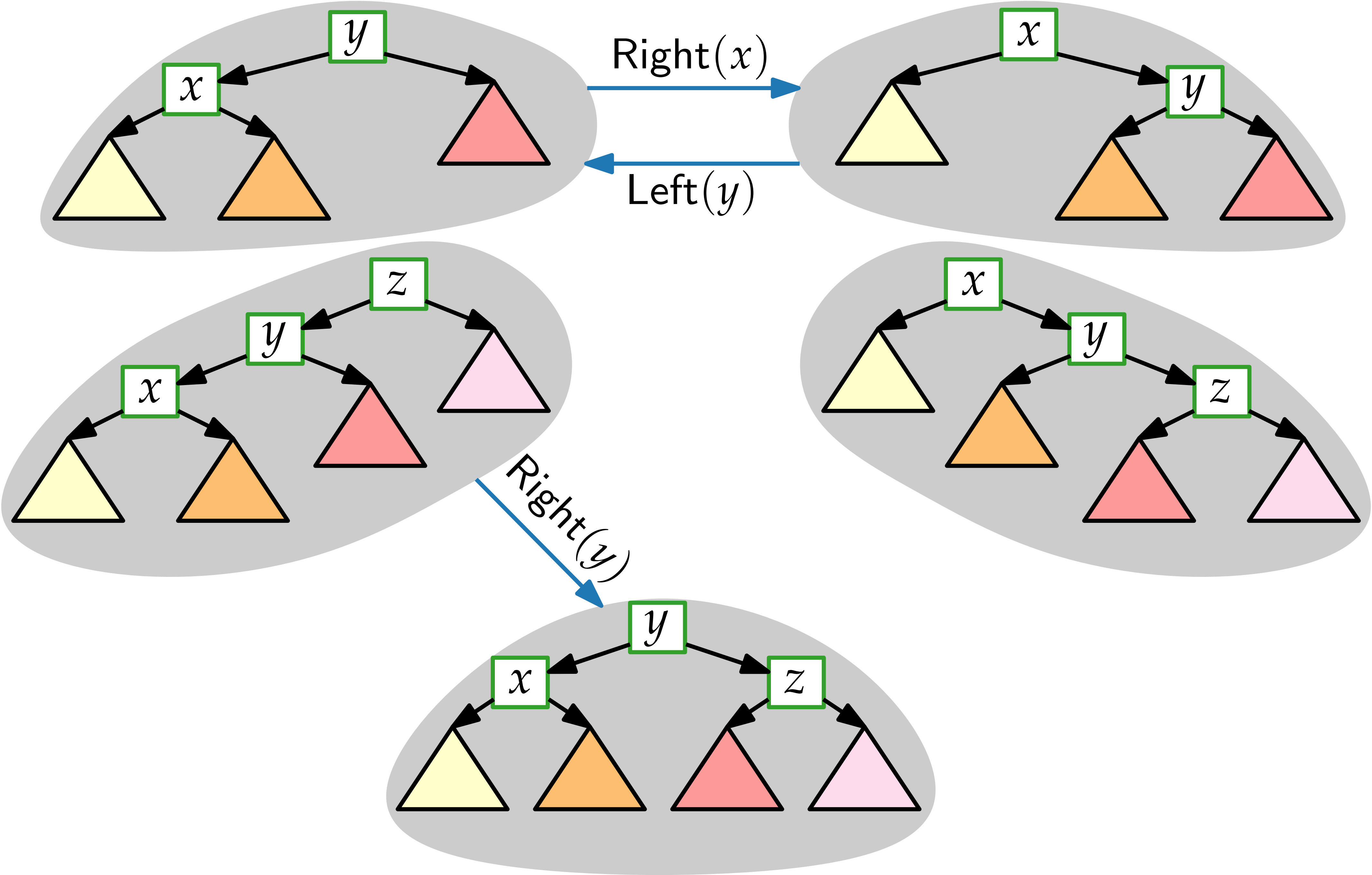
Right(x)

Left(y)

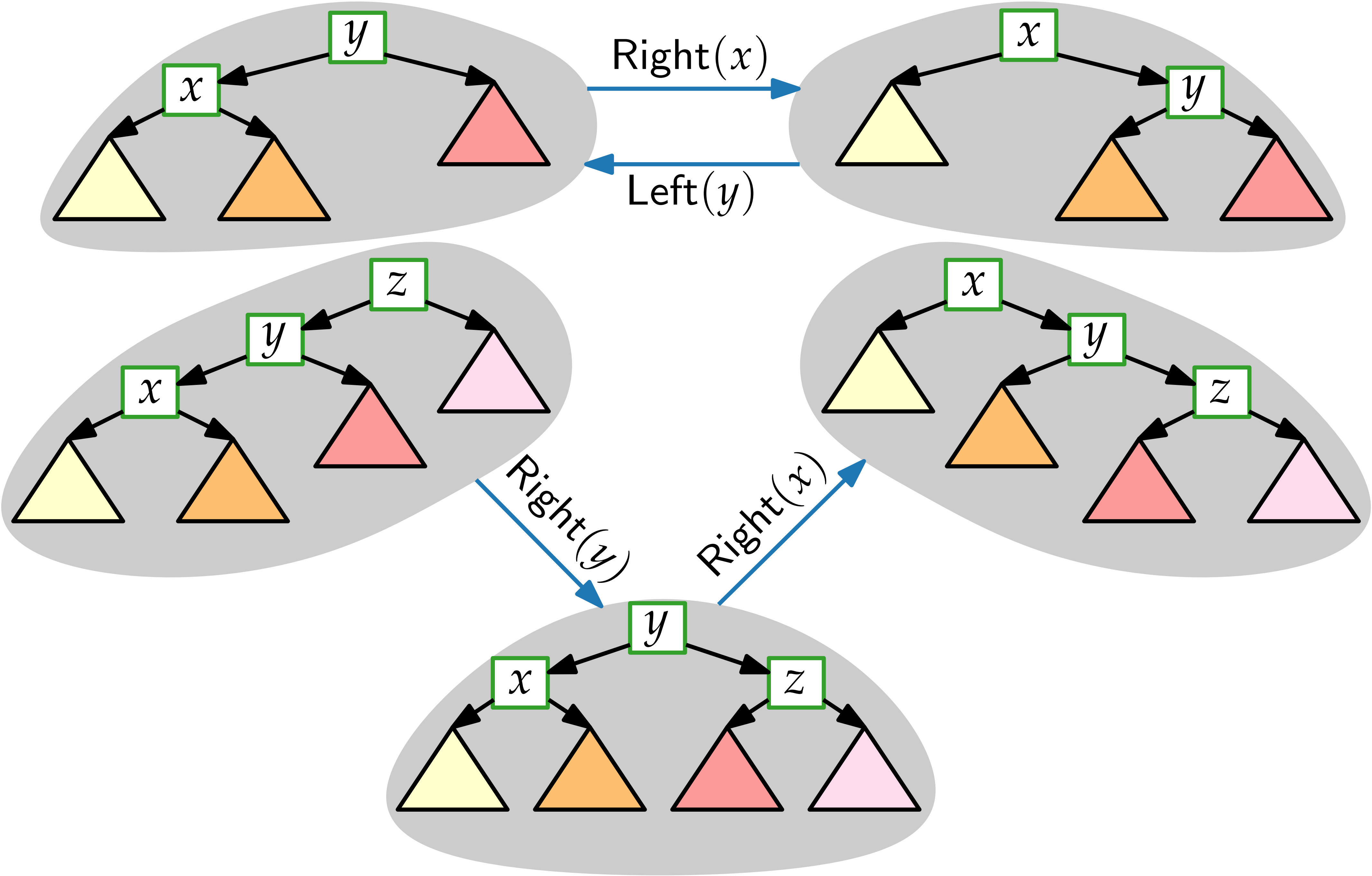
Rotations II



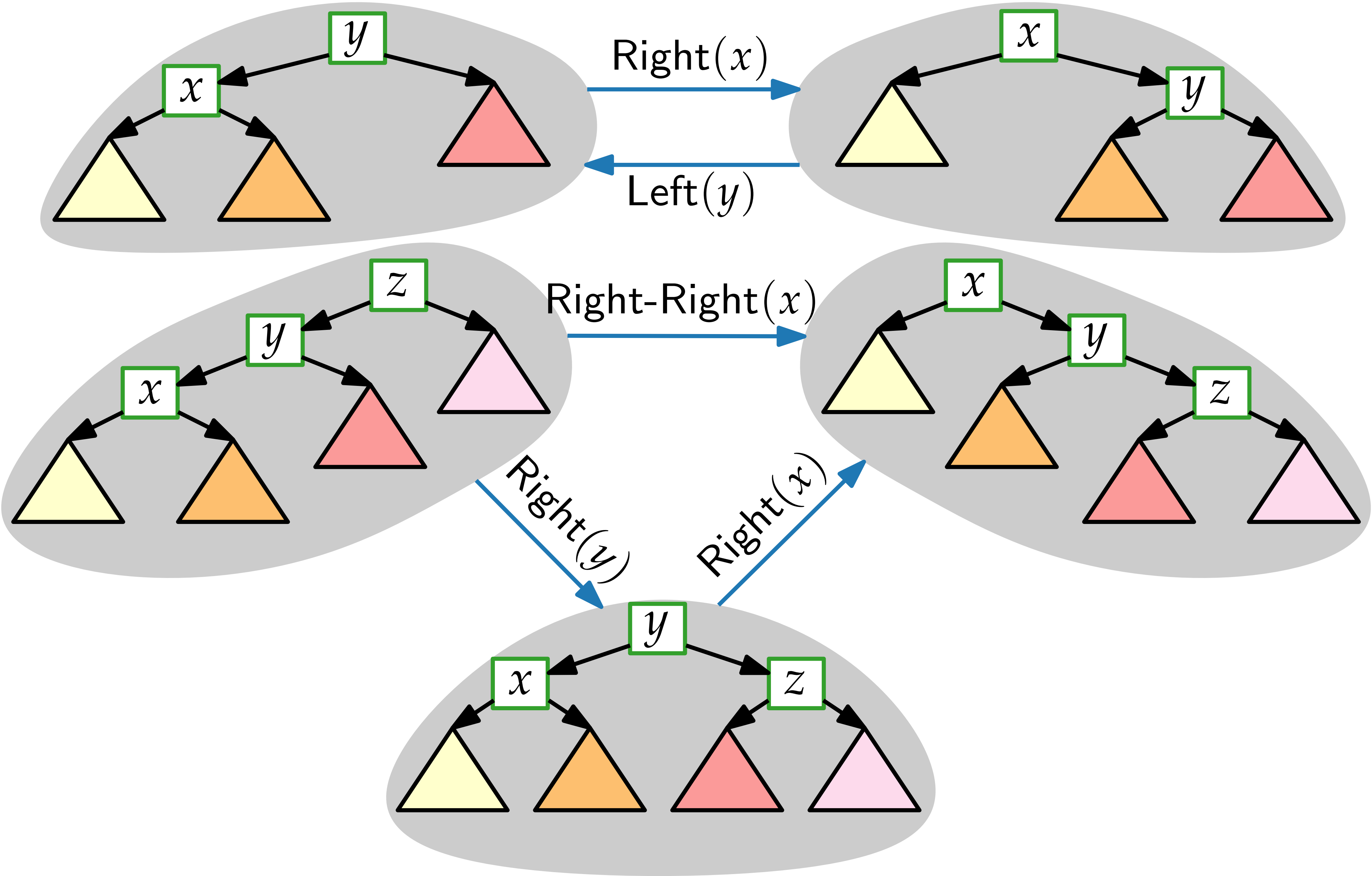
Rotations II



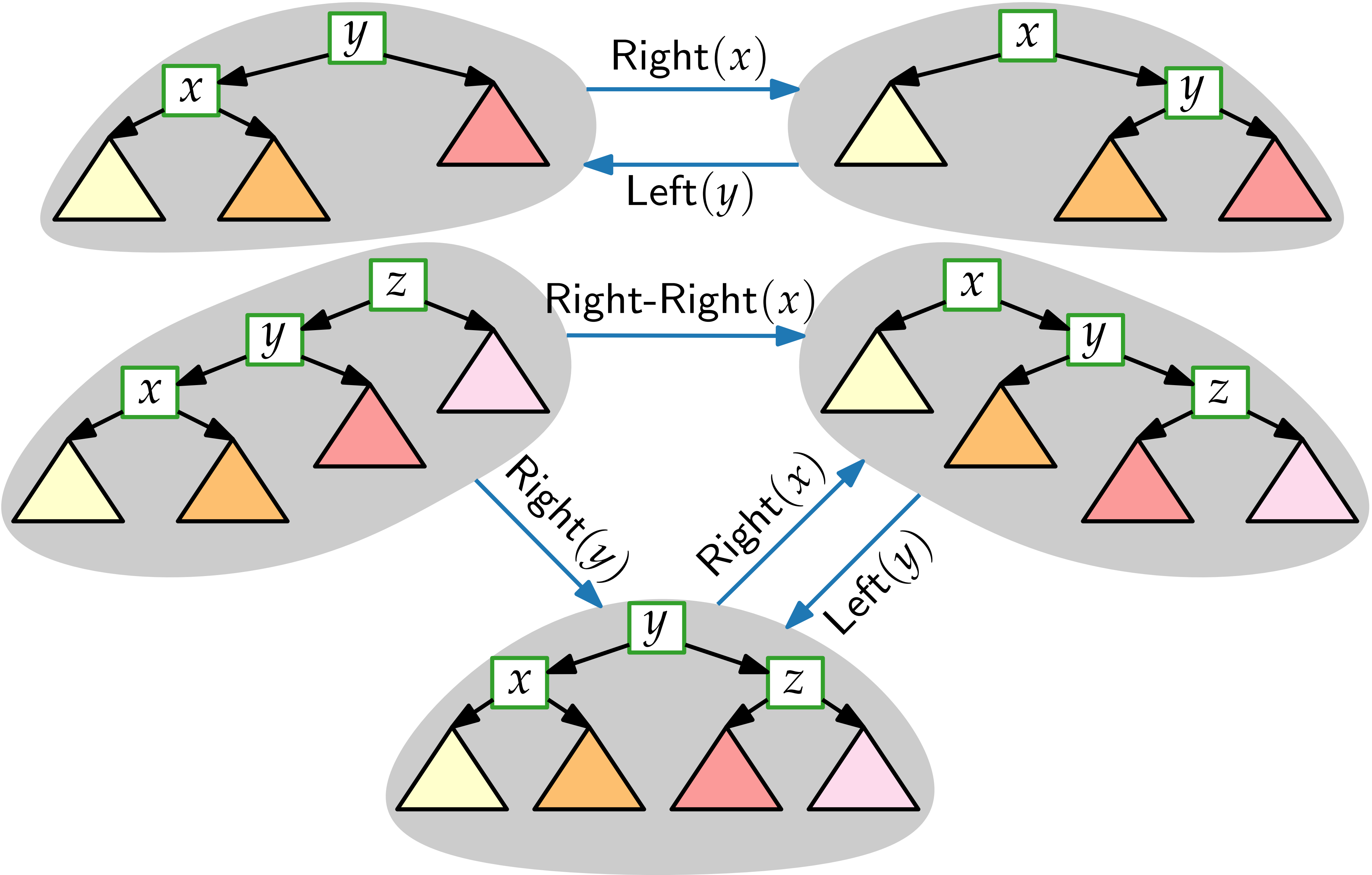
Rotations II



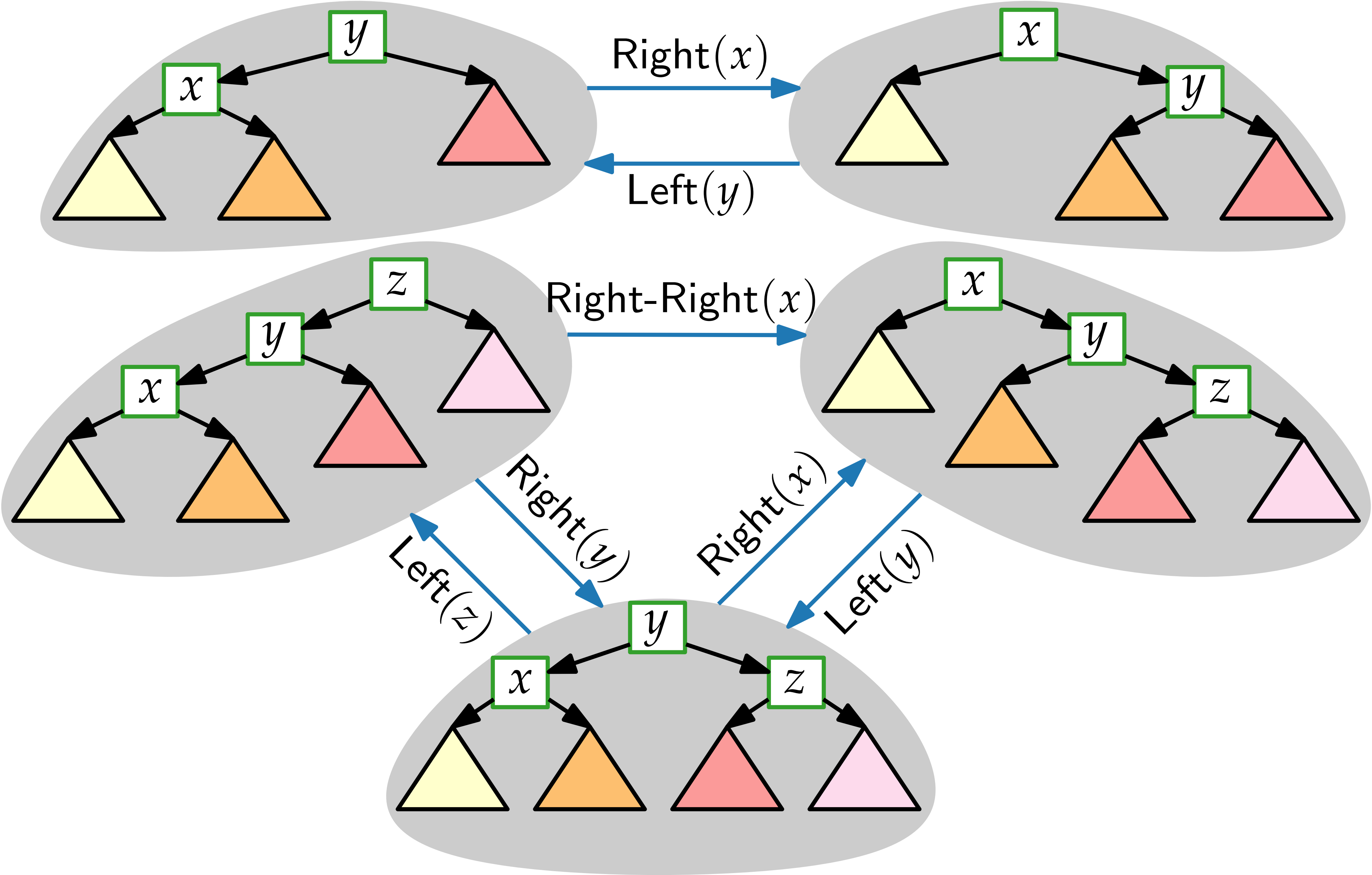
Rotations II



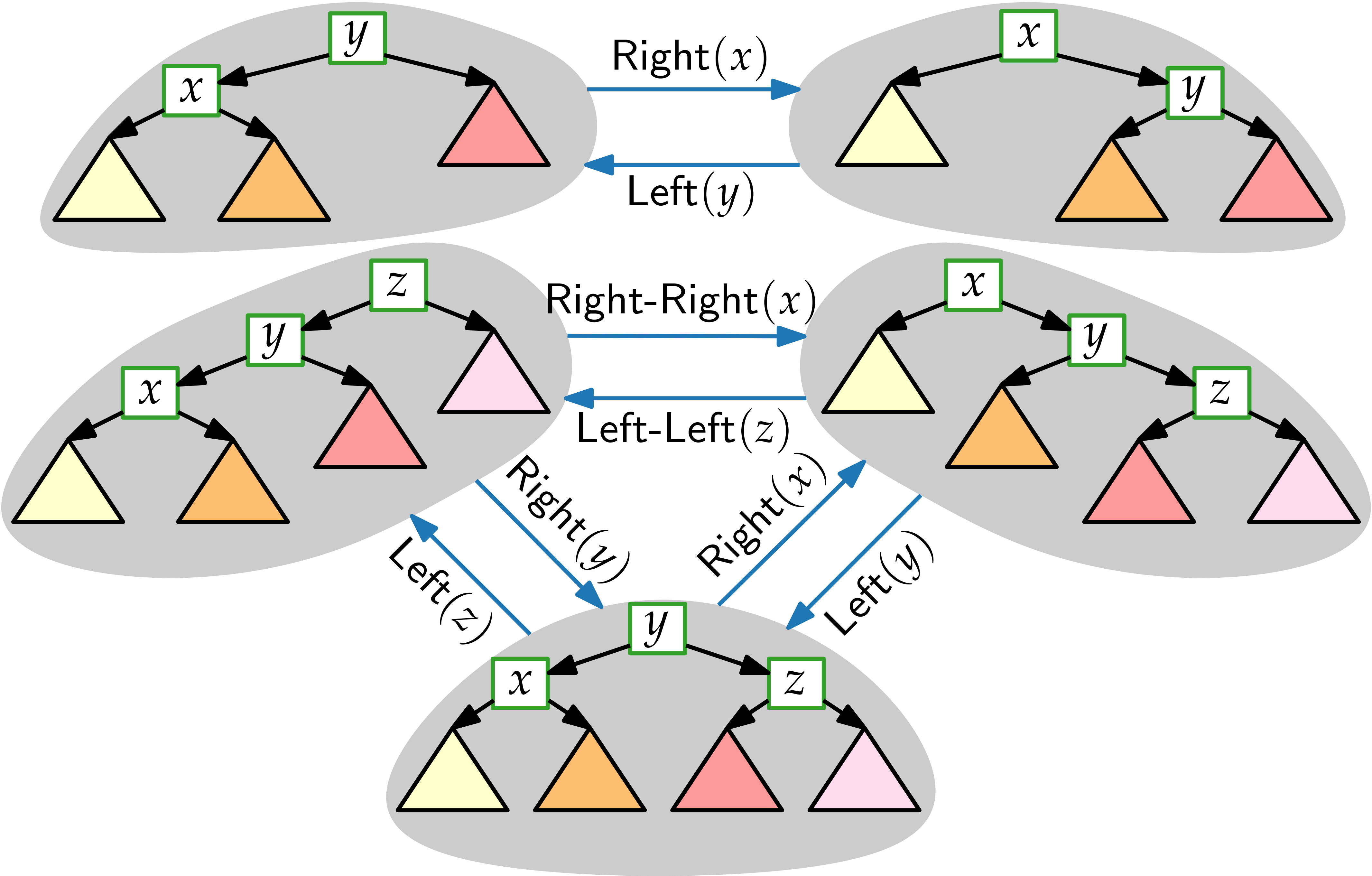
Rotations II



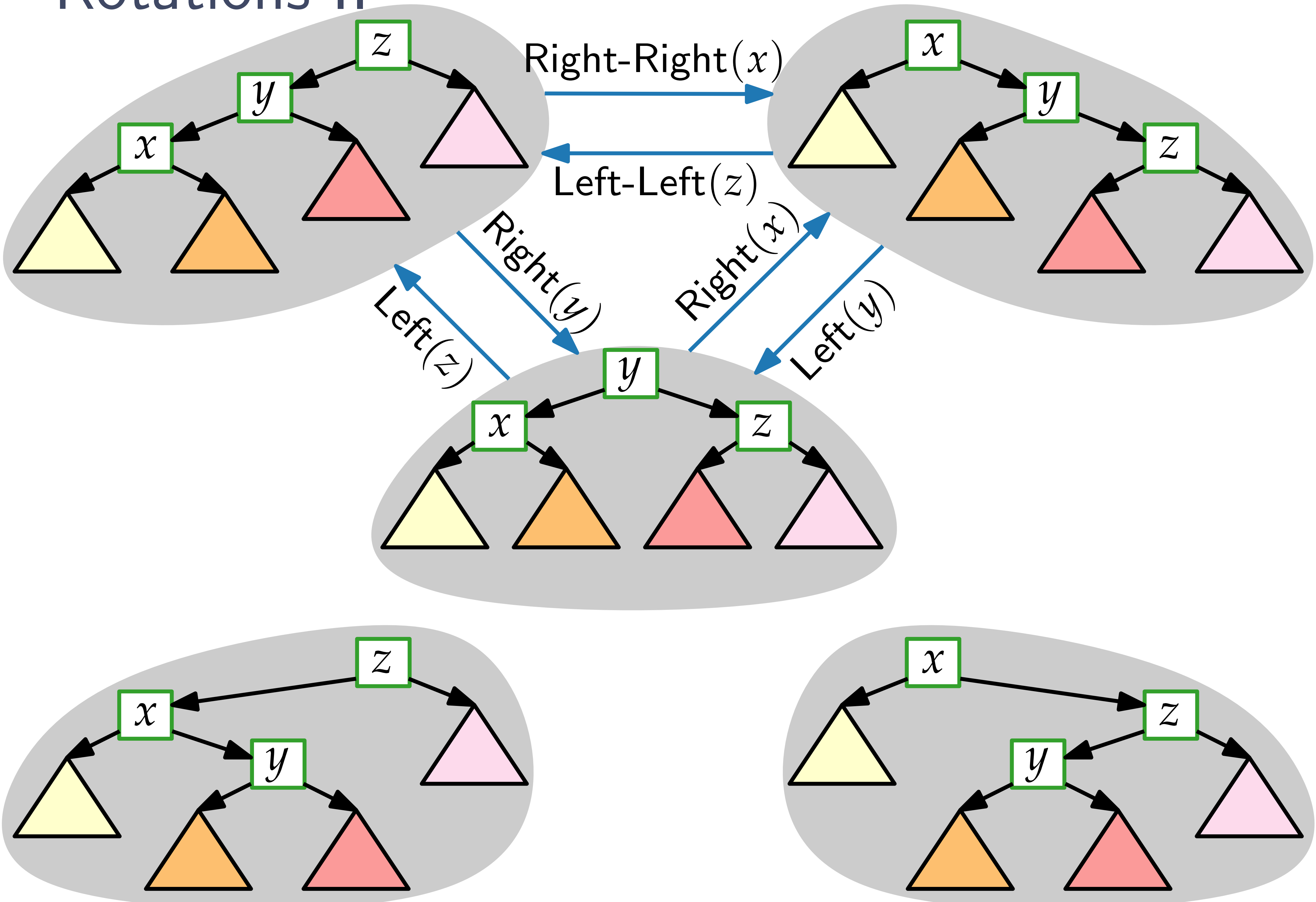
Rotations II



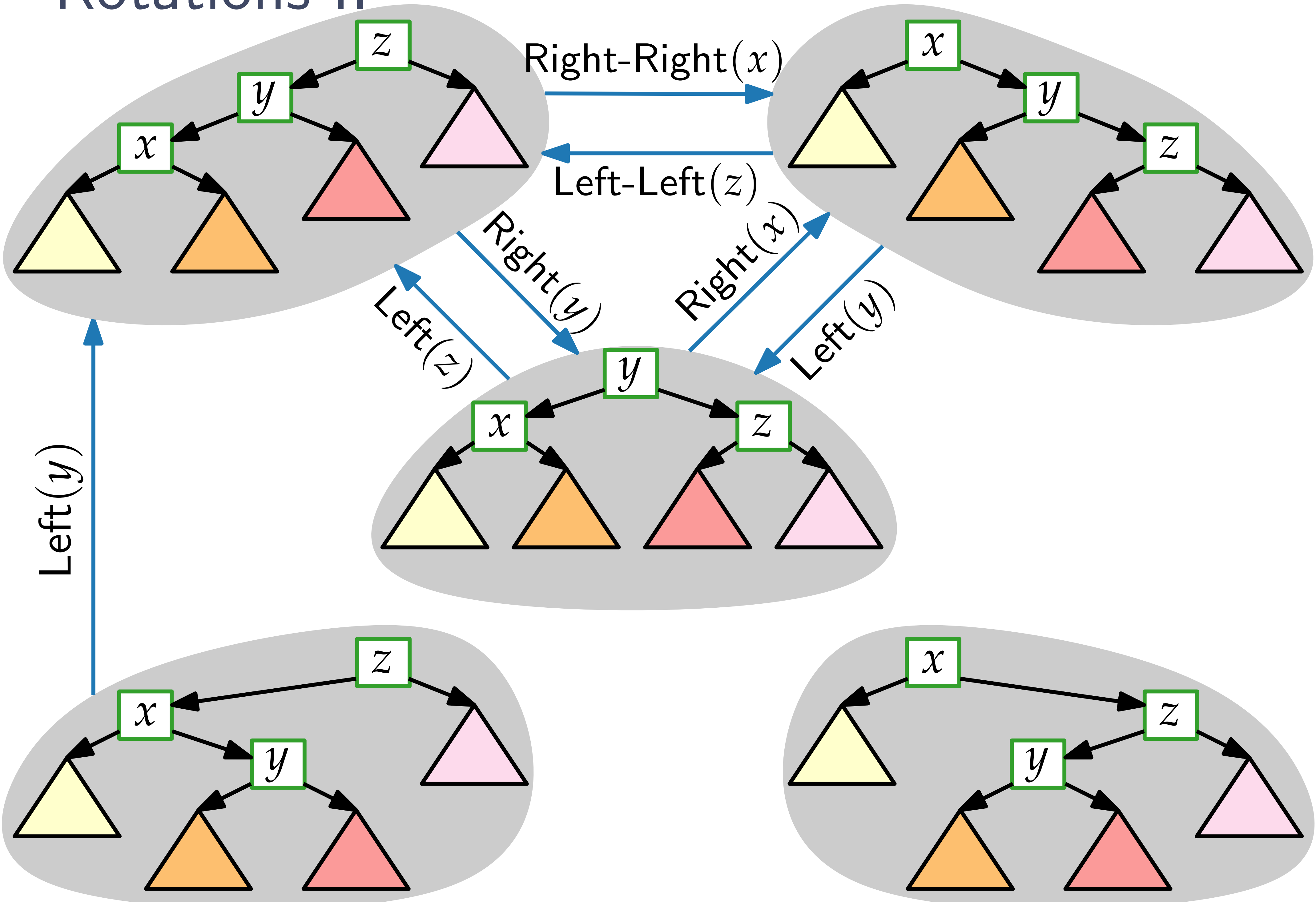
Rotations II



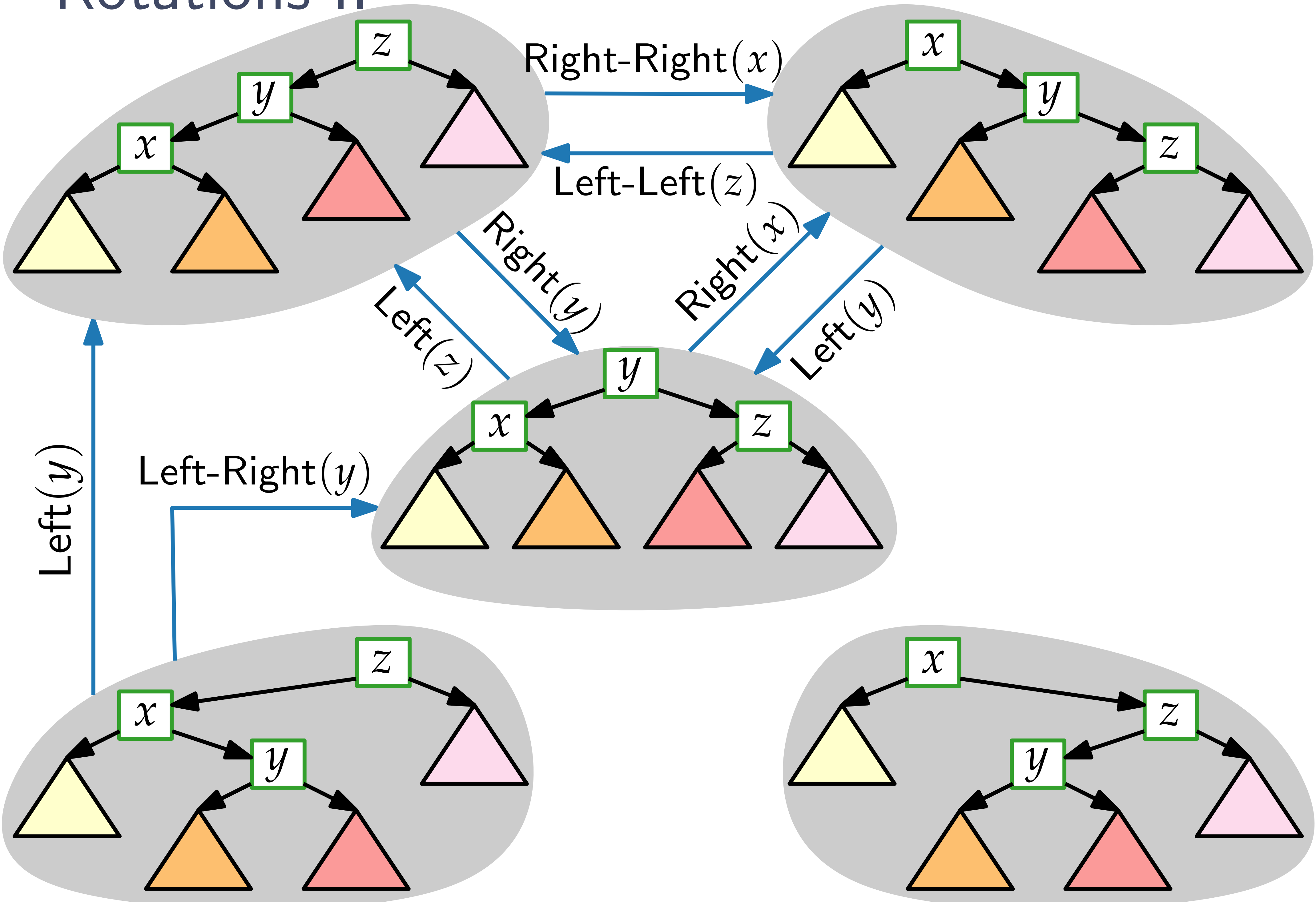
Rotations II



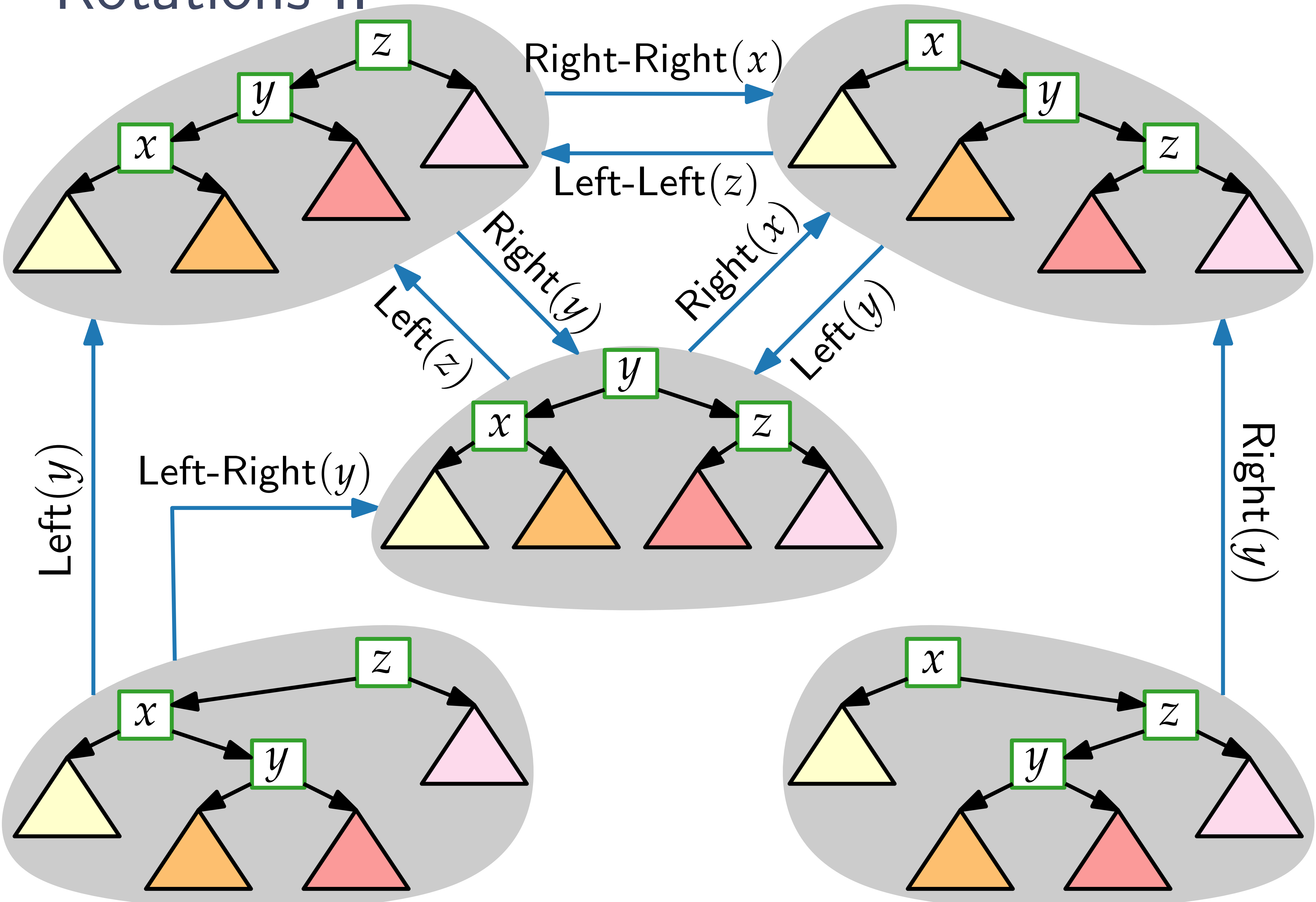
Rotations II



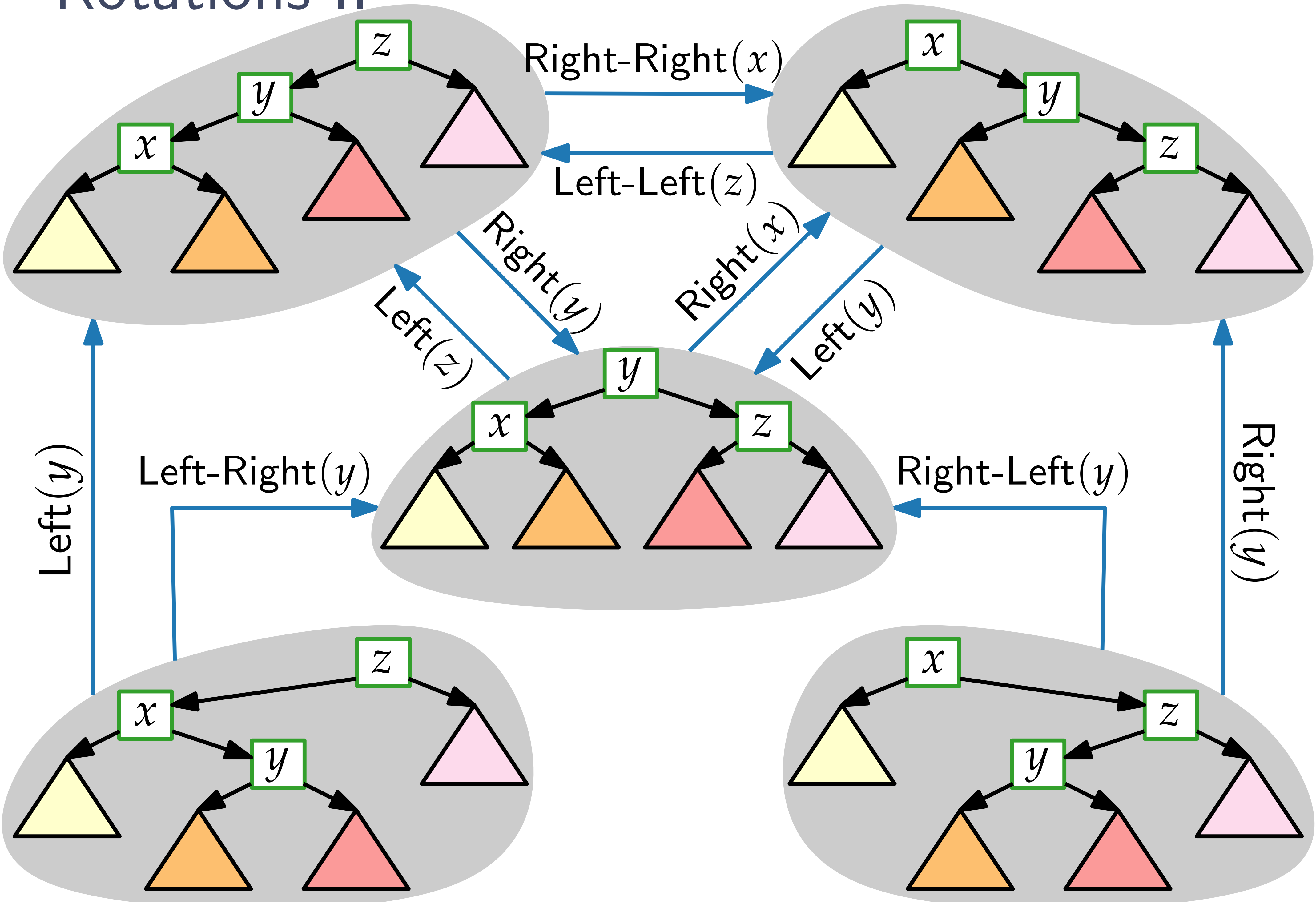
Rotations II



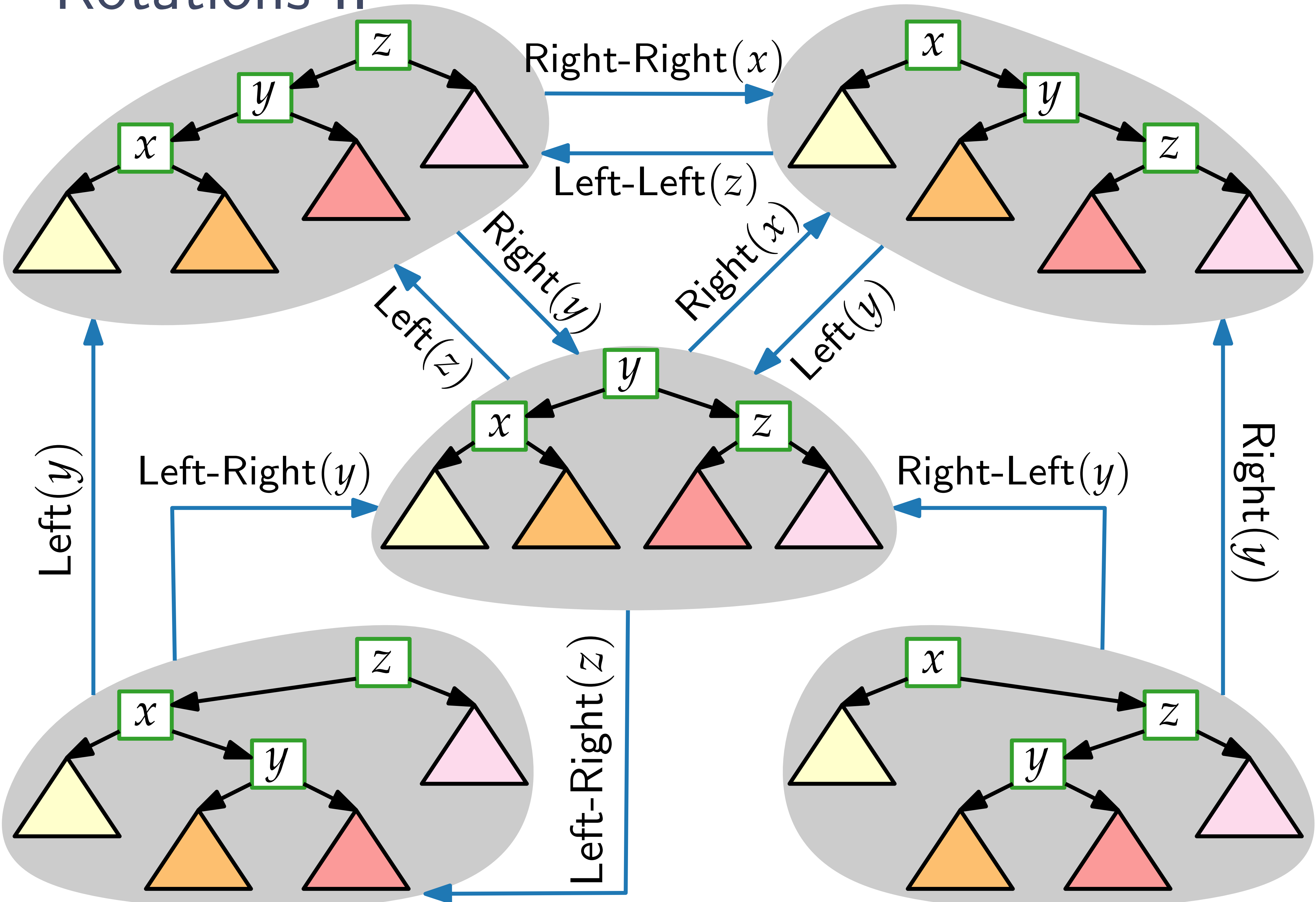
Rotations II



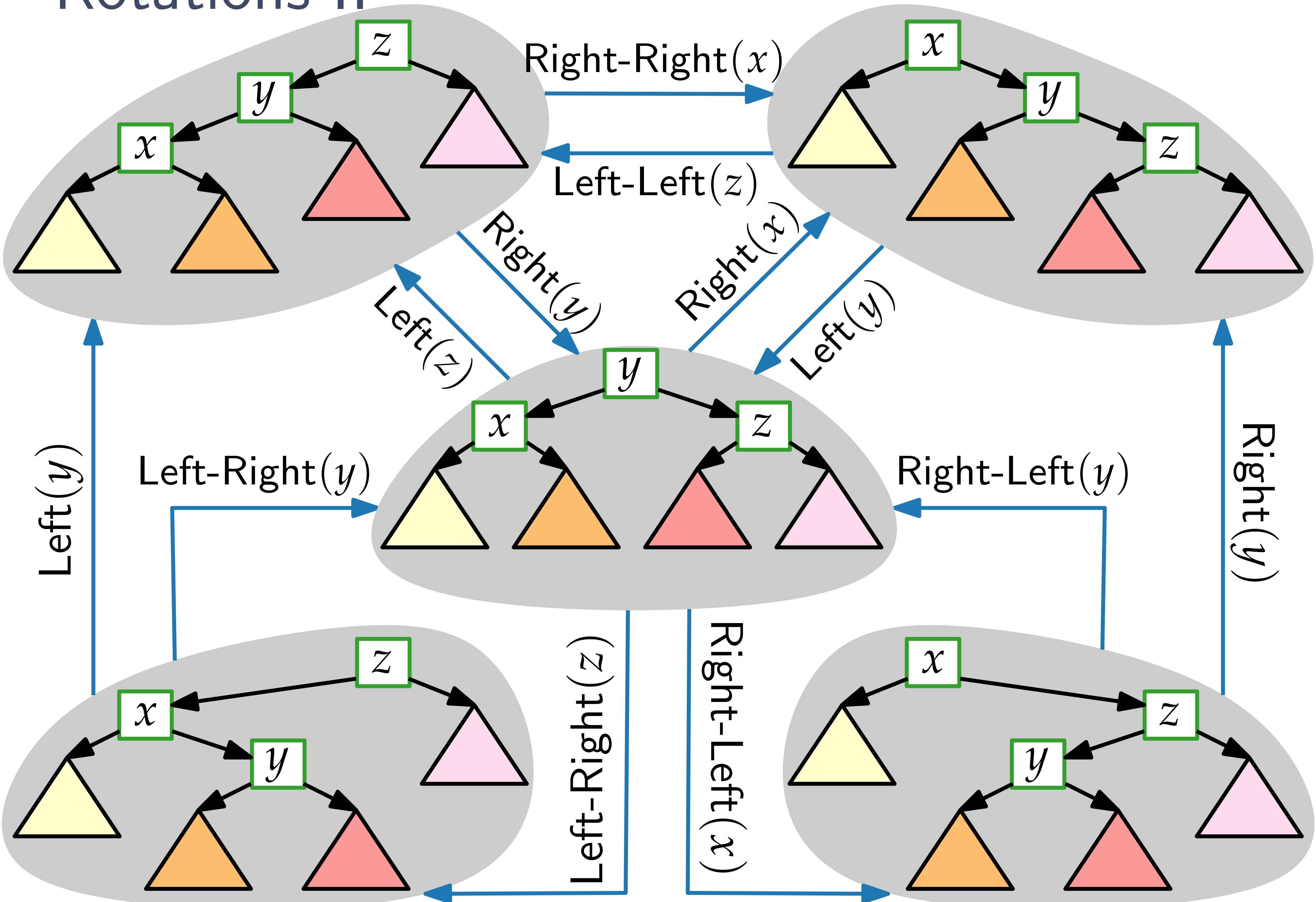
Rotations II



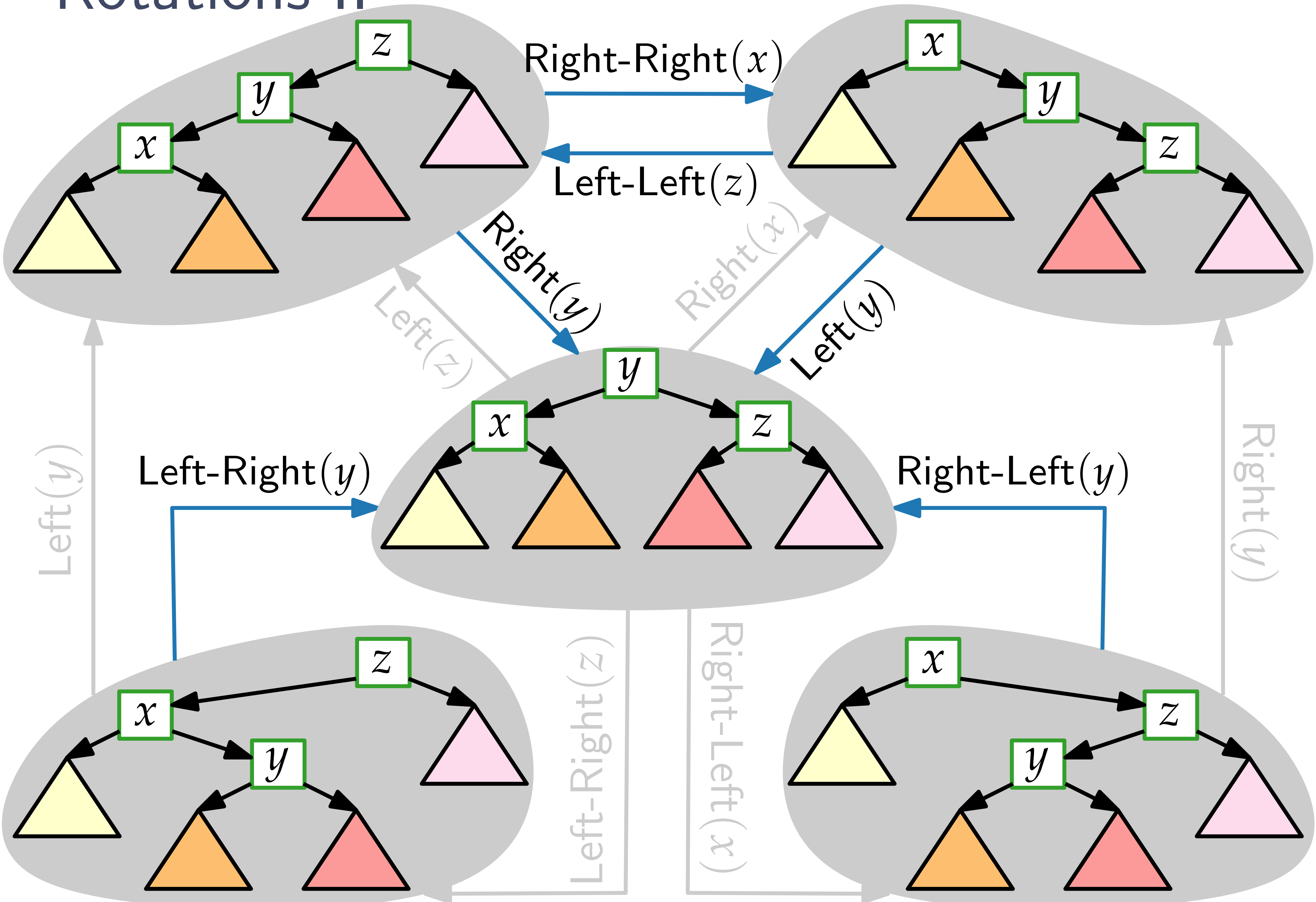
Rotations II



Rotations II



Rotations II



Splay

Algorithm: Splay(x)

Splay

```
**Algorithm:** Splay($x$)
**if** $x \neq root$ **then**
```

Splay

```
Algorithm: Splay($x$)
if $x \neq root$ then
    $y =$ parent of $x$
```

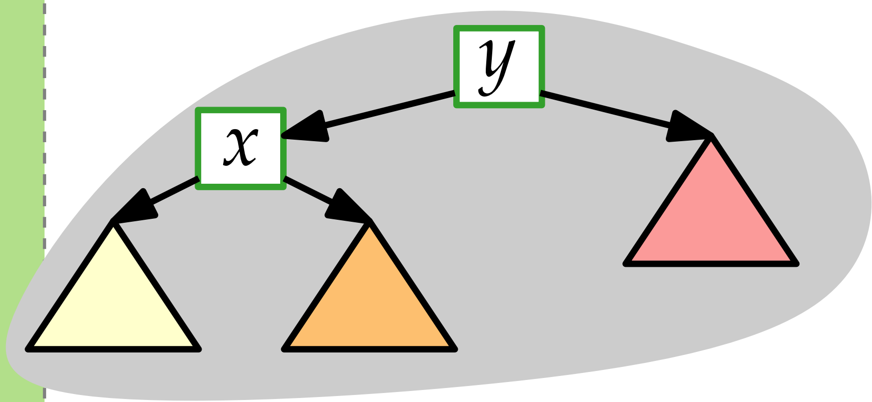
Splay

```
Algorithm: Splay($x$)
if $x \neq root$ then
    $y =$ parent of $x$
    if $y = root$ then
```

Splay

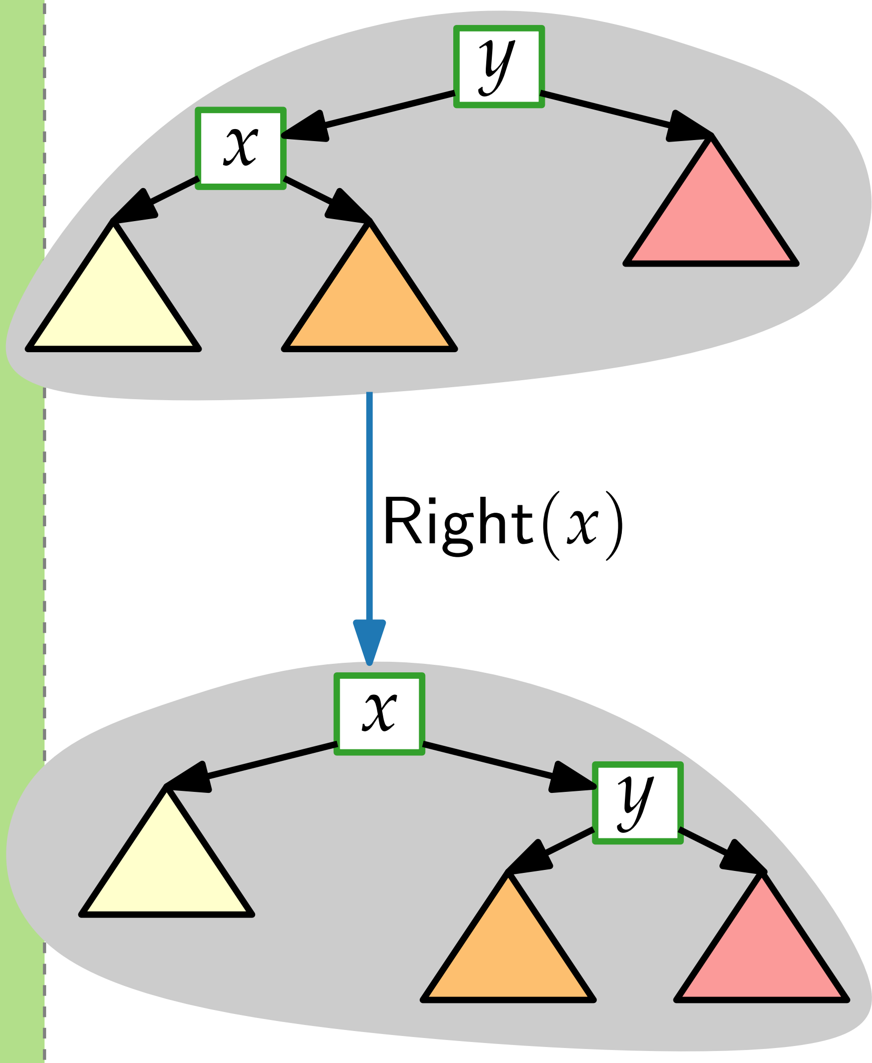
Algorithm: Splay(x)

```
if $x \neq root$ then
    $y =$ parent of $x$
    if $y = root$ then
        if $x < y$ then
```



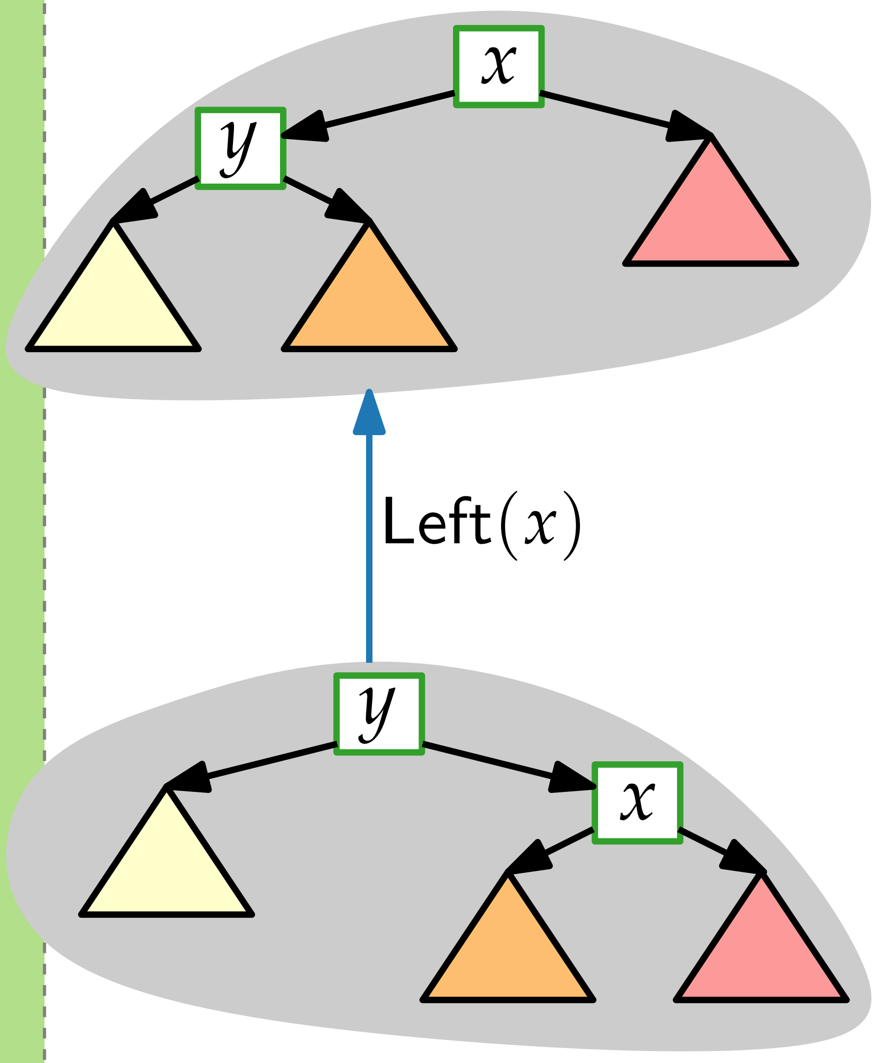
Splay

```
Algorithm: Splay($x$)
if $x \neq root$ then
    $y$ = parent of $x$
    if $y = root$ then
        if $x < y$ then Right($x$)
```



Splay

```
Algorithm: Splay($x$)
if $x \neq root$ then
    $y$ = parent of $x$
    if $y = root$ then
        if $x < y$ then Right($x$)
        if $y < x$ then Left($x$)
```

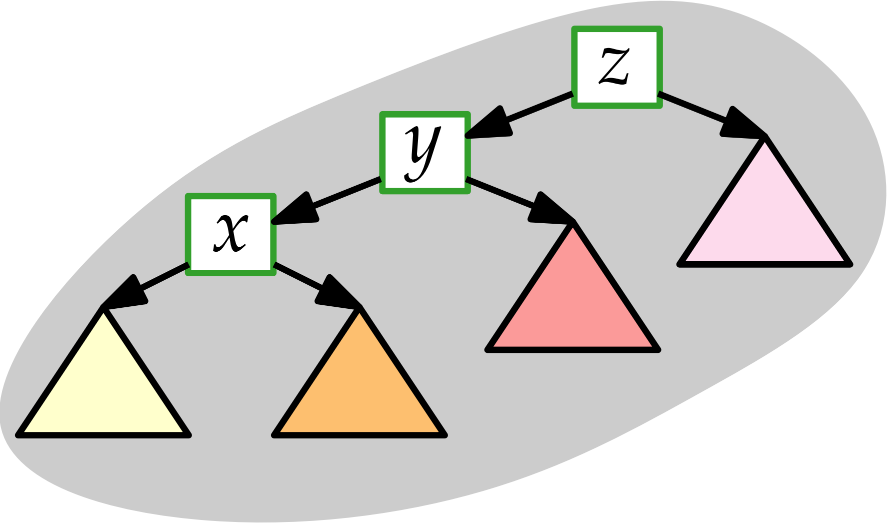


Splay

```
Algorithm: Splay($x$)
if $x \neq root$ then
    $y$ = parent of $x$
    if $y = root$ then
        if $x < y$ then Right($x$)
        if $y < x$ then Left($x$)
    else
        $z$ = parent of $y$
```

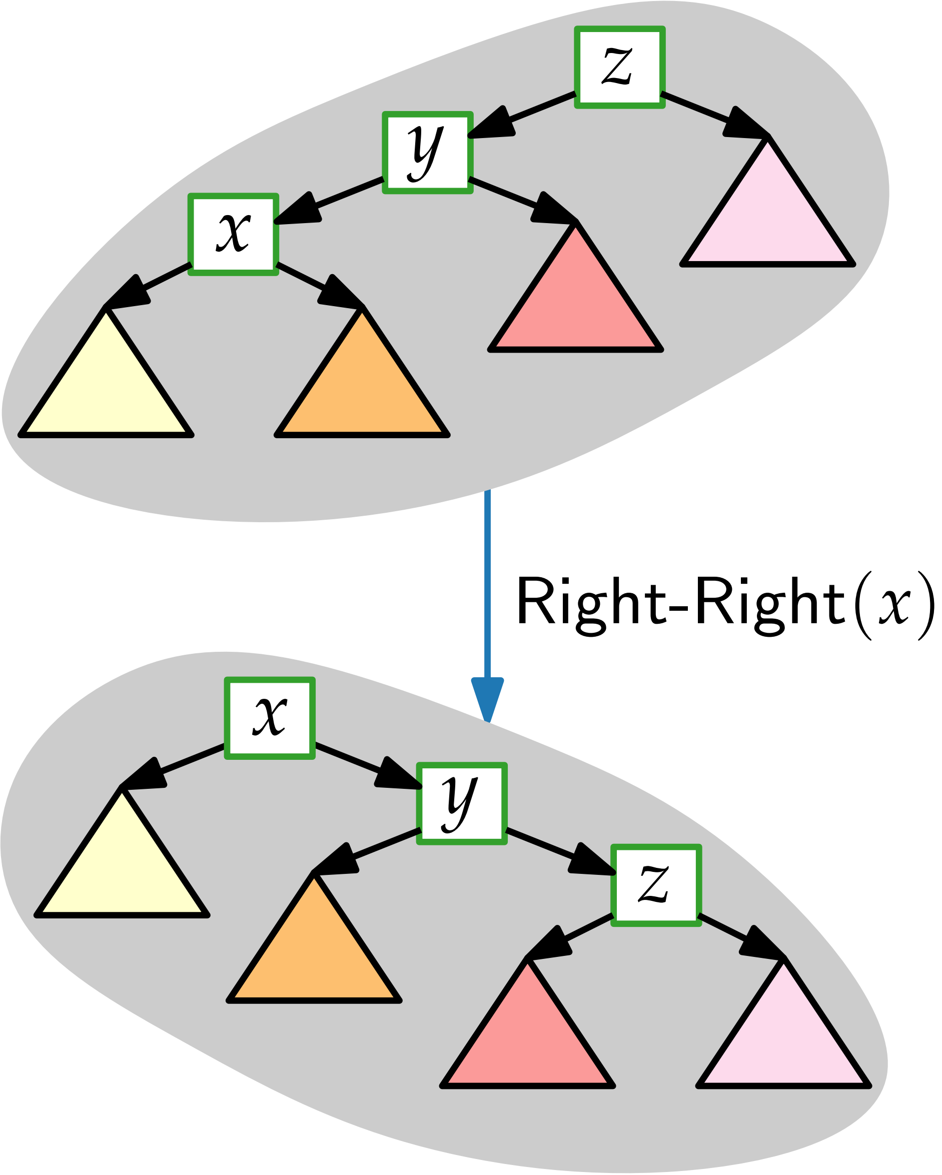
Splay

```
Algorithm: Splay($x$)
if $x \neq root$ then
    $y$ = parent of $x$
    if $y = root$ then
        if $x < y$ then Right($x$)
        if $y < x$ then Left($x$)
    else
        $z$ = parent of $y$
        if $x < y < z$ then
```



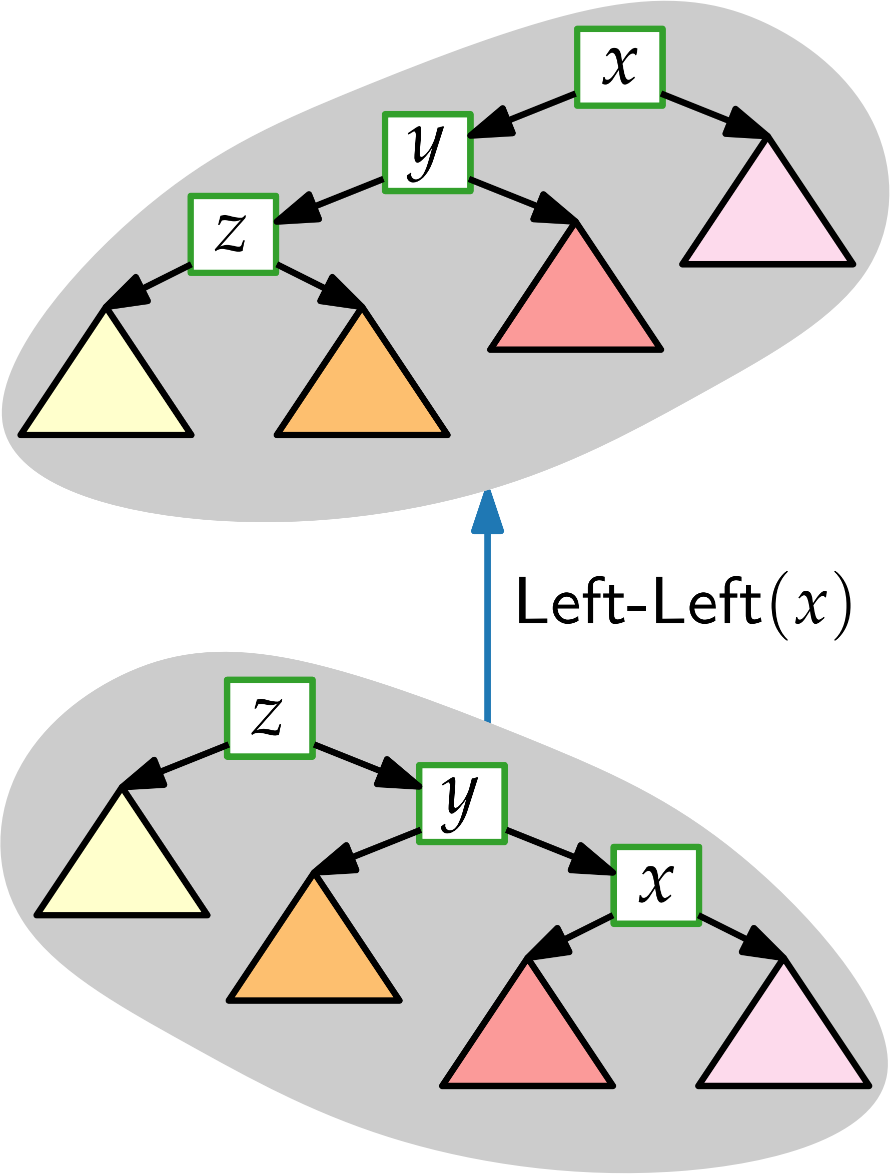
Splay

```
Algorithm: Splay($(x)$)
if $x \neq root$ then
    $y$ = parent of $x$
    if $y = root$ then
        if $x < y$ then Right$(x)$
        if $y < x$ then Left$(x)$
    else
        $z$ = parent of $y$
        if $x < y < z$ then Right-Right$(x)$
```



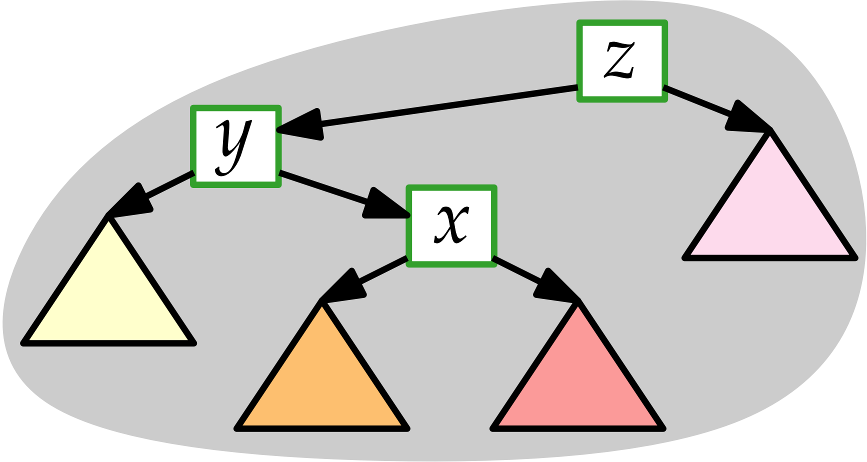
Splay

```
Algorithm: Splay($x$)
if $x \neq root$ then
    $y$ = parent of $x$
    if $y = root$ then
        if $x < y$ then Right($x$)
        if $y < x$ then Left($x$)
    else
        $z$ = parent of $y$
        if $x < y < z$ then Right-Right($x$)
        if $z < y < x$ then Left-Left($x$)
```



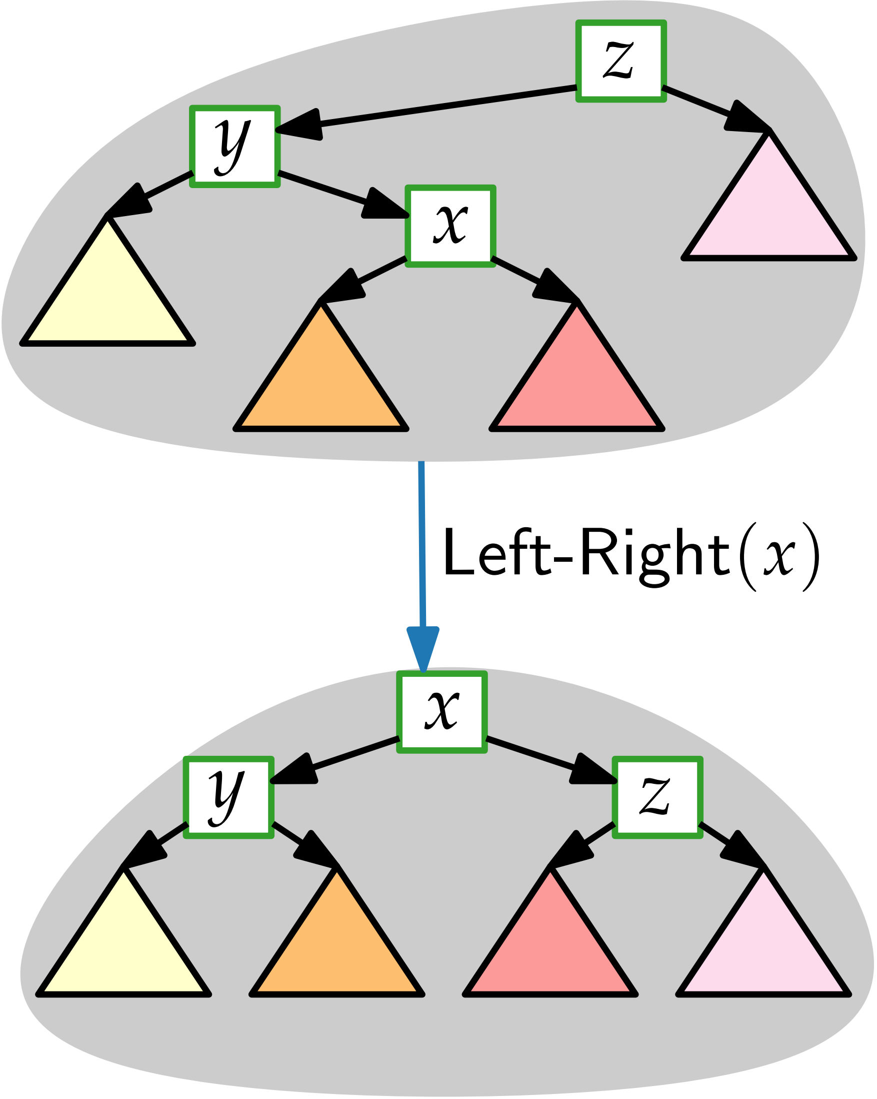
Splay

```
Algorithm: Splay($x$)
if $x \neq root$ then
    $y$ = parent of $x$
    if $y = root$ then
        if $x < y$ then Right($x$)
        if $y < x$ then Left($x$)
    else
        $z$ = parent of $y$
        if $x < y < z$ then Right-Right($x$)
        if $z < y < x$ then Left-Left($x$)
        if $y < x < z$ then
```



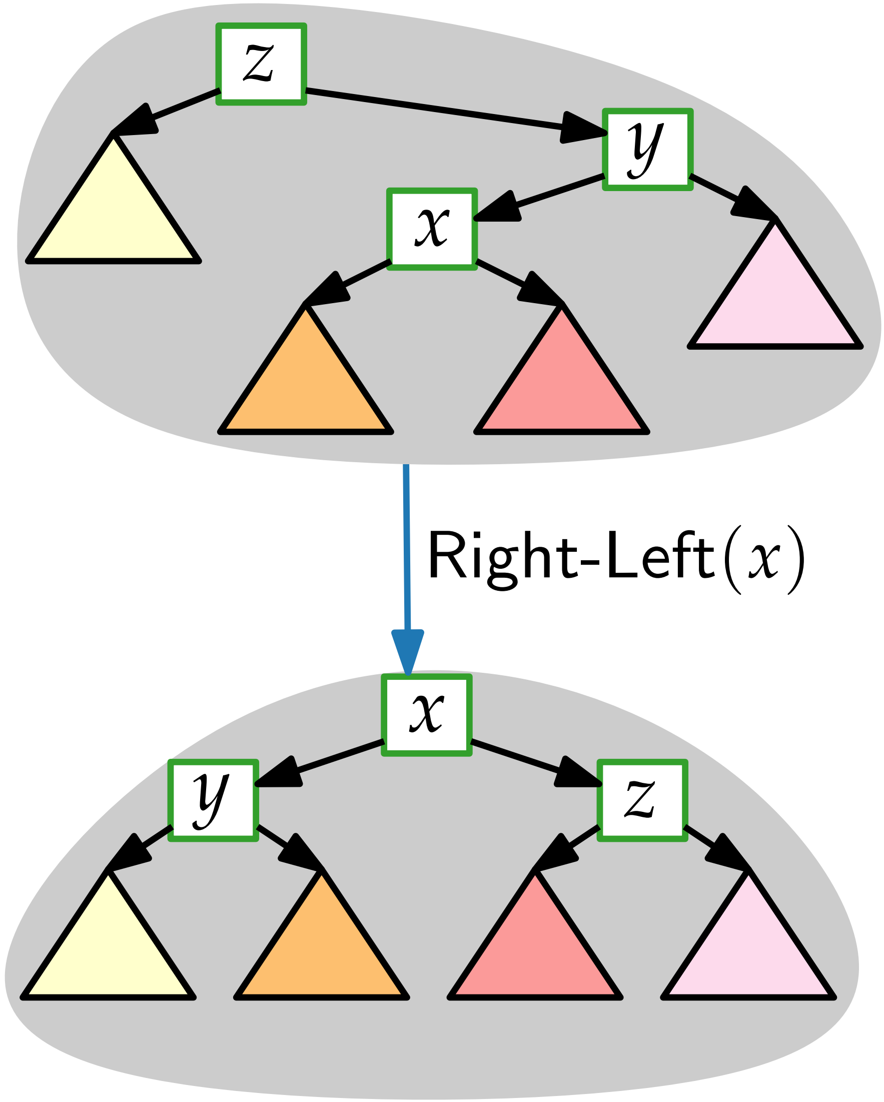
Splay

```
**Algorithm:** Splay($x$)
if $x \neq root$ then
    $y$ = parent of $x$
    if $y = root$ then
        if $x < y$ then Right($x$)
        if $y < x$ then Left($x$)
    else
        $z$ = parent of $y$
        if $x < y < z$ then Right-Right($x$)
        if $z < y < x$ then Left-Left($x$)
        if $y < x < z$ then Left-Right($x$)
```



Splay

```
Algorithm: Splay($x$)
if $x \neq root$ then
    $y$ = parent of $x$
    if $y = root$ then
        if $x < y$ then Right($x$)
        if $y < x$ then Left($x$)
    else
        $z$ = parent of $y$
        if $x < y < z$ then Right-Right($x$)
        if $z < y < x$ then Left-Left($x$)
        if $y < x < z$ then Left-Right($x$)
        if $z < x < y$ then Right-Left($x$)
```



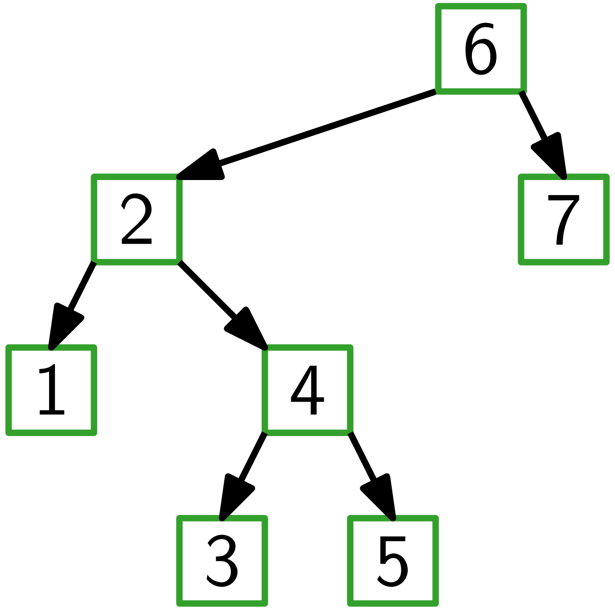
Splay

```
**Algorithm:** Splay($x$)
if $x \neq root$ then
    $y =$ parent of $x$
    if $y = root$ then
        if $x < y$ then Right($x$)
        if $y < x$ then Left($x$)
    else
        $z =$ parent of $y$
        if $x < y < z$ then Right-Right($x$)
        if $z < y < x$ then Left-Left($x$)
        if $y < x < z$ then Left-Right($x$)
        if $z < x < y$ then Right-Left($x$)
    Splay($x$)
```

Splay

```
Algorithm: Splay($(x)$)
if $x \neq root$ then
    $y =$ parent of $x$
    if $y = root$ then
        if $x < y$ then Right($x$)
        if $y < x$ then Left($x$)
    else
        $z =$ parent of $y$
        if $x < y < z$ then Right-Right($x$)
        if $z < y < x$ then Left-Left($x$)
        if $y < x < z$ then Left-Right($x$)
        if $z < x < y$ then Right-Left($x$)
    Splay($x$)
```

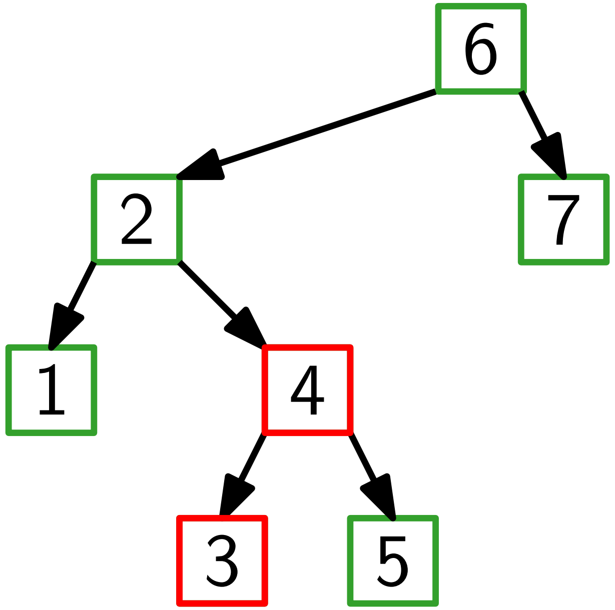
Splay(3):



Splay

```
Algorithm: Splay($x$)
if $x \neq root$ then
    $y$ = parent of $x$
    if $y = root$ then
        if $x < y$ then Right($x$)
        if $y < x$ then Left($x$)
    else
        $z$ = parent of $y$
        if $x < y < z$ then Right-Right($x$)
        if $z < y < x$ then Left-Left($x$)
        if $y < x < z$ then Left-Right($x$)
        if $z < x < y$ then Right-Left($x$)
    Splay($x$)
```

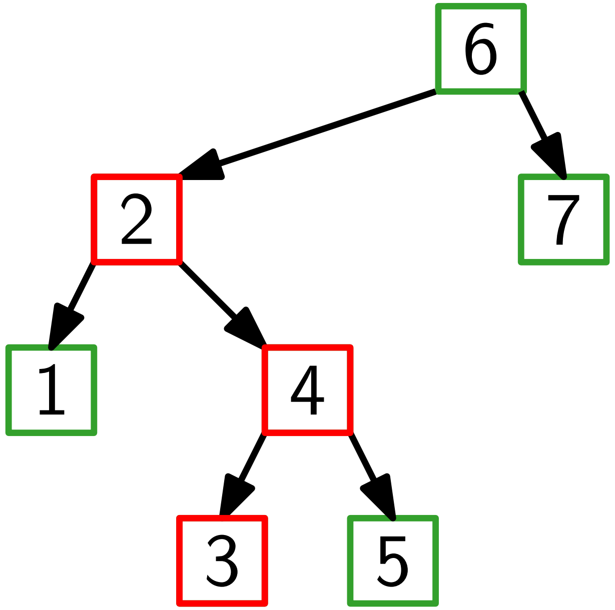
Splay(3):



Splay

```
Algorithm: Splay($x$)
if $x \neq root$ then
    $y$ = parent of $x$
    if $y = root$ then
        if $x < y$ then Right($x$)
        if $y < x$ then Left($x$)
    else
        $z$ = parent of $y$
        if $x < y < z$ then Right-Right($x$)
        if $z < y < x$ then Left-Left($x$)
        if $y < x < z$ then Left-Right($x$)
        if $z < x < y$ then Right-Left($x$)
    Splay($x$)
```

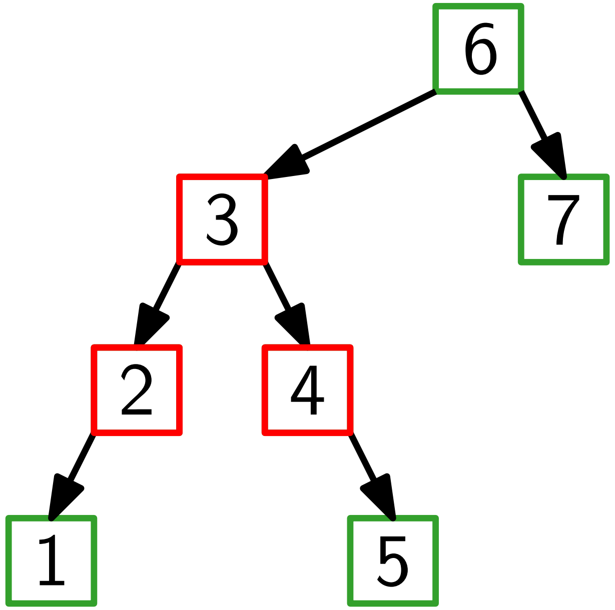
Splay(3):



Splay

```
Algorithm: Splay($x$)
if $x \neq root$ then
    $y$ = parent of $x$
    if $y = root$ then
        if $x < y$ then Right($x$)
        if $y < x$ then Left($x$)
    else
        $z$ = parent of $y$
        if $x < y < z$ then Right-Right($x$)
        if $z < y < x$ then Left-Left($x$)
        if $y < x < z$ then Left-Right($x$)
        if $z < x < y$ then Right-Left($x$)
    Splay($x$)
```

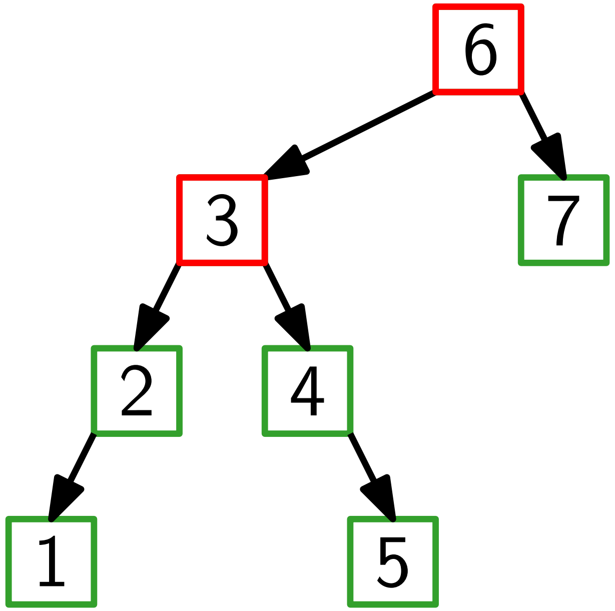
Splay(3):



Splay

```
Algorithm: Splay($(x)$)
if $x \neq root$ then
    $y$ = parent of $x$
    if $y = root$ then
        if $x < y$ then Right($(x)$)
        if $y < x$ then Left($(x)$)
    else
        $z$ = parent of $y$
        if $x < y < z$ then Right-Right($(x)$)
        if $z < y < x$ then Left-Left($(x)$)
        if $y < x < z$ then Left-Right($(x)$)
        if $z < x < y$ then Right-Left($(x)$)
    Splay($(x)$)
```

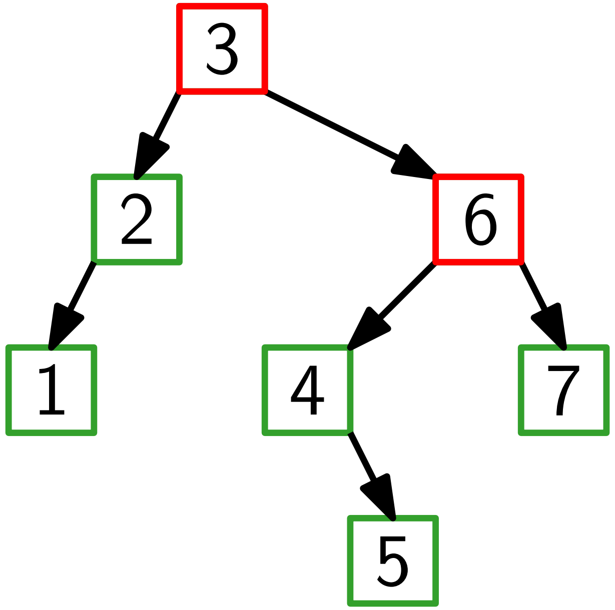
Splay(3):



Splay

```
Algorithm: Splay($x$)
if $x \neq root$ then
    $y$ = parent of $x$
    if $y = root$ then
        if $x < y$ then Right($x$)
        if $y < x$ then Left($x$)
    else
        $z$ = parent of $y$
        if $x < y < z$ then Right-Right($x$)
        if $z < y < x$ then Left-Left($x$)
        if $y < x < z$ then Left-Right($x$)
        if $z < x < y$ then Right-Left($x$)
    Splay($x$)
```

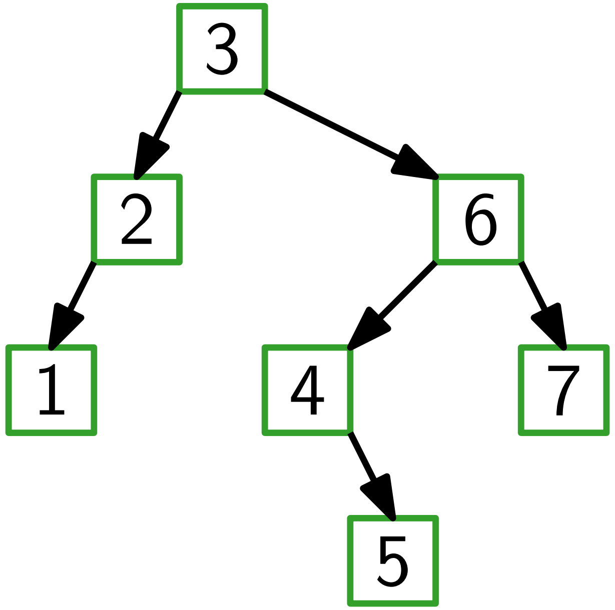
Splay(3):



Splay

```
Algorithm: Splay($x$)
if $x \neq root$ then
    $y$ = parent of $x$
    if $y = root$ then
        if $x < y$ then Right($x$)
        if $y < x$ then Left($x$)
    else
        $z$ = parent of $y$
        if $x < y < z$ then Right-Right($x$)
        if $z < y < x$ then Left-Left($x$)
        if $y < x < z$ then Left-Right($x$)
        if $z < x < y$ then Right-Left($x$)
    Splay($x$)
```

Splay(3):

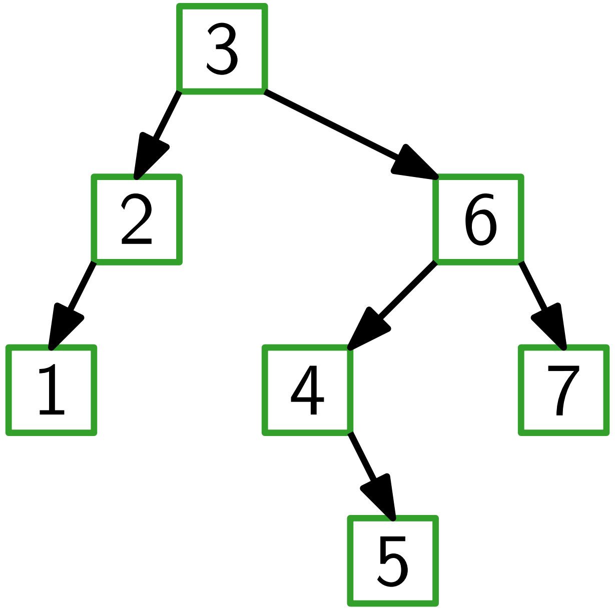


Call Splay(x):

Splay

```
Algorithm: Splay($x$)
if $x \neq root$ then
    $y$ = parent of $x$
    if $y = root$ then
        if $x < y$ then Right($x$)
        if $y < x$ then Left($x$)
    else
        $z$ = parent of $y$
        if $x < y < z$ then Right-Right($x$)
        if $z < y < x$ then Left-Left($x$)
        if $y < x < z$ then Left-Right($x$)
        if $z < x < y$ then Right-Left($x$)
    Splay($x$)
```

Splay(3):



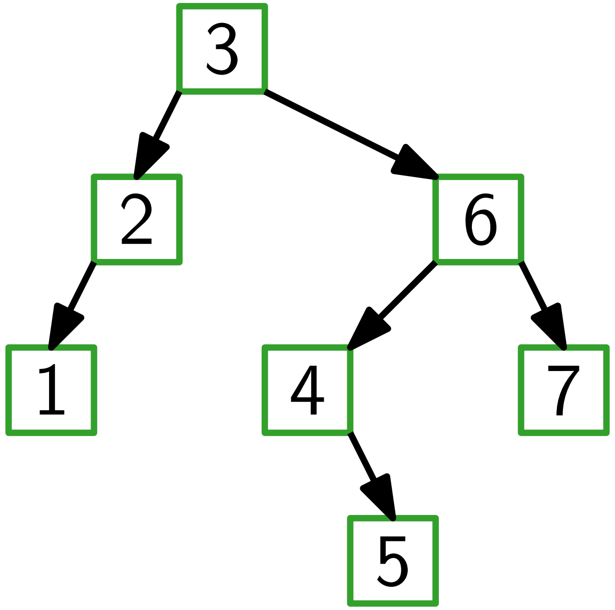
Call Splay(x):

- 🟦 after Search(x)

Splay

```
Algorithm: Splay($x$)
if $x \neq root$ then
    $y$ = parent of $x$
    if $y = root$ then
        if $x < y$ then Right($x$)
        if $y < x$ then Left($x$)
    else
        $z$ = parent of $y$
        if $x < y < z$ then Right-Right($x$)
        if $z < y < x$ then Left-Left($x$)
        if $y < x < z$ then Left-Right($x$)
        if $z < x < y$ then Right-Left($x$)
    Splay($x$)
```

Splay(3):



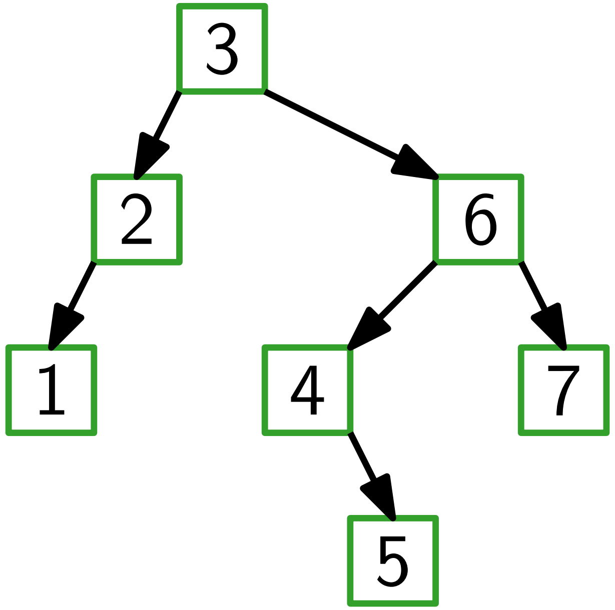
Call Splay(x):

- after Search(x)
- after Insert(x)

Splay

```
**Algorithm:** Splay($x$)
if $x \neq root$ then
    $y$ = parent of $x$
    if $y = root$ then
        if $x < y$ then Right($x$)
        if $y < x$ then Left($x$)
    else
        $z$ = parent of $y$
        if $x < y < z$ then Right-Right($x$)
        if $z < y < x$ then Left-Left($x$)
        if $y < x < z$ then Left-Right($x$)
        if $z < x < y$ then Right-Left($x$)
    Splay($x$)
```

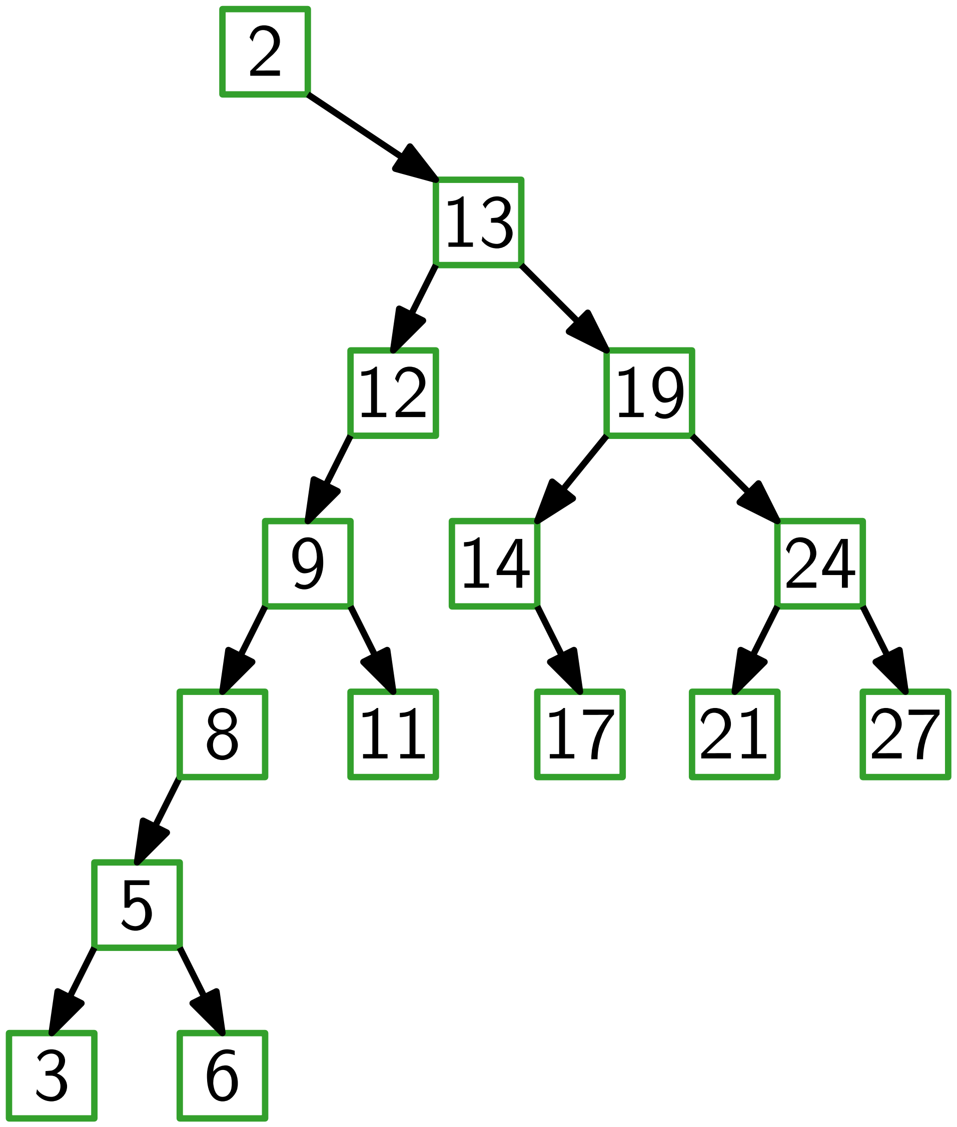
Splay(3):



Call Splay(x):

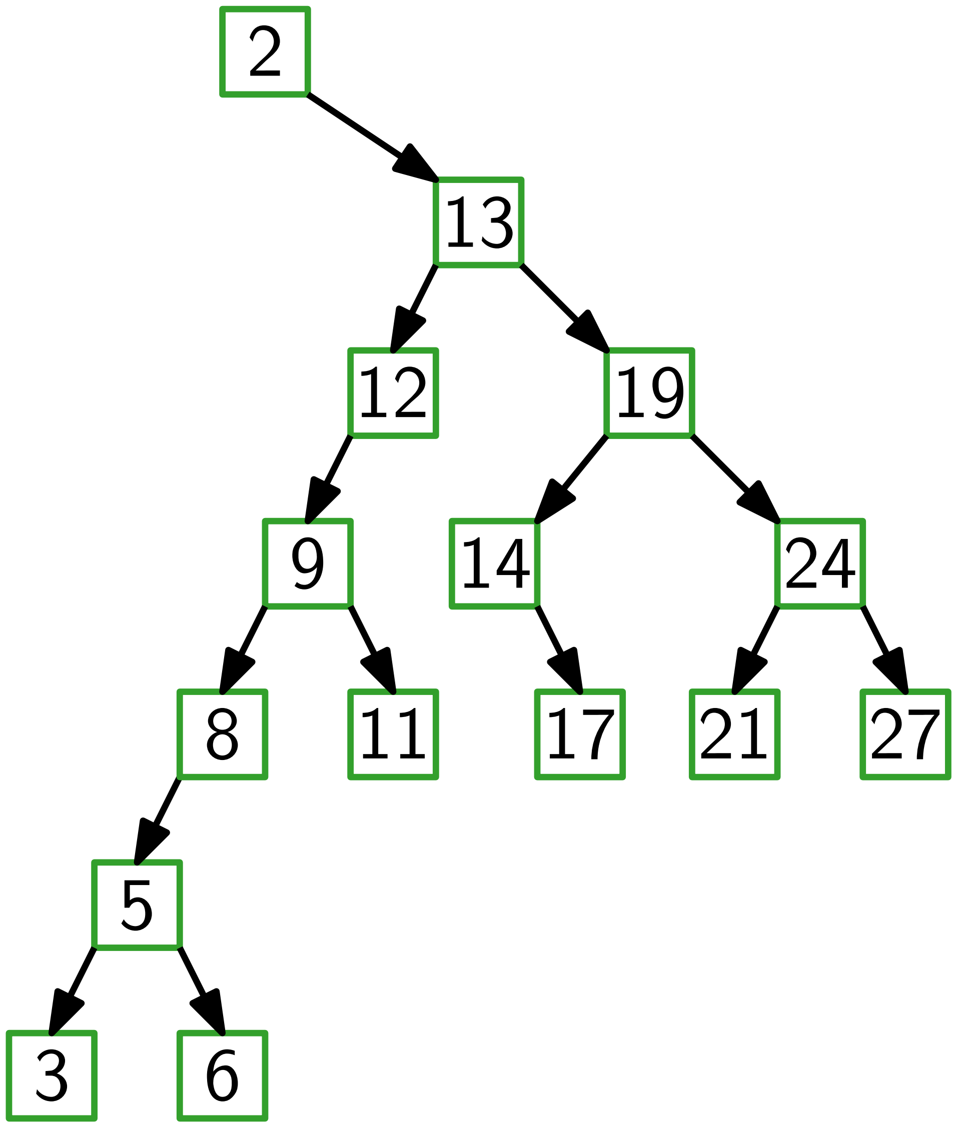
- after Search(x)
- after Insert(x)
- before Delete(x)

Why is Splay Fast?



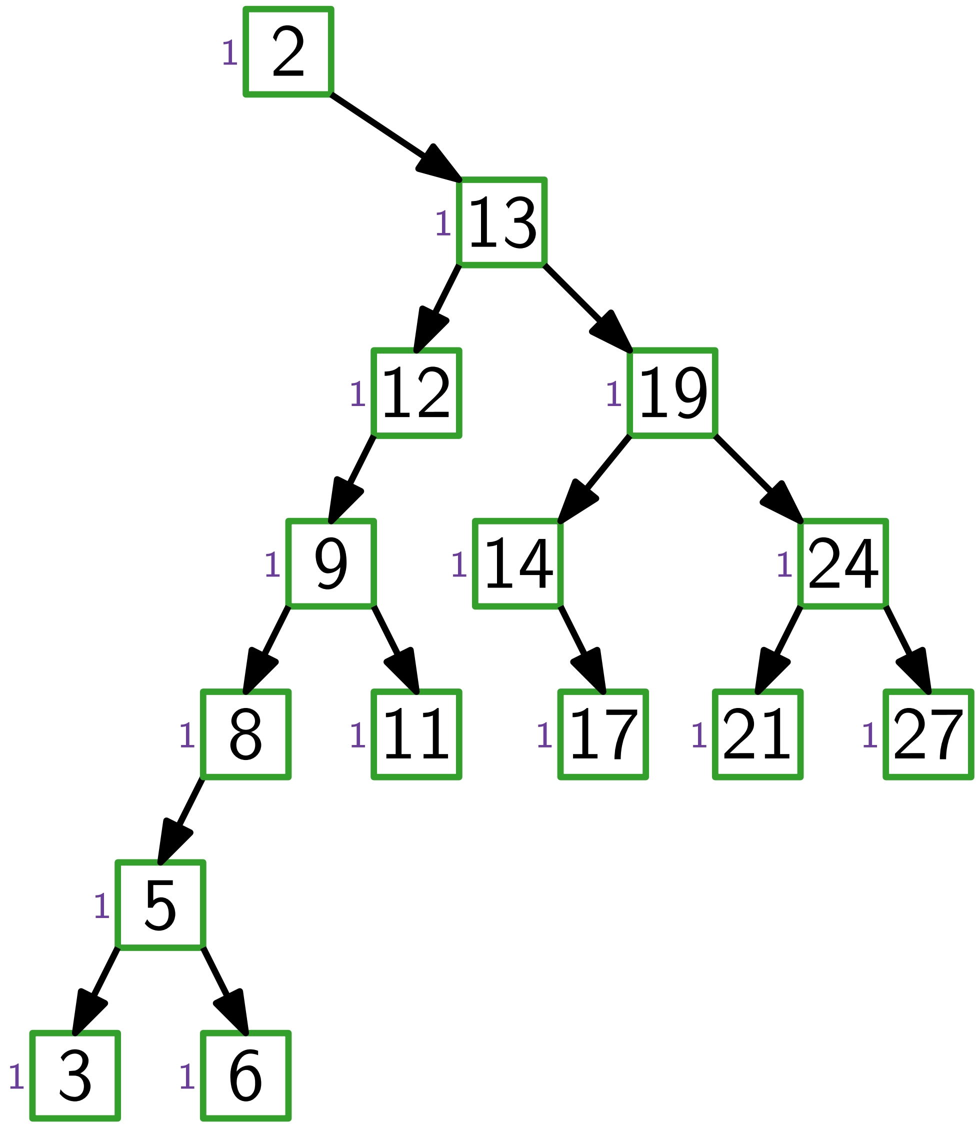
Why is Splay Fast?

$w(x)$: weight of x (here 1), $W = \sum w(x)$ (here n)



Why is Splay Fast?

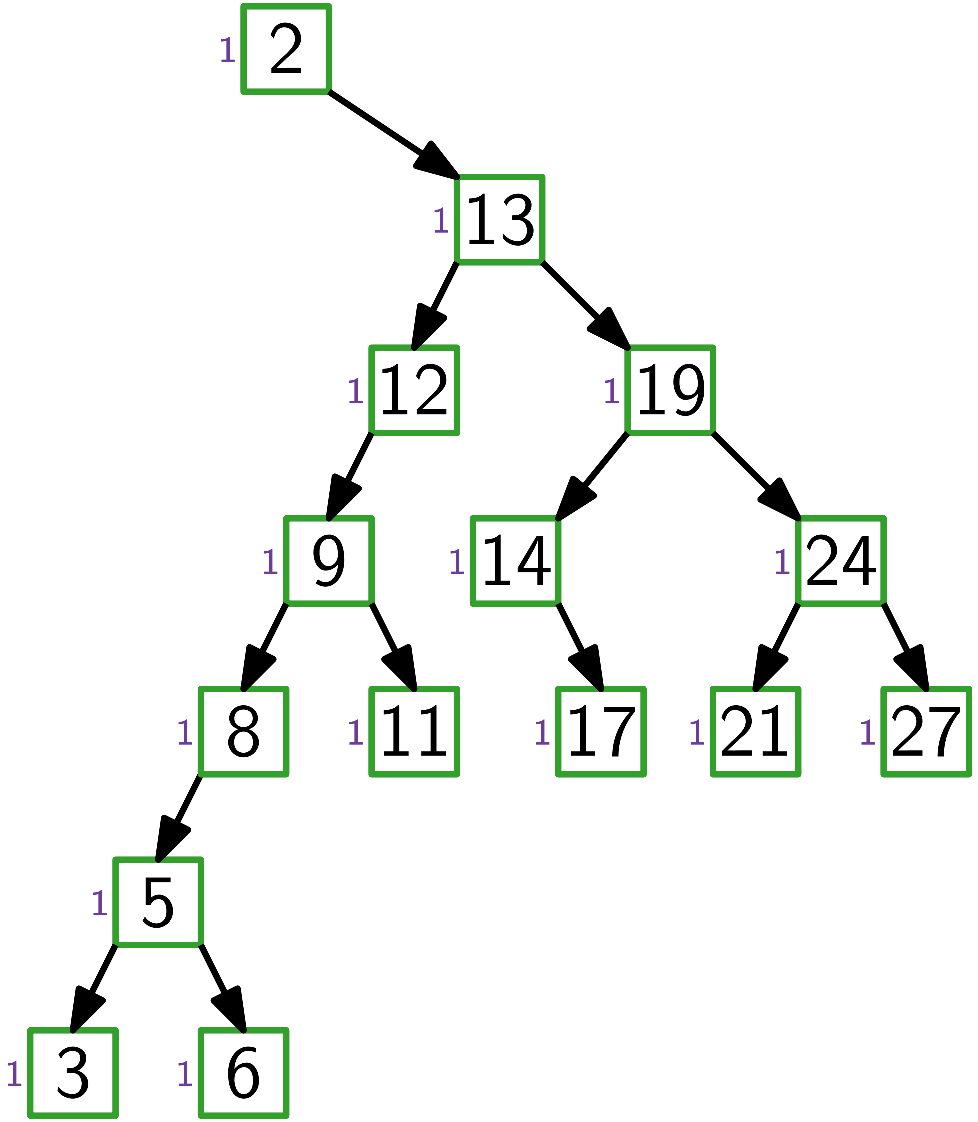
$w(x)$: weight of x (here 1), $W = \sum w(x)$ (here n)



Why is Splay Fast?

$w(x)$: weight of x (here 1), $W = \sum w(x)$ (here n)

$s(x)$: sum of all $w(x)$ in subtree of x



Why is Splay Fast?

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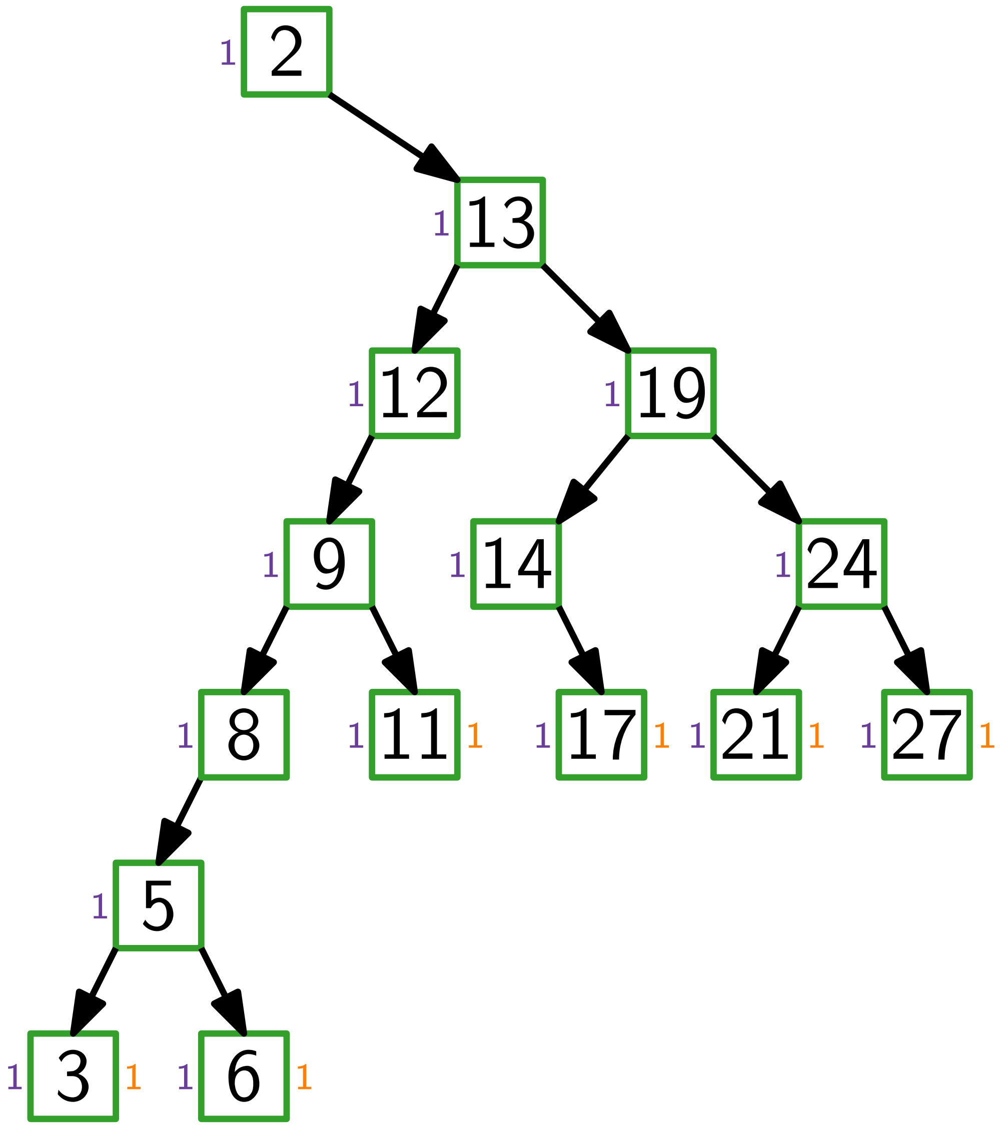

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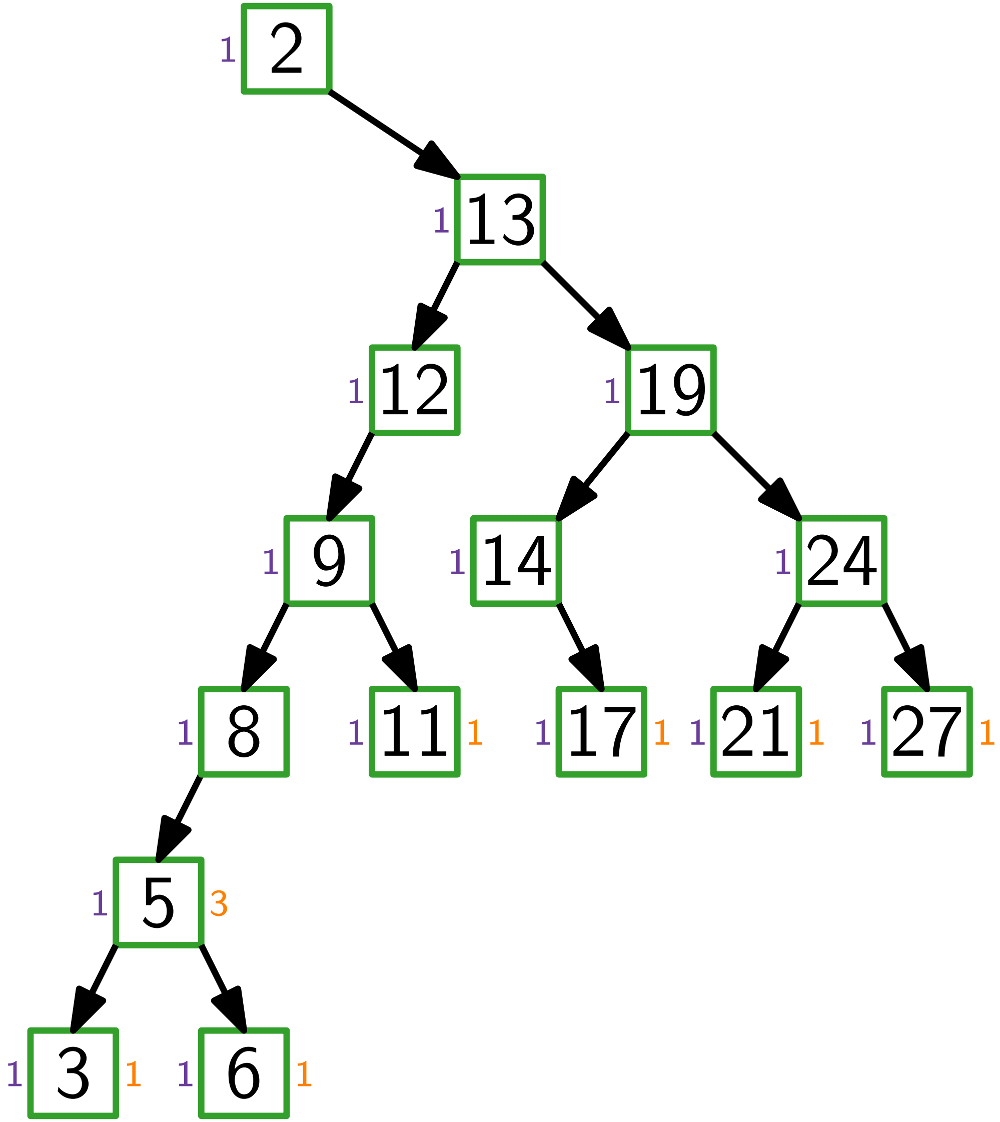

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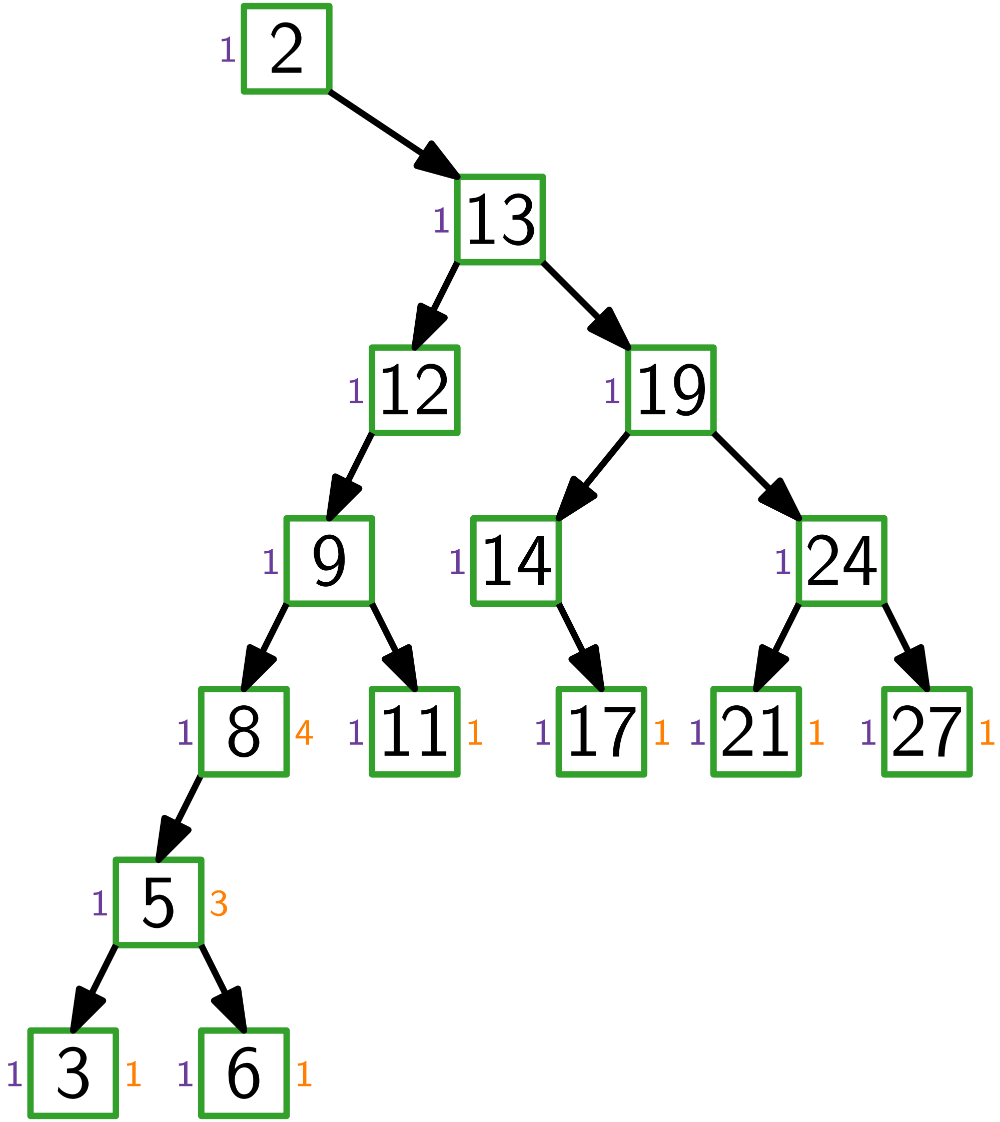

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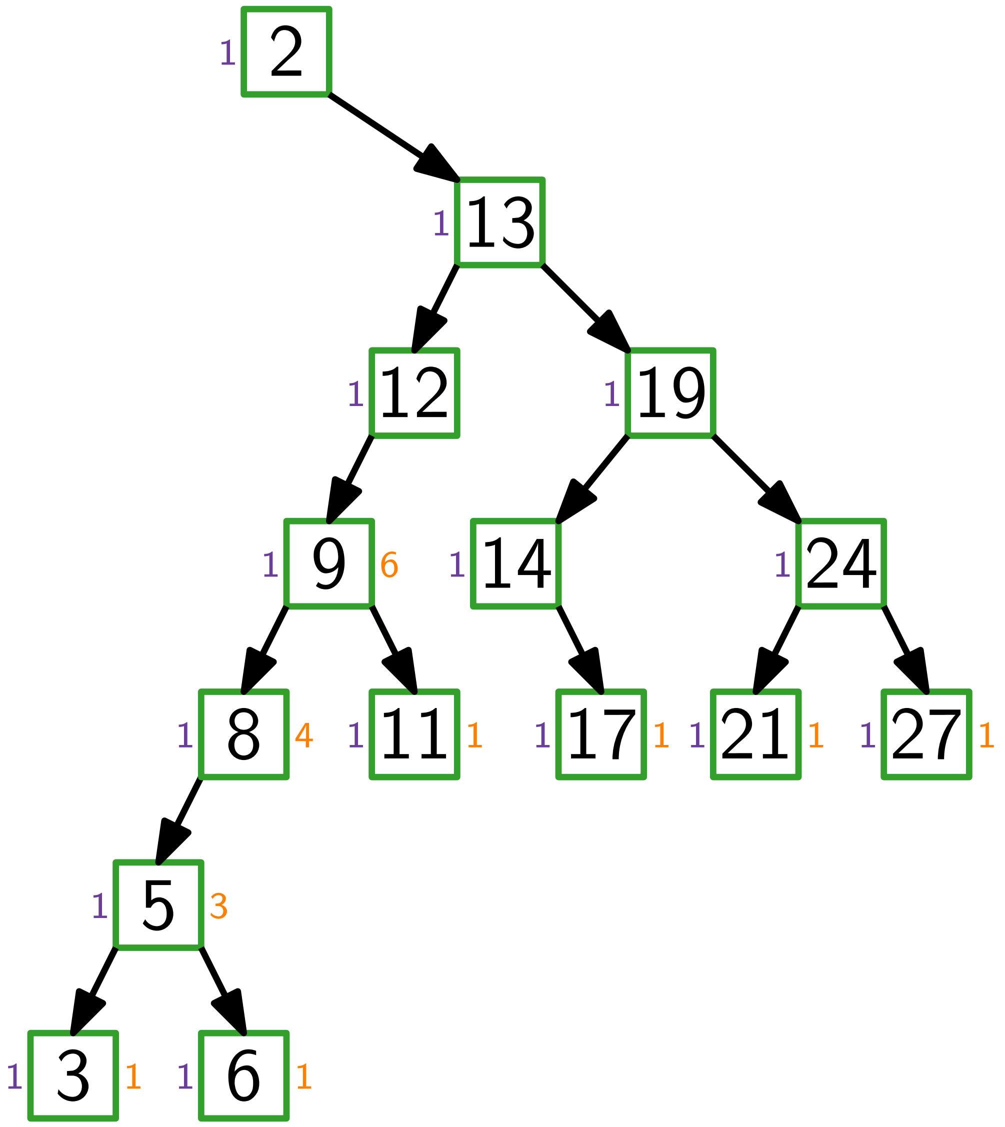

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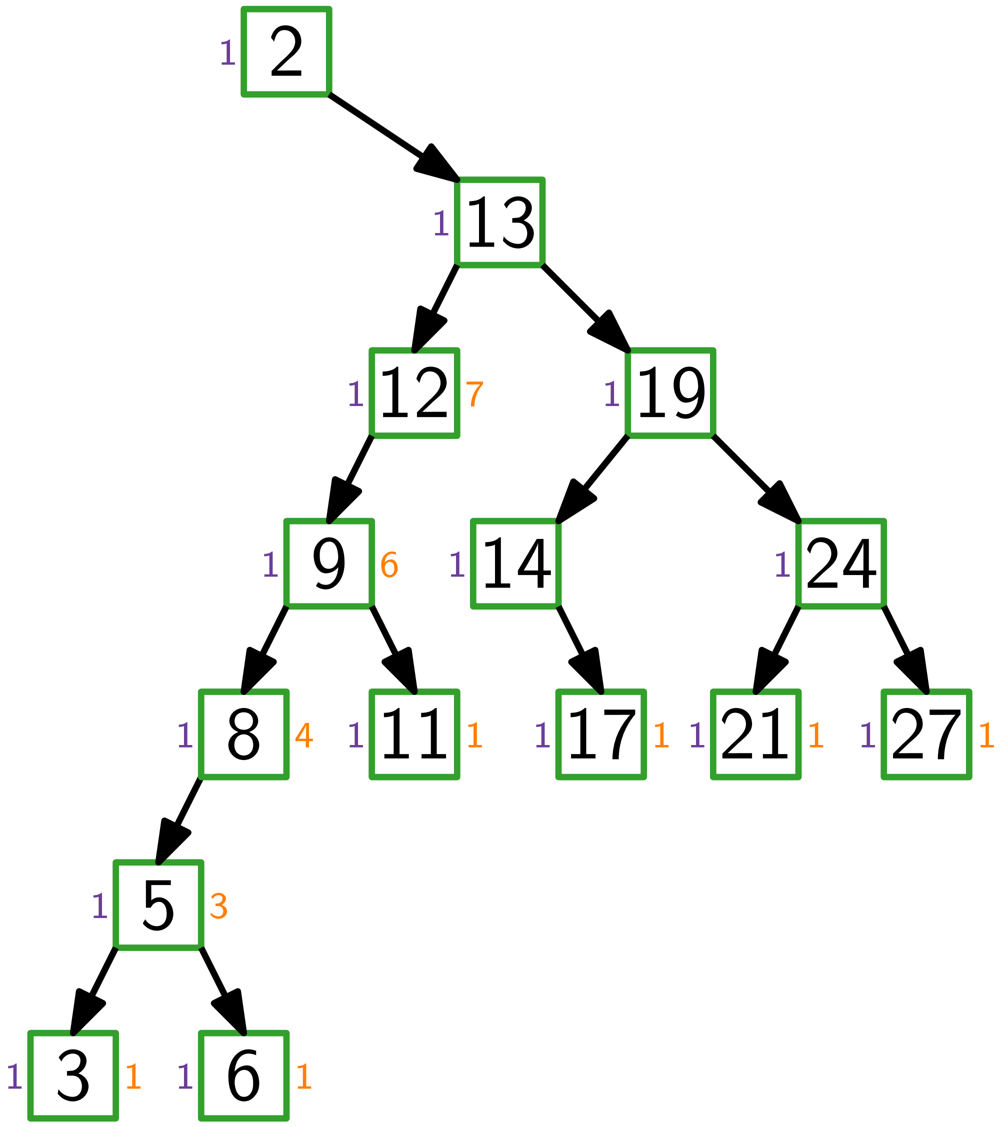

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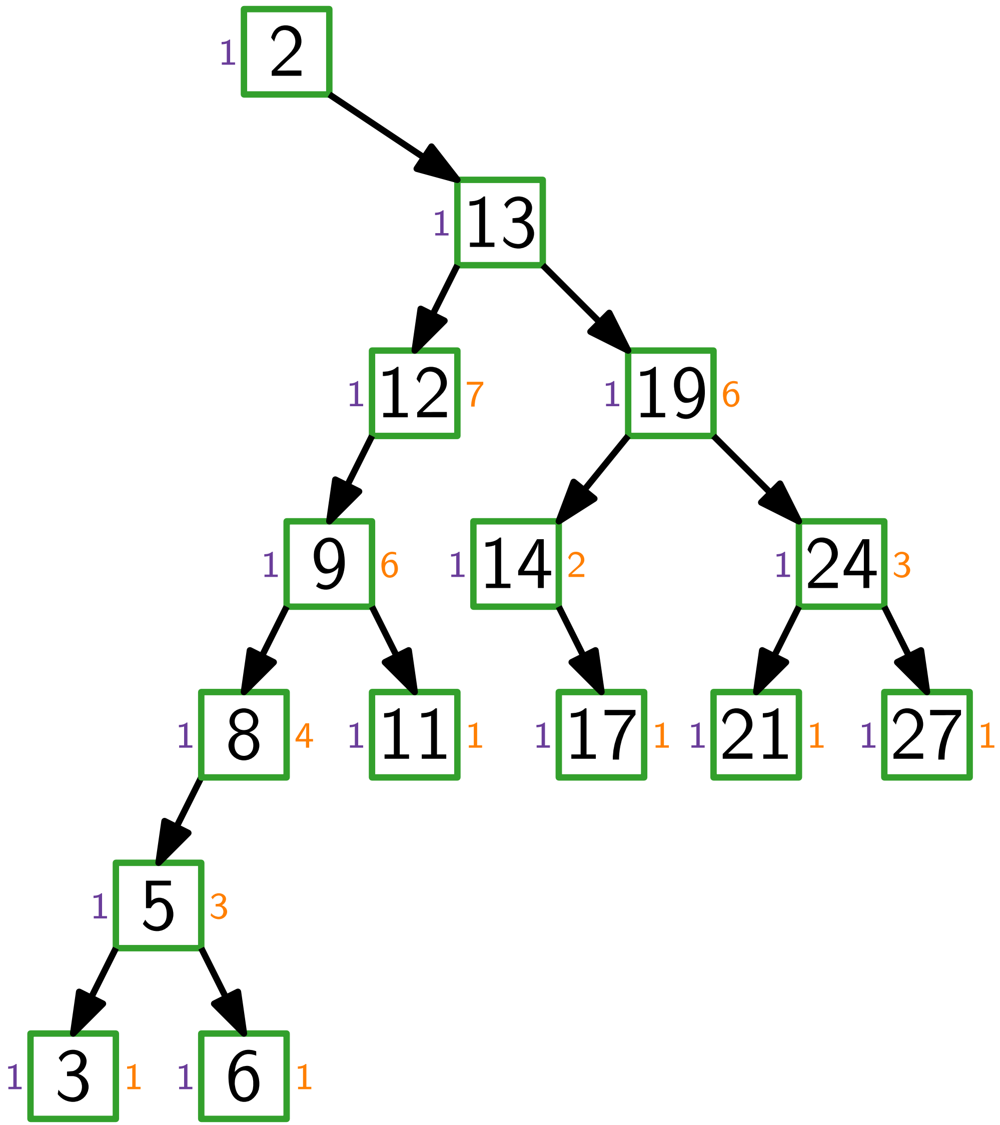

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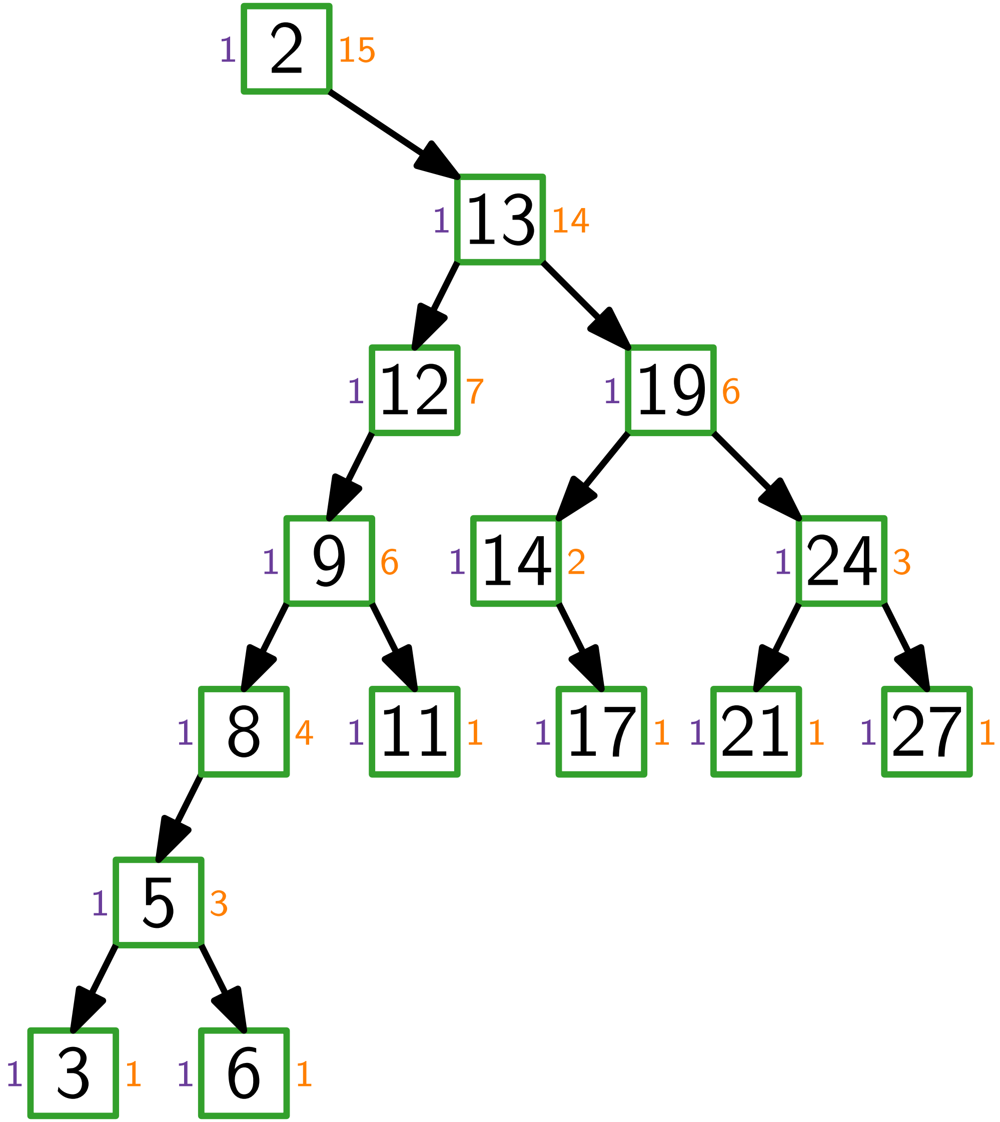

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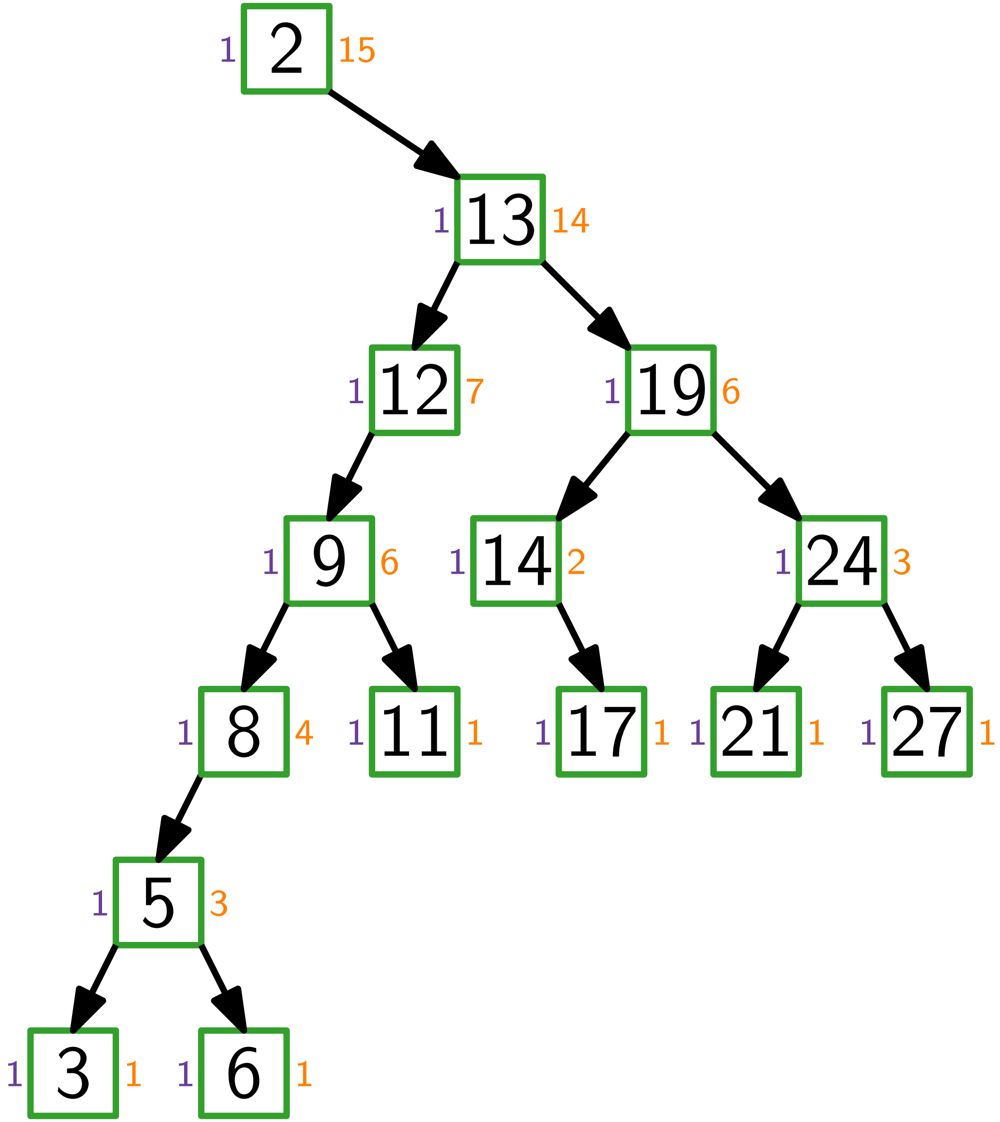

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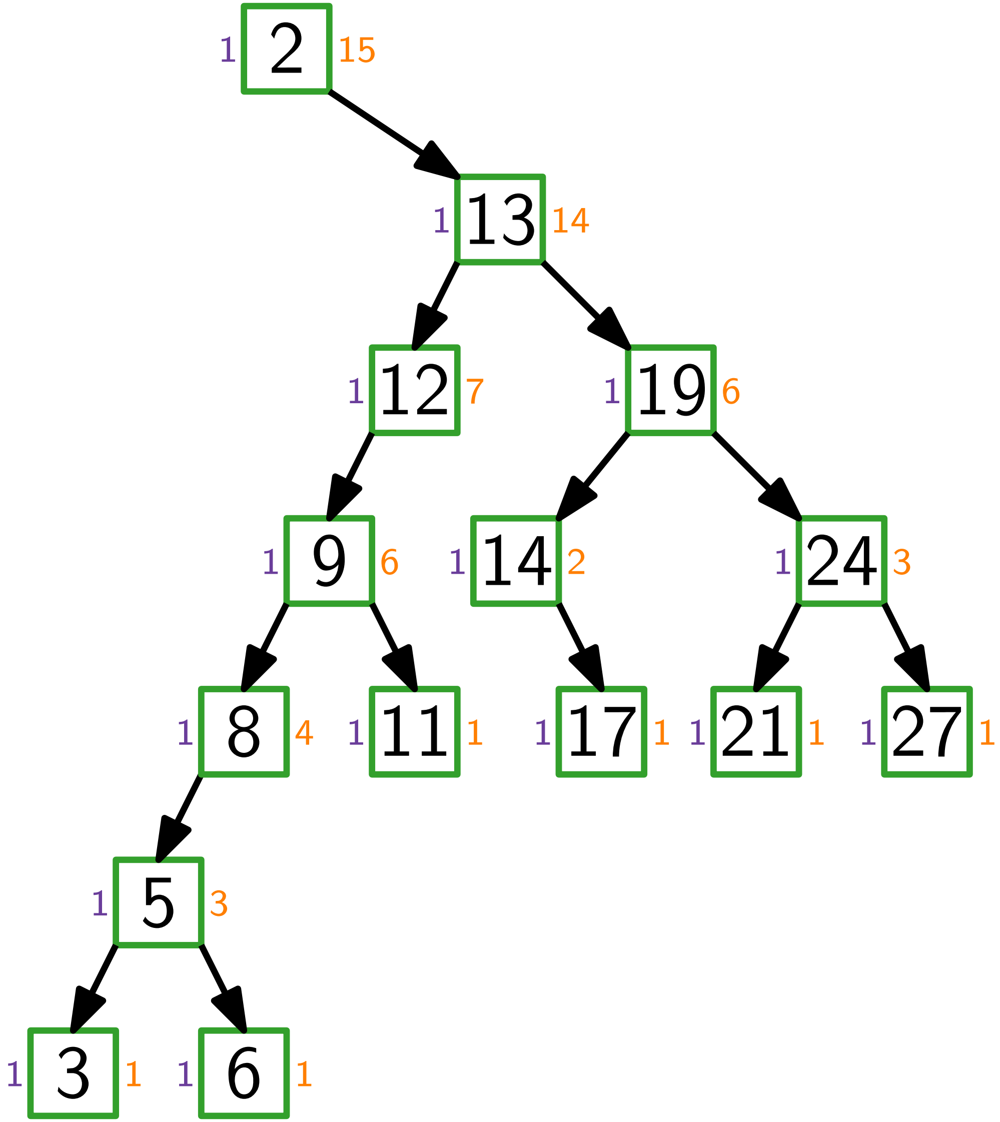

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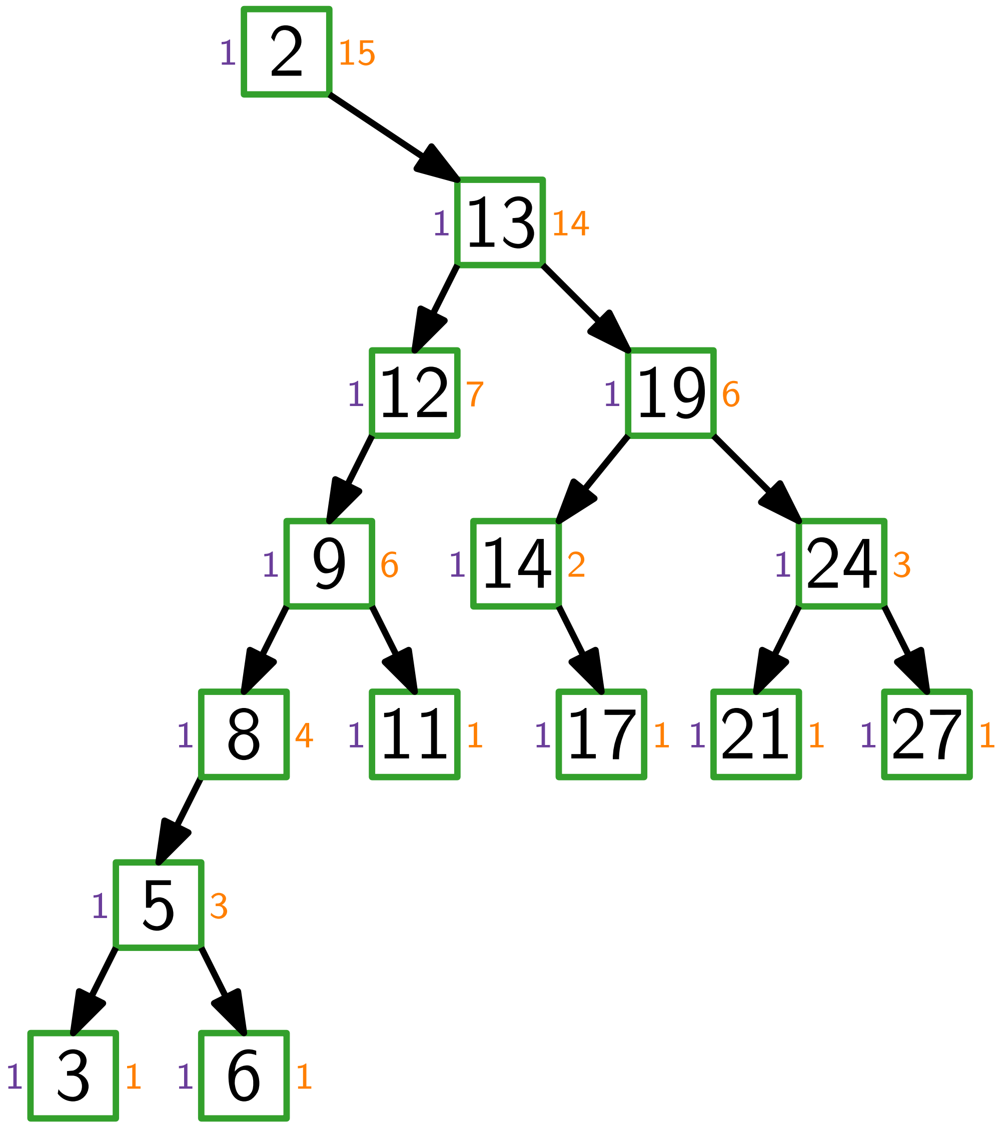

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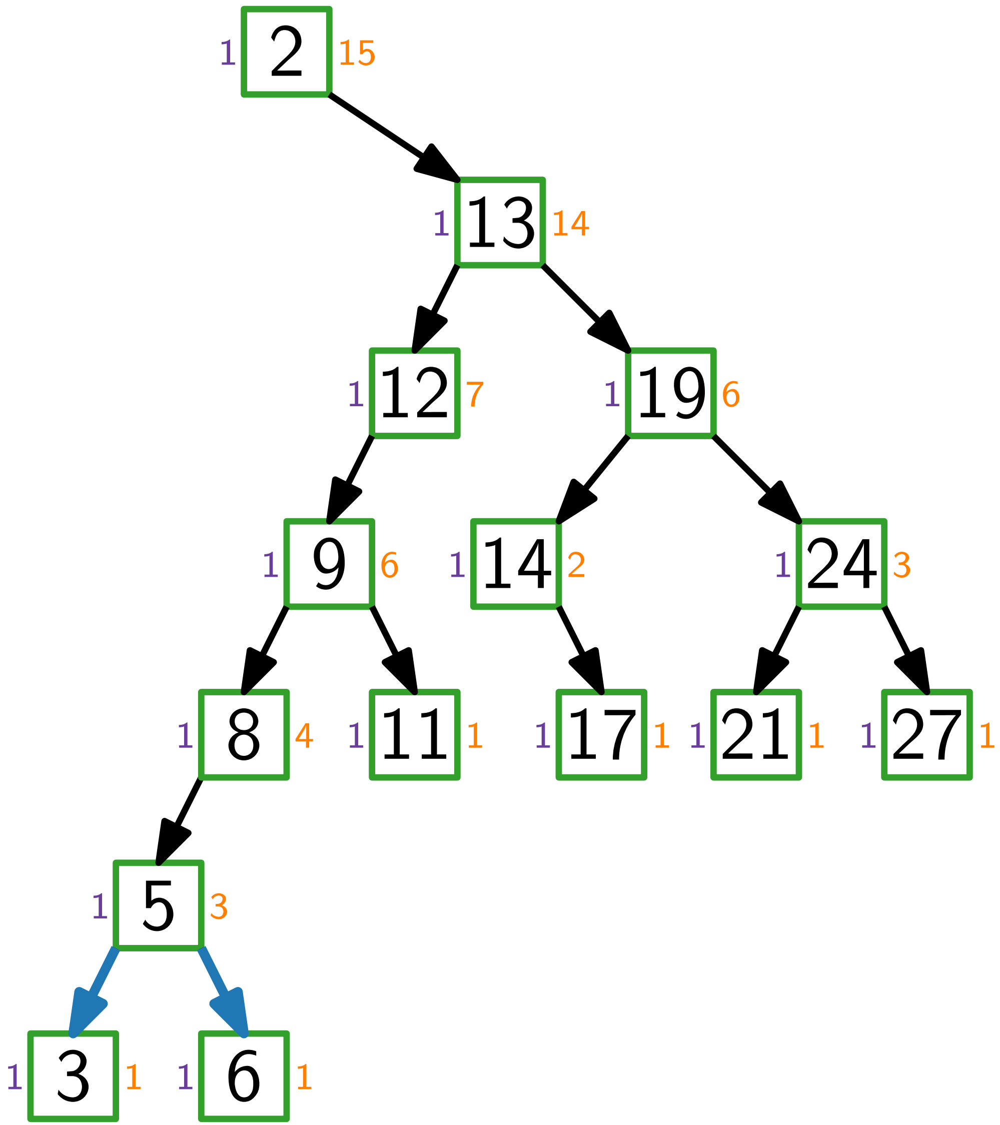

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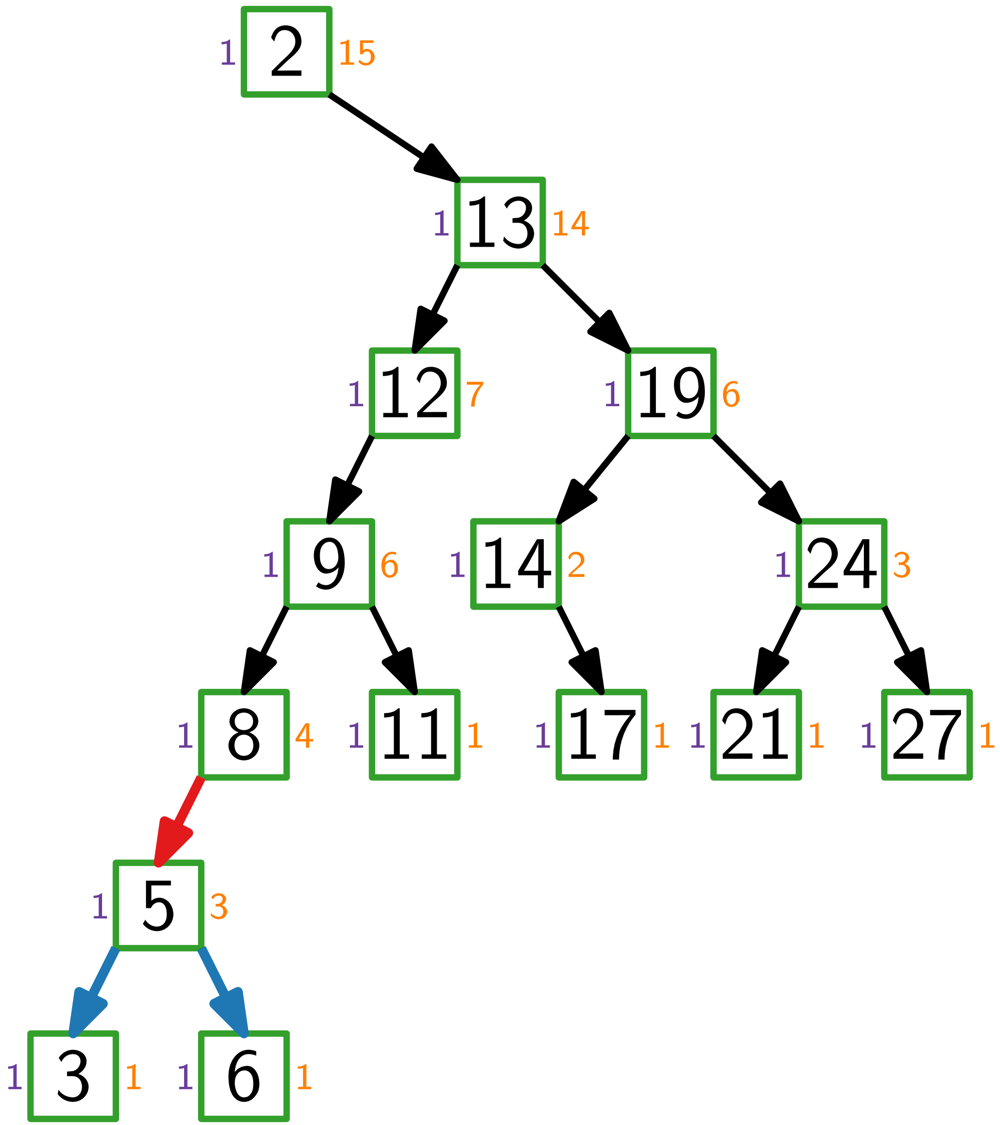

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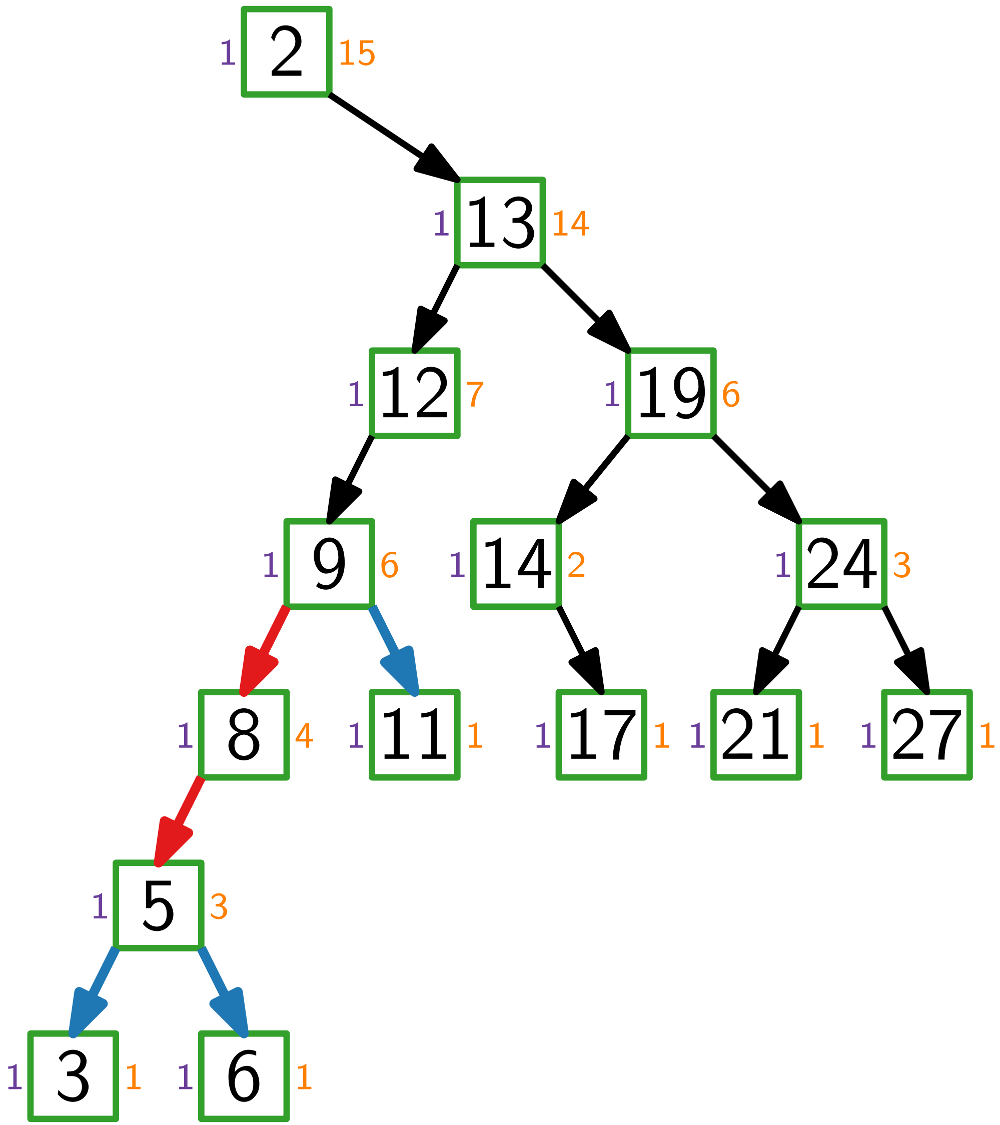

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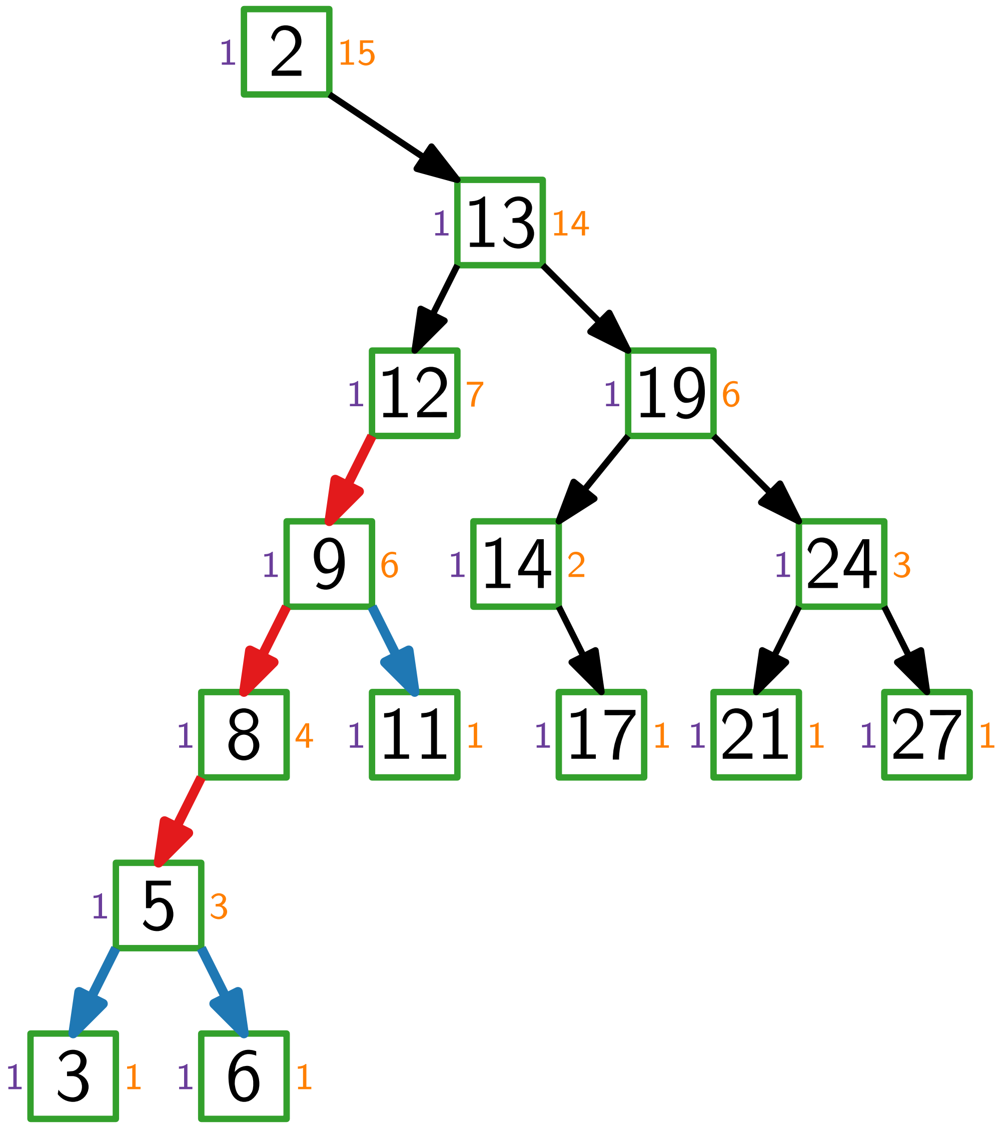

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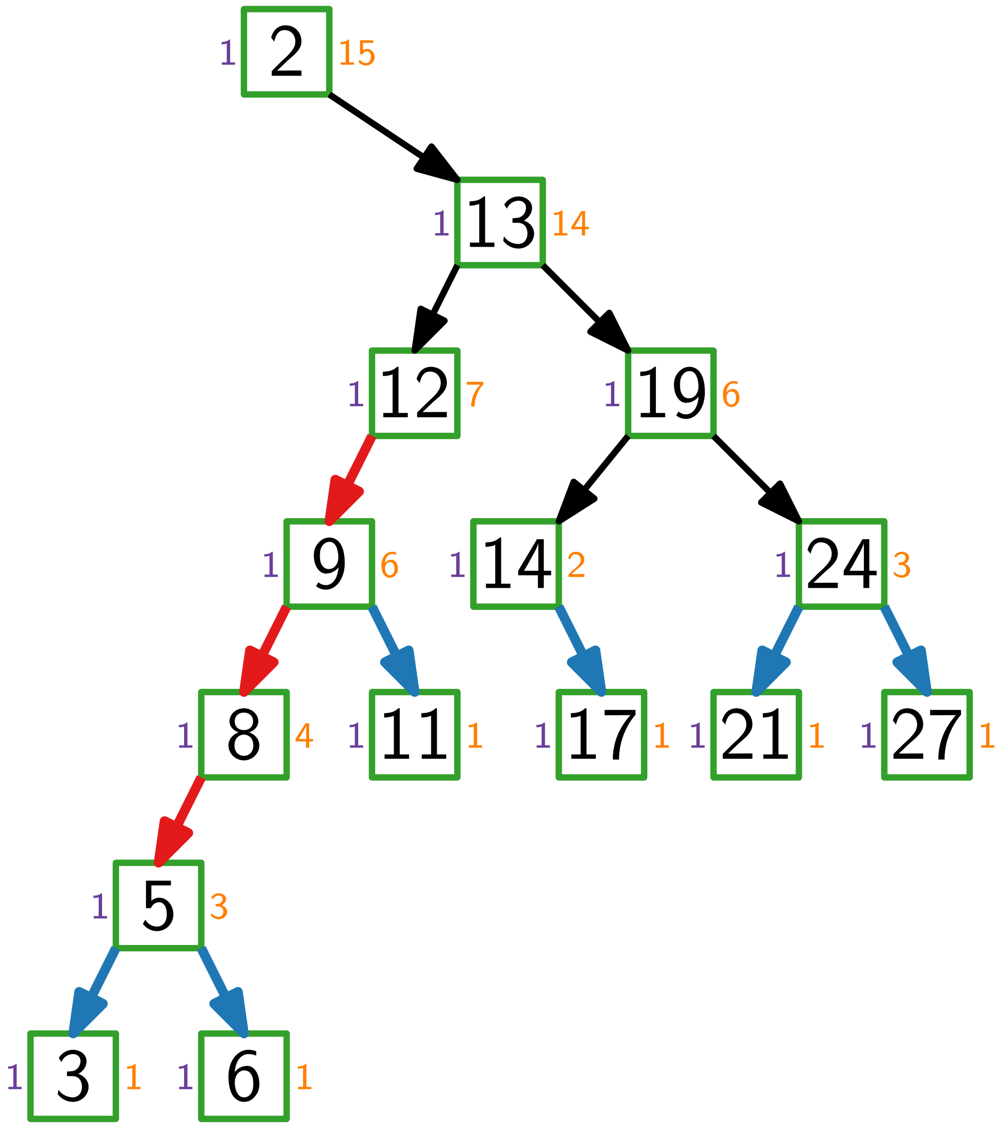

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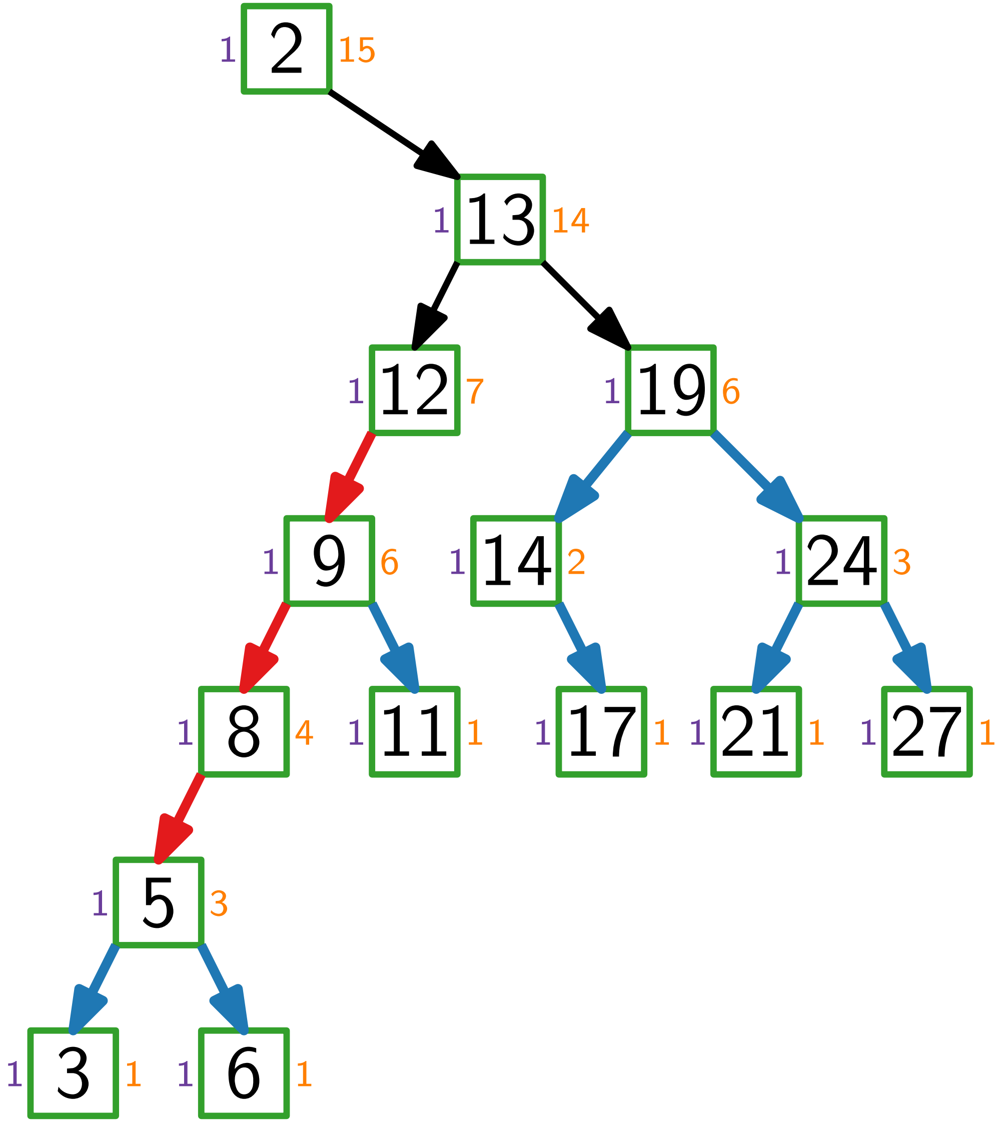

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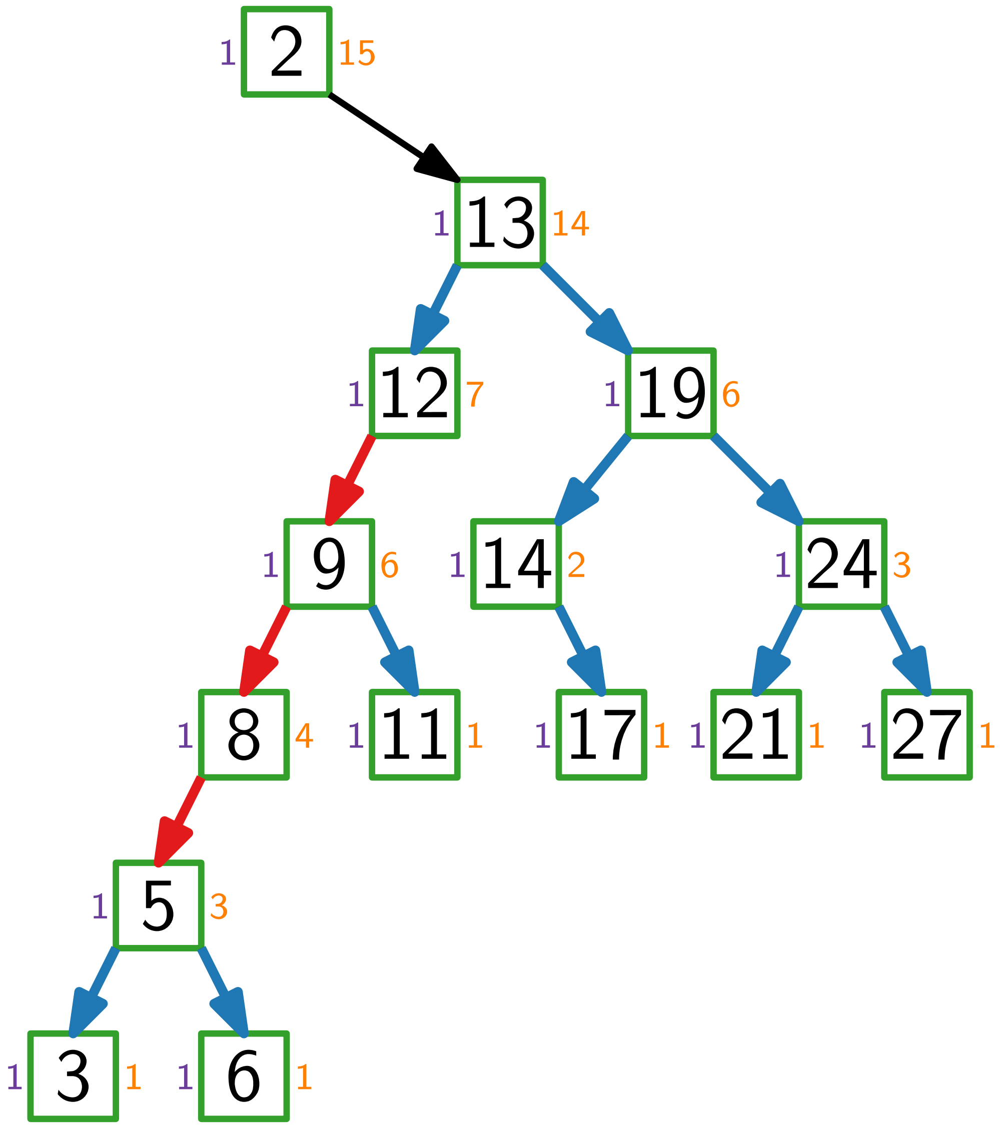

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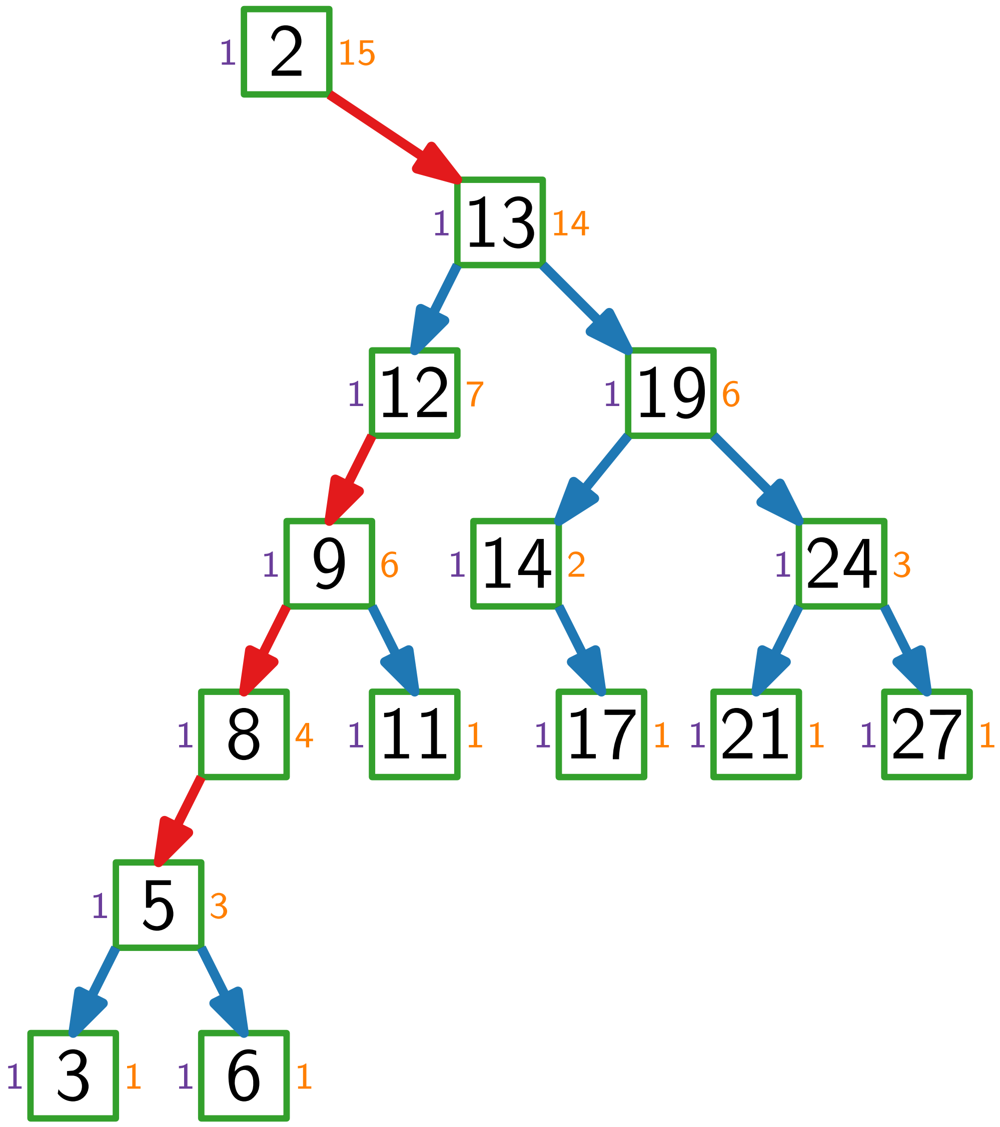

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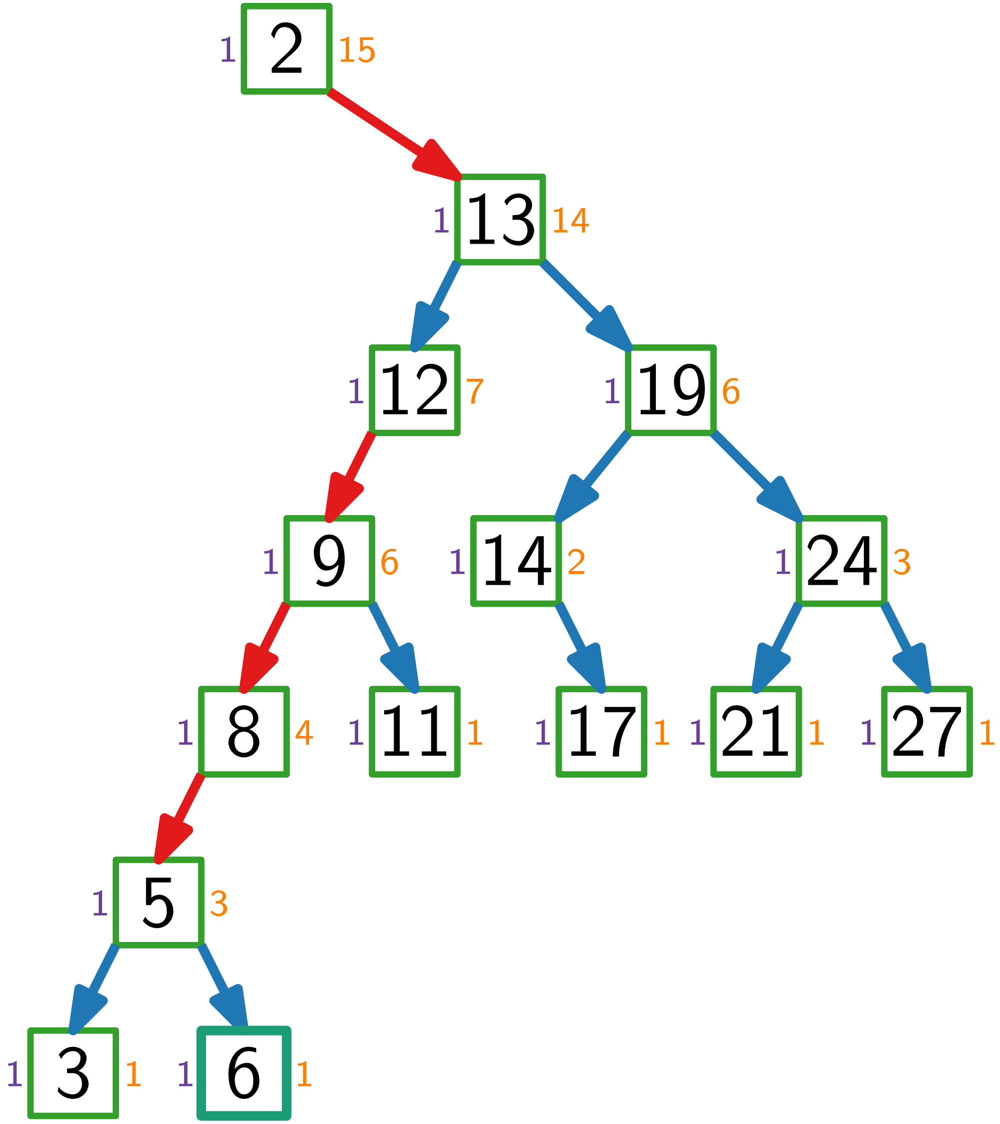

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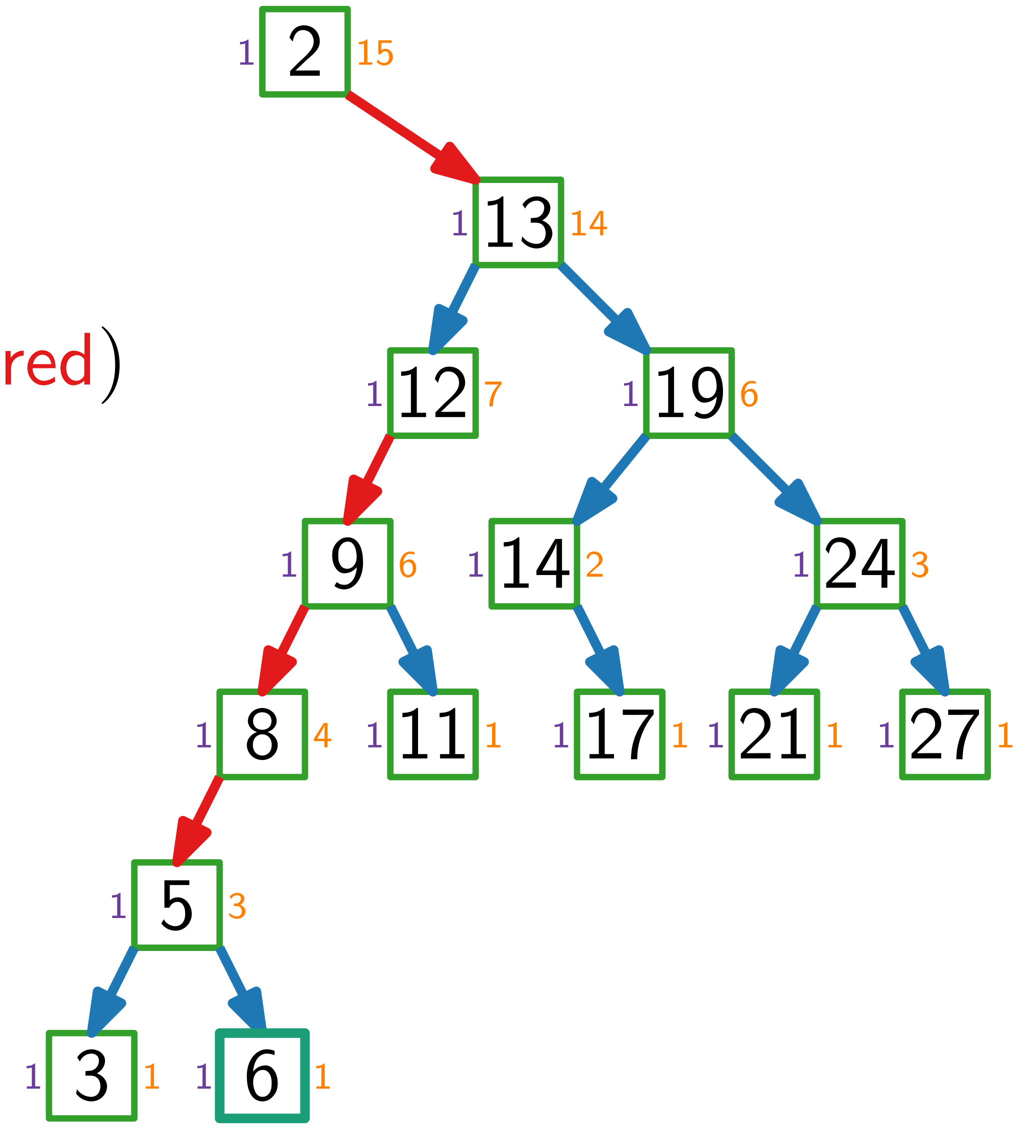
Cost to query x: $O(\#\text{blue} + \#\text{red})$



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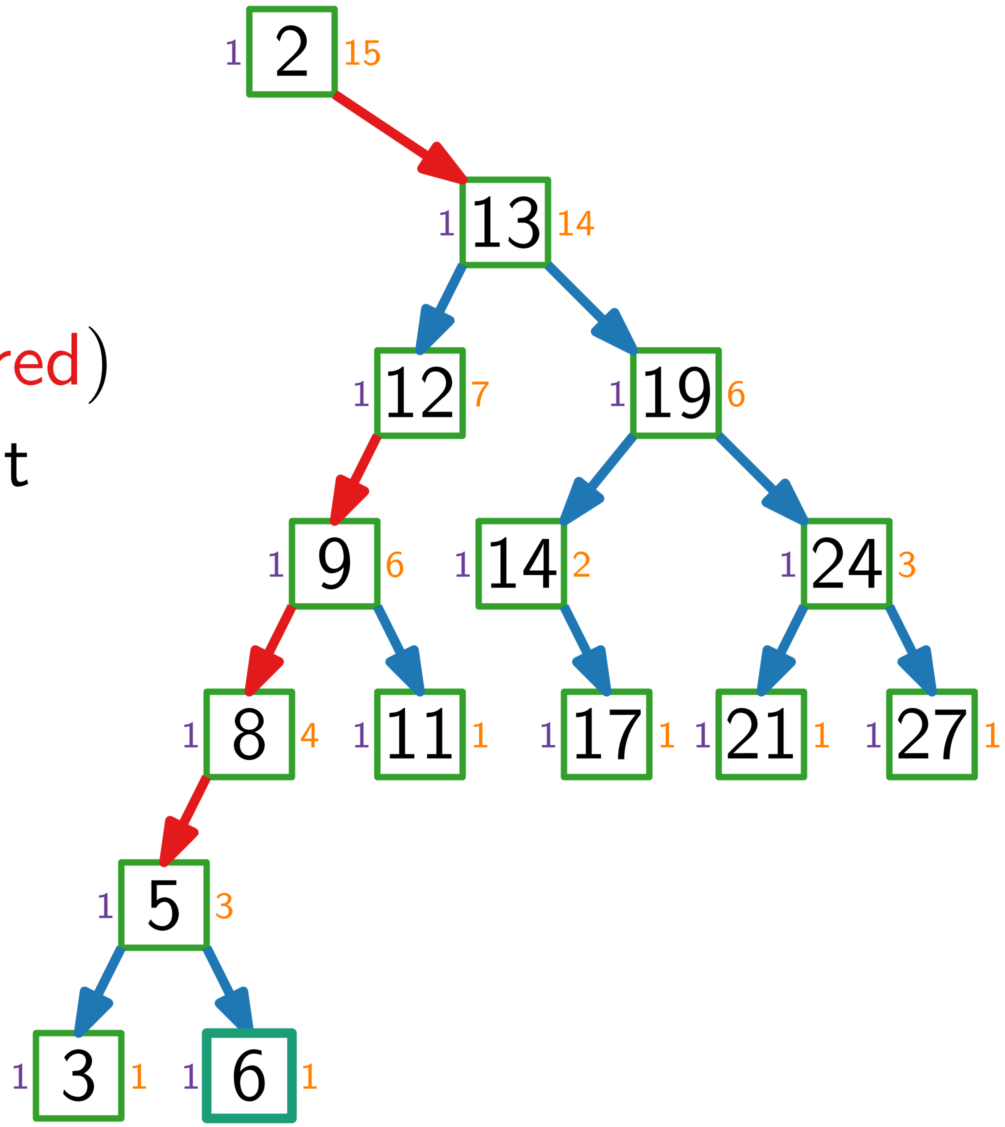

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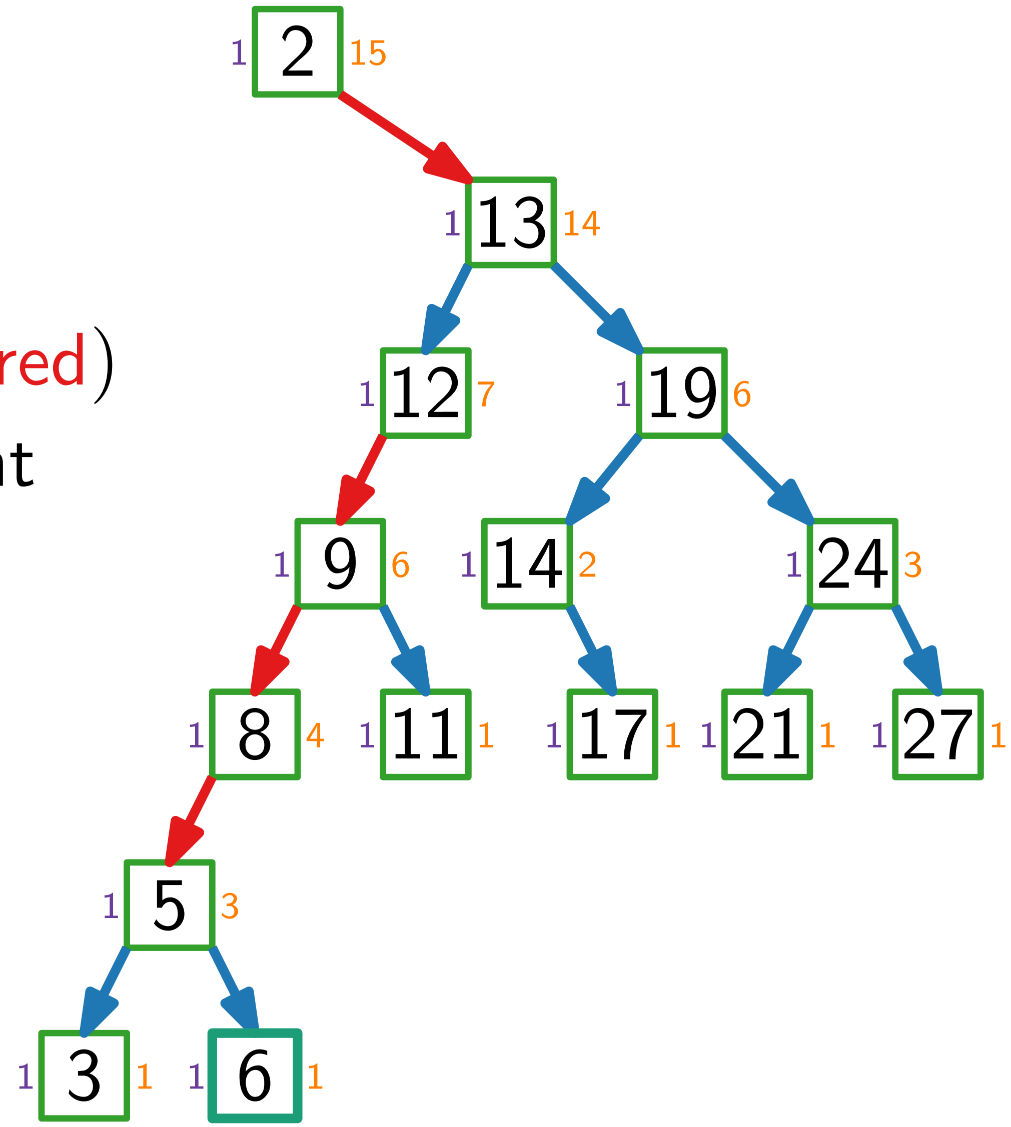

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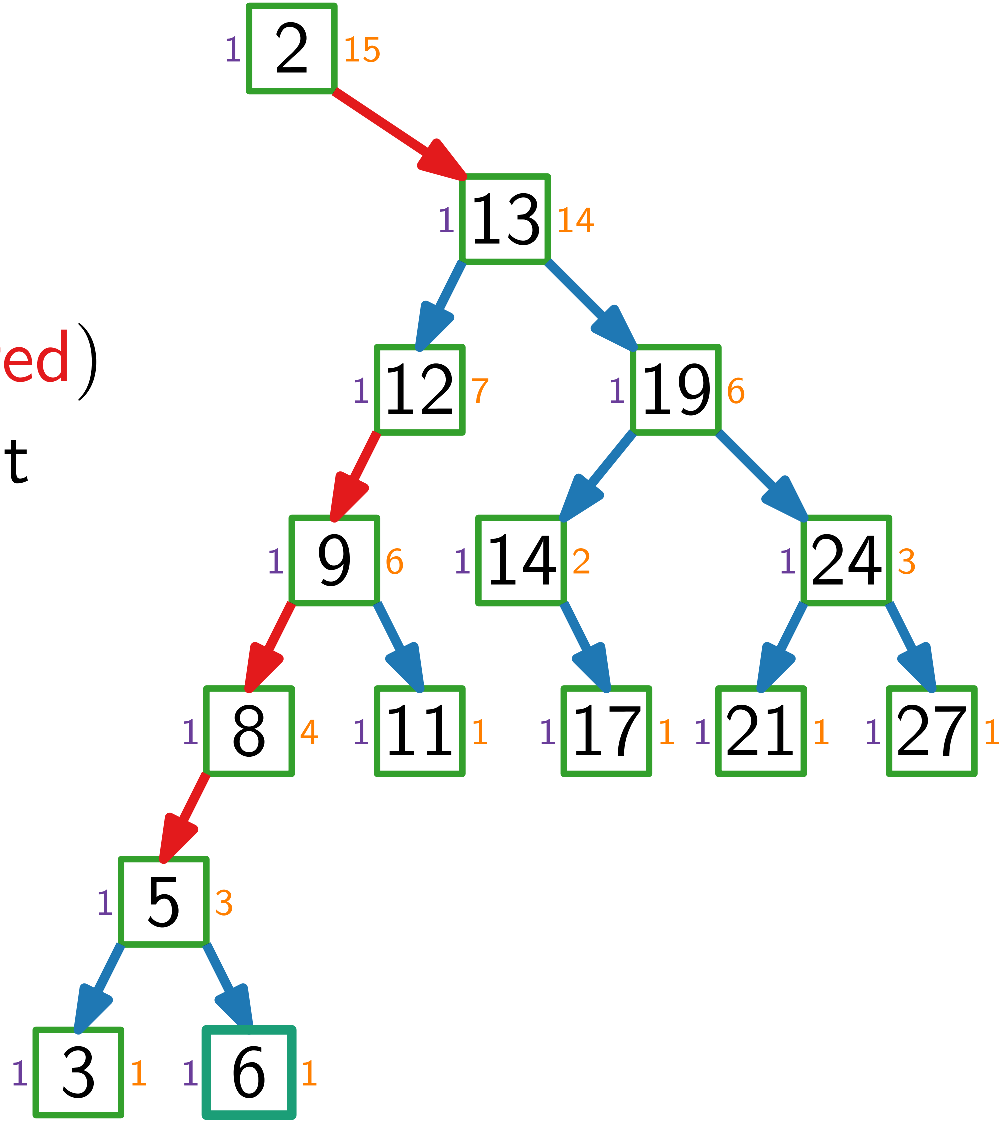

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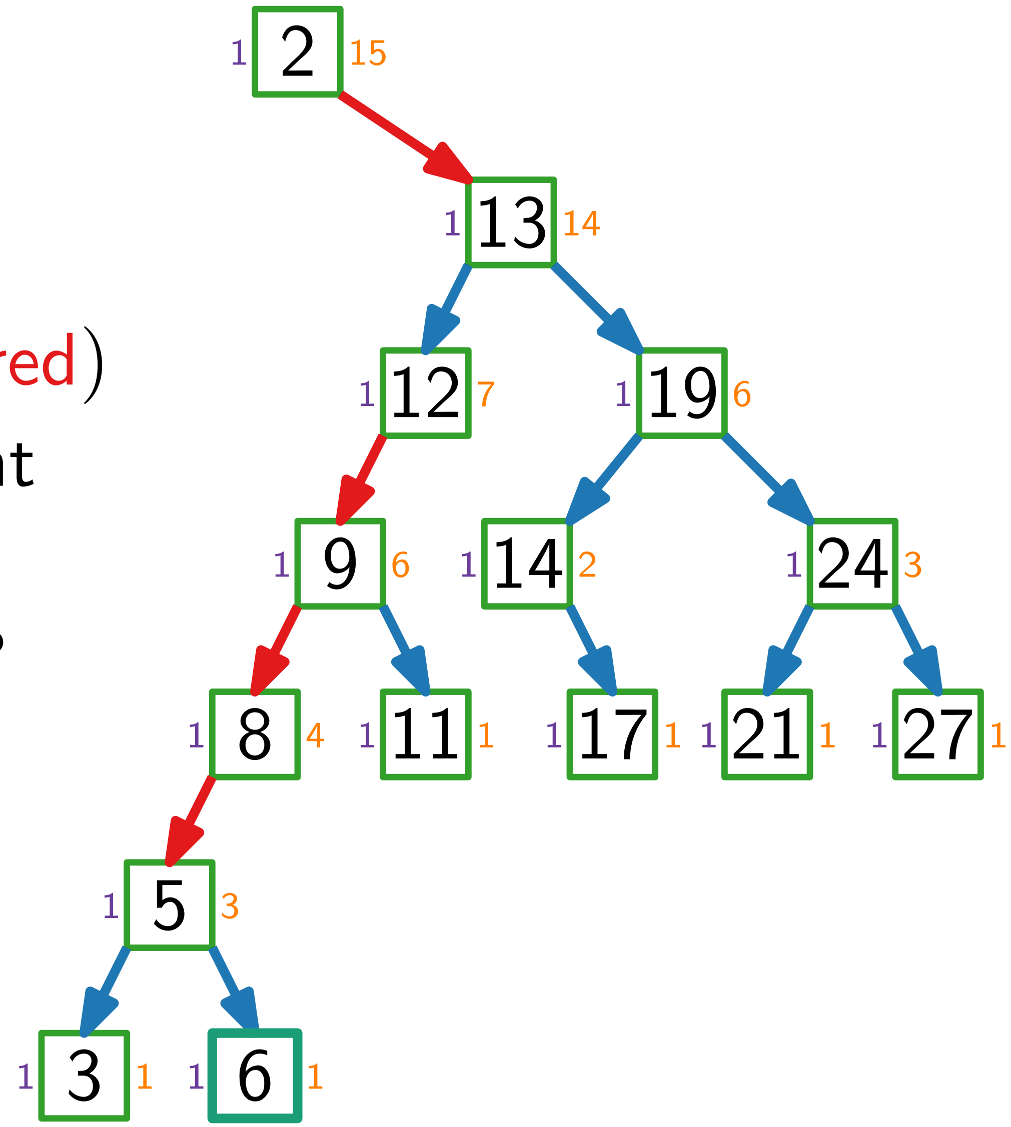

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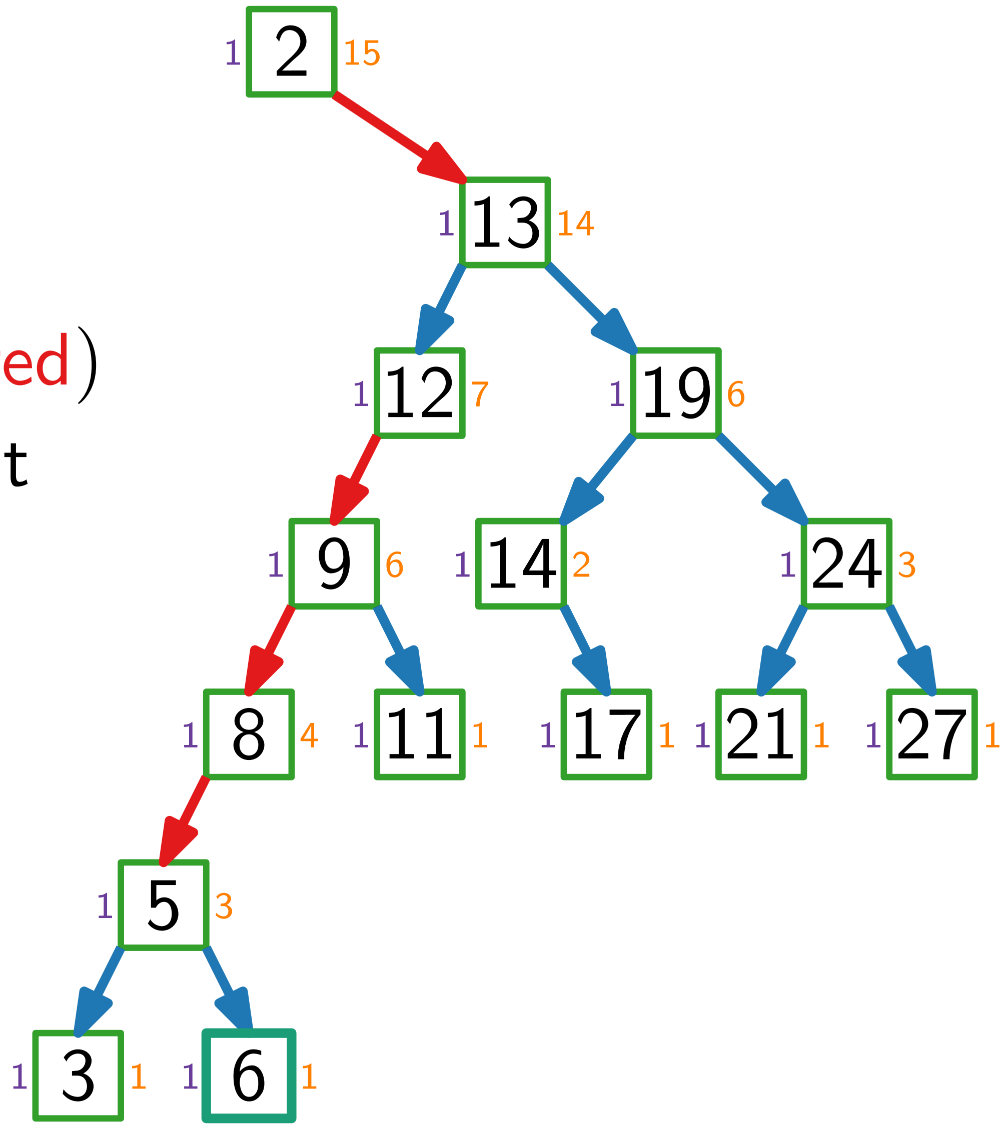

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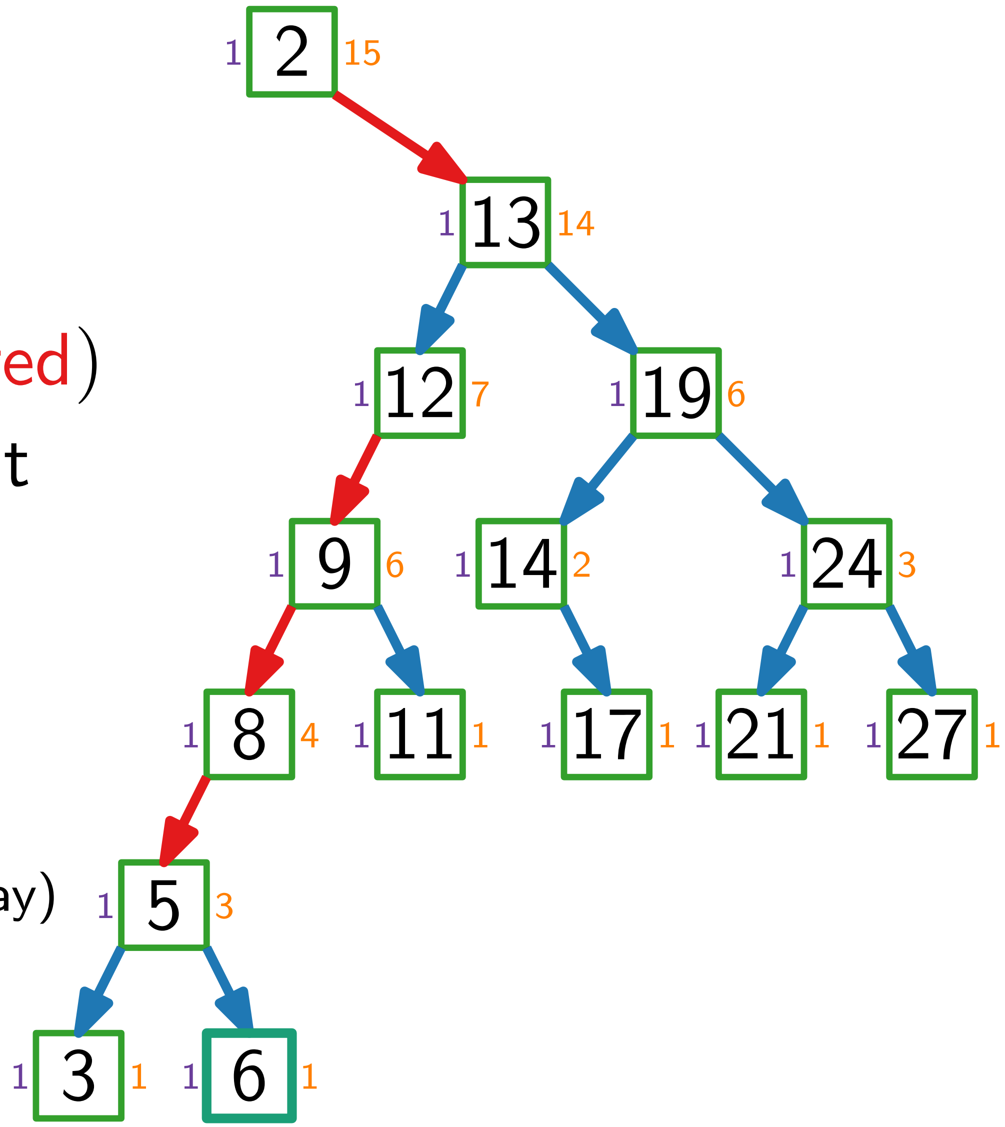
Amortized cost:

(potential before splay)

real cost $+ \Phi_{+} - \Phi$

(potential after splay)



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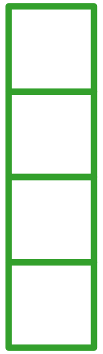

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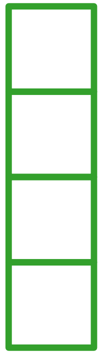

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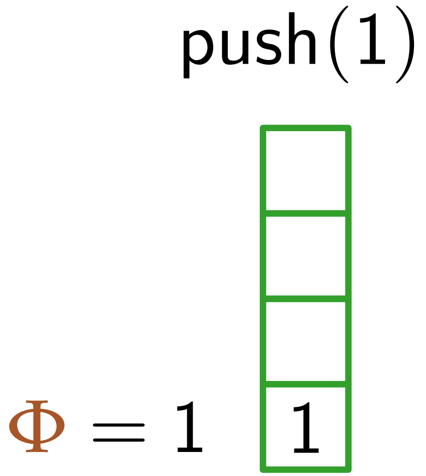

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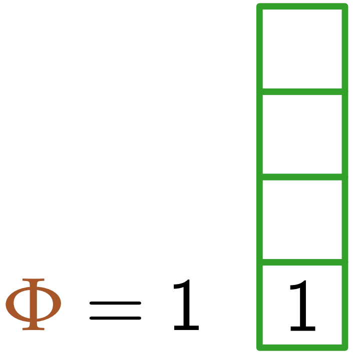
push(2)



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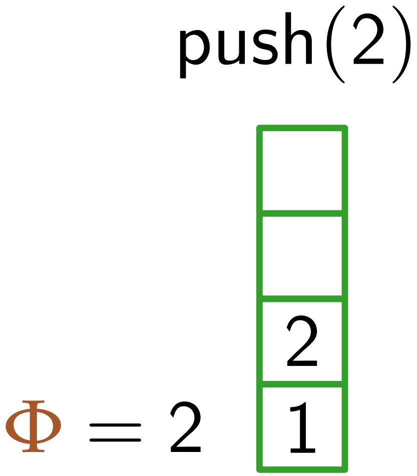

push(2)

$\Phi = 2$

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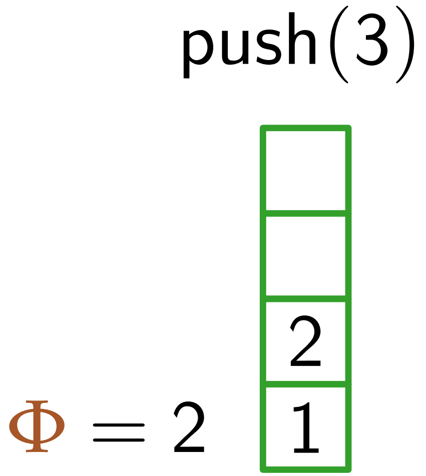

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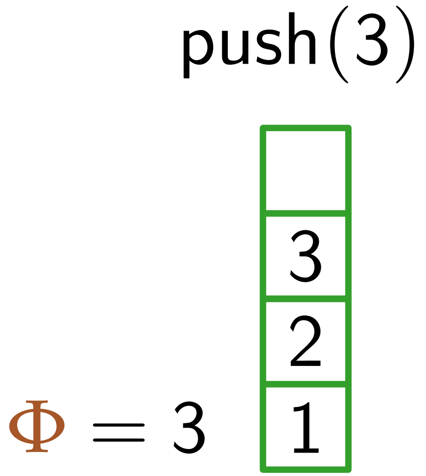

push(3)

$\Phi = 3$

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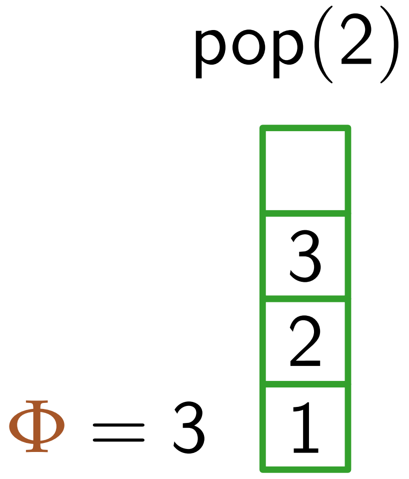

pop(2)

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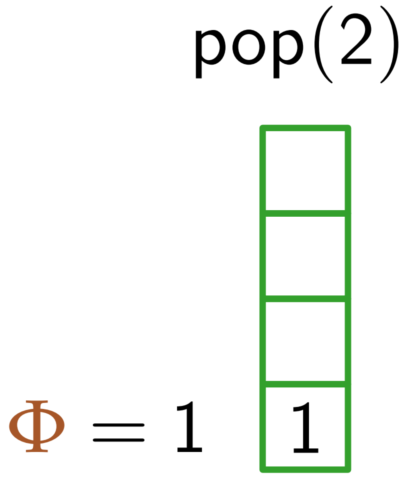

pop(2)

$\Phi = 1$

1

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Example (from ADS): Stack with multipop

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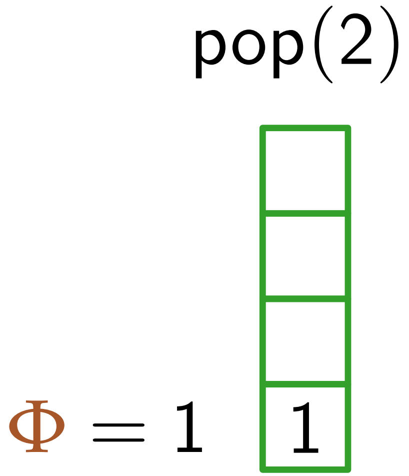
push:

pop(k):



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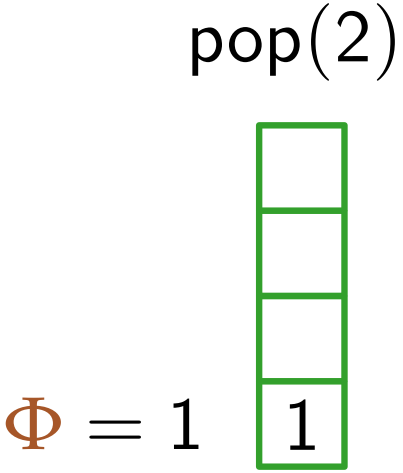

pop(2)

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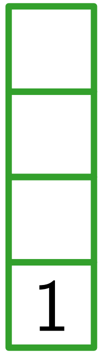

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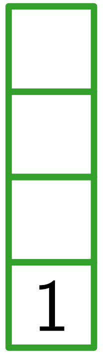

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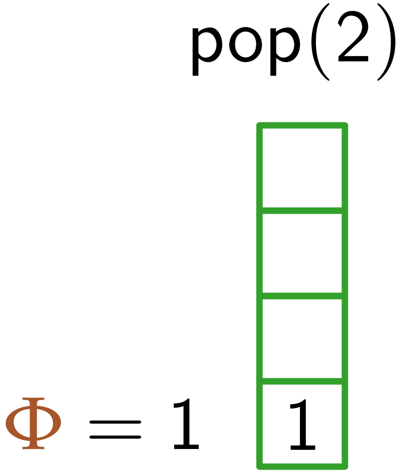

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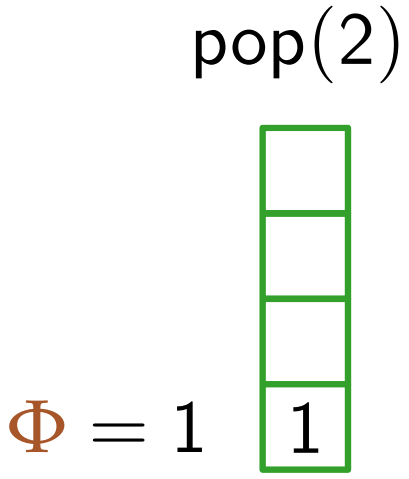

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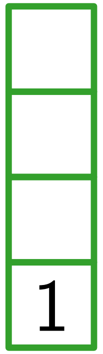

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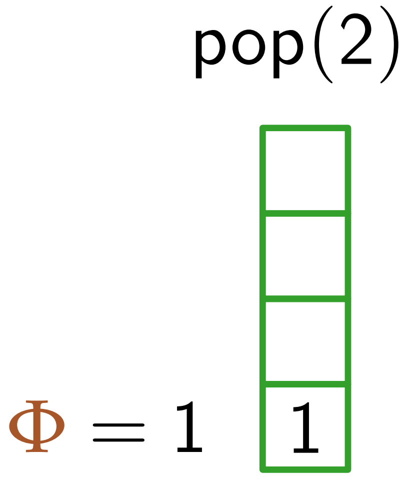

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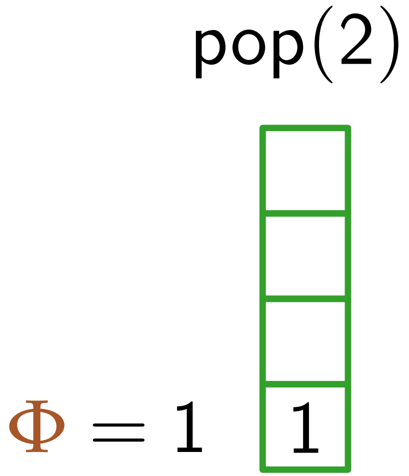

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mark edges:

→ $s(\text{child}) \leq s(\text{parent})/2$

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Cost to query x_i: $O(\log W + \#\text{red})$

Idea: blue edges halve the weight

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How can we amortize red edges?

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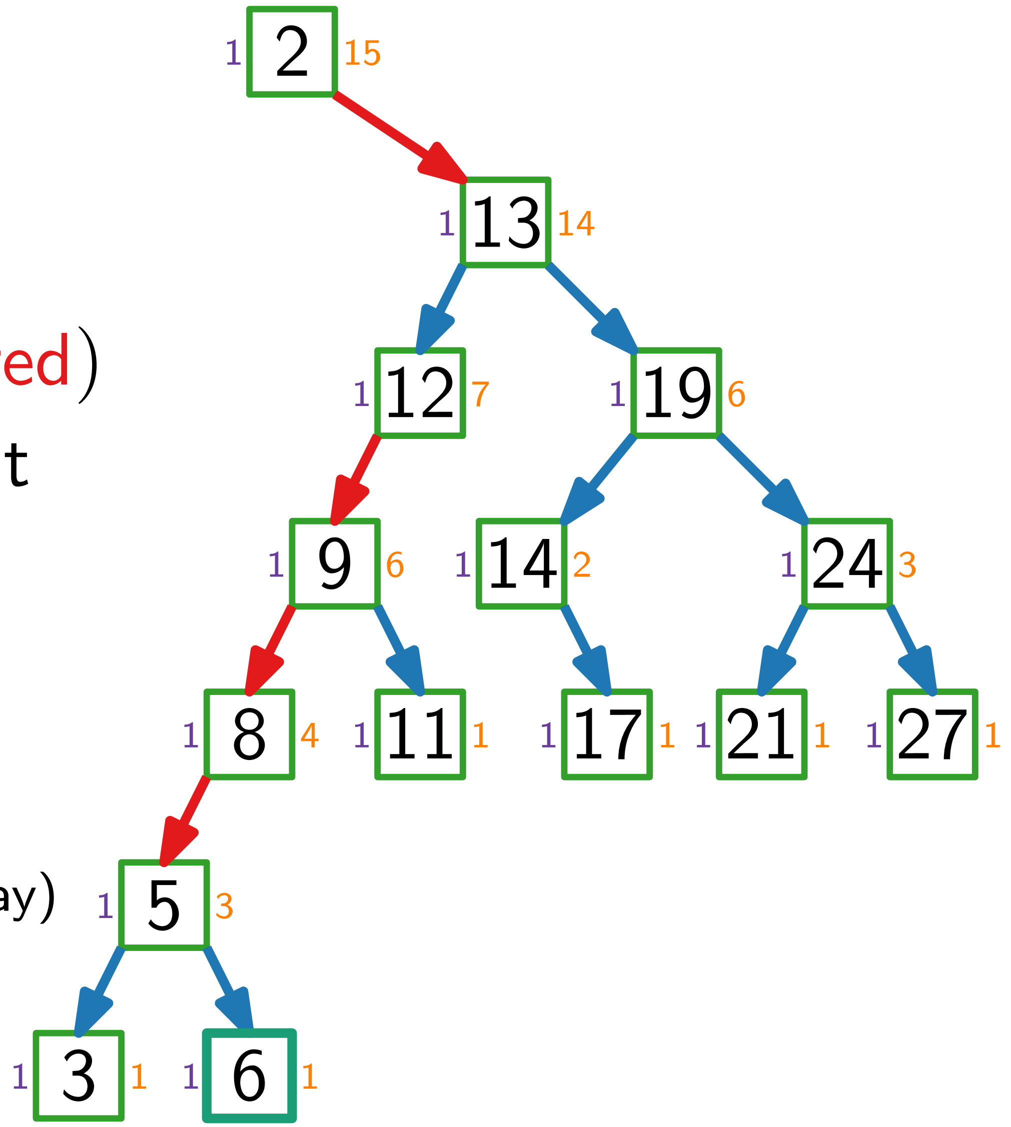

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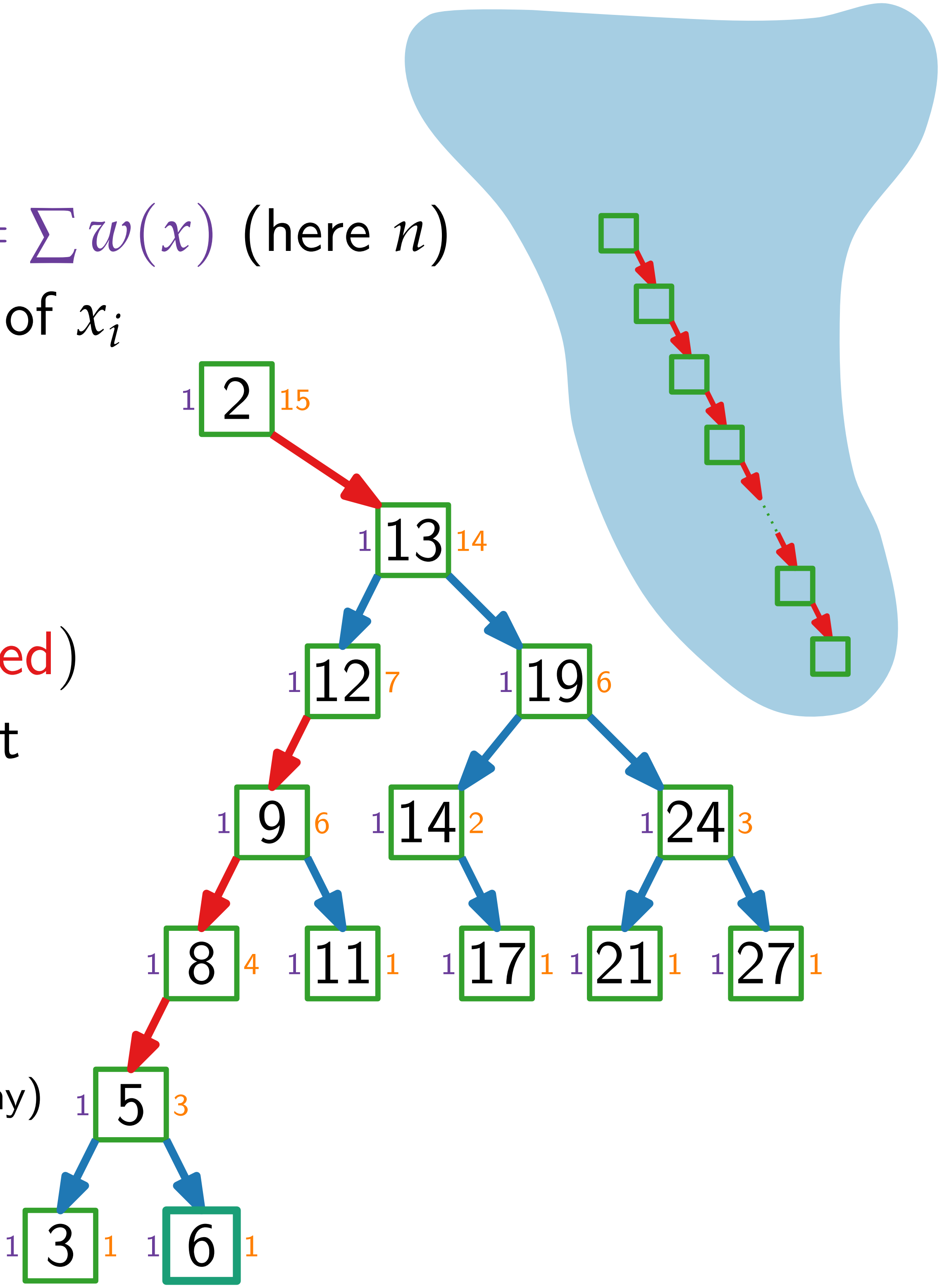

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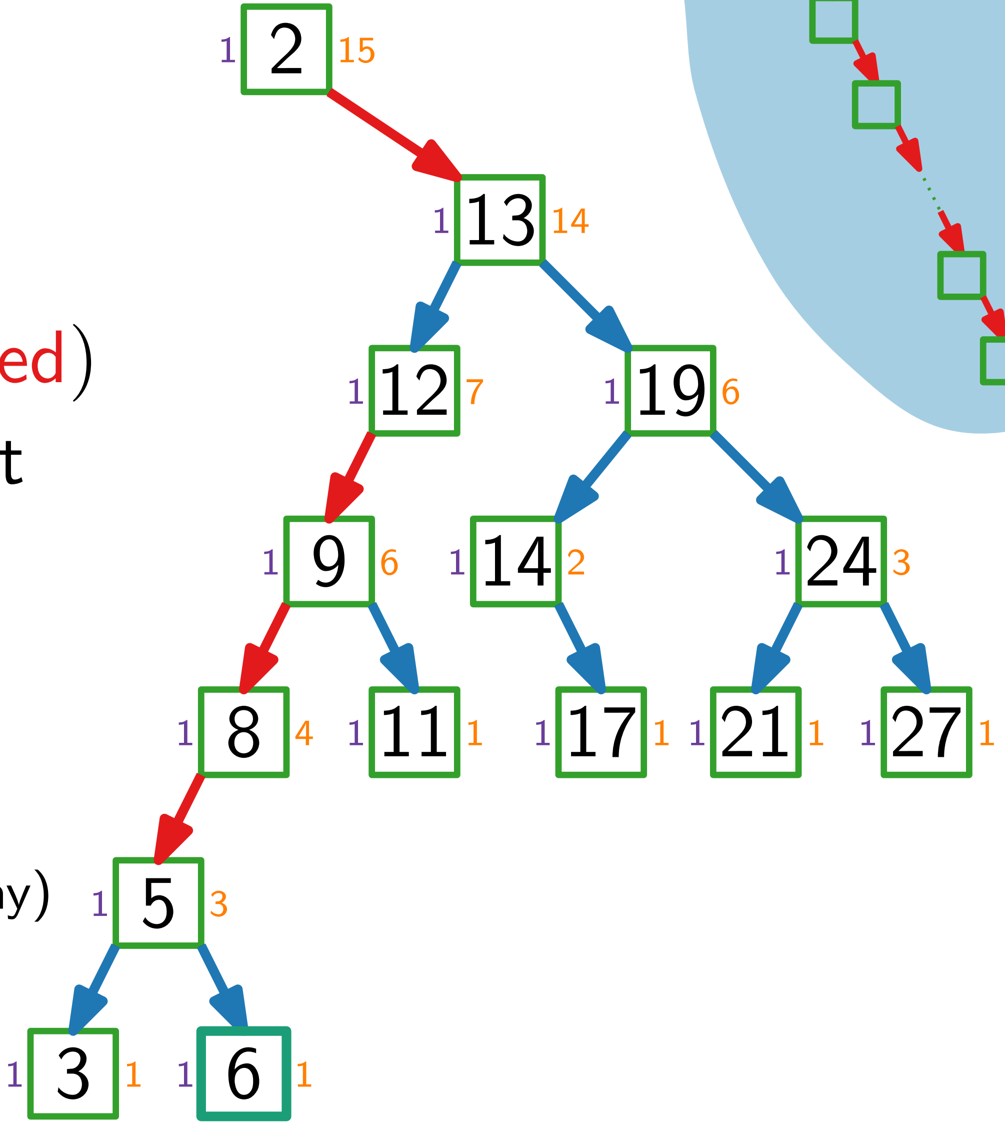

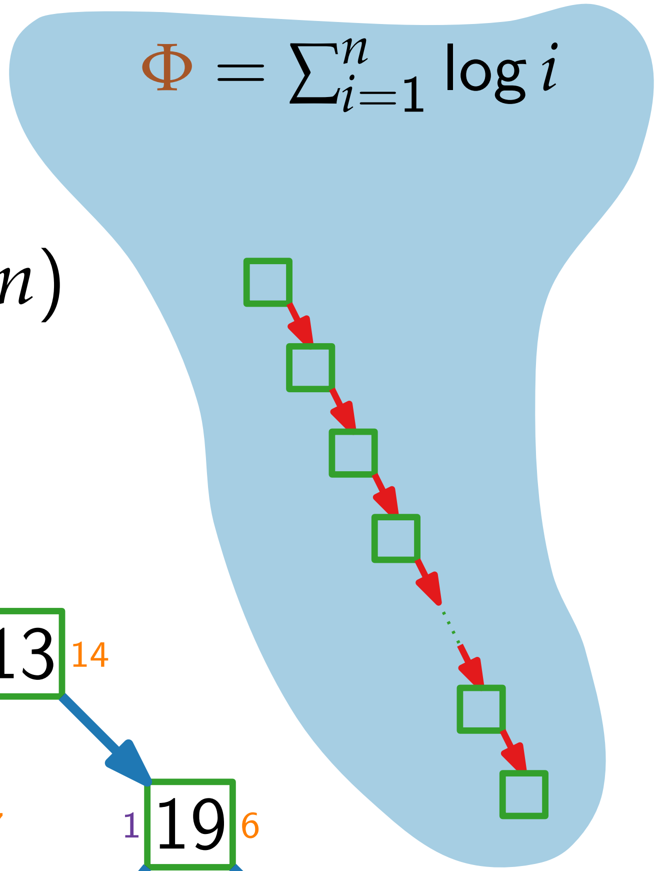

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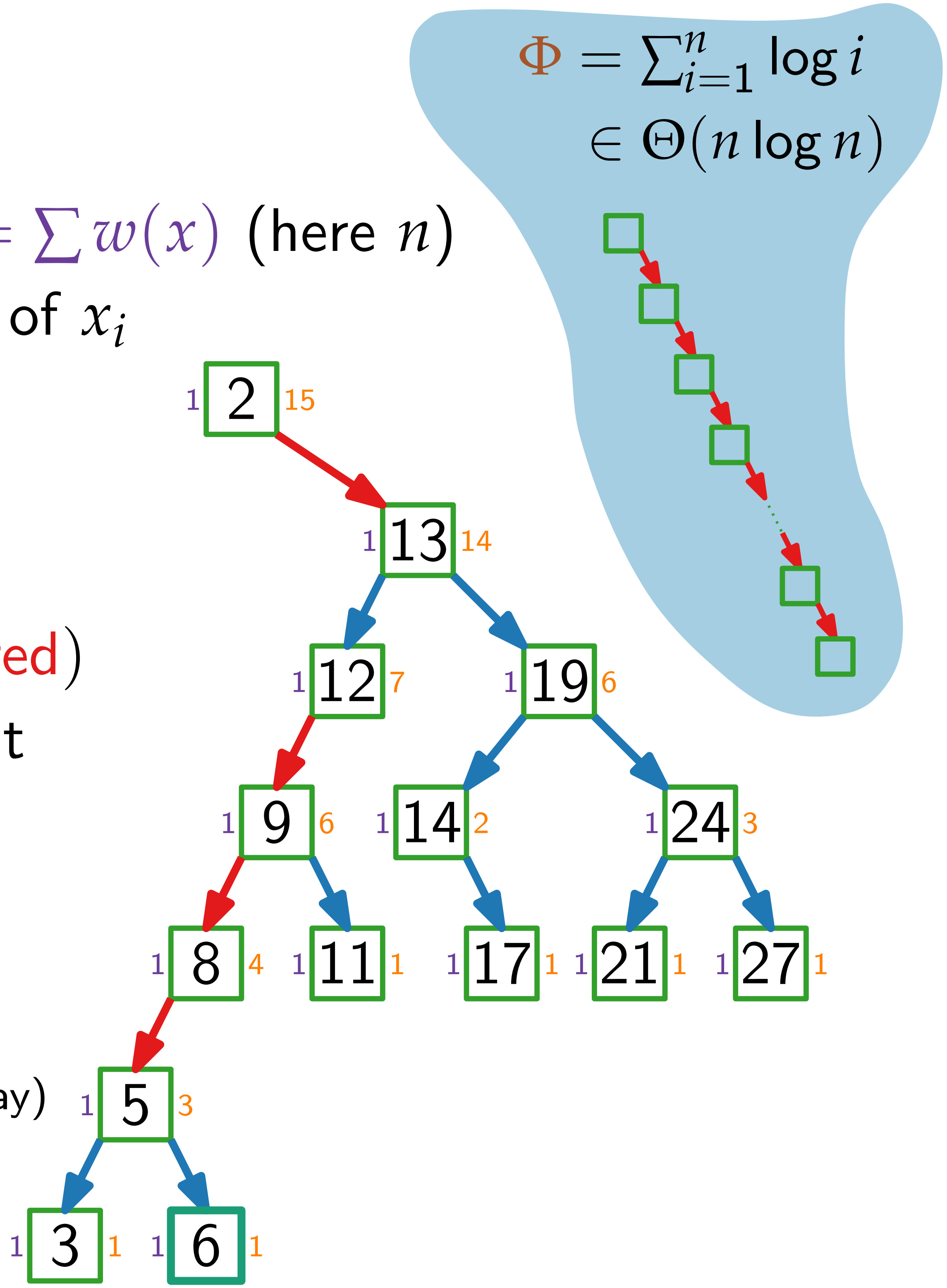

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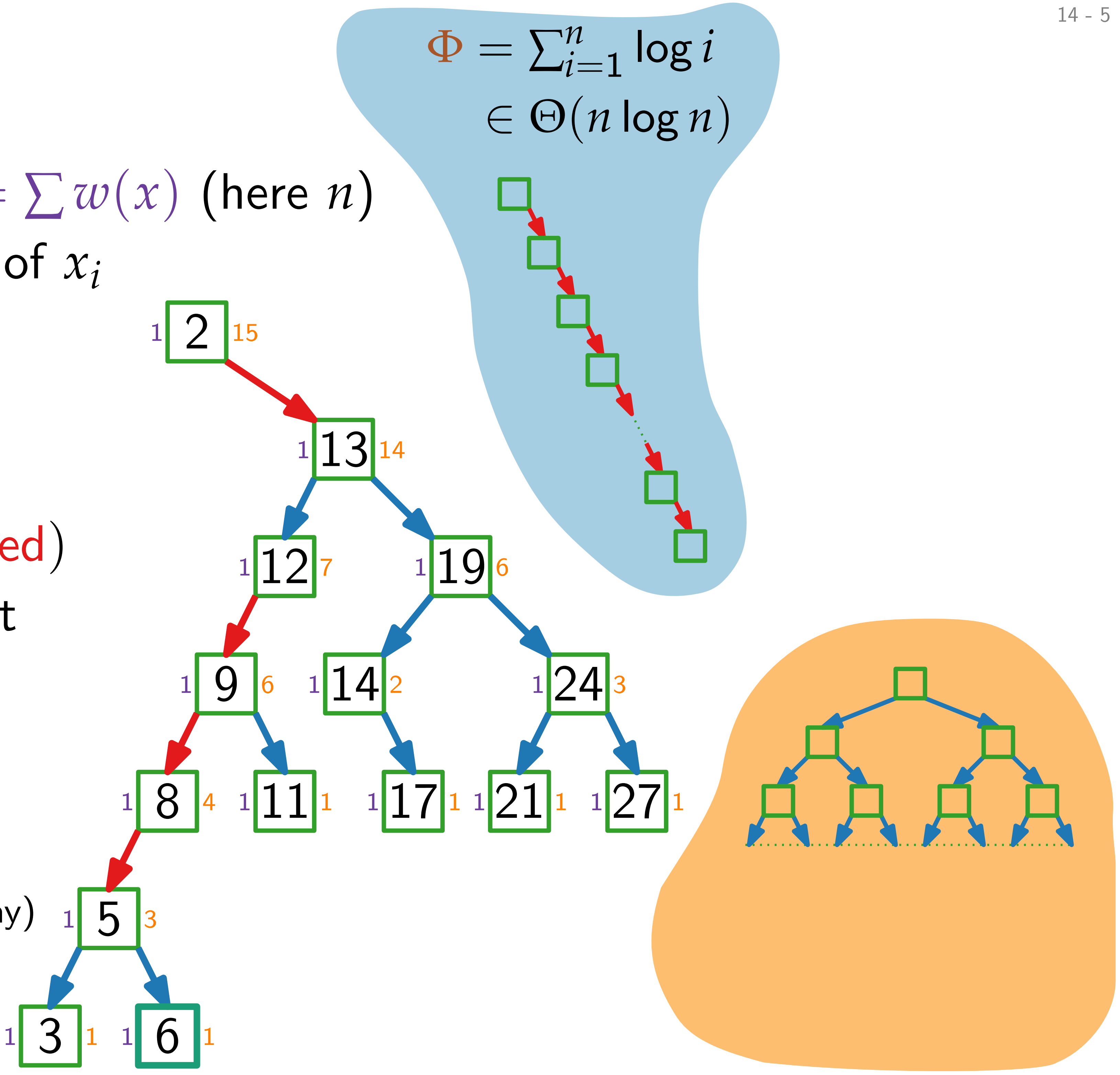

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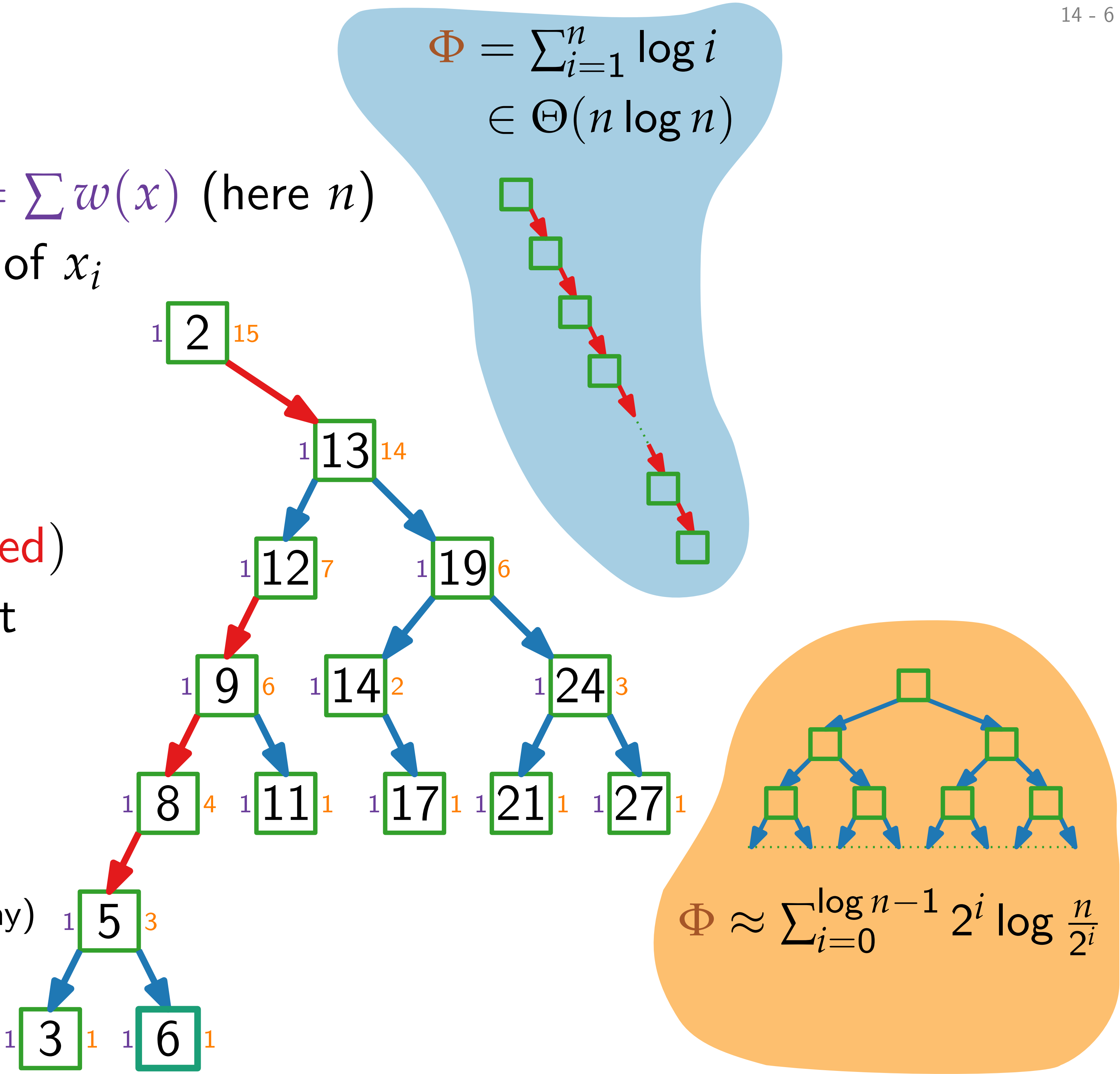

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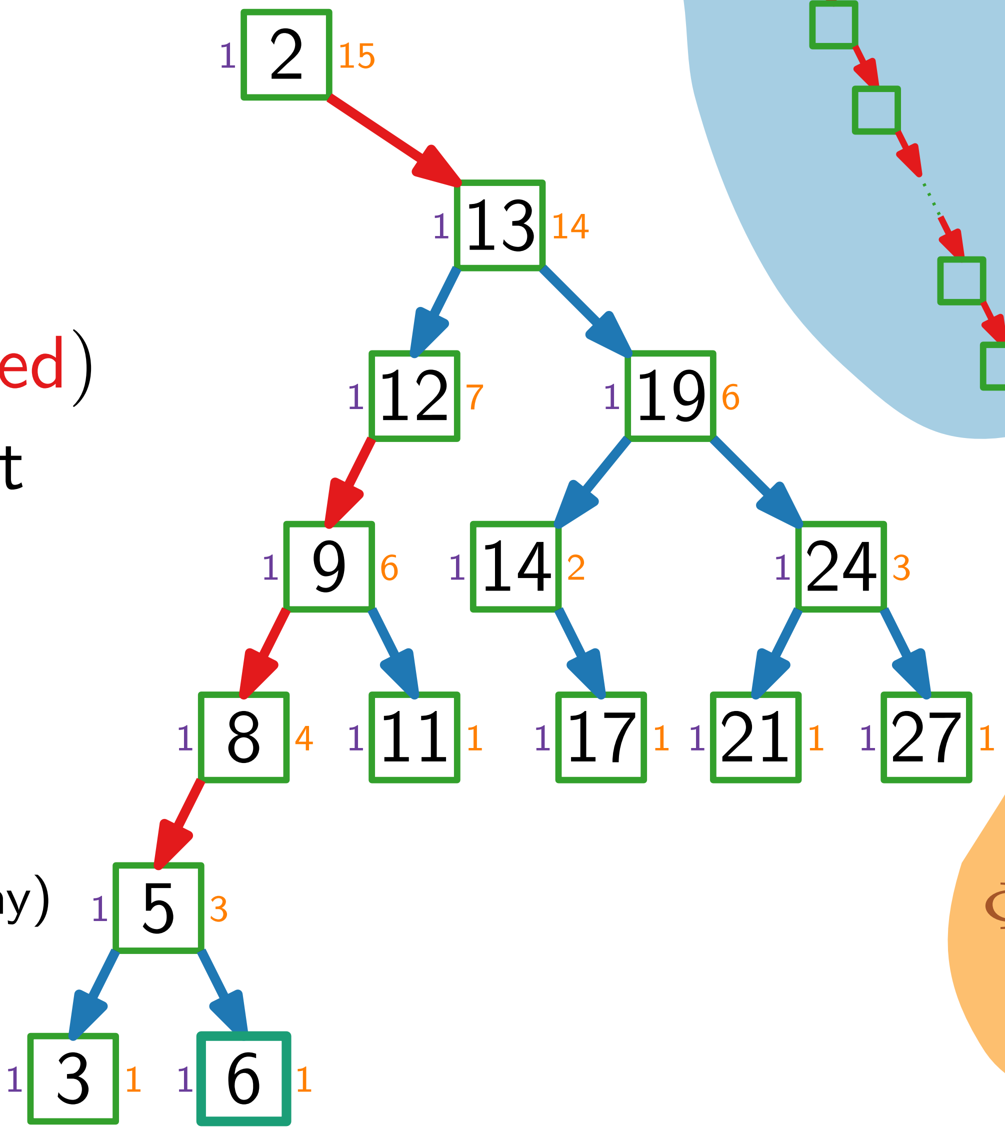

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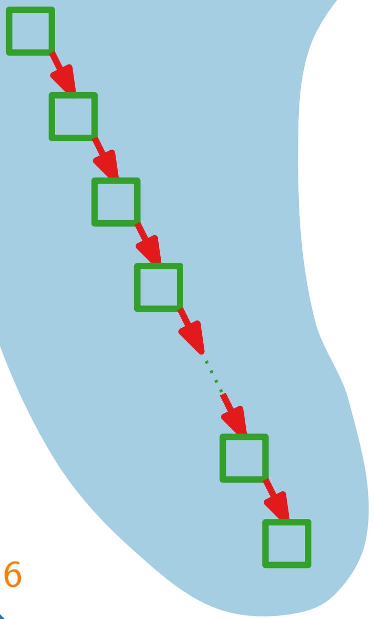

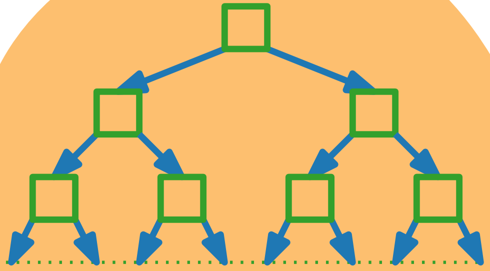

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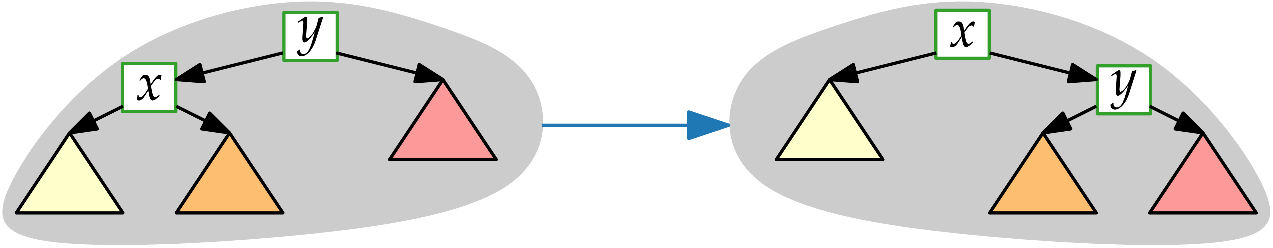

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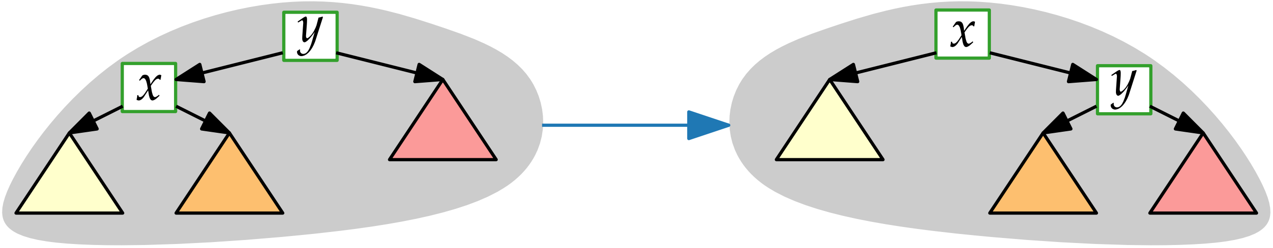

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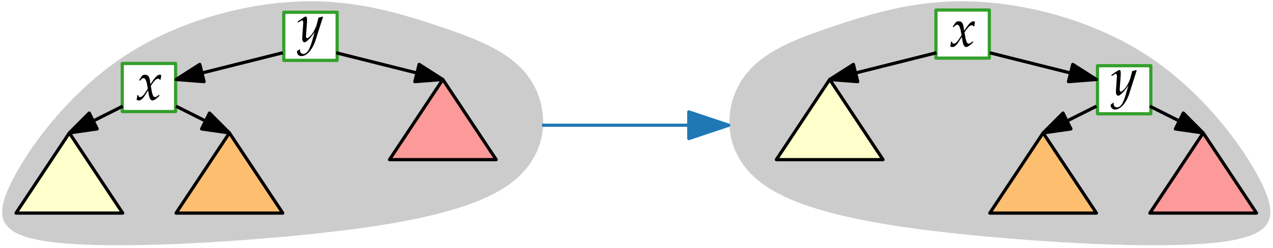

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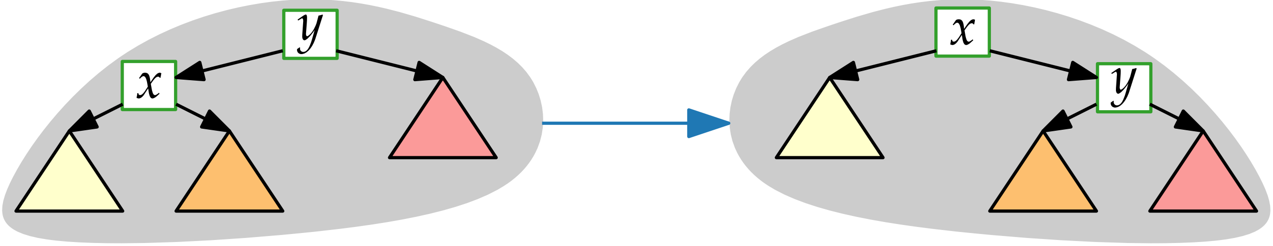

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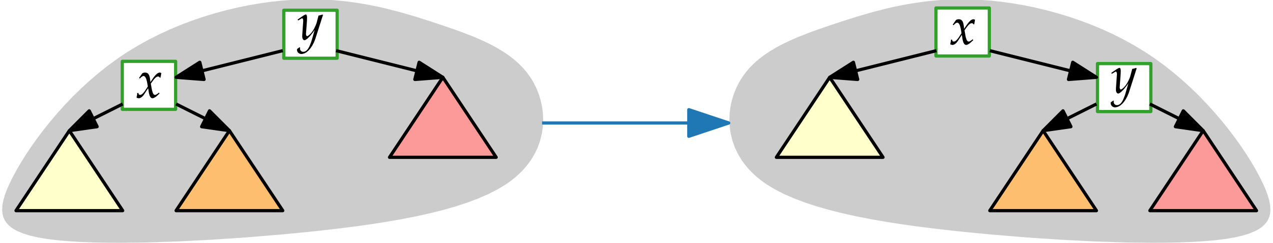

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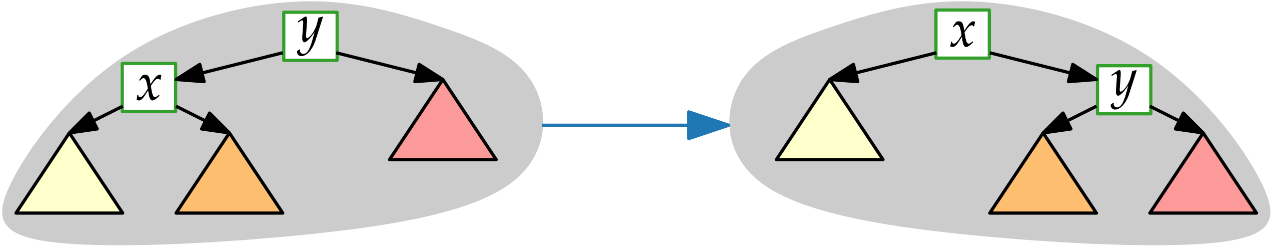

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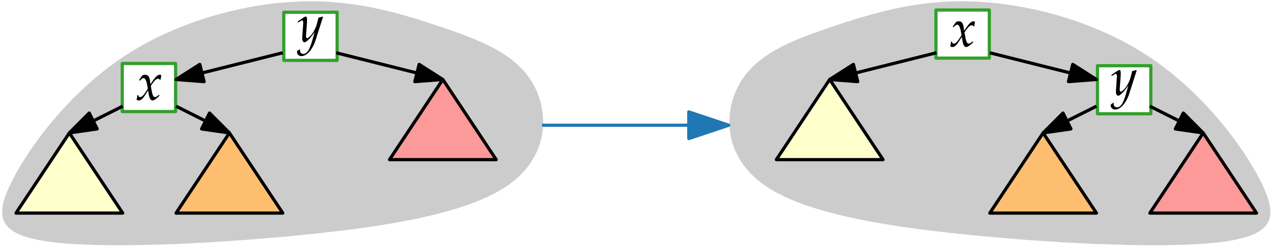

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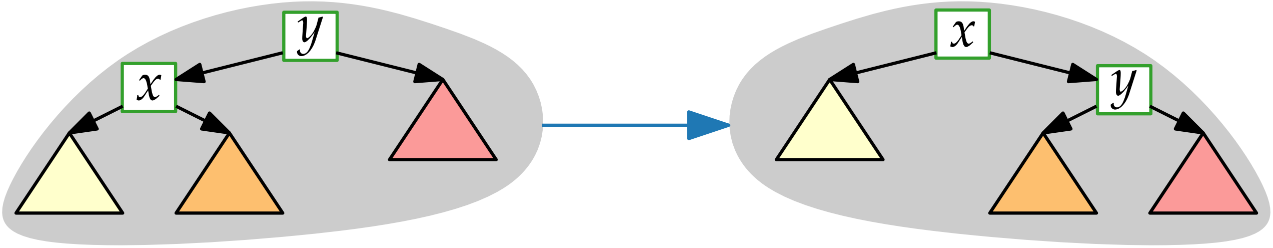

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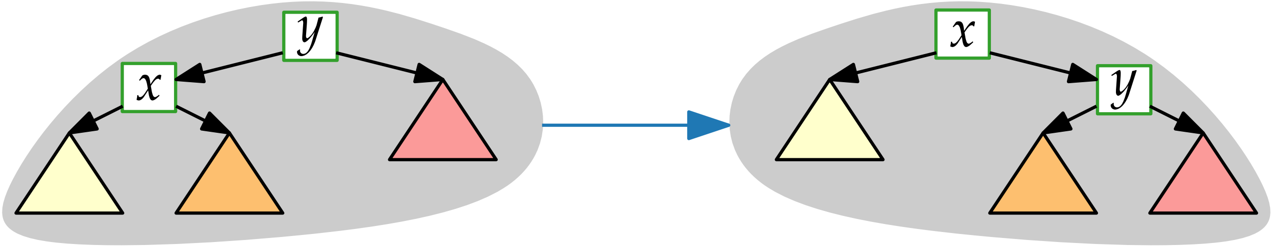

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Left(x) analogue ✓

□

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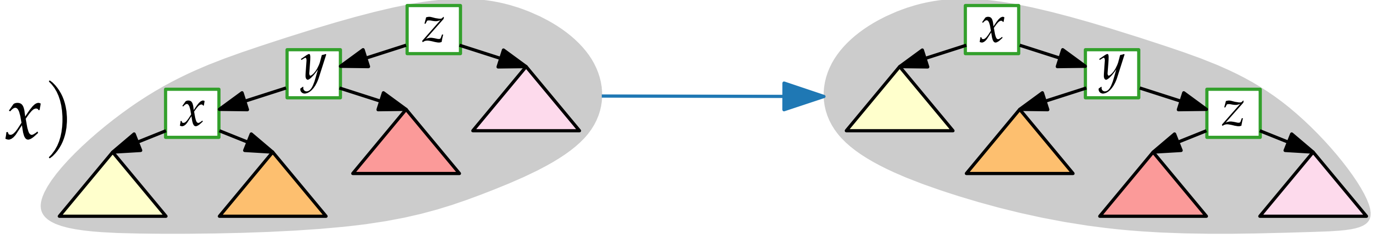

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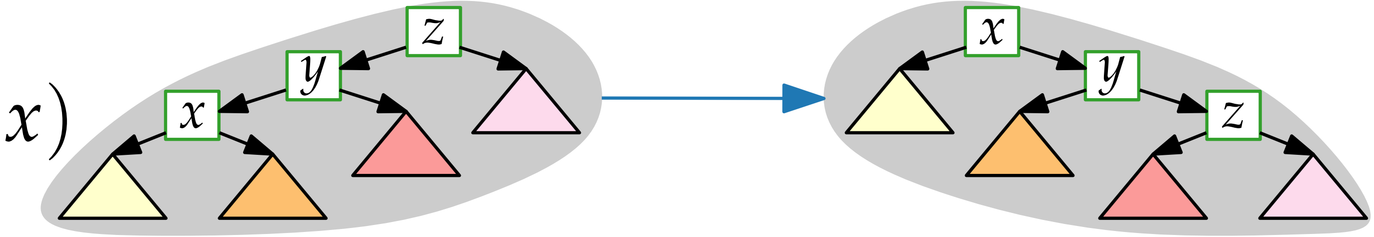

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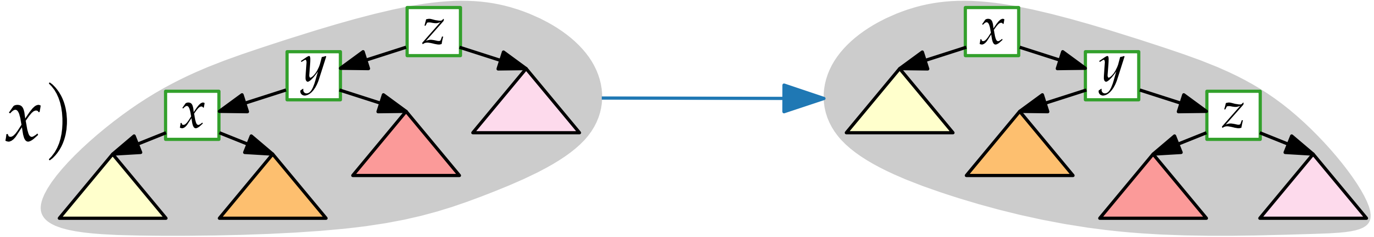

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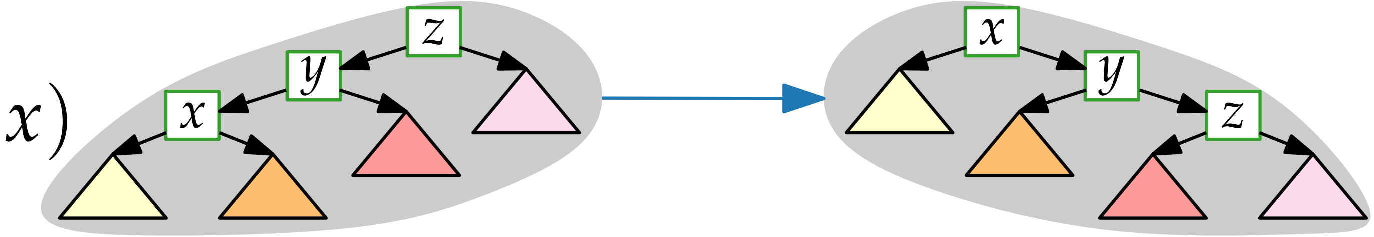
Potential after Rotation

Consider any rotation; $s(x)$ before rotation, $s_+(x)$ afterwards

Lemma. After a double rotation, the potential increases by $\leq 3\,(\log s_+(x) - \log s(x)) - 2$.

Proof.

Case 1. Right-Right(x)



pot. change $= \log s_+(x) + \log s_+(y) + \log s_+(z)$
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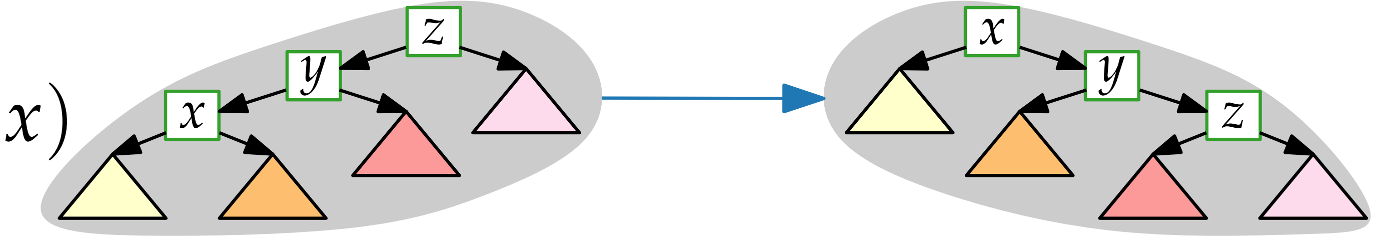

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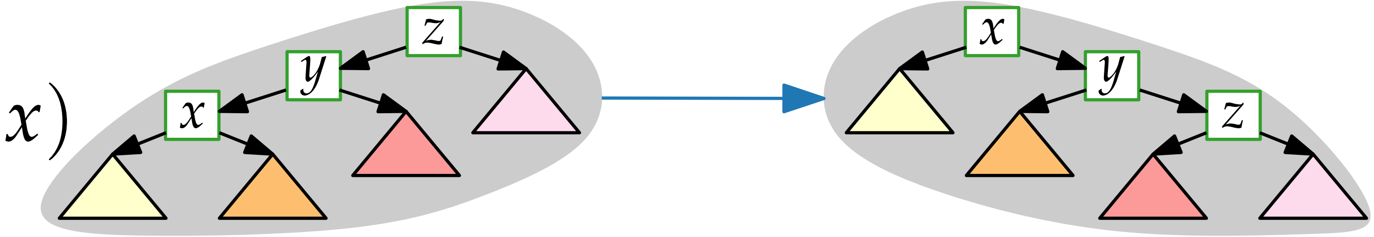

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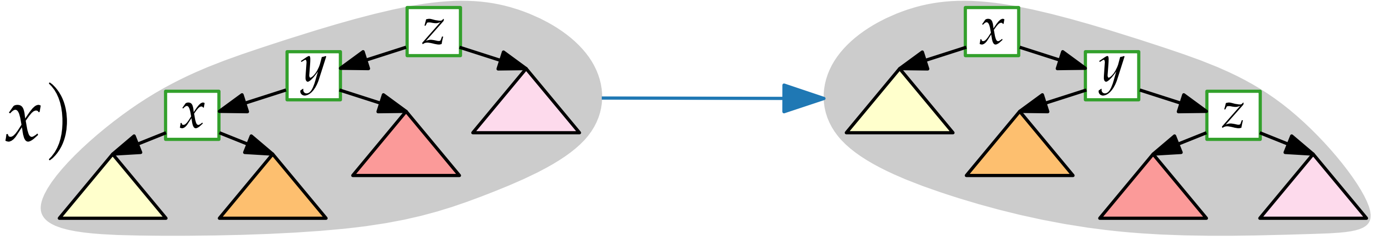

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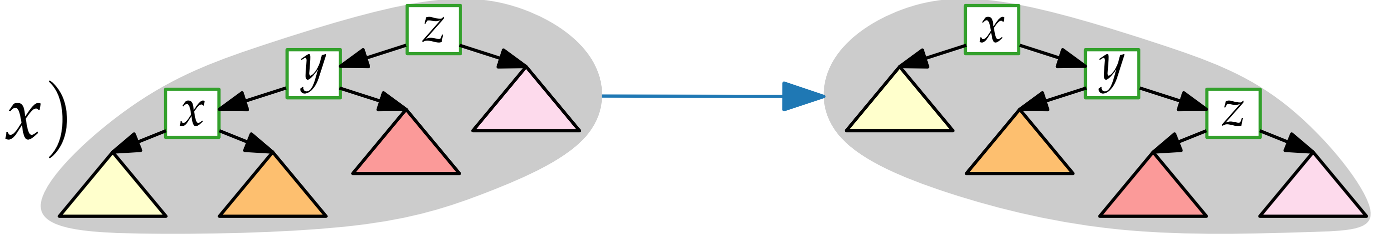

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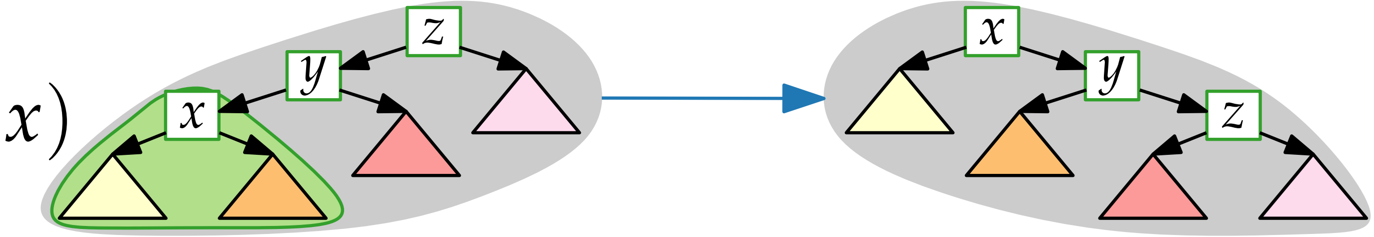

$$
\begin{aligned}
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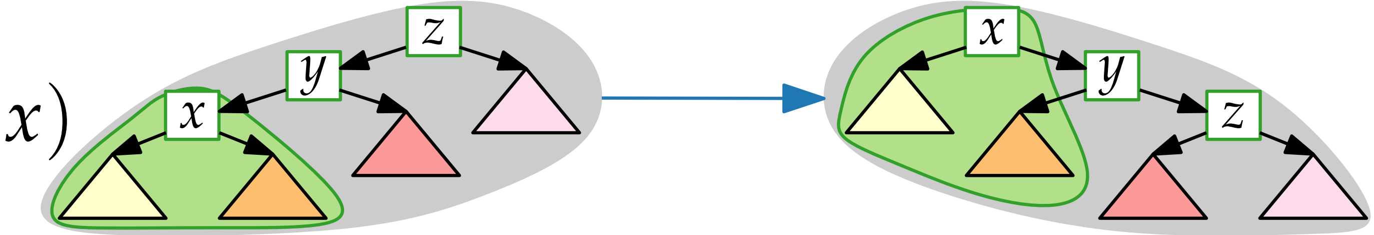

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Potential after Rotation

Consider any rotation; $s(x)$ before rotation, $s_+(x)$ afterwards

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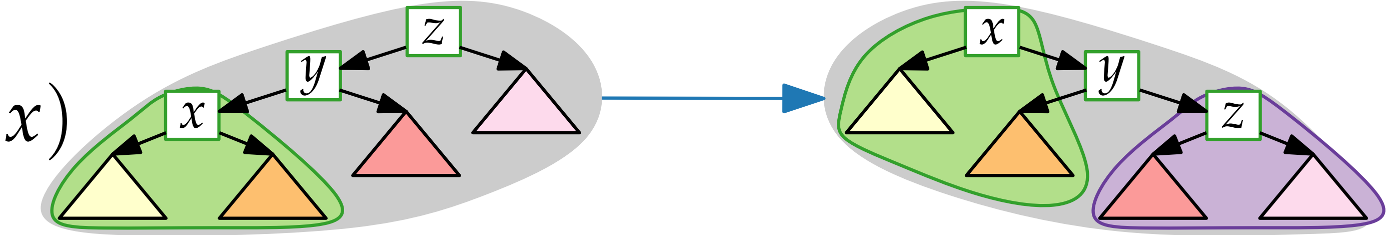

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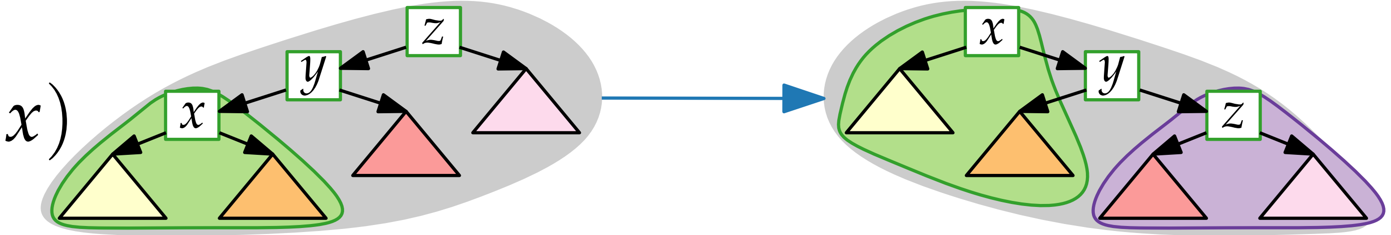

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Case 1.

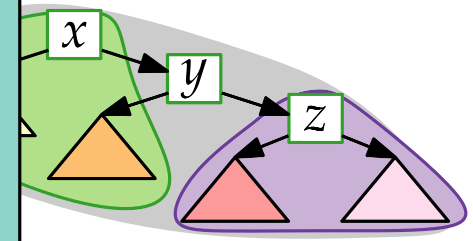
Inequality of arithmetic and geometric means (AM-GM):

$$\frac{x_1+x_2+\cdots+x_k}{k} \geq \sqrt[k]{x_1 \cdot x_2 \cdot \ldots \cdot x_k}$$

(arithmetic mean) (geometric mean)

for $k = 2$:

$$\frac{x+y}{2} \geq \sqrt{xy} \;\Rightarrow xy \leq \left(\frac{x+y}{2}\right)^2$$



pot. cha

$(s_+(x) =$ g $s(y)$

$(s(x) \leq s$

$(s_+(y) \leq$

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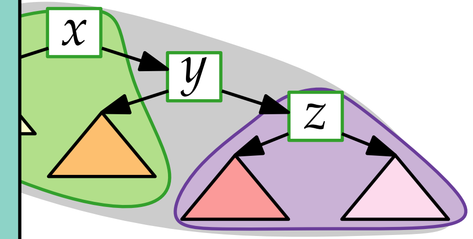

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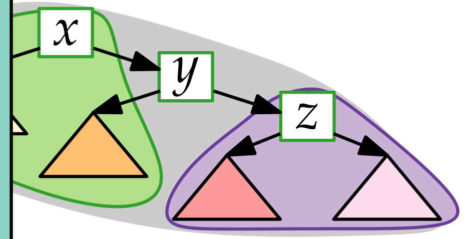

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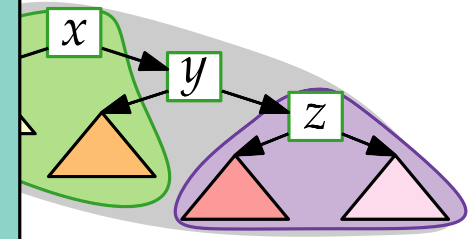

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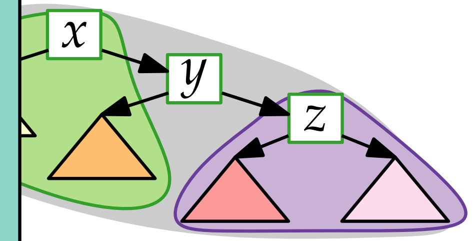

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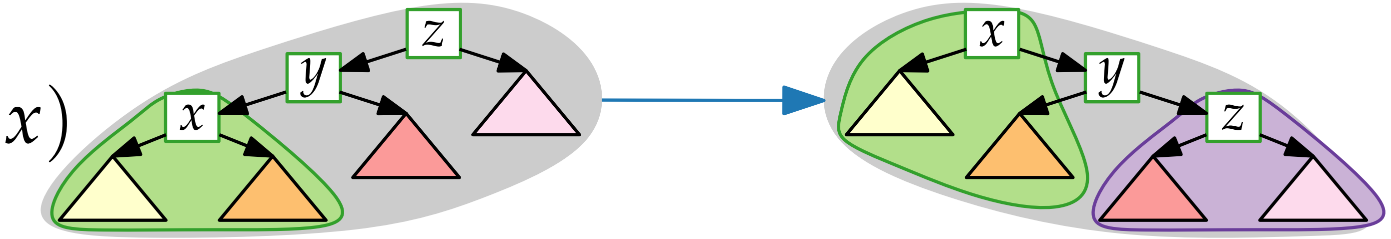

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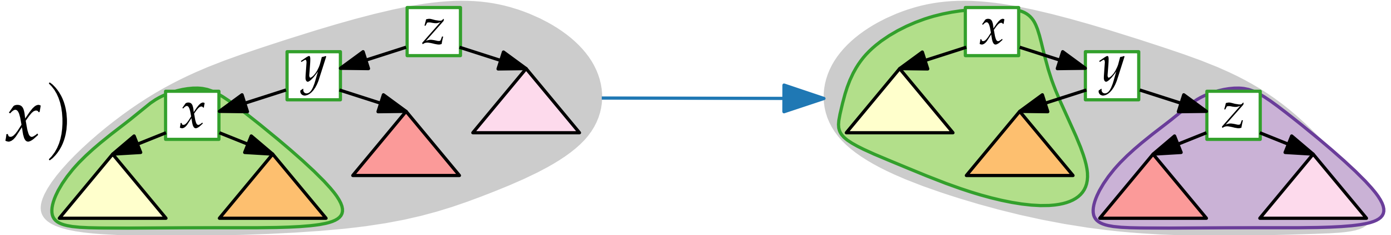

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$$
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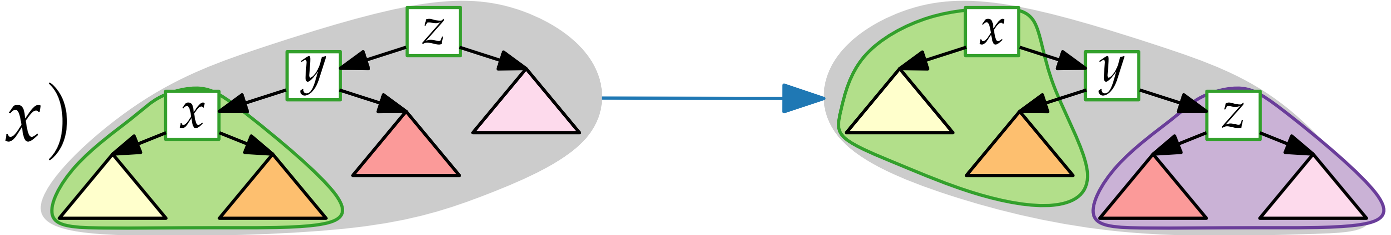

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\end{aligned}$$

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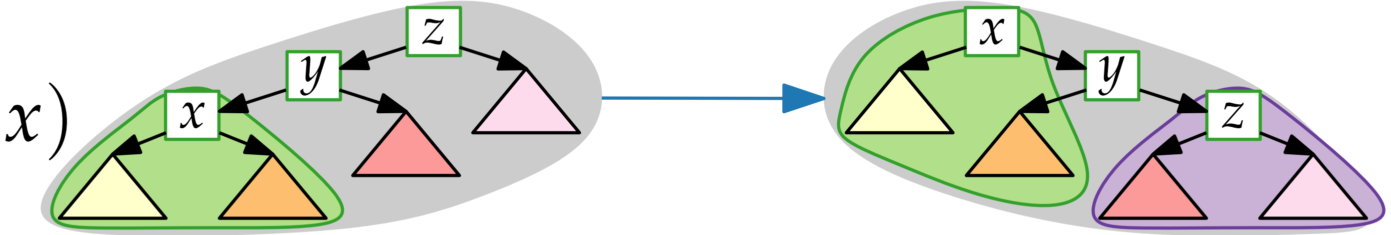

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\end{aligned}
$$

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$$
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\end{aligned}
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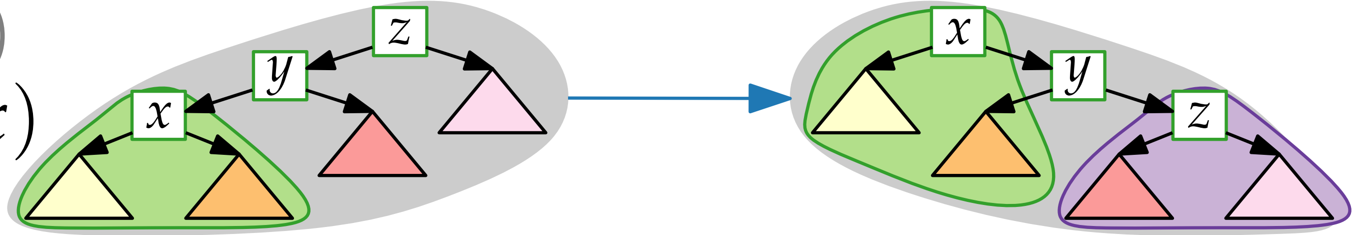
Potential after Rotation

Consider any rotation; $s(x)$ before rotation, $s_+(x)$ afterwards

Lemma. After a double rotation, the potential increases by $\leq 3\,(\log s_+(x) - \log s(x)) - 2$.

Proof. / Left-Left(x)

Case 1. Right-Right(x)



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(s(x) \leq s(y)) \quad &\leq \log s_+(y) + \log s_+(z) - 2\log s(x) \\
(s_+(y) \leq s_+(x)) \quad &\leq \log s_+(x) + \log s_+(z) - 2\log s(x) \\
&\leq 3\log s_+(x) - 3\log s(x) - 2 \quad \checkmark
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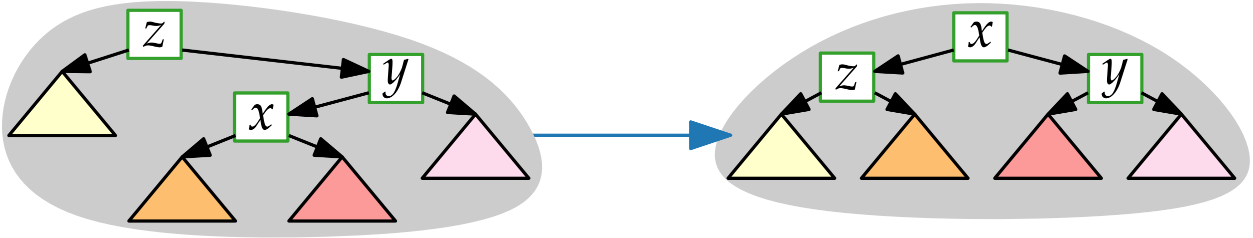
Potential after Rotation

Consider any rotation; $s(x)$ before rotation, $s_+(x)$ afterwards

Lemma. After a double rotation, the potential increases by $\leq 3\,(\log s_+(x) - \log s(x)) - 2$.

Proof.

Case 2. Right-Left(x)



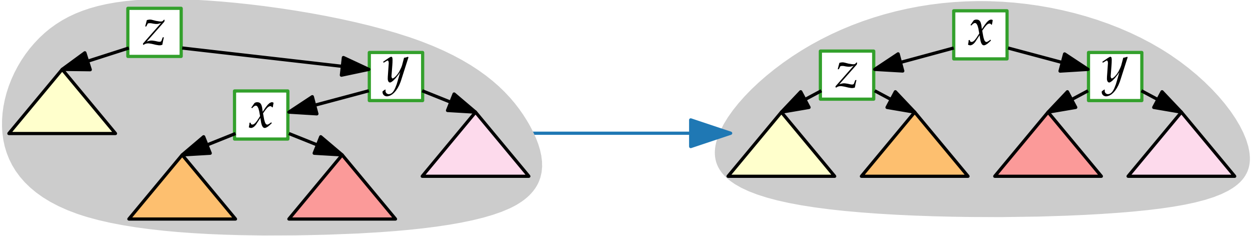
Potential after Rotation

Consider any rotation; $s(x)$ before rotation, $s_+(x)$ afterwards

Lemma. After a double rotation, the potential increases by $\leq 3\,(\log s_+(x) - \log s(x)) - 2$.

Proof.

Case 2. Right-Left(x)



pot. change $= \log s_+(x) + \log s_+(y) + \log s_+(z)$
$- \log s(x) - \log s(y) - \log s(z)$

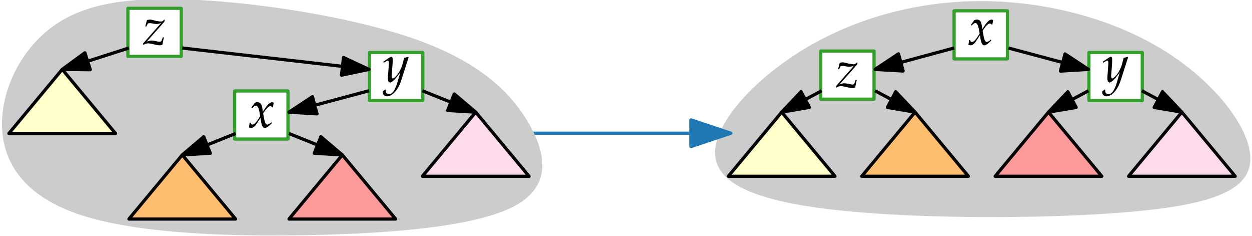
Potential after Rotation

Consider any rotation; $s(x)$ before rotation, $s_+(x)$ afterwards

Lemma. After a double rotation, the potential increases by $\leq 3\,(\log s_+(x) - \log s(x)) - 2$.

Proof.

Case 2. Right-Left(x)



$$\text{pot. change} \quad = \log s_+(x) + \log s_+(y) + \log s_+(z) - \log s(x) - \log s(y) - \log s(z)$$

$$(s_+(x) = s(z)) \quad = \log s_+(y) + \log s_+(z) - \log s(x) - \log s(y)$$

Potential after Rotation

Consider any rotation; $s(x)$ before rotation, $s_+(x)$ afterwards

Lemma. After a double rotation, the potential increases by $\leq 3\,(\log s_+(x) - \log s(x)) - 2$.

Proof.

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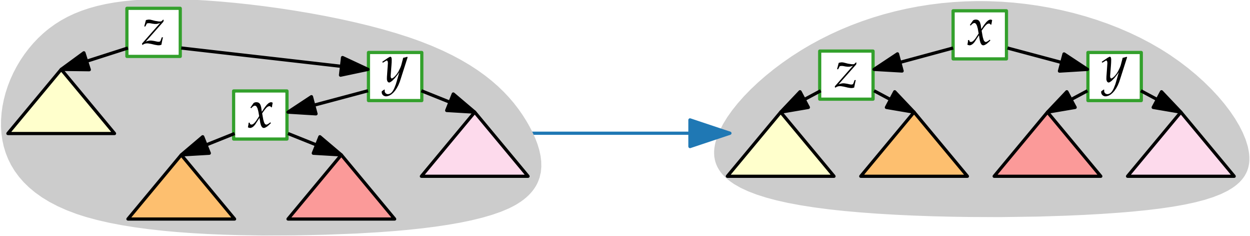


pot. change $= \log s_+(x) + \log s_+(y) + \log s_+(z)$
$\quad - \log s(x) - \log s(y) - \log s(z)$

$(s_+(x) = s(z))$ $\quad = \log s_+(y) + \log s_+(z) - \log s(x) - \log s(y)$

$(s(x) \leq s(y))$

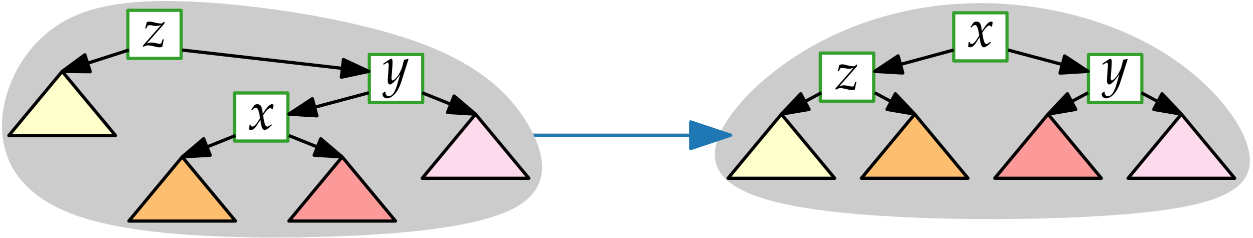
Potential after Rotation

Consider any rotation; $s(x)$ before rotation, $s_+(x)$ afterwards

Lemma. After a double rotation, the potential increases by $\leq 3\,(\log s_+(x) - \log s(x)) - 2$.

Proof.

Case 2. Right-Left(x)



$$\text{pot. change} \quad = \log s_+(x) + \log s_+(y) + \log s_+(z) - \log s(x) - \log s(y) - \log s(z)$$

$$(s_+(x) = s(z)) \quad = \log s_+(y) + \log s_+(z) - \log s(x) - \log s(y)$$

$$(s(x) \leq s(y)) \quad \leq \log s_+(y) + \log s_+(z) - 2\log s(x)$$

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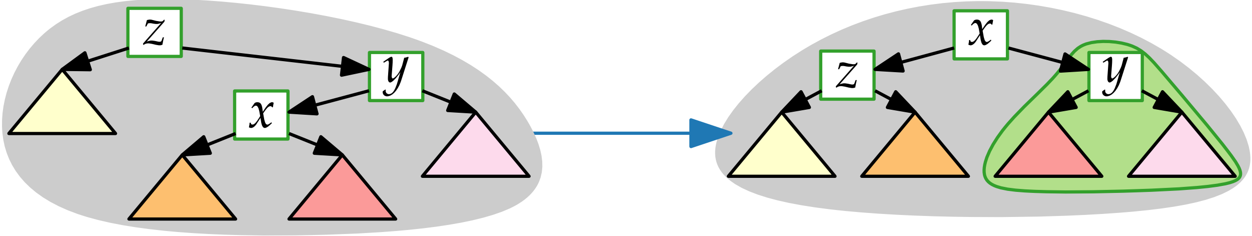

$$\begin{aligned}
\text{pot. change} \quad &= \log s_+(x) + \log s_+(y) + \log s_+(z) \\
&\quad - \log s(x) - \log s(y) - \log s(z) \\
(s_+(x) = s(z)) \quad &= \log s_+(y) + \log s_+(z) - \log s(x) - \log s(y) \\
(s(x) \leq s(y)) \quad &\leq \log s_+(y) + \log s_+(z) - 2\log s(x)
\end{aligned}$$

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Consider any rotation; $s(x)$ before rotation, $s_+(x)$ afterwards

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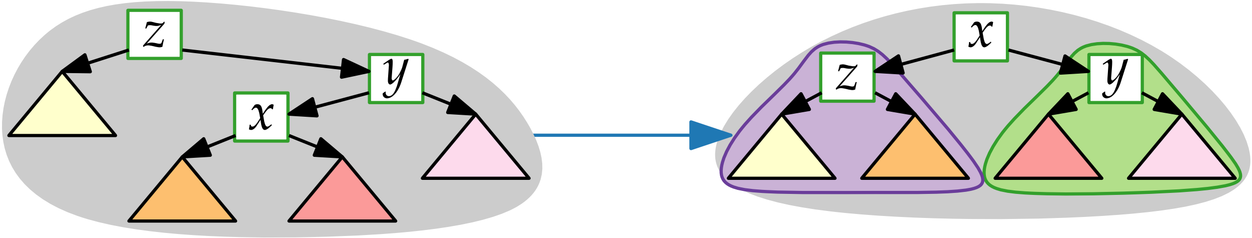

$$
\begin{aligned}
\text{pot. change} \quad &= \log s_+(x) + \log s_+(y) + \log s_+(z) \\
&\quad - \log s(x) - \log s(y) - \log s(z) \\
(s_+(x) = s(z)) \quad &= \log s_+(y) + \log s_+(z) - \log s(x) - \log s(y) \\
(s(x) \leq s(y)) \quad &\leq \log s_+(y) + \log s_+(z) - 2\log s(x)
\end{aligned}
$$

Potential after Rotation

Consider any rotation; $s(x)$ before rotation, $s_+(x)$ afterwards

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Proof.

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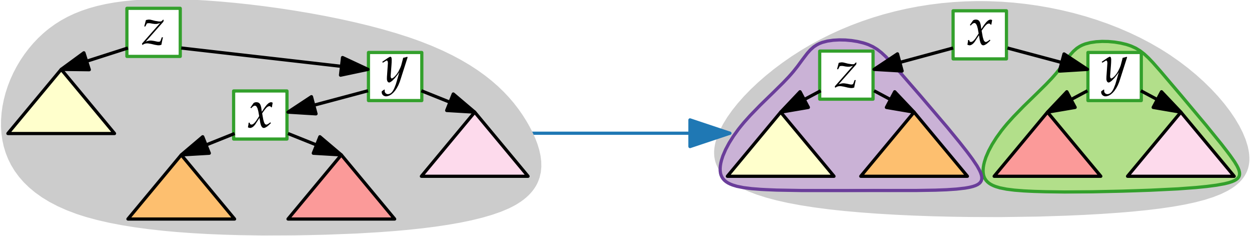

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$\quad - \log s(x) - \log s(y) - \log s(z)$

$(s_+(x) = s(z))$ $\quad = \log s_+(y) + \log s_+(z) - \log s(x) - \log s(y)$

$(s(x) \leq s(y))$ $\quad \leq \log s_+(y) + \log s_+(z) - 2\log s(x)$

$(\star)$: $s_+(y) + s_+(z) \leq s_+(x)$

Potential after Rotation

Consider any rotation; $s(x)$ before rotation, $s_+(x)$ afterwards

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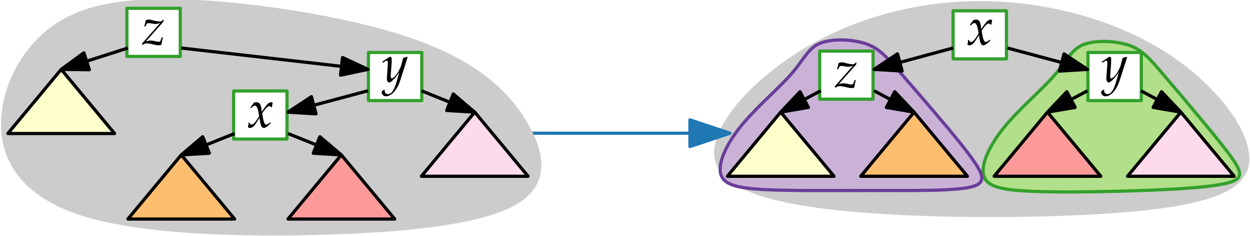

$$\begin{aligned}
\text{pot. change} \quad &= \log s_+(x) + \log s_+(y) + \log s_+(z) \\
&\quad - \log s(x) - \log s(y) - \log s(z) \\
(s_+(x) = s(z)) \quad &= \log s_+(y) + \log s_+(z) - \log s(x) - \log s(y) \\
(s(x) \leq s(y)) \quad &\leq \log s_+(y) + \log s_+(z) - 2\log s(x)
\end{aligned}$$

$(\star)$: $s_+(y) + s_+(z) \leq s_+(x)$ | $\log s_+(y) + \log s_+(z) \underset{\substack{\text{(AM-GM)} \\ (\star)}}{\leq} 2\log s_+(x) - 2$

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Consider any rotation; $s(x)$ before rotation, $s_+(x)$ afterwards

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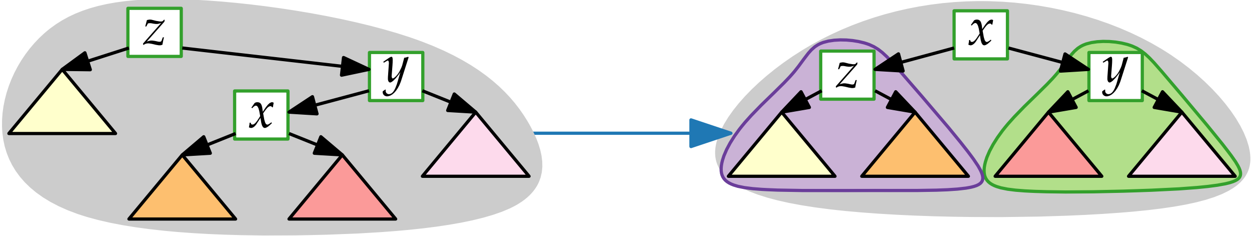

$$
\begin{aligned}
\text{pot. change} \quad &= \log s_+(x) + \log s_+(y) + \log s_+(z) \\
&\quad - \log s(x) - \log s(y) - \log s(z) \\
(s_+(x) = s(z)) \quad &= \log s_+(y) + \log s_+(z) - \log s(x) - \log s(y) \\
(s(x) \leq s(y)) \quad &\leq \log s_+(y) + \log s_+(z) - 2\log s(x)
\end{aligned}
$$

$(\star)$: $s_+(y) + s_+(z) \leq s_+(x)$ $\quad\Big|\quad$ $\log s_+(y) + \log s_+(z) \underset{(\star)}{\underset{\text{(AM-GM)}}{\leq}} 2\log s_+(x) - 2$

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Proof.

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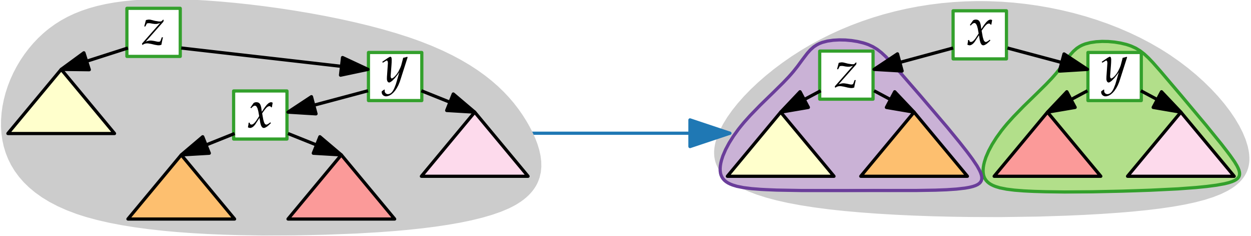

$$\begin{aligned}
\text{pot. change} \quad &= \log s_+(x) + \log s_+(y) + \log s_+(z) \\
&\quad - \log s(x) - \log s(y) - \log s(z) \\
(s_+(x) = s(z)) \quad &= \log s_+(y) + \log s_+(z) - \log s(x) - \log s(y) \\
(s(x) \leq s(y)) \quad &\leq \log s_+(y) + \log s_+(z) - 2\log s(x) \\
&\leq 2\log s_+(x) - 2\log s(x) - 2
\end{aligned}$$

$(\star)$: $s_+(y) + s_+(z) \leq s_+(x)$ | $\log s_+(y) + \log s_+(z) \underset{\substack{\text{(AM-GM)} \\ (\star)}}{\leq} 2\log s_+(x) - 2$

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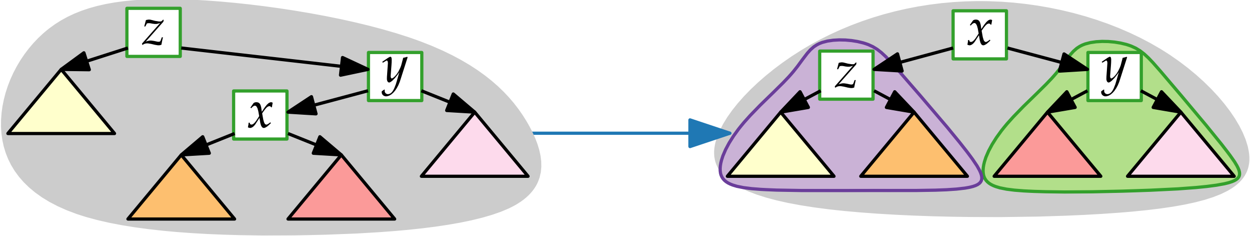

pot. change $= \log s_+(x) + \log s_+(y) + \log s_+(z) - \log s(x) - \log s(y) - \log s(z)$

$(s_+(x) = s(z))$ $= \log s_+(y) + \log s_+(z) - \log s(x) - \log s(y)$

$(s(x) \leq s(y))$ $\leq \log s_+(y) + \log s_+(z) - 2\log s(x)$

$\leq 2\log s_+(x) - 2\log s(x) - 2$

$(s_+(x) > s(x))$

$(\star)$: $s_+(y) + s_+(z) \leq s_+(x)$ | $\log s_+(y) + \log s_+(z) \underset{\substack{\text{(AM-GM)}\\(\star)}}{\leq} 2\log s_+(x) - 2$

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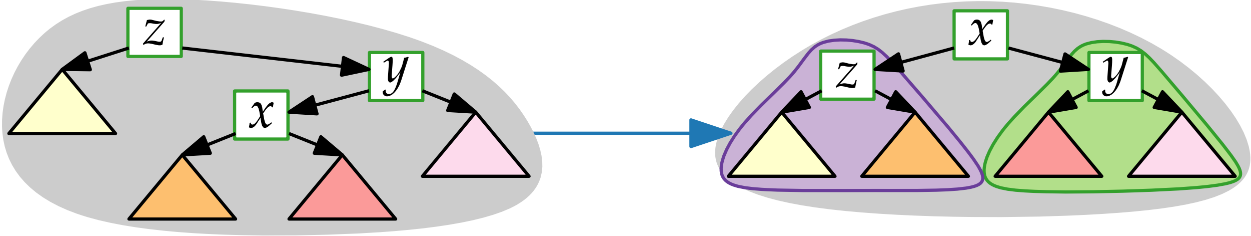

$$
\begin{aligned}
\text{pot. change} \quad &= \log s_+(x) + \log s_+(y) + \log s_+(z) \\
&\quad - \log s(x) - \log s(y) - \log s(z) \\
(s_+(x) = s(z)) \quad &= \log s_+(y) + \log s_+(z) - \log s(x) - \log s(y) \\
(s(x) \leq s(y)) \quad &\leq \log s_+(y) + \log s_+(z) - 2\log s(x) \\
&\leq 2\log s_+(x) - 2\log s(x) - 2 \\
(s_+(x) > s(x)) \quad &\leq 3\log s_+(x) - 3\log s(x) - 2
\end{aligned}
$$

$(\star)$: $s_+(y) + s_+(z) \leq s_+(x)$ | $\log s_+(y) + \log s_+(z) \underset{(\star)}{\underset{\text{(AM-GM)}}{\leq}} 2\log s_+(x) - 2$

Potential after Rotation

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Proof.

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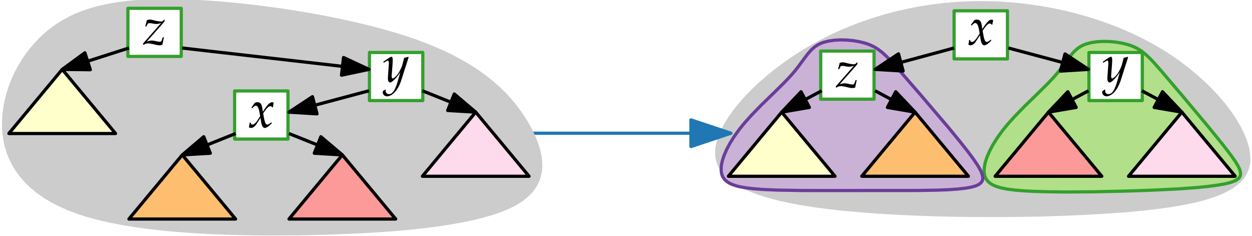


$$\begin{aligned}
&\text{pot. change} && = \log s_+(x) + \log s_+(y) + \log s_+(z) \\
& && \quad - \log s(x) - \log s(y) - \log s(z) \\
&(s_+(x) = s(z)) && = \log s_+(y) + \log s_+(z) - \log s(x) - \log s(y) \\
&(s(x) \le s(y)) && \le \log s_+(y) + \log s_+(z) - 2\log s(x) \\
& && \le 2\log s_+(x) - 2\log s(x) - 2 \\
&(s_+(x) > s(x)) && \le 3\log s_+(x) - 3\log s(x) - 2 \qquad \checkmark
\end{aligned}$$

$(\star)$: $s_+(y) + s_+(z) \le s_+(x)$ | $\log s_+(y) + \log s_+(z) \underset{(\star)}{\underset{\text{(AM-GM)}}{\le}} 2\log s_+(x) - 2$

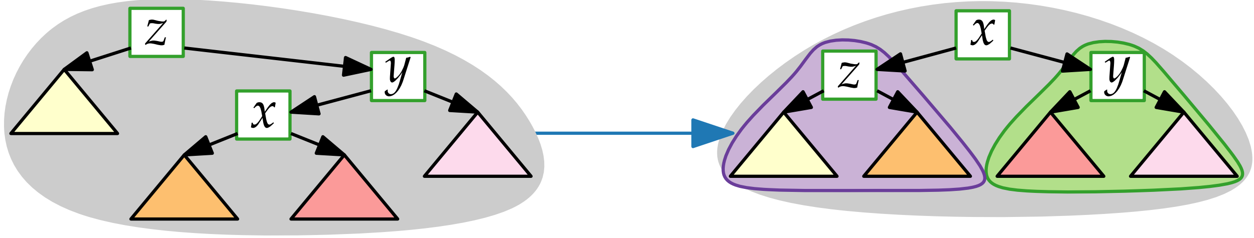
Potential after Rotation

Consider any rotation; $s(x)$ before rotation, $s_+(x)$ afterwards

Lemma. After a double rotation, the potential increases by $\leq 3\,(\log s_+(x) - \log s(x)) - 2$.

Proof. / Left-Right(x)

Case 2. Right-Left(x)



$$
\begin{aligned}
\text{pot. change} \quad &= \log s_+(x) + \log s_+(y) + \log s_+(z) \\
&\quad - \log s(x) - \log s(y) - \log s(z) \\
(s_+(x) = s(z)) \quad &= \log s_+(y) + \log s_+(z) - \log s(x) - \log s(y) \\
(s(x) \leq s(y)) \quad &\leq \log s_+(y) + \log s_+(z) - 2 \log s(x) \\
&\leq 2 \log s_+(x) - 2 \log s(x) - 2 \\
(s_+(x) > s(x)) \quad &\leq 3 \log s_+(x) - 3 \log s(x) - 2 \qquad \checkmark
\end{aligned}
$$

□

$(\star)$: $s_+(y) + s_+(z) \leq s_+(x)$ | $\log s_+(y) + \log s_+(z) \underset{\substack{\text{(AM-GM)}\\(\star)}}{\leq} 2 \log s_+(x) - 2$

Access Lemma

Lemma. After a single rotation, the potential increases by $\leq 3\,(\log s_+(x) - \log s(x))$.
After a double rotation, the potential increases by $\leq 3\,(\log s_+(x) - \log s(x)) - 2$.

Access Lemma

Lemma. After a single rotation, the potential increases by $\leq 3\,(\log s_+(x) - \log s(x))$.
After a double rotation, the potential increases by $\leq 3\,(\log s_+(x) - \log s(x)) - 2$.

Lemma. The (amortized) cost of Splay(x) is $c(\text{Splay}(x)) \leq 1 + 3\log(W/w(x))$.

Access Lemma

Lemma. After a single rotation, the potential increases by $\leq 3\,(\log s_+(x) - \log s(x))$.
After a double rotation, the potential increases by $\leq 3\,(\log s_+(x) - \log s(x)) - 2$.

Lemma. The (amortized) cost of $\mathsf{Splay}(x)$ is $c(\mathsf{Splay}(x)) \leq 1 + 3\log(W / w(x))$.

Proof.

Access Lemma

Lemma. After a single rotation, the potential increases by $\leq 3\,(\log s_+(x) - \log s(x))$.
After a double rotation, the potential increases by $\leq 3\,(\log s_+(x) - \log s(x)) - 2$.

Lemma. The (amortized) cost of Splay(x) is $c(\text{Splay}(x)) \leq 1 + 3\log(W/w(x))$.

Proof. W.l.o.g. k double rotations and 1 single rotation.

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After a double rotation, the potential increases by $\leq 3\,(\log s_+(x) - \log s(x)) - 2$.

Lemma. The (amortized) cost of Splay(x) is $c(\text{Splay}(x)) \leq 1 + 3\log(W/w(x))$.

Proof. W.l.o.g. k double rotations and 1 single rotation.
Let $s_i(x)$ be $s(x)$ after i single/double rotations.

Access Lemma

Lemma. After a single rotation, the potential increases by $\leq 3\left(\log s_+(x) - \log s(x)\right)$.
After a double rotation, the potential increases by $\leq 3\left(\log s_+(x) - \log s(x)\right) - 2$.

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Proof. W.l.o.g. k double rotations and 1 single rotation.
Let $s_i(x)$ be $s(x)$ after i single/double rotations.
Potential increases by at most

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Proof. W.l.o.g. k double rotations and 1 single rotation.
Let $s_i(x)$ be $s(x)$ after i single/double rotations.
Potential increases by at most
$\sum_{i=1}^{k} (3\,(\log s_i(x) - \log s_{i-1}(x)) - 2)$

Access Lemma

Lemma. After a single rotation, the potential increases by $\leq 3\,(\log s_+(x) - \log s(x))$.
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$\sum_{i=1}^{k} (3\,(\log s_i(x) - \log s_{i-1}(x)) - 2)$
$+3\,(\log s_{k+1}(x) - \log s_k(x))$

Access Lemma

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Let $s_i(x)$ be $s(x)$ after i single/double rotations.
Potential increases by at most

$$\sum_{i=1}^{k} \left(3\left(\log s_i(x) - \log s_{i-1}(x)\right) - 2\right)$$
$$+3\left(\log s_{k+1}(x) - \log s_k(x)\right)$$
$$\text{(id. entries rem.)} = 3\left(\log s_{k+1}(x) - \log s(x)\right) - 2k$$

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Lemma. After a single rotation, the potential increases by $\leq 3\,(\log s_+(x) - \log s(x))$.
After a double rotation, the potential increases by $\leq 3\,(\log s_+(x) - \log s(x)) - 2$.

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$$\textstyle\sum_{i=1}^{k} \left(3\left(\log s_i(x) - \log s_{i-1}(x)\right) - 2\right)$$
$$+3\left(\log s_{k+1}(x) - \log s_k(x)\right)$$

root!

(id. entries rem.) $= 3\left(\log s_{k+1}(x) - \log s(x)\right) - 2k$

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Lemma. After a single rotation, the potential increases by $\leq 3\,(\log s_+(x) - \log s(x))$.
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Potential increases by at most

$$\textstyle\sum_{i=1}^{k} \left(3\,(\log s_i(x) - \log s_{i-1}(x)) - 2\right)$$
$$+3\,(\log s_{k+1}(x) - \log s_k(x))$$

root!

(id. entries rem.) $= 3\,(\log s_{k+1}(x) - \log s(x)) - 2k$

$= 3\,(\log W - \log s(x)) - 2k$

Access Lemma

Lemma. After a single rotation, the potential increases by $\leq 3\,(\log s_+(x) - \log s(x))$.
After a double rotation, the potential increases by $\leq 3\,(\log s_+(x) - \log s(x)) - 2$.

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Proof. W.l.o.g. k double rotations and 1 single rotation.
Let $s_i(x)$ be $s(x)$ after i single/double rotations.
Potential increases by at most

$$\begin{aligned}
& \textstyle\sum_{i=1}^{k} (3\,(\log s_i(x) - \log s_{i-1}(x)) - 2) \\
& +3\,(\log s_{k+1}(x) - \log s_k(x)) \\
\text{(id. entries rem.)} \quad & = 3\,(\log s_{k+1}(x) - \log s(x)) - 2k \\
& = 3\,(\log W - \log s(x)) - 2k
\end{aligned}$$

root!

$(s(x) \geq w(x))$

Access Lemma

Lemma. After a single rotation, the potential increases by $\leq 3\left(\log s_+(x) - \log s(x)\right)$.
After a double rotation, the potential increases by $\leq 3\left(\log s_+(x) - \log s(x)\right) - 2$.

Lemma. The (amortized) cost of Splay(x) is
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$$\sum_{i=1}^{k} \left(3\left(\log s_i(x) - \log s_{i-1}(x)\right) - 2\right)$$
$$+3\left(\log s_{k+1}(x) - \log s_k(x)\right)$$

root!

$$\begin{aligned}
\text{(id. entries rem.)} \quad &= 3\left(\log s_{k+1}(x) - \log s(x)\right) - 2k \\
&= 3\left(\log W - \log s(x)\right) - 2k \\
(s(x) \geq w(x)) \quad &\leq 3\left(\log W - \log w(x)\right) - 2k
\end{aligned}$$

Access Lemma

Lemma. After a single rotation, the potential increases by $\leq 3\left(\log s_+(x) - \log s(x)\right)$.
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Potential increases by at most

$$\begin{aligned}
& \textstyle\sum_{i=1}^{k} \left(3\left(\log s_i(x) - \log s_{i-1}(x)\right) - 2\right) \\
\text{root!} \quad & +3\left(\log s_{k+1}(x) - \log s_k(x)\right) \\
\text{(id. entries rem.)} \quad & = 3\left(\log s_{k+1}(x) - \log s(x)\right) - 2k \\
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Lemma. After a single rotation, the potential increases by $\leq 3\,(\log s_+(x) - \log s(x))$.
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Proof. W.l.o.g. k double rotations and 1 single rotation.
Let $s_i(x)$ be $s(x)$ after i single/double rotations.
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All These Models . . .

Balanced: Queries take (amortized) $O(\log n)$ time

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Dynamic Finger: Queries take $O(\log \delta_i)$ time (δ_i: rank diff.)

Working Set: Queries take $O(\log t)$ time (t: recency)

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... is there one BST to rule them all?

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All of these properties can be shown by chosing the weight function accordingly.
Note that the actual algorithm is always the same!

Yes!

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Let S be a sequence of queries.

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$\Rightarrow$ as long as every key is queried at least once, it doesn't change the asymptotic running time.

Balance

Lemma. The (amortized) cost of Splay(x) is $c(\text{Splay}(x)) \leq 1 + 3\log(W/w(x))$.

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Definition. A BST has the **entropy property** if queries take expected $O(1 - \sum_{i=1}^{n} p_i \log p_i)$ time.

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⇒ expected query time:

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□

Static Optimality

Given a sequence S of queries.

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Let $T_S^\star$ be an *optimal* static tree with the shortest query time OPT_S for S.

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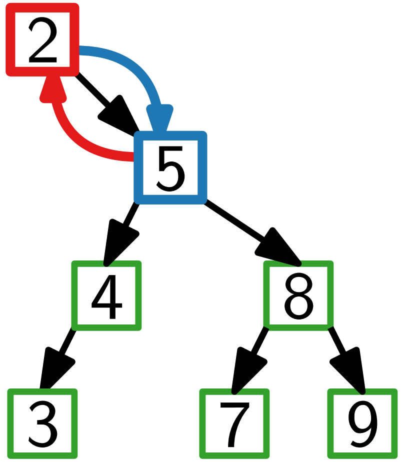

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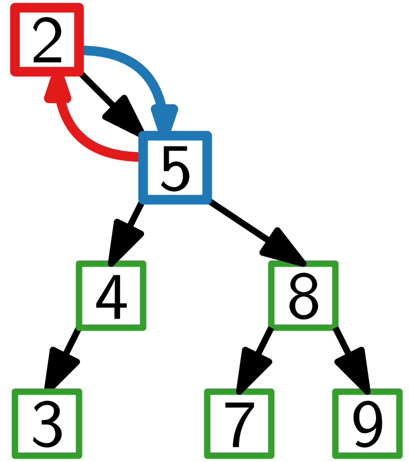

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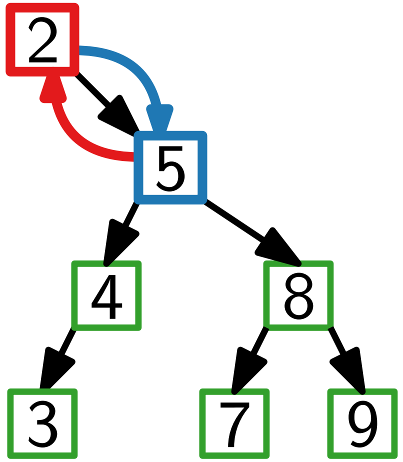

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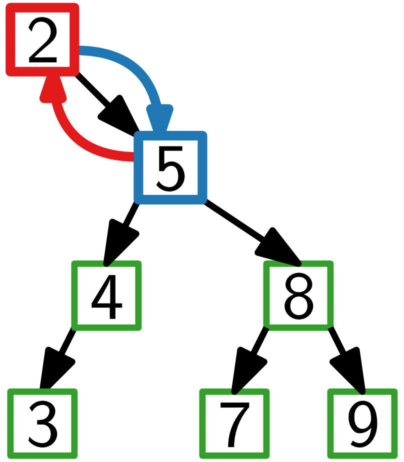

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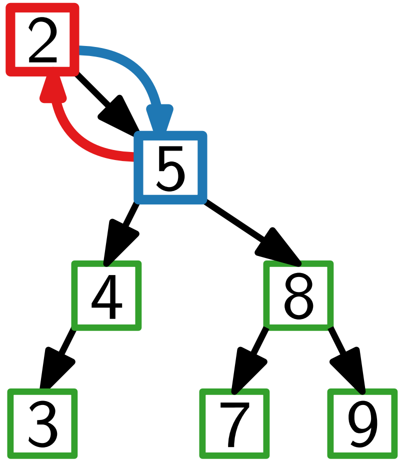

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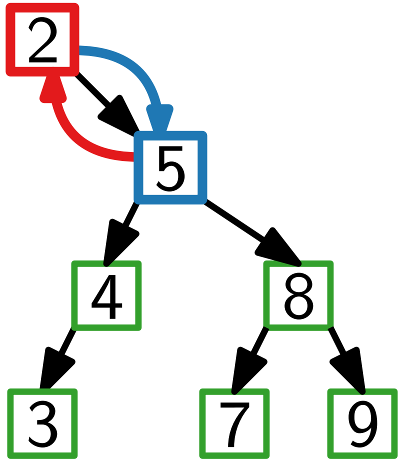

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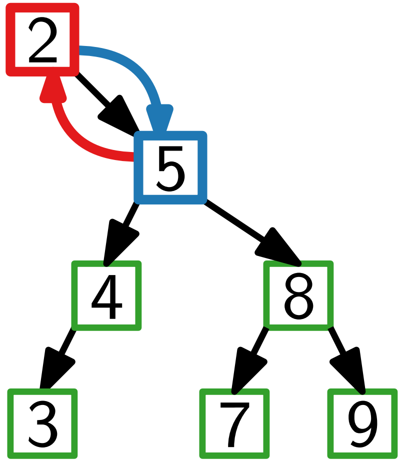

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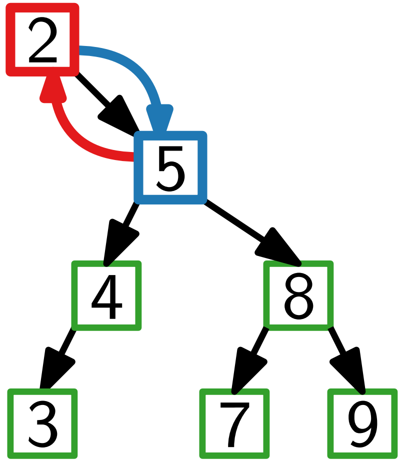

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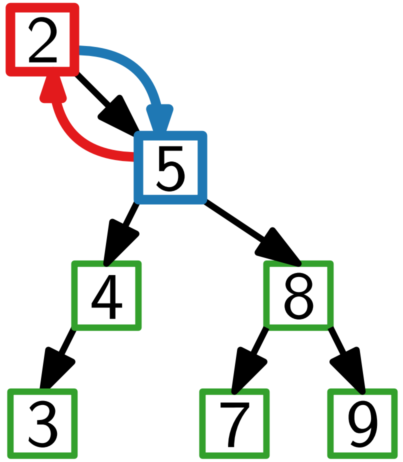

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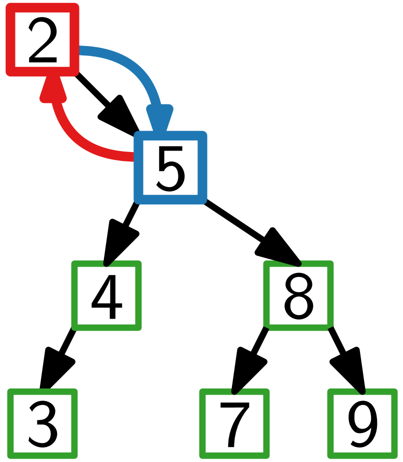

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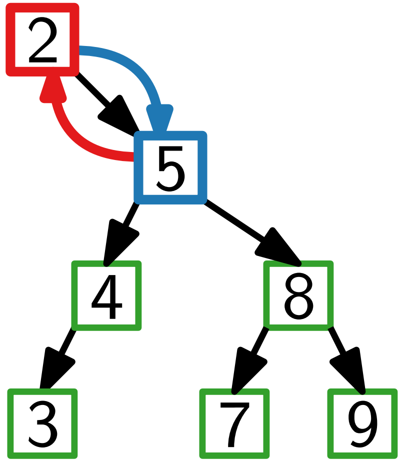

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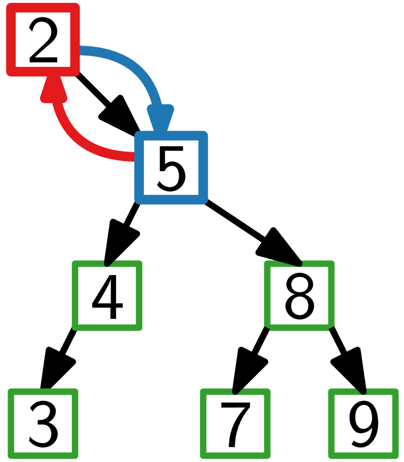

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Conjecture. Splay Trees are dynamically optimal.