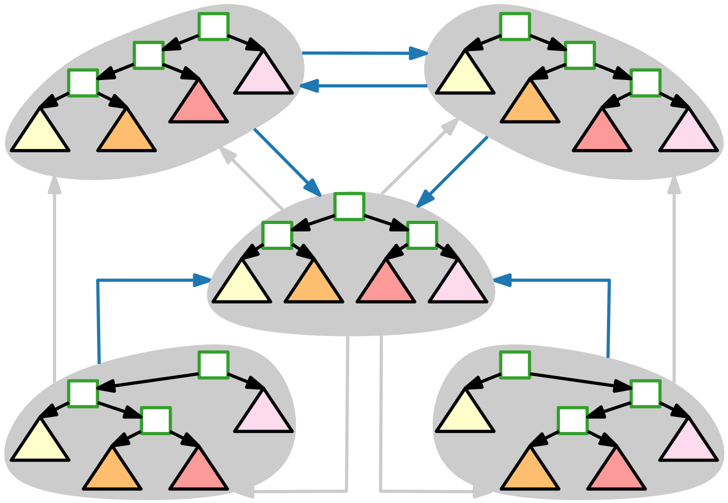
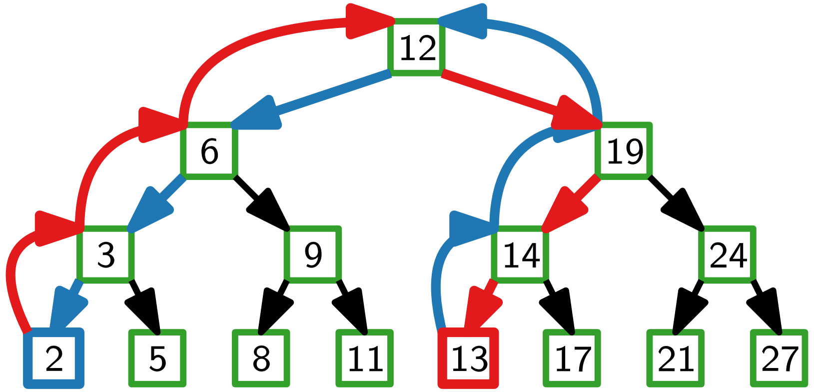
Advanced Algorithms

Optimal Binary Search Trees

Splay Trees

Johannes Zink · WS23/24





How Good is a Binary Search Tree?

optimal

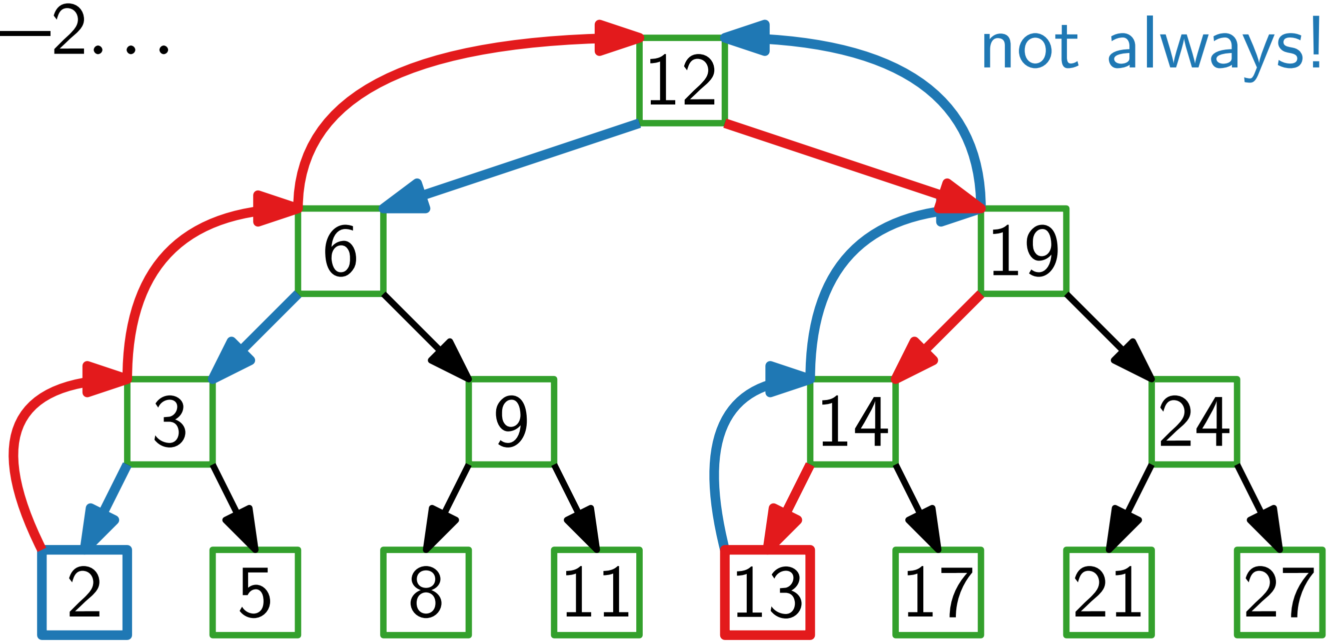
Binary search tree (BST): w.c. query time $\Theta(n)$

Balanced binary search tree: w.c. query time $\Theta(\log n)$
(e.g., Red-Black-Tree, AVL-Tree)

What if we *know* the query before? w.c. query time 1

Sequence of queries? $O(\log n)$ per query
e.g. 2—13—5
or 2—13—2—13—2...

optimal?
not always!



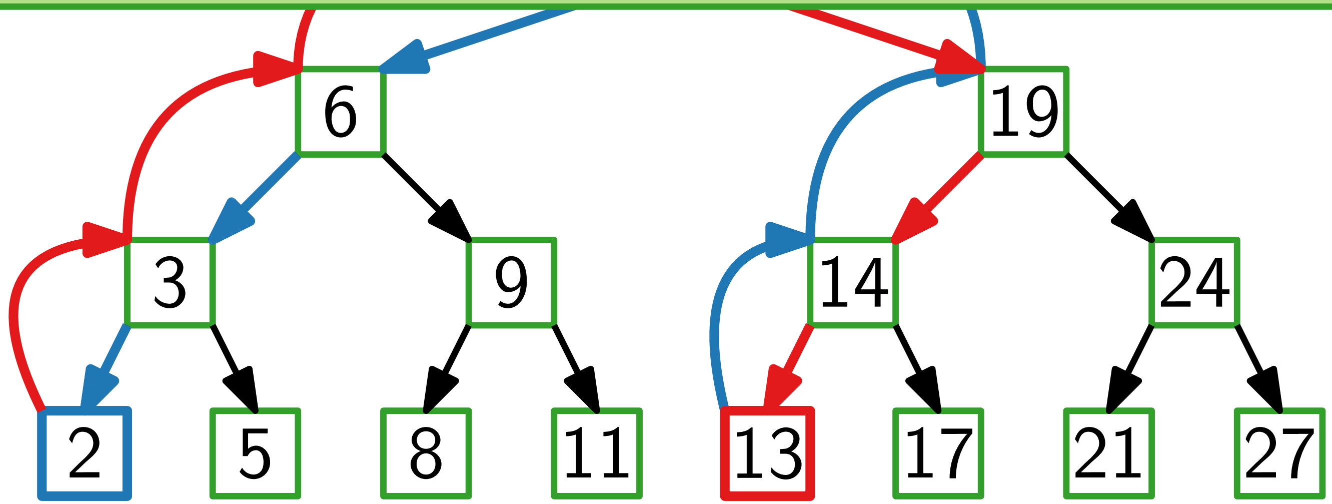
The performance of a BST depends on the model!

Model 1: Malicious Queries

Given a BST, what is the worst sequence of queries?

Lemma. The worst-case malicious query cost in any BST with n nodes is at least $\Omega(\log n)$ per query.

Definition. A BST is **balanced** if the cost of *any* sequence of m queries is $O(m \log n + n \log n)$.
$\Rightarrow$ the (amortized) cost of each query is $O(\log n)$ (for at least n queries)

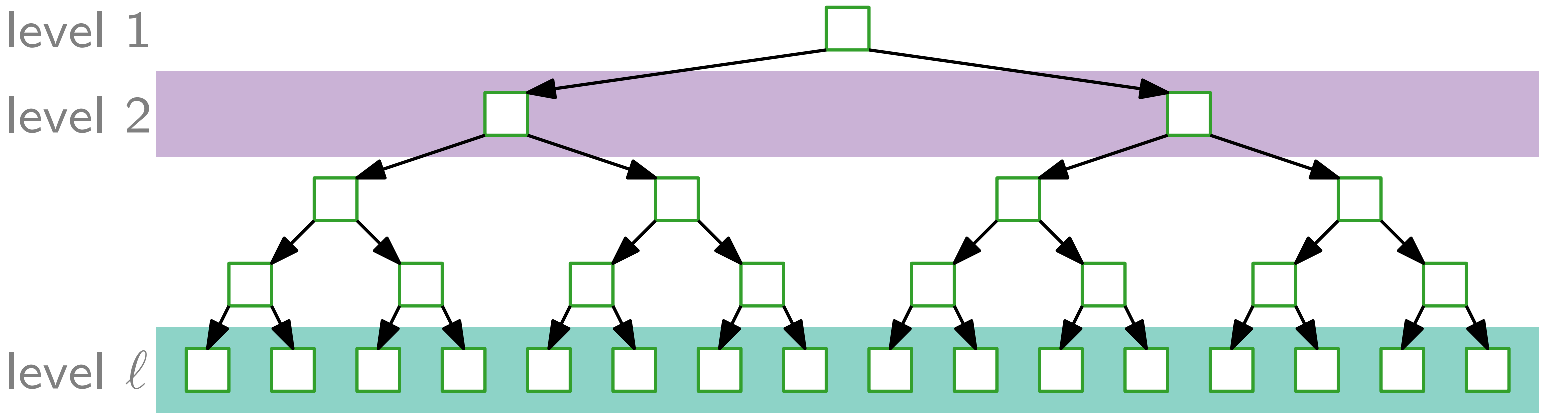


Model 2: Known Probability Distribution

Access Probabilities:

2	3	5	6	8	9	11
2%	20%	30%	8%	20%	15%	5%

Idea: Place nodes with higher probability higher in the tree.



prob. $\leq 1/2$

OPT: prob. $p \Rightarrow$ level $\leq 1 - \log_2 p$

prob. $\leq 1/2^{\ell-1}$

Lemma. The expected query cost in any BST is at least $\Omega(1+H)$ per query with $H = \sum_{i=1}^{n} -p_i \log p_i$.

Definition. A BST has the **entropy property** if it reaches this bound, i.e., the expected query cost is in $O(1+H)$.

$$p_i = 1/n \Rightarrow H = \sum_{i=1}^{n} 1/n \cdot \log n = \log n$$

$$p_1 \approx 1, p_i \approx 0 \Rightarrow H \approx -\log 1 = 0$$

Model 3: Spacial Locality

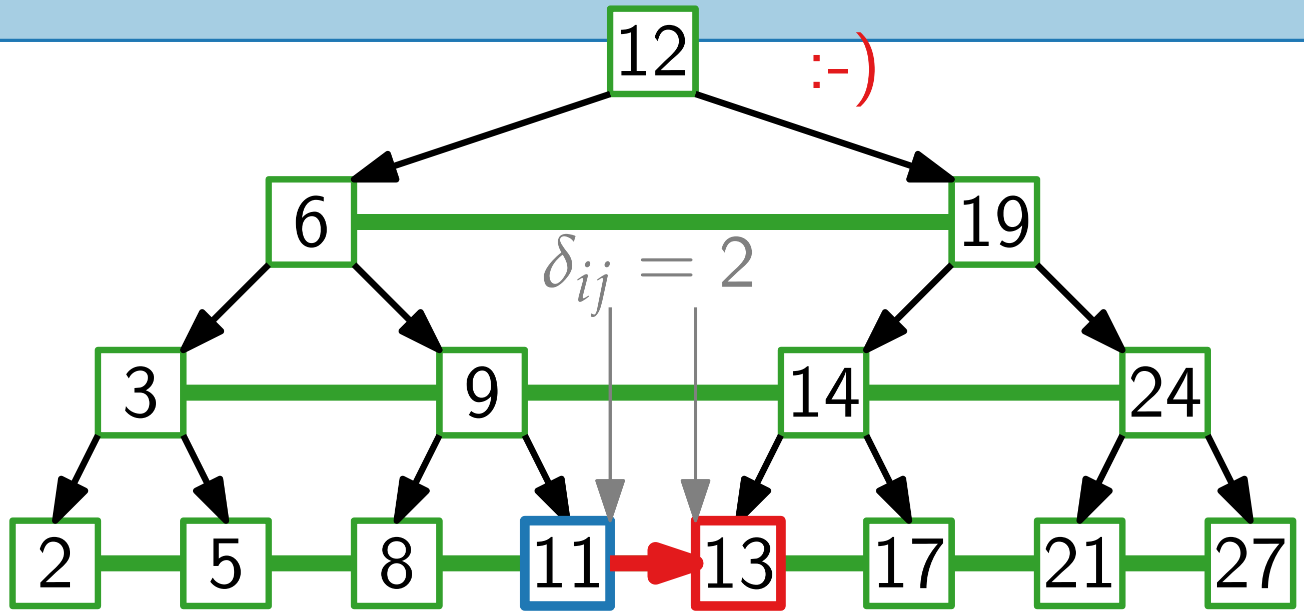
If a key is queried, then keys with nearby values are more likely to be queried.

Suppose we queried key x_i and want to query key x_j next.
Let $\delta_{ij} = |\,\mathrm{rank}(x_j) - \mathrm{rank}(x_i)|$.

Definition. A BST has the **dynamic finger property** if the (amortized) cost of queries are $O(\log \delta_{ij})$.

Lemma. A level-linked Red-Black-Tree has the dynamic finger property.

:-)



Model 4: Temporal Locality

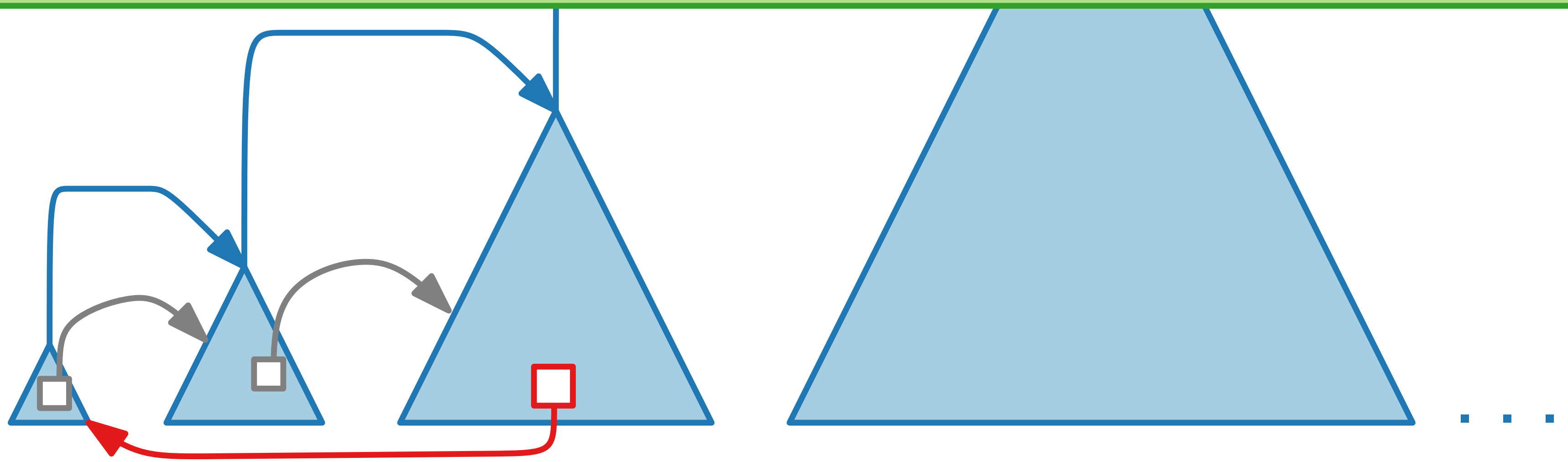
If a key is queried, then it is likely to be queried again soon.

A static tree will have a hard time... What if we can move elements?

Idea: Use a sequence of trees

Move queried key to first tree, then kick out oldest key.

Definition. A BST has the **working set property** if the (amortized) cost of a query for key x is $O(\log t)$, where t is the number of keys queried more recently than x.



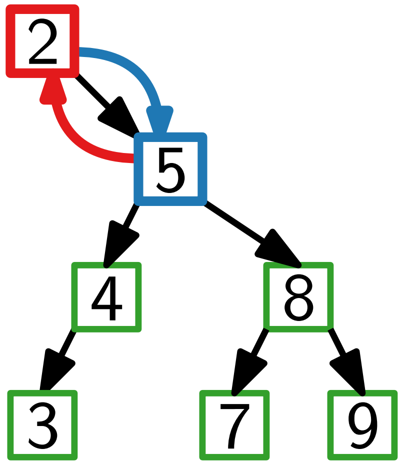
Model 5: Static Optimality

Given a sequence S of queries.

e.g. $S = 2, 5, 2, 5, 2, \ldots, 5$

$T_S^\star$:

OPT: $|S|$



Let $T_S^\star$ be an *optimal* static tree with the shortest query time OPT_S for S.

Definition. A BST is **statically optimal** if queries take (amortized) $O(\mathrm{OPT}_S)$ time for every S.

All These Models . . .

Balanced: Queries take (amortized) $O(\log n)$ time

Entropy: Queries take expected $O(1 + H)$ time

Dynamic Finger: Queries take $O(\log \delta_i)$ time (δ_i: rank diff.)

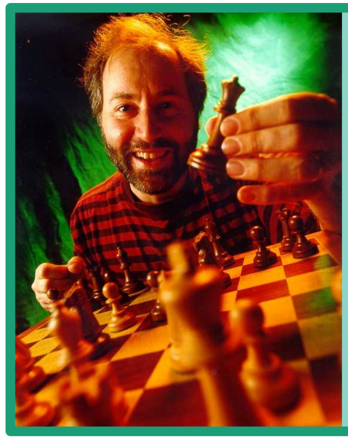
Working Set: Queries take $O(\log t)$ time (t: recency)

Static Optimality: Queries take (amortized) $O(\mathrm{OPT}_S)$ time.

... is there one BST to rule them all?

Yes!

Splay Trees

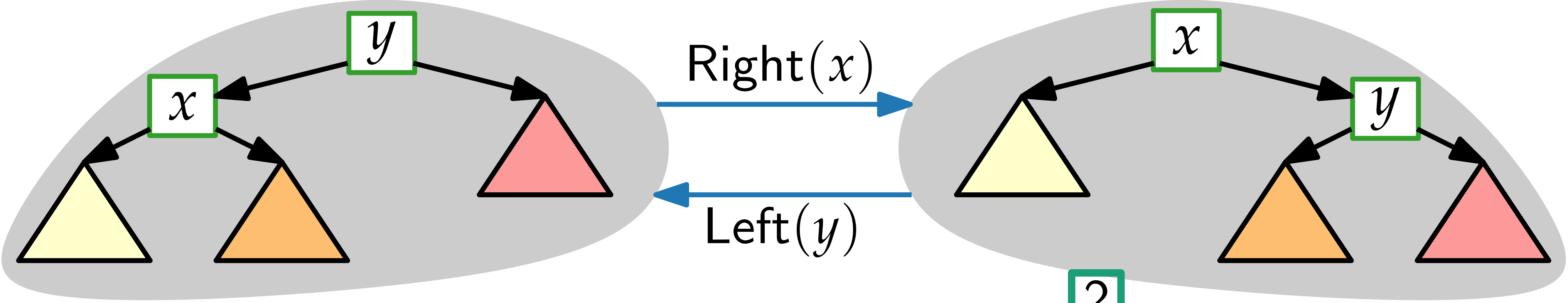


Daniel D. Sleator Robert E. Tarjan

J. ACM 1985

Idea: Whenever we query a key, rotate it to the root.

Known from the lecture *algorithms and data structures* (ADS):



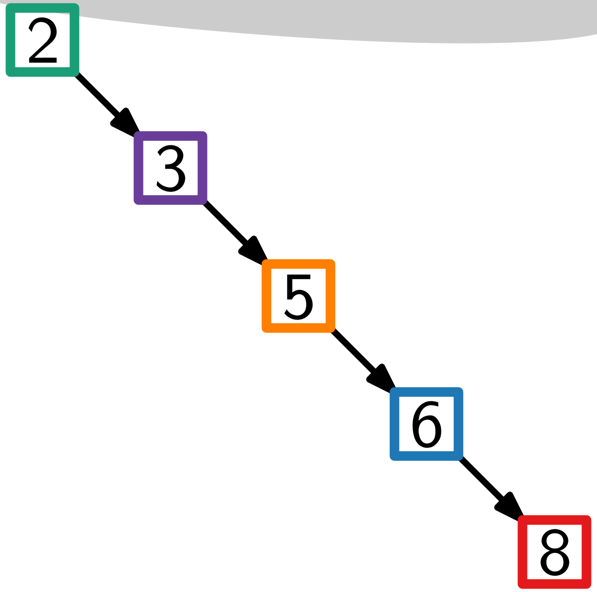
New:

Splay(x): Rotate x to the root

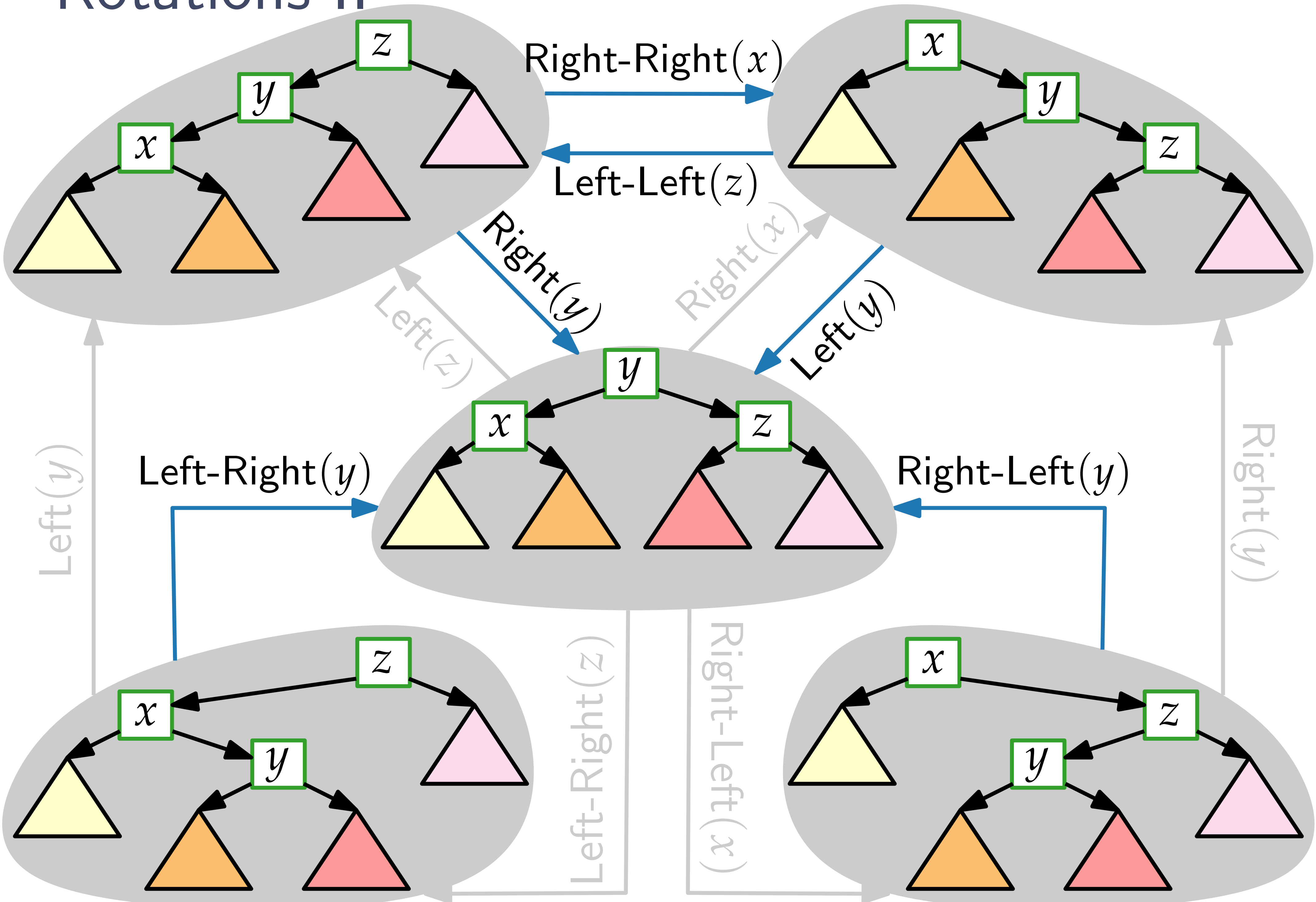
Query(x): Splay(x), then return root

Query(8) Query(6) Query(5)
Query(3)
Query(2)

We're back at the start...
and we did $\Theta(n^2)$ rotations



Rotations II

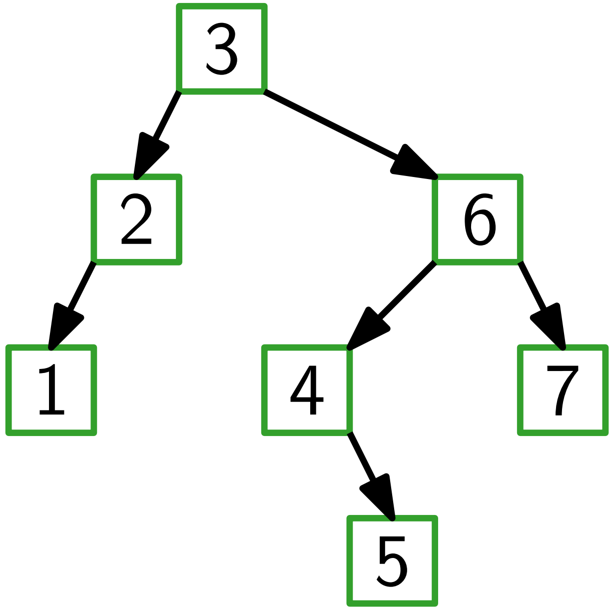


Splay

Algorithm: Splay(x)

if $x \neq \mathit{root}$ **then**
- $y =$ parent of x
- **if** $y = \mathit{root}$ **then**
  - **if** $x < y$ **then** Right(x)
  - **if** $y < x$ **then** Left(x)
- **else**
  - $z =$ parent of y
  - **if** $x < y < z$ **then** Right-Right(x)
  - **if** $z < y < x$ **then** Left-Left(x)
  - **if** $y < x < z$ **then** Left-Right(x)
  - **if** $z < x < y$ **then** Right-Left(x)
- Splay(x)

Splay(3):



Call Splay(x):

- after Search(x)
- after Insert(x)
- before Delete(x)

Why is Splay Fast?

$w(x)$: weight of x (here 1), $W = \sum w(x)$ (here n)

$s(x)$: sum of all $w(x)$ in subtree of x

mark edges:

→ $s(\text{child}) \leq s(\text{parent})/2$

→ $s(\text{child}) > s(\text{parent})/2$

Cost to query x: $O(\log W + \#\text{red})$

Idea: blue edges halve the weight

$\Rightarrow \#\text{blue} \in O(\log W)$

How can we amortize red edges?

Use sum-of-logs potential

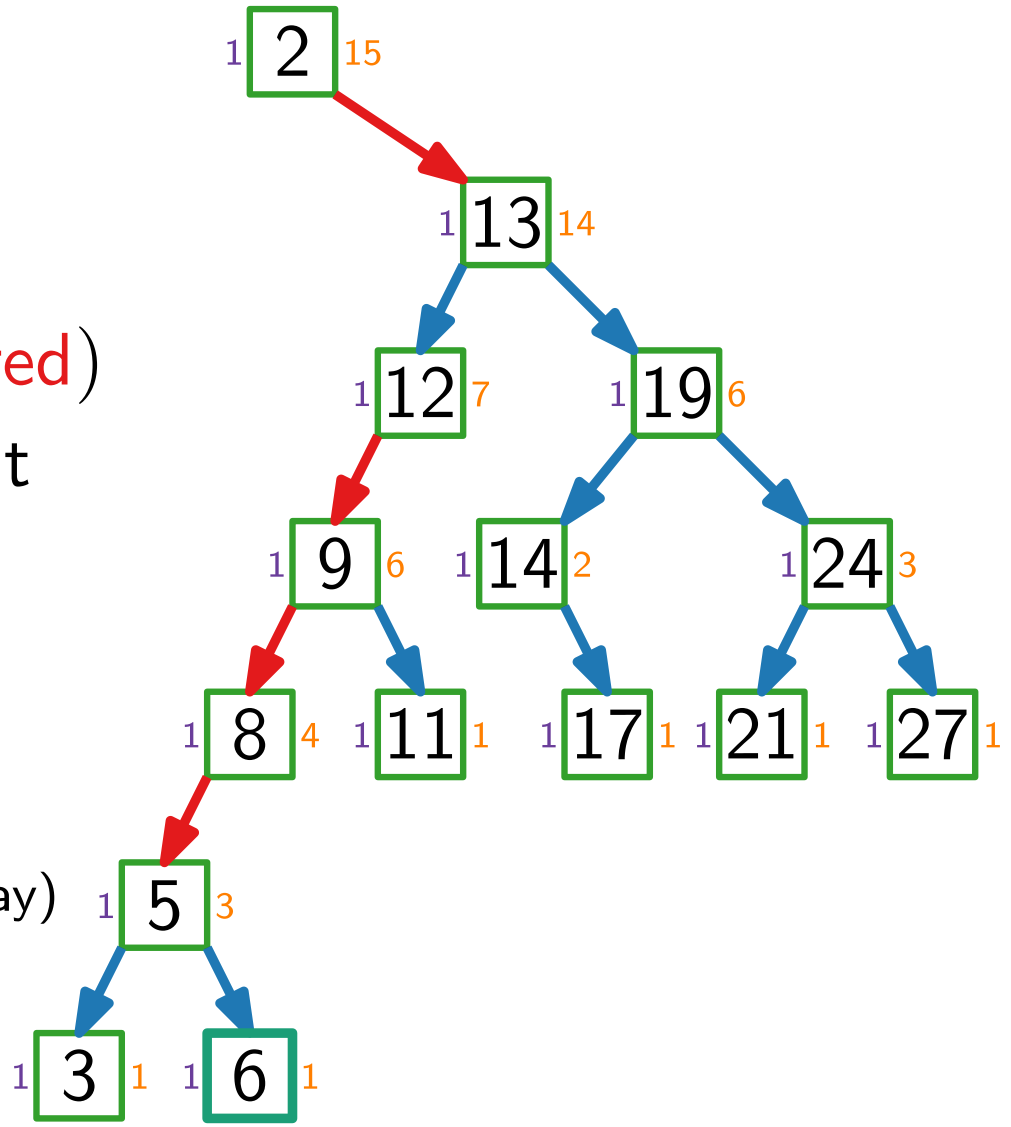
$\Phi = \sum \log s(x)$

Amortized cost:

(potential before splay)

real cost $+ \Phi_{+} - \Phi$

(potential after splay)



What is Potential?

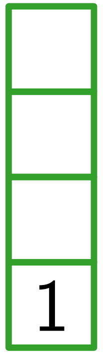
Φ represents work that has been “paid for” but not yet performed.

amortized cost per step: real cost $+\, \Phi_+ - \Phi$

total cost $= \Phi_0 - \Phi_{\text{end}} + \sum$ amortized cost

(initial potential) (potential at the end)

pop(2)



$\Phi = 1$

Example (from ADS): Stack with multipop

$\Phi :=$ size of the stack

push: $\quad 1 + \Phi_+ - \Phi \ = 2$

pop(k): $\quad k + \Phi_+ - \Phi \ = 0$

$$
\begin{aligned}
\text{total cost} &= \Phi_0 - \Phi_{\text{end}} + \text{amortized cost} \\
&\le \Phi_0 - \Phi_{\text{end}} + 2n \\
&\le 2n \in O(n)
\end{aligned}
$$

Why is Splay Fast?

$w(x)$: weight of x (here 1), $W = \sum w(x)$ (here n)
$s(x)$: sum of all $w(x)$ in subtree of x_i

mark edges:
→ $s(\text{child}) \leq s(\text{parent})/2$ (blue)
→ $s(\text{child}) > s(\text{parent})/2$ (red)

Cost to query x_i: $O(\log W + \#\text{red})$

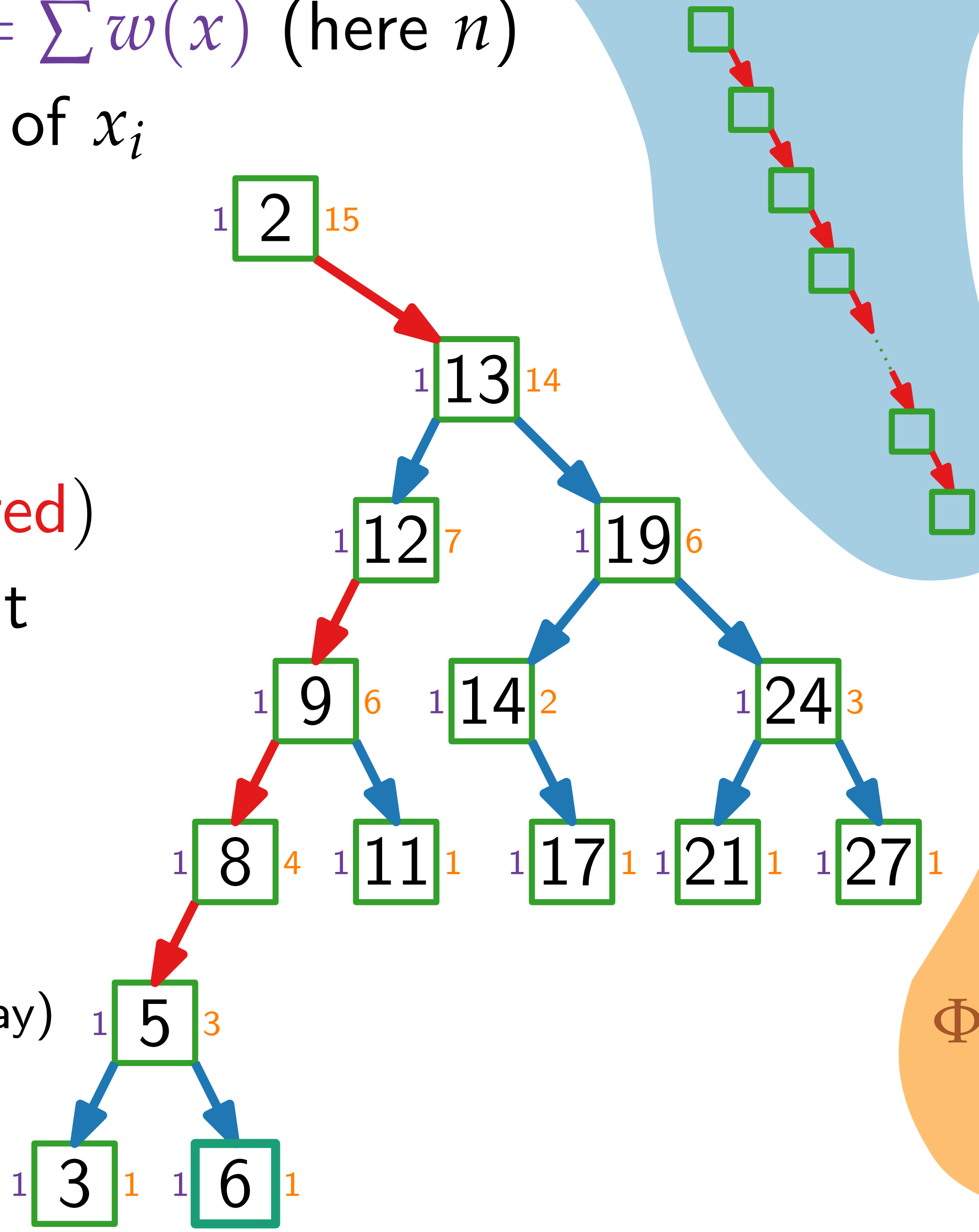
Idea: blue edges halve the weight
$\Rightarrow \#\text{blue} \in O(\log W)$

How can we amortize red edges?

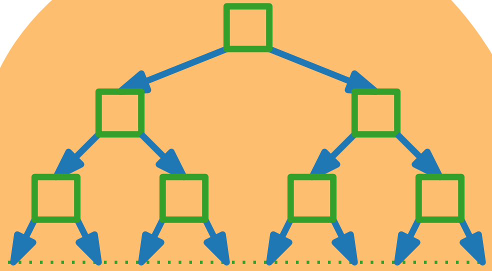
Use sum-of-logs potential
$\Phi = \sum \log s(x)$

Amortized cost:
real cost $+ \Phi_+ - \Phi$

(potential before splay)
(potential after splay)



$\Phi = \sum_{i=1}^{n} \log i$
$\in \Theta(n \log n)$



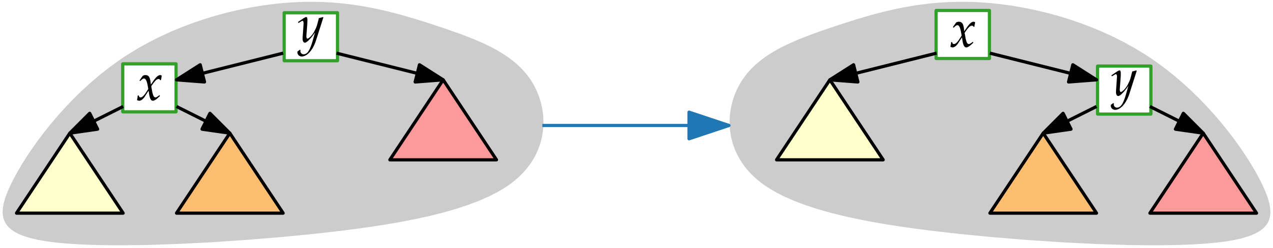
$\Phi \approx \sum_{i=0}^{\log n - 1} 2^i \log \frac{n}{2^i}$
$\in \Theta(n)$

Potential after Rotation

Consider any rotation; $s(x)$ before rotation, $s_+(x)$ afterwards

Lemma. After a single rotation, the potential increases by $\leq 3\,(\log s_+(x) - \log s(x))$.

Proof. Right(x)



Observe: Only $s(x)$ and $s(y)$ change.

pot. change $= \log s_+(x) + \log s_+(y) - \log s(x) - \log s(y)$

$(s_+(y) \leq s(y))$ $\leq \log s_+(x) - \log s(x)$

$(s_+(x) > s(x))$ $\leq 3\,(\log s_+(x) - \log s(x))$ ✓

Left(x) analogue ✓

□

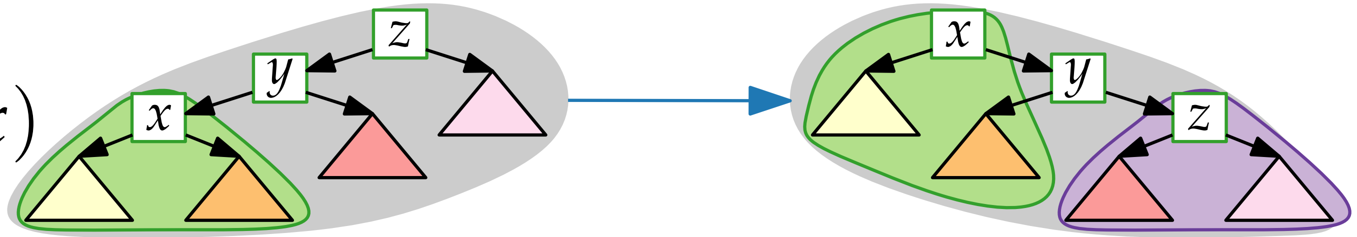
Potential after Rotation

Consider any rotation; $s(x)$ before rotation, $s_+(x)$ afterwards

Lemma. After a double rotation, the potential increases by $\leq 3\,(\log s_+(x) - \log s(x)) - 2$.

Proof.

Case 1. Right-Right(x)



$$
\begin{aligned}
\text{pot. change} \quad &= \log s_+(x) + \log s_+(y) + \log s_+(z) \\
&\quad - \log s(x) - \log s(y) - \log s(z) \\
(s_+(x) = s(z)) \quad &= \log s_+(y) + \log s_+(z) - \log s(x) - \log s(y) \\
(s(x) \leq s(y)) \quad &\leq \log s_+(y) + \log s_+(z) - 2\log s(x) \\
(s_+(y) \leq s_+(x)) \quad &\leq \log s_+(x) + \log s_+(z) - 2\log s(x)
\end{aligned}
$$

$(\star)$: $s(x) + s_+(z) \leq s_+(x)$

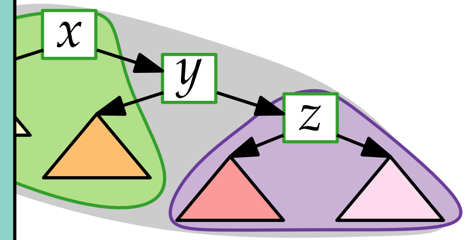
Potential after Rotation

Consider any rotation; $s(x)$ before rotation, $s_+(x)$ afterwards

Lemma. After a double rotation, the potential increases by $\leq 3\left(\log s_+(x) - \log s(x)\right) - 2$.

Proof.

Case 1.



pot. cha

$(s_+(x) =$ ... $g\, s(y)$

$(s(x) \leq s$

$(s_+(y) \leq$

Inequality of arithmetic and geometric means (AM-GM):

$$\frac{x_1+x_2+\dots+x_k}{k} \geq \sqrt[k]{x_1 \cdot x_2 \cdot \ldots \cdot x_k}$$

(arithmetic mean) (geometric mean)

for $k = 2$:

$$\frac{x+y}{2} \geq \sqrt{xy} \;\Rightarrow xy \leq \left(\frac{x+y}{2}\right)^2$$

$(\star)$: $s(x) + s_+(z) \leq s_+(x)$

$$\log s(x) + \log s_+(z) = \log(s(x)s_+(z)) \underset{\text{(AM-GM)}}{\leq} \log(((s(x) + s_+(z))/2)^2) \underset{(\star)}{\leq} \log((s_+(x)/2)^2)$$

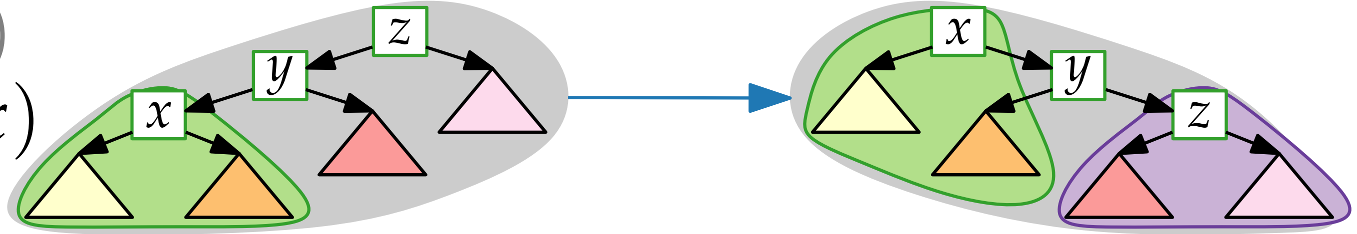
Potential after Rotation

Consider any rotation; $s(x)$ before rotation, $s_+(x)$ afterwards

Lemma. After a double rotation, the potential increases by $\leq 3\,(\log s_+(x) - \log s(x)) - 2$.

Proof. / Left-Left(x)

Case 1. Right-Right(x)



$$
\begin{aligned}
\text{pot. change} \quad &= \log s_+(x) + \log s_+(y) + \log s_+(z) \\
&\quad - \log s(x) - \log s(y) - \log s(z) \\
(s_+(x) = s(z)) \quad &= \log s_+(y) + \log s_+(z) - \log s(x) - \log s(y) \\
(s(x) \leq s(y)) \quad &\leq \log s_+(y) + \log s_+(z) - 2\log s(x) \\
(s_+(y) \leq s_+(x)) \quad &\leq \log s_+(x) + \log s_+(z) - 2\log s(x) \\
&\leq 3\log s_+(x) - 3\log s(x) - 2 \quad \checkmark
\end{aligned}
$$

$(\star)$: $s(x) + s_+(z) \leq s_+(x)$

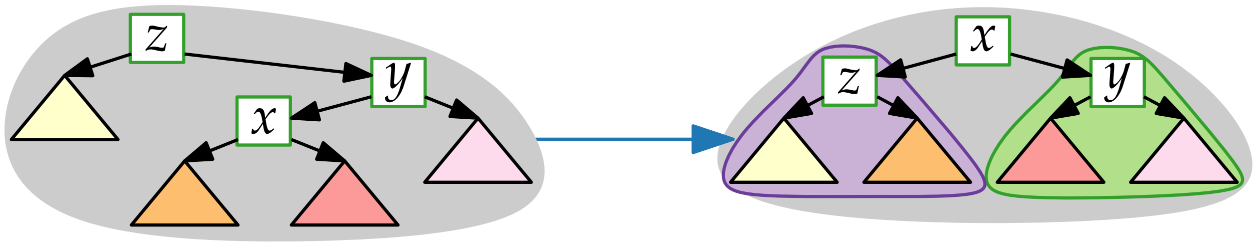
$$
\log s(x) + \log s_+(z) = \log(s(x)s_+(z)) \underset{\text{(AM-GM)}}{\leq} \log(((s(x) + s_+(z))/2)^2) \underset{(\star)}{\leq} \log((s_+(x)/2)^2) = 2\log s_+(x) - 2
$$

Potential after Rotation

Consider any rotation; $s(x)$ before rotation, $s_+(x)$ afterwards

Lemma. After a double rotation, the potential increases by $\leq 3\left(\log s_+(x) - \log s(x)\right) - 2$.

Proof. / Left-Right(x)
Case 2. Right-Left(x)



$$
\begin{aligned}
\text{pot. change} \quad & = \log s_+(x) + \log s_+(y) + \log s_+(z) \\
& \quad - \log s(x) - \log s(y) - \log s(z) \\
(s_+(x) = s(z)) \quad & = \log s_+(y) + \log s_+(z) - \log s(x) - \log s(y) \\
(s(x) \leq s(y)) \quad & \leq \log s_+(y) + \log s_+(z) - 2\log s(x) \\
& \leq 2\log s_+(x) - 2\log s(x) - 2 \\
(s_+(x) > s(x)) \quad & \leq 3\log s_+(x) - 3\log s(x) - 2 \quad \checkmark
\end{aligned}
$$

$\square$

$(\star)$: $s_+(y) + s_+(z) \leq s_+(x)$ | $\log s_+(y) + \log s_+(z) \underset{(\star)}{\overset{}{\underset{\text{(AM-GM)}}{\leq}}} 2\log s_+(x) - 2$

Access Lemma

Lemma. After a single rotation, the potential increases by $\leq 3\,(\log s_+(x) - \log s(x))$.
After a double rotation, the potential increases by $\leq 3\,(\log s_+(x) - \log s(x)) - 2$.

Lemma. The (amortized) cost of Splay(x) is $c(\text{Splay}(x)) \leq 1 + 3\log(W/w(x))$.

Proof. W.l.o.g. k double rotations and 1 single rotation.
Let $s_i(x)$ be $s(x)$ after i single/double rotations.
Potential increases by at most

$$\sum_{i=1}^{k} (3\,(\log s_i(x) - \log s_{i-1}(x)) - 2)$$
$$+3\,(\log s_{k+1}(x) - \log s_k(x))$$

root!

(id. entries rem.) $= 3\,(\log s_{k+1}(x) - \log s(x)) - 2k$
$= 3\,(\log W - \log s(x)) - 2k$
$(s(x) \geq w(x))$ $\leq 3\,(\log W - \log w(x)) - 2k \;= 3\log(W/w(x)) - 2k$

$2k+1$ rotations $\Rightarrow$ (amort.) cost $c(\text{Splay}(x)) \leq 1 + 3\log(W/w(x))$ $\square$

All These Models . . .

Balanced: Queries take (amortized) $O(\log n)$ time

Entropy: Queries take expected $O(1 + H)$ time

Dynamic Finger: Queries take $O(\log \delta_i)$ time (δ_i: rank diff.)

Working Set: Queries take $O(\log t)$ time (t: recency)

Static Optimality: Queries take (amortized) $O(\mathrm{OPT}_S)$ time.

... is there one BST to rule them all?

All of these properties can be shown by chosing the weight function accordingly.
Note that the actual algorithm is always the same!

Yes!

Querying a Sequence

Let S be a sequence of queries.

What is the *real* cost of querying S?

Let Φ_i be the potential after query i.

(amort. cost to execute Splay(x))

$\Rightarrow$ total cost $= \Phi_0 - \Phi_{|S|} + \sum_{x \in S} c(\text{Splay}(x))$

How can we bound $\Phi_0 - \Phi_{|S|}$?

Reminder: $\Phi = \sum \log s(x)$

$s(x) \geq w(x) \quad\quad \Rightarrow \Phi_{|S|} \geq \sum_{x \in T} \log w(x)$

$s(\text{root}) = \log W \quad\quad \Rightarrow \Phi_0 \leq \sum_{x \in T} \log W$

$\Rightarrow \Phi_0 - \Phi_{|S|} \leq \sum_{x \in T} (\log W - \log w(x)) \leq \sum_{x \in T} O\left(c(\text{Splay}(x))\right)$

$\Rightarrow$ as long as every key is queried at least once, it doesn't change the asymptotic running time.

Balance

Lemma. The (amortized) cost of Splay(x) is $c(\text{Splay}(x)) \leq 1 + 3\log(W/w(x))$.

Definition. A BST is **balanced** if the (amortized) cost of *any* query is $O(\log n)$ (for at least n queries in total).

Theorem. Splay Trees are balanced.

Proof. Choose $w(x) = 1$ for each x $\Rightarrow W = n$

Splay(x) costs at least as much as finding x

$$\begin{aligned}\Rightarrow \text{total time} &= \Phi_0 - \Phi_{|S|} + \textstyle\sum_{x \in S} c(\text{Splay}(x)) \\ &\leq \textstyle\sum_{x \in T}(\log W - \log w(x)) + \sum_{x \in S} c(\text{Splay}(x)) \\ &\leq n \log n + \textstyle\sum_{x \in S} (1 + 3\log(W/w(x))) \\ &\leq n \log n + |S| + 3|S| \log n \quad \in O(|S| \log n)\end{aligned}$$

$\Rightarrow$ Queries take (amort.) $O(\log n)$ time. $\square$

Entropy

Lemma. The (amortized) cost of Splay(x) is $c(\text{Splay}(x)) \leq 1 + 3\log(W/w(x))$.

Definition. A BST has the **entropy property** if queries take expected $O(1 - \sum_{i=1}^{n} p_i \log p_i)$ time.

Theorem. Splay Trees have the entropy property.

Proof. Choose $w(x_i) = p_i \quad \Rightarrow W = 1$

Amortized cost to query x_i:

$$\leq 1 + 3\log(W/w(x_i))$$
$$= 1 + 3\log(1/p_i)$$
$$= 1 - 3\log p_i$$

$\Rightarrow$ expected query time:

$$O(\textstyle\sum_{i=1}^{n} p_i(1 - 3\log p_i)) = O(1 - \sum_{i=1}^{n} p_i \log p_i)$$

$\square$

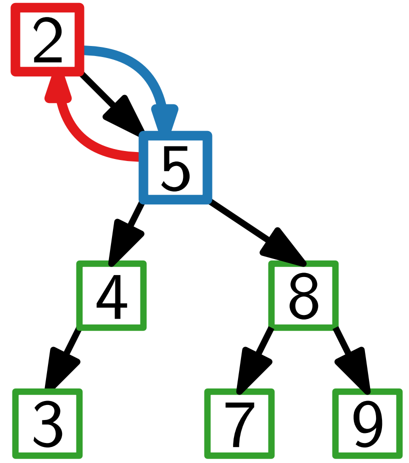
Static Optimality

Given a sequence S of queries.
Let $T^\star_S$ be an *optimal* static tree with the shortest query time OPT_S for S.

e.g. $S = 2, 5, 2, 5, 2, \dots, 5$

$T^\star$:

OPT: $|S|$



Definition. A BST is **statically optimal** if queries take (amort.) $O(\mathrm{OPT}_S)$ time for every S.

Theorem. Splay Trees are statically optimal.

Proof. Let f_i be the depth of x_i in $T^\star$ (root has depth 1).
Let $w_i := 3^{-f_i}$. $\Rightarrow W \leq 1$

$$\Rightarrow c(\mathrm{Splay}(x_i)) = 1 + 3\log(W / w(x_i))$$
$$\leq 1 + 3\log 3^{f_i} \in O(f_i)$$

$\square$

Dynamic Optimality

Given a sequence S of queries.

Let $D_S^\star$ be an optimal *dynamic* tree with the shortest query time $\mathrm{OPT}_S^\star$ for S.

(That is, modifications are allowed, e.g., rotations)

Definition. A BST is **dynamically optimal** if queries take (amort.) $O(\mathrm{OPT}_S^\star)$ time for every S.

Splay Trees: Queries take $O(\mathrm{OPT}_S^\star \cdot \log n)$ time.

Tango Trees: Queries take $O(\mathrm{OPT}_S^\star \cdot \log\log n)$ time.

[Demaine, Harmon, Iacono, Pătrașcu '04]

Open Problem. Does a dynamically optimal BST exist?

This is one of the biggest open problems in algorithms.

Conjecture. Splay Trees are dynamically optimal.