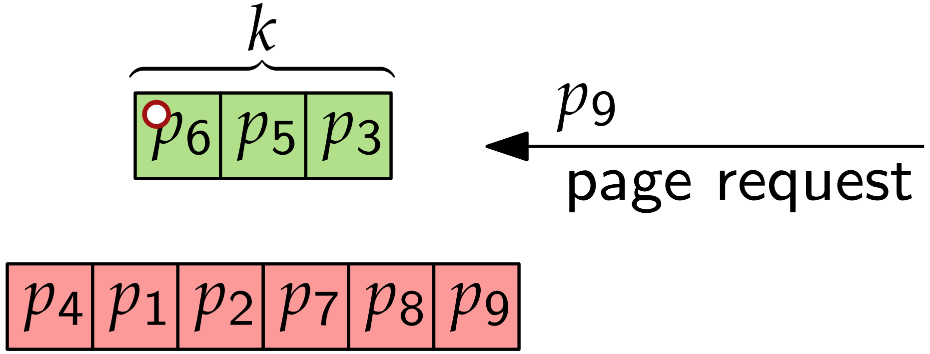
Julius-Maximilians-
UNIVERSITÄT
WÜRZBURG

Advanced Algorithms

Online Algorithms

Ski-Rental Problem and Paging

Johannes Zink · WS23/24



Introduction

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- Is it worth **buying** new skis?
- Or should we rather **rent** them?

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- But what if there is not always enough snow? Or snow but “bad” weather?
- Is it worth **buying** new skis?
- Or should we rather **rent** them?
- We don’t know the weather (much) in advance.

Ski-Rental Problem – Definition

Behavior.

- Every day when there is "good" weather, you go skiing.
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Task.

- Not knowing T, devise a strategy if and when to buy skis.

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- Case $T = \alpha M$: $\frac{E[c_{\text{StrategyIV}}]}{c_{\text{OPT}}} = \frac{\frac{1}{2}\cdot\alpha M+\frac{1}{2}\cdot((1+\alpha)M-1)}{\alpha M} = 1 + \frac{1}{2\alpha} - \frac{1}{2\alpha M} \overset{M\rightsquigarrow\infty}{=} 1 + \frac{1}{2\alpha} = 2$

not better than the deterministic Strategy III

Ski-Rental Problem – Strategy IV

Renting costs 1 per day
Buying costs M
T good days

Can we get below this bound using randomization? – Let's try!

Strategy IV: throw a coin; **HEADS:** buy on the **M**-th good day
TAILS: buy on the α**M**-th good day ($\alpha \in (0,1)$)

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- Case $T = \alpha M$: $\frac{E[c_{\text{StrategyIV}}]}{c_{\text{OPT}}} = \frac{\frac{1}{2}\cdot\alpha M+\frac{1}{2}\cdot((1+\alpha)M-1)}{\alpha M} = 1 + \frac{1}{2\alpha} - \frac{1}{2\alpha M} \overset{M\rightsquigarrow\infty}{=} 1 + \frac{1}{2\alpha}$
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$\Rightarrow$ Strategy IV (with $\alpha = \frac{\sqrt{5}-1}{2} \approx 0.62$) is 1.81-competitive, randomized, and better than any deterministic strategy.

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$\Rightarrow$ Strategy IV (with $\alpha = \frac{\sqrt{5}-1}{2} \approx 0.62$) is 1.81-competitive, randomized, and better than any deterministic strategy.

- With a more sophisticated probability distribution for the time we buy skis, we can expect even a competitive ratio of $\frac{e}{e-1} \approx 1.58$.

Online vs. Offline Algorithms

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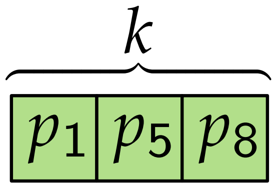

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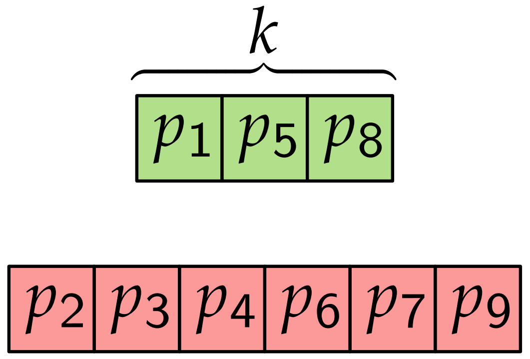

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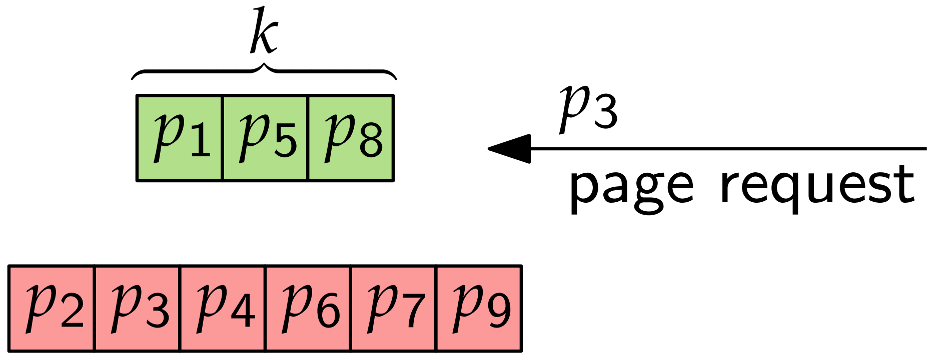

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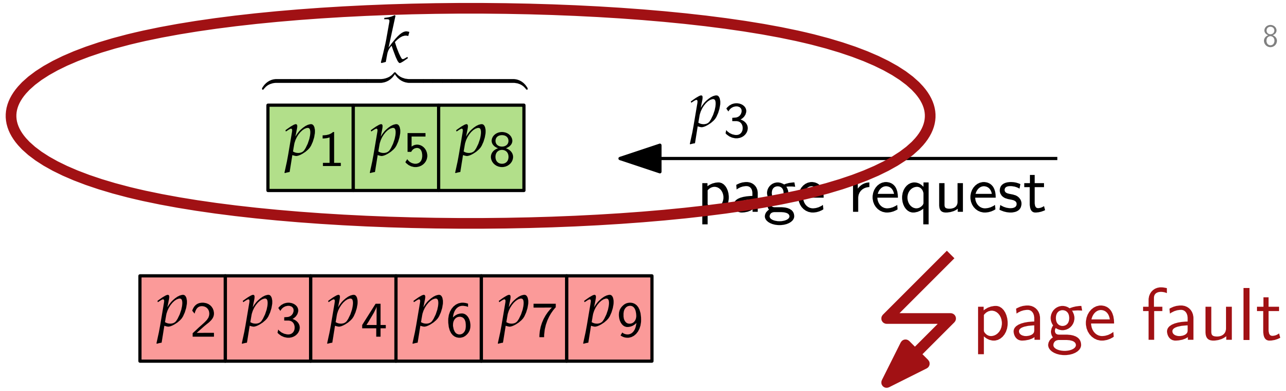

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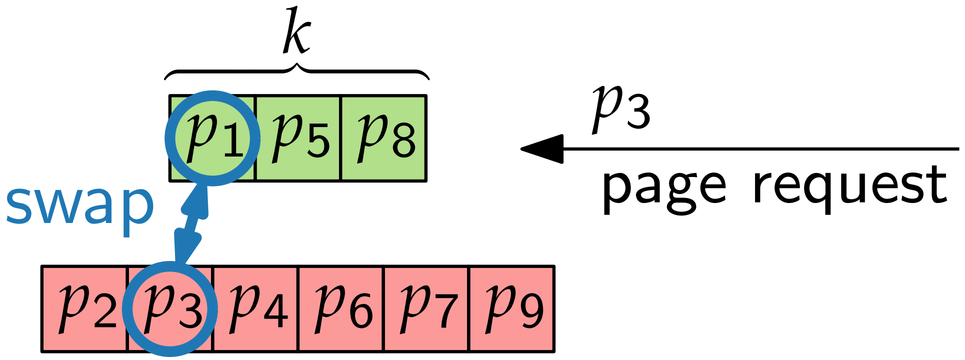

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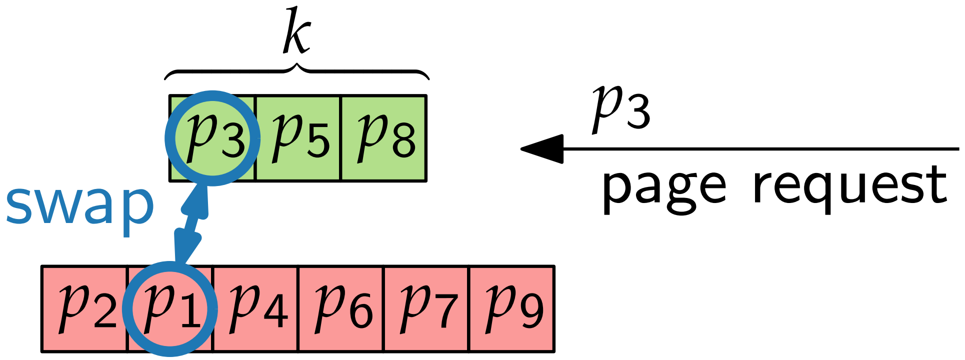

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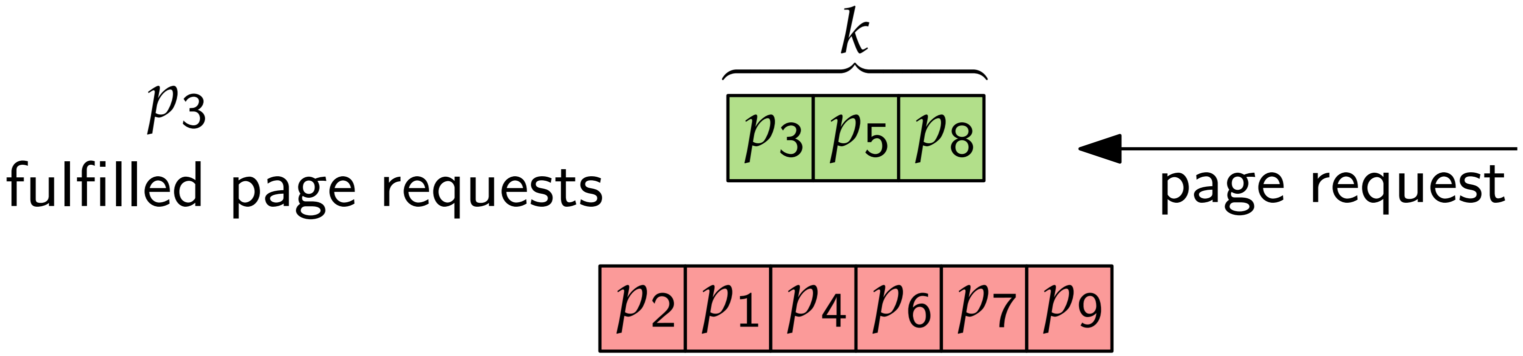

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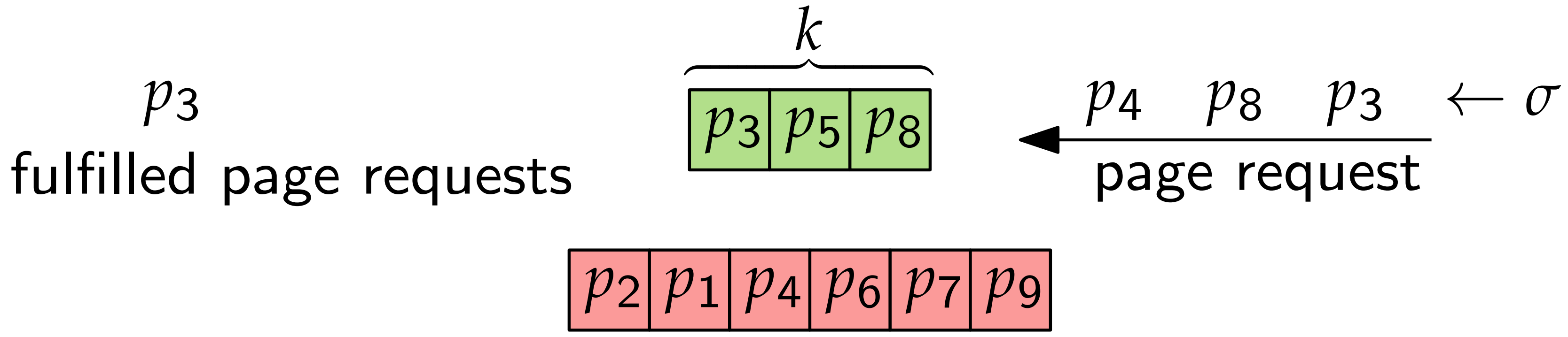

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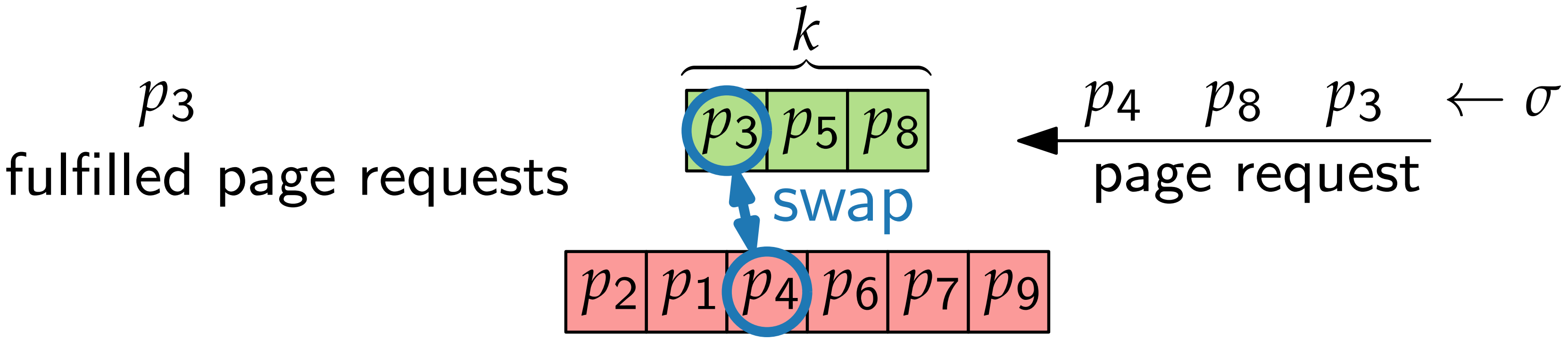

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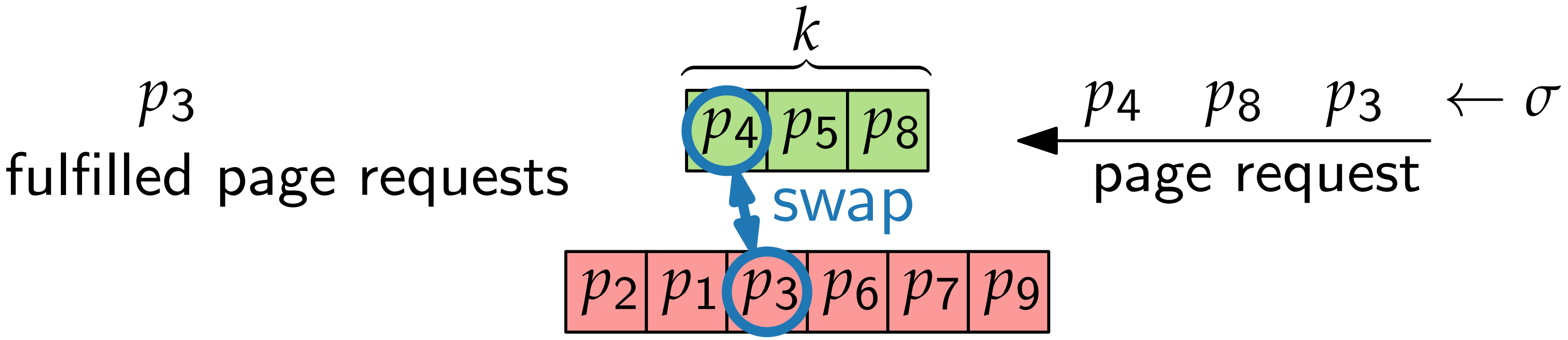

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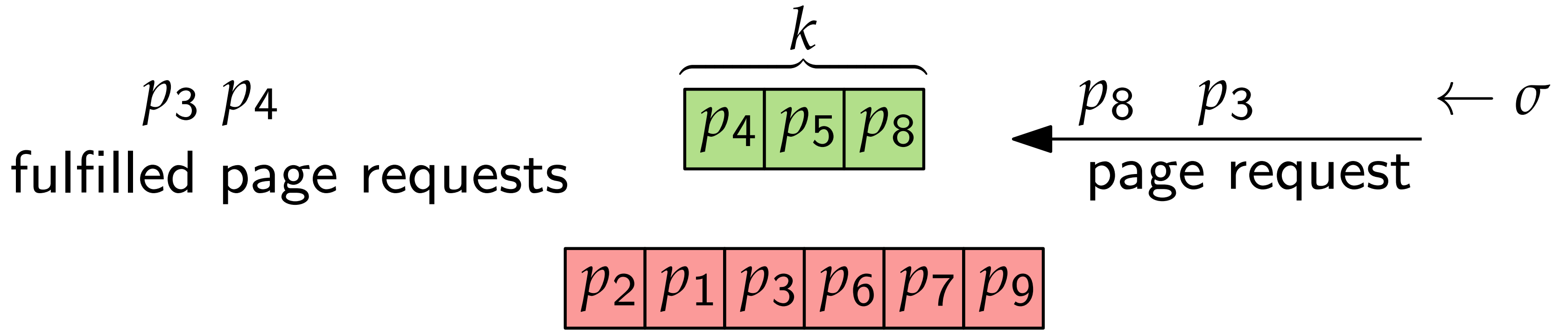

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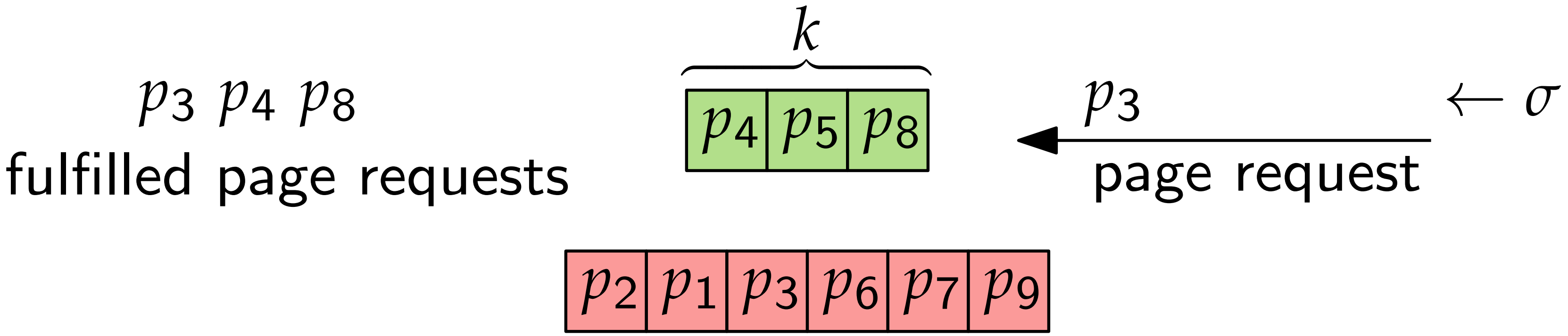

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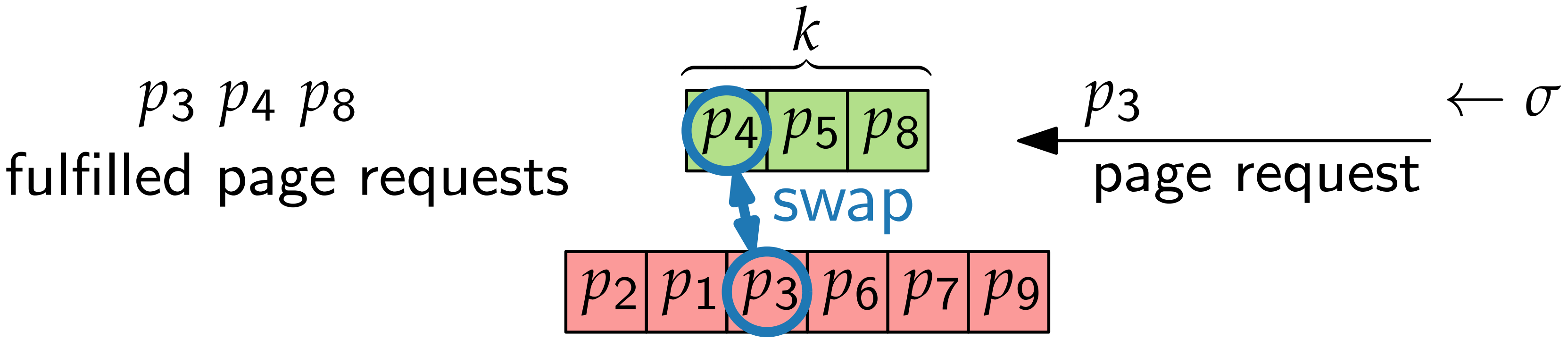

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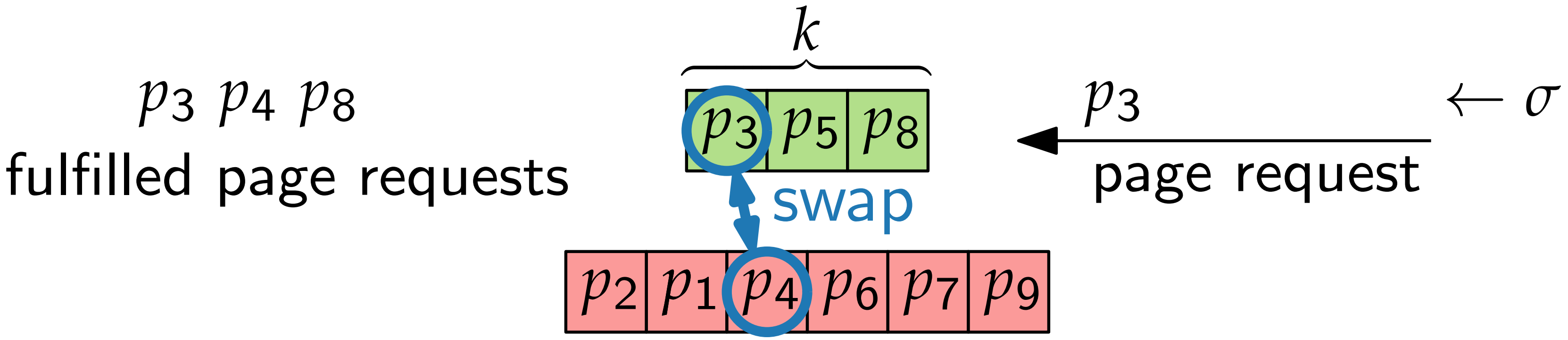

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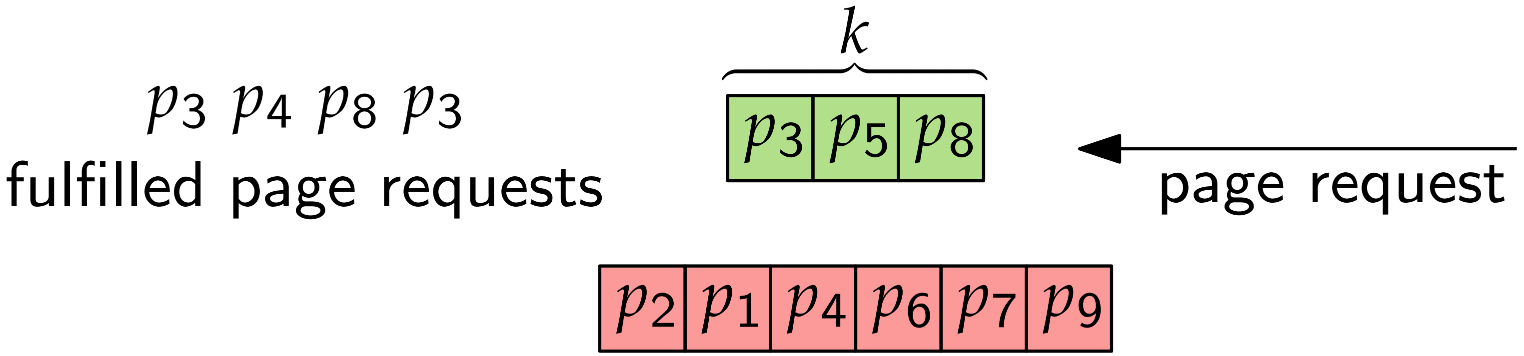

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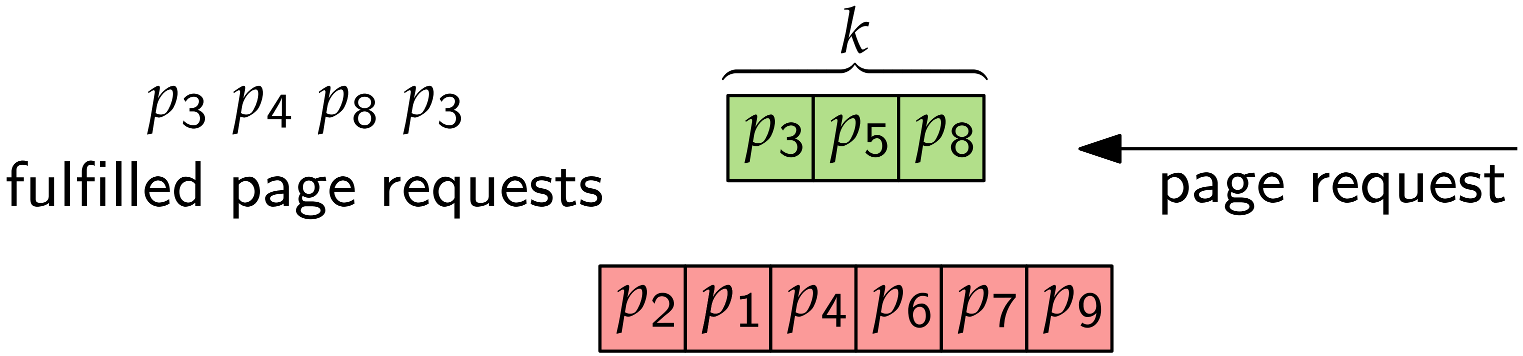

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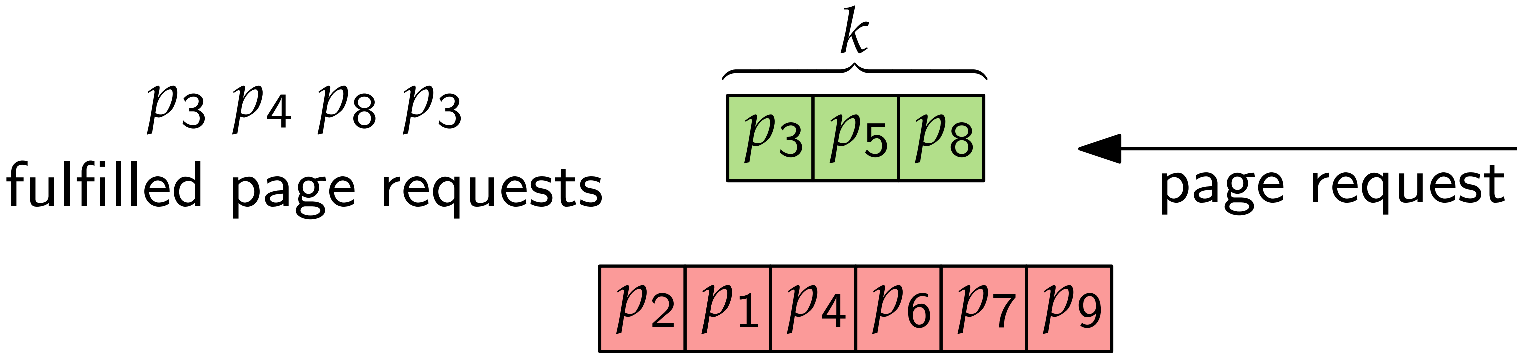

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- Minimize the number of page faults while fulfilling σ.

Paging – Det. Strat.

- On a page fault, a Paging algorithm chooses which page to evict from the cache.

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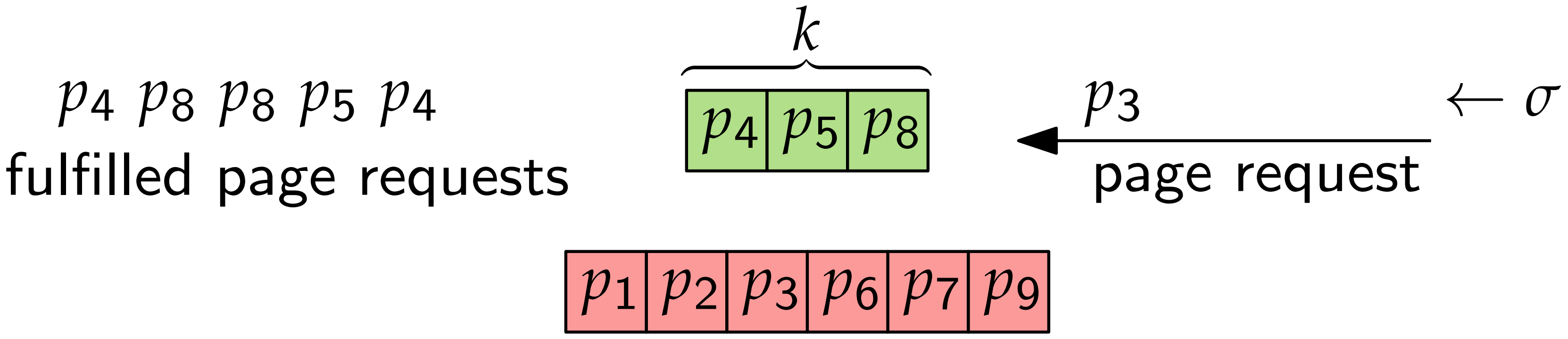

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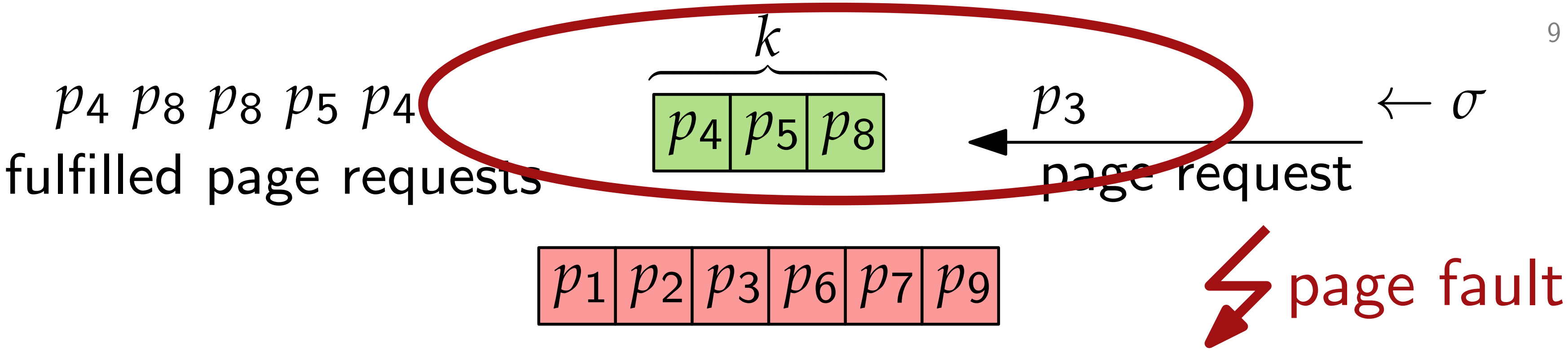

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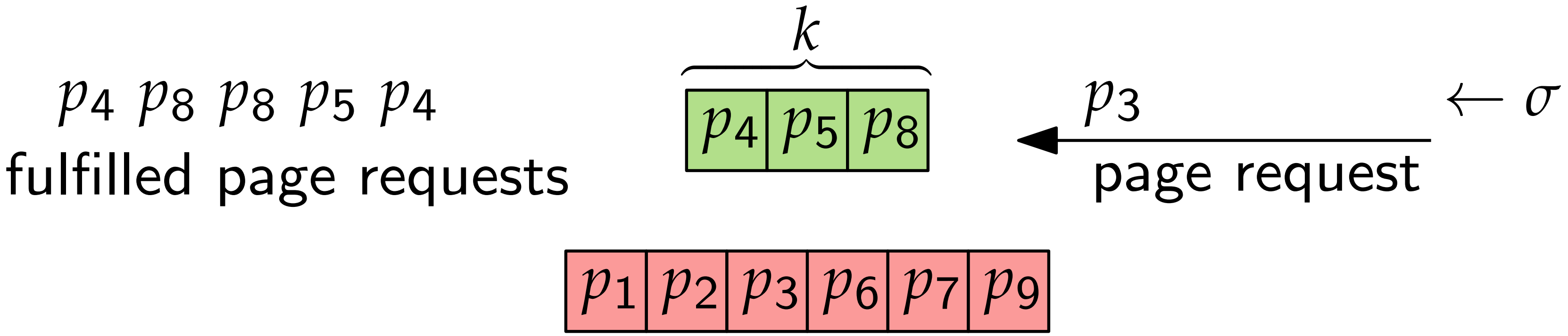

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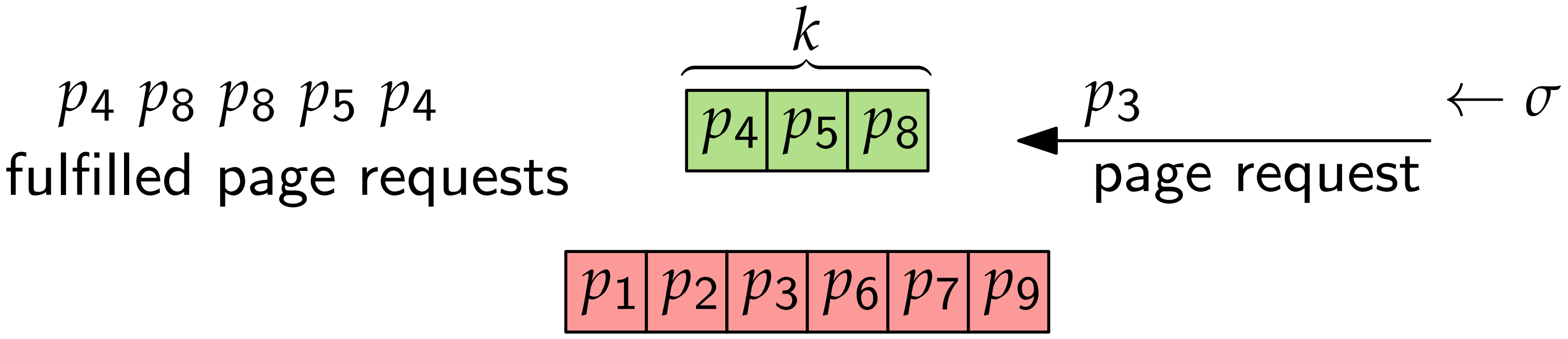

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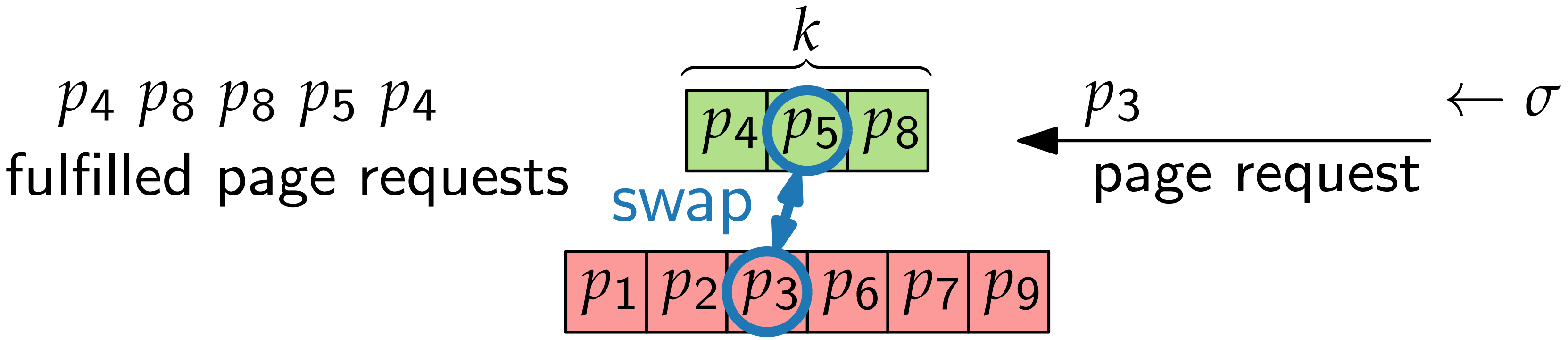

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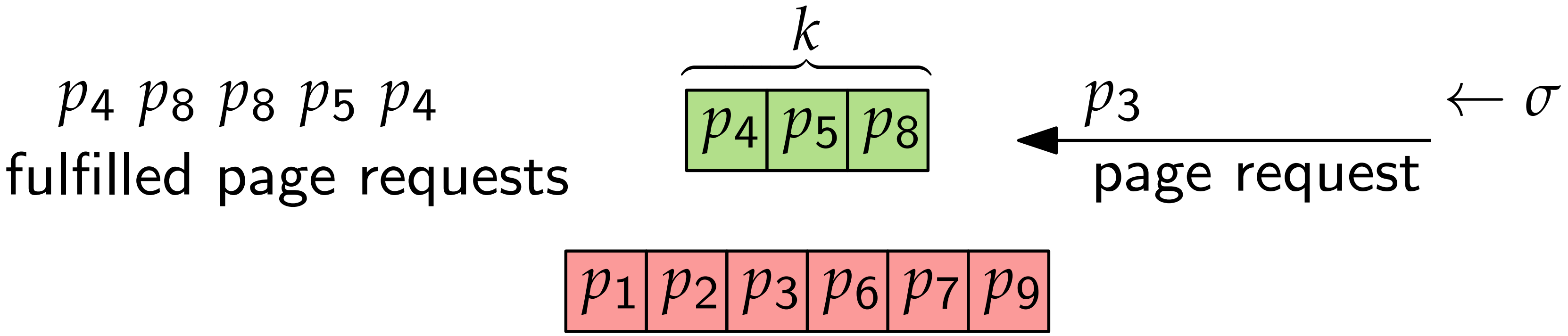

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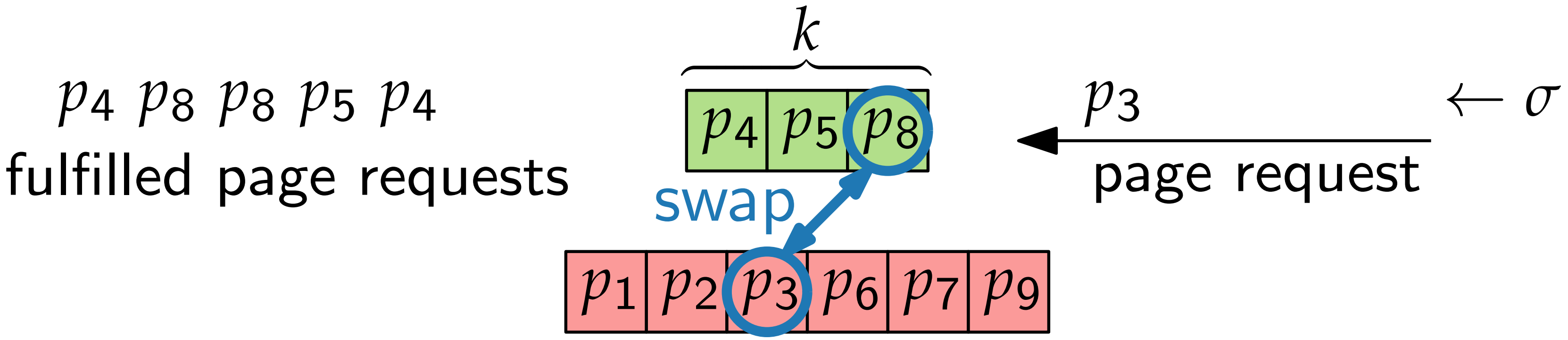

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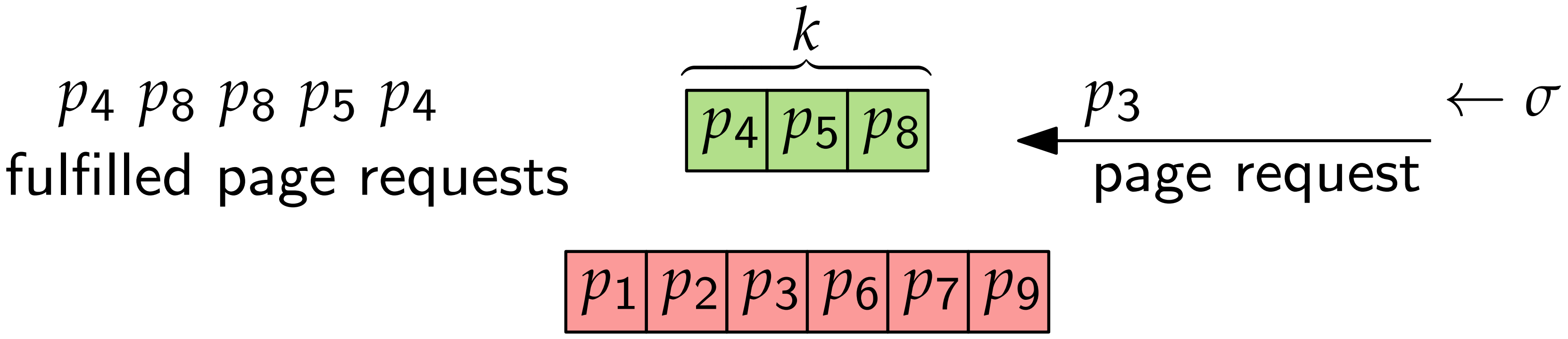

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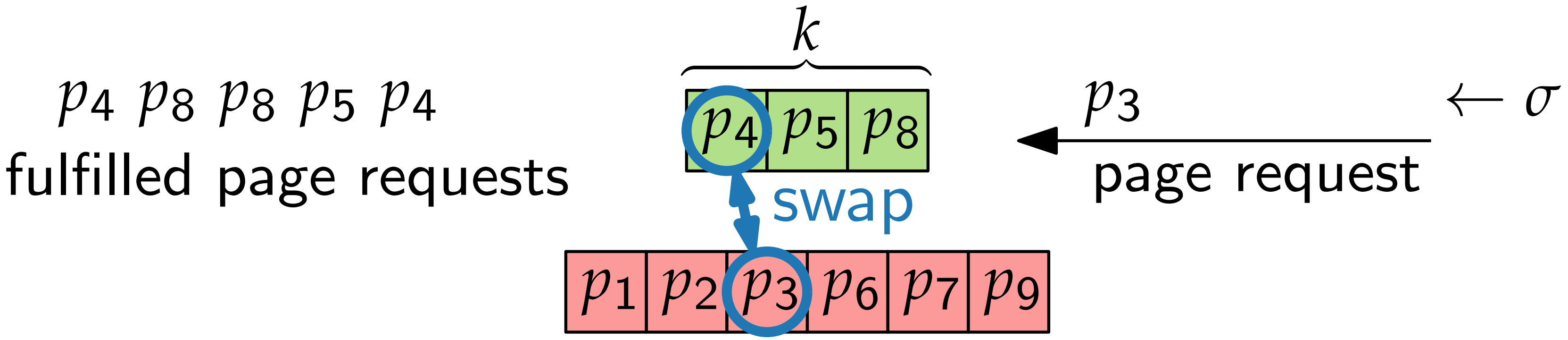
Paging – Det. Strat.



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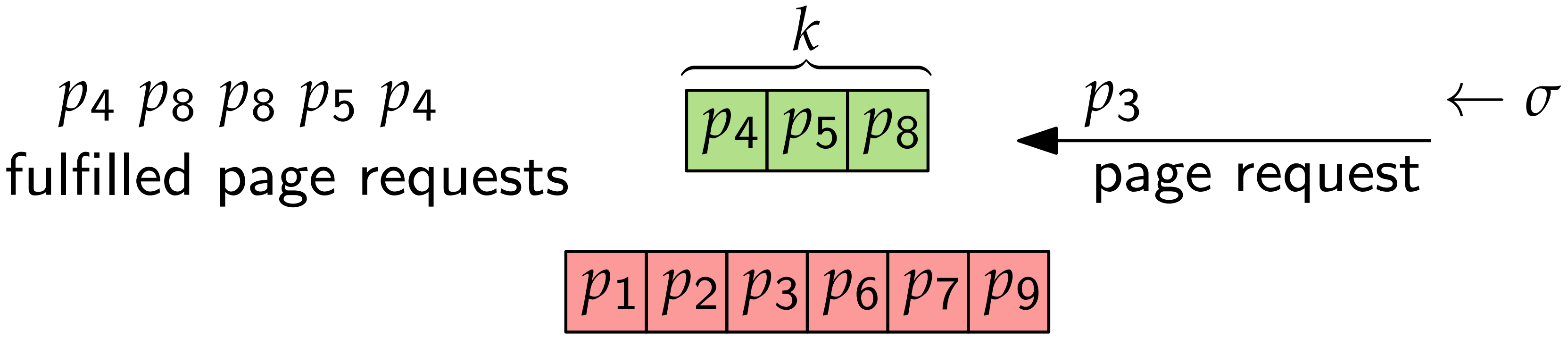

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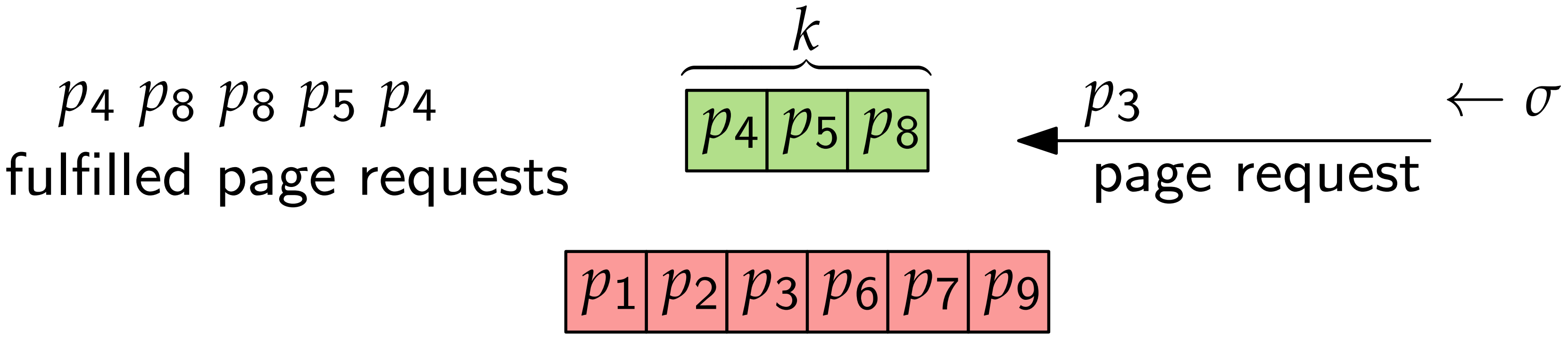

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$\Rightarrow$ The competitive ratio cannot be better than $\frac{|\sigma^\star|}{\left\lceil \frac{|\sigma^\star|}{k} \right\rceil} \overset{|\sigma^\star| \rightsquigarrow \infty}{=} k$.

□

Paging – Rand. Strat.

Randomized strategy: MARKING

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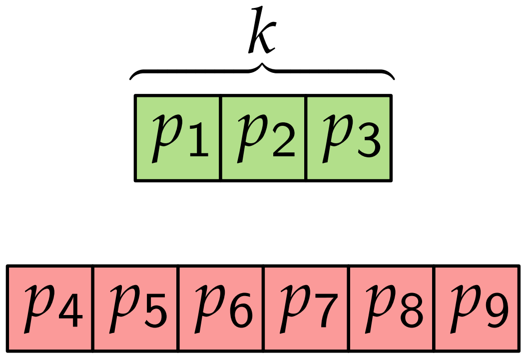

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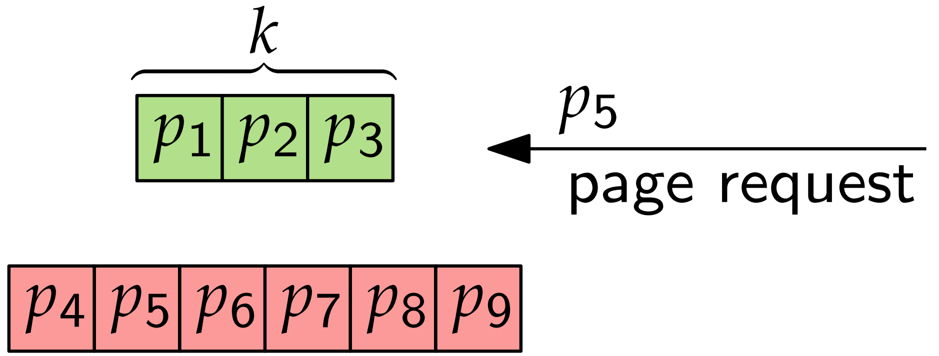

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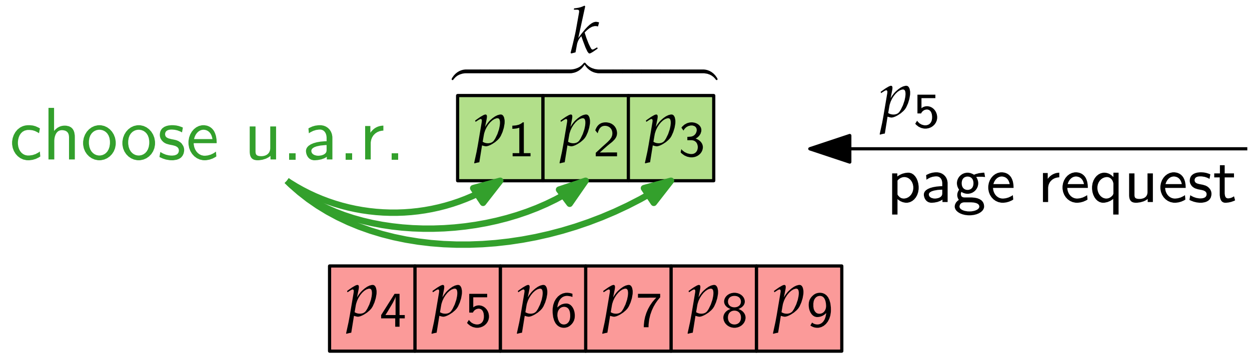

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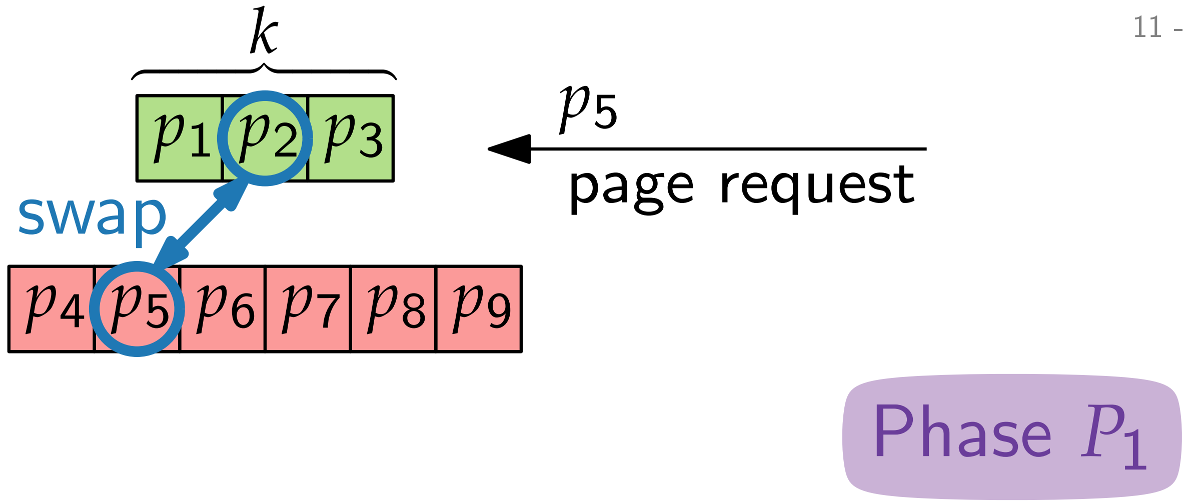

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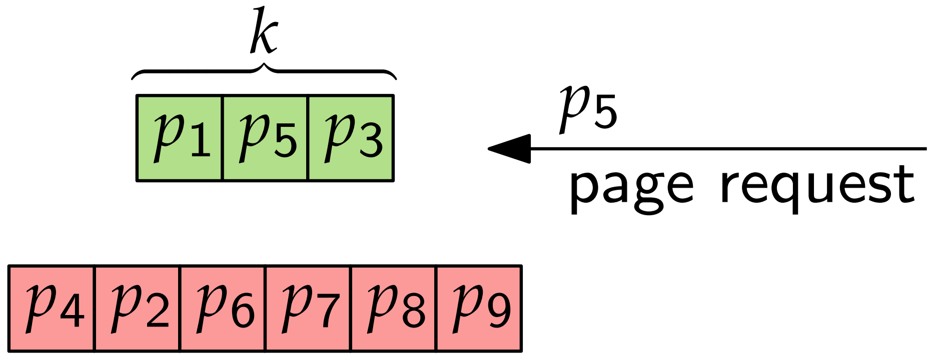

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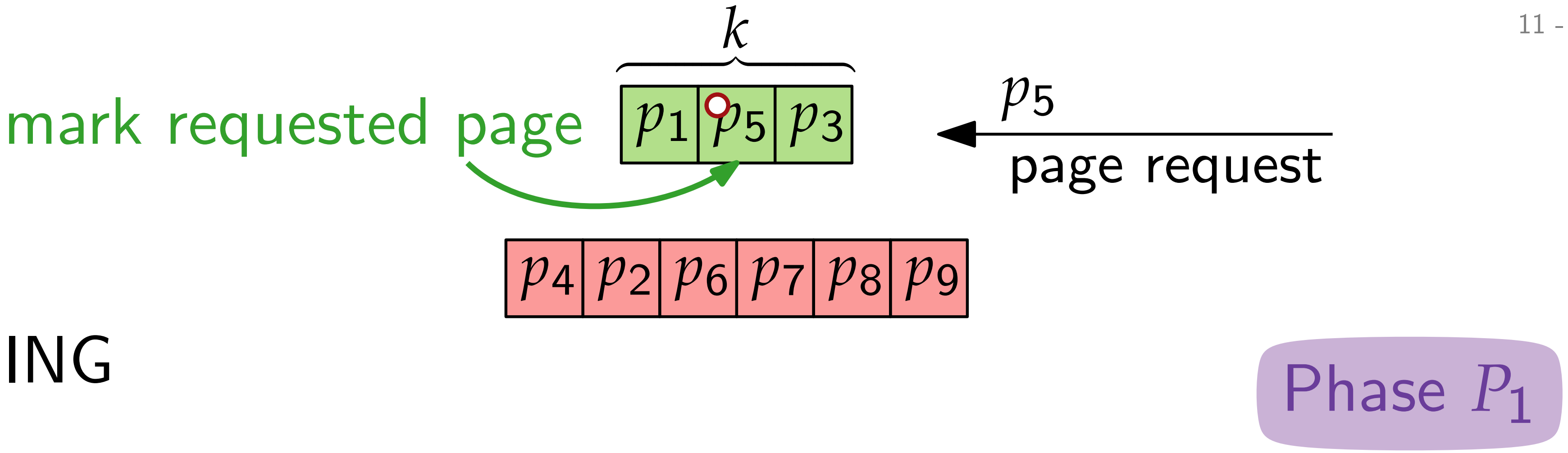

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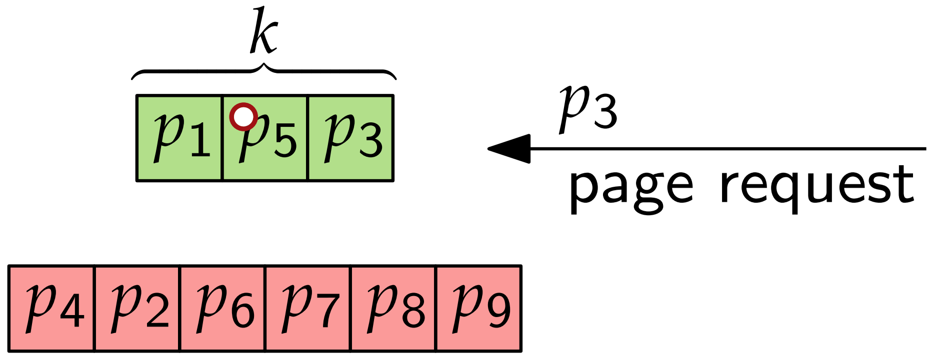

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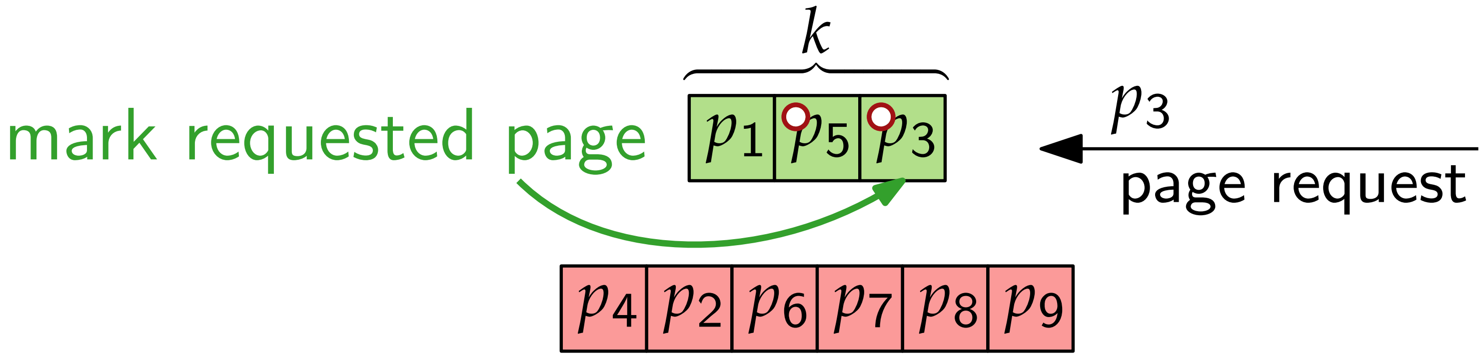

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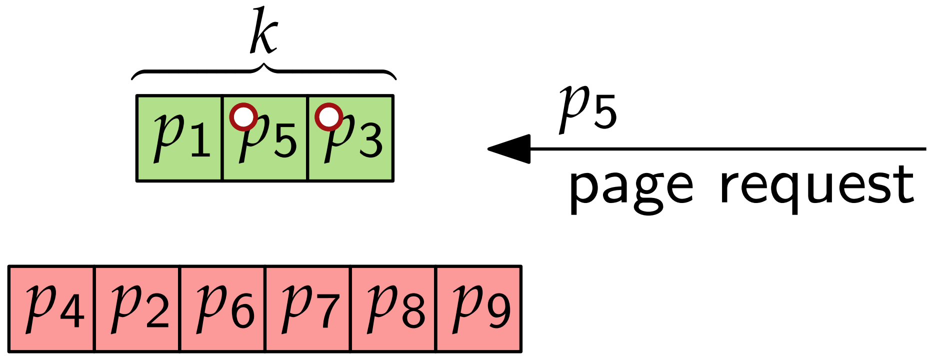

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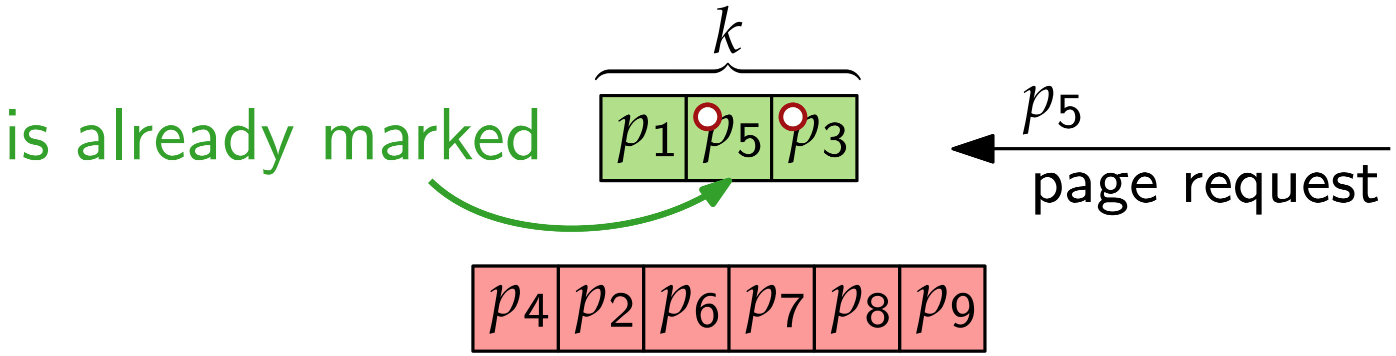

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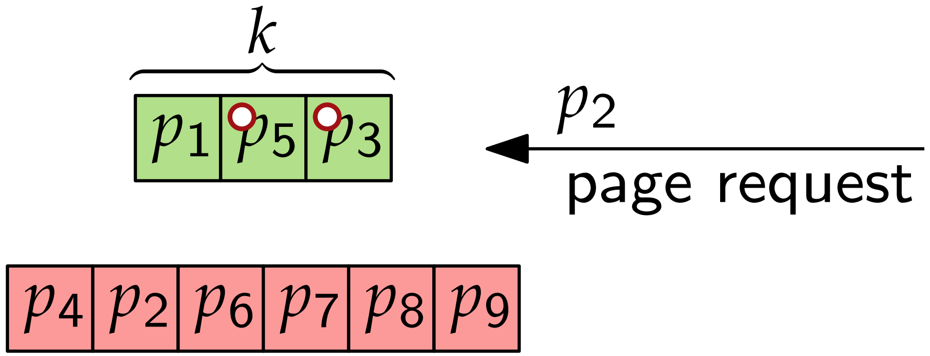

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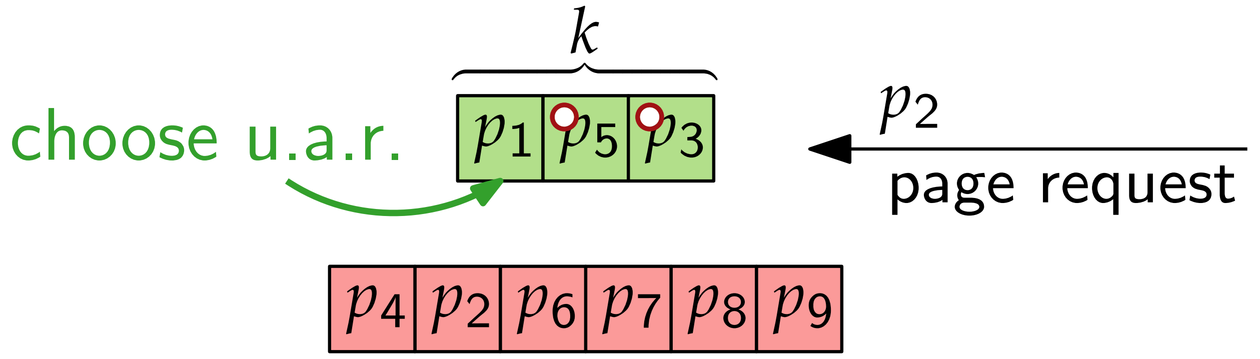

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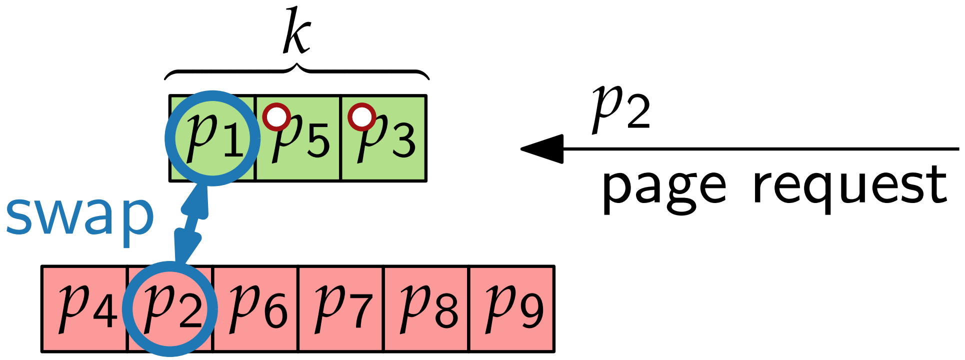

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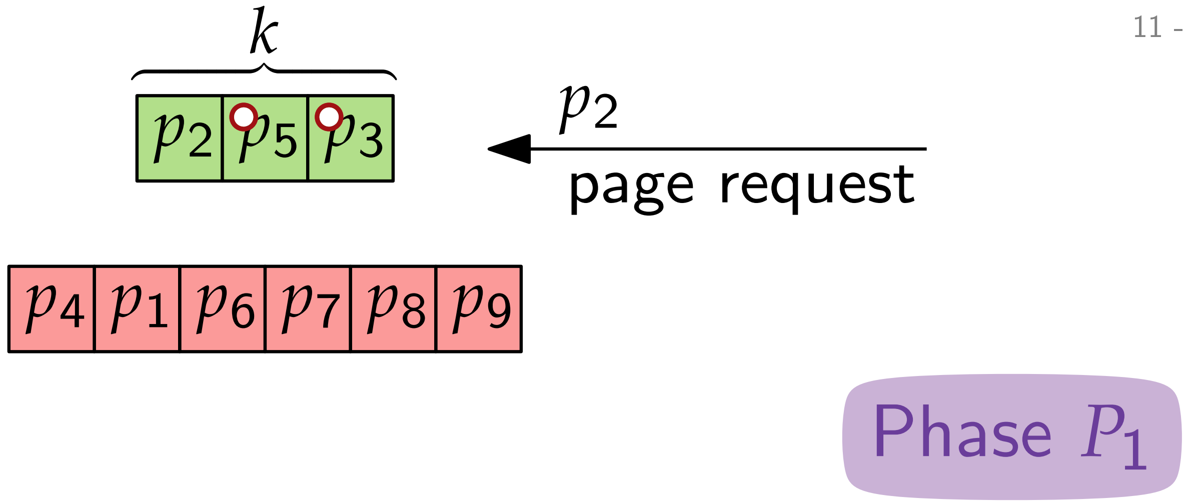

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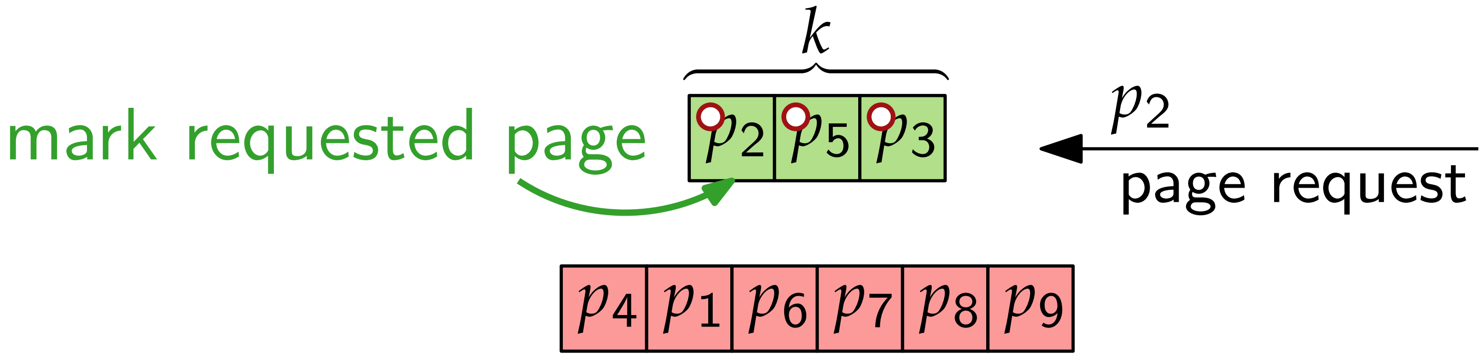

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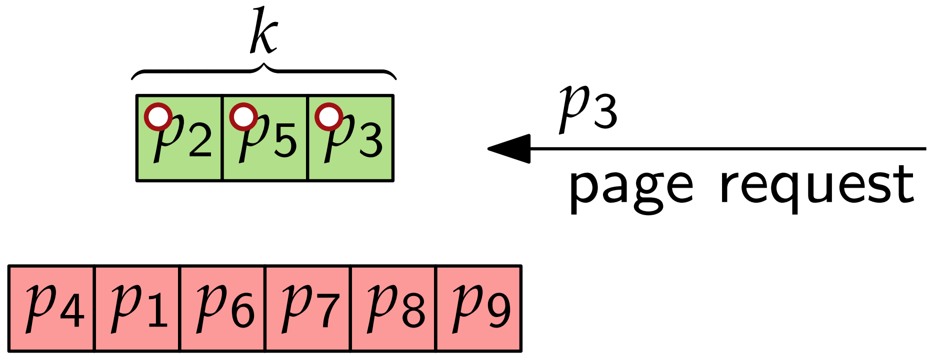

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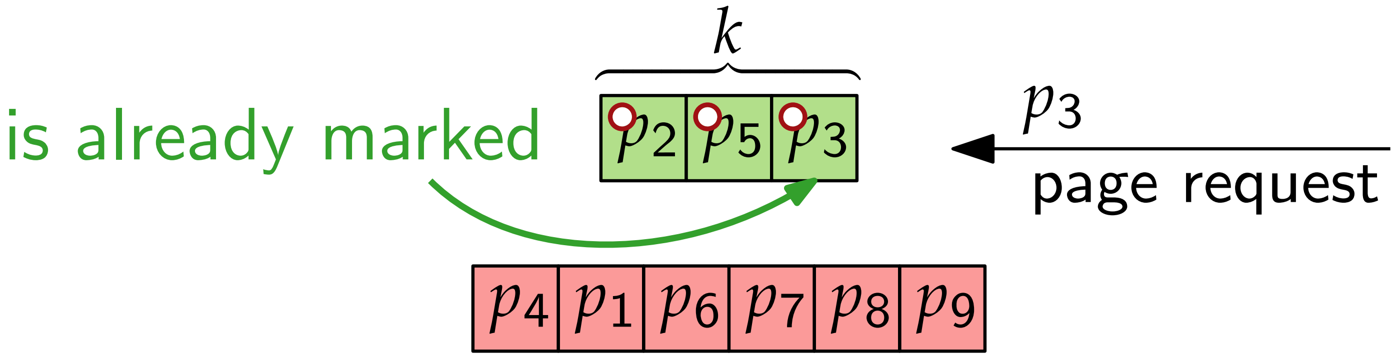

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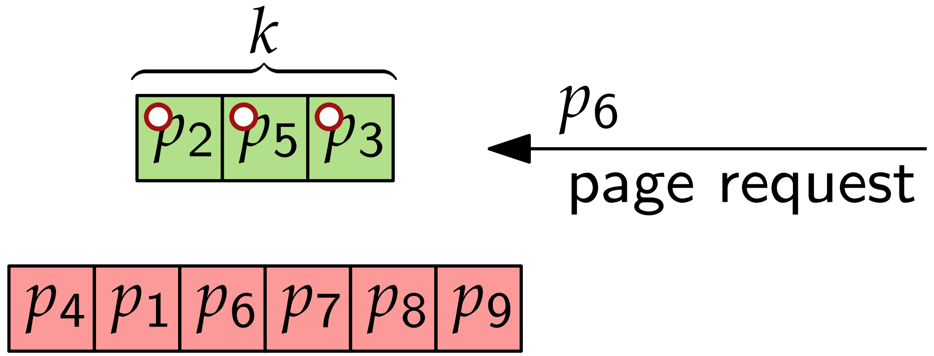

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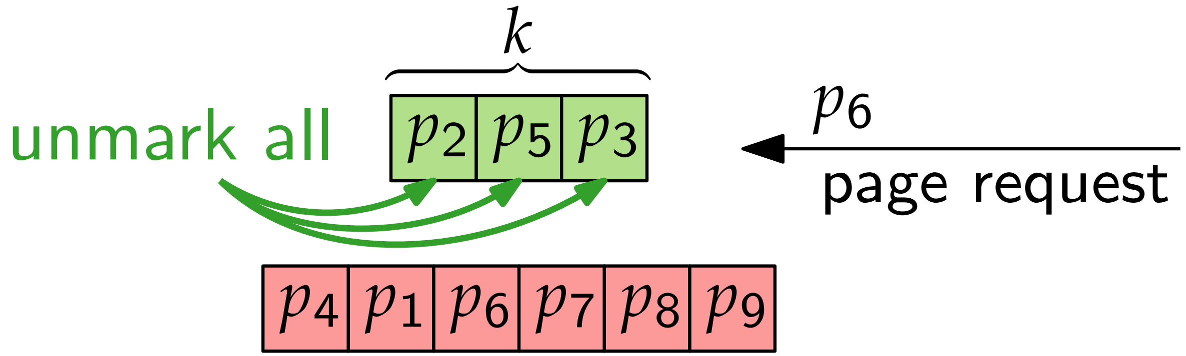

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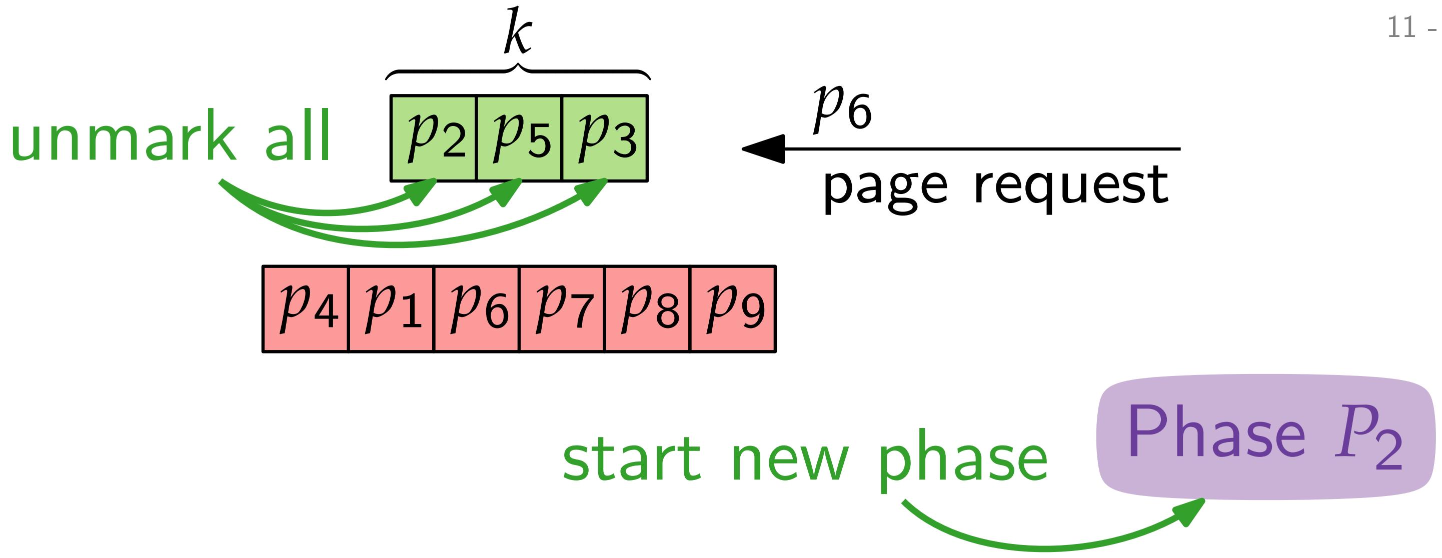

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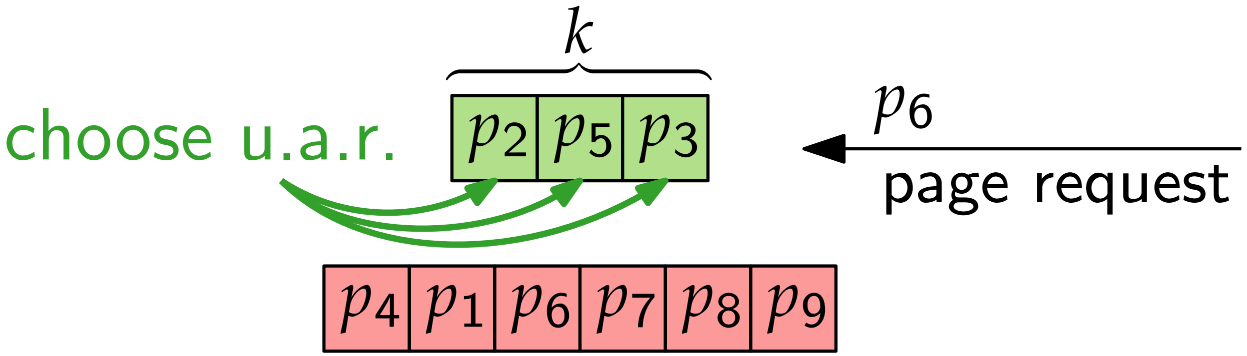

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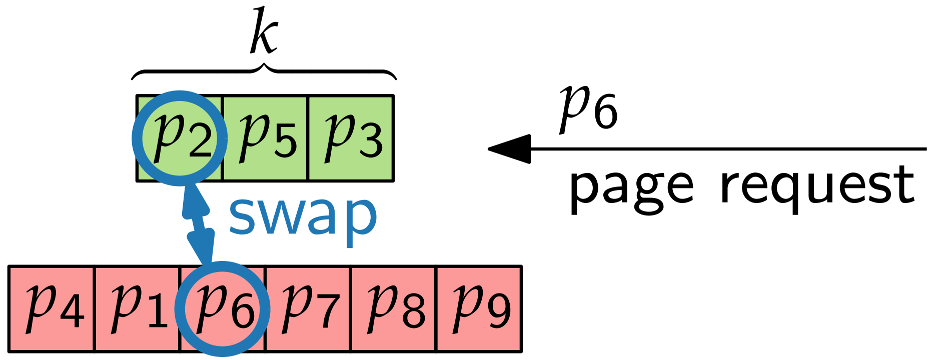

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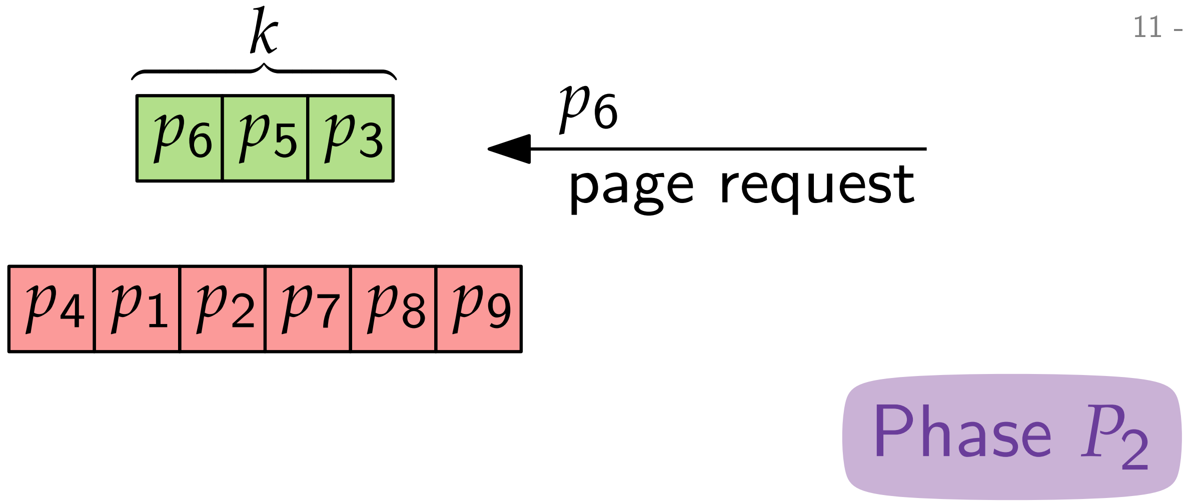

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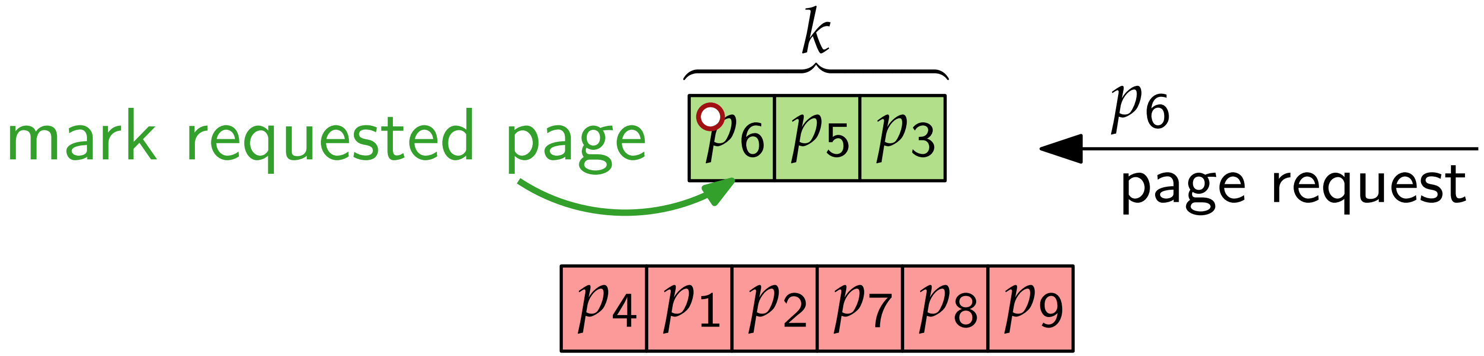

Phase P_2

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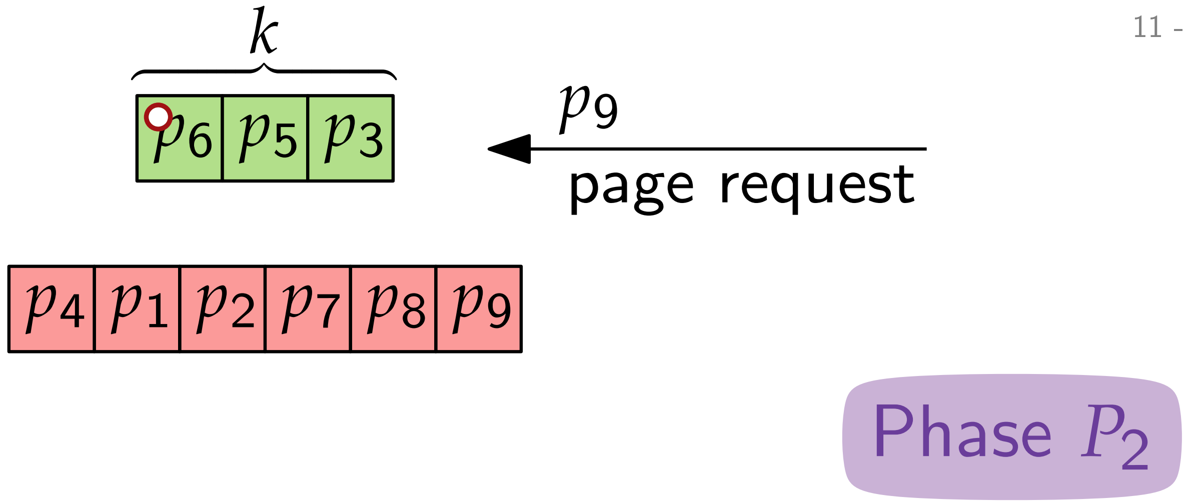

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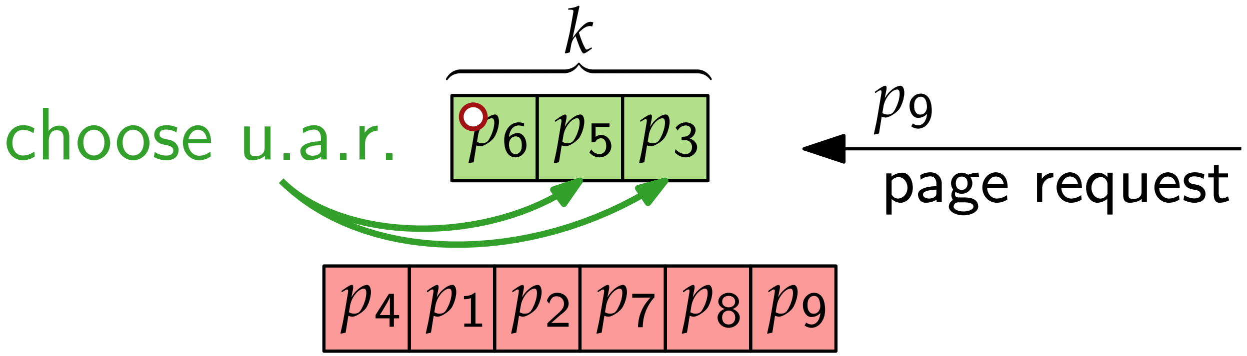

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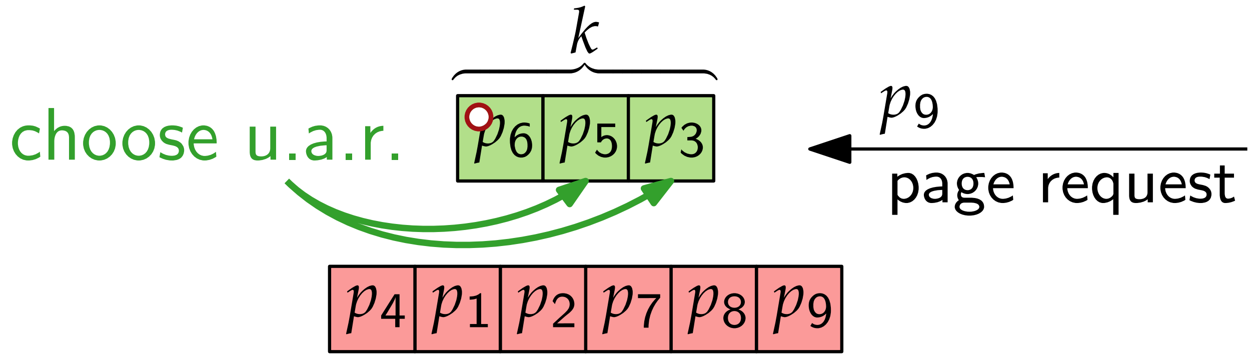

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Theorem 3. MARKING is $2H_k$-competitive.

Remark.

$H_k = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{k}$ is the k-th harmonic number and for $k \geq 2$: $H_k < \ln(k) + 1$.

Paging – Rand. Strategy – Analysis

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  Over all phases, all $\frac{d_{\text{begin}}}{2}$ and $\frac{d_{\text{end}}}{2}$ cancel out, except the first $\frac{d_{\text{begin}}}{2}$ and the last $\frac{d_{\text{end}}}{2}$.

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- Since the first $d_{\text{begin}} = 0$, MIN has at least $\frac{c}{2}$ faults per phase.

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- So the competitive ratio of MARKING is at most $\frac{c + c(H_k - 1)}{c/2} = 2H_k \in O(\log k)$ $\square$

Paging – Rand. Strategy – Analysis

Reminder.
No deterministic strategy is better than k-competitive.

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Reminder.
No deterministic strategy is better than k-competitive.
MARKING is $O(\log k)$-competitive
$\Rightarrow$ **exponential improvement!**

Discussion

- Online algorithms operate in a setting different from that of classical algorithms. However, this setting of incomplete information is very natural and occurs often in real-world applications. Can you think of further examples?

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- We might also transform a classical problem with incomplete information into an online problem. E.g.: Matching problem for ride sharing.

- Randomization can help to improve our behavior on worst-case instances. You may also think of: we are less predictable for an adversary.

Literature

Main source:

- Sabine Storandt's lecture script "Randomized Algorithms" (2016–2017)

Original papers:

- [Belady '66] "A Study of Replacement Algorithms for Virtual-Storage Computer."
- [Sleator, Tarjan '85] "Amortized Efficiency of List Update and Paging Rules."
- [Fiat, Karp, Luby, McGeoch, Sleator, Young '91] "Competitive Paging Algorithms."