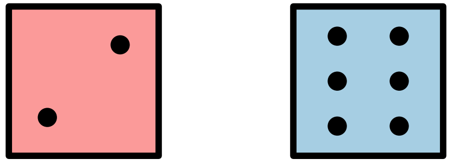
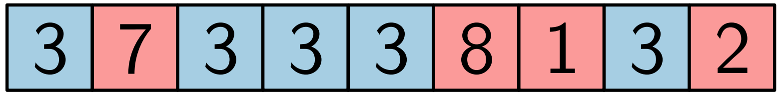
Advanced Algorithms

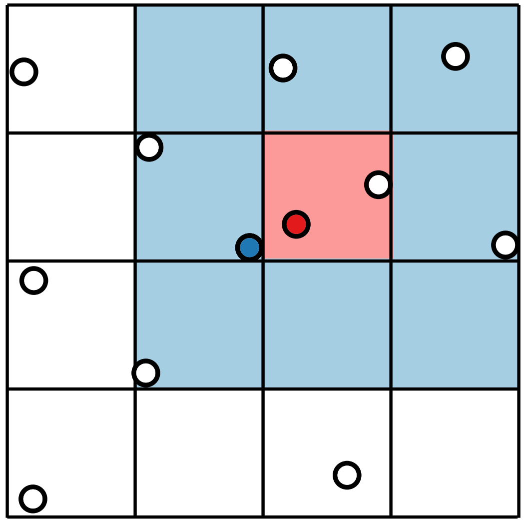
Randomized Algorithms

An introduction

Diana Sieper · WS23/24







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A **discrete probability space** $(\Omega, \Pr)$ is used to model random experiments.

Ω is a countable set of **elementary events** (= outcomes of the experiment).

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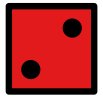

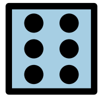

Example. Rolling a red and a blue fair six-sided die.

$\Omega = \{(1,1),(1,2),(1,3),\dots,(6,6)\}, \quad \Pr((i,j)) = \frac{1}{6} \cdot \frac{1}{6} = \frac{1}{36}$ for each $(i,j) \in \Omega$

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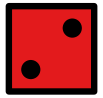
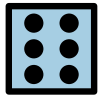

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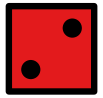
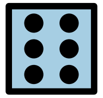

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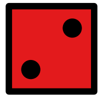
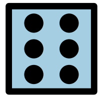

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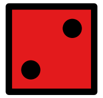

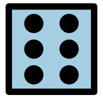

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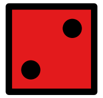 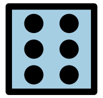

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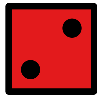

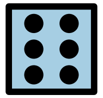

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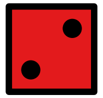
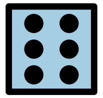

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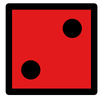
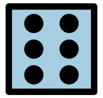

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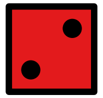
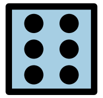

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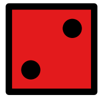

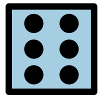

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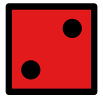
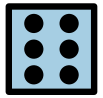

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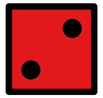
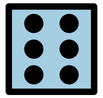

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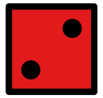
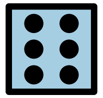

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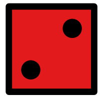
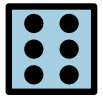

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$\mathrm{E}[X] = 1 \cdot \frac{1}{36} + 2 \cdot \frac{3}{36} + 3 \cdot \frac{5}{36} + \cdots + 6 \cdot \frac{11}{36} \approx 4.5$

Linearity of Expectation

For each set of random variables $X_1, X_2, \ldots, X_n : \Omega \longrightarrow \mathbb{R}$,
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$$\begin{aligned} \mathsf{E}[(X_1 + X_2 + \cdots + X_n)] &= \sum_{\omega\in\Omega} \Pr[\omega] \cdot \sum_{i=1}^{n} X_i(\omega) = \sum_{\omega\in\Omega} \sum_{i=1}^{n} \Pr[\omega] \cdot X_i(\omega) \\ &= \sum_{i=1}^{n} \sum_{\omega\in\Omega} \Pr[\omega] \cdot X_i(\omega) \end{aligned}$$

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For each set of random variables $X_1, X_2, \ldots, X_n : \Omega \longrightarrow \mathbb{R}$,
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Using Indicator Random Variables (I)

Let A be an array filled with n pairwise distinct integers.

$\textsc{FindMax}(A)$
  $m := A[1]$
  for $i = 2, 3, \ldots, n$
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H_n is the n-th harmonic number; $\ln(n+1) \leq H_n \leq \ln(n) + 1$.

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  for i = 1, 2, ..., k
    randomly choose r ∈ {1, 2, ..., n}
    if A[r] > ℓ
      ℓ := A[r]
  return ℓ
```

FindLarge has error probability $\leq \frac{1}{2^k}$.

Set $k = c \log_2 n$ for some constant $c > 1$.

$\Rightarrow$ Error probability $\leq \frac{1}{n^c}$

runtime $\mathcal{O}(\log n)$

Finding a Large Number

Given: An array A of n pairwise distinct natural numbers.

Task: Determine an integer $A[j]$ that is at least as large as the median.

Deterministic approach: Go through all elements, return maximum.

(Actually, it suffices to go through $\lfloor n/2 \rfloor + 1$ elements.)

runtime $\Theta(n)$

Randomized approach:

$\textsc{FindLarge}(A, k \in \mathbb{N})$

- $\ell := 0$
- **for** $i = 1, 2, \ldots, k$
  - randomly choose $r \in \{1, 2, \ldots, n\}$
  - **if** $A[r] > \ell$
    - $\ell := A[r]$
- **return** ℓ

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Remark. We traded correctness for running time.

Finding a Repeated Element

Given: An array A of n natural numbers such that $\lceil \frac{n}{2} \rceil$ of them are identical and $\lfloor \frac{n}{2} \rfloor$ of them are pairwise distinct.

Task: Find the repeated element.

3	7	3	3	3	8	1	3	2

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Deterministic approaches:

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Deterministic approaches:

Compare each element with every predecessor $\Theta(n^2)$ time

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Randomized approach:

$\textsc{FindRepeated}(A)$

while true **do**
 randomly choose $i \in \{1, \ldots, n\}$
 randomly choose $j \in \{1, \ldots, n\} \setminus \{i\}$
 if $A[i] = A[j]$ **then return** $A[i]$

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Deterministic approaches:

- Compare each element with every predecessor — $\Theta(n^2)$ time
- Sort the array, then perform a linear sweep. — $\Theta(n \log n)$ time
- Compute and report the median. — $\Theta(n)$ time

Randomized approach:

Success probability in each step

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Compute and report the median.	$\Theta(n)$ time

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Remark. The algorithm only returns correct answers, but may run forever.

Las Vegas and Monte Carlo Algorithms

Las Vegas algorithm. Returns a correct result, but the running time (and possibly the required space) are random variables.

Examples. FindRepeated, RandomizedQuicksort

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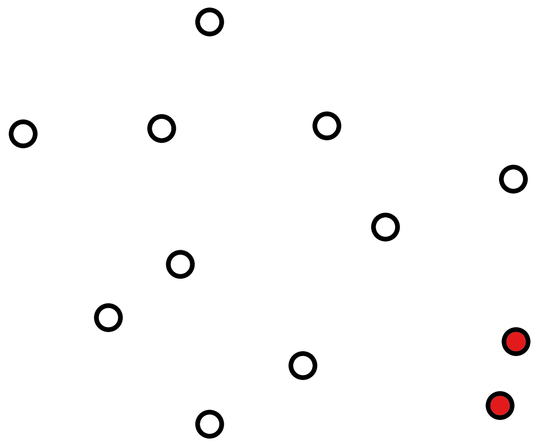
Remark. A Monte Carlo algorithm can often be turned into a Las Vegas algorithm and vice versa.

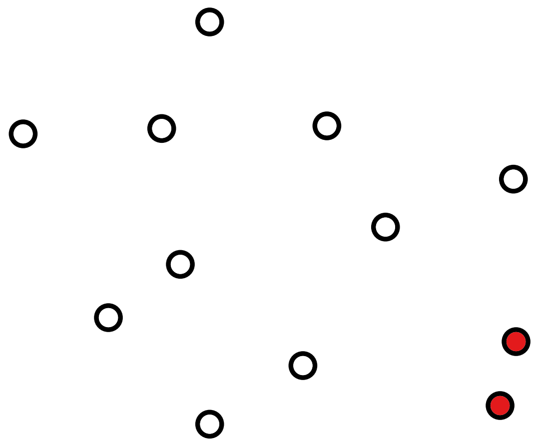
Closest Pair

Given: (multi-)set of points $P = \{p_1, p_2, \ldots, p_n\} \subseteq \mathbb{R}^2$.

Task: Find a pair of distinct elements $p_a, p_b \in P$ such that the Euclidean distance $\delta = ||p_a, p_b||$ is minimum.



Closest Pair



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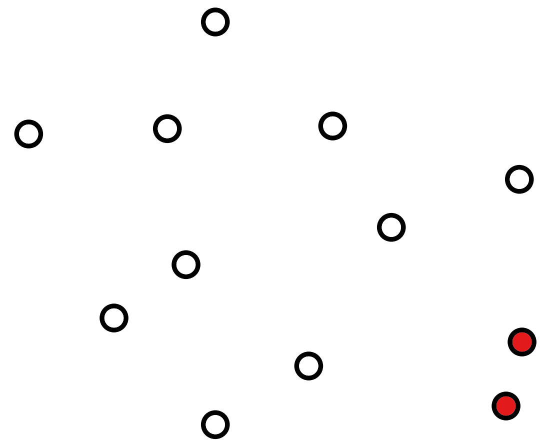
Deterministic approaches:

Brute-force $\Theta(n^2)$

Closest Pair



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Deterministic approaches:

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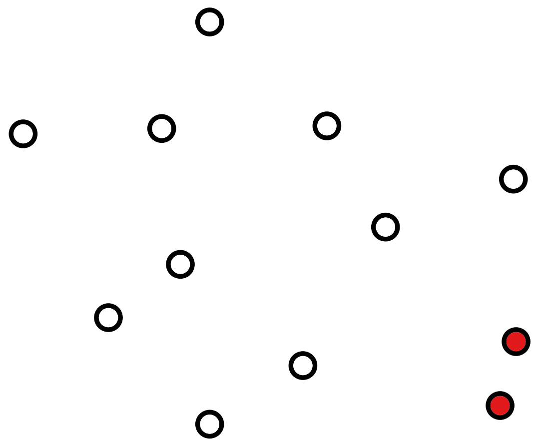
Divide and conquer (recall from ADS)	$\Theta(n \log n)$

Closest Pair



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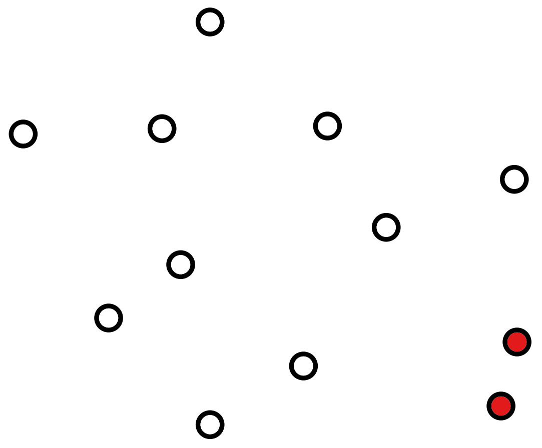
Lower bound:

Element Uniqueness: Given numbers $a_1, a_2, \ldots, a_n$. Are they pairwise distinct?

There is no $o(n \log n)$ time algorithm for Element Uniqueness.

(under some assumption concerning the arithmetic model)

Closest Pair



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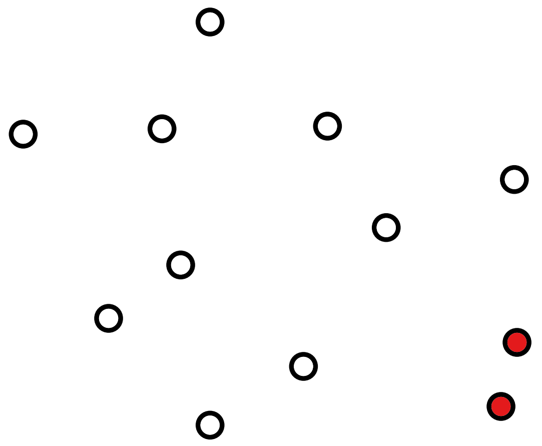
Closest Pair



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$\Rightarrow$ There is no $o(n \log n)$ time algorithm for Closest pair.

(under the same assumption concerning the arithmetic model)

Reduction: map each a_i to a point (a_i, a_i) and test if the minimum distance is 0.

A Randomized Incremental Algorithm for Closest Pair

Define $P_i = \{p_1, p_2, \ldots, p_i\}$ and let δ_i be the distance of a closest pair in P_i.

A Randomized Incremental Algorithm for CLOSEST PAIR

Define $P_i = \{p_1, p_2, \ldots, p_i\}$ and let δ_i be the distance of a closest pair in P_i.

Idea: $\delta_2 = ||p_1, p_2||$. Compute $\delta_3, \delta_4, \ldots, \delta_n$ by adding the points iteratively.

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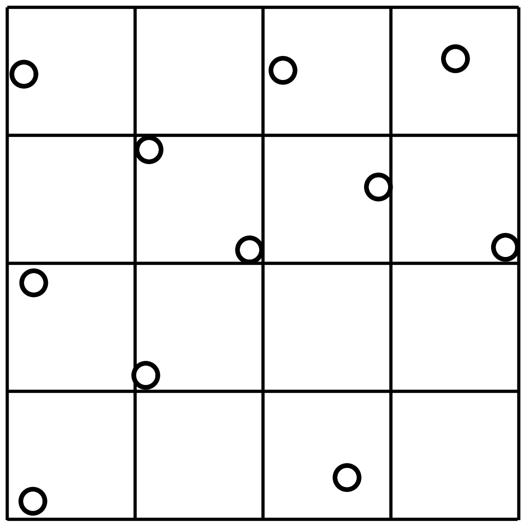

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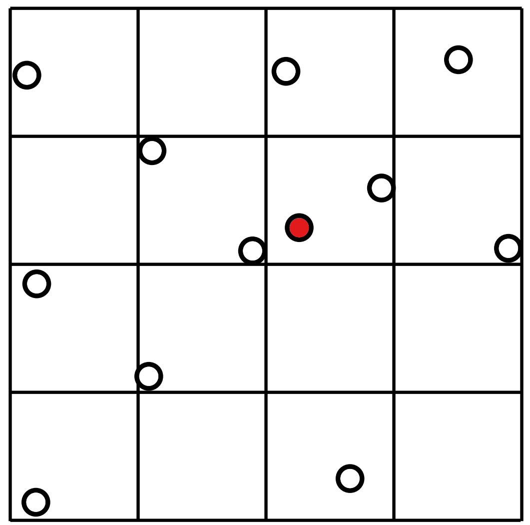
Add the point p_i. If $\delta_i < \delta_{i-1}$, then p_i must be part of each closest pair p_i, p_j.



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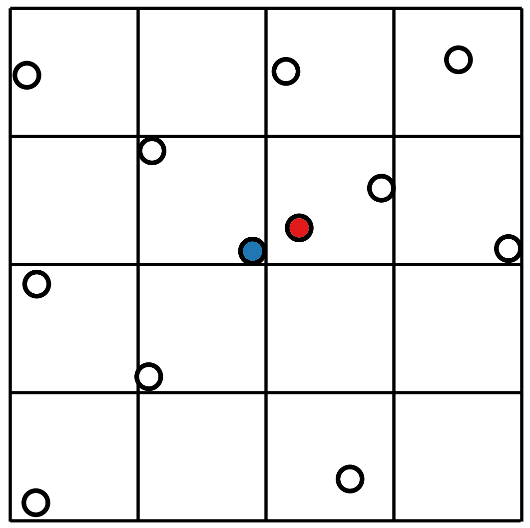

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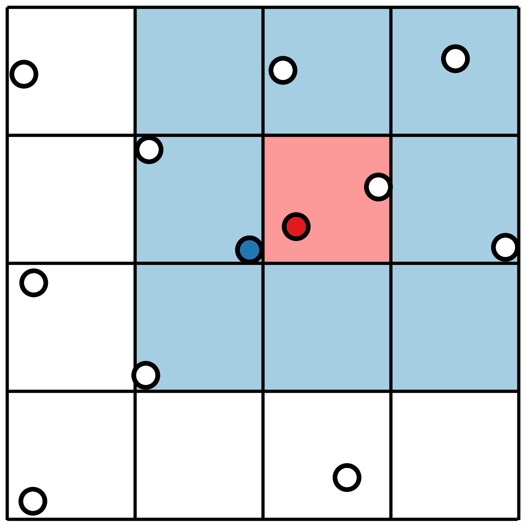
Add the point p_i. If $\delta_i < \delta_{i-1}$, then p_i must be part of each closest pair p_i, p_j.

Moreover, p_j must lie in the cell of p_i or one of the adjacent cells.



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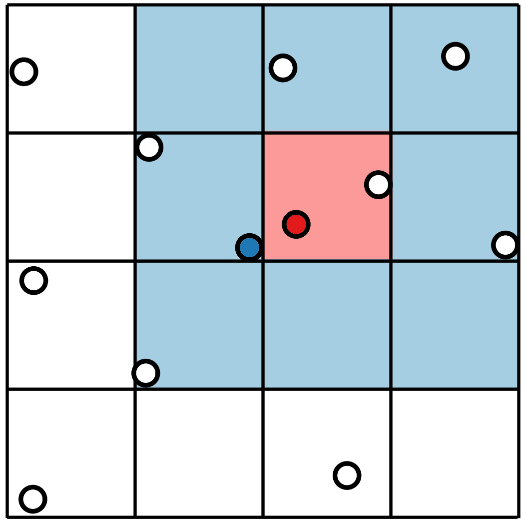

(simple) exercise

Each of these cells contains at most $\mathcal{O}(1)$ points of P_{i-1} ($\Leftarrow$ packing argument).

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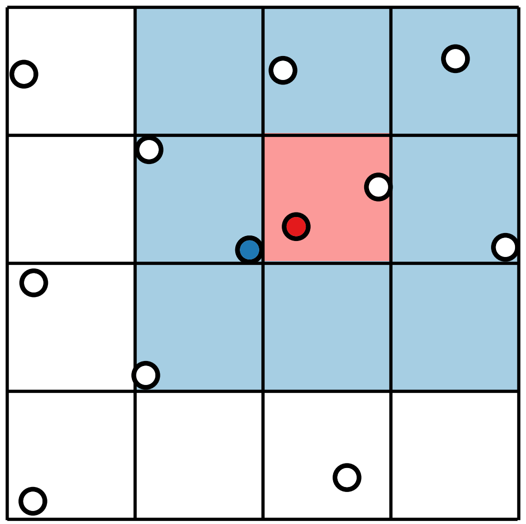

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The coordinates of the cell of p_i can be determined in $\mathcal{O}(1)$ time assuming the floor function can be computed in $\mathcal{O}(1)$ time.

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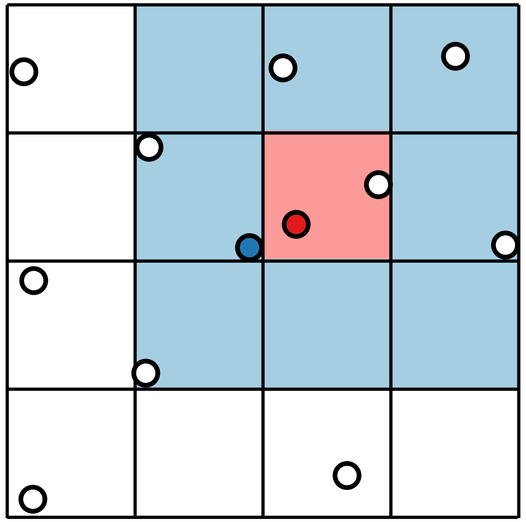

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The coordinates of the cell of p_i can be determined in $\mathcal{O}(1)$ time assuming the floor function can be computed in $\mathcal{O}(1)$ time.

$\Rightarrow$ The test $\delta_i < \delta_{i-1}$ can be performed in $\mathcal{O}(1)$ time assuming P_{i-1} is stored in a suitable dictionary for the nonempty cells (implementable via dynamic perfect hashing).

Backwards Analysis

If $\delta_i = \delta_{i-1}$, we add p_i to the dictionary in $\mathcal{O}(1)$ time.

If $\delta_i < \delta_{i-1}$, the cell size changes and we have to rebuild the dictionary in $\mathcal{O}(i)$ time.

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How many points p in P_i have the property
that the minimum distance in $P_i \setminus \{p\}$ is larger than in P_i?

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$=$ Probability that deleting p_i from P_i increases the minimum distance

How many points p in P_i have the property
that the minimum distance in $P_i \setminus \{p\}$ is larger than in P_i?

- If the closest distance in P_i is unique: 2 points

Backwards Analysis

If $\delta_i = \delta_{i-1}$, we add p_i to the dictionary in $\mathcal{O}(1)$ time.
If $\delta_i < \delta_{i-1}$, the cell size changes and we have to rebuild the dictionary in $\mathcal{O}(i)$ time.
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- If there are at least two disjoint closest pairs: 0 points

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- provide a good strategy for games or search in unknown environments.