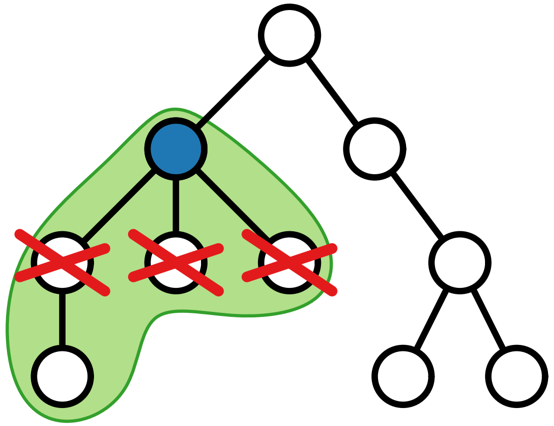
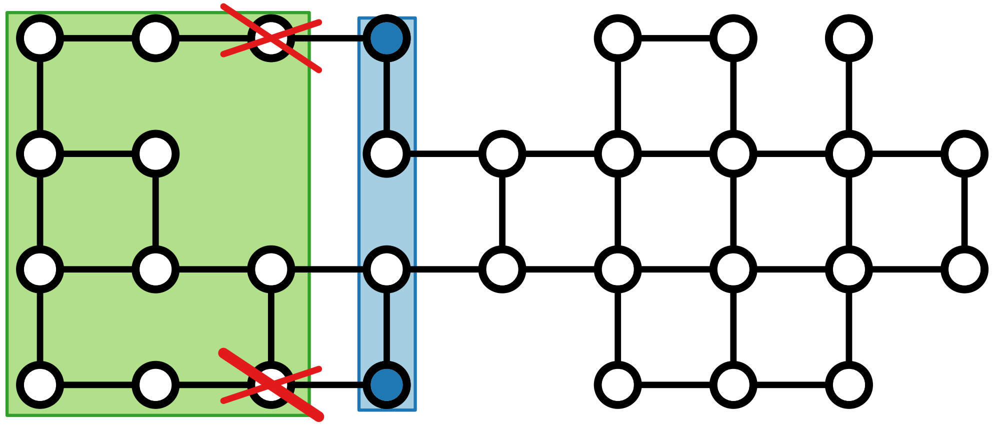
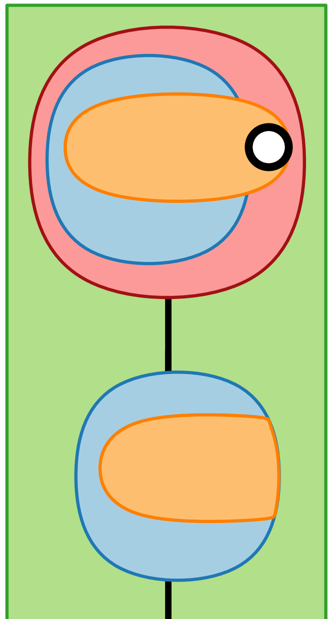
Advanced Algorithms

Parameterized Algorithms
Structural Parametrization

Johannes Zink · WS23/24



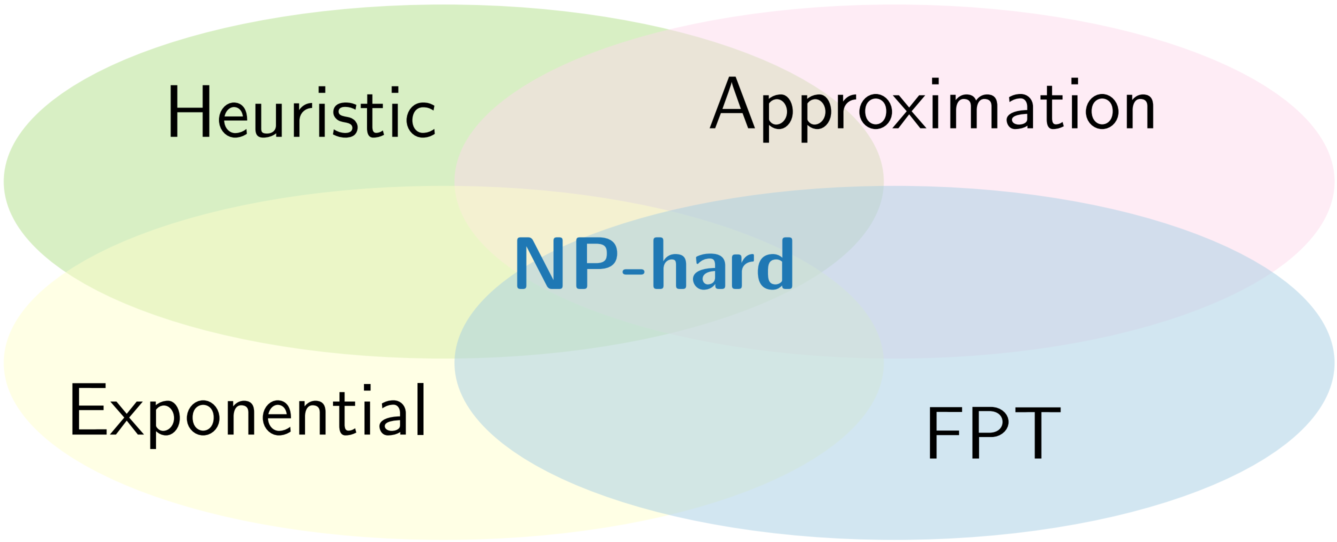




Dealing with NP-Hard Problems

What should we do?

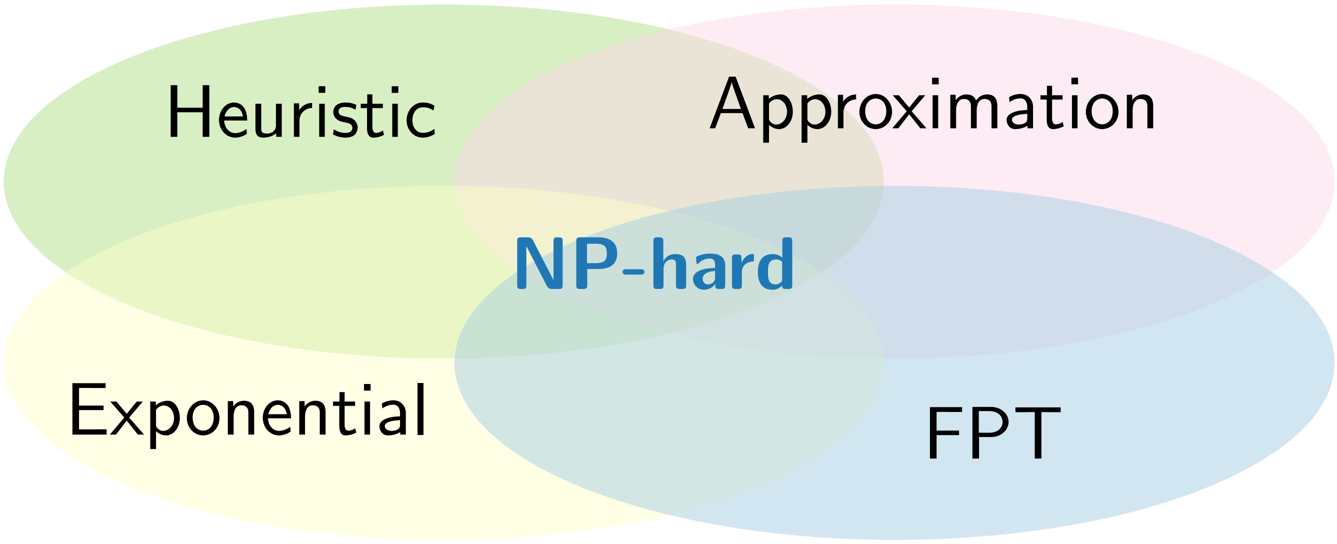
- Sacrifice optimality for speed
  - Heuristics
  - Approximation Algorithms
- Optimal Solutions
  - Exact exponential-time algorithms
  - Fine-grained analysis – parameterized algorithms



Dealing with NP-Hard Problems

What should we do?

- Sacrifice optimality for speed
  - Heuristics
  - Approximation Algorithms
- Optimal Solutions
  - Exact exponential-time algorithms
  - Fine-grained analysis – parameterized algorithms ← this lecture



Parameterized Algorithms

Classical complexity theory:

Running time is expressed as a function in the input size.

Parameterized algorithmics:

Running time is expressed as a function in the input size, as well as one or more additional parameter(s).

Parameterized Algorithms

Classical complexity theory:

Running time is expressed as a function in the input size.

Parameterized algorithmics:

Running time is expressed as a function in the input size, as well as one or more additional parameter(s).

Example: (recall from AGT)

k-VERTEX COVER

Input Graph $G = (V, E)$, $k \in \mathbb{N}$

Question Is there a set $C \subseteq V$ with $|C| \leq k$ s.t. $\forall \{u, v\} \in E \colon \{u, v\} \cap C \neq \emptyset$?

Parameterized Algorithms

Classical complexity theory:

Running time is expressed as a function in the input size.

Parameterized algorithmics:

Running time is expressed as a function in the input size, as well as one or more additional parameter(s).

Example: (recall from AGT)

k-VERTEX COVER

Input Graph $G = (V, E)$, $k \in \mathbb{N}$

Question Is there a set $C \subseteq V$ with $|C| \leq k$ s.t. $\forall \{u, v\} \in E \colon \{u, v\} \cap C \neq \emptyset$?

- NP-complete,
- but there is an algorithm with runtime $\mathcal{O}(2^k \cdot k \cdot (|V| + |E|))$.

Parameterized Algorithms

Classical complexity theory:

Running time is expressed as a function in the input size.

Parameterized algorithmics:

Running time is expressed as a function in the input size, as well as one or more additional parameter(s).

Example: (recall from AGT)

k-VERTEX COVER

Input Graph $G = (V, E)$, $k \in \mathbb{N}$

Question Is there a set $C \subseteq V$ with $|C| \leq k$ s.t. $\forall \{u, v\} \in E \colon \{u, v\} \cap C \neq \emptyset$?

- NP-complete,
- but there is an algorithm with runtime $\mathcal{O}(2^k \cdot k \cdot (|V| + |E|))$.

Idea: If $k \in \mathcal{O}(1)$, then $\mathcal{O}(2^k \cdot k \cdot (|V| + |E|)) \subseteq \mathcal{O}(|V| + |E|)$

Parameterized Algorithms

Classical complexity theory:

Running time is expressed as a function in the input size.

Parameterized algorithmics:

Running time is expressed as a function in the input size, as well as one or more additional parameter(s).

Example: (recall from AGT)

k-Vertex Cover

Input Graph $G = (V, E)$, $k \in \mathbb{N}$

Question Is there a set $C \subseteq V$ with $|C| \leq k$ s.t. $\forall \{u, v\} \in E\colon \{u, v\} \cap C \neq \emptyset$?

- NP-complete,
- but there is an algorithm with runtime $\mathcal{O}(2^k \cdot k \cdot (|V| + |E|))$.

Idea: If $k \in \mathcal{O}(1)$, then $\mathcal{O}(2^k \cdot k \cdot (|V| + |E|)) \subseteq \mathcal{O}(|V| + |E|)$, in other words, if we assume the **parameter** k to be **fixed**, k-Vertex Cover becomes **tractable**.

Parameterized Complexity Classes

Definition.

Let Π be a decision problem. If there is

- an algorithm $\mathcal{A}$ and
- a computable function f

such that, given an instance I of Π and a parameter $k \in \mathbb{N}$,
the algorithm $\mathcal{A}$ provides the correct answer to I in time $f(k) \cdot |I|^{\mathcal{O}(1)}$,
then $\mathcal{A}$ (and Π) are called **fixed-parameter tractable (FPT)** with respect to k.

Parameterized Complexity Classes

Definition.

Let Π be a decision problem. If there is

- an algorithm $\mathcal{A}$ and
- a computable function f

such that, given an instance I of Π and a parameter $k \in \mathbb{N}$,
the algorithm $\mathcal{A}$ provides the correct answer to I in time $f(k) \cdot |I|^{\mathcal{O}(1)}$,
then $\mathcal{A}$ (and Π) are called **fixed-parameter tractable (FPT)** with respect to k.

If $\mathcal{A}$ provides the correct answer to I in time $|I|^{f(k)}$, then $\mathcal{A}$ (and Π) are called **slice-wise polynomial (XP)** with respect to k. (Note that FPT $\subsetneq$ XP.)

Parameterized Complexity Classes

Definition.

Let Π be a decision problem. If there is

- an algorithm $\mathcal{A}$ and
- a computable function f

such that, given an instance I of Π and a parameter $k \in \mathbb{N}$,
the algorithm $\mathcal{A}$ provides the correct answer to I in time $f(k) \cdot |I|^{\mathcal{O}(1)}$,
then $\mathcal{A}$ (and Π) are called **fixed-parameter tractable (FPT)** with respect to k.

If $\mathcal{A}$ provides the correct answer to I in time $|I|^{f(k)}$, then $\mathcal{A}$ (and Π) are called **slice-wise polynomial (XP)** with respect to k. (Note that FPT $\subsetneq$ XP.)

Example. k-Vertex Cover can be solved in time $\mathcal{O}(2^k \cdot k \cdot (|V| + |E|))$.

Parameterized Complexity Classes

Definition.

Let Π be a decision problem. If there is

- an algorithm $\mathcal{A}$ and
- a computable function f

such that, given an instance I of Π and a parameter $k \in \mathbb{N}$,
the algorithm $\mathcal{A}$ provides the correct answer to I in time $f(k) \cdot |I|^{\mathcal{O}(1)}$,
then $\mathcal{A}$ (and Π) are called **fixed-parameter tractable (FPT)** with respect to k.

If $\mathcal{A}$ provides the correct answer to I in time $|I|^{f(k)}$, then $\mathcal{A}$ (and Π) are called **slice-wise polynomial (XP)** with respect to k. (Note that FPT $\subsetneq$ XP.)

Example. k-Vertex Cover can be solved in time $\mathcal{O}(\underbrace{2^k \cdot k}_{f(k)} \cdot \underbrace{(|V| + |E|)}_{|I|^{\mathcal{O}(1)}})$.

Parameterized Complexity Classes

Definition.

Let Π be a decision problem. If there is
- an algorithm $\mathcal{A}$ and
- a computable function f

such that, given an instance I of Π and a parameter $k \in \mathbb{N}$,
the algorithm $\mathcal{A}$ provides the correct answer to I in time $f(k) \cdot |I|^{\mathcal{O}(1)}$,
then $\mathcal{A}$ (and Π) are called **fixed-parameter tractable (FPT)** with respect to k.

If $\mathcal{A}$ provides the correct answer to I in time $|I|^{f(k)}$, then $\mathcal{A}$ (and Π) are called **slice-wise polynomial (XP)** with respect to k. (Note that FPT $\subsetneq$ XP.)

Example. k-Vertex Cover can be solved in time $\mathcal{O}(\underbrace{2^k \cdot k}_{f(k)} \cdot \underbrace{(|V| + |E|)}_{|I|^{\mathcal{O}(1)}})$.

$\Rightarrow$ k-Vertex Cover is FPT (and therefore also XP) with respect to k.

Examples and Counterexamples

k-Vertex Cover

- NP-complete
- but FPT with respect to k

Examples and Counterexamples

k-VERTEX COVER

- NP-complete
- but FPT with respect to k

k-CLIQUE

- NP-complete
- but XP with respect to k
- Under common assumptions, k-CLIQUE is not FPT with respect to k (namely, k-CLIQUE is $W[1]$-complete with respect to k; $\rightarrow$ Section 13 in [1])

Examples and Counterexamples

k-Vertex Cover

- NP-complete
- but FPT with respect to k

k-Clique

- NP-complete
- but XP with respect to k
- Under common assumptions, k-Clique is not FPT with respect to k (namely, k-Clique is $W[1]$-complete with respect to k; $\rightarrow$ Section 13 in [1])

Vertex k-Coloring

- NP-complete for every $k \geq 3$
- $\Rightarrow$ neither FPT nor XP with respect to k, unless $P = NP$

Examples and Counterexamples

In all these examples, k is the *natural* parameter that comes with the decision problem.

k-Vertex Cover

- NP-complete
- but FPT with respect to k

k-Clique

- NP-complete
- but XP with respect to k
- Under common assumptions, k-Clique is not FPT with respect to k (namely, k-Clique is $W[1]$-complete with respect to k; $\rightarrow$ Section 13 in [1])

Vertex k-Coloring

- NP-complete for every $k \geq 3$
- $\Rightarrow$ neither FPT nor XP with respect to k, unless $P = NP$

Examples and Counterexamples

k-VERTEX COVER

- NP-complete
- but FPT with respect to k

k-CLIQUE

- NP-complete
- but XP with respect to k
- Under common assumptions, k-CLIQUE is not FPT with respect to k (namely, k-CLIQUE is $W[1]$-complete with respect to k; $\rightarrow$ Section 13 in [1])

In all these examples, k is the *natural* parameter that comes with the decision problem.

We can also study other types of parameters!

VERTEX k-COLORING

- NP-complete for every $k \geq 3$
- $\Rightarrow$ neither FPT nor XP with respect to k, unless $P = NP$

Examples and Counterexamples

k-Vertex Cover
- NP-complete
- but FPT with respect to k

k-Clique
- NP-complete
- but XP with respect to k
- Under common assumptions, k-Clique is not FPT with respect to k (namely, k-Clique is $W[1]$-complete with respect to k; $\rightarrow$ Section 13 in [1])
- There is an $\mathcal{O}(2^{\Delta} \cdot \Delta^2 \cdot (|V| + |E|))$ time algorithm for k-Clique, where Δ is the maximum degree of the input graph $\Rightarrow$ k-Clique is FPT with respect to Δ.

Vertex k-Coloring
- NP-complete for every $k \geq 3$
- $\Rightarrow$ neither FPT nor XP with respect to k, unless $P = NP$

In all these examples, k is the *natural* parameter that comes with the decision problem.

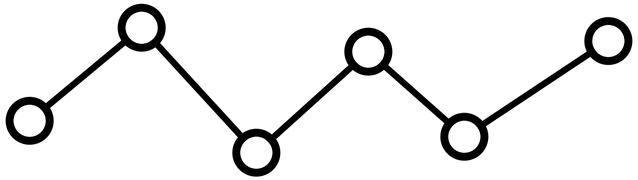
We can also study other types of parameters!

Pathwidth and Treewidth (Intuition)

Pathwidth describes how *path-like* a graph is.

Pathwidth and Treewidth (Intuition)

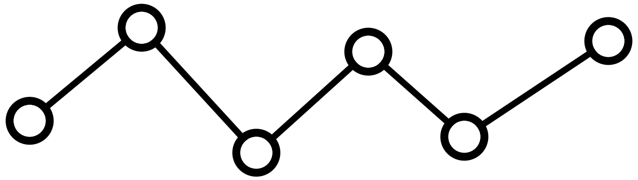
Pathwidth describes how *path-like* a graph is.



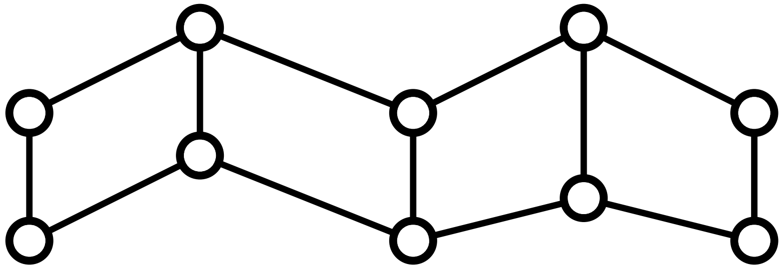
1

Pathwidth and Treewidth (Intuition)

Pathwidth describes how *path-like* a graph is.



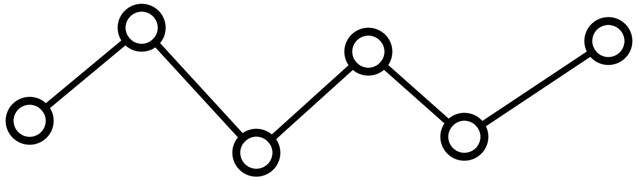
1



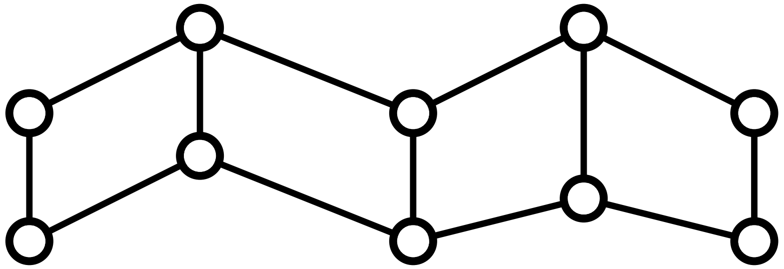
2

Pathwidth and Treewidth (Intuition)

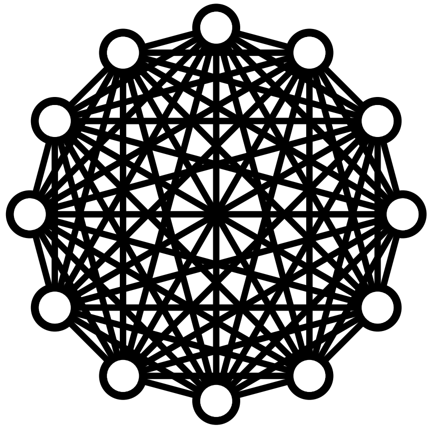
Pathwidth describes how *path-like* a graph is.



1



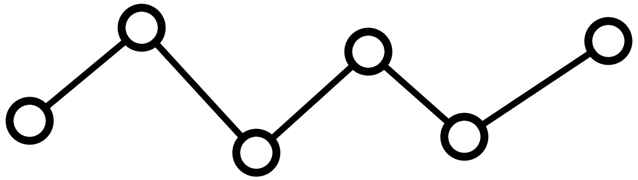
2



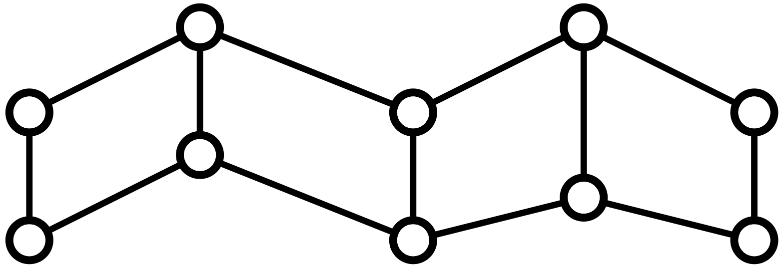
$n - 1$

Pathwidth and Treewidth (Intuition)

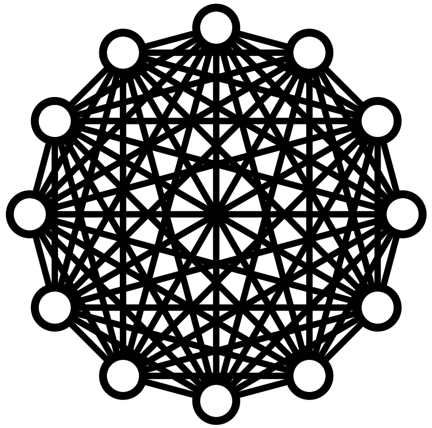
Pathwidth describes how *path-like* a graph is.



1



2

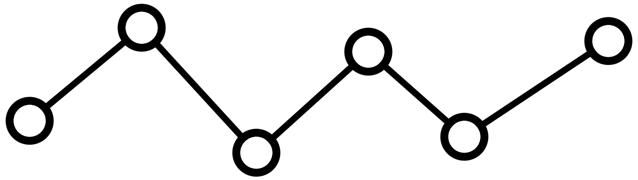


$n-1$

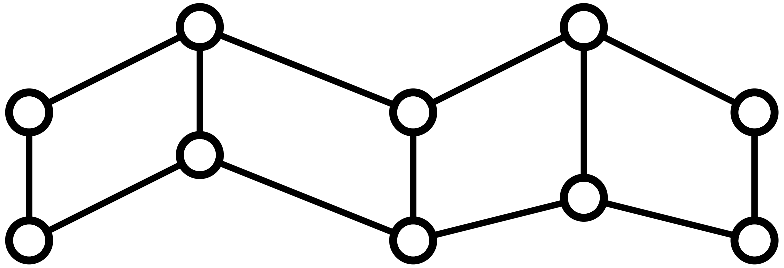
Treewidth describes how *tree-like* a graph is.

Pathwidth and Treewidth (Intuition)

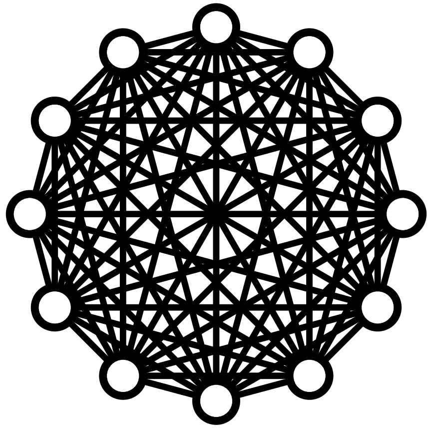
Pathwidth describes how *path-like* a graph is.



1

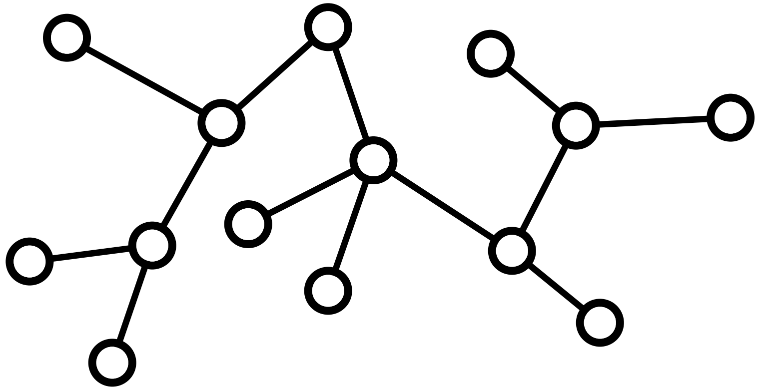


2



$n - 1$

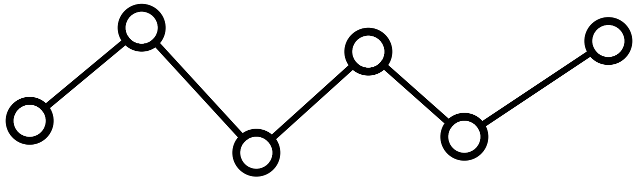
Treewidth describes how *tree-like* a graph is.



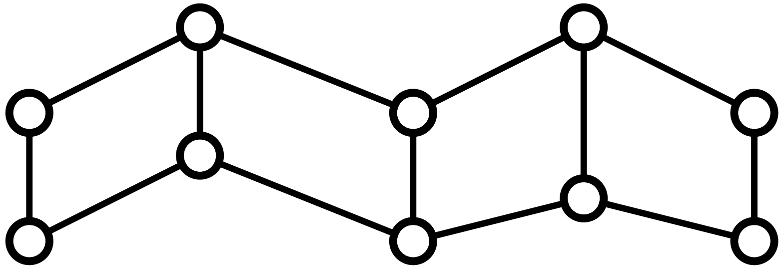
1

Pathwidth and Treewidth (Intuition)

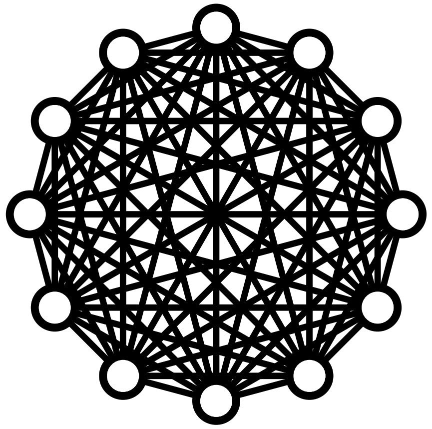
Pathwidth describes how *path-like* a graph is.



1

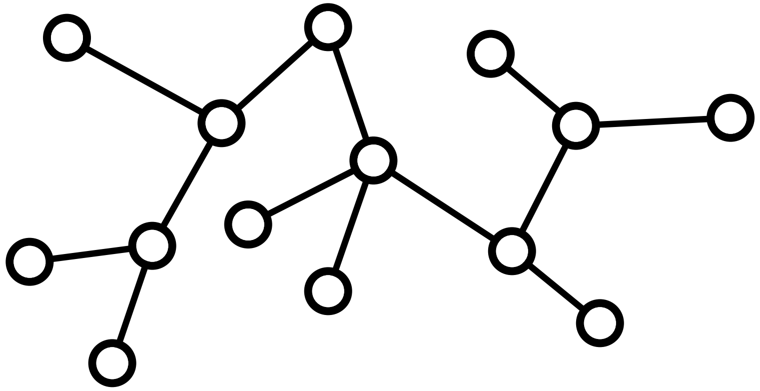


2

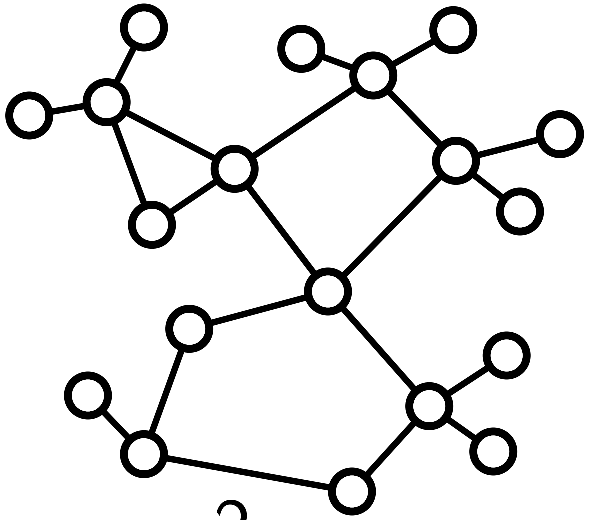


$n-1$

Treewidth describes how *tree-like* a graph is.



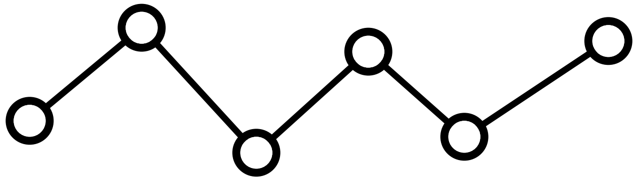
1



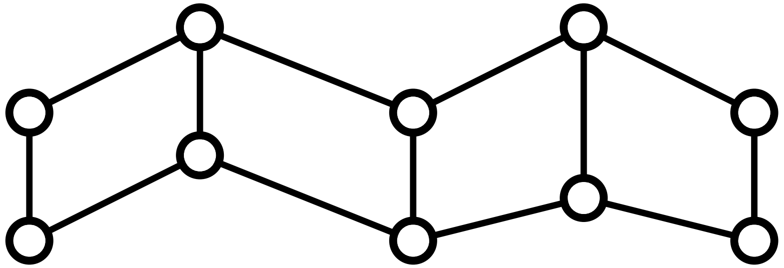
2

Pathwidth and Treewidth (Intuition)

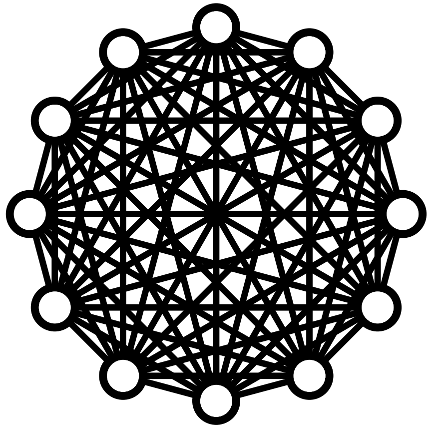
Pathwidth describes how *path-like* a graph is.



1

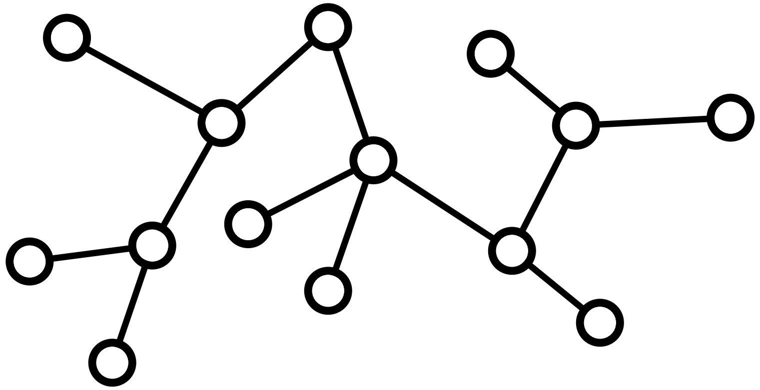


2

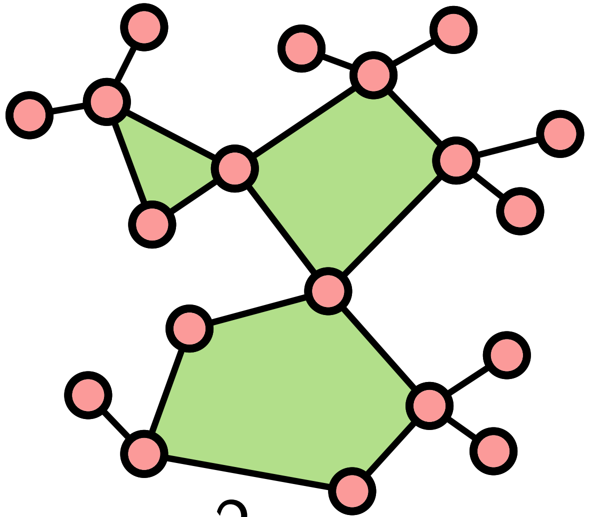


$n - 1$

Treewidth describes how *tree-like* a graph is.



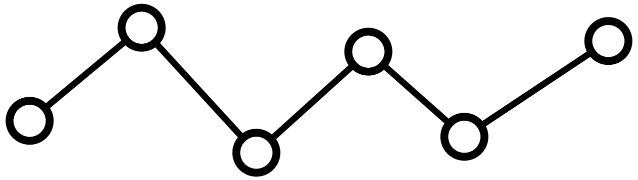
1



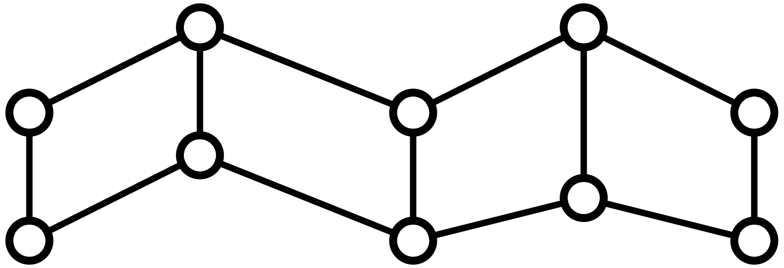
2

Pathwidth and Treewidth (Intuition)

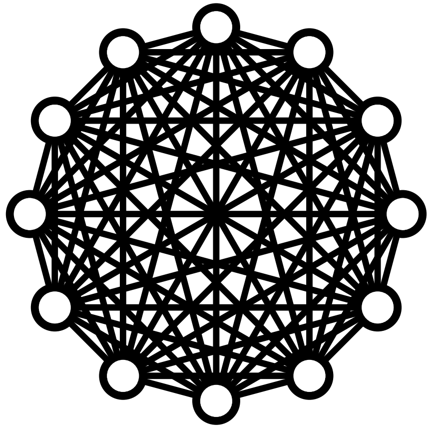
Pathwidth describes how *path-like* a graph is.



1

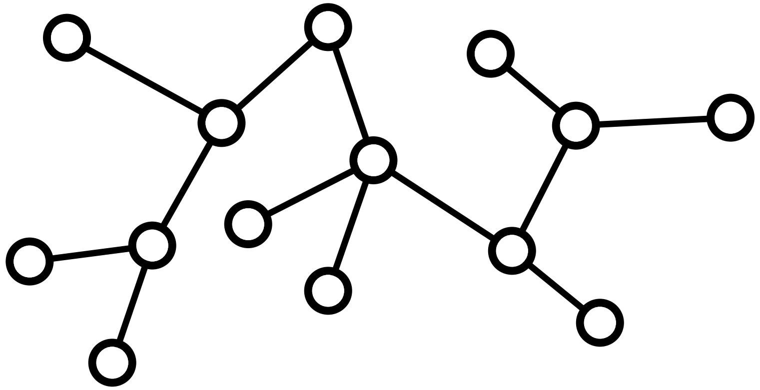


2

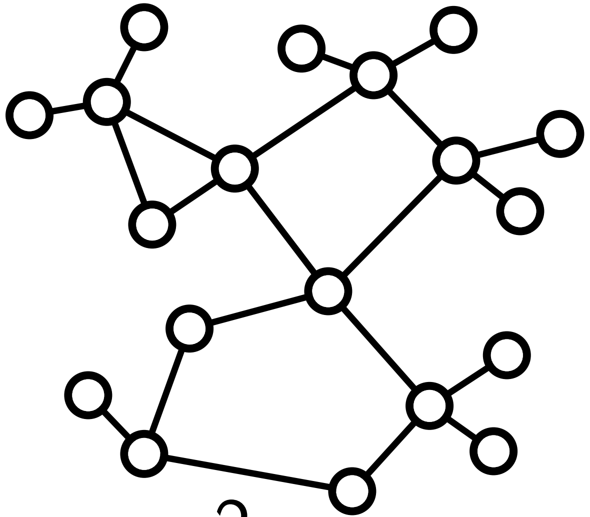


$n-1$

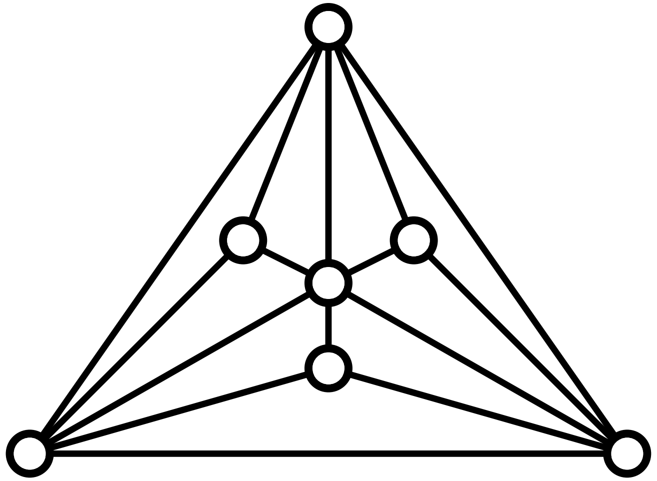
Treewidth describes how *tree-like* a graph is.



1



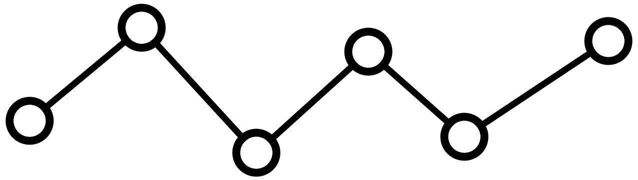
2



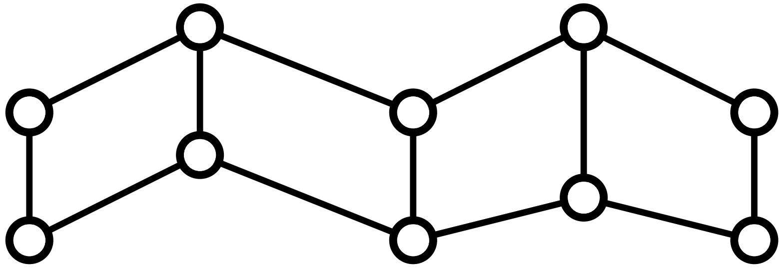
3

Pathwidth and Treewidth (Intuition)

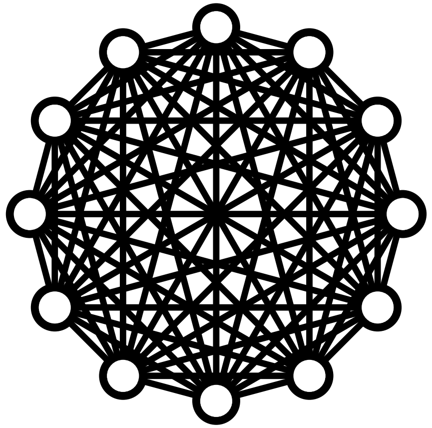
Pathwidth describes how *path-like* a graph is.



1

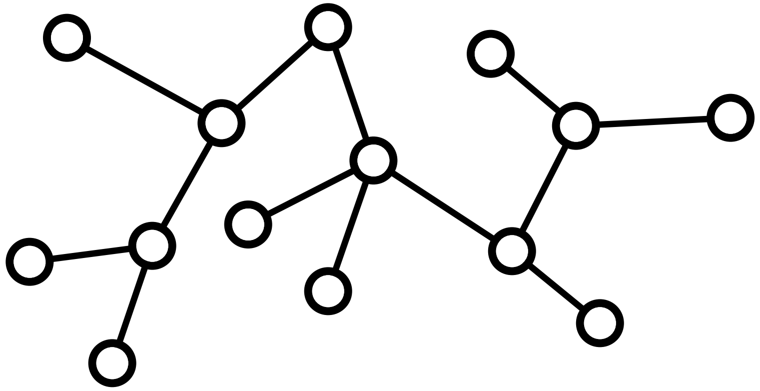


2

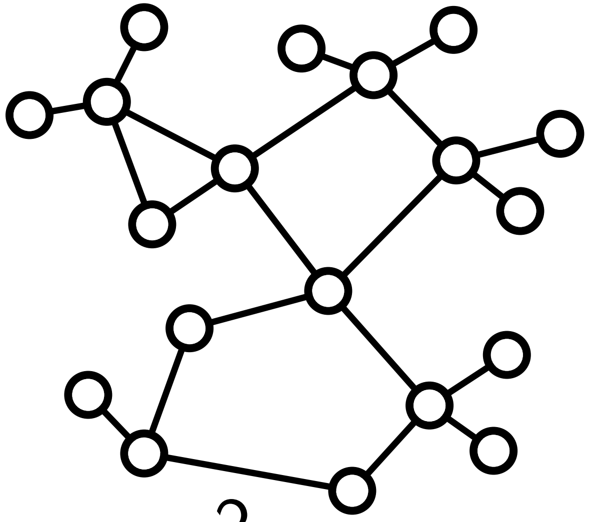


$n-1$

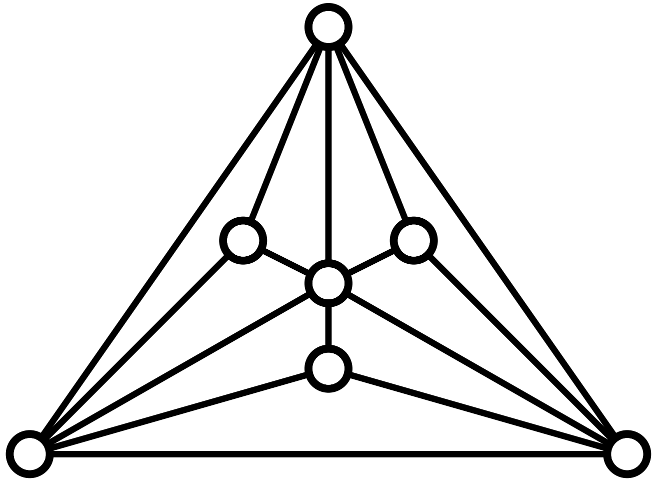
Treewidth describes how *tree-like* a graph is.



1



2



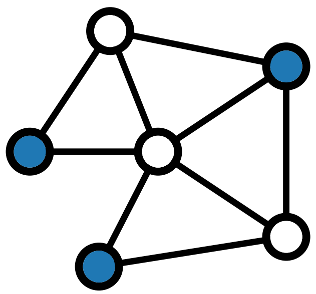
3

Path-/tree-like structure can be useful for designing dynamic programming algorithms.

(Weighted) Independent Set

Input. A graph $G = (V, E)$. Weight function $w : V \to \mathbb{N}$.

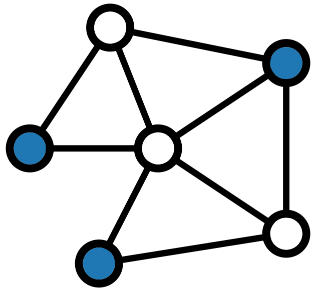
Output. A set $I \subseteq V$ that is **independent**, i.e., $\forall u, v \in I \colon \{u, v\} \notin E$, and has **maximum weight**, i.e., $w(I) := \sum_{v \in I} w(v)$ is maximized.

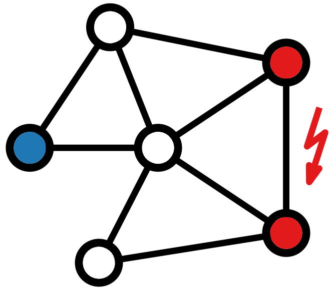


(Weighted) Independent Set

Input. A graph $G = (V, E)$. Weight function $w : V \to \mathbb{N}$.

Output. A set $I \subseteq V$ that is **independent**, i.e., $\forall u, v \in I\colon \{u, v\} \notin E$,
and has **maximum weight**, i.e., $w(I) := \sum_{v \in I} w(v)$ is maximized.

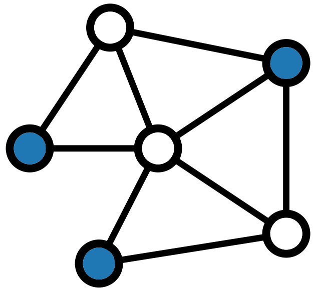


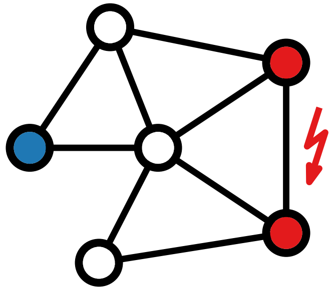


(Weighted) Independent Set

Input. A graph $G = (V, E)$. Weight function $w : V \to \mathbb{N}$.

Output. A set $I \subseteq V$ that is **independent**, i.e., $\forall u, v \in I \colon \{u, v\} \notin E$, and has **maximum weight**, i.e., $w(I) := \sum_{v \in I} w(v)$ is maximized.



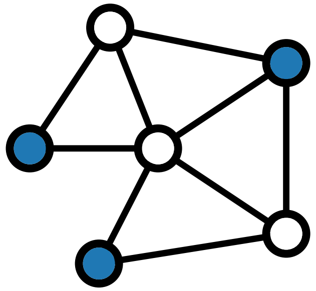


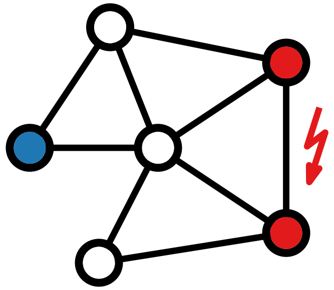
- (Already unweighted) Independent Set is NP-complete,

(Weighted) Independent Set

Input. A graph $G = (V, E)$. Weight function $w : V \to \mathbb{N}$.

Output. A set $I \subseteq V$ that is **independent**, i.e., $\forall u, v \in I \colon \{u, v\} \notin E$, and has **maximum weight**, i.e., $w(I) := \sum_{v \in I} w(v)$ is maximized.



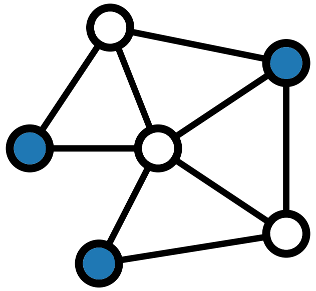


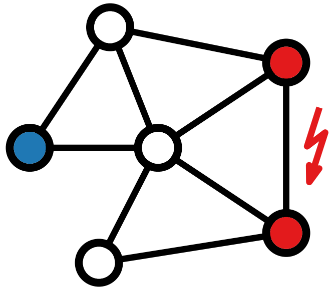
- (Already unweighted) Independent Set is NP-complete,
- but can be solved efficiently on tree-like graphs (also when weighted).

(Weighted) Independent Set

Input. A graph $G = (V, E)$. Weight function $w : V \to \mathbb{N}$.

Output. A set $I \subseteq V$ that is **independent**, i.e., $\forall u, v \in I \colon \{u, v\} \notin E$, and has **maximum weight**, i.e., $w(I) := \sum_{v \in I} w(v)$ is maximized.

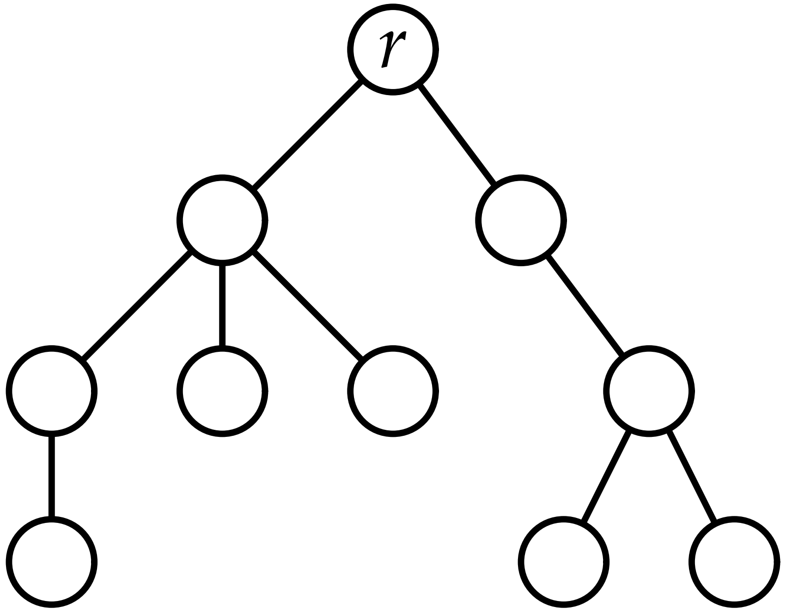




- (Already unweighted) Independent Set is NP-complete,
- but can be solved efficiently on tree-like graphs (also when weighted).
- On trees, (Weighted) Independent Set can be solved in linear time.

Independent Set in Trees

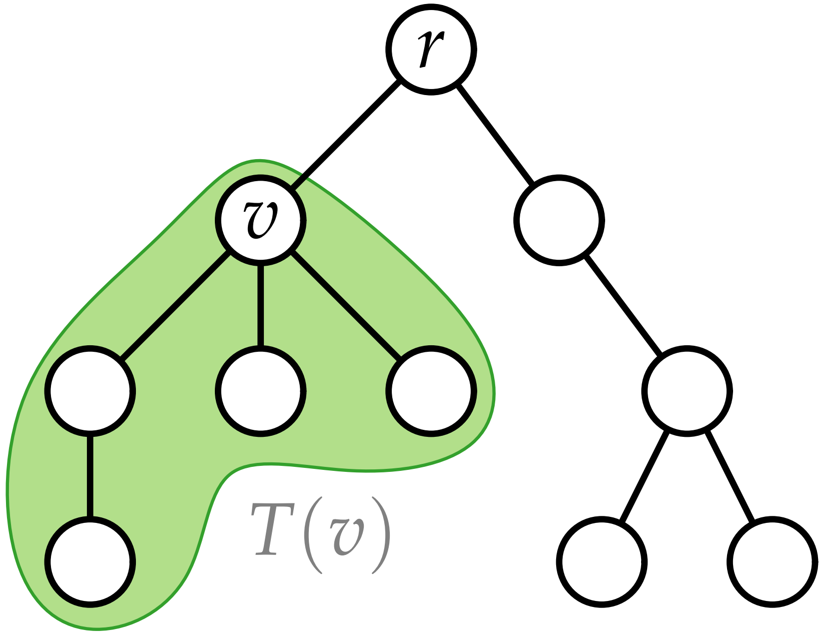
Choose an arbitrary root r.



Independent Set in Trees

Choose an arbitrary root r.

Let $T(v) :=$ subtree rooted at v

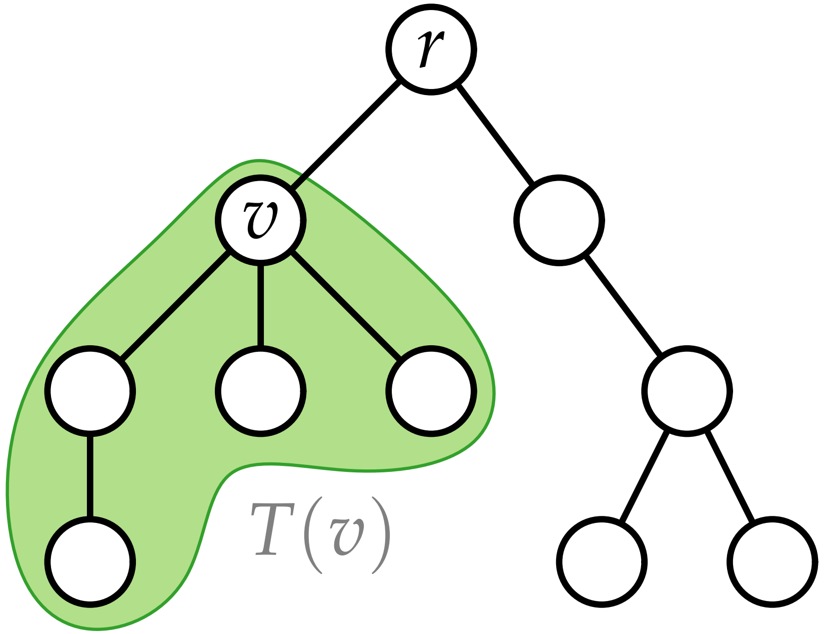


Independent Set in Trees

Choose an arbitrary root r.

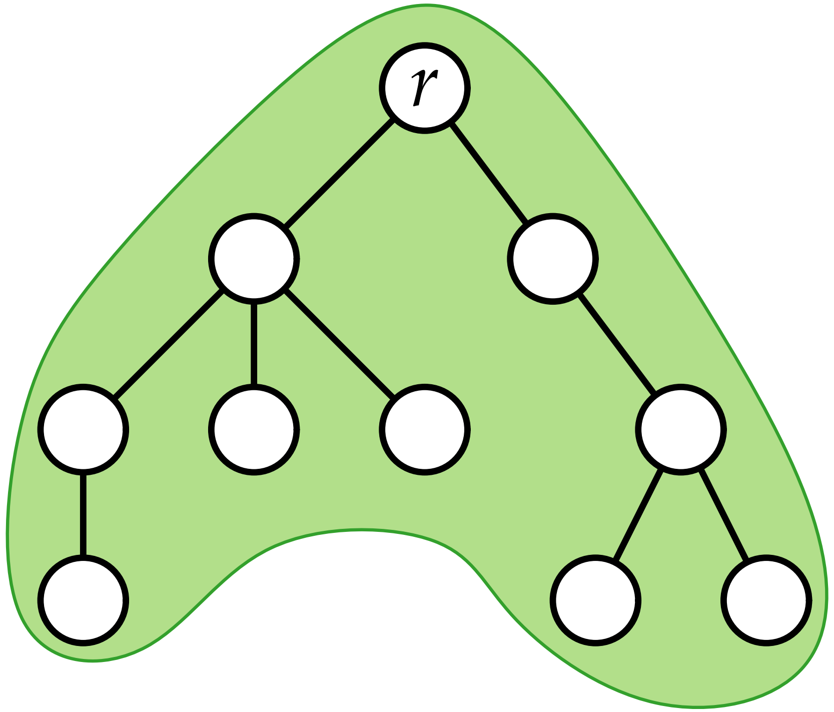
Let $T(v)$:= subtree rooted at v

Let $A(v)$:= maximum weight of an independent set I in $T(v)$



Independent Set in Trees

$A(r)$ = solution



Choose an arbitrary root r.

Let $T(v)$:= subtree rooted at v

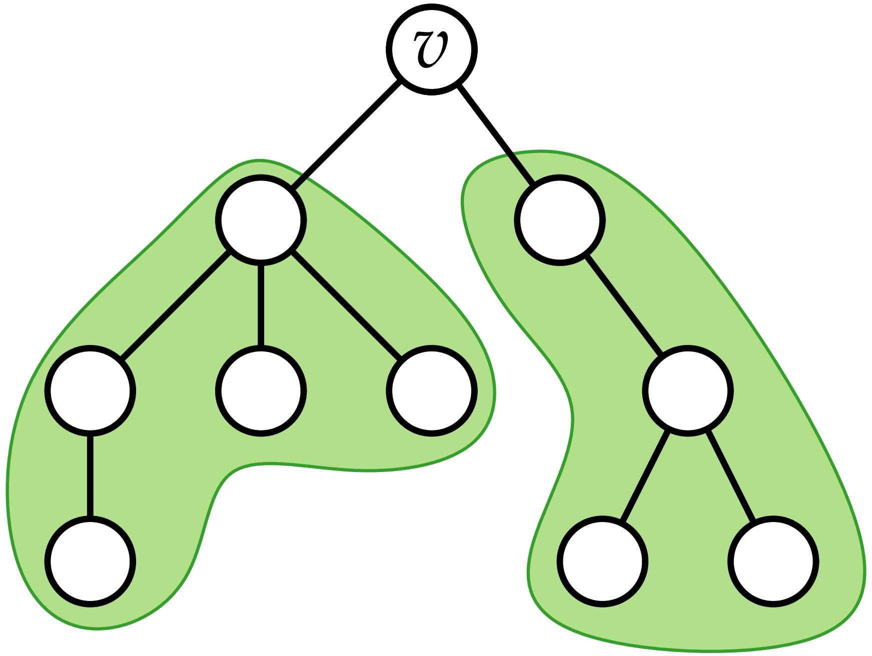
Let $A(v)$:= maximum weight of an independent set I in $T(v)$

INDEPENDENT SET in Trees

Choose an arbitrary root r.

Let $T(v) :=$ subtree rooted at v

Let $A(v) :=$ maximum weight of an independent set I in $T(v)$

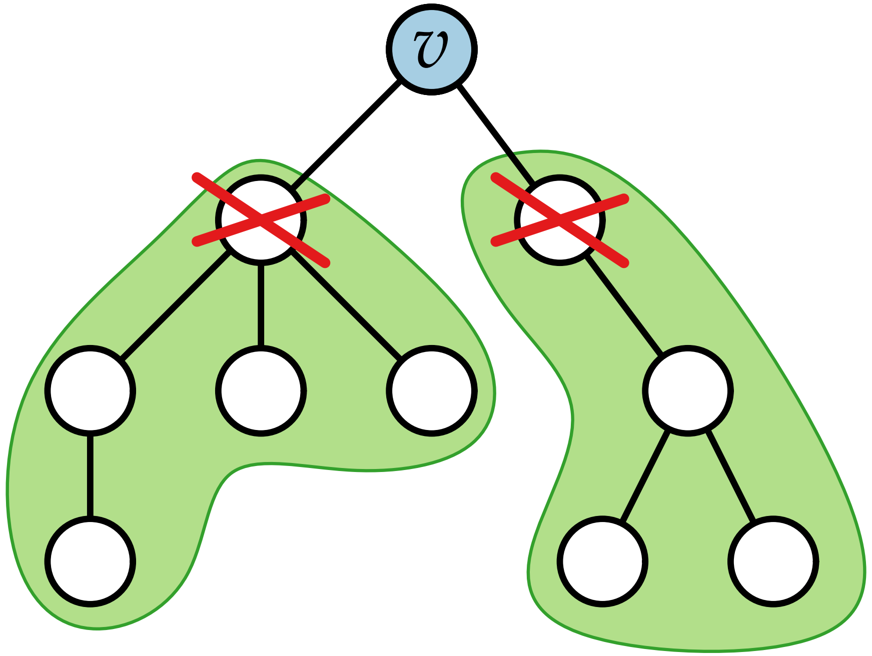


Independent Set in Trees

Choose an arbitrary root r.

Let $T(v)$:= subtree rooted at v

Let $A(v)$:= maximum weight of an independent set I in $T(v)$



If $v \in V$ is part of the indepent set I,
then none of its neighbors $\mathsf{N}(v)$ is also in I.

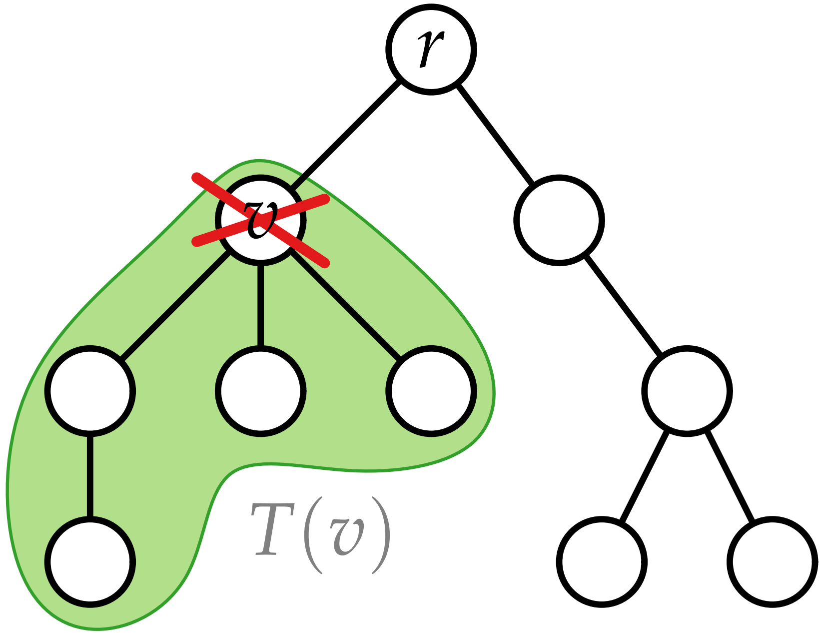
Independent Set in Trees

Choose an arbitrary root r.

Let $T(v)$:= subtree rooted at v

Let $A(v)$:= maximum weight of an independent set I in $T(v)$

Let $B(v)$:= maximum weight of an independent set I in $T(v)$ where $v \notin I$



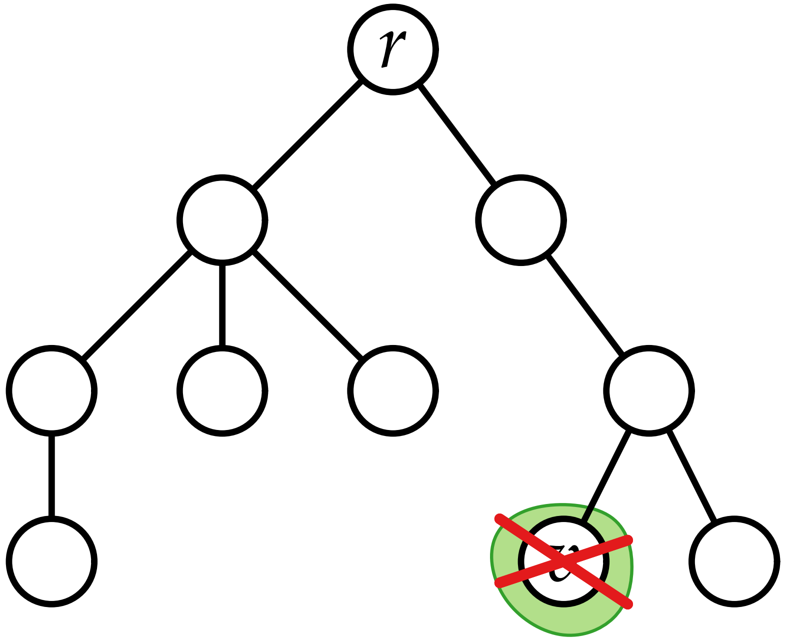
INDEPENDENT SET in Trees

Choose an arbitrary root r.

Let $T(v) :=$ subtree rooted at v

Let $A(v) :=$ maximum weight of an independent set I in $T(v)$

Let $B(v) :=$ maximum weight of an independent set I in $T(v)$ where $v \notin I$



- If v is a leaf: $B(v) =$

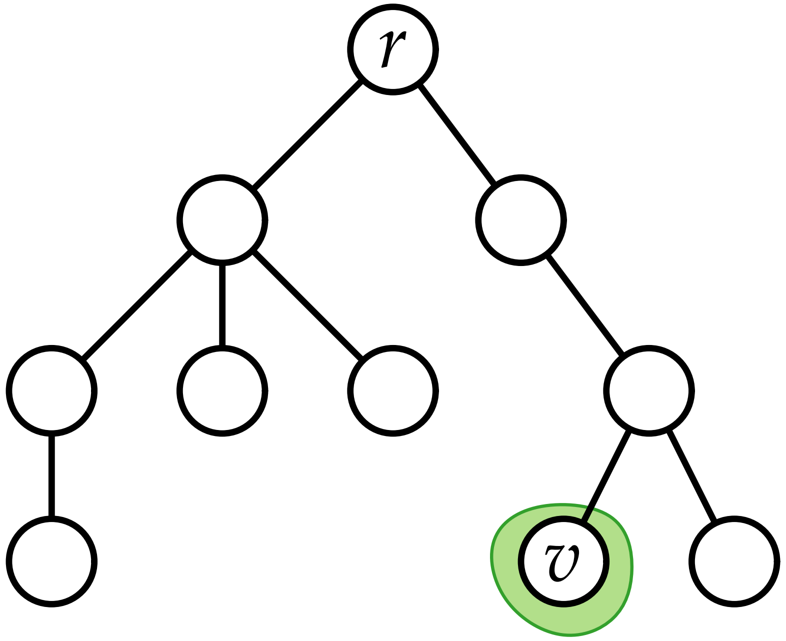
Independent Set in Trees

Choose an arbitrary root r.

Let $T(v) :=$ subtree rooted at v

Let $A(v) :=$ maximum weight of an independent set I in $T(v)$

Let $B(v) :=$ maximum weight of an independent set I in $T(v)$ where $v \notin I$



- If v is a leaf: $B(v) = 0$ and $A(v) =$

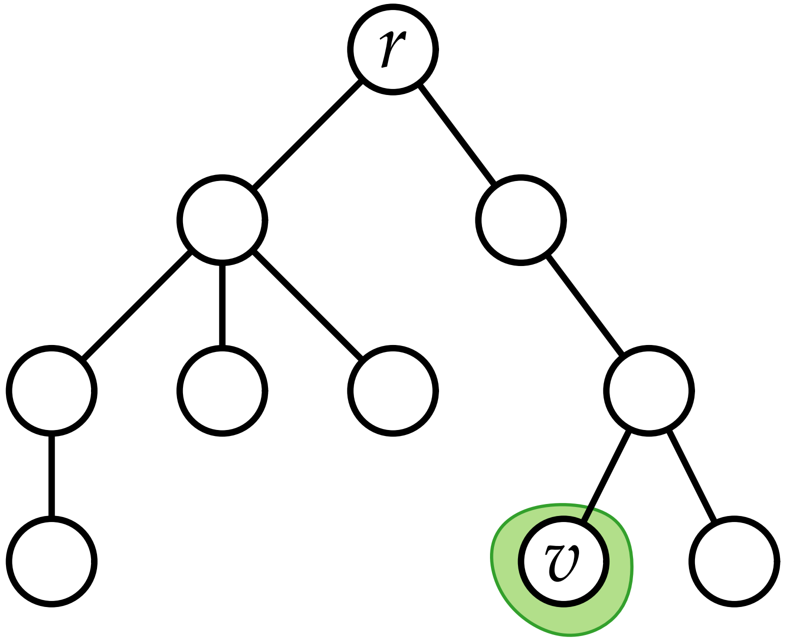
Independent Set in Trees

Choose an arbitrary root r.

Let $T(v) :=$ subtree rooted at v

Let $A(v) :=$ maximum weight of an independent set I in $T(v)$

Let $B(v) :=$ maximum weight of an independent set I in $T(v)$ where $v \notin I$



- If v is a leaf: $B(v) = 0$ and $A(v) = w(v)$

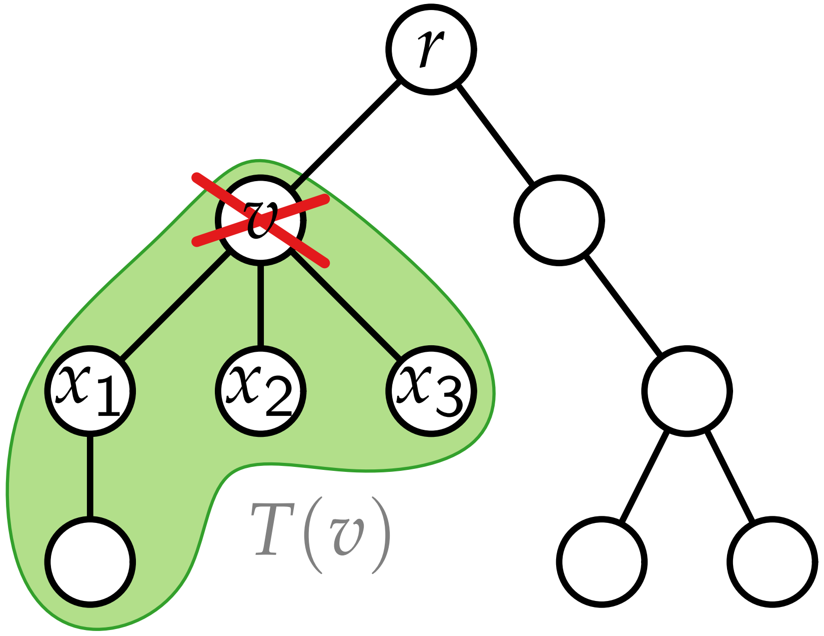
Independent Set in Trees

Choose an arbitrary root r.

Let $T(v)$:= subtree rooted at v

Let $A(v)$:= maximum weight of an independent set I in $T(v)$

Let $B(v)$:= maximum weight of an independent set I in $T(v)$ where $v \notin I$



- If v is a leaf: $B(v) = 0$ and $A(v) = w(v)$
- If v has children $x_1, \dots, x_\ell$:

  $B(v) =$

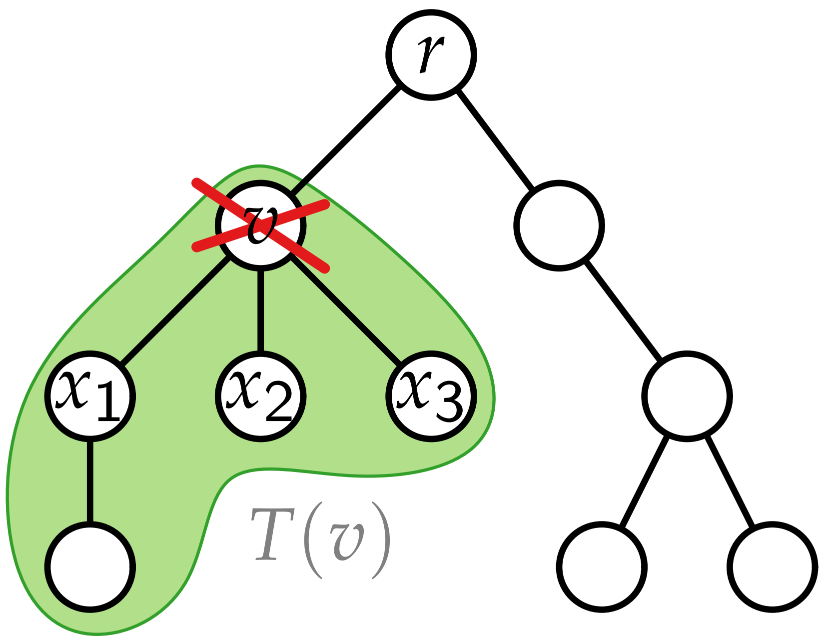
Independent Set in Trees

Choose an arbitrary root r.

Let $T(v)$:= subtree rooted at v

Let $A(v)$:= maximum weight of an independent set I in $T(v)$

Let $B(v)$:= maximum weight of an independent set I in $T(v)$ where $v \notin I$



- If v is a leaf: $B(v) = 0$ and $A(v) = w(v)$
- If v has children $x_1, \dots, x_\ell$:
  $B(v) = \sum_{i=1}^{\ell} A(x_i)$;

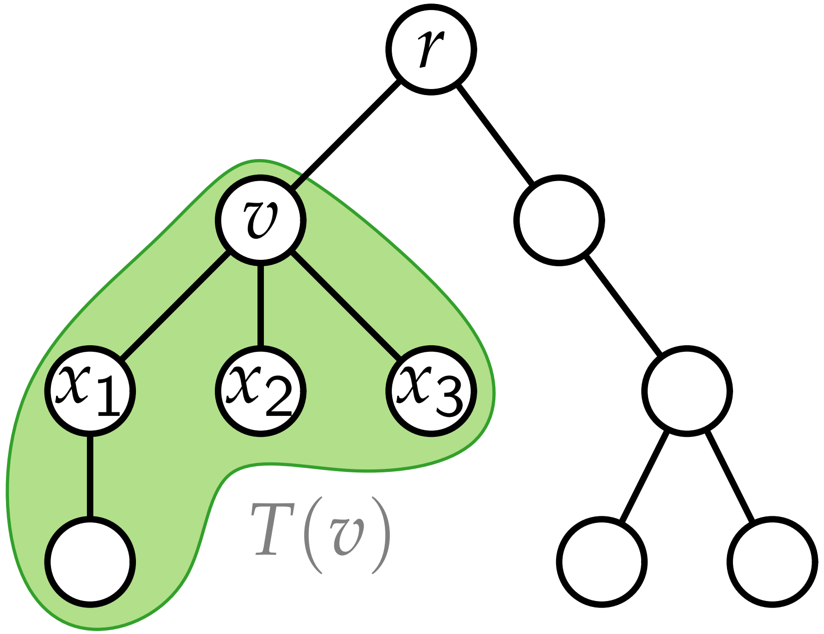
Independent Set in Trees

Choose an arbitrary root r.

Let $T(v)$:= subtree rooted at v

Let $A(v)$:= maximum weight of an independent set I in $T(v)$

Let $B(v)$:= maximum weight of an independent set I in $T(v)$ where $v \notin I$



- If v is a leaf: $B(v) = 0$ and $A(v) = w(v)$
- If v has children $x_1, \ldots, x_\ell$:
  $B(v) = \sum_{i=1}^{\ell} A(x_i)$; $A(v) =$

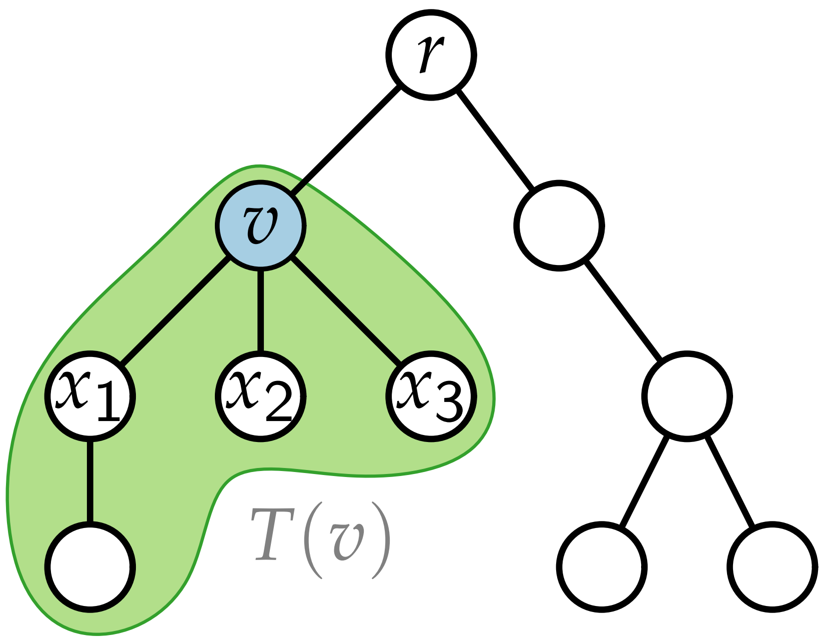
INDEPENDENT SET in Trees

Choose an arbitrary root r.

Let $T(v)$:= subtree rooted at v

Let $A(v)$:= maximum weight of an independent set I in $T(v)$

Let $B(v)$:= maximum weight of an independent set I in $T(v)$ where $v \notin I$



- If v is a leaf: $B(v) = 0$ and $A(v) = w(v)$
- If v has children $x_1, \ldots, x_\ell$:
  $B(v) = \sum_{i=1}^{\ell} A(x_i)$; $A(v) =$ $w(v)$

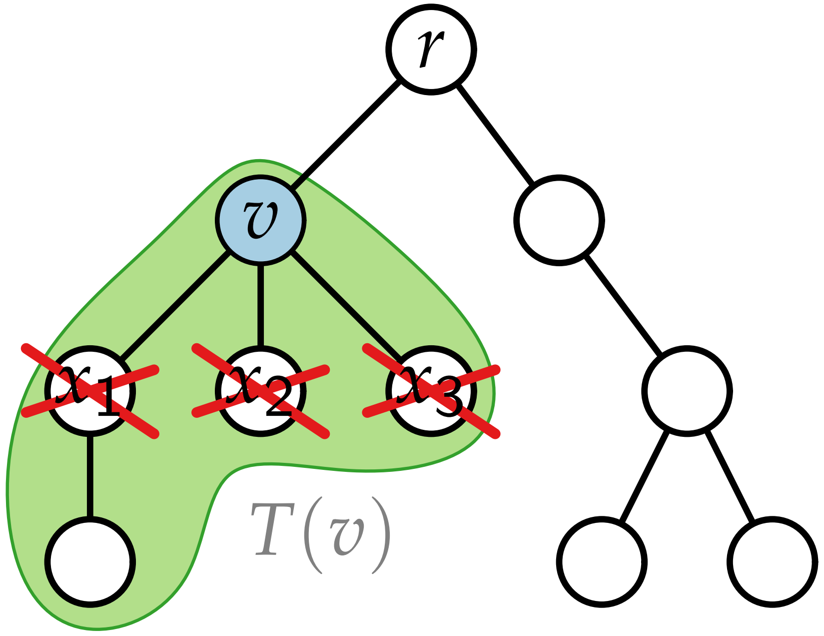
Independent Set in Trees

Choose an arbitrary root r.

Let $T(v)$:= subtree rooted at v

Let $A(v)$:= maximum weight of an independent set I in $T(v)$

Let $B(v)$:= maximum weight of an independent set I in $T(v)$ where $v \notin I$



- If v is a leaf: $B(v) = 0$ and $A(v) = w(v)$
- If v has children $x_1, \ldots, x_\ell$:

  $B(v) = \sum_{i=1}^{\ell} A(x_i)$; $A(v) = \quad w(v) + \sum_{i=1}^{\ell} B(x_i)$

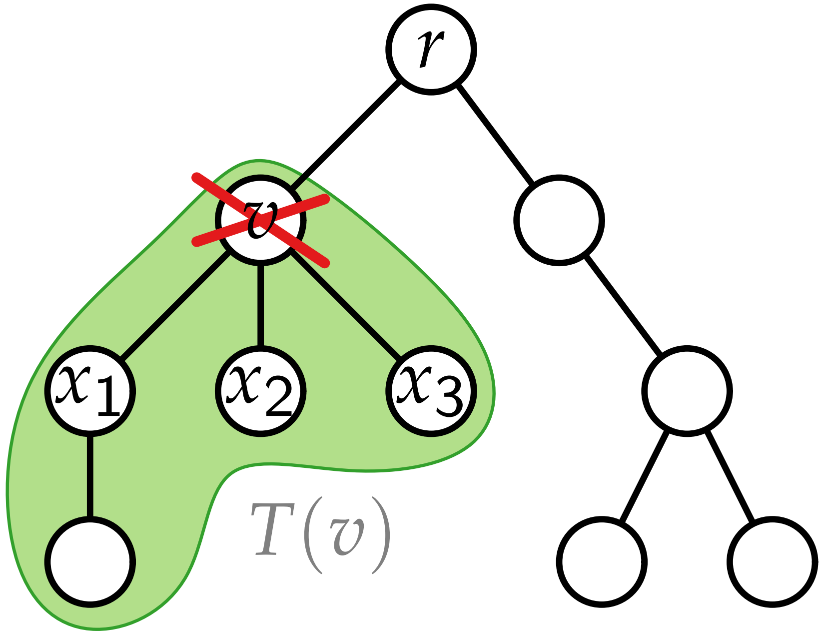
Independent Set in Trees

Choose an arbitrary root r.

Let $T(v) :=$ subtree rooted at v

Let $A(v) :=$ maximum weight of an independent set I in $T(v)$

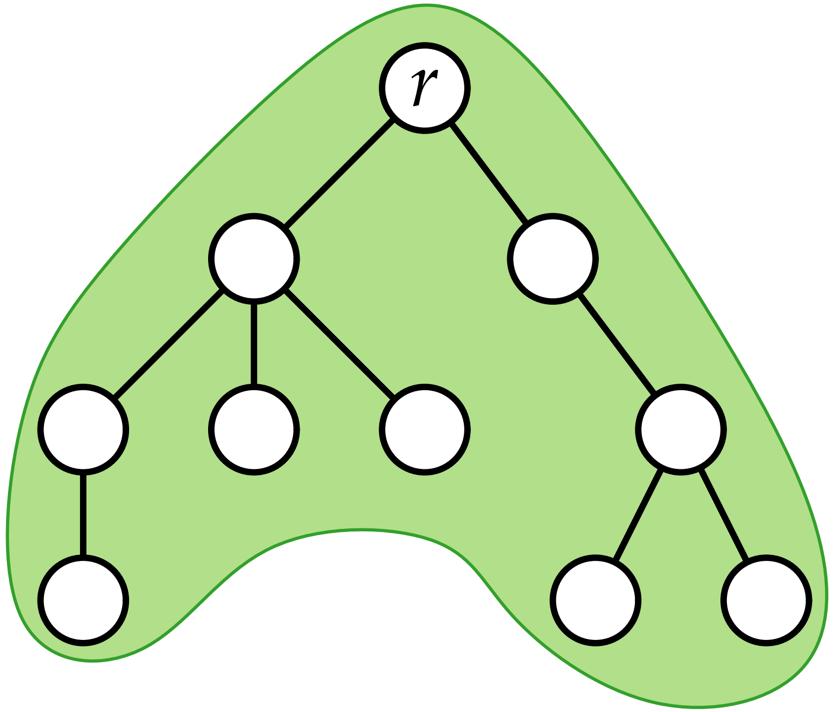
Let $B(v) :=$ maximum weight of an independent set I in $T(v)$ where $v \notin I$



- If v is a leaf: $B(v) = 0$ and $A(v) = w(v)$
- If v has children $x_1, \ldots, x_\ell$:
  $B(v) = \sum_{i=1}^{\ell} A(x_i)$; $A(v) = \max\{B(v), w(v) + \sum_{i=1}^{\ell} B(x_i)\}$

Independent Set in Trees

$A(r)$ = solution



Choose an arbitrary root r.

Let $T(v)$:= subtree rooted at v

Let $A(v)$:= maximum weight of an independent set I in $T(v)$

Let $B(v)$:= maximum weight of an independent set I in $T(v)$ where $v \notin I$

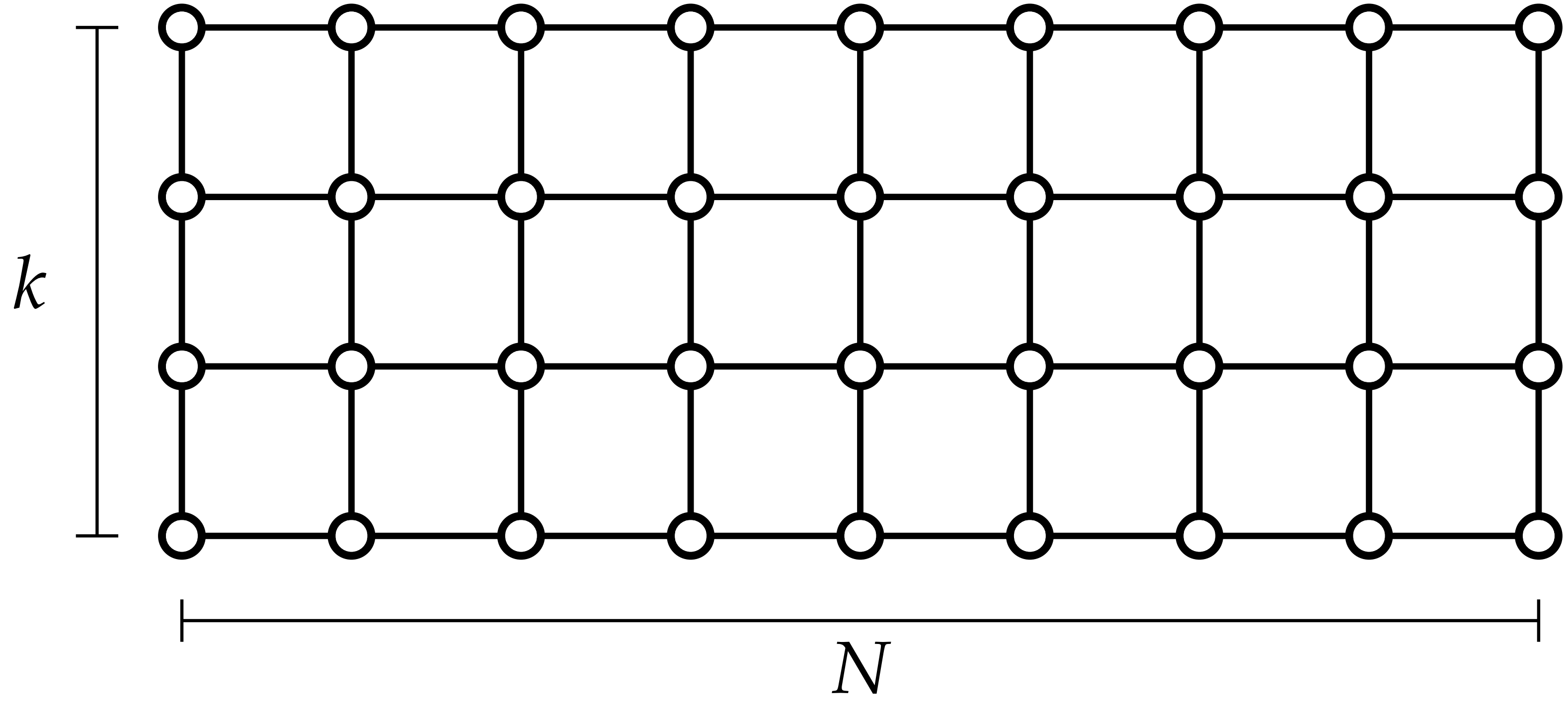
- If v is a leaf: $B(v) = 0$ and $A(v) = w(v)$
- If v has children $x_1, \ldots, x_\ell$:
  $B(v) = \sum_{i=1}^{\ell} A(x_i)$; $A(v) = \max\{B(v), w(v) + \sum_{i=1}^{\ell} B(x_i)\}$

Algorithm: Compute $A(\cdot)$ and $B(\cdot)$ bottom-up, return $A(r)$.

Grid Graphs

In a $k \times N$ **grid graph**

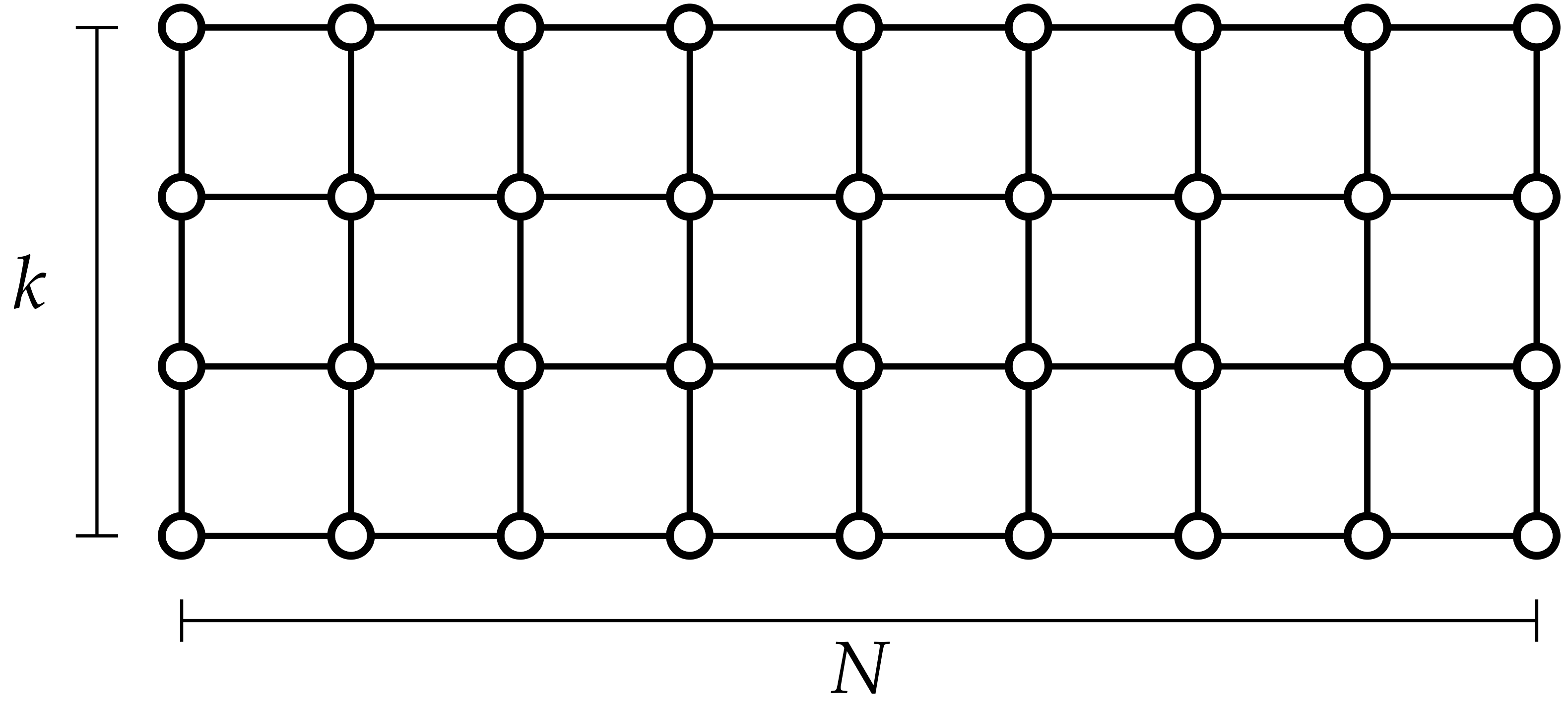
- the vertex set consist of all pairs (i, j) where $1 \leq i \leq k$ and $1 \leq j \leq N$, and
- two vertices (i_1, j_1) and (i_2, j_2) are adjacent if and only if $|i_1 - i_2| + |j_1 - j_2| = 1$.



Grid Graphs

In a $k \times N$ **grid graph**

- the vertex set consist of all pairs (i, j) where $1 \leq i \leq k$ and $1 \leq j \leq N$, and
- two vertices (i_1, j_1) and (i_2, j_2) are adjacent if and only if $|i_1 - i_2| + |j_1 - j_2| = 1$.

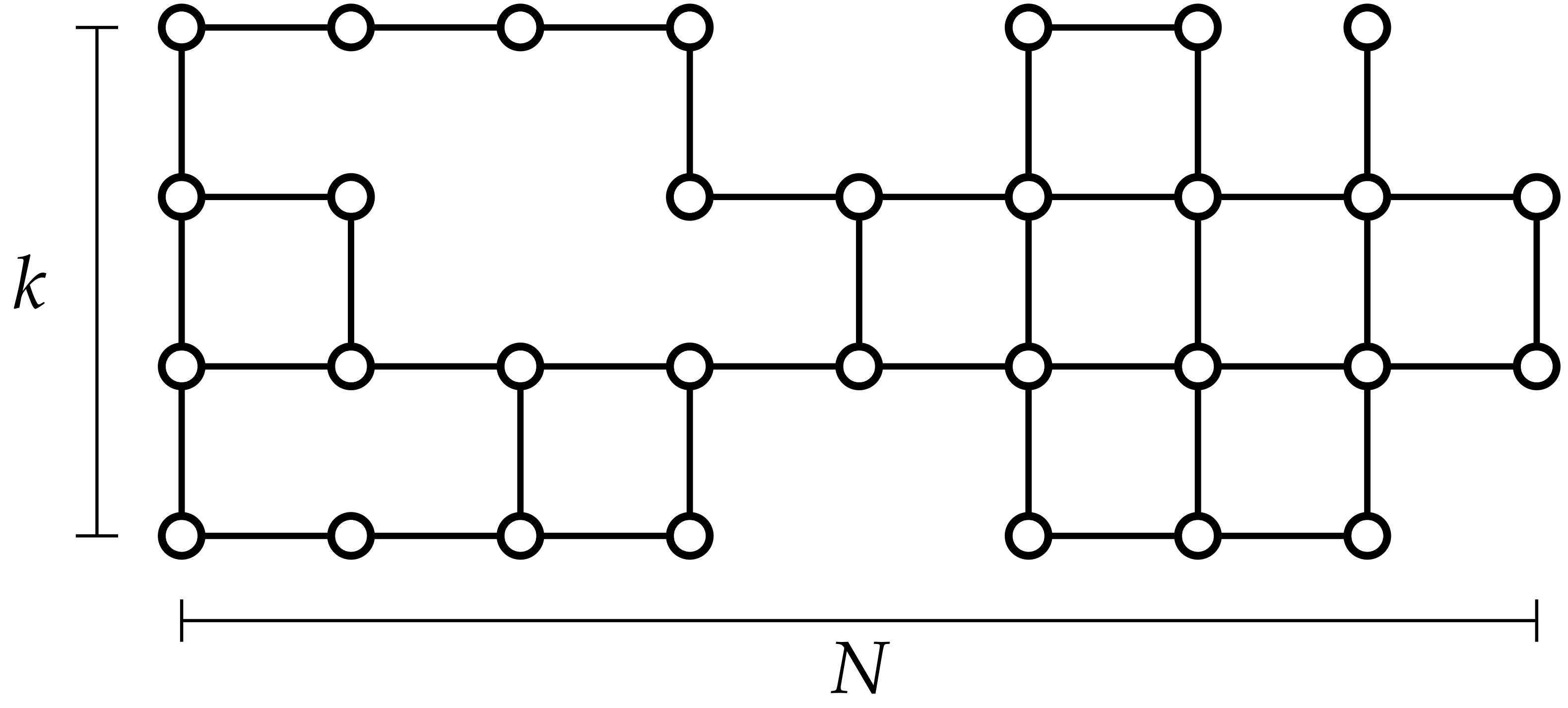


We will study INDEPENDENT SET in subgraphs of $k \times N$ grid graphs.

Grid Graphs

In a $k \times N$ **grid graph**

- the vertex set consist of all pairs (i, j) where $1 \le i \le k$ and $1 \le j \le N$, and
- two vertices (i_1, j_1) and (i_2, j_2) are adjacent if and only if $|i_1 - i_2| + |j_1 - j_2| = 1$.

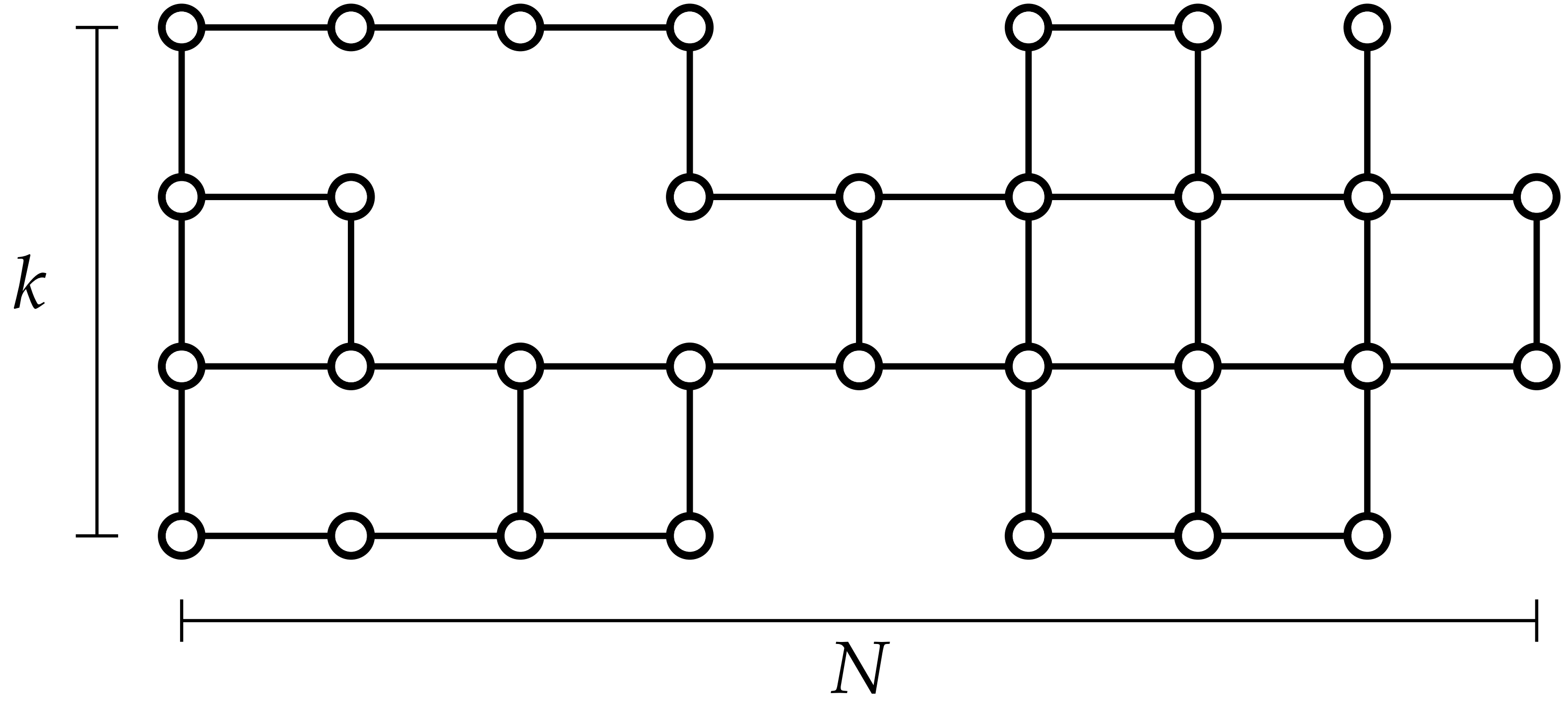


We will study Independent Set in subgraphs of $k \times N$ grid graphs.

Grid Graphs

In a $k \times N$ **grid graph**

- the vertex set consist of all pairs (i, j) where $1 \leq i \leq k$ and $1 \leq j \leq N$, and
- two vertices (i_1, j_1) and (i_2, j_2) are adjacent if and only if $|i_1 - i_2| + |j_1 - j_2| = 1$.

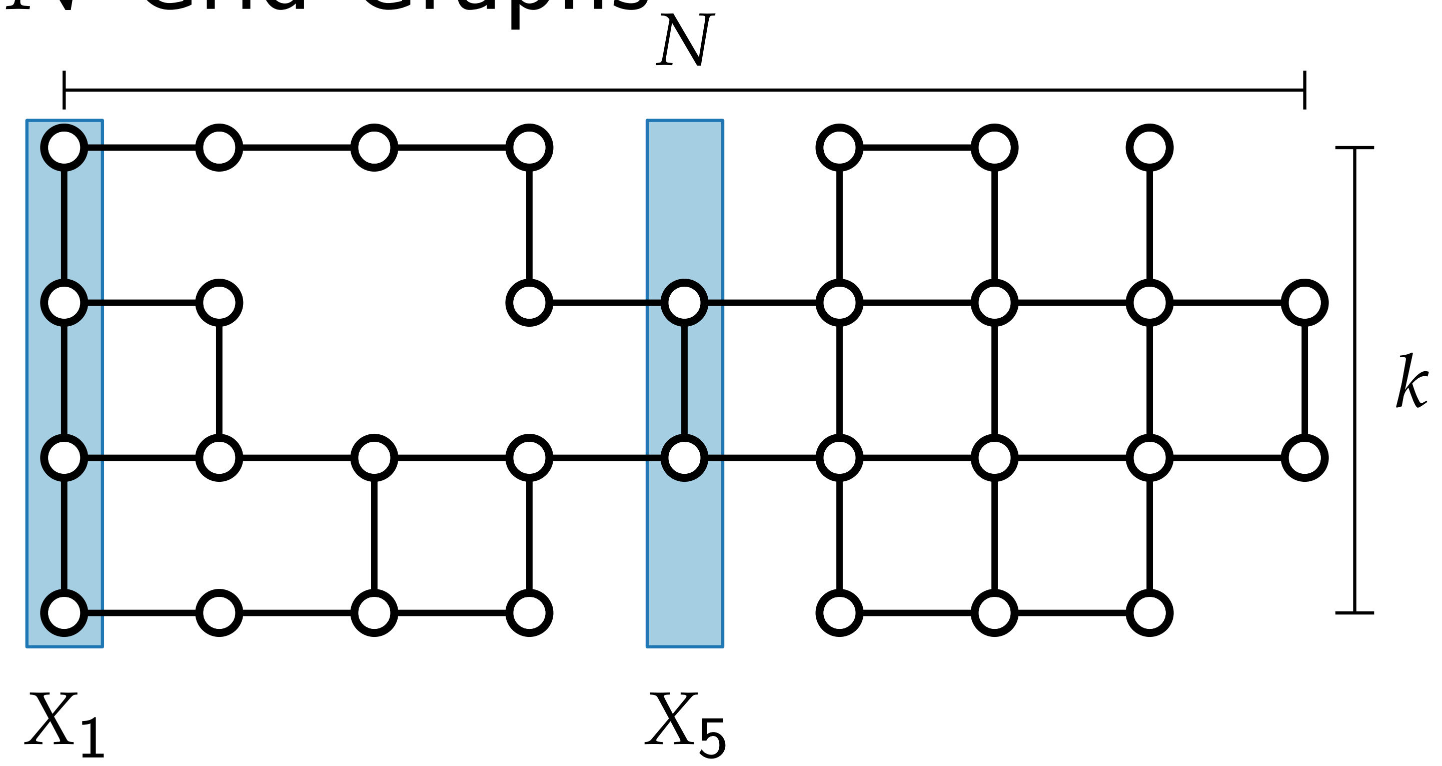


We will study Independent Set in subgraphs of $k \times N$ grid graphs.

Goal: An FPT algorithm with respect to the parameter k.

INDENPENDENT SET in $k \times N$ Grid Graphs

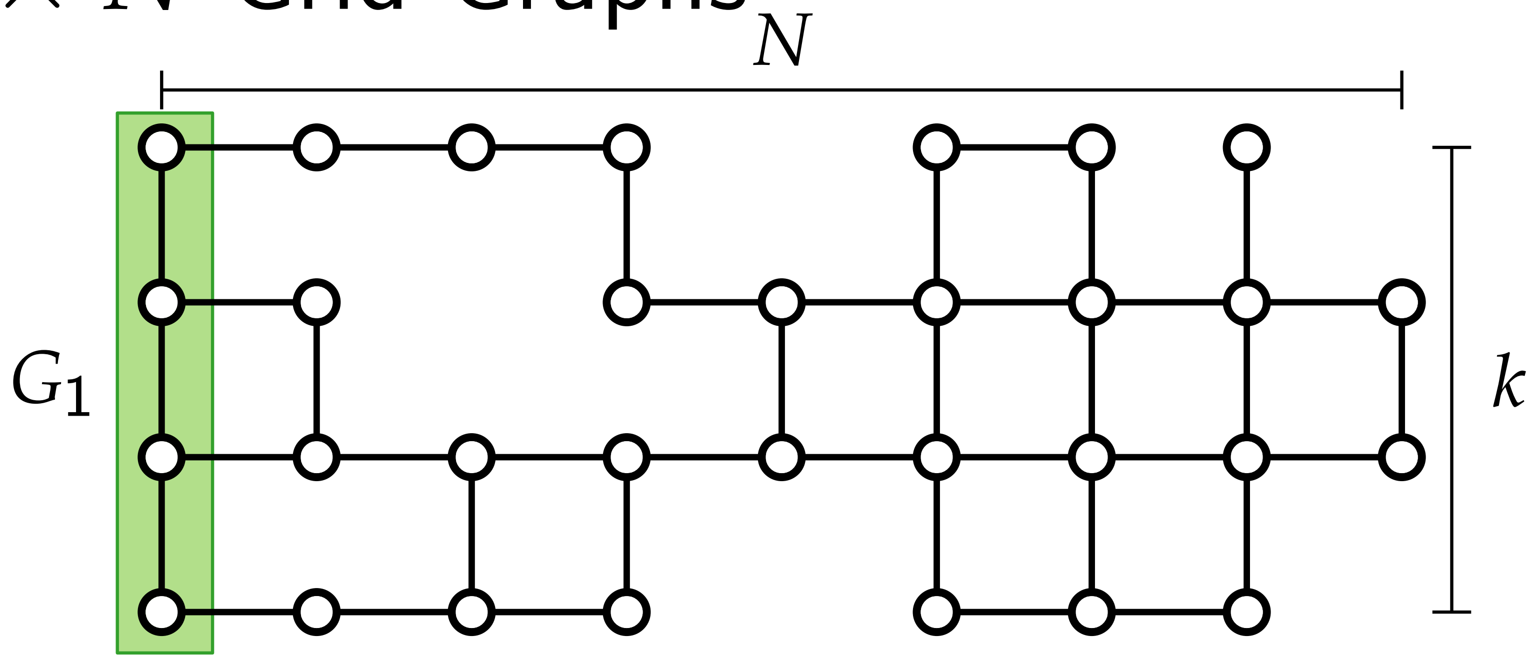
Let X_j be the **j-th column**, that is,
$X_j = \mathsf{V}(G) \cap \{(i,j) \mid 1 \leq i \leq k\}$.



INDENPENDENT SET in $k \times N$ Grid Graphs

Let X_j be the **j-th column**, that is, $X_j = \mathsf{V}(G) \cap \{(i,j) \mid 1 \leq i \leq k\}$.

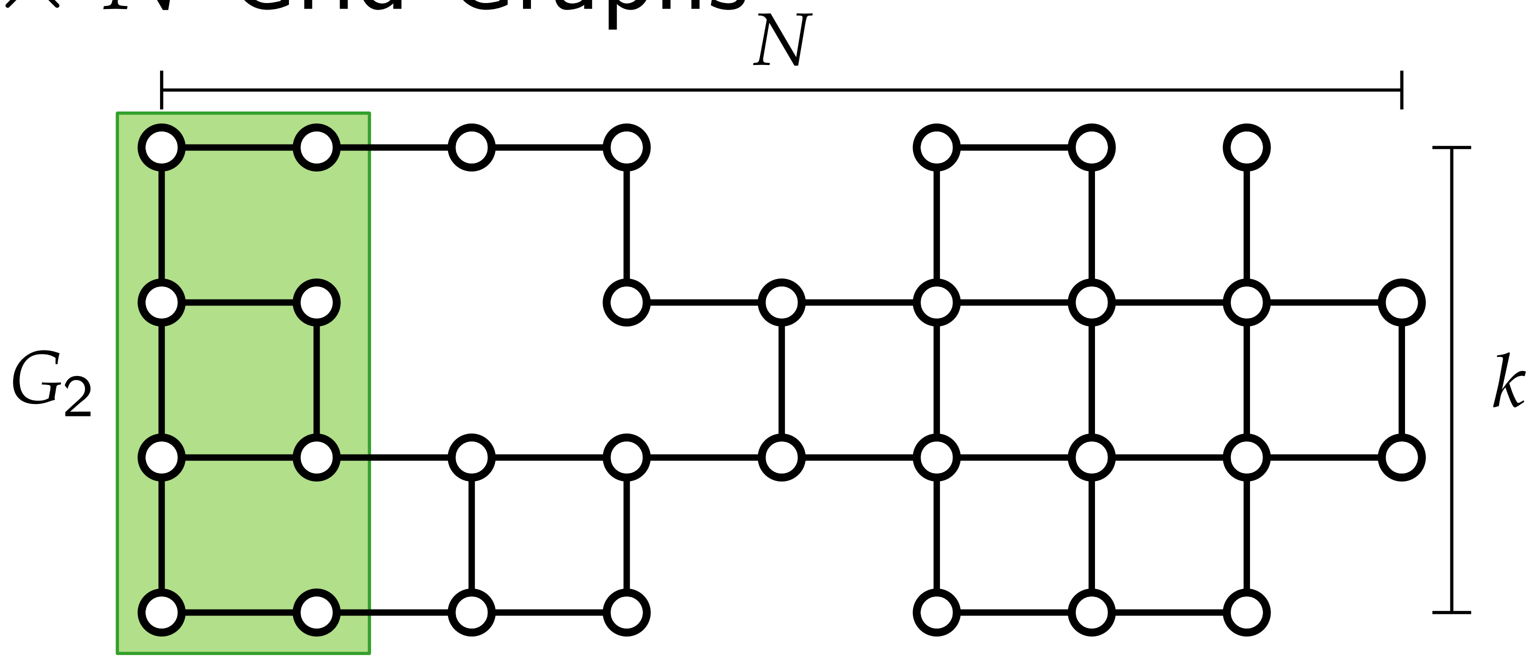
Let G_j be the graph induced by the first j columns $X_1 \cup X_2 \cup \ldots X_j$.



Indenpendent Set in $k \times N$ Grid Graphs

Let X_j be the **j-th column**, that is, $X_j = \mathsf{V}(G) \cap \{(i, j) \mid 1 \leq i \leq k\}$.

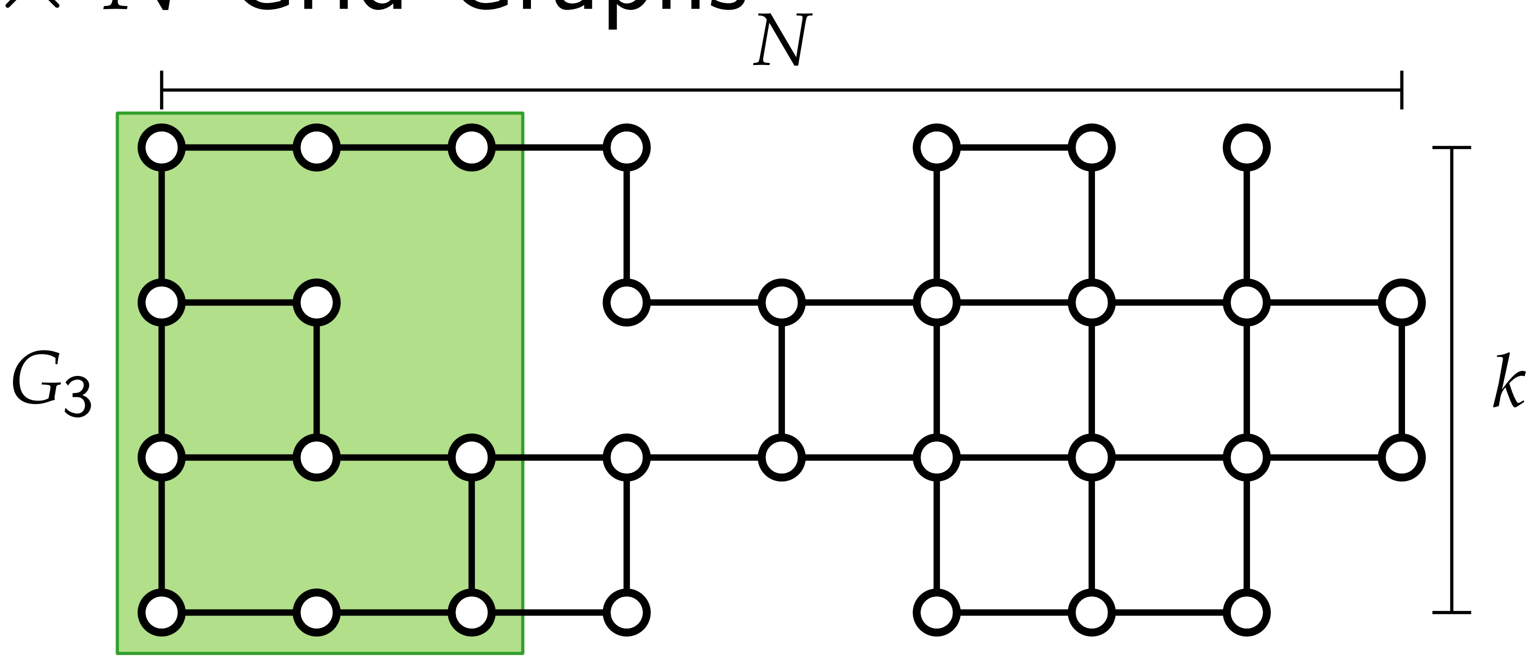
Let G_j be the graph induced by the first j columns $X_1 \cup X_2 \cup \ldots X_j$.



INDENPENDENT SET in $k \times N$ Grid Graphs

Let X_j be the **j-th column**, that is, $X_j = \mathsf{V}(G) \cap \{(i,j) \mid 1 \leq i \leq k\}$.

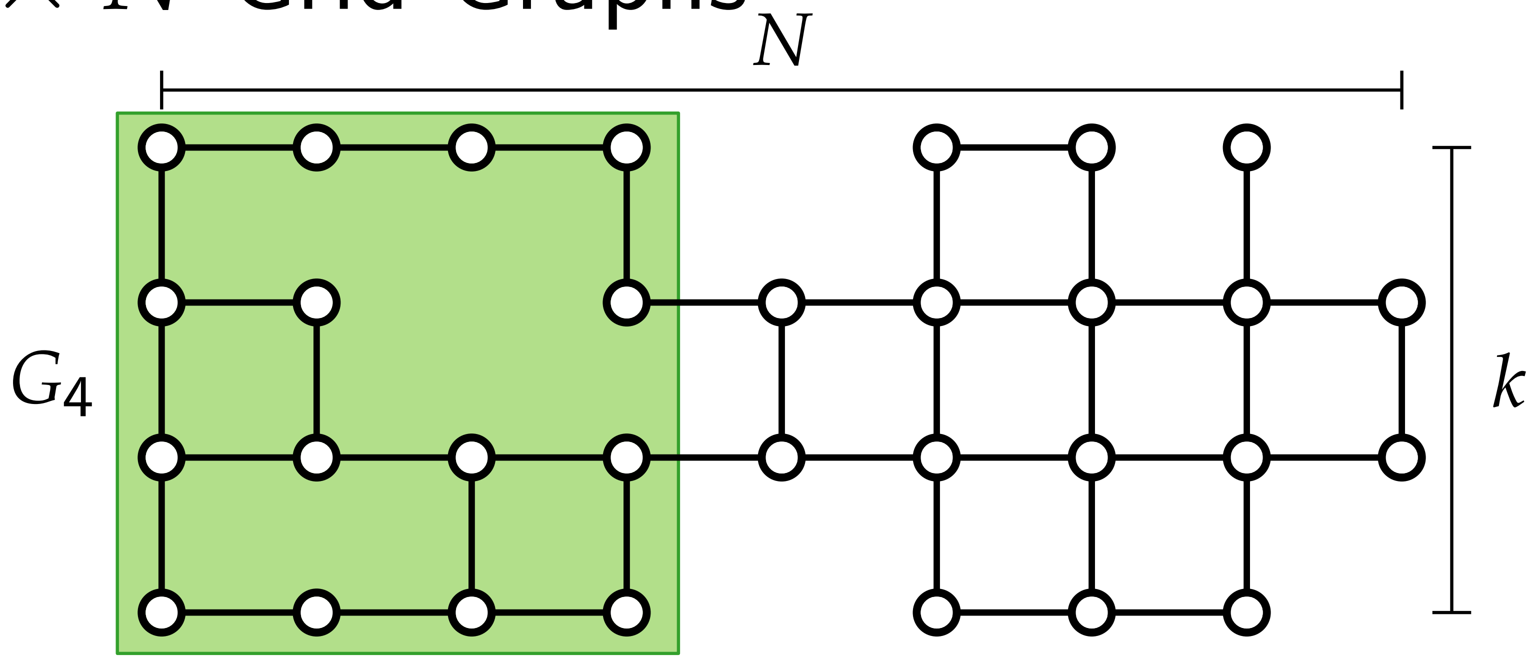
Let G_j be the graph induced by the first j columns $X_1 \cup X_2 \cup \ldots X_j$.



INDENPENDENT SET in $k \times N$ Grid Graphs

Let X_j be the **j-th column**, that is, $X_j = \mathsf{V}(G) \cap \{(i,j) \mid 1 \leq i \leq k\}$.

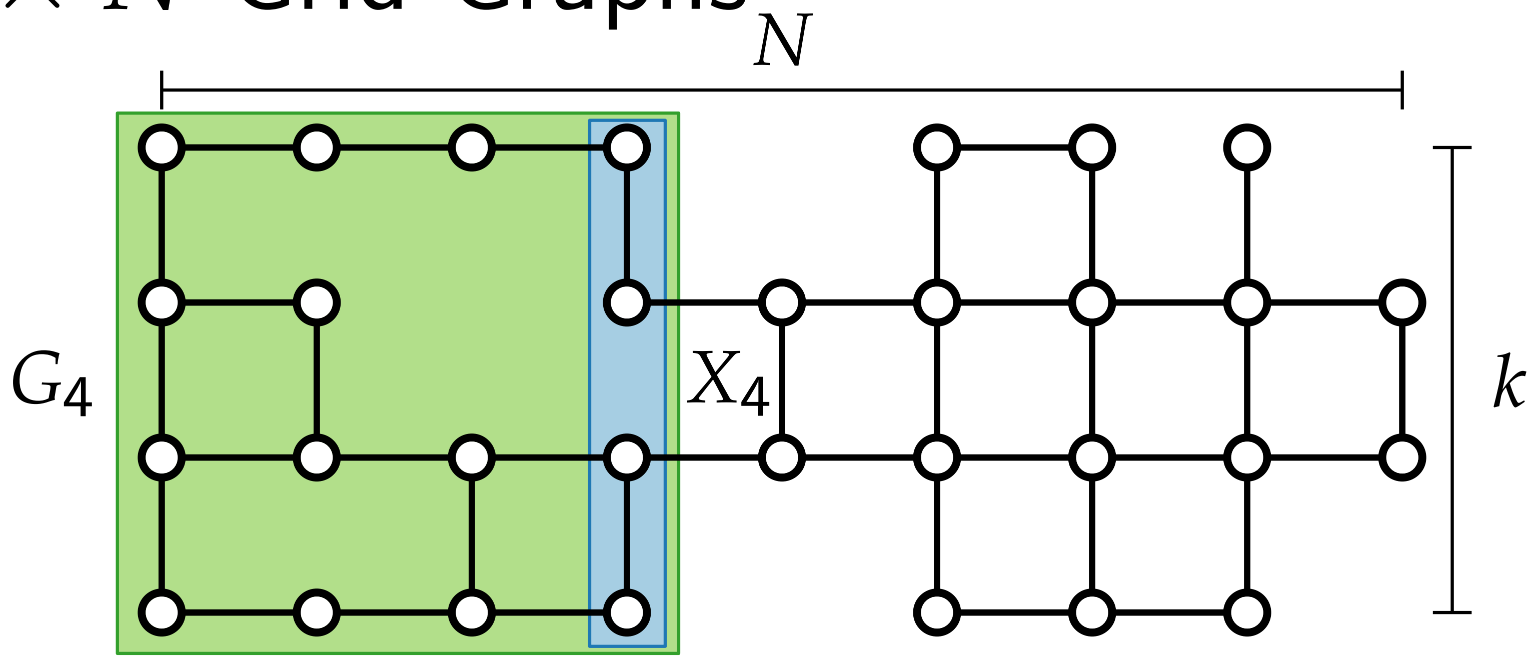
Let G_j be the graph induced by the first j columns $X_1 \cup X_2 \cup \ldots X_j$.



INDENPENDENT SET in $k \times N$ Grid Graphs

Let X_j be the ***j*-th column**, that is, $X_j = \mathsf{V}(G) \cap \{(i,j) \mid 1 \leq i \leq k\}$.

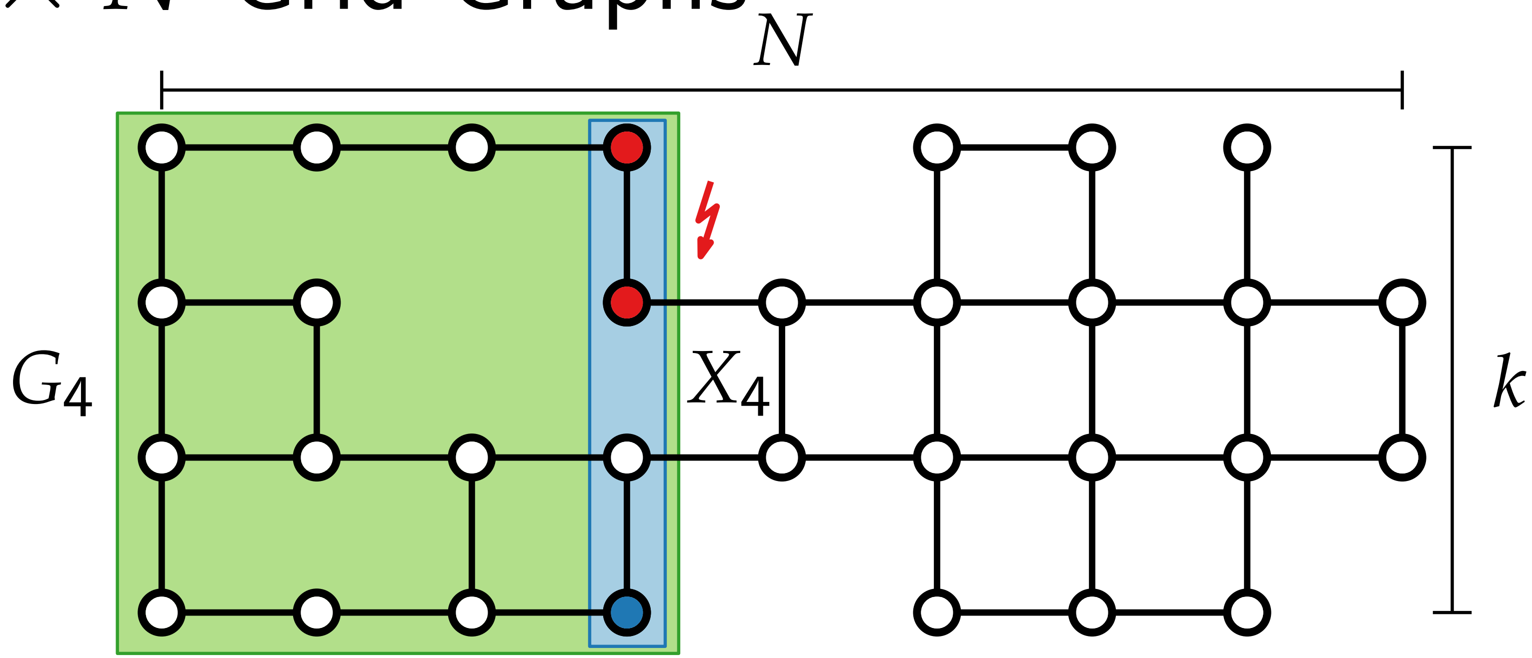
Let G_j be the graph induced by the first j columns $X_1 \cup X_2 \cup \ldots X_j$.



Indenpendent Set in $k \times N$ Grid Graphs

Let X_j be the **j-th column**, that is, $X_j = \mathsf{V}(G) \cap \{(i,j) \mid 1 \leq i \leq k\}$.

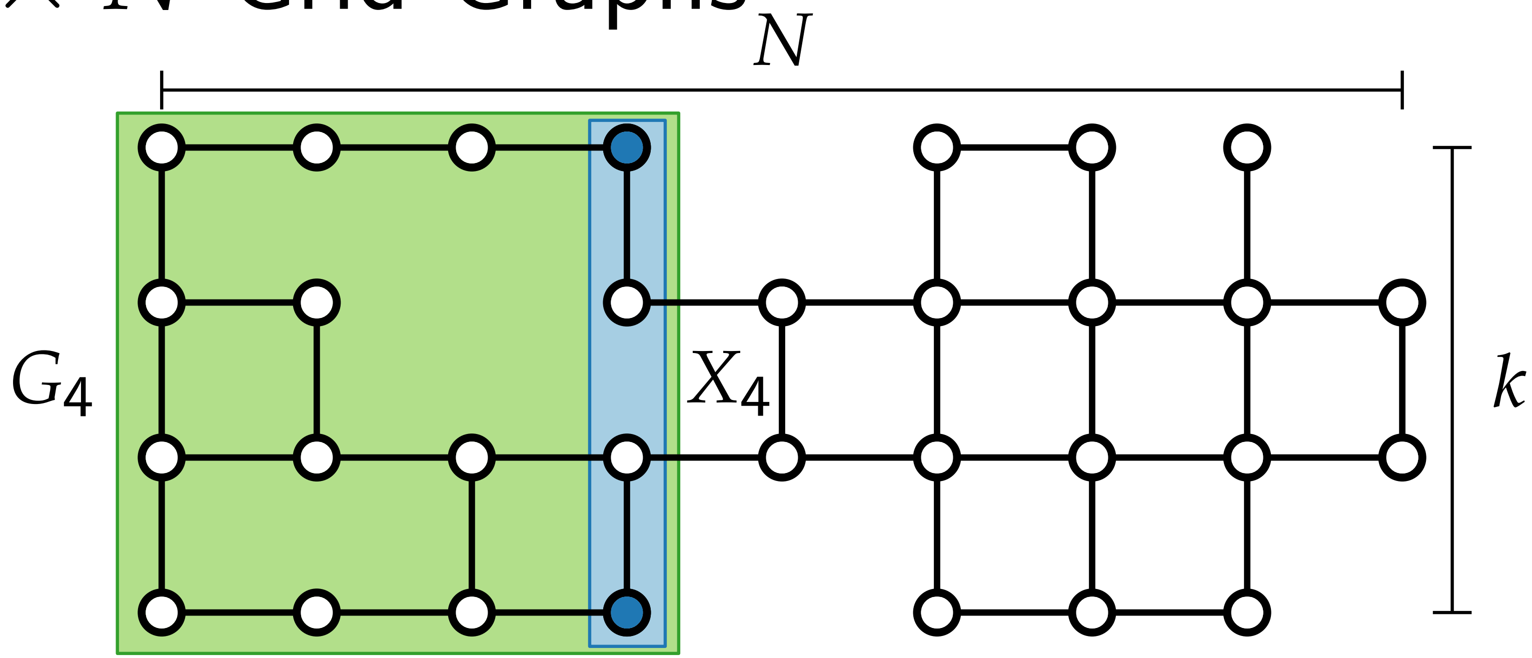
Let G_j be the graph induced by the first j columns $X_1 \cup X_2 \cup \ldots X_j$.



INDENPENDENT SET in $k \times N$ Grid Graphs

Let X_j be the **j-th column**, that is, $X_j = \mathsf{V}(G) \cap \{(i,j) \mid 1 \leq i \leq k\}$.

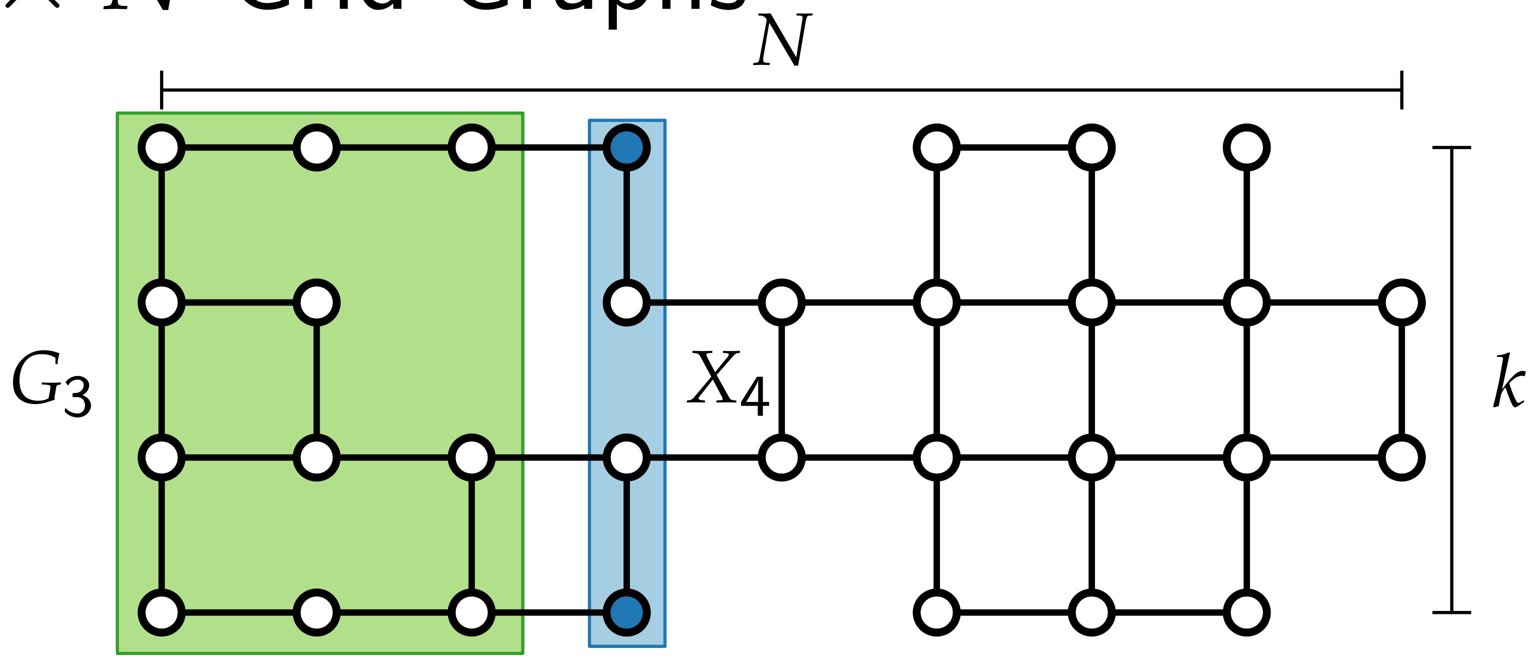
Let G_j be the graph induced by the first j columns $X_1 \cup X_2 \cup \ldots X_j$.



Indenpendent Set in $k \times N$ Grid Graphs

Let X_j be the **j-th column**, that is, $X_j = \mathsf{V}(G) \cap \{(i,j) \mid 1 \leq i \leq k\}$.

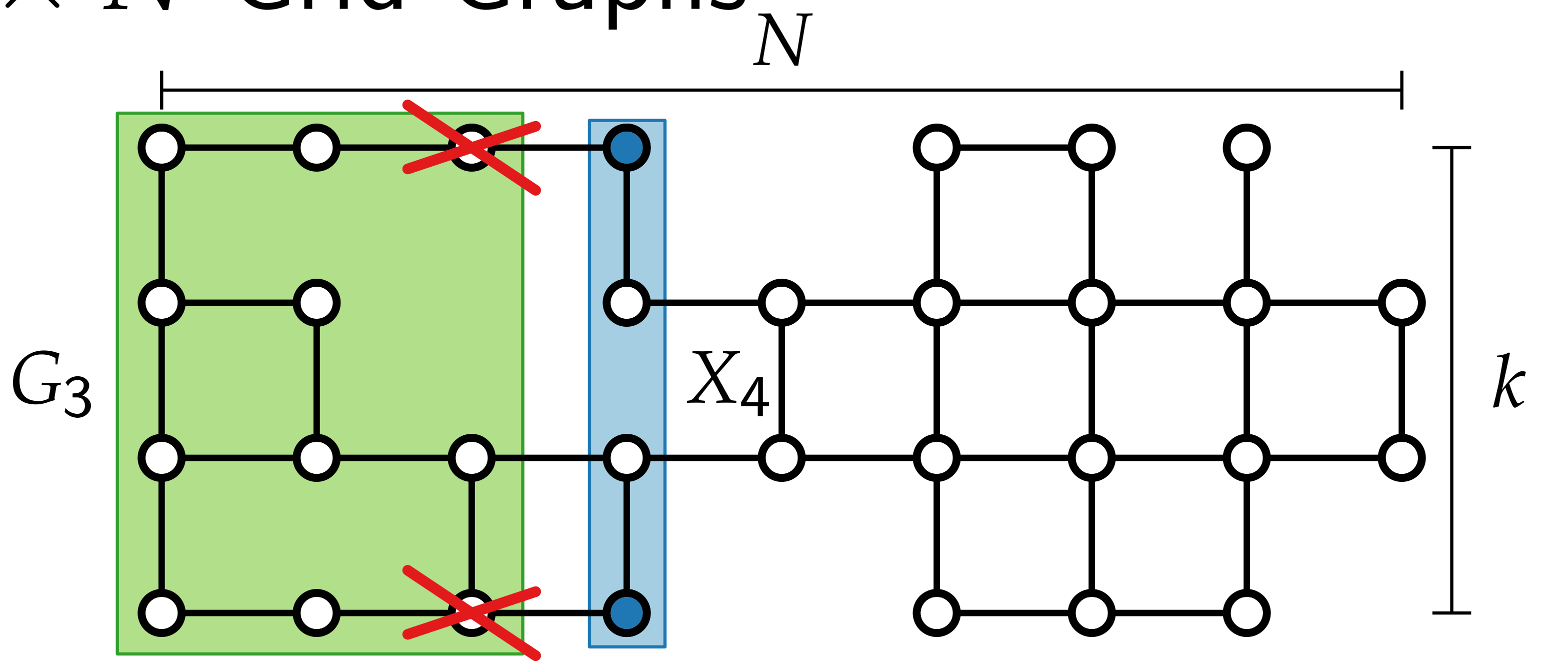
Let G_j be the graph induced by the first j columns $X_1 \cup X_2 \cup \ldots X_j$.



INDENPENDENT SET in $k \times N$ Grid Graphs

Let X_j be the **j-th column**, that is, $X_j = \mathsf{V}(G) \cap \{(i,j) \mid 1 \leq i \leq k\}$.

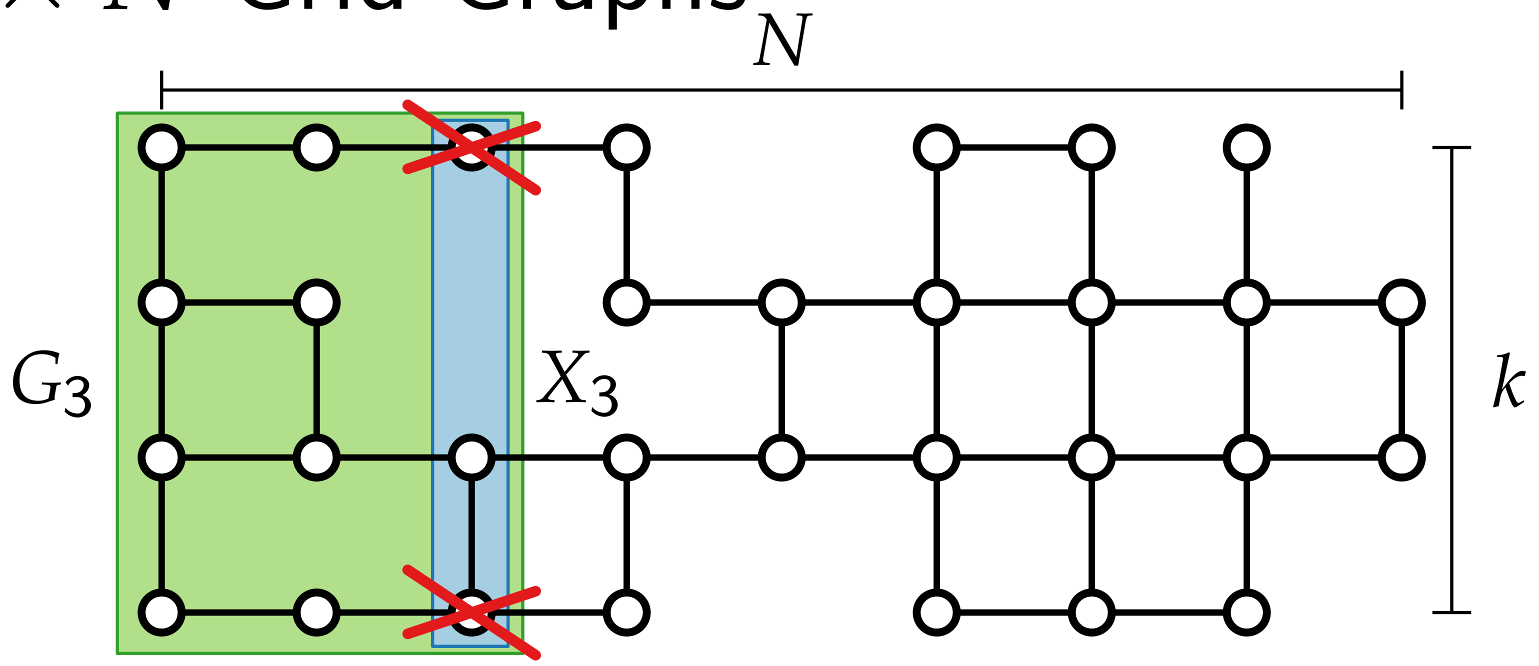
Let G_j be the graph induced by the first j columns $X_1 \cup X_2 \cup \ldots X_j$.



Indenpendent Set in $k \times N$ Grid Graphs

Let X_j be the **j-th column**, that is, $X_j = \mathsf{V}(G) \cap \{(i,j) \mid 1 \le i \le k\}$.

Let G_j be the graph induced by the first j columns $X_1 \cup X_2 \cup \ldots X_j$.

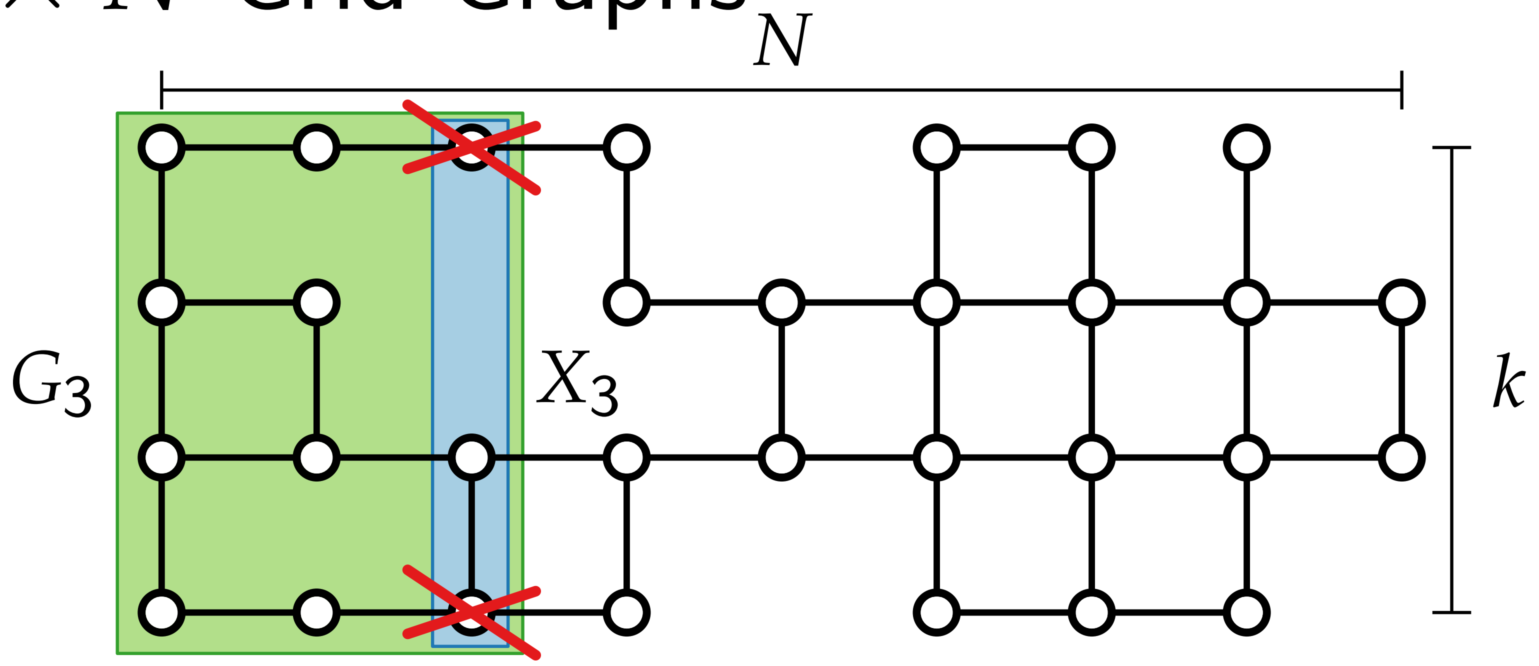


Indenpendent Set in $k \times N$ Grid Graphs

Let X_j be the **j-th column**, that is, $X_j = \mathrm{V}(G) \cap \{(i,j) \mid 1 \leq i \leq k\}$.

Let G_j be the graph induced by the first j columns $X_1 \cup X_2 \cup \ldots X_j$.

Let $1 \leq j \leq N$. For each $Y \subseteq X_j$ let

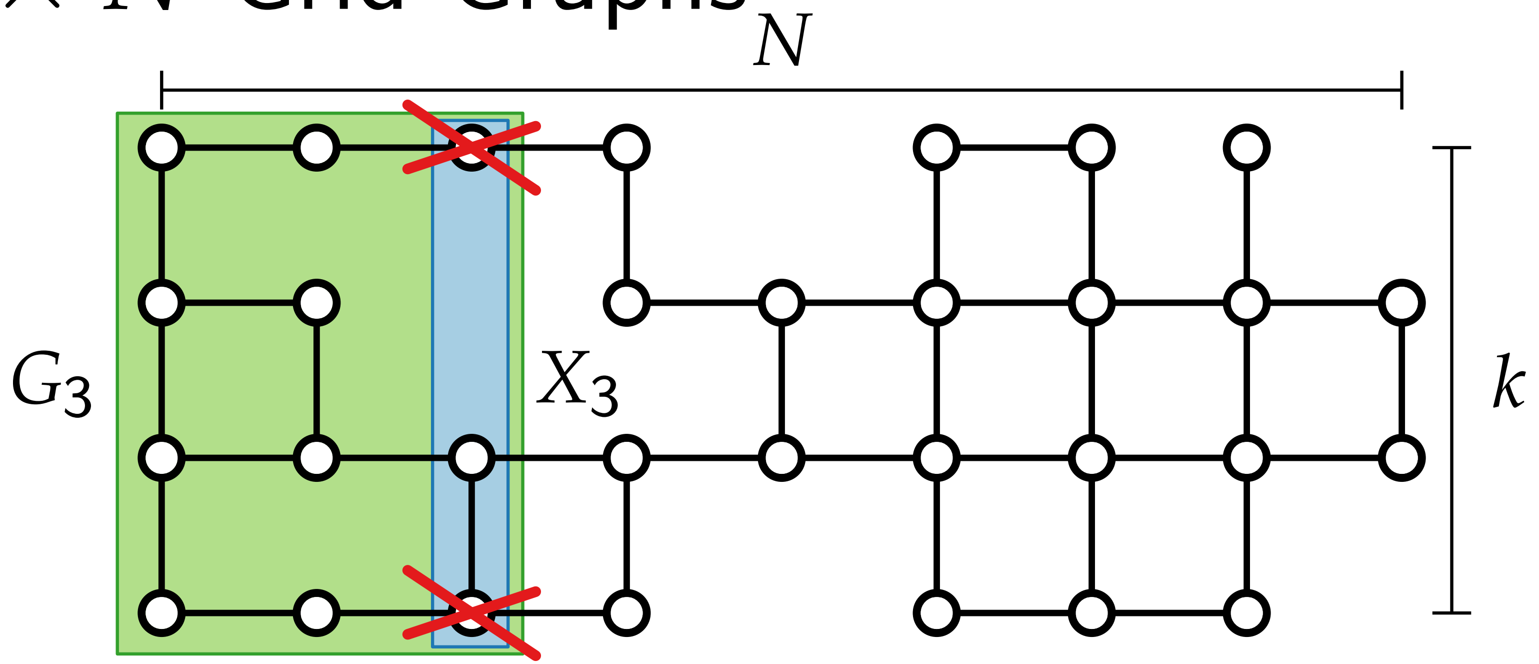


INDENPENDENT SET in $k \times N$ Grid Graphs

Let X_j be the **j-th column**, that is,
$X_j = \mathsf{V}(G) \cap \{(i,j) \mid 1 \le i \le k\}$.

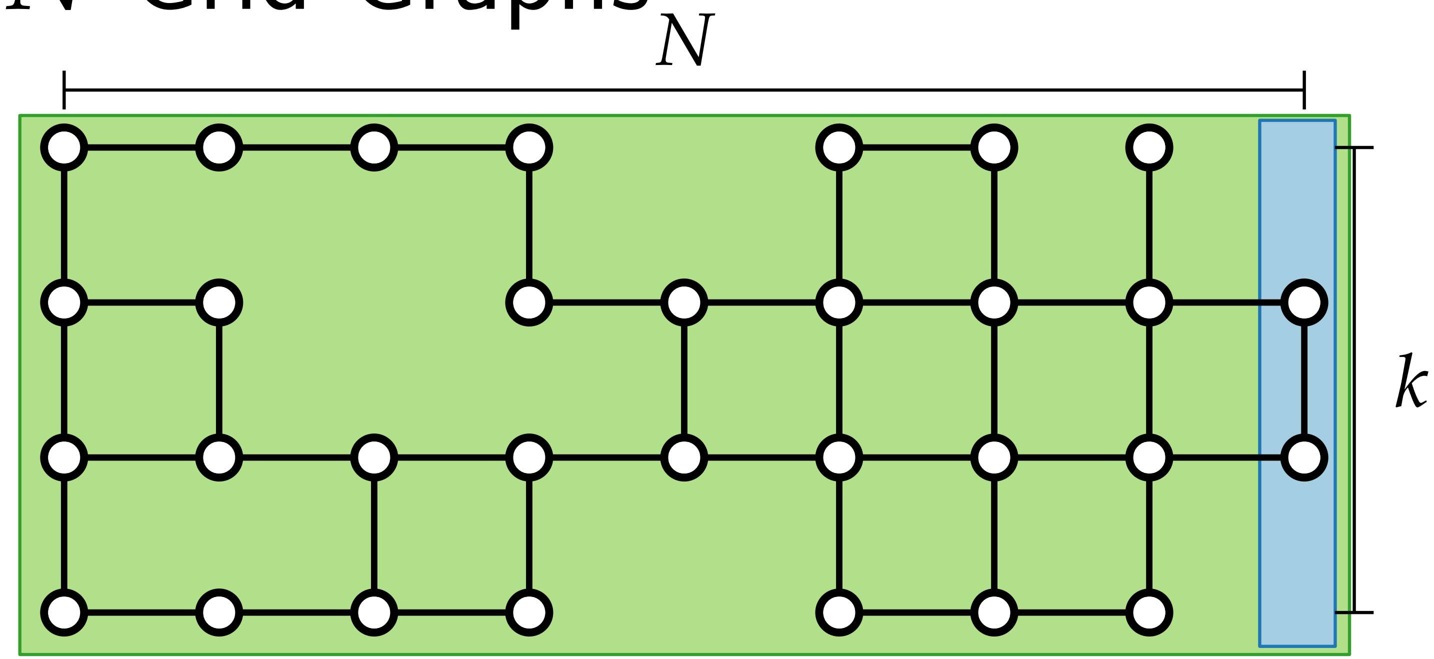
Let G_j be the graph induced by the first j columns $X_1 \cup X_2 \cup \ldots X_j$.

Let $1 \le j \le N$. For each $Y \subseteq X_j$ let



$C[j, Y] :=$ maximum weight of an independent set I in G_j such that $I \cap Y = \emptyset$

Indenpendent Set in $k \times N$ Grid Graphs



Let X_j be the **j-th column**, that is, $X_j = \mathsf{V}(G) \cap \{(i,j) \mid 1 \leq i \leq k\}$.

Let G_j be the graph induced by the first j columns $X_1 \cup X_2 \cup \ldots X_j$.

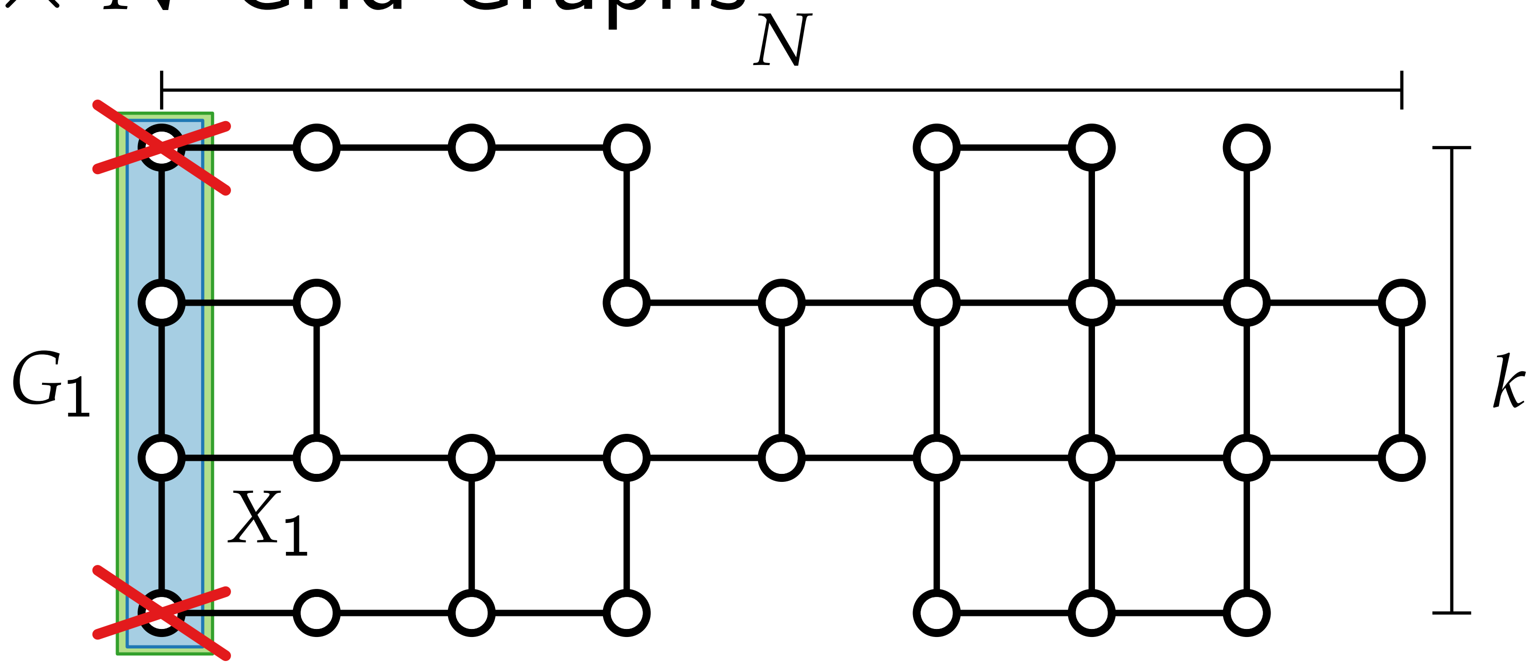
Let $1 \leq j \leq N$. For each $Y \subseteq X_j$ let

$C[j, Y] :=$ maximum weight of an independent set I in G_j such that $I \cap Y = \emptyset$

$C[N, \emptyset] =$ solution

Indenpendent Set in $k \times N$ Grid Graphs

Let X_j be the **j-th column**, that is, $X_j = \mathsf{V}(G) \cap \{(i,j) \mid 1 \leq i \leq k\}$.



Let G_j be the graph induced by the first j columns $X_1 \cup X_2 \cup \ldots X_j$.

Let $1 \leq j \leq N$. For each $Y \subseteq X_j$ let

$C[j, Y] :=$ maximum weight of an independent set I in G_j such that $I \cap Y = \emptyset$

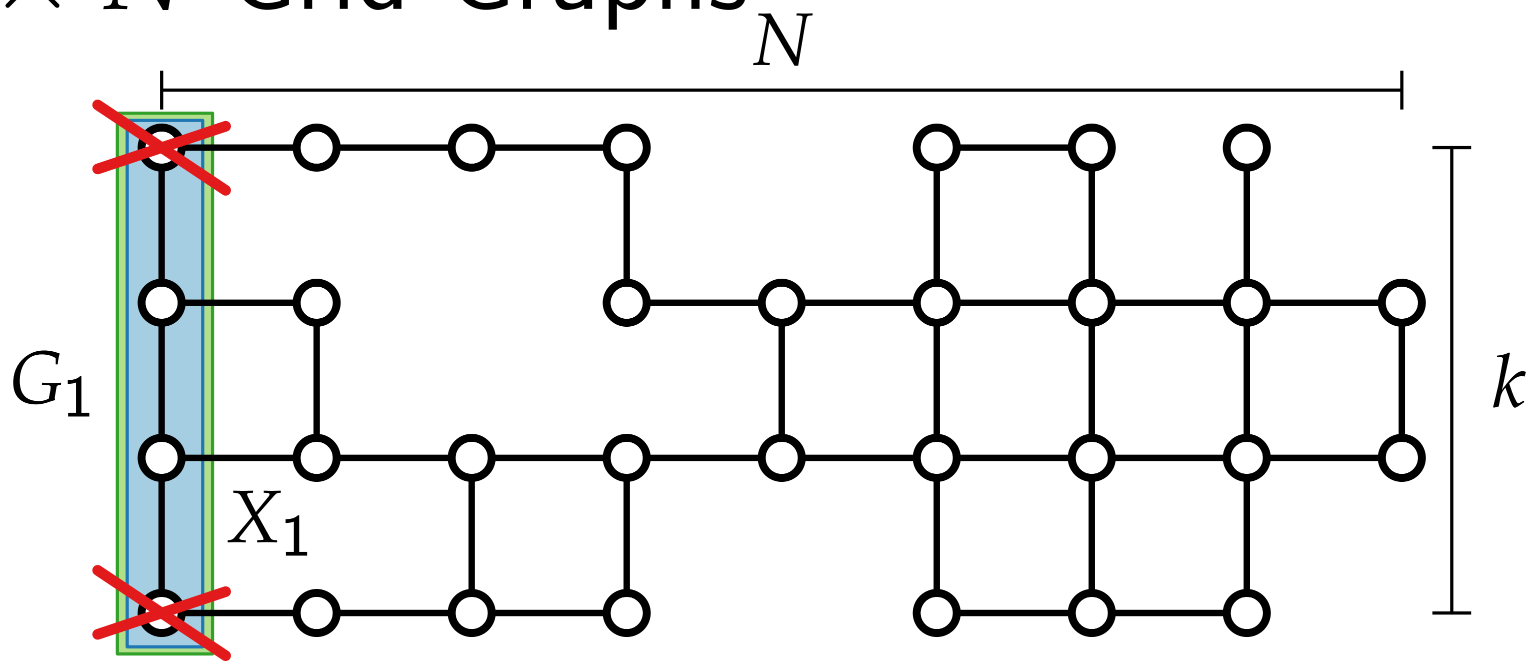
$C[1, Y] =$

INDENPENDENT SET in $k \times N$ Grid Graphs

Let X_j be the **j-th column**, that is, $X_j = \mathsf{V}(G) \cap \{(i,j) \mid 1 \le i \le k\}$.

Let G_j be the graph induced by the first j columns $X_1 \cup X_2 \cup \ldots X_j$.

Let $1 \le j \le N$. For each $Y \subseteq X_j$ let



$C[j, Y] :=$ maximum weight of an independent set I in G_j such that $I \cap Y = \emptyset$

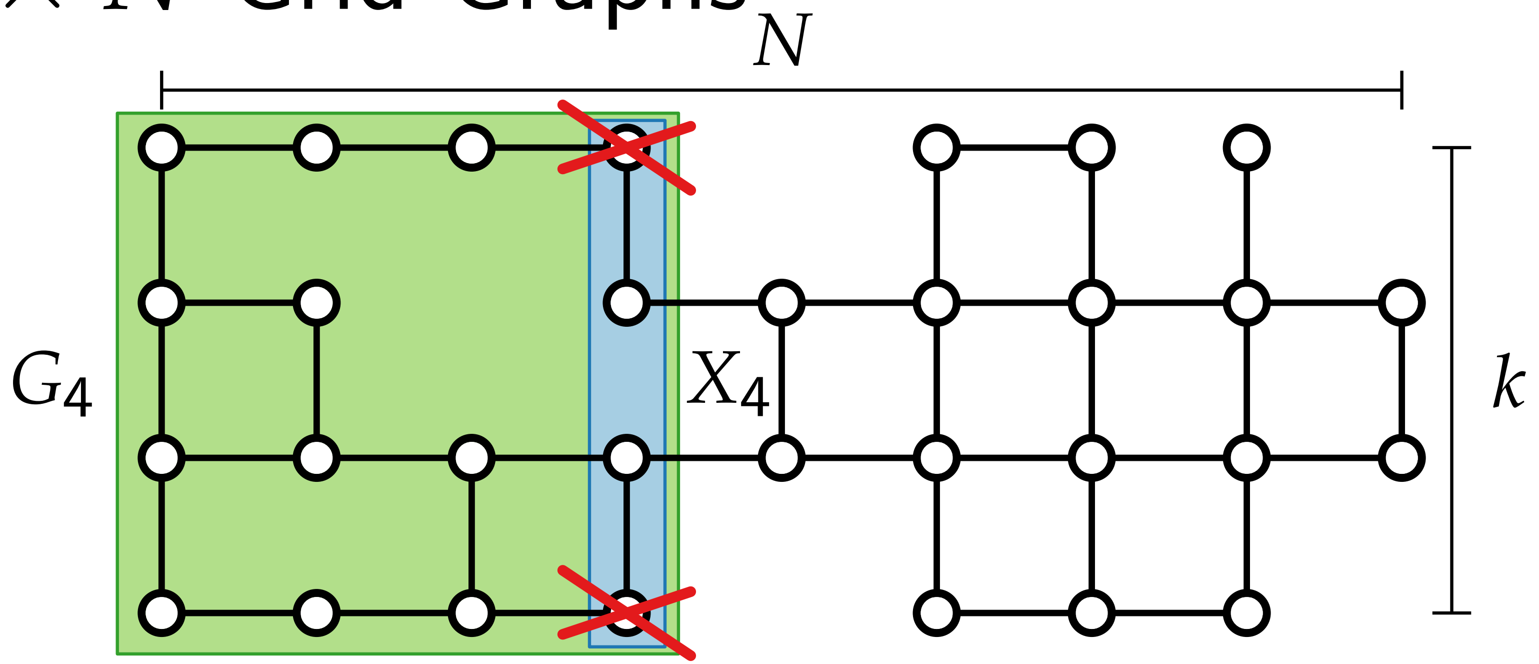
$C[1, Y] = \max_{I \subseteq X_1 \setminus Y \text{ where } I \text{ independent}} \{w(I)\}$

INDENPENDENT SET in $k \times N$ Grid Graphs

Let X_j be the **j-th column**, that is,
$X_j = \mathsf{V}(G) \cap \{(i,j) \mid 1 \leq i \leq k\}$.

Let G_j be the graph induced by the first j columns $X_1 \cup X_2 \cup \ldots X_j$.

Let $1 \leq j \leq N$. For each $Y \subseteq X_j$ let



$C[j, Y] :=$ maximum weight of an independent set I in G_j such that $I \cap Y = \emptyset$

$C[1, Y] = \max_{I \subseteq X_1 \setminus Y \text{ where } I \text{ independent}} \{w(I)\}$

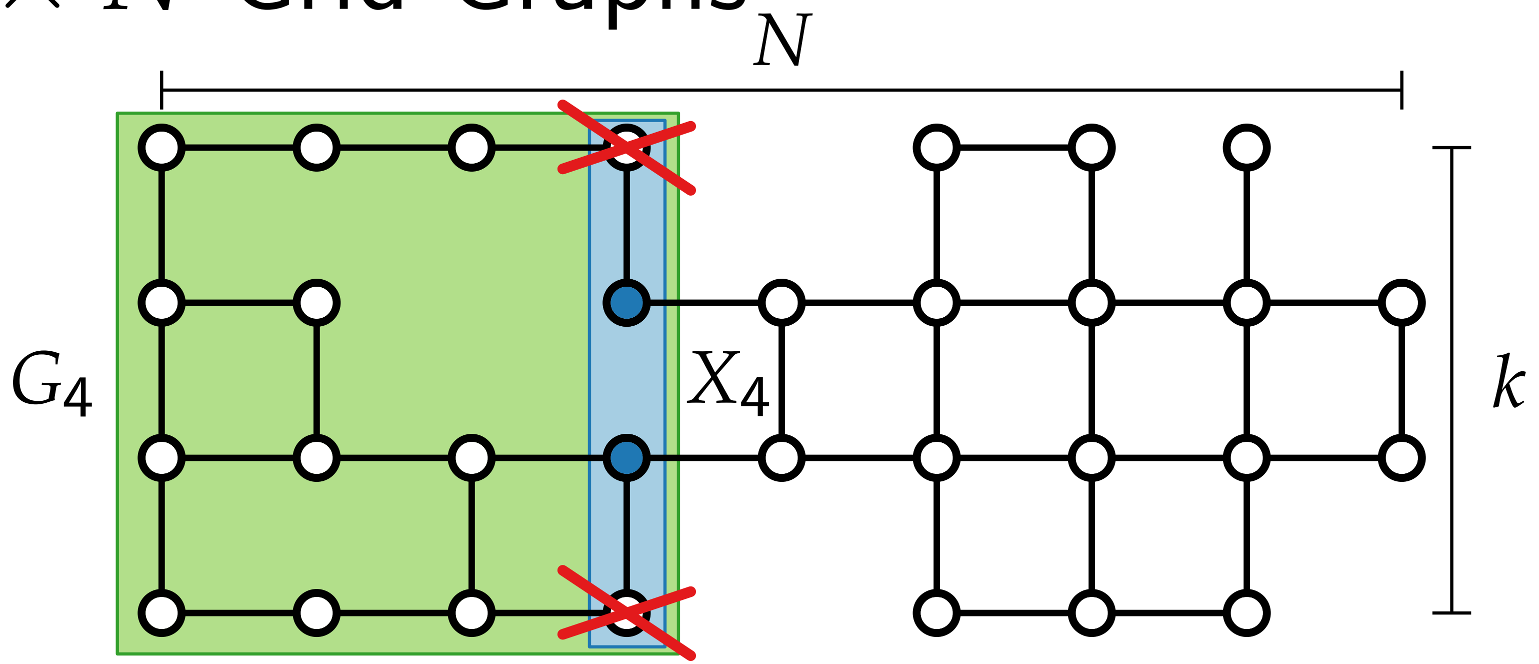
$C[j, Y] =$

Indenpendent Set in $k \times N$ Grid Graphs

Let X_j be the **j-th column**, that is, $X_j = \mathsf{V}(G) \cap \{(i,j) \mid 1 \le i \le k\}$.

Let G_j be the graph induced by the first j columns $X_1 \cup X_2 \cup \ldots X_j$.

Let $1 \le j \le N$. For each $Y \subseteq X_j$ let



$C[j, Y] :=$ maximum weight of an independent set I in G_j such that $I \cap Y = \emptyset$

$C[1, Y] = \max_{I \subseteq X_1 \setminus Y \text{ where } I \text{ independent}} \{w(I)\}$

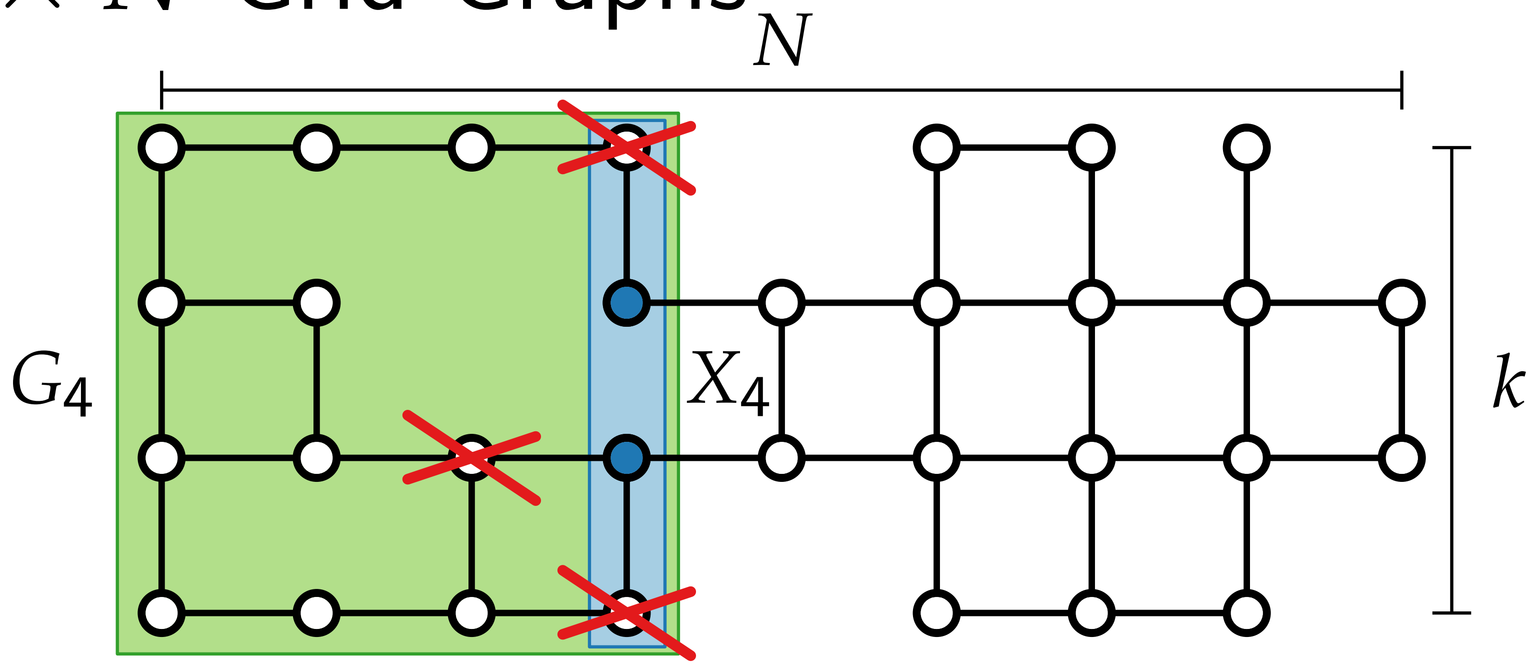
$C[j, Y] =$

Indenpendent Set in $k \times N$ Grid Graphs

Let X_j be the **j-th column**, that is, $X_j = \mathsf{V}(G) \cap \{(i,j) \mid 1 \le i \le k\}$.

Let G_j be the graph induced by the first j columns $X_1 \cup X_2 \cup \ldots X_j$.

Let $1 \le j \le N$. For each $Y \subseteq X_j$ let



$C[j, Y] :=$ maximum weight of an independent set I in G_j such that $I \cap Y = \emptyset$

$C[1, Y] = \max_{I \subseteq X_1 \setminus Y \text{ where } I \text{ independent}} \{w(I)\}$

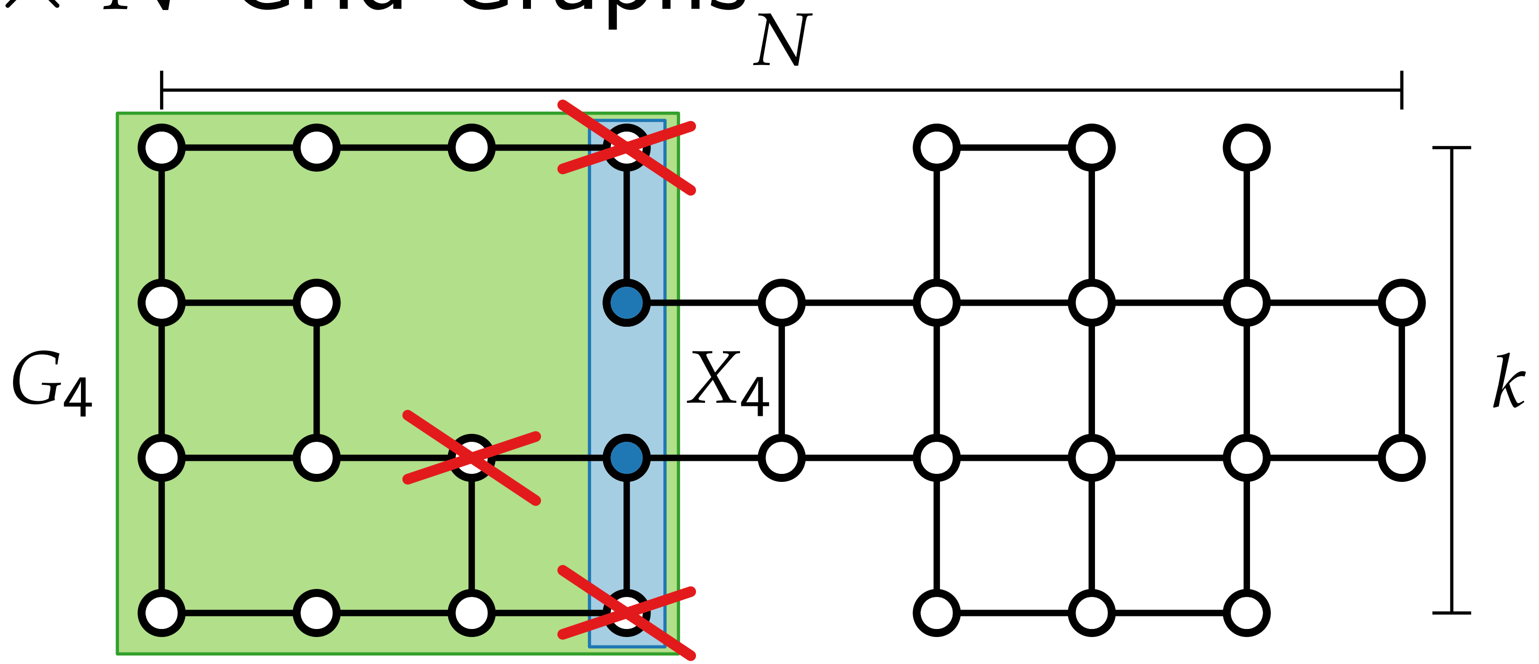
$C[j, Y] =$

Indenpendent Set in $k \times N$ Grid Graphs

Let X_j be the **j-th column**, that is, $X_j = \mathsf{V}(G) \cap \{(i,j) \mid 1 \le i \le k\}$.

Let G_j be the graph induced by the first j columns $X_1 \cup X_2 \cup \ldots X_j$.

Let $1 \le j \le N$. For each $Y \subseteq X_j$ let



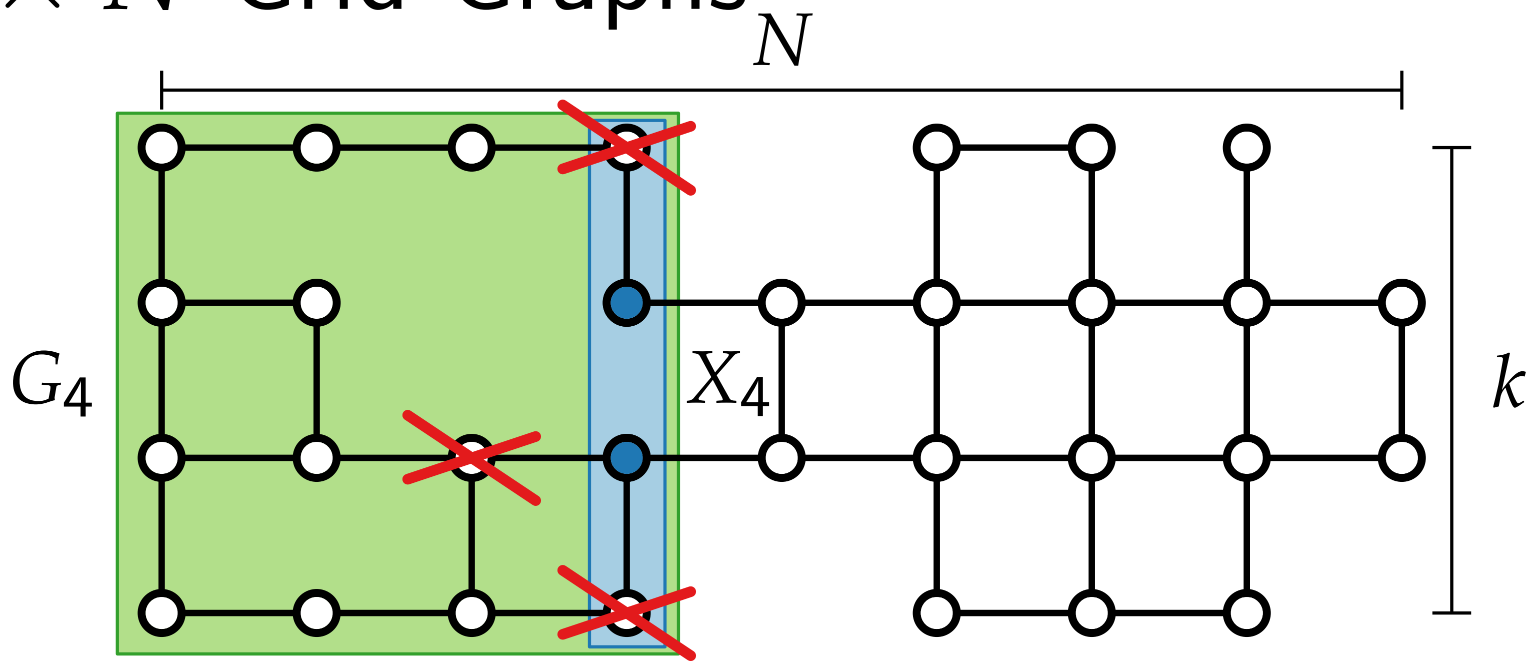
$C[j, Y] :=$ maximum weight of an independent set I in G_j such that $I \cap Y = \emptyset$

$C[1, Y] = \max_{I \subseteq X_1 \setminus Y \text{ where } I \text{ independent}} \{w(I)\}$

$C[j, Y] = \max_{I \subseteq X_j \setminus Y \text{ where } I \text{ independent}} \{w(I) + C[j-1, X_{j-1} \cap \mathsf{N}(I)]\}$

INDENPENDENT SET in $k \times N$ Grid Graphs

Let X_j be the **j-th column**, that is, $X_j = \mathsf{V}(G) \cap \{(i, j) \mid 1 \leq i \leq k\}$.



Let G_j be the graph induced by the first j columns $X_1 \cup X_2 \cup \ldots X_j$.

Let $1 \leq j \leq N$. For each $Y \subseteq X_j$ let

$C[j, Y] :=$ maximum weight of an independent set I in G_j such that $I \cap Y = \emptyset$

$C[1, Y] = \max_{I \subseteq X_1 \setminus Y \text{ where } I \text{ independent}} \{w(I)\}$

$C[j, Y] = \max_{I \subseteq X_j \setminus Y \text{ where } I \text{ independent}} \{w(I) + C[j-1, X_{j-1} \cap \mathsf{N}(I)]\}$

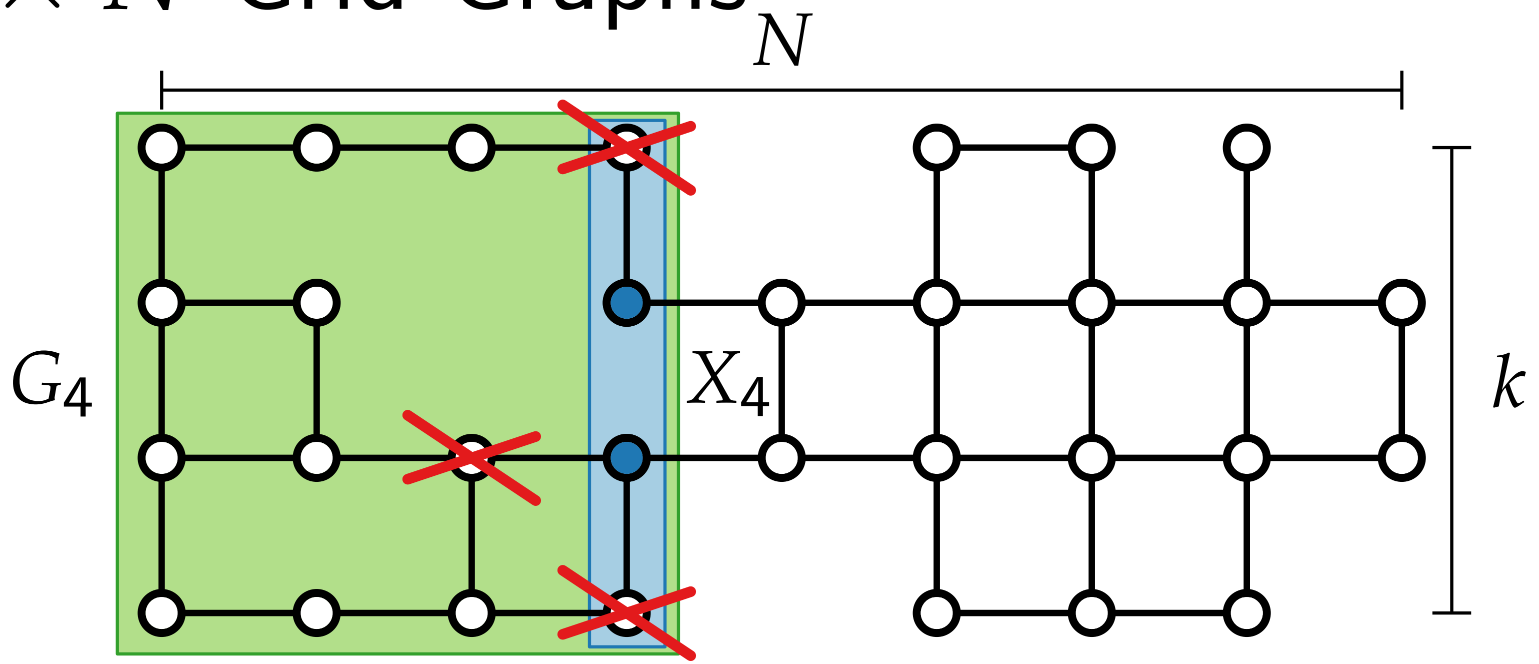
For each j there are $\leq 2^k$ choices of Y, and for each Y there are $2^{|X_j \setminus Y|}$ choices of I.

INDENPENDENT SET in $k \times N$ Grid Graphs

Let X_j be the **j-th column**, that is, $X_j = \mathsf{V}(G) \cap \{(i,j) \mid 1 \le i \le k\}$.

Let G_j be the graph induced by the first j columns $X_1 \cup X_2 \cup \ldots X_j$.

Let $1 \le j \le N$. For each $Y \subseteq X_j$ let



$C[j, Y] :=$ maximum weight of an independent set I in G_j such that $I \cap Y = \emptyset$

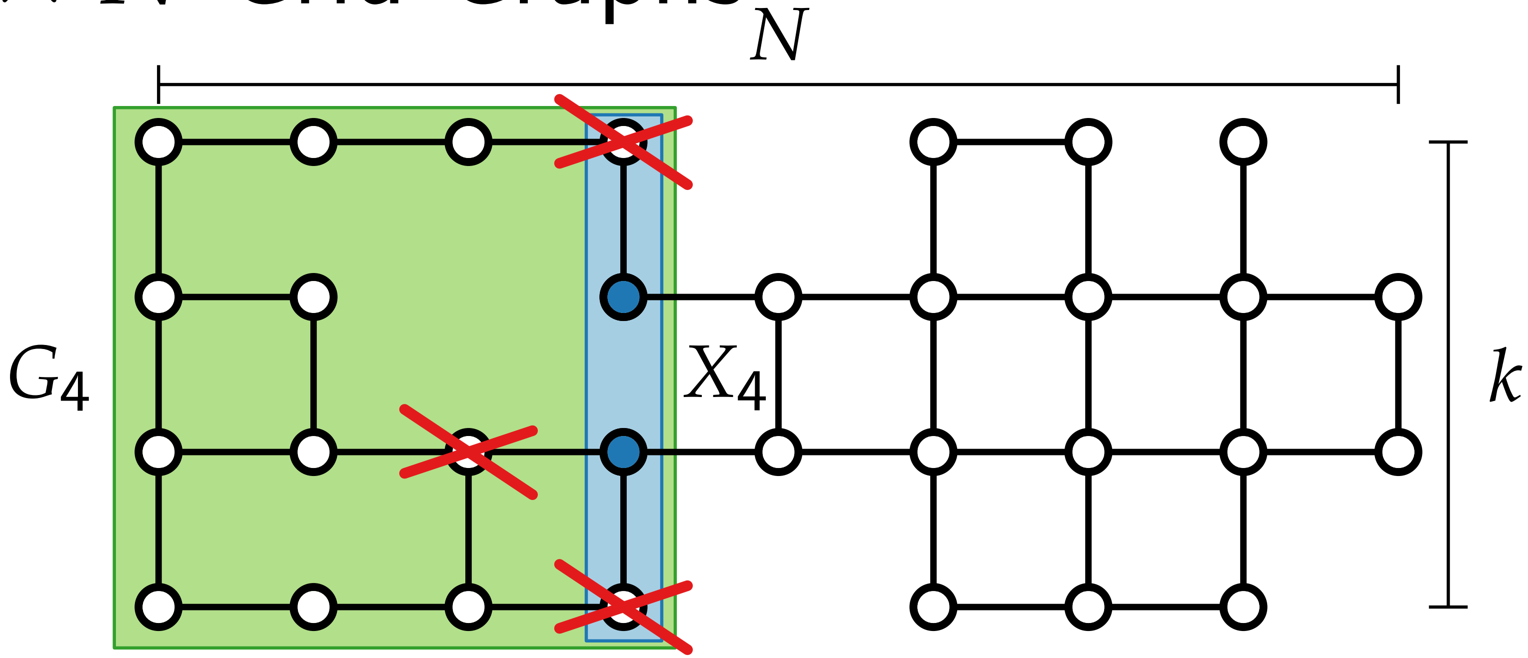
$C[1, Y] = \max_{I \subseteq X_1 \setminus Y \text{ where } I \text{ independent}} \{w(I)\}$

$C[j, Y] = \max_{I \subseteq X_j \setminus Y \text{ where } I \text{ independent}} \{w(I) + C[j-1, X_{j-1} \cap \mathsf{N}(I)]\}$

For each j there are $\le 2^k$ choices of Y, and for each Y there are $2^{|X_j \setminus Y|}$ choices of I.

For each of these $\le N4^k$ choices of I, we need to test if I is independent.

Indenpendent Set in $k \times N$ Grid Graphs



Let X_j be the **j-th column**, that is, $X_j = \mathsf{V}(G) \cap \{(i,j) \mid 1 \le i \le k\}$.

Let G_j be the graph induced by the first j columns $X_1 \cup X_2 \cup \ldots X_j$.

Let $1 \le j \le N$. For each $Y \subseteq X_j$ let

$C[j, Y] :=$ maximum weight of an independent set I in G_j such that $I \cap Y = \emptyset$

$$C[1, Y] = \max_{I \subseteq X_1 \setminus Y \text{ where } I \text{ independent}} \{w(I)\}$$

$$C[j, Y] = \max_{I \subseteq X_j \setminus Y \text{ where } I \text{ independent}} \{w(I) + C[j-1, X_{j-1} \cap \mathsf{N}(I)]\}$$

For each j there are $\le 2^k$ choices of Y, and for each Y there are $2^{|X_j \setminus Y|}$ choices of I.

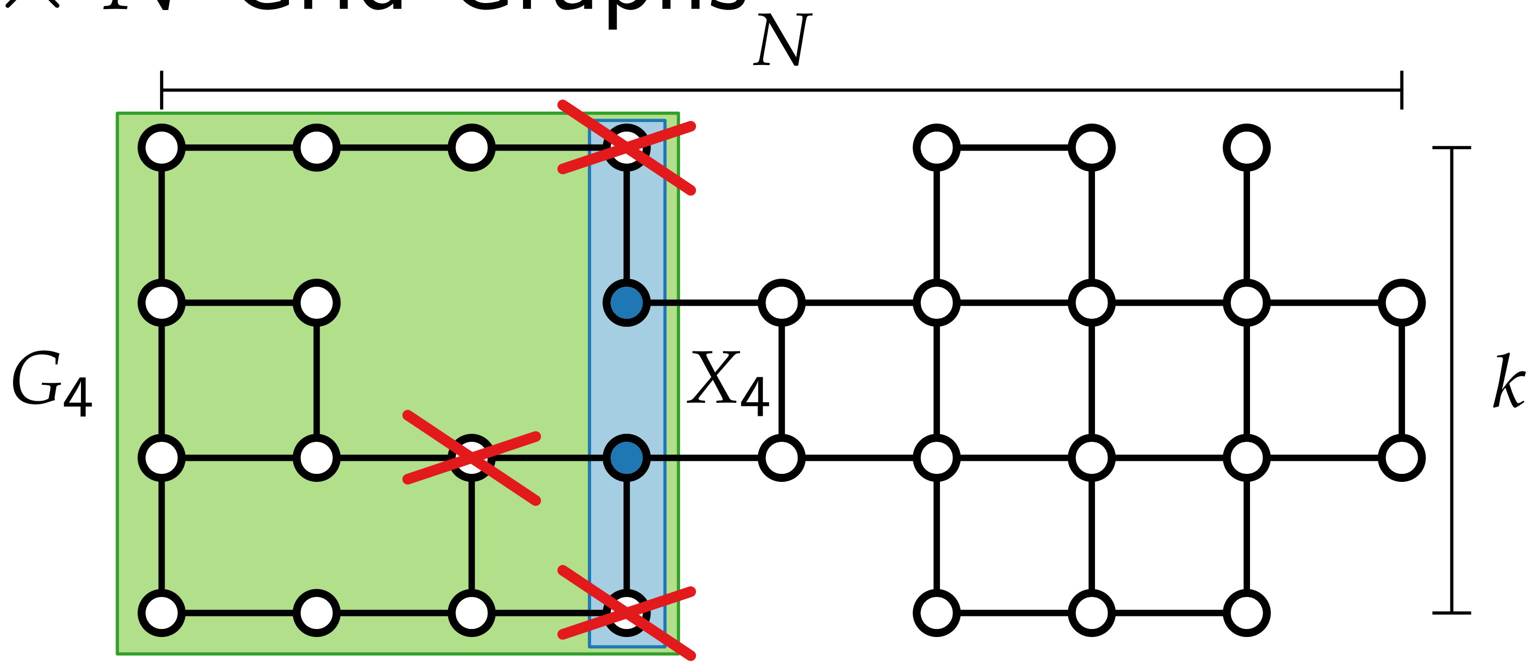
For each of these $\le N3^k$ choices of I, we need to test if I is independent.

each element in a column has one of three options: being in Y or I or none of them

INDENPENDENT SET in $k \times N$ Grid Graphs

Let X_j be the **j-th column**, that is, $X_j = \mathsf{V}(G) \cap \{(i,j) \mid 1 \le i \le k\}$.

Let G_j be the graph induced by the first j columns $X_1 \cup X_2 \cup \ldots X_j$.



Let $1 \le j \le N$. For each $Y \subseteq X_j$ let

$C[j, Y] :=$ maximum weight of an independent set I in G_j such that $I \cap Y = \emptyset$

$$C[1, Y] = \max_{I \subseteq X_1 \setminus Y \text{ where } I \text{ independent}} \{w(I)\}$$

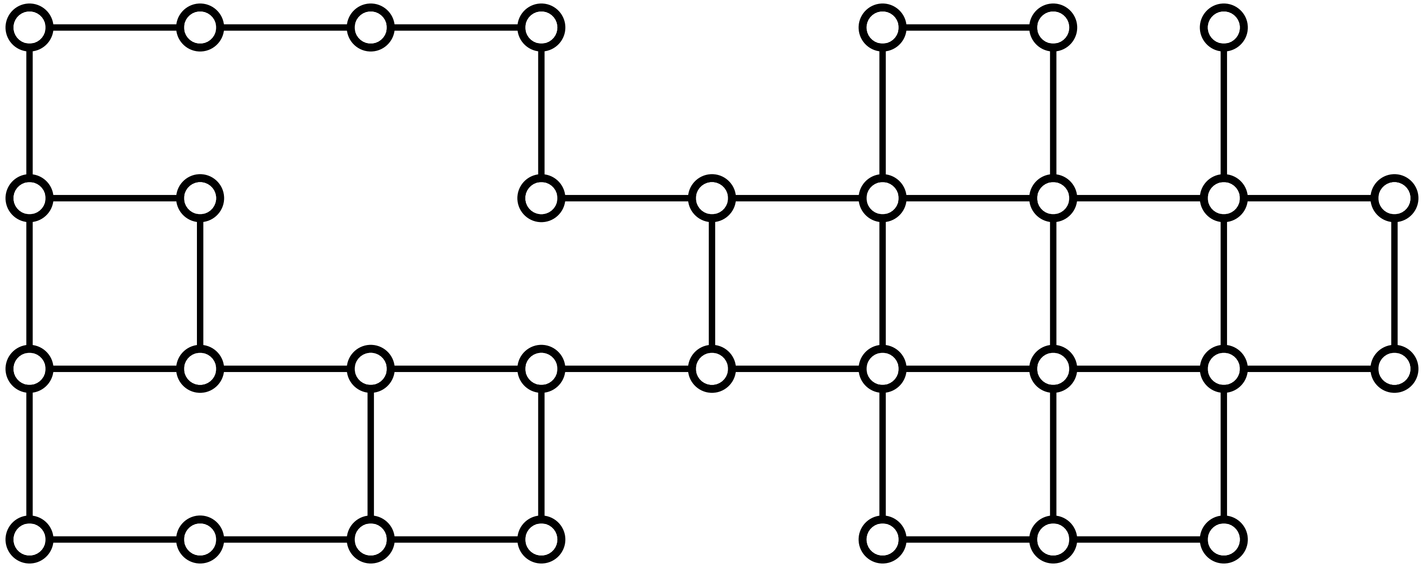
$$C[j, Y] = \max_{I \subseteq X_j \setminus Y \text{ where } I \text{ independent}} \{w(I) + C[j-1, X_{j-1} \cap \mathsf{N}(I)]\}$$

For each j there are $\le 2^k$ choices of Y, and for each Y there are $2^{|X_j \setminus Y|}$ choices of I.

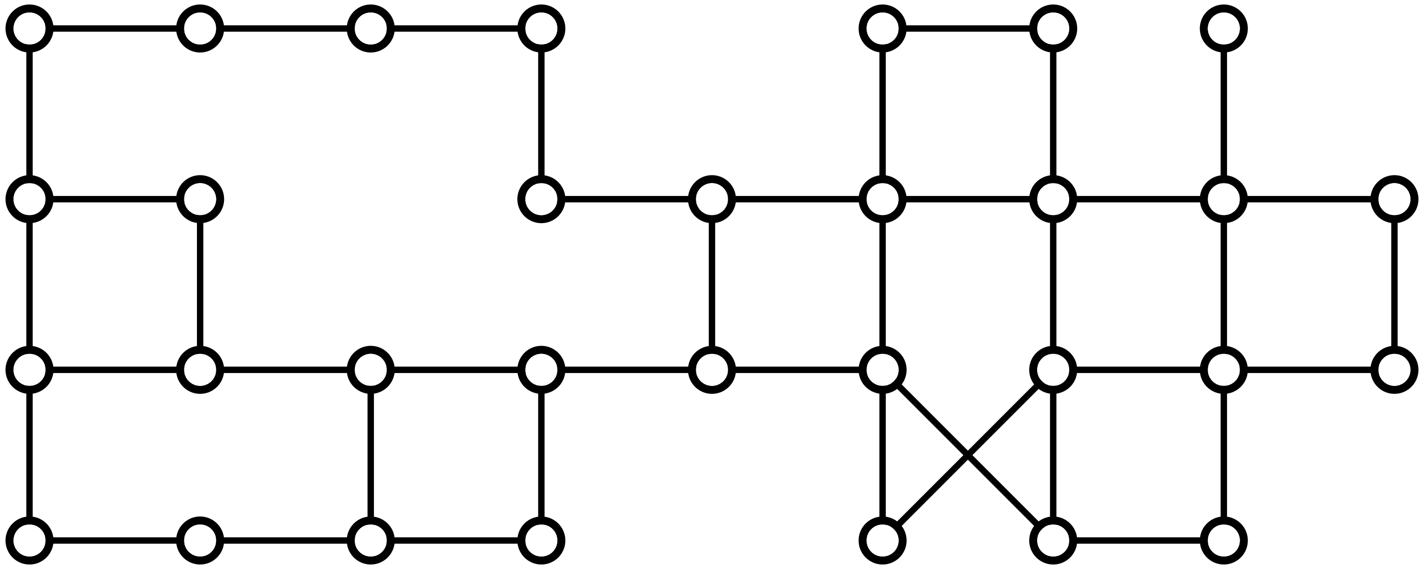
For each of these $\le N3^k$ choices of I, we need to test if I is independent.

$\rightarrow$ total running time $\le 3^k k^{\mathcal{O}(1)} N$.

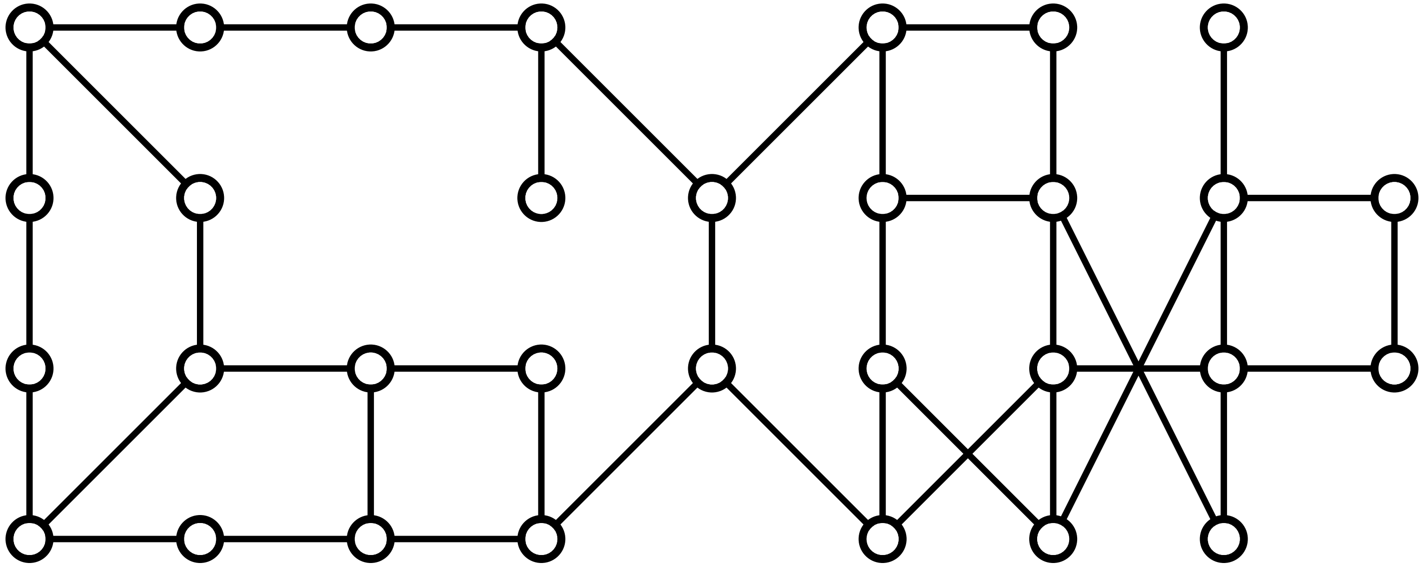
Can We Apply This Approach to Other Graphs?



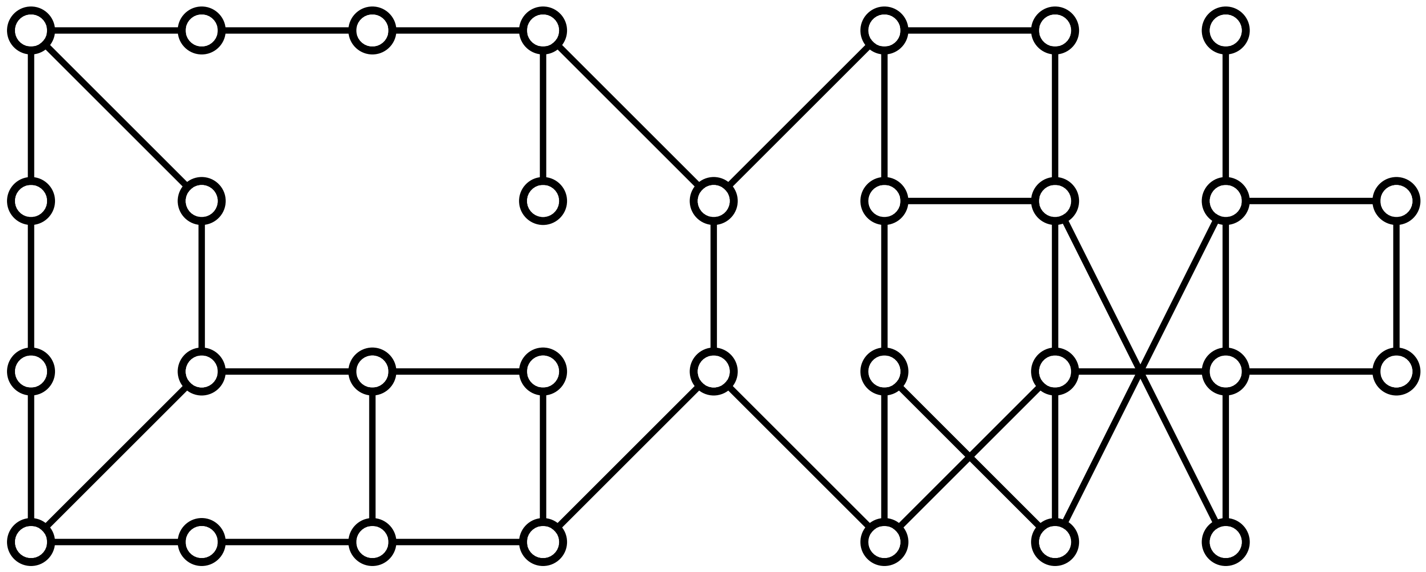
Can We Apply This Approach to Other Graphs?



Can We Apply This Approach to Other Graphs?

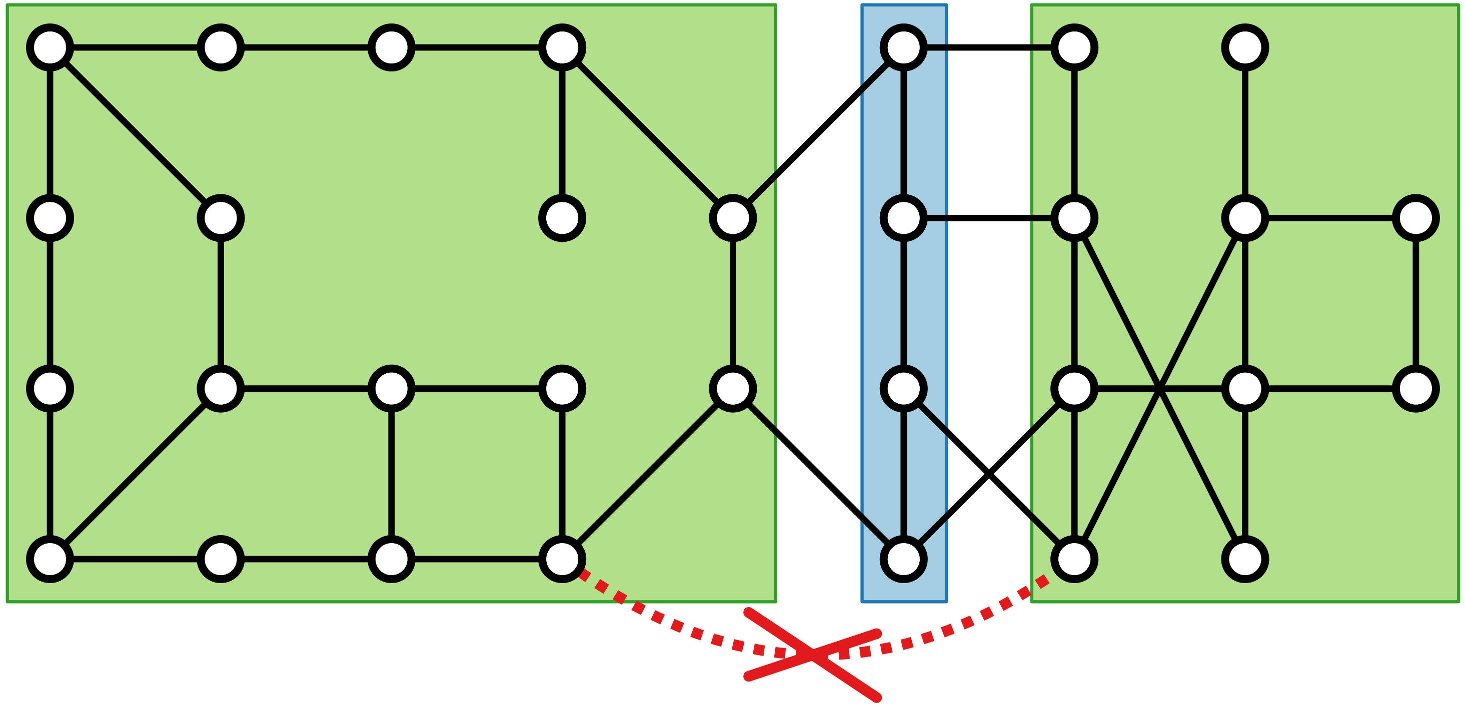


Can We Apply This Approach to Other Graphs?



Yes!

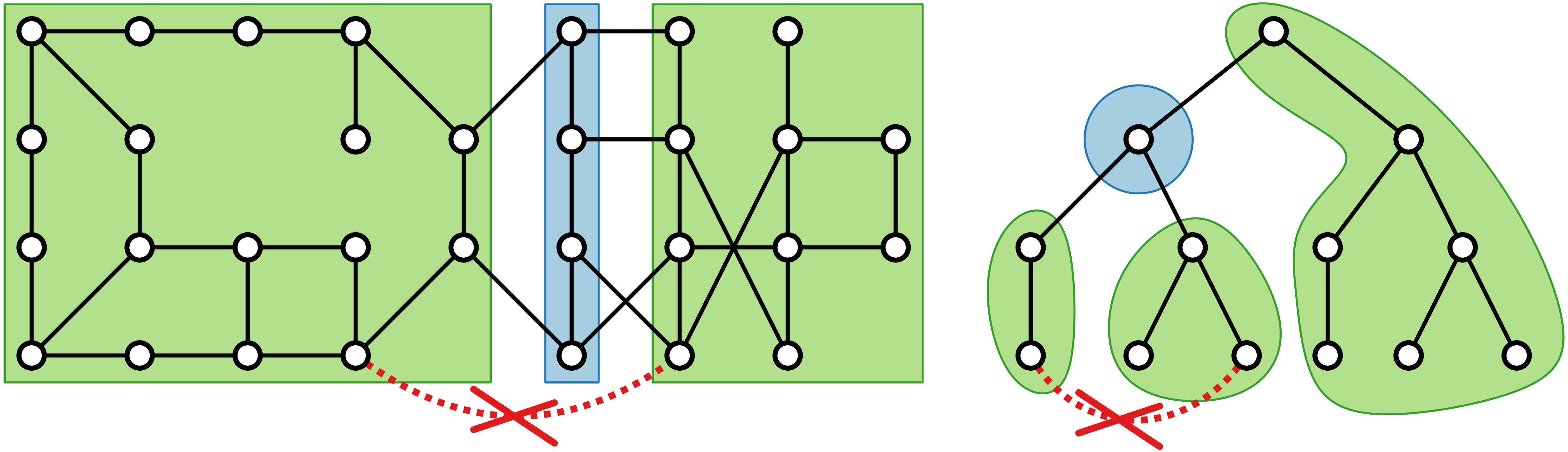
Can We Apply This Approach to Other Graphs?



Yes!

We mainly used the fact that the graph consists of a **sequence of small separators**.

Can We Apply This Approach to Other Graphs?

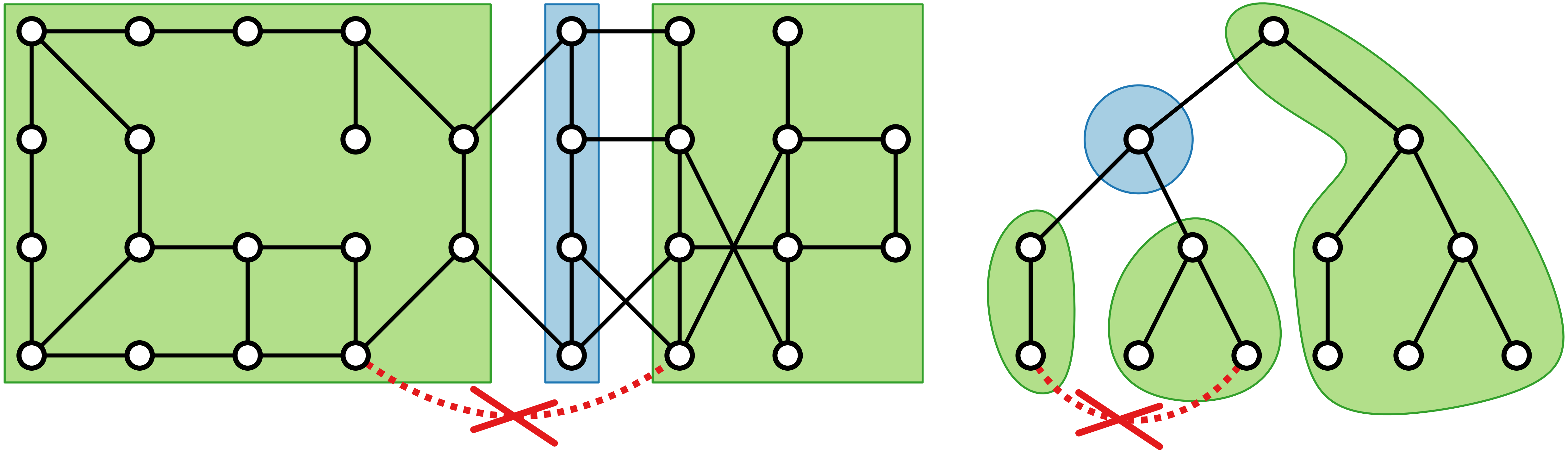


Yes!

We mainly used the fact that the graph consists of a **sequence of small separators**.

A similiar fact was used in the algorithm for trees.

Can We Apply This Approach to Other Graphs?



Yes!

We mainly used the fact that the graph consists of a **sequence of small separators**.

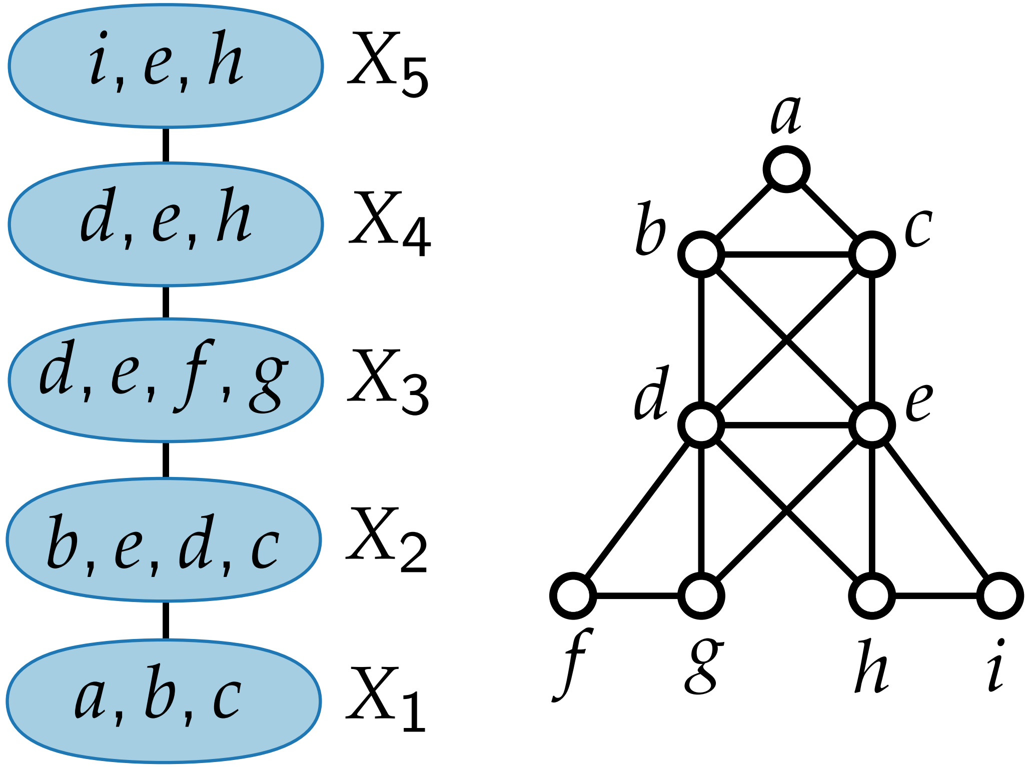
A similiar fact was used in the algorithm for trees.

Goal: Define a more general graph class featuring a structure that is suited for this kind of dynamic programming approach.

Path Decompositions

Let $G = (V, E)$ be a graph.

A **path decomposition** of G is a sequence $P = (X_1, X_2, \ldots, X_r)$ of **bags**, where $X_i \subseteq V$, such that

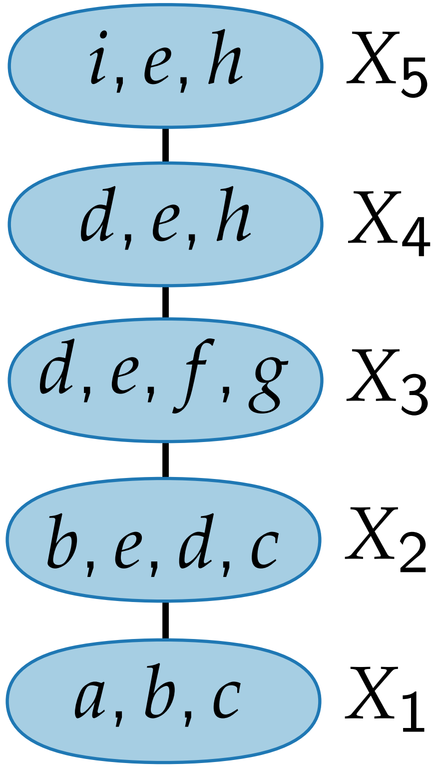


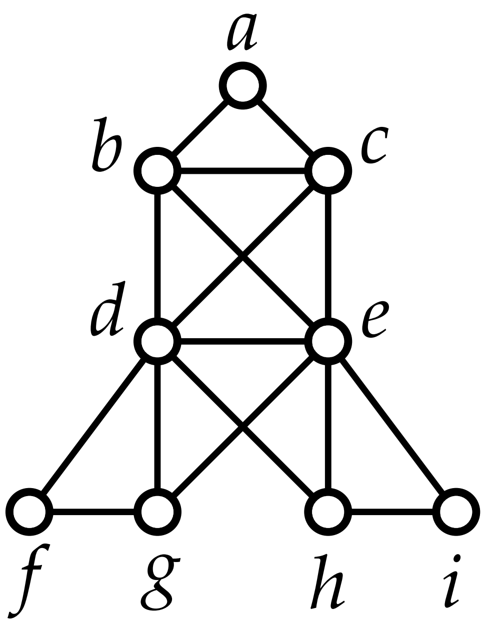
Path Decompositions

Let $G = (V, E)$ be a graph.

A **path decomposition** of G is a sequence $P = (X_1, X_2, \ldots, X_r)$ of **bags**, where $X_i \subseteq V$, such that

(P1) $\bigcup_{i=1}^{r} X_i = V$



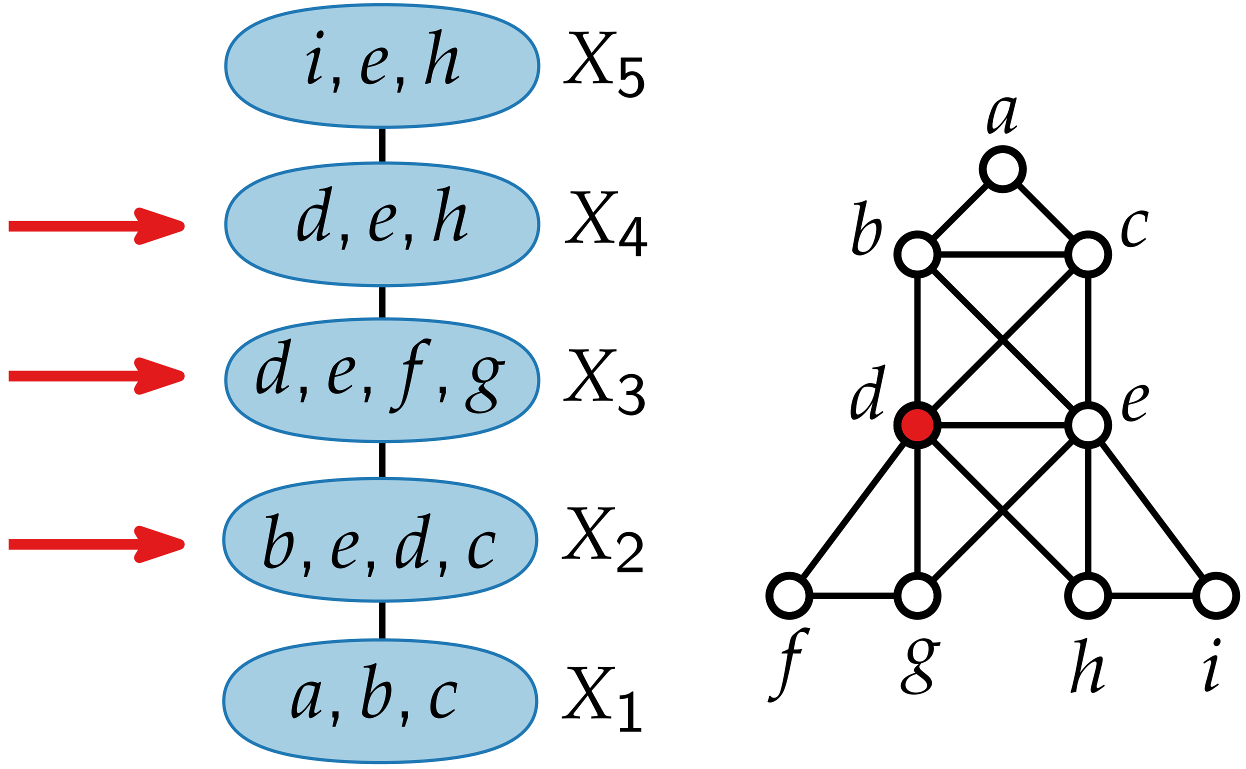


Path Decompositions

Let $G = (V, E)$ be a graph.

A **path decomposition** of G is a sequence $P = (X_1, X_2, \ldots, X_r)$ of **bags**, where $X_i \subseteq V$, such that

(P1) $\bigcup_{i=1}^{r} X_i = V$

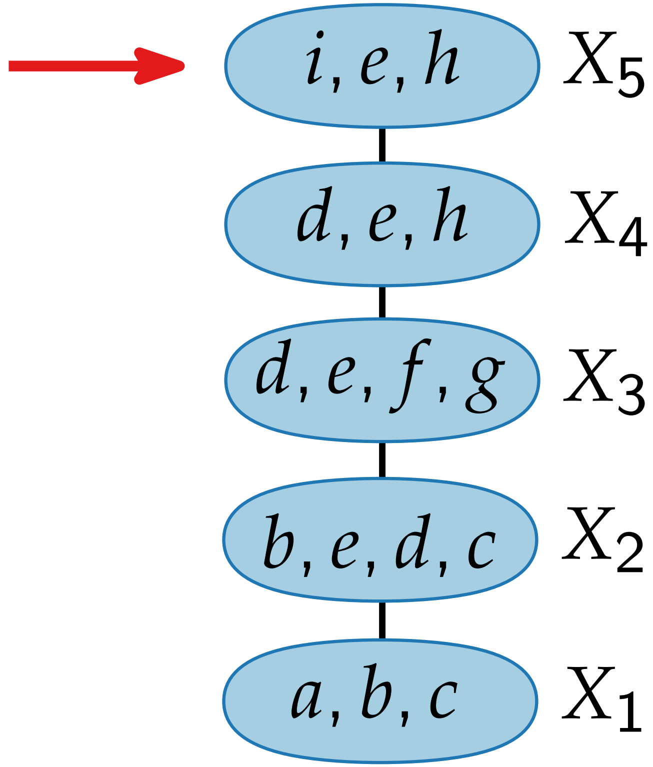


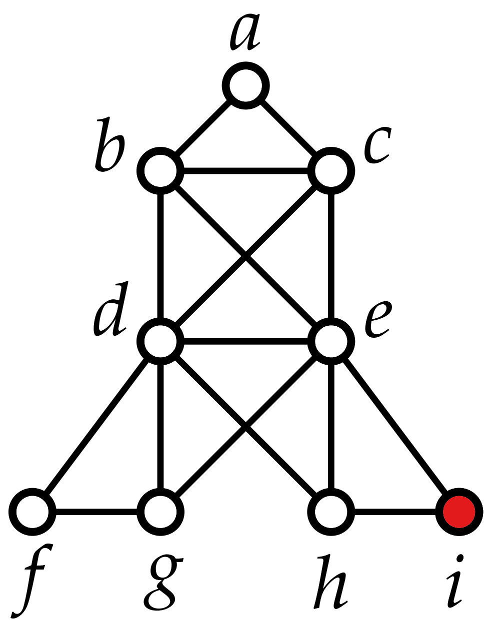
Path Decompositions

Let $G = (V, E)$ be a graph.

A **path decomposition** of G is a sequence $P = (X_1, X_2, \ldots, X_r)$ of **bags**, where $X_i \subseteq V$, such that

(P1) $\bigcup_{i=1}^{r} X_i = V$





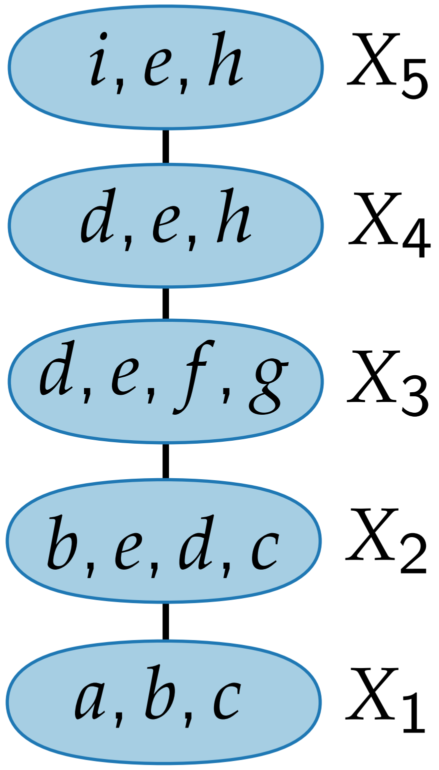
Path Decompositions

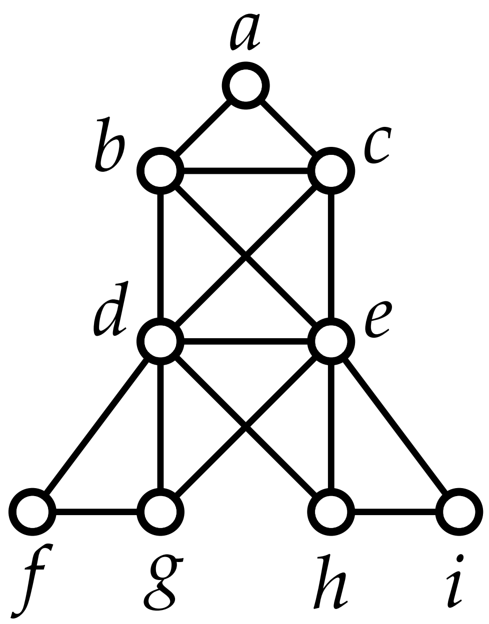
Let $G = (V, E)$ be a graph.

A **path decomposition** of G is a sequence $P = (X_1, X_2, \ldots, X_r)$ of **bags**, where $X_i \subseteq V$, such that

(P1) $\bigcup_{i=1}^{r} X_i = V$

(P2) $\forall \{u, v\} \in E \ \exists i \in \{1, 2, \ldots, r\} : u, v \in X_i$





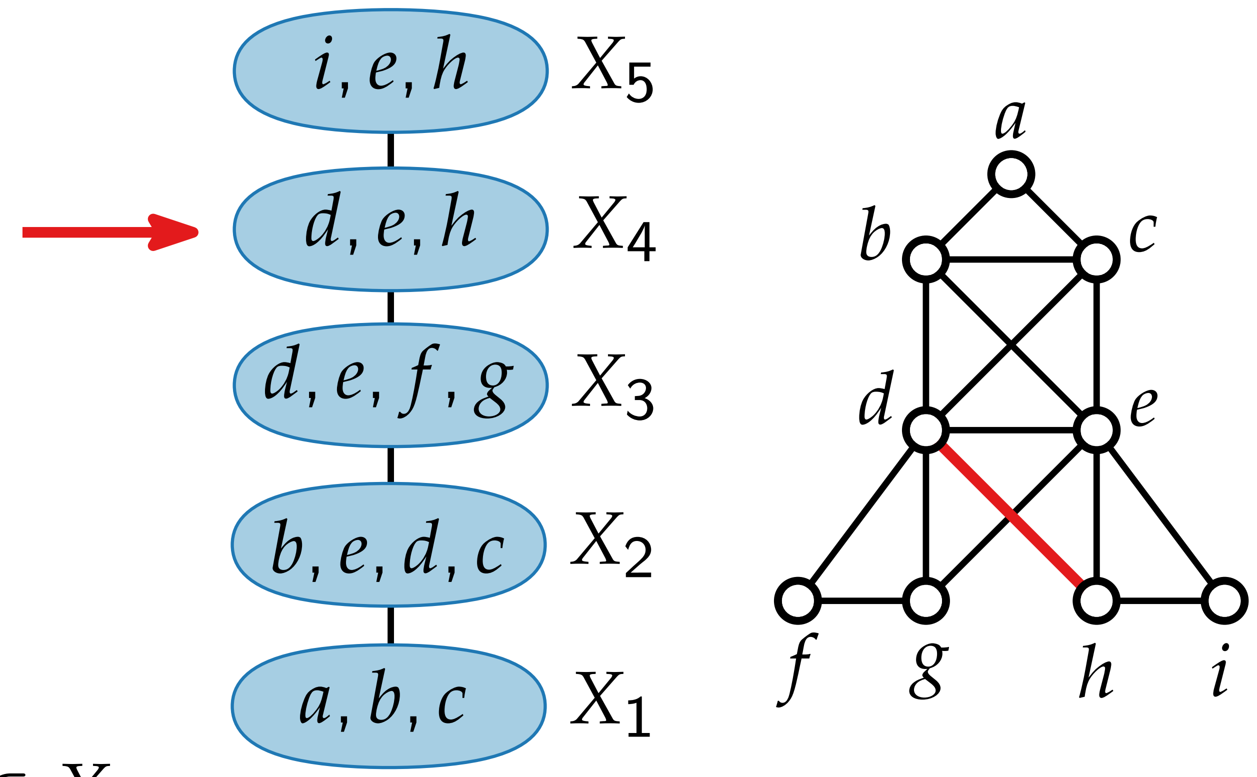
Path Decompositions

Let $G = (V, E)$ be a graph.

A **path decomposition** of G is a sequence $P = (X_1, X_2, \ldots, X_r)$ of **bags**, where $X_i \subseteq V$, such that

(P1) $\bigcup_{i=1}^{r} X_i = V$

(P2) $\forall \{u, v\} \in E \; \exists i \in \{1, 2, \ldots, r\} : u, v \in X_i$



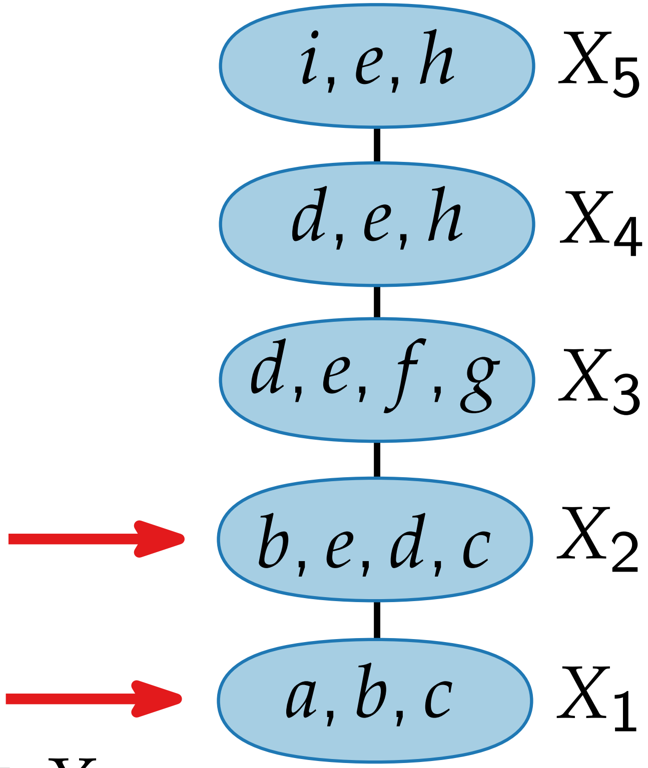
Path Decompositions

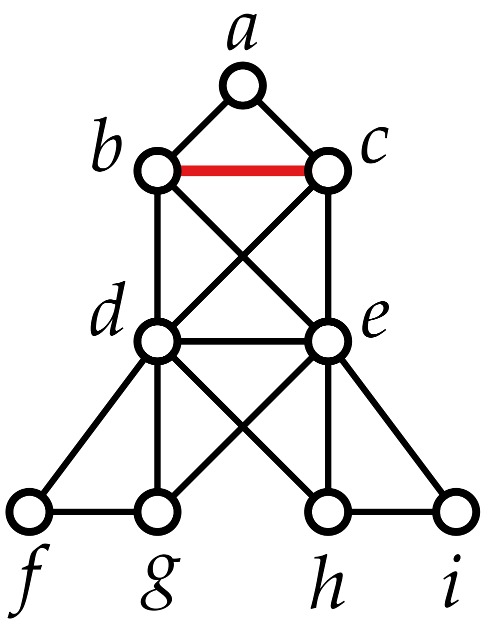
Let $G = (V, E)$ be a graph.

A **path decomposition** of G is a sequence $P = (X_1, X_2, \ldots, X_r)$ of **bags**, where $X_i \subseteq V$, such that

(P1) $\bigcup_{i=1}^{r} X_i = V$

(P2) $\forall \{u, v\} \in E \; \exists i \in \{1, 2, \ldots, r\} : u, v \in X_i$





Path Decompositions

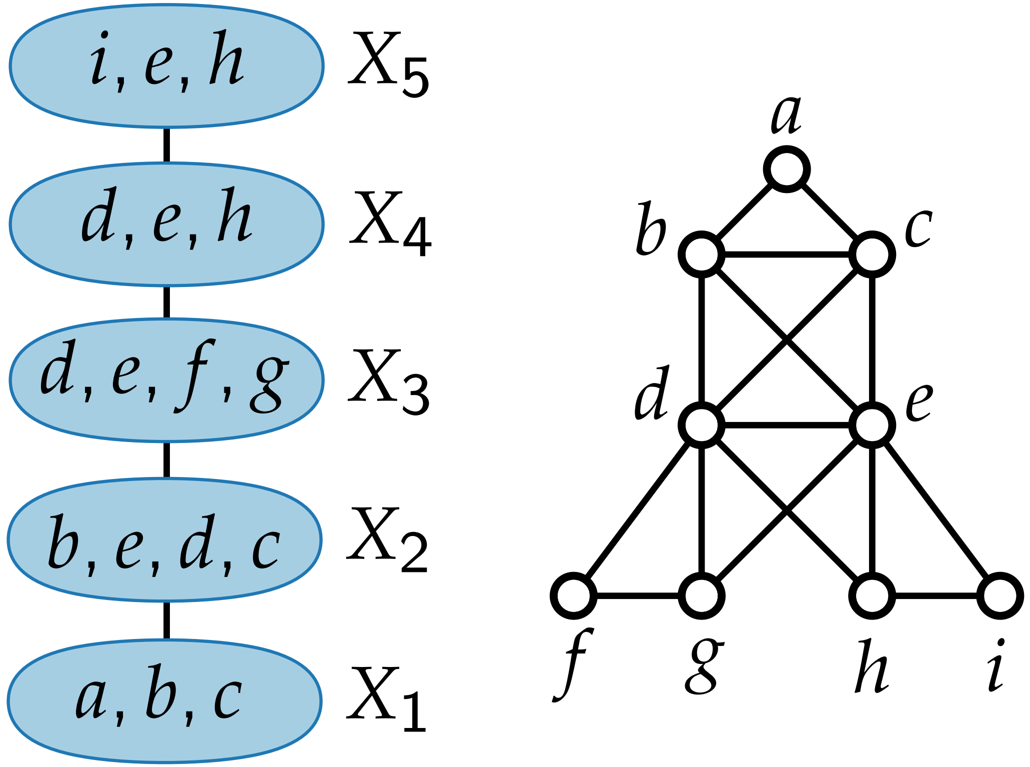
Let $G = (V, E)$ be a graph.

A **path decomposition** of G is a sequence $P = (X_1, X_2, \ldots, X_r)$ of **bags**, where $X_i \subseteq V$, such that

(P1) $\bigcup_{i=1}^{r} X_i = V$

(P2) $\forall \{u, v\} \in E \; \exists i \in \{1, 2, \ldots, r\} : u, v \in X_i$

(P3) $\forall v \in V$, if $v \in X_i \cap X_j$ with $i \leq j$, then $v \in X_i \cap X_{i+1} \cap \cdots \cap X_j$

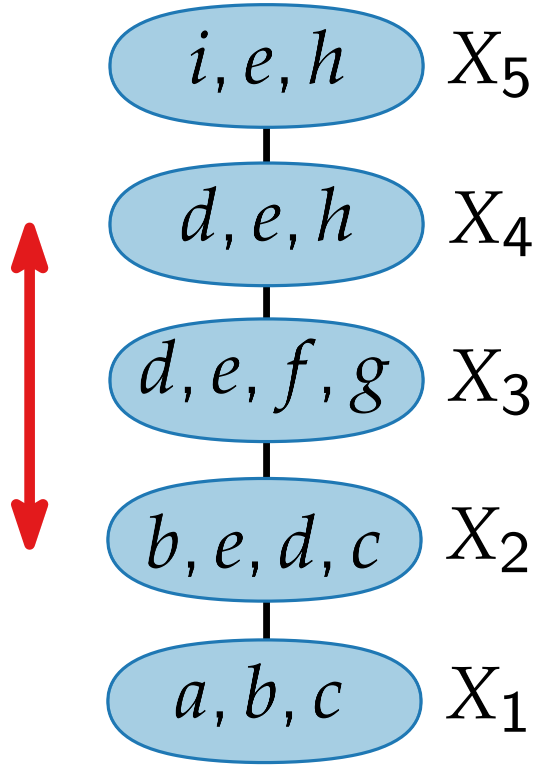


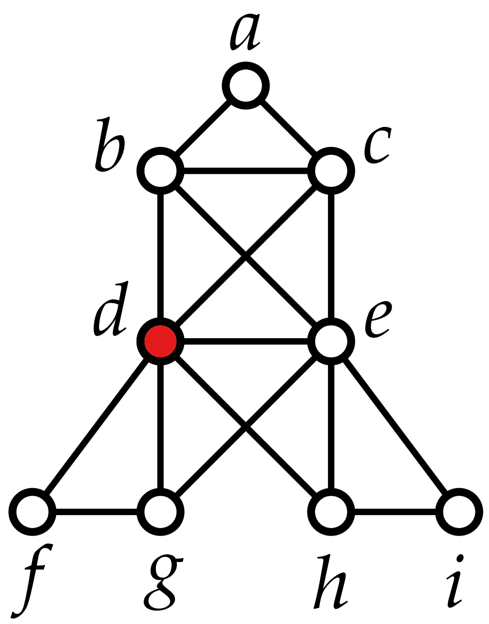
Path Decompositions

Let $G = (V, E)$ be a graph.

A **path decomposition** of G is a sequence $P = (X_1, X_2, \ldots, X_r)$ of **bags**, where $X_i \subseteq V$, such that

- **(P1)** $\bigcup_{i=1}^{r} X_i = V$
- **(P2)** $\forall \{u, v\} \in E\ \exists i \in \{1, 2, \ldots, r\} : u, v \in X_i$
- **(P3)** $\forall v \in V$, if $v \in X_i \cap X_j$ with $i \leq j$, then $v \in X_i \cap X_{i+1} \cap \cdots \cap X_j$





Path Decompositions

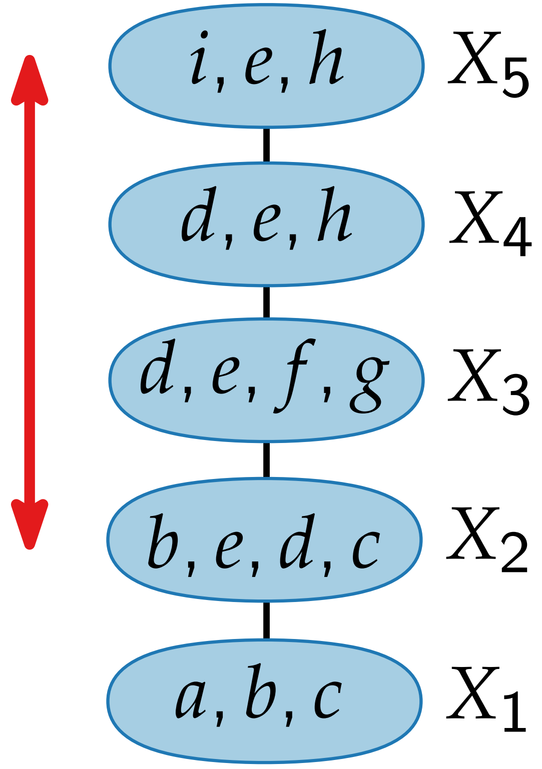
Let $G = (V, E)$ be a graph.

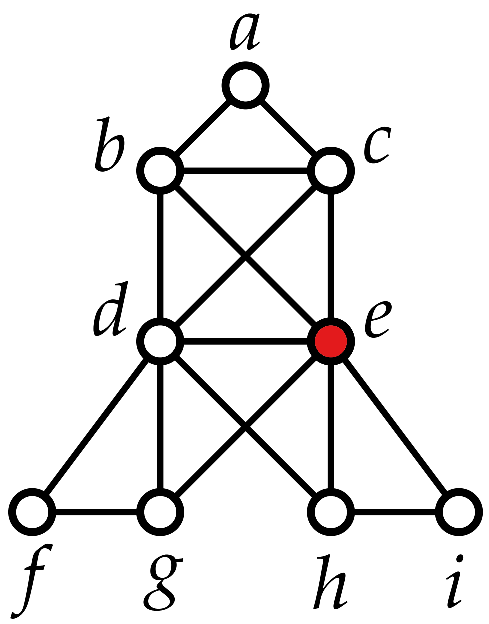
A **path decomposition** of G is a sequence $P = (X_1, X_2, \dots, X_r)$ of **bags**, where $X_i \subseteq V$, such that

(P1) $\bigcup_{i=1}^{r} X_i = V$

(P2) $\forall \{u, v\} \in E \; \exists i \in \{1, 2, \dots, r\} : u, v \in X_i$

(P3) $\forall v \in V$, if $v \in X_i \cap X_j$ with $i \leq j$, then $v \in X_i \cap X_{i+1} \cap \dots \cap X_j$



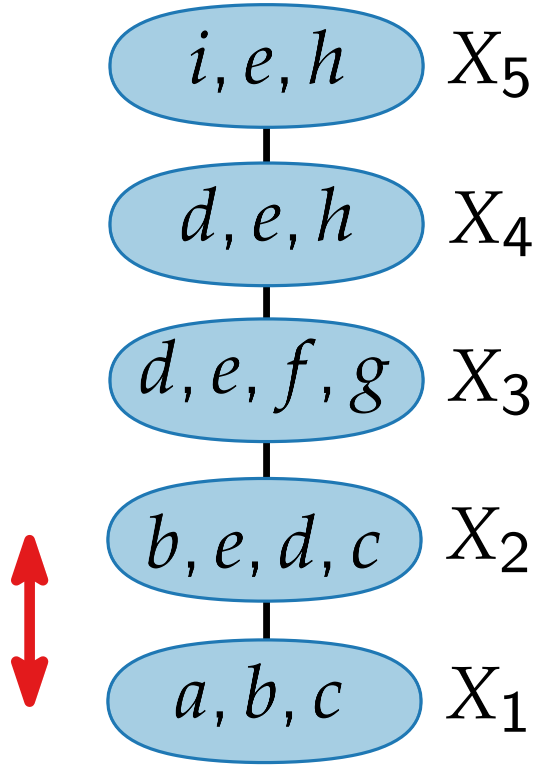


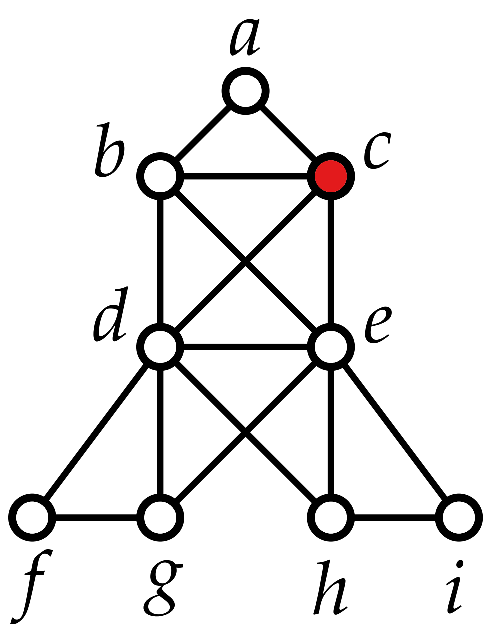
Path Decompositions

Let $G = (V, E)$ be a graph.

A **path decomposition** of G is a sequence $P = (X_1, X_2, \ldots, X_r)$ of **bags**, where $X_i \subseteq V$, such that

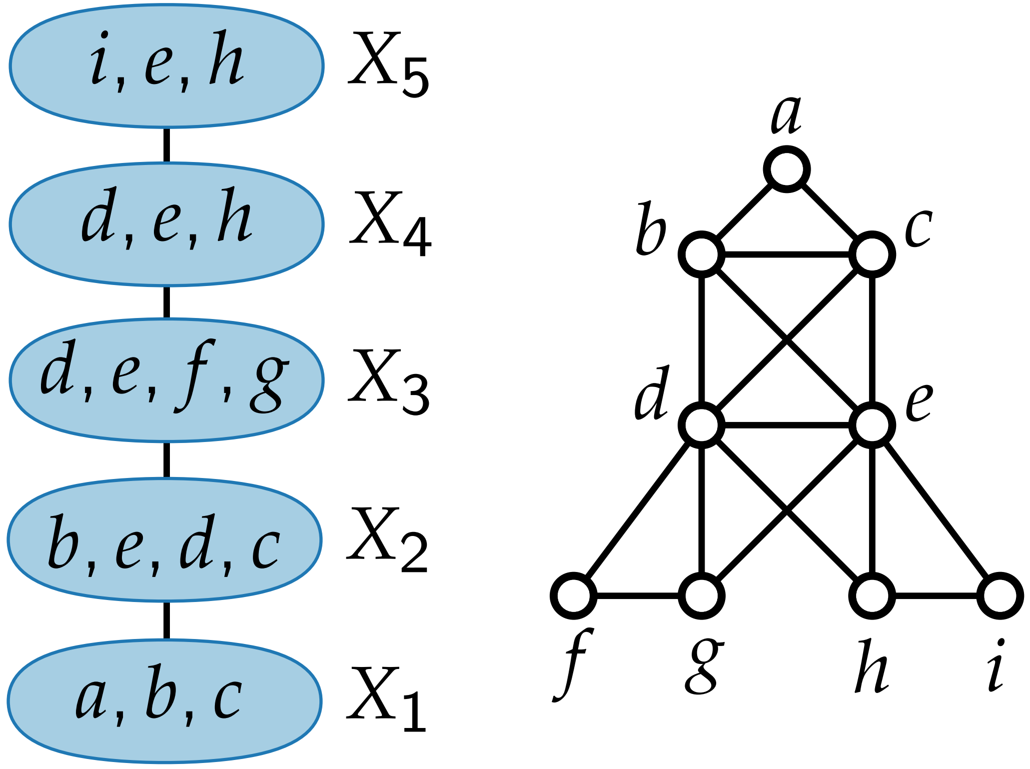
- **(P1)** $\bigcup_{i=1}^{r} X_i = V$
- **(P2)** $\forall \{u, v\} \in E \; \exists i \in \{1, 2, \ldots, r\} : u, v \in X_i$
- **(P3)** $\forall v \in V$, if $v \in X_i \cap X_j$ with $i \leq j$, then $v \in X_i \cap X_{i+1} \cap \cdots \cap X_j$





Path Decompositions

$\mathrm{w}(P) = 3$



Let $G = (V, E)$ be a graph.

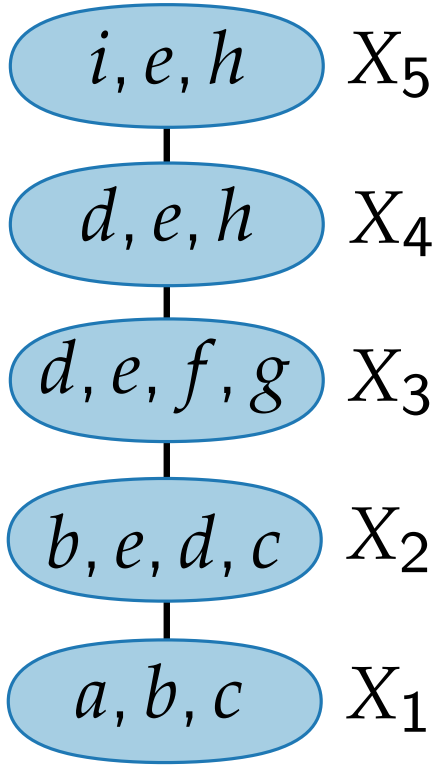
A **path decomposition** of G is a sequence $P = (X_1, X_2, \dots, X_r)$ of **bags**, where $X_i \subseteq V$, such that

- **(P1)** $\bigcup_{i=1}^{r} X_i = V$
- **(P2)** $\forall \{u, v\} \in E \; \exists i \in \{1, 2, \dots, r\} : u, v \in X_i$
- **(P3)** $\forall v \in V$, if $v \in X_i \cap X_j$ with $i \leq j$, then $v \in X_i \cap X_{i+1} \cap \dots \cap X_j$

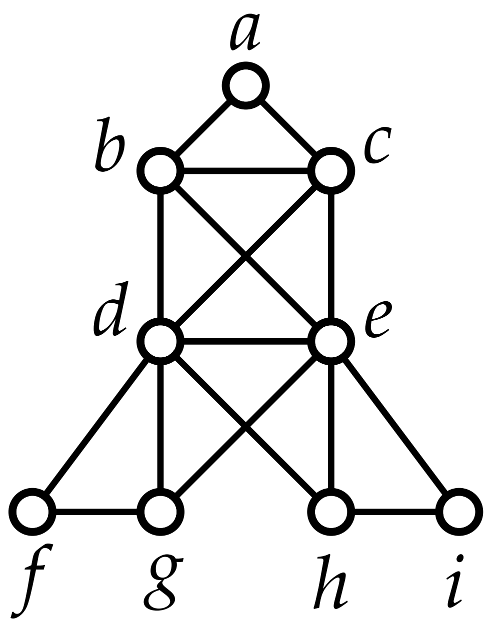
The **width** of P is $\mathrm{w}(P) = \max_{1 \leq I \leq r} |X_i| - 1$.

Path Decompositions

$\mathrm{w}(P) = 3$



$\mathrm{pw}(G) \leq 3$



Let $G = (V, E)$ be a graph.

A **path decomposition** of G is a sequence $P = (X_1, X_2, \dots, X_r)$ of **bags**, where $X_i \subseteq V$, such that

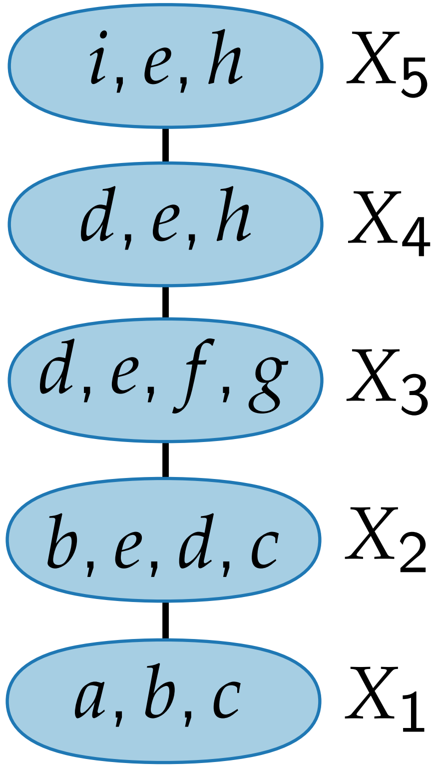
- **(P1)** $\bigcup_{i=1}^{r} X_i = V$
- **(P2)** $\forall \{u, v\} \in E\ \exists i \in \{1, 2, \dots, r\} : u, v \in X_i$
- **(P3)** $\forall v \in V$, if $v \in X_i \cap X_j$ with $i \leq j$, then $v \in X_i \cap X_{i+1} \cap \dots \cap X_j$

The **width** of P is $\mathrm{w}(P) = \max_{1 \leq I \leq r} |X_i| - 1$.

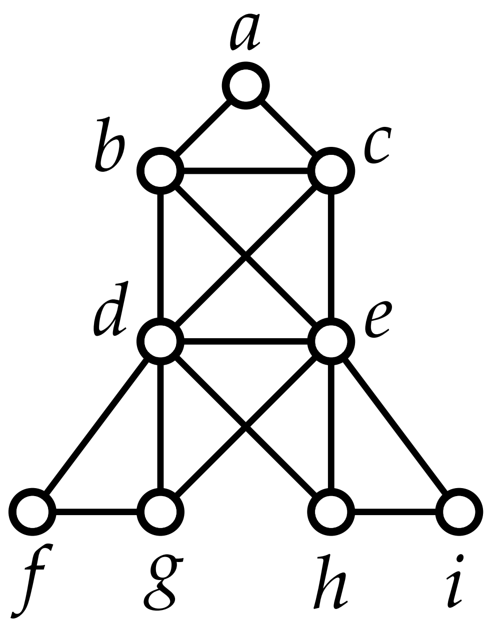
The **pathwidth** $\mathrm{pw}(G)$ of G is the minimum width of a path decomposition of G.

Path Decompositions

$w(P) = 3$



$pw(G) \leq 3$



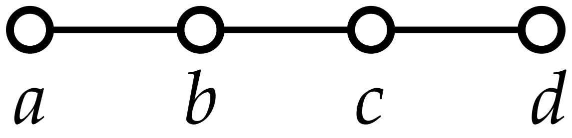
Let $G = (V, E)$ be a graph.

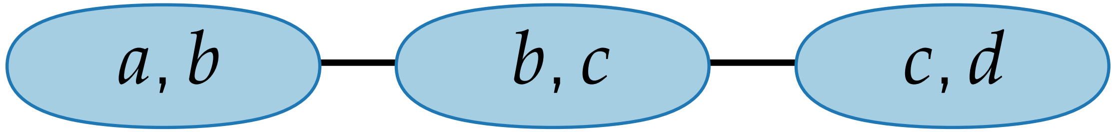
A **path decomposition** of G is a sequence $P = (X_1, X_2, \dots, X_r)$ of **bags**, where $X_i \subseteq V$, such that

- **(P1)** $\bigcup_{i=1}^{r} X_i = V$
- **(P2)** $\forall \{u, v\} \in E \; \exists i \in \{1, 2, \dots, r\} : u, v \in X_i$
- **(P3)** $\forall v \in V$, if $v \in X_i \cap X_j$ with $i \leq j$, then $v \in X_i \cap X_{i+1} \cap \dots \cap X_j$

The **width** of P is $w(P) = \max_{1 \leq I \leq r} |X_i| - 1$.

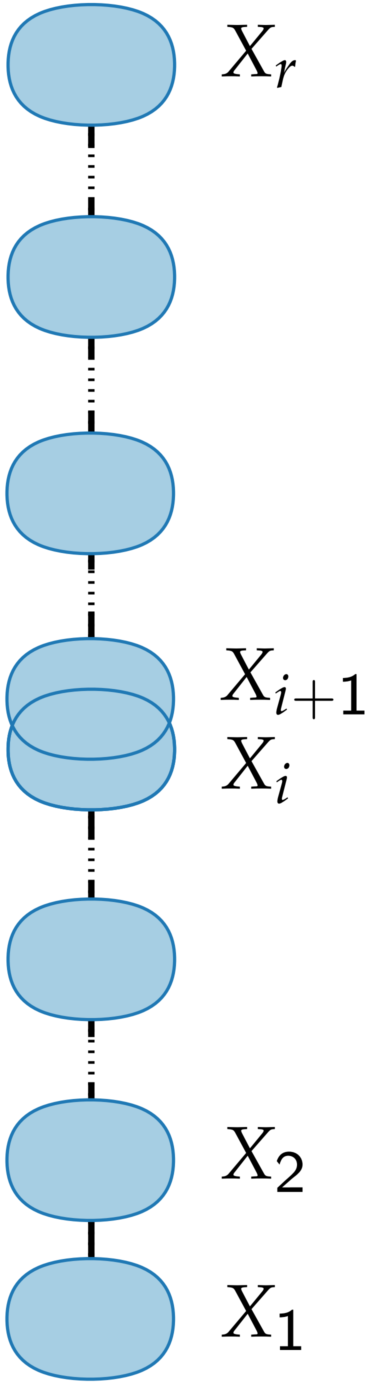
The **pathwidth** $pw(G)$ of G is the minimum width of a path decomposition of G.





$pw(G) = 1$

Okay – But Where Are the Separators?

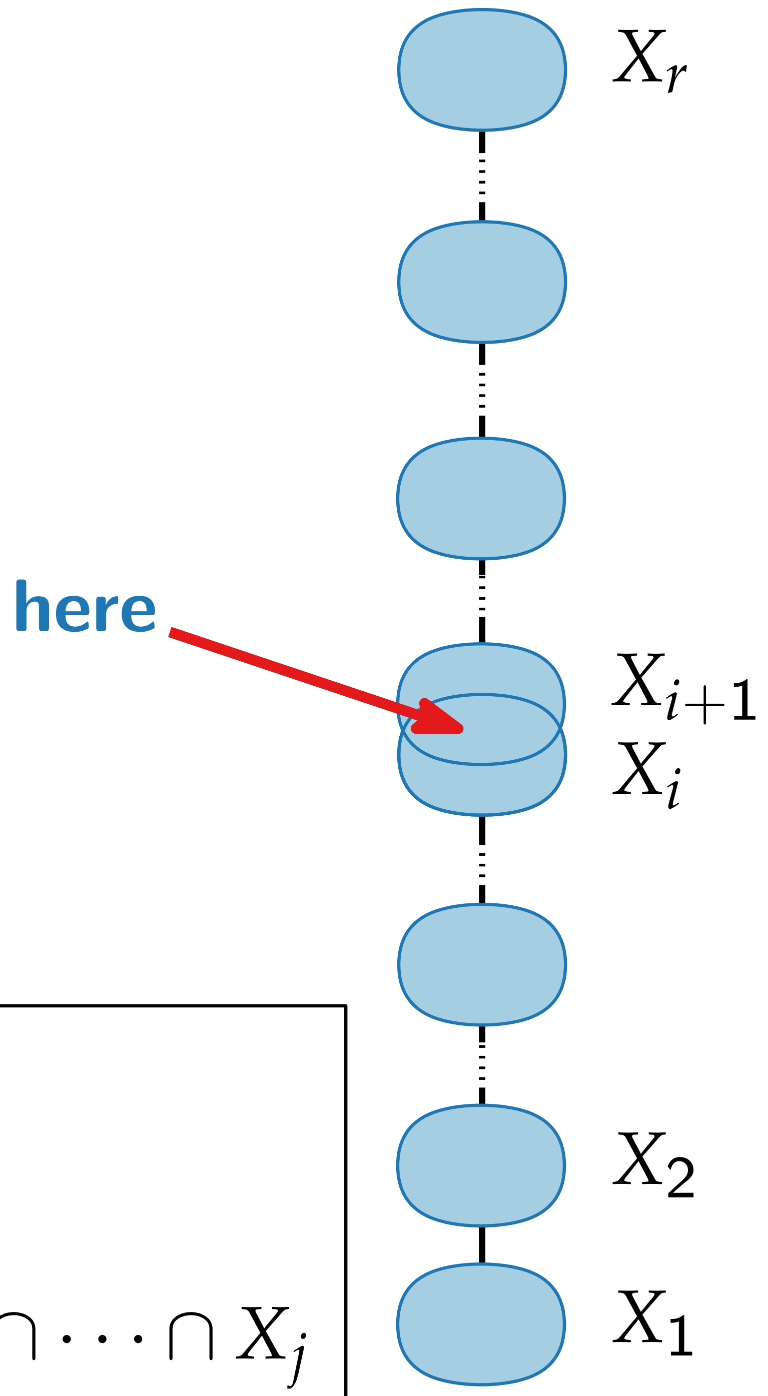


(P1) $\bigcup_{i=1}^{r} X_i = V$

(P2) $\forall \{u, v\} \in E \; \exists i \in \{1, 2, \dots, r\} : u, v \in X_i$

(P3) $\forall v \in V$, if $v \in X_i \cap X_j$ with $i \leq j$, then $v \in X_i \cap X_{i+1} \cap \dots \cap X_j$

Okay – But Where Are the Separators?



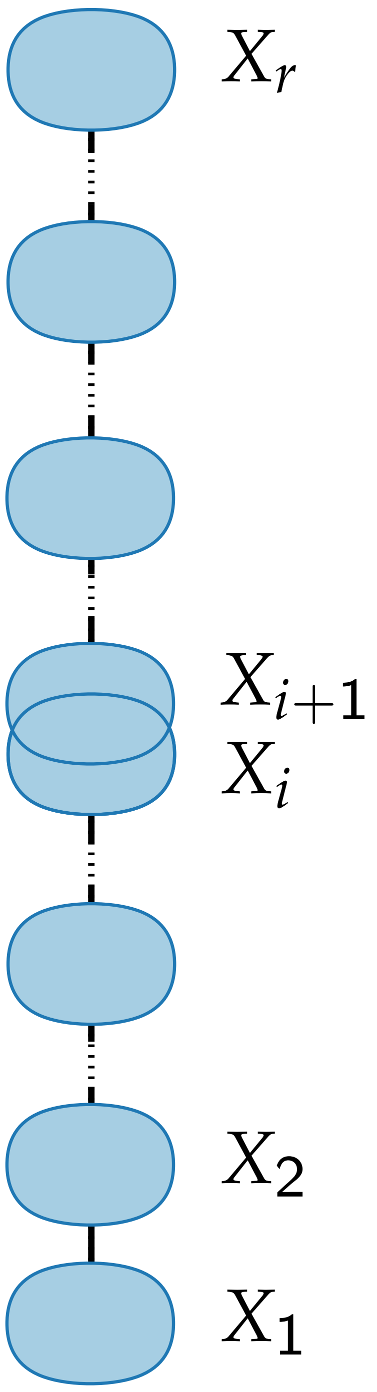
(P1) $\bigcup_{i=1}^{r} X_i = V$

(P2) $\forall \{u, v\} \in E\ \exists i \in \{1, 2, \dots, r\} : u, v \in X_i$

(P3) $\forall v \in V$, if $v \in X_i \cap X_j$ with $i \leq j$, then $v \in X_i \cap X_{i+1} \cap \dots \cap X_j$

Okay – But Where Are the Separators?

Lemma. Let $i < r$. Then there is no edge between
$A = (X_1 \cup X_2 \cup \cdots \cup X_i) \setminus (X_i \cap X_{i+1})$ and
$B = (X_{i+1} \cup X_{i+2} \cup \cdots \cup X_r) \setminus (X_i \cap X_{i+1})$.



(P1) $\bigcup_{i=1}^{r} X_i = V$

(P2) $\forall \{u, v\} \in E \; \exists i \in \{1, 2, \ldots, r\} : u, v \in X_i$

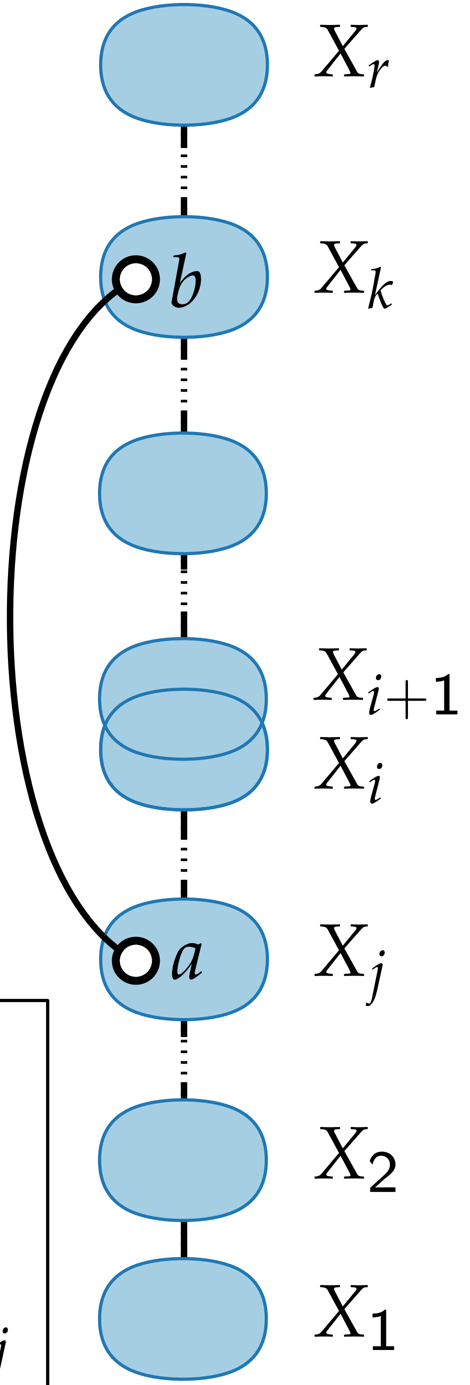
(P3) $\forall v \in V$, if $v \in X_i \cap X_j$ with $i \leq j$, then $v \in X_i \cap X_{i+1} \cap \cdots \cap X_j$

Okay – But Where Are the Separators?

Lemma. Let $i < r$. Then there is no edge between
$A = (X_1 \cup X_2 \cup \cdots \cup X_i) \setminus (X_i \cap X_{i+1})$ and
$B = (X_{i+1} \cup X_{i+2} \cup \cdots \cup X_r) \setminus (X_i \cap X_{i+1})$.

Proof. Assume there are $a \in A$ and $b \in B$ s.t. $\{a, b\} \in E$.

Let $j \leq i$ s.t. $a \in X_j$ and let $k \geq i + 1$ s.t. $b \in X_k$.



(P1) $\bigcup_{i=1}^{r} X_i = V$

(P2) $\forall \{u, v\} \in E \; \exists i \in \{1, 2, \ldots, r\} : u, v \in X_i$

(P3) $\forall v \in V$, if $v \in X_i \cap X_j$ with $i \leq j$, then $v \in X_i \cap X_{i+1} \cap \cdots \cap X_j$

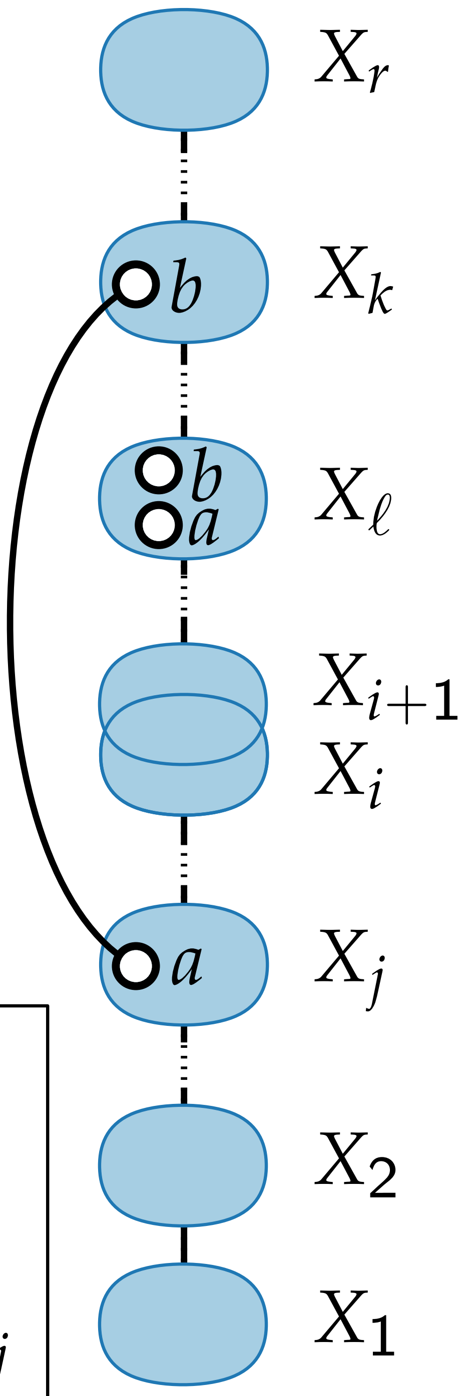
Okay – But Where Are the Separators?

Lemma. Let $i < r$. Then there is no edge between
$A = (X_1 \cup X_2 \cup \cdots \cup X_i) \setminus (X_i \cap X_{i+1})$ and
$B = (X_{i+1} \cup X_{i+2} \cup \cdots \cup X_r) \setminus (X_i \cap X_{i+1})$.

Proof. Assume there are $a \in A$ and $b \in B$ s.t. $\{a, b\} \in E$.

Let $j \leq i$ s.t. $a \in X_j$ and let $k \geq i+1$ s.t. $b \in X_k$.

(P2) $\Rightarrow$ there is a bag X_ℓ s.t. $a, b \in X_\ell$, w.l.o.g. let $\ell \geq i+1$.



(P1) $\bigcup_{i=1}^{r} X_i = V$

(P2) $\forall \{u, v\} \in E \ \exists i \in \{1, 2, \ldots, r\} : u, v \in X_i$

(P3) $\forall v \in V$, if $v \in X_i \cap X_j$ with $i \leq j$, then $v \in X_i \cap X_{i+1} \cap \cdots \cap X_j$

Okay – But Where Are the Separators?

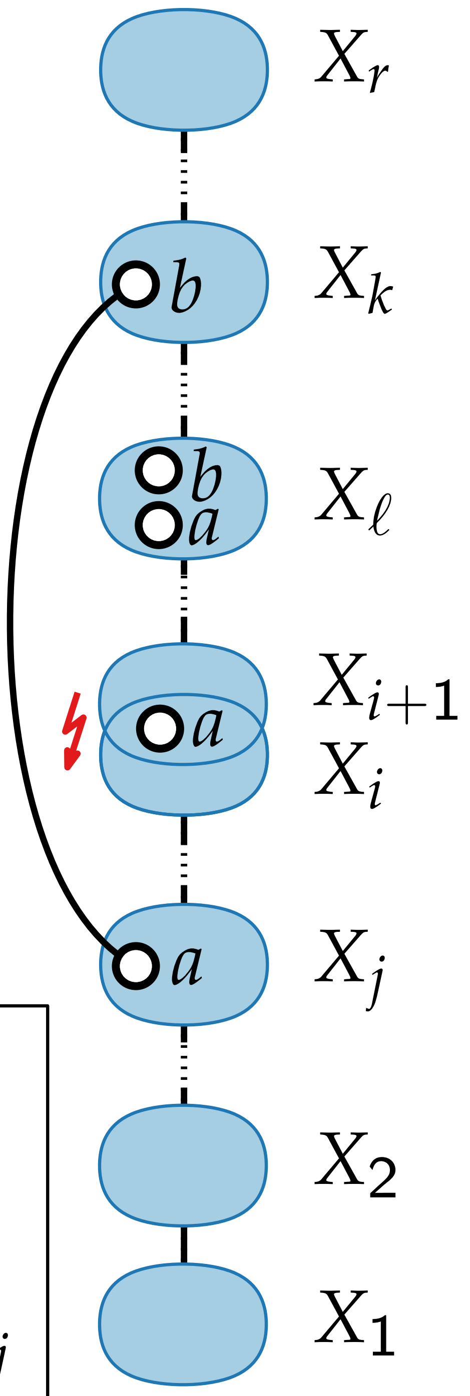
Lemma. Let $i < r$. Then there is no edge between
$A = (X_1 \cup X_2 \cup \cdots \cup X_i) \setminus (X_i \cap X_{i+1})$ and
$B = (X_{i+1} \cup X_{i+2} \cup \cdots \cup X_r) \setminus (X_i \cap X_{i+1})$.

Proof. Assume there are $a \in A$ and $b \in B$ s.t. $\{a, b\} \in E$.

Let $j \leq i$ s.t. $a \in X_j$ and let $k \geq i+1$ s.t. $b \in X_k$.

(P2) $\Rightarrow$ there is a bag X_ℓ s.t. $a, b \in X_\ell$, w.l.o.g. let $\ell \geq i+1$.

(P3) $\Rightarrow$ $a \in X_i \cap X_{i+1}$; contradiction to $a \in A$. □



(P1) $\bigcup_{i=1}^{r} X_i = V$

(P2) $\forall \{u, v\} \in E \; \exists i \in \{1, 2, \ldots, r\} : u, v \in X_i$

(P3) $\forall v \in V$, if $v \in X_i \cap X_j$ with $i \leq j$, then $v \in X_i \cap X_{i+1} \cap \cdots \cap X_j$

Computing Path Decompositions

k-PATHWIDTH

Input. Graph $G = (V, E)$, $k \in \mathbb{N}$

Question. Is the pathwidth of G at most k?

- NP-complete
- FPT in k
  - The algorithm constructs a path decomposition of width $\leq k$.
  - Its runtime depends linearly on $|V| + |E|$.

Computing Path Decompositions

k-PATHWIDTH

Input. Graph $G = (V, E)$, $k \in \mathbb{N}$

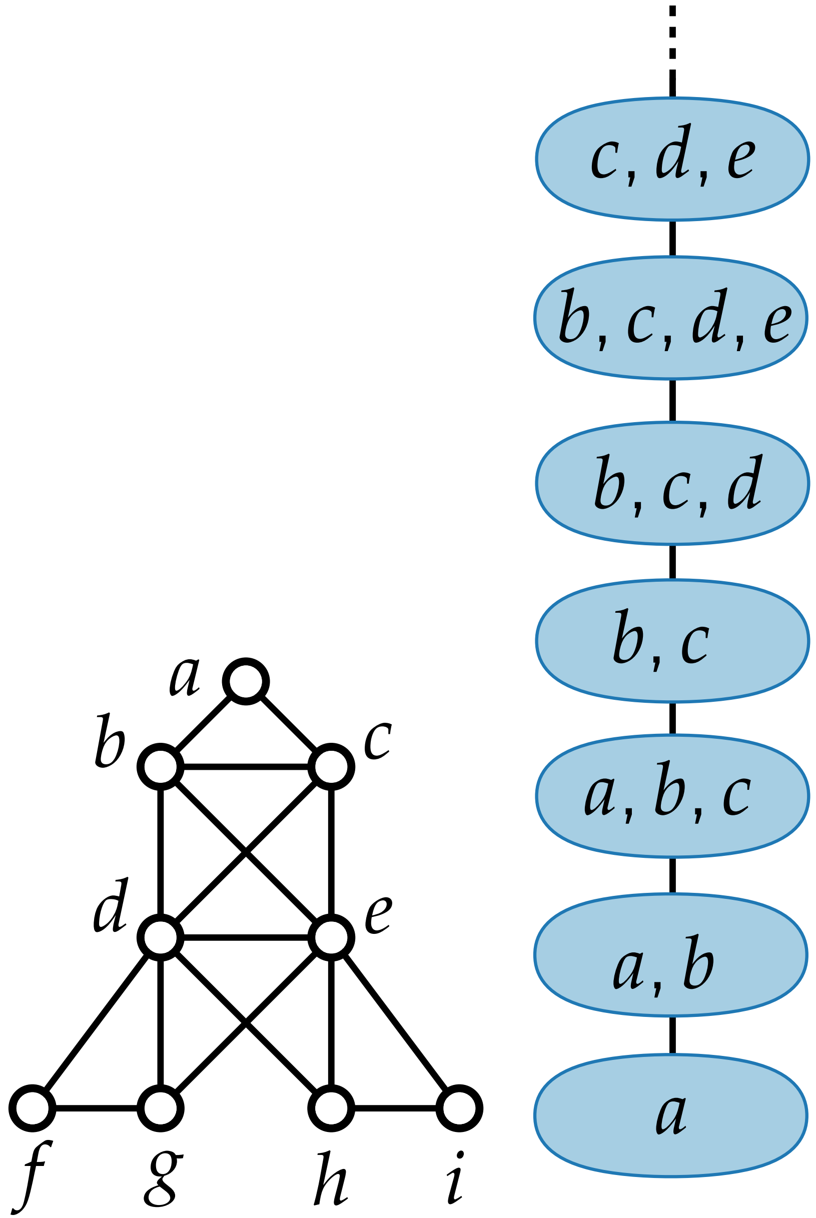
Question. Is the pathwidth of G at most k?

- NP-complete
- FPT in k
  - The algorithm constructs a path decomposition of width $\leq k$.
  - Its runtime depends linearly on $|V| + |E|$.

$\Rightarrow$ When designing FPT algorithms with respect to the pathwidth, we may assume to be given a path decomposition!

Nice Path Decompositions

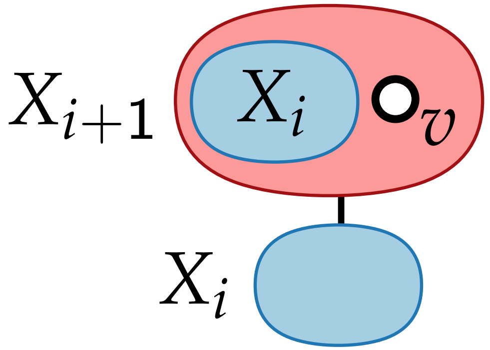
A path decomposition is **nice** if $|X_1| = 1$ and each other bag has one of two **types**:



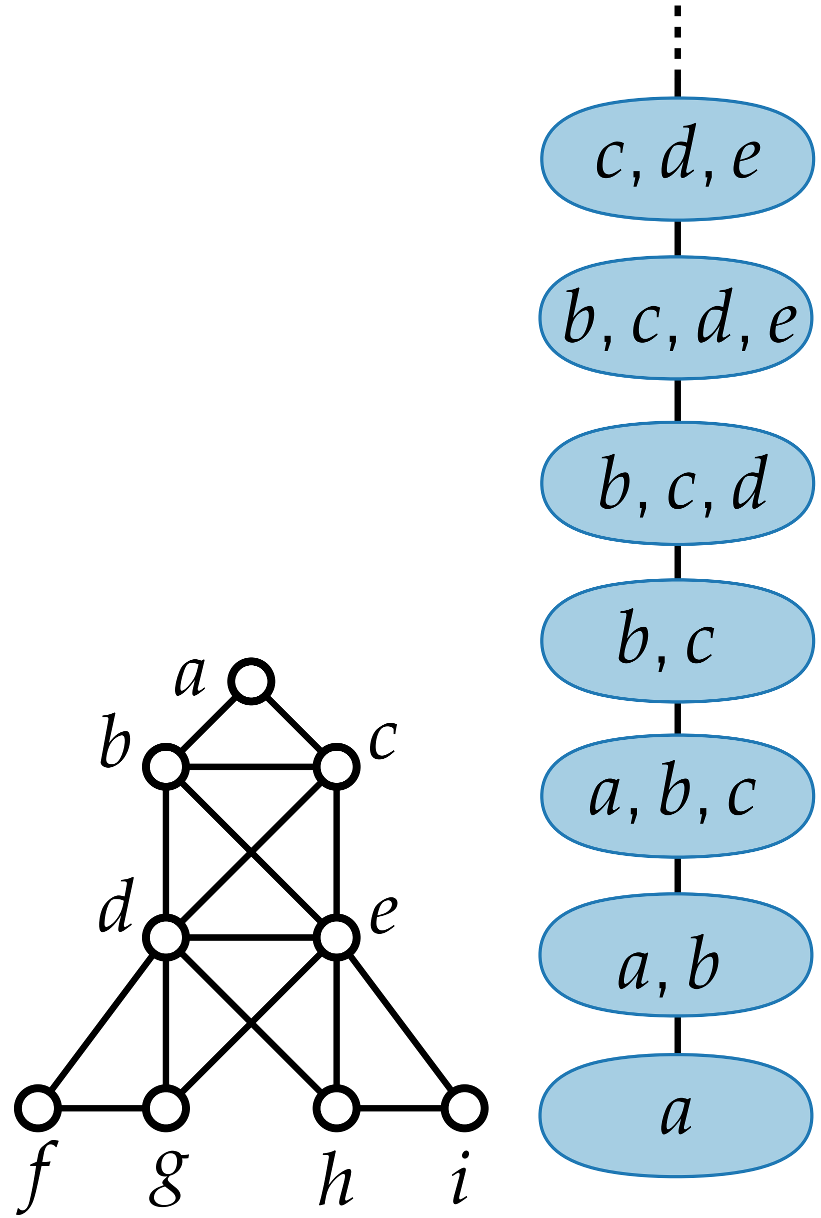
Nice Path Decompositions

A path decomposition is **nice** if $|X_1| = 1$ and each other bag has one of two **types**:

X_{i+1} is of type **Introduce** if



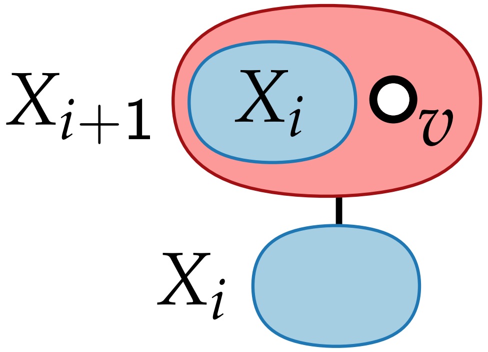
$X_{i+1} = X_i \cup \{v\}$ where $v \notin X_i$



Nice Path Decompositions

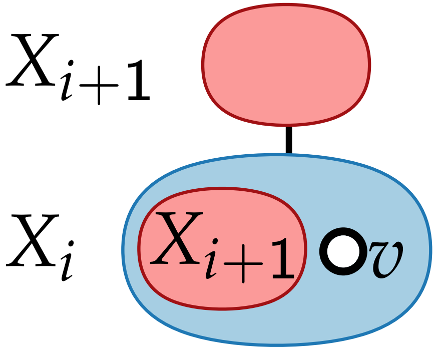
A path decomposition is **nice** if $|X_1| = 1$ and each other bag has one of two **types**:

X_{i+1} is of type **Introduce** if

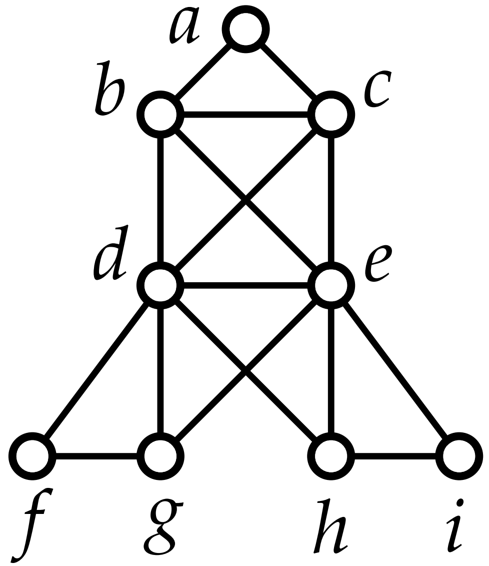


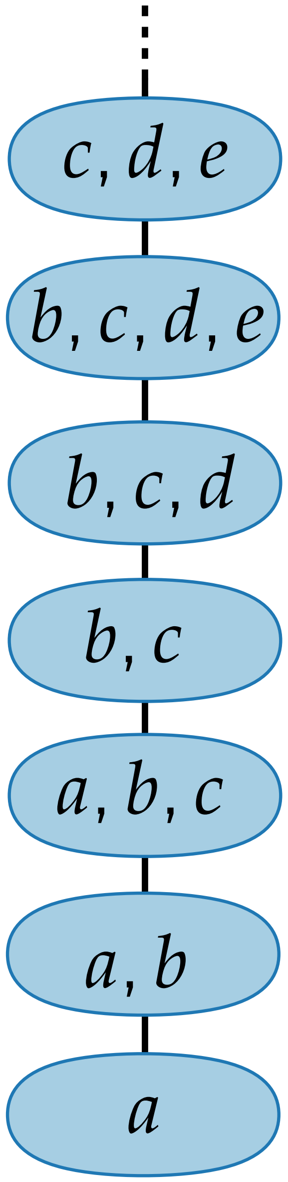
$X_{i+1} = X_i \cup \{v\}$ where $v \notin X_i$

X_{i+1} is of type **Forget** if



$X_i = X_{i+1} \cup \{v\}$ where $v \notin X_{i+1}$

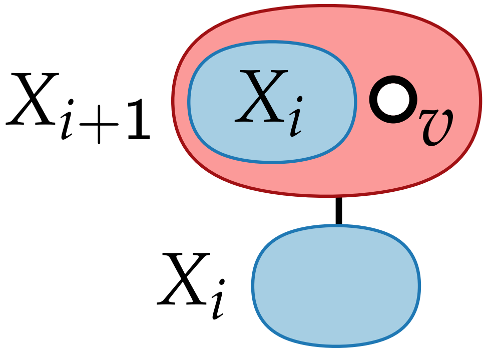




Nice Path Decompositions

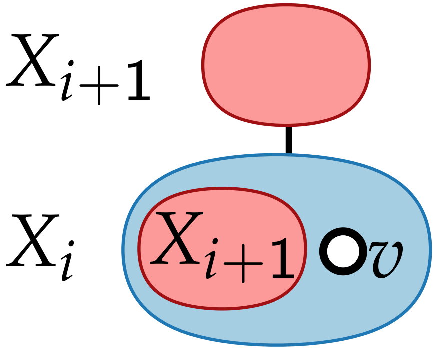
A path decomposition is **nice** if $|X_1| = 1$ and each other bag has one of two **types**:

X_{i+1} is of type **Introduce** if



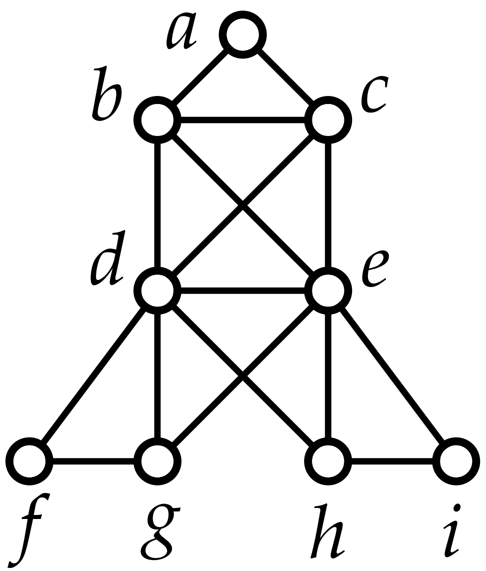
$X_{i+1} = X_i \cup \{v\}$ where $v \notin X_i$

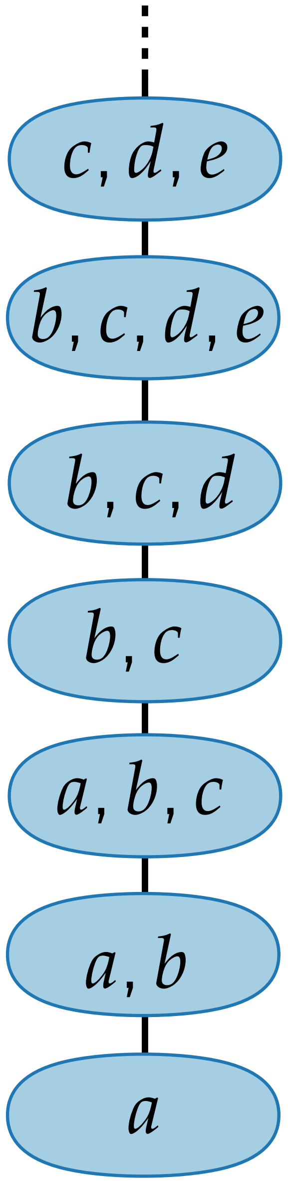
X_{i+1} is of type **Forget** if



$X_i = X_{i+1} \cup \{v\}$ where $v \notin X_{i+1}$

Observation. The number of bags is $r \leq 2|V| - 1$.

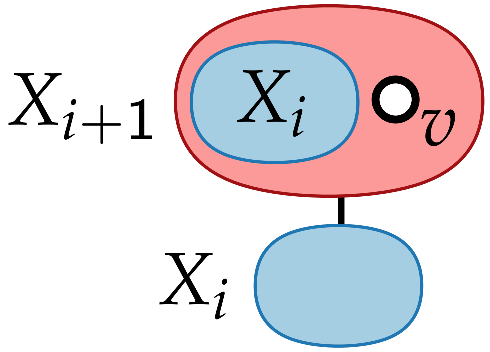




Nice Path Decompositions

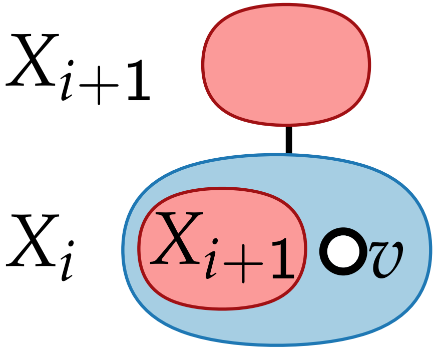
A path decomposition is **nice** if $|X_1| = 1$ and each other bag has one of two **types**:

X_{i+1} is of type **Introduce** if



$X_{i+1} = X_i \cup \{v\}$ where $v \notin X_i$

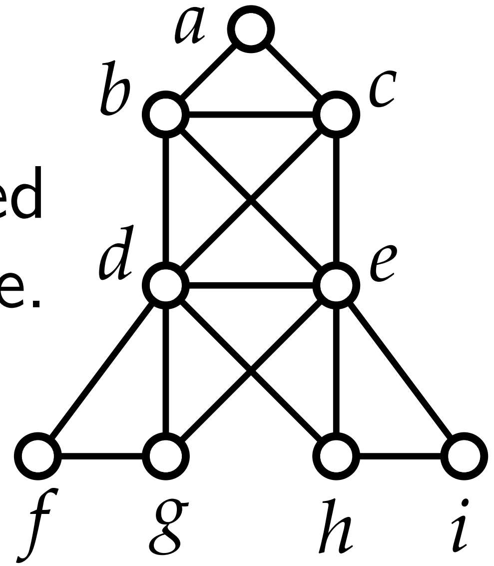
X_{i+1} is of type **Forget** if

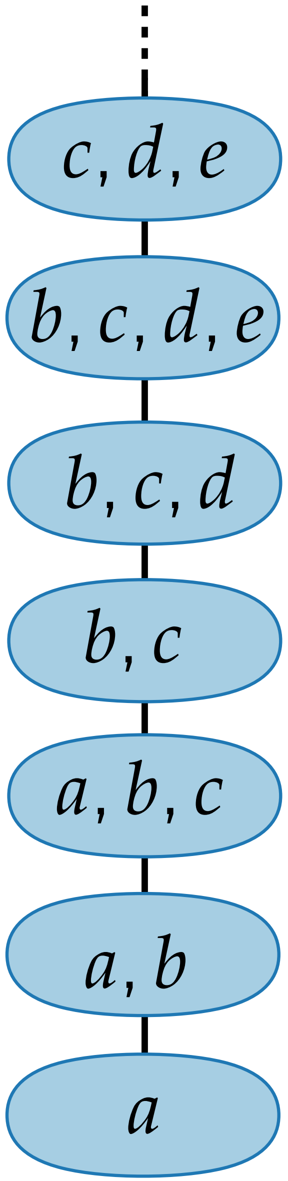


$X_i = X_{i+1} \cup \{v\}$ where $v \notin X_{i+1}$

Observation. The number of bags is $r \leq 2|V| - 1$.

Lemma. A path decomposition of width k can be transformed into a nice path decomposition of width k in polynomial time.

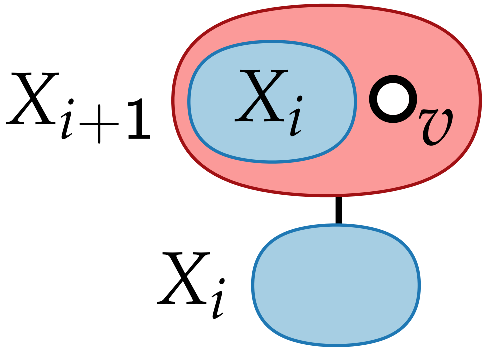




Nice Path Decompositions

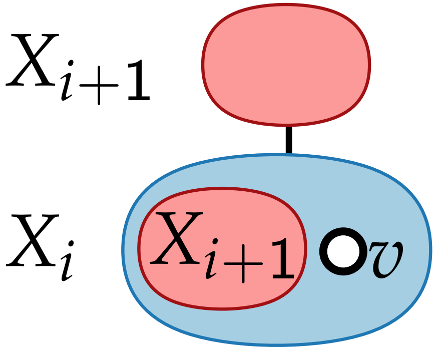
A path decomposition is **nice** if $|X_1| = 1$ and each other bag has one of two **types**:

X_{i+1} is of type **Introduce** if

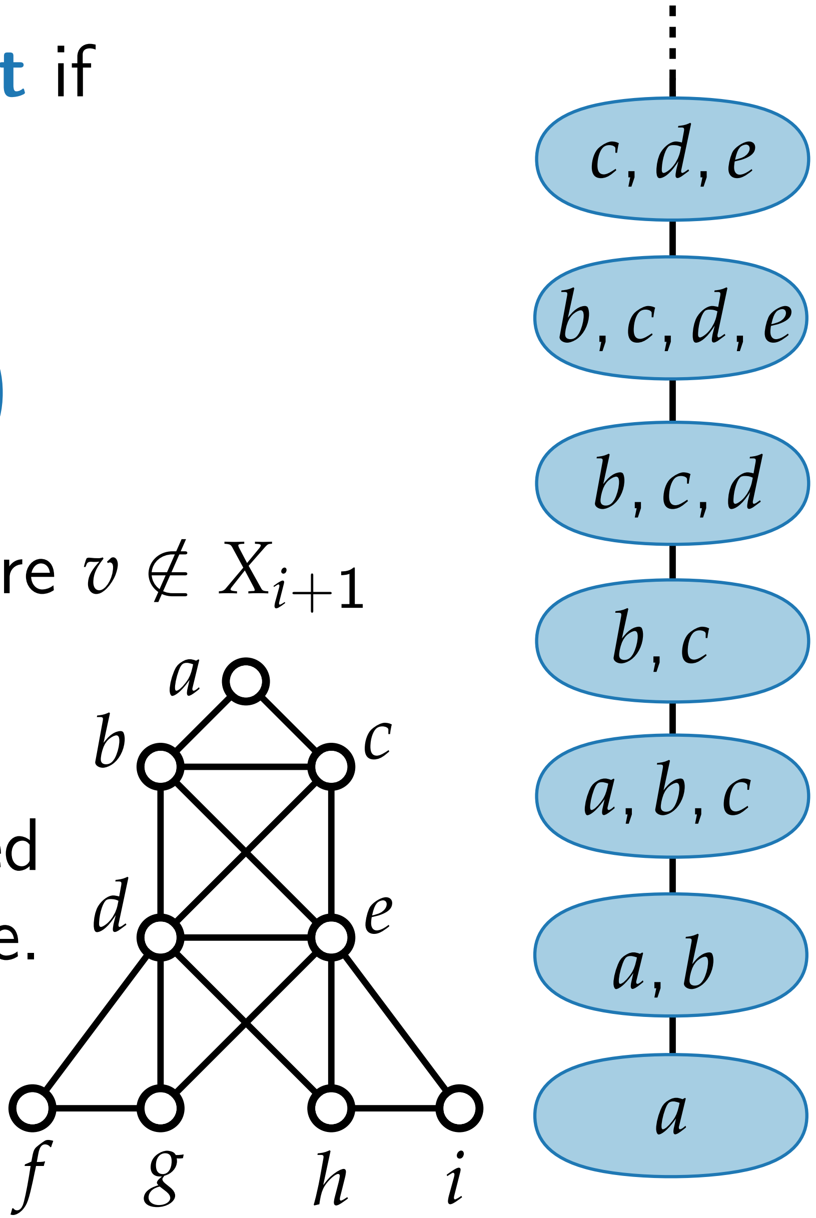


$X_{i+1} = X_i \cup \{v\}$ where $v \notin X_i$

X_{i+1} is of type **Forget** if



$X_i = X_{i+1} \cup \{v\}$ where $v \notin X_{i+1}$



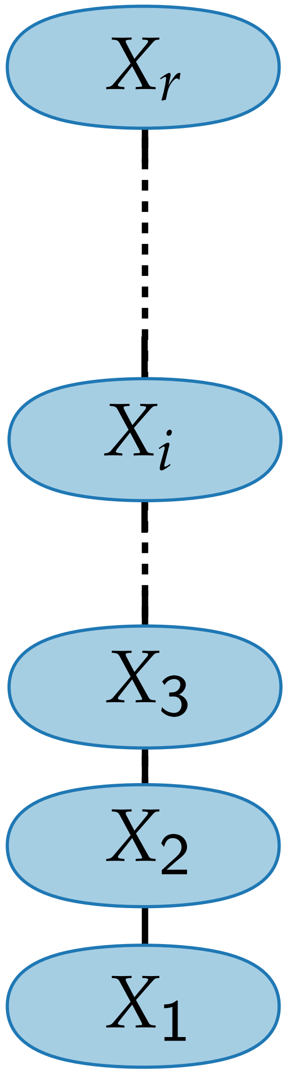
Observation. The number of bags is $r \leq 2|V| - 1$.

Lemma. A path decomposition of width k can be transformed into a nice path decomposition of width k in polynomial time.

$\Rightarrow$ When designing FPT algorithms w.r.t. the pathwidth, we may assume to be given a *nice* path decomposition.

Independent Set in Graphs of Bounded Pathwidth

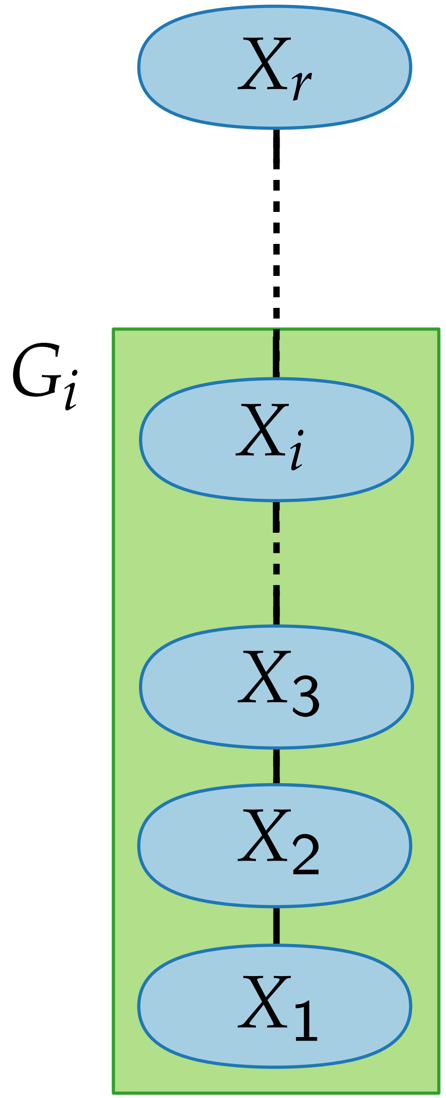
Assume we are given a nice path decomposition $P = (X_1, X_2, \ldots, X_r)$ of width k.



INDEPENDENT SET in Graphs of Bounded Pathwidth

Assume we are given a nice path decomposition $P = (X_1, X_2, \dots, X_r)$ of width k.

Let G_i be the graph induced by $X_1 \cup X_2 \cup \dots \cup X_i$ for some $i \in \{1, 2, \dots, r\}$.



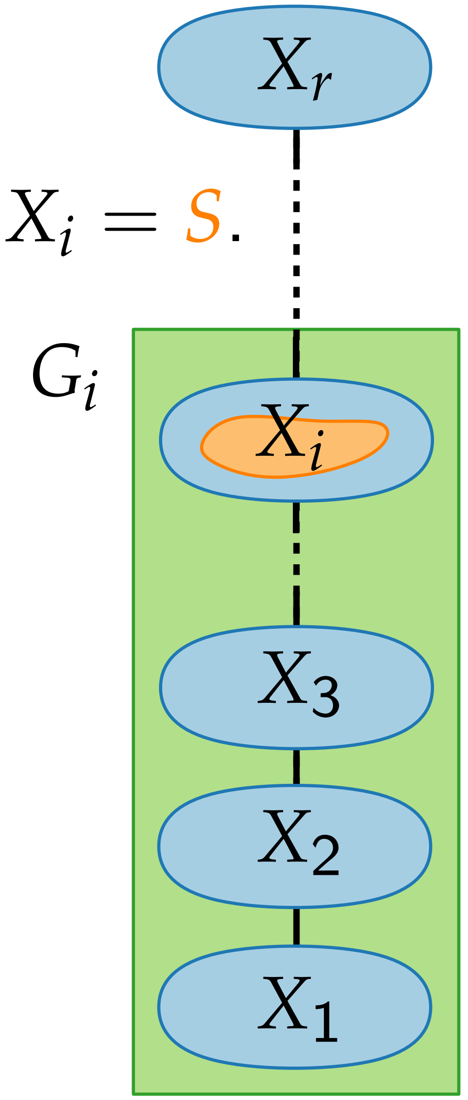
INDEPENDENT SET in Graphs of Bounded Pathwidth

Assume we are given a nice path decomposition $P = (X_1, X_2, \ldots, X_r)$ of width k.

Let G_i be the graph induced by $X_1 \cup X_2 \cup \cdots \cup X_i$ for some $i \in \{1, 2, \ldots, r\}$.

For each $S \subseteq X_i$ let

$D[i, S]$:= maximum weight of an independent set I in G_i such that $I \cap X_i = S$.



INDEPENDENT SET in Graphs of Bounded Pathwidth

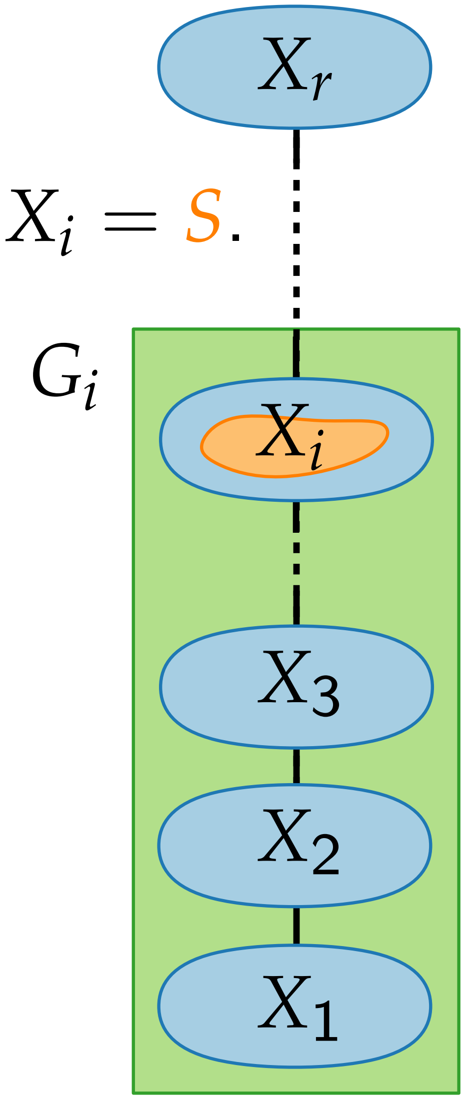
Assume we are given a nice path decomposition $P = (X_1, X_2, \ldots, X_r)$ of width k.

Let G_i be the graph induced by $X_1 \cup X_2 \cup \cdots \cup X_i$ for some $i \in \{1, 2, \ldots, r\}$.

For each $S \subseteq X_i$ let

$D[i, S]$:= maximum weight of an independent set I in G_i such that $I \cap X_i = S$.

(P1) $\Rightarrow$ $G_r = G$ $\Rightarrow$ solution = $\max_{S \subseteq X_r} D[r, S]$



INDEPENDENT SET in Graphs of Bounded Pathwidth

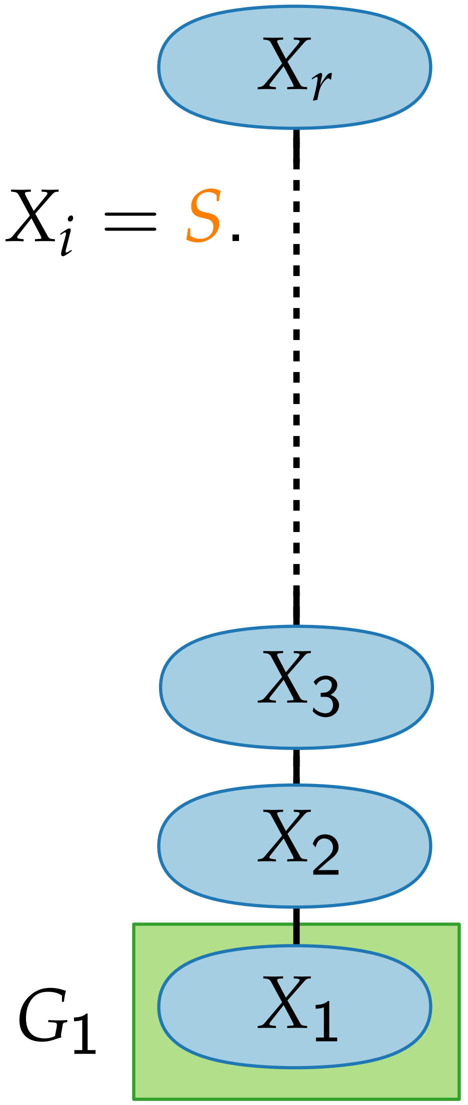
Assume we are given a nice path decomposition $P = (X_1, X_2, \ldots, X_r)$ of width k.

Let G_i be the graph induced by $X_1 \cup X_2 \cup \cdots \cup X_i$ for some $i \in \{1, 2, \ldots, r\}$.

For each $S \subseteq X_i$ let

$D[i, S] :=$ maximum weight of an independent set I in G_i such that $I \cap X_i = S$.

$$D[1, S] = \begin{cases} 0 & \text{, if } S = \emptyset \\ w(v) & \text{, if } S = \{v\} \end{cases}$$



INDEPENDENT SET in Graphs of Bounded Pathwidth

Assume we are given a nice path decomposition $P = (X_1, X_2, \ldots, X_r)$ of width k.

Let G_i be the graph induced by $X_1 \cup X_2 \cup \cdots \cup X_i$ for some $i \in \{1, 2, \ldots, r\}$.

For each $S \subseteq X_i$ let

$D[i, S] :=$ maximum weight of an independent set I in G_i such that $I \cap X_i = S$.

Assume that $i > 1$. If S is not independent, $D[i, S] = -\infty$.

INDEPENDENT SET in Graphs of Bounded Pathwidth

Assume we are given a nice path decomposition $P = (X_1, X_2, \ldots, X_r)$ of width k.

Let G_i be the graph induced by $X_1 \cup X_2 \cup \cdots \cup X_i$ for some $i \in \{1, 2, \ldots, r\}$.

For each $S \subseteq X_i$ let

$D[i, S] :=$ maximum weight of an independent set I in G_i such that $I \cap X_i = S$.

Assume that $i > 1$. If S is not independent, $D[i, S] = -\infty$.

Otherwise, we distinguish between the two types of X_i.

Independent Set in Graphs of Bounded Pathwidth

Assume we are given a nice path decomposition $P = (X_1, X_2, \ldots, X_r)$ of width k.

Let G_i be the graph induced by $X_1 \cup X_2 \cup \cdots \cup X_i$ for some $i \in \{1, 2, \ldots, r\}$.

For each $S \subseteq X_i$ let

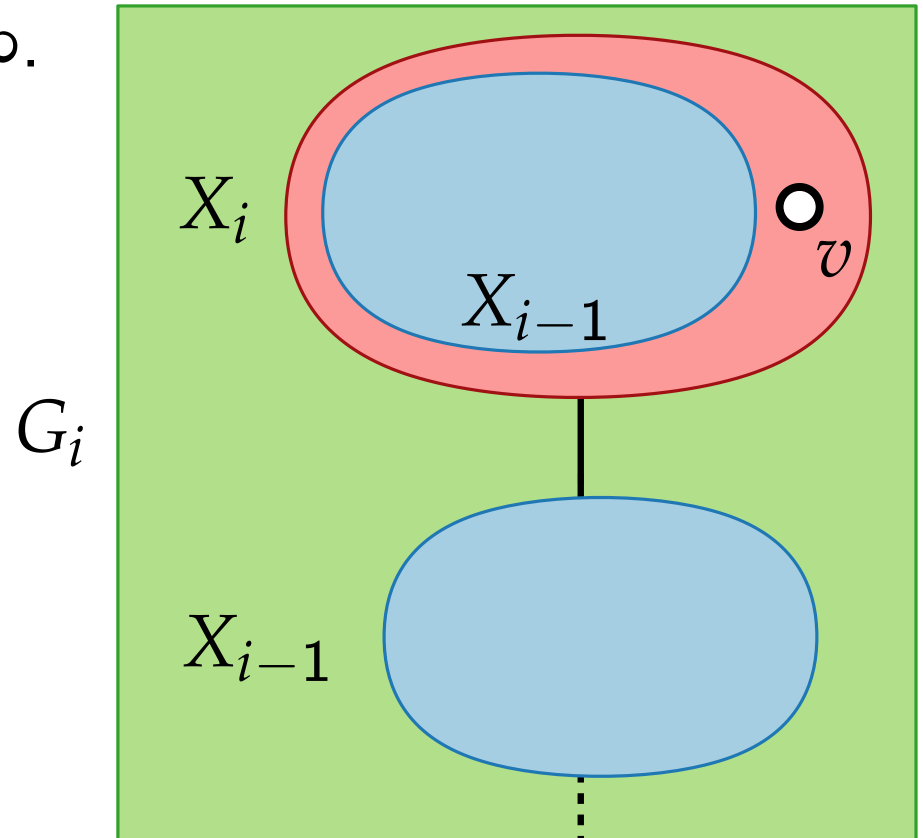
$D[i, S]$:= maximum weight of an independent set I in G_i such that $I \cap X_i = S$.

Assume that $i > 1$. If S is not independent, $D[i, S] = -\infty$.

Otherwise, we distinguish between the two types of X_i.

If X_i is **Introduce**, then

$$D[i, S] =$$



INDEPENDENT SET in Graphs of Bounded Pathwidth

Assume we are given a nice path decomposition $P = (X_1, X_2, \ldots, X_r)$ of width k.

Let G_i be the graph induced by $X_1 \cup X_2 \cup \cdots \cup X_i$ for some $i \in \{1, 2, \ldots, r\}$.

For each $S \subseteq X_i$ let

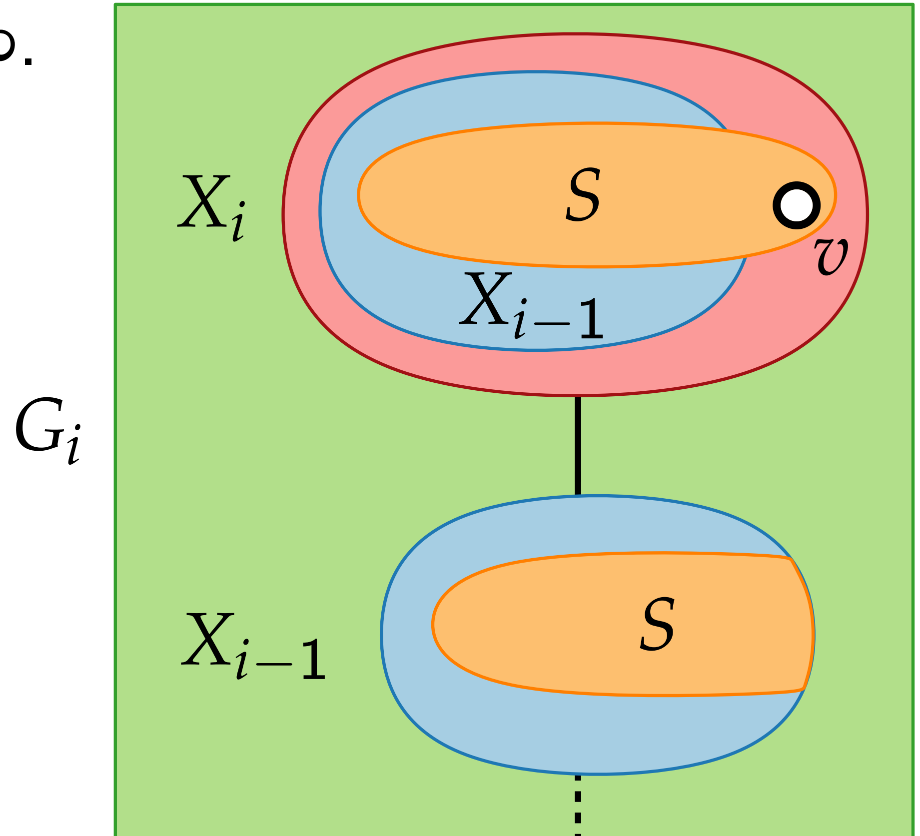
$D[i, S]$:= maximum weight of an independent set I in G_i such that $I \cap X_i = S$.

Assume that $i > 1$. If S is not independent, $D[i, S] = -\infty$.

Otherwise, we distinguish between the two types of X_i.

If X_i is **Introduce**, then

$$D[i, S] =$$



INDEPENDENT SET in Graphs of Bounded Pathwidth

Assume we are given a nice path decomposition $P = (X_1, X_2, \ldots, X_r)$ of width k.

Let G_i be the graph induced by $X_1 \cup X_2 \cup \cdots \cup X_i$ for some $i \in \{1, 2, \ldots, r\}$.

For each $S \subseteq X_i$ let

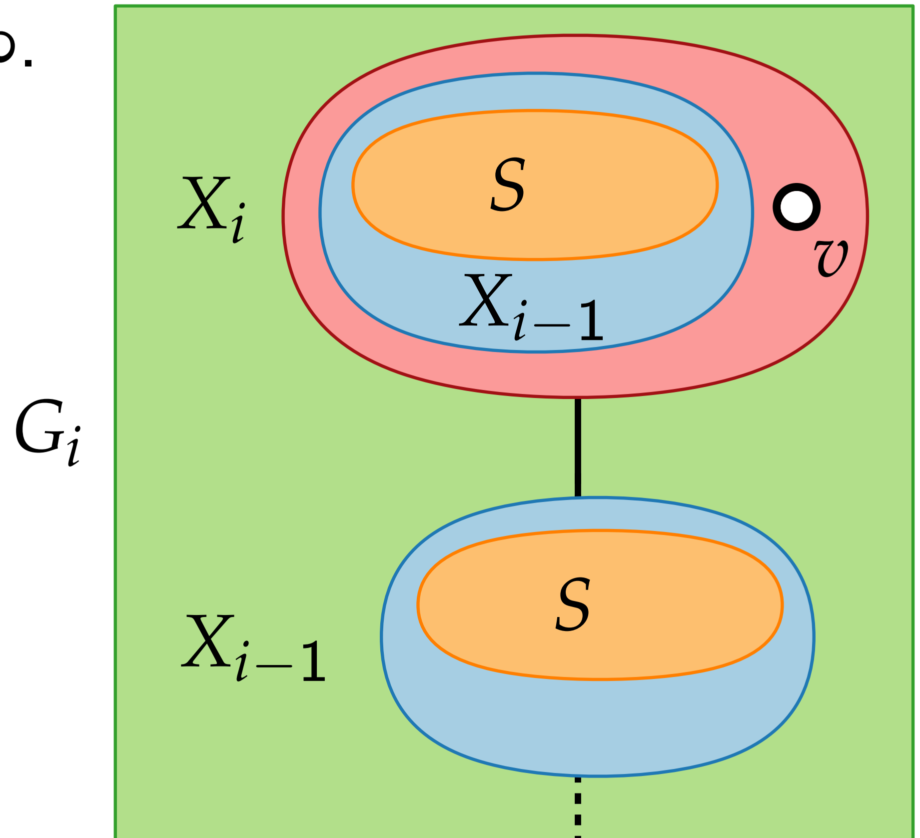
$D[i, S]$:= maximum weight of an independent set I in G_i such that $I \cap X_i = S$.

Assume that $i > 1$. If S is not independent, $D[i, S] = -\infty$.

Otherwise, we distinguish between the two types of X_i.

If X_i is **Introduce**, then

$$D[i, S] =$$



Independent Set in Graphs of Bounded Pathwidth

Assume we are given a nice path decomposition $P = (X_1, X_2, \dots, X_r)$ of width k.

Let G_i be the graph induced by $X_1 \cup X_2 \cup \dots \cup X_i$ for some $i \in \{1, 2, \dots, r\}$.

For each $S \subseteq X_i$ let

$D[i, S]$:= maximum weight of an independent set I in G_i such that $I \cap X_i = S$.

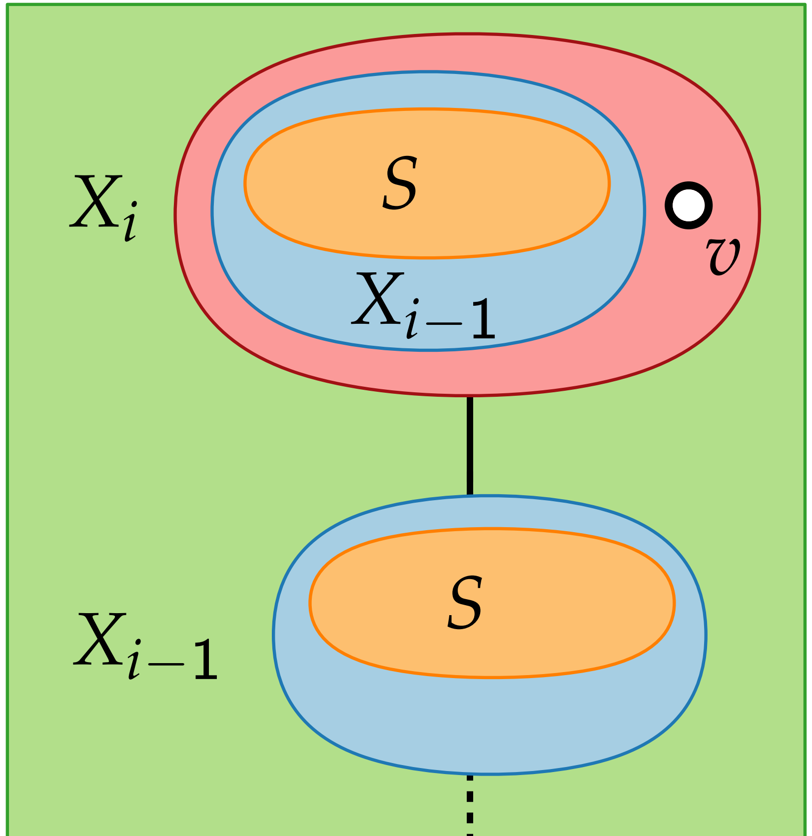
Assume that $i > 1$. If S is not independent, $D[i, S] = -\infty$.

Otherwise, we distinguish between the two types of X_i.

If X_i is **Introduce**, then

$$D[i, S] = \begin{cases} & \text{, if } v \notin S \\ & \text{, if } v \in S \end{cases}$$

G_i



Independent Set in Graphs of Bounded Pathwidth

Assume we are given a nice path decomposition $P = (X_1, X_2, \ldots, X_r)$ of width k.

Let G_i be the graph induced by $X_1 \cup X_2 \cup \cdots \cup X_i$ for some $i \in \{1, 2, \ldots, r\}$.

For each $S \subseteq X_i$ let

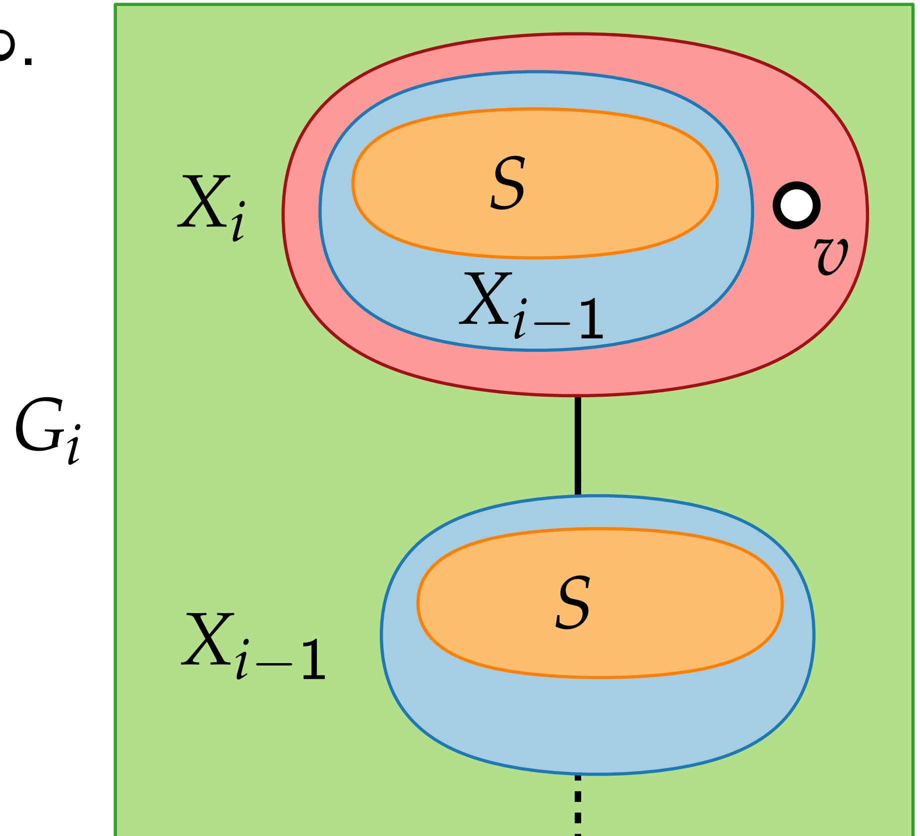
$D[i, S]$:= maximum weight of an independent set I in G_i such that $I \cap X_i = S$.

Assume that $i > 1$. If S is not independent, $D[i, S] = -\infty$.

Otherwise, we distinguish between the two types of X_i.

If X_i is **Introduce**, then

$$D[i, S] = \begin{cases} D[i-1, S] & \text{, if } v \notin S \\ & \text{, if } v \in S \end{cases}$$



INDEPENDENT SET in Graphs of Bounded Pathwidth

Assume we are given a nice path decomposition $P = (X_1, X_2, \ldots, X_r)$ of width k.

Let G_i be the graph induced by $X_1 \cup X_2 \cup \cdots \cup X_i$ for some $i \in \{1, 2, \ldots, r\}$.

For each $S \subseteq X_i$ let

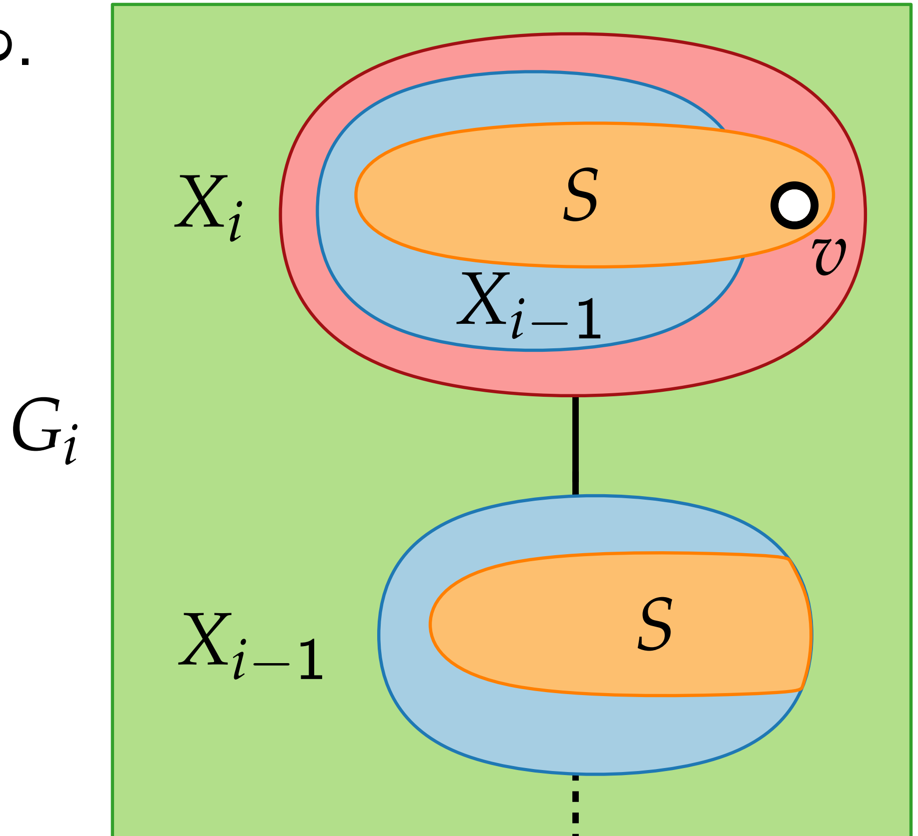
$D[i, S]$:= maximum weight of an independent set I in G_i such that $I \cap X_i = S$.

Assume that $i > 1$. If S is not independent, $D[i, S] = -\infty$.

Otherwise, we distinguish between the two types of X_i.

If X_i is **Introduce**, then

$$D[i, S] = \begin{cases} D[i-1, S] & \text{, if } v \notin S \\ & \text{, if } v \in S \end{cases}$$



Independent Set in Graphs of Bounded Pathwidth

Assume we are given a nice path decomposition $P = (X_1, X_2, \ldots, X_r)$ of width k.

Let G_i be the graph induced by $X_1 \cup X_2 \cup \cdots \cup X_i$ for some $i \in \{1, 2, \ldots, r\}$.

For each $S \subseteq X_i$ let

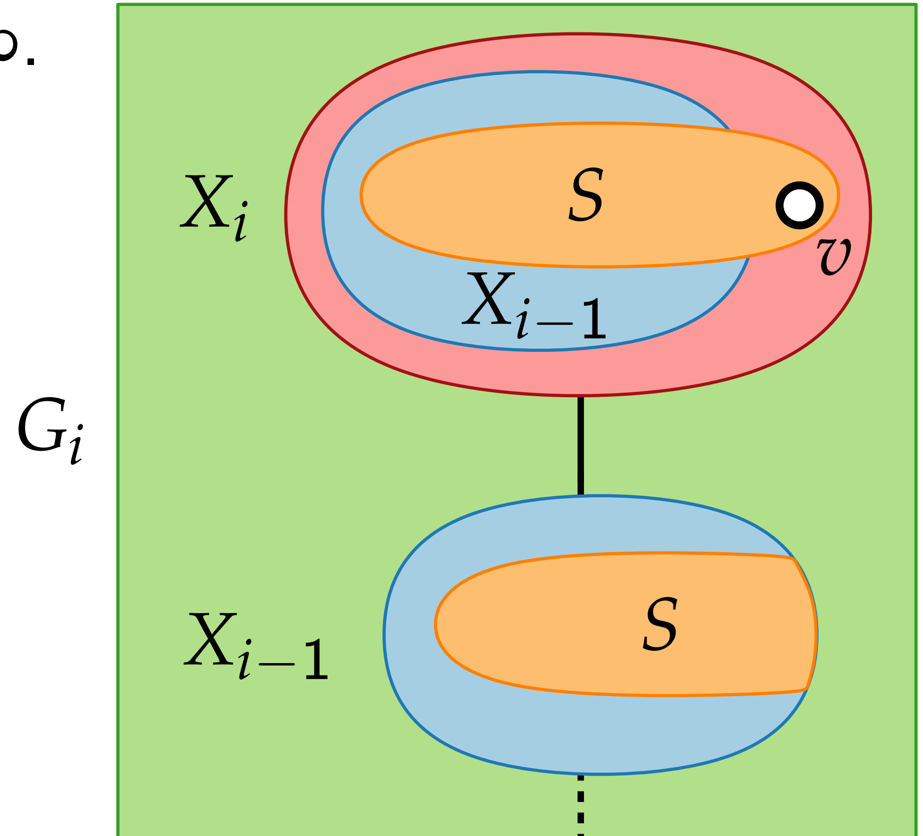
$D[i, S]$:= maximum weight of an independent set I in G_i such that $I \cap X_i = S$.

Assume that $i > 1$. If S is not independent, $D[i, S] = -\infty$.

Otherwise, we distinguish between the two types of X_i.

If X_i is **Introduce**, then

$$D[i, S] = \begin{cases} D[i-1, S] & \text{, if } v \notin S \\ w(v) + D[i-1, S \setminus \{v\}] & \text{, if } v \in S \end{cases}$$



Independent Set in Graphs of Bounded Pathwidth

Assume we are given a nice path decomposition $P = (X_1, X_2, \dots, X_r)$ of width k.

Let G_i be the graph induced by $X_1 \cup X_2 \cup \dots \cup X_i$ for some $i \in \{1, 2, \dots, r\}$.

For each $S \subseteq X_i$ let

$D[i, S]$:= maximum weight of an independent set I in G_i such that $I \cap X_i = S$.

Assume that $i > 1$. If S is not independent, $D[i, S] = -\infty$.

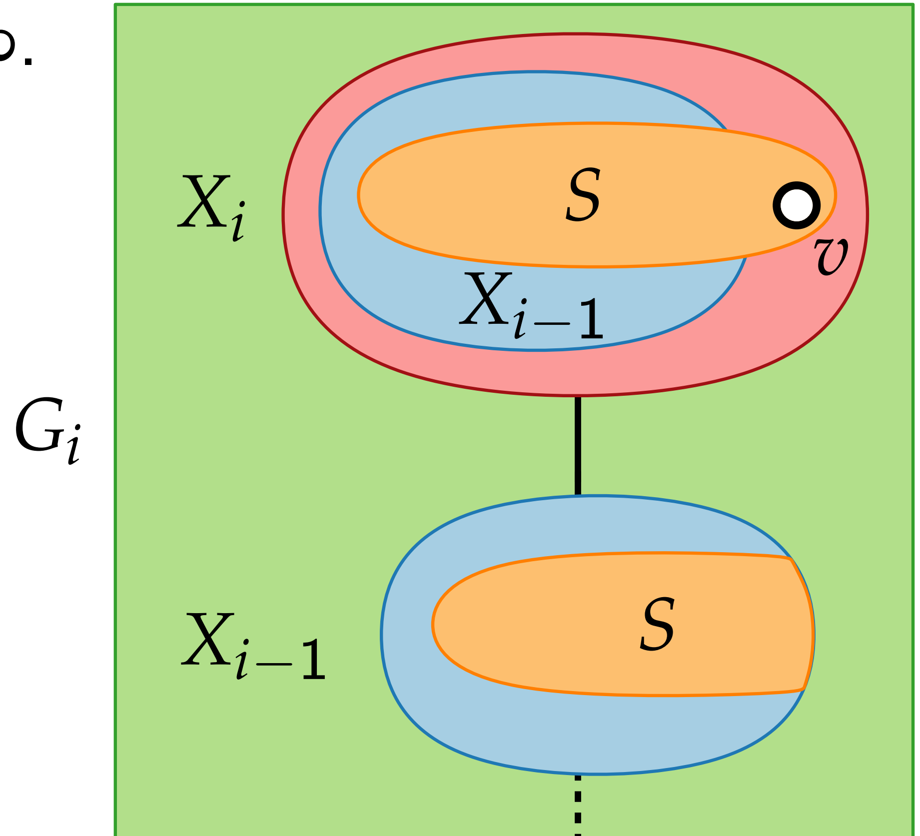
Otherwise, we distinguish between the two types of X_i.

If X_i is **Introduce**, then

$$D[i, S] = \begin{cases} D[i-1, S] & \text{, if } v \notin S \\ w(v) + D[i-1, S \setminus \{v\}] & \text{, if } v \in S \end{cases}$$

Let I' denote the independent set corresponding to

Why is $I' \cup \{v\}$ independent?



Independent Set in Graphs of Bounded Pathwidth

Assume we are given a nice path decomposition $P = (X_1, X_2, \ldots, X_r)$ of width k.

Let G_i be the graph induced by $X_1 \cup X_2 \cup \cdots \cup X_i$ for some $i \in \{1, 2, \ldots, r\}$.

For each $S \subseteq X_i$ let

$D[i, S] :=$ maximum weight of an independent set I in G_i such that $I \cap X_i = S$.

Assume that $i > 1$. If S is not independent, $D[i, S] = -\infty$.

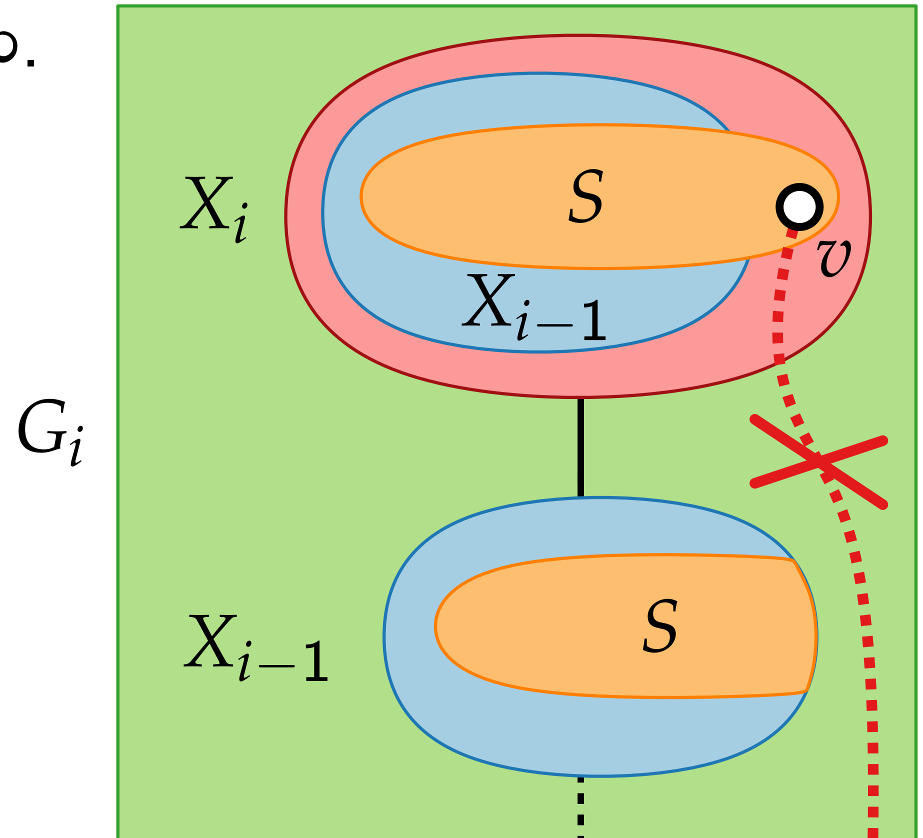
Otherwise, we distinguish between the two types of X_i.

If X_i is **Introduce**, then

$$D[i, S] = \begin{cases} D[i-1, S] & \text{, if } v \notin S \\ w(v) + D[i-1, S \setminus \{v\}] & \text{, if } v \in S \end{cases}$$

Let I' denote the independent set corresponding to

Why is $I' \cup \{v\}$ independent? due to Lemma 1!



INDEPENDENT SET in Graphs of Bounded Pathwidth

Assume we are given a nice path decomposition $P = (X_1, X_2, \ldots, X_r)$ of width k.

Let G_i be the graph induced by $X_1 \cup X_2 \cup \cdots \cup X_i$ for some $i \in \{1, 2, \ldots, r\}$.

For each $S \subseteq X_i$ let

$D[i, S] :=$ maximum weight of an independent set I in G_i such that $I \cap X_i = S$.

Assume that $i > 1$. If S is not independent, $D[i, S] = -\infty$.

Otherwise, we distinguish between the two types of X_i.

INDEPENDENT SET in Graphs of Bounded Pathwidth

Assume we are given a nice path decomposition $P = (X_1, X_2, \ldots, X_r)$ of width k.

Let G_i be the graph induced by $X_1 \cup X_2 \cup \cdots \cup X_i$ for some $i \in \{1, 2, \ldots, r\}$.

For each $S \subseteq X_i$ let

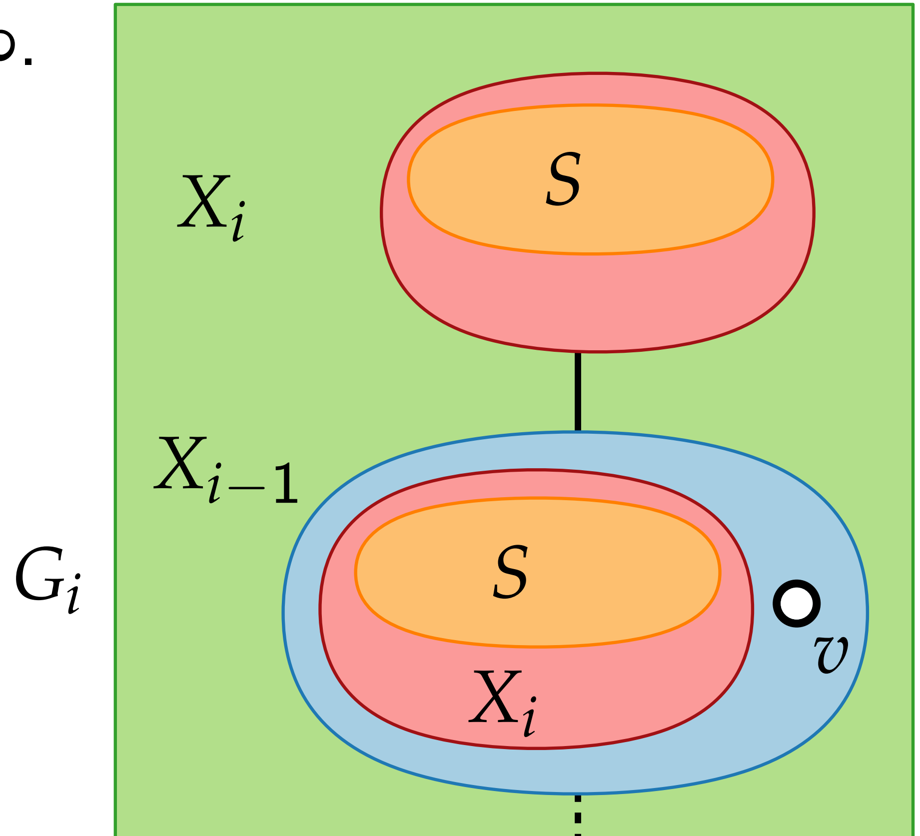
$D[i, S] :=$ maximum weight of an independent set I in G_i such that $I \cap X_i = S$.

Assume that $i > 1$. If S is not independent, $D[i, S] = -\infty$.

Otherwise, we distinguish between the two types of X_i.

If X_i is **Forget**, then

$D[i, S] =$



INDEPENDENT SET in Graphs of Bounded Pathwidth

Assume we are given a nice path decomposition $P = (X_1, X_2, \ldots, X_r)$ of width k.

Let G_i be the graph induced by $X_1 \cup X_2 \cup \cdots \cup X_i$ for some $i \in \{1, 2, \ldots, r\}$.

For each $S \subseteq X_i$ let

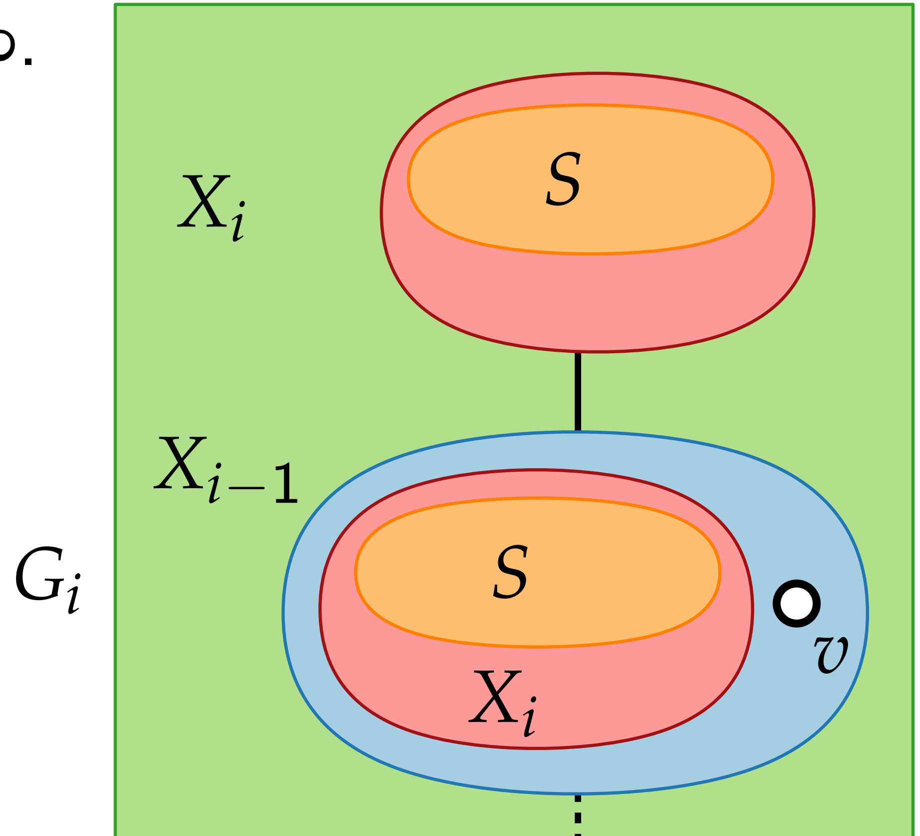
$D[i, S]$:= maximum weight of an independent set I in G_i such that $I \cap X_i = S$.

Assume that $i > 1$. If S is not independent, $D[i, S] = -\infty$.

Otherwise, we distinguish between the two types of X_i.

If X_i is **Forget**, then

$$D[i, S] = \max\{ \qquad , \qquad \}$$



INDEPENDENT SET in Graphs of Bounded Pathwidth

Assume we are given a nice path decomposition $P = (X_1, X_2, \ldots, X_r)$ of width k.

Let G_i be the graph induced by $X_1 \cup X_2 \cup \cdots \cup X_i$ for some $i \in \{1, 2, \ldots, r\}$.

For each $S \subseteq X_i$ let

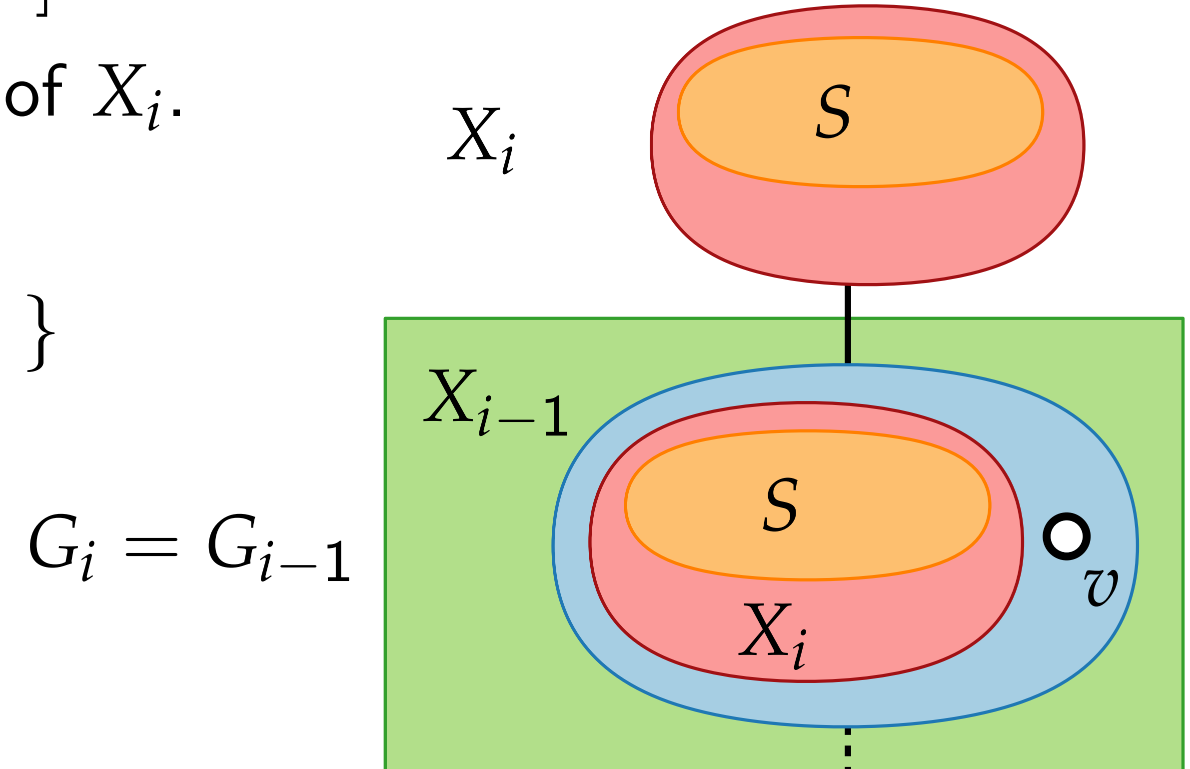
$D[i, S]$:= maximum weight of an independent set I in G_i such that $I \cap X_i = S$.

Assume that $i > 1$. If S is not independent, $D[i, S] = -\infty$.

Otherwise, we distinguish between the two types of X_i.

If X_i is **Forget**, then

$D[i, S] = \max\{ \quad , \quad \}$



INDEPENDENT SET in Graphs of Bounded Pathwidth

Assume we are given a nice path decomposition $P = (X_1, X_2, \dots, X_r)$ of width k.

Let G_i be the graph induced by $X_1 \cup X_2 \cup \dots \cup X_i$ for some $i \in \{1, 2, \dots, r\}$.

For each $S \subseteq X_i$ let

$D[i, S] :=$ maximum weight of an independent set I in G_i such that $I \cap X_i = S$.

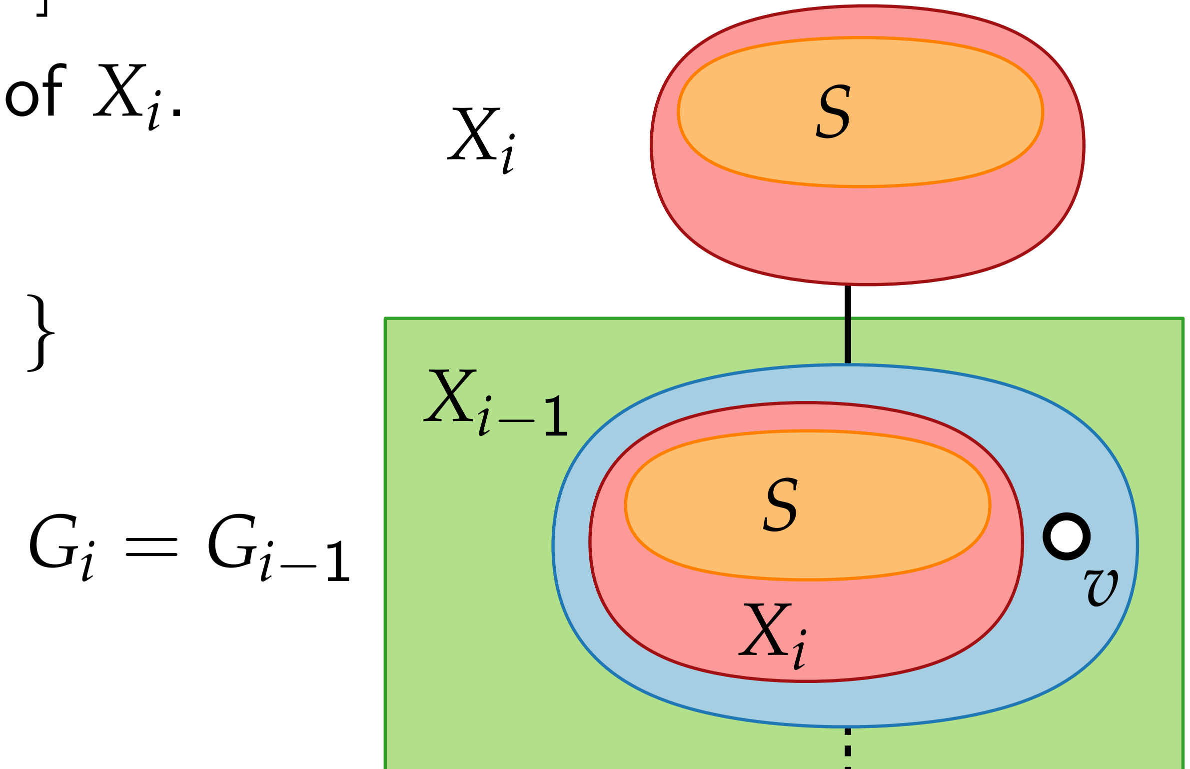
Assume that $i > 1$. If S is not independent, $D[i, S] = -\infty$.

Otherwise, we distinguish between the two types of X_i.

If X_i is **Forget**, then

$$D[i, S] = \max\{ \qquad , \qquad \}$$

$v \notin I \Rightarrow I \cap X_{i-1} = S$



Independent Set in Graphs of Bounded Pathwidth

Assume we are given a nice path decomposition $P = (X_1, X_2, \ldots, X_r)$ of width k.

Let G_i be the graph induced by $X_1 \cup X_2 \cup \cdots \cup X_i$ for some $i \in \{1, 2, \ldots, r\}$.

For each $S \subseteq X_i$ let

$D[i, S]$:= maximum weight of an independent set I in G_i such that $I \cap X_i = S$.

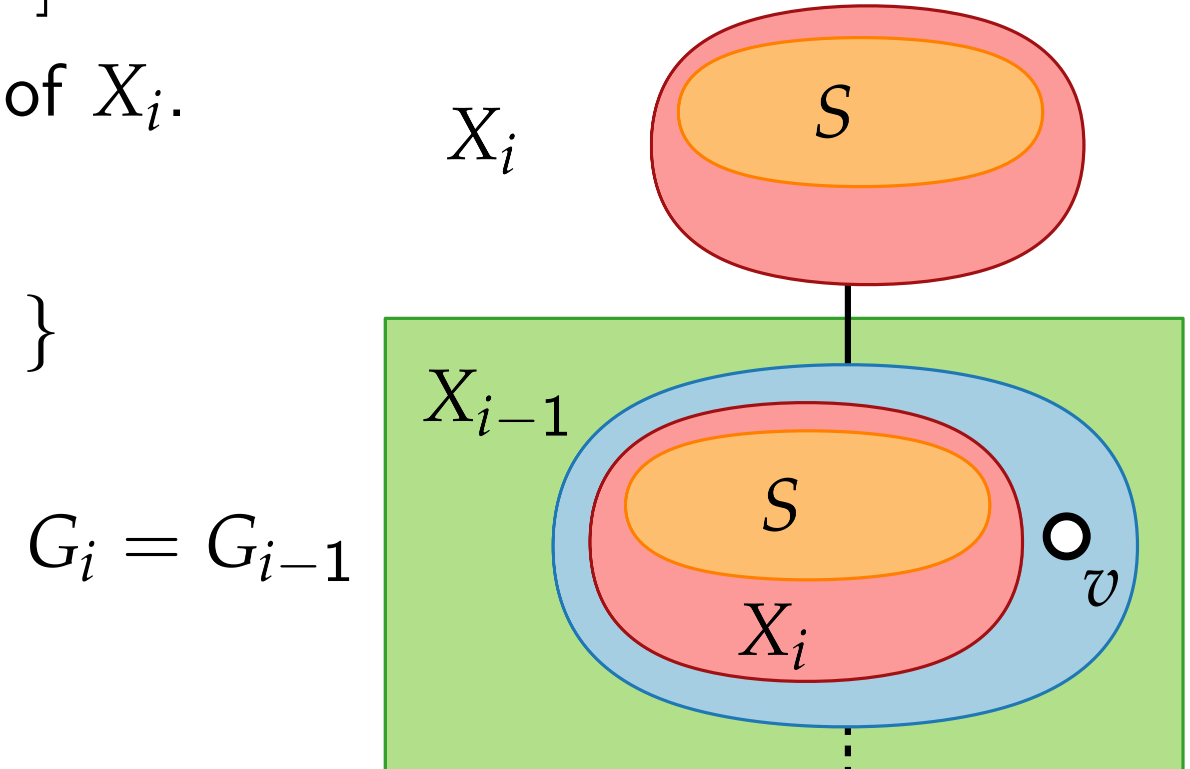
Assume that $i > 1$. If S is not independent, $D[i, S] = -\infty$.

Otherwise, we distinguish between the two types of X_i.

If X_i is **Forget**, then

$$D[i, S] = \max\{\ D[i-1, S]\ ,\ \qquad\qquad \}$$

$v \notin I \Rightarrow I \cap X_{i-1} = S$



INDEPENDENT SET in Graphs of Bounded Pathwidth

Assume we are given a nice path decomposition $P = (X_1, X_2, \ldots, X_r)$ of width k.

Let G_i be the graph induced by $X_1 \cup X_2 \cup \cdots \cup X_i$ for some $i \in \{1, 2, \ldots, r\}$.

For each $S \subseteq X_i$ let

$D[i, S] :=$ maximum weight of an independent set I in G_i such that $I \cap X_i = S$.

Assume that $i > 1$. If S is not independent, $D[i, S] = -\infty$.

Otherwise, we distinguish between the two types of X_i.

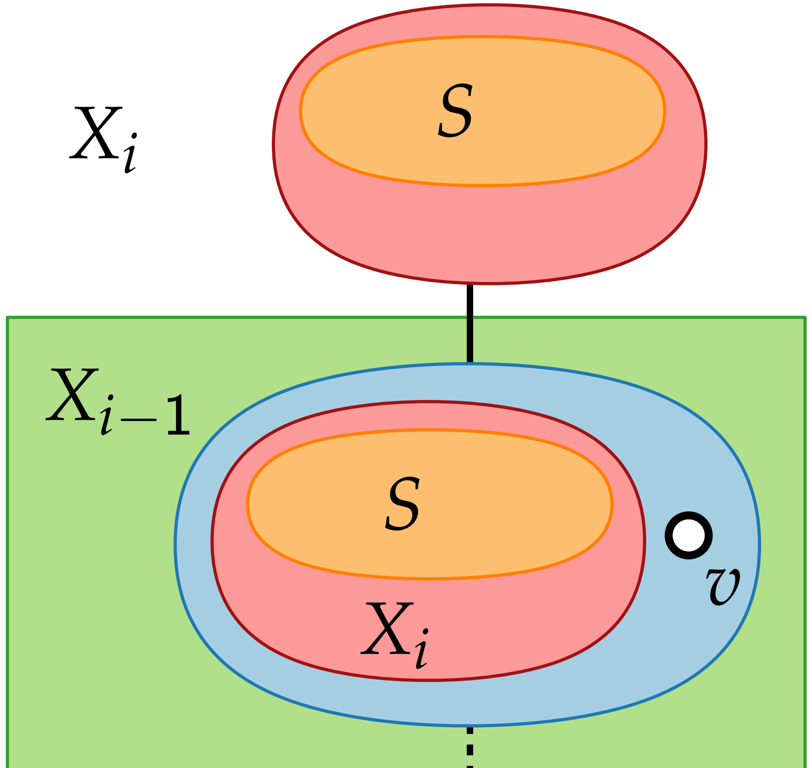
If X_i is **Forget**, then

$$D[i, S] = \max\{\ D[i-1, S]\ ,\ \qquad\qquad\qquad \}$$

$v \notin I \Rightarrow I \cap X_{i-1} = S$

$v \in I \Rightarrow I \cap X_{i-1} = S \cup \{v\}$

$G_i = G_{i-1}$



Independent Set in Graphs of Bounded Pathwidth

Assume we are given a nice path decomposition $P = (X_1, X_2, \ldots, X_r)$ of width k.

Let G_i be the graph induced by $X_1 \cup X_2 \cup \cdots \cup X_i$ for some $i \in \{1, 2, \ldots, r\}$.

For each $S \subseteq X_i$ let

$D[i, S] :=$ maximum weight of an independent set I in G_i such that $I \cap X_i = S$.

Assume that $i > 1$. If S is not independent, $D[i, S] = -\infty$.

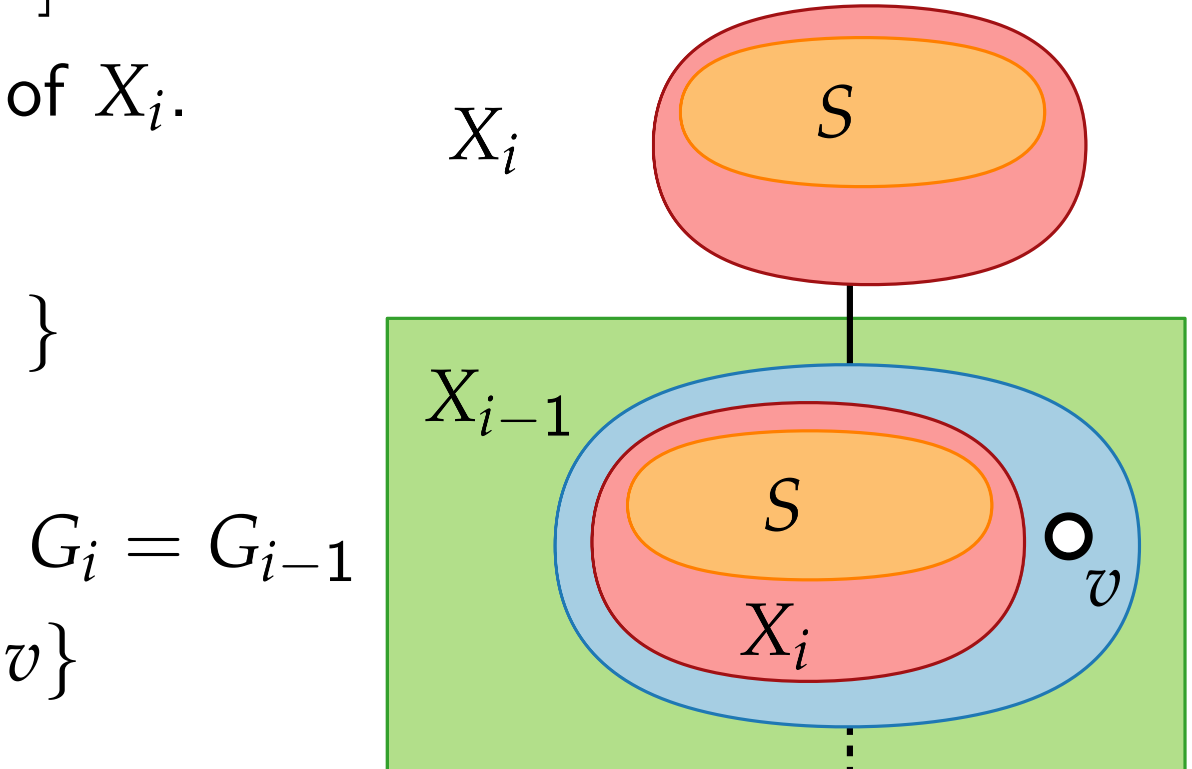
Otherwise, we distinguish between the two types of X_i.

If X_i is **Forget**, then

$$D[i, S] = \max\{ D[i-1, S] , D[i-1, S \cup \{v\}] \}$$

$v \notin I \Rightarrow I \cap X_{i-1} = S$

$v \in I \Rightarrow I \cap X_{i-1} = S \cup \{v\}$



INDEPENDENT SET in Graphs of Bounded Pathwidth

Assume we are given a nice path decomposition $P = (X_1, X_2, \ldots, X_r)$ of width k.

Let G_i be the graph induced by $X_1 \cup X_2 \cup \cdots \cup X_i$ for some $i \in \{1, 2, \ldots, r\}$.

For each $S \subseteq X_i$ let

$D[i, S] :=$ maximum weight of an independent set I in G_i such that $I \cap X_i = S$.

Assume that $i > 1$. If S is not independent, $D[i, S] = -\infty$.

Otherwise, we distinguish between the two types of X_i.

INDEPENDENT SET in Graphs of Bounded Pathwidth

Assume we are given a nice path decomposition $P = (X_1, X_2, \ldots, X_r)$ of width k.

Let G_i be the graph induced by $X_1 \cup X_2 \cup \cdots \cup X_i$ for some $i \in \{1, 2, \ldots, r\}$.

For each $S \subseteq X_i$ let

$D[i, S] :=$ maximum weight of an independent set I in G_i such that $I \cap X_i = S$.

Assume that $i > 1$. If S is not independent, $D[i, S] = -\infty$.

Otherwise, we distinguish between the two types of X_i.

For each of the $\leq 2|V| - 1$ many bags, there are $\leq 2^{k+1}$ choices for S.

Independent Set in Graphs of Bounded Pathwidth

Assume we are given a nice path decomposition $P = (X_1, X_2, \ldots, X_r)$ of width k.

Let G_i be the graph induced by $X_1 \cup X_2 \cup \cdots \cup X_i$ for some $i \in \{1, 2, \ldots, r\}$.

For each $S \subseteq X_i$ let

$D[i, S] :=$ maximum weight of an independent set I in G_i such that $I \cap X_i = S$.

Assume that $i > 1$. If S is not independent, $D[i, S] = -\infty$.

Otherwise, we distinguish between the two types of X_i.

For each of the $\leq 2|V| - 1$ many bags, there are $\leq 2^{k+1}$ choices for S.

For each of these choices, we need to test if S is independent, which can be done in $k^{\mathcal{O}(1)}$ time ($\rightarrow$ Section 7.3.1 in [1]).

INDEPENDENT SET in Graphs of Bounded Pathwidth

Assume we are given a nice path decomposition $P = (X_1, X_2, \ldots, X_r)$ of width k.

Let G_i be the graph induced by $X_1 \cup X_2 \cup \cdots \cup X_i$ for some $i \in \{1, 2, \ldots, r\}$.

For each $S \subseteq X_i$ let

$D[i, S]$:= maximum weight of an independent set I in G_i such that $I \cap X_i = S$.

Assume that $i > 1$. If S is not independent, $D[i, S] = -\infty$.

Otherwise, we distinguish between the two types of X_i.

For each of the $\leq 2|V| - 1$ many bags, there are $\leq 2^{k+1}$ choices for S.

For each of these choices, we need to test if S is independent, which can be done in $k^{\mathcal{O}(1)}$ time ($\rightarrow$ Section 7.3.1 in [1]).

$\Rightarrow$ total running time $\leq 2^{k+2} k^{\mathcal{O}(1)} |V|$

INDEPENDENT SET in Graphs of Bounded Pathwidth

Assume we are given a nice path decomposition $P = (X_1, X_2, \ldots, X_r)$ of width k.

Let G_i be the graph induced by $X_1 \cup X_2 \cup \cdots \cup X_i$ for some $i \in \{1, 2, \ldots, r\}$.

For each $S \subseteq X_i$ let

$D[i, S] :=$ maximum weight of an independent set I in G_i such that $I \cap X_i = S$.

Assume that $i > 1$. If S is not independent, $D[i, S] = -\infty$.

Otherwise, we distinguish between the two types of X_i.

For each of the $\leq 2|V| - 1$ many bags, there are $\leq 2^{k+1}$ choices for S.

For each of these choices, we need to test if S is independent, which can be done in $k^{\mathcal{O}(1)}$ time ($\rightarrow$ Section 7.3.1 in [1]).

$\Rightarrow$ total running time $\leq 2^{k+2} k^{\mathcal{O}(1)} |V|$

Theorem. INDEPENDENT SET is FPT with respect to the pathwidth.

Discussion

- The fixed-parameter tractability of a problem may be studied with respect to various structural parameters.
- The assumption that the chosen parameter is small should be plausible!

Discussion

- The fixed-parameter tractability of a problem may be studied with respect to various structural parameters.
- The assumption that the chosen parameter is small should be plausible!
- **Treewidth** is among the most studied parameters.
  - It is defined like pathwidth, except that the bags form a tree instead of a path.
  - **Nice** tree decomposition only have one additional bag type ...
  - ... and can be constructed efficiently from a tree decomposition.

Discussion

- The fixed-parameter tractability of a problem may be studied with respect to various structural parameters.
- The assumption that the chosen parameter is small should be plausible!
- **Treewidth** is among the most studied parameters.
  - It is defined like pathwidth, except that the bags form a tree instead of a path.
  - **Nice** tree decomposition only have one additional bag type ...
  - ... and can be constructed efficiently from a tree decomposition.
- Our $\leq 2^{\mathrm{pw}(G)} \mathrm{pw}(G)^{\mathcal{O}(1)} |V|$-time algorithm for Independent Set can easily be turned into an algorithm with running time $\leq 2^{\mathrm{tw}(G)} \mathrm{tw}(G)^{\mathcal{O}(1)} |V|$.

Theorem. Independent Set is FPT with respect to the treewidth.

References and Literature

[1] Parameterized Algorithms,
M. Cygan, F. Fomin, Ł. Kowalik, D. Lokshtanov, D. Marx, M. Pilipczuk,
M. Pilipczuk, S. Saurabh, Springer International Publishing 2015.

Sections 1, 7.1, 7.2, 7.3