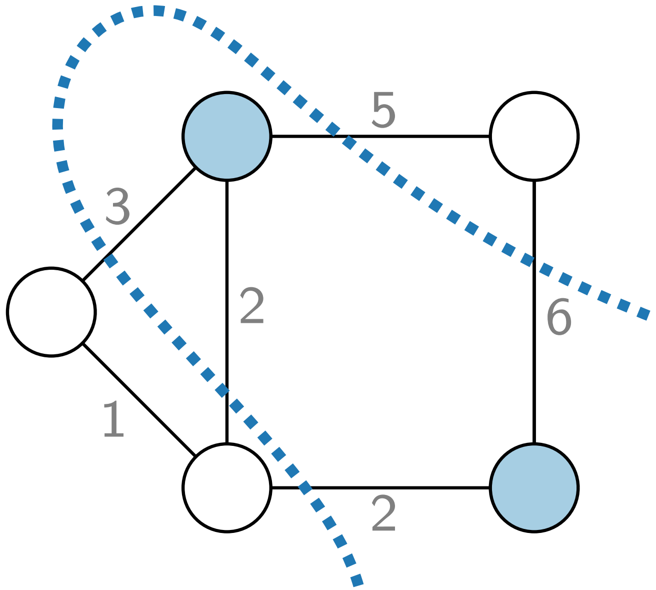
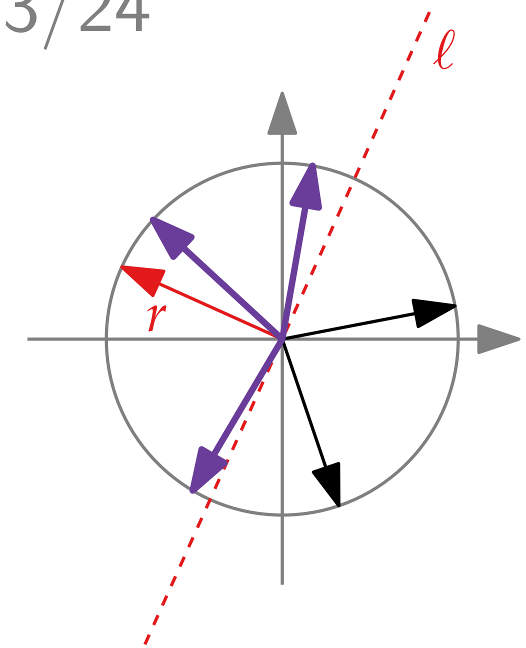
Julius-Maximilians-UNIVERSITÄT WÜRZBURG

Advanced Algorithms

QP-Relaxation
for MaxCut

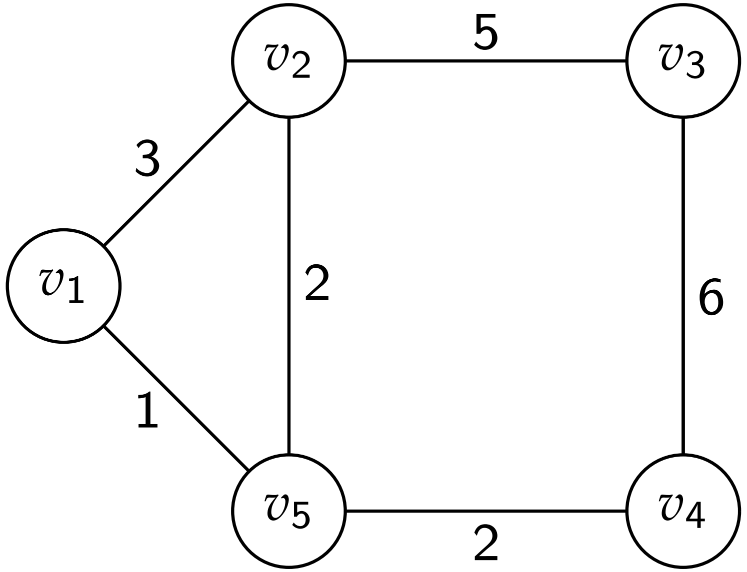
Johannes Zink · WS23/24





Cut

- Let $G = (V, E)$ be a graph with edge weights $w \colon E \to \mathbb{N}$.



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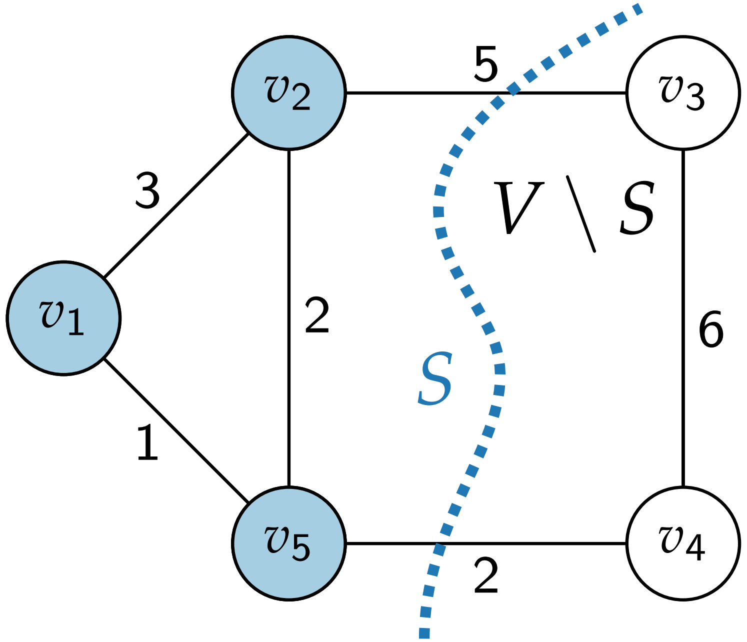

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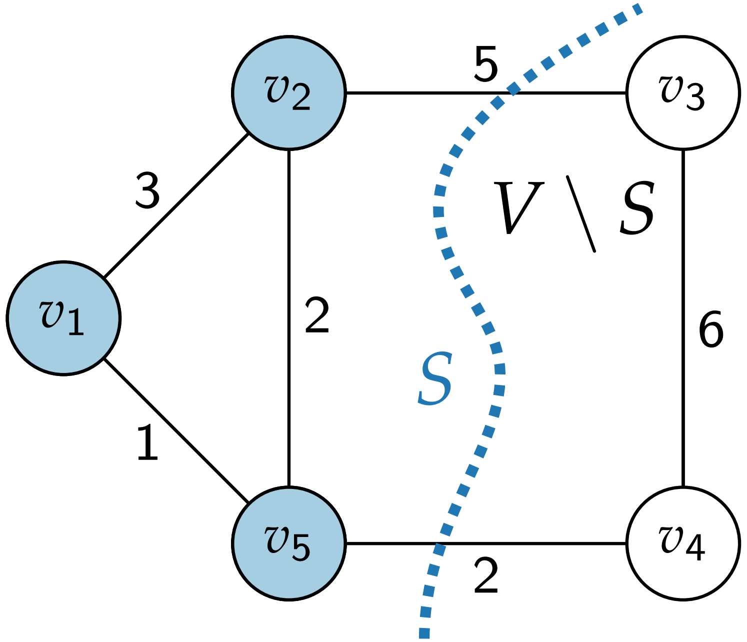
$$w(S, V \setminus S) = \sum_{\substack{uv \in E,\\ u \in S, v \in V \setminus S}} w(uv)$$



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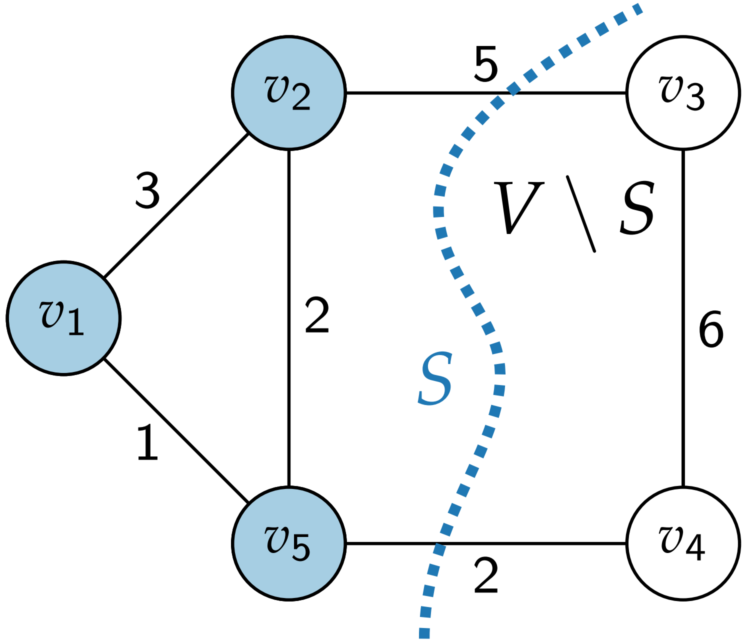

$$w(\{v_1, v_2, v_5\}, \{v_3, v_4\})$$
$$=$$

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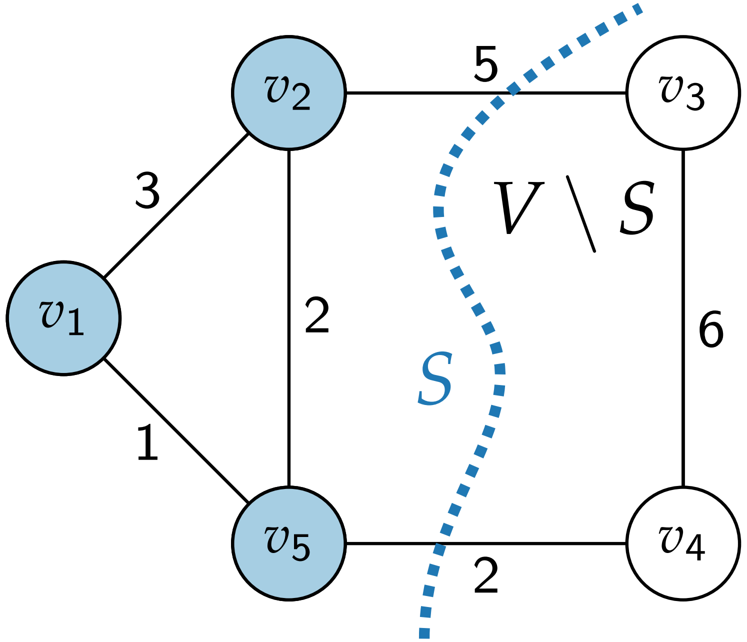

$$w(\{v_1, v_2, v_5\}, \{v_3, v_4\}) \\ = w(v_2 v_3) + w(v_4 v_5)$$

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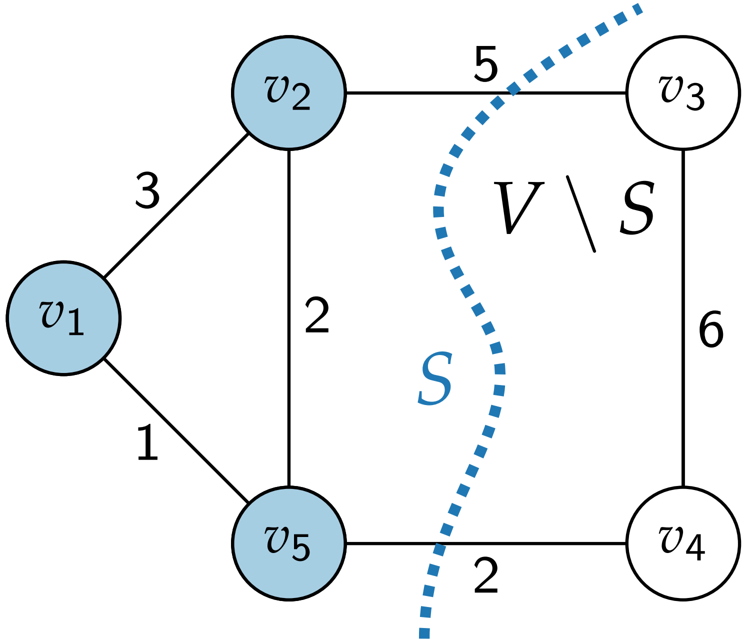
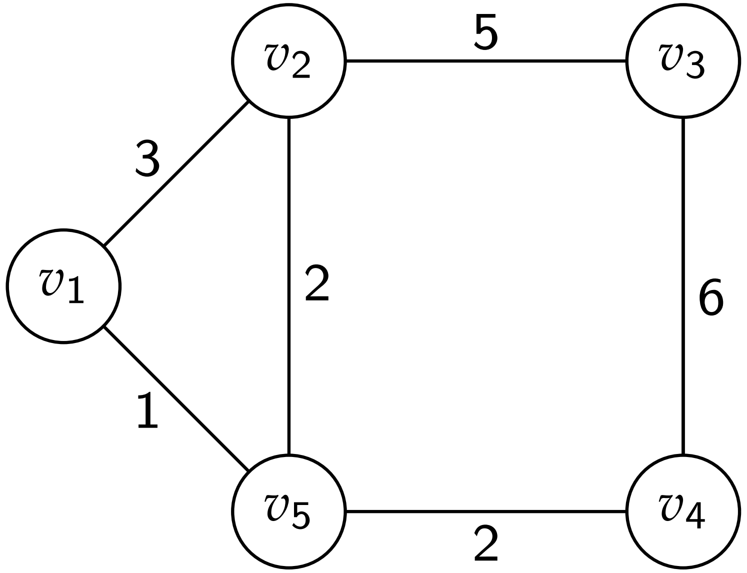


$$w(\{v_1, v_2, v_5\}, \{v_3, v_4\})$$
$$= w(v_2v_3) + w(v_4v_5) = 7$$

The **MinCut** Problem

Input. Graph $G = (V, E)$, edge weights $w\colon E \to \mathbb{N}$.

Output. Cut $(S, V \setminus S)$ of G with **minimum** weight.



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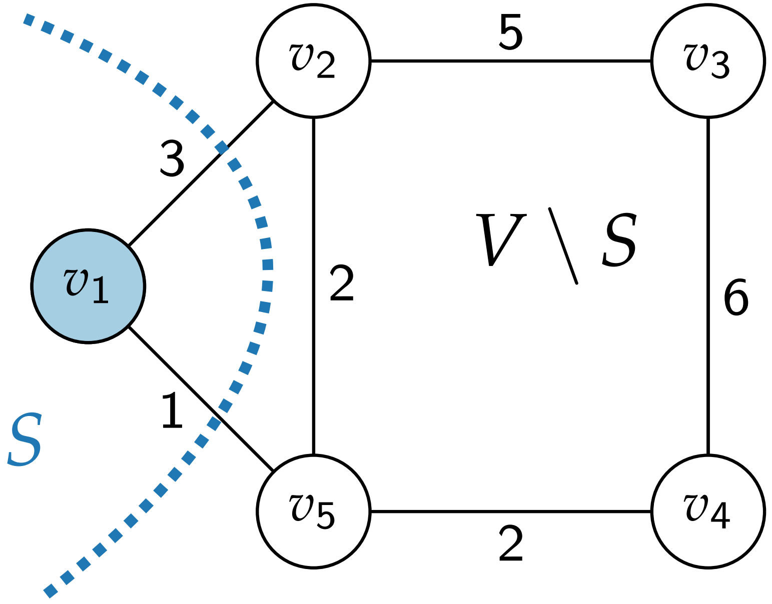

$$w(S, V \setminus S) = 4$$

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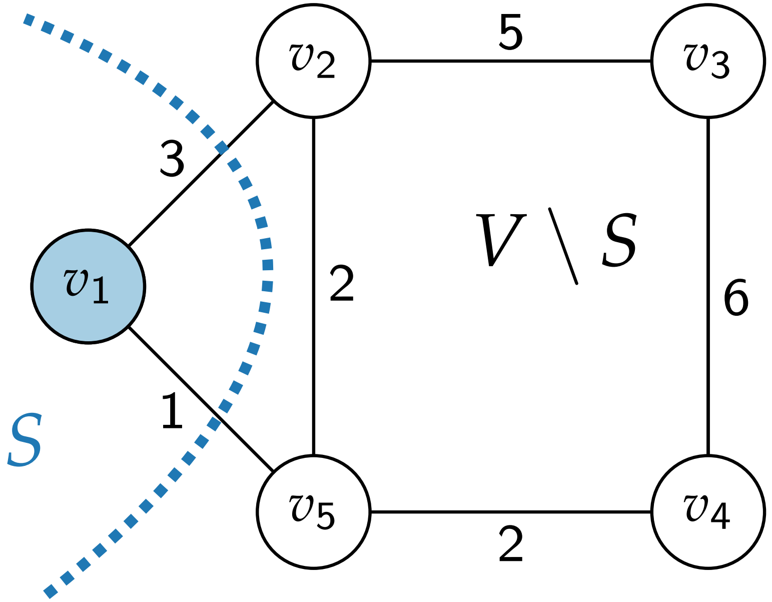

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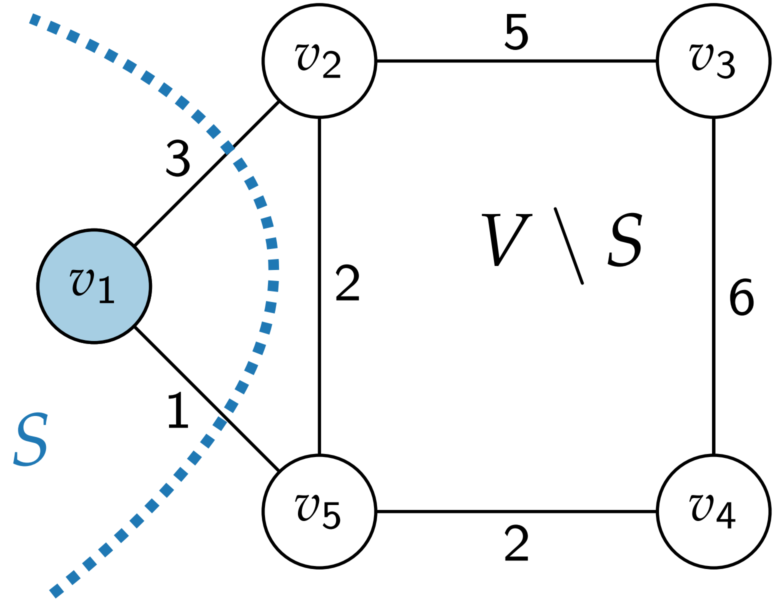
- Can be solved optimally in polynomial time, e.g., by the Stoer–Wagner algorithm.



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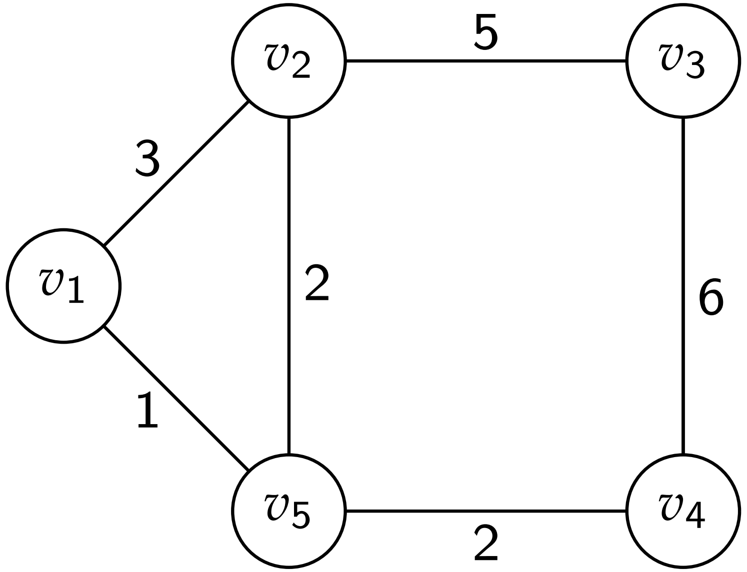

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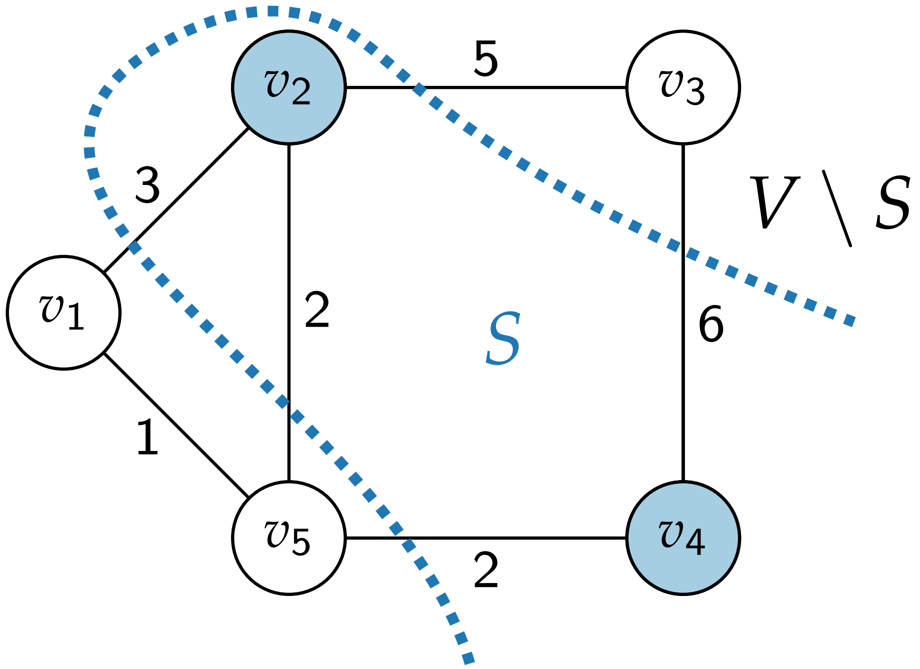

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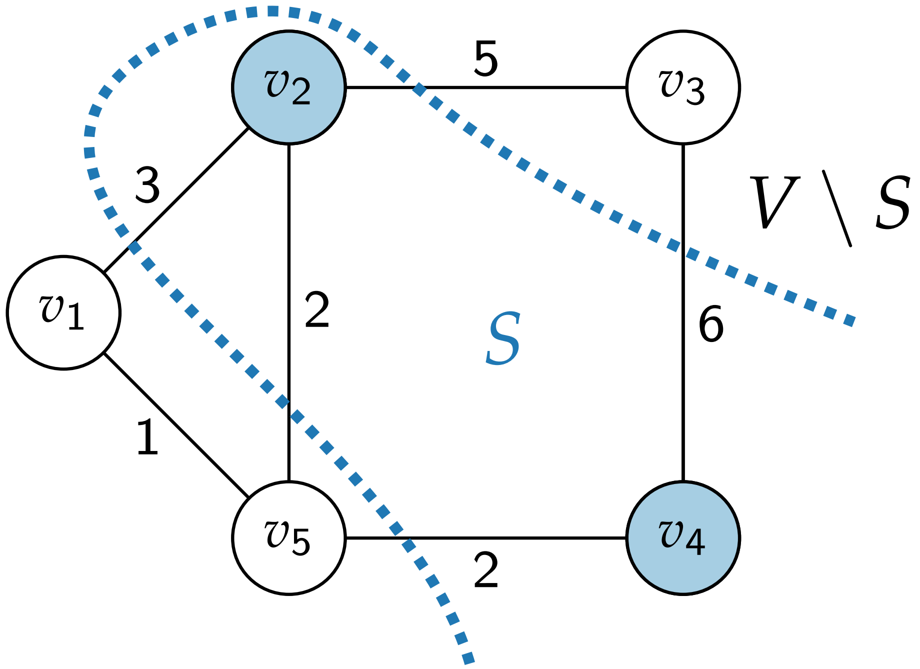

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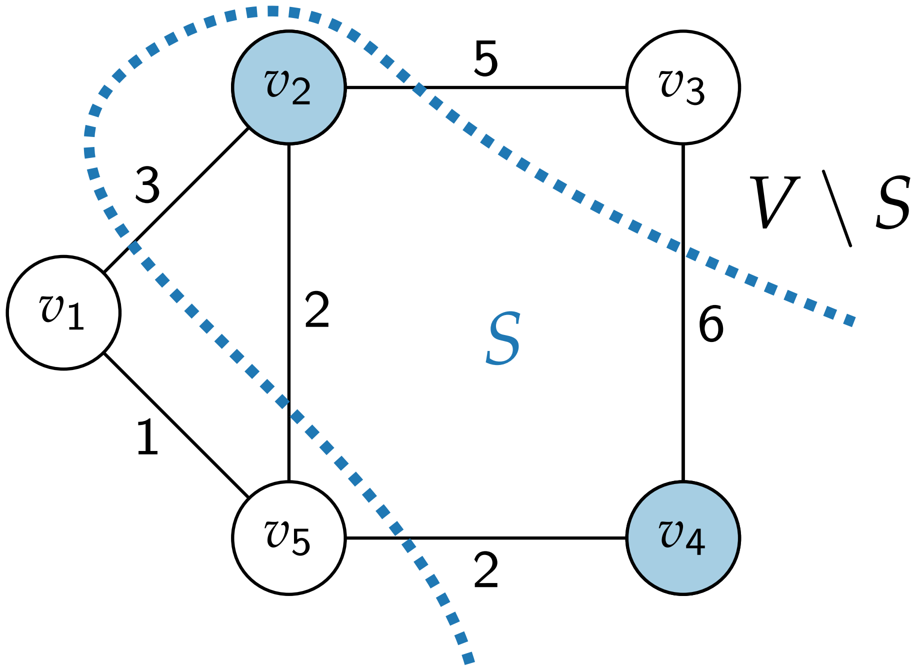
- NP-complete to find a cut of maximum weight.



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Randomized 0.5-Approximation for (Unweighted) MaxCut

$\textsc{CoinFlipMaxCut}(G, w\colon E \to 1)$

$S \leftarrow \emptyset$

foreach $v \in V$ **do**

  if coin flip shows $\textsc{Heads}$ **then**

    $S \leftarrow S \cup \{v\}$

return $w(S, V \setminus S), S$

Randomized 0.5-Approximation for (Unweighted) MaxCut

Theorem 1.
CoinFlipMaxCut is a randomized 0.5-approximation algorithm for MaxCut.

```
CoinFlipMaxCut(G, w: E → 1)
  S ← ∅
  foreach v ∈ V do
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Proof.

- Runs in $O(n+m)$, where $n = |V|, m = |E|$.

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$$\mathrm{E}[w(\textsc{CoinFlipMaxCut}(G))]$$

$$\geq \frac{1}{2}\mathrm{OPT}(G)$$

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- Can be "de-randomized". Exercise.

LP-Relaxation

Integer Linear Program

$$\begin{array}{llll} \textbf{maximize} & c^{\mathsf{T}} x & & \\ \textbf{subject to} & Ax & \leq & b \\ & x & \geq & 0 \\ & x & \in & \mathbb{Z}^n \end{array}$$

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↓ Solve in polynomial time

Solution for LP

$$x^\star$$

LP-Relaxation

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$$\begin{aligned} \textbf{maximize} \quad & c^\top x \\ \textbf{subject to} \quad & Ax \le b \\ & x \ge 0 \\ & x \in \mathbb{Z}^n \end{aligned}$$

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Solve in polynomial time ↓

Solution for LP

$x^\star$

← e.g. rounding

Assignment for ILP

$x^\star$

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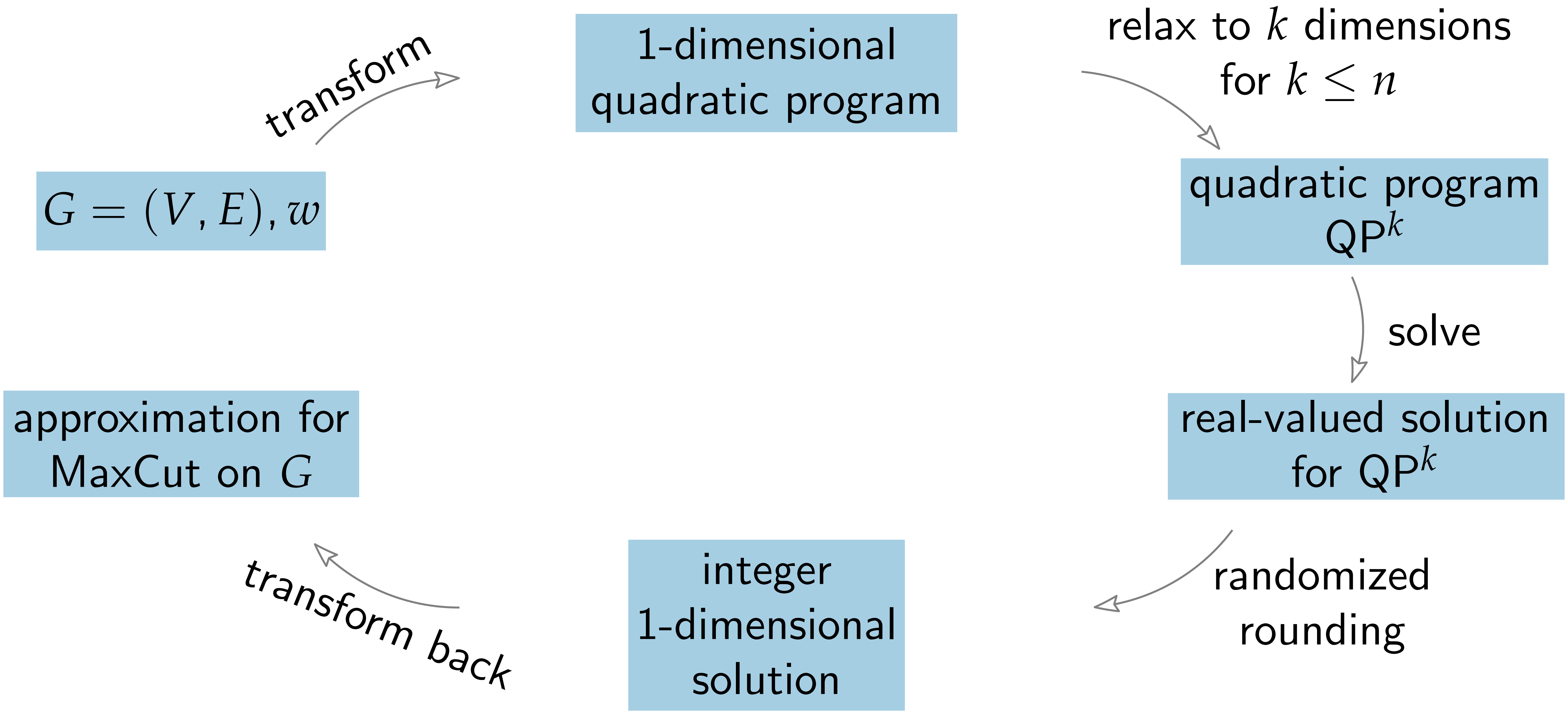
↑ Solution, approximation, or bound

Goemans-Williamson Algorithm for MaxCut



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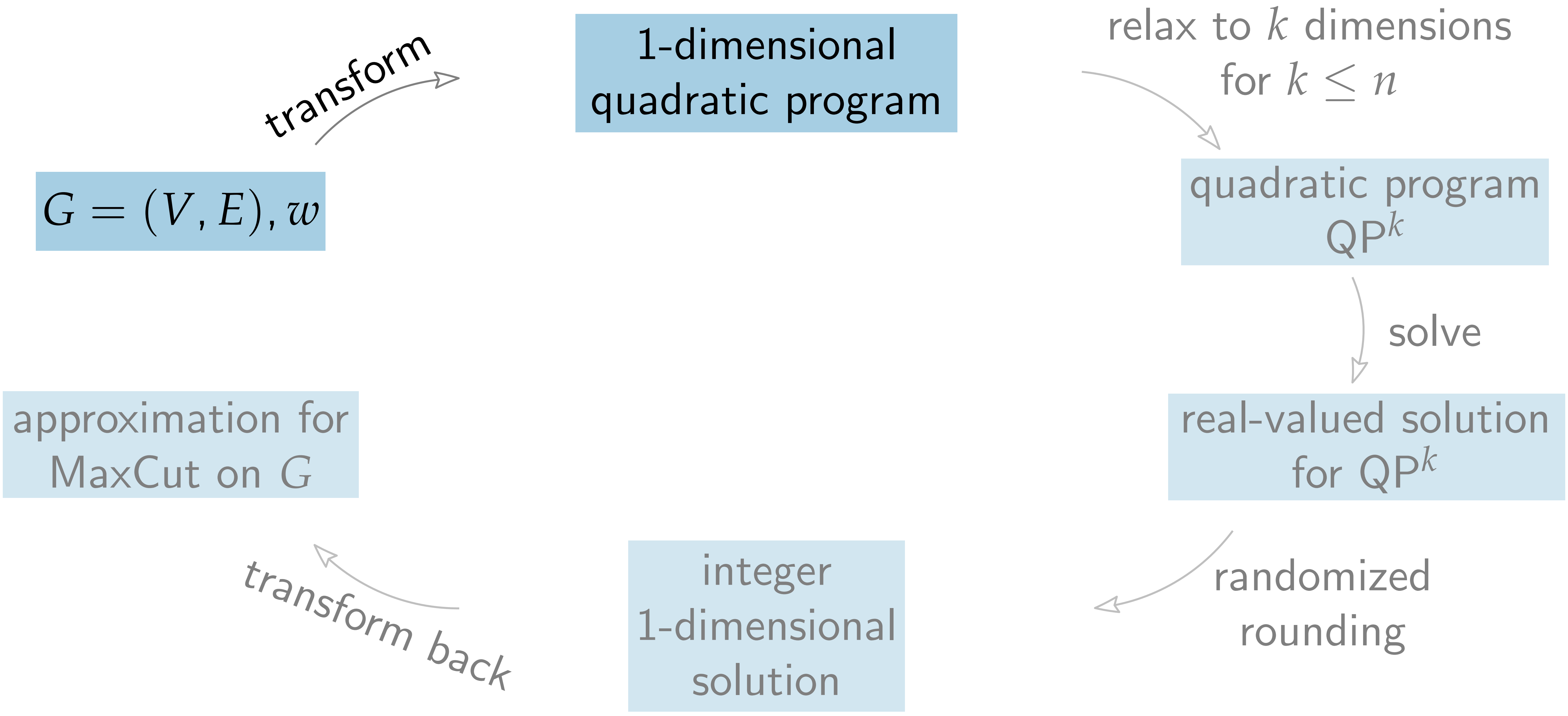

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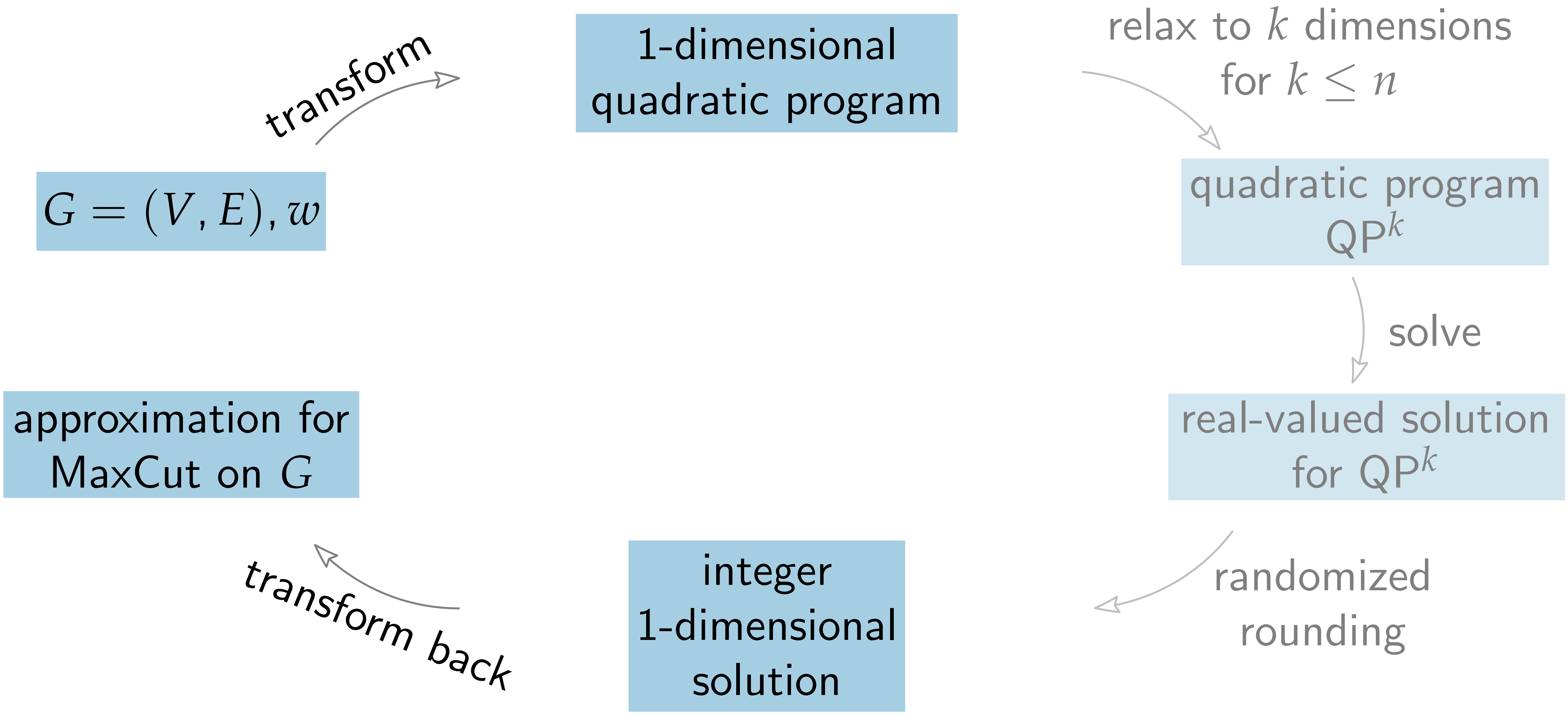
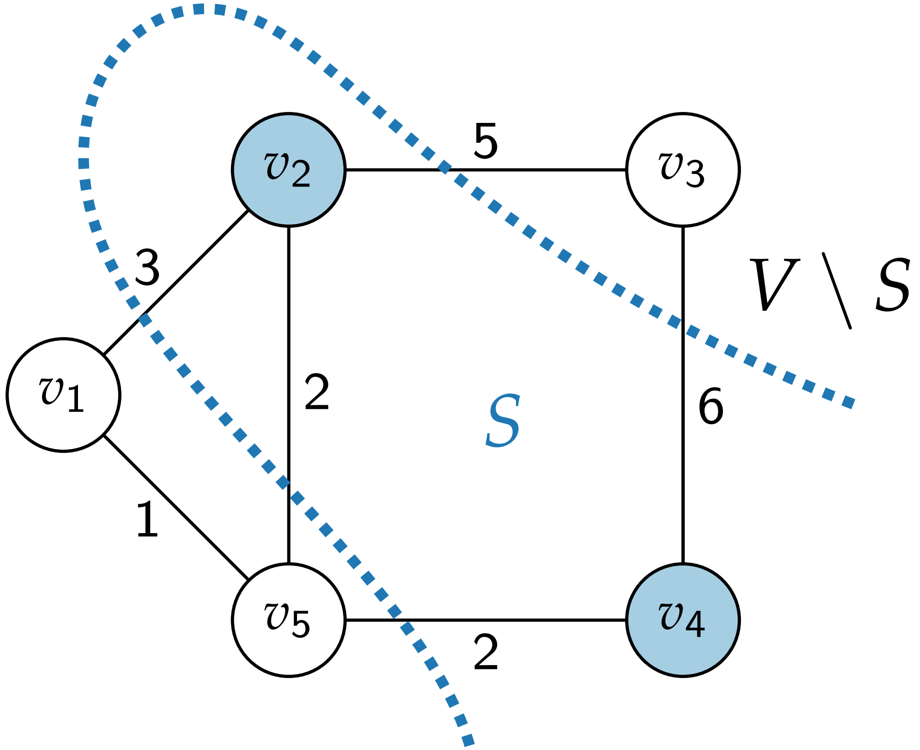


QP(G, w)

Idea.



QP(G, w)

maximize

subject to

$\text{QP}(G, w)$

Idea.

- Indicator variable for each vertex v_i: $x_i \in \{1, -1\}$

QP (G, w)

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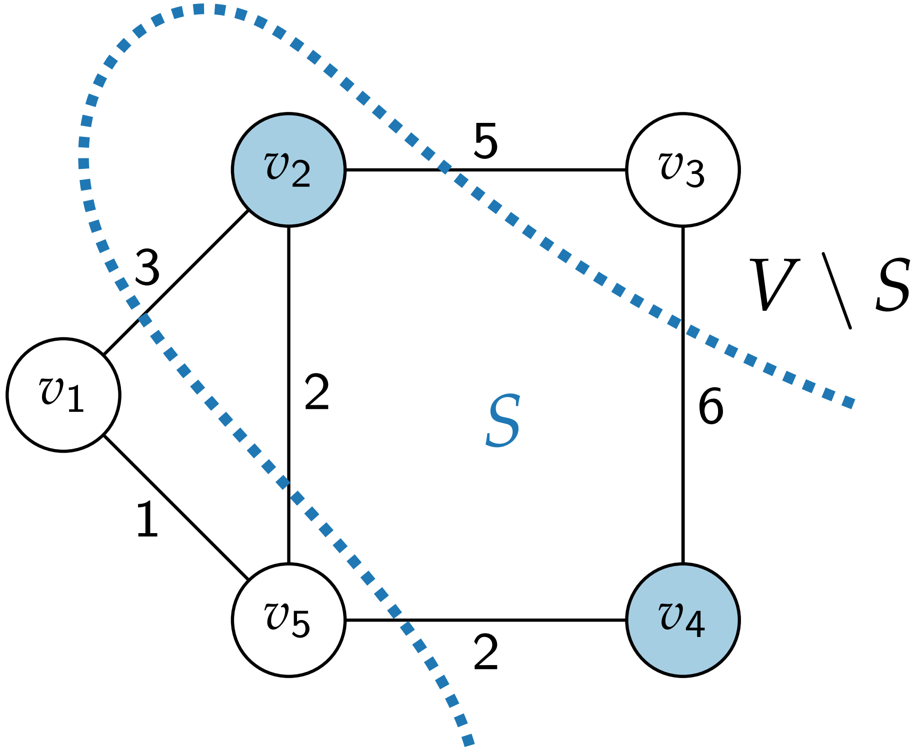

QP(G, w)

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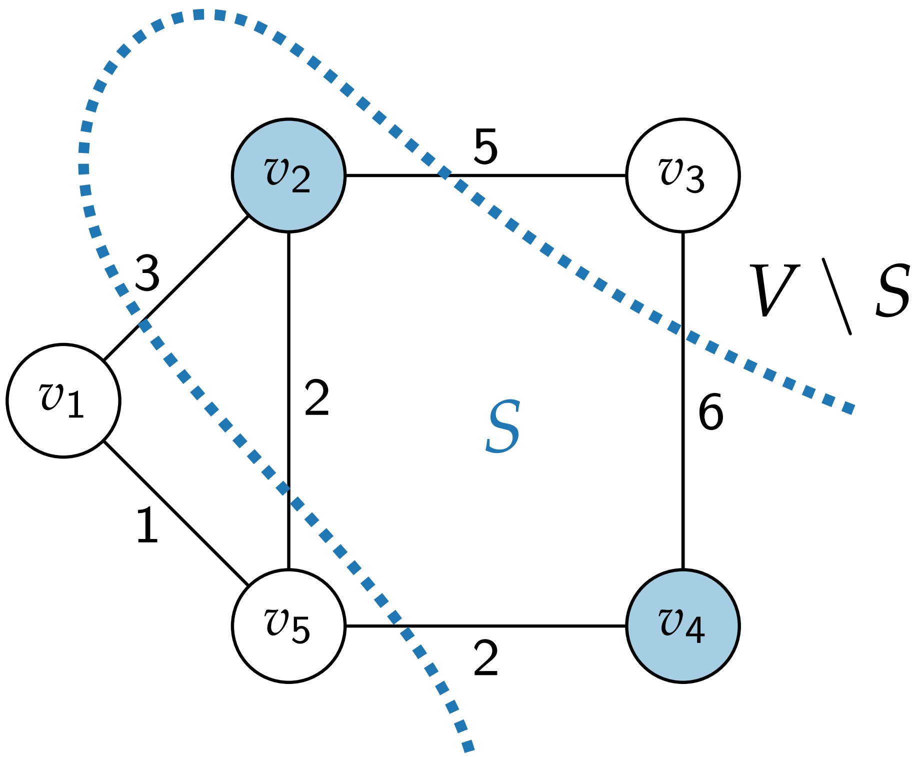
- Indicator variable for each vertex v_i: $x_i \in \{1, -1\}$

QP(G, w)

maximize

subject to $x_i^2 = 1$



QP(G, w)

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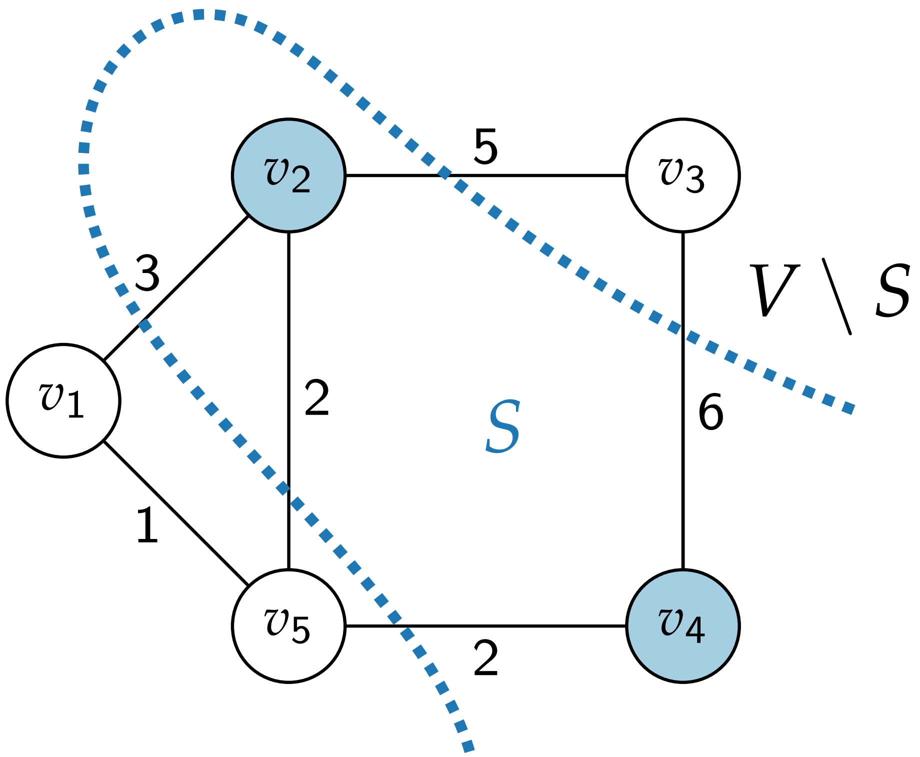
- Indicator variable for each vertex v_i: $x_i \in \{1, -1\}$
- $x_i \cdot x_j = \begin{cases} 1 & \text{if } i, j \text{ in same partition} \\ -1 & \text{otherwise} \end{cases}$



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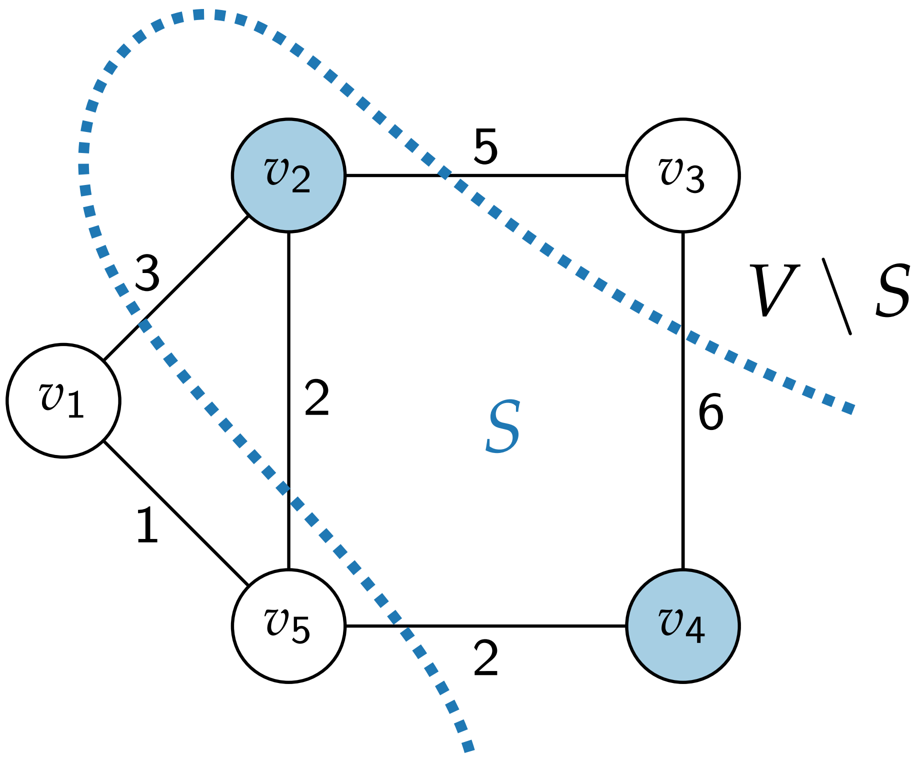

QP(G, w)

maximize $(1 - x_i x_j)$

subject to $x_i^2 = 1$

QP(G, w)

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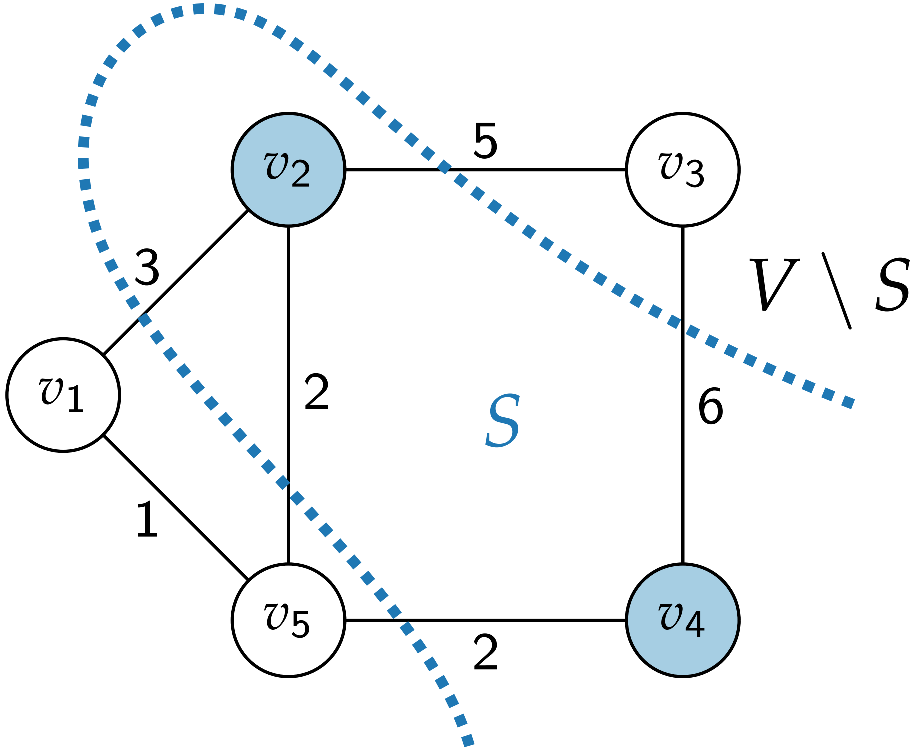
QP(G, w)

maximize $w_{ij}(1 - x_i x_j)$

subject to $x_i^2 = 1$



- Weight matrix w_{ij}

	1	2	3	4	5
1		3			1
2	3		5		2
3		5		6	
4			6		2
5	1	2		2	

QP(G, w)

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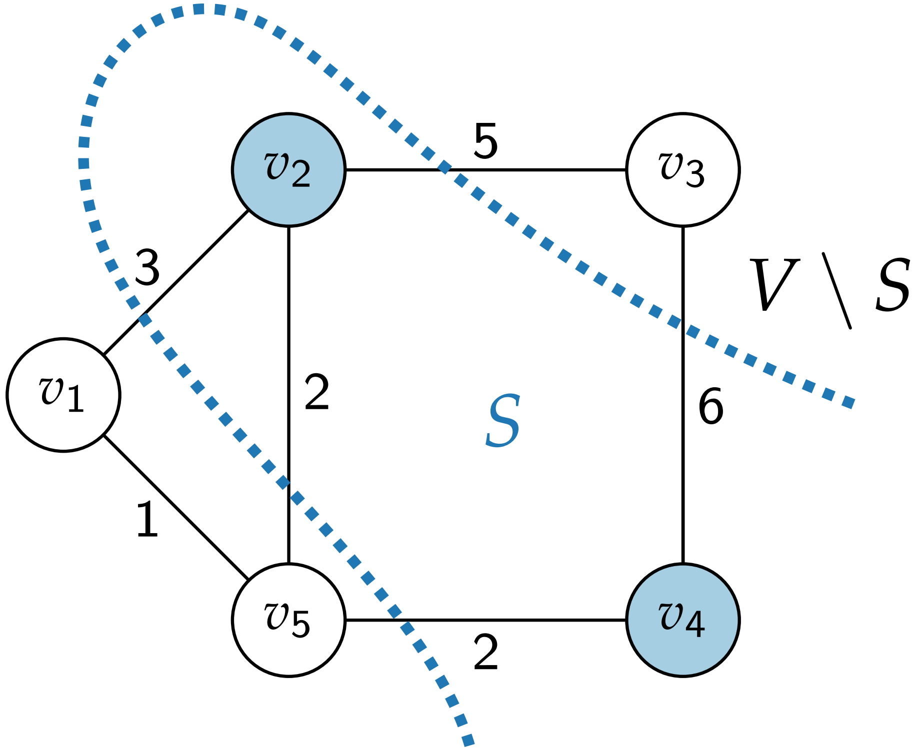
QP(G, w)

maximize $\frac{1}{2} \sum_{j=1}^{n} \sum_{i=1}^{j-1} w_{ij}(1 - x_i x_j)$

subject to $x_i^2 = 1$



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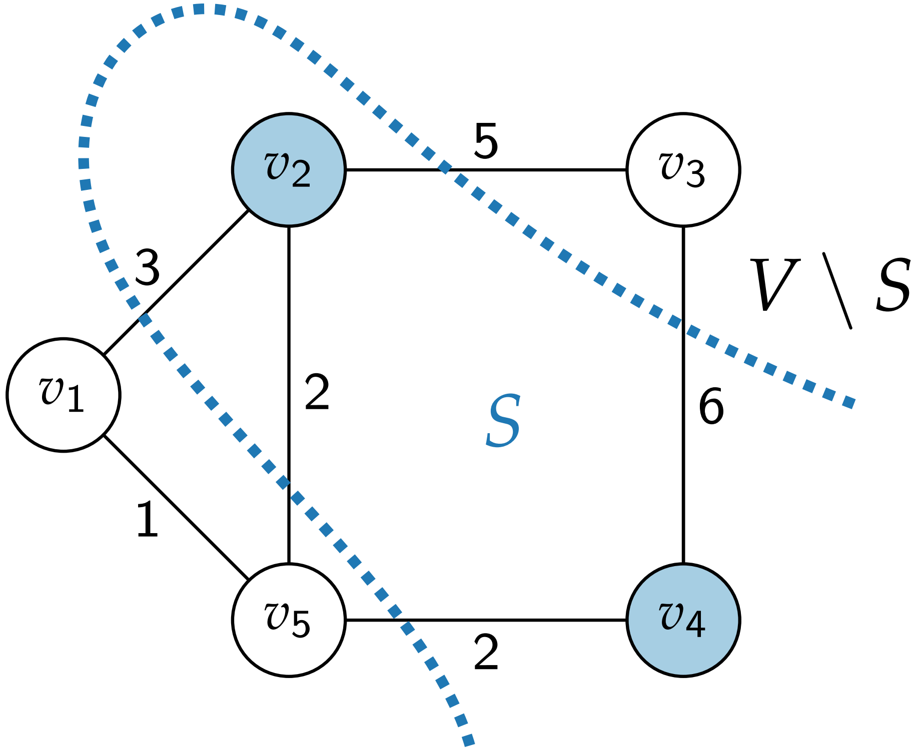

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- Solution

$x_2 = x_4 = 1$

$x_1 = x_3 = x_5 = -1$

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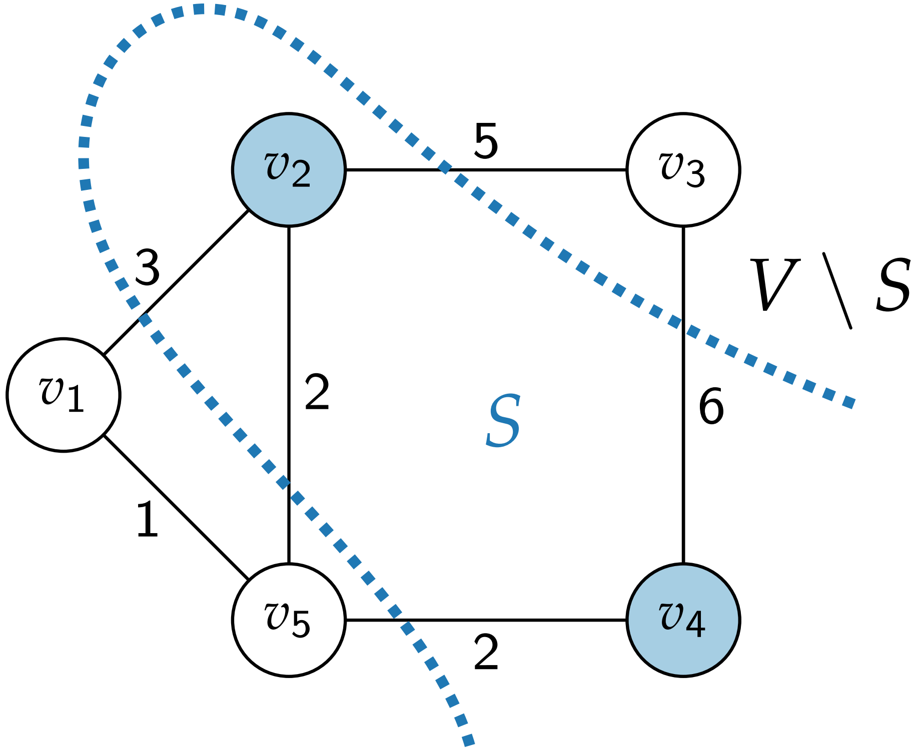

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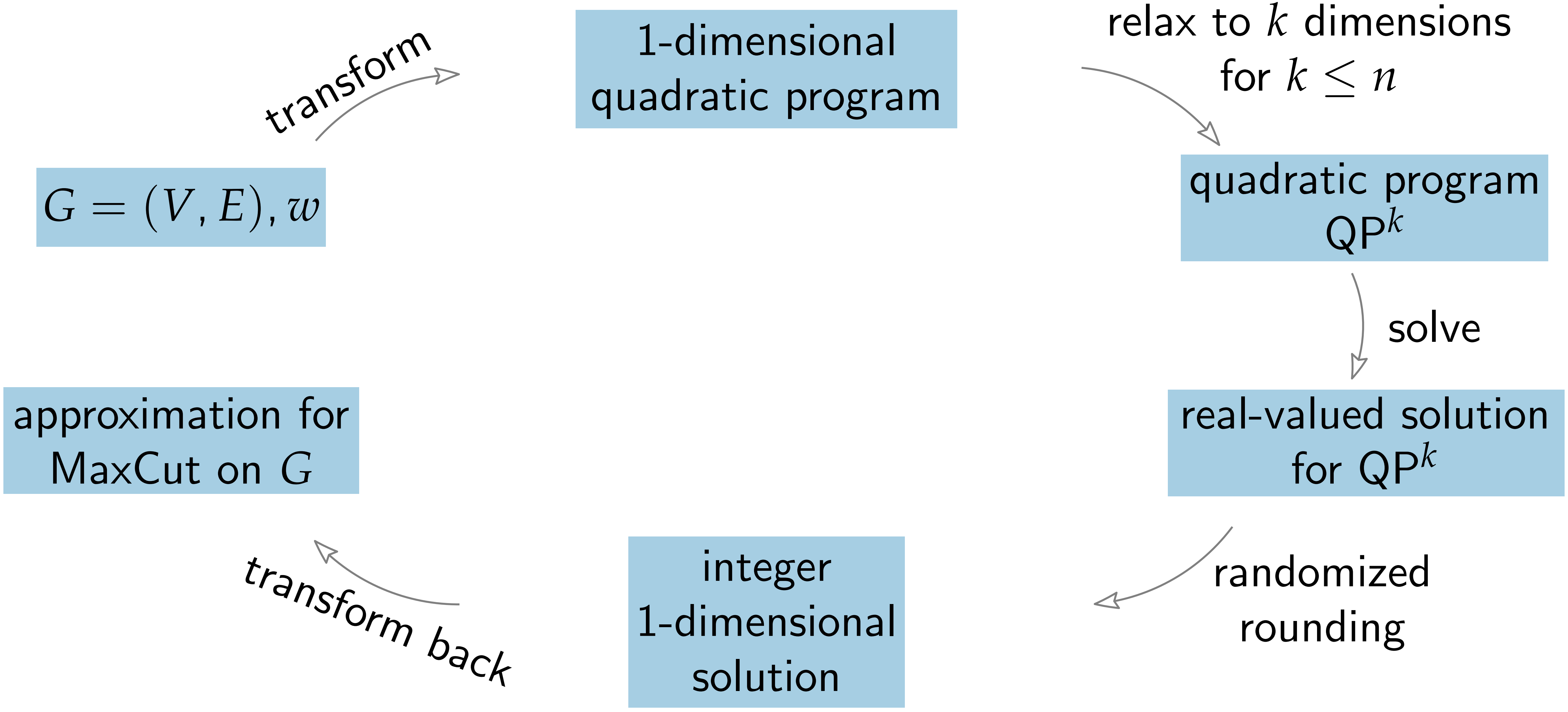
Note.

- Solving QP(G, w) is NP-hard.
- Otherwise MaxCut would not be NP-hard.

Goemans-Williamson Algorithm for MaxCut



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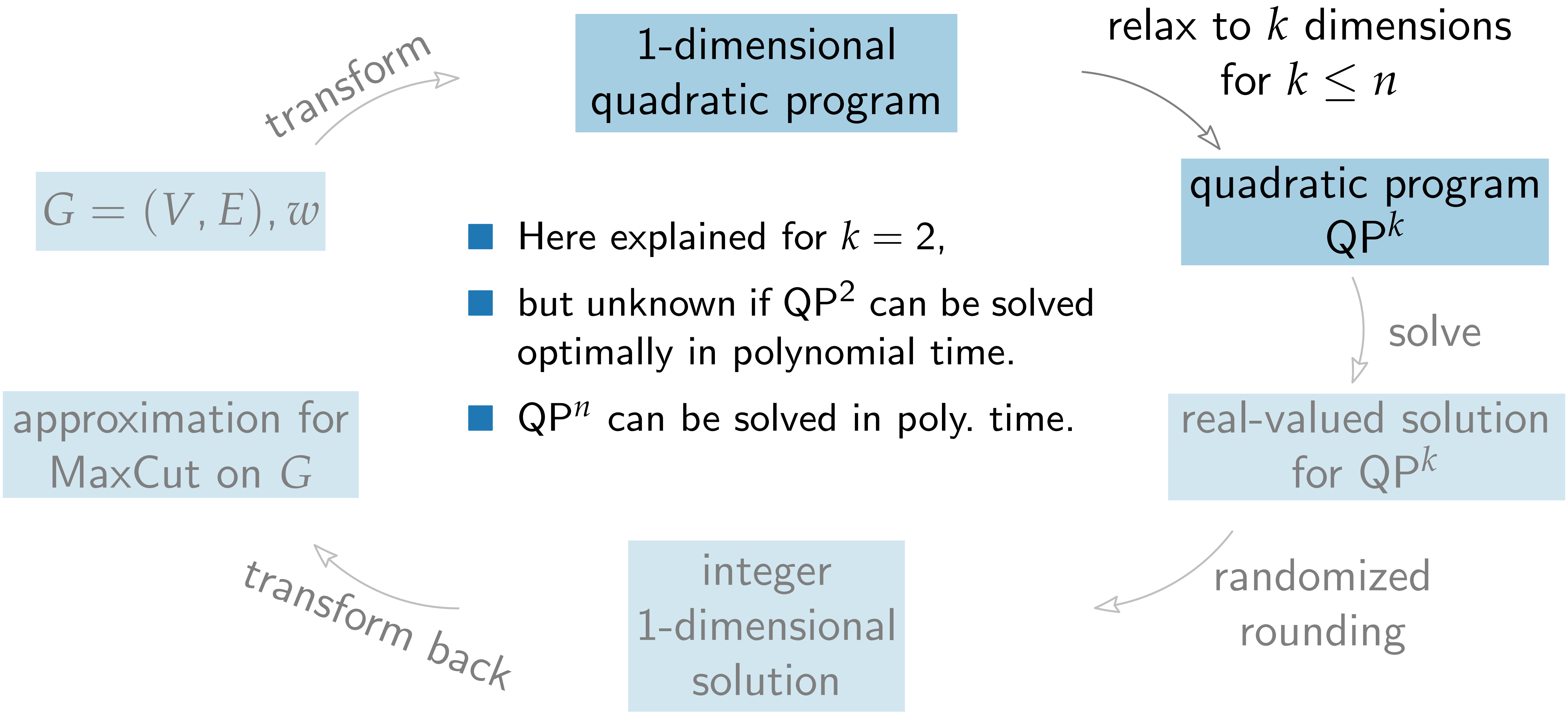

- Here explained for $k = 2$,
- but unknown if QP^2 can be solved optimally in polynomial time.
- QP^n can be solved in poly. time.

Relaxation of QP(G, w)

QP$^2(G, w)$

maximize $\frac{1}{2} \sum_{j=1}^{n} \sum_{i=1}^{j-1} w_{ij}(1 - x^i \cdot x^j)$

subject to

$$x^i \cdot x^i = 1$$

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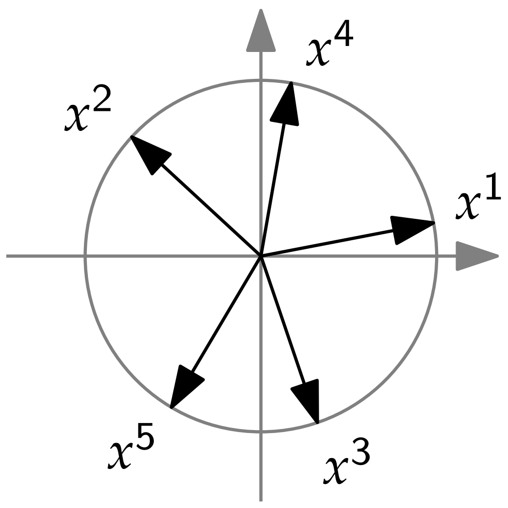

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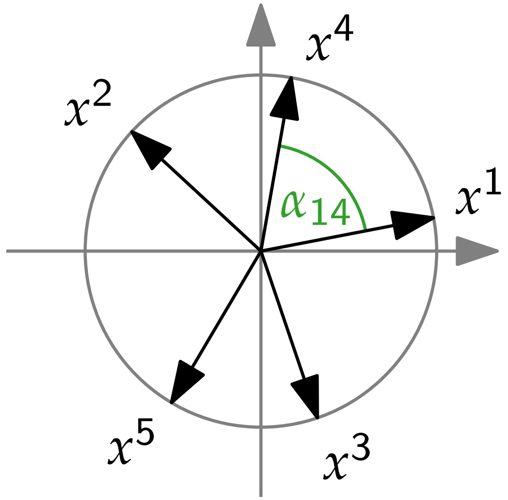

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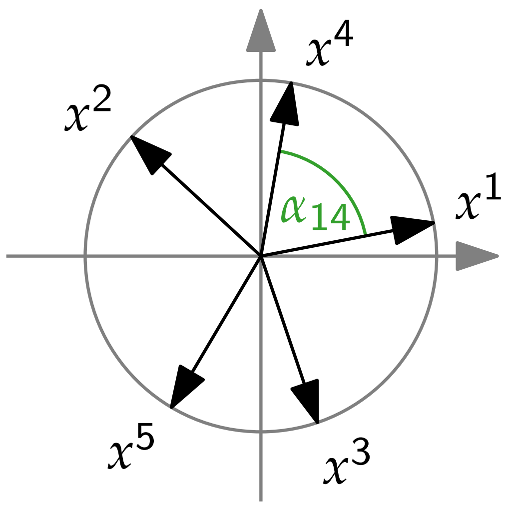

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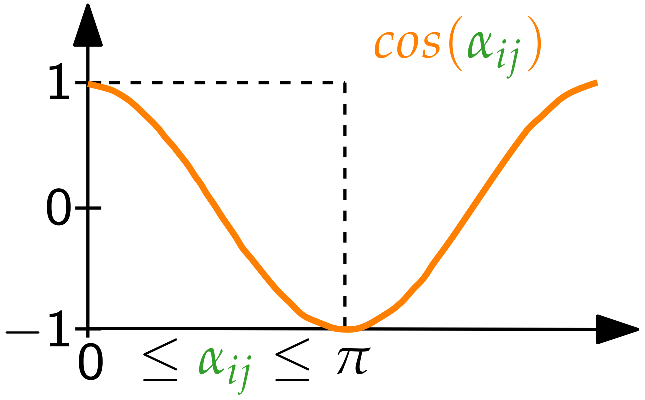

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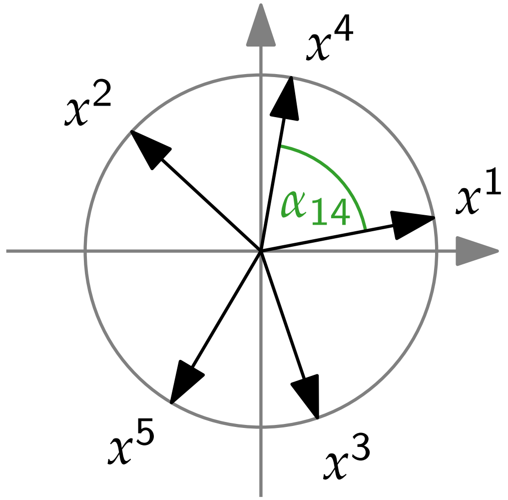

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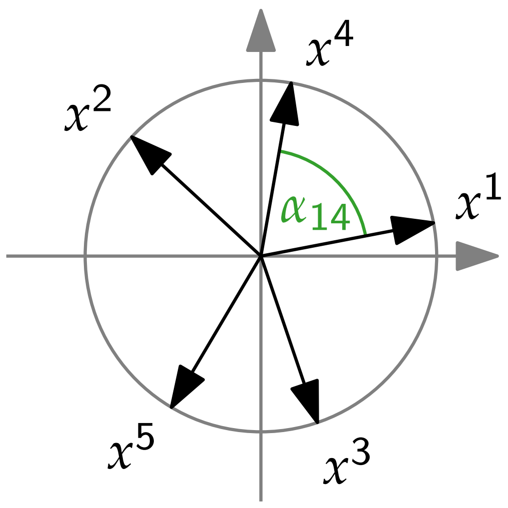

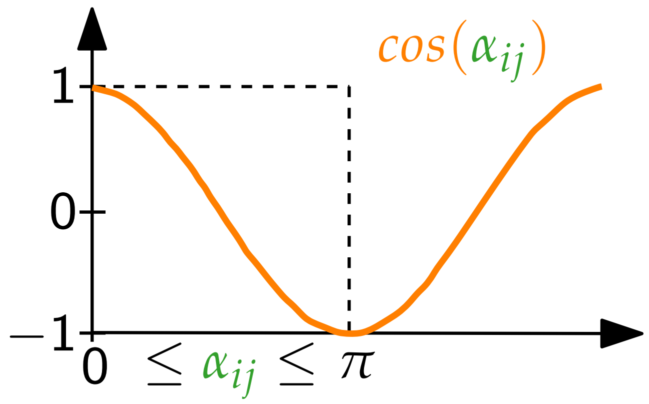

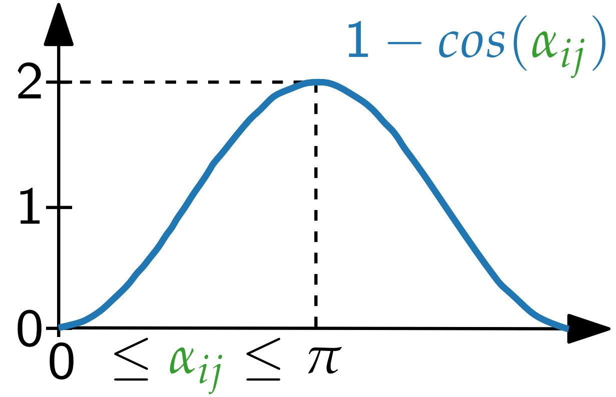

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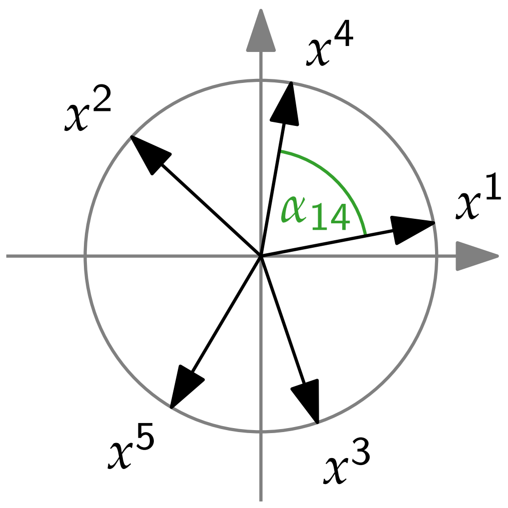
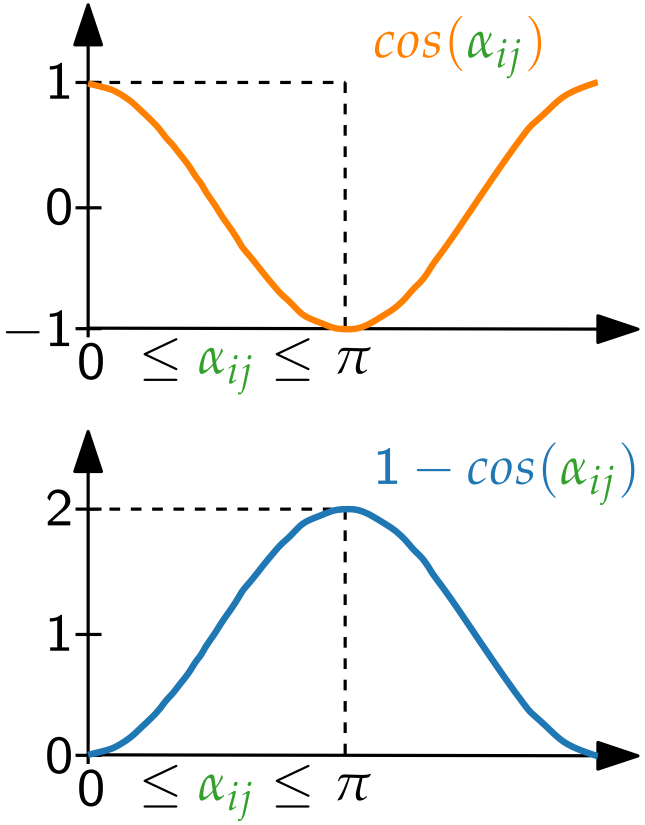
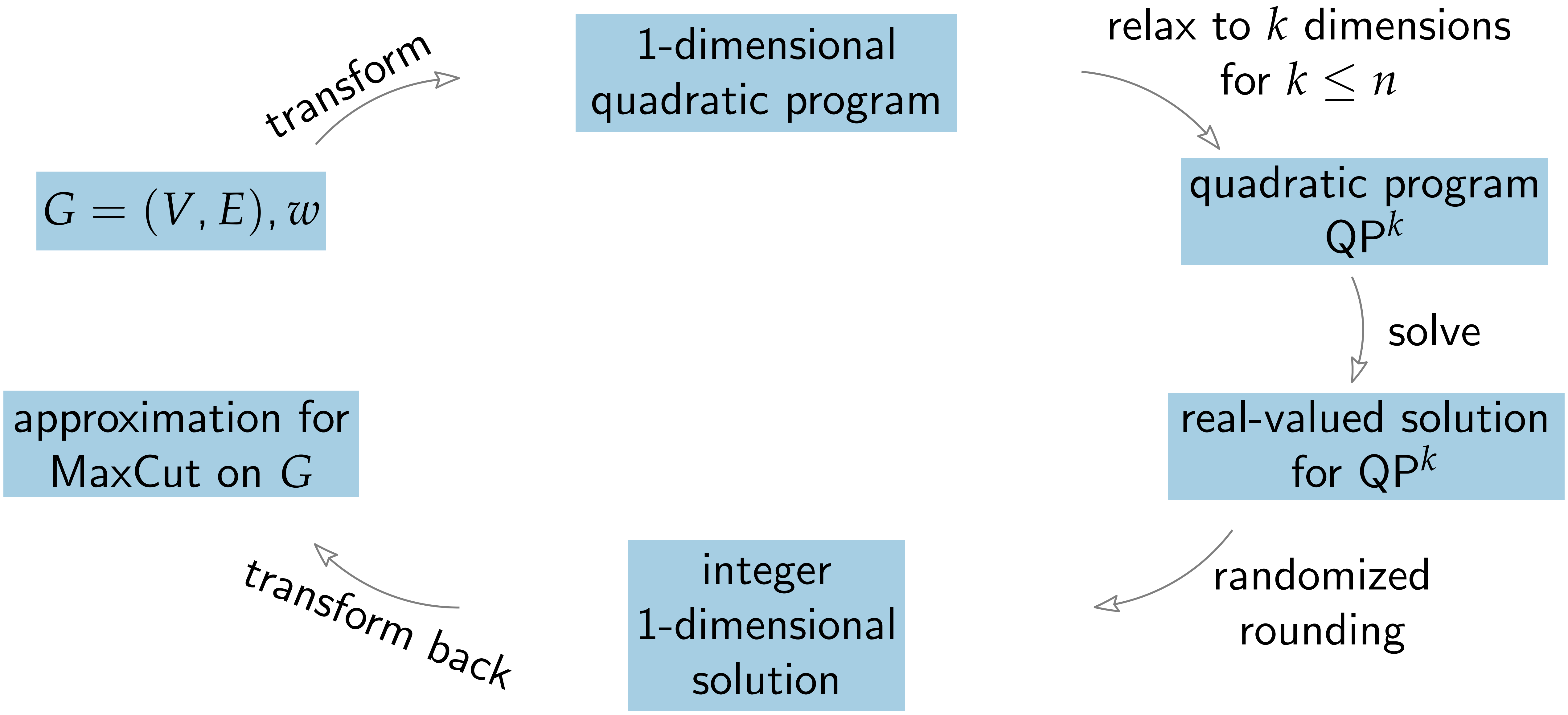


- The variables are 2-dimensional vectors.
- We maximize angles α_{ij} since larger α_{ij} increase the contribution of w_{ij}.
- Hence, our objective is:

$$\frac{1}{2} \sum_{j=1}^{n} \sum_{i=1}^{j-1} w_{ij}(1 - \cos(\alpha_{ij}))$$



Goemans-Williamson Algorithm for MaxCut



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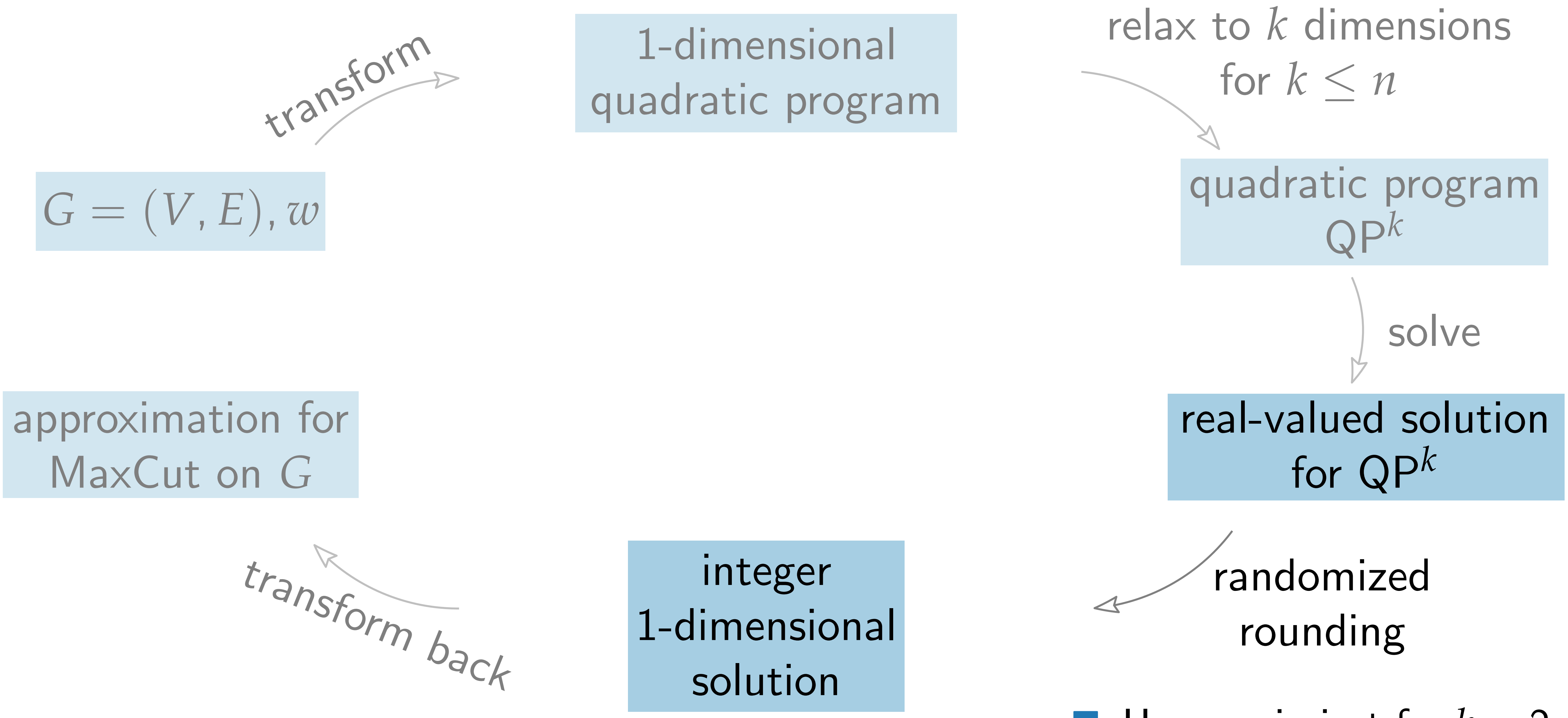

- Here again just for $k = 2$.

Algorithm RandomizedMaxCut

RandomizedMaxCut(G, w)

Compute optimal solution $(\tilde{x}^1, \dots, \tilde{x}^n)$ for QP$^2(G, w)$

Pick random vector $r \in \mathbb{R}^2$

$S \leftarrow \{v_i \in V : \tilde{x}^i \cdot r \geq 0\}$

return $c(S, V \setminus S)$

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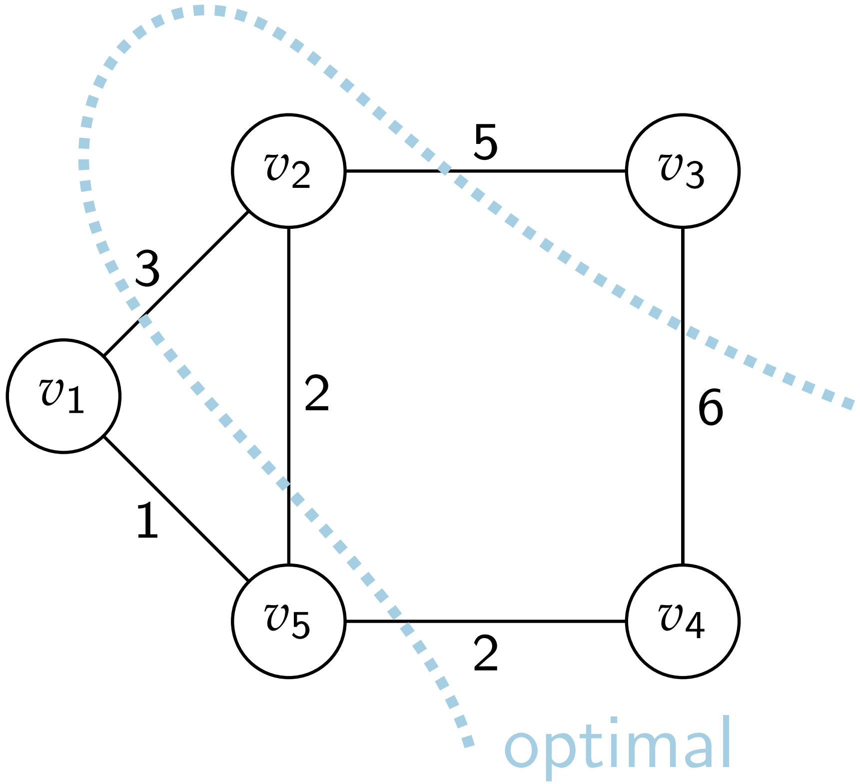

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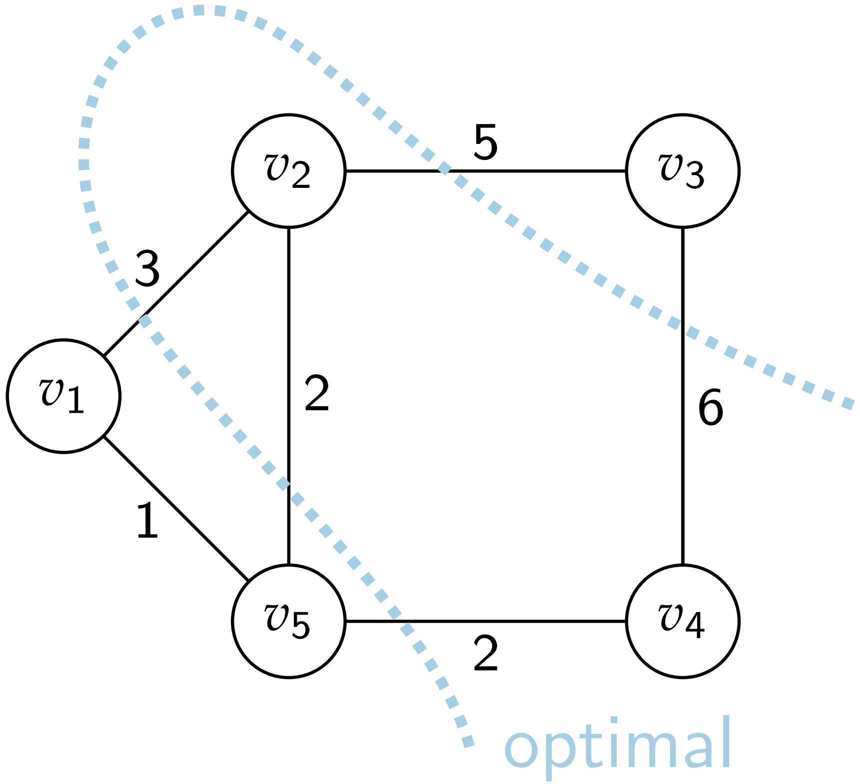

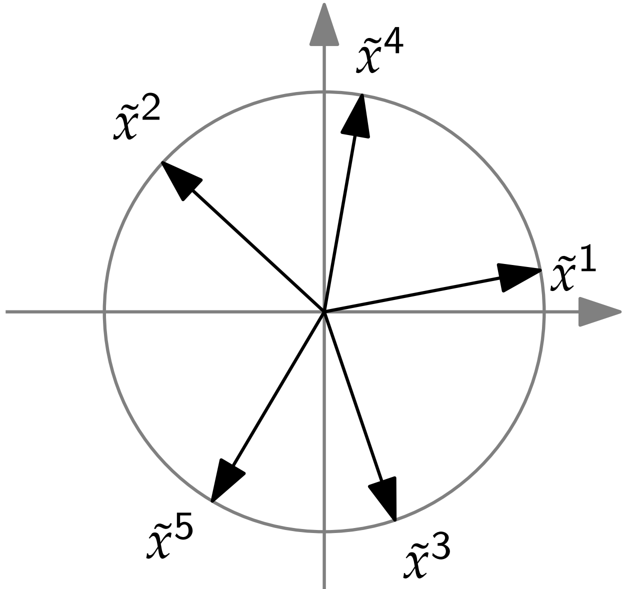

(Sketch)

Algorithm RANDOMIZEDMAXCUT

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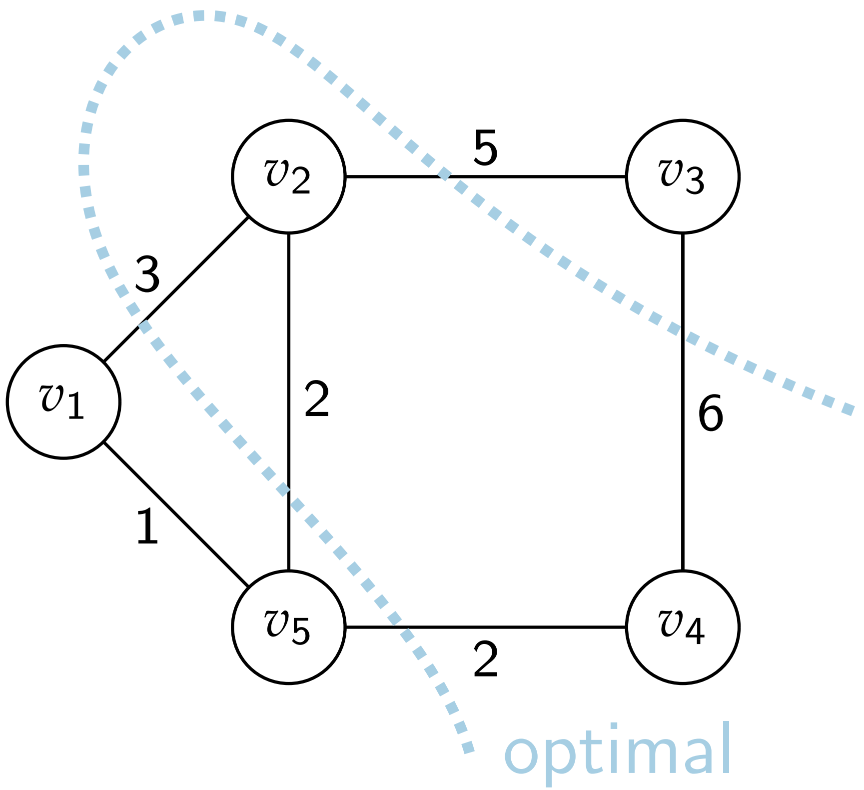

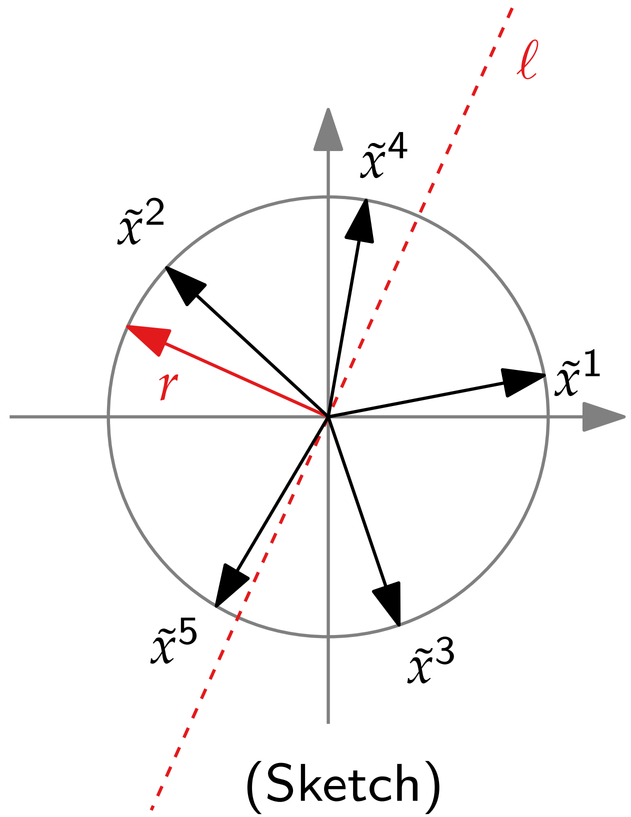

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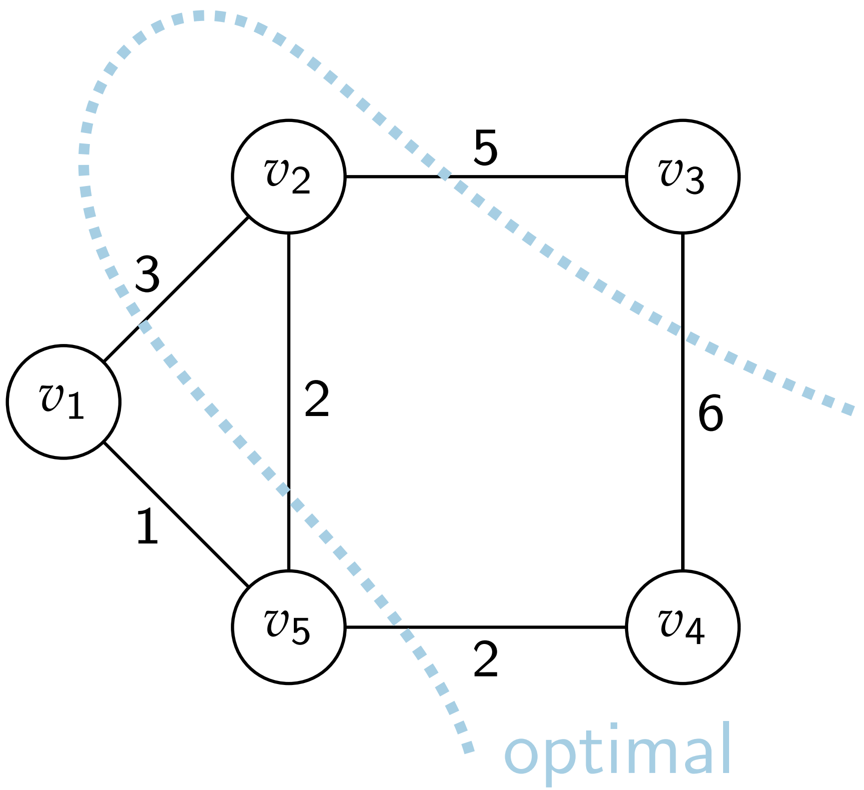
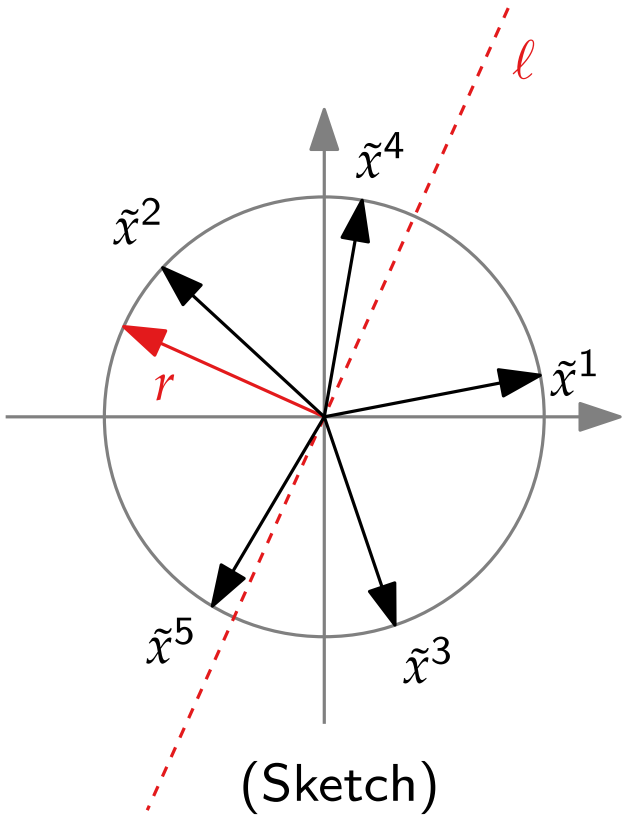
- $\tilde{x}^i$ lies above the line ℓ orthogonal to r





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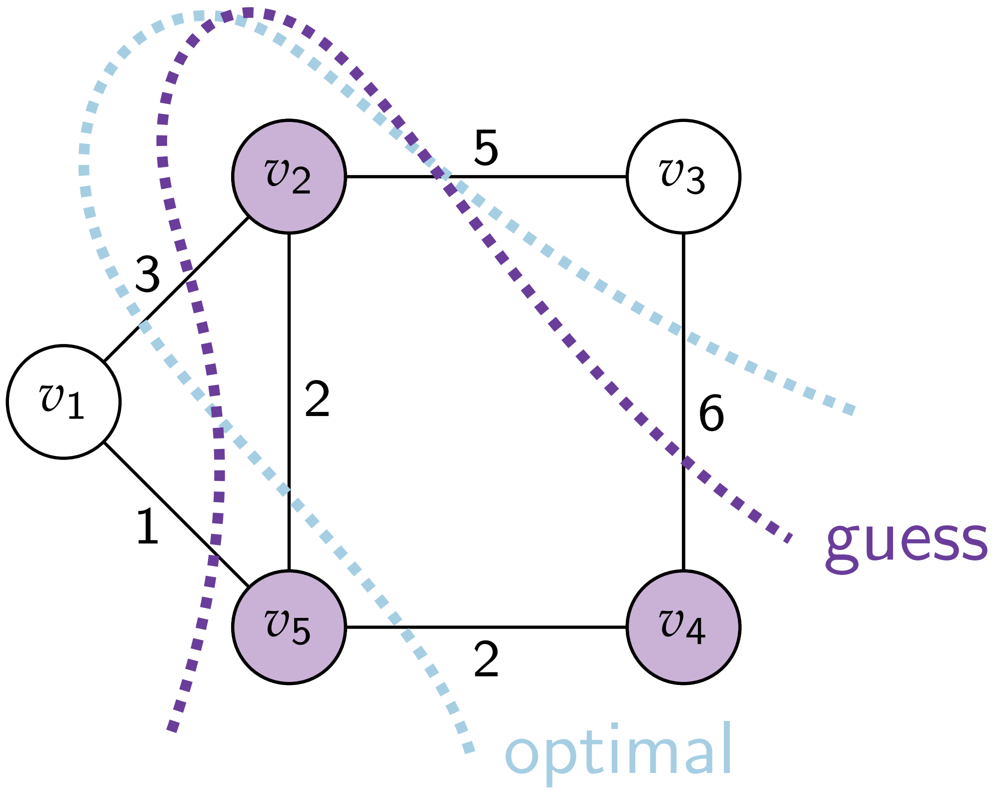

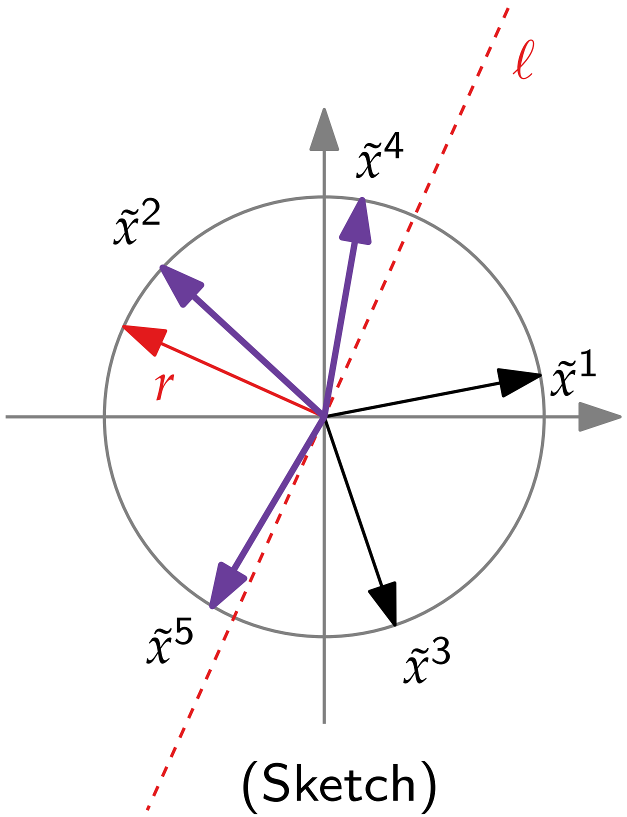

RandomMaxCut – Expected Value

Lemma 2.
Let X be the solution of RandomizedMaxCut(G, w).
If r is picked uniformally at random, then

$$\mathsf{E}[X] = \sum_{j=1}^{n} \sum_{i=1}^{j-1} w_{ij} \frac{\alpha_{ij}}{\pi} \,.$$

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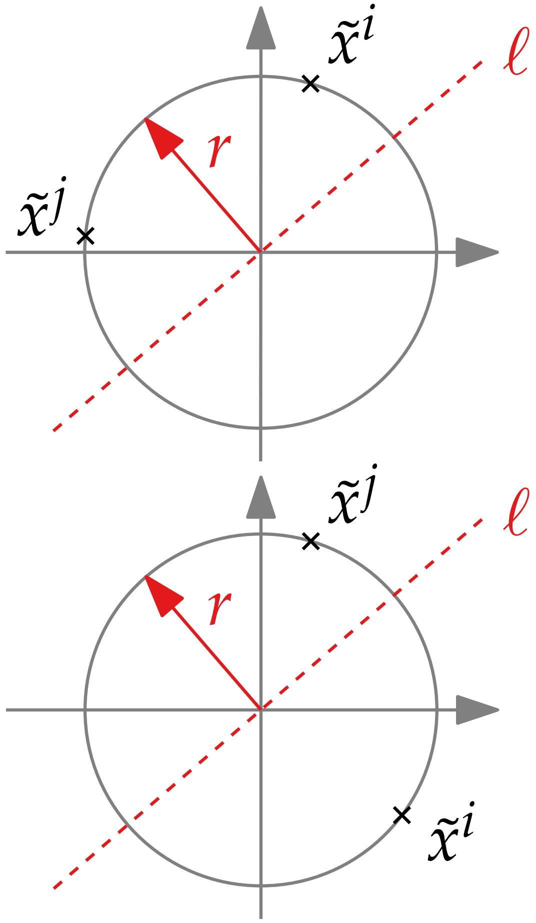

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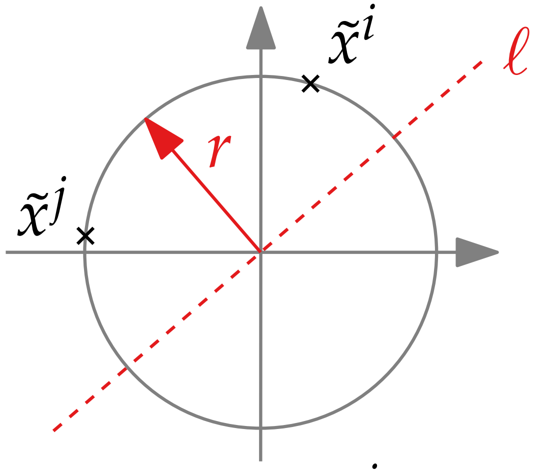

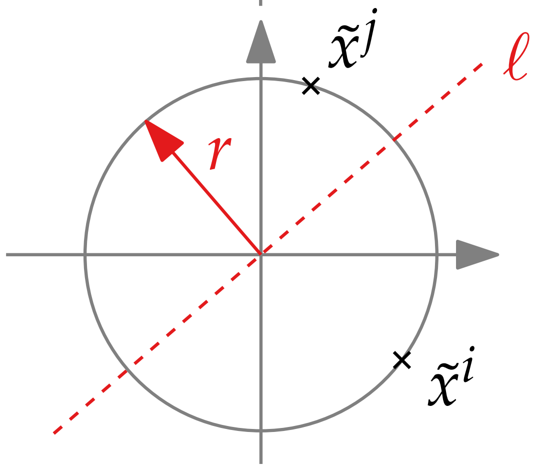

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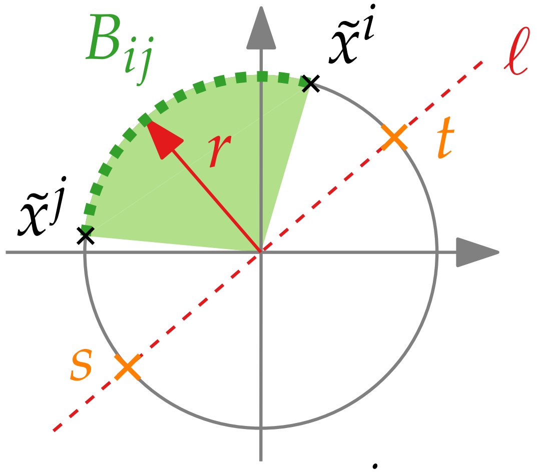

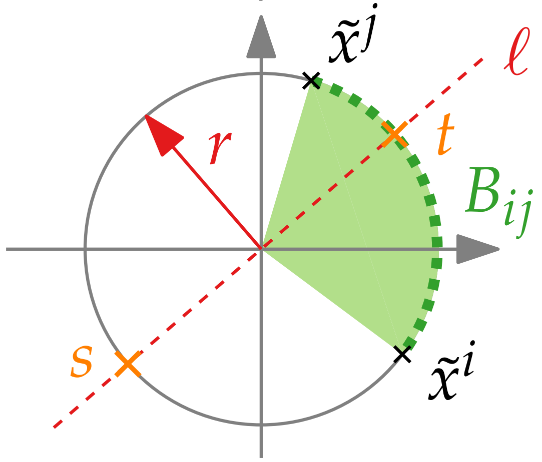

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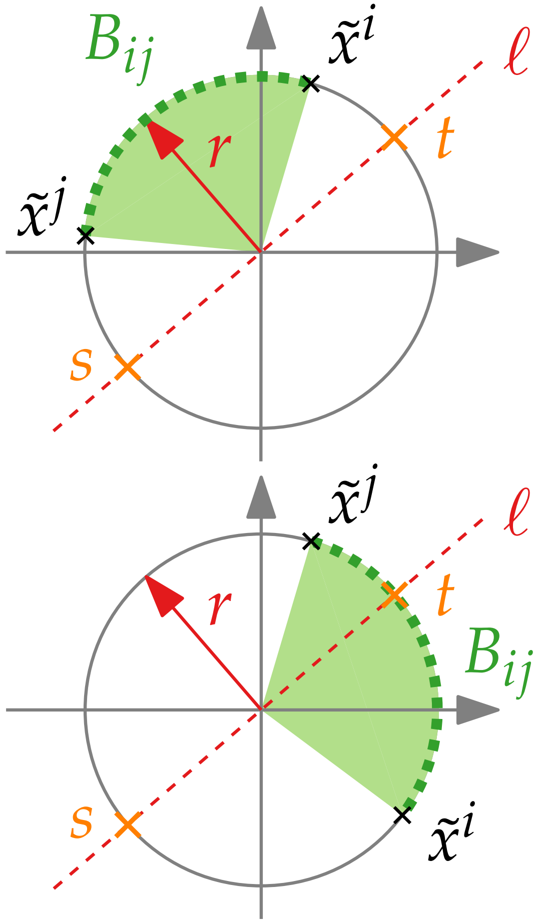

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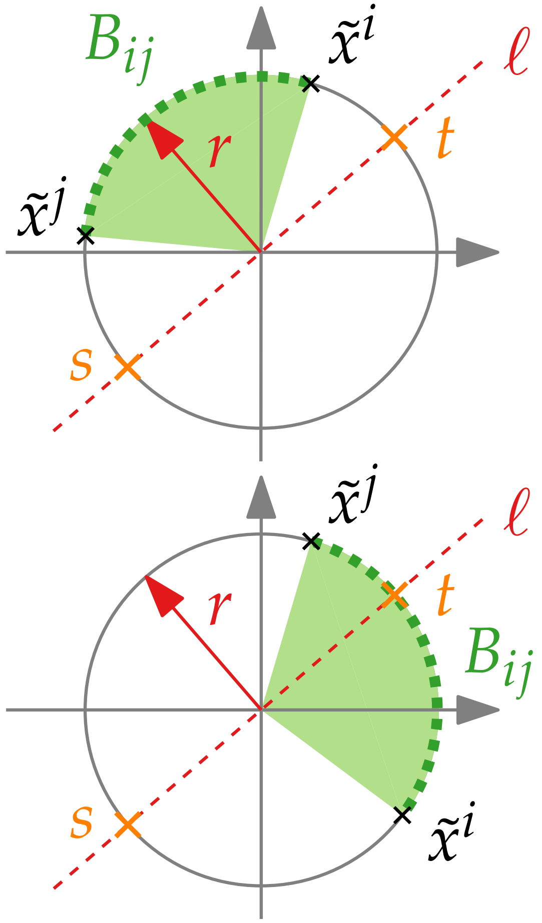

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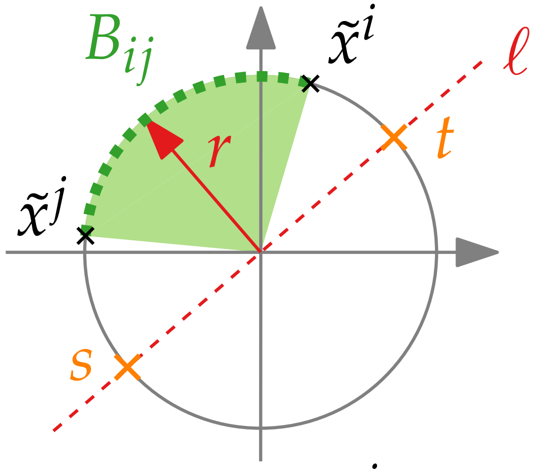

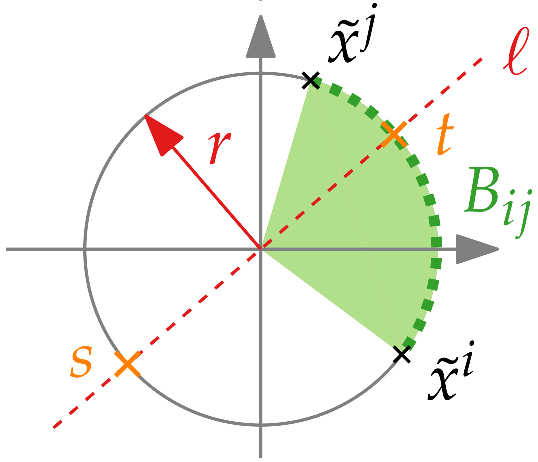

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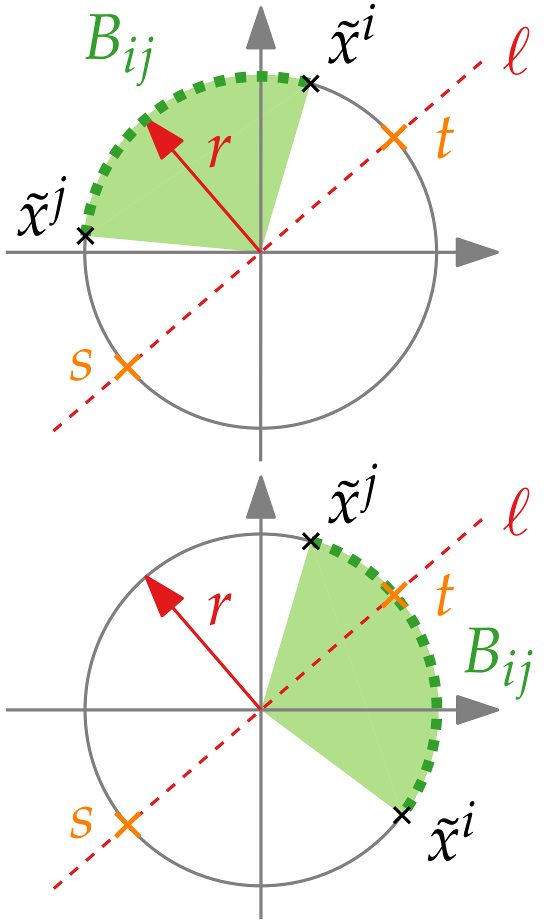

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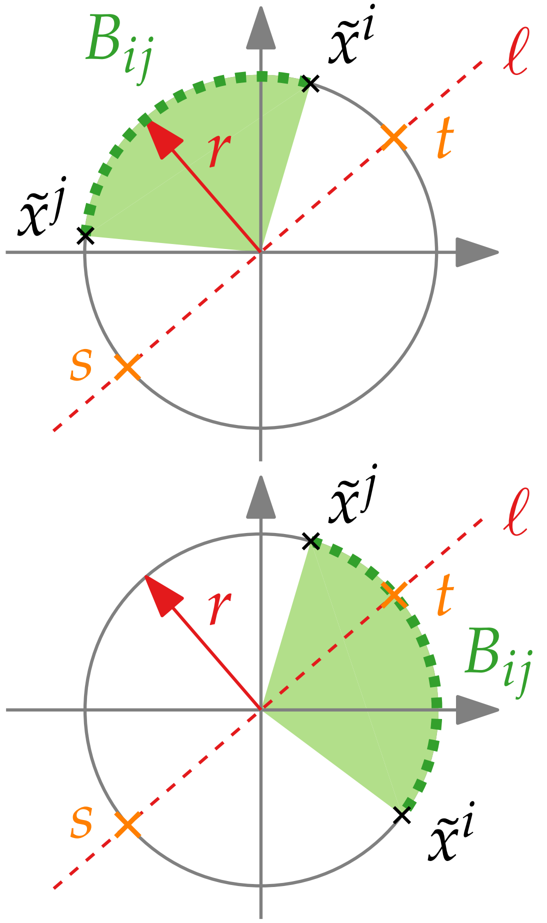

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- If $\tilde{x}^i$ (or $\tilde{x}^j$) lies $\leq \alpha_{ij}$ before s or t on the perimter of the unit disk, s or t lies on B_{ij}.

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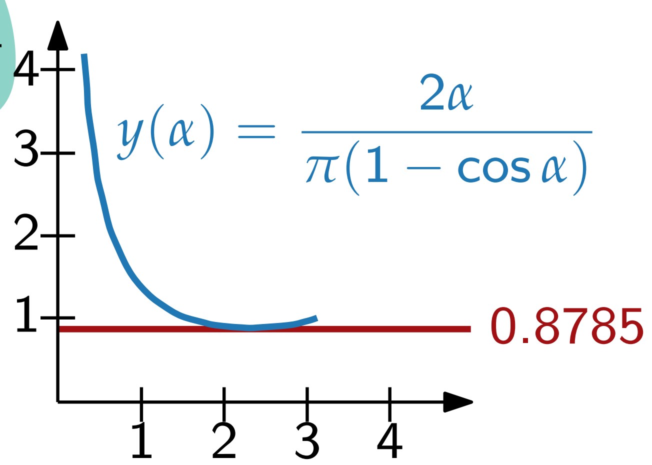

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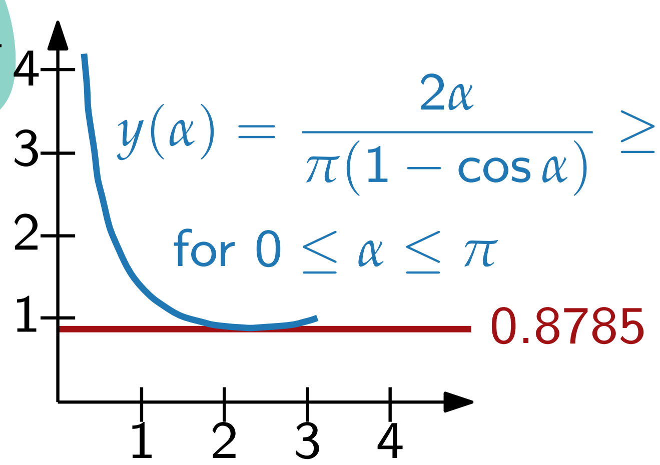

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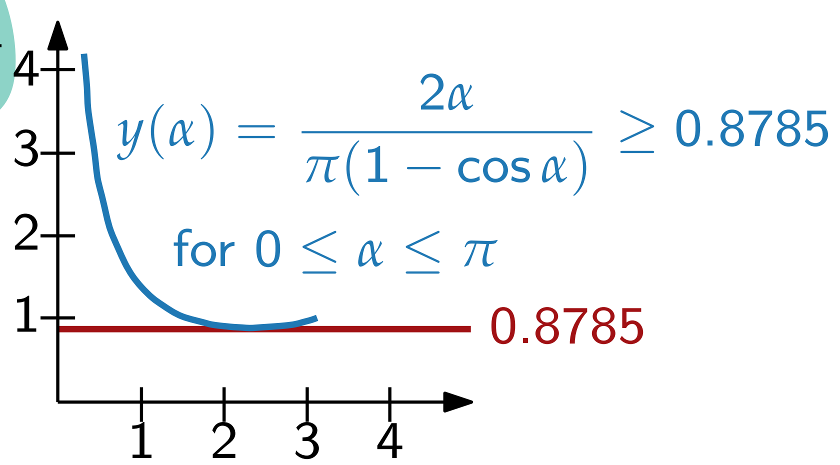

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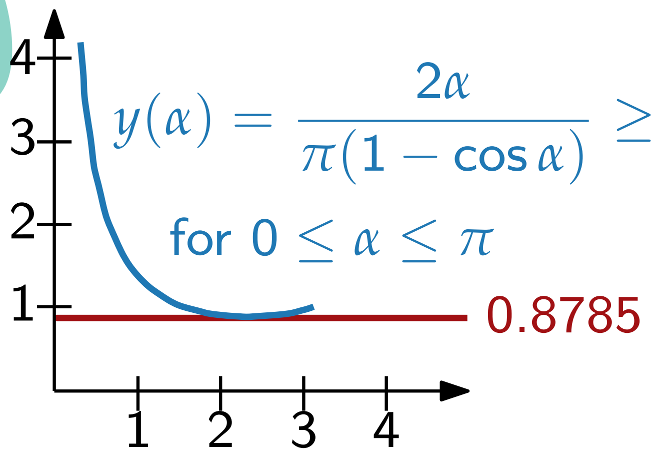

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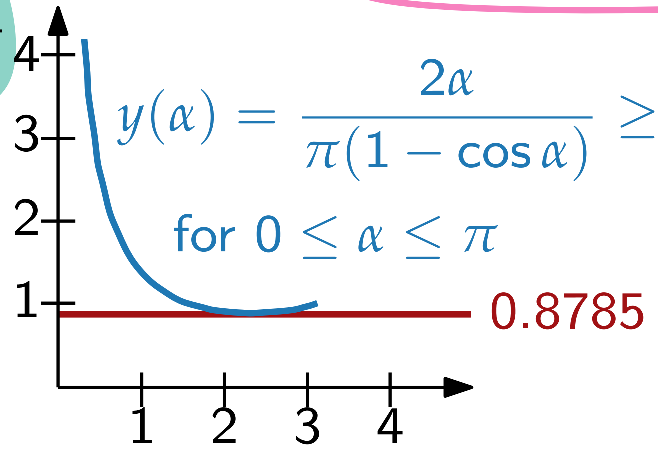

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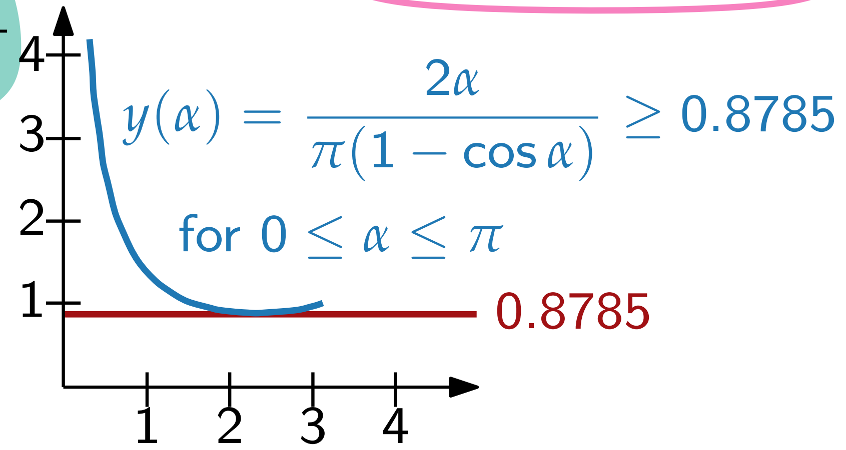
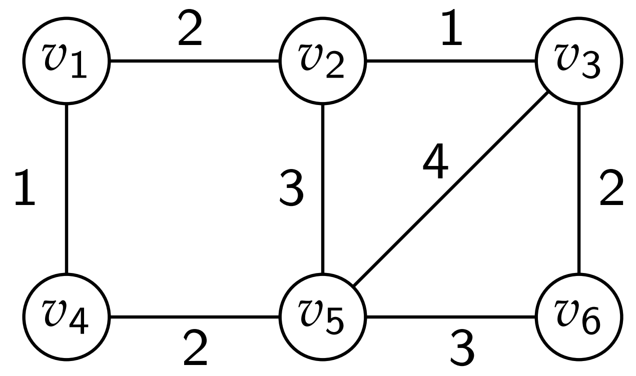


Example



Example

1. Step: Build QP

maximize $\frac{1}{2}\sum_{j=1}^{6}\sum_{i=1}^{j-1} w_{ij}(1 - x_i x_j)$

subject to $x_i^2 = 1$

Weight matrix w_{ij}

	1	2	3	4	5	6
1		2		1		
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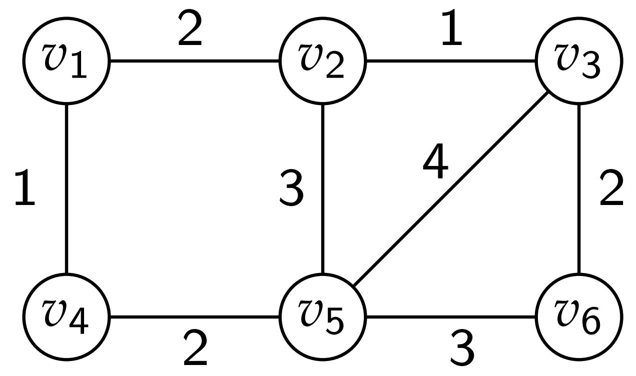

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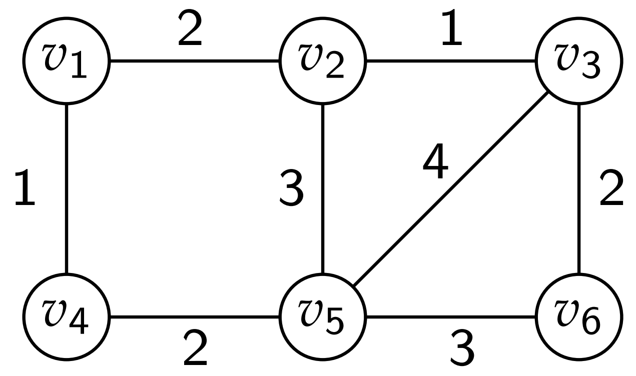

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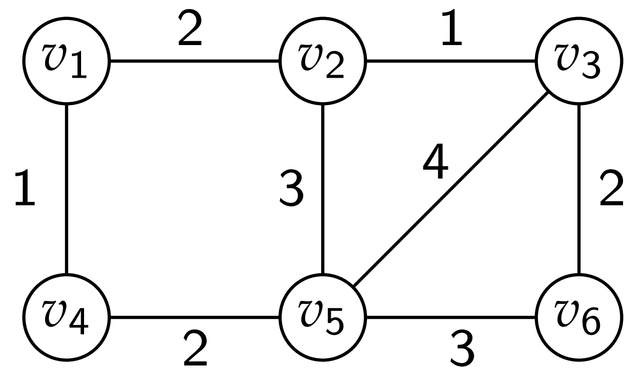
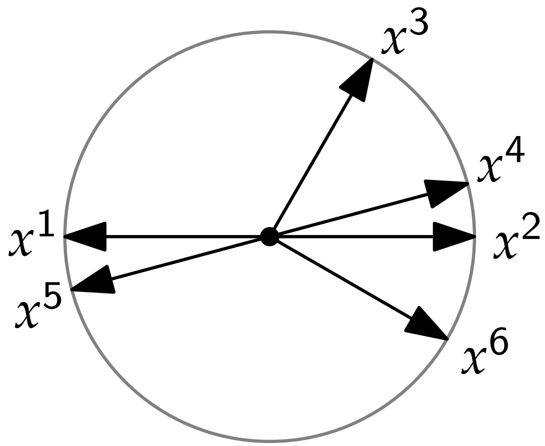
Variable	x^1	x^2	x^3	x^4	x^5	x^6
Angle	0	180	120	165	345	210





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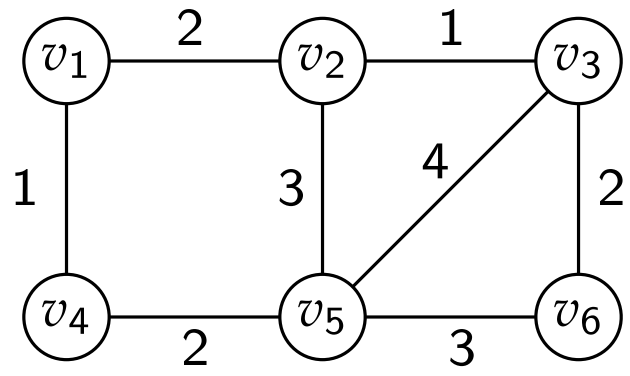

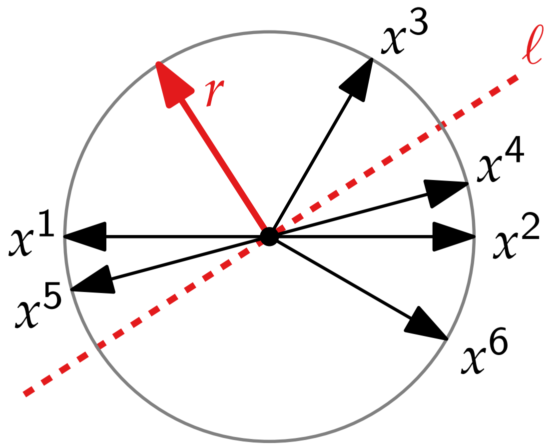

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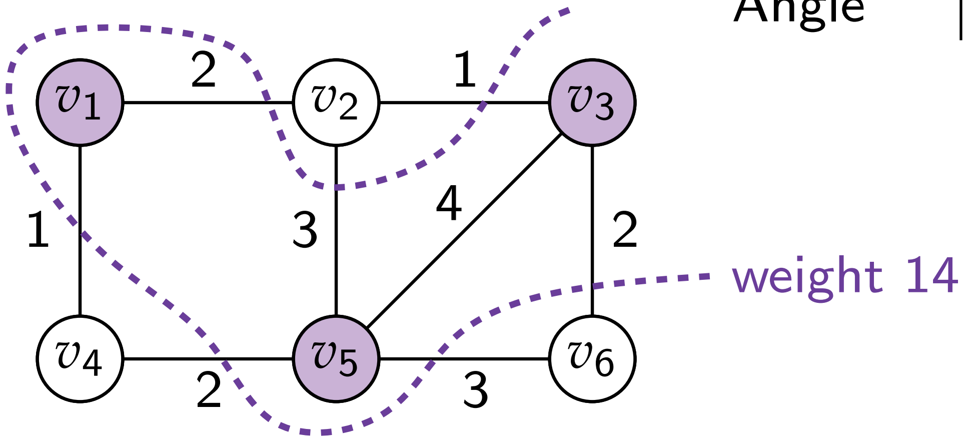

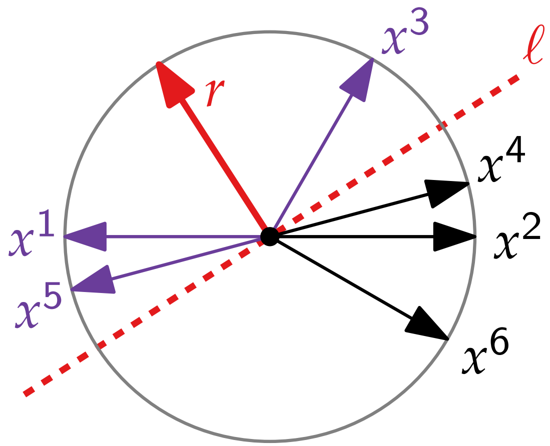

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6			2		3	

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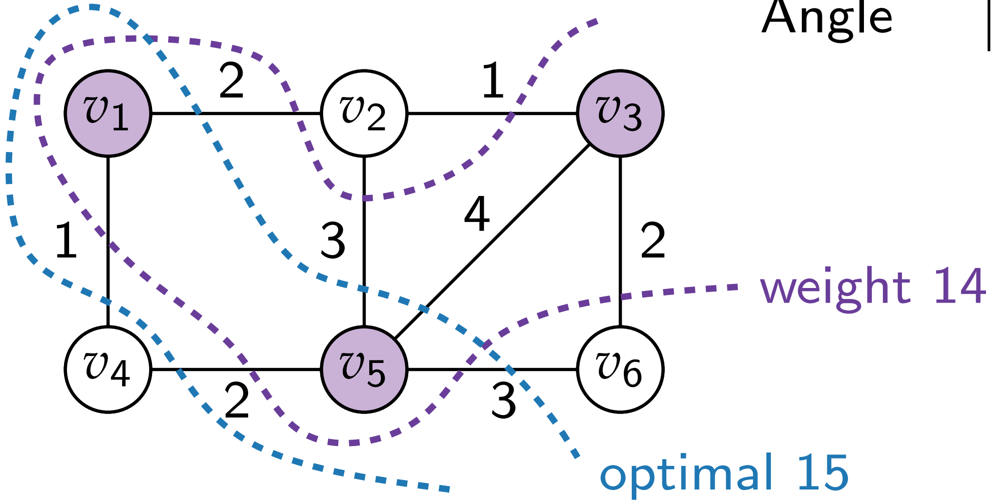
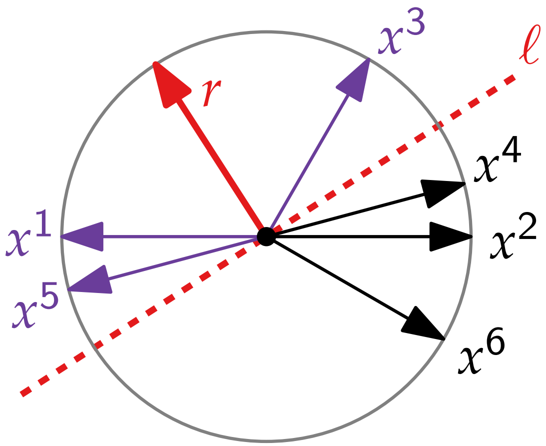
$$\begin{aligned} \textbf{maximize} \quad & \frac{1}{2}\sum_{j=1}^{6}\sum_{i=1}^{j-1} w_{ij}(1 - x^i \cdot x^j) \\ \textbf{subject to} \quad & x^i \cdot x^i = 1 \\ & x^i = (x^i_1, x^i_2) \in \mathbb{R}^2 \end{aligned}$$

3. Step: Solve QP^2

Variable	x^1	x^2	x^3	x^4	x^5	x^6
Angle	0	180	120	165	345	210

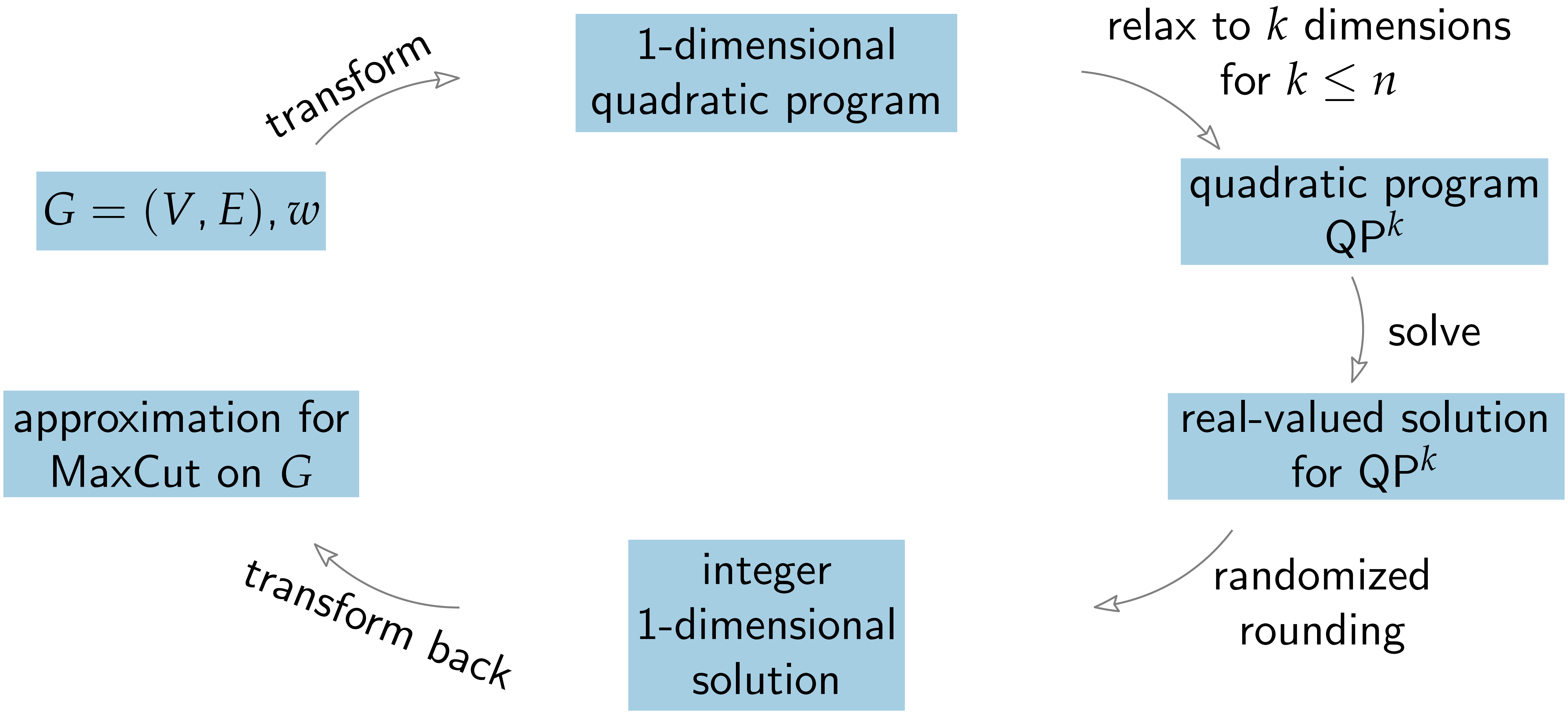




4. Step: Guess r

5. Step: Derive S

Goemans-Williamson Algorithm for MaxCut



Goemans-Williamson Algorithm for MaxCut

$G = (V, E), w$

transform

1-dimensional quadratic program

relax to k dimensions for $k \leq n$

quadratic program QP^k

solve

real-valued solution for QP^k

randomized rounding

integer 1-dimensional solution

transform back

approximation for MaxCut on G

- So far, $k = 2$.
- QP^n can be solved in polynomial time.

$\mathrm{QP}^n(G, w)$

$\mathbf{QP}^2(G, w)$

maximize $\frac{1}{2} \sum_{j=1}^{n} \sum_{i=1}^{j-1} w_{ij}(1 - x^i \cdot x^j)$

subject to $x^i \cdot x^i = 1$

$x^i = (x_1^i, x_2^i) \in \mathbb{R}^2$

$\mathbf{QP}^n(G, w)$

maximize $\frac{1}{2} \sum_{j=1}^{n} \sum_{i=1}^{j-1} w_{ij}(1 - x^i \cdot x^j)$

subject to $x^i \cdot x^i = 1$

$x^i \in \mathbb{R}^n$

$\mathrm{QP}^n(G, w)$

$\mathrm{QP}^2(G, w)$

$$\begin{aligned} \textbf{maximize} \quad & \tfrac{1}{2}\sum_{j=1}^{n}\sum_{i=1}^{j-1} w_{ij}(1 - x^i \cdot x^j) \\ \textbf{subject to} \quad & x^i \cdot x^i = 1 \\ & x^i = (x^i_1, x^i_2) \in \mathbb{R}^2 \end{aligned}$$

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- A matrix M is called **positive semidefinite** if for any vector $v \in \mathbb{R}^n$:

$$v^\intercal \cdot M \cdot v \geq 0$$

$\mathrm{QP}^n(G, w)$

QP^2 (G, w)

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- $M = (m_{ij}) = (x^i \cdot x^j)$ is positive semidefinite.

$\mathrm{QP}^n(G, w)$

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maximize $\frac{1}{2} \sum_{j=1}^{n} \sum_{i=1}^{j-1} w_{ij}(1 - x^i \cdot x^j)$

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$\mathbf{QP}^n(G, w)$

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$$v^\intercal \cdot M \cdot v \geq 0$$
- $M = (m_{ij}) = (x^i \cdot x^j)$ is positive semidefinite.
- $\mathrm{QP}^n(G, w)$ becomes the problem $\textsc{SemiDefiniteCut}(G, w)$.
  - Can be approximated in time polynomial in (G, w) and $1/\varepsilon$ with additive guarantee ε.
  - Note that the approximation of $\mathrm{QP}(G, w)$ is an extra step we have seen before. (The approximation of $\mathrm{QP}(G, w)$ with factor 0.8785 works for $\mathrm{QP}^n(G, w)$, too)

Discussion

- If the *Unique Games Conjecture* is true, then the approximation ratio of ≈ 0.8785 achieved by $\textsc{SemiDefiniteCut}$ (and $\textsc{RandomizedMaxCut}$) is best possible.
- Otherwise, no approximation ratio better than $\frac{16}{17} \approx 0.941$ is possible.
  In particular no polynomial-time approximation scheme (PTAS) exists.
- On planar graphs, the MaxCut problem can be solved optimally in polynomial time.

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- Whole book on this topic:
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- Using randomness is another tool to design approximation algorithms.

$\rightarrow$ See future lectures, in particular the next lecture!

Literature

Original paper:

- [GW '95] "Improved approximation algorithms for maximum cut and satisfiability problems using semidefinite programming"

Source:

- [Vazirani Ch26] "Approximation Algorithms"

Whole book on this topic:

- [Gärtner, Matoušek] "Approximation Algorithms and Semidefinite Progamming"