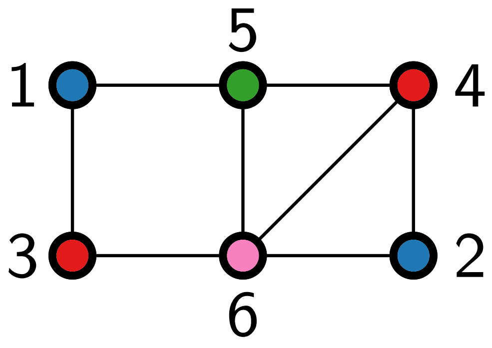
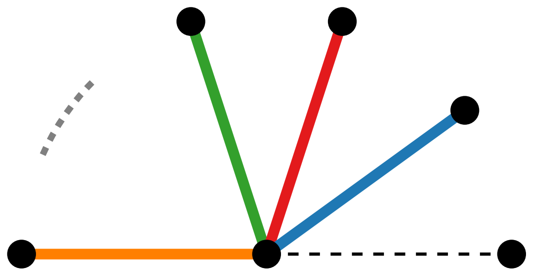
Advanced Algorithms

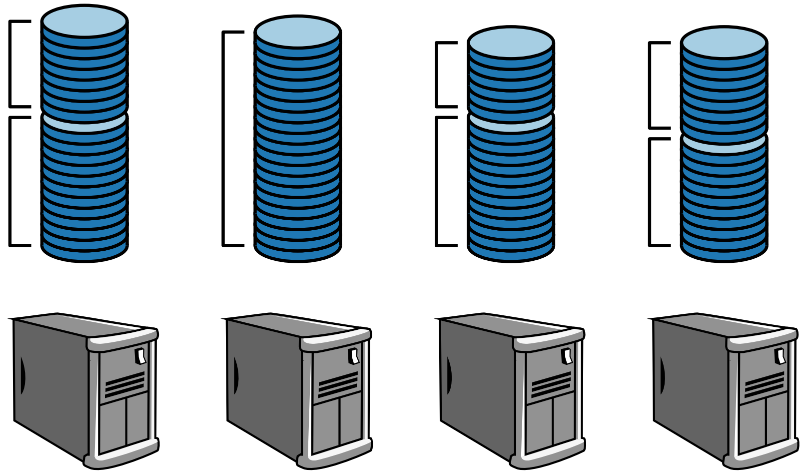
Approximation Algorithms

Coloring and Scheduling Problems

Johannes Zink · WS23/24



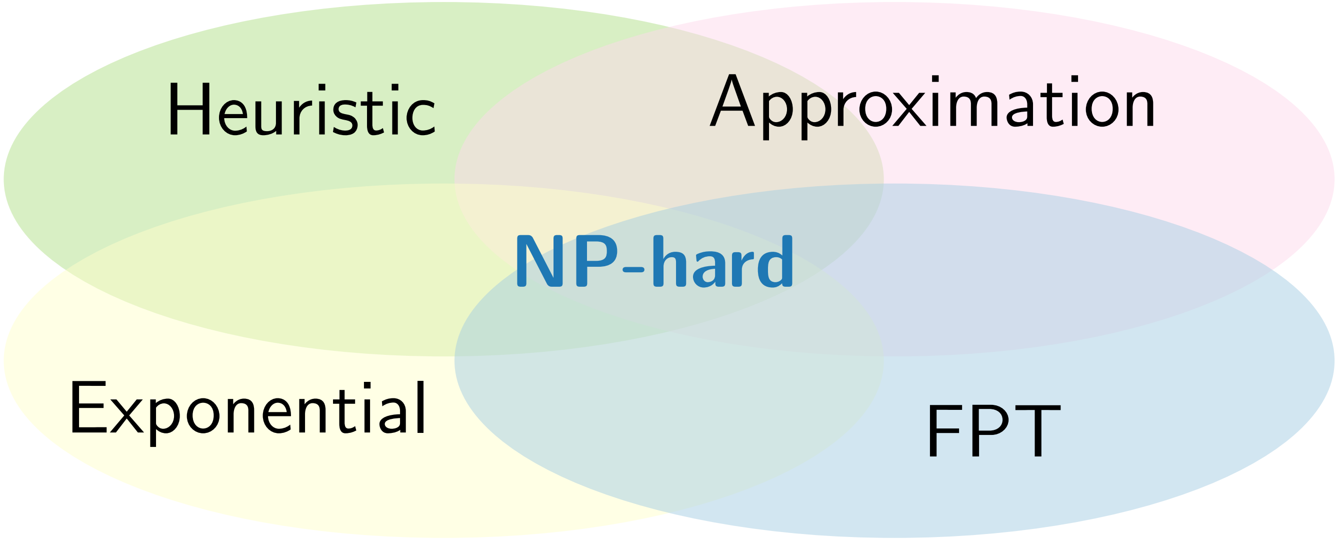




Dealing with NP-Hard Optimization Problems

What should we do?

- Sacrifice optimality for speed
  - Heuristics
  - Approximation algorithms
- Optimal solutions
  - Exact exponential-time algorithms
  - Fine-grained analysis – parameterized algorithms



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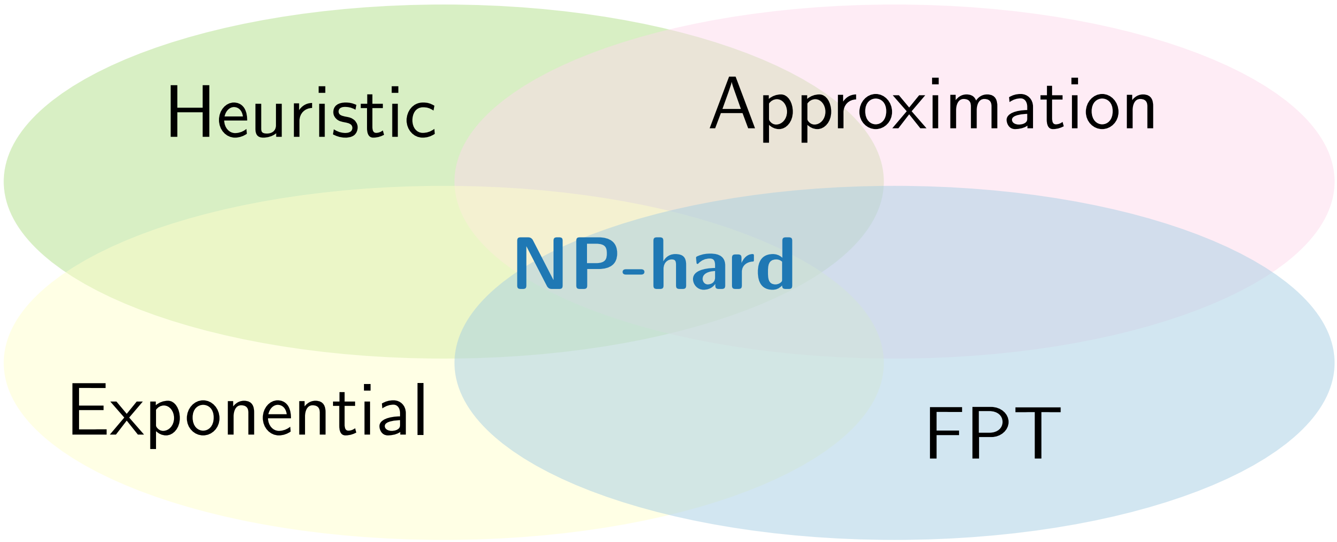

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PTAS (*polynomial-time approximation scheme*)

Approximation with Additive Guarantee

Definition.
Let Π be an optimization problem,
let $\mathcal{A}$ be a polynomial-time algorithm for Π,
let I be an instance of Π, and
let ALG(I) be the value of the objective function of the solution that $\mathcal{A}$ computes given I.

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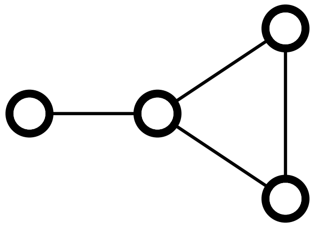
- Most problems that we know do not admit an approximation algorithm with additive guarantee.

Minimum Vertex Coloring



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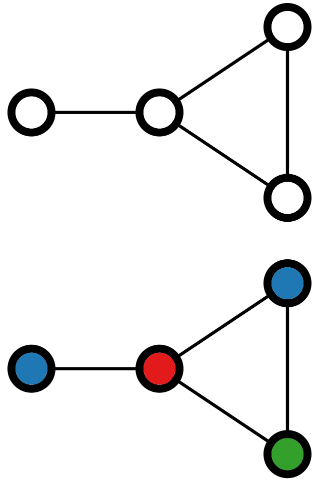

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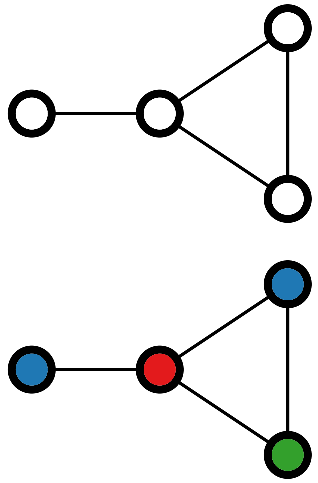

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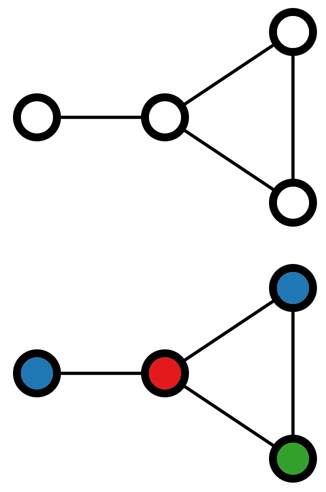

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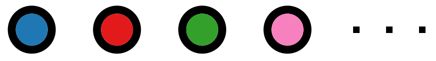

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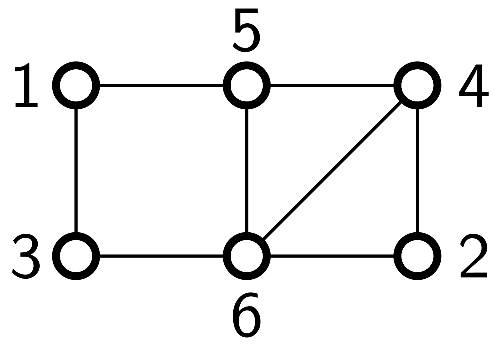
  Color vertices in some order with the lowest feasible color.



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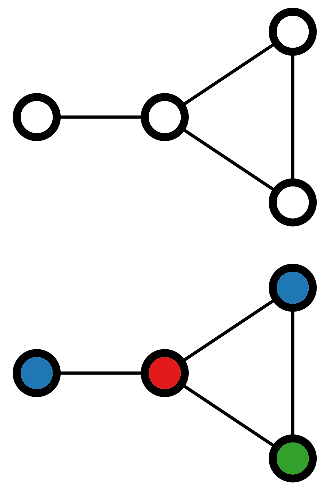

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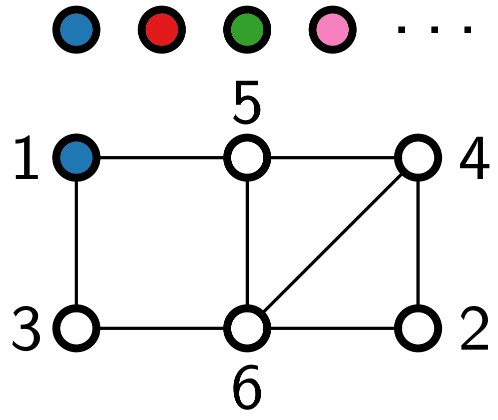

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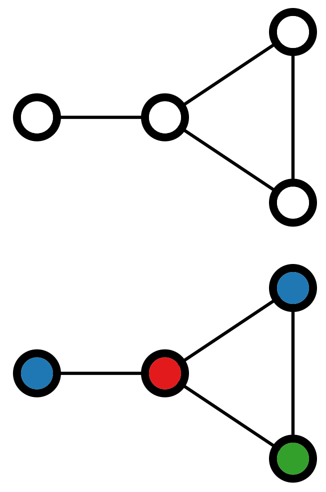

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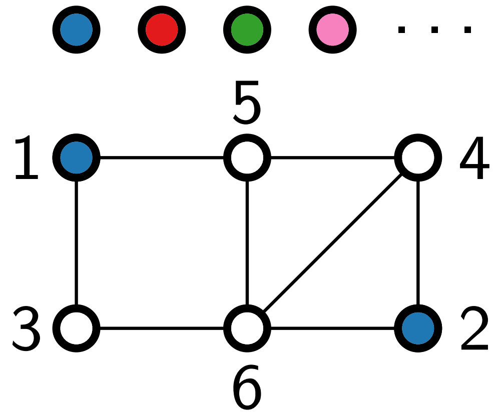

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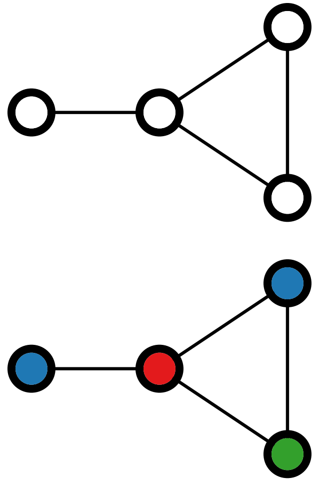

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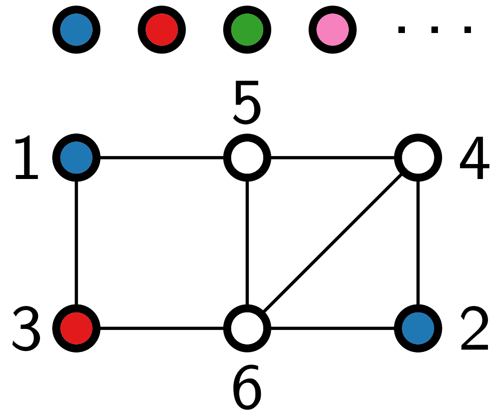

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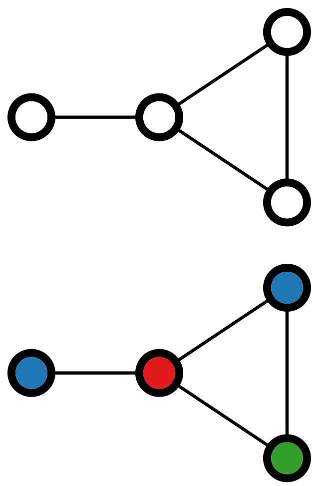

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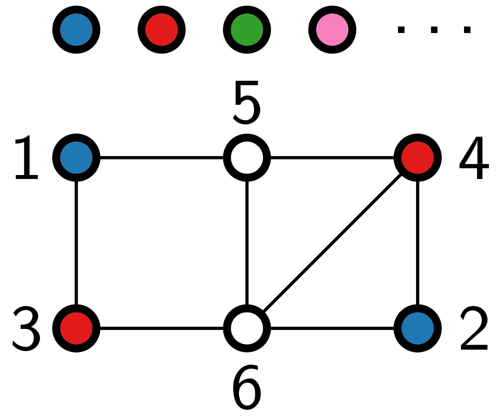

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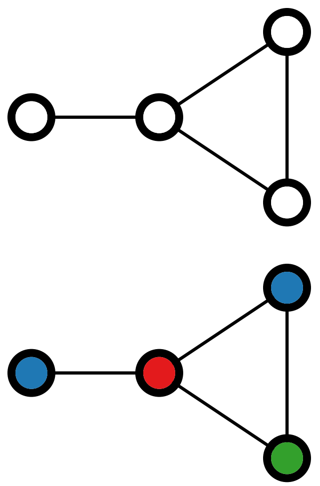

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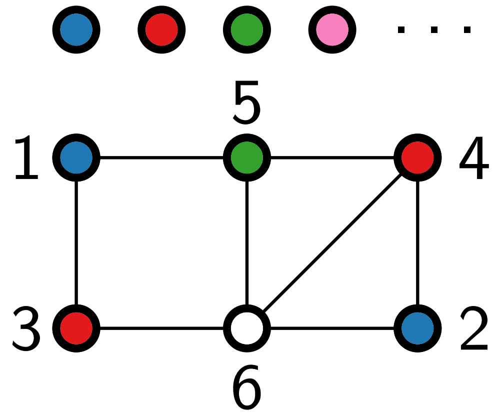

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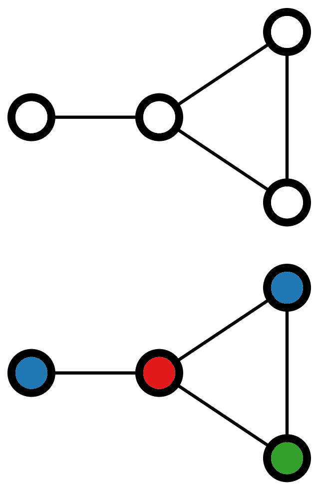

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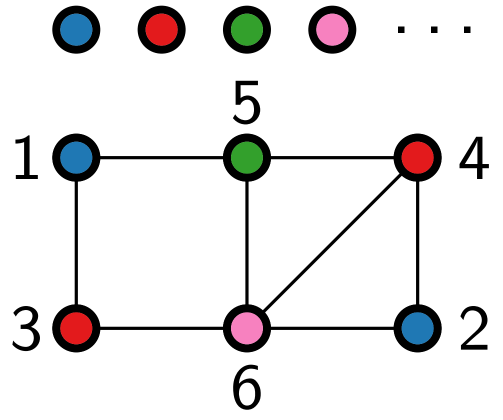

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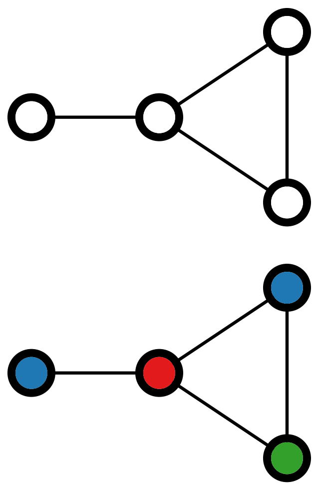

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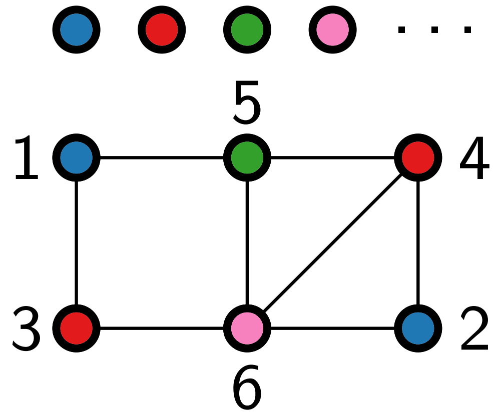


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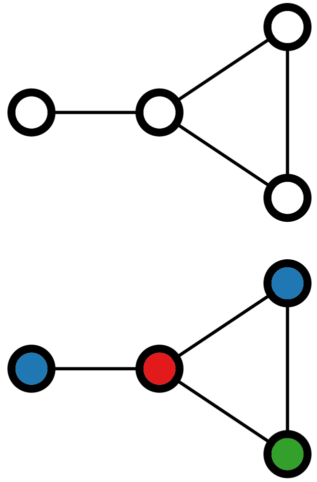

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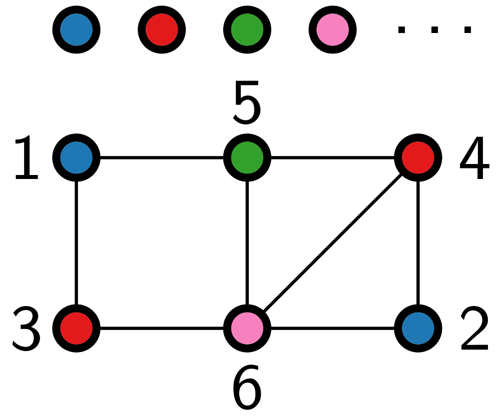

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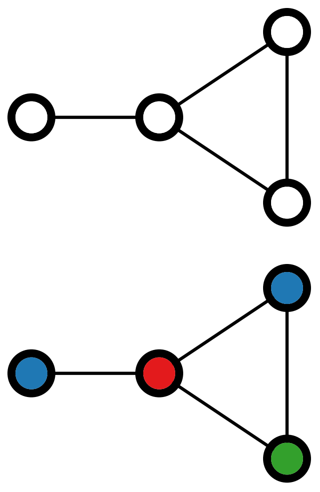

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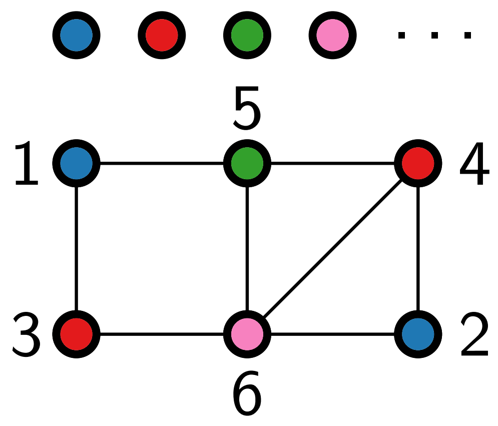

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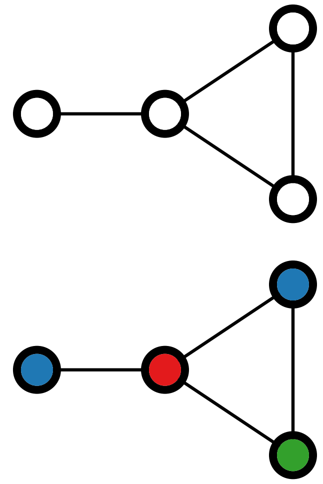

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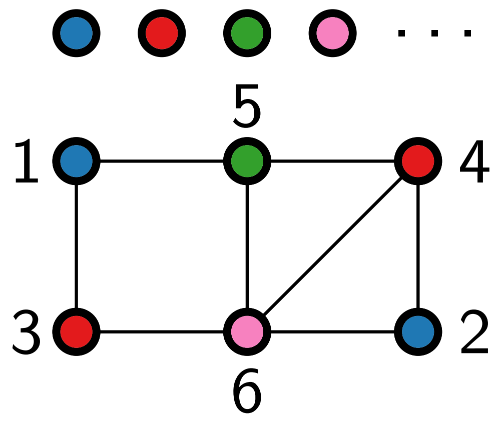

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We can even get $\Delta - 2$ if we return a 2-coloring whenever G is bipartite.

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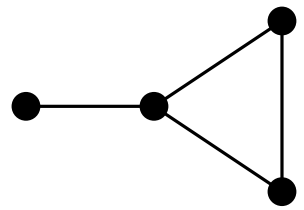

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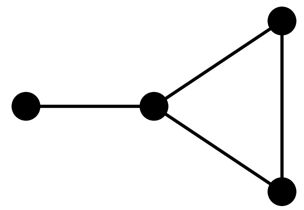

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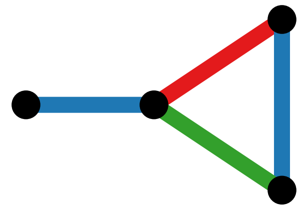

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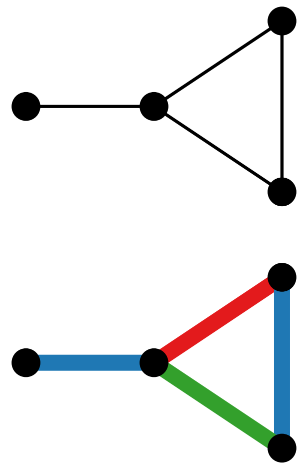

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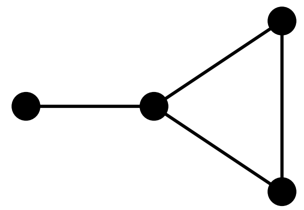

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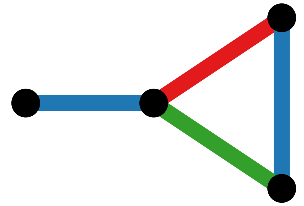

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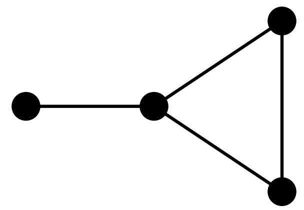

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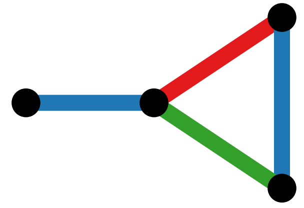

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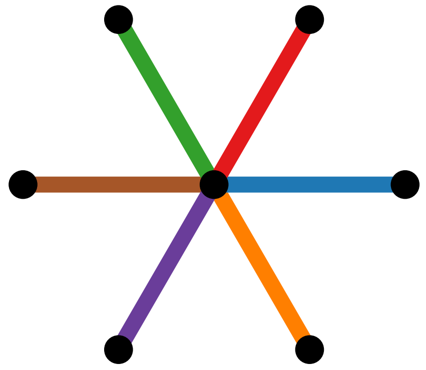

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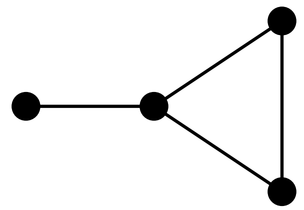

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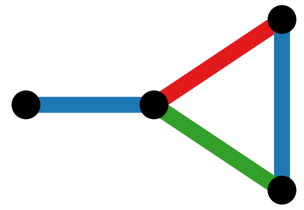

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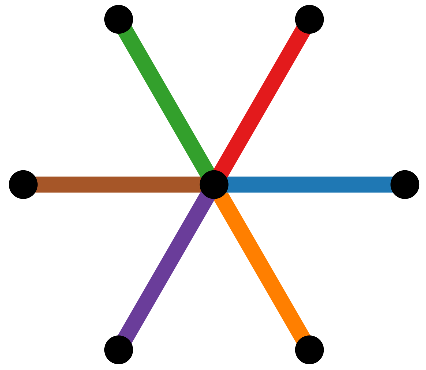
- We show that $\chi'(G) \leq \Delta + 1$.



Minimum Edge Coloring – Upper Bound

Vizing's Theorem.
For every graph $G = (V, E)$ with maximum degree Δ, it holds that $\Delta \leq \chi'(G) \leq \Delta + 1$.

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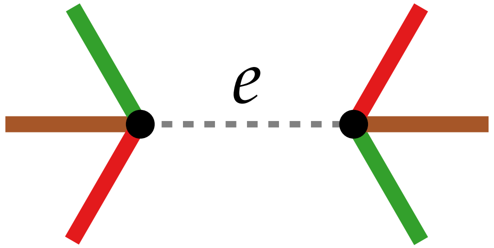
Let G be a graph on m edges, and let $e = uv$ be an edge of G.

- By induction, $G - e$ has a $(\Delta(G - e) + 1)$-edge coloring.



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For every graph $G = (V, E)$ with maximum degree Δ, it holds that $\Delta \leq \chi'(G) \leq \Delta + 1$.

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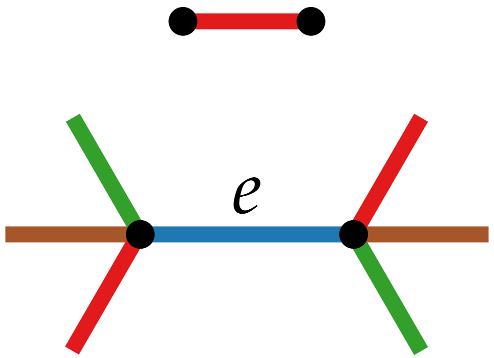
Vadim G. Vizing
(Kiew 1937 – 2017 Odessa)



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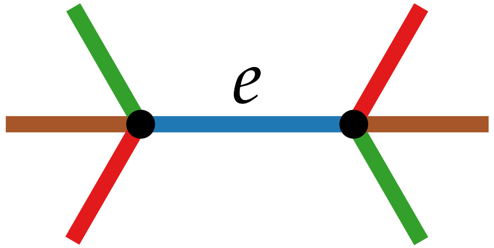

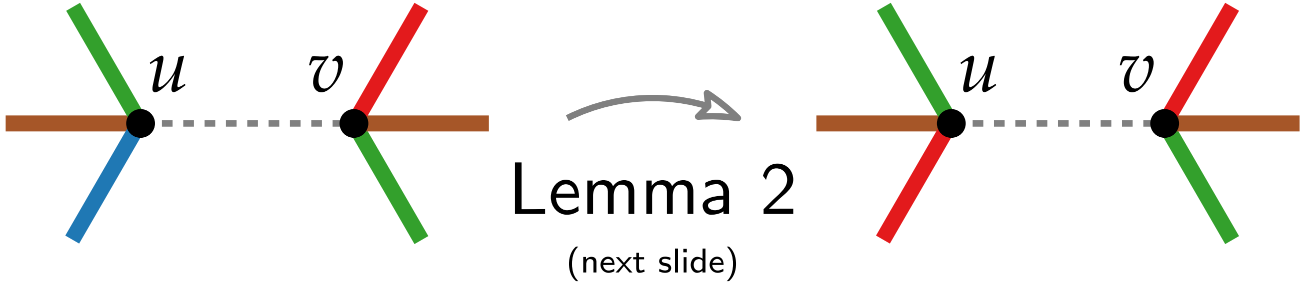

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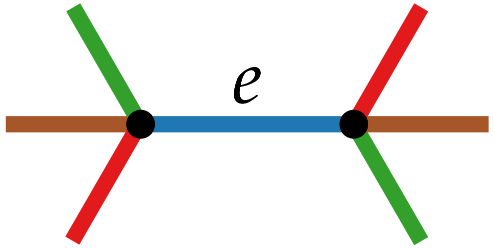

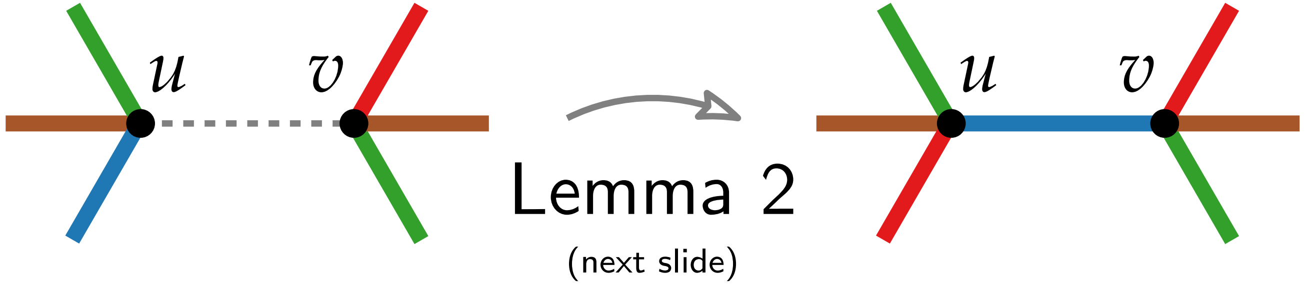

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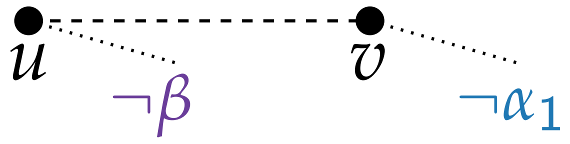

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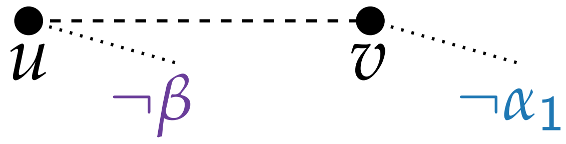

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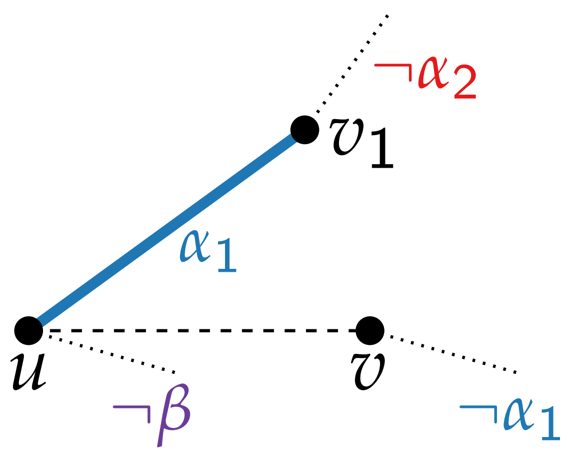

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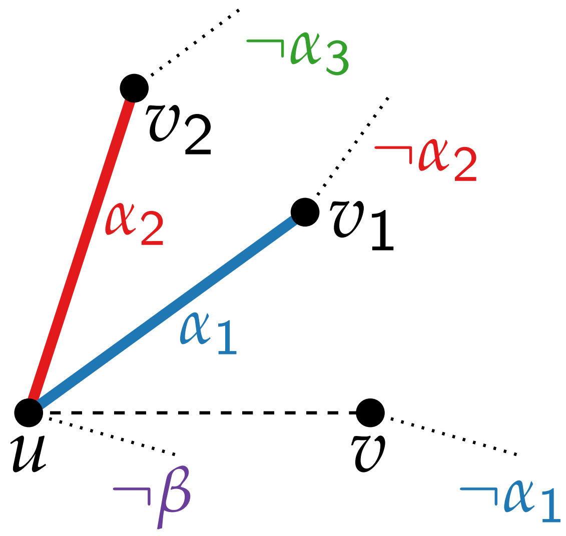

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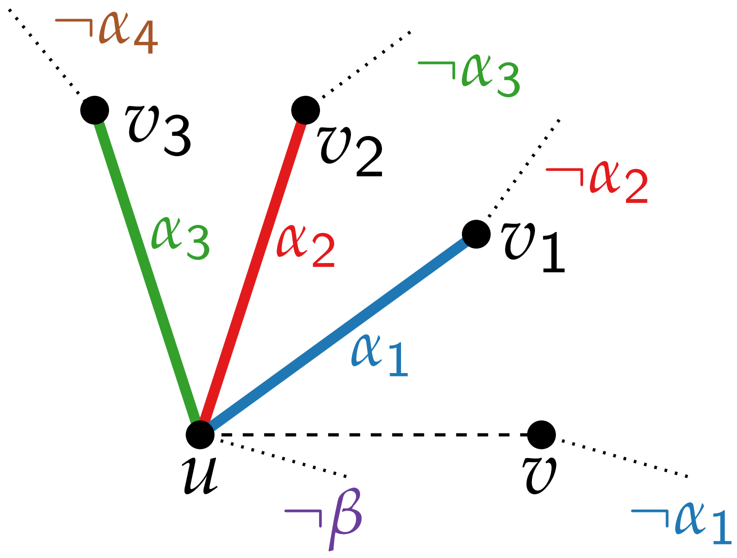

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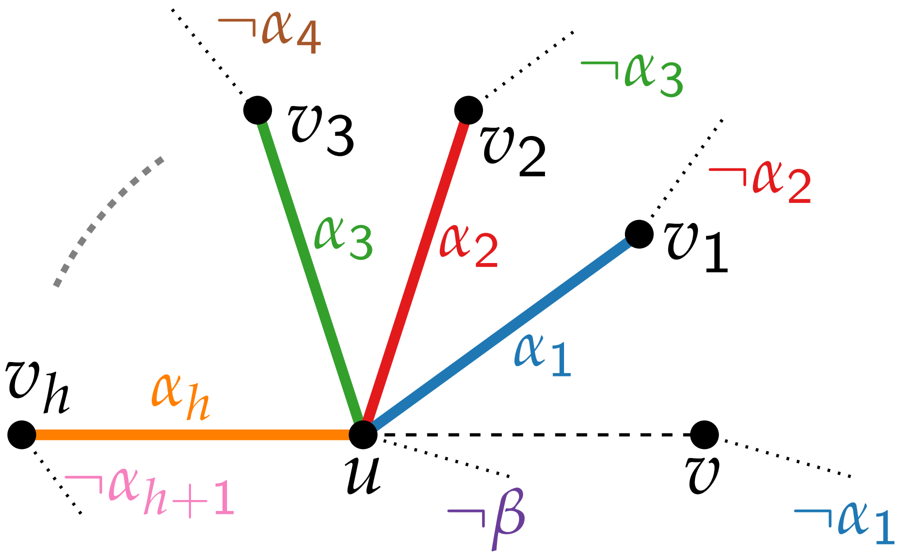

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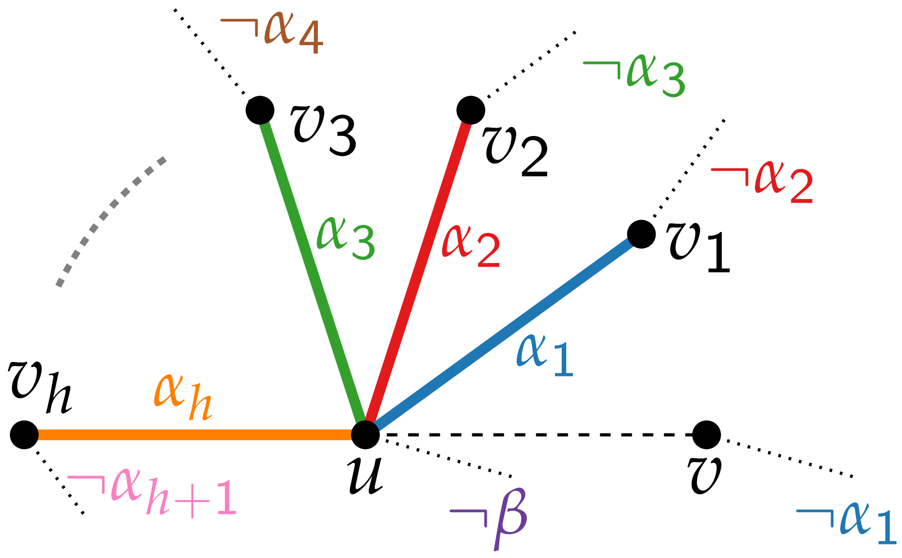

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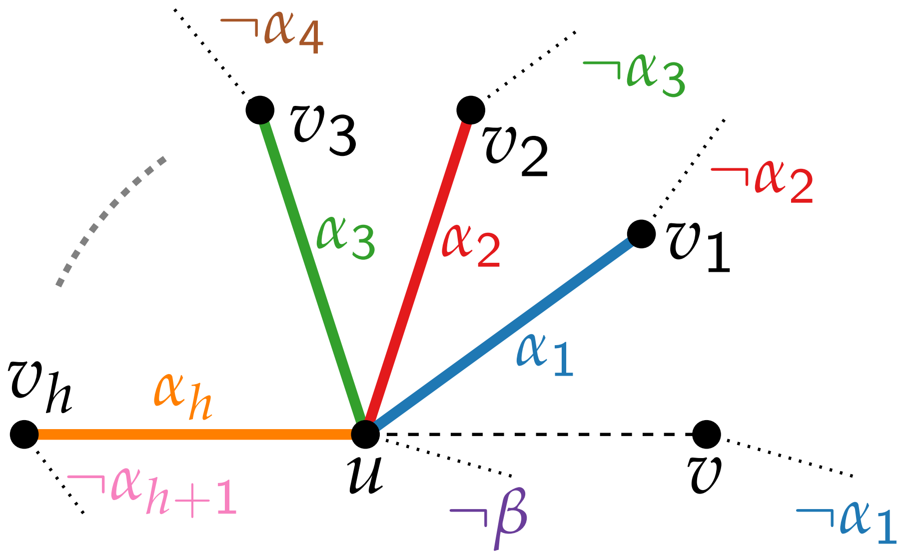

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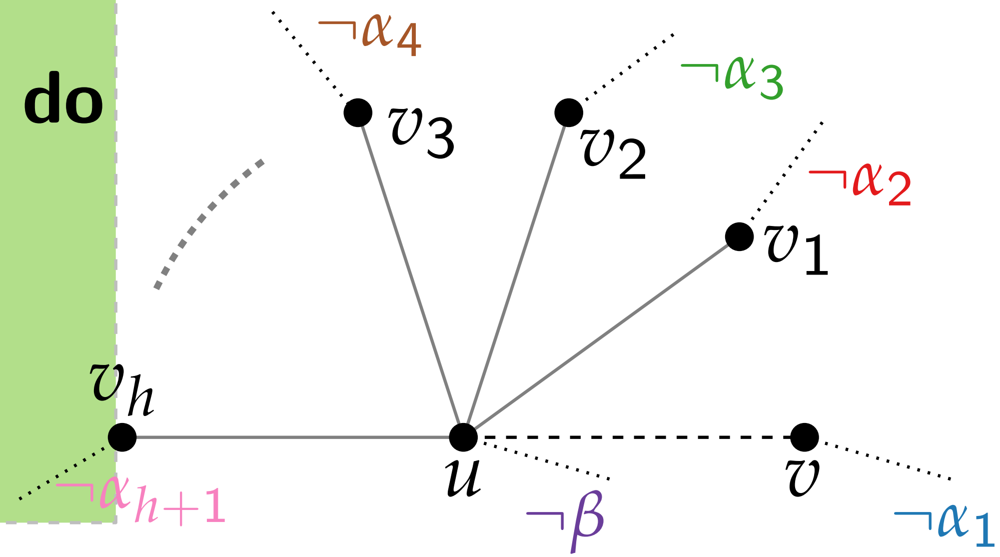

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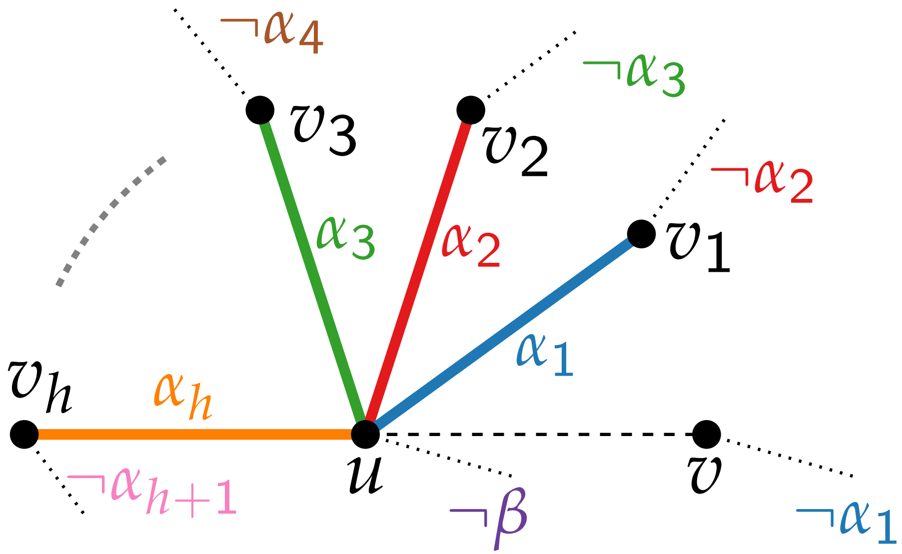

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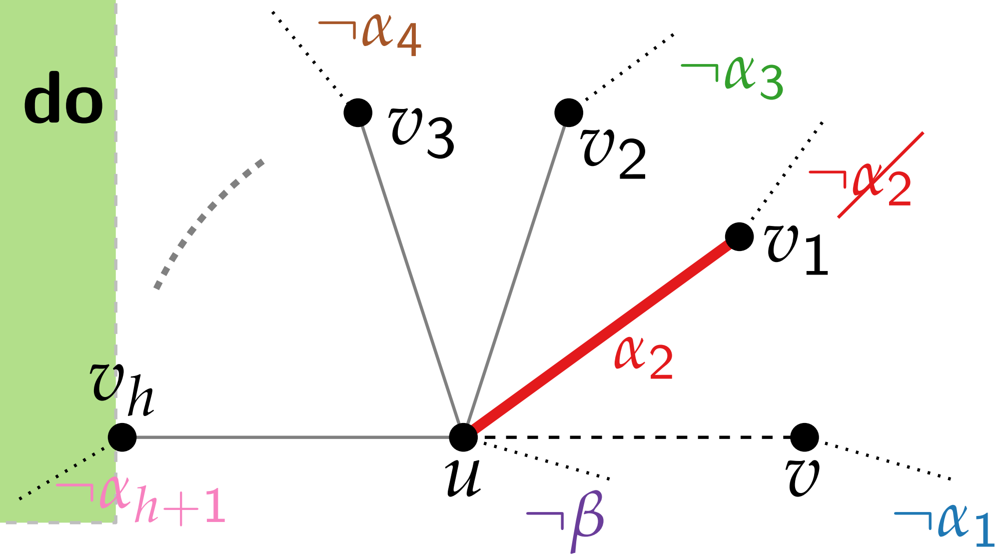

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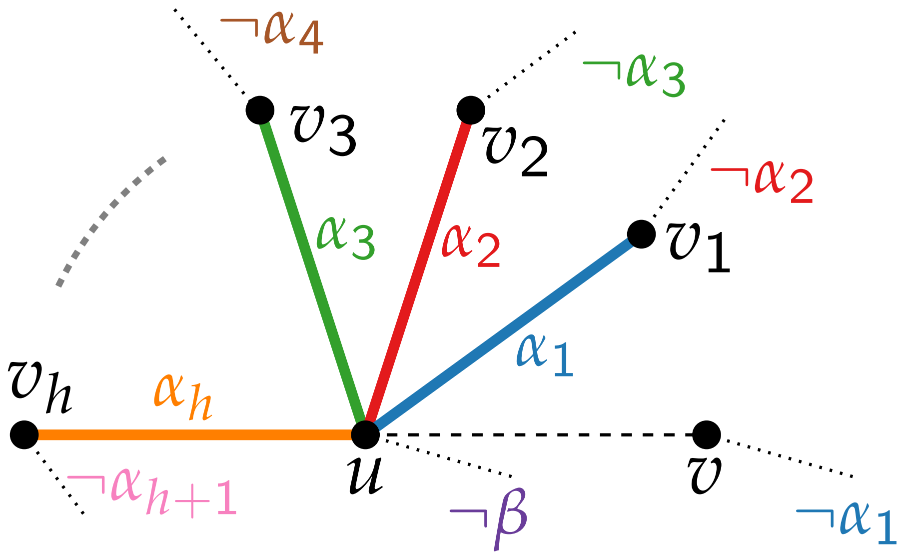

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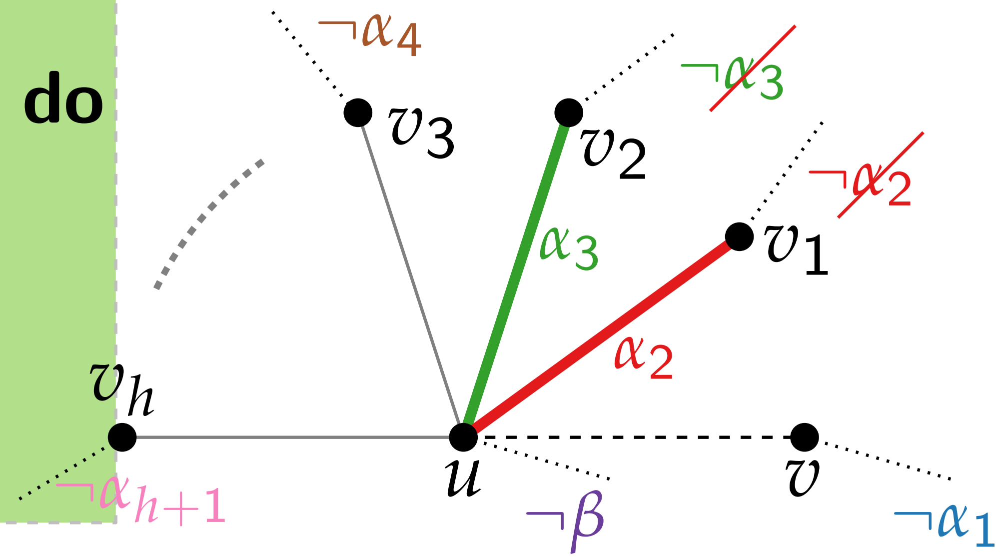

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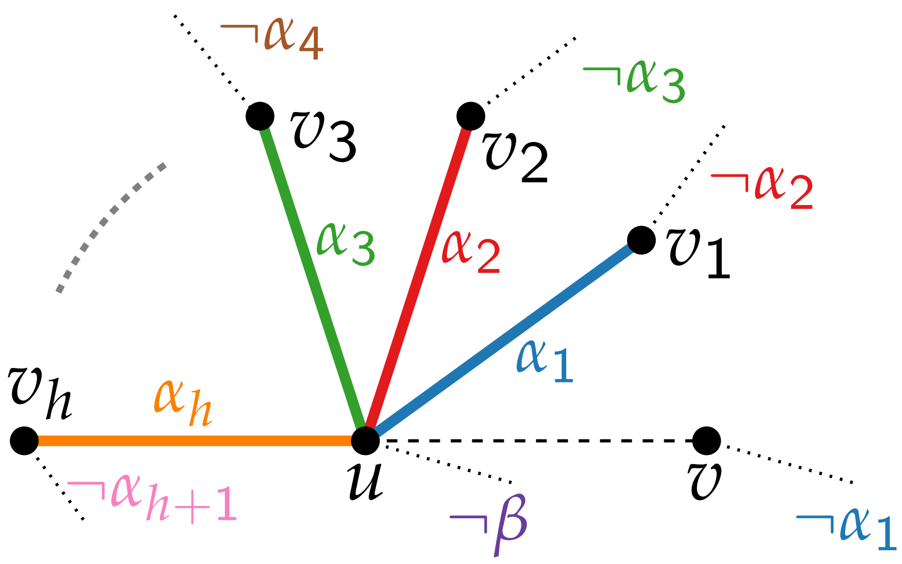


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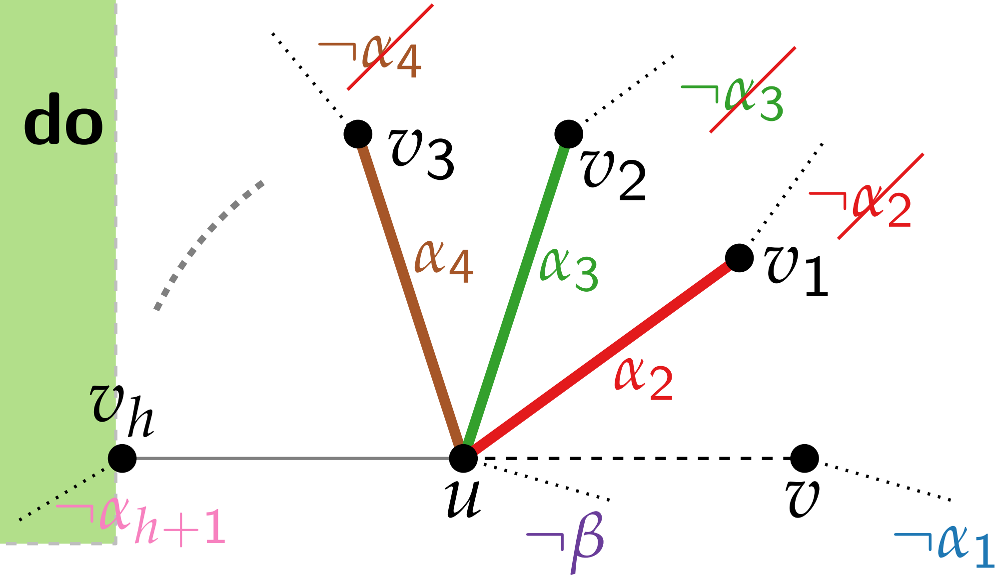

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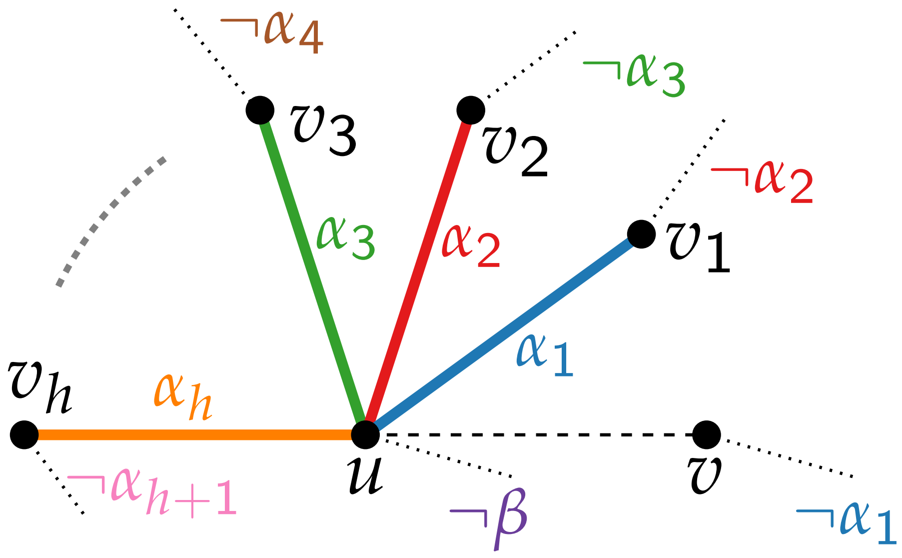

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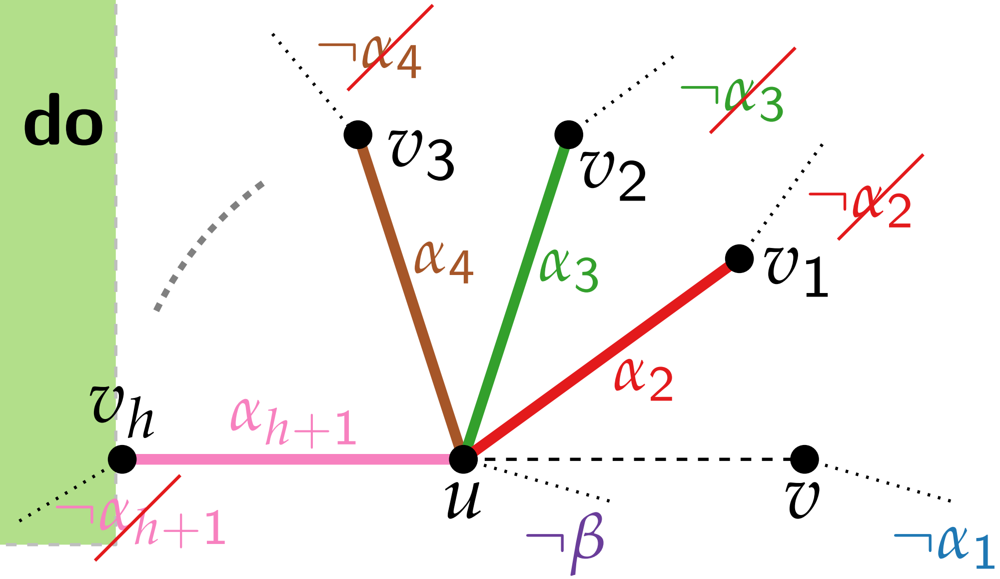

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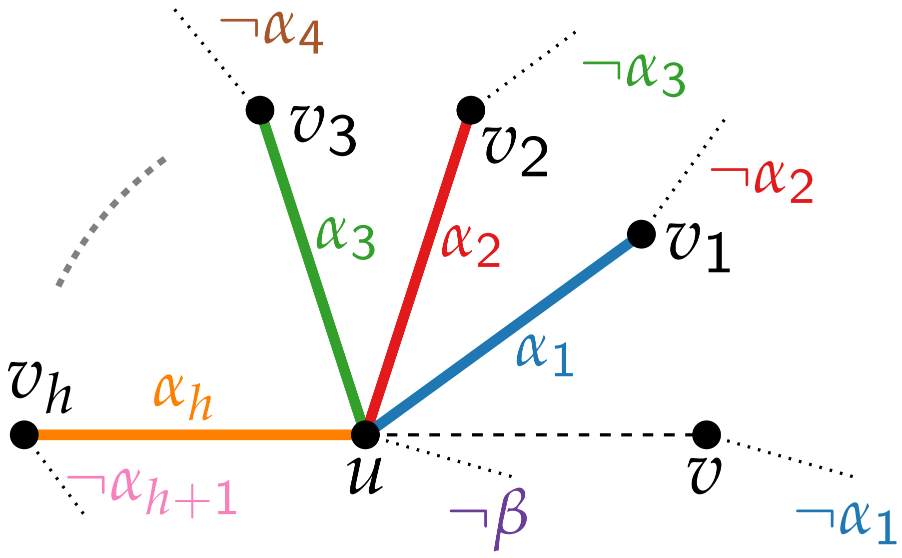

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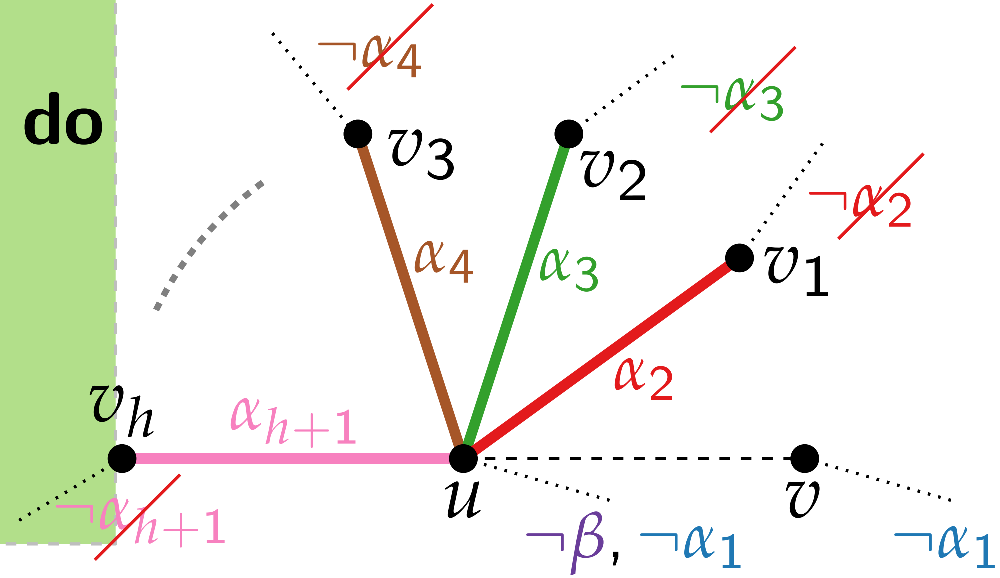

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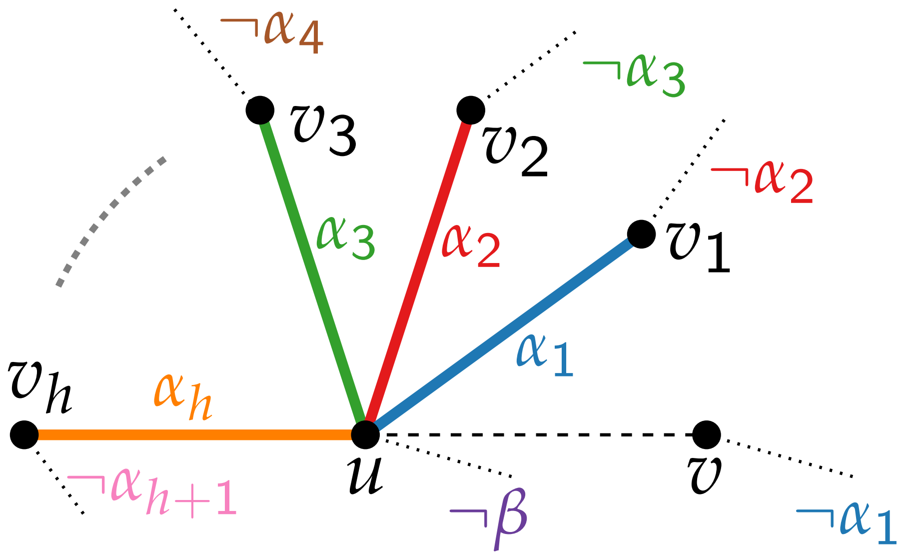

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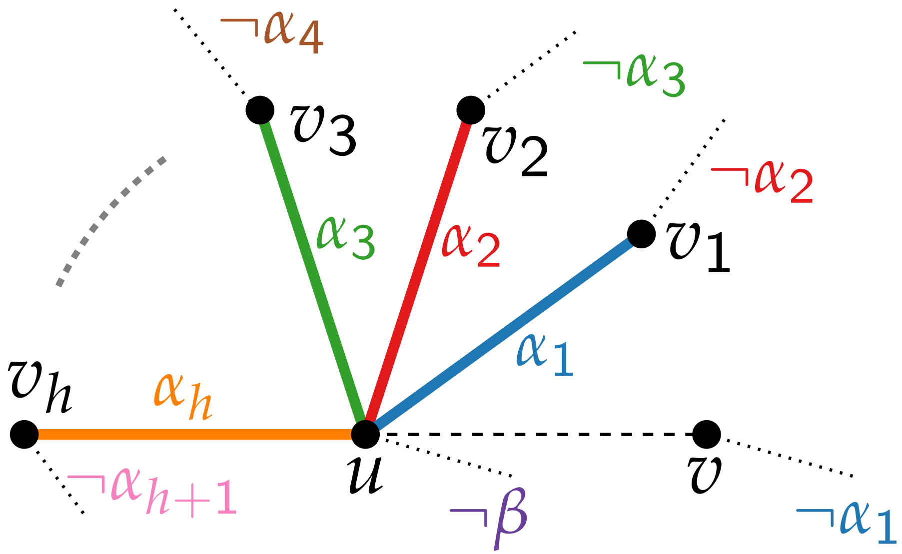

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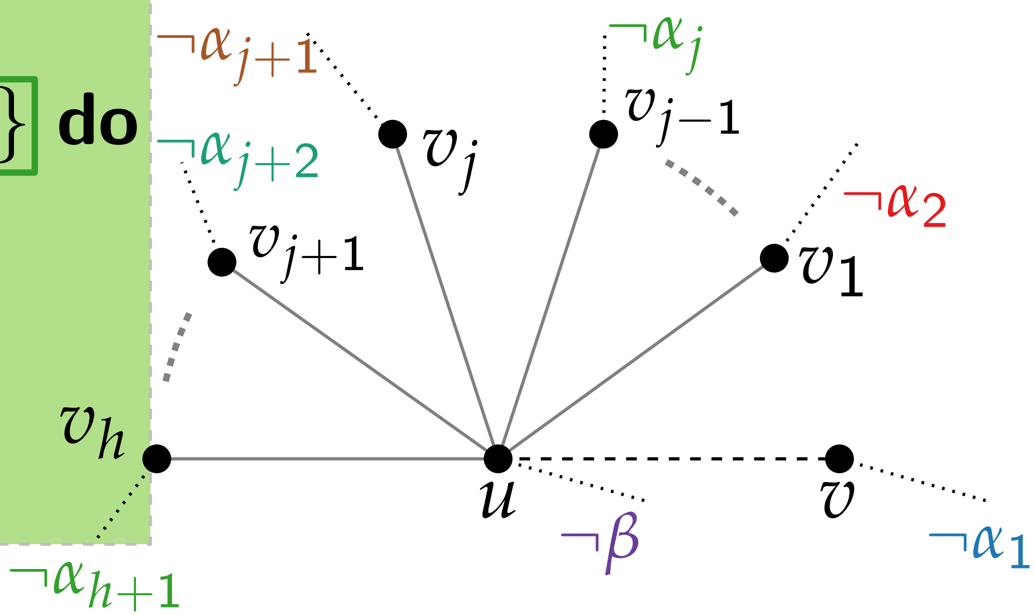

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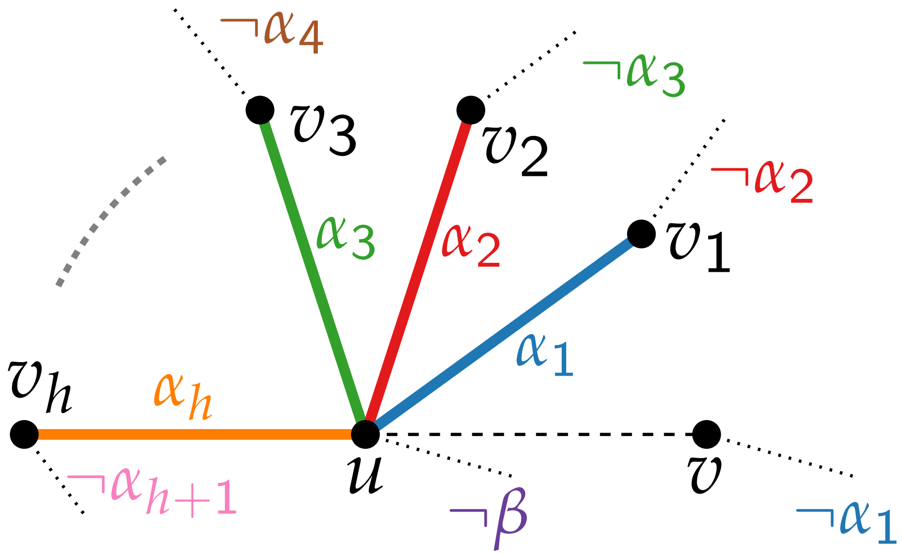

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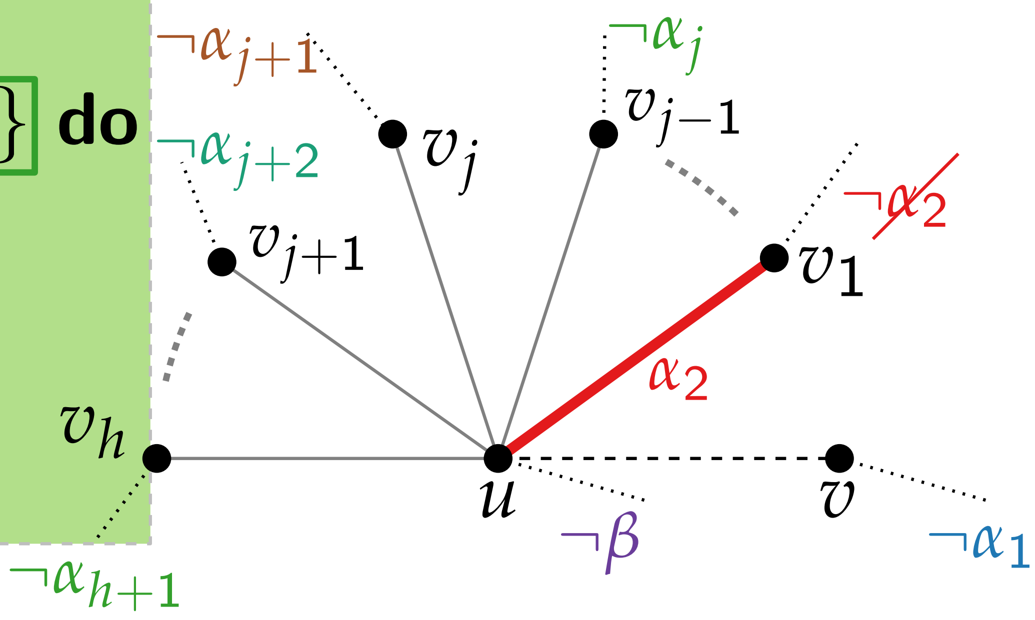

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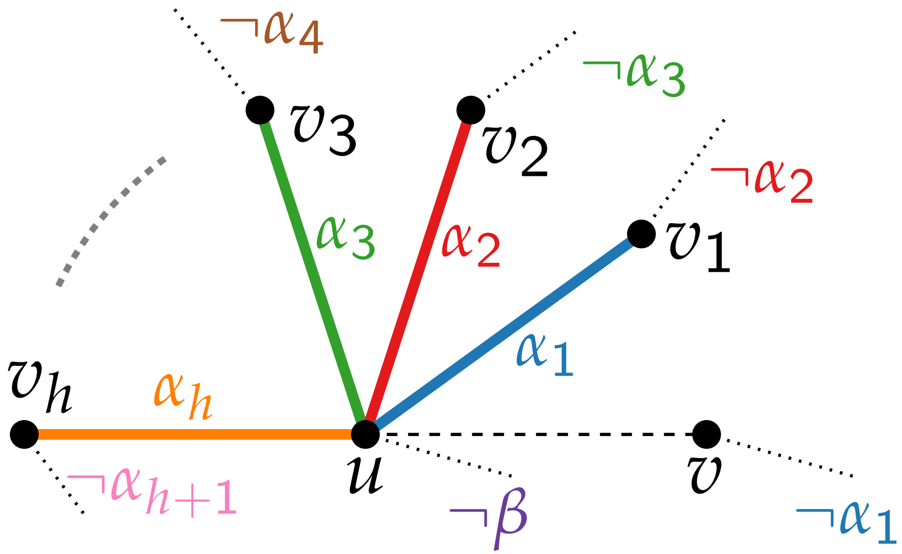

Proof. Note that every vertex is **missing** a color.
Let u miss β and v miss α_1; apply the following algorithm:

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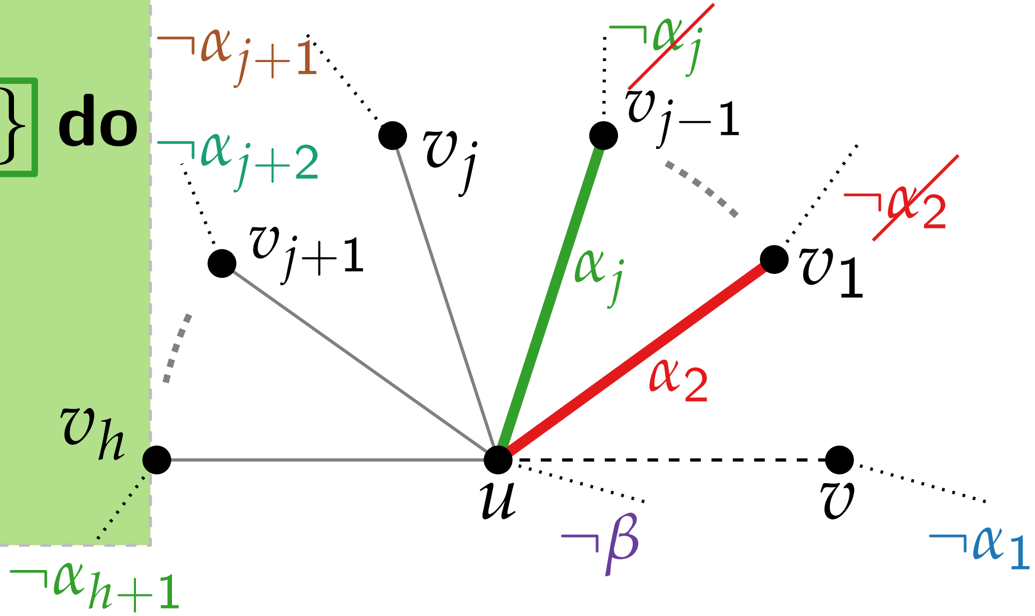
  $i \leftarrow 1$
  while $\exists w \in N(u)\colon c(uw) = \alpha_i \wedge w \notin \{v_1, \dots, v_{i-1}\}$ **do**
    $v_i \leftarrow w$
    $\alpha_{i+1} \leftarrow$ min color missing at w
    $i \leftarrow i + 1$
  return $v_1, \dots, v_i; \alpha_1, \dots, \alpha_{i+1}$

Case 2: $\alpha_{h+1} = \alpha_j$, $j < h$.



Minimum Edge Coloring – Recoloring

Lemma 2.
Let G be a graph with a $(\Delta+1)$-edge coloring c,
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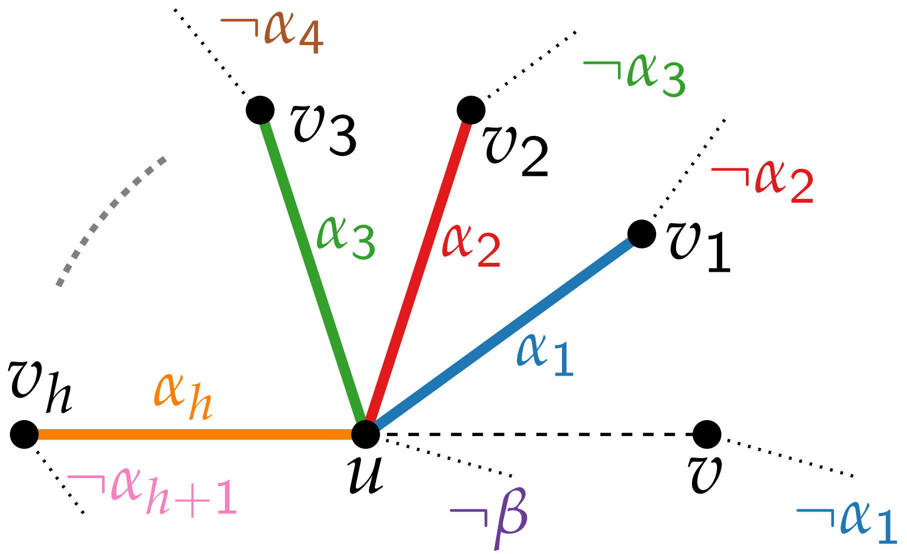
Then c can be changed s.t. u and v miss the same color.



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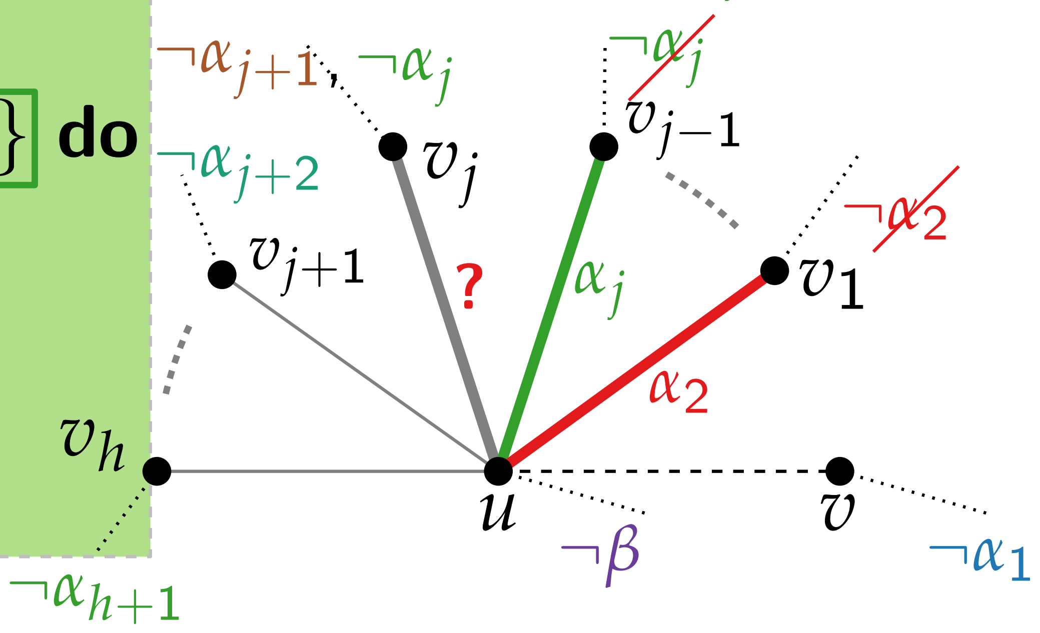

Minimum Edge Coloring – Recoloring

Lemma 2.
Let G be a graph with a $(\Delta + 1)$-edge coloring c,
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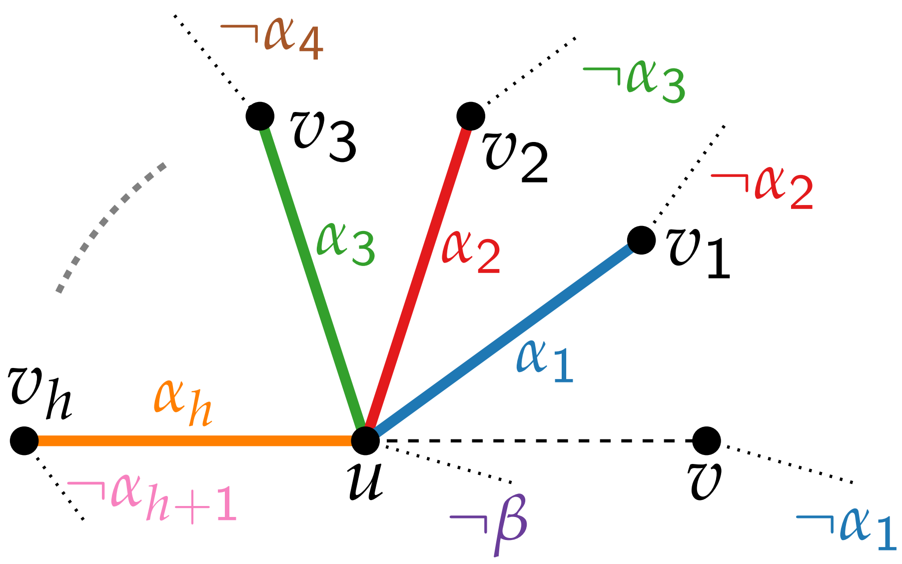
Then c can be changed s.t. u and v miss the same color.



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  i ← 1
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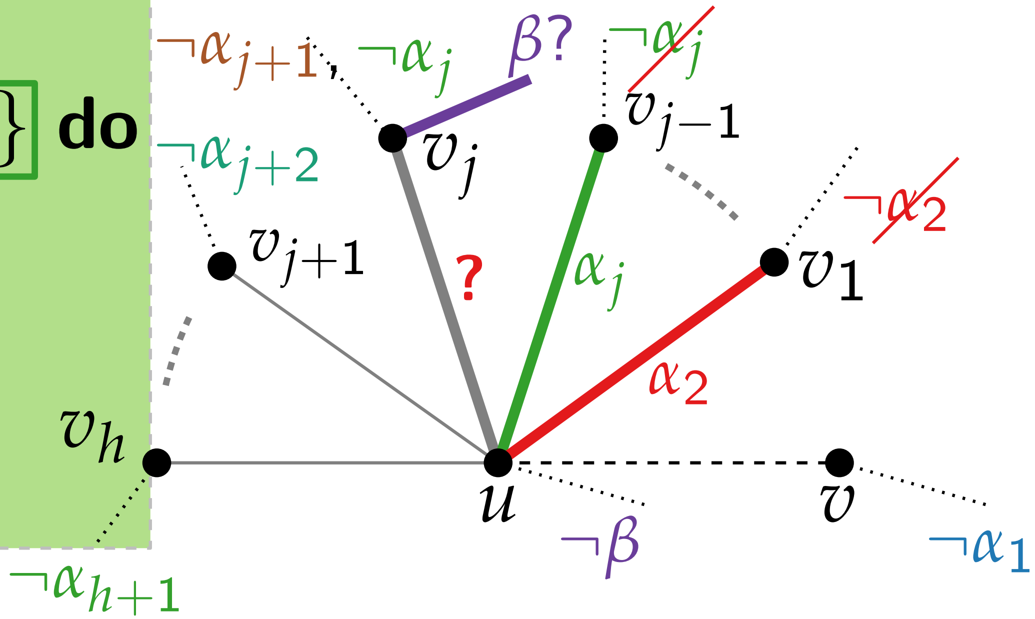

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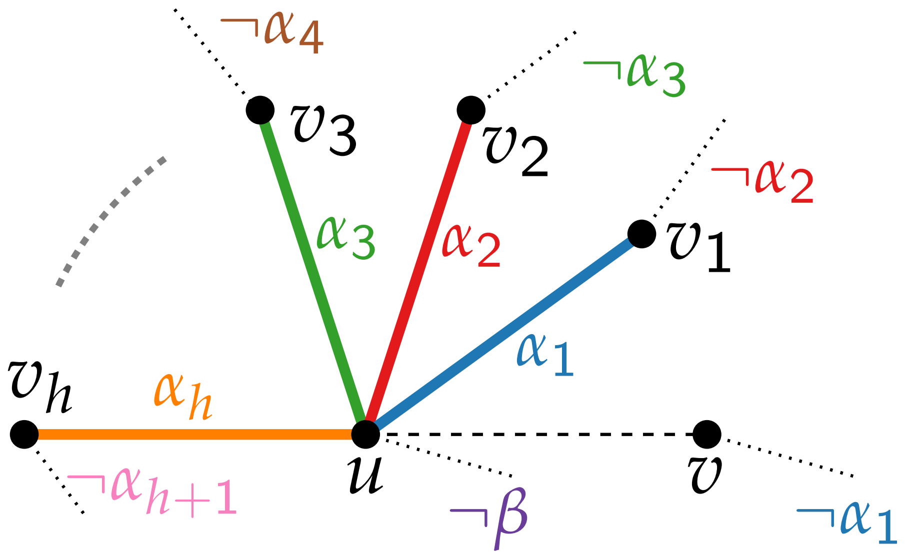

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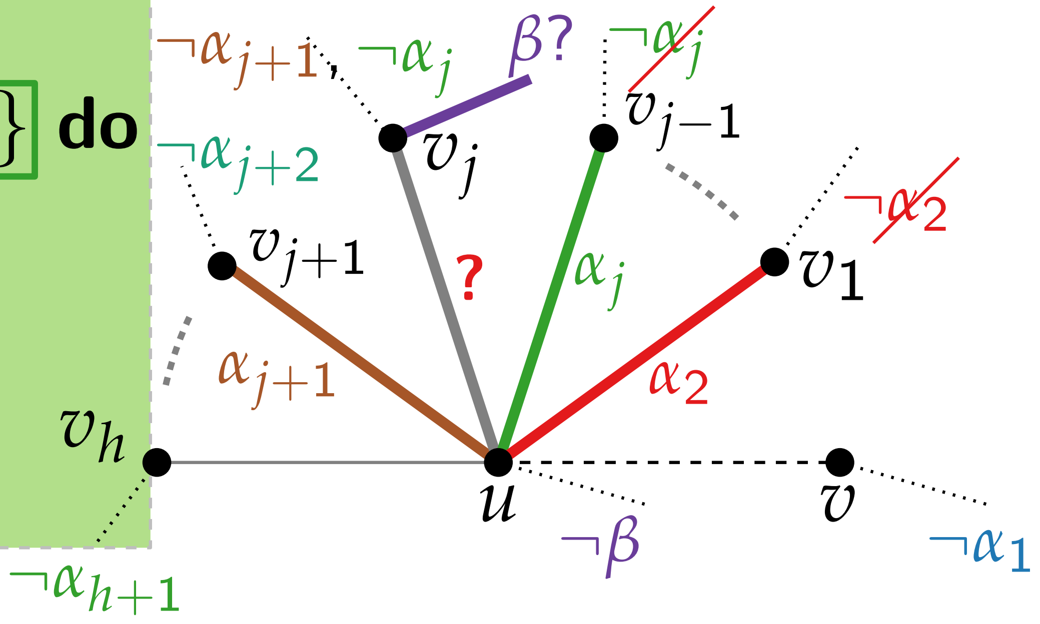

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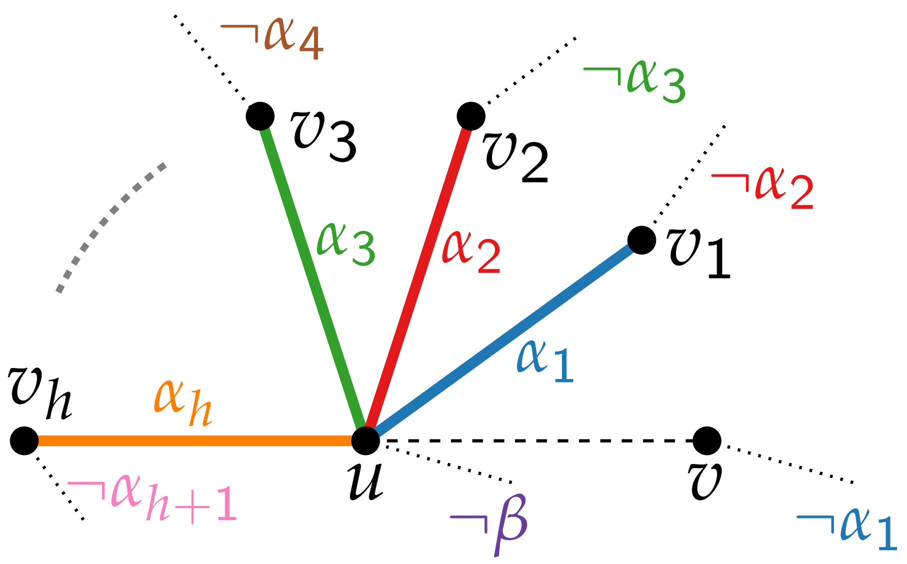

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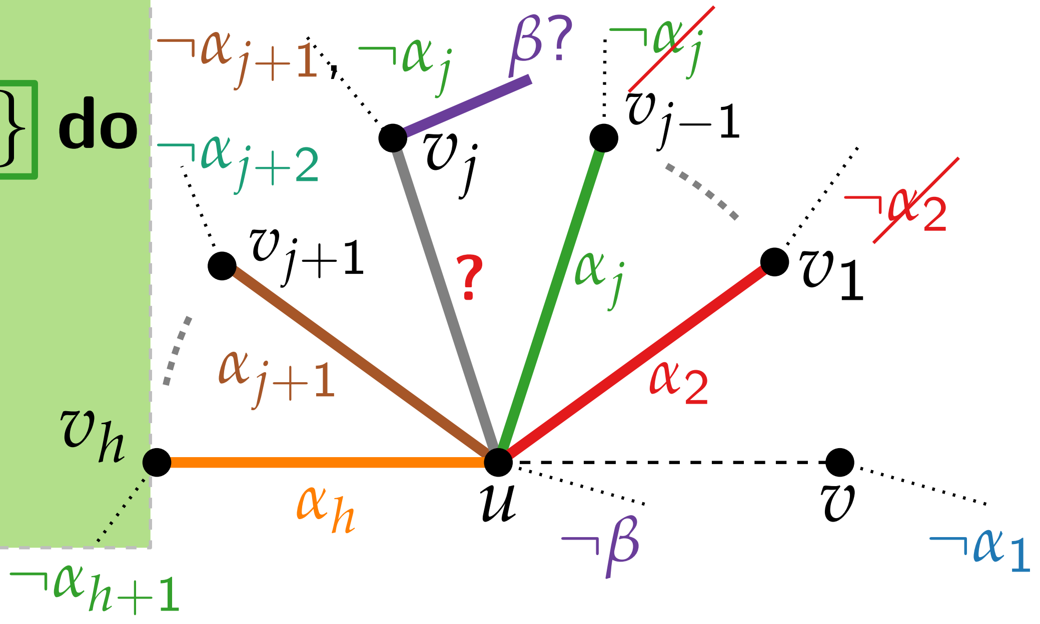

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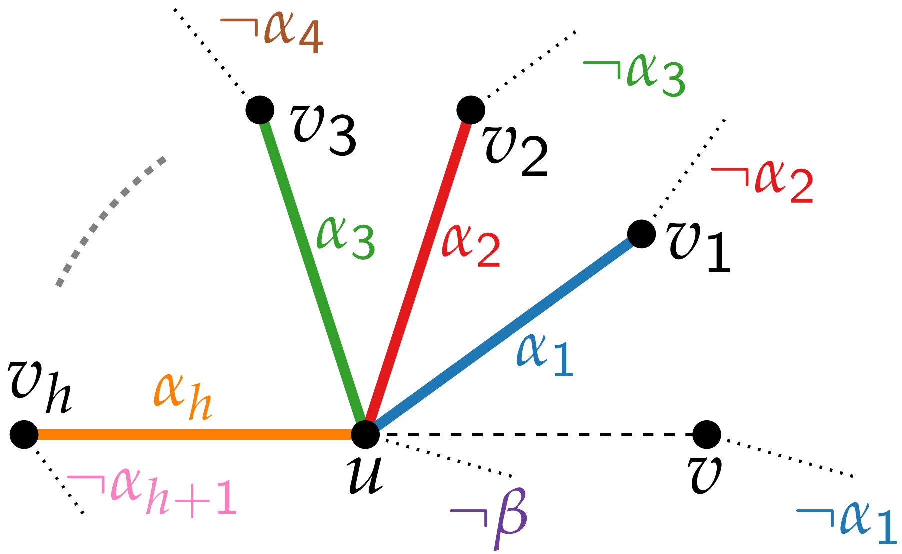

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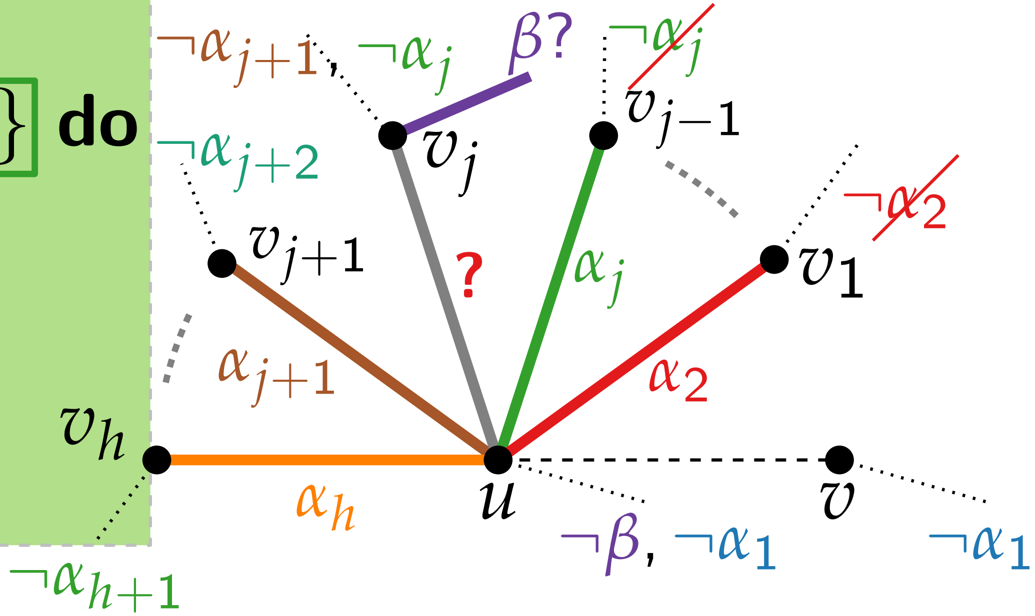

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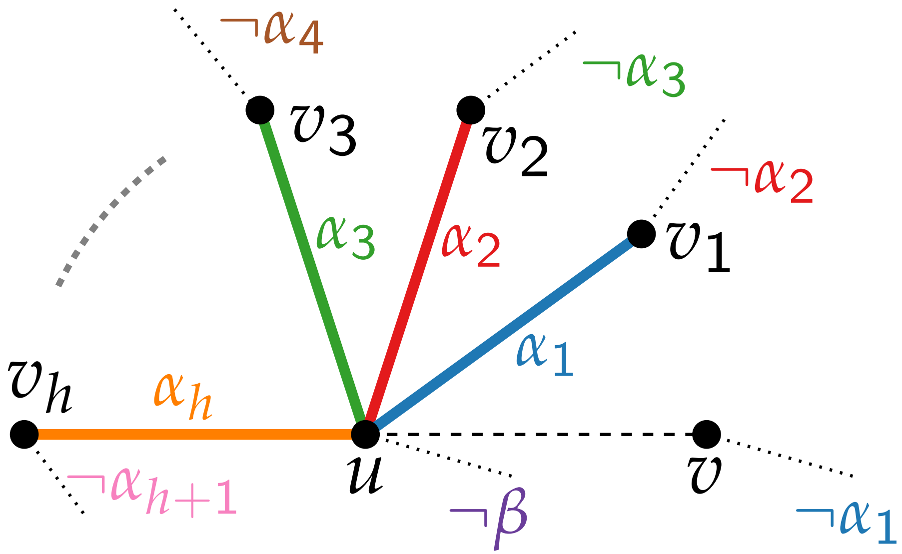

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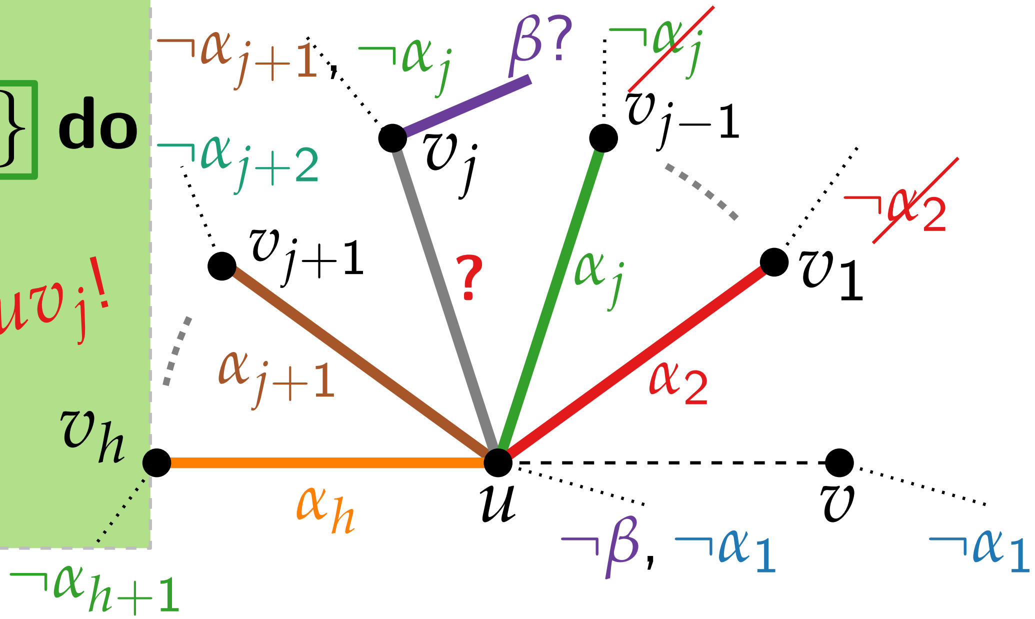
```

Need color for edge uv_j!

Case 2: $\alpha_{h+1} = \alpha_j$, $j < h$.



Minimum Edge Coloring – Recoloring

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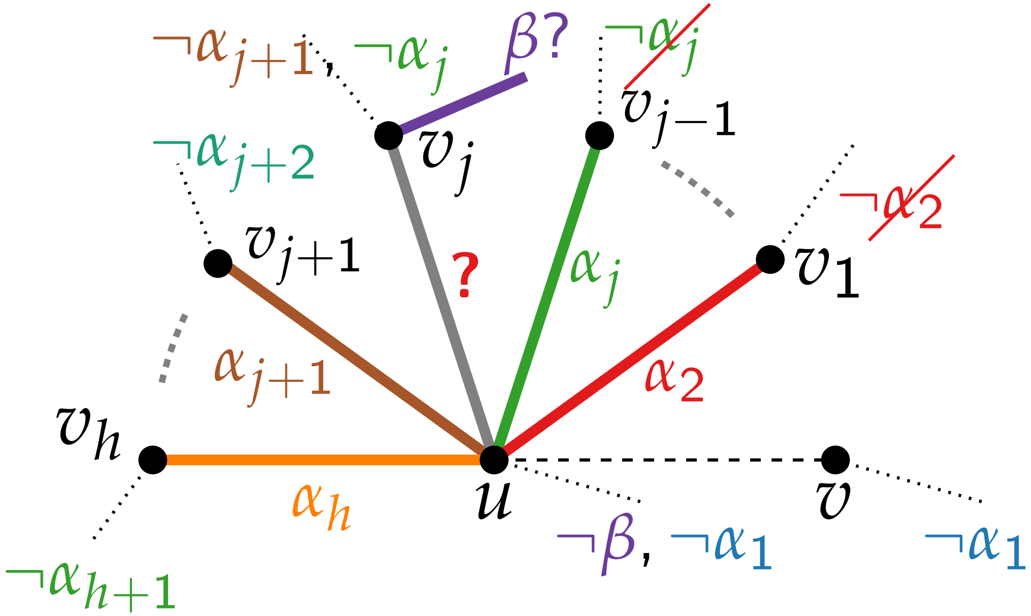

Minimum Edge Coloring – Recoloring

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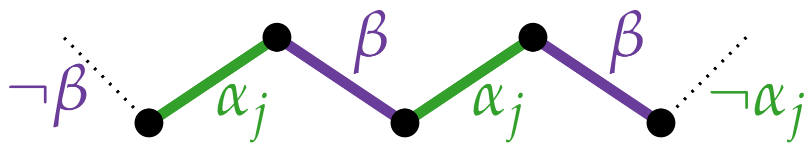
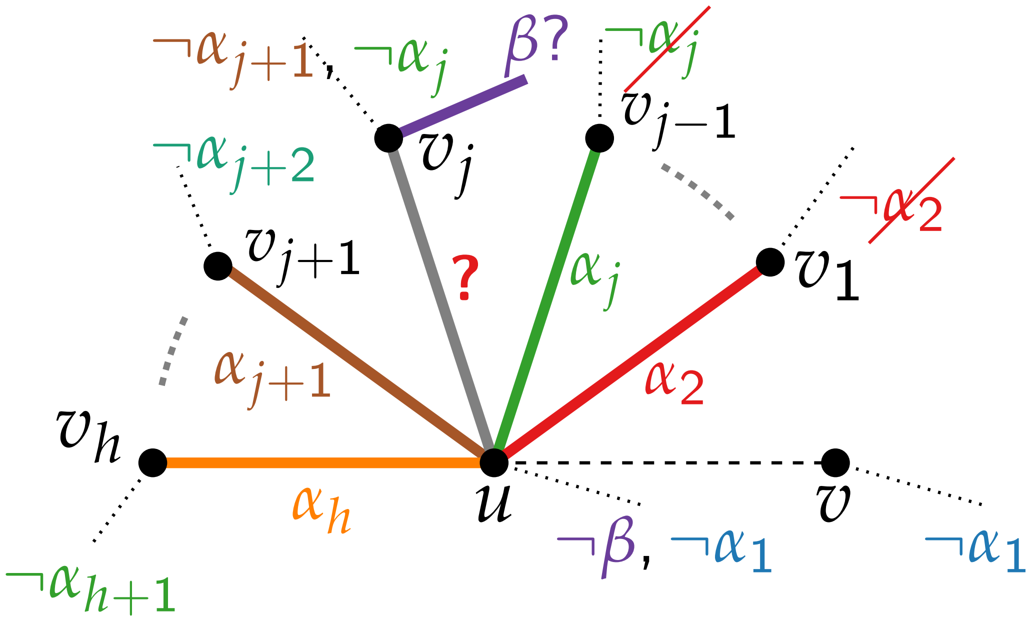
- Consider subgraph G' of G induced by the edges of colors β and α_j.





Minimum Edge Coloring – Recoloring

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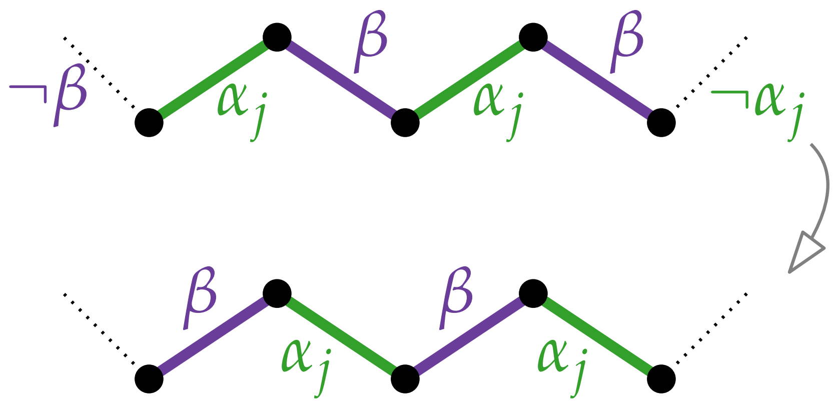
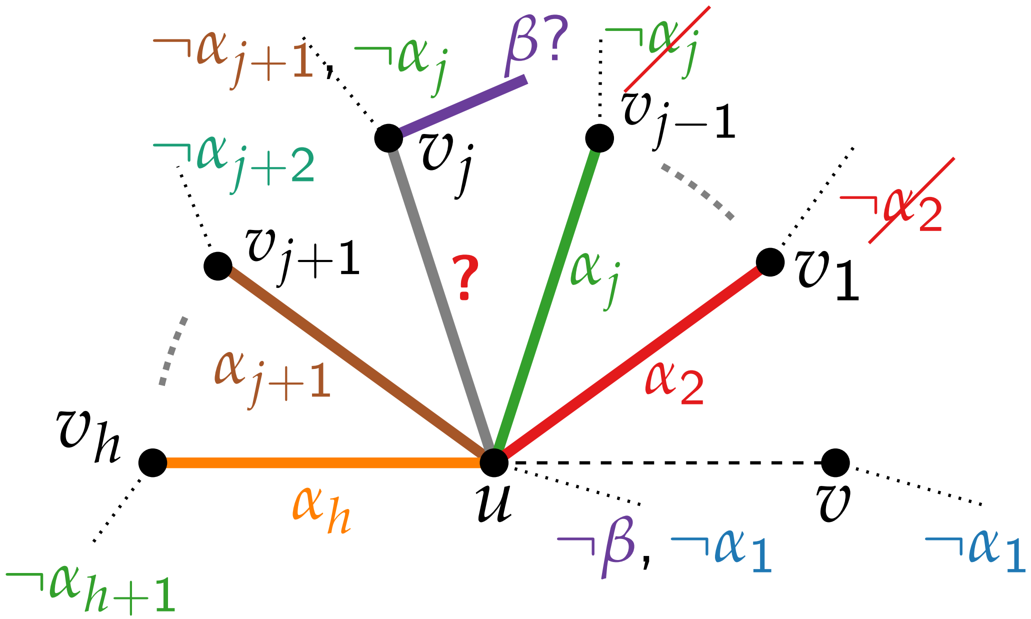
- Consider subgraph G' of G induced by the edges of colors β and α_j.
- Since $\Delta(G') \leq 2$, we can recolor components.





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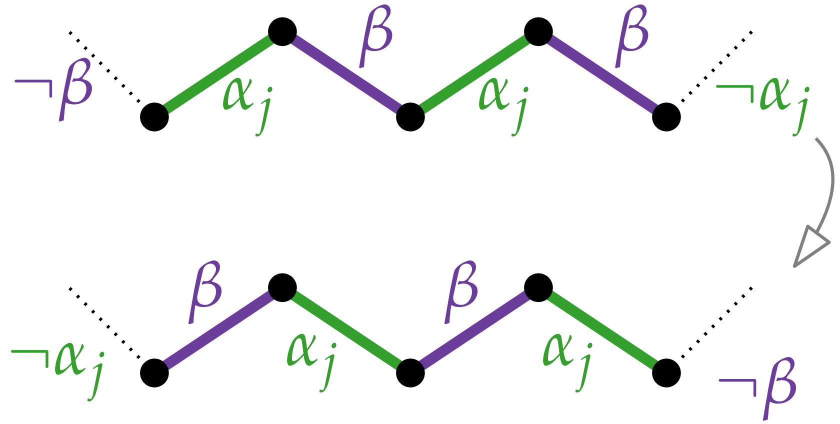

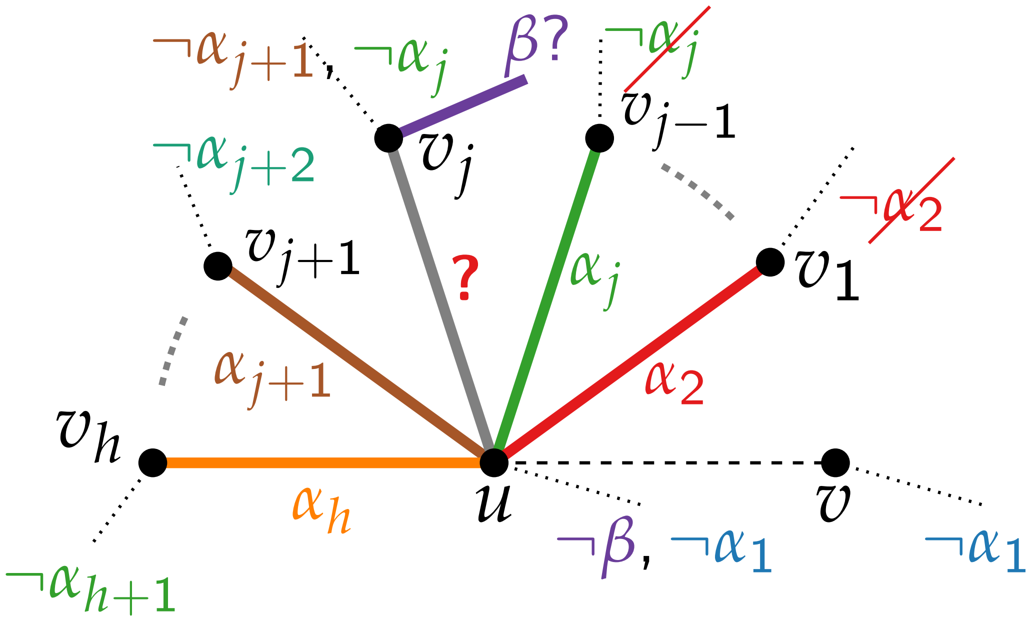

Minimum Edge Coloring – Recoloring

Proof continued for Case 2: $\alpha_{h+1} = \alpha_j$, $j < h$, and we need to find a color for edge uv_j.

- Consider subgraph G' of G induced by the edges of colors β and α_j.
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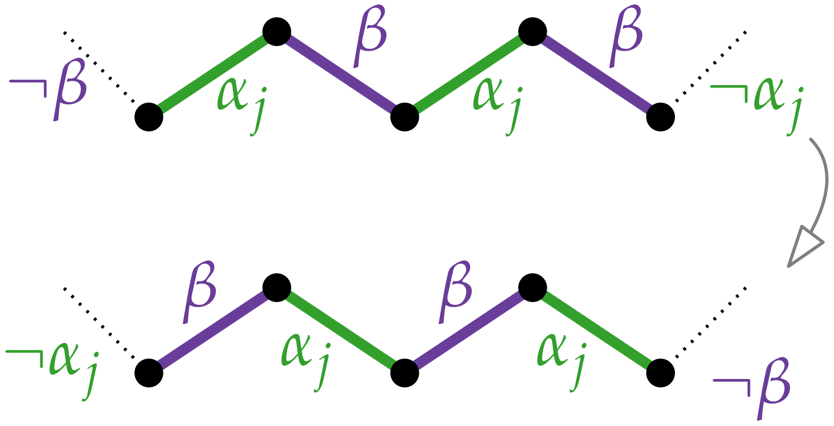
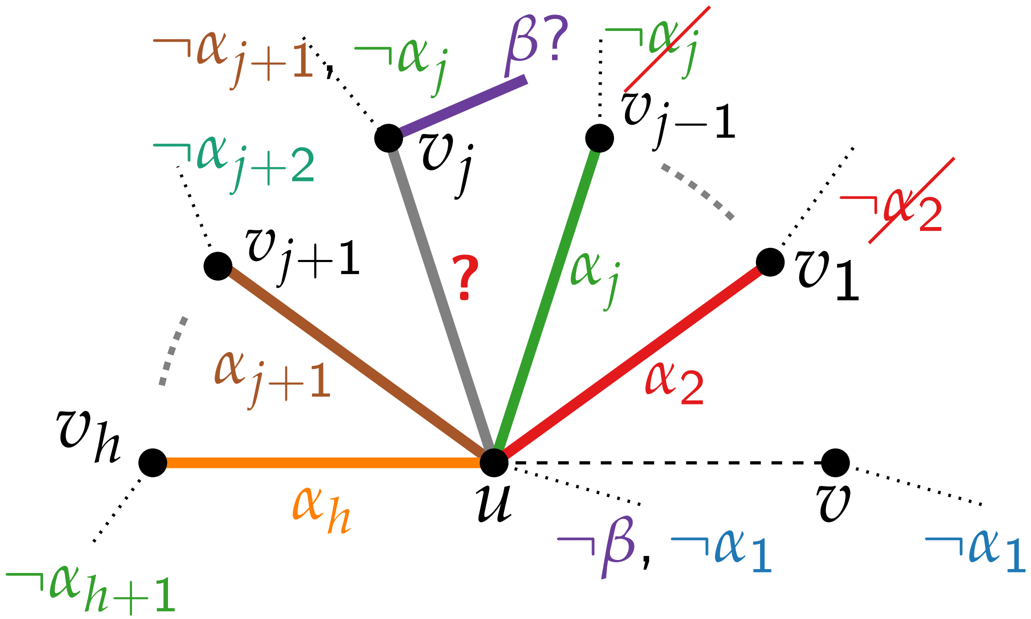
- Nodes u, v_j, v_h are all leaves in G'.
  $\Rightarrow$ They are not all in the same component of G'.





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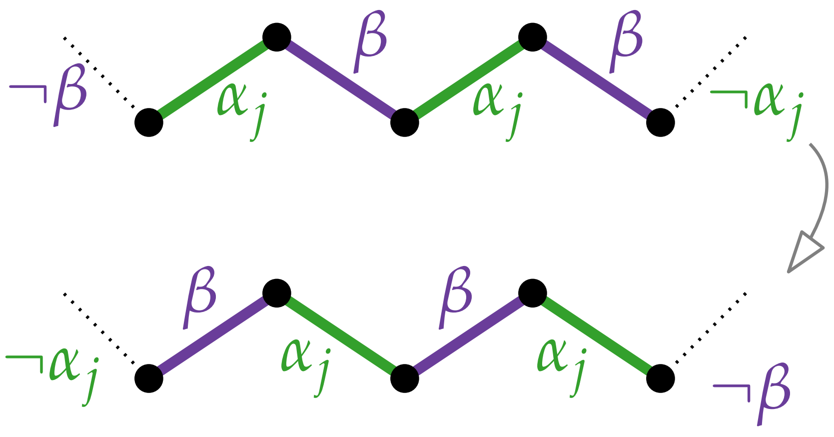

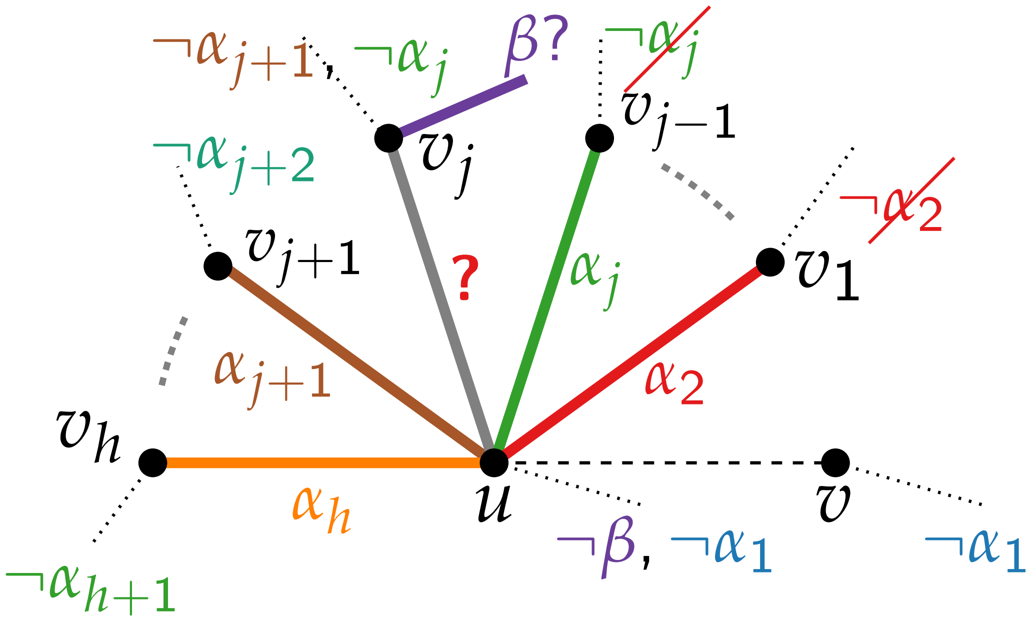

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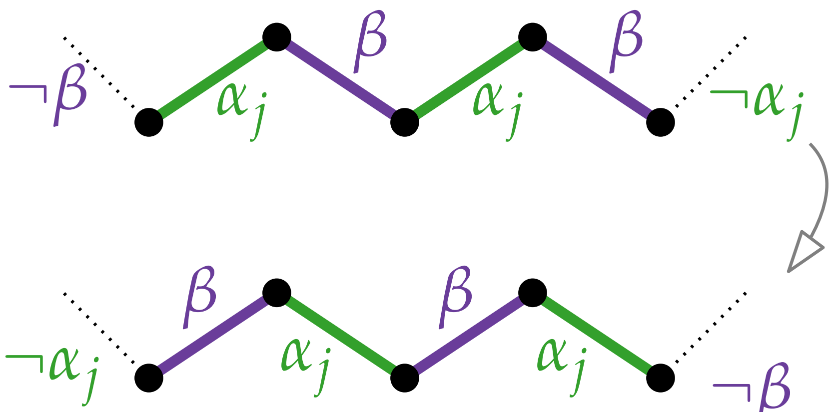

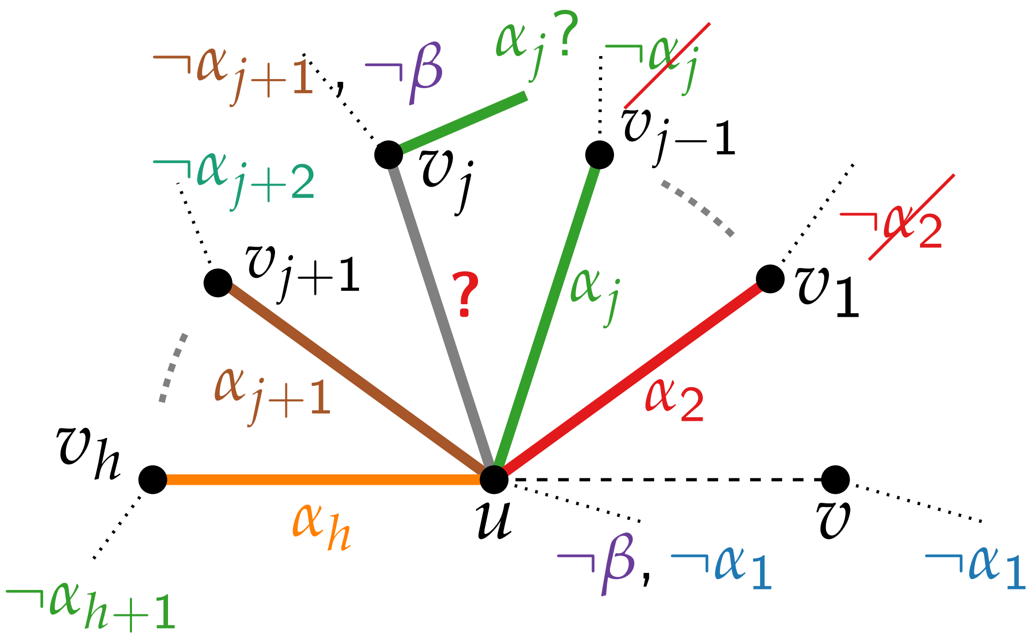

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- If u and v_j are not in the same component:
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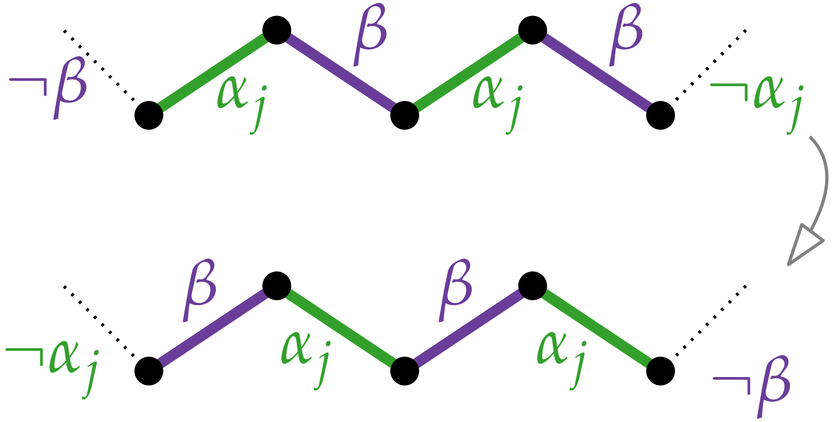

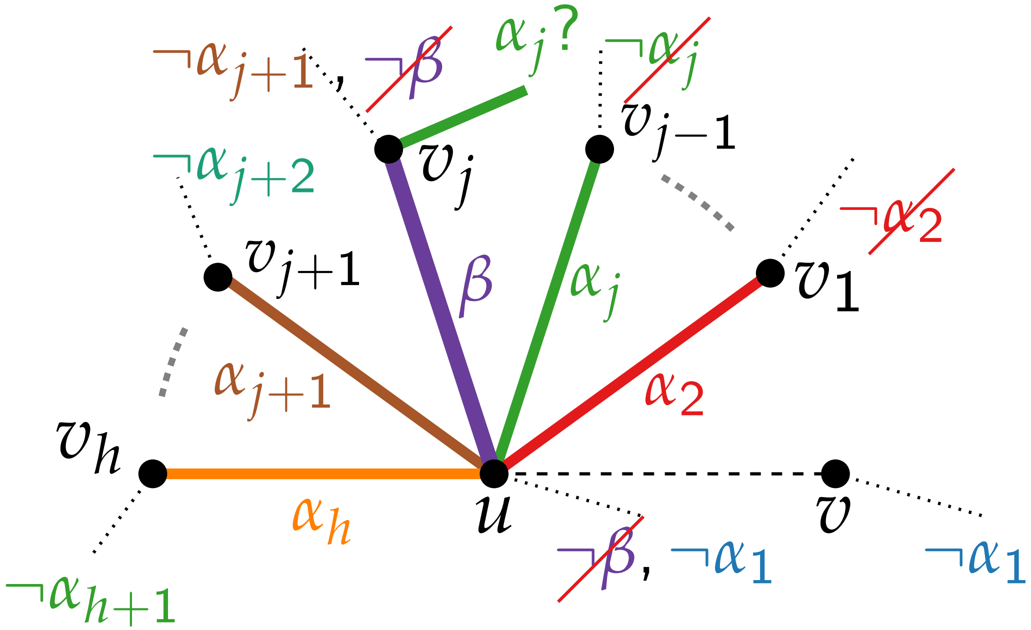

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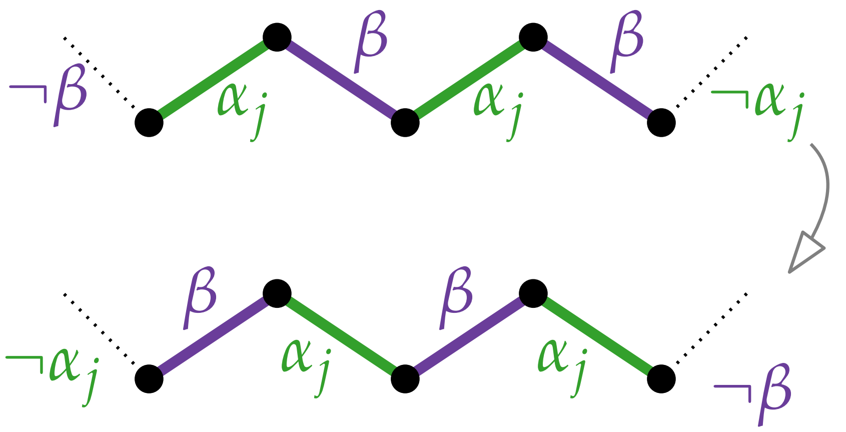
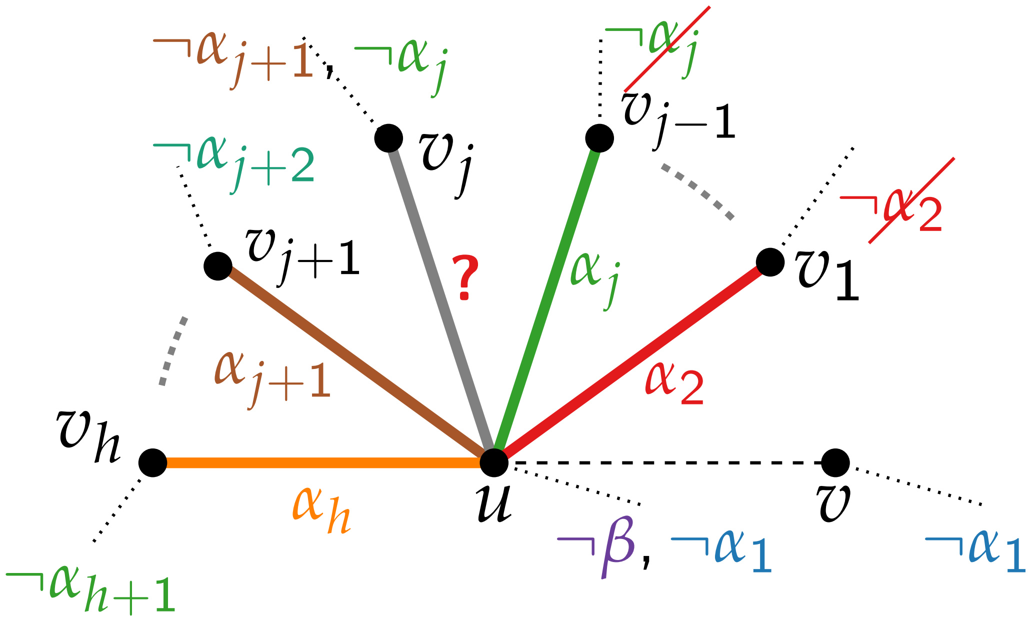
- What if u and v_j are in the same component?





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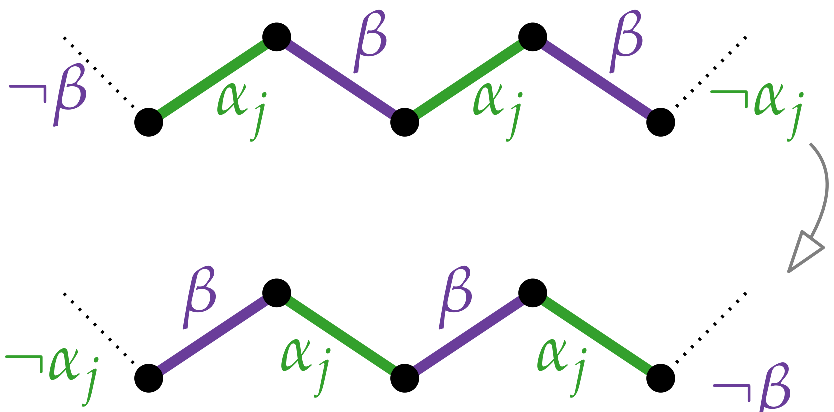

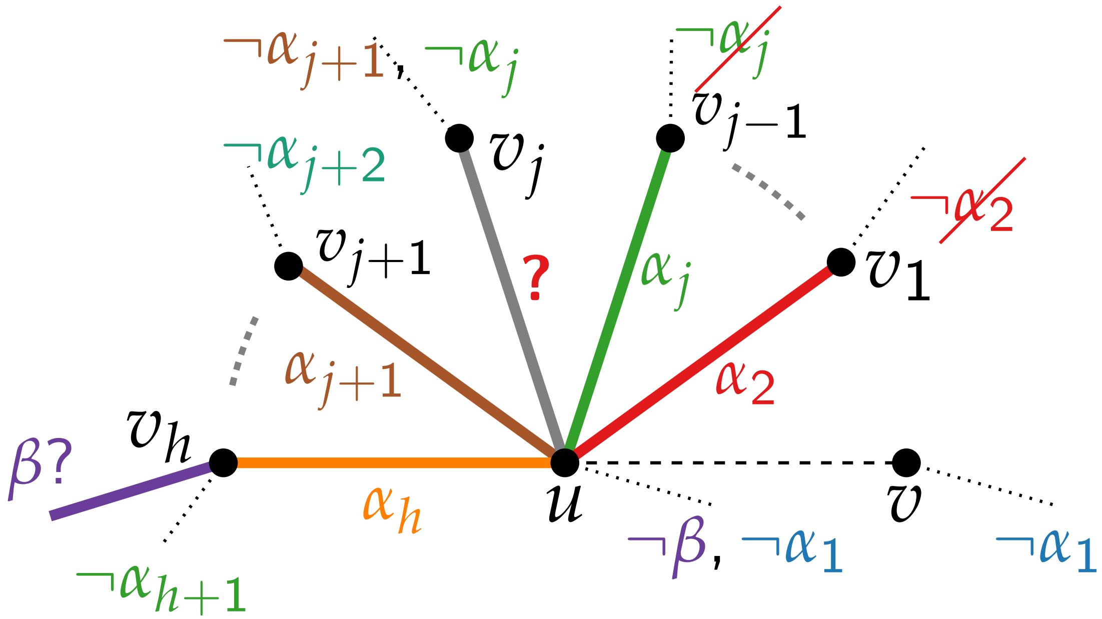

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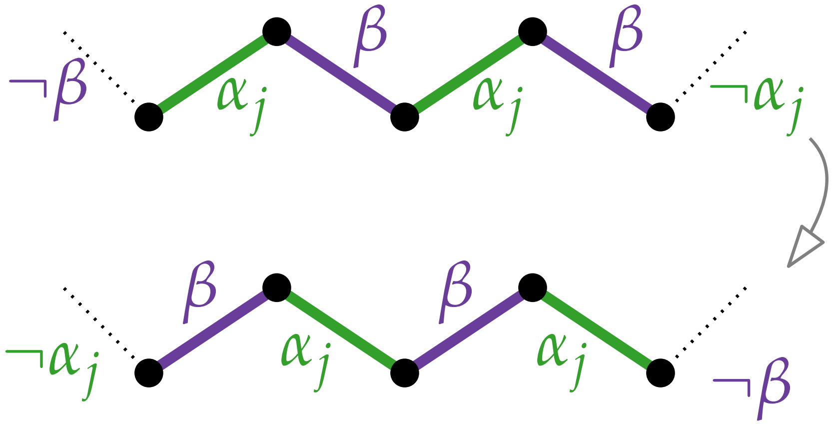

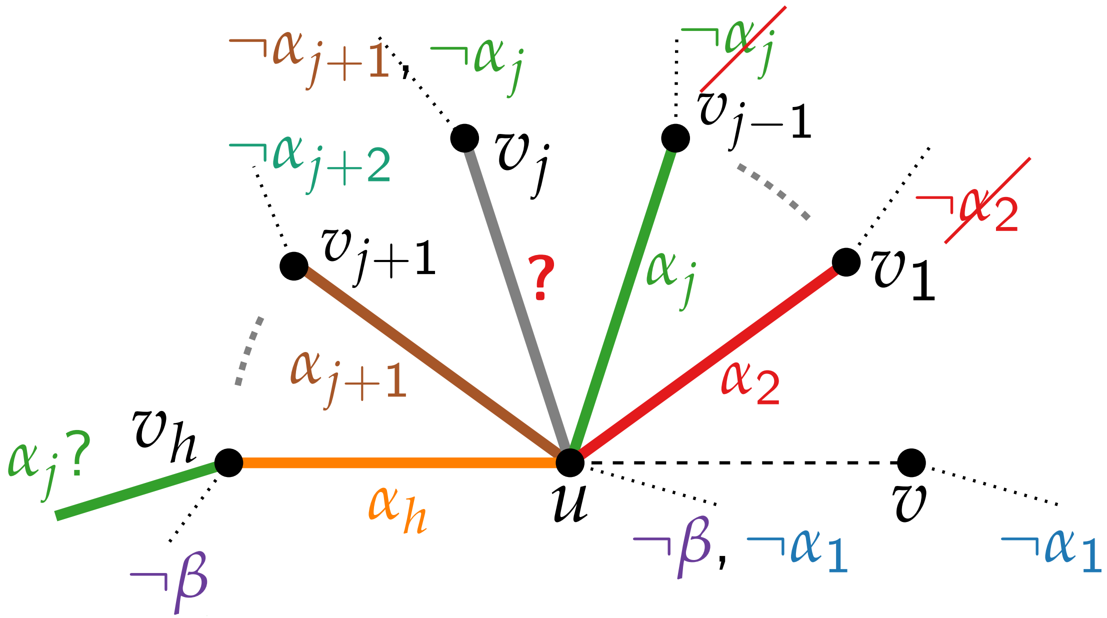

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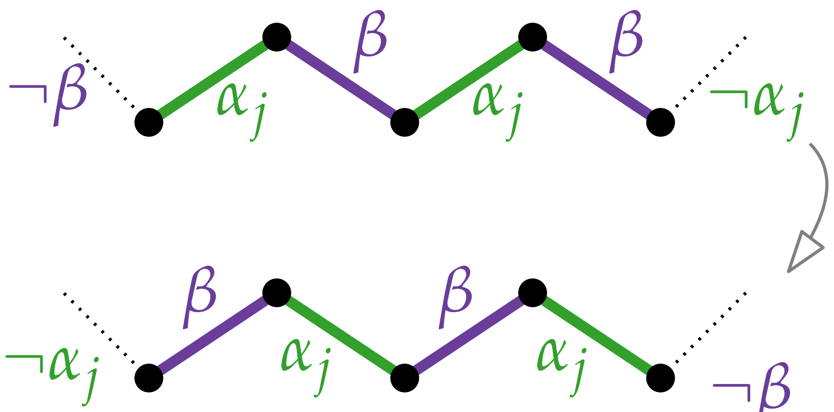
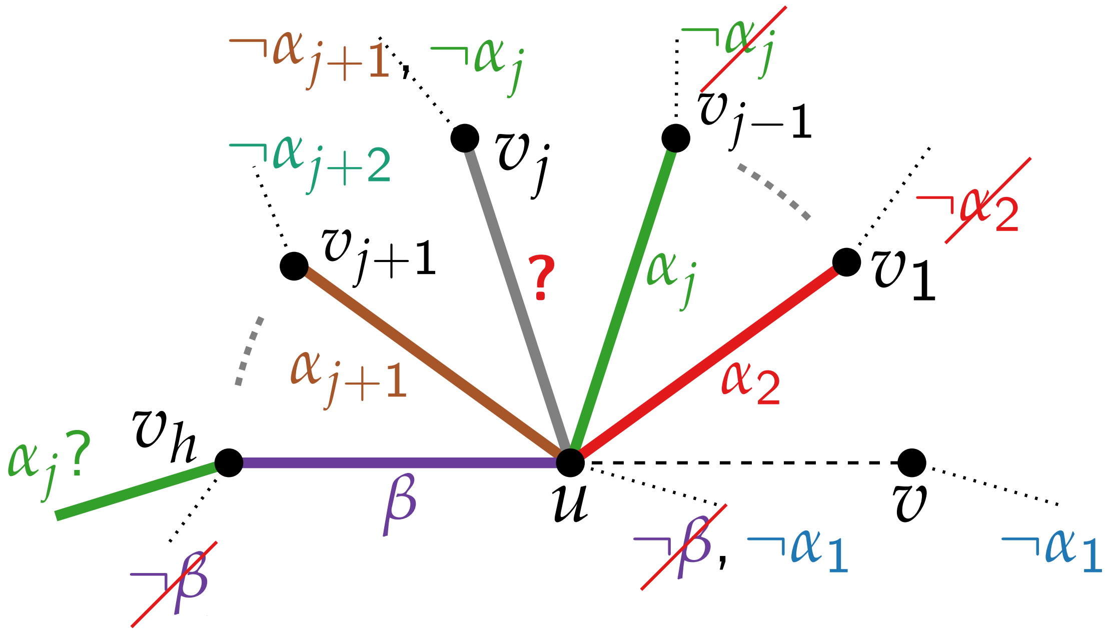
  - color uv_h with β;





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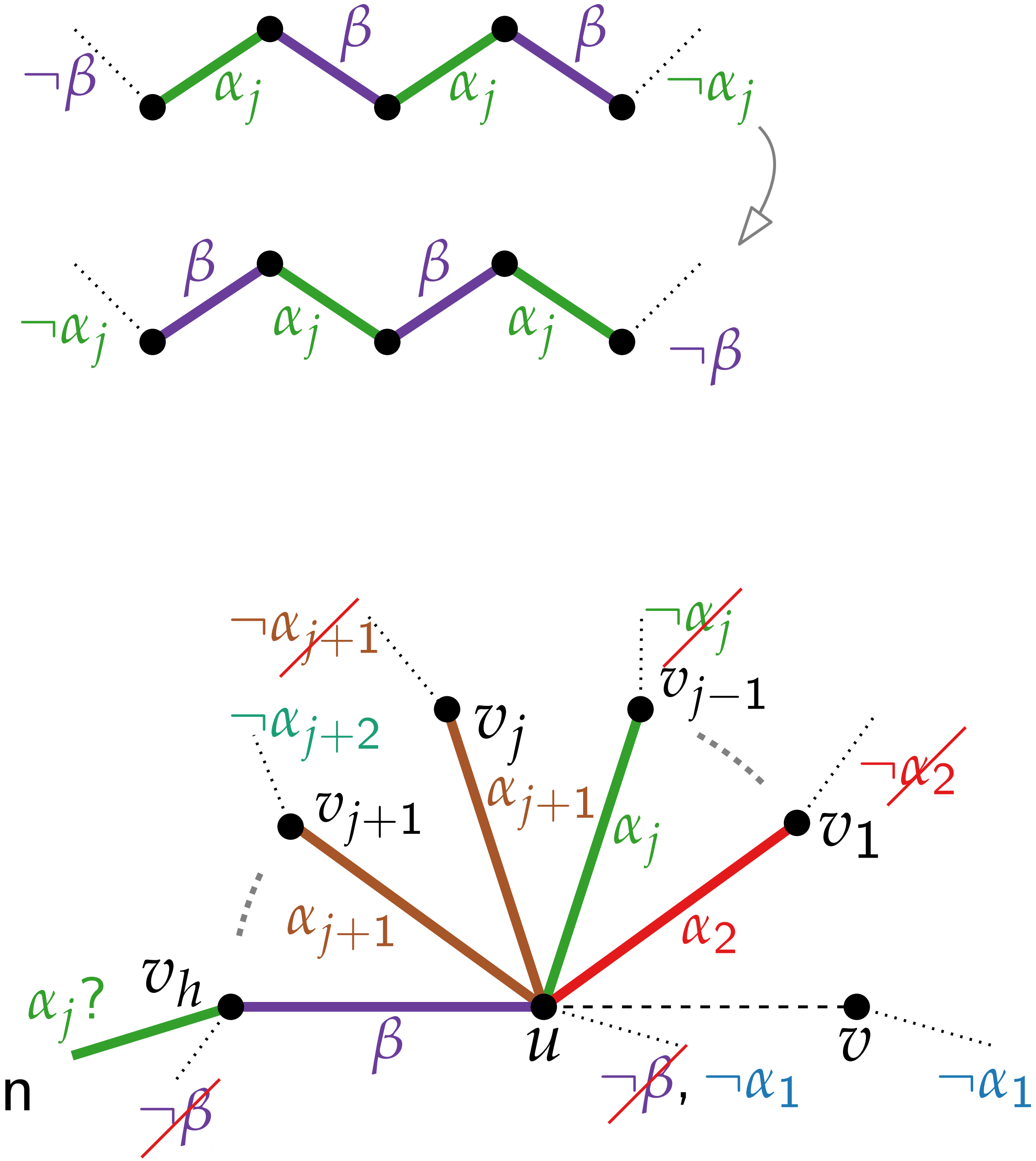

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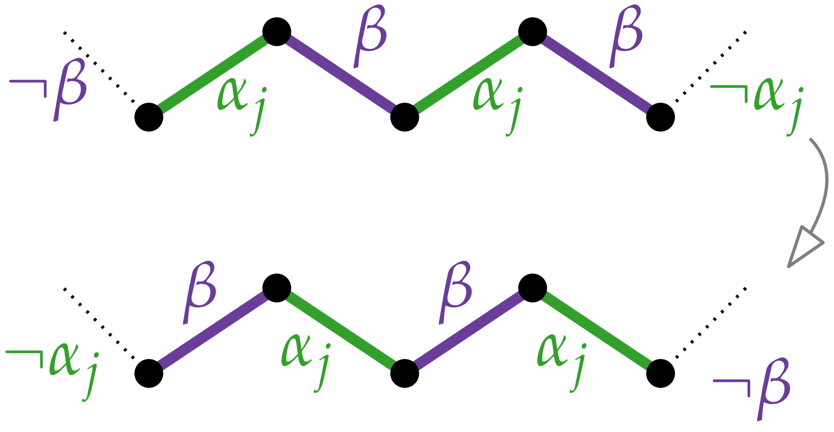
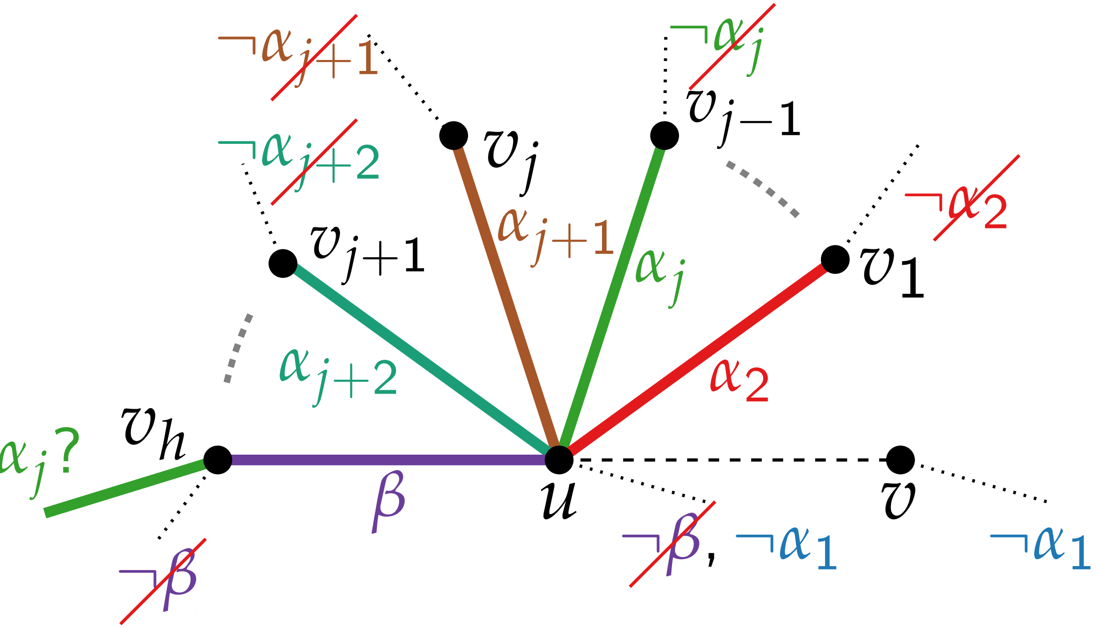
  - color uv_h with β; color uv_j with α_{j+1} and so on





Minimum Edge Coloring – Algorithm

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VizingEdgeColoring(graph G, coloring c ≡ 0)
  if E(G) ≠ ∅ then
      Let e = uv be an arbitrary edge of G.
      G_e ← G − e
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      if Δ(G_e) < Δ(G) then
          Color e with lowest free color.
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          Recolor G_e as in Lemma 2.
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Theorem 4.
VizingEdgeColoring is an approximation algorithm with additive approximation guarantee $\mathrm{ALG}(G) - \mathrm{OPT}(G) \leq 1$.

Approximation with Relative Factor

- An additive approximation guarantee can rarely be achieved; but sometimes, there is a multiplicative approximation!

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$$\frac{\mathrm{ALG}(I)}{\mathrm{OPT}(I)} \leq \alpha.$$

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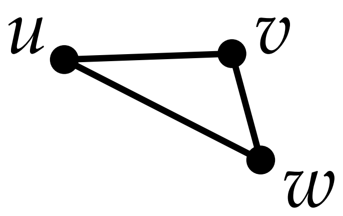
2-Approximation for Metric TSP (from AGT)

Input. Complete graph $G = (V, E)$ and a distance function $d\colon E \to \mathbb{R}_{\geq 0}$ that satisfies the triangle inequality, i.e., $\forall u, v, w \in V\colon d(u, w) \leq d(u, v) + d(v, w)$.



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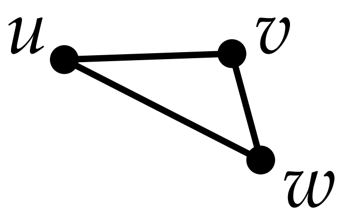

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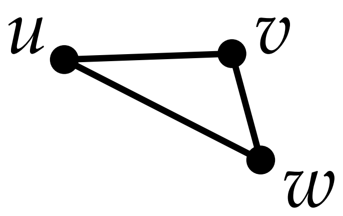

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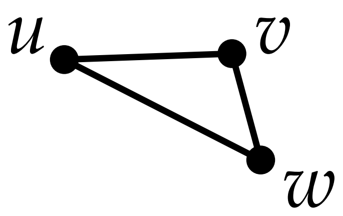

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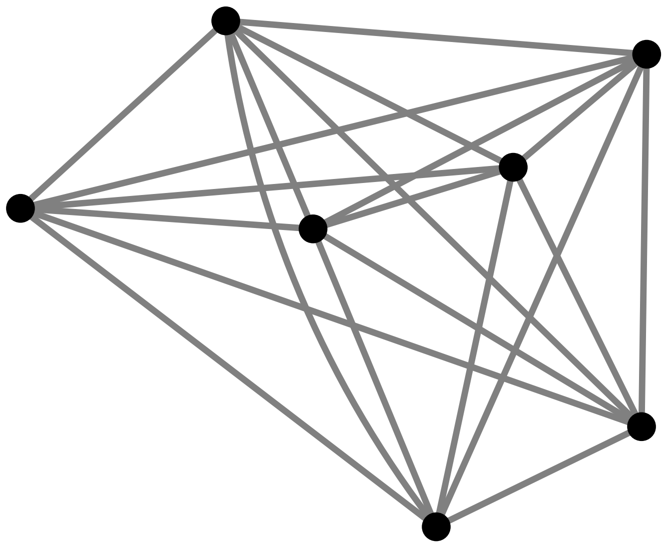

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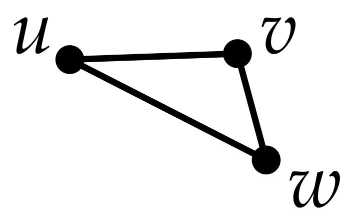

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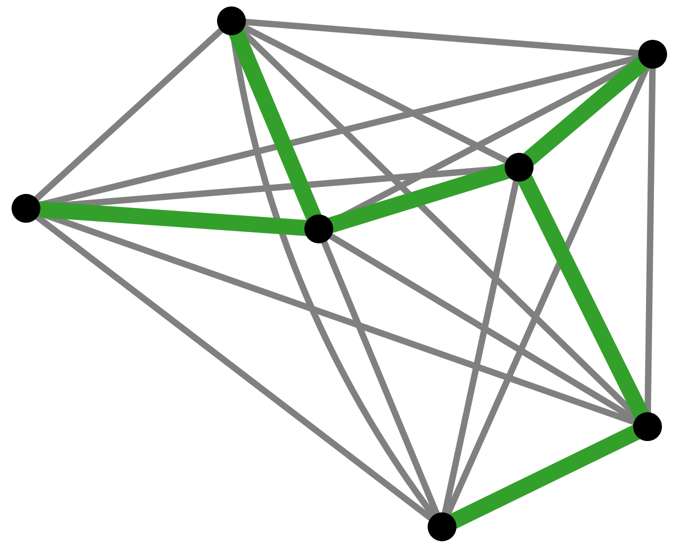

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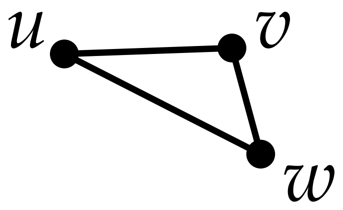

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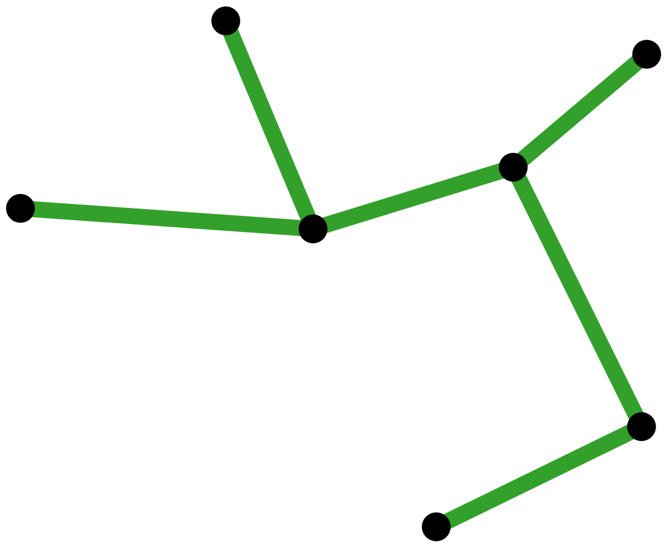

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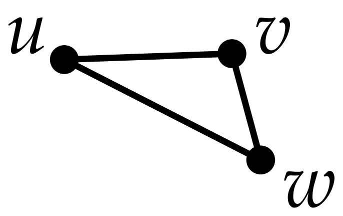

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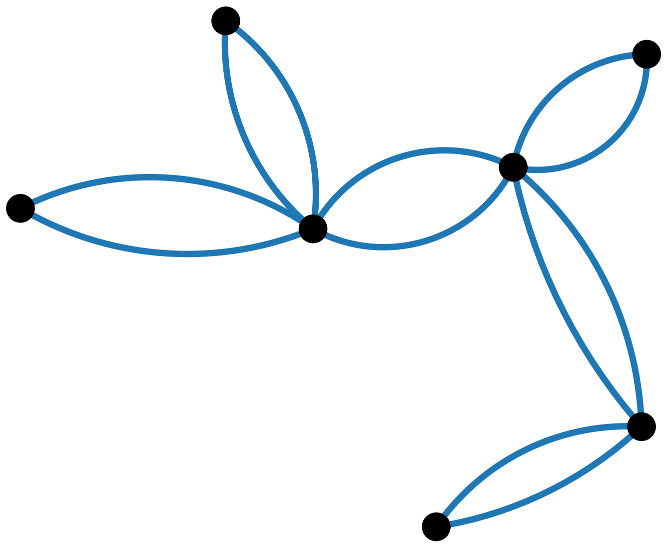

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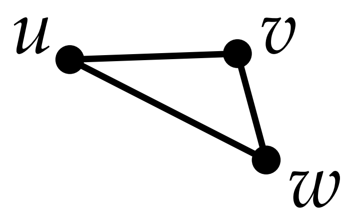

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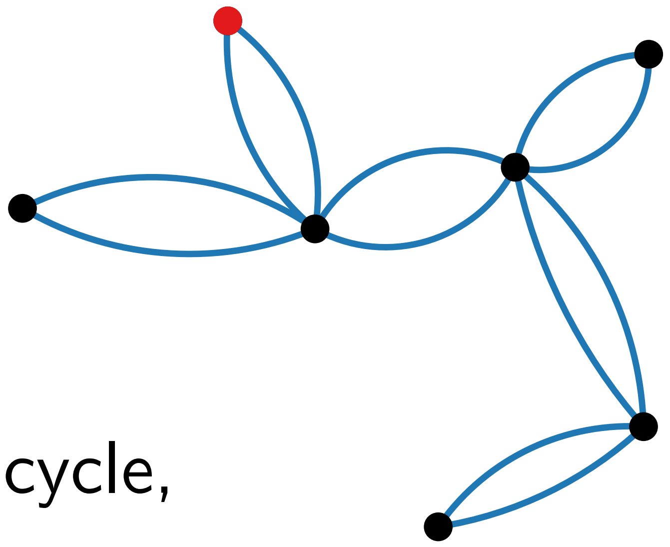

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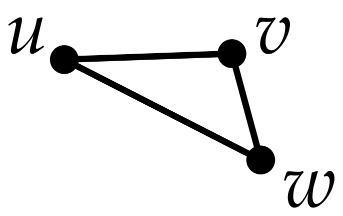

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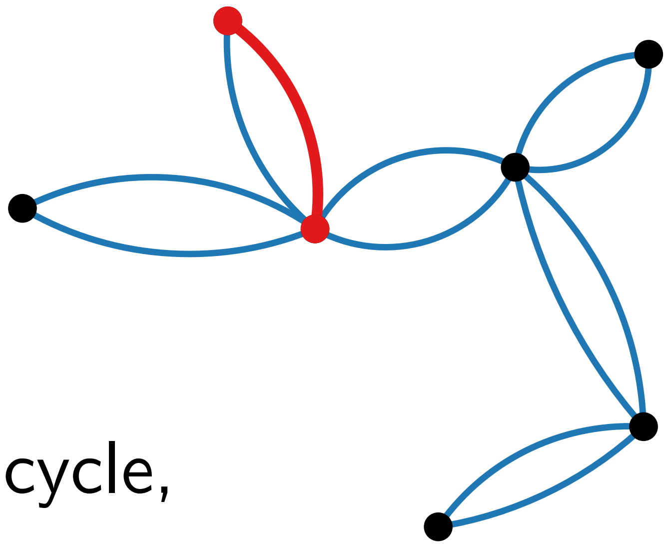

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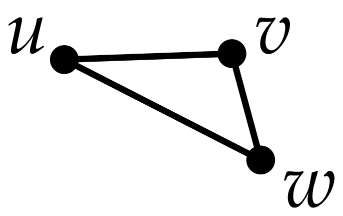

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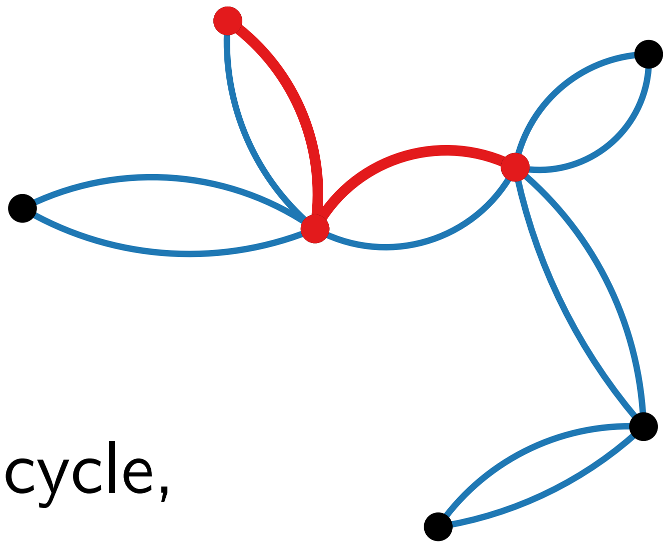

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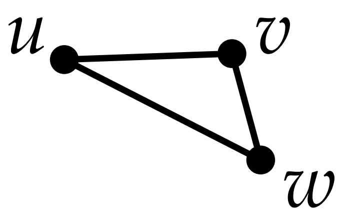

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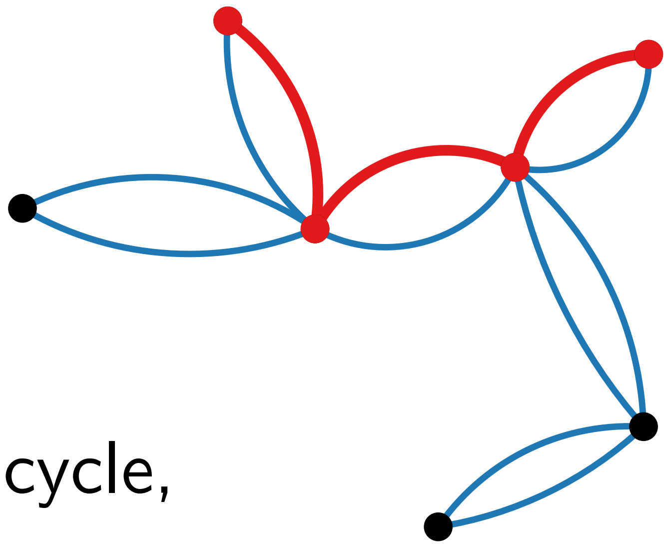

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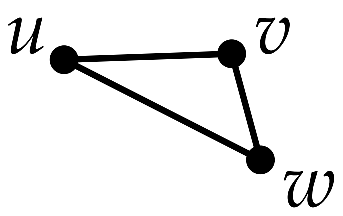

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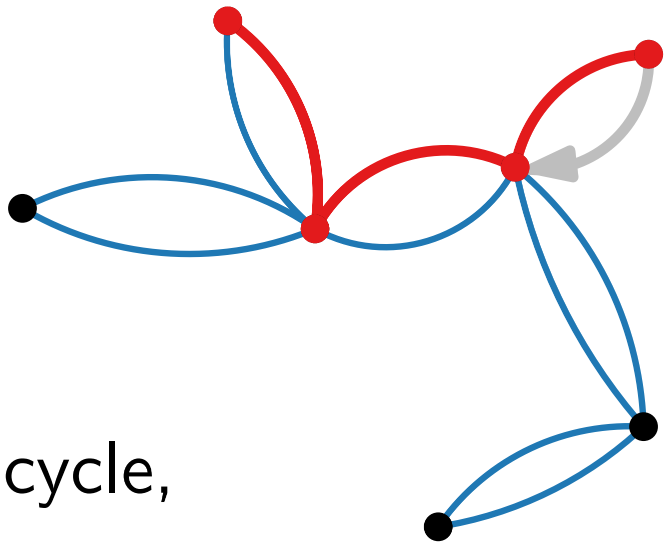

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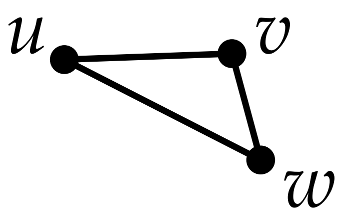

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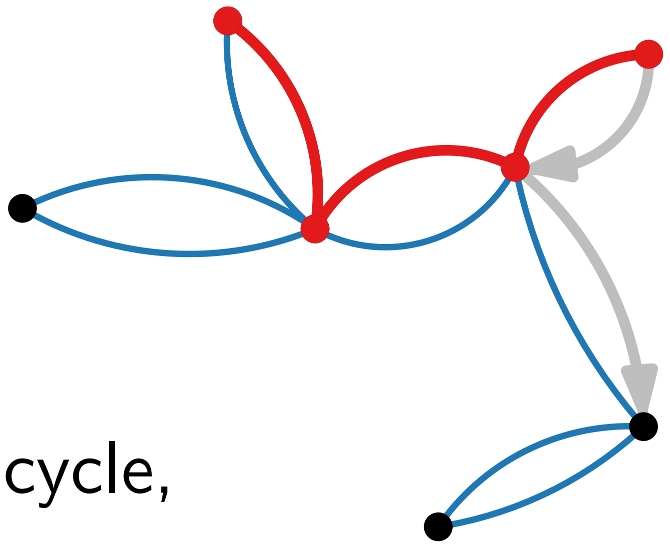

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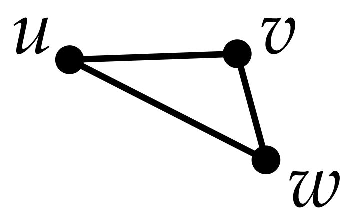

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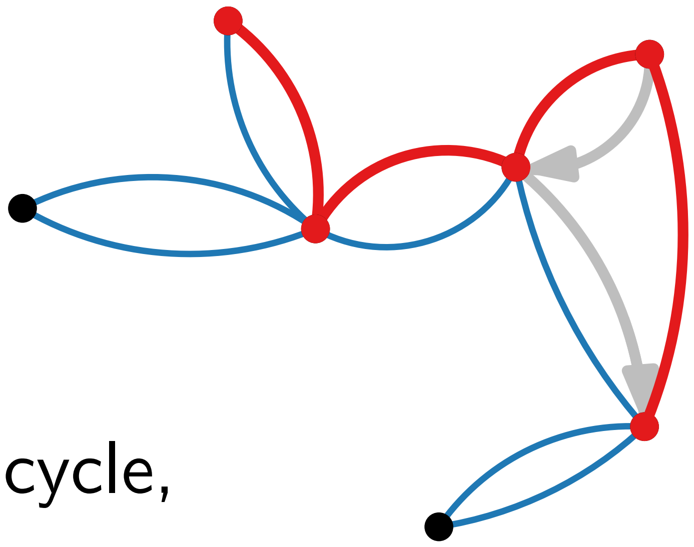

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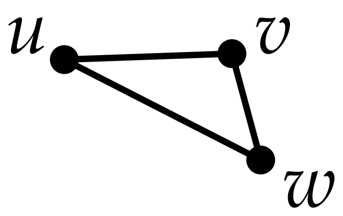

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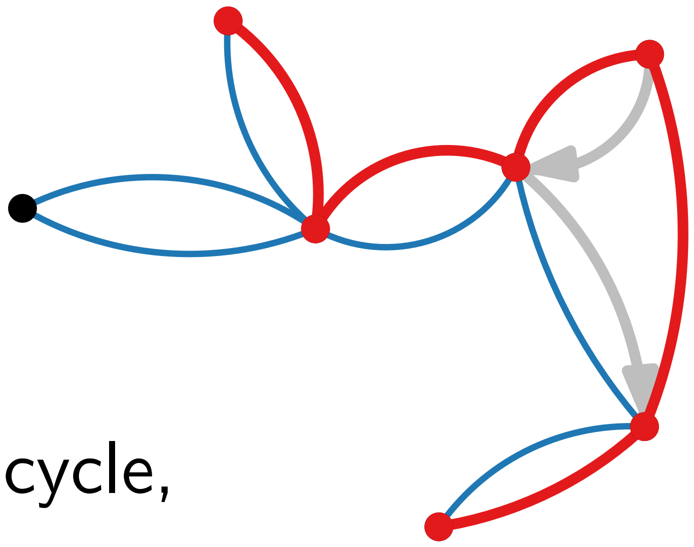

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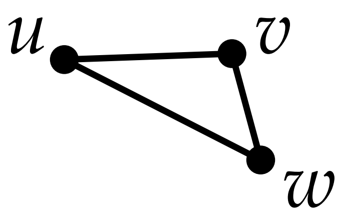

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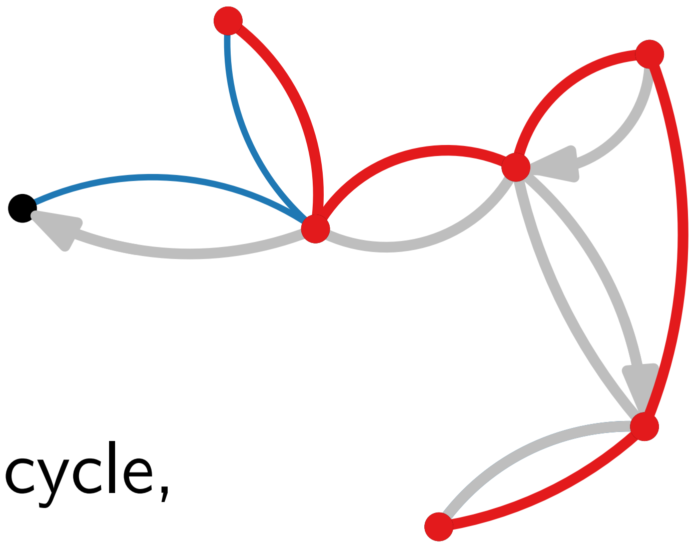

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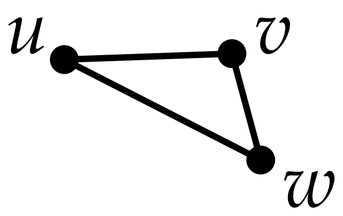

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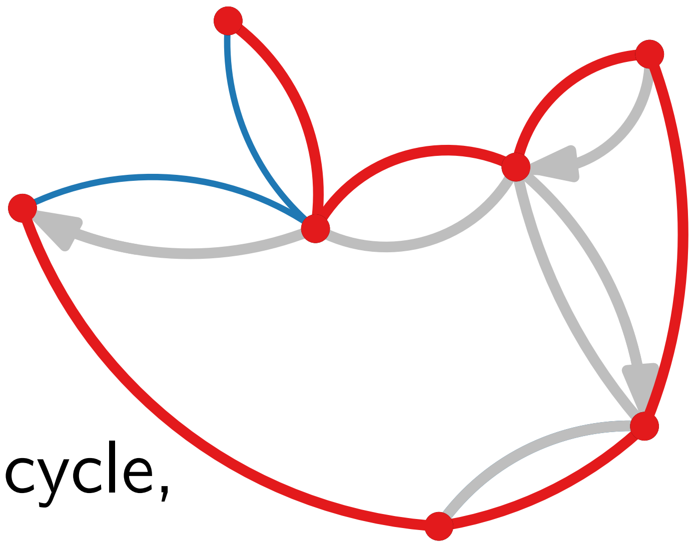

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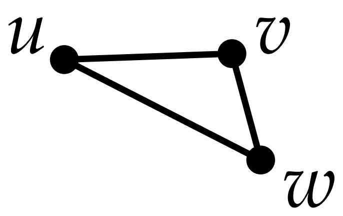

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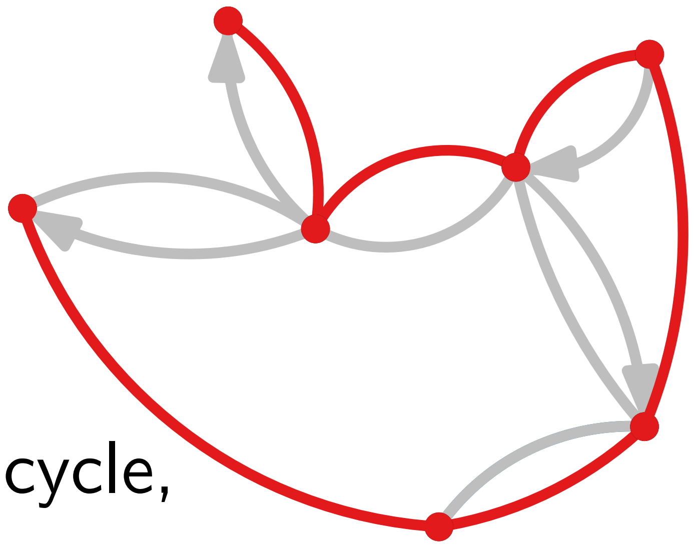

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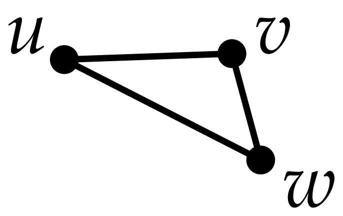

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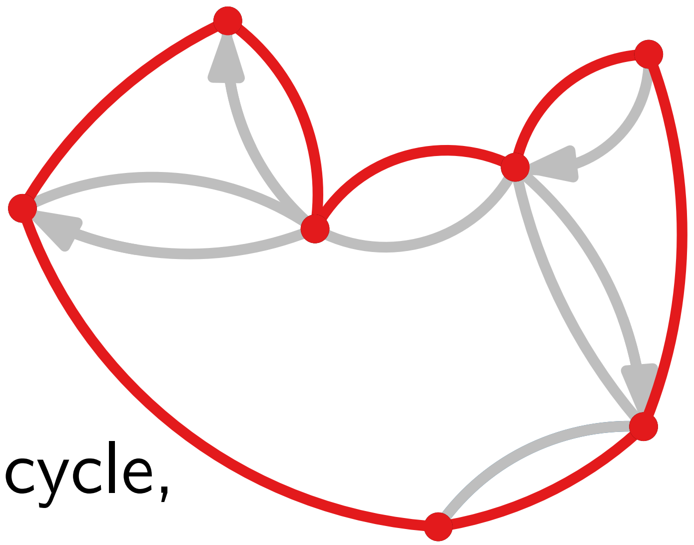

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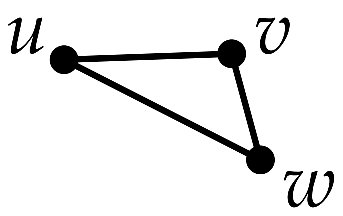

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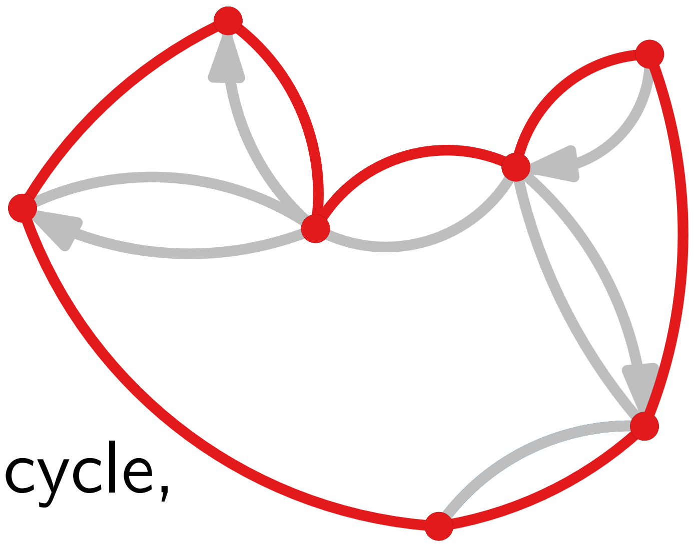

Theorem 5.
The MST edge doubling algorithm is a 2-approximation algorithm for metric TSP.

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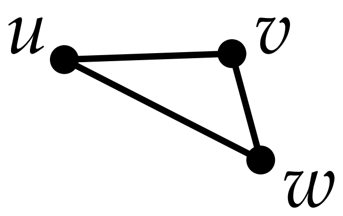

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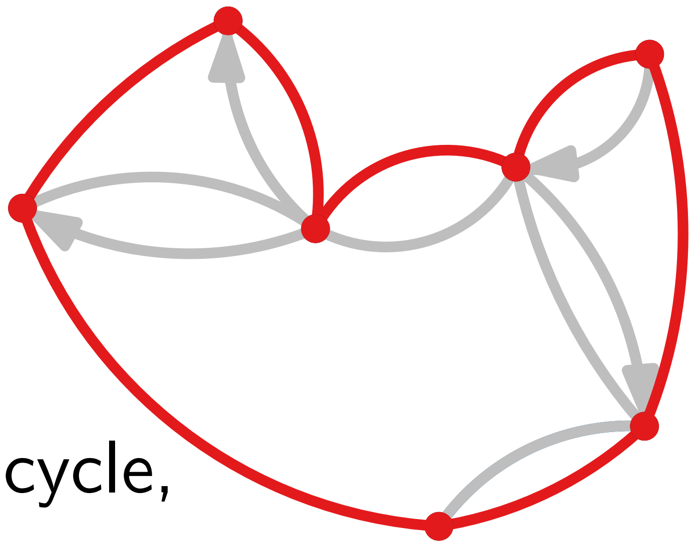

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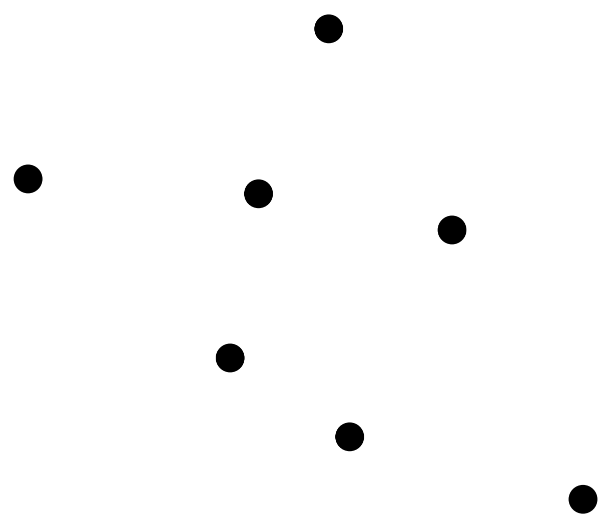
Proof.
$\text{ALG} \leq d(\text{cycle}) = 2d(\text{MST}) \leq 2\text{OPT}$.

Nearest Addition Algorithm for Metric TSP

`NearestAdditionAlgorithm`$(G = (V, E), d)$

Find closest pair, say i and k.
Set tour T to go from i to k to i (clockwise).
while $T \subsetneq V$ **do**
  Find pair $(i, j) \in T \times (V \setminus T)$ minimizing $d(i, j)$.
  Let k be vertex after i in T.
  Add j between i and k.



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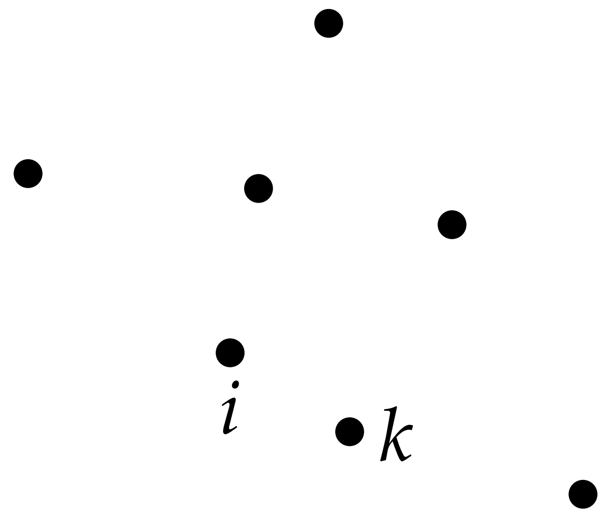

Nearest Addition Algorithm for Metric TSP

```
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```

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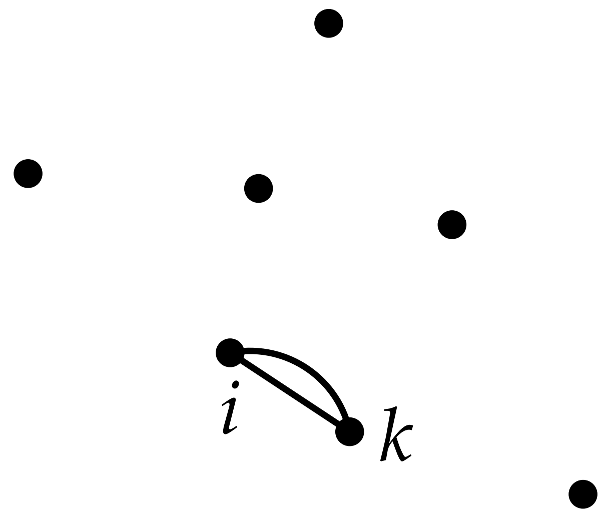

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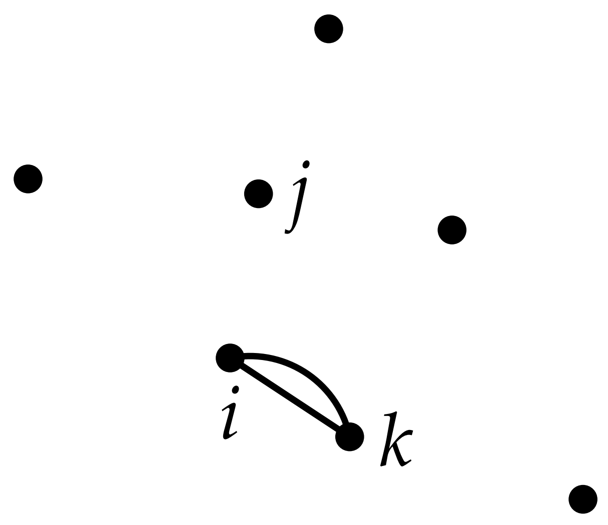

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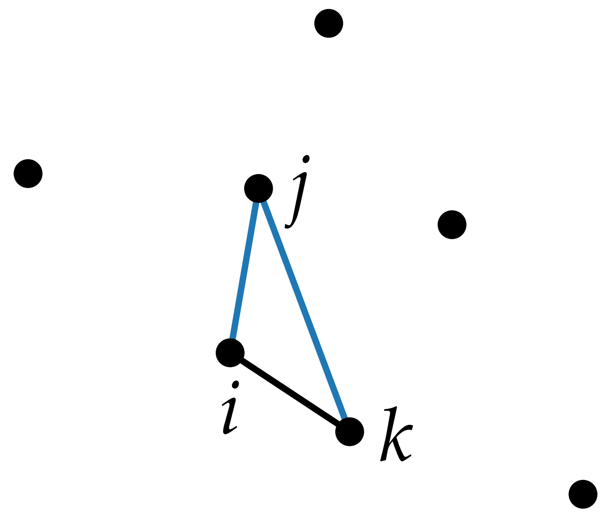

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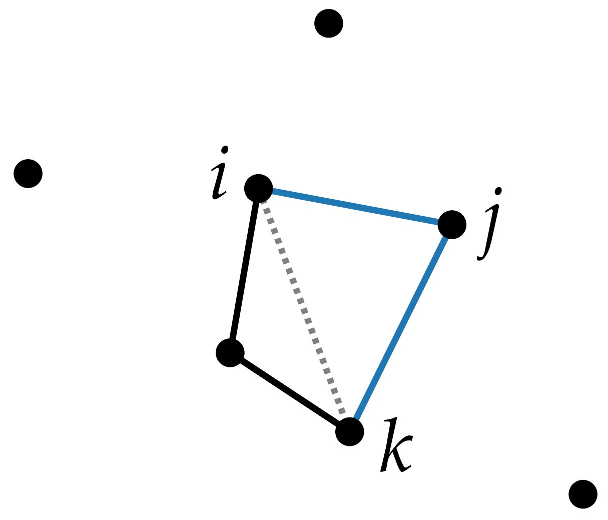

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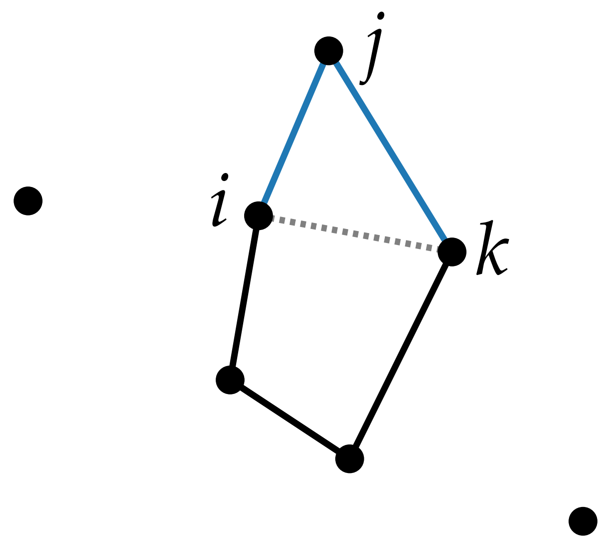

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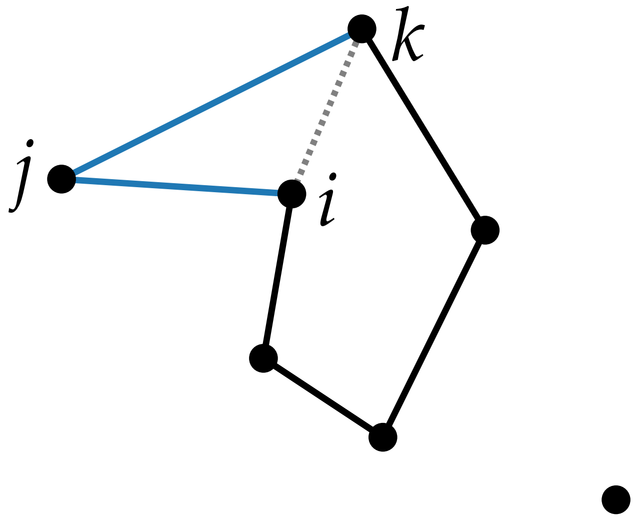

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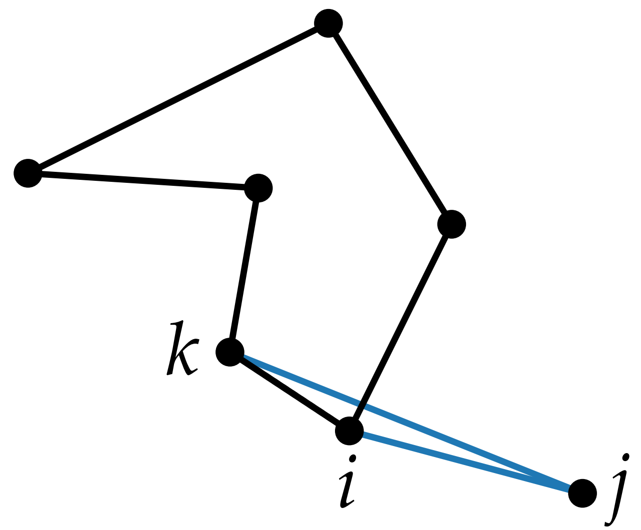

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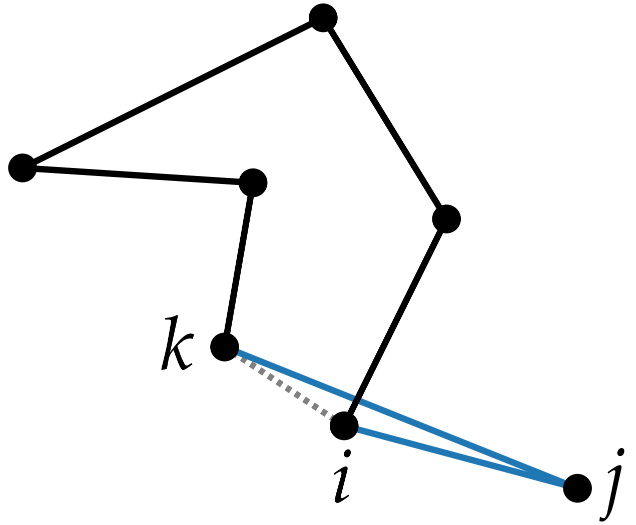

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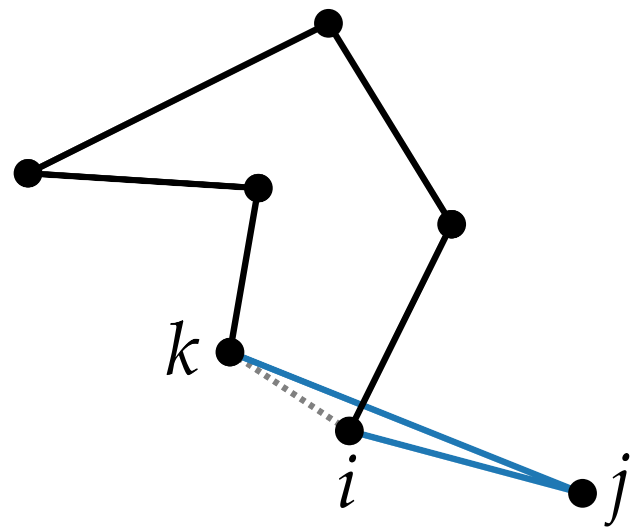

Theorem 6.
`NearestAdditionAlgorithm` is a 2-approximation algorithm for metric TSP.

Nearest Addition Algorithm for Metric TSP

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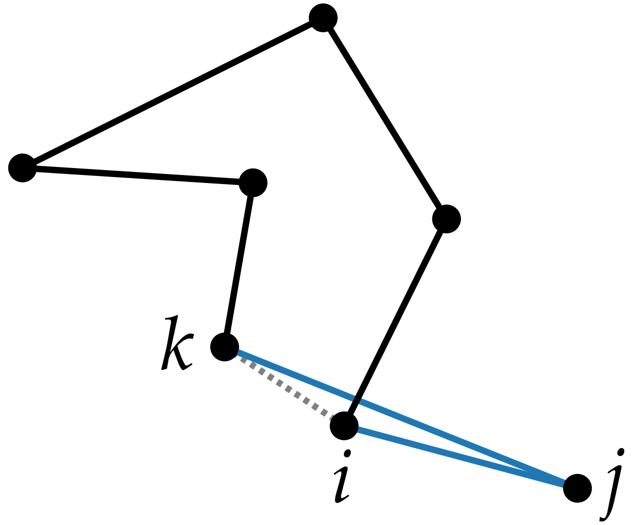

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Proof.

- Exercise.
- *Hints:* MST and Prim's algorithm.

Approximation Schemes

- In some cases, we can get arbitrarily good approximations.

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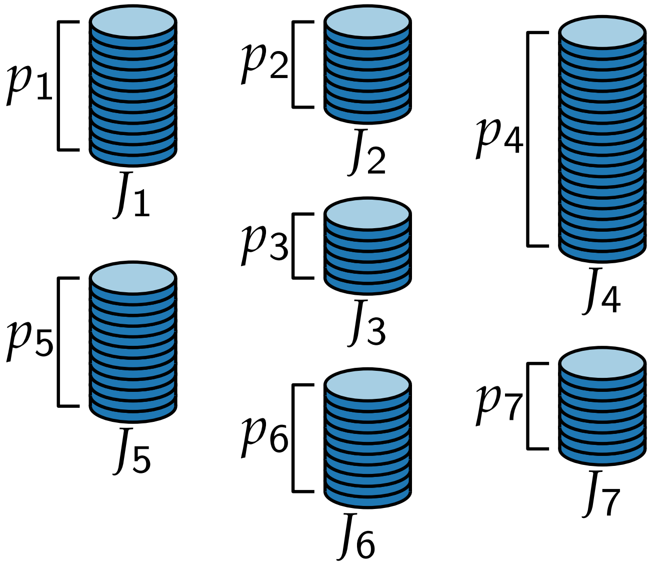
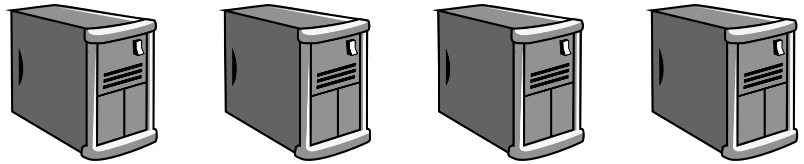
Multiprocessor Scheduling

Input.

- n jobs $J_1, \ldots, J_n$ with durations $p_1, \ldots, p_n$.



- m identical machines ($m < n$)



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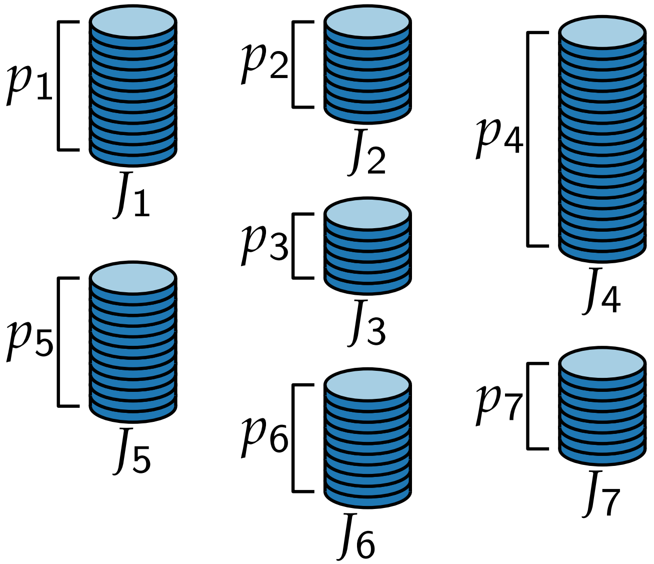

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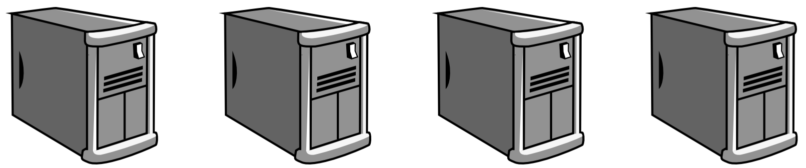

Output. Assignment of jobs to machines such that the time when all jobs have been processed is minimum.
This is called the **makespan** of the assignment.

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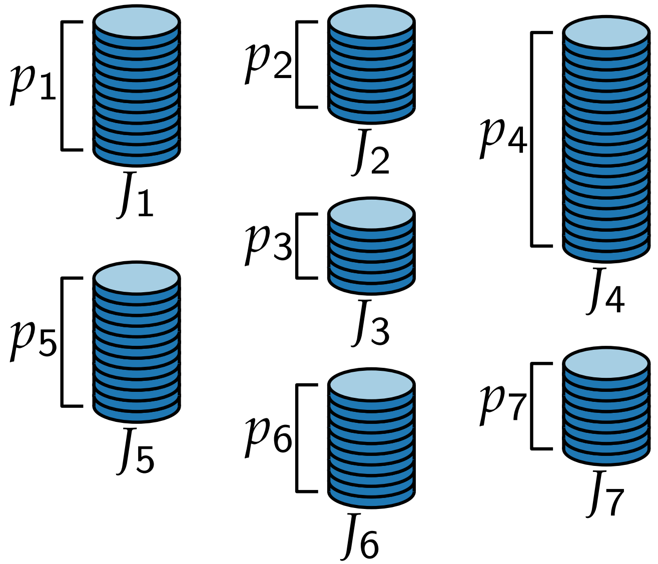

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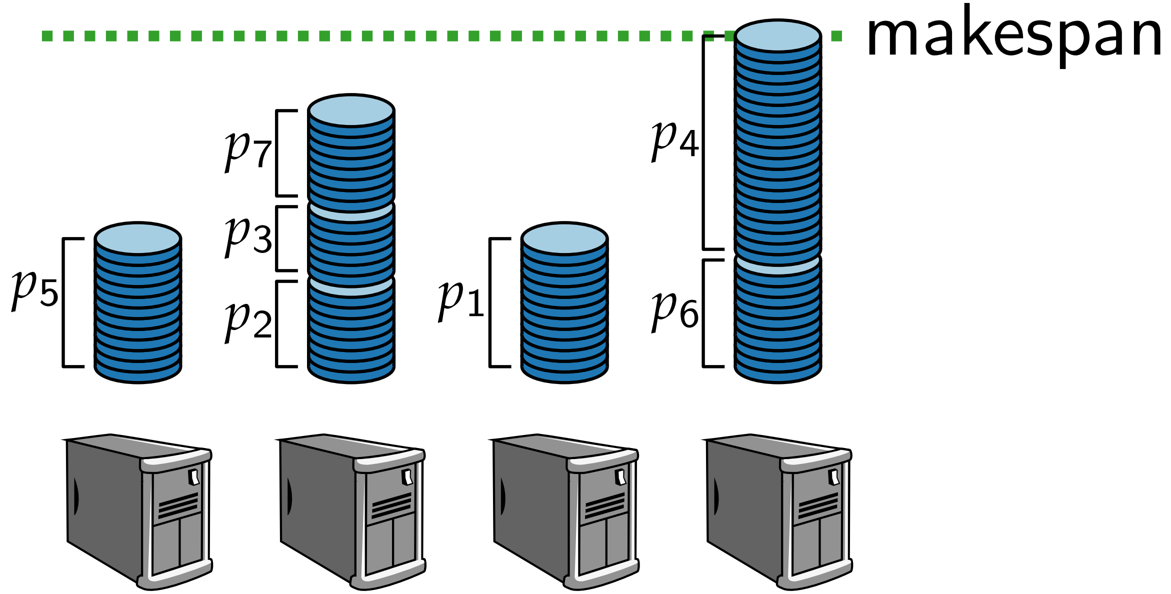

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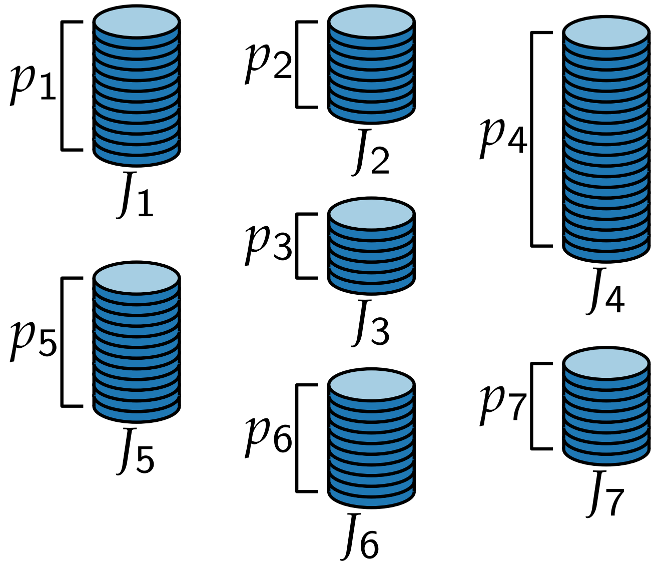

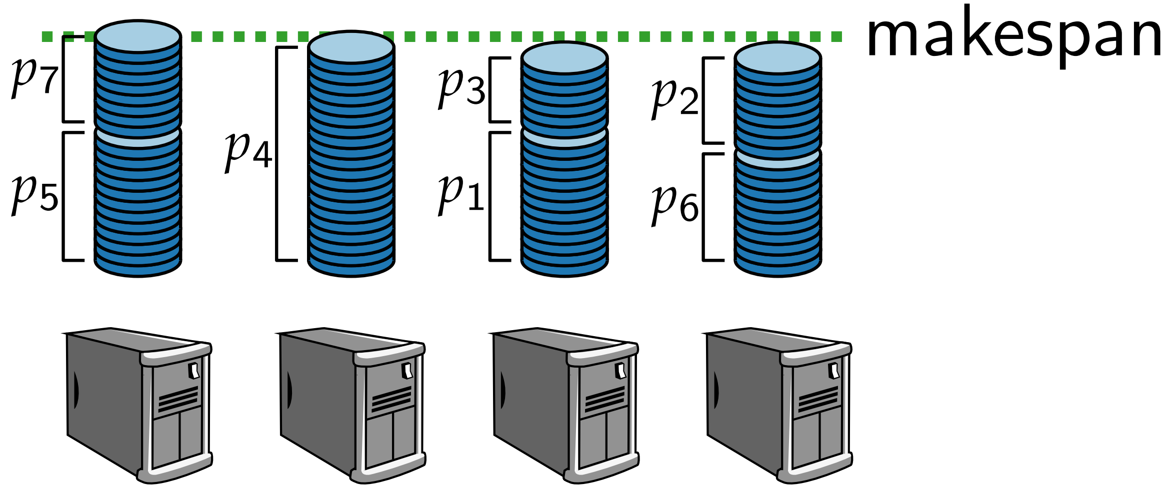

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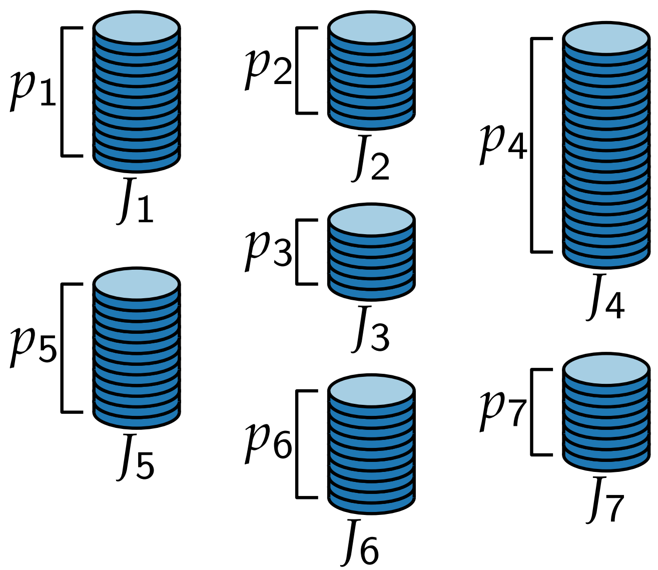

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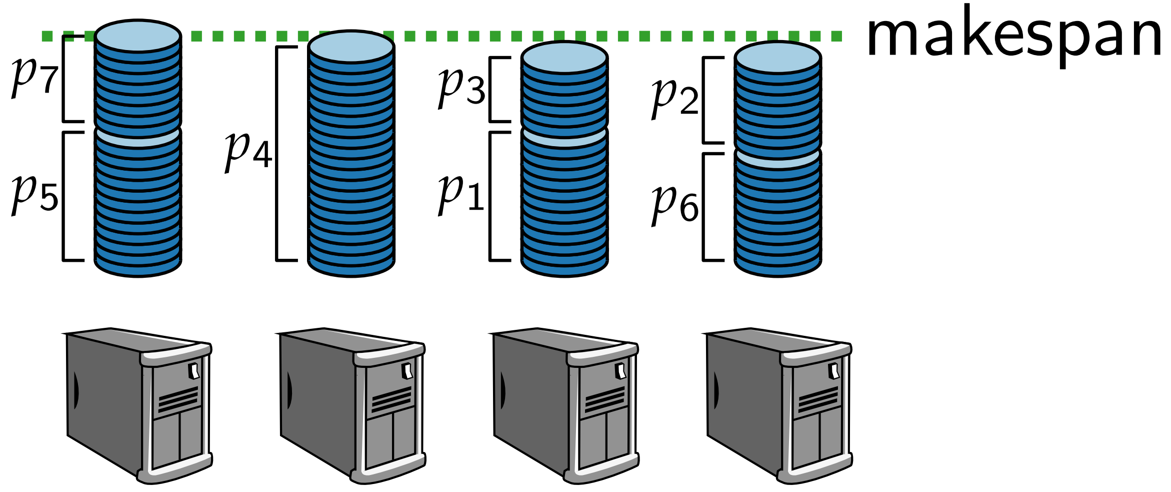

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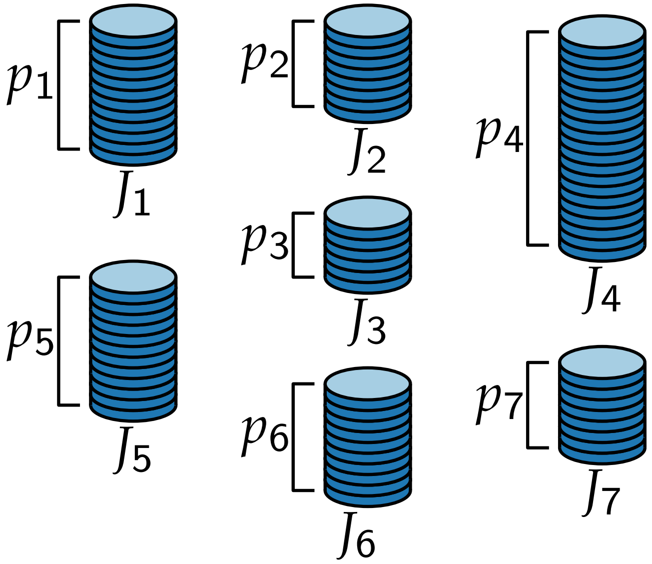
- Multiprocessor scheduling is NP-hard.

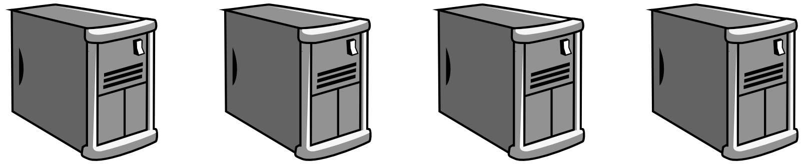
Multiprocessor Scheduling – List Scheduling

$\textsc{ListScheduling}(J_1, \ldots, J_n, m)$

Put the first m jobs on the m machines.
Put the next job on the first free machine.

Example.





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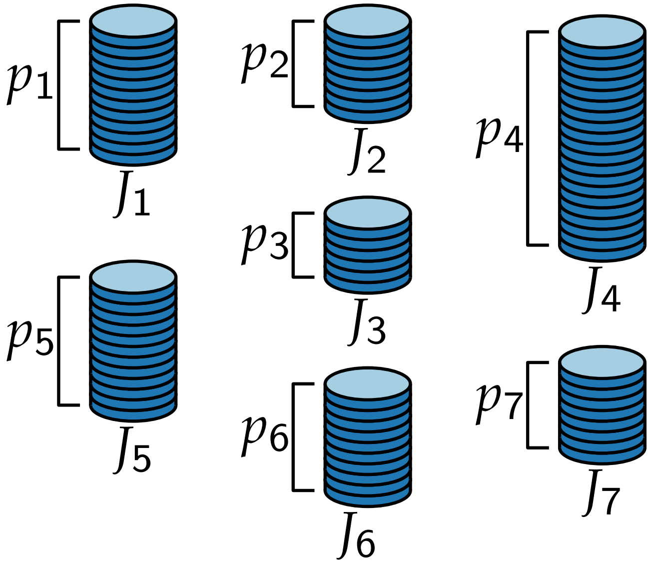

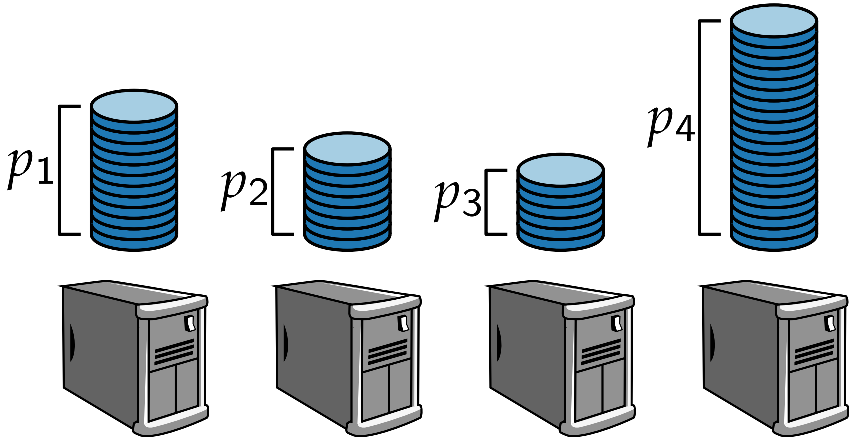

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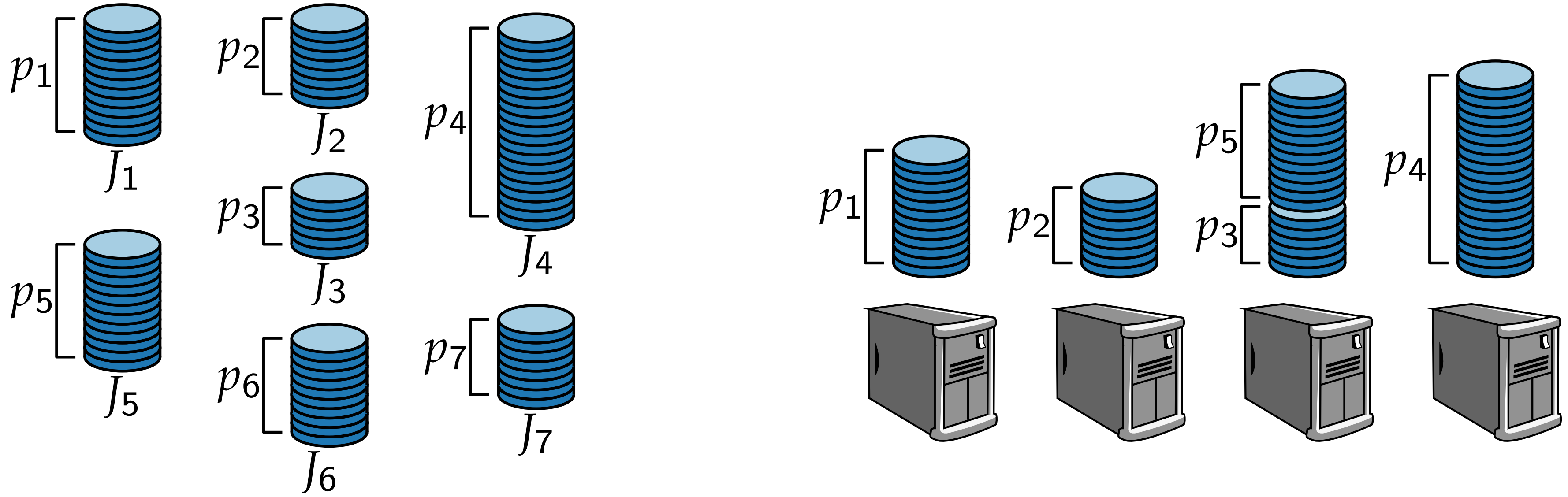

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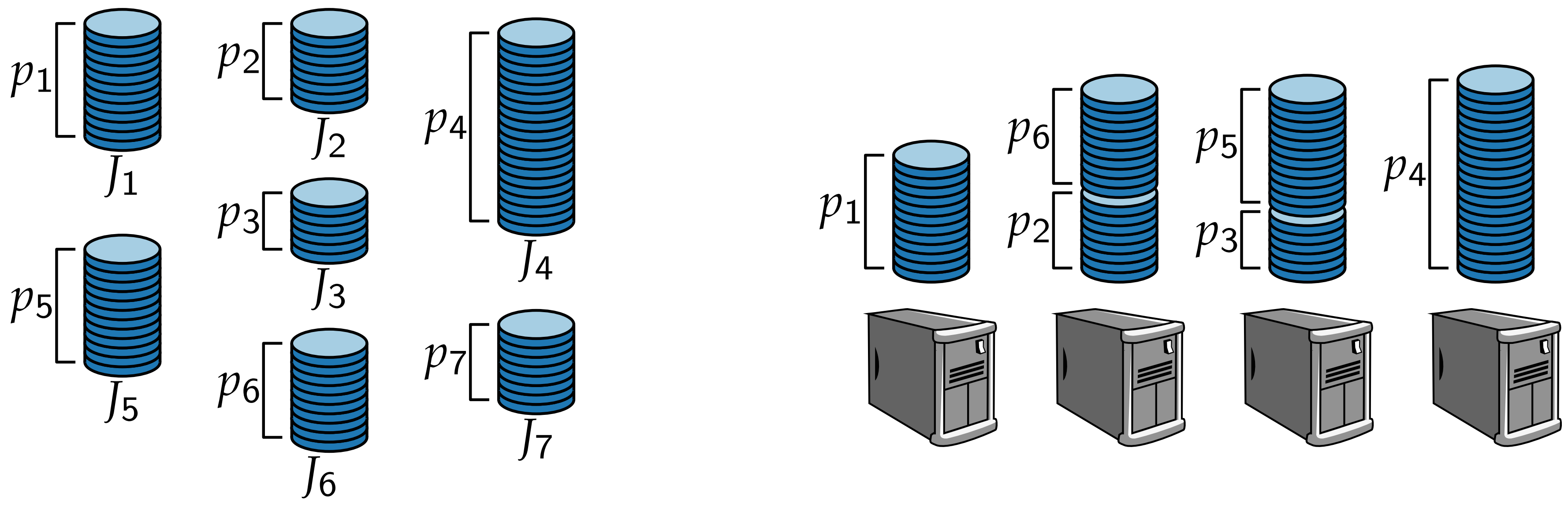

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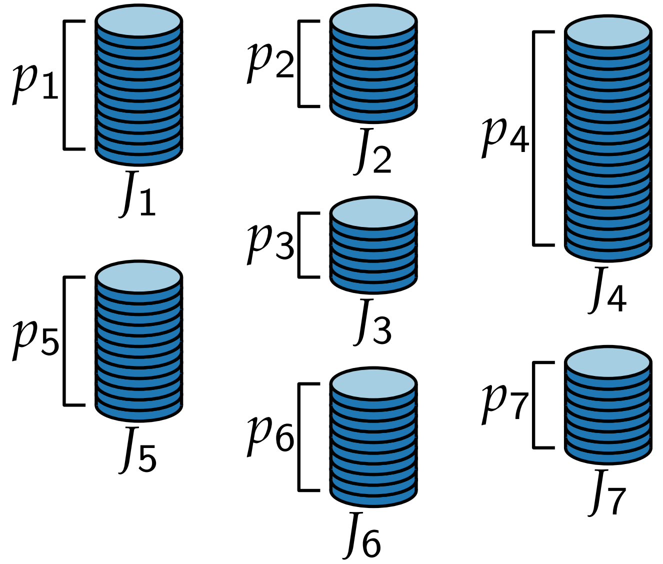

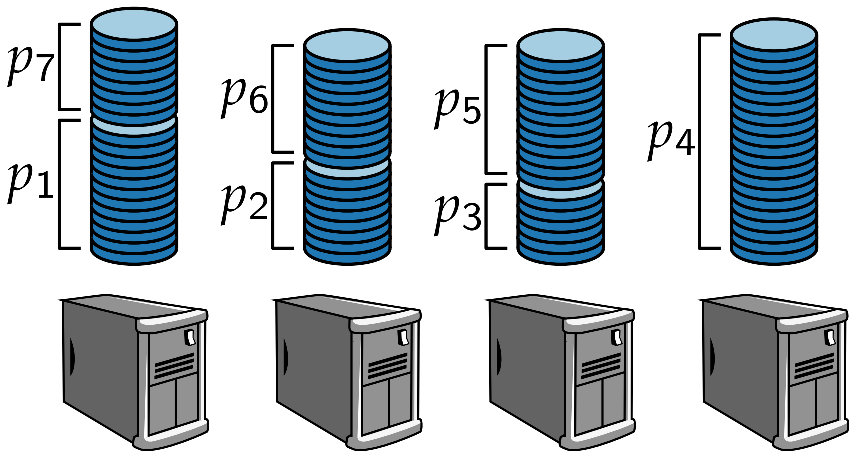

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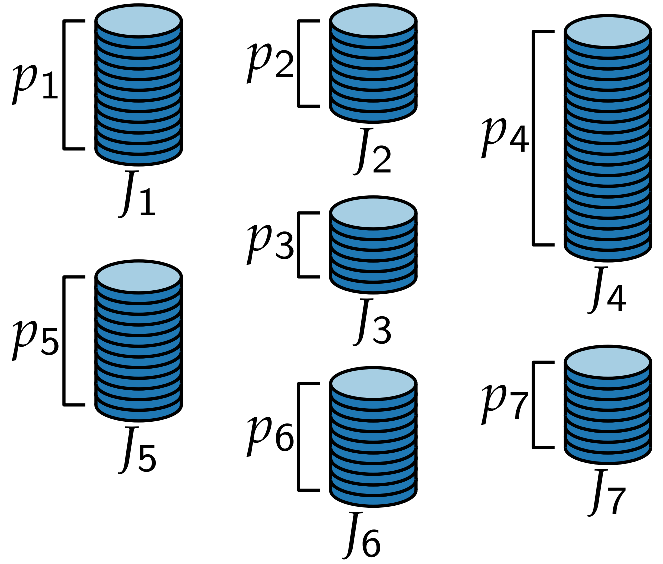

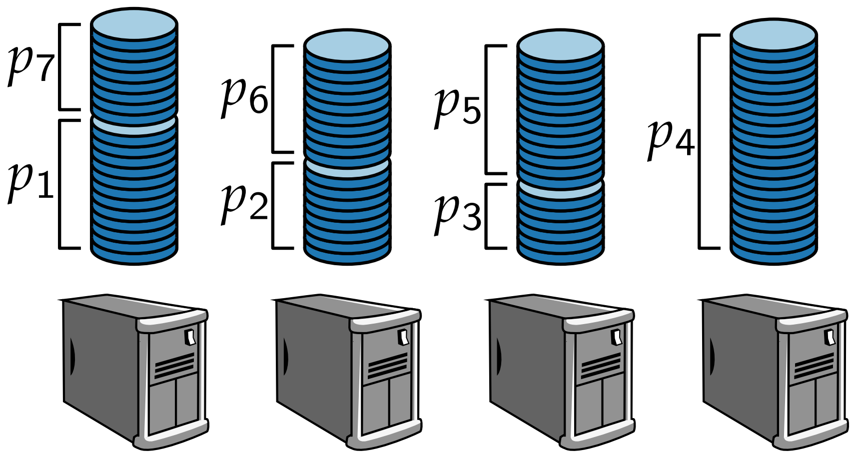

- ListScheduling runs in time.

Multiprocessor Scheduling – List Scheduling

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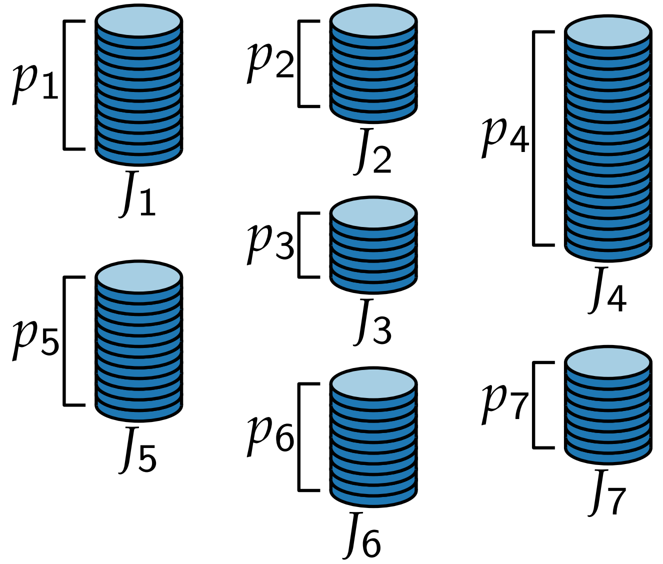

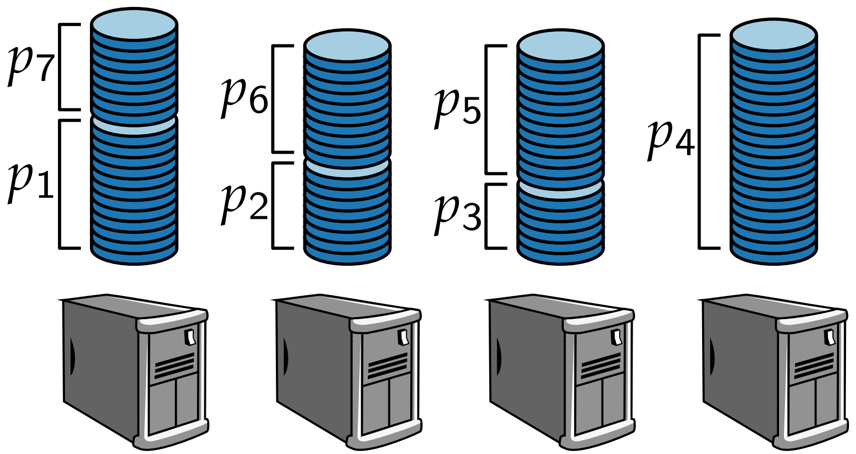

- ListScheduling runs in $\mathcal{O}(n \log m)$ time.

Multiprocessor Scheduling – List Scheduling

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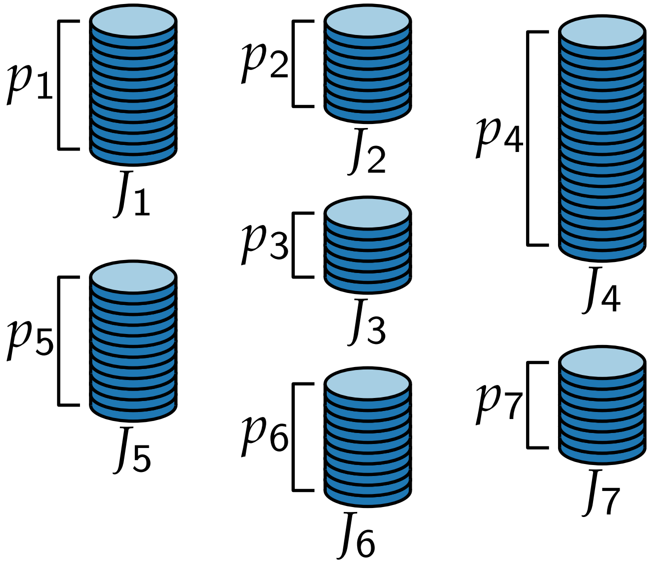

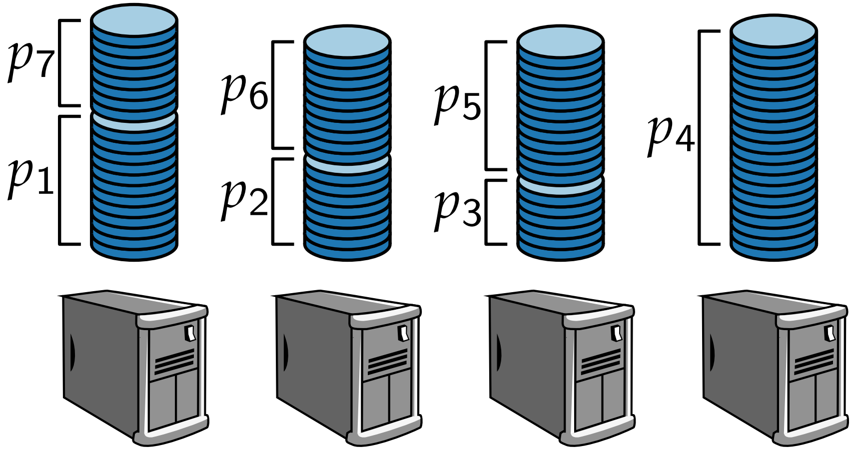

- LISTSCHEDULING runs in $\mathcal{O}(n \log m)$ time.

  Iterate over n jobs while maintaining a priority queue for the machines where each machine has its current completion time as its priority.

Multiprocessor Scheduling – List Scheduling

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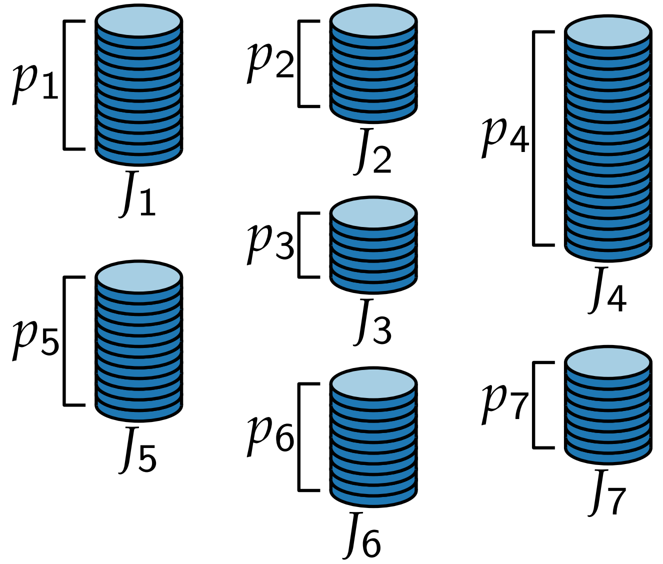
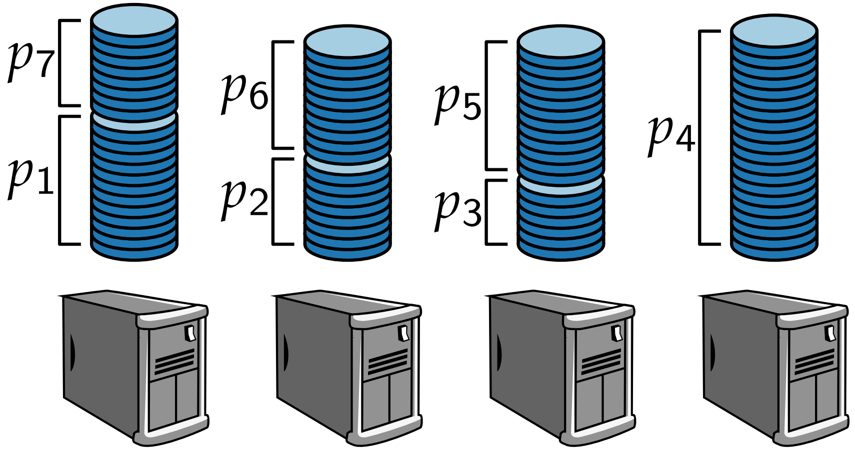
Theorem 7.
ListScheduling is a factor-approximation algorithm.

Example.





- ListScheduling runs in $\mathcal{O}(n \log m)$ time.

  Iterate over n jobs while maintaining a priority queue for the machines where each machine has its current completion time as its priority.

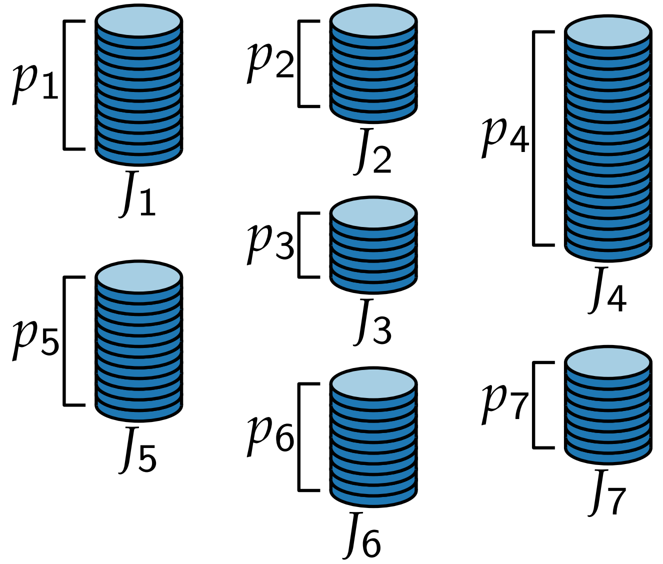
Multiprocessor Scheduling – List Scheduling

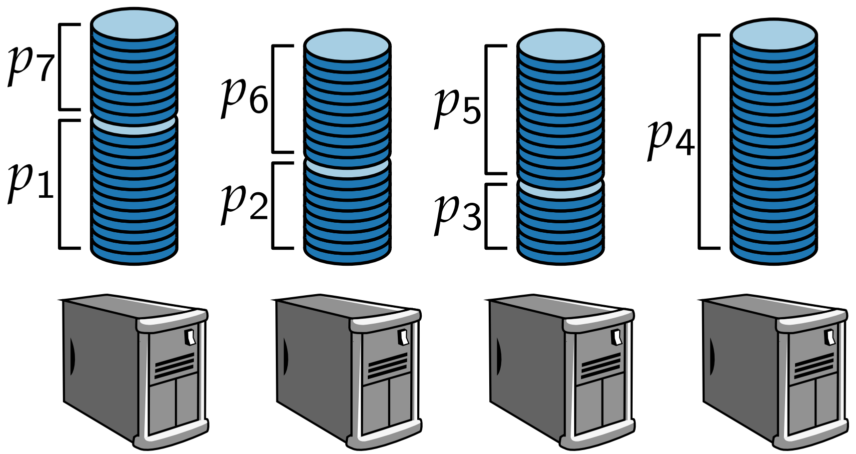
ListScheduling$(J_1, \ldots, J_n, m)$

Put the first m jobs on the m machines.
Put the next job on the first free machine.

Theorem 7.
ListScheduling is a factor-$\left(2 - \frac{1}{m}\right)$ approximation algorithm.

Example.





- ListScheduling runs in $\mathcal{O}(n \log m)$ time.

  Iterate over n jobs while maintaining a priority queue for the machines where each machine has its current completion time as its priority.

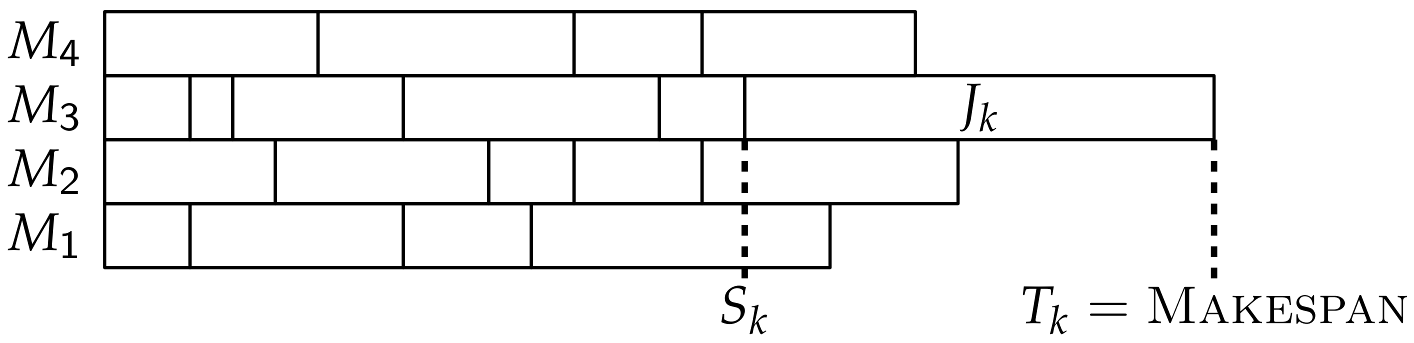
Multiprocessor Scheduling – List Scheduling (Proof)

ListScheduling($J_1, \dots, J_n, m$)

Put the first m jobs on the m machines.
Put the next job on the first free machine.

Theorem 7.
ListScheduling is a $(2 - \frac{1}{m})$-approximation alg.

Proof. Let $J_k = (S_k, T_k)$ be the last job, that is, T_k determines the makespan.



Multiprocessor Scheduling – List Scheduling (Proof)

ListScheduling($J_1, \dots, J_n, m$)

Put the first m jobs on the m machines.
Put the next job on the first free machine.

Theorem 7.
ListScheduling is a $(2 - \frac{1}{m})$-approximation alg.

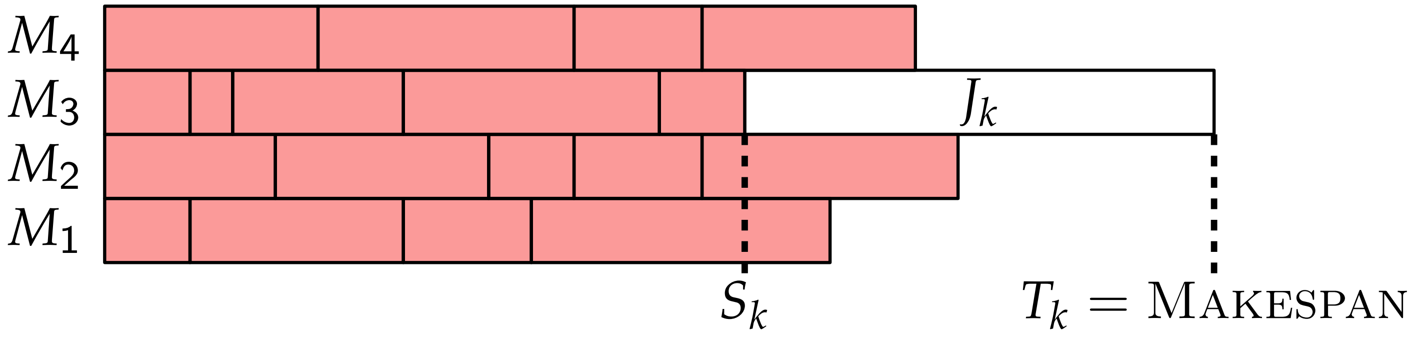
Proof. Let $J_k = (S_k, T_k)$ be the last job, that is, T_k determines the makespan.

- No machine idles at time S_k.

$$S_k \leq \frac{1}{m} \sum_{i \neq k} p_i$$

weight of all jobs but J_k

evenly distributed on m machines



Multiprocessor Scheduling – List Scheduling (Proof)

ListScheduling($J_1, \dots, J_n, m$)

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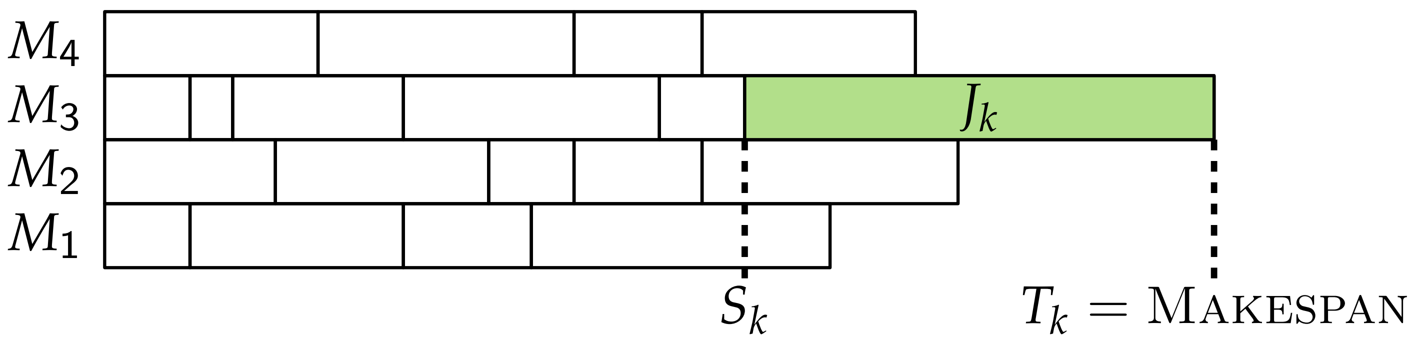
weight of all jobs but J_k

evenly distributed on m machines

- For the optimal makespan T_{OPT}, we have:
- $T_{\mathrm{OPT}} \geq p_k$



Multiprocessor Scheduling – List Scheduling (Proof)

ListScheduling($J_1, \ldots, J_n, m$)

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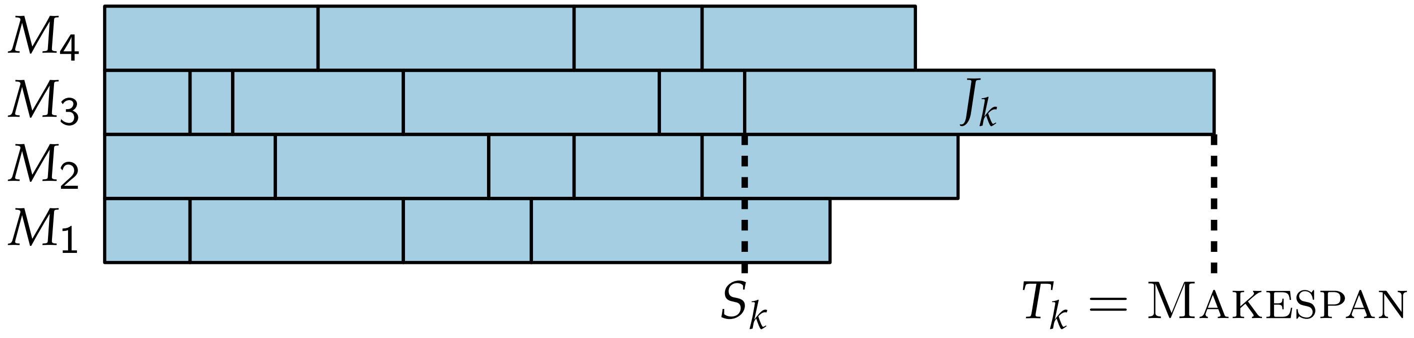
weight of all jobs but J_k evenly distributed on m machines

- For the optimal makespan T_{OPT}, we have:

- $T_{\text{OPT}} \geq p_k$
- $T_{\text{OPT}} \geq \frac{1}{m} \sum_{i=1}^{n} p_i$ weight of all jobs evenly distributed



Multiprocessor Scheduling – List Scheduling (Proof)

ListScheduling($J_1, \dots, J_n, m$)

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Put the next job on the first free machine.

Theorem 7.
ListScheduling is a $(2 - \frac{1}{m})$-approximation alg.

Proof. Let $J_k = (S_k, T_k)$ be the last job, that is, T_k determines the makespan.

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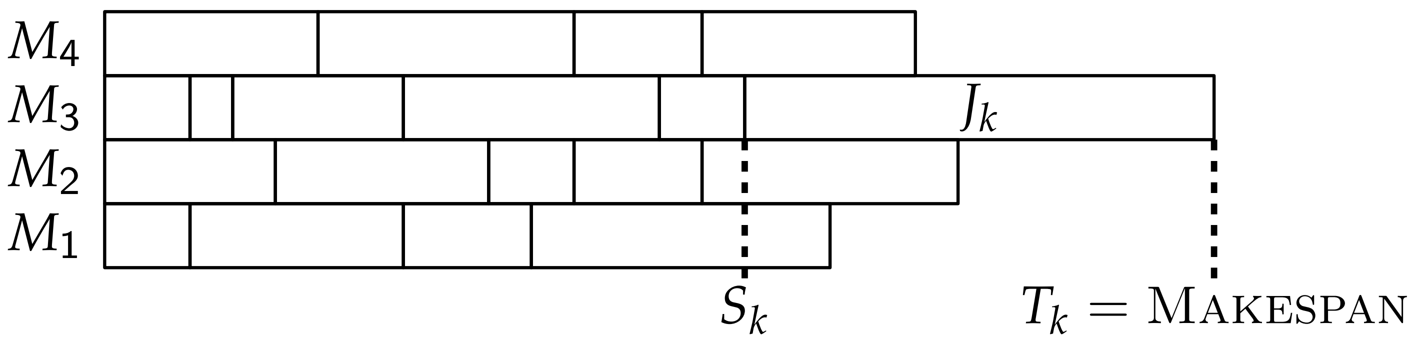
weight of all jobs but J_k evenly distributed on m machines

- For the optimal makespan T_{OPT}, we have:
  - $T_{\text{OPT}} \geq p_k$
  - $T_{\text{OPT}} \geq \frac{1}{m} \sum_{i=1}^{n} p_i$ weight of all jobs evenly distributed

- Hence:

$$T_k = S_k + p_k$$



Multiprocessor Scheduling – List Scheduling (Proof)

$\textsc{ListScheduling}(J_1, \dots, J_n, m)$

Put the first m jobs on the m machines.
Put the next job on the first free machine.

Theorem 7.
$\textsc{ListScheduling}$ is a $(2 - \frac{1}{m})$-approximation alg.

Proof. Let $J_k = (S_k, T_k)$ be the last job, that is, T_k determines the makespan.

- No machine idles at time S_k.

$$S_k \leq \frac{1}{m} \sum_{i \neq k} p_i$$

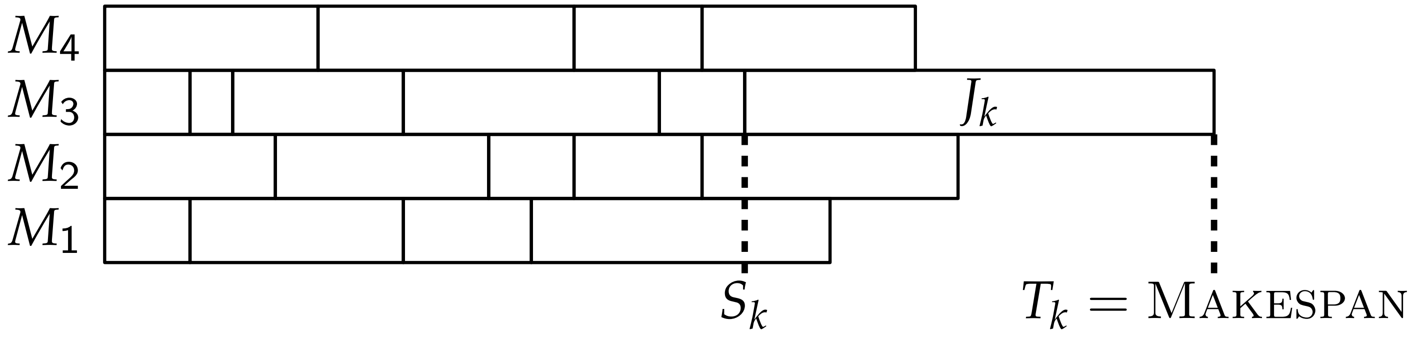
weight of all jobs but J_k evenly distributed on m machines

- Hence:

$$T_k = S_k + p_k \leq \frac{1}{m} \cdot \sum_{i \neq k} p_i + p_k$$

- For the optimal makespan T_{OPT}, we have:

- $T_{\text{OPT}} \geq p_k$
- $T_{\text{OPT}} \geq \frac{1}{m} \sum_{i=1}^{n} p_i$ weight of all jobs evenly distributed



Multiprocessor Scheduling – List Scheduling (Proof)

ListScheduling$(J_1, \ldots, J_n, m)$

Put the first m jobs on the m machines.
Put the next job on the first free machine.

Theorem 7.
ListScheduling is a $(2 - \frac{1}{m})$-approximation alg.

Proof. Let $J_k = (S_k, T_k)$ be the last job, that is, T_k determines the makespan.

- No machine idles at time S_k.

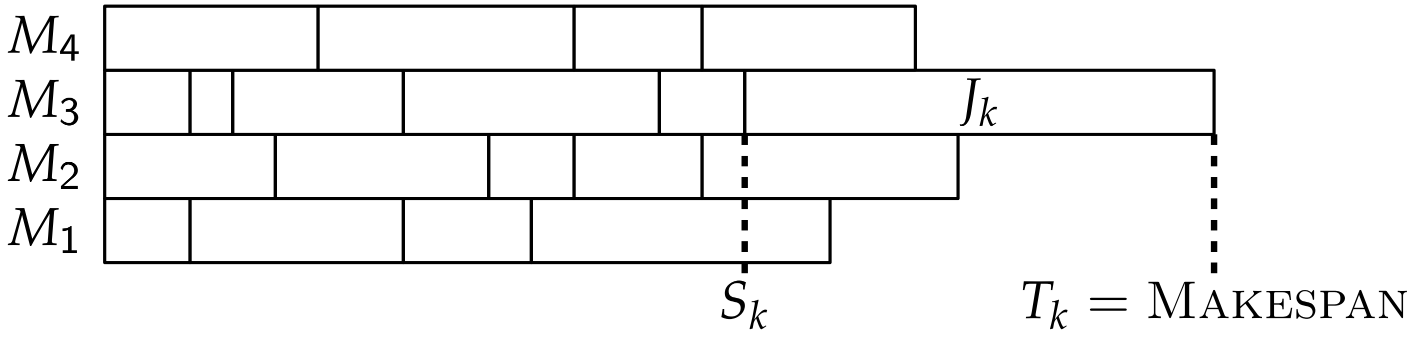
$$S_k \leq \frac{1}{m} \sum_{i \neq k} p_i$$

weight of all jobs but J_k evenly distributed on m machines

- For the optimal makespan T_{OPT}, we have:
- $T_{\text{OPT}} \geq p_k$
- $T_{\text{OPT}} \geq \frac{1}{m} \sum_{i=1}^{n} p_i$ weight of all jobs evenly distributed

- Hence:

$$T_k = S_k + p_k$$
$$\leq \frac{1}{m} \cdot \sum_{i \neq k} p_i + p_k$$
$$= \frac{1}{m} \cdot \sum_{i=1}^{n} p_i + \left(1 - \frac{1}{m}\right) \cdot p_k$$



□

Multiprocessor Scheduling – List Scheduling (Proof)

ListScheduling($J_1, \ldots, J_n, m$)

Put the first m jobs on the m machines.
Put the next job on the first free machine.

Theorem 7.
ListScheduling is a $(2 - \frac{1}{m})$-approximation alg.

Proof. Let $J_k = (S_k, T_k)$ be the last job, that is, T_k determines the makespan.

- No machine idles at time S_k.

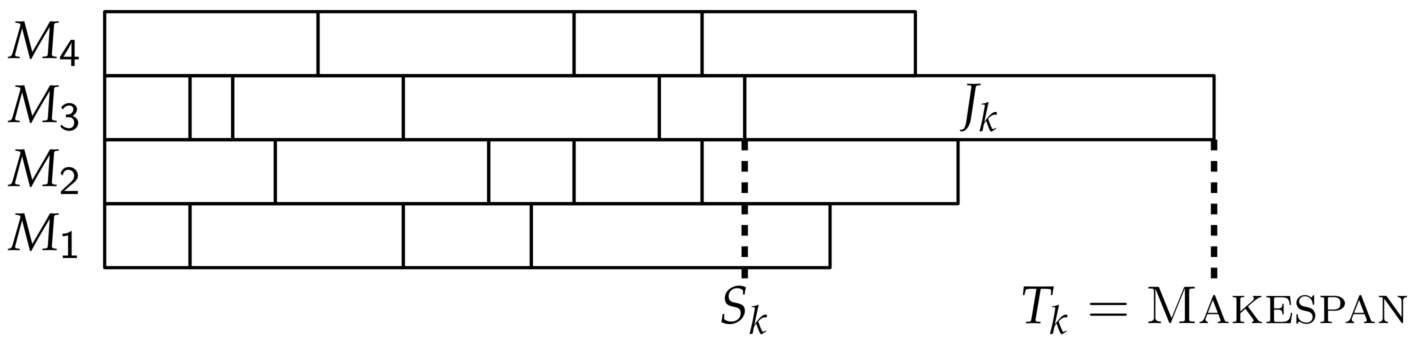
$$S_k \leq \frac{1}{m} \sum_{i \neq k} p_i$$

weight of all jobs but J_k evenly distributed on m machines

- For the optimal makespan T_{OPT}, we have:
  - $T_{\mathrm{OPT}} \geq p_k$
  - $T_{\mathrm{OPT}} \geq \frac{1}{m} \sum_{i=1}^{n} p_i$ weight of all jobs evenly distributed

- Hence:

$$\begin{aligned} T_k &= S_k + p_k \\ &\leq \frac{1}{m} \cdot \sum_{i \neq k} p_i + p_k \\ &= \frac{1}{m} \cdot \sum_{i=1}^{n} p_i + \left(1 - \frac{1}{m}\right) \cdot p_k \\ &\leq T_{\mathrm{OPT}} + \left(1 - \frac{1}{m}\right) \cdot T_{\mathrm{OPT}} \end{aligned}$$



Multiprocessor Scheduling – List Scheduling (Proof)

ListScheduling$(J_1, \dots, J_n, m)$

Put the first m jobs on the m machines.
Put the next job on the first free machine.

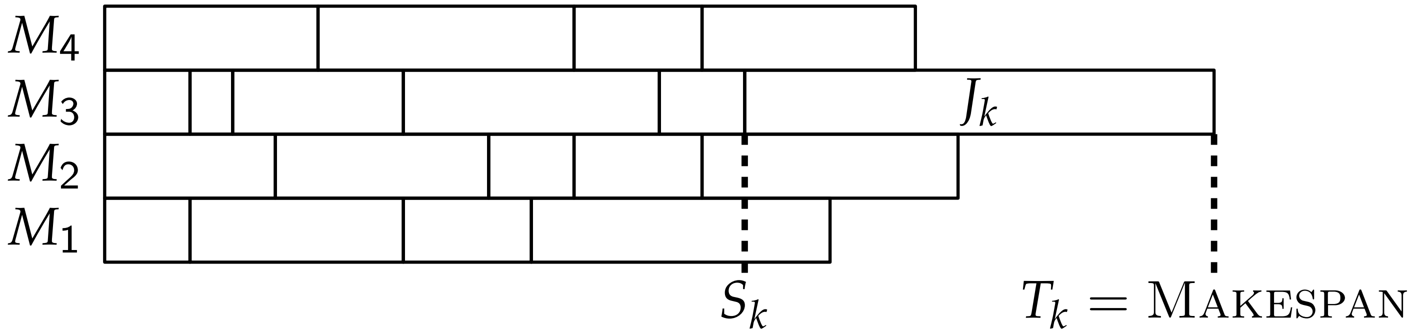
Theorem 7.
ListScheduling is a $(2 - \frac{1}{m})$-approximation alg.

Proof. Let $J_k = (S_k, T_k)$ be the last job, that is, T_k determines the makespan.

- No machine idles at time S_k.

$S_k \leq \frac{1}{m} \sum_{i \neq k} p_i$ — weight of all jobs but J_k evenly distributed on m machines

- For the optimal makespan T_{OPT}, we have:
  - $T_{\mathrm{OPT}} \geq p_k$
  - $T_{\mathrm{OPT}} \geq \frac{1}{m} \sum_{i=1}^{n} p_i$ — weight of all jobs evenly distributed



- Hence:

$$
\begin{aligned}
T_k &= S_k + p_k \\
&\leq \frac{1}{m} \cdot \sum_{i \neq k} p_i + p_k \\
&= \frac{1}{m} \cdot \sum_{i=1}^{n} p_i + \left(1 - \frac{1}{m}\right) \cdot p_k \\
&\leq T_{\mathrm{OPT}} + \left(1 - \frac{1}{m}\right) \cdot T_{\mathrm{OPT}} \\
&= \left(2 - \frac{1}{m}\right) \cdot T_{\mathrm{OPT}}
\end{aligned}
$$

□

Multiprocessor Scheduling – PTAS

For a constant ℓ $(1 \leq \ell \leq n)$ define the algorithm $\mathcal{A}_\ell$ as follows.

$\mathcal{A}_\ell(J_1, \ldots, J_n, m)$

Sort jobs in descending order of runtime.

Schedule the ℓ longest jobs $J_1, \ldots, J_\ell$ optimally.

Use ListScheduling for the remaining jobs $J_{\ell+1}, \ldots, J_n$.

Multiprocessor Scheduling – PTAS

For a constant ℓ $(1 \leq \ell \leq n)$ define the algorithm $\mathcal{A}_\ell$ as follows.

$\mathcal{A}_\ell(J_1, \ldots, J_n, m)$

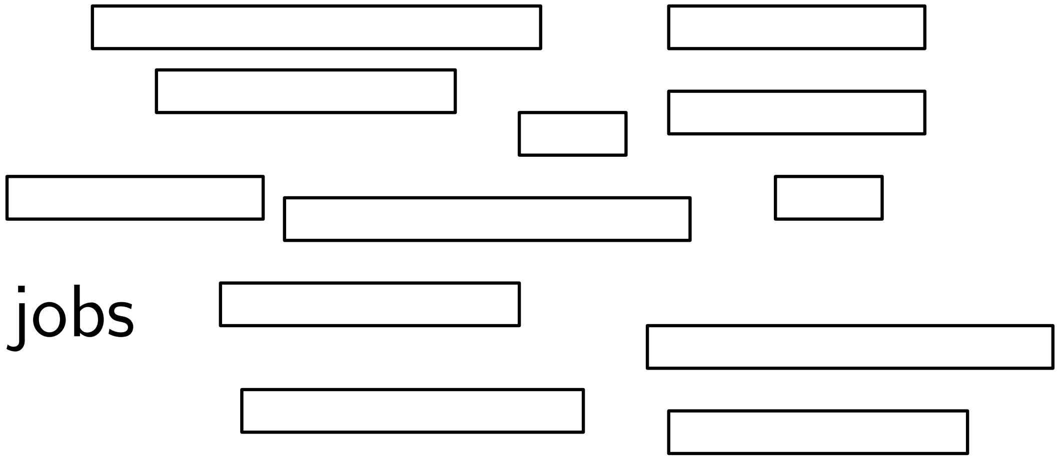
Sort jobs in descending order of runtime.

Schedule the ℓ longest jobs $J_1, \ldots, J_\ell$ optimally.

Use ListScheduling for the remaining jobs $J_{\ell+1}, \ldots, J_n$.

Example.

$\ell = 6$



Multiprocessor Scheduling – PTAS

For a constant ℓ $(1 \leq \ell \leq n)$ define the algorithm $\mathcal{A}_\ell$ as follows.

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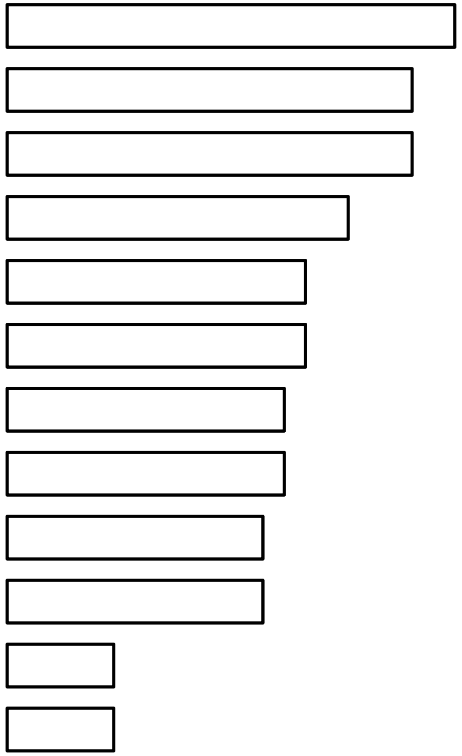
Example.

$\ell = 6$

sorted jobs



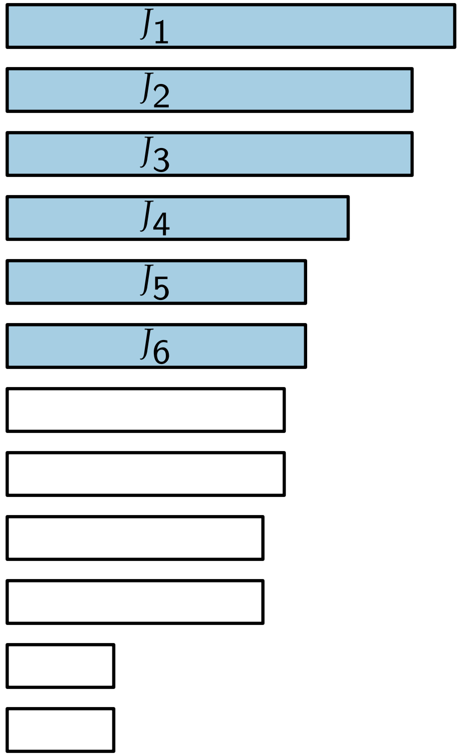
Multiprocessor Scheduling – PTAS

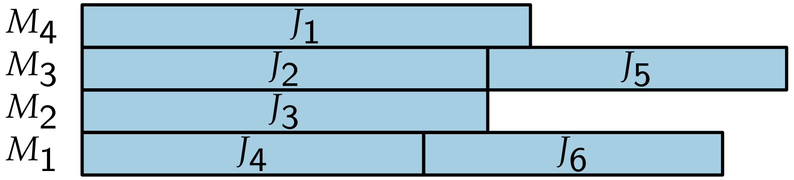
For a constant ℓ $(1 \leq \ell \leq n)$ define the algorithm $\mathcal{A}_\ell$ as follows.

$\mathcal{A}_\ell(J_1, \ldots, J_n, m)$
  Sort jobs in descending order of runtime.
  Schedule the ℓ longest jobs $J_1, \ldots, J_\ell$ optimally.
  Use ListScheduling for the remaining jobs $J_{\ell+1}, \ldots, J_n$.

Example.
$\ell = 6$

sorted jobs





Multiprocessor Scheduling – PTAS

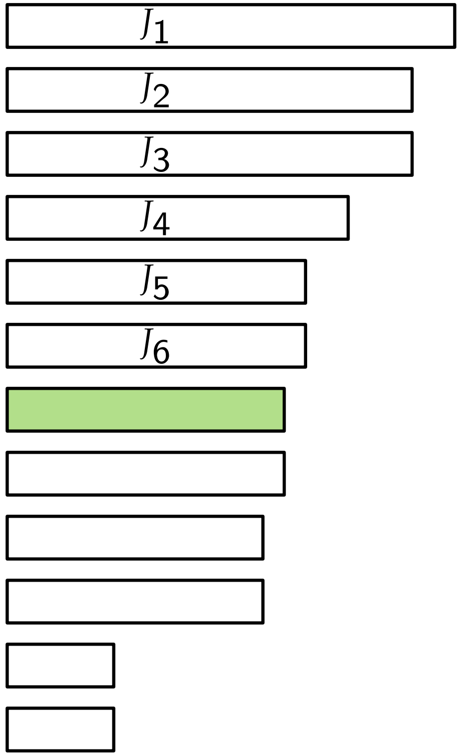
For a constant ℓ ($1 \leq \ell \leq n$) define the algorithm $\mathcal{A}_\ell$ as follows.

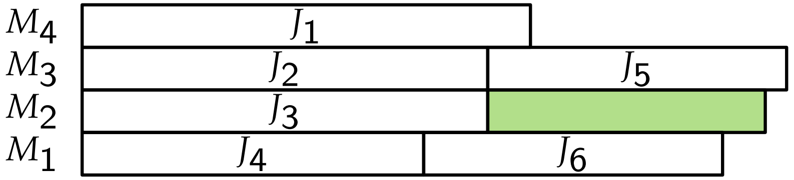
$\mathcal{A}_\ell(J_1, \ldots, J_n, m)$
- Sort jobs in descending order of runtime.
- Schedule the ℓ longest jobs $J_1, \ldots, J_\ell$ optimally.
- Use ListScheduling for the remaining jobs $J_{\ell+1}, \ldots, J_n$.

Example.

$\ell = 6$

sorted jobs





Multiprocessor Scheduling – PTAS

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Example.
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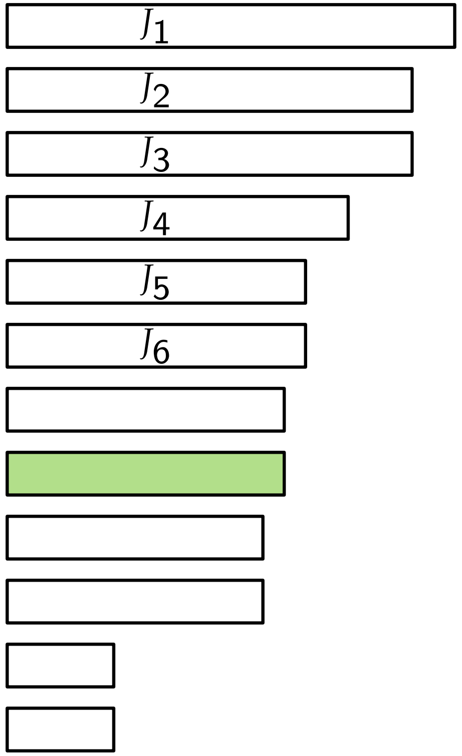
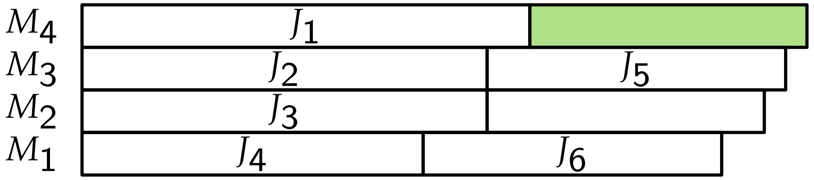
sorted jobs





Multiprocessor Scheduling – PTAS

For a constant ℓ $(1 \leq \ell \leq n)$ define the algorithm $\mathcal{A}_\ell$ as follows.

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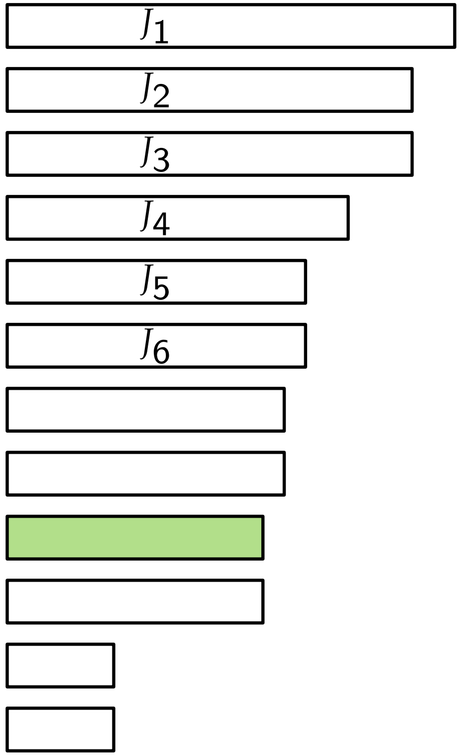
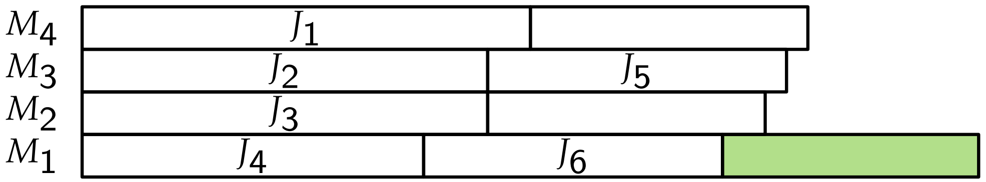
Example.
$\ell = 6$

sorted jobs





Multiprocessor Scheduling – PTAS

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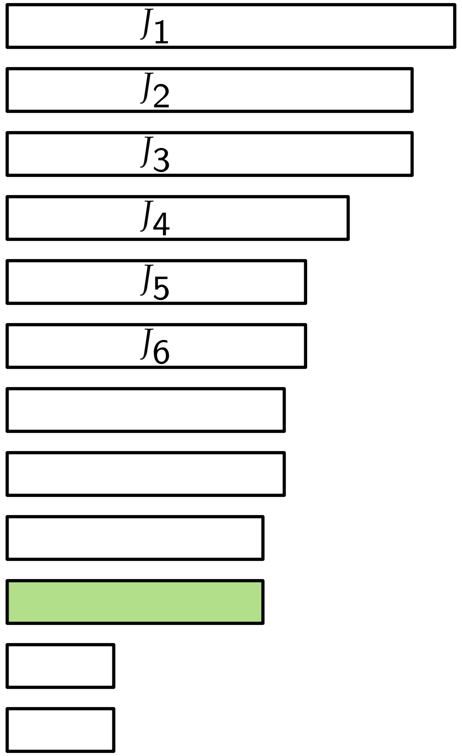
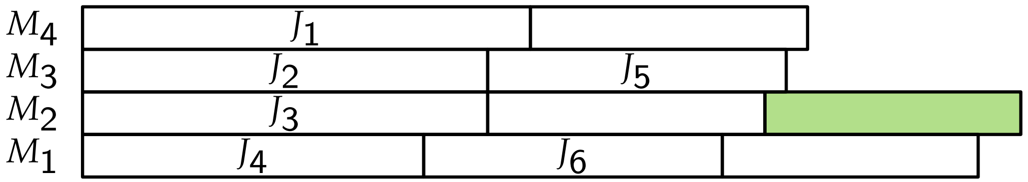
sorted jobs





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For a constant ℓ $(1 \leq \ell \leq n)$ define the algorithm $\mathcal{A}_\ell$ as follows.

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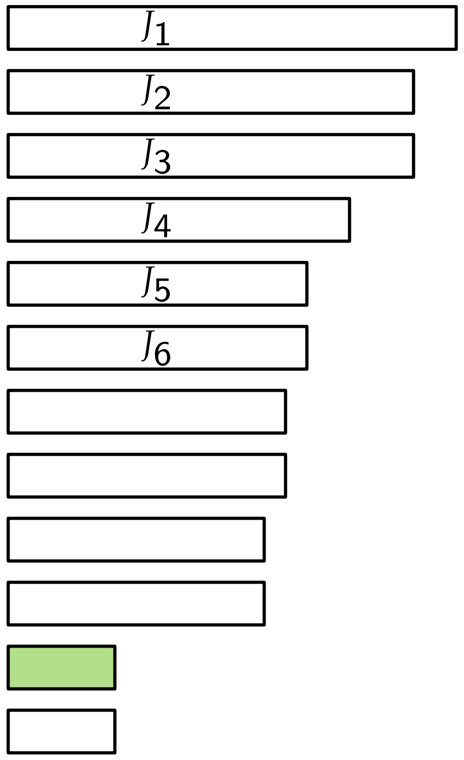
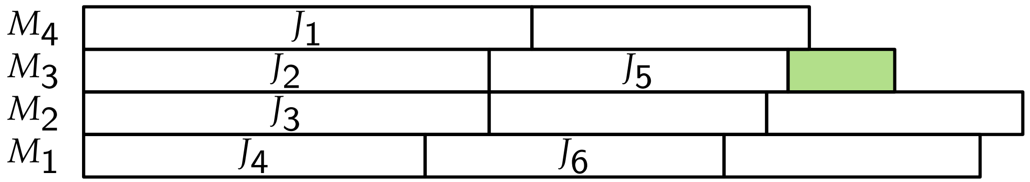
Example.

$\ell = 6$

sorted jobs





Multiprocessor Scheduling – PTAS

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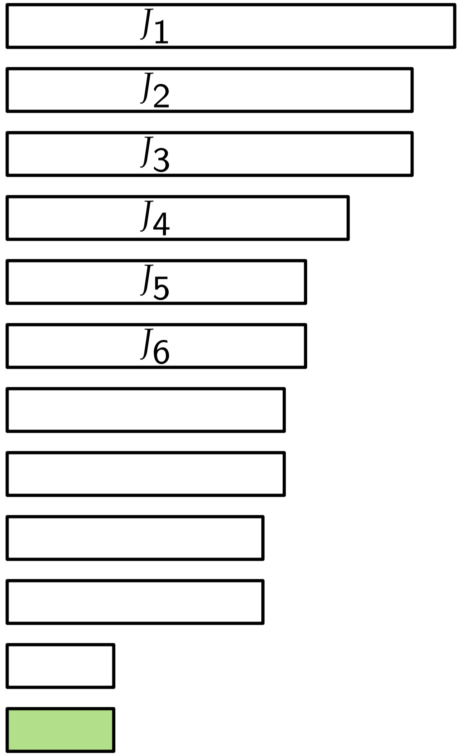
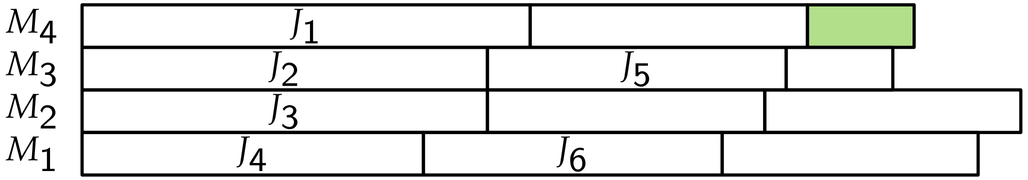
Example.
$\ell = 6$

sorted jobs





Multiprocessor Scheduling – PTAS

For a constant ℓ $(1 \leq \ell \leq n)$ define the algorithm $\mathcal{A}_\ell$ as follows.

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  Sort jobs in descending order of runtime.
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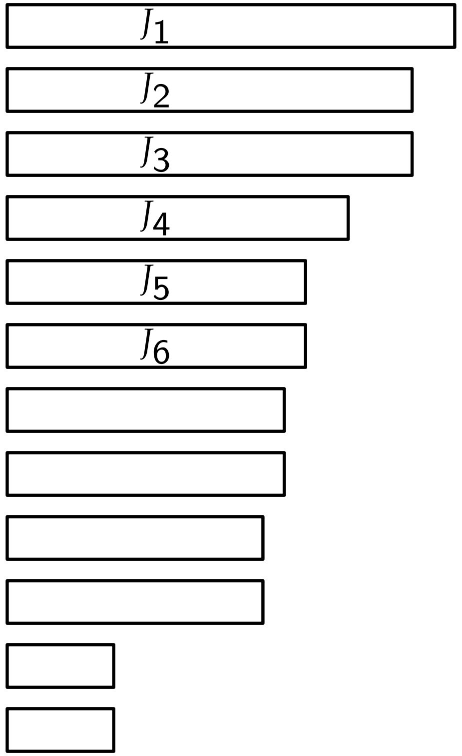
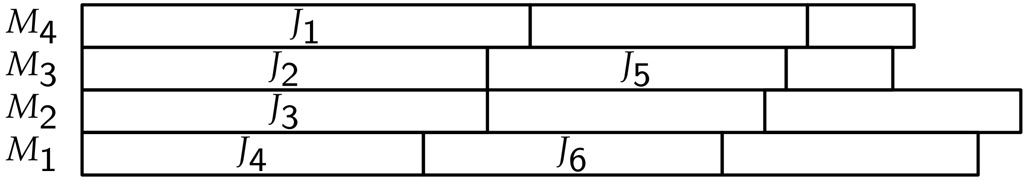
  Use ListScheduling for the remaining jobs $J_{\ell+1}, \ldots, J_n$.

Example.
$\ell = 6$

sorted jobs





Multiprocessor Scheduling – PTAS

For a constant ℓ $(1 \leq \ell \leq n)$ define the algorithm $\mathcal{A}_\ell$ as follows.

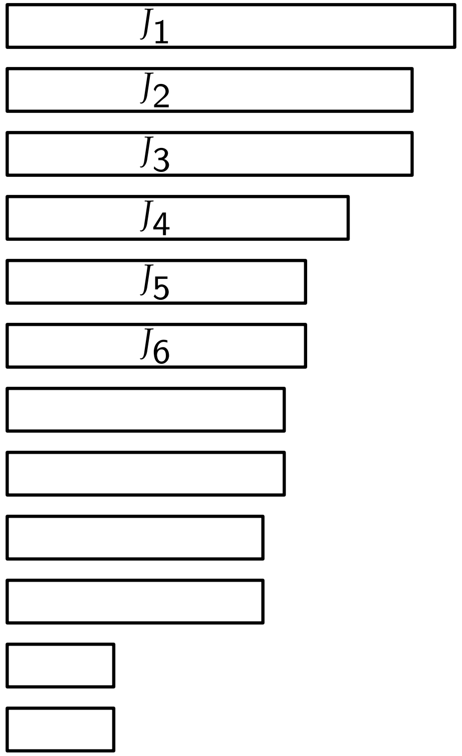
$\mathcal{A}_\ell(J_1, \ldots, J_n,\ m)$
- Sort jobs in descending order of runtime. $\mathcal{O}(n \log n)$
- Schedule the ℓ longest jobs $J_1, \ldots, J_\ell$ optimally. $\mathcal{O}(m^\ell)$
- Use ListScheduling for the remaining jobs $J_{\ell+1}, \ldots, J_n$. $\mathcal{O}(n \log m)$

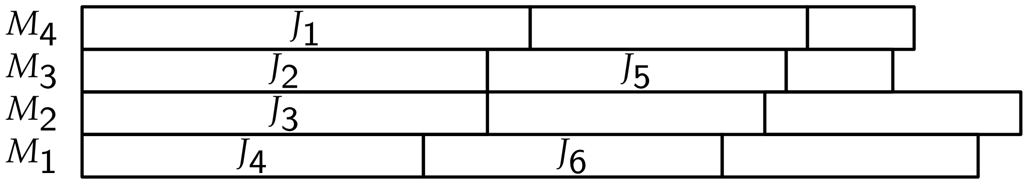
- Polynomial time for constant ℓ: $\mathcal{O}(m^\ell + n \log n)$

Example.

$\ell = 6$

sorted jobs





Multiprocessor Scheduling – PTAS

For a constant ℓ $(1 \leq \ell \leq n)$ define the algorithm $\mathcal{A}_\ell$ as follows.

$\mathcal{A}_\ell(J_1, \ldots, J_n,\ m)$

- Sort jobs in descending order of runtime. $\mathcal{O}(n \log n)$
- Schedule the ℓ longest jobs $J_1, \ldots, J_\ell$ optimally. $\mathcal{O}(m^\ell)$
- Use ListScheduling for the remaining jobs $J_{\ell+1}, \ldots, J_n$. $\mathcal{O}(n \log m)$

- Polynomial time for constant ℓ: $\mathcal{O}(m^\ell + n \log n)$

Theorem 8.
For constant $\ell \in \{1, \ldots, n\}$, algorithm $\mathcal{A}_\ell$ is a $1 + \frac{1 - \frac{1}{m}}{1 + \lfloor \frac{\ell}{m} \rfloor}$-approximation algorithm.

Multiprocessor Scheduling – PTAS

For a constant ℓ $(1 \leq \ell \leq n)$ define the algorithm $\mathcal{A}_\ell$ as follows.

$\mathcal{A}_\ell(J_1, \ldots, J_n, m)$
- Sort jobs in descending order of runtime. $\mathcal{O}(n \log n)$
- Schedule the ℓ longest jobs $J_1, \ldots, J_\ell$ optimally. $\mathcal{O}(m^\ell)$
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- Polynomial time for constant ℓ: $\mathcal{O}(m^\ell + n \log n)$

Theorem 8.
For constant $\ell \in \{1, \ldots, n\}$, algorithm $\mathcal{A}_\ell$ is a $1 + \frac{1 - \frac{1}{m}}{1 + \lfloor \frac{\ell}{m} \rfloor}$-approximation algorithm.

- For $\varepsilon > 0$, choose ℓ such that $\mathcal{A}_\varepsilon = \mathcal{A}_{\ell(\varepsilon)}$ is a $(1 + \varepsilon)$-approximation algorithm.

Corollary 9.
For a constant number of machines, $\{\mathcal{A}_\varepsilon \mid \varepsilon > 0\}$ is a PTAS.

Multiprocessor Scheduling – PTAS

For a constant ℓ $(1 \leq \ell \leq n)$ define the algorithm $\mathcal{A}_\ell$ as follows.

$\mathcal{A}_\ell(J_1, \dots, J_n, m)$
- Sort jobs in descending order of runtime. — $\mathcal{O}(n \log n)$
- Schedule the ℓ longest jobs $J_1, \dots, J_\ell$ optimally. — $\mathcal{O}(m^\ell)$
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- $\{\mathcal{A}_\varepsilon \mid \varepsilon > 0\}$ is not an FPTAS since the running time is not polynomial in $\frac{1}{\varepsilon}$.

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Multiprocessor Scheduling – PTAS (Proof)

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Proof. Let $J_k = (S_k, T_k)$ be the last job, that is, T_k determines the makespan.

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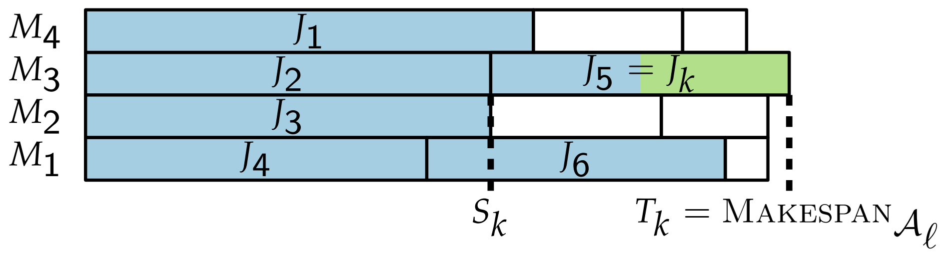

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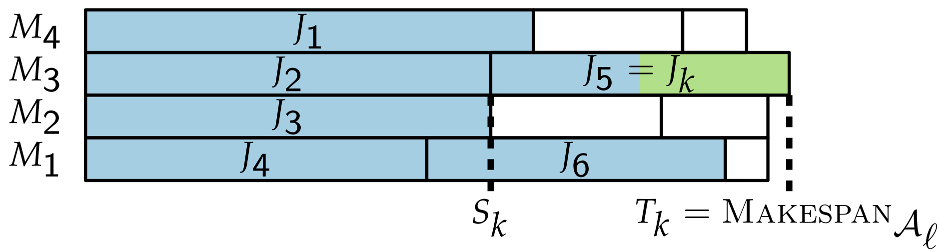
- Hence, solution is optimal for $J_1, \ldots, J_n$



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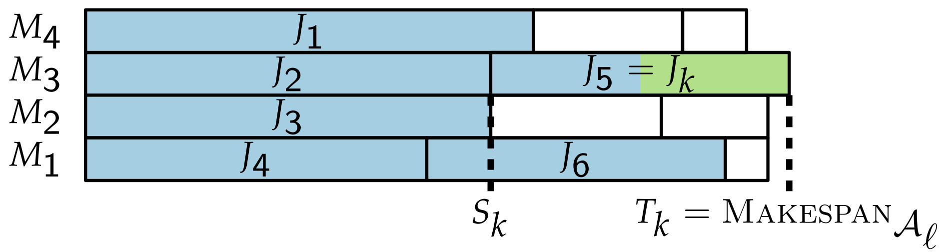

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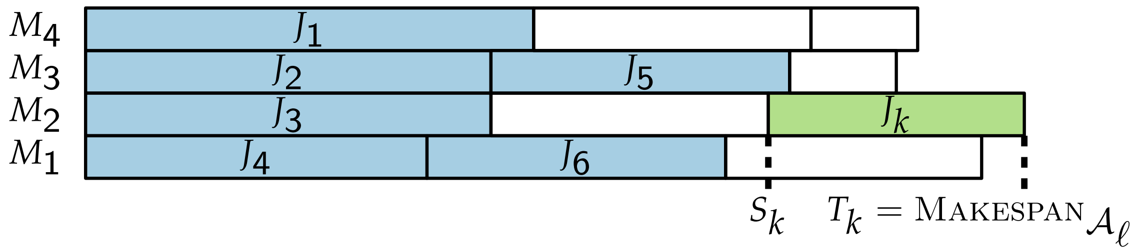

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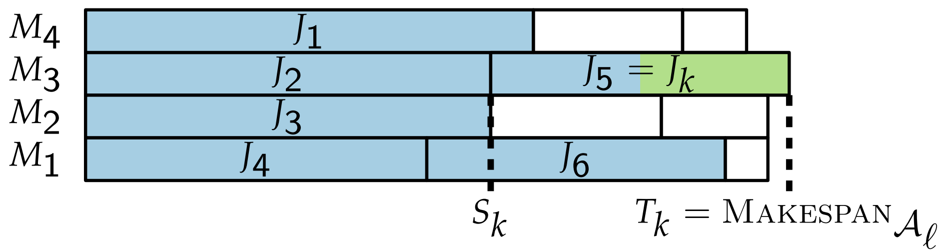
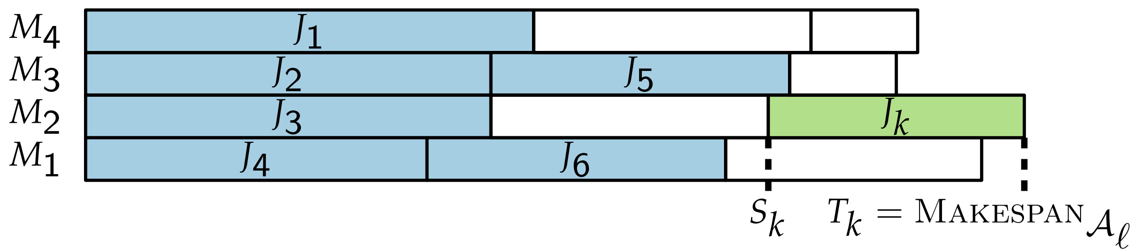


Case 2. J_k is not one of the longest ℓ jobs $J_1, \dots, J_\ell$.

- Similar analysis to ListScheduling
- Use that there are $\ell + 1$ jobs that are at least as long as J_k (including J_k).



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Proof of **Case 2**.

- $S_k \leq \frac{1}{m} \sum_{i \neq k} p_i$
- $T_{\text{OPT}} \geq \frac{1}{m} \sum_{i=1}^{n} p_i$
- $T_{\text{OPT}} \geq p_k$

$$T_k = S_k + p_k$$

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can we do better?

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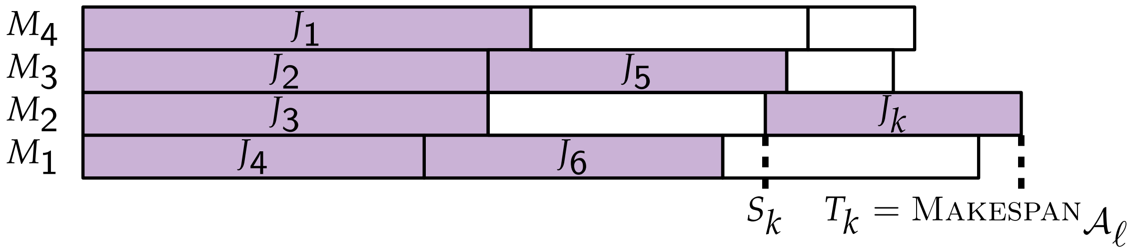

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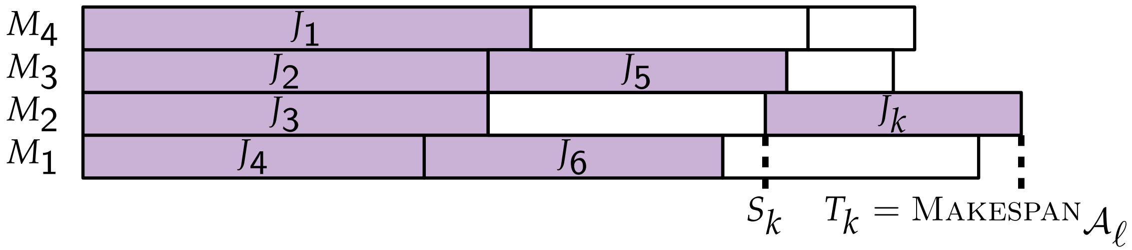

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one machine has this many jobs*

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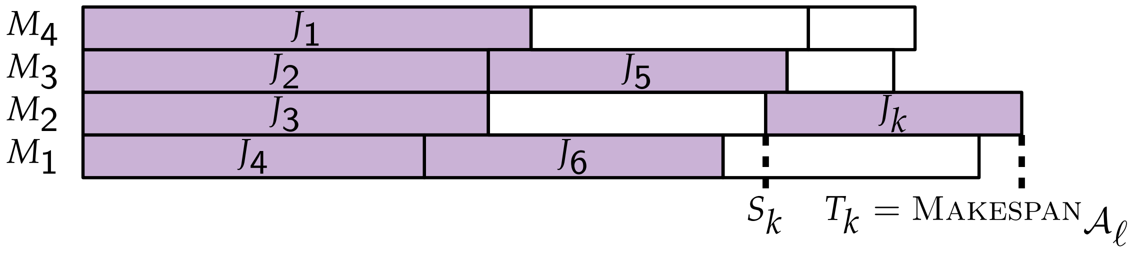

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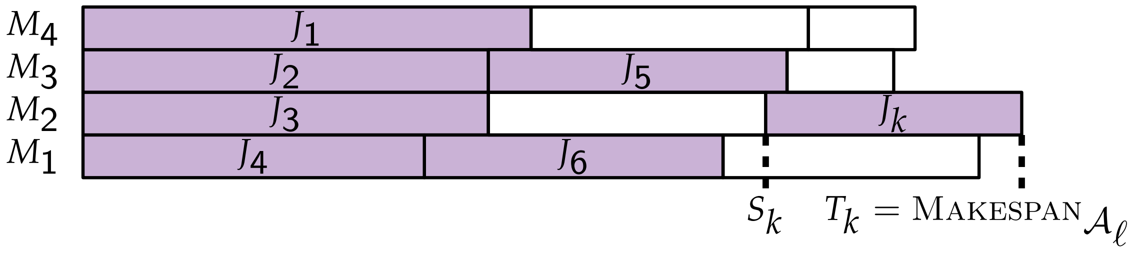

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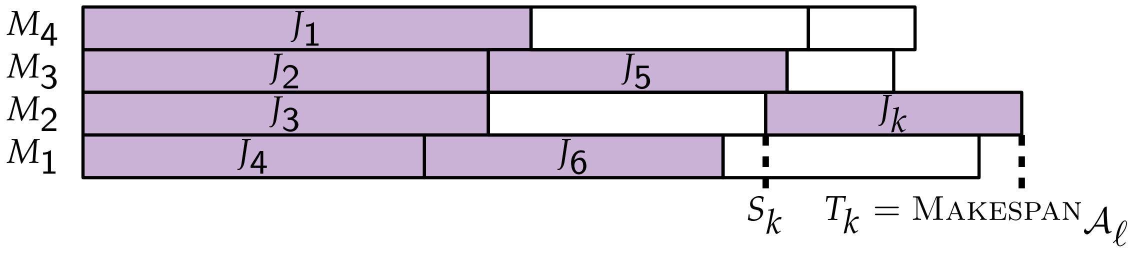

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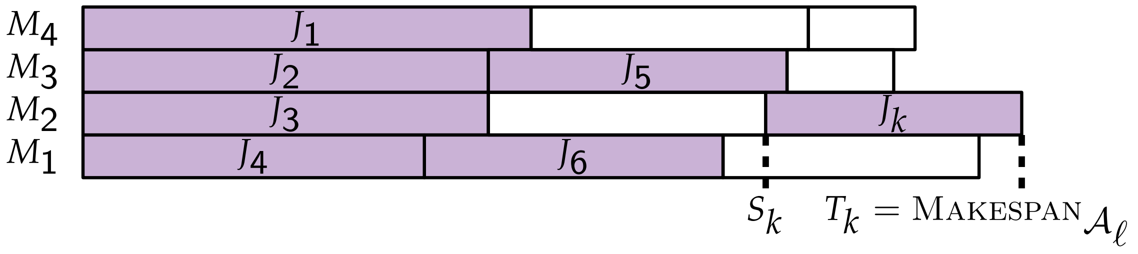

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$T_{\mathrm{OPT}} \geq p_k \cdot \left(1 + \left\lfloor \frac{\ell}{m} \right\rfloor\right)$ — one machine has this many jobs*; each has length $\geq p_k$

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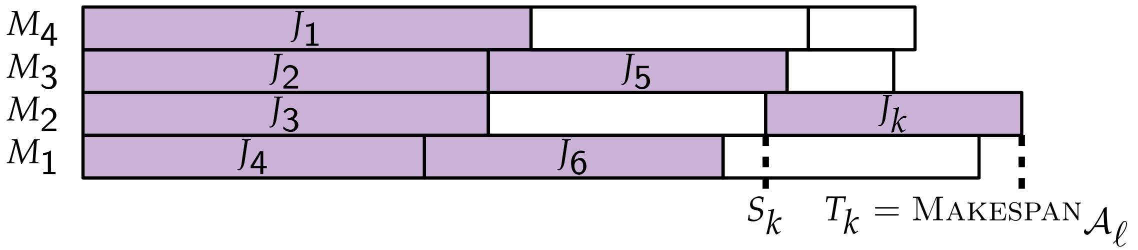

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- Only “easy” NP-hard problems admit FPTAS (PTAS).
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- There is a whole lecture on approximation algorithms this semester!
  `https://wuecampus.uni-wuerzburg.de/moodle/course/view.php?id=62943`

Literature

Main references

- [Jansen & Margraf, 2008: Ch3] "Approximative Algorithmen und Nichtapproximierbarkeit"
- [Williamson & Shmoys, 2011: Ch3] "The Design of Approximation Algorithms"

Another book recommendation:

- [Vazirani, 2013] "Approximation Algorithms"