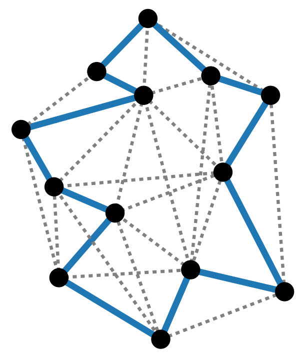
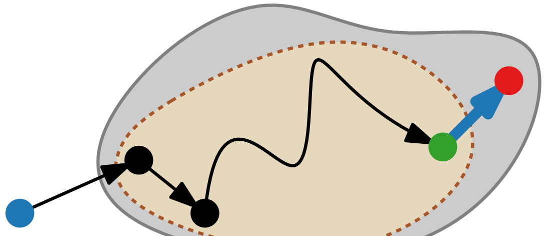
Advanced Algorithms

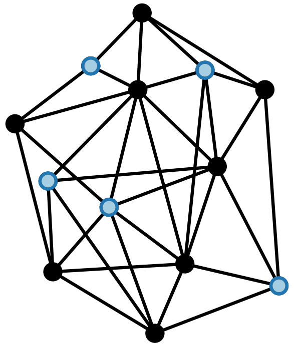
Exact Algorithms for NP-Hard Problems

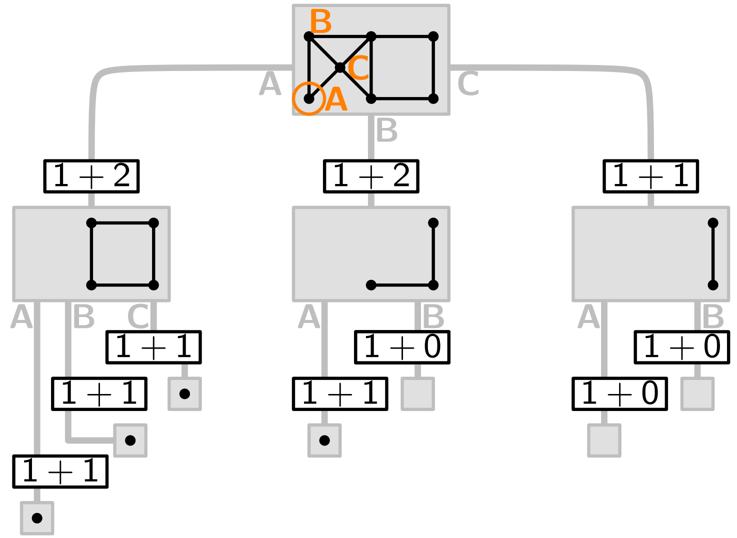
Traveling Salesman Problem and Maximal Independent Set

Johannes Zink · WS23/24









Examples of NP-Hard Problems

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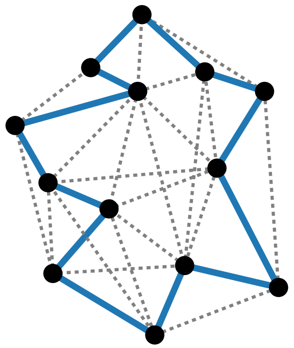
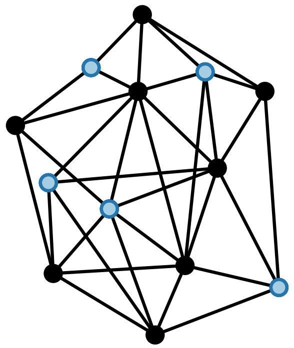
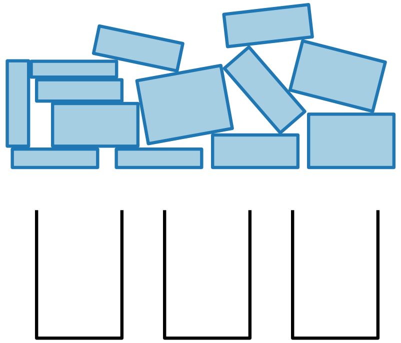


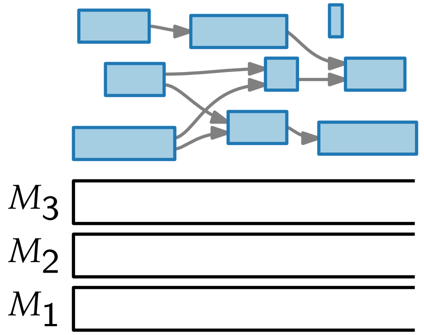
TSP



MIS



Bin Packing



Scheduling

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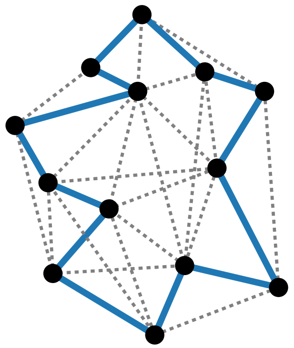

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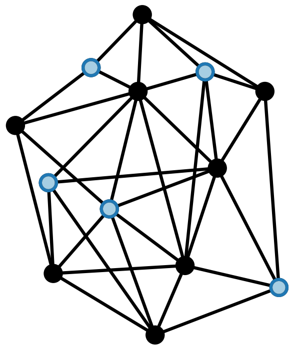

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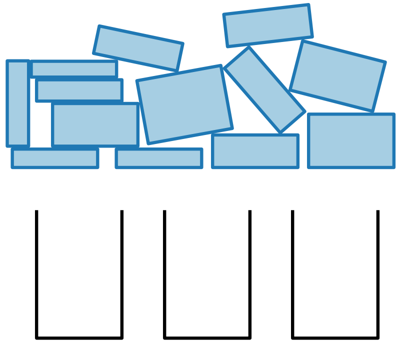

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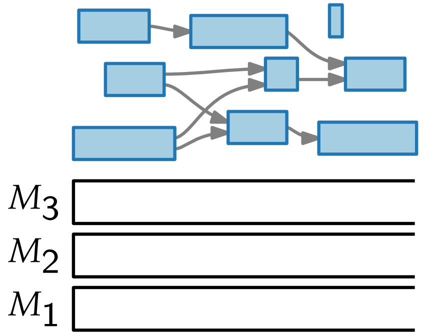
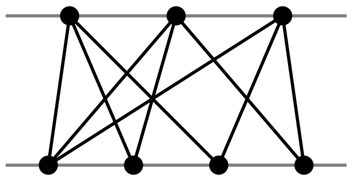


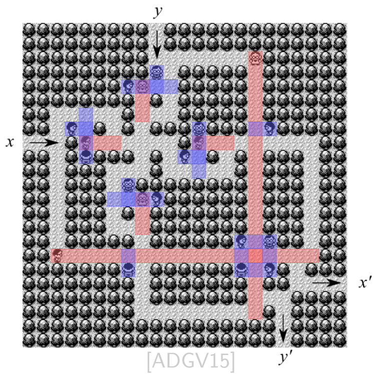
Scheduling

$(x_1 \vee x_2 \vee \neg x_4) \wedge$
$(\neg x_2 \vee x_3 \vee \neg x_4) \wedge$
$(x_3 \vee x_7 \vee \neg x_8) \wedge$
$\ldots$

SAT



Graph Drawing



Games

...

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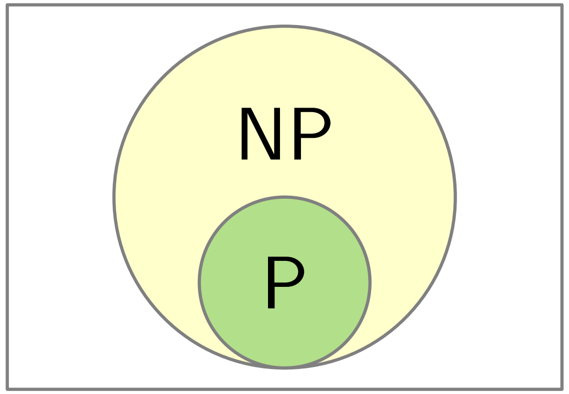

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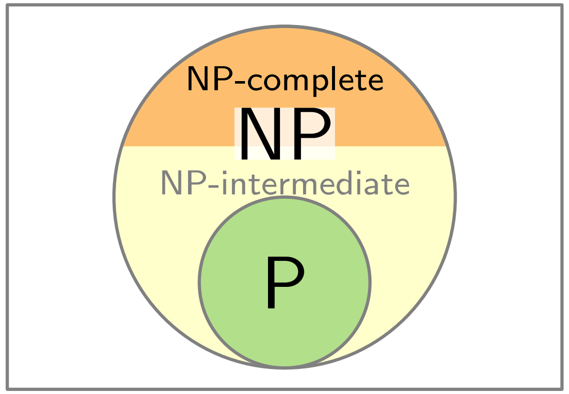

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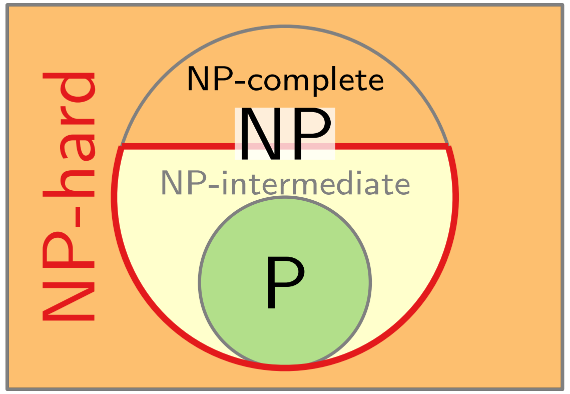

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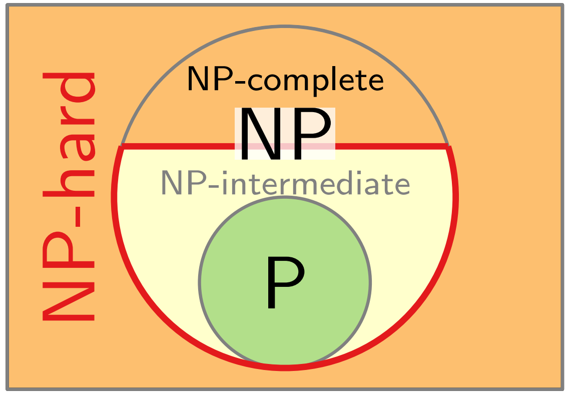

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- Assuming P $\neq$ NP, NP-hard problems cannot be solved in polynomial time.

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Common misconceptions [Mann '17]

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- For solving NP-hard problems, the only practical possibility is the use of heuristics.

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- Sacrifice optimality for speed
  - Heuristics
    (Simulated Annealing,
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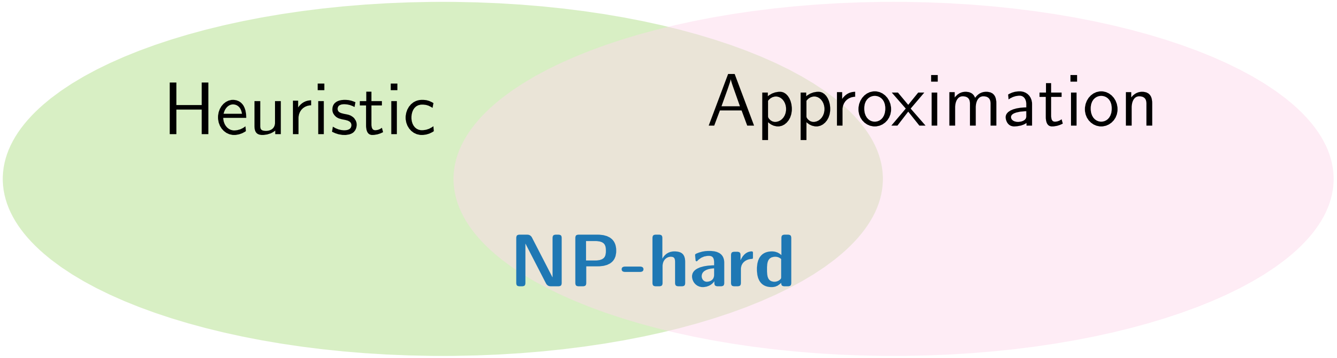

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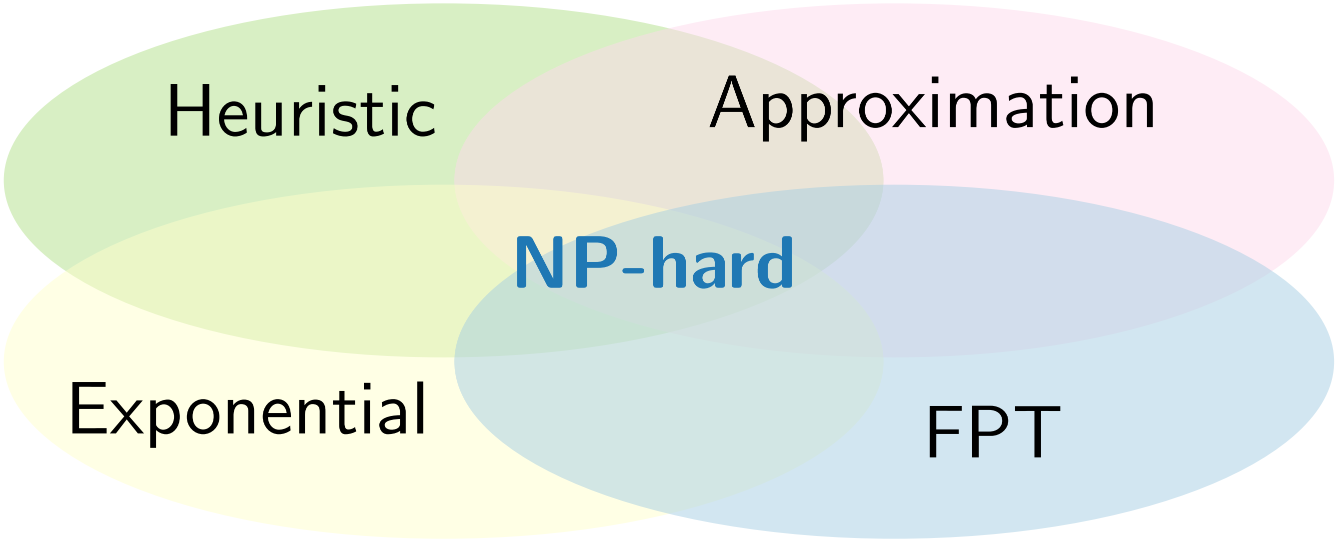

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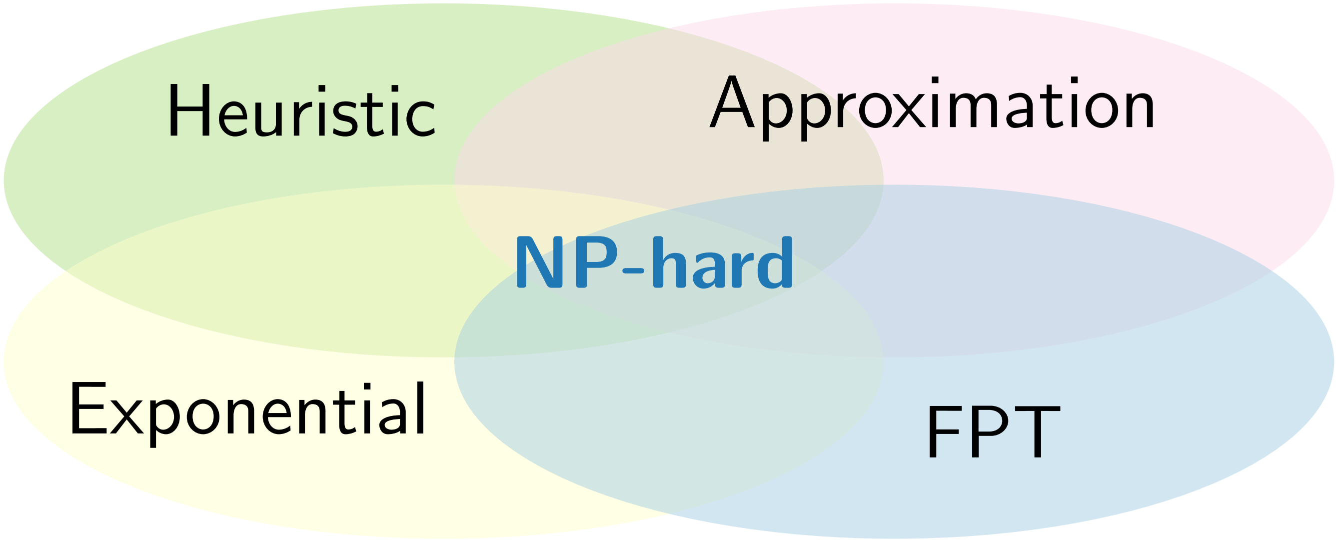
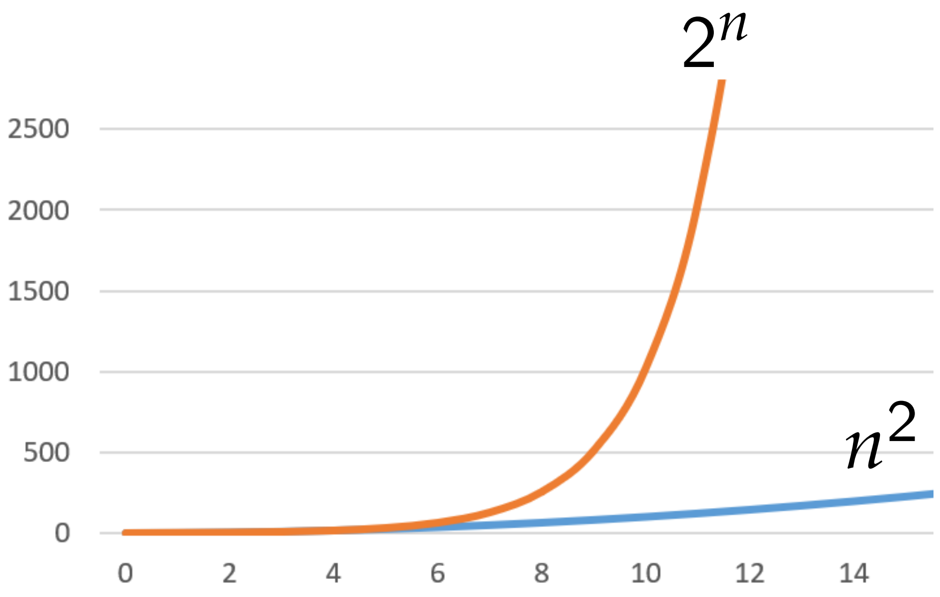


this lecture

Motivation



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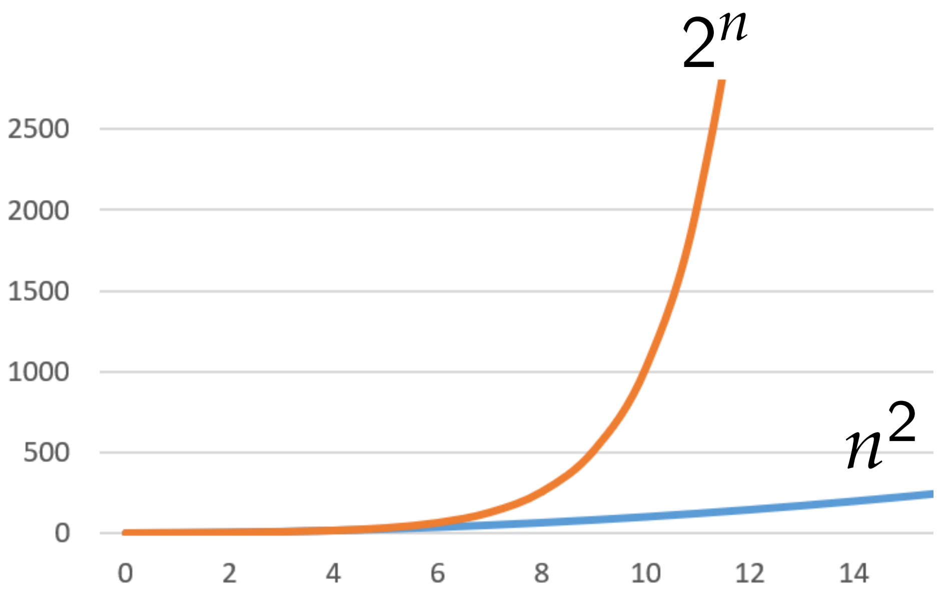

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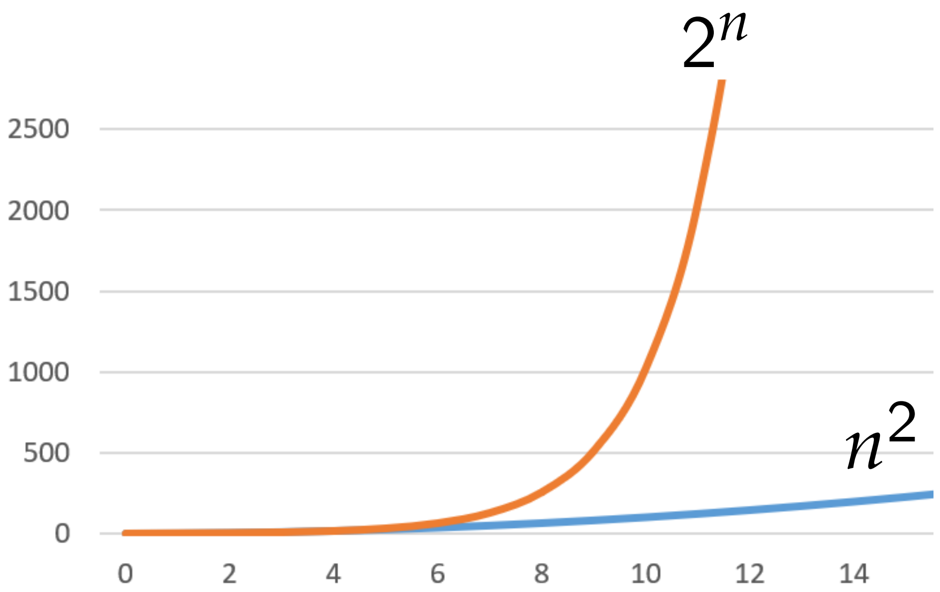

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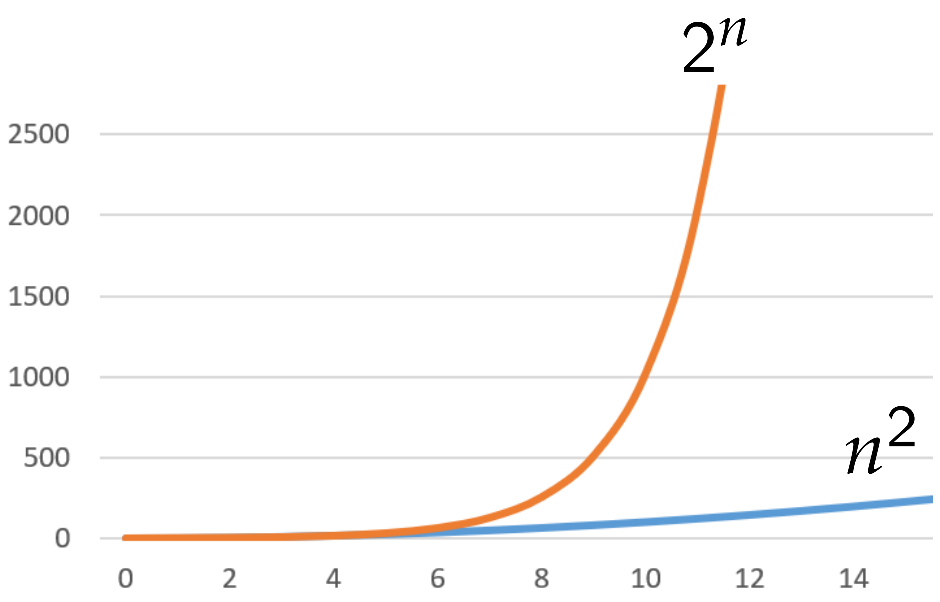

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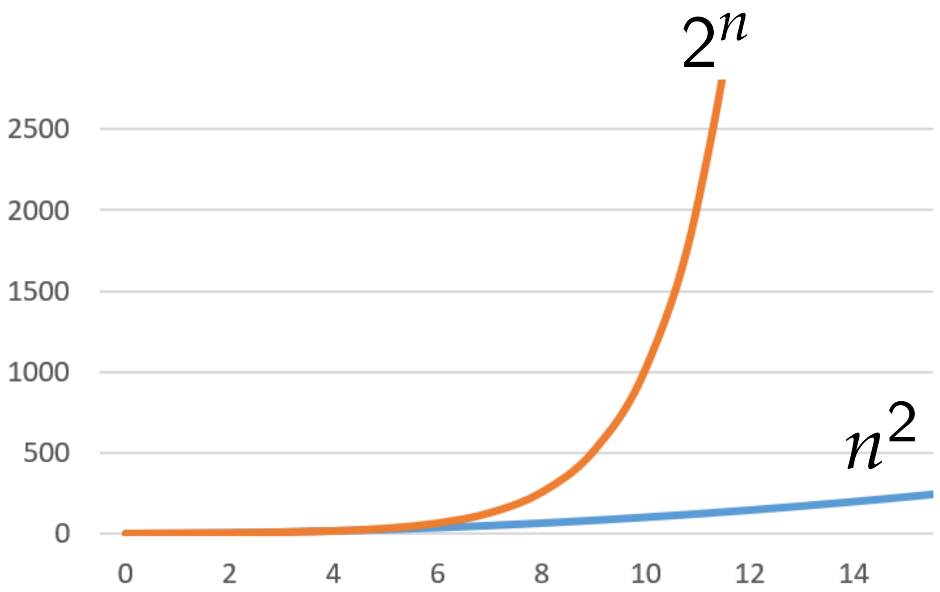

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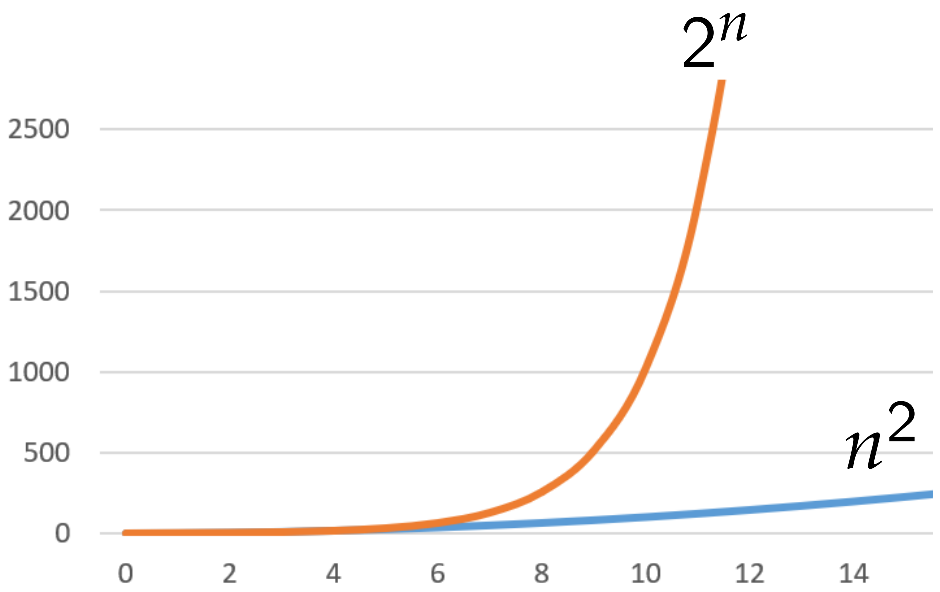

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  - TSP solvable exactly for $n \leq 2000$ and specialized instances with $n \leq 85900$

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- Reducing the base of the runtime to $b < a$ results in a *multiplicative* increase:

$$b^{n_0'} = a^{n_0} \rightsquigarrow n_0' = n_0 \cdot \log_b a$$

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- $\textsc{Coloring}$: Lawler gave an $\mathcal{O}(n(1+\sqrt[3]{3})^n)$ algorithm compared to $\mathcal{O}(n^{n+1})$-time brute-force search.
- $\textsc{SAT}$: No better algorithm than trivial brute-force search known.

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- typical result

Approach	Runtime in $\mathcal{O}$-Notation	$\mathcal{O}^*$-Notation
Brute-Force	$\mathcal{O}(2^n)$	$\mathcal{O}^*(2^n)$
Algorithm A	$\mathcal{O}(1.5^n \cdot n)$	$\mathcal{O}^*(1.5^n)$
Algorithm B	$\mathcal{O}(1.4^n \cdot n^2)$	$\mathcal{O}^*(1.4^n)$

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- Runtime: $\Theta(n! \cdot n) = n \cdot 2^{\Theta(n \log n)}$

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Idea.

- Dynamic programming means re-using optimal substructures (typically stored in a "table"). We store optimal partial tour lengths.

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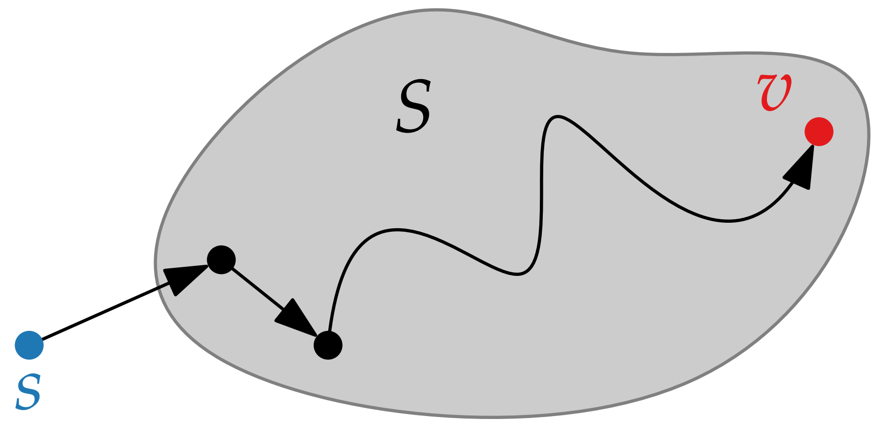

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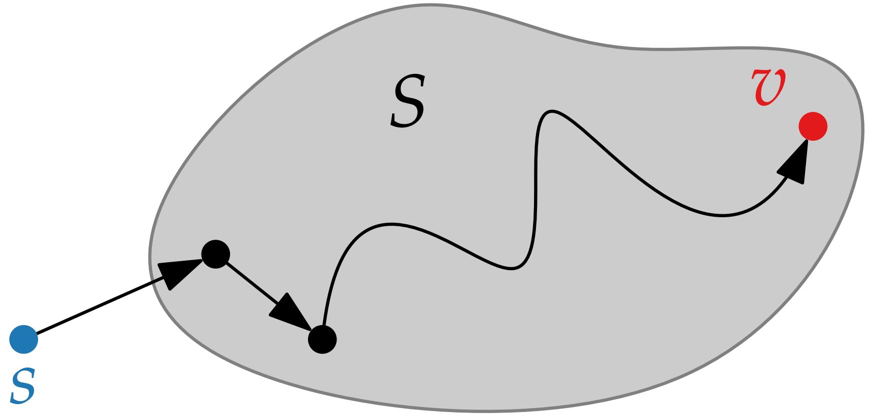

- Use OPT$[S - v, u]$ to compute OPT$[S, v]$.

Richard M. Karp

Richard E. Bellman

TSP – Dynamic Programming

Details.

- The base case $S = \{v\}$ is easy: $\mathrm{OPT}[\{v\}, v] =$

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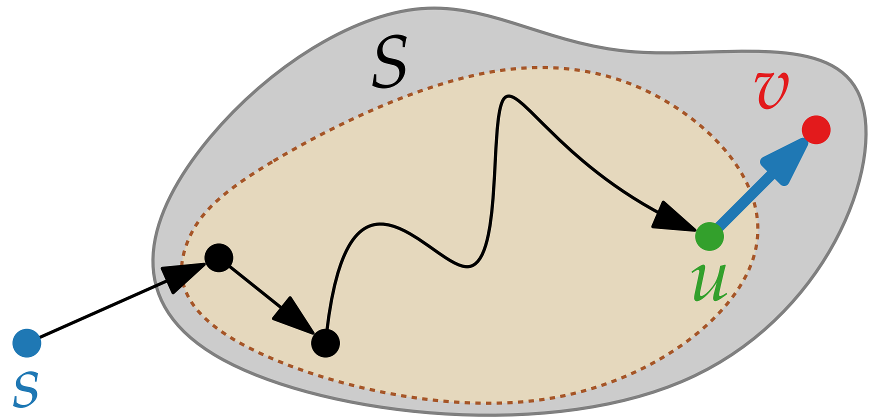

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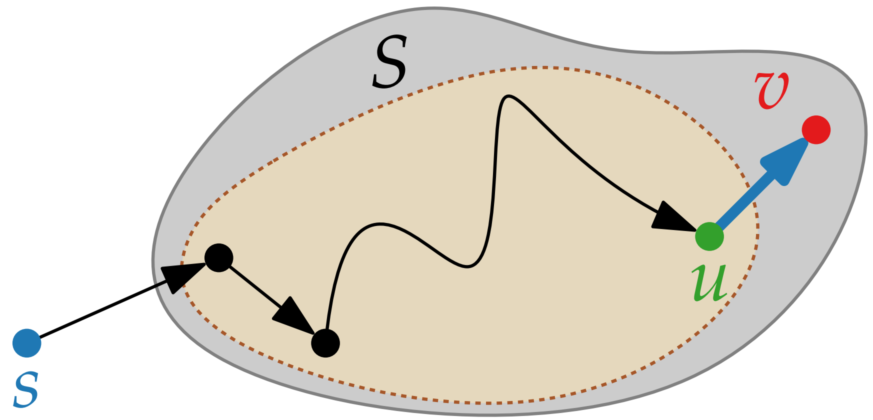

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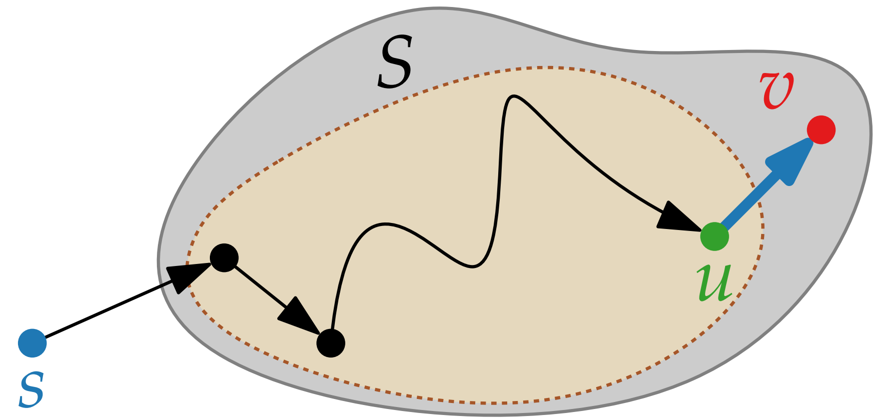

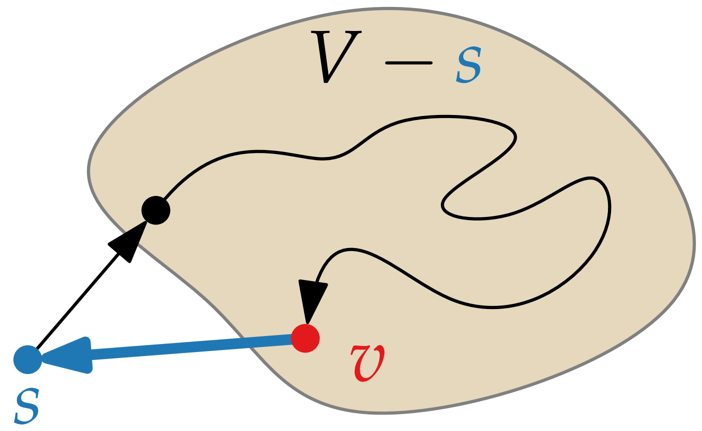

- After computing $\text{OPT}[S, v]$ for each $S \subseteq V - s$ and each $v \in V - s$, the optimal solution is easily obtained as follows:

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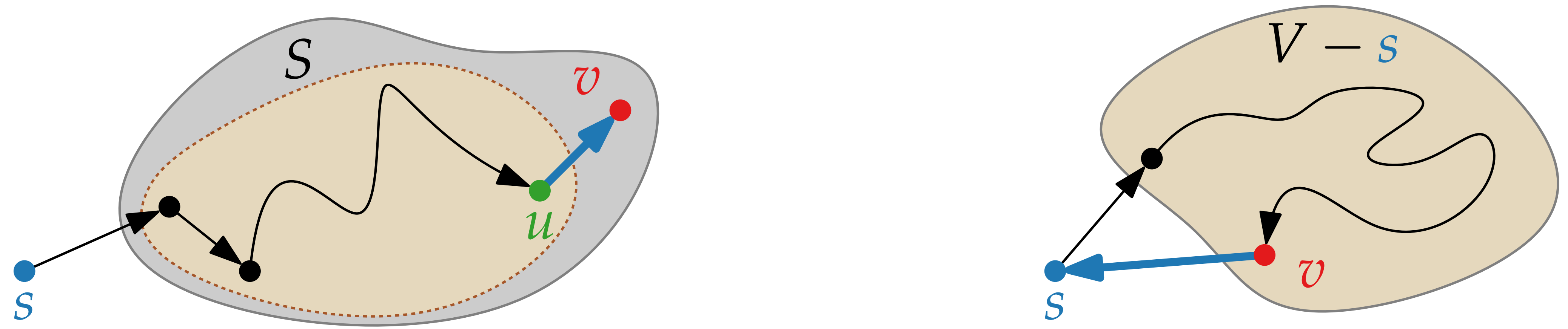

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$$\text{OPT} = \min\{\text{OPT}[V - s, v]\} + d(v, s) \mid v \in V - s\}$$

TSP – Dynamic Programming

Pseudocode.

Bellmann-Held-Karp(G, d):

```
foreach v ∈ V − s do
    OPT[{v}, v] = d(s, v)
for j = 2 to n − 1 do
    foreach S ⊆ V − s with |S| = j do
        foreach v ∈ S do
            OPT[S, v] = min{ OPT[S − v, u]
                             +d(u, v) | u ∈ S − v }
return min{ OPT[V − s, v] + d(v, s) | v ∈ V − s }
```

TSP – Dynamic Programming

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- running time for the central for-loop is in $\mathcal{O}(2^n n^2) \subseteq \mathcal{O}^*(2^n)$
- Space usage in $\Theta(2^n \cdot n)$
- Or actually better? What table values do we need to store?

TSP – Discussion

- DP algorithm that runs in $\mathcal{O}^*(2^n)$ time and $\mathcal{O}^*(2^n)$ space.
- Brute-force runs in $2^{\mathcal{O}(n \log n)}$ time and $\mathcal{O}(n^2)$ space.

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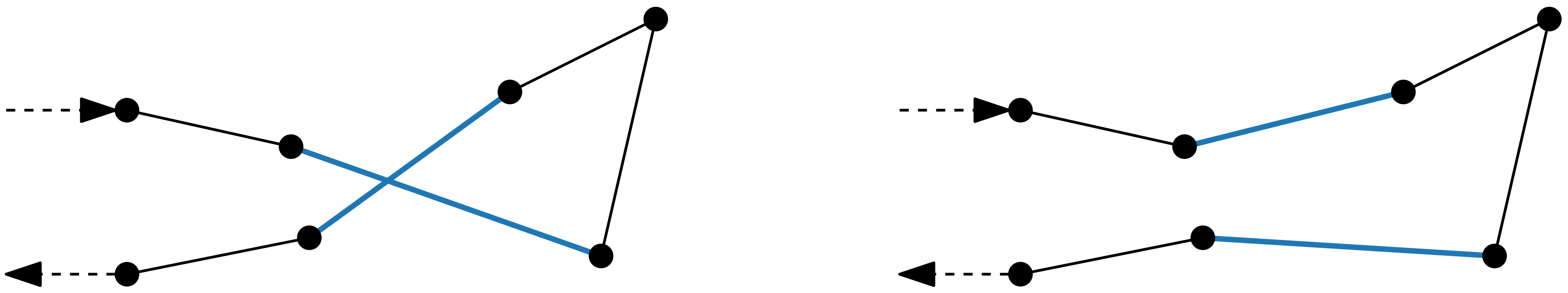
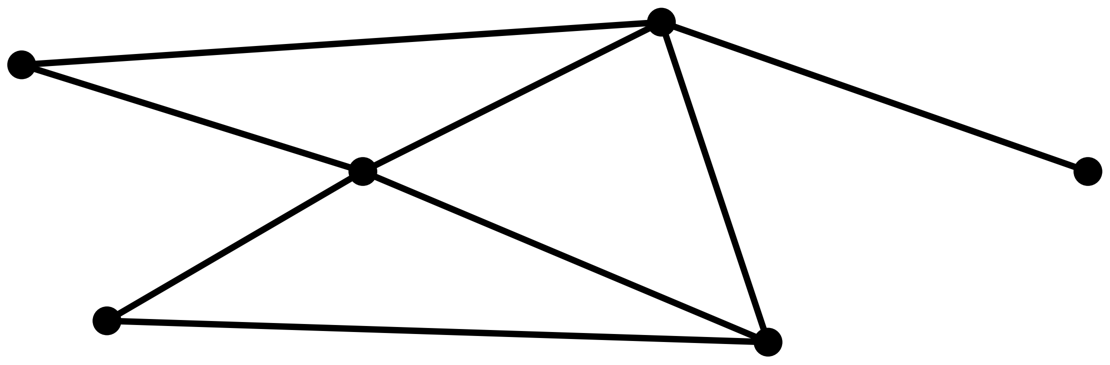
- Eucledian TSP is considered in the course Approximation Algorithms.
- In practice, one successful approach is to start with a greedily computed Hamiltonian cycle and then use 2-OPT and 3-OPT swaps to improve it.



Maximum Independent Set (MIS)

Input. Graph $G = (V, E)$ with n vertices.



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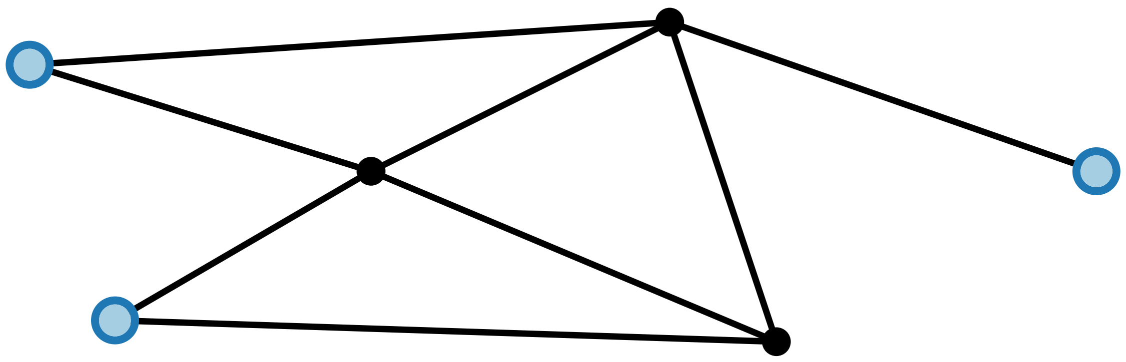

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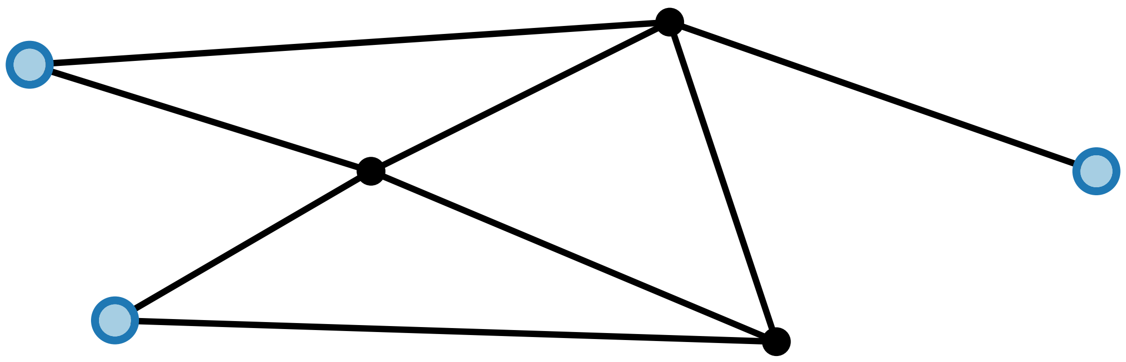

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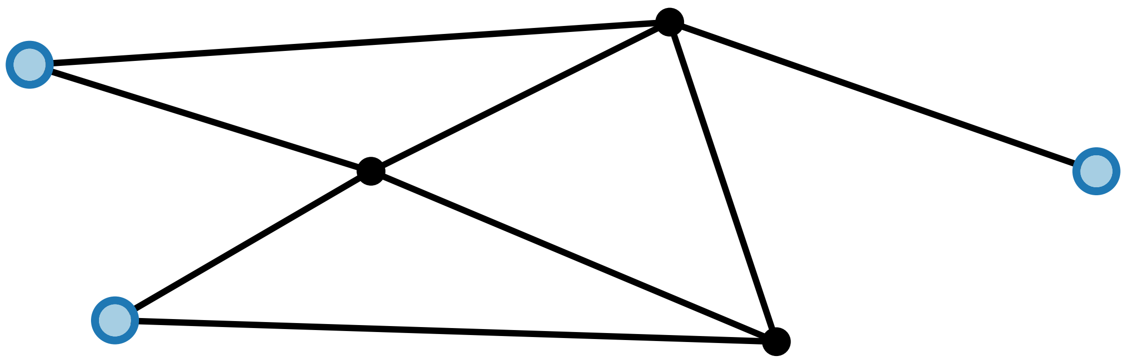

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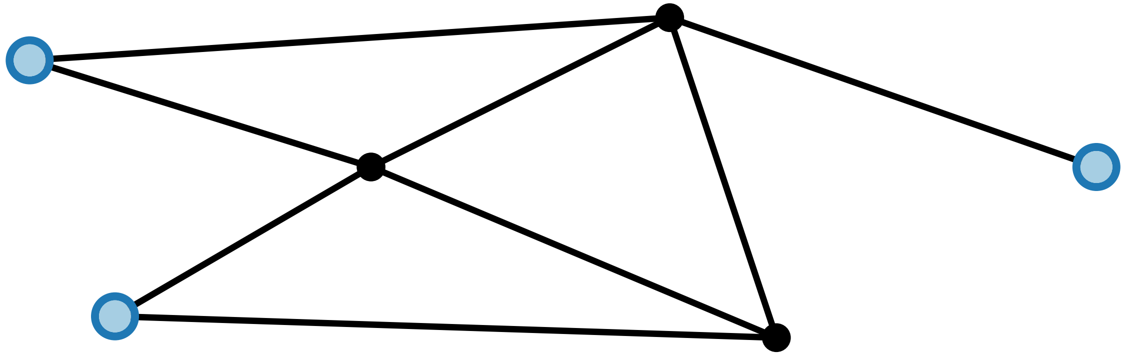

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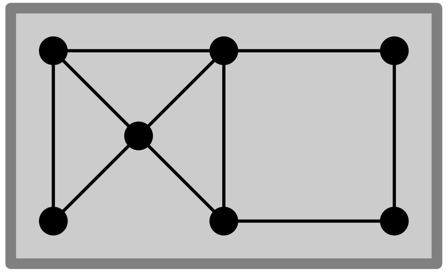
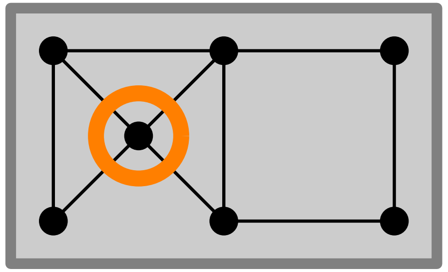
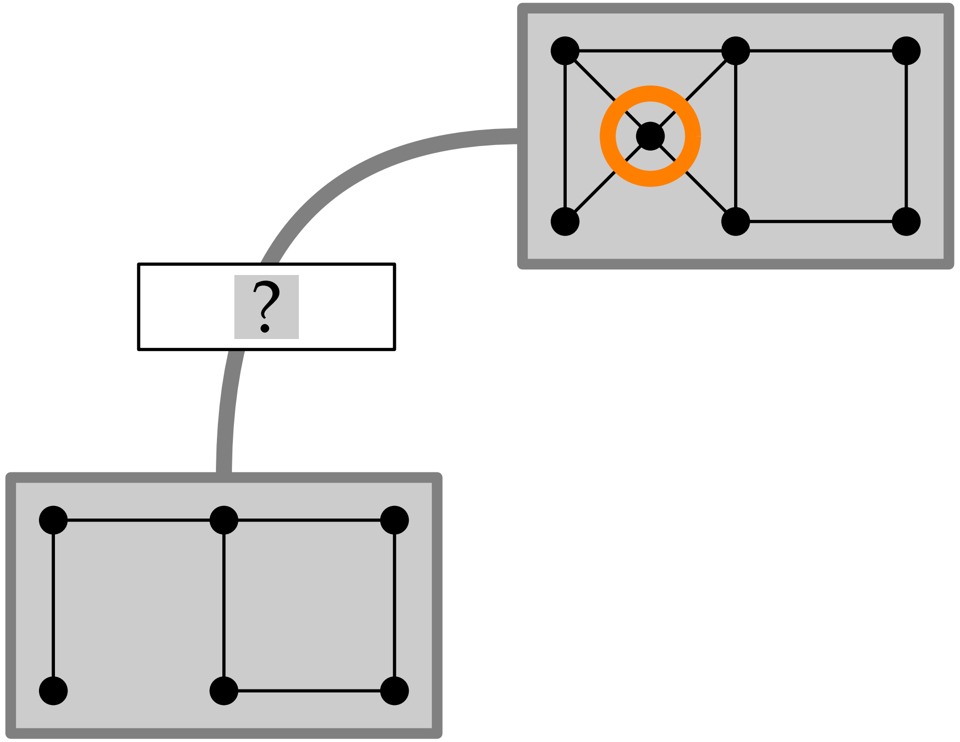
NaiveMIS(G):

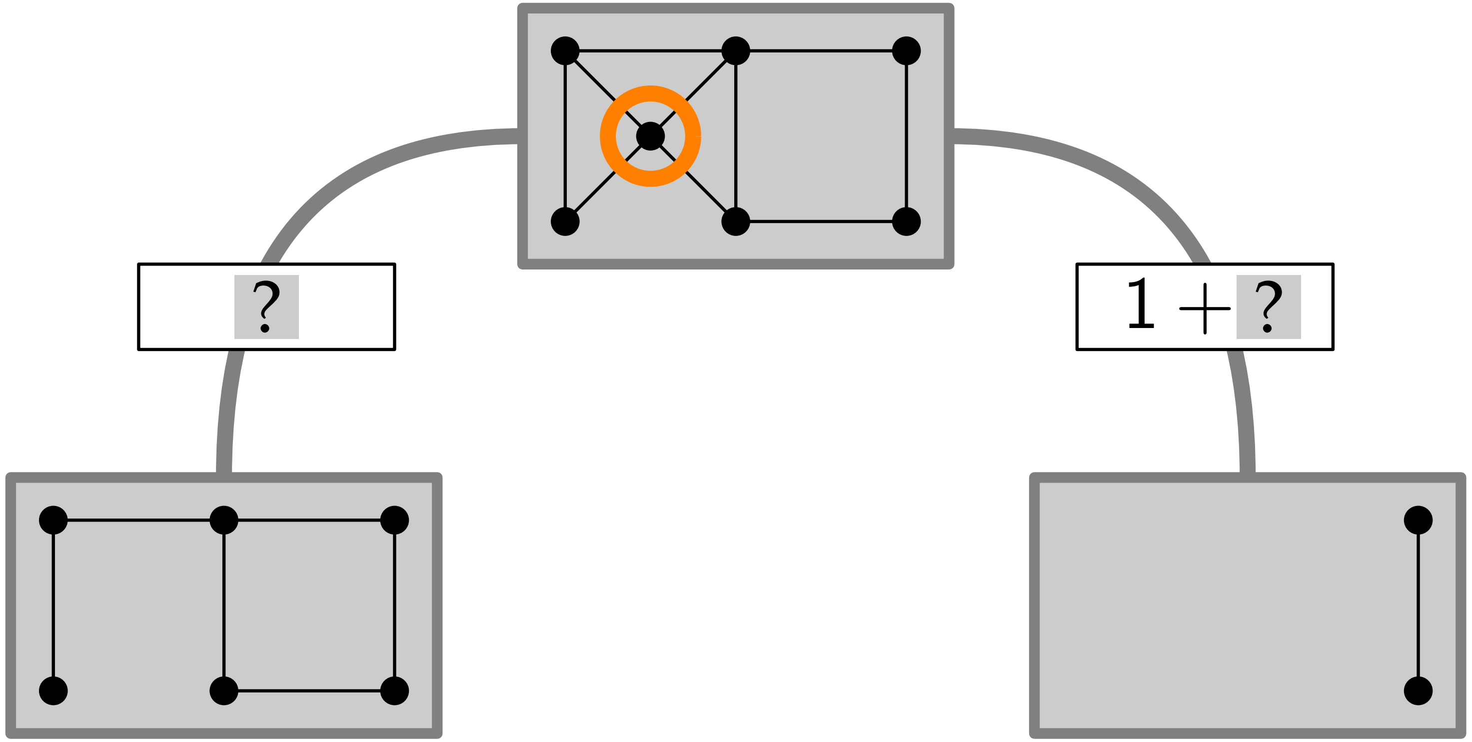
if $V == \emptyset$ **then**
  return 0

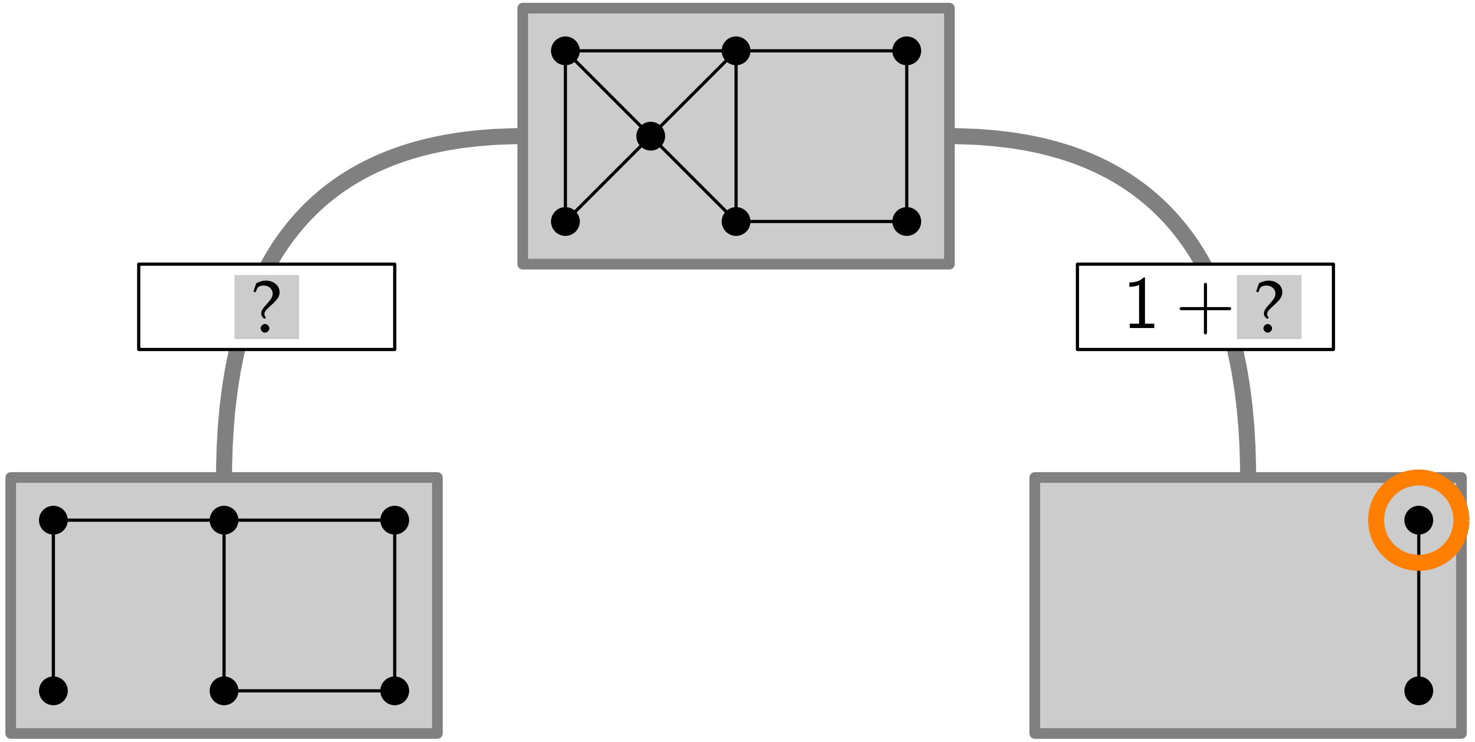
$v =$ arbitrary vertex in $V(G)$
return $\max\{1+$ NaiveMIS$(G - N(v) - \{v\})$, NaiveMIS$(G - \{v\})\}$








?
1 + ?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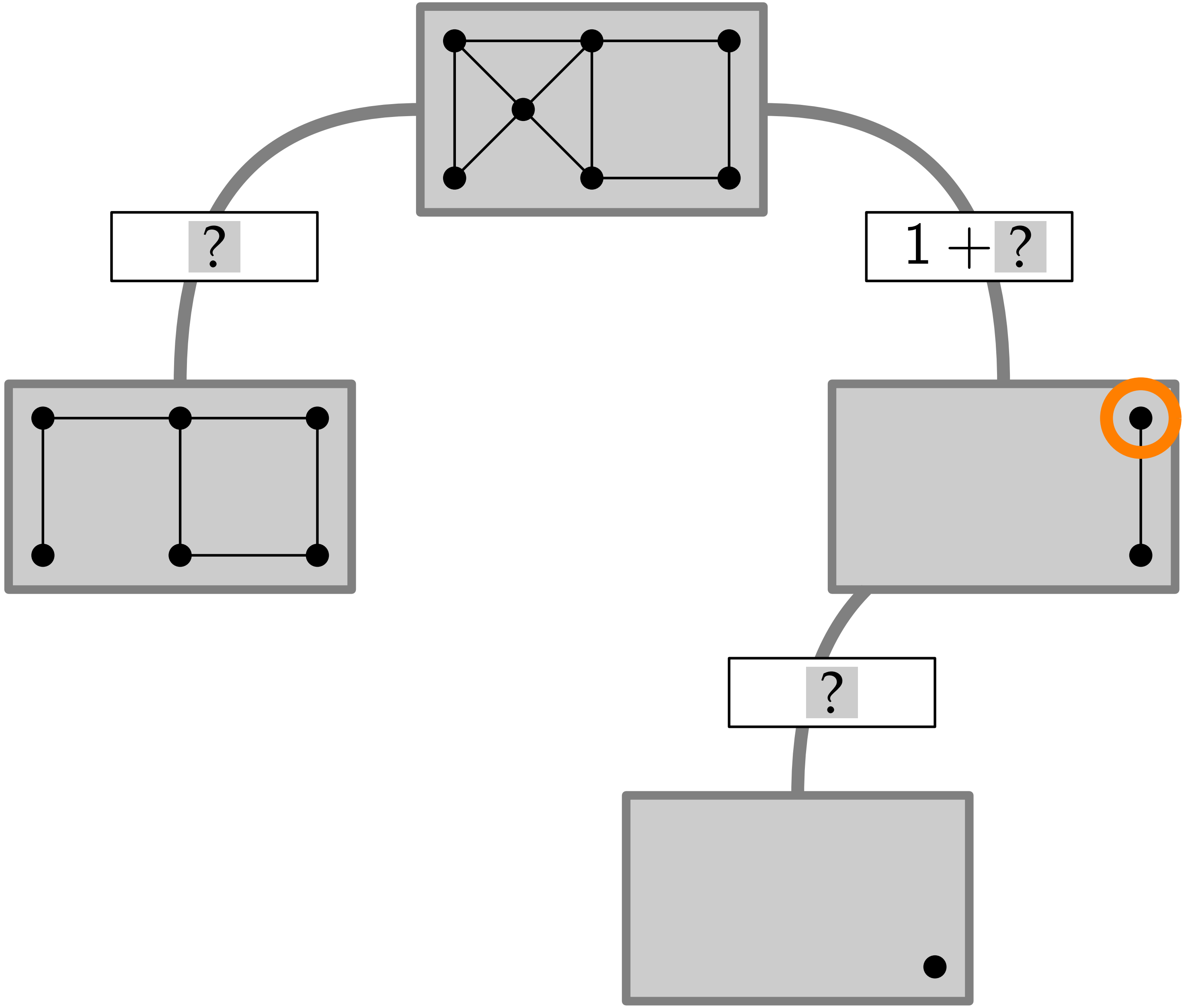
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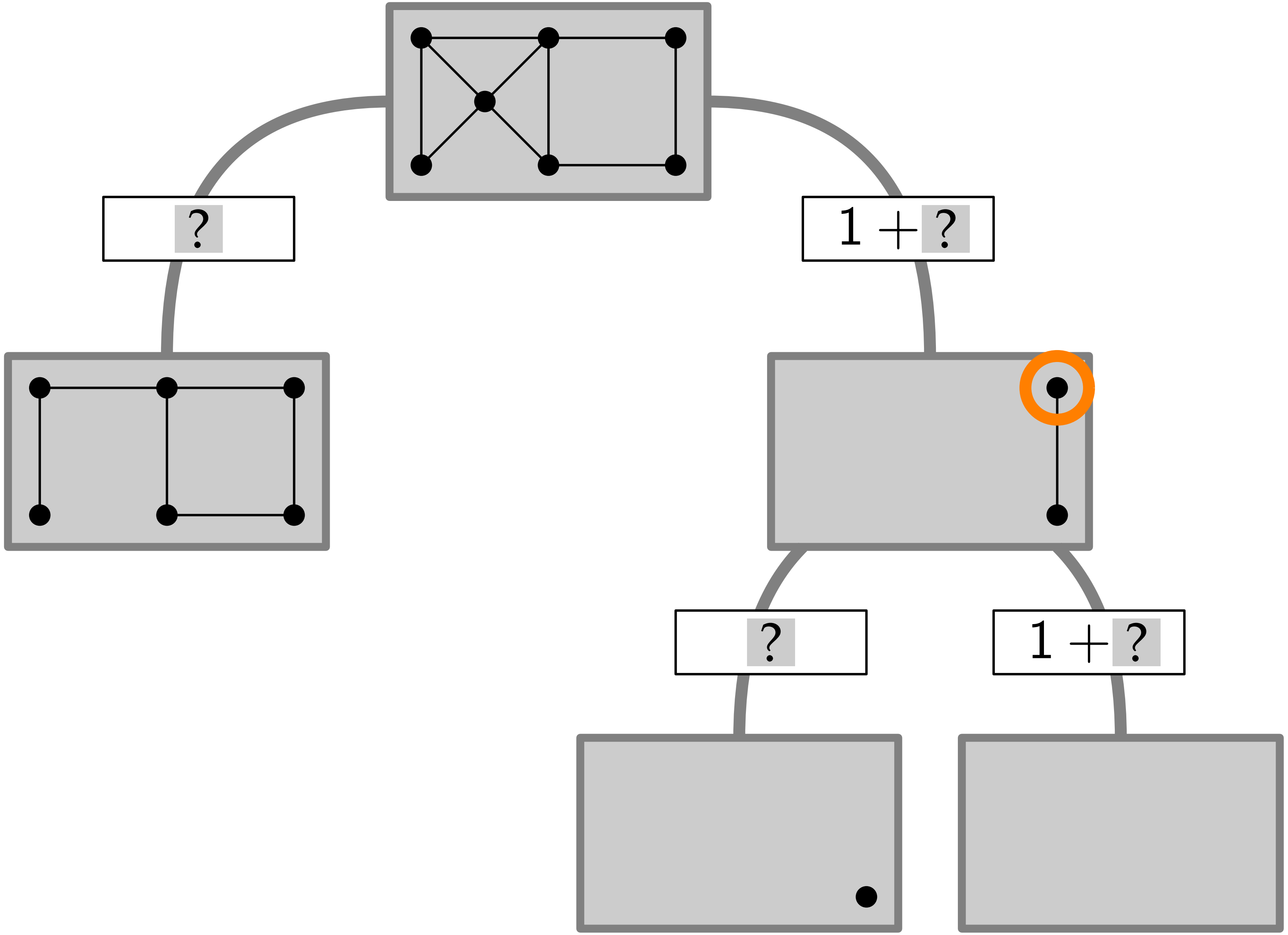
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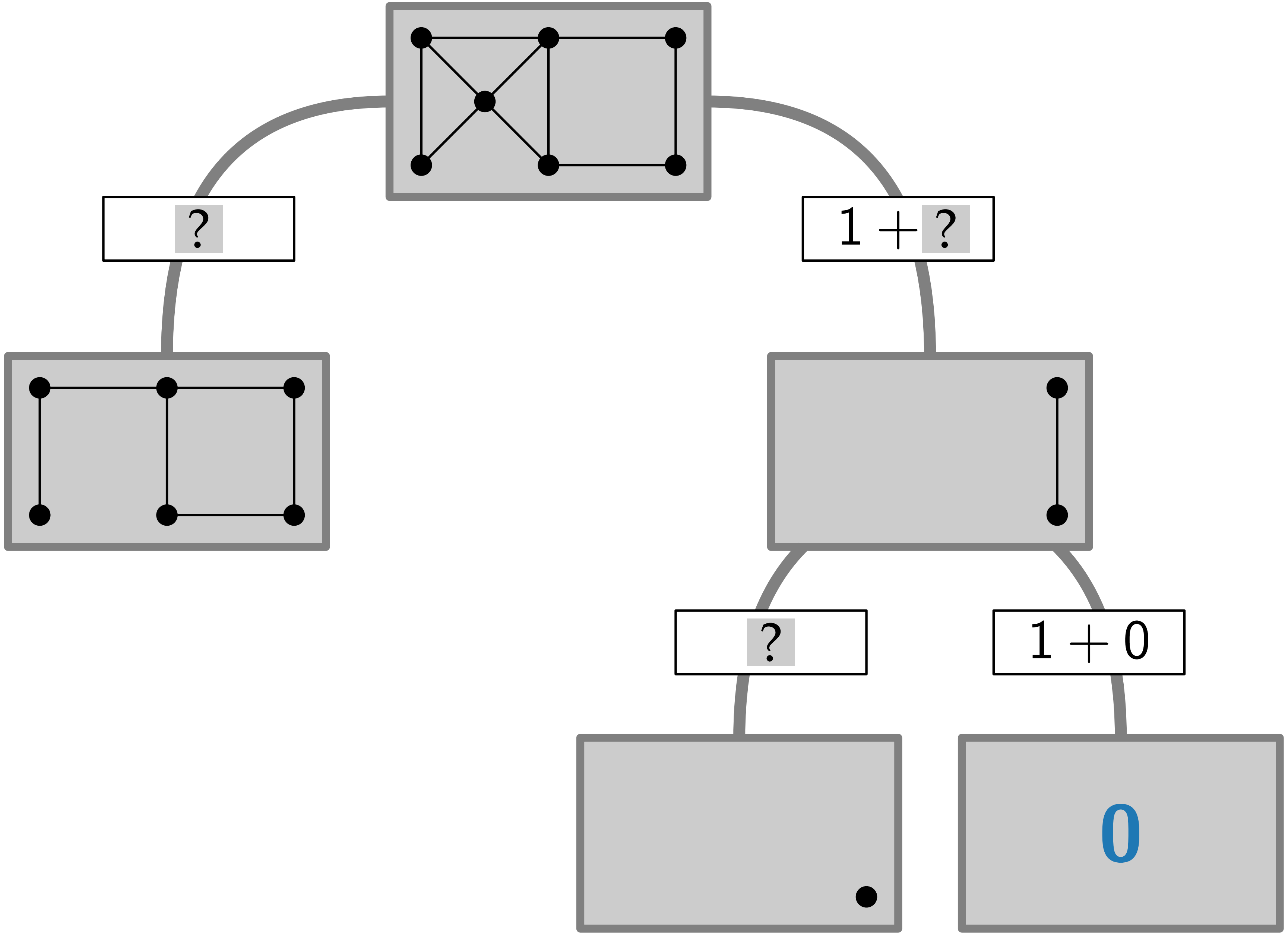
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1 + 0
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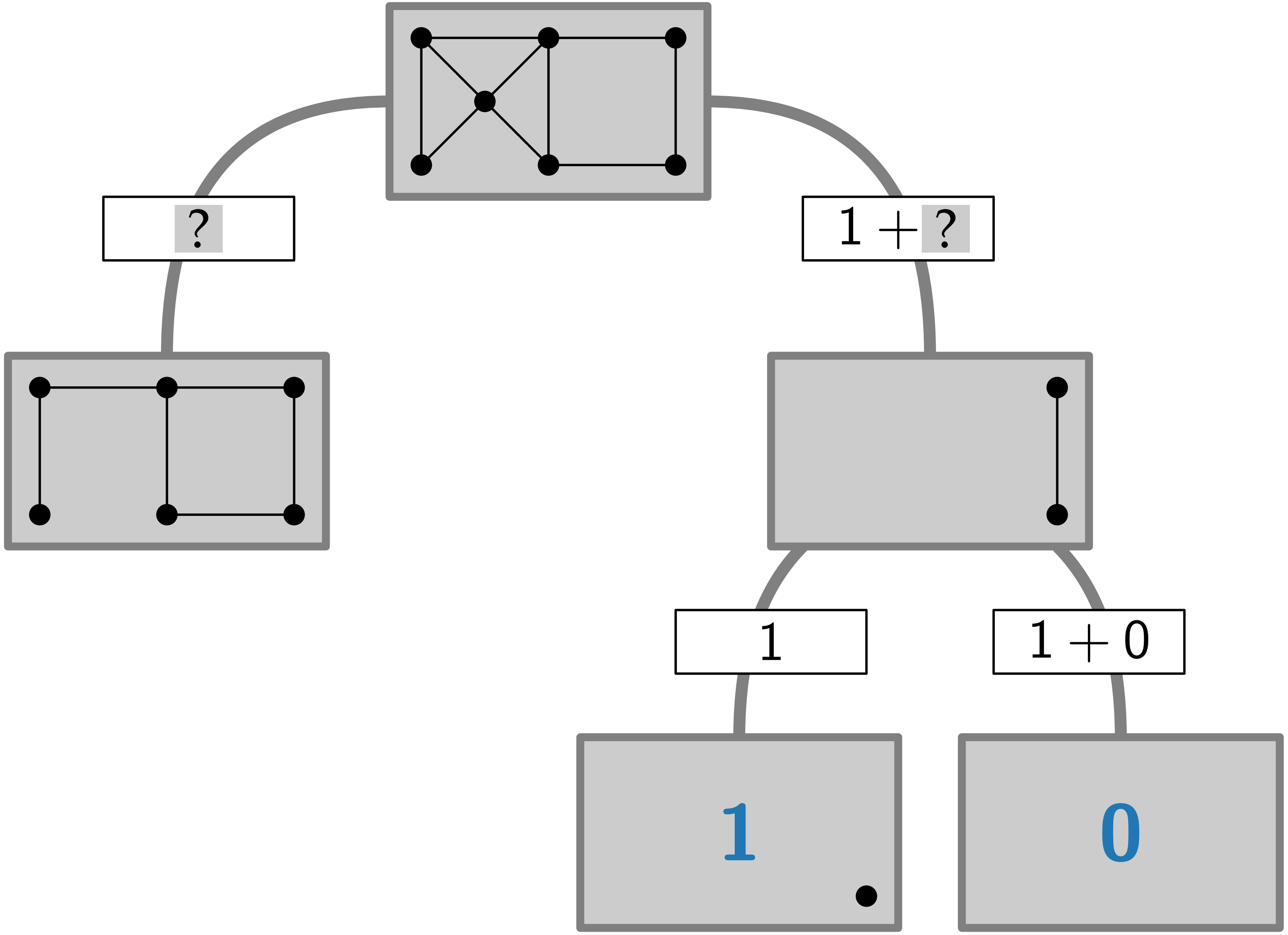
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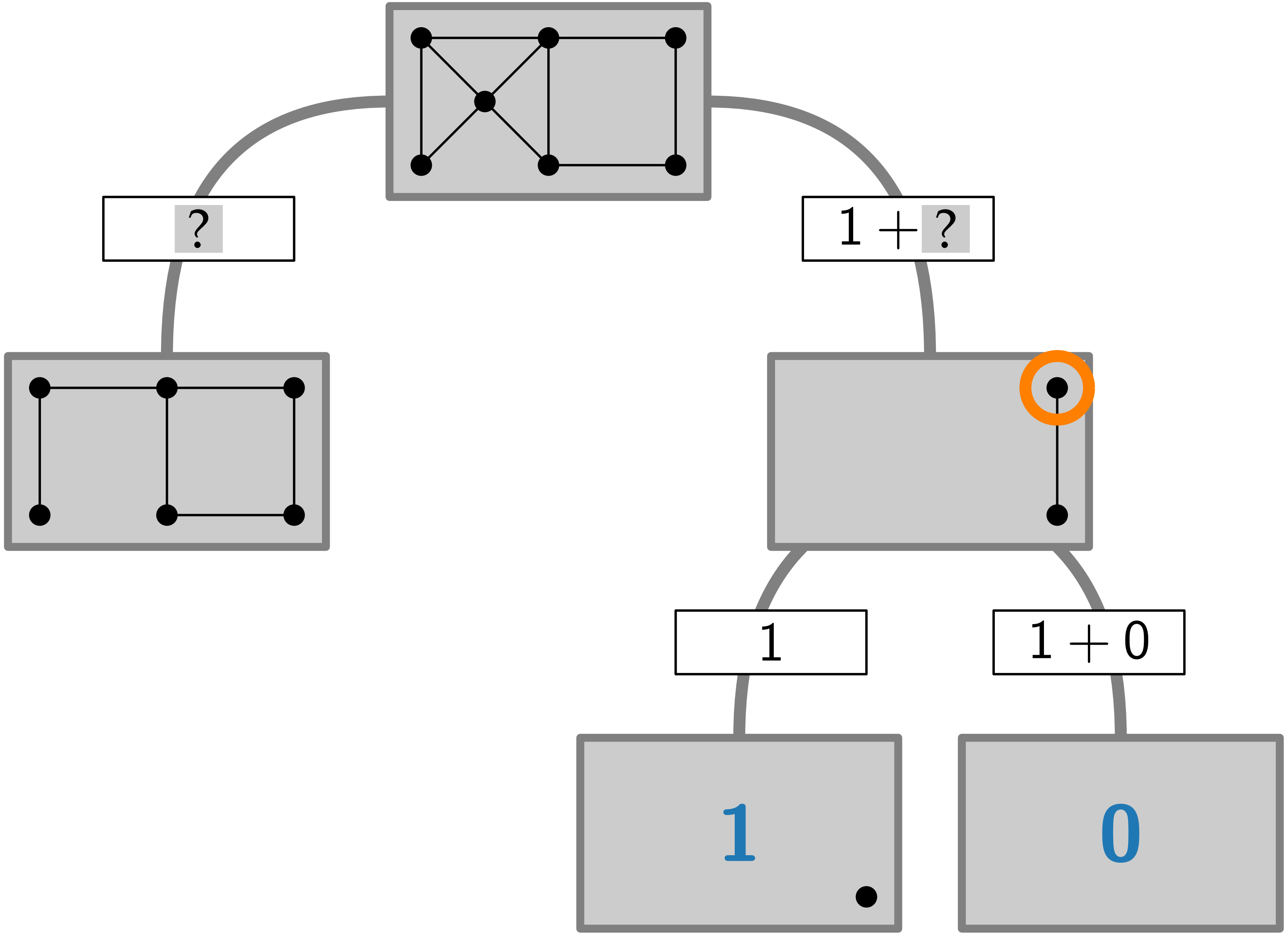
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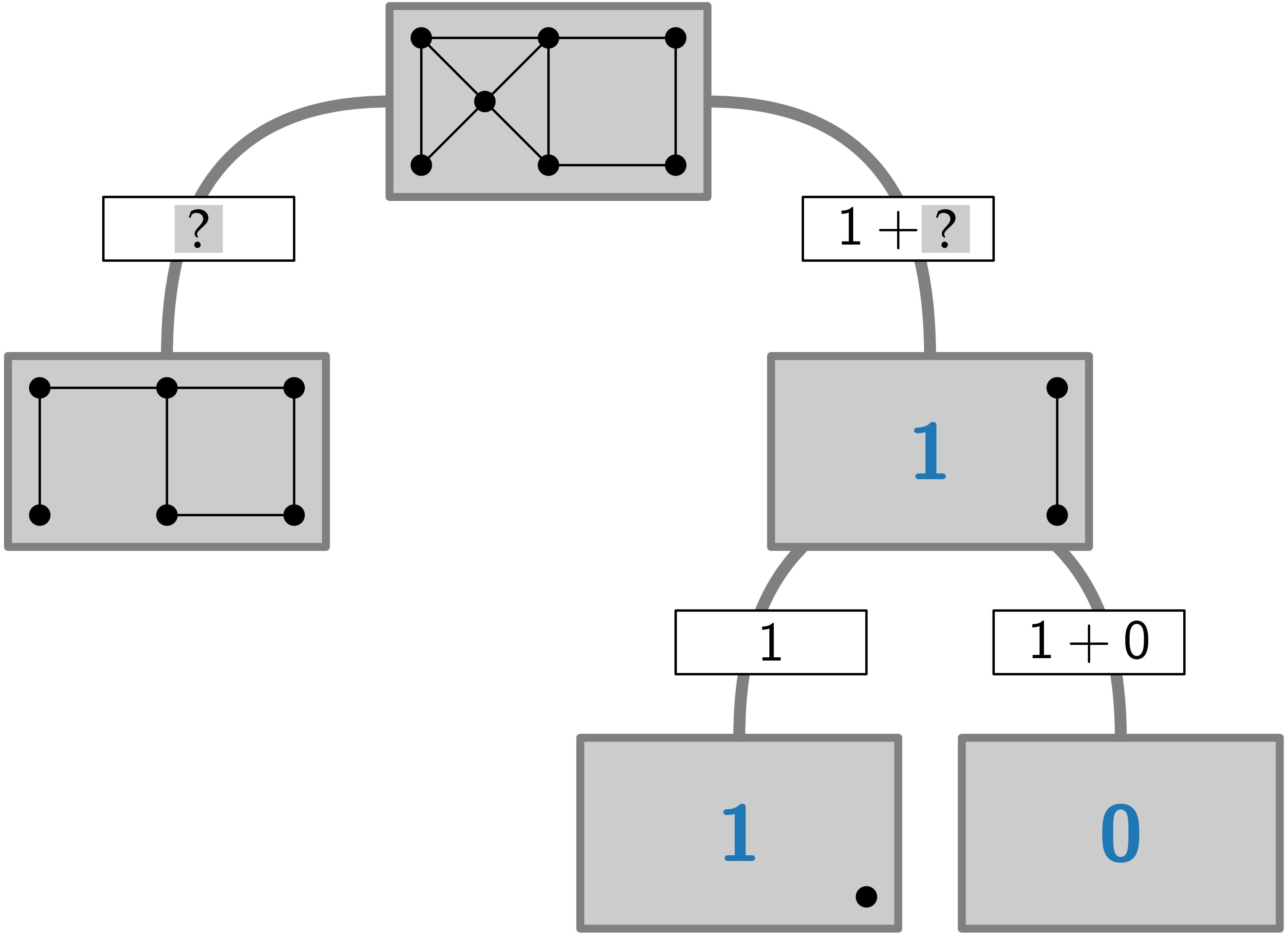
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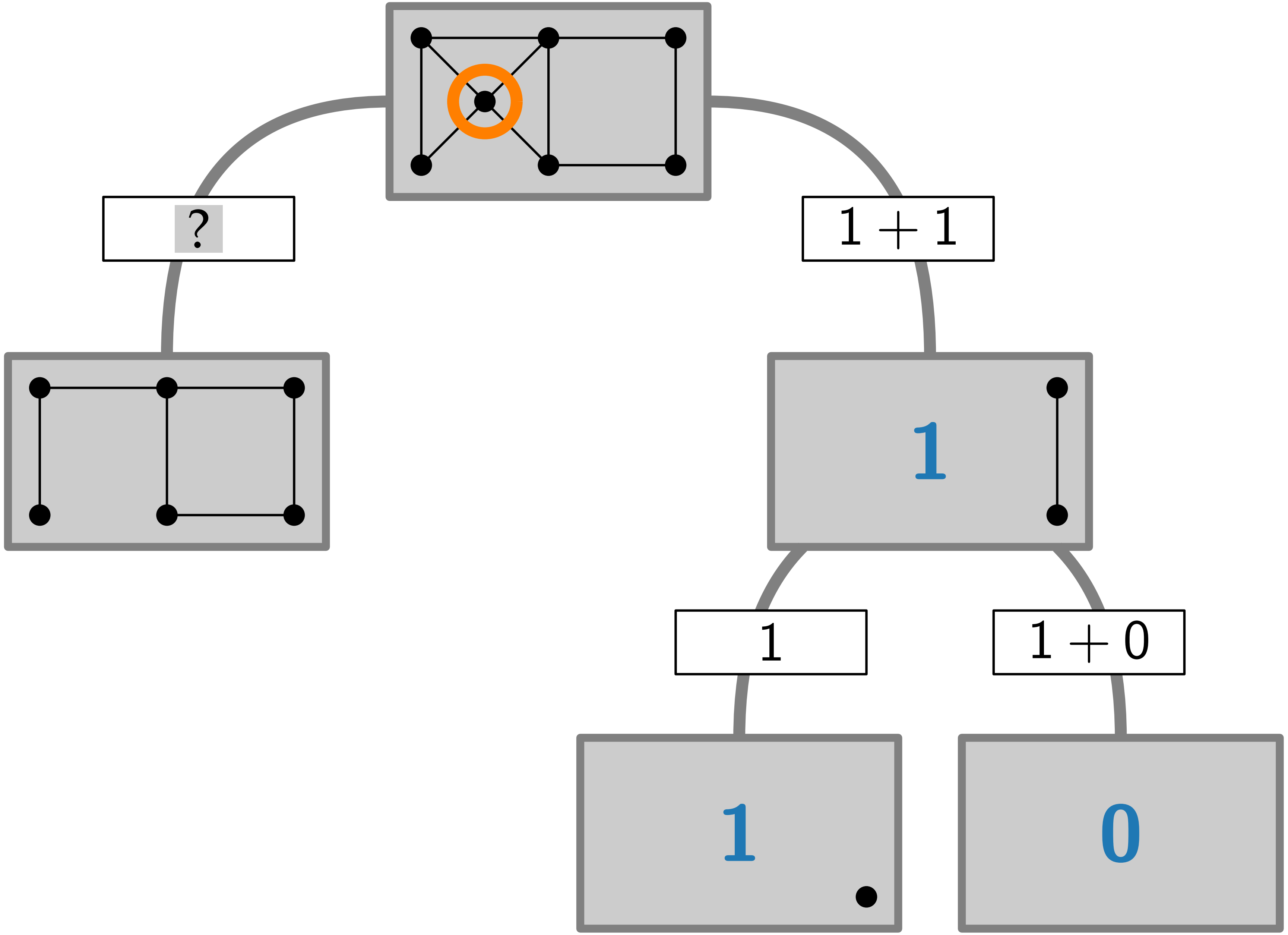
?
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1 + 0
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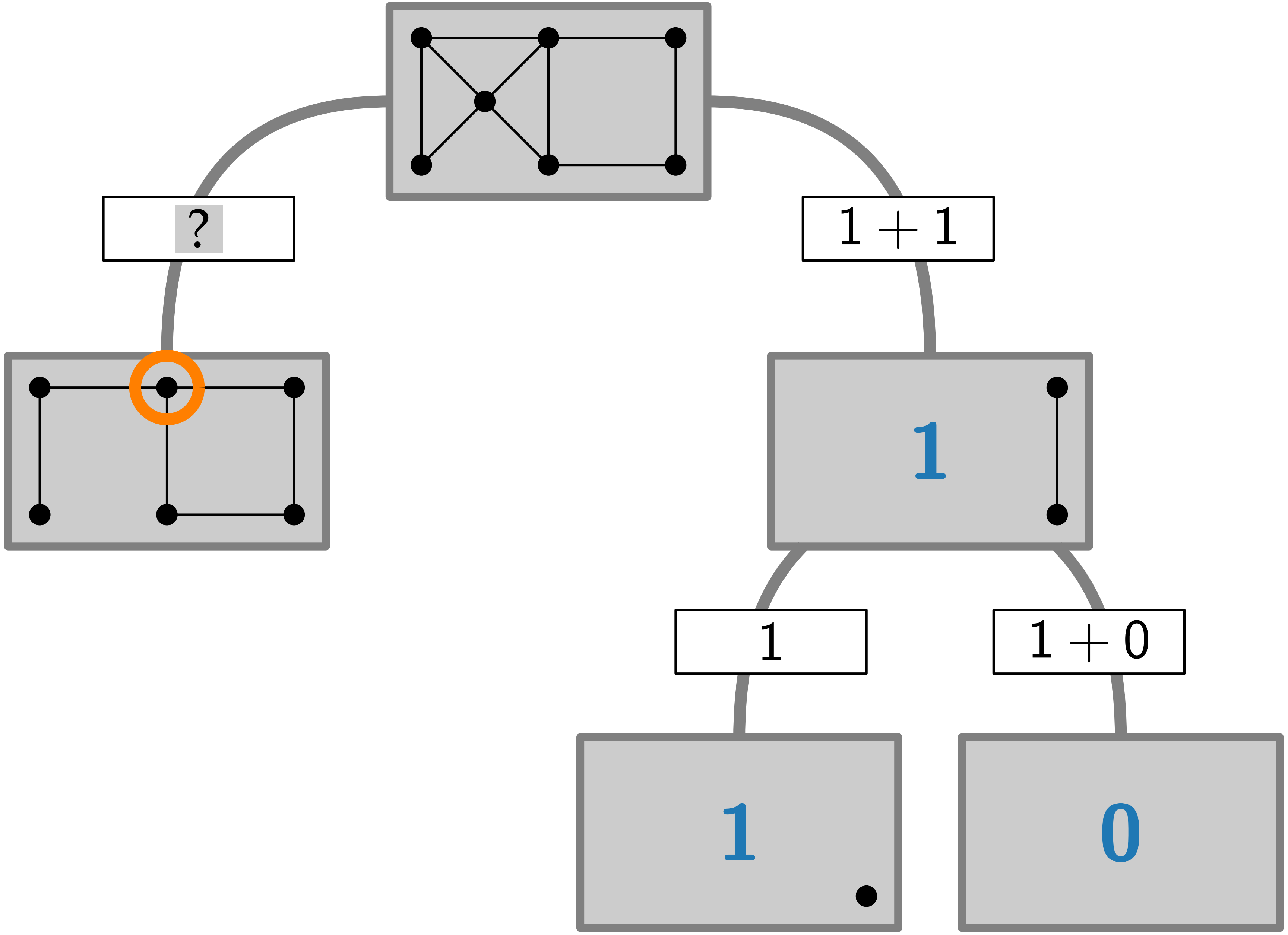
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1 + 0
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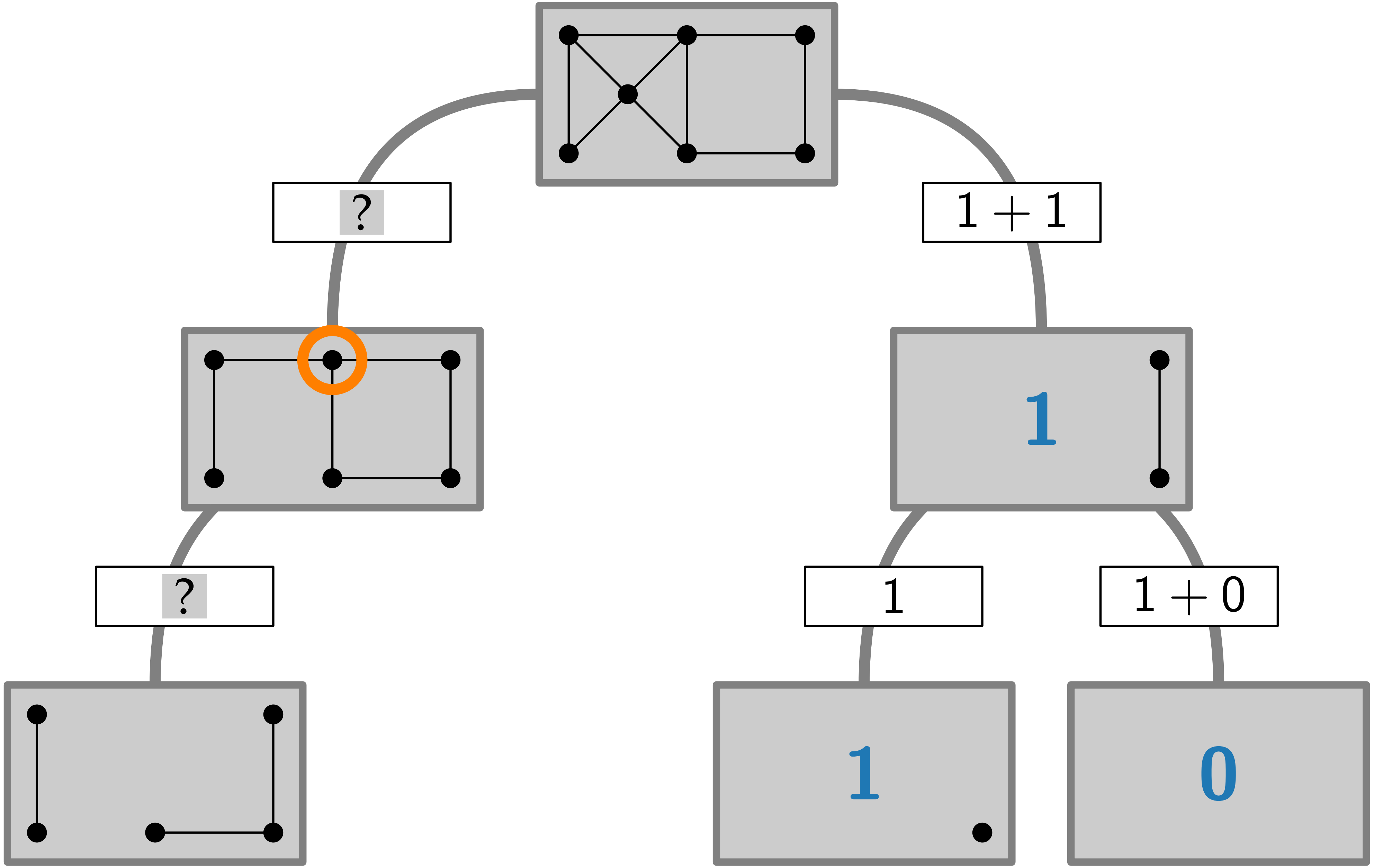
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1 + 1
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1
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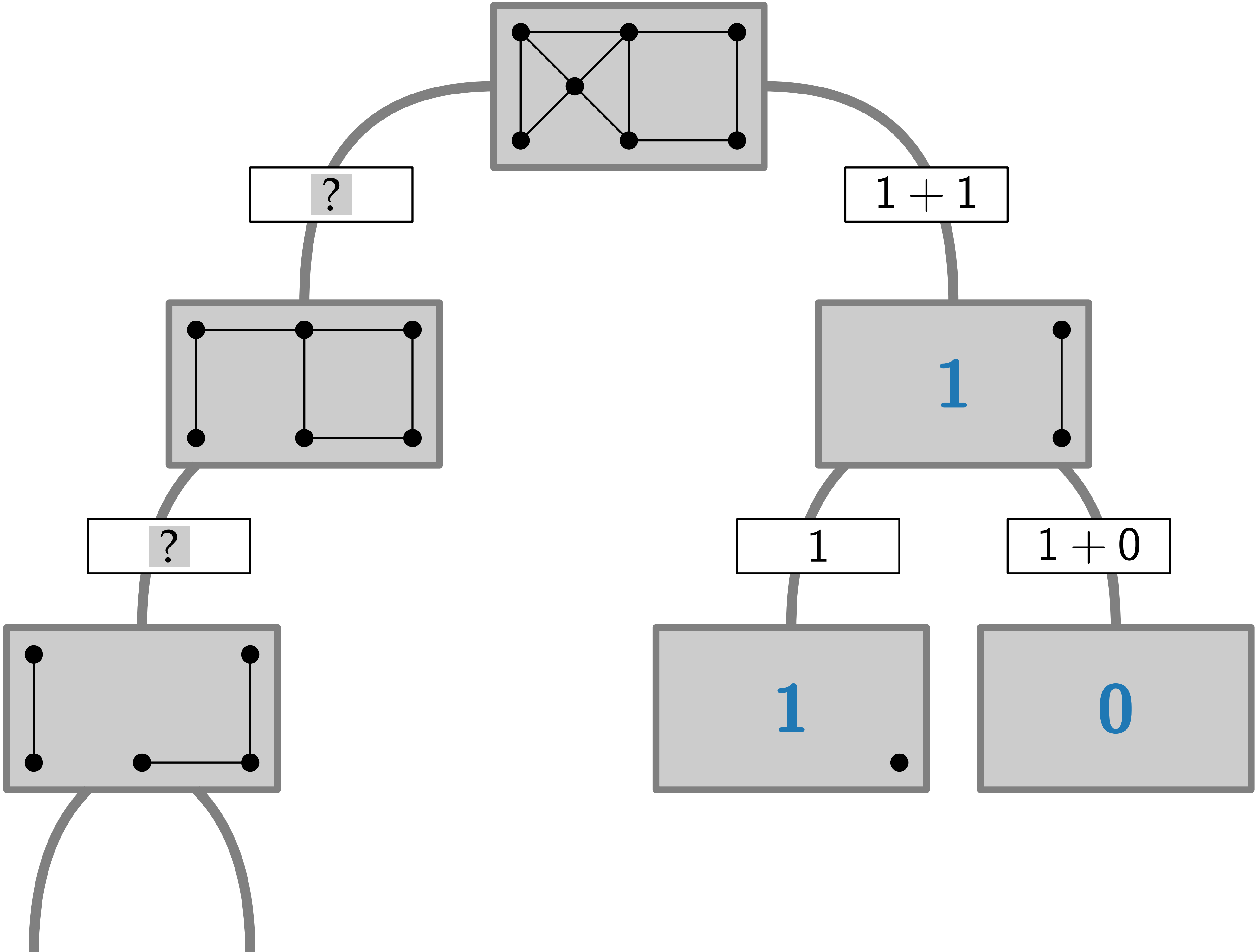
0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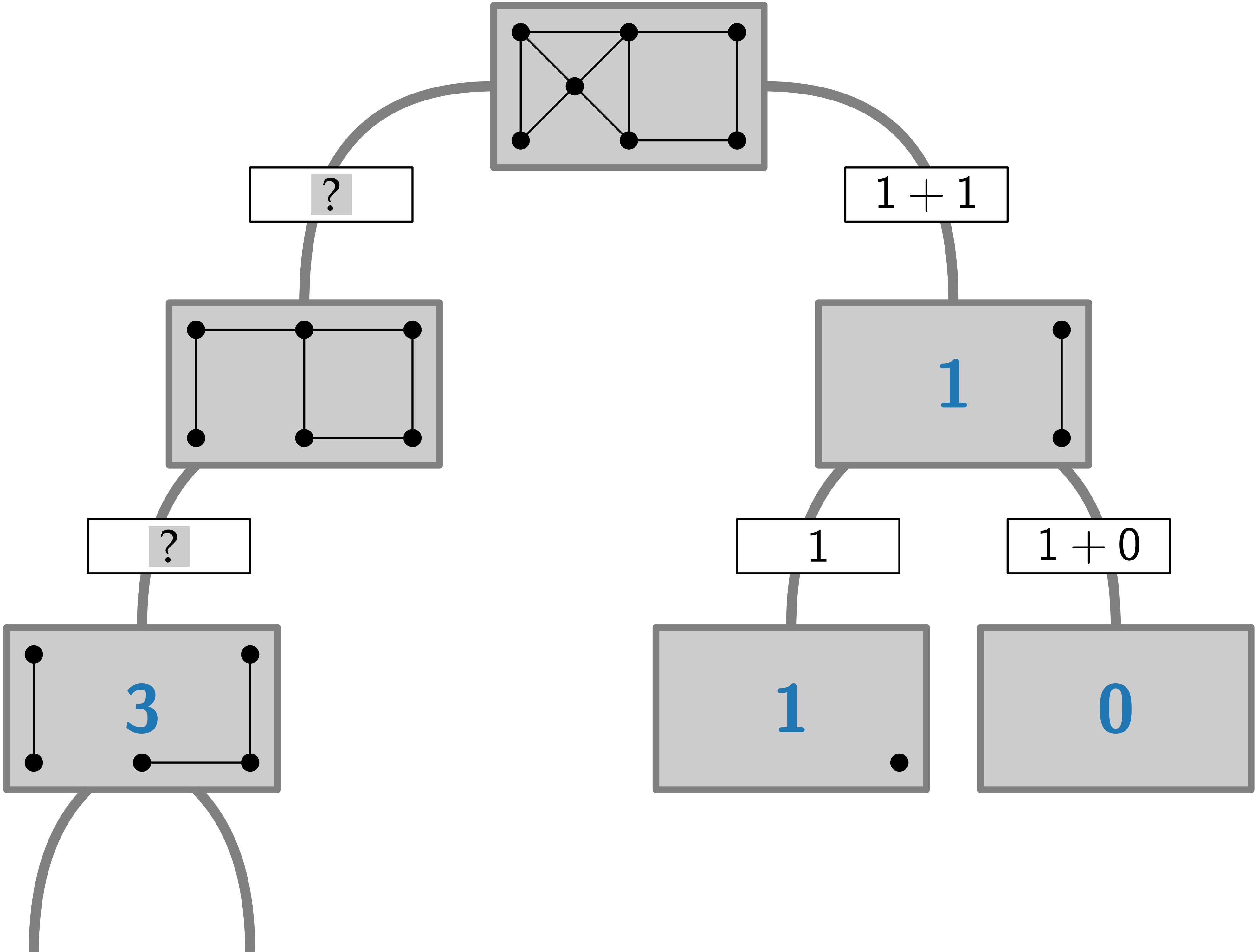
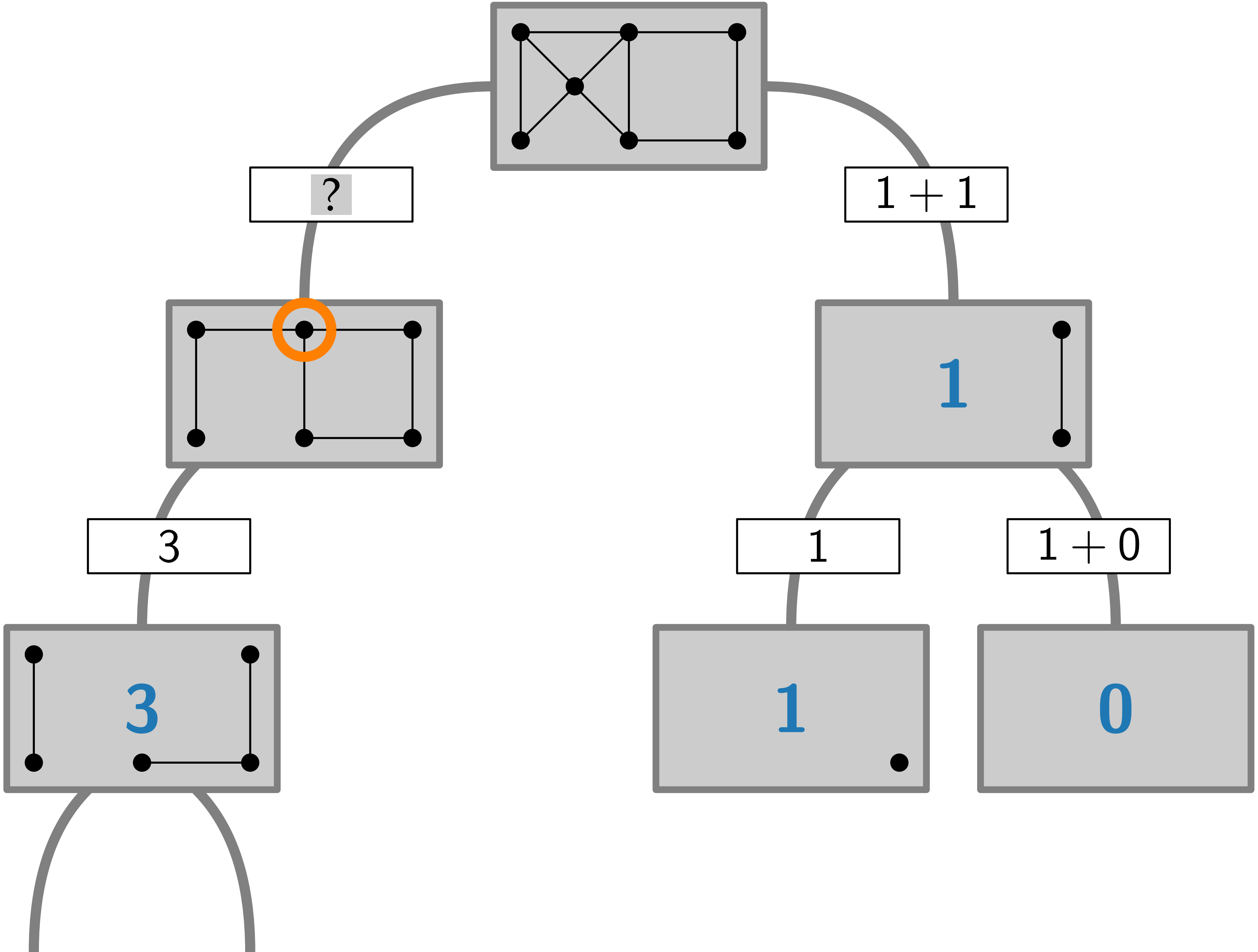
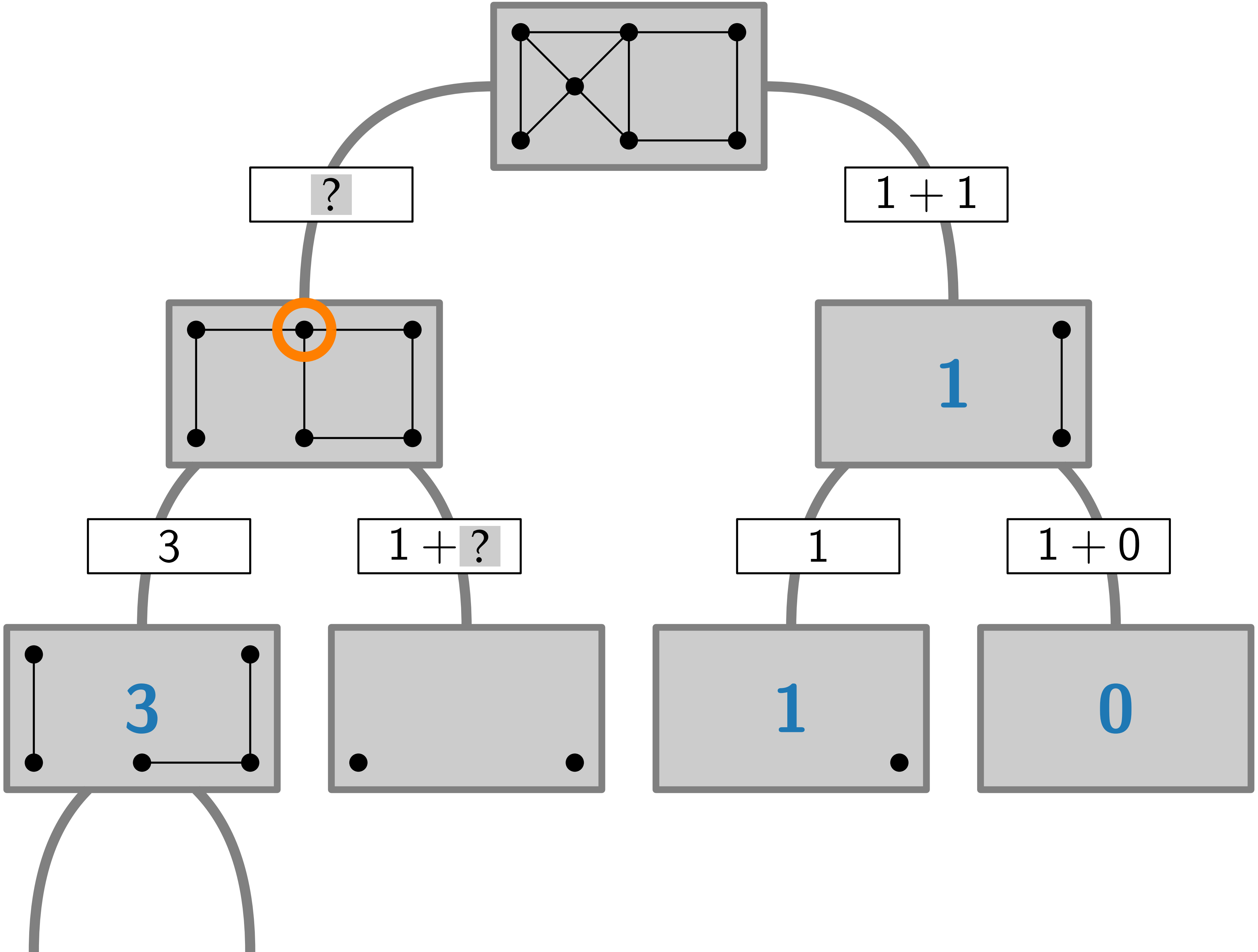
1 + 1
1
?
1
1 + 0
1
0


?
1 + 1
1
?
1
1 + 0
3
1
0



?
1 + 1
1
3
1
1 + 0
3
1
0


?
1 + 1
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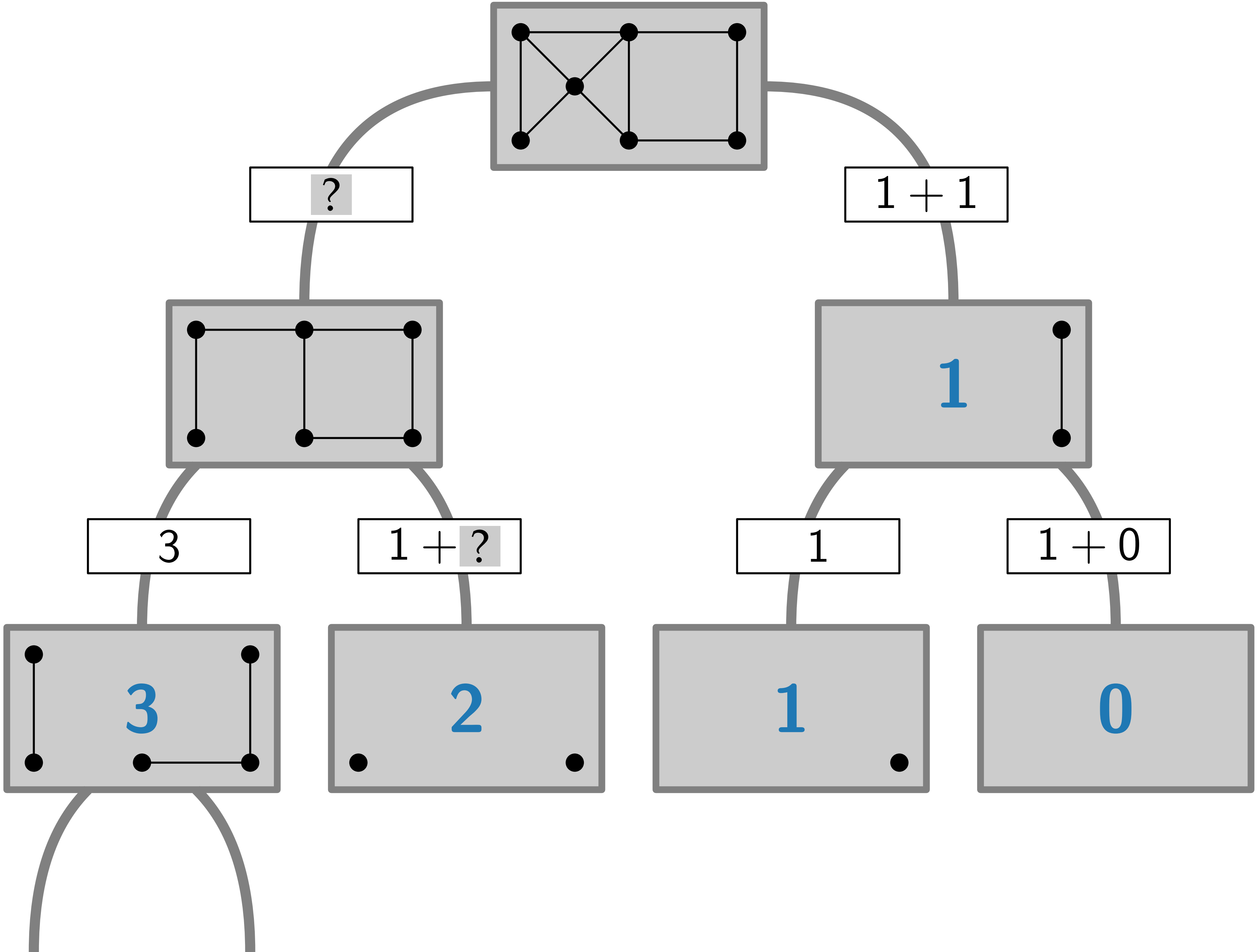
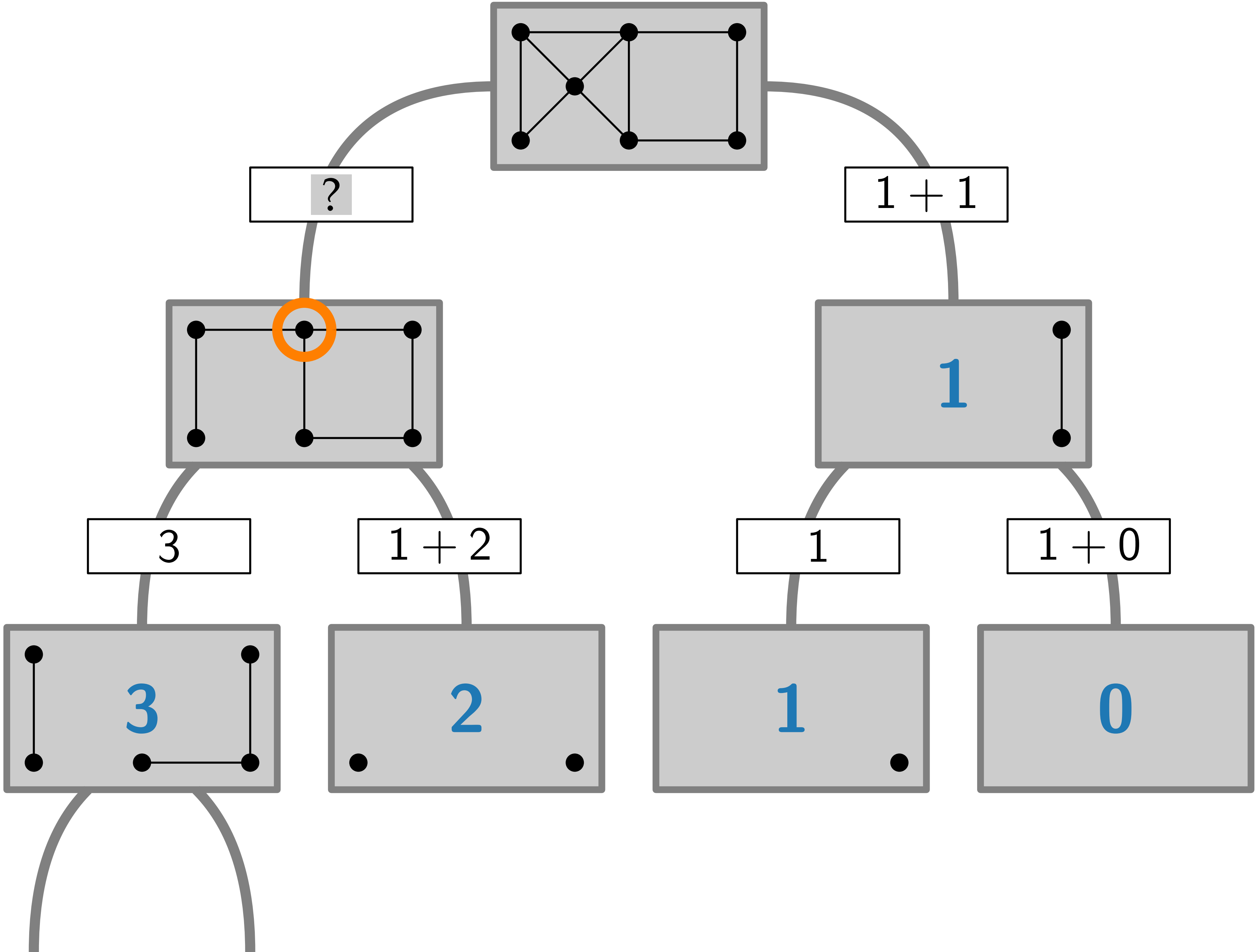
3
1 + ?
1
1 + 0
3
1
0



?
1 + 1
1
3
1 + ?
1
1 + 0
3
2
1
0


?
1 + 1
1
3
1 + 2
1
1 + 0
3
2
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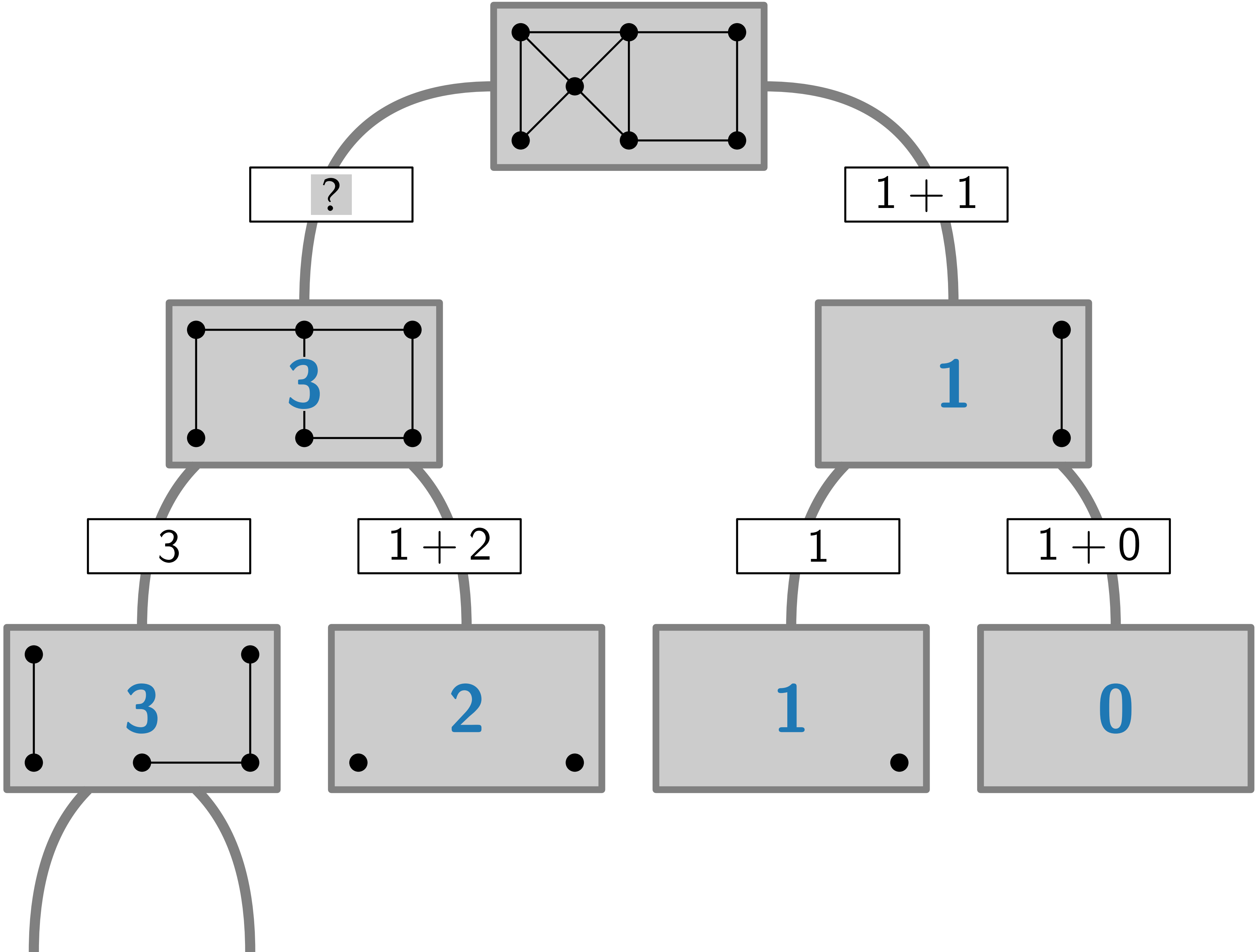
?
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1 + 2
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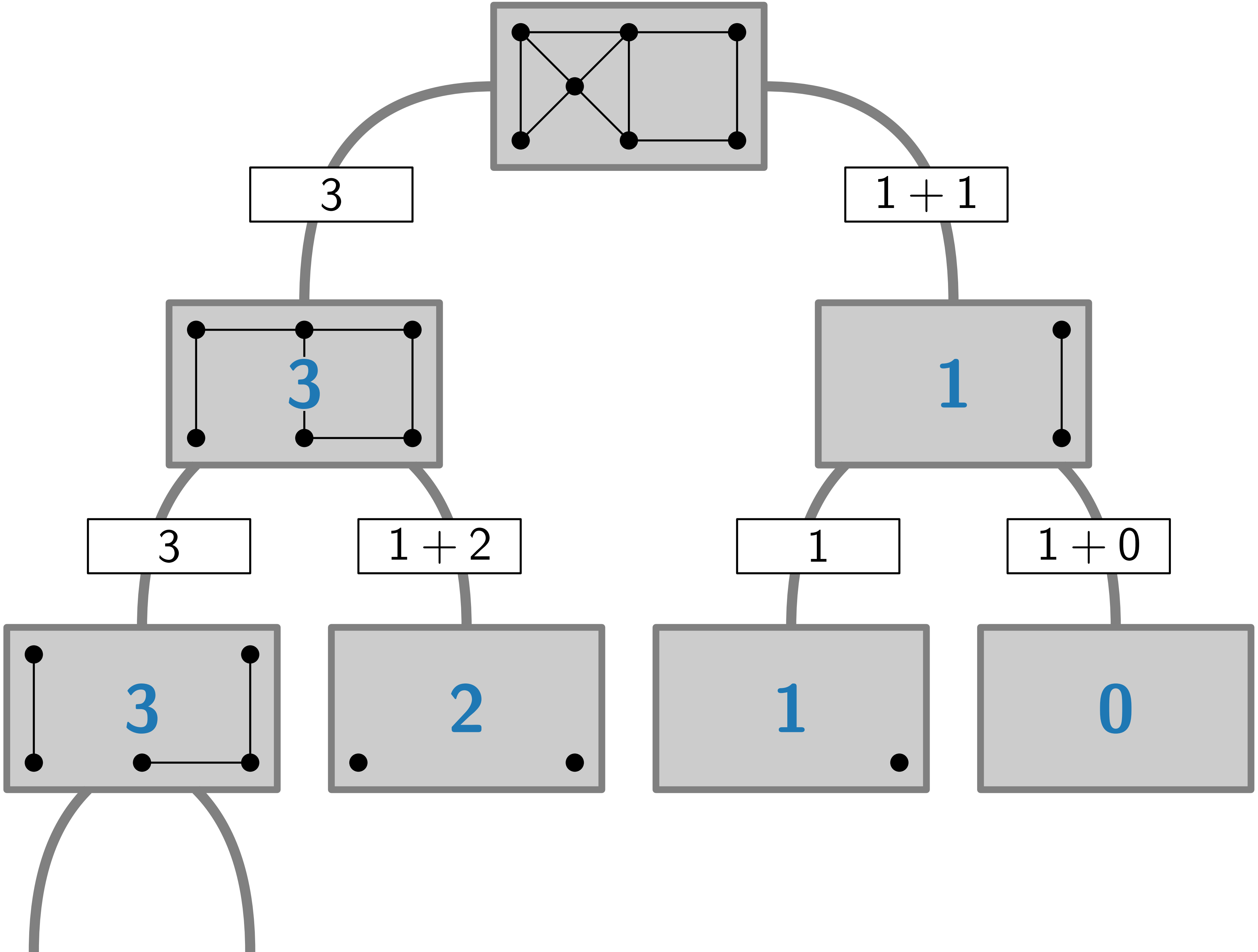
3
2
1
0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1 + 1
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1 + 2
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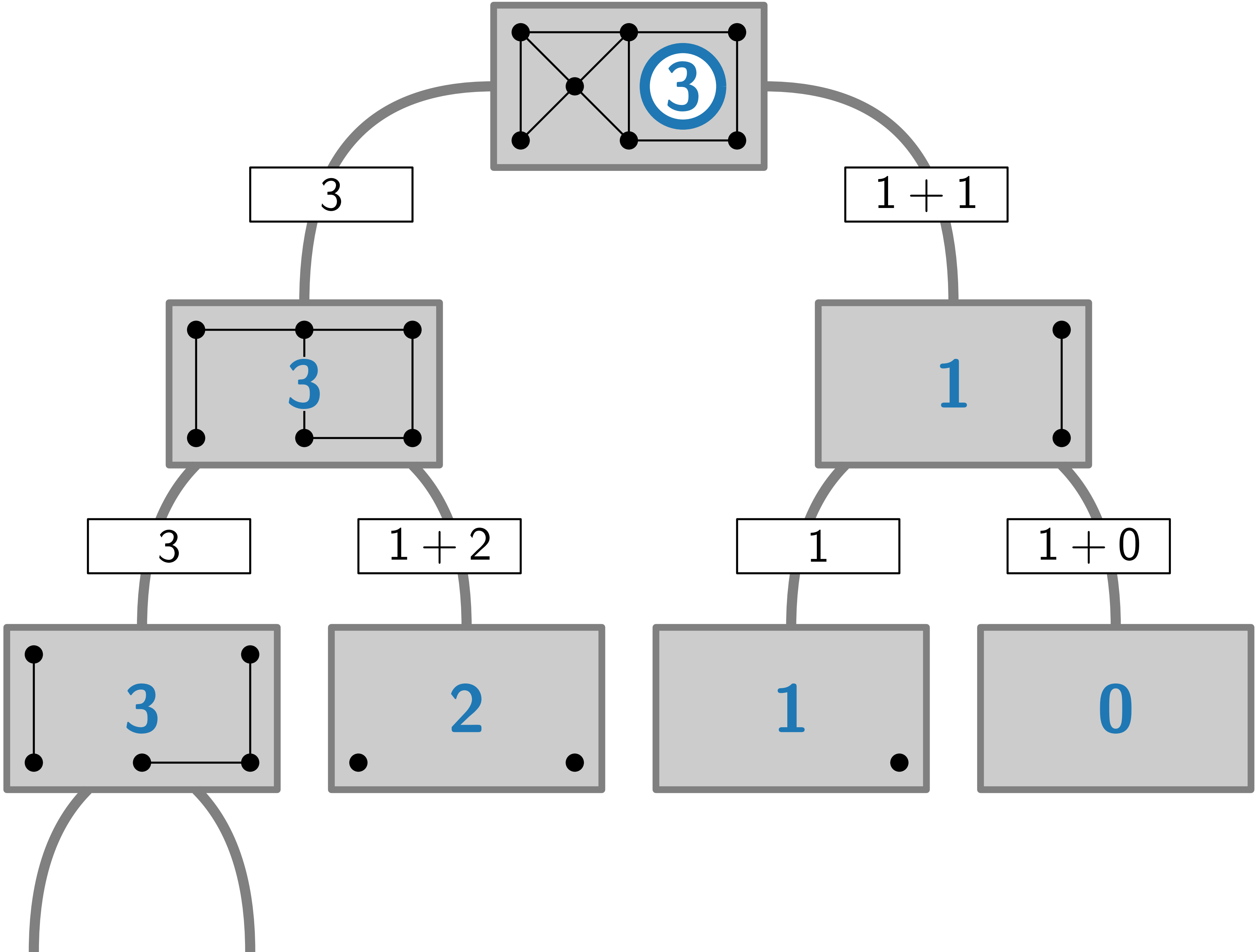

3
1 + 1
3
1
3
1 + 2
1
1 + 0
3
2
1
0

MIS – Smarter Branching

Lemma.
Let U be a maximum independent set in G. Then for each $v \in V$:

1. $v \in U \Rightarrow N(v) \cap U = \emptyset$
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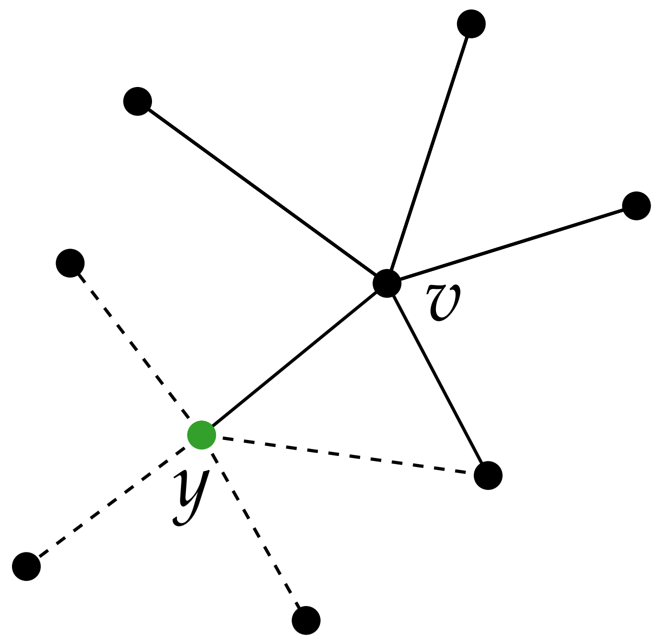
Thus, $N[v] := N(v) \cup \{v\}$ contains some $y \in U$ and no other vertex of $N[y]$ is in U.



MIS – Smarter Branching

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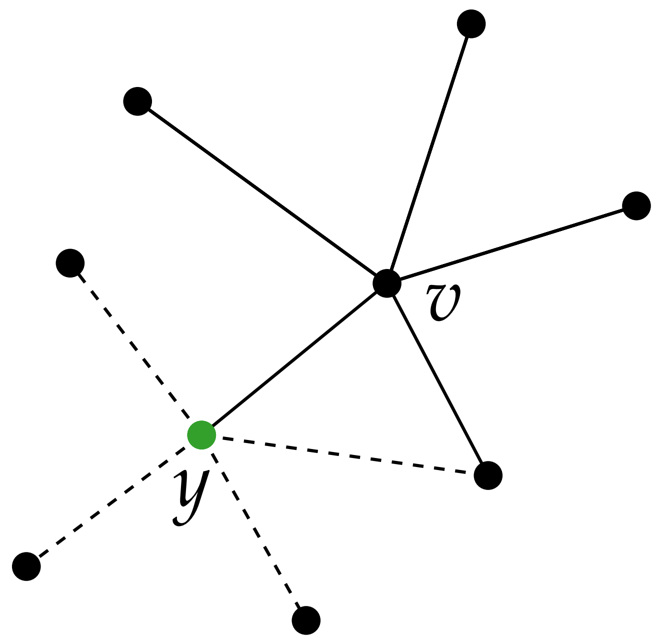

Smarter MIS branching.

- For some vertex v, branch on vertices in $N[v]$.

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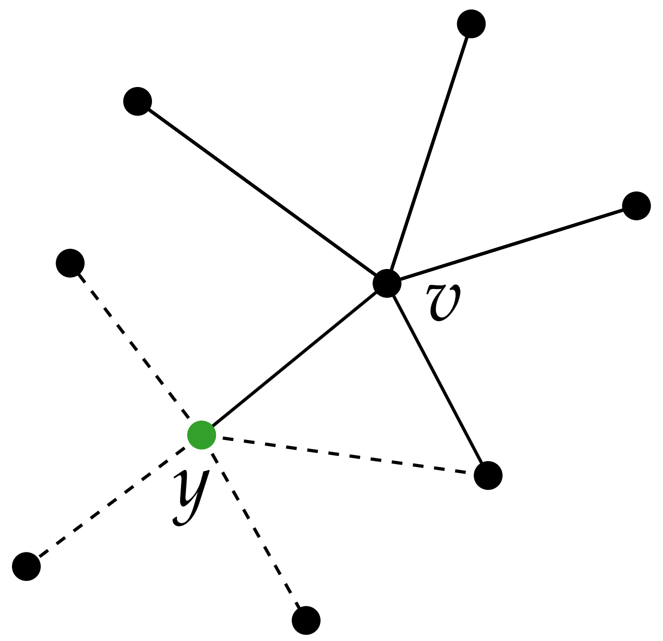

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SmarterMIS(G):

if $V == \emptyset$ **then**
  return 0

$v =$ vertex of minimum degree in $V(G)$
return $1 + \max\{\text{MIS}(G - N[y]) \mid y \in N[v]\}$

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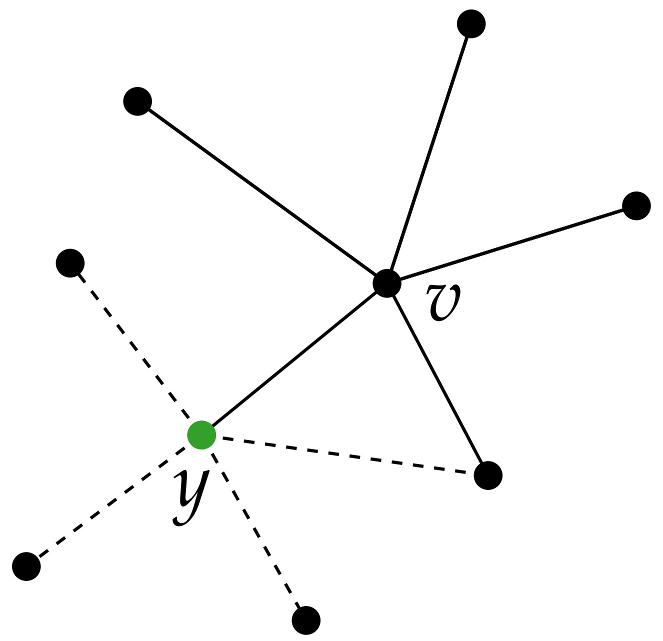

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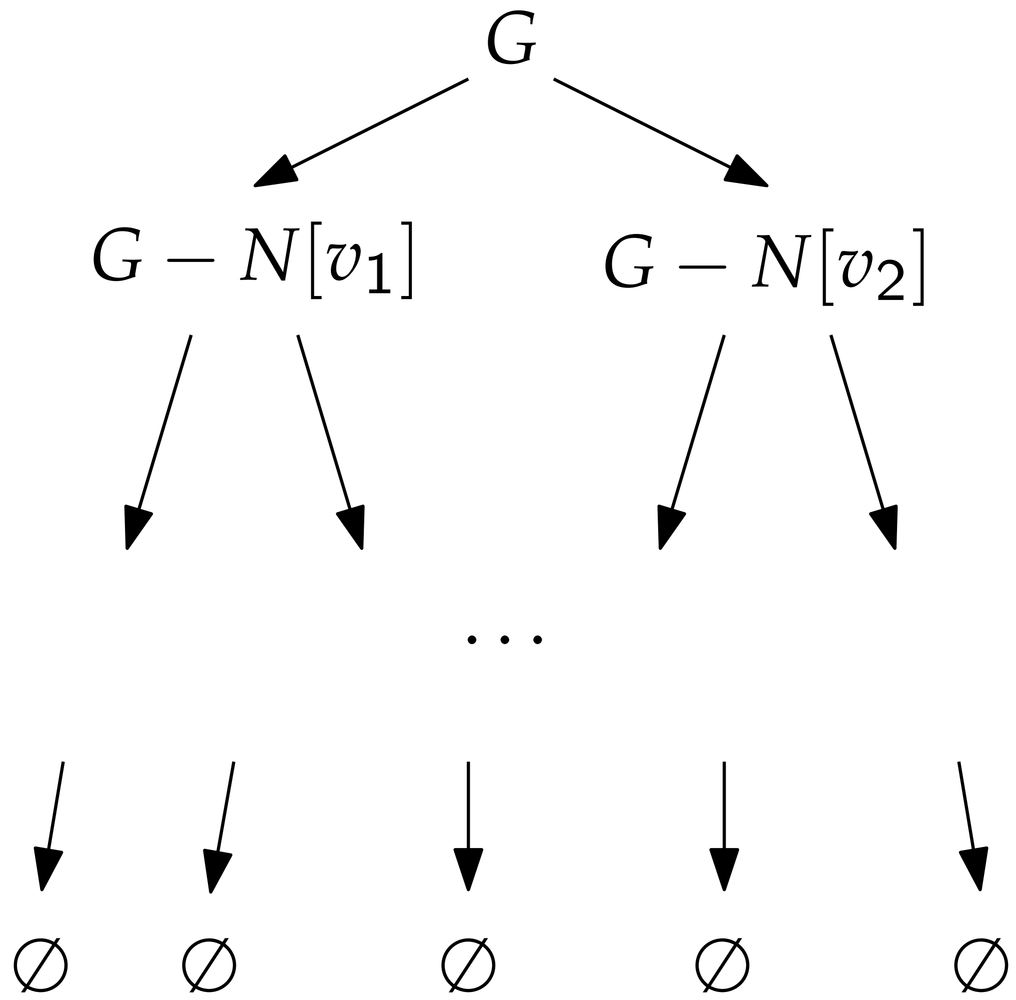
- Correctness follows from the lemma.
- We prove a runtime of $\mathcal{O}^*(3^{n/3}) = \mathcal{O}^*(1.4423^n)$.

MIS – Branching Analysis

Execution corresponds to a **search tree** whose vertices are labeled with the input of the respective recursive call.



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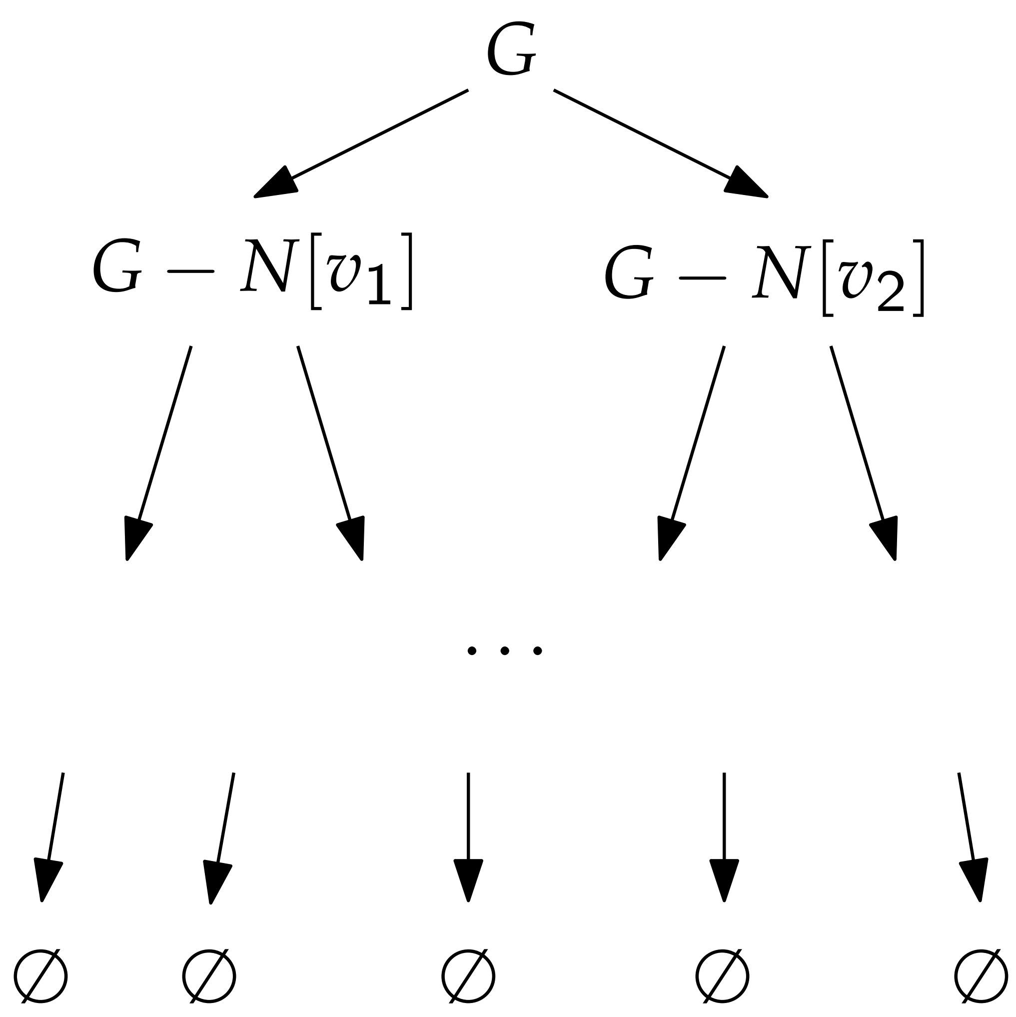
- Let $B(n)$ be the maximum number of leaves of a search tree for a graph with n vertices.



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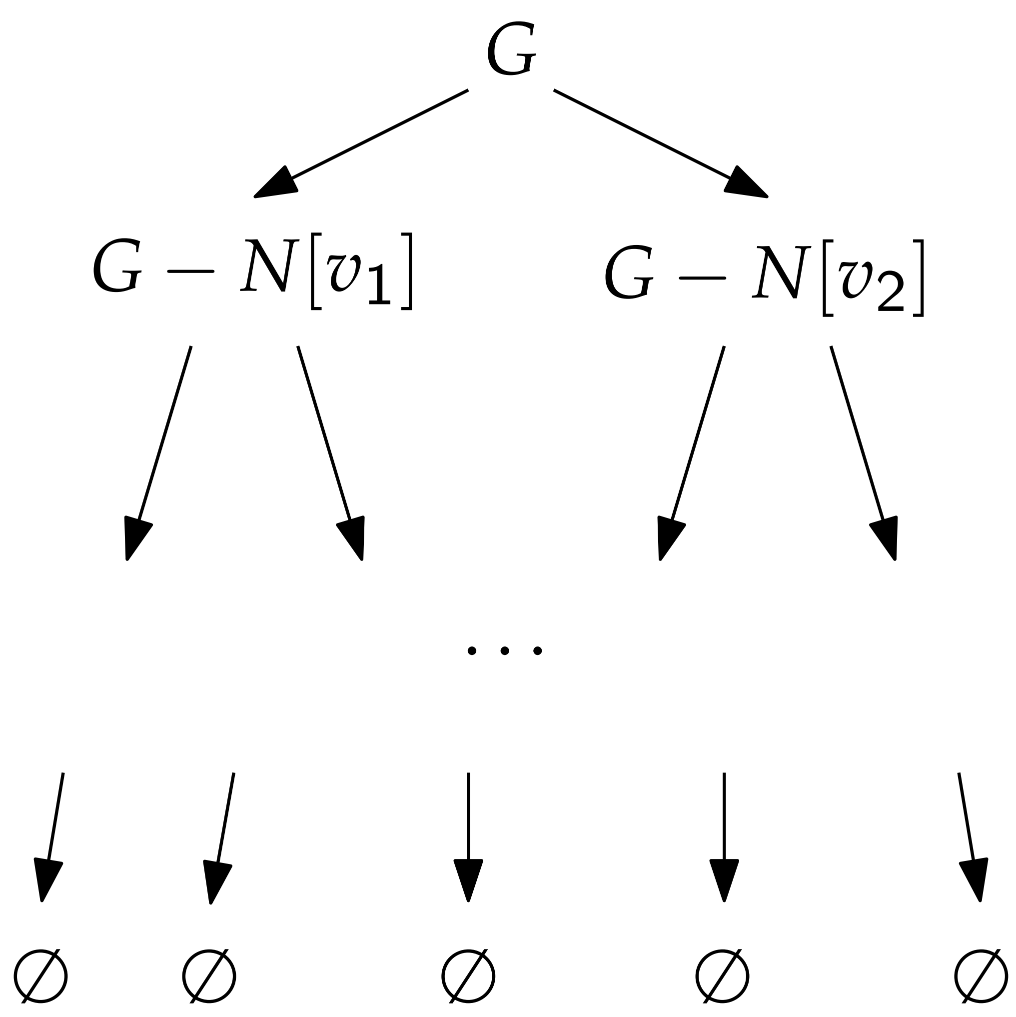
- Let $B(n)$ be the maximum number of leaves of a search tree for a graph with n vertices.
- Search-tree has height $\leq n$.



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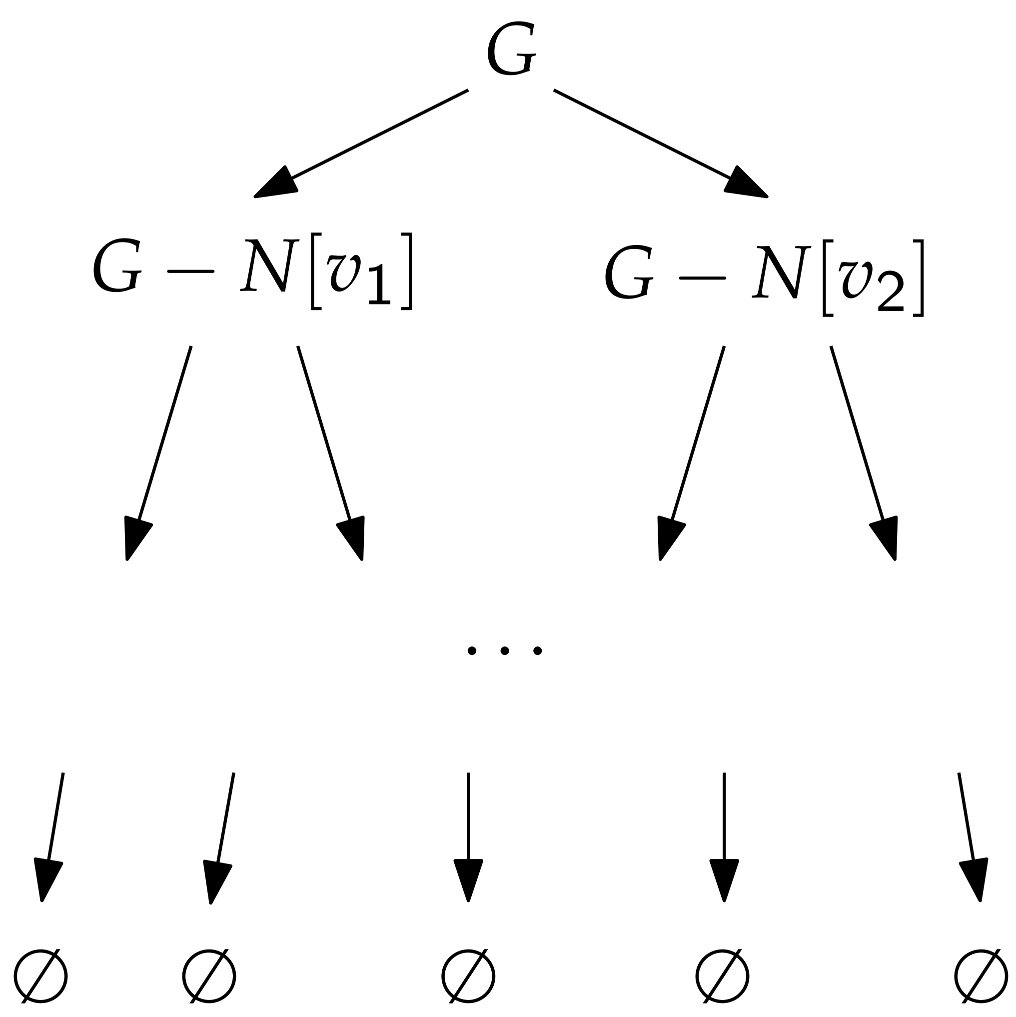
$\rightsquigarrow$ The runtime of the algorithm is

$$T(n) \in \mathcal{O}(nB(n)) = \mathcal{O}^*(B(n)).$$



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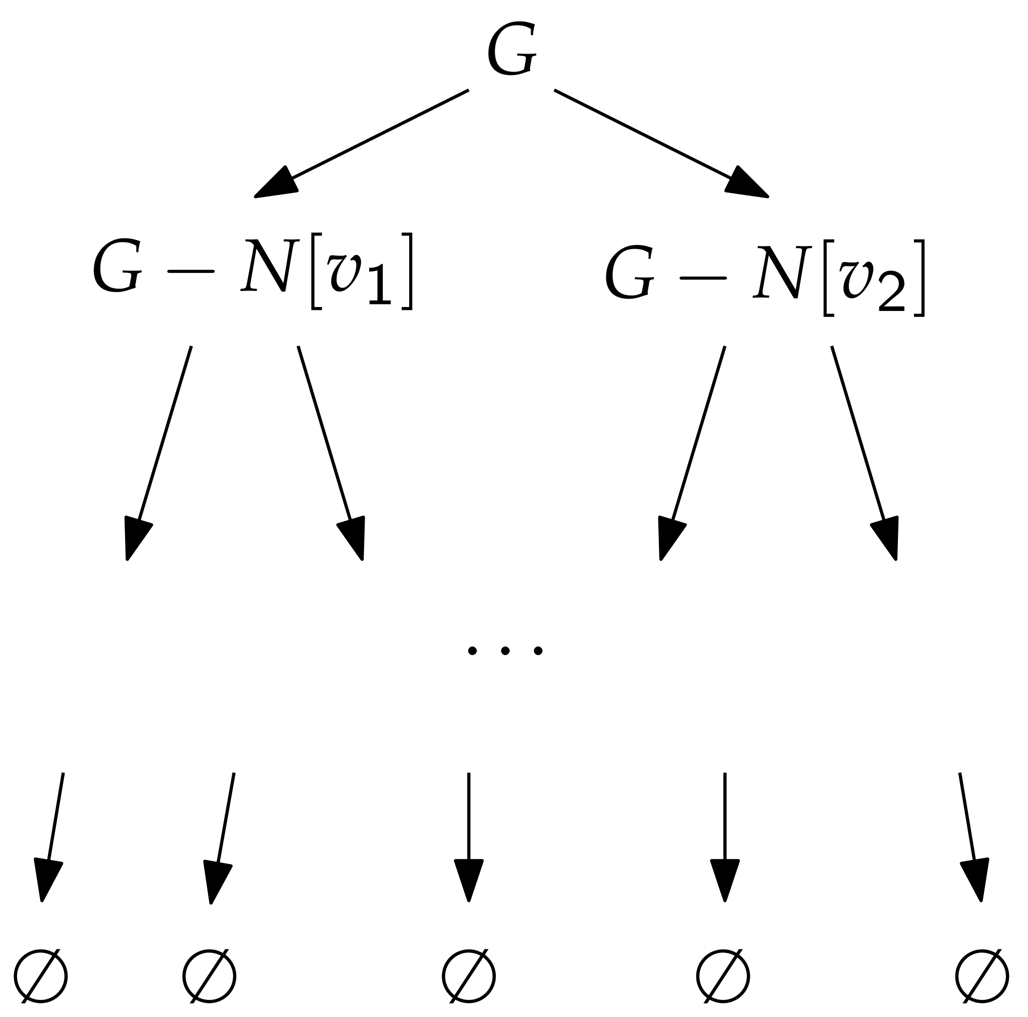
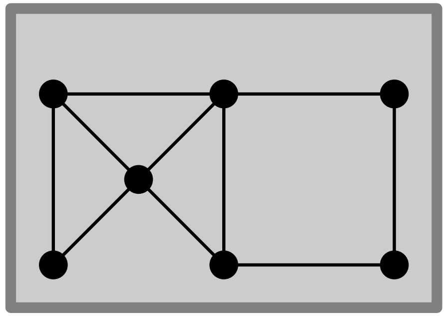
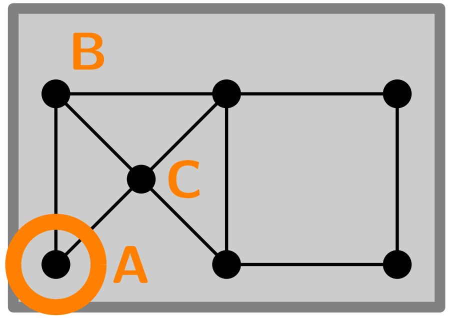
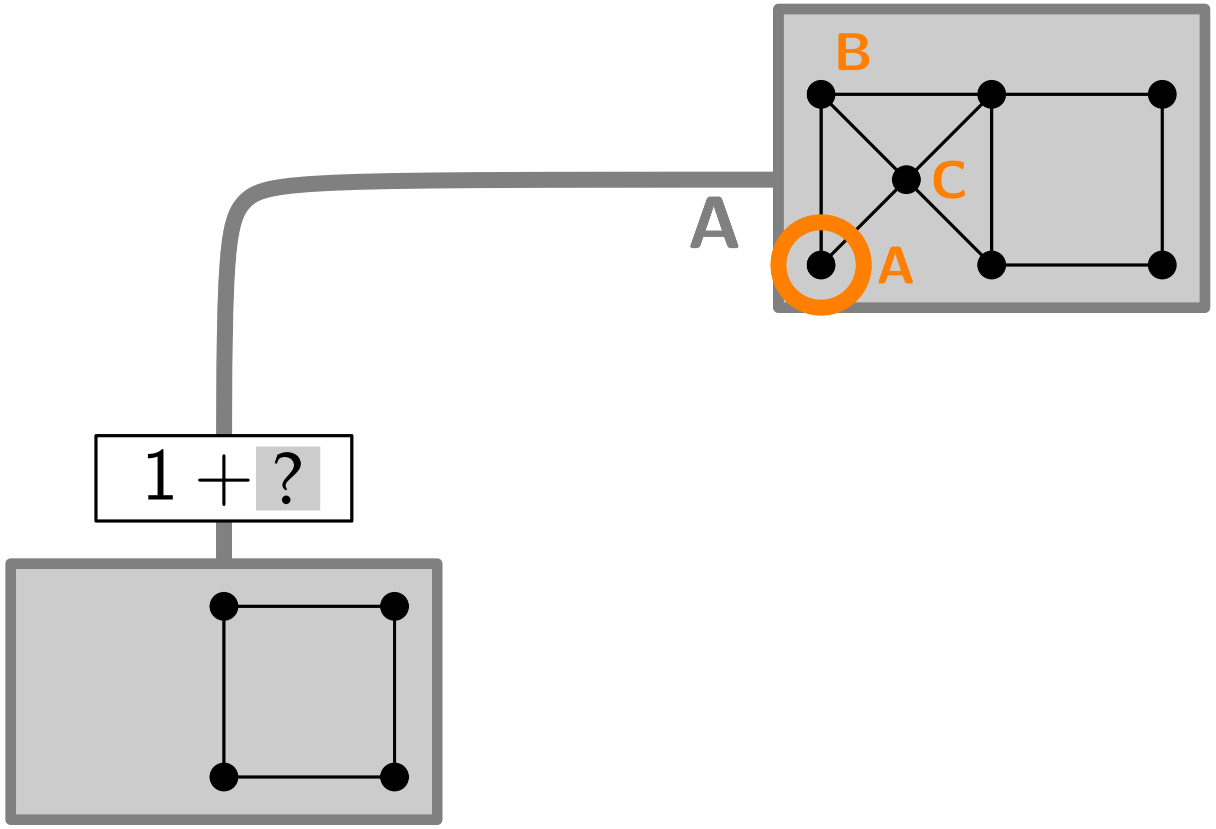
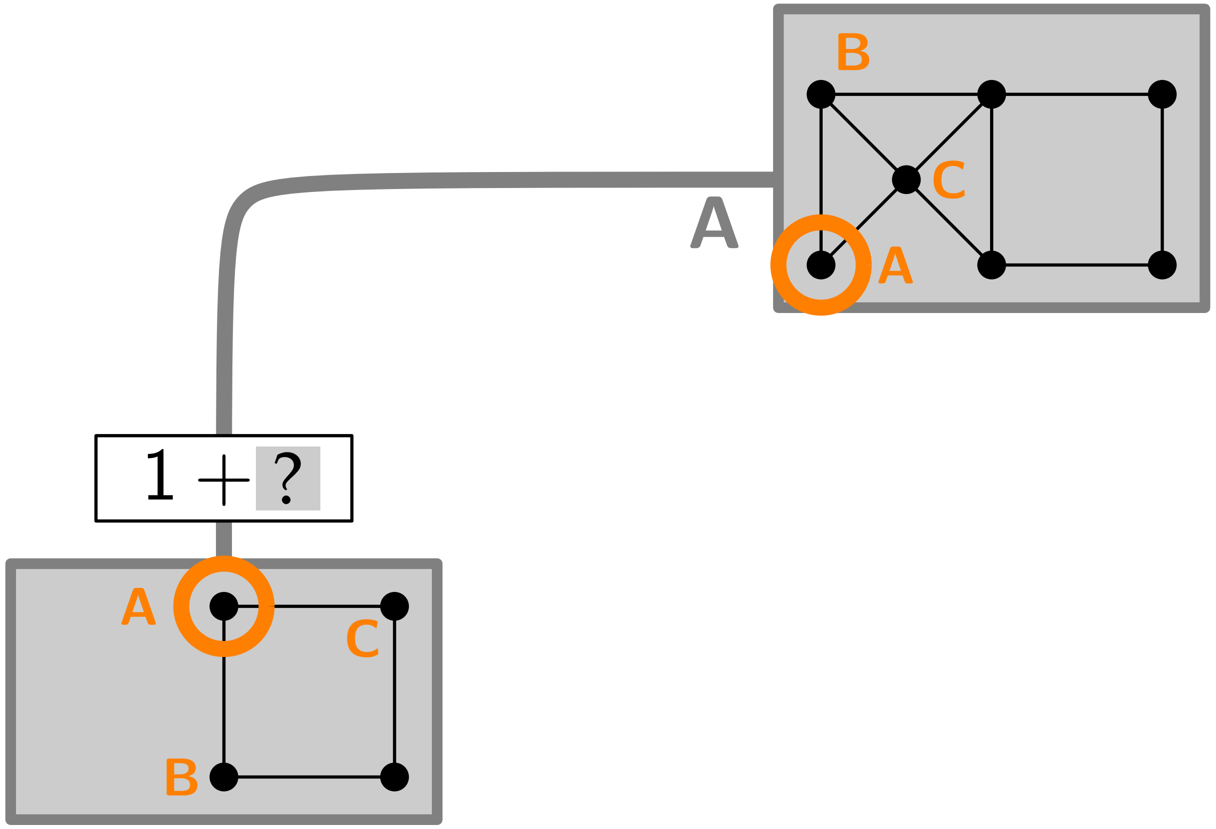
- Let's consider an example run.

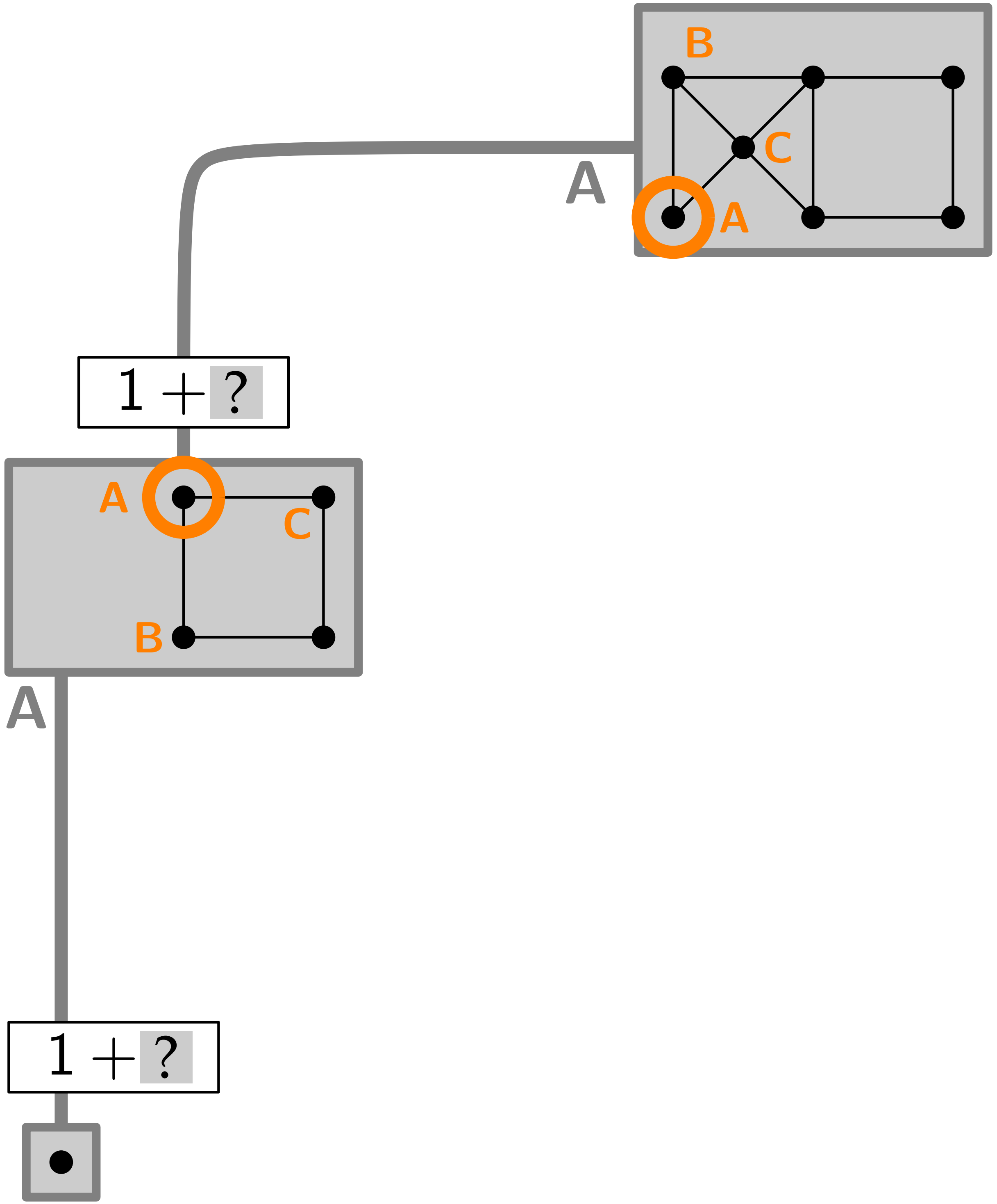


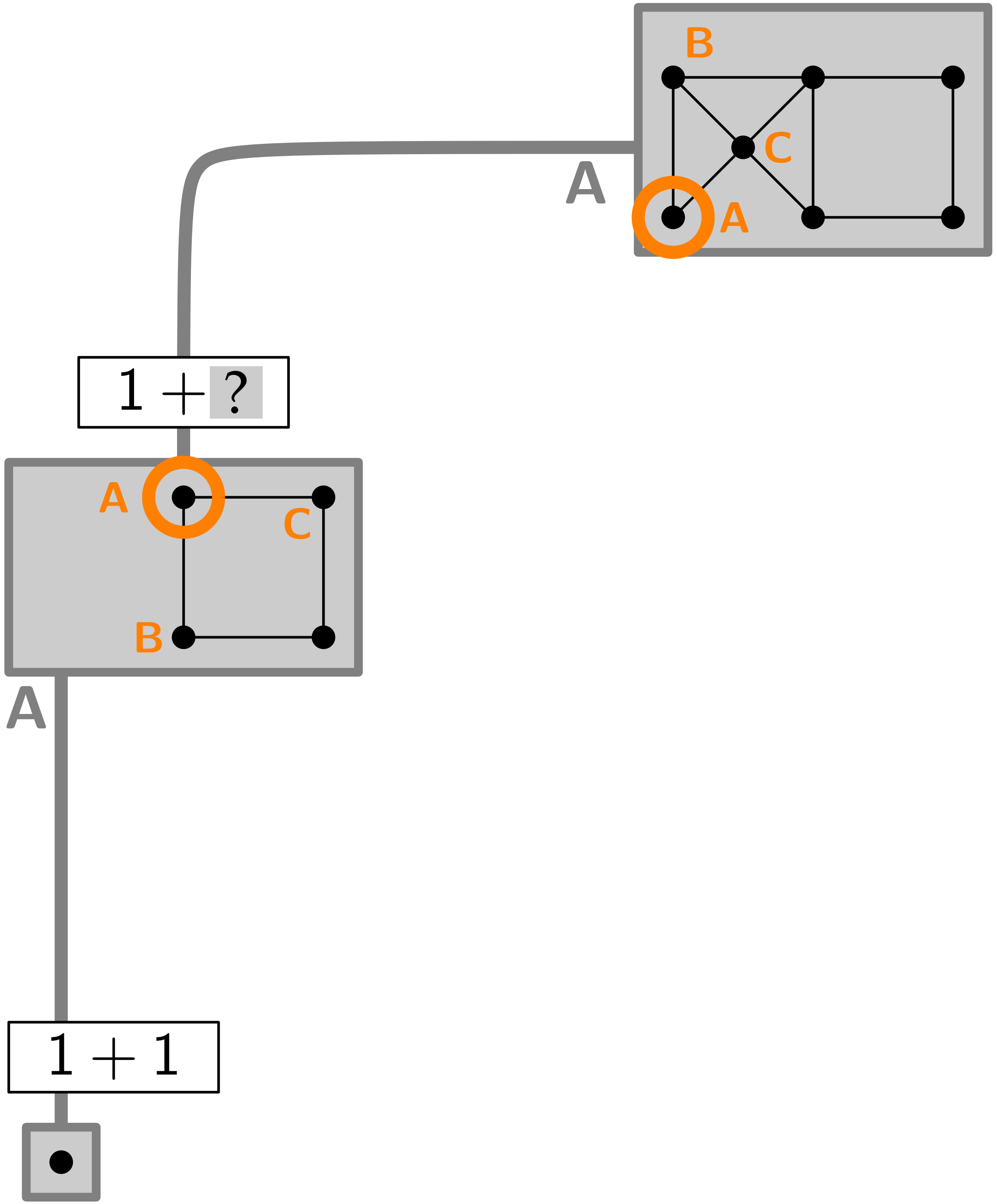


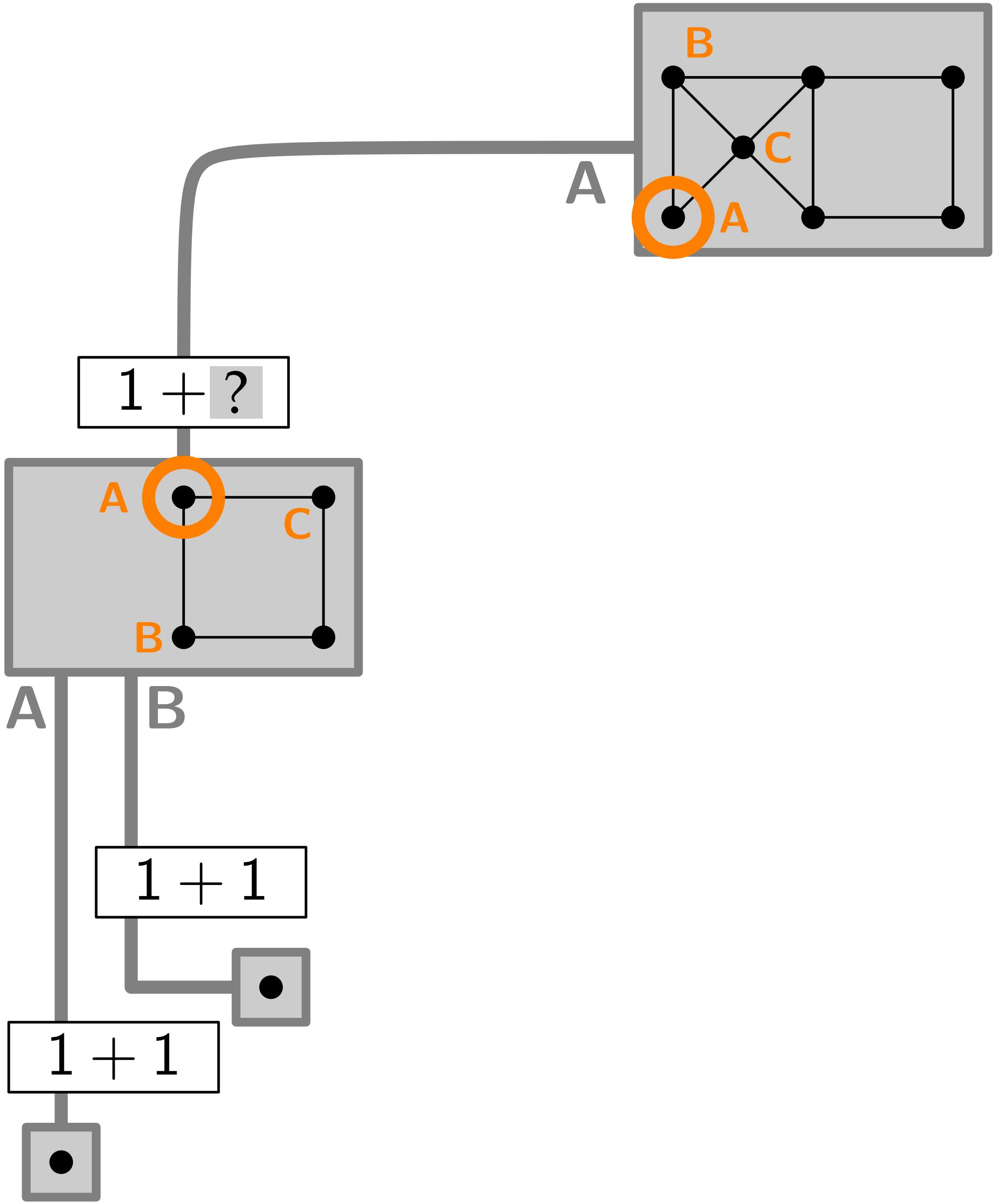

B
C
A

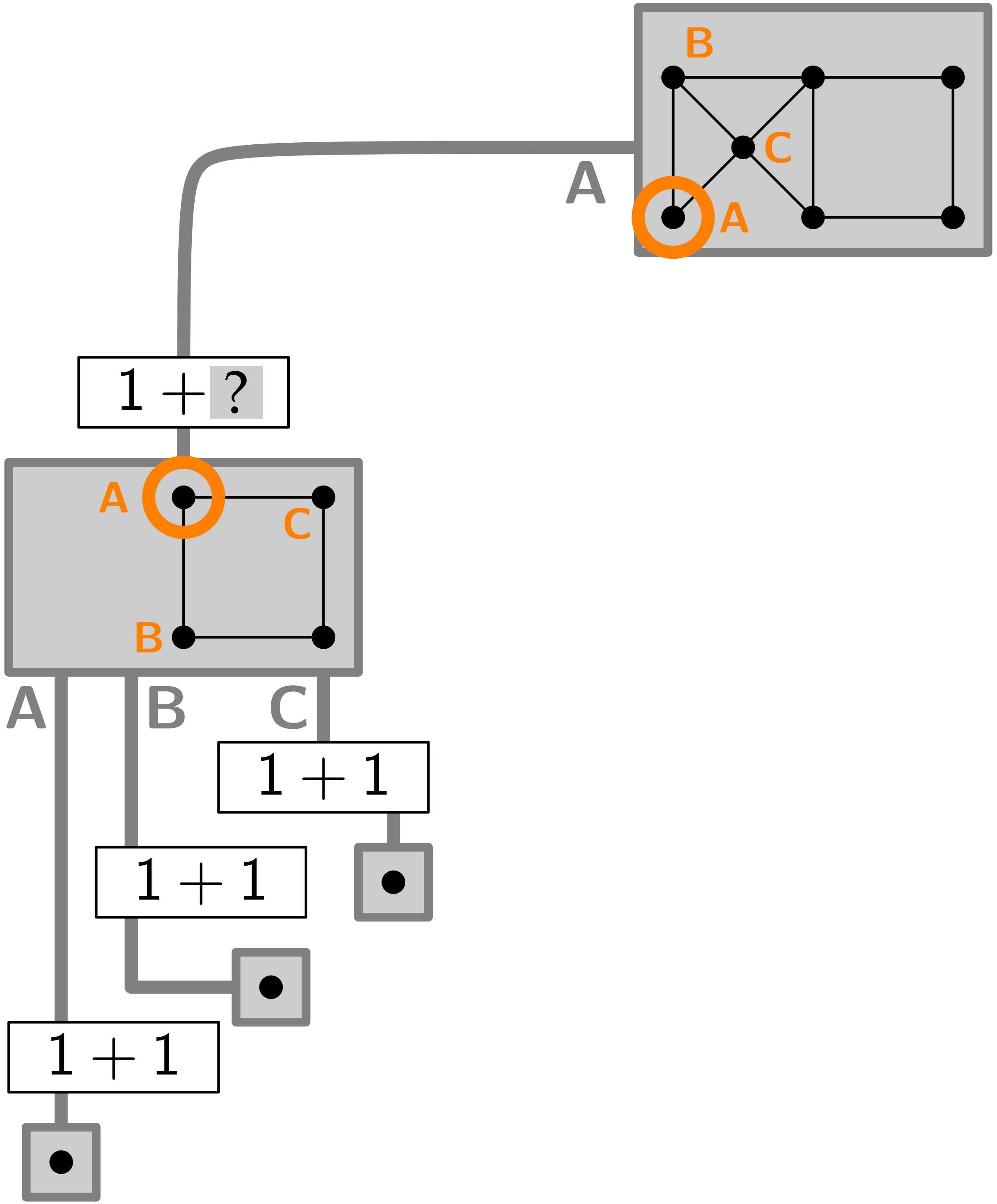

B
C
A
A
1 + ?

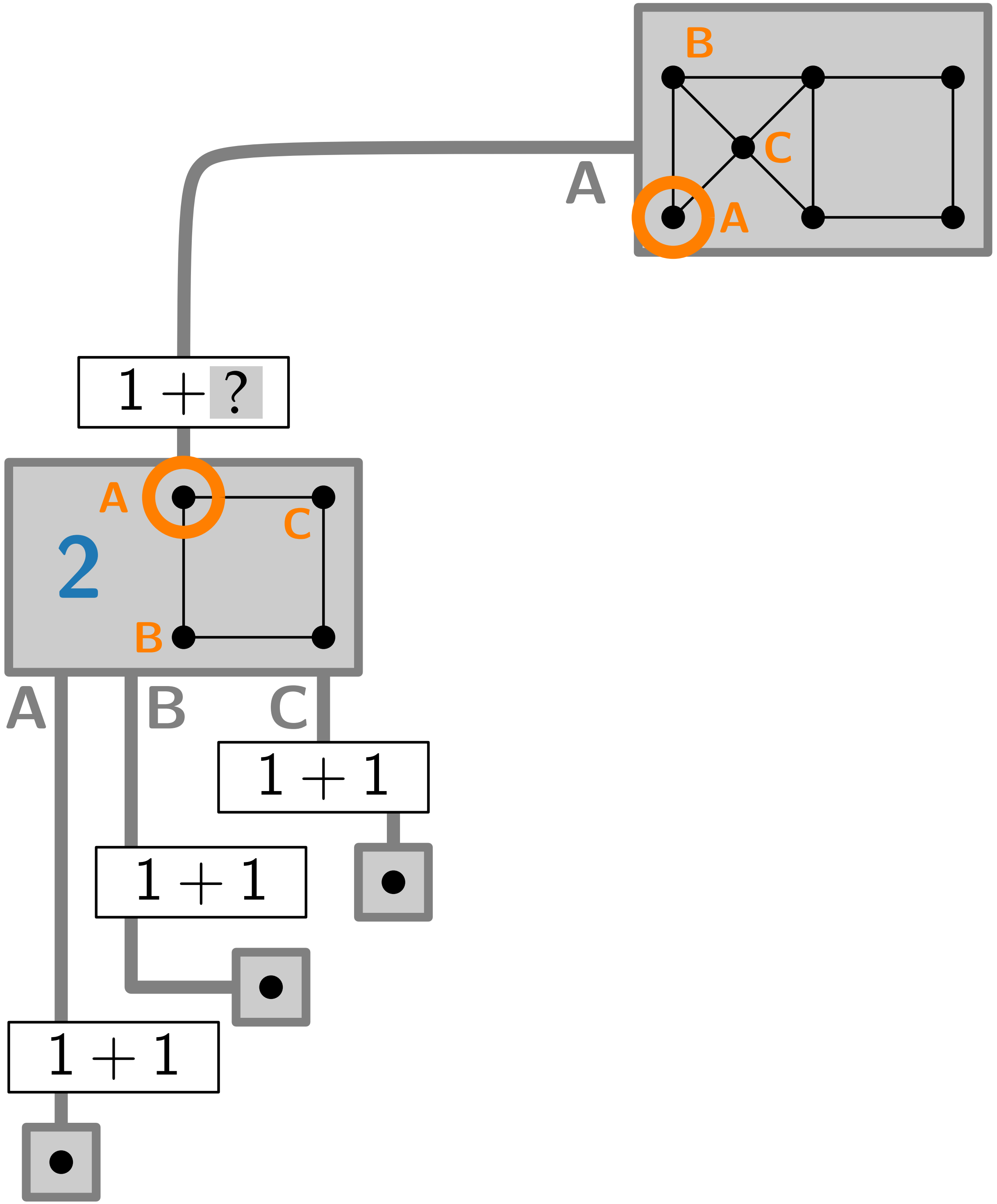

B
C
A
A
1 + ?
A
C
B

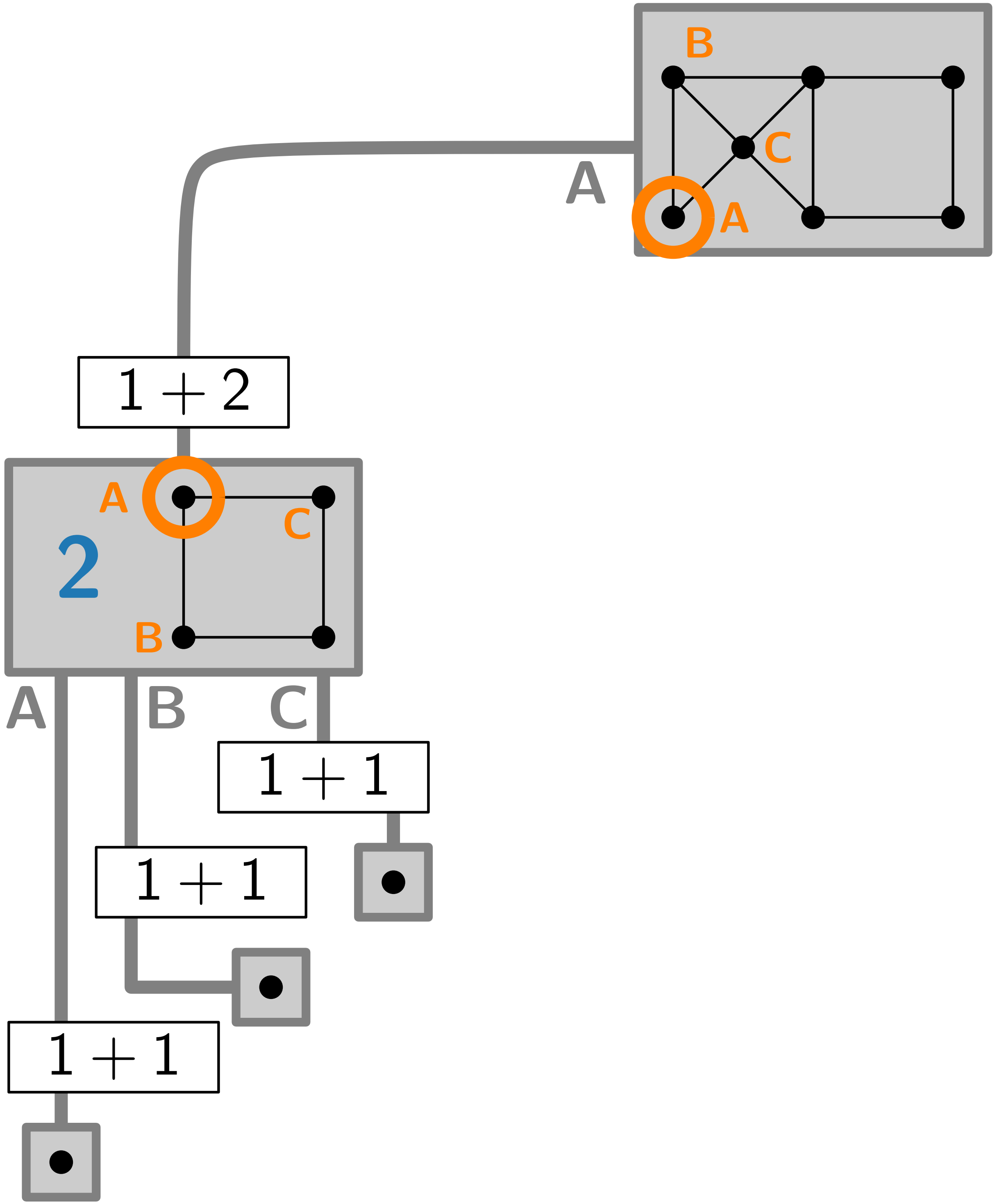

B
C
A
A
1 + ?
A
C
B
A
1 + ?

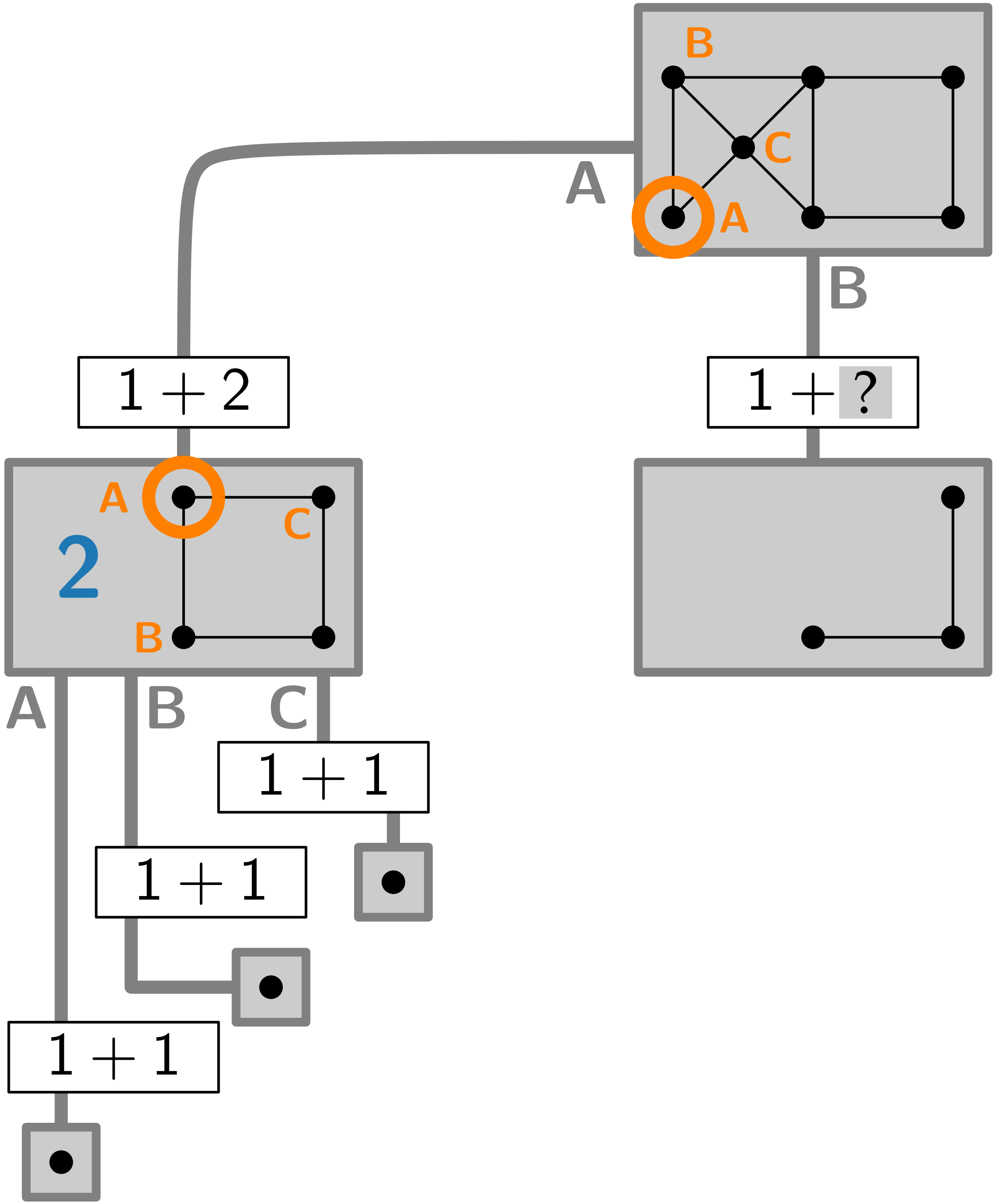

B
C
A
A
$1 + ?$
A
C
B
A
$1 + 1$

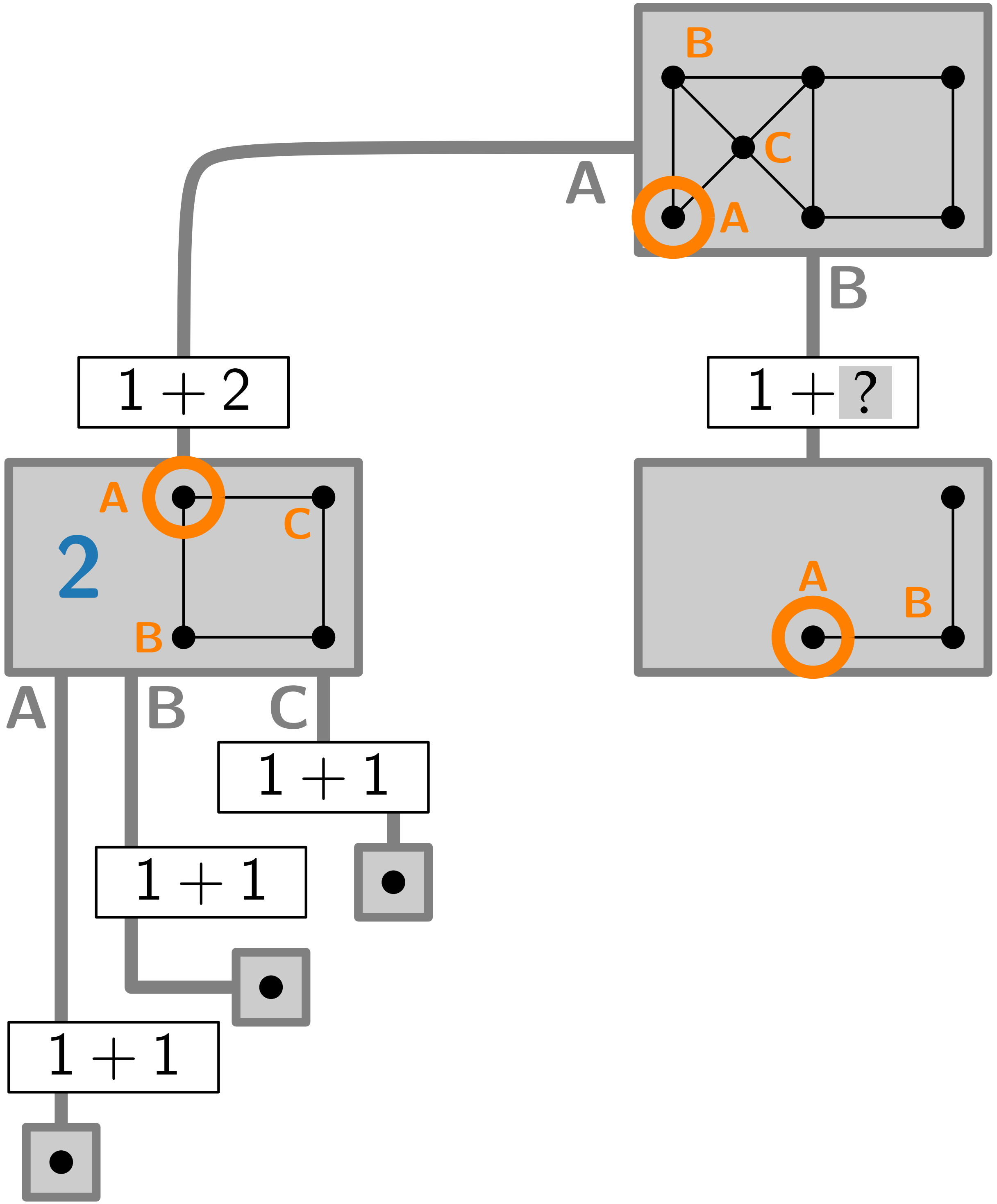

B
C
A
A
1 + ?
A
C
B
A
B
1 + 1
1 + 1

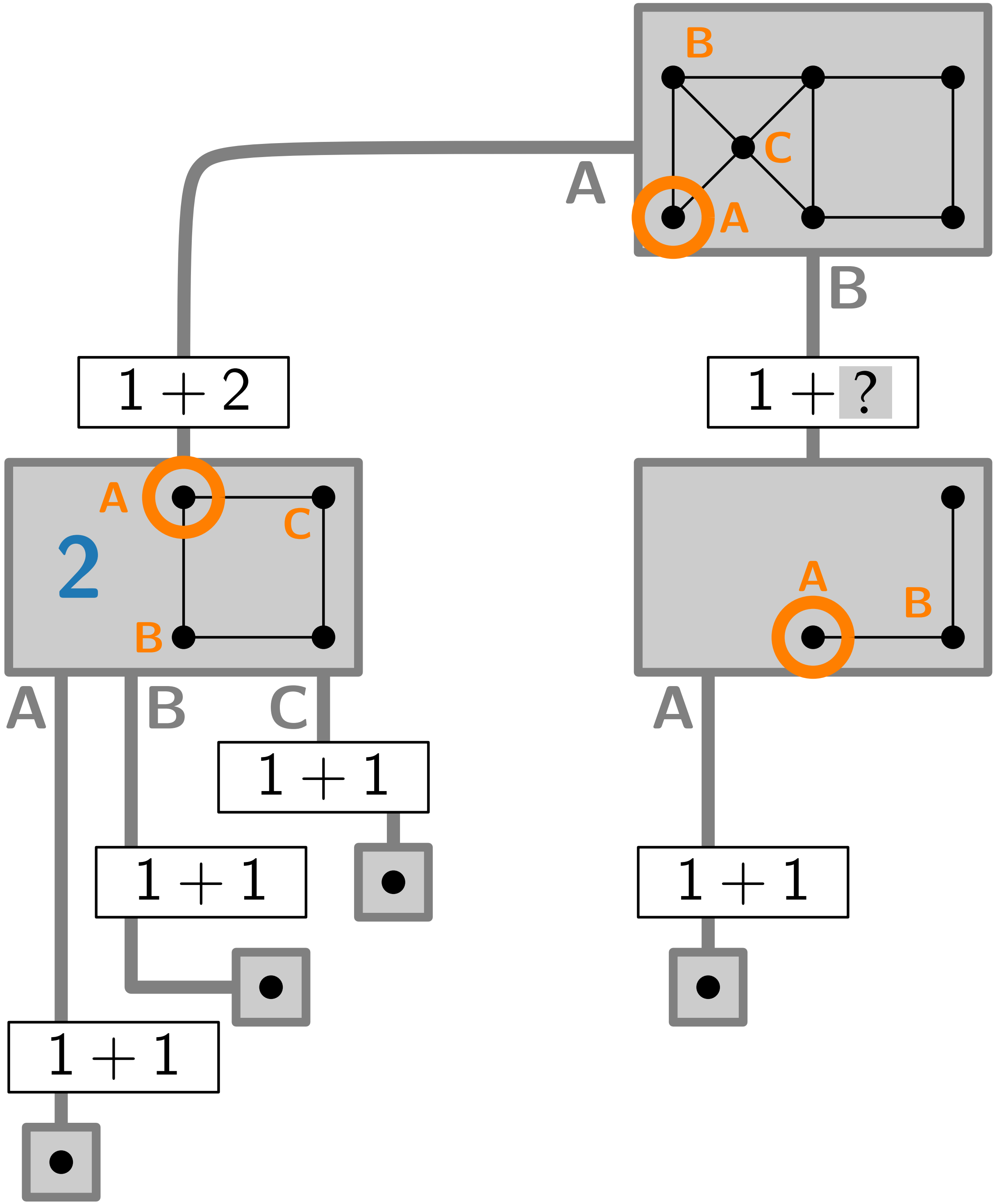

B
C
A
A
1 + ?
A
C
B
A
B
C
1 + 1
1 + 1
1 + 1

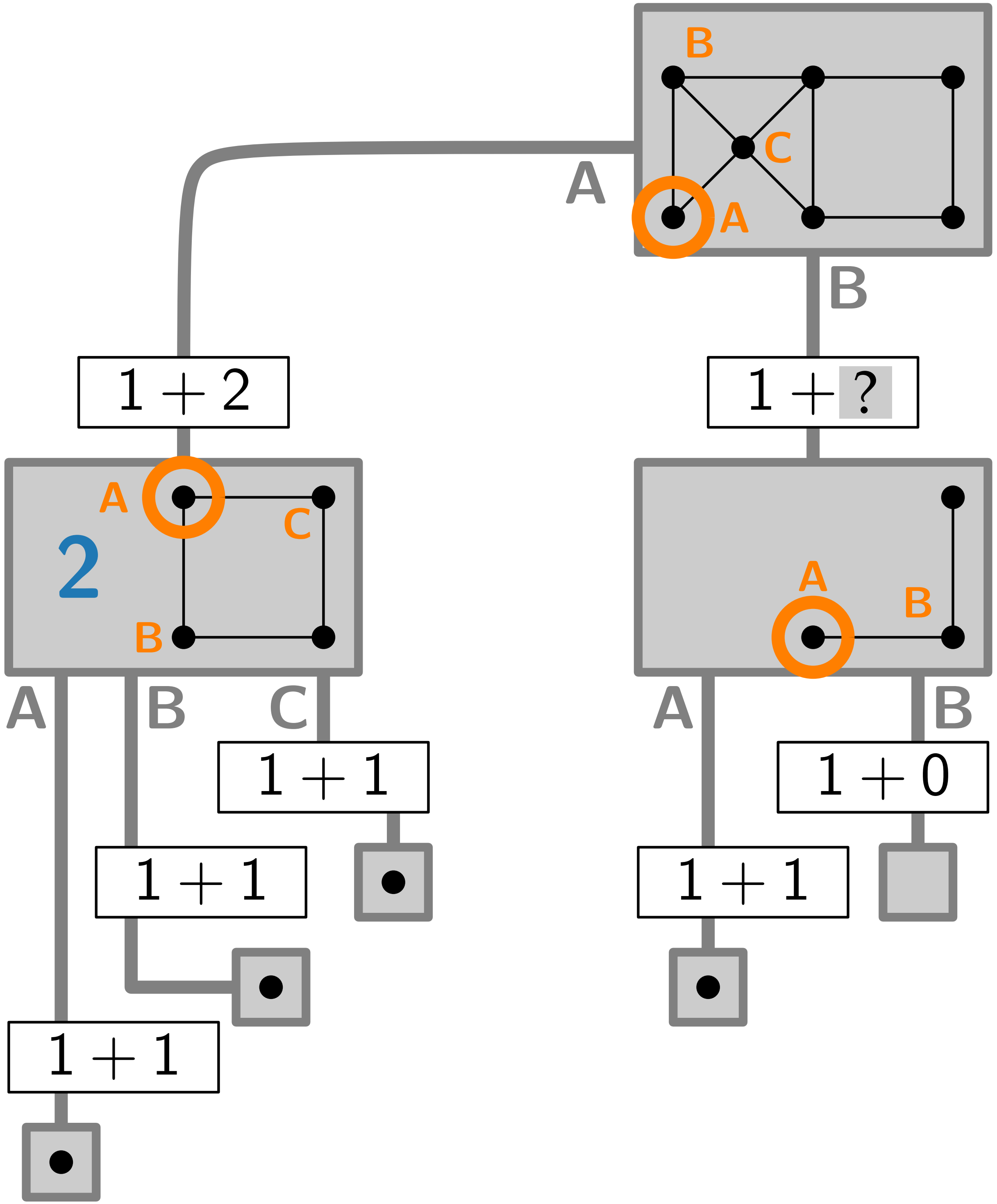

B
C
A
A
1 + ?
A
C
2
B
A
B
C
1 + 1
1 + 1
1 + 1


B
C
A
A
1 + 2
A
C
2
B
A
B
C
1 + 1
1 + 1
1 + 1

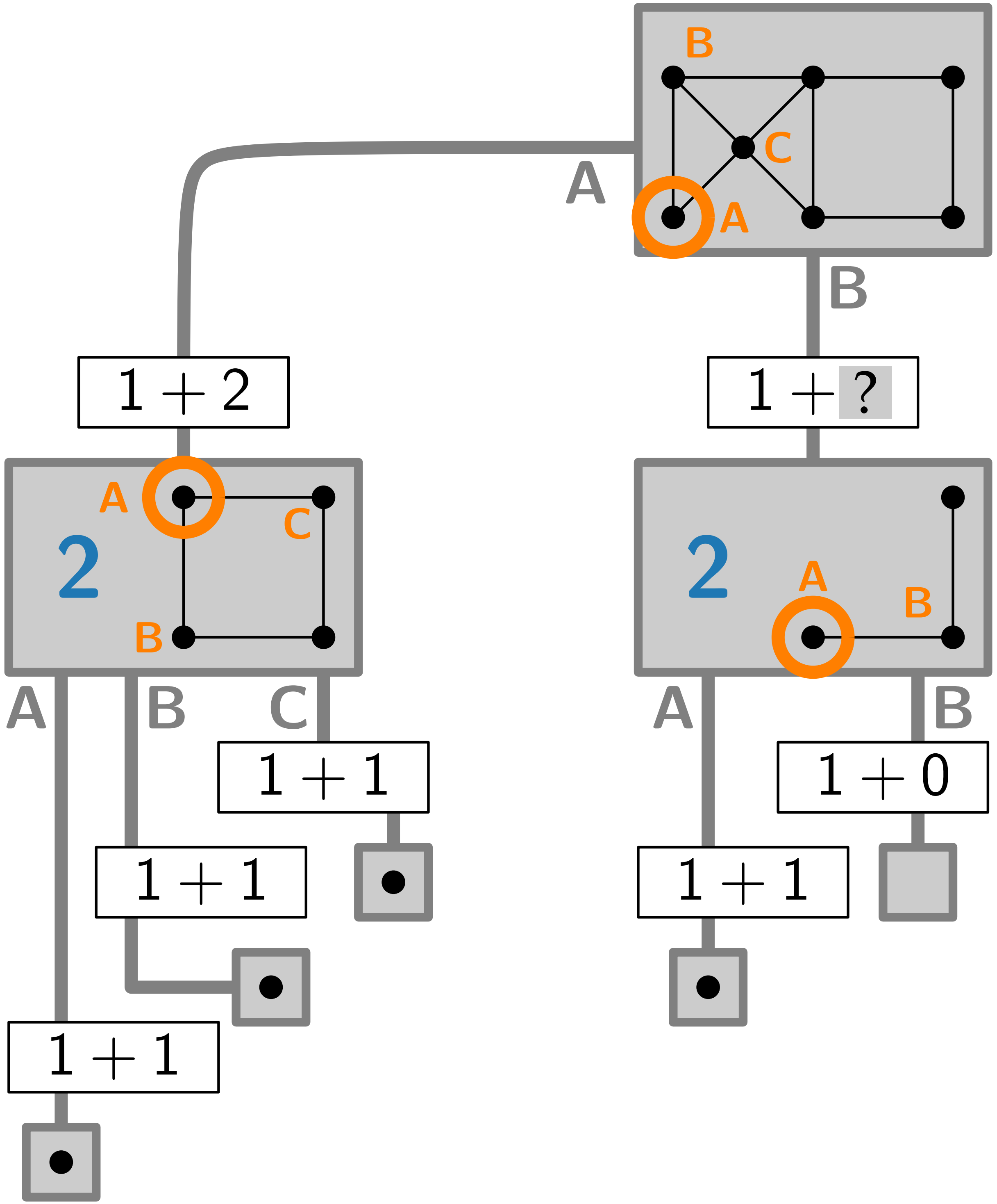

B
C
A
A
B
1 + 2
1 + ?
A
C
2
B
A
B
C
1 + 1
1 + 1
1 + 1


B
C
A
A
B
1 + 2
1 + ?
A
C
2
B
A
B
A
B
C
1 + 1
1 + 1
1 + 1


B
C
A
A
B
1 + 2
1 + ?
A
C
2
B
A
B
A
B
C
A
1 + 1
1 + 1
1 + 1
1 + 1



B
C
A
A
B
1 + 2
1 + ?
A
C
2
B
A
B
A
B
C
A
B
1 + 1
1 + 0
1 + 1
1 + 1
1 + 1


B
C
A
A
B
1 + 2
1 + ?
A
2
C
B
2
A
B
A
B
C
A
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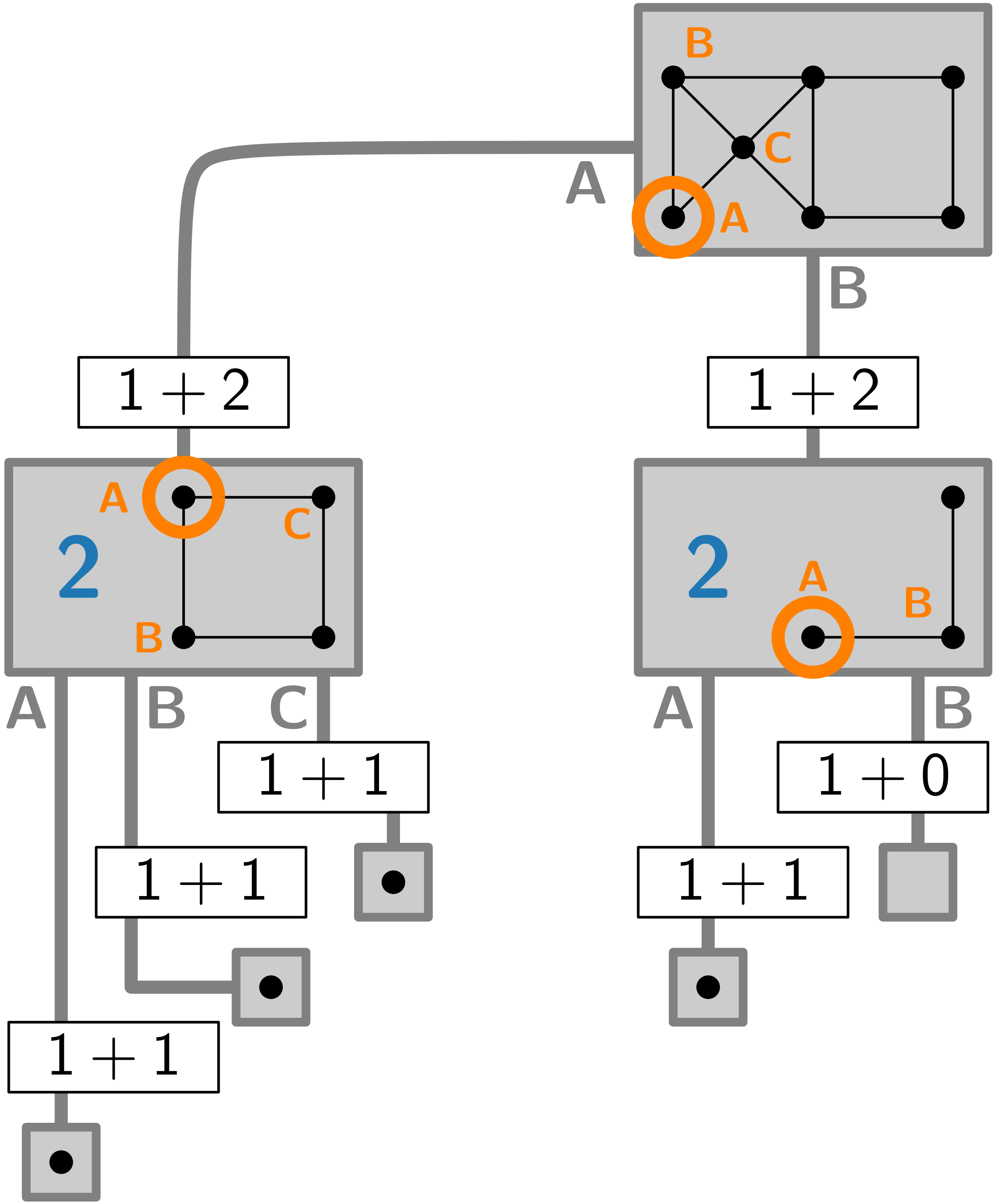
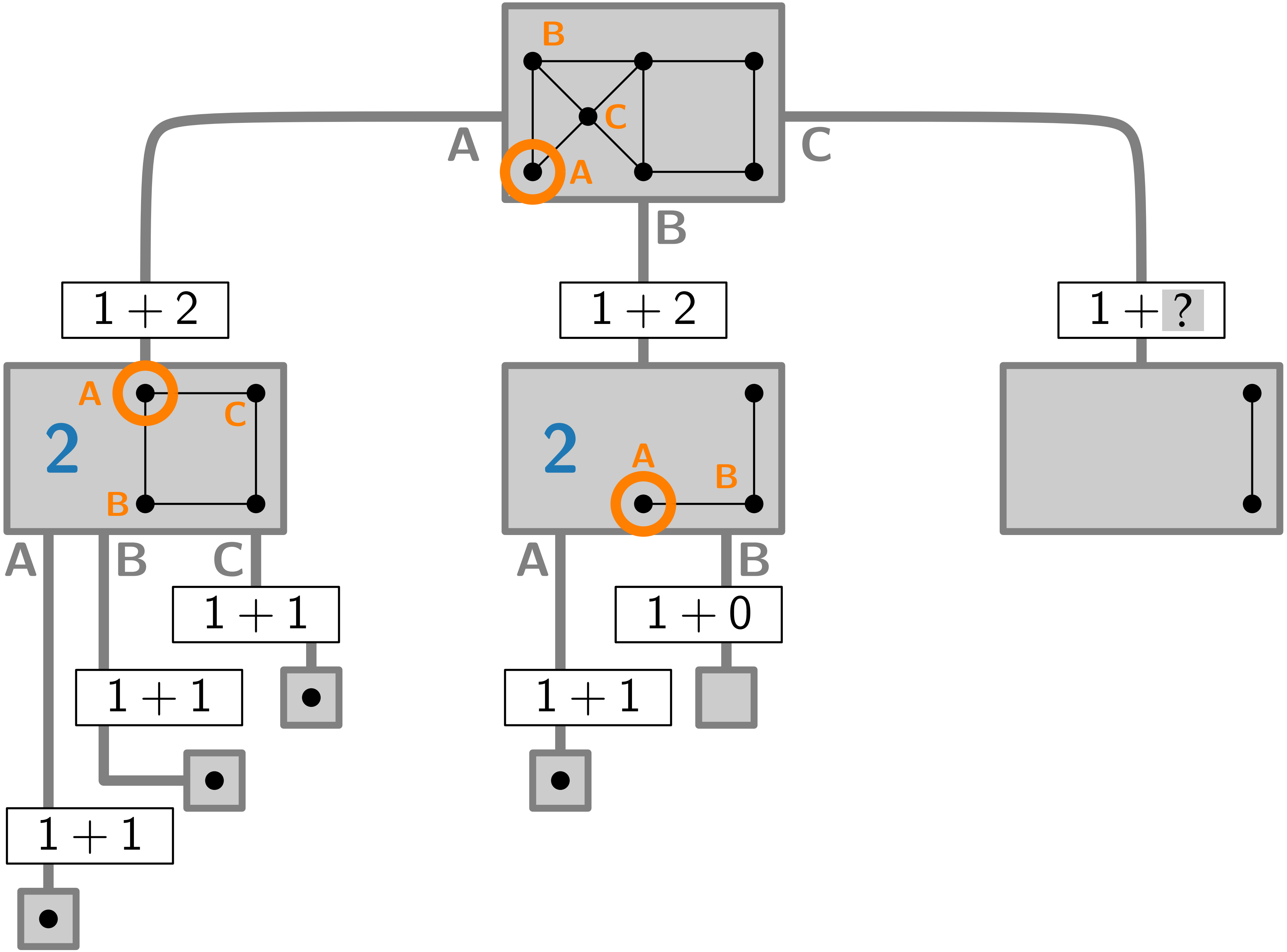
1 + 0
1 + 1
1 + 1
1 + 1



A
B
C
A
B
$1+2$
$1+2$
2
A
C
B
2
A
B
A
B
C
A
B
$1+1$
$1+0$
$1+1$
$1+1$
$1+1$


B
C
A
A
B
C
1 + 2
1 + 2
1 + ?
A
C
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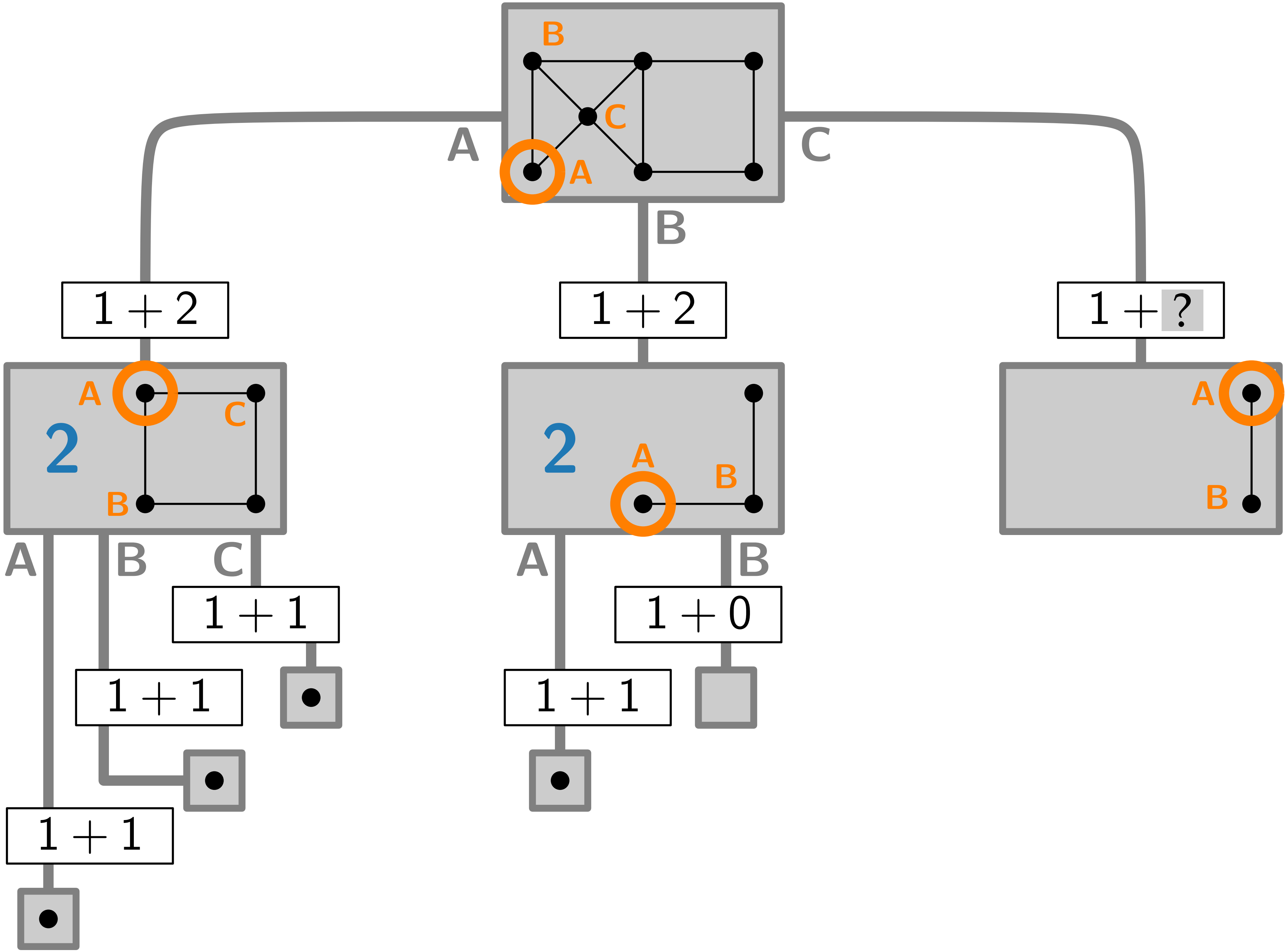
A
B
1 + 1
1 + 0
1 + 1
1 + 1
1 + 1


B
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A
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B
C
1 + 2
1 + 2
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C
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1 + 0
1 + 1
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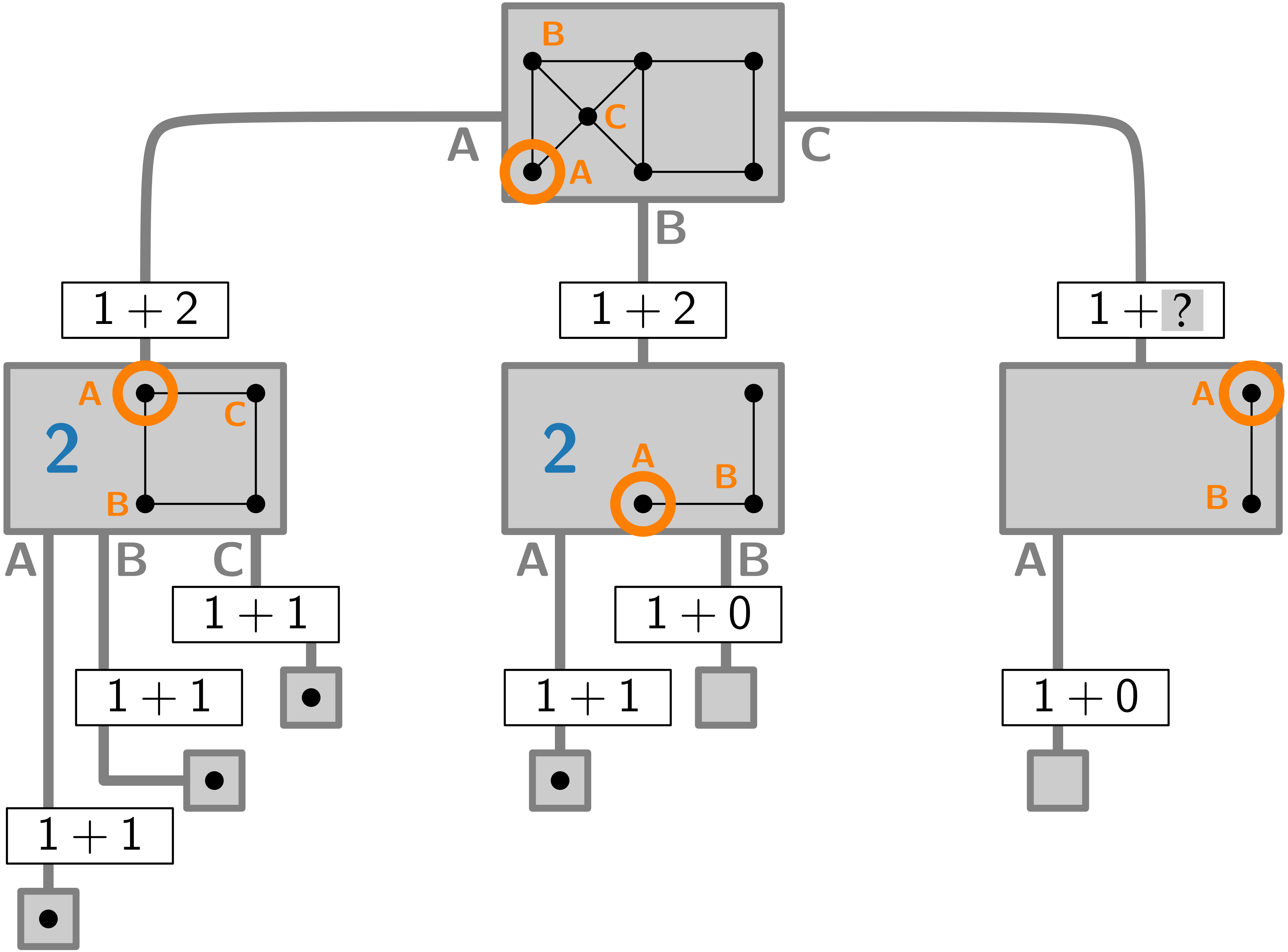

B
C
A
A
B
C
1 + 2
1 + 2
1 + ?
A
C
B
2
A
B
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C
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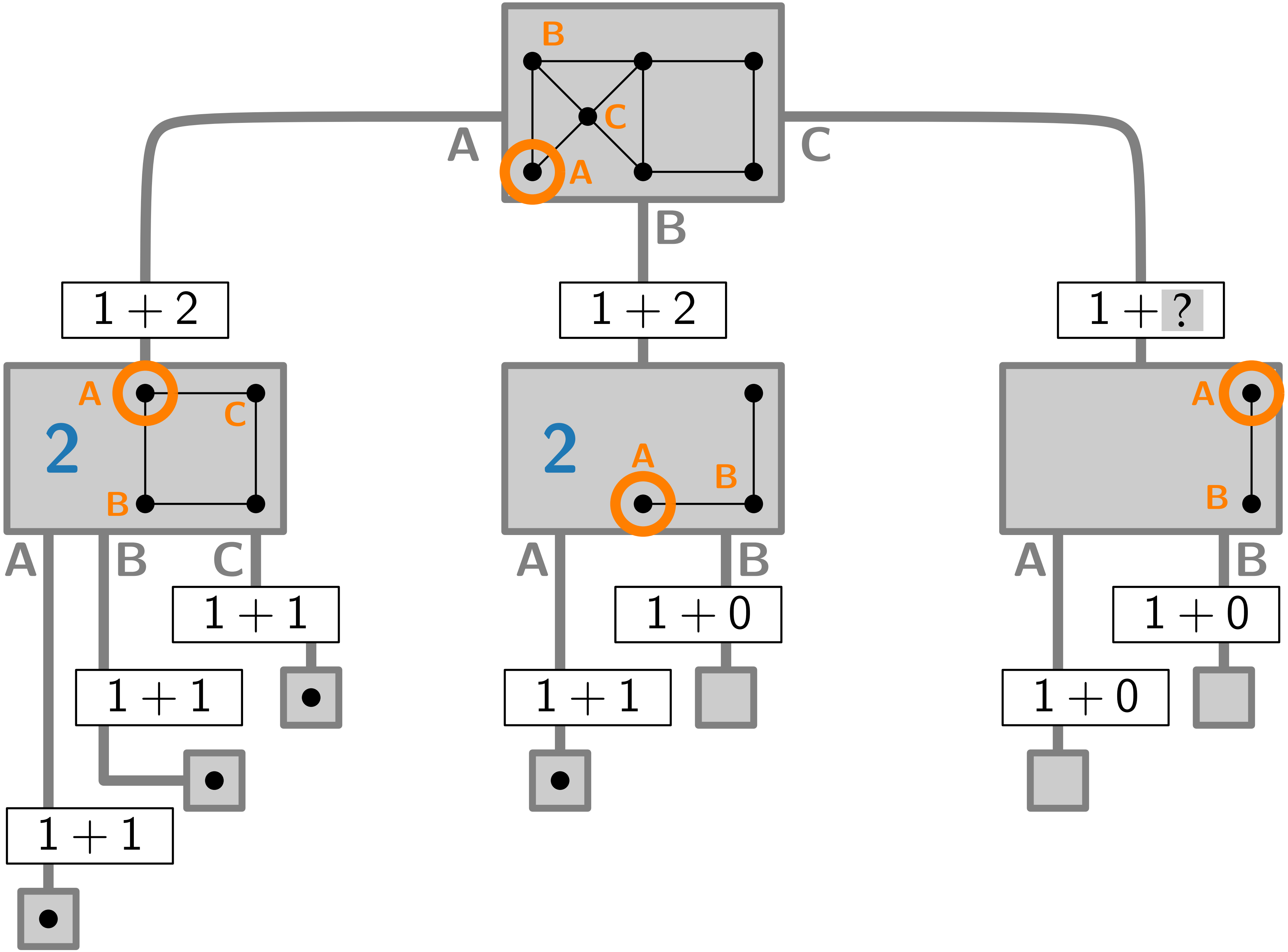
B
A
1 + 1
1 + 0
1 + 1
1 + 1
1 + 0
1 + 1


B
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B
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1 + 2
1 + 2
1 + ?
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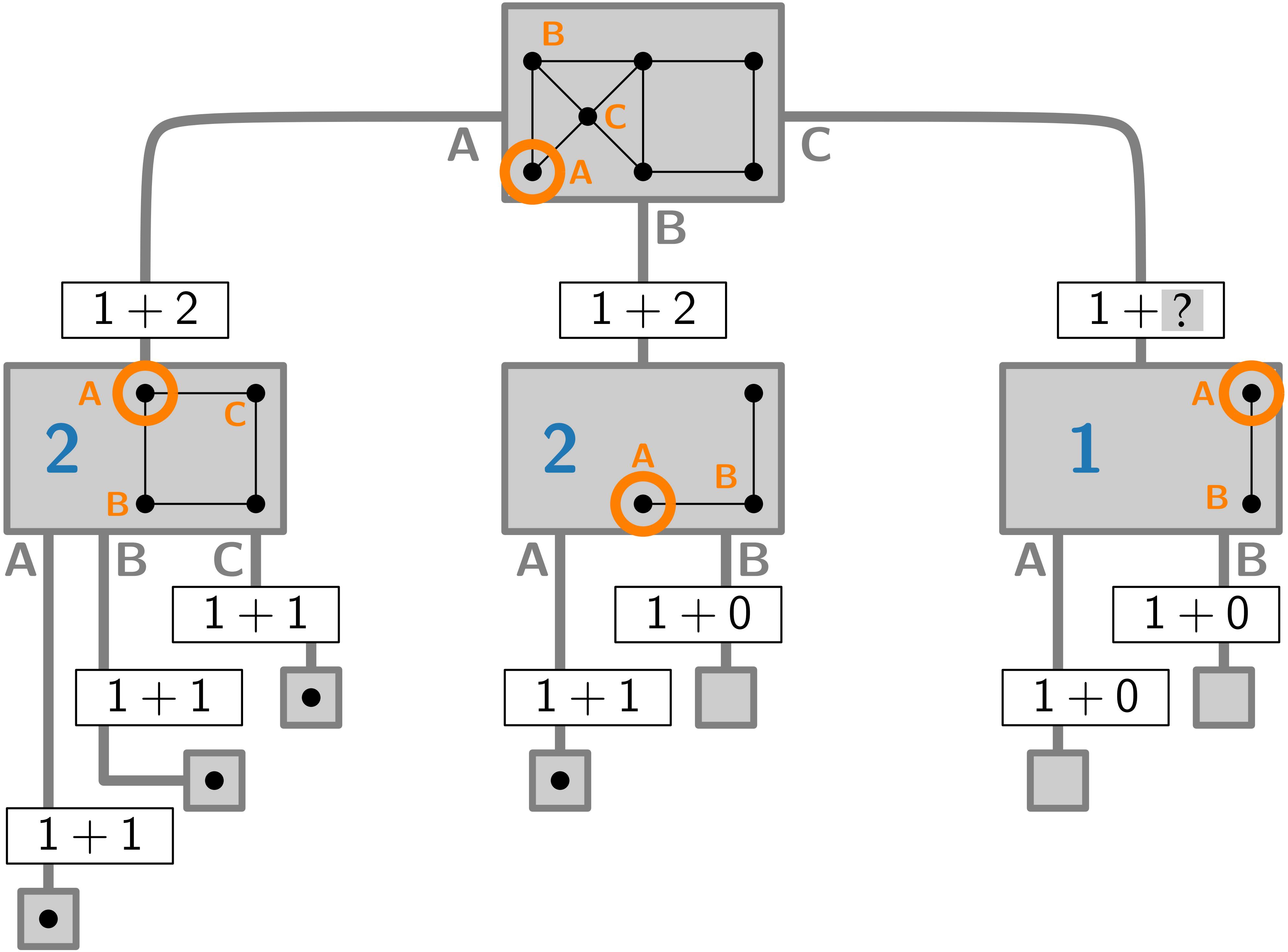
1 + 0
1 + 1
1 + 1
1 + 0
1 + 1


B
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1 + 2
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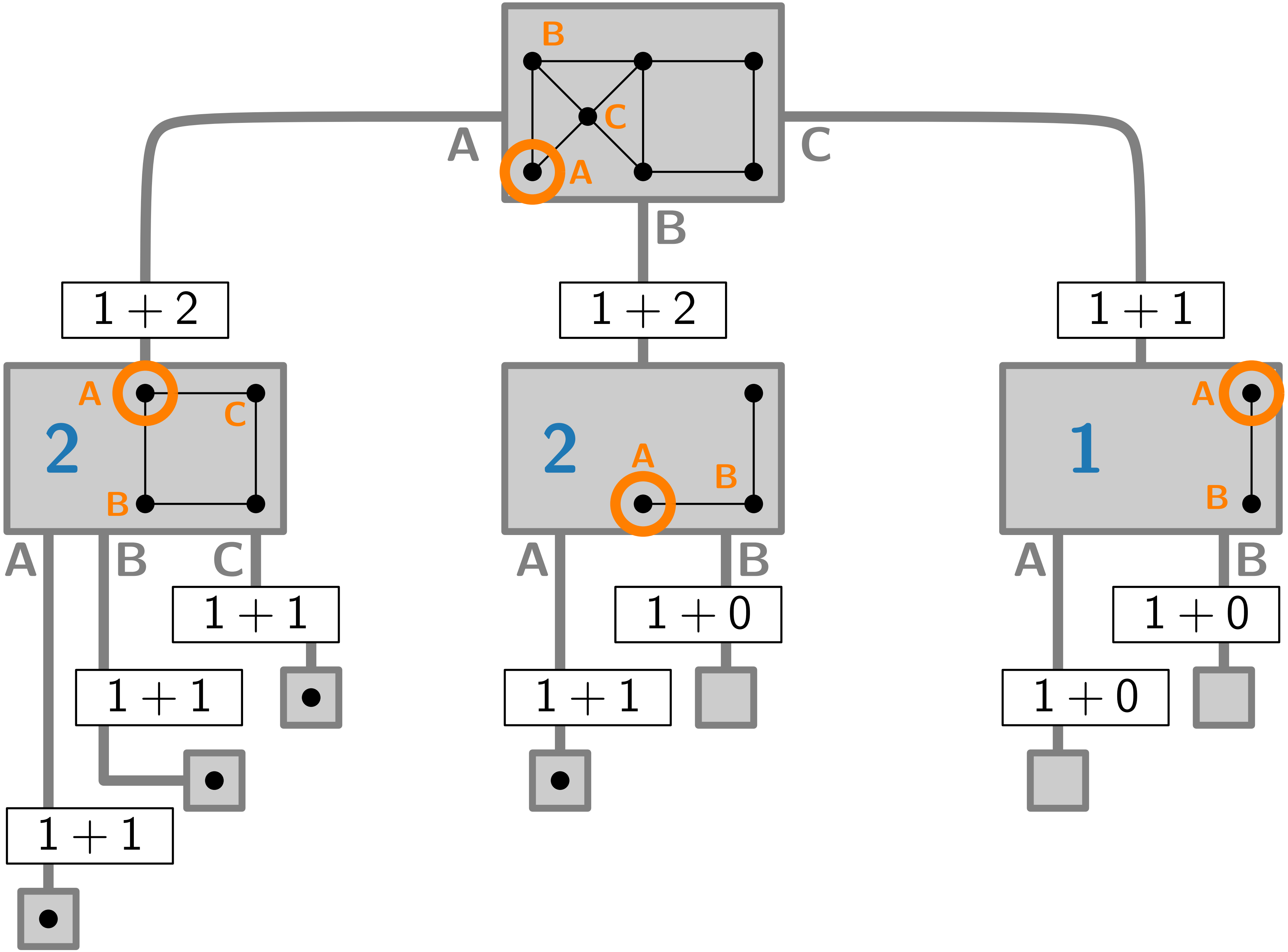
1 + 0
1 + 1


B
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1 + 2
1 + 2
1 + 1
A
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2
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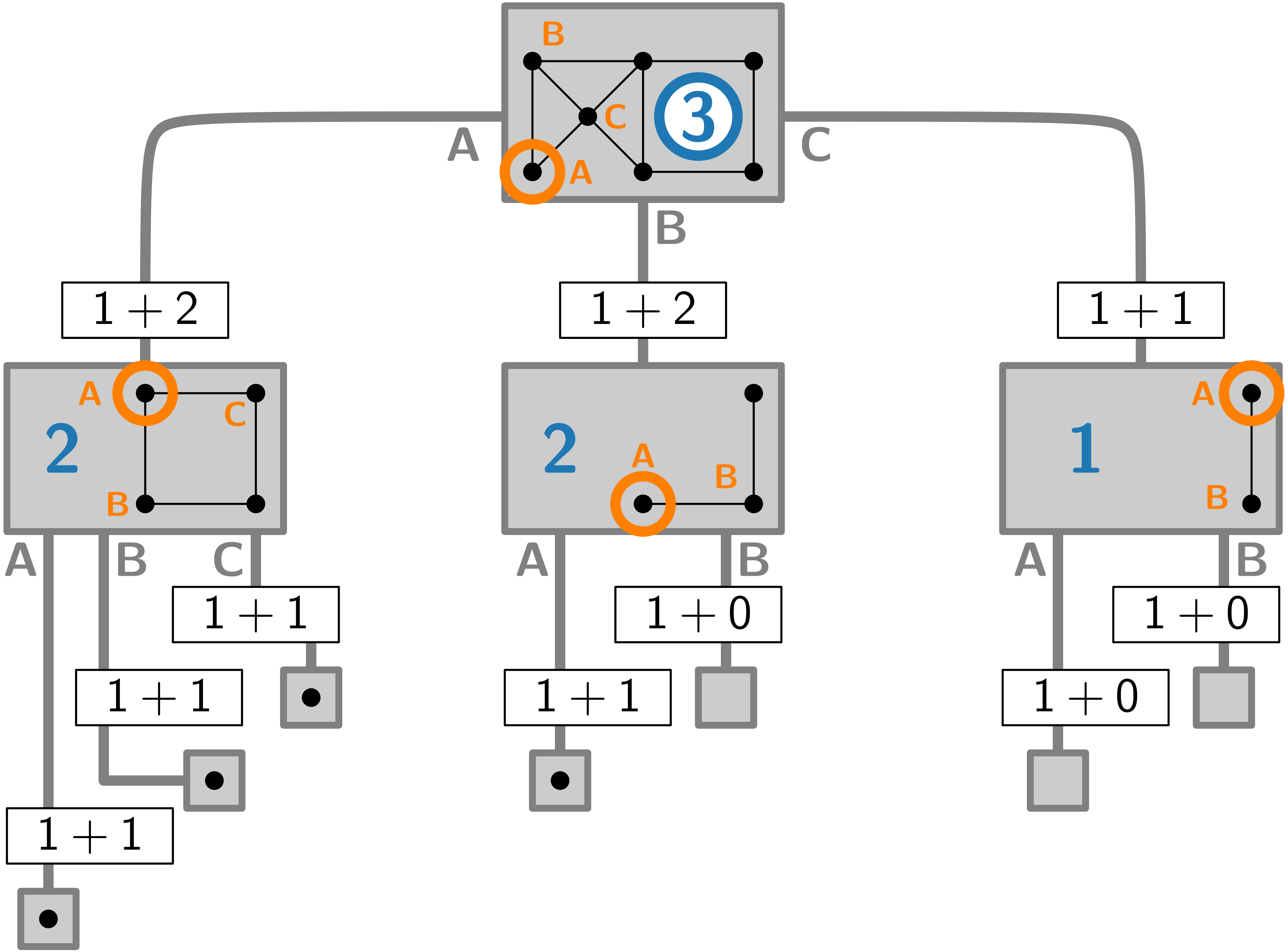
B
A
B
1 + 1
1 + 0
1 + 0
1 + 1
1 + 1
1 + 0
1 + 1


B
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A
3
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B
C
1 + 2
1 + 2
1 + 1
A
C
B
2
A
C
B
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2
B
A
B
1
A
B
C
1 + 1
1 + 1
1 + 1
A
B
1 + 0
1 + 1
A
B
1 + 0
1 + 0

MIS – Runtime Analysis

For a worst-case n-vertex graph G ($n \geq 1$):

$$B(n) \leq \sum_{y \in N[v]} B(n - (\deg(y) + 1))$$

where v is a minimum degree vertex of G, and $B(n') \leq B(n)$ for any $n' \leq n$.

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- Induc. step: for $n \geq 1$, set $s = \deg(v) + 1$.

MIS – Runtime Analysis

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$B(n) \leq s \cdot B(n - s)$

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$B(n) \leq s \cdot B(n - s) \leq s \cdot 3^{(n-s)/3}$

MIS – Runtime Analysis

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$$B(n) \leq s \cdot B(n - s) \leq s \cdot 3^{(n-s)/3} = \frac{s}{3^{s/3}} \cdot 3^{n/3}$$

MIS – Runtime Analysis

For a worst-case n-vertex graph G ($n \geq 1$):

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$$B(n) \leq s \cdot B(n - s) \leq s \cdot 3^{(n-s)/3} = \frac{s}{3^{s/3}} \cdot 3^{n/3} \overset{?}{\leq} 3^{n/3}$$

MIS – Runtime Analysis

For a worst-case n-vertex graph G $(n \geq 1)$:

$$B(n) \leq \sum_{y \in N[v]} B(n - (\deg(y)+1)) \leq (\deg(v)+1) \cdot B(\ n - (\deg(v)+1)\ )$$

where v is a minimum degree vertex of G, and $B(n') \leq B(n)$ for any $n' \leq n$.

We prove by induction that $B(n) \leq 3^{n/3}$.

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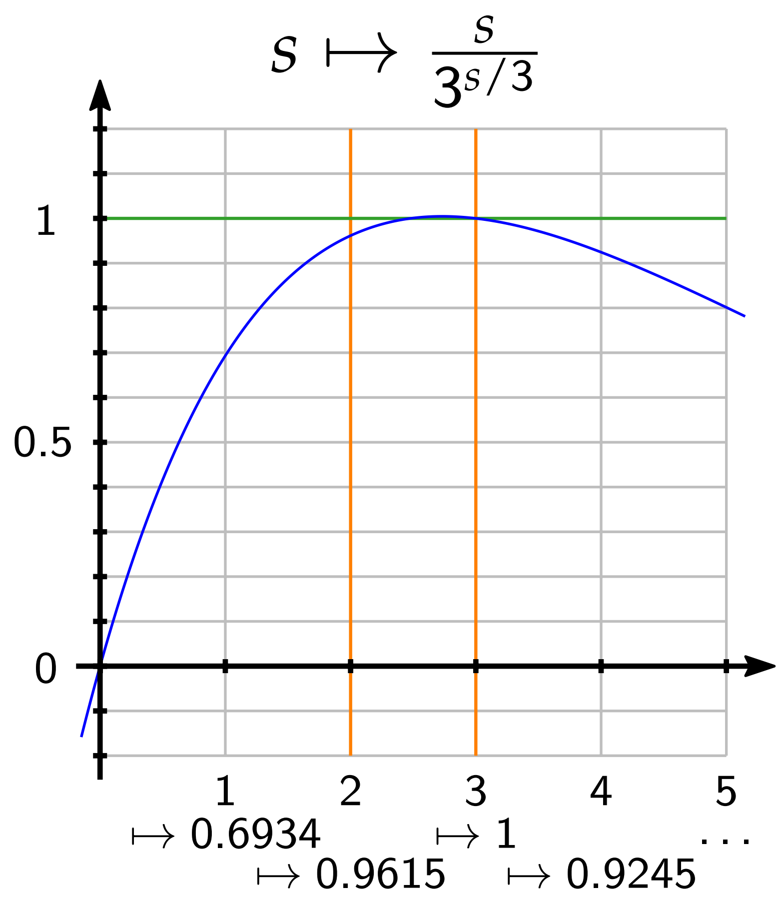

MIS – Runtime Analysis

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$$B(n) \leq \sum_{y \in N[v]} B(n - (\deg(y)+1)) \leq (\deg(v)+1) \cdot B(\ n - (\deg(v)+1)\ )$$

where v is a minimum degree vertex of G, and $B(n') \leq B(n)$ for any $n' \leq n$.

We prove by induction that $B(n) \leq 3^{n/3}$.

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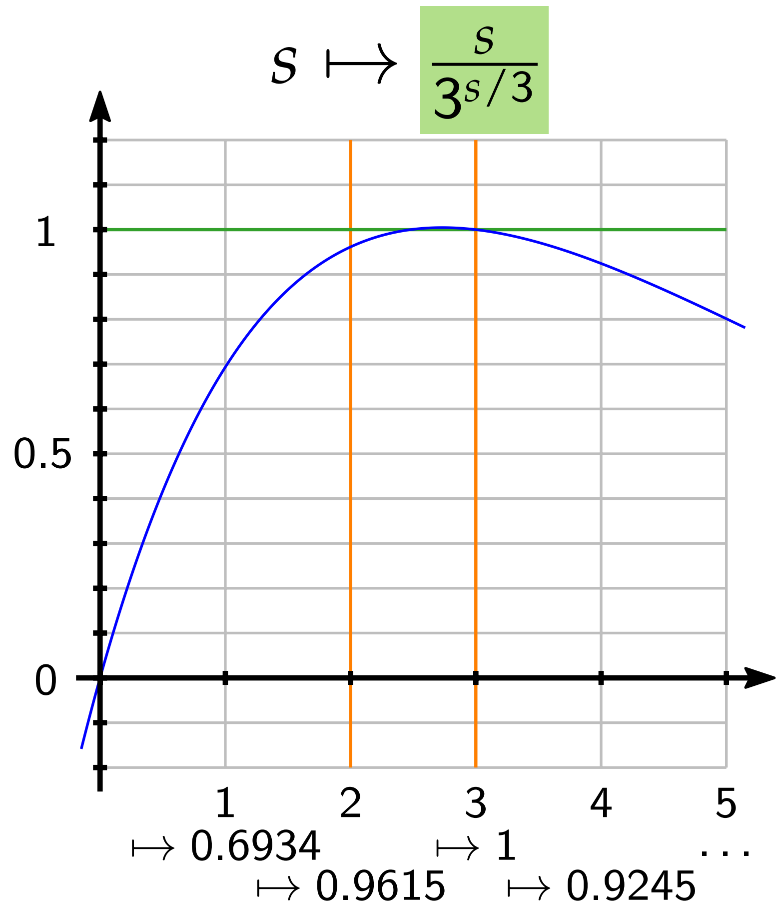

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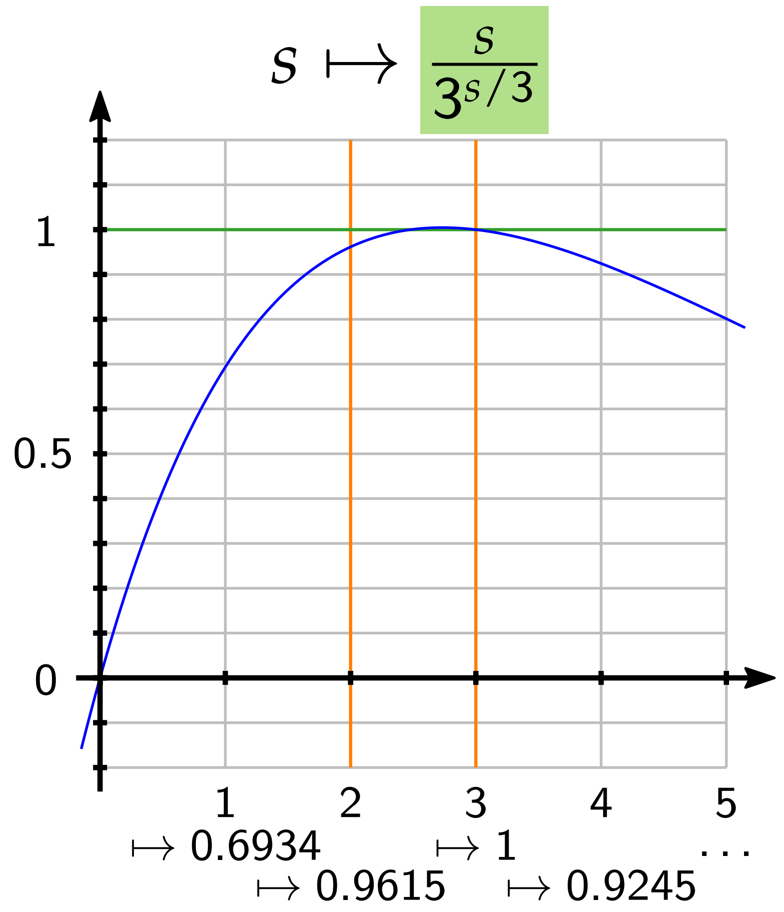
$$B(n) \in \mathcal{O}^*(\sqrt[3]{3}^{\,n}) \subseteq \mathcal{O}^*(1.44225^n)$$



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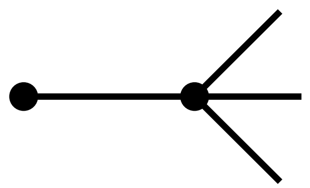

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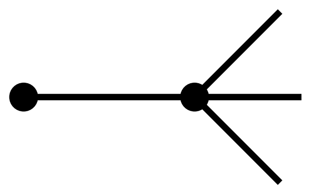

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- **Exercise**: Edge-branching for MIS

Literature

Main source:

- [Fomin, Kratsch Ch1] “Exact Exponential Algorithms”

Referenced papers:

- [ADMV ’15] Classic Nintendo Games are (Computationally) Hard
- [Mann ’17] The Top Eight Misconceptions about NP-Hardness