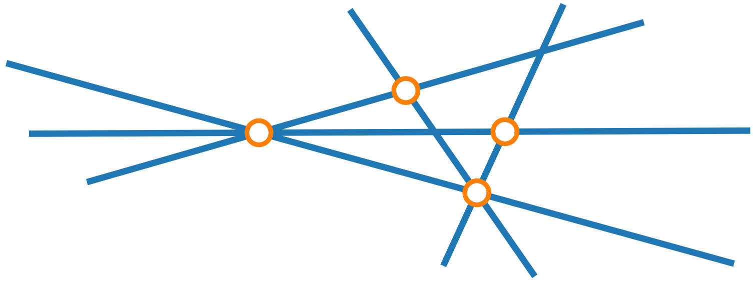
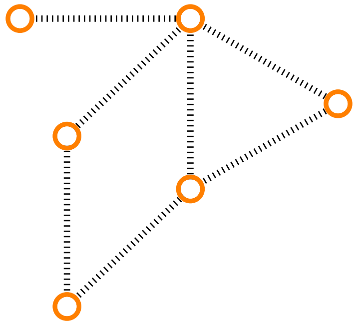
Visualization of Graphs

Lecture 11:
The Crossing Lemma
and Its Applications





Johannes Zink

Crossing Number and Topological Graphs

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Example.
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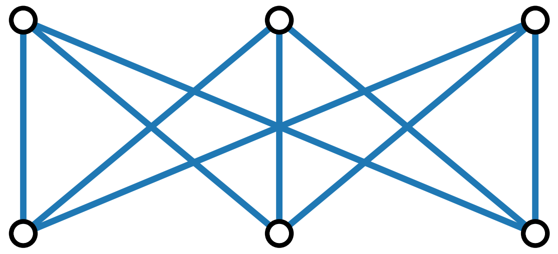

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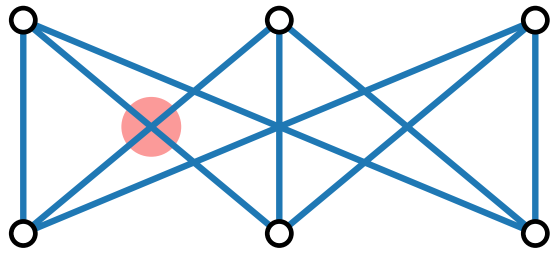

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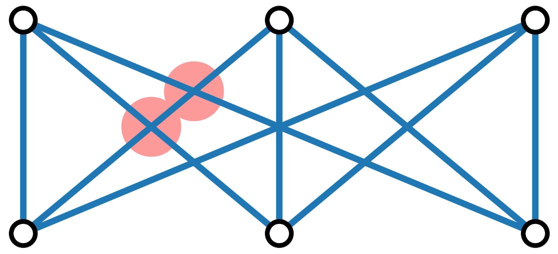

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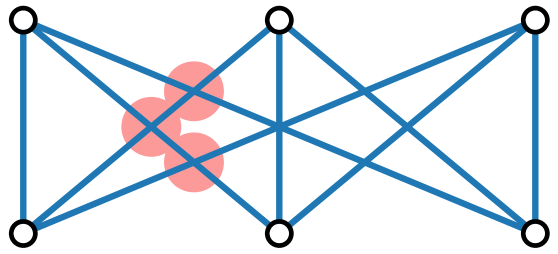

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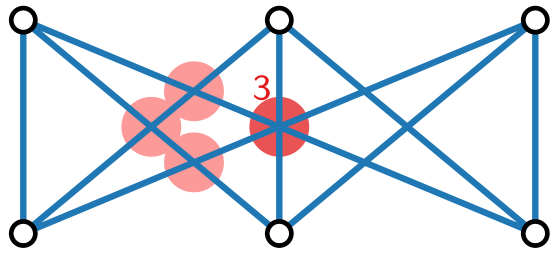

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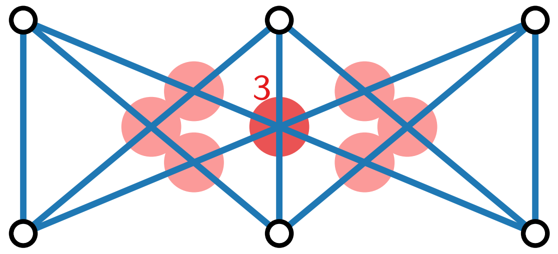

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Example.
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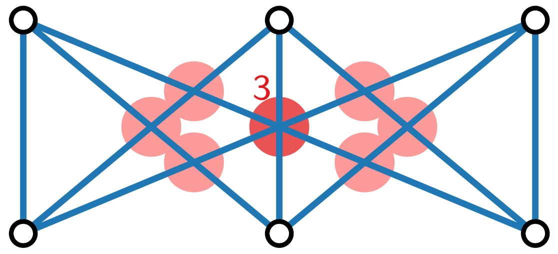

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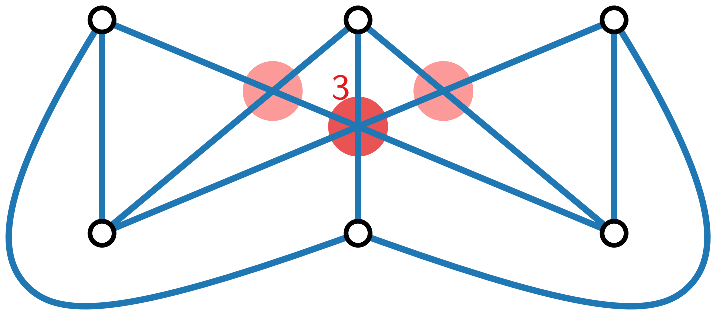

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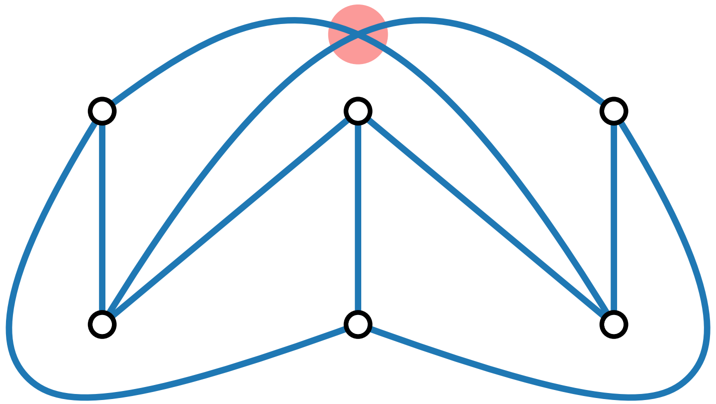

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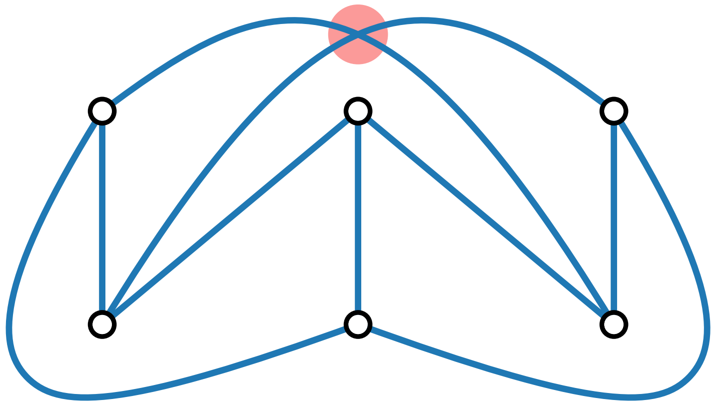

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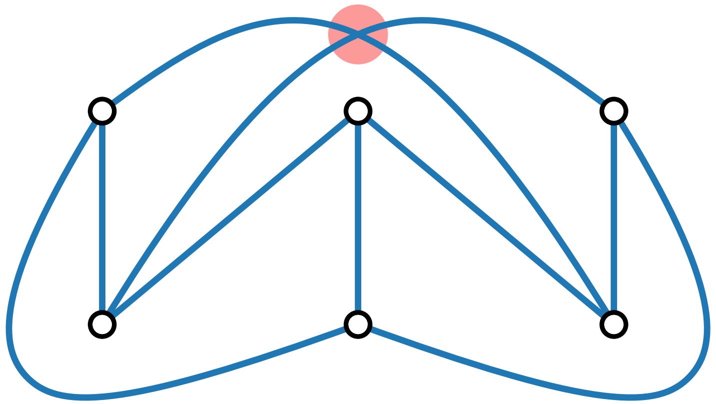

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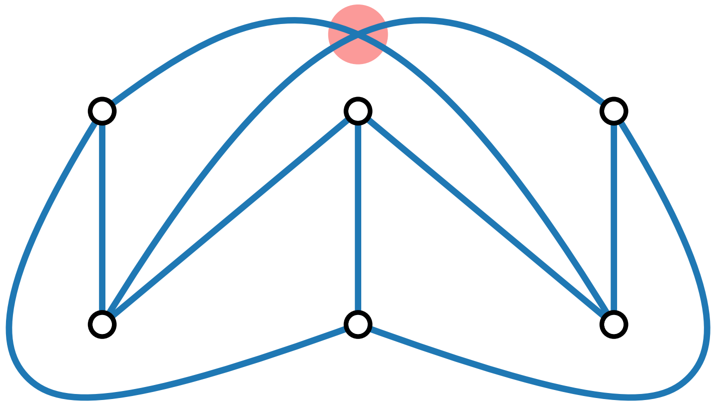

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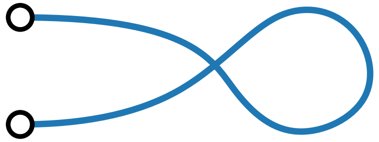

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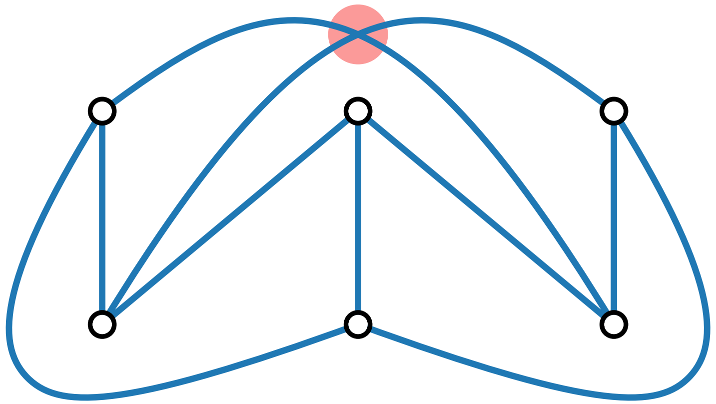

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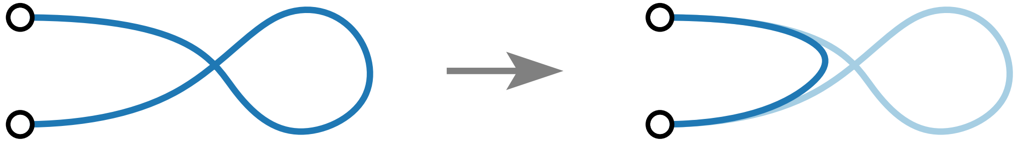

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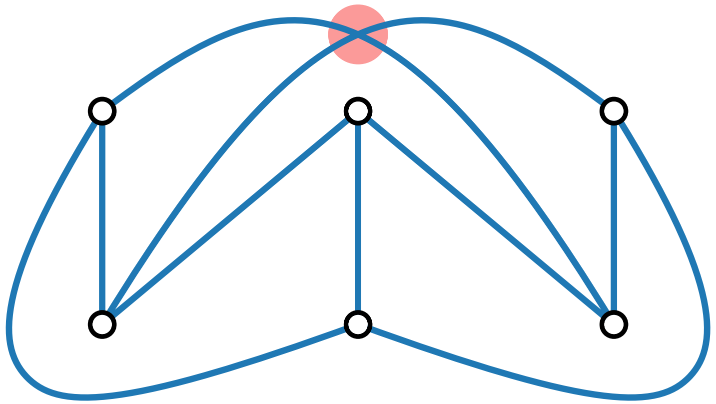

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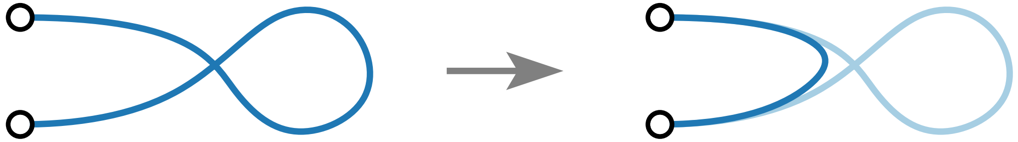

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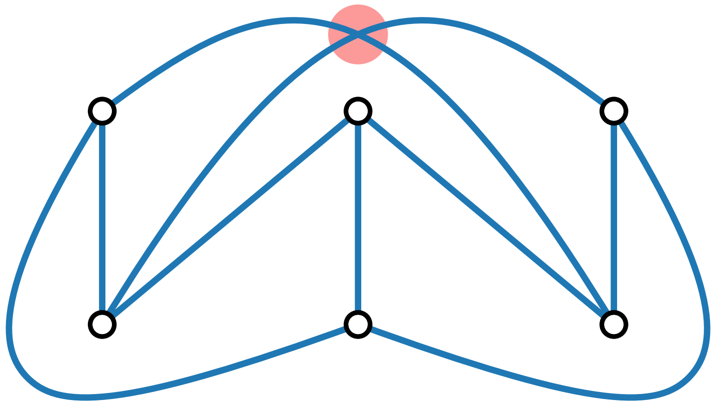

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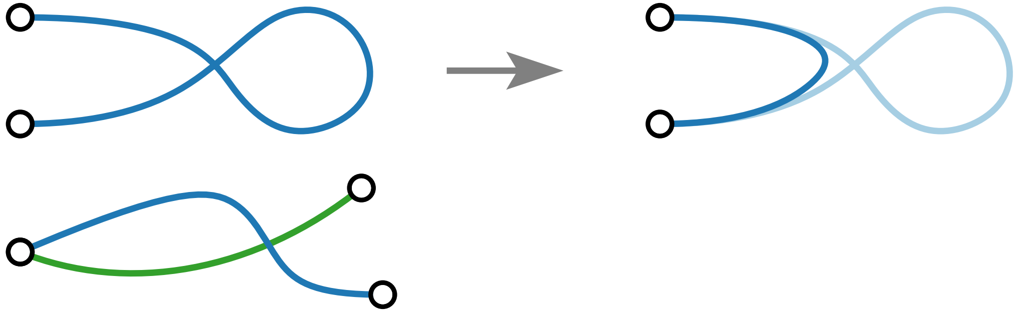

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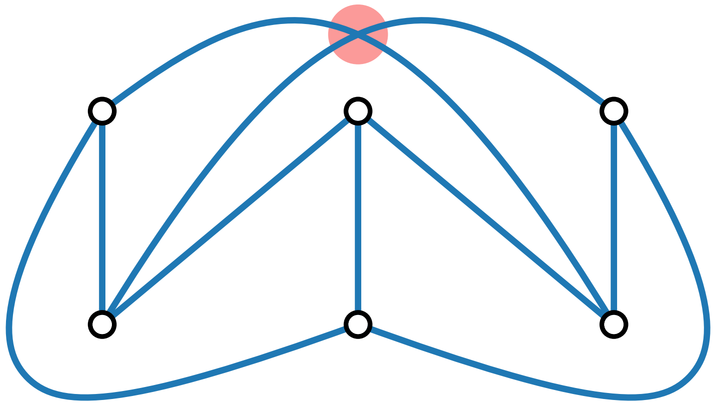

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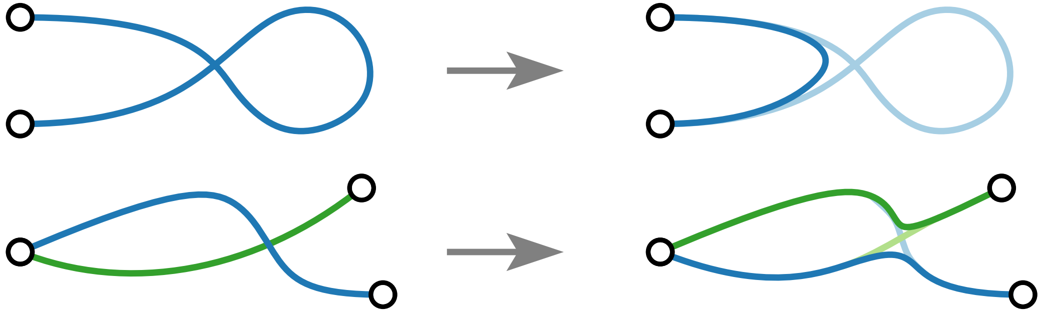

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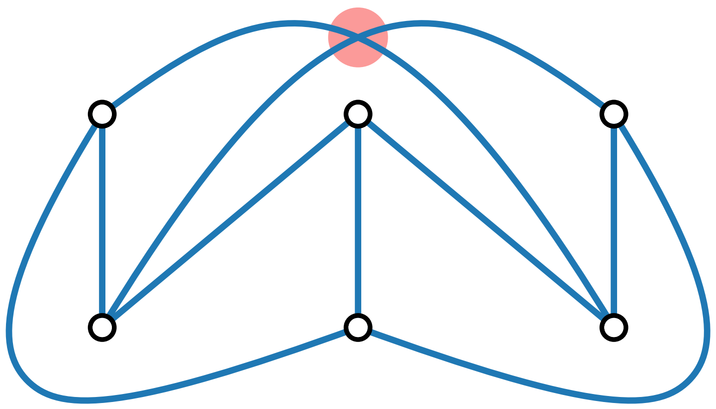

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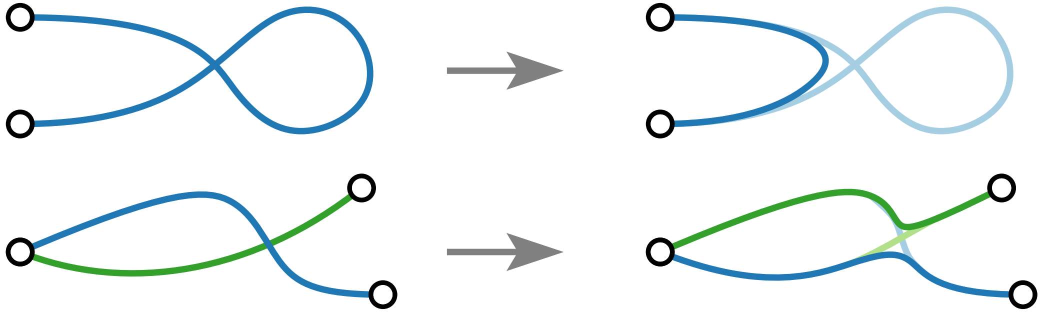

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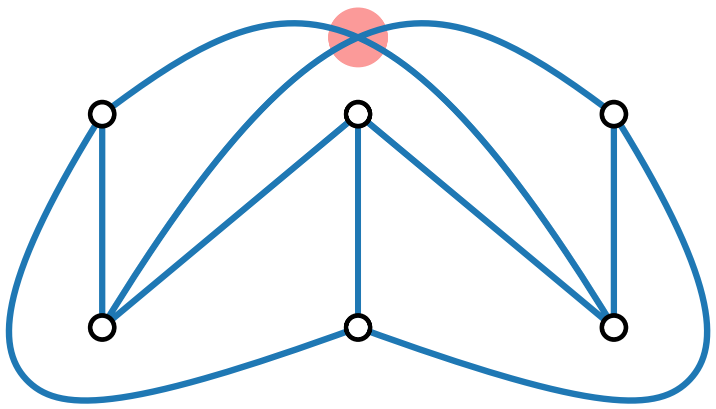

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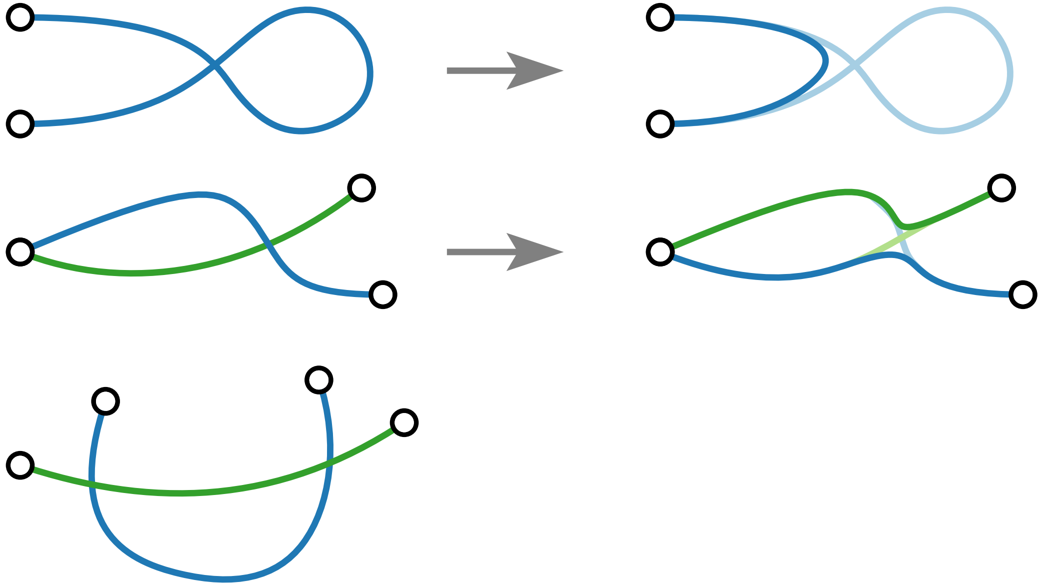

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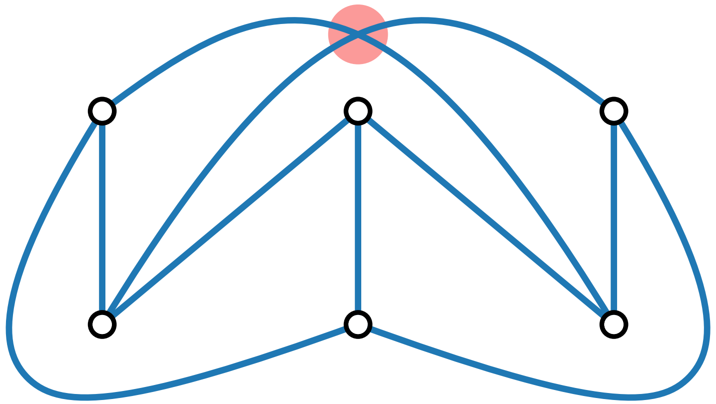

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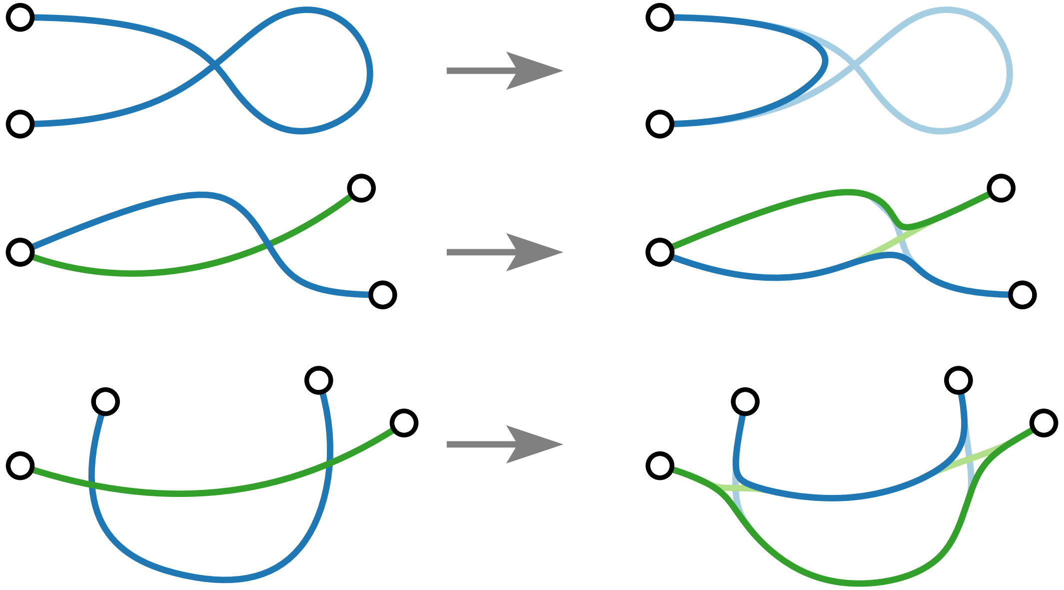

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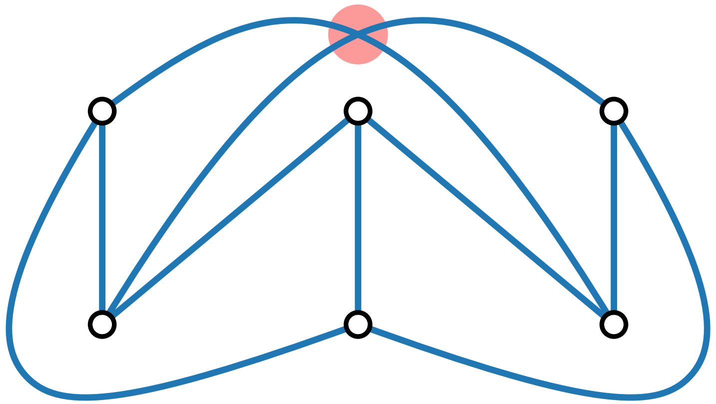

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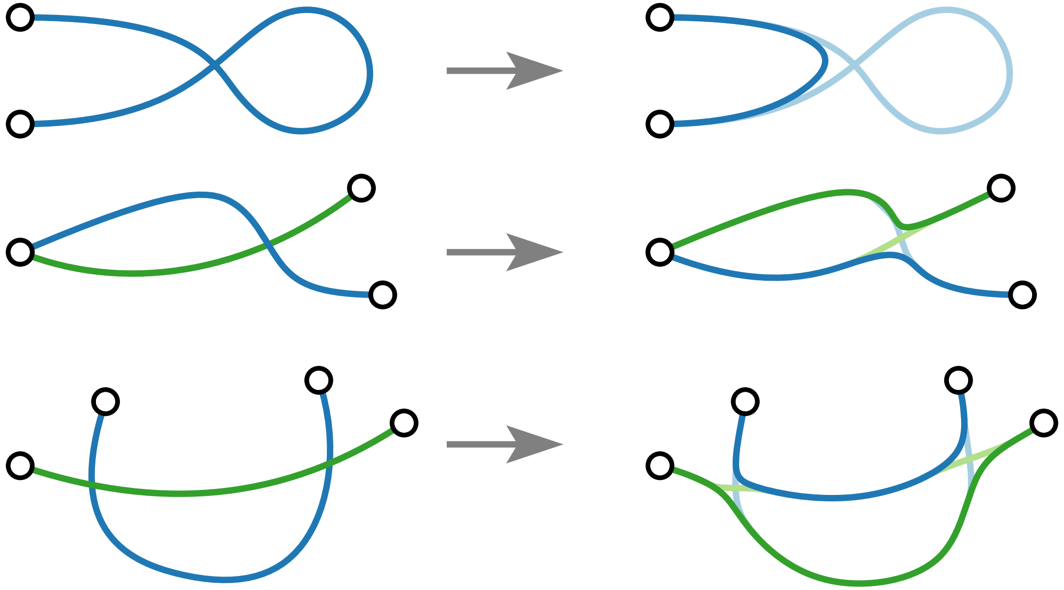

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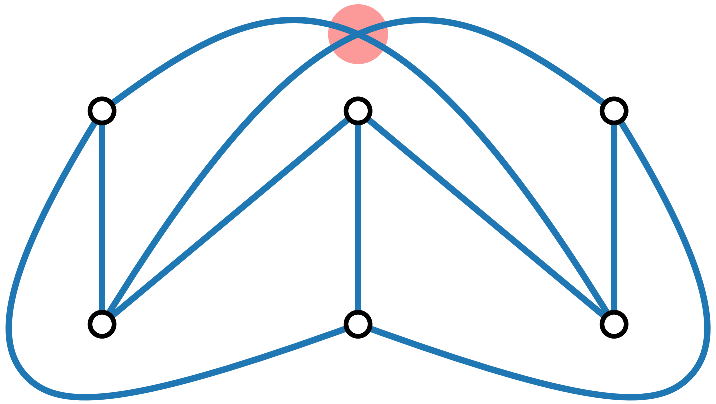

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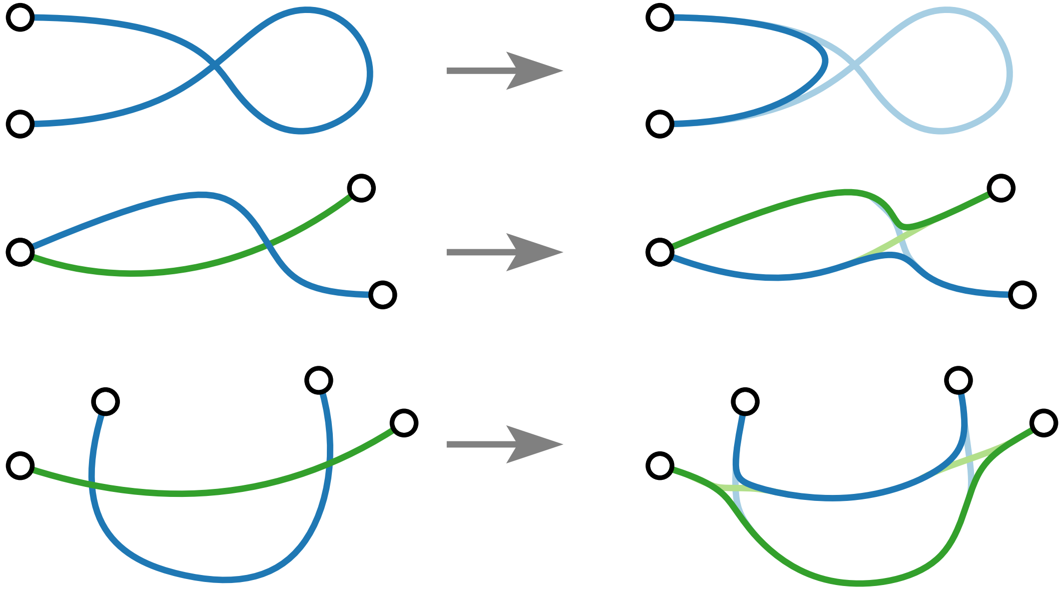

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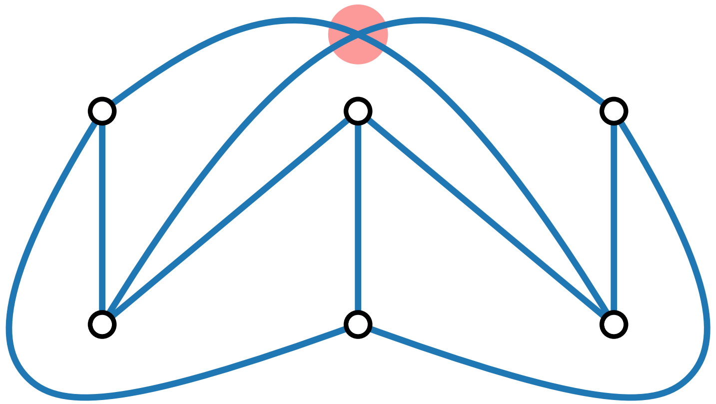

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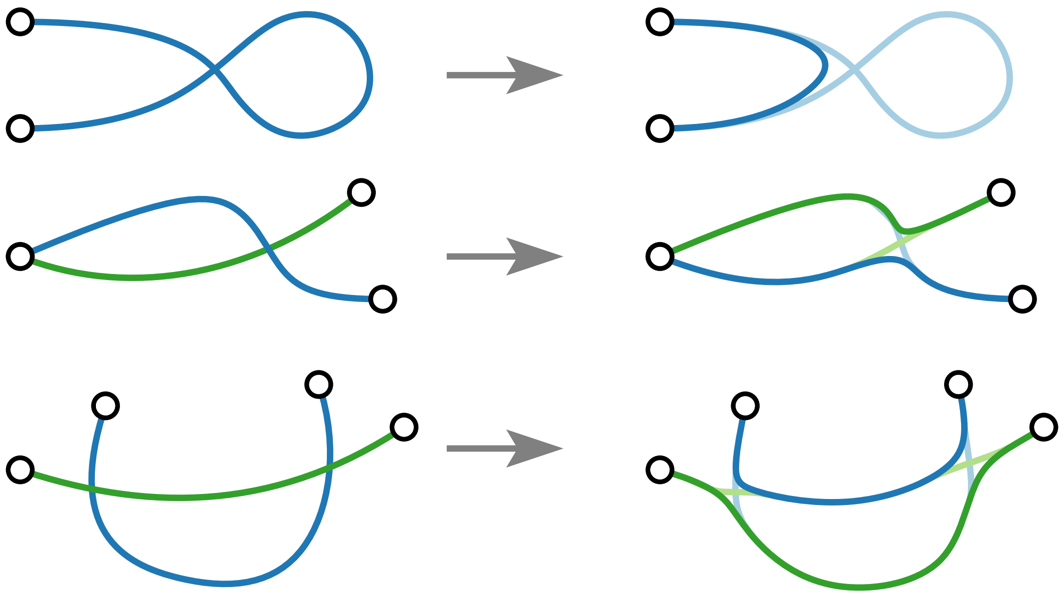

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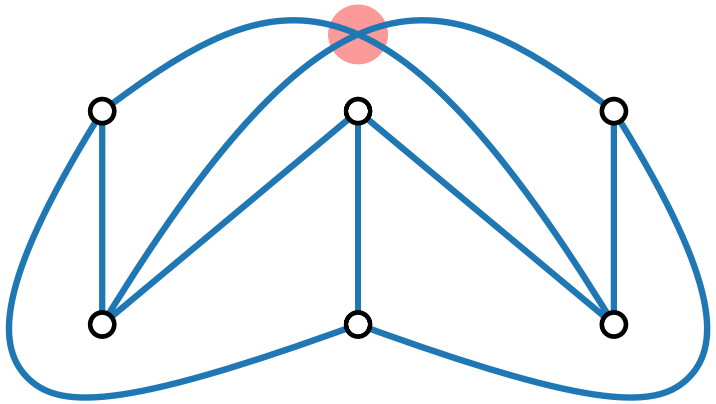

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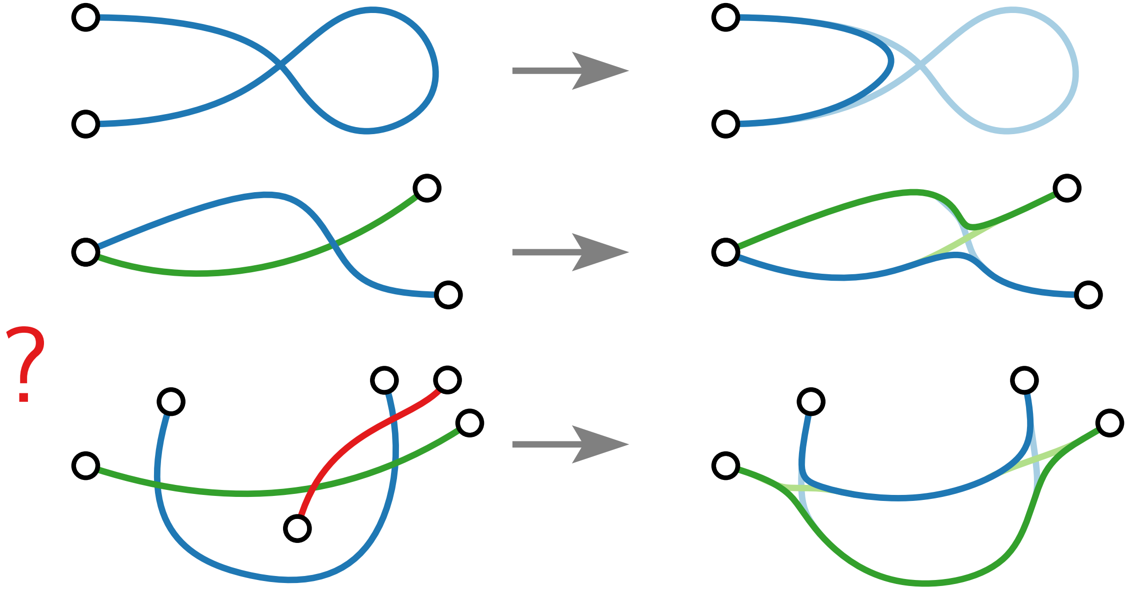

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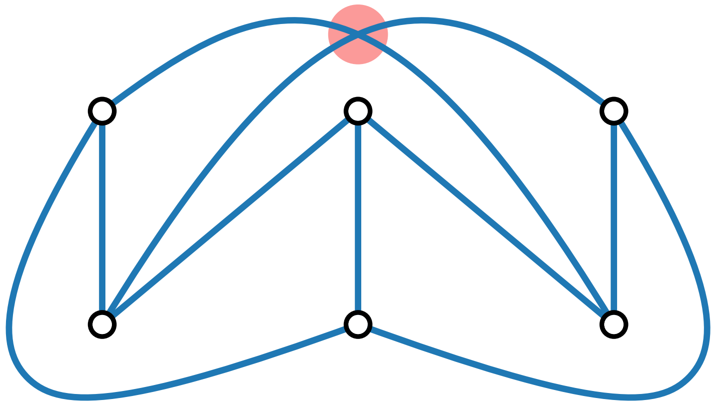

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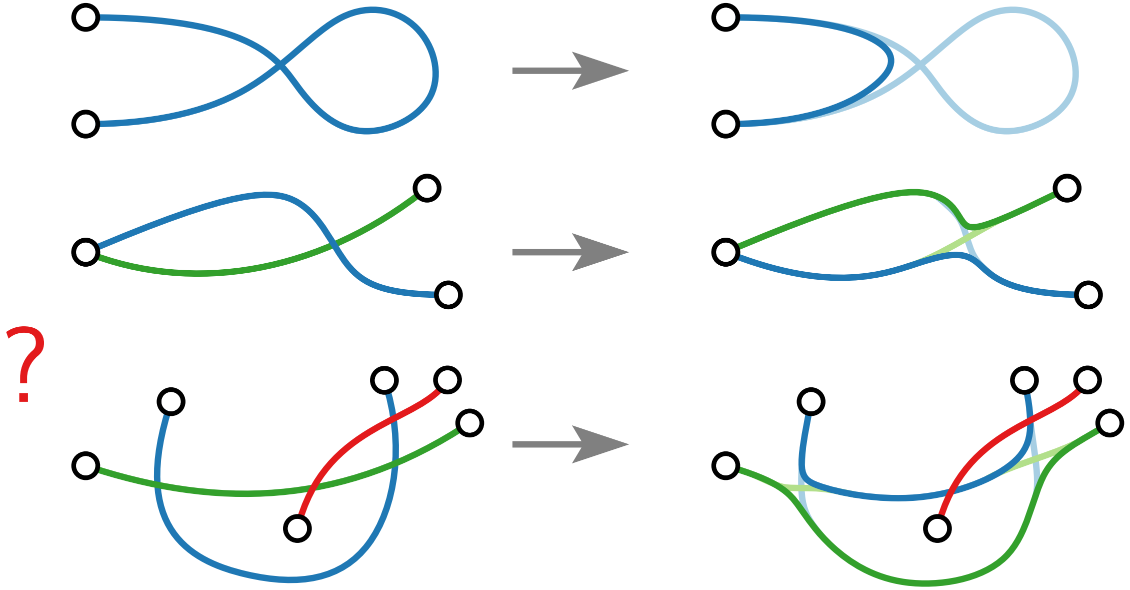

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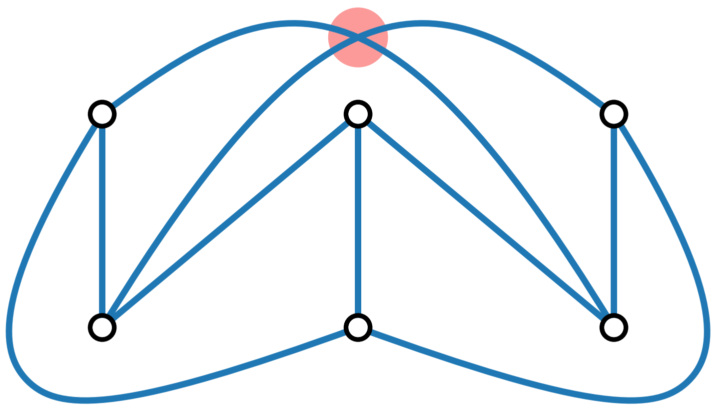

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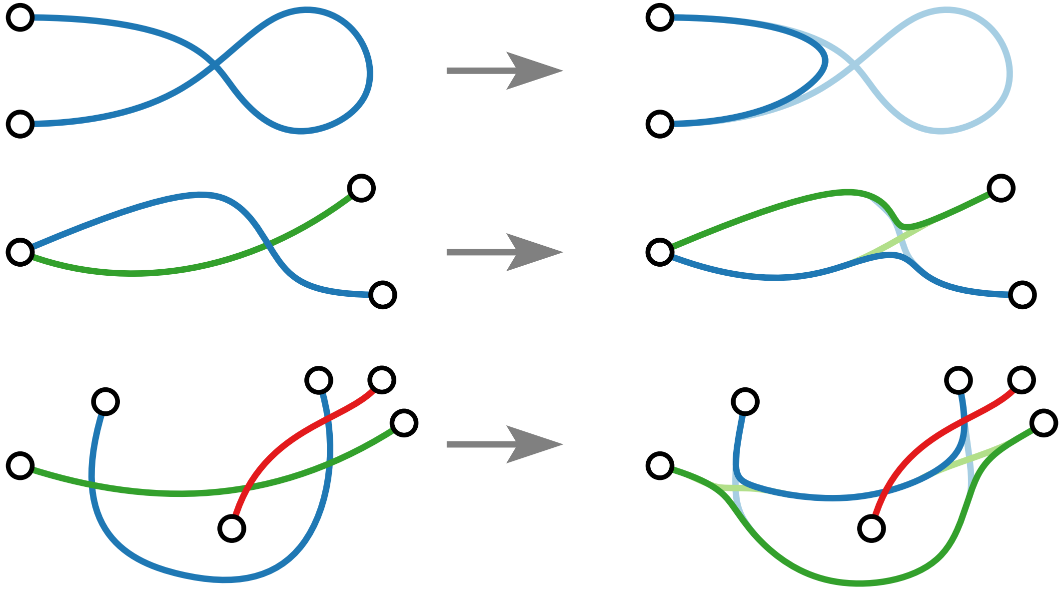

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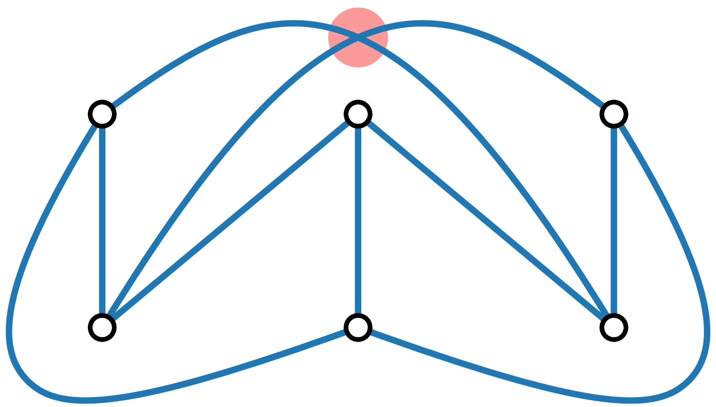

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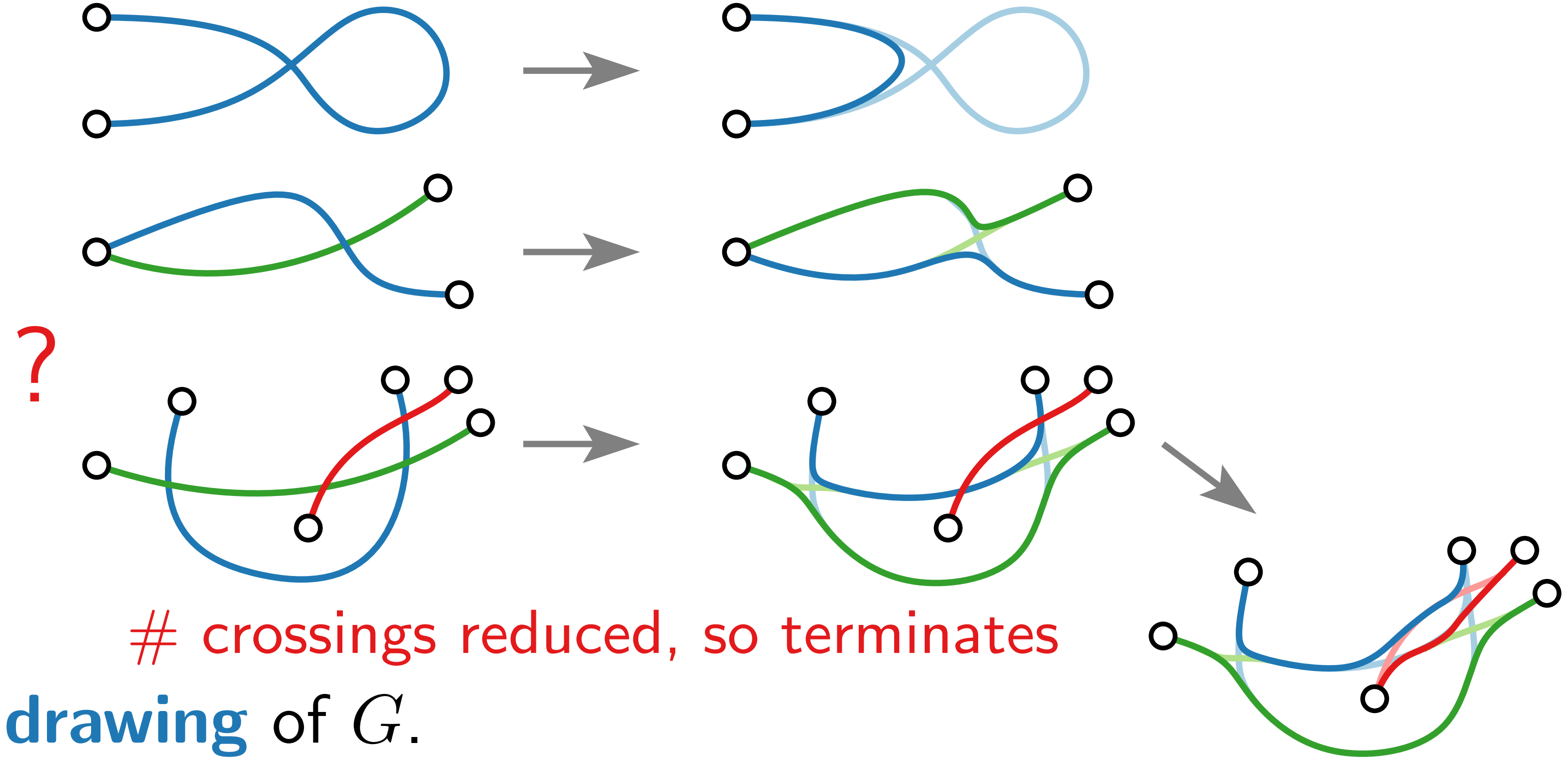

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Theorem. [Hanani '43, Tutte '70]
A graph is planar if and only if it has a drawing in which all pairs of vertex-disjoint edges cross an even number of times.

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Proof sketch.
Hanani showed that every drawing of K_5 and $K_{3,3}$ must have a pair of edges that crosses an odd number of times.

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Every non-planar graph has K_5 or $K_{3,3}$ as a minor, so there are two paths that cross an odd number of times.

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Hence, there must be two edges on these paths that cross an odd number of times. $\square$

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The **odd crossing number** $\mathrm{ocr}(G)$ of G is the smallest number of pairs of edges that cross oddly in a drawing of G.

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Corollary. $\mathrm{ocr}(G) = 0 \Rightarrow \mathrm{cr}(G) = 0$

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Is $\text{ocr}(G) = \text{cr}(G)$?

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Is $\text{ocr}(G) = \text{cr}(G)$? No!

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- For planarization, where we replace crossings with dummy vertices, also only heuristic approaches are known.

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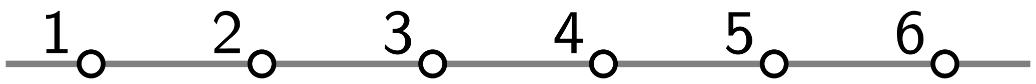

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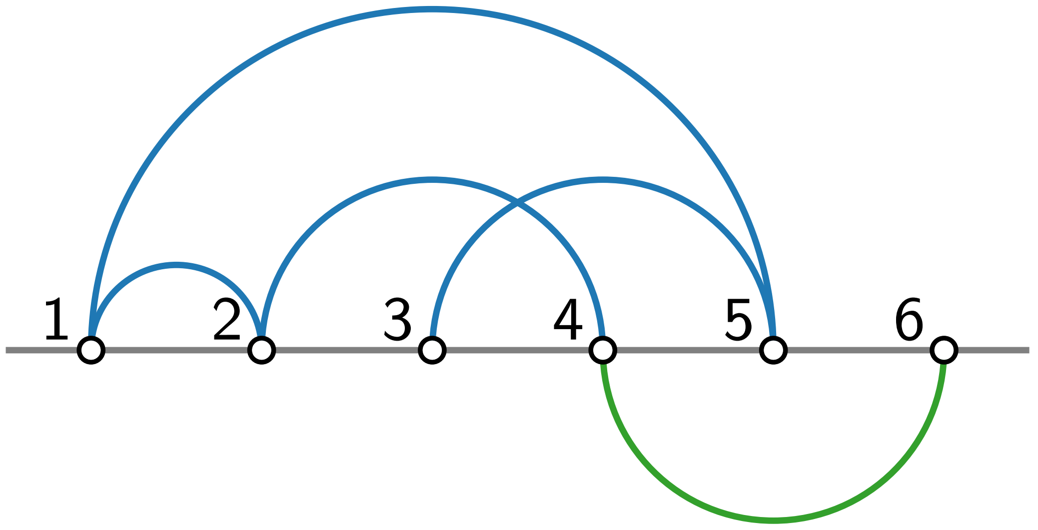

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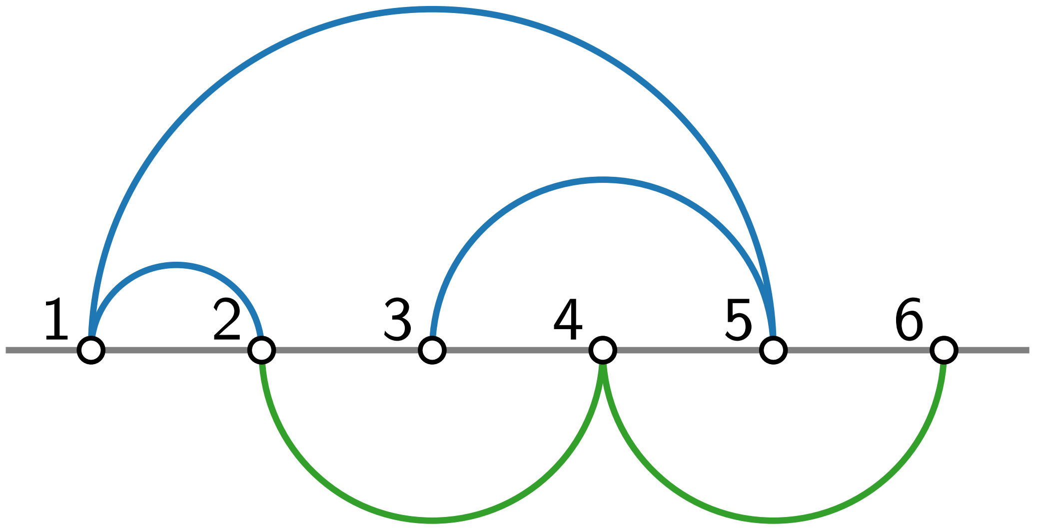

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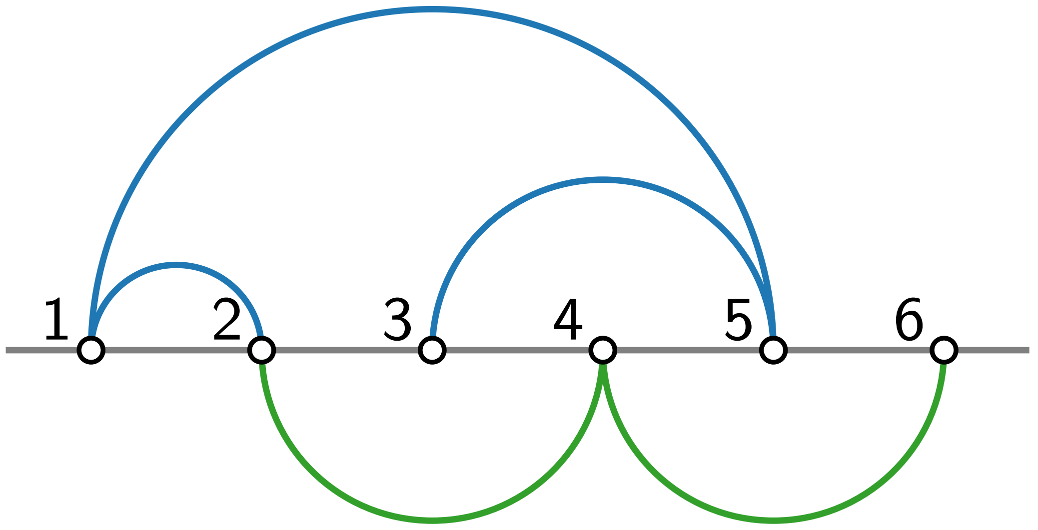

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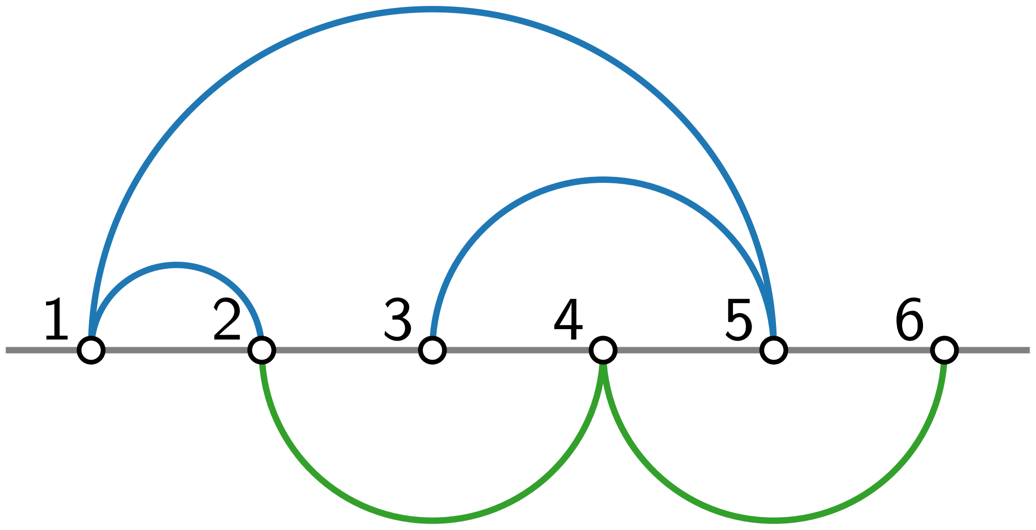

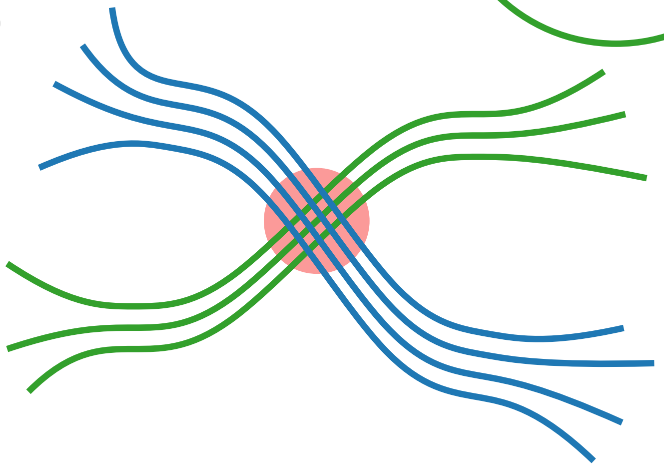

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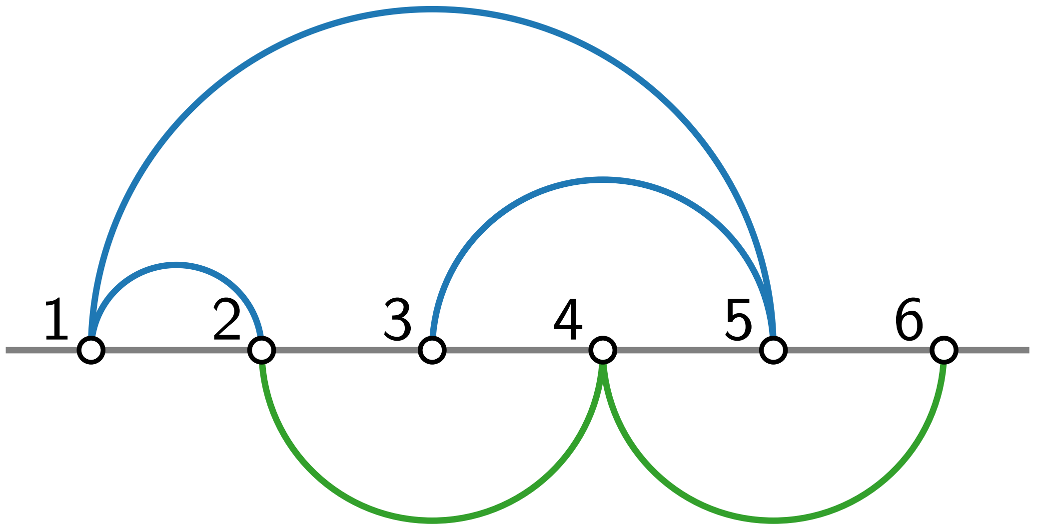

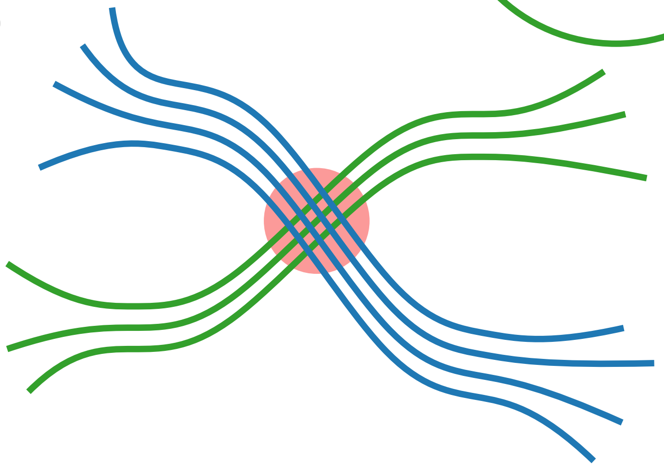

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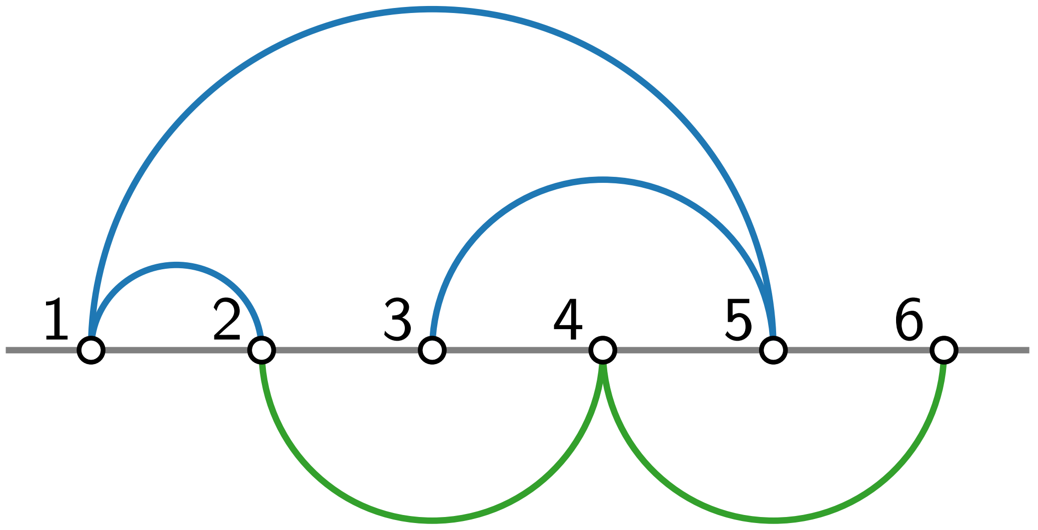

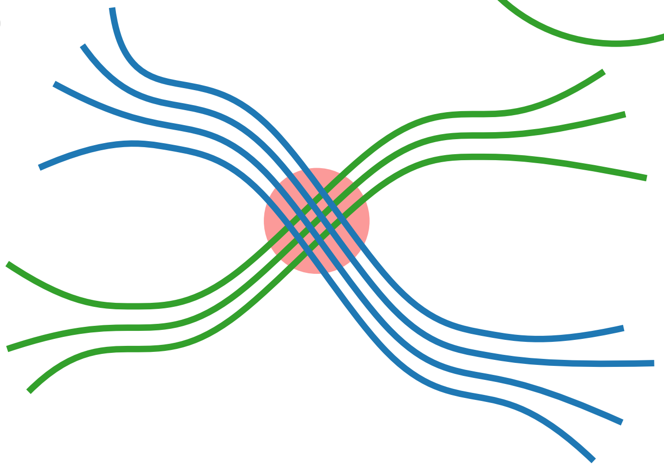

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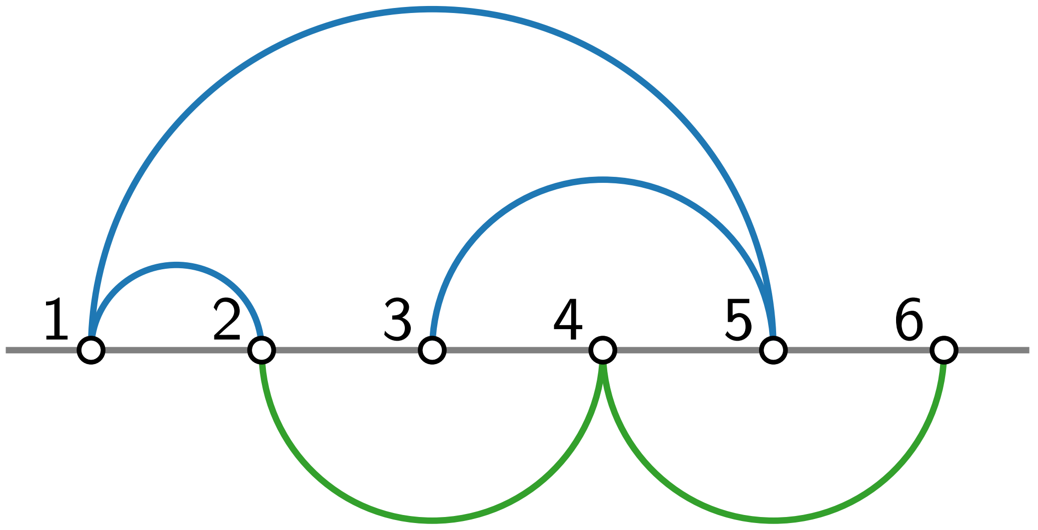

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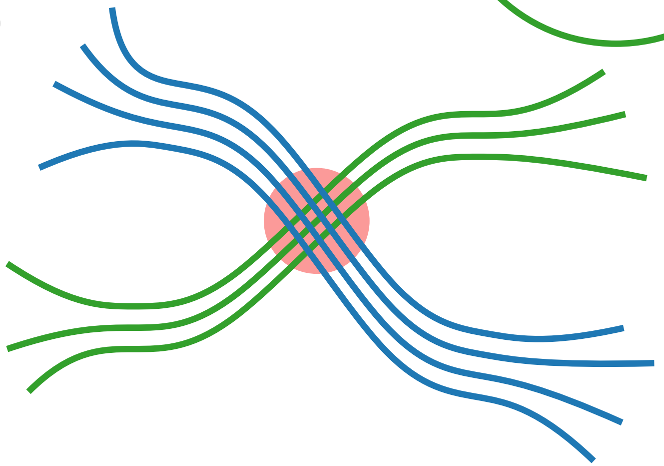

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- Crossing minimization is NP-hard for most variants.

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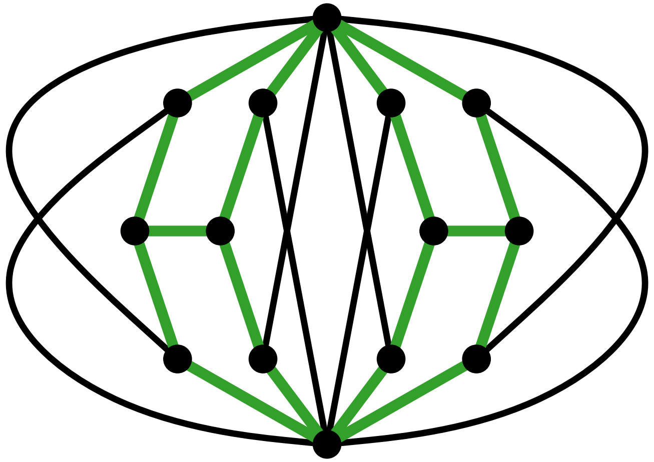

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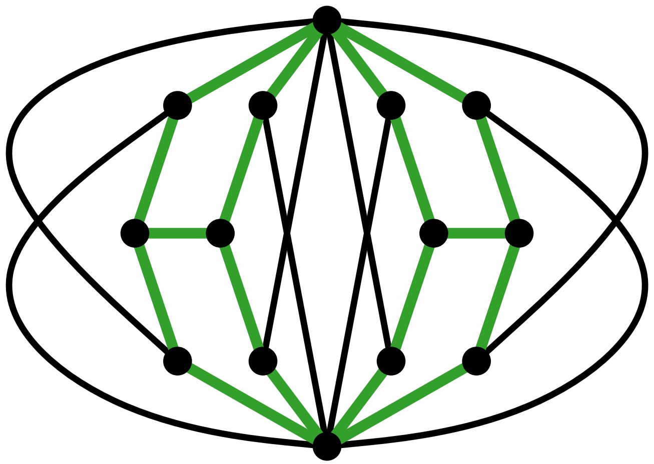

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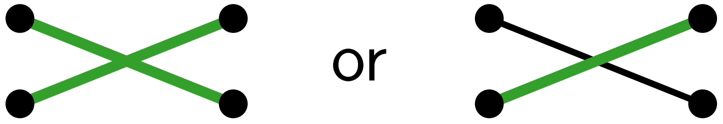

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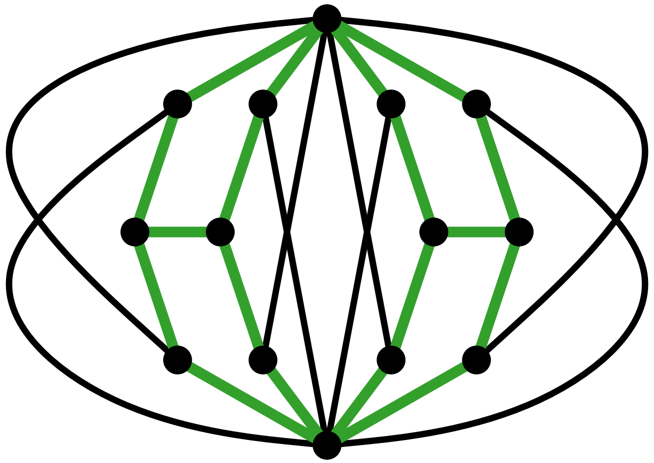

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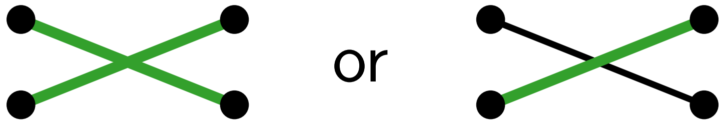
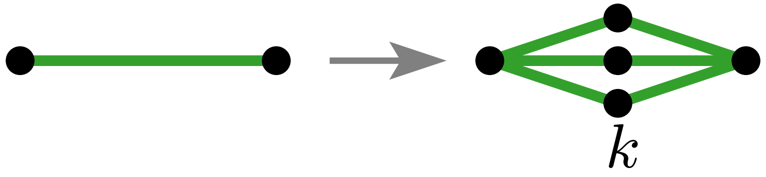


- From G_1 to G_k do



Bounds for Complete Graphs

Theorem. [Guy '60]

$$\mathrm{cr}(K_n) \leq \frac{1}{4} \left\lceil \frac{n}{2} \right\rceil \left\lceil \frac{n-1}{2} \right\rceil \left\lceil \frac{n-2}{2} \right\rceil \left\lceil \frac{n-3}{2} \right\rceil = \frac{3}{8} \binom{n}{4} + O(n^3)$$

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Turán's brick factory problem (1944)

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Exact numbers are known for $n \leq 27$.

Bounds for Complete Graphs

Theorem. **Conjecture.** [Guy '60]

$$\mathrm{cr}(K_n) \not\leq \underline{\underline{}} \frac{1}{4}\left\lceil\frac{n}{2}\right\rceil\left\lceil\frac{n-1}{2}\right\rceil\left\lceil\frac{n-2}{2}\right\rceil\left\lceil\frac{n-3}{2}\right\rceil = \frac{3}{8}\binom{n}{4} + O(n^3)$$

Bound is tight for $n \leq 12$.

Theorem. **Conjecture.** [Zarankiewicz '54, Urbaník '55]

$$\mathrm{cr}(K_{m,n}) \not\leq \underline{\underline{}} \frac{1}{4}\left\lceil\frac{n}{2}\right\rceil\left\lceil\frac{n-1}{2}\right\rceil\left\lceil\frac{m}{2}\right\rceil\left\lceil\frac{m-1}{2}\right\rceil$$

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Check out `http://www.ist.tugraz.at/staff/aichholzer/crossings.html`

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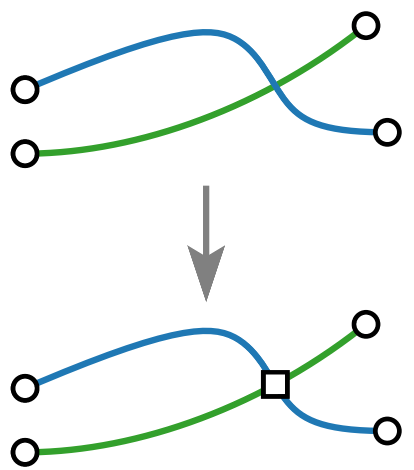

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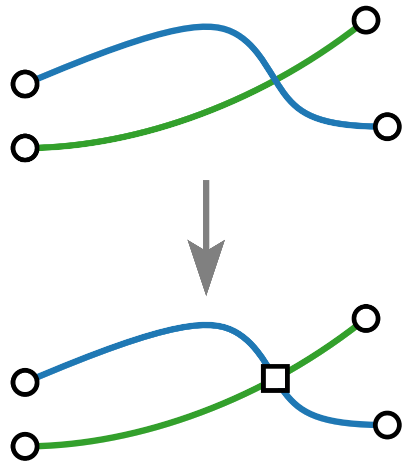

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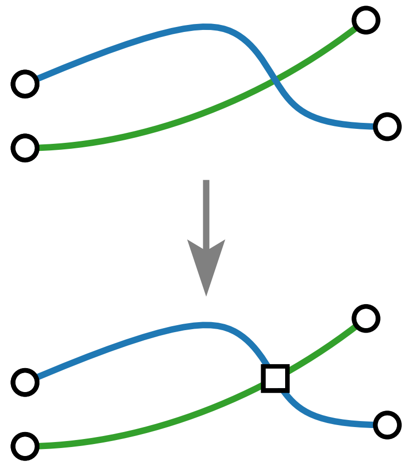

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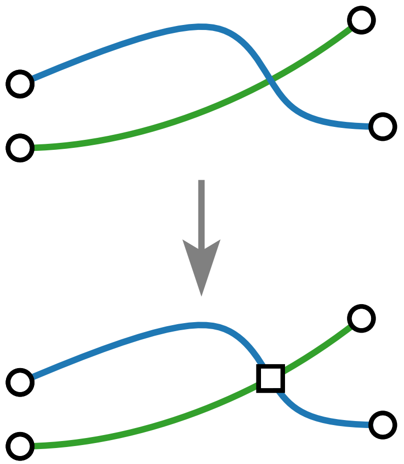

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- Let n_p, m_p, X_p be the random variables counting the numbers of vertices / edges / crossings of Γ_p, resp.
- By Lemma 2, $\mathrm{cr}(G_p) - m_p + 3n_p \geq 6$.
  $\Rightarrow \mathrm{E}(X_p - m_p + 3n_p) \geq 0$.
- $\mathrm{E}(n_p) = pn$ and $\mathrm{E}(m_p) = p^2 m$
- $\mathrm{E}(X_p) = p^4 \mathrm{cr}(G)$
- $0 \leq \mathrm{E}(X_p) - \mathrm{E}(m_p) + 3\mathrm{E}(n_p)$
  $= p^4 \mathrm{cr}(G) - p^2 m + 3pn$
- $\mathrm{cr}(G) \geq \frac{p^2 m - 3pn}{p^4} = \frac{m}{p^2} - \frac{3n}{p^3}$
- Set $p = \frac{4n}{m}$.
- $\mathrm{cr}(G) \geq \frac{m^3}{16n^2} - \frac{3m^3}{64n^2} = \frac{1}{64} \frac{m^3}{n^2}$ $\square$

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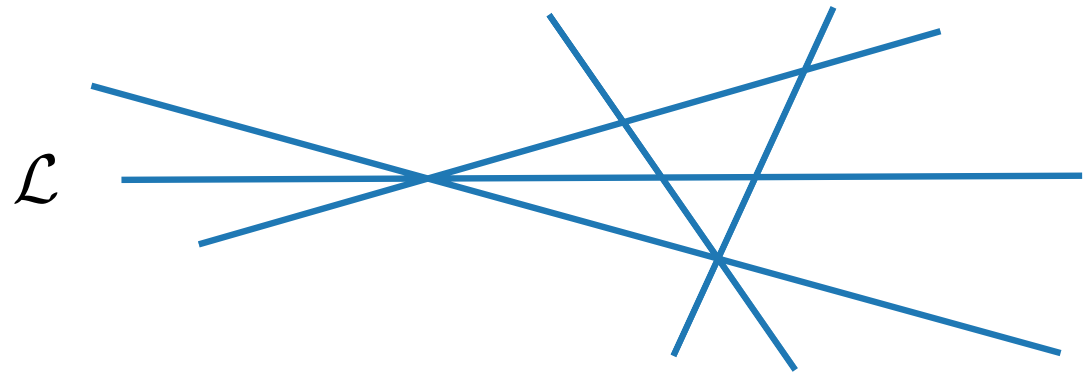

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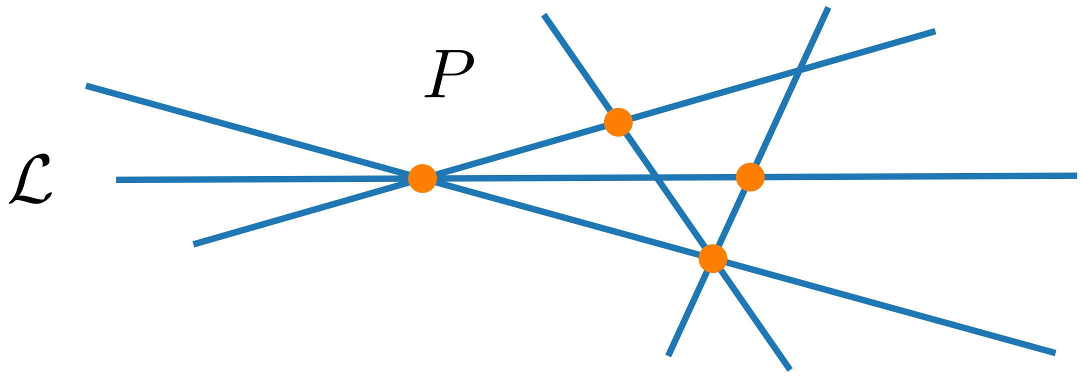

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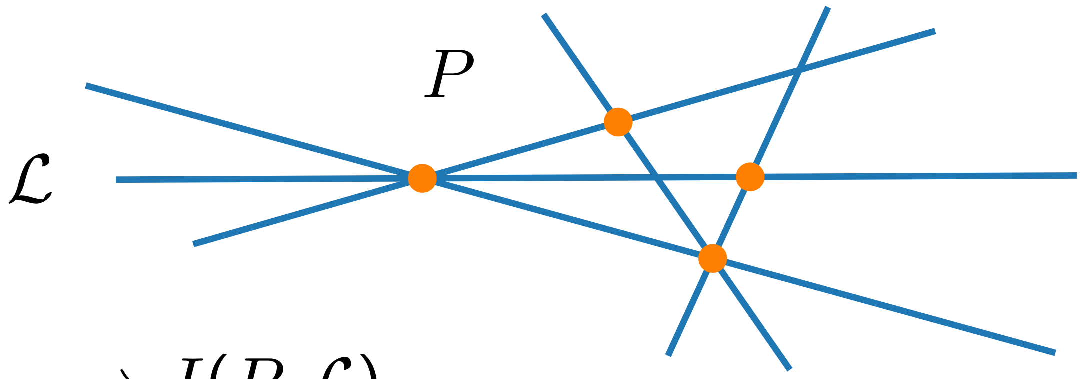

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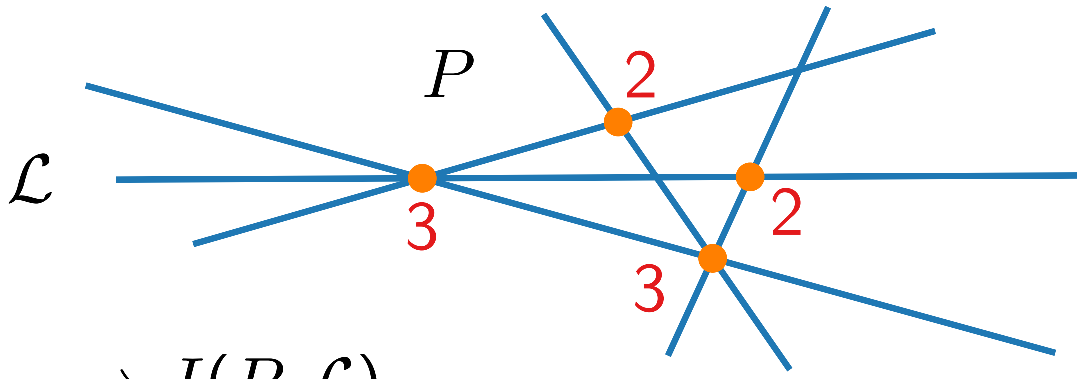

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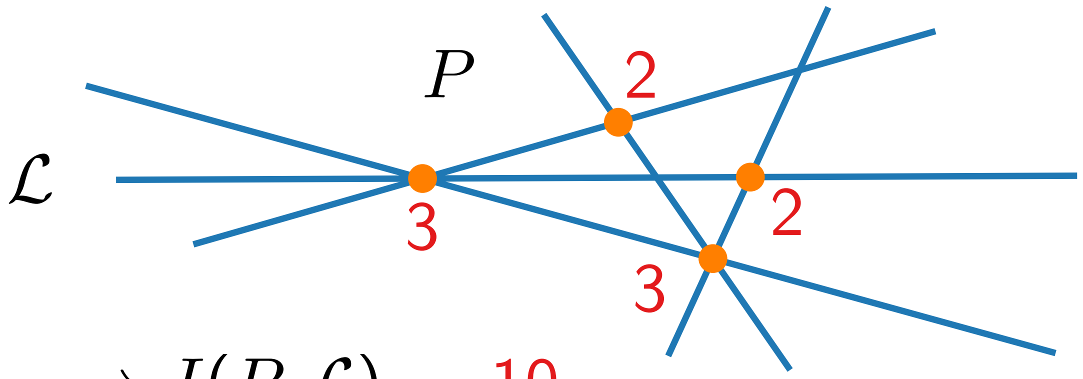

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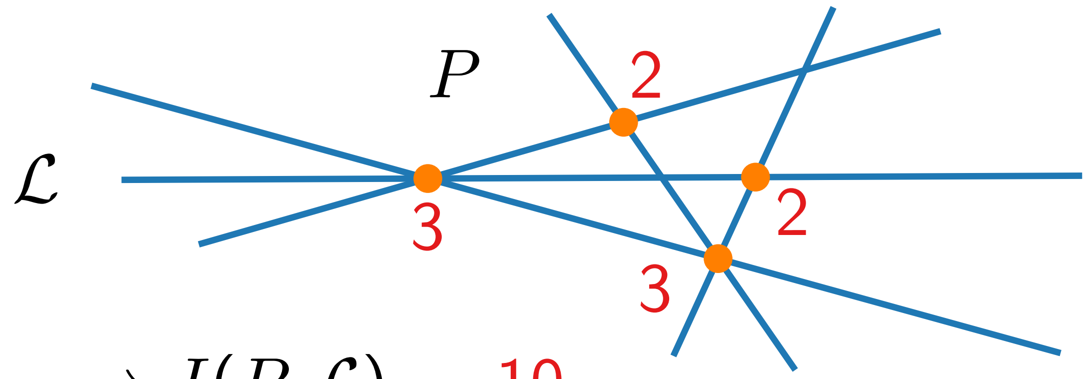

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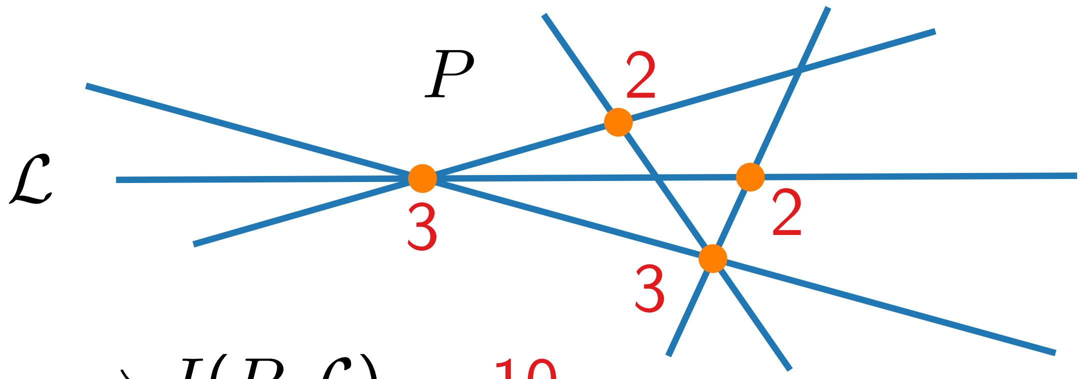

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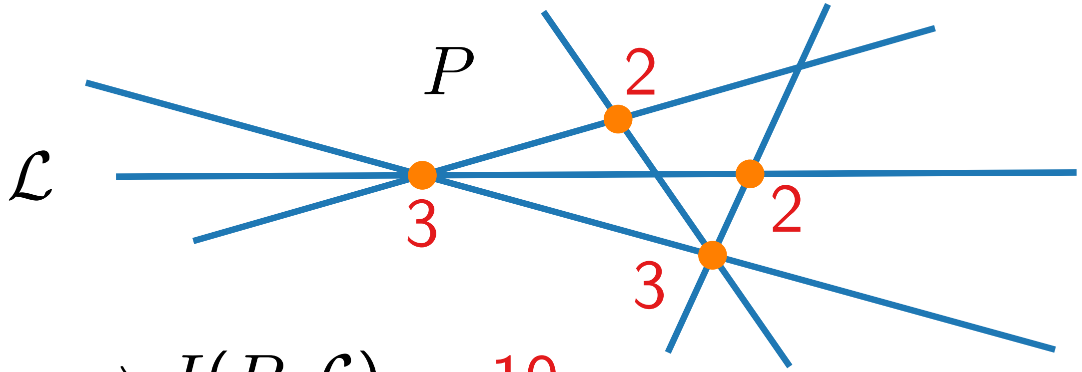

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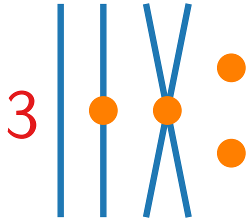

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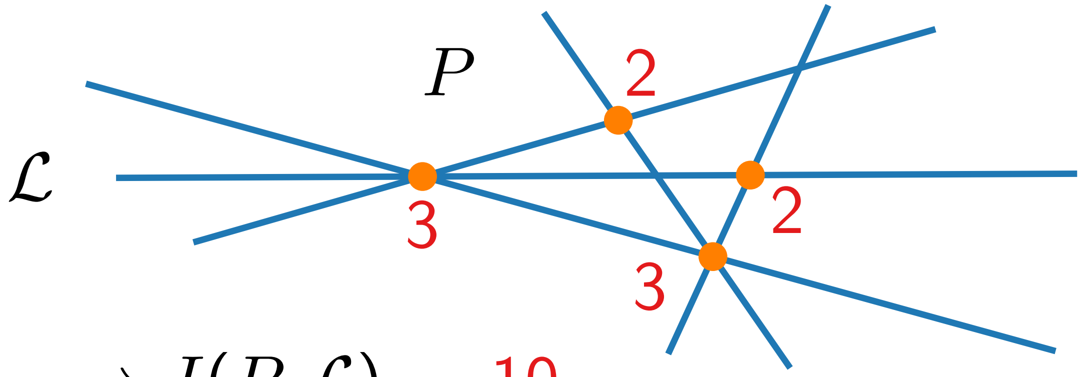

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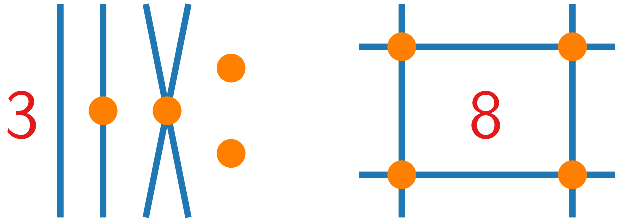

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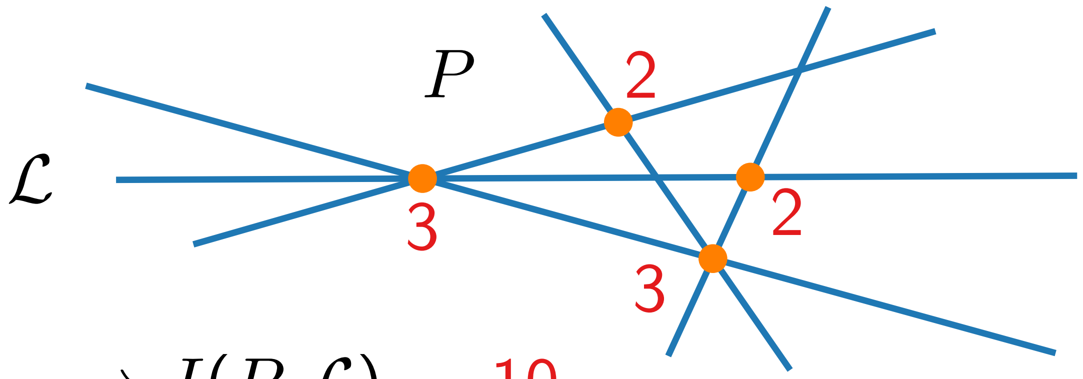

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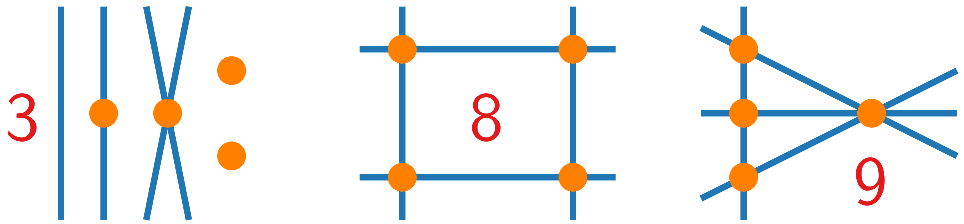

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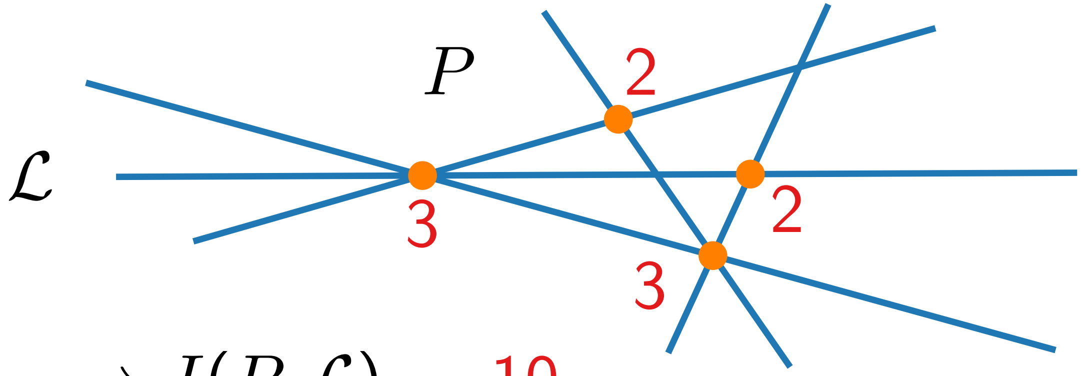

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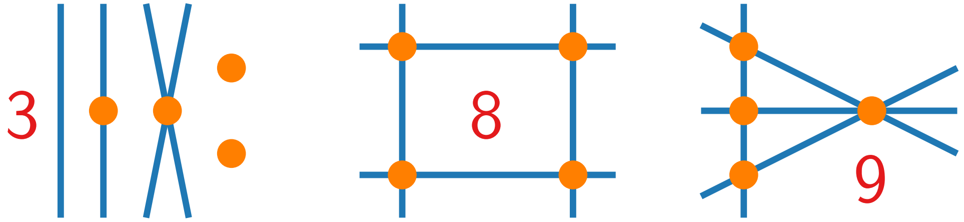

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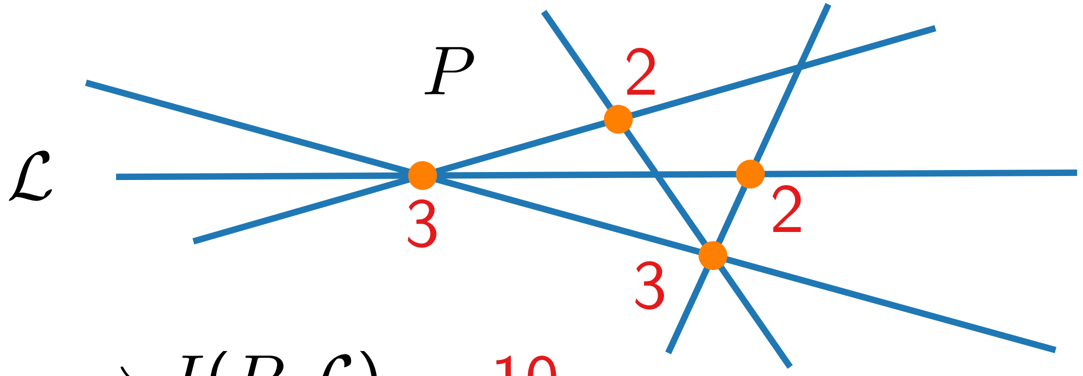

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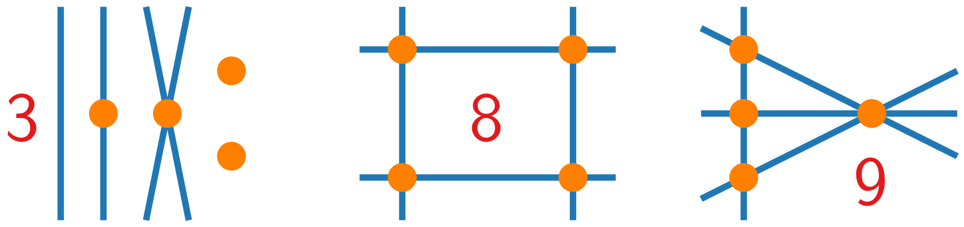

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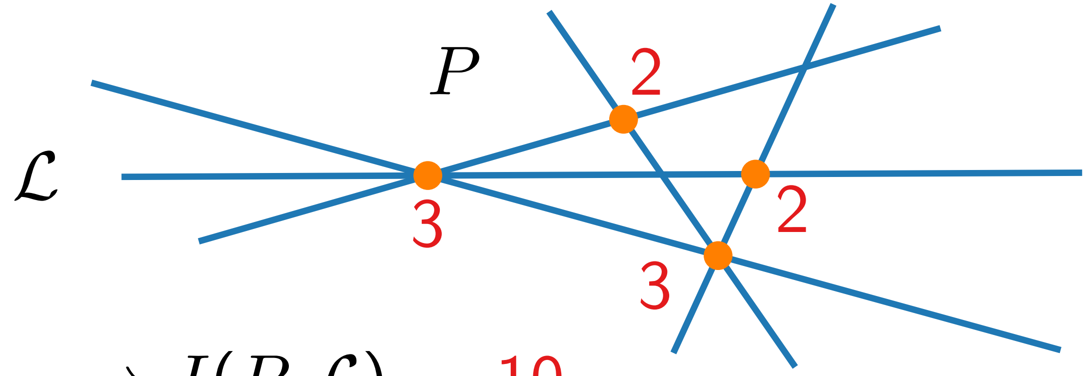

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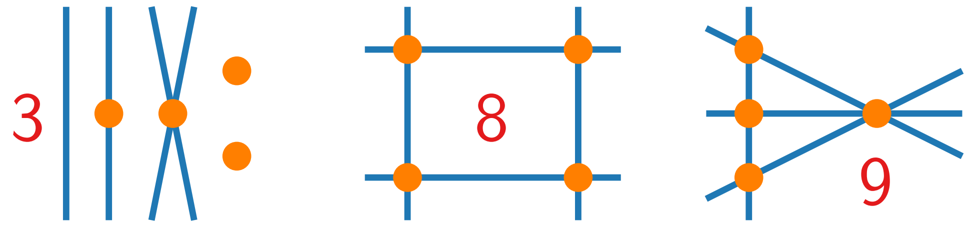

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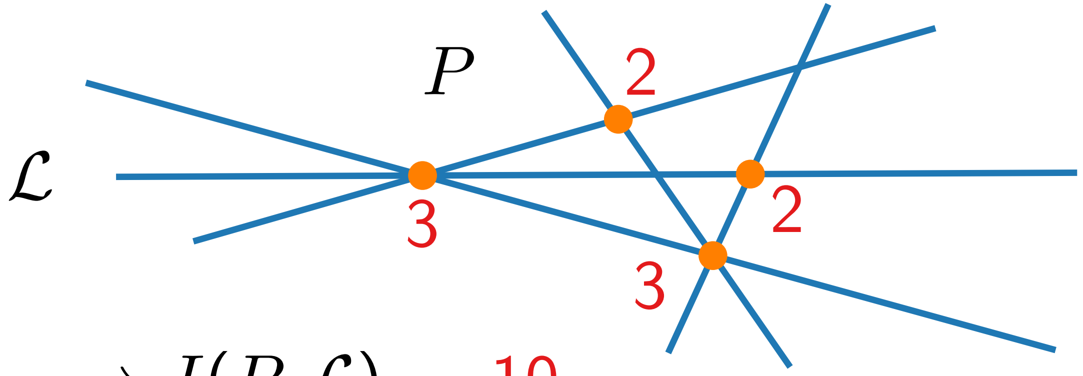

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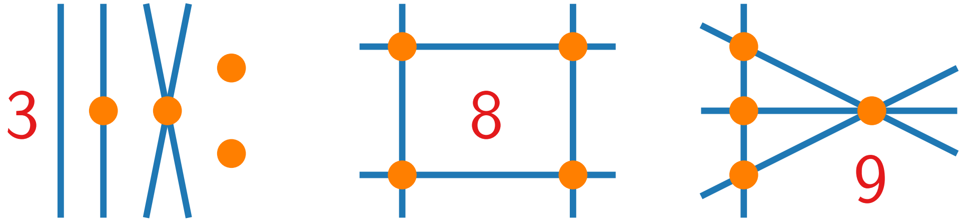

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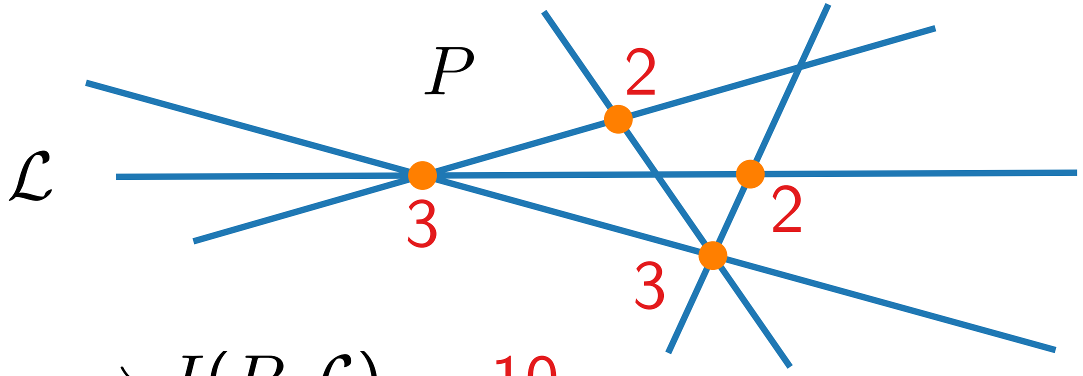

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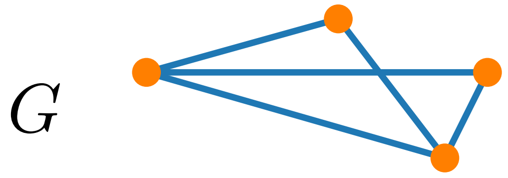

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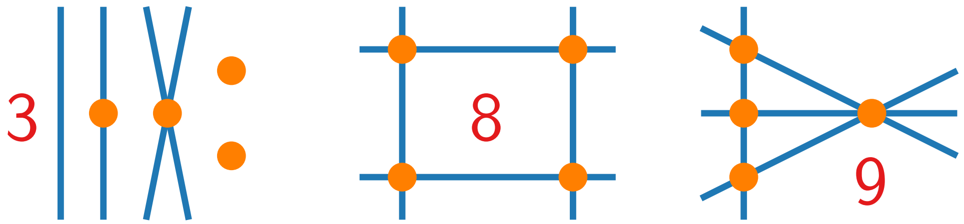

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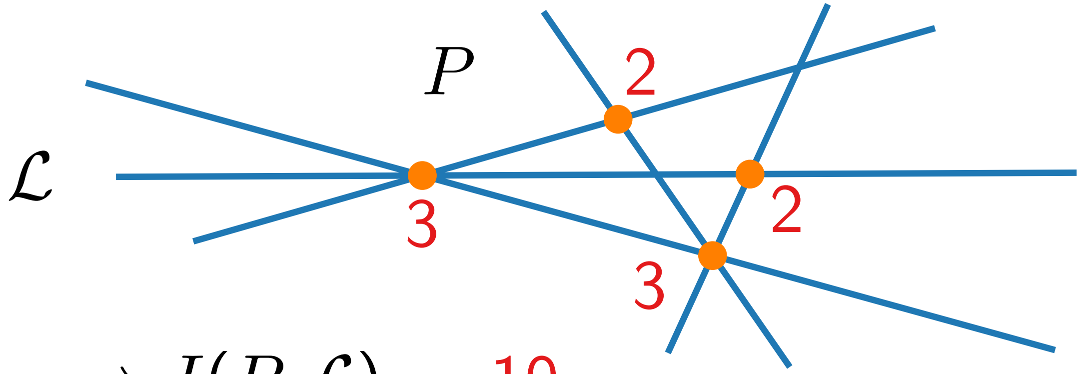

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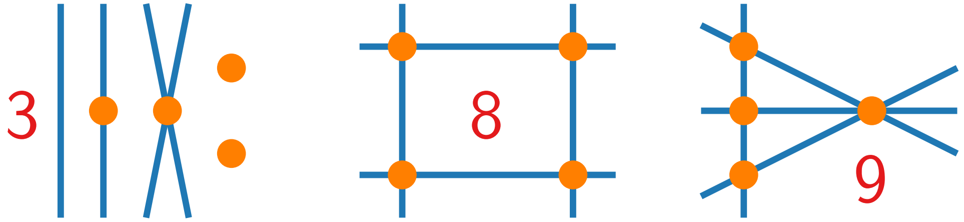

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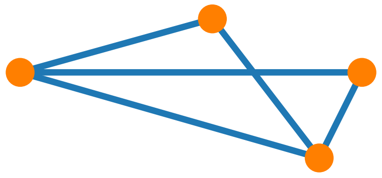

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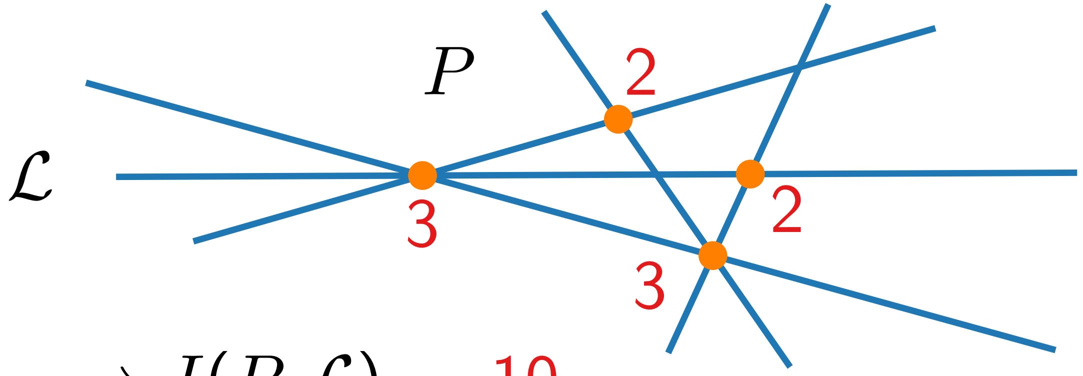

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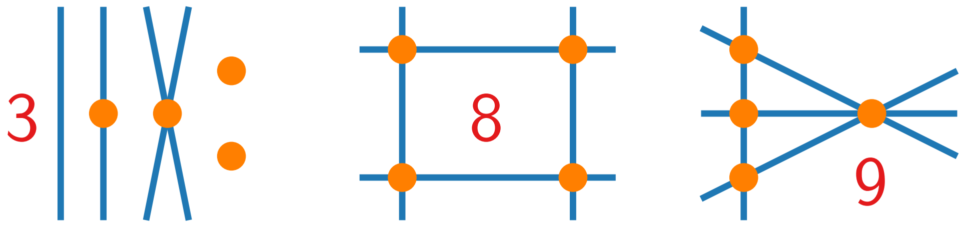

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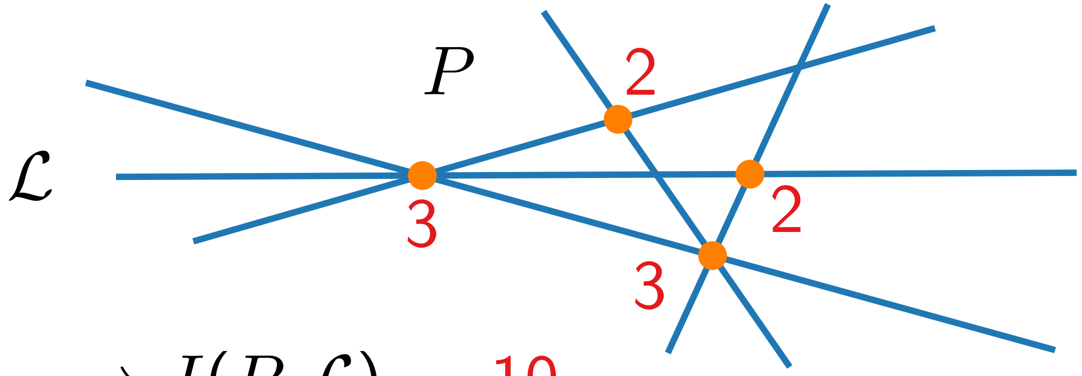

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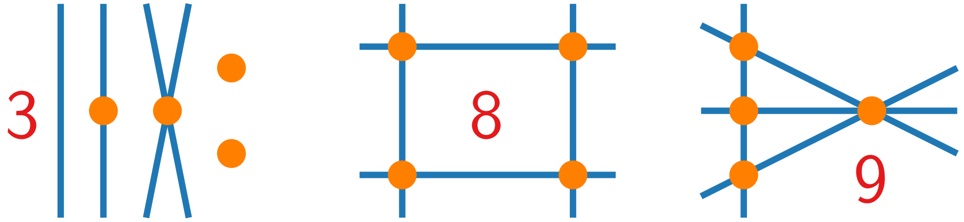

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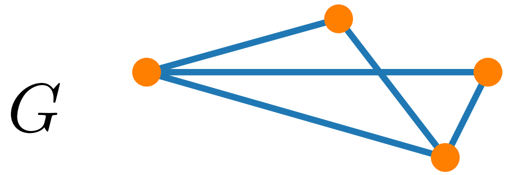

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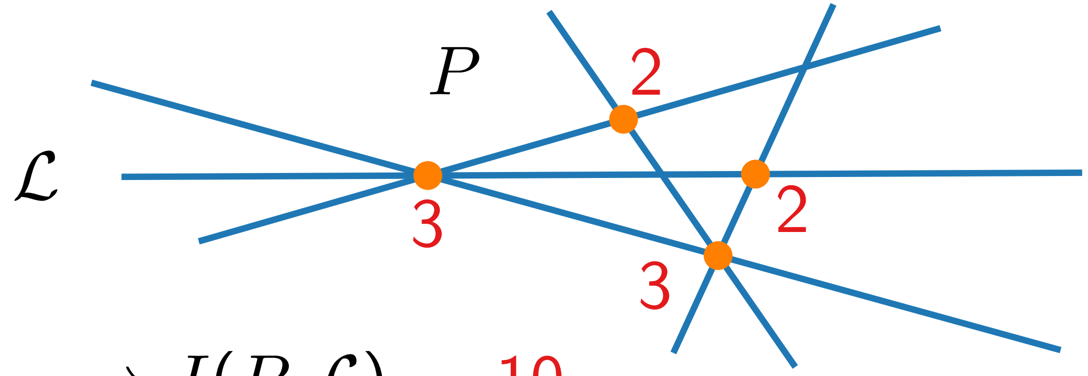

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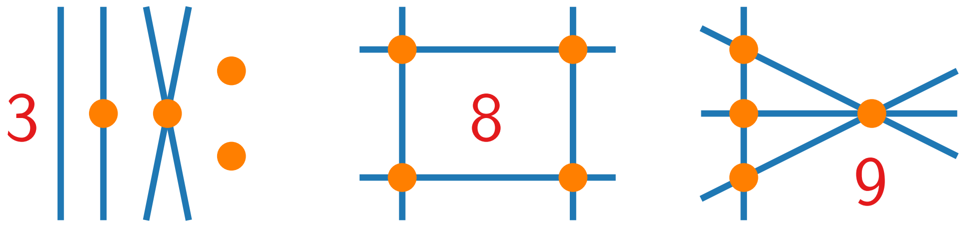

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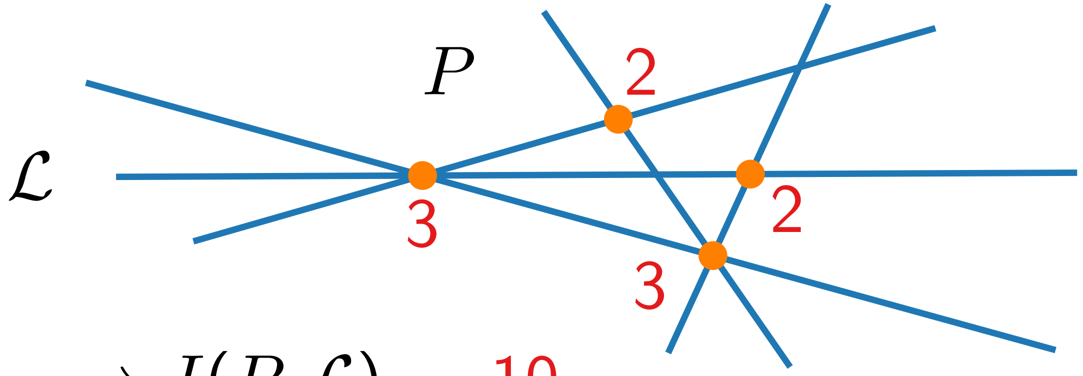

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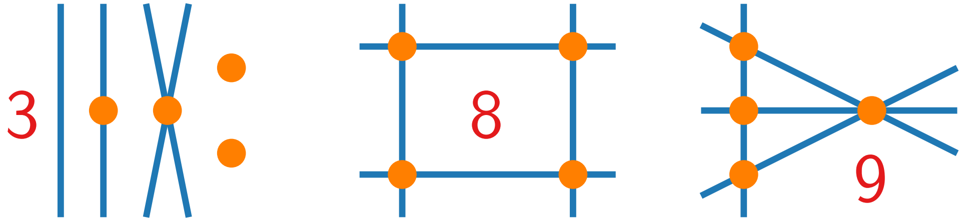

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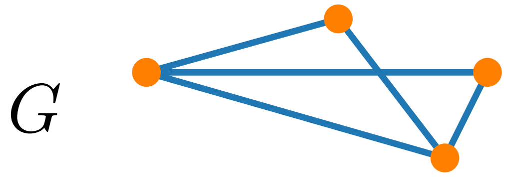

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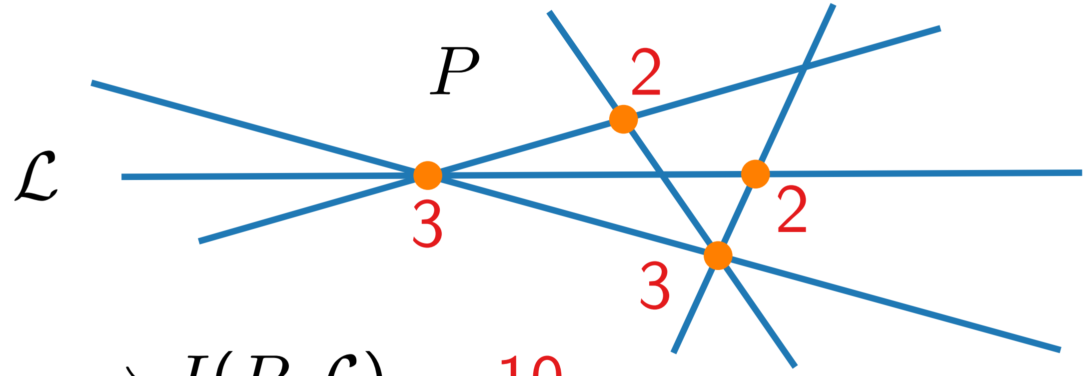

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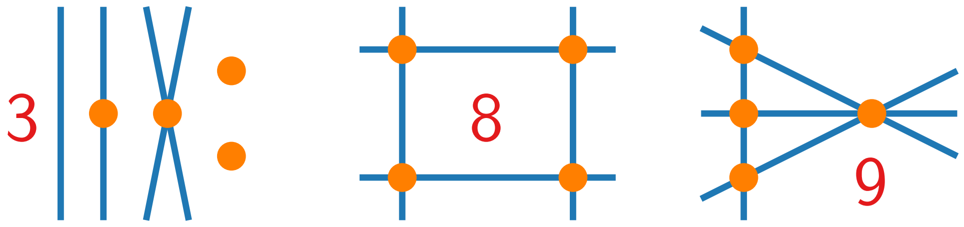

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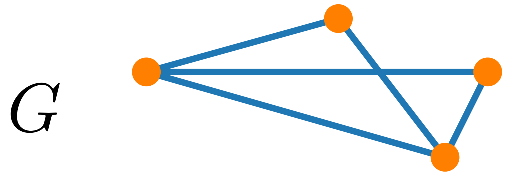

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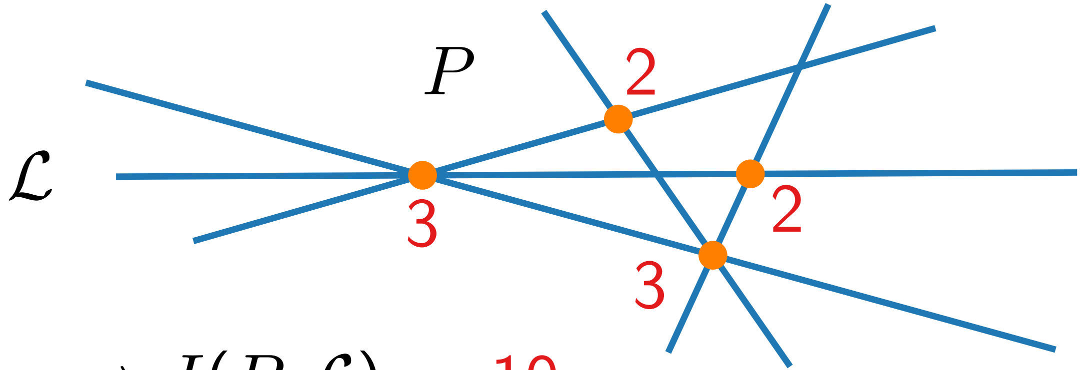

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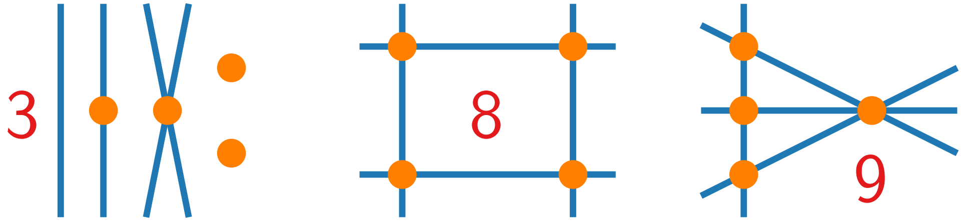

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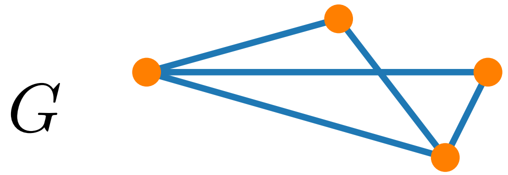

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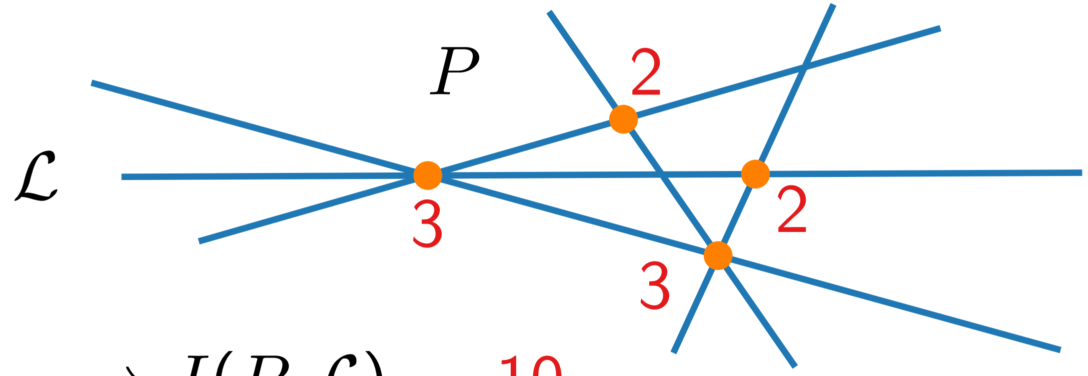

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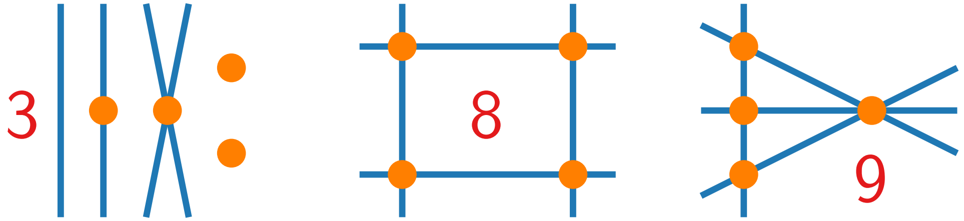

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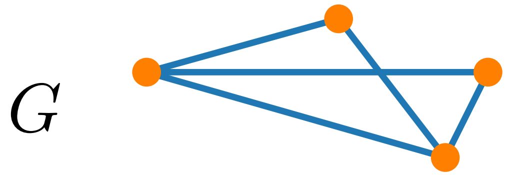

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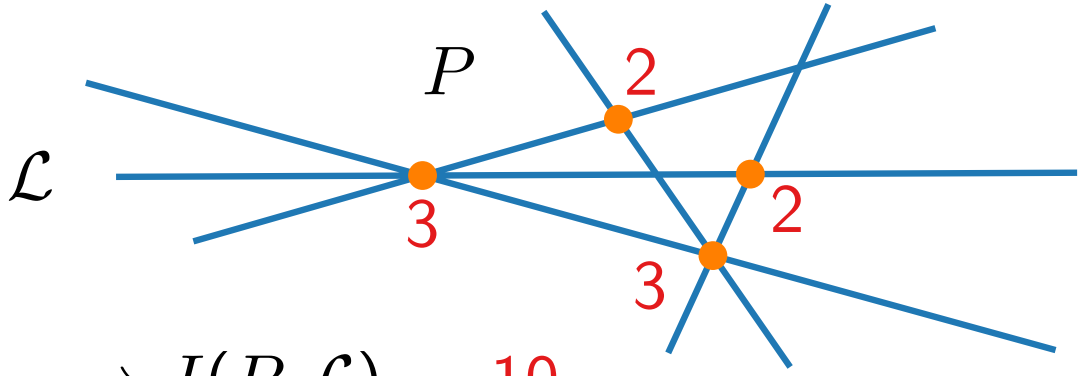

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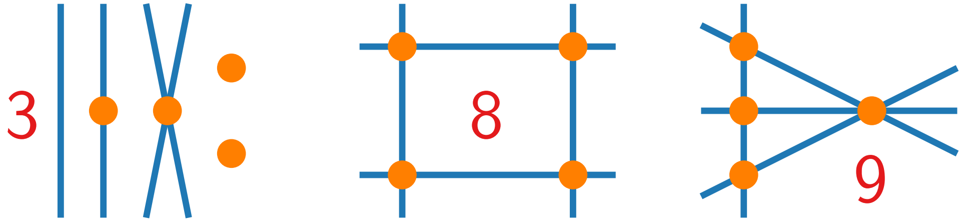

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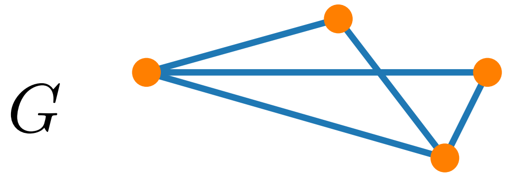

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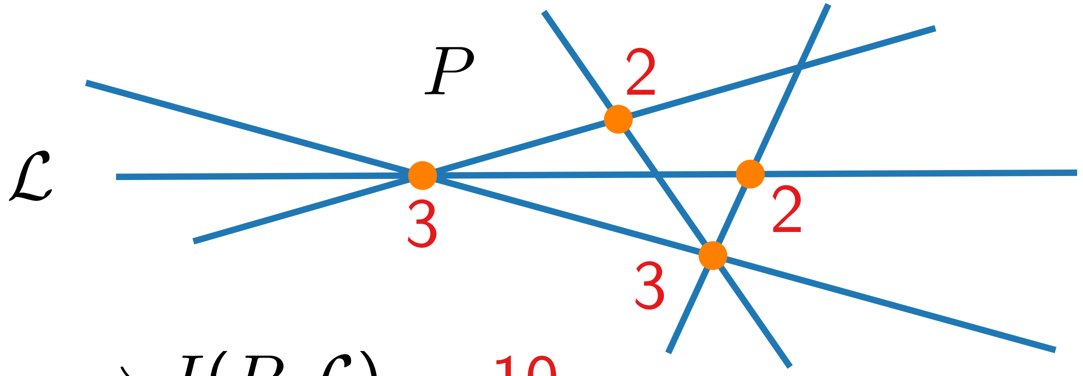

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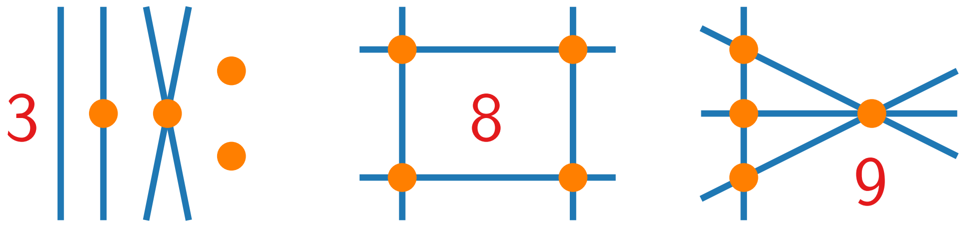

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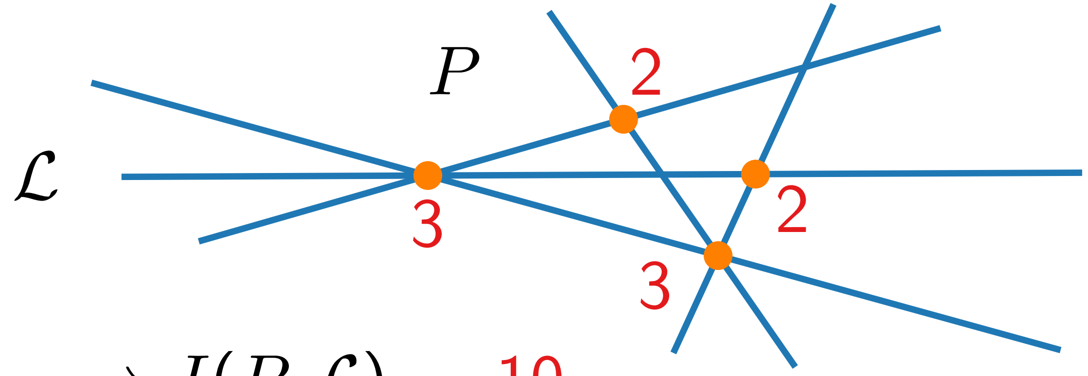

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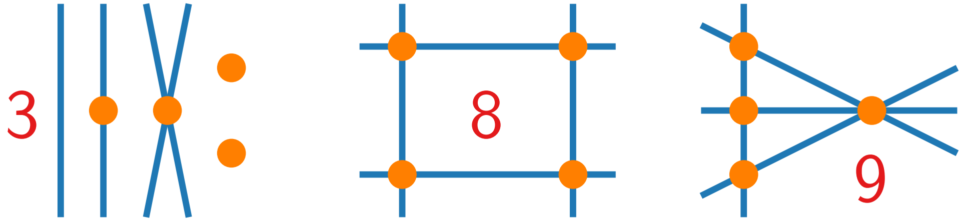

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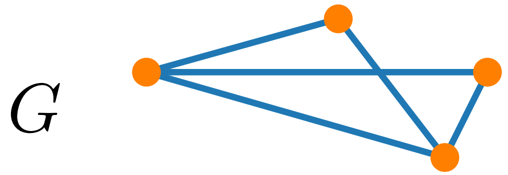

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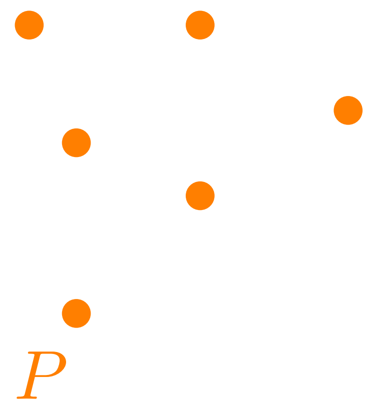

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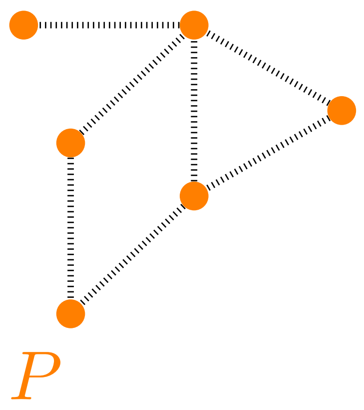

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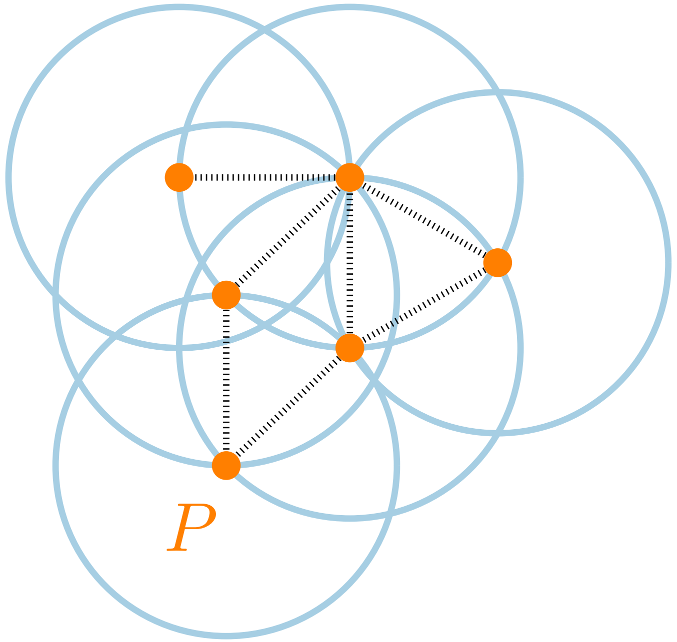

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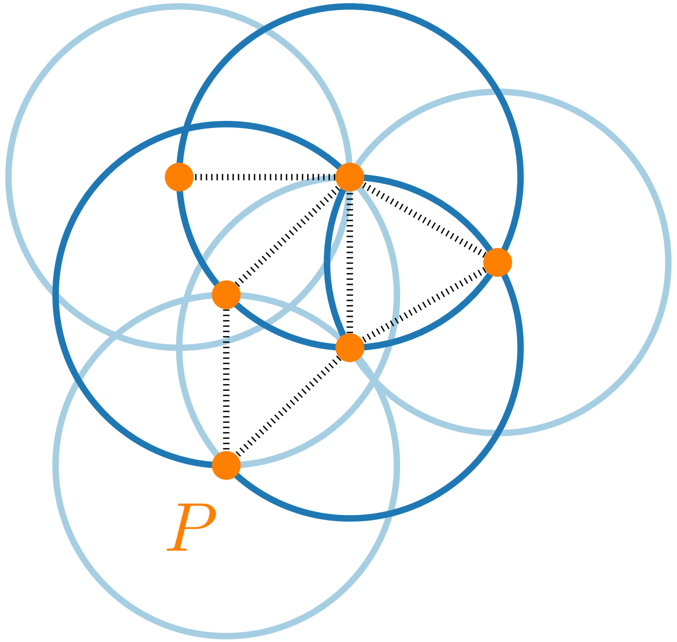

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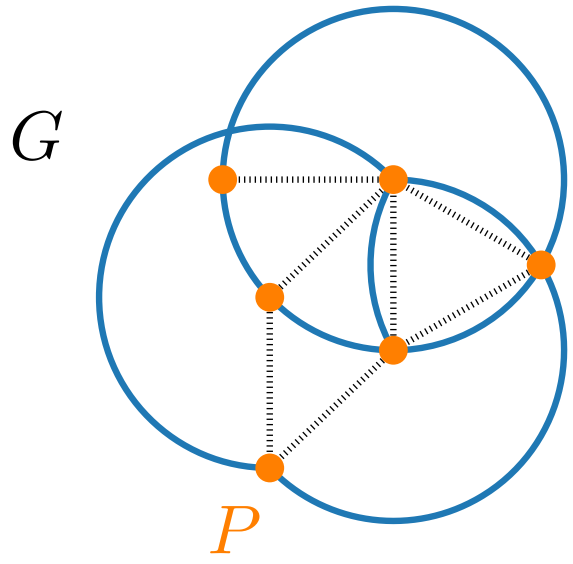

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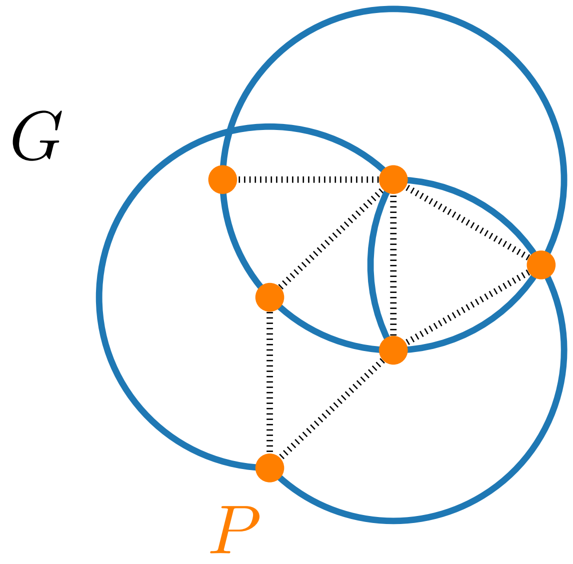

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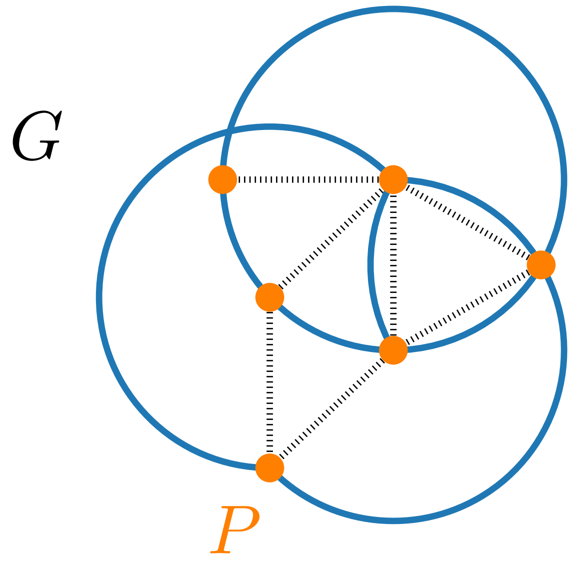

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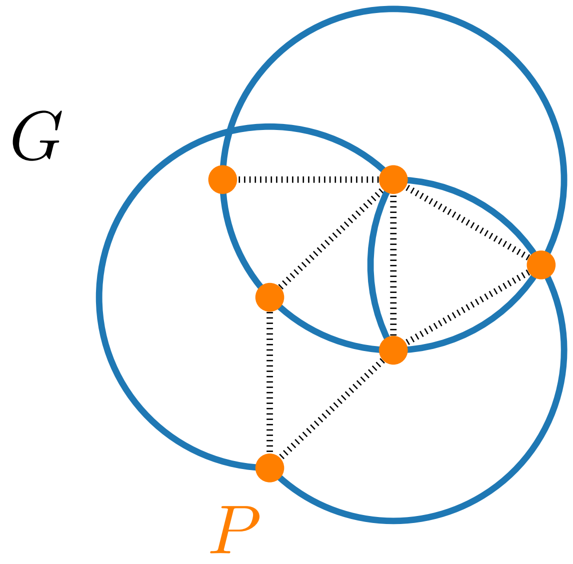

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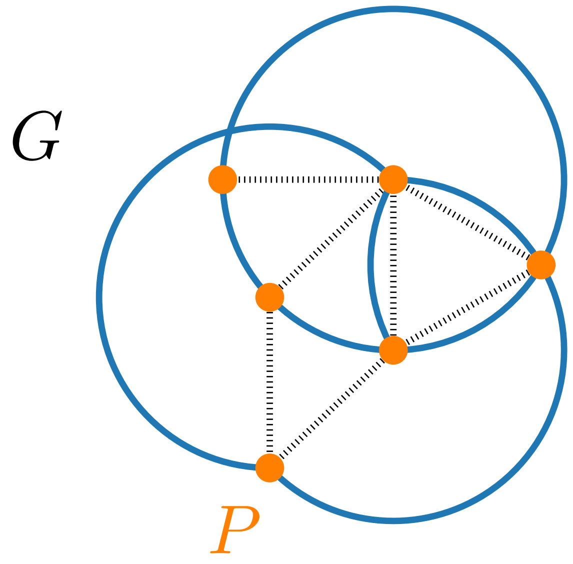

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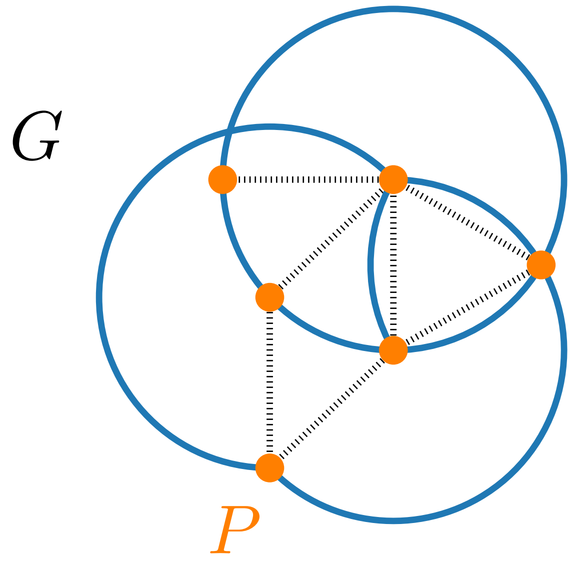

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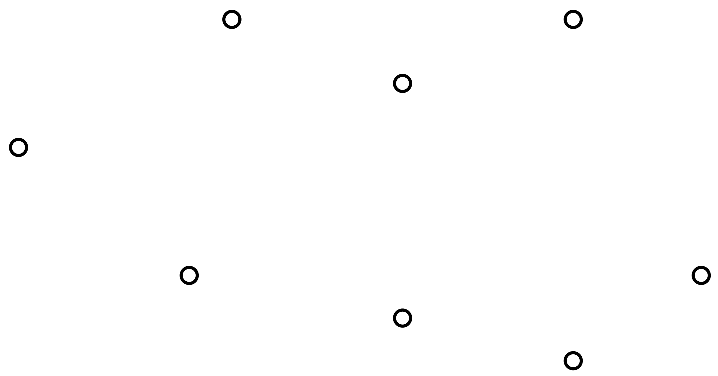

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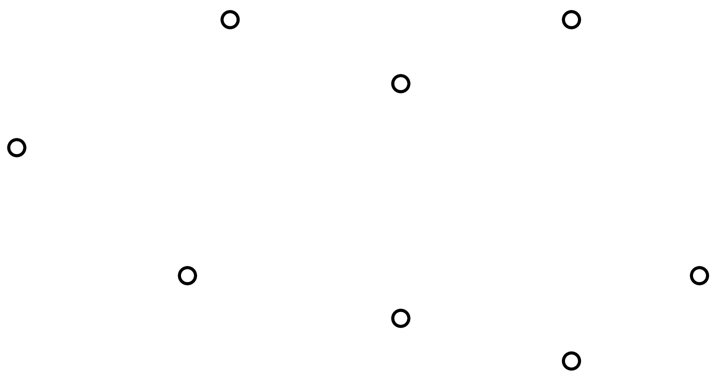

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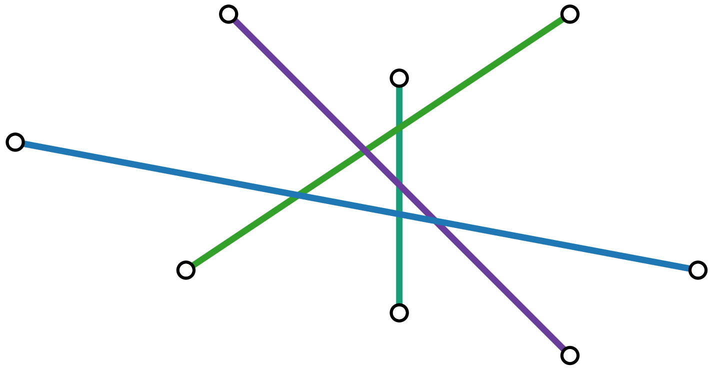

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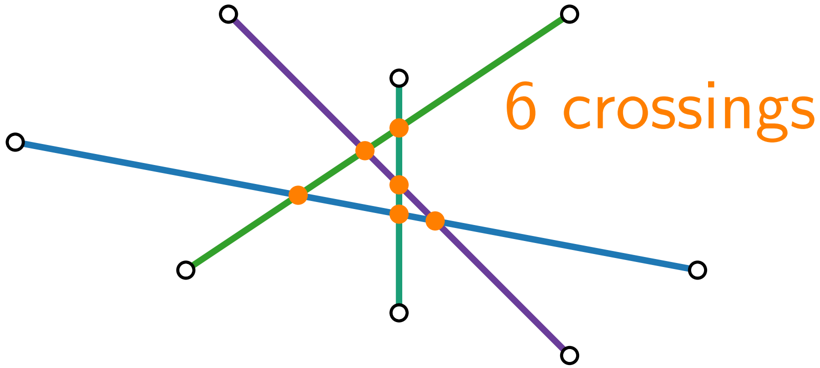

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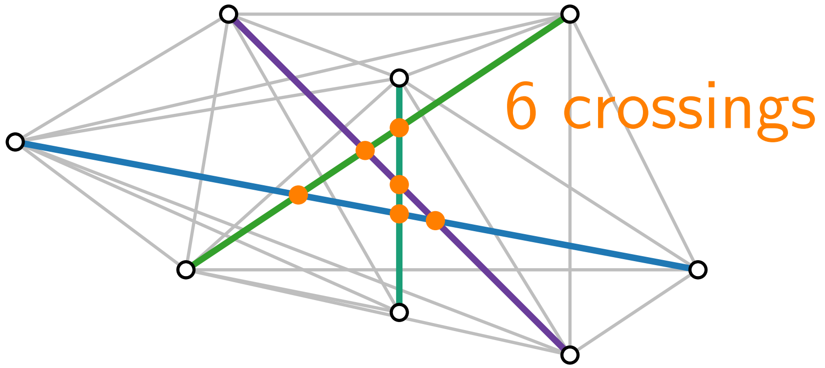

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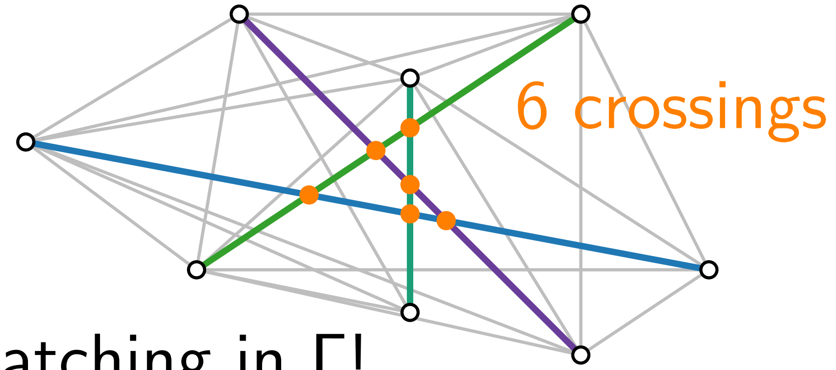

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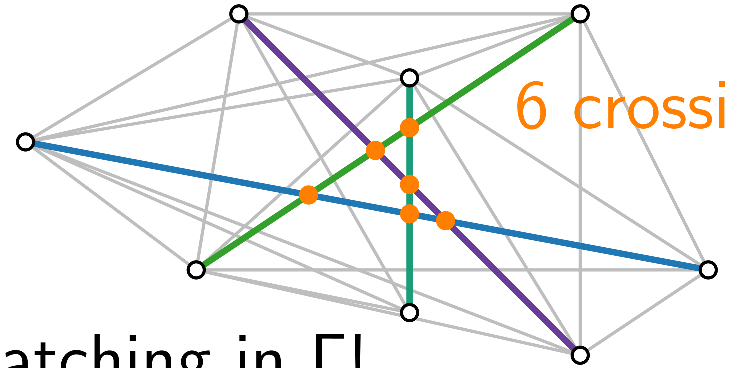

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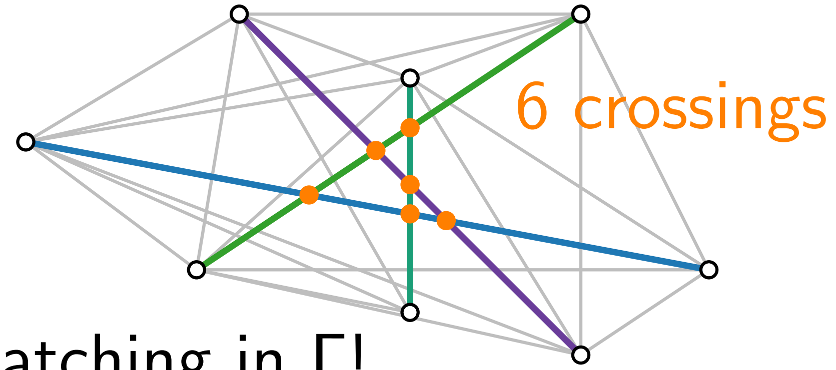

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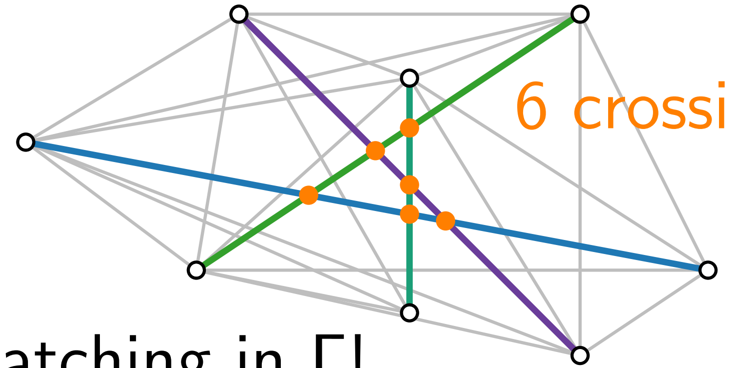

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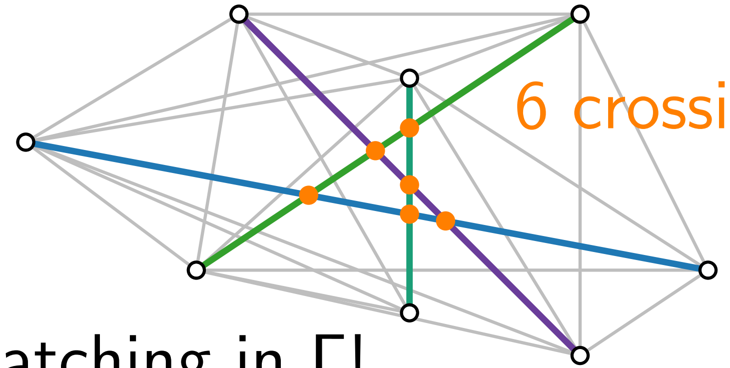

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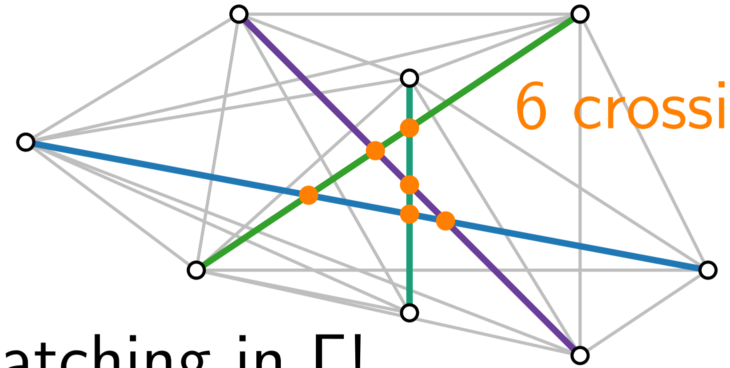

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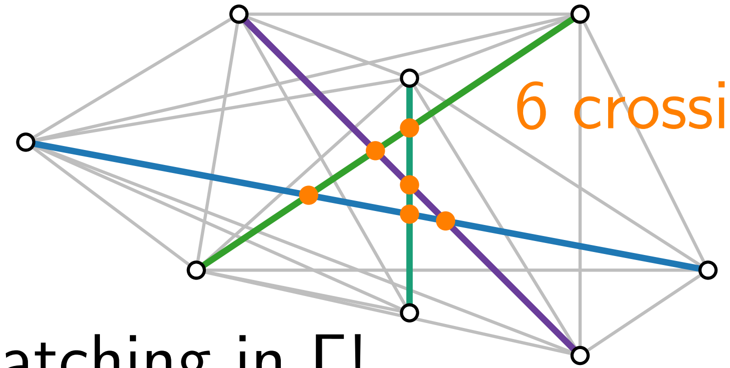

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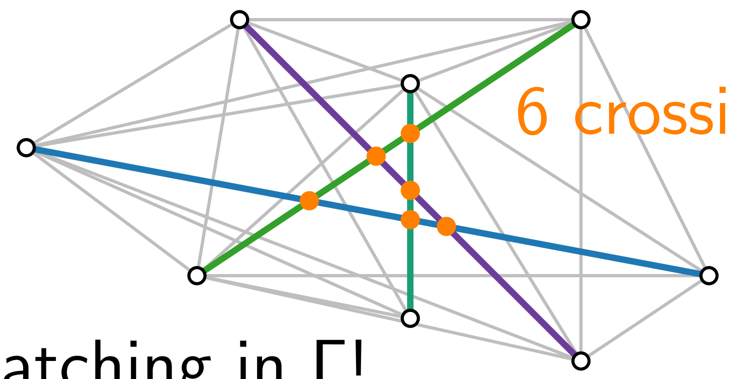

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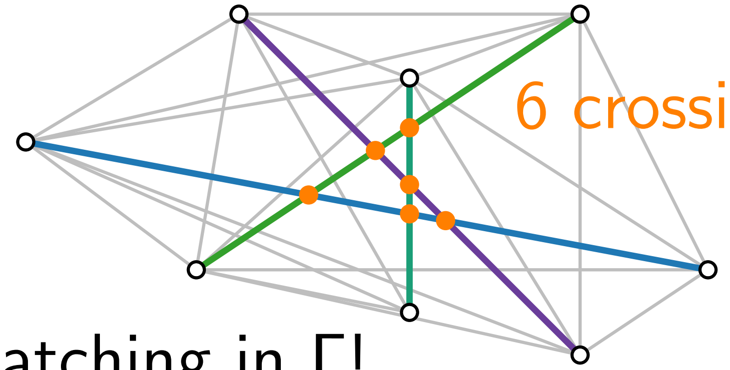

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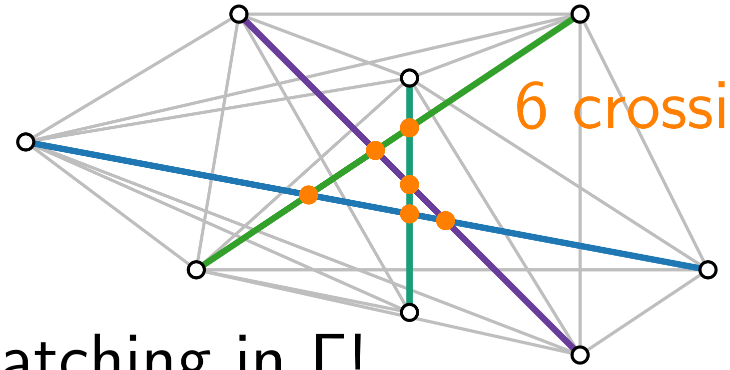

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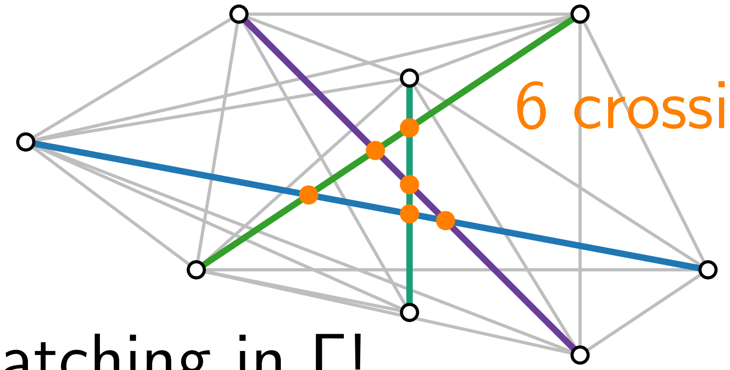

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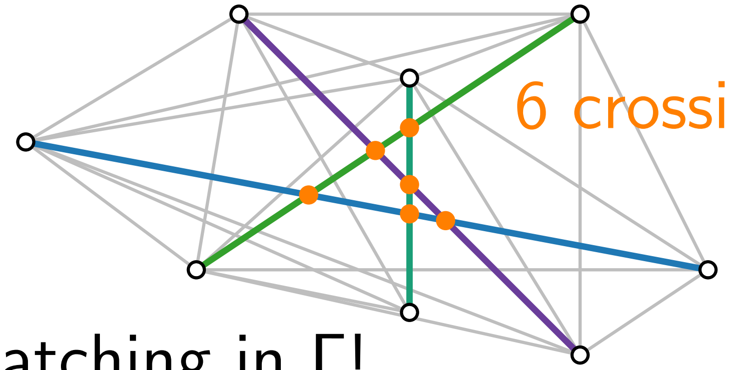

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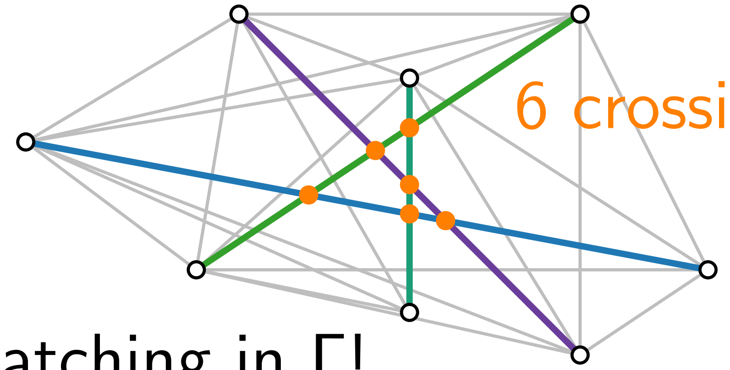

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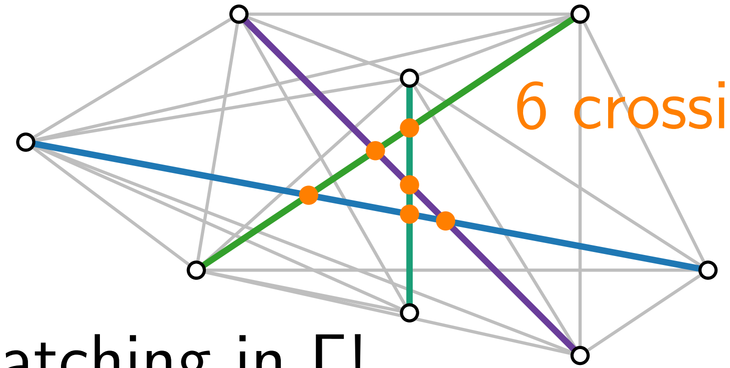

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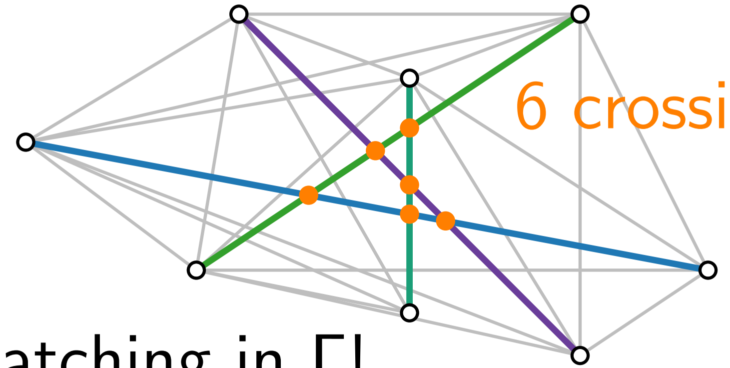

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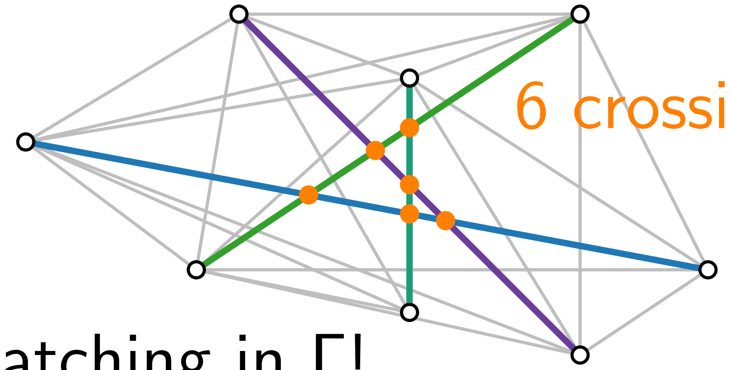

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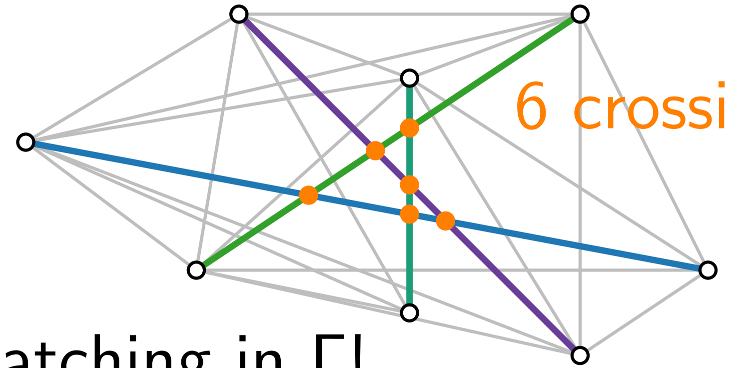

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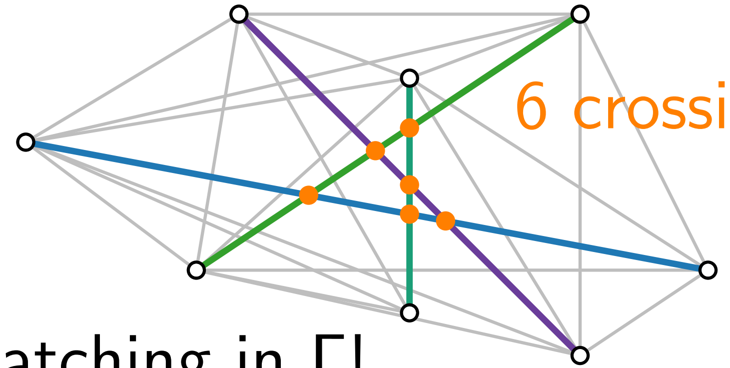

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Literature

- [Aigner, Ziegler] Proofs from THE BOOK [https://doi.org/10.1007/978-3-662-57265-8]
- [Schaefer '20] The Graph Crossing Number and its Variants: A Survey
- Terrence Tao's blog post "The crossing number inequality" from 2007
- [Garey, Johnson '83] Crossing number is NP-complete
- [Bienstock, Dean '93] Bounds for rectilinear crossing numbers
- [Székely '97] Crossing Numbers and Hard Erdős Problems in Discrete Geometry
- Documentary/Biography "N Is a Number: A Portrait of Paul Erdős"
- Exact computations of crossing numbers: `http://crossings.uos.de`