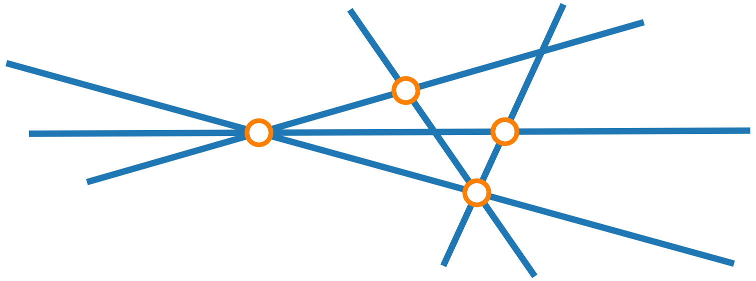
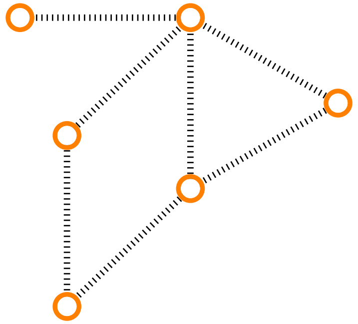
Visualization of Graphs

Lecture 11:
The Crossing Lemma
and Its Applications



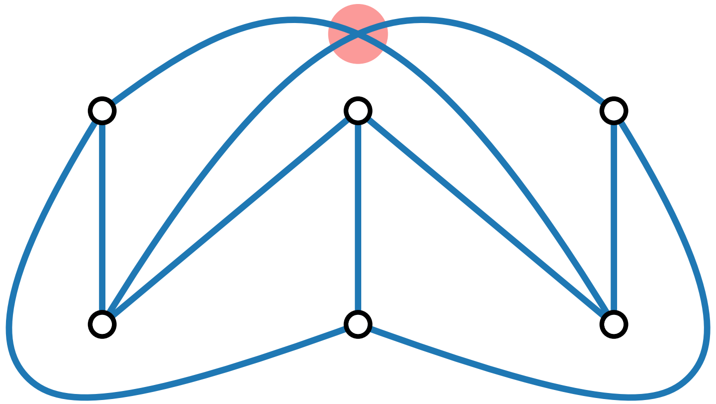


Johannes Zink

Crossing Number and Topological Graphs

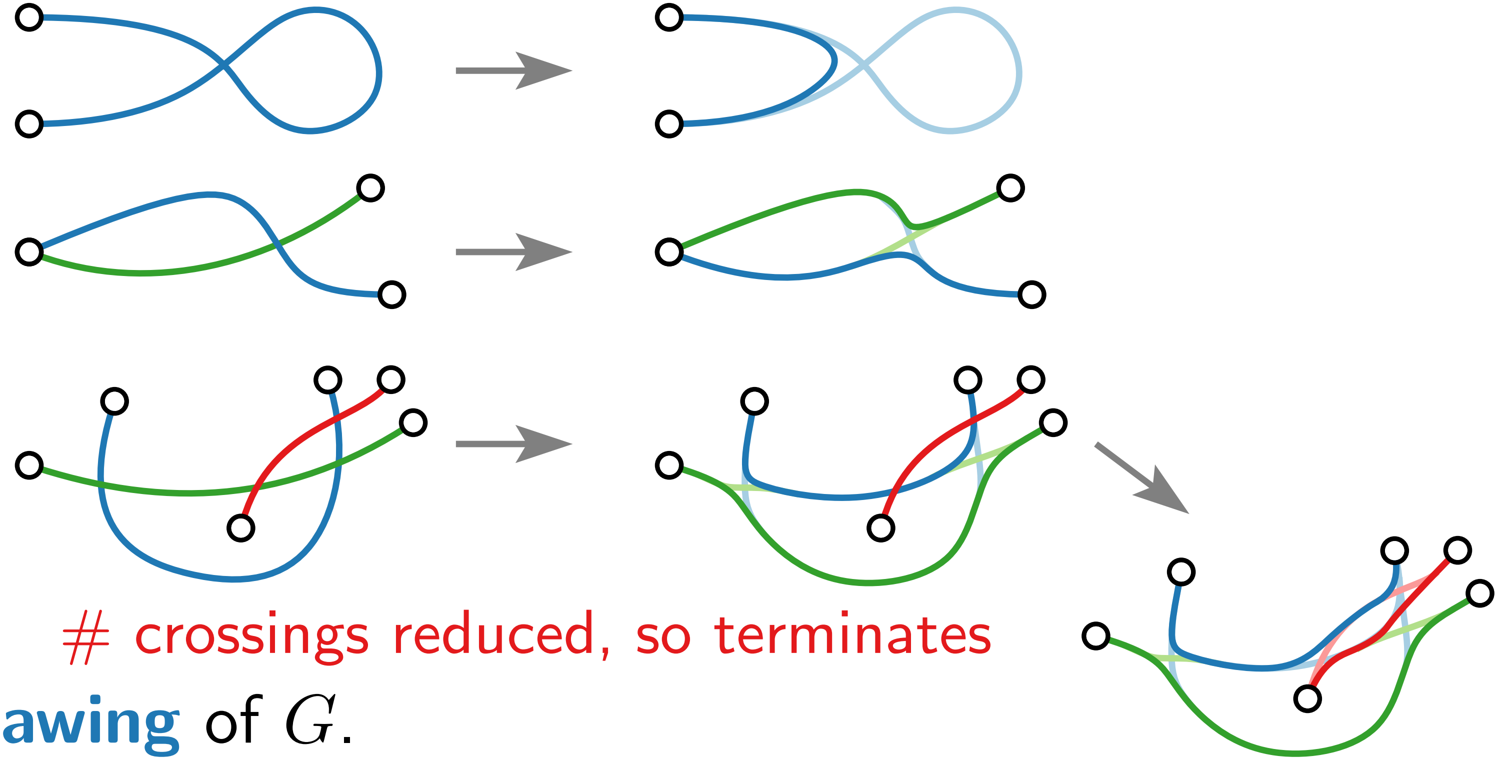
For a graph G, the **crossing number** cr(G) is the smallest number of pair-wise edge crossings in a drawing of G (in the plane).

Example.
cr$(K_{3,3}) = 1$



In a crossing-minimal drawing of G

- no edge is self-intersecting,
- edges with common endpoints do not intersect,
- two edges intersect at most once, ?
- and, w.l.o.g., at most two edges intersect at the same point.



crossings reduced, so terminates

Such a drawing is called a **topological drawing** of G.

Hanani–Tutte Theorem

Theorem. [Hanani '43, Tutte '70]
A graph is planar if and only if it has a drawing in which all pairs of vertex-disjoint edges cross an even number of times.

Proof sketch.
Hanani showed that every drawing of K_5 and $K_{3,3}$ must have a pair of edges that crosses an odd number of times.

Every non-planar graph has K_5 or $K_{3,3}$ as a minor, so there are two paths that cross an odd number of times.

Hence, there must be two edges on these paths that cross an odd number of times. $\square$

Hanani–Tutte Theorem

Theorem. [Hanani '43, Tutte '70]
A graph is planar if and only if it has a drawing in which all pairs of vertex-disjoint edges cross an even number of times.

The **odd crossing number** ocr(G) of G is the smallest number of pairs of edges that cross oddly in a drawing of G.

Corollary. $\text{ocr}(G) = 0 \Rightarrow \text{cr}(G) = 0$

Is $\text{ocr}(G) = \text{cr}(G)$? No!

Theorem. [Pelsmajer, Schaefer & Štefankovič '08, Tóth '08]
There is a graph G with $\text{ocr}(G) < \text{cr}(G) \leq 10$

Theorem. [Pelsmajer, Schaefer & Štefankovič '08] [Pach & Tóth '00]
If Γ is a drawing of G and E_0 is the set of edges with only even numbers of crossings in Γ, then G can be drawn such that no edge in E_0 is involved in any crossings and no new pairs of edges cross.

Hanani–Tutte Theorem

Theorem. [Hanani '43, Tutte '70]
A graph is planar if and only if it has a drawing in which all pairs of vertex-disjoint edges cross an even number of times.

The **odd crossing number** $\text{ocr}(G)$ of G is the smallest number of pairs of edges that cross oddly in a drawing of G.

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Is $\text{ocr}(G) = \text{cr}(G)$? No!

Theorem. [Pelsmajer, Schaefer & Štefankovič '08, Tóth '08]
There is a graph G with $\text{ocr}(G) < \text{cr}(G) \leq 10$

The **pairwise crossing number** $\text{pcr}(G)$ of G is the smallest number of pairs of edges that cross in a drawing of G.

By definition $\text{ocr}(G) \leq \text{pcr}(G) \leq \text{cr}(G)$
Is $\text{pcr}(G) = \text{cr}(G)$? Open!

Computing the Crossing Number

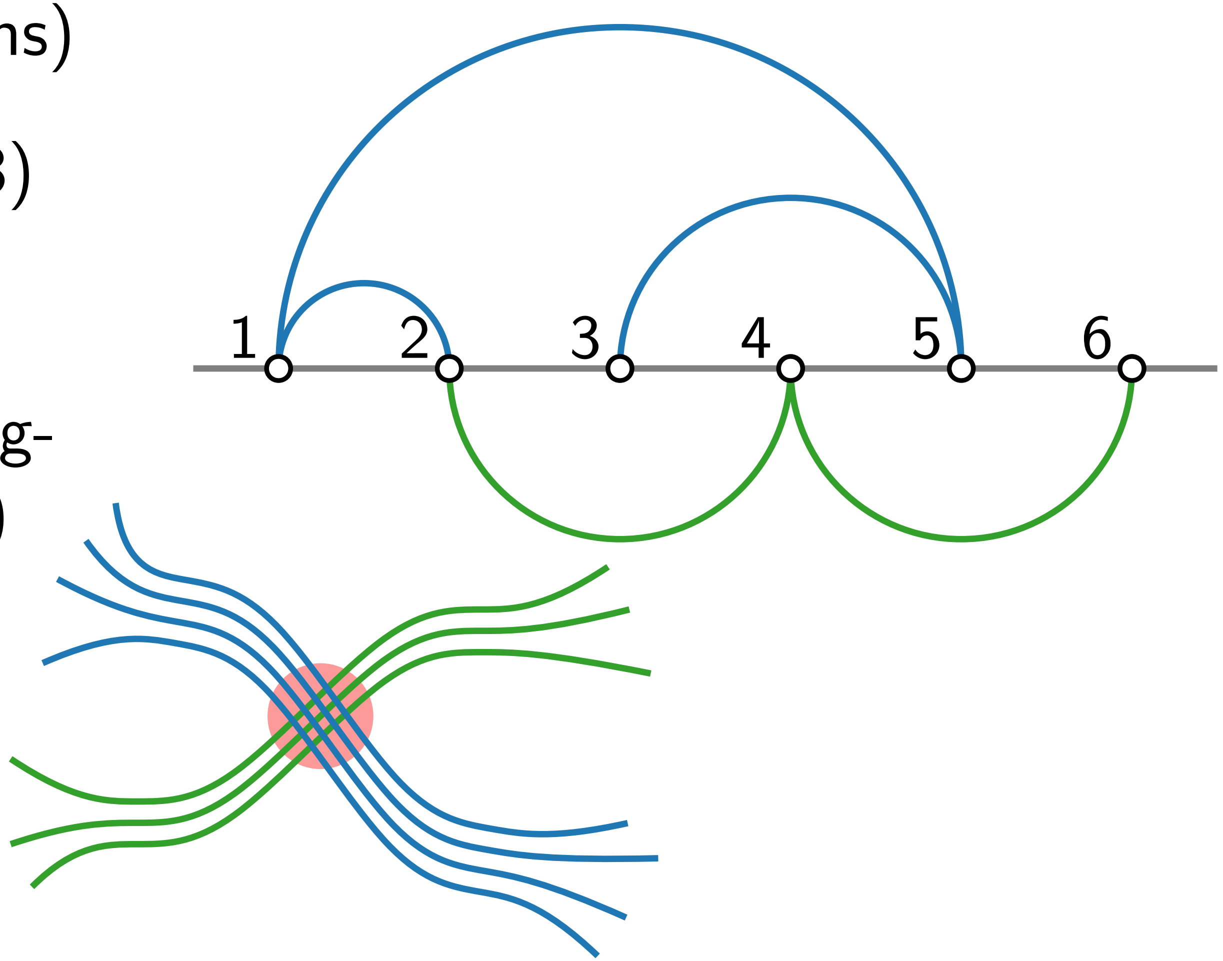
- Computing cr(G) is NP-hard. [Garey & Johnson '83]
  ... even if G is a planar graph plus one edge! [Cabello & Mohar '08]
- cr(G) often unknown, only conjectures exist
  (for K_n it is only known for up to $\approx$ 12 vertices)
- In practice, cr(G) is often not computed directly but rather drawings of G are optimized with
  - force-based methods,
  - multidimensional scaling,
  - heuristics, . . .

For exact computations,
check out `http://crossings.uos.de`!

- cr(G) is a measure of how far G is from being planar.
- For planarization, where we replace crossings with dummy vertices, also only heuristic approaches are known.

Other Crossing Numbers

- Schaefer [Schae20] gives a huge survey on different crossings numbers (and more precise definitions)
- One-sided crossing minimization (see lecture 8)
- Fixed linear crossing number
- Book embeddings (vertices on a line, edges assigned to few "pages" where edges do not cross)
- Crossings of edge bundles
- On other surfaces, such as donuts
- Weighted crossings
- Crossing minimization is NP-hard for most variants.



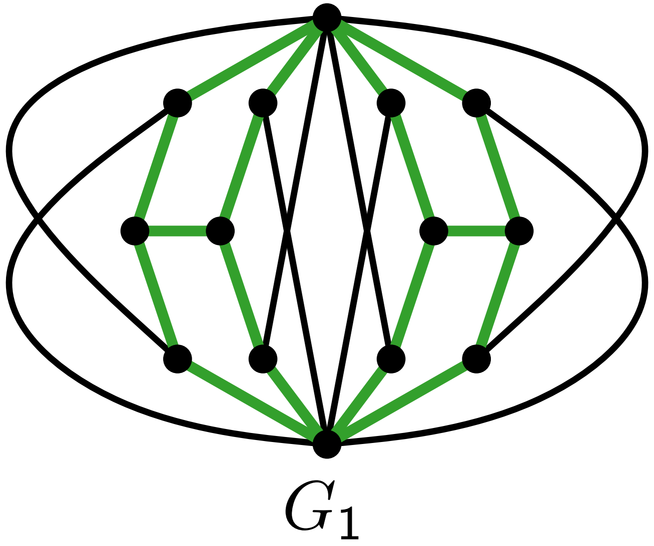
Rectilinear Crossing Number

Definition.
For a graph G, the **rectilinear (straight-line) crossing number** $\overline{\text{cr}}(G)$ is the smallest number of crossings in a straight-line drawing of G.

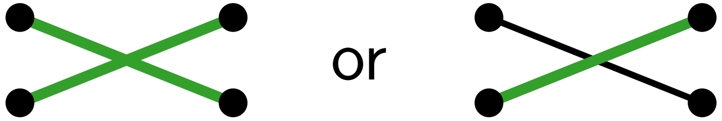
Separation.
$\text{cr}(K_8) = 18$, but $\overline{\text{cr}}(K_8) = 19$.

Even more . . .

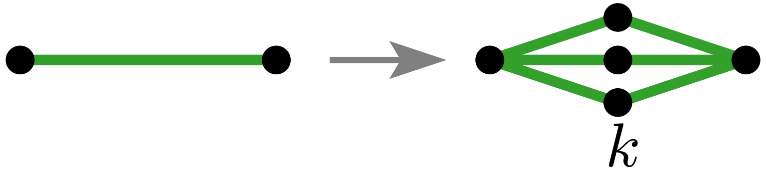
Lemma 1. [Bienstock, Dean '93]
For $k \geq 4$, there exists a graph G_k with $\text{cr}(G_k) = 4$ and $\overline{\text{cr}}(G_k) \geq k$.



- Each straight-line drawing of G_1 has at least one crossing of the following types:



- From G_1 to G_k do



Bounds for Complete Graphs

Theorem. **Conjecture.** [Guy '60]

$$\mathrm{cr}(K_n) \underset{=}{\cancel{\leq}} \frac{1}{4}\left\lceil\frac{n}{2}\right\rceil\left\lceil\frac{n-1}{2}\right\rceil\left\lceil\frac{n-2}{2}\right\rceil\left\lceil\frac{n-3}{2}\right\rceil = \frac{3}{8}\binom{n}{4} + O(n^3)$$

Bound is tight for $n \leq 12$.

Theorem. **Conjecture.** [Zarankiewicz '54, Urbaník '55]

$$\mathrm{cr}(K_{m,n}) \underset{=}{\cancel{\leq}} \frac{1}{4}\left\lceil\frac{n}{2}\right\rceil\left\lceil\frac{n-1}{2}\right\rceil\left\lceil\frac{m}{2}\right\rceil\left\lceil\frac{m-1}{2}\right\rceil$$

Turán's brick factory problem (1944)

Pál Turán
*1910 – 1976
Budapest, Hungary

© TruckinTim

Bounds for Complete Graphs

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Theorem. [Lovász et al. '04, Aichholzer et al. '06]

$$\left(\frac{3}{8}+\varepsilon\right)\binom{n}{4} + O(n^3) < \overline{\mathrm{cr}}(K_n) < 0.3807\binom{n}{4} + O(n^3)$$

Exact numbers are known for $n \leq 27$.

Check out `http://www.ist.tugraz.at/staff/aichholzer/crossings.html`

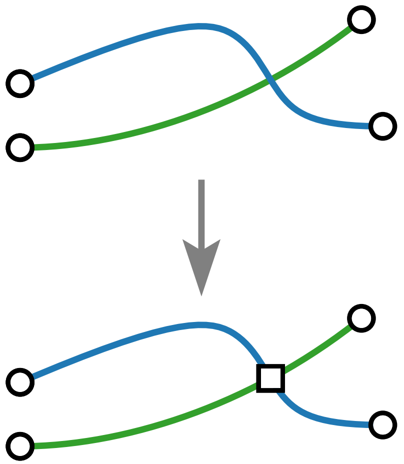
First Lower Bounds on cr(G)

Lemma 2.
For a graph G with n vertices and m edges,

$$\text{cr}(G) \geq m - 3n + 6.$$

Consider this bound for graphs with $\Theta(n)$ and $\Theta(n^2)$ many edges.

Proof.



- Consider a drawing of G with cr(G) crossings.
- Obtain a graph H by turning crossings into dummy vertices.
- H has $n + \text{cr}(G)$ vertices and $m + 2\text{cr}(G)$ edges.
- H is planar, so

$$m + 2\text{cr}(G) \leq 3(n + \text{cr}(G)) - 6. \qquad \square$$

First Lower Bounds on cr(G)

Lemma 3.
For a non-planar graph G with n vertices and m edges,

$$\text{cr}(G) \geq r \cdot \binom{\lfloor m/r \rfloor}{2} \in \Omega\left(\frac{m^2}{n}\right)$$

where $r \leq 3n - 6$ is the maximum number of edges in a planar subgraph of G.

Consider this bound for graphs with $\Theta(n)$ and $\Theta(n^2)$ many edges.

Proof sketch.

- Take $\lfloor m/r \rfloor$ edge-disjoint subgraphs of G with r edges.
- In the best case, they are all planar.
- For every $i < j$, any edge of G_j induces at least one crossings with G_i. (Otherwise, we could add an edge to G_i and obtain a planar subgraph of G with $r + 1$ edges.)

The Crossing Lemma

- 1973 Erdős and Guy conjectured that $\text{cr}(G) \in \Omega(m^3/n^2)$.
- In 1982 Leighton and, independently, Ajtai, Chávtal, Newborn, and Szemerédi showed that

$$\text{cr}(G) \geq \frac{1}{64} \cdot \frac{m^3}{n^2}.$$

- Bound is asymptotically tight.
- Result stayed hardly known until Székely demonstrated its usefulness (in 1997).
- We go through the proof from "THE BOOK" by Chazelle, Sharir, and Welzl.
- Factor $\frac{1}{64}$ was later (with intermediate steps) improved to $\frac{1}{29}$ by Ackerman in 2013.

Consider this bound for graphs with $\Theta(n)$ and $\Theta(n^2)$ many edges.

The Crossing Lemma

Crossing Lemma.
For a graph G with n vertices and m edges, $m \geq 4n$,

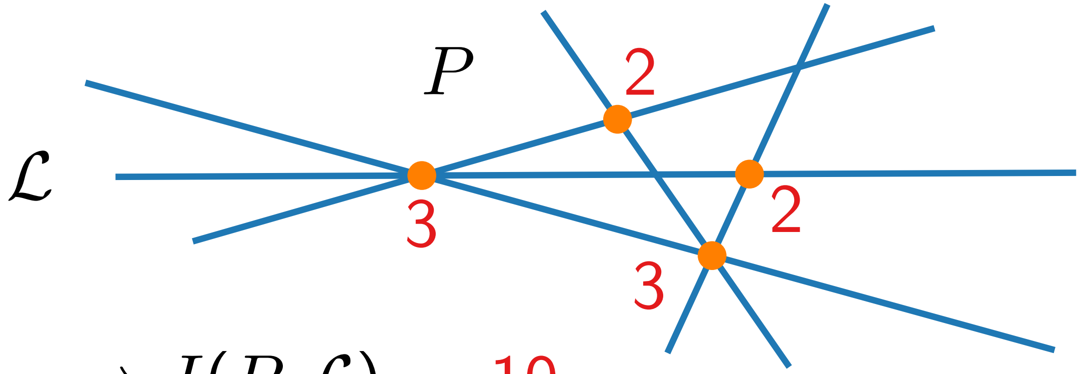
$$\text{cr}(G) \geq \frac{1}{64} \cdot \frac{m^3}{n^2}.$$

Proof.

- Consider a crossing-minimal drawing of G.
- Let p be a number in $(0, 1]$.
- Keep every vertex of G independently with probability p.
- $G_p =$ remaining graph (with drawing Γ_p).
- Let n_p, m_p, X_p be the random variables counting the numbers of vertices / edges / crossings of Γ_p, resp.
- By Lemma 2, $\text{cr}(G_p) - m_p + 3n_p \geq 6$.
  $\Rightarrow \text{E}(X_p - m_p + 3n_p) \geq 0$.
- $\text{E}(n_p) = pn$ and $\text{E}(m_p) = p^2 m$
- $\text{E}(X_p) = p^4 \text{cr}(G)$
- $0 \leq \text{E}(X_p) - \text{E}(m_p) + 3\text{E}(n_p)$
  $= p^4 \text{cr}(G) - p^2 m + 3pn$
- $\text{cr}(G) \geq \frac{p^2 m - 3pn}{p^4} = \frac{m}{p^2} - \frac{3n}{p^3}$
- Set $p = \frac{4n}{m}$.
- $\text{cr}(G) \geq \frac{m^3}{16n^2} - \frac{3m^3}{64n^2} = \frac{1}{64} \frac{m^3}{n^2}$ $\square$

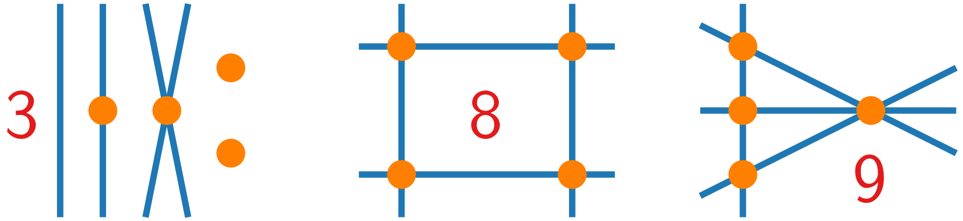
Application 1: Point–Line Incidences

For a set $P \subset \mathbb{R}^2$ of points and a set $\mathcal{L}$ of lines, let $I(P, \mathcal{L})$ = number of point–line incidences in $(P, \mathcal{L})$.



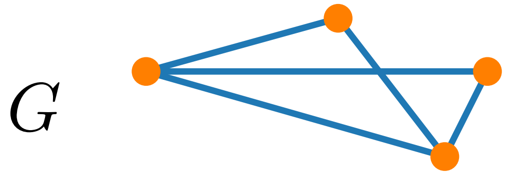
$\Rightarrow I(P, \mathcal{L}) = 10$

- Define $I(n, k) = \max\limits_{|P|=n, |\mathcal{L}|=k} I(P, \mathcal{L})$.
- For example: $I(4, 4) = 9$



Theorem 1.
[Szemerédi, Trotter '83, Székely '97]
$I(n, k) \le c(n^{2/3}k^{2/3} + n + k)$.

Proof.



- $\mathrm{cr}(G) \le k^2$
- $\#(\text{points on } \ell) - 1 = \#(\text{edges on } \ell)$
  $\Rightarrow I(n, k) - k = m$ (sum up over $\mathcal{L}$ in an "optimal" instance)
- If $m \le 4n$, then $I(n, k) - k = m \le 4n$.
  $\Rightarrow I(n, k) \le 4n + k \le c(n + k + n^{2/3}k^{2/3})$
- Otherwise, employ the Crossing Lemma:

$$\frac{1}{64}\frac{m^3}{n^2} \le \mathrm{cr}(G) \le k^2 \quad \Rightarrow \frac{1}{64}\frac{(I(n,k)-k)^3}{n^2} \le k^2$$

$$\Leftrightarrow I(n, k) \le c(n^{2/3}k^{2/3} + k)$$

$$\le c(n^{2/3}k^{2/3} + k + n)$$

□

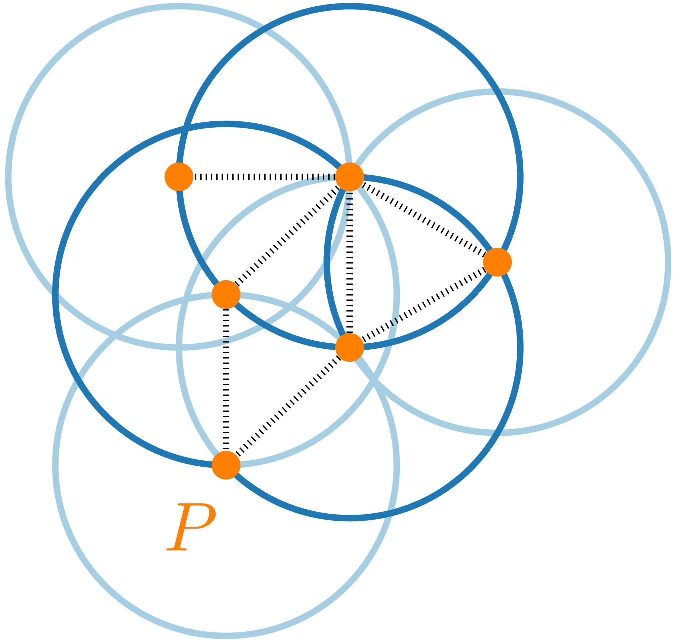
Application 2: Unit Distances

For a set $P \subset \mathbb{R}^2$ of points, define

- $U(P) =$ number of pairs in P at unit distance and
- $U(n) = \max_{|P|=n} U(P)$.

Theorem 2.
[Spencer, Szemerédi, Trotter '84, Székely '97]
$U(n) < 6.7n^{4/3}$

Proof Sketch.



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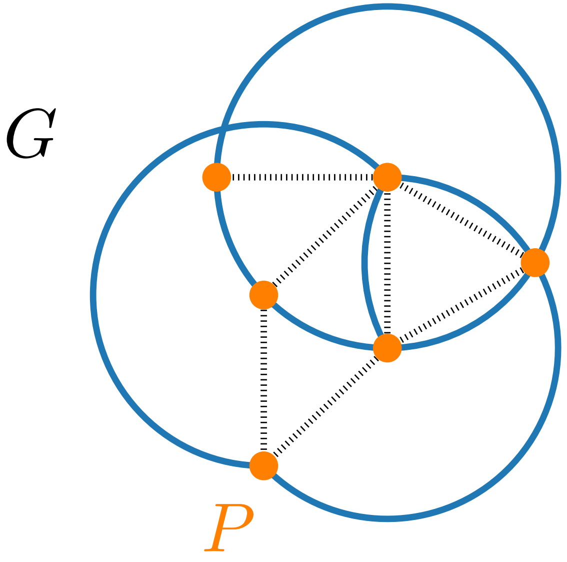
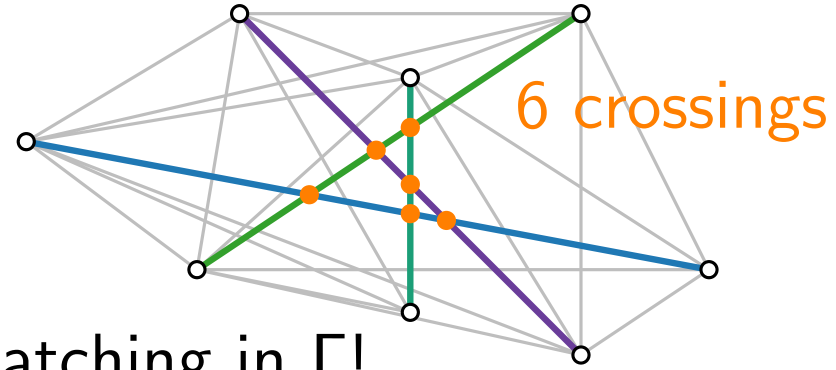


- $U(P) \leq \underbrace{c''}_{\text{some constant}}\underbrace{m}_{\text{number of edges in } G}$
- $\operatorname{cr}(G) \leq 2n^2$ (circles intersecte each other at most twice)
- $c' \frac{U(P)^3}{n^2} \leq \operatorname{cr}(G) \leq 2n^2$ by the Crossing Lemma.

Application 3: Expected Number of Crossings in a Matching



Given set of n points (in general position, n even) –
what is the average number of crossings in a perfect matching?

Point set spans drawing Γ of K_n.
We will analyze the number of crossings in a **random** perfect matching in Γ!

Number of crossings in Γ $\geq \overline{\text{cr}}(K_n) > \frac{3}{8}\binom{n}{4}$
Number of edges in K_n: $\binom{n}{2}$
Number of *potential crossings* (all pairs of edges): $\text{pot}(K_n) = \binom{\binom{n}{2}}{2} \approx 3\binom{n}{4}$

Pick two random edges e_1 and e_2.
$\Pr[e_1 \text{ and } e_2 \text{ cross}] \geq \overline{\text{cr}}(K_n)/\text{pot}(K_n) > \frac{1}{8}$.

Pick random perfect matching M; it has $n/2$ edges, so $\binom{n/2}{2} = \frac{1}{8}n(n-2)$ pairs of edges.
By linearity of expectation,
the expected number of crossings in M is $> \frac{1}{8}\binom{n/2}{2} = \frac{1}{64}n(n-2) \in \Theta(n^2)$. ☐

Literature

- [Aigner, Ziegler] Proofs from THE BOOK [https://doi.org/10.1007/978-3-662-57265-8]
- [Schaefer '20] The Graph Crossing Number and its Variants: A Survey
- Terrence Tao's blog post "The crossing number inequality" from 2007
- [Garey, Johnson '83] Crossing number is NP-complete
- [Bienstock, Dean '93] Bounds for rectilinear crossing numbers
- [Székely '97] Crossing Numbers and Hard Erdős Problems in Discrete Geometry
- Documentary/Biography "N Is a Number: A Portrait of Paul Erdős"
- Exact computations of crossing numbers: `http://crossings.uos.de`