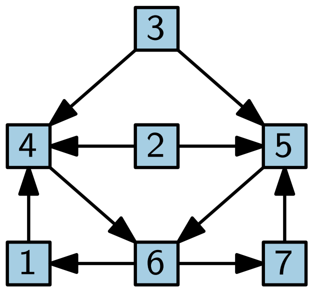
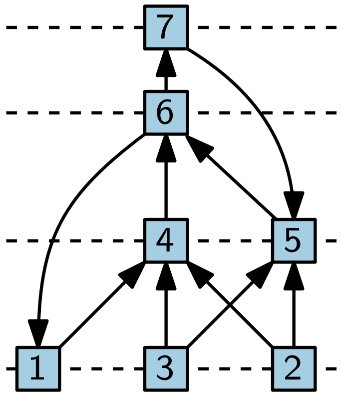
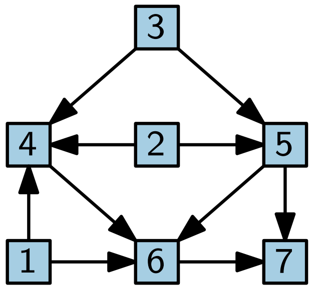
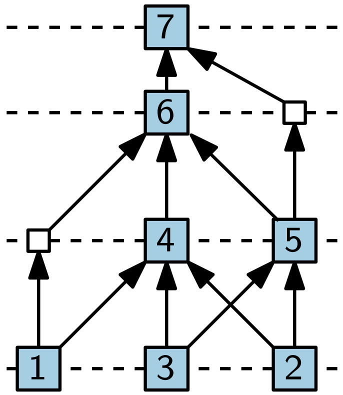


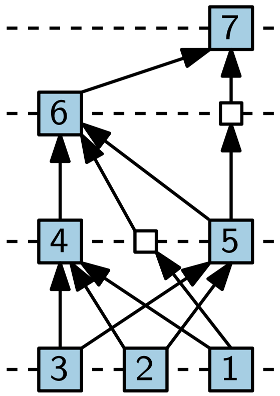


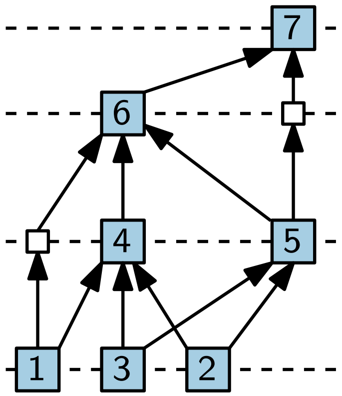
Visualization of Graphs



Lecture 8:
Hierarchical Layouts:
Sugiyama Framework

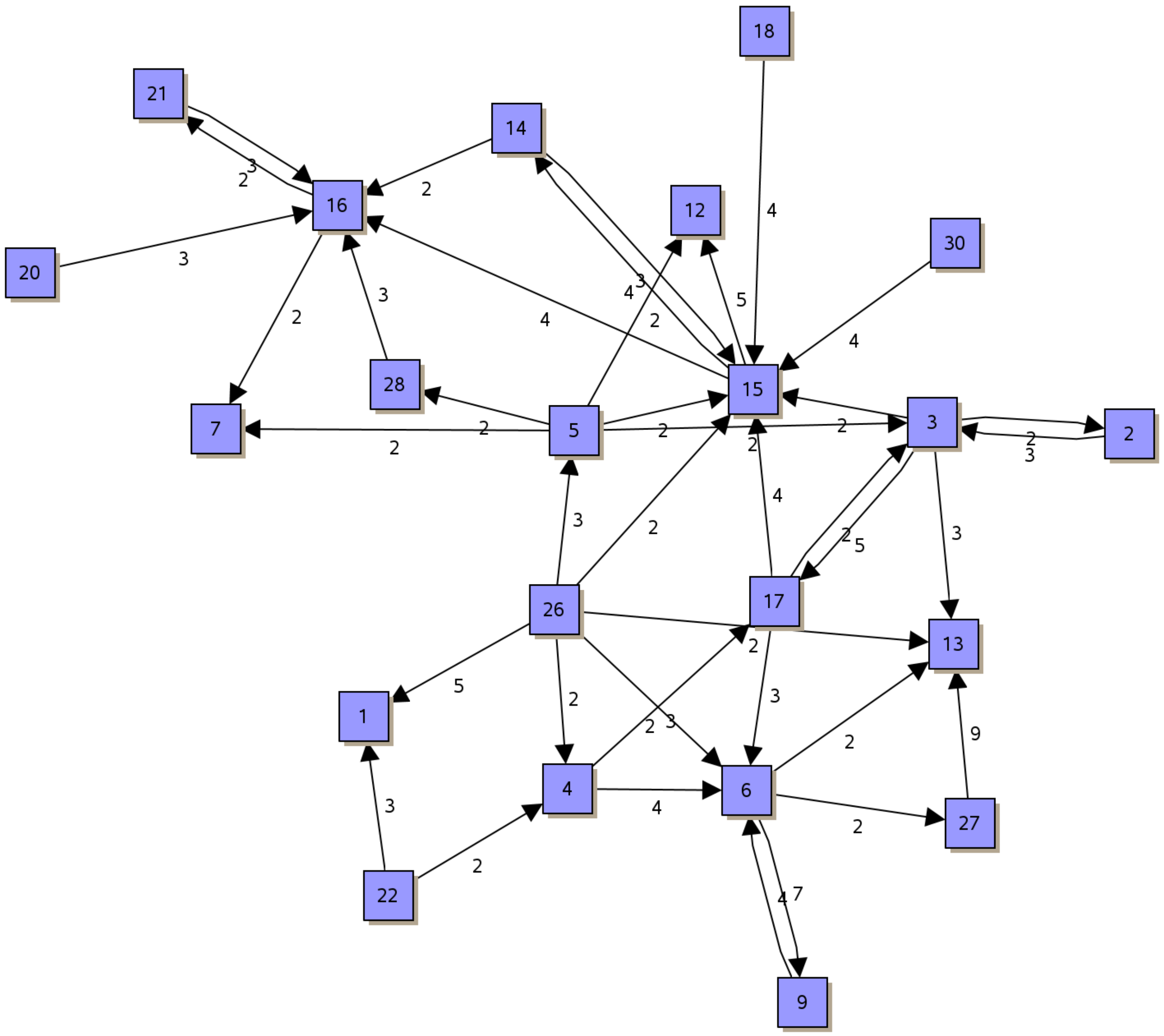




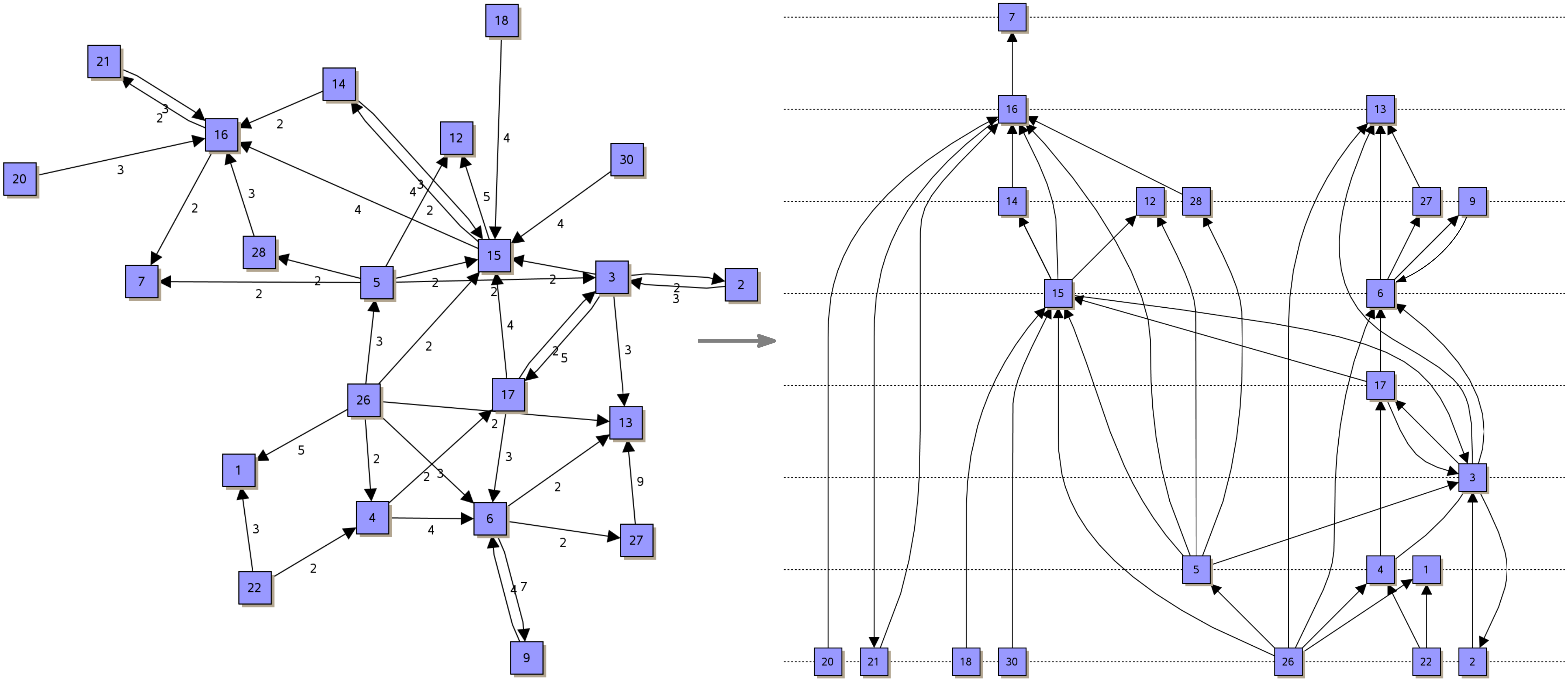


Johannes Zink

Hierarchical Drawings – Motivation



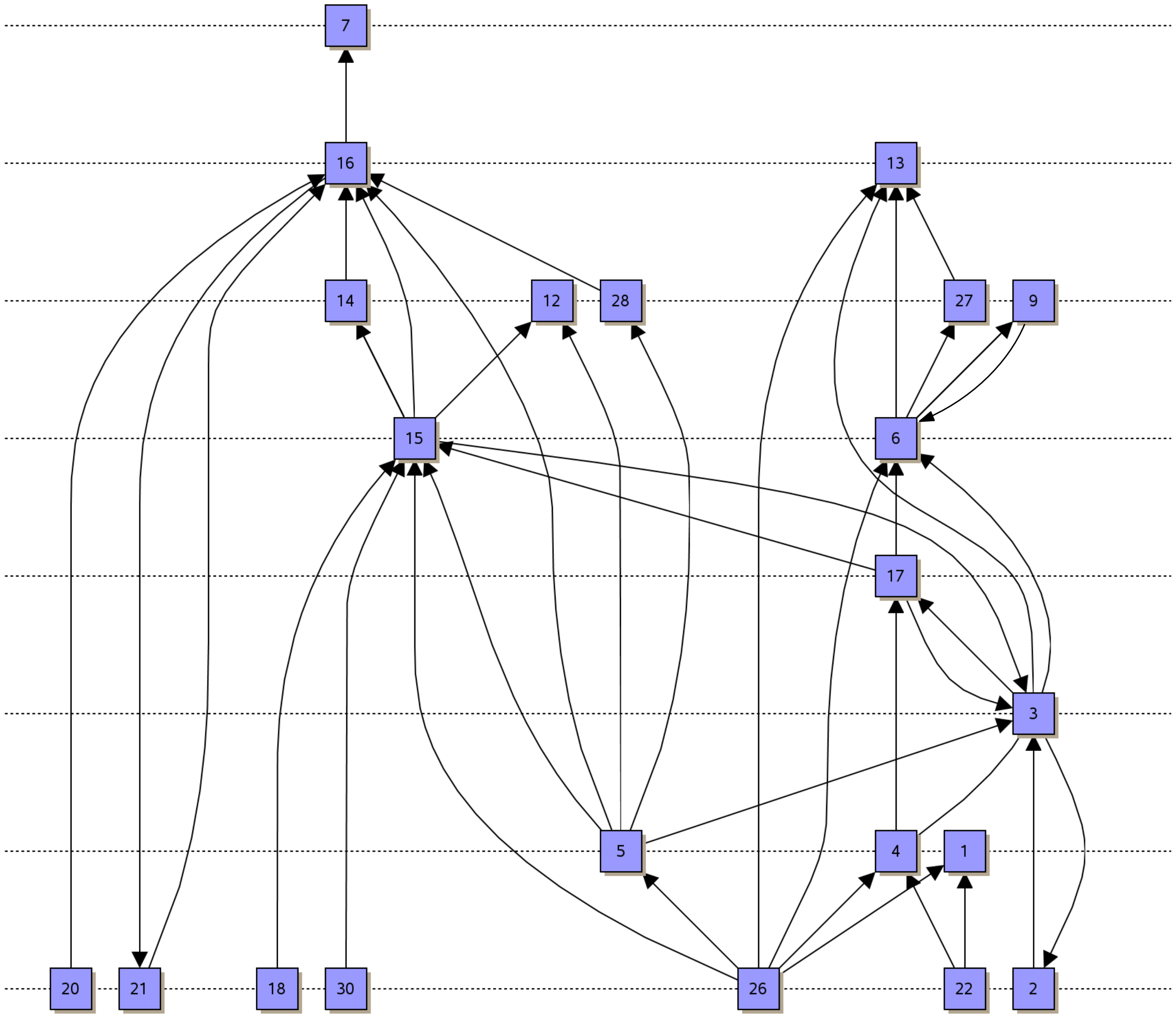
Hierarchical Drawings – Motivation



Hierarchical Drawing

Problem Statement.

- Input: digraph $G = (V, E)$
- Output: drawing of G that "closely" reproduces the hierarchical properties of G

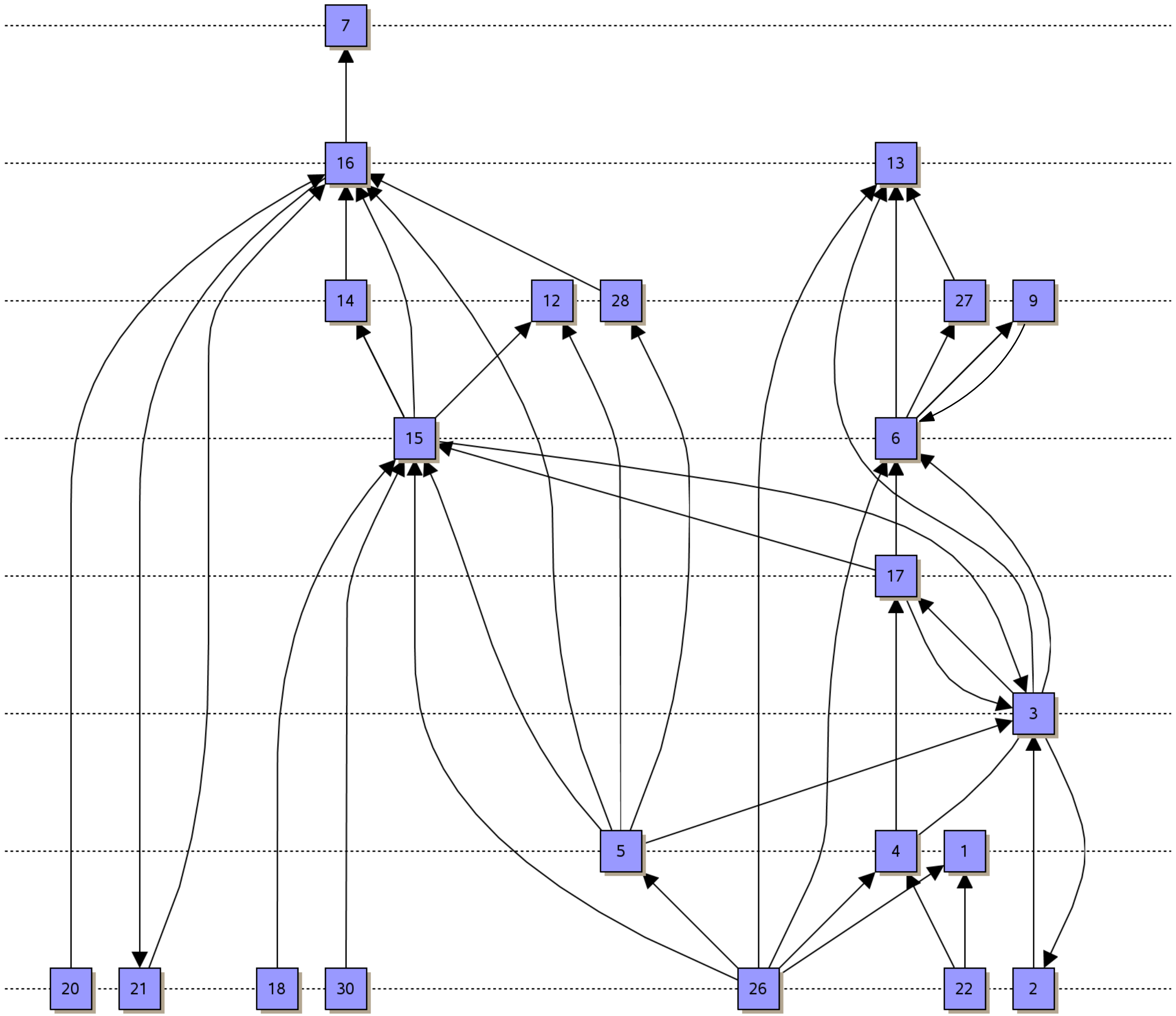


Hierarchical Drawing

Problem Statement.

- Input: digraph $G = (V, E)$
- Output: drawing of G that "closely" reproduces the hierarchical properties of G

Desirable Properties.



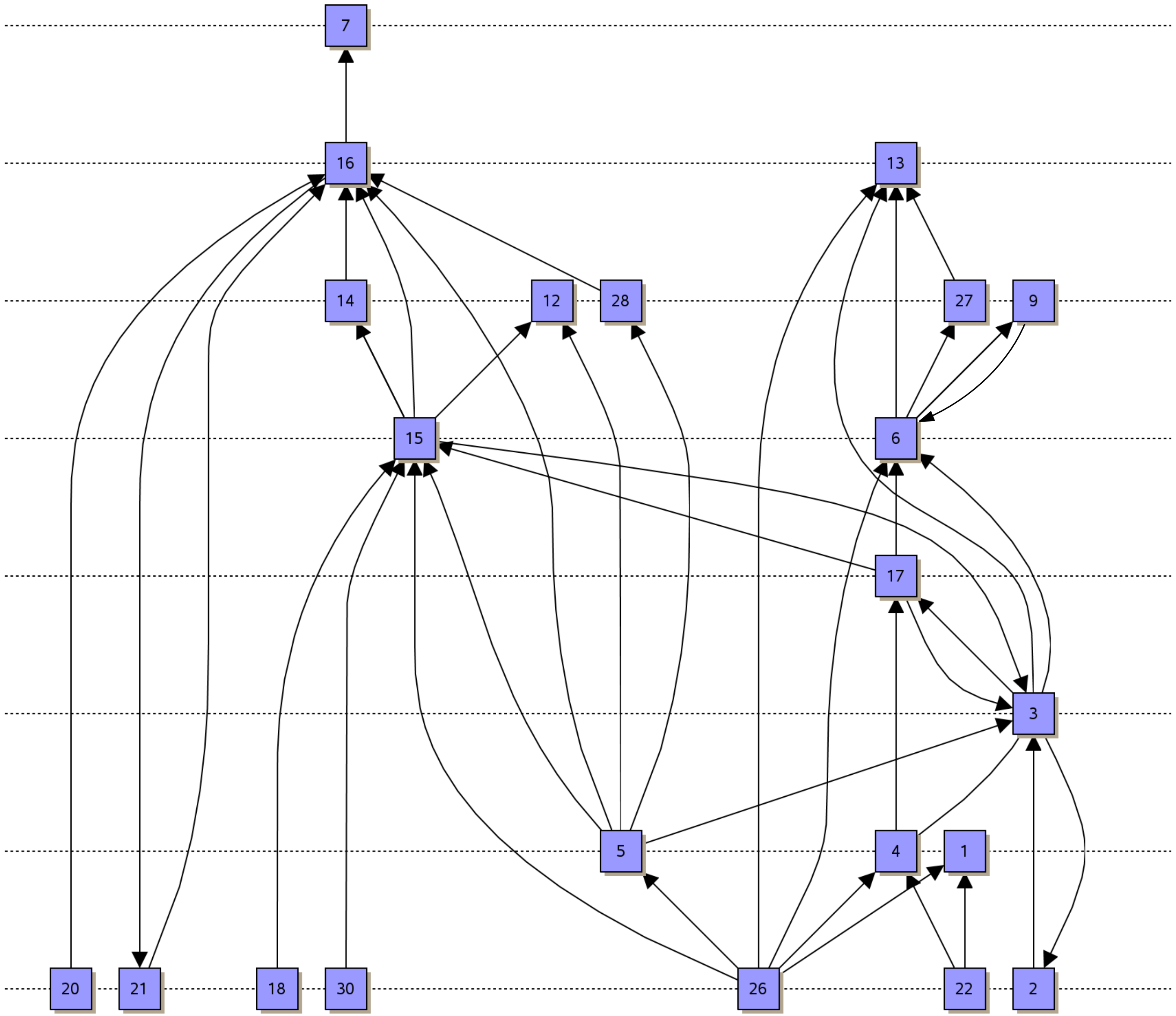
Hierarchical Drawing

Problem Statement.

- Input: digraph $G = (V, E)$
- Output: drawing of G that "closely" reproduces the hierarchical properties of G

Desirable Properties.

- edges directed upwards



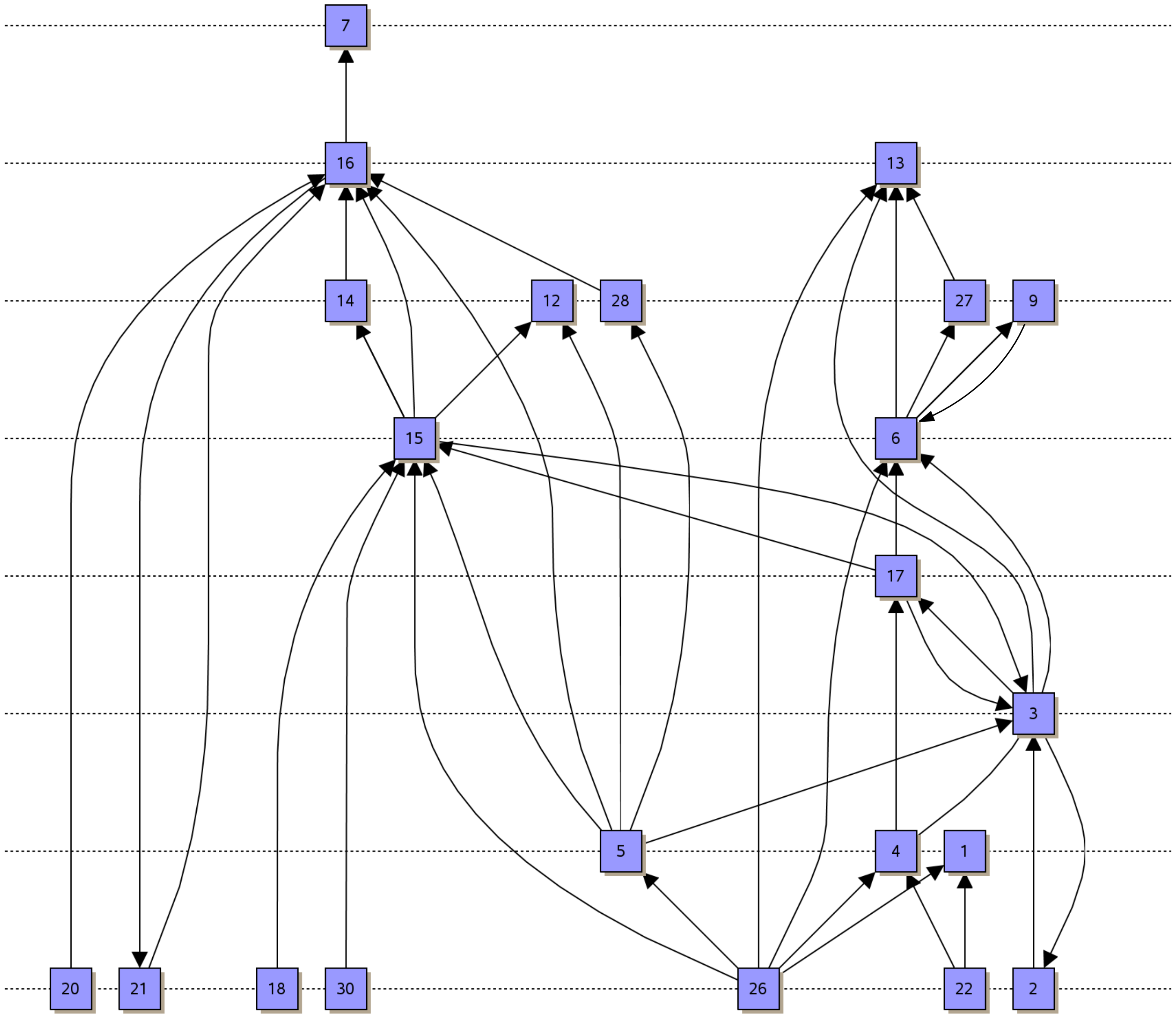
Hierarchical Drawing

Problem Statement.

- Input: digraph $G = (V, E)$
- Output: drawing of G that "closely" reproduces the hierarchical properties of G

Desirable Properties.

- edges directed upwards
- vertices occur on (few) horizontal lines



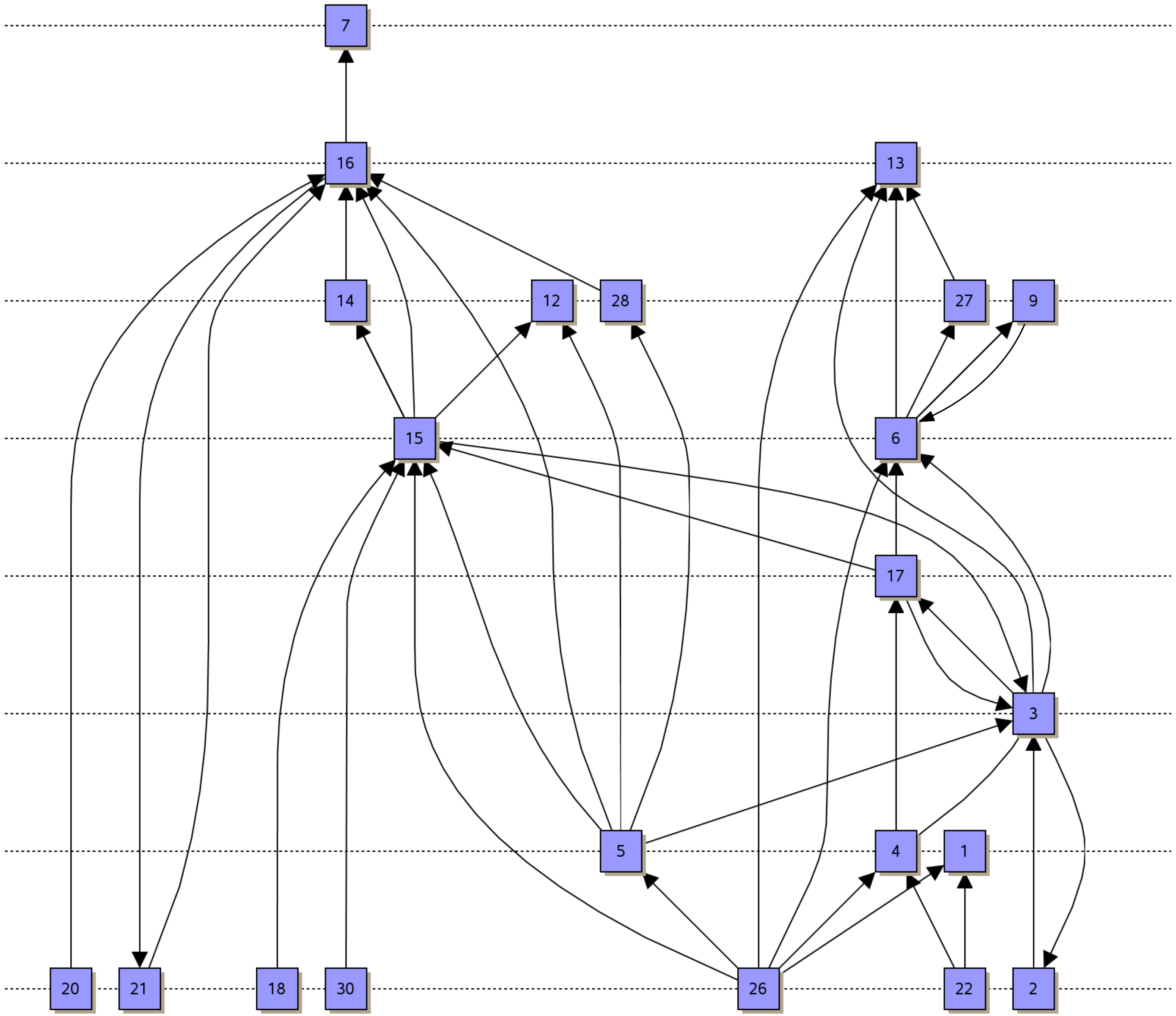
Hierarchical Drawing

Problem Statement.

- Input: digraph $G = (V, E)$
- Output: drawing of G that “closely” reproduces the hierarchical properties of G

Desirable Properties.

- edges directed upwards
- vertices occur on (few) horizontal lines
- edge crossings minimized



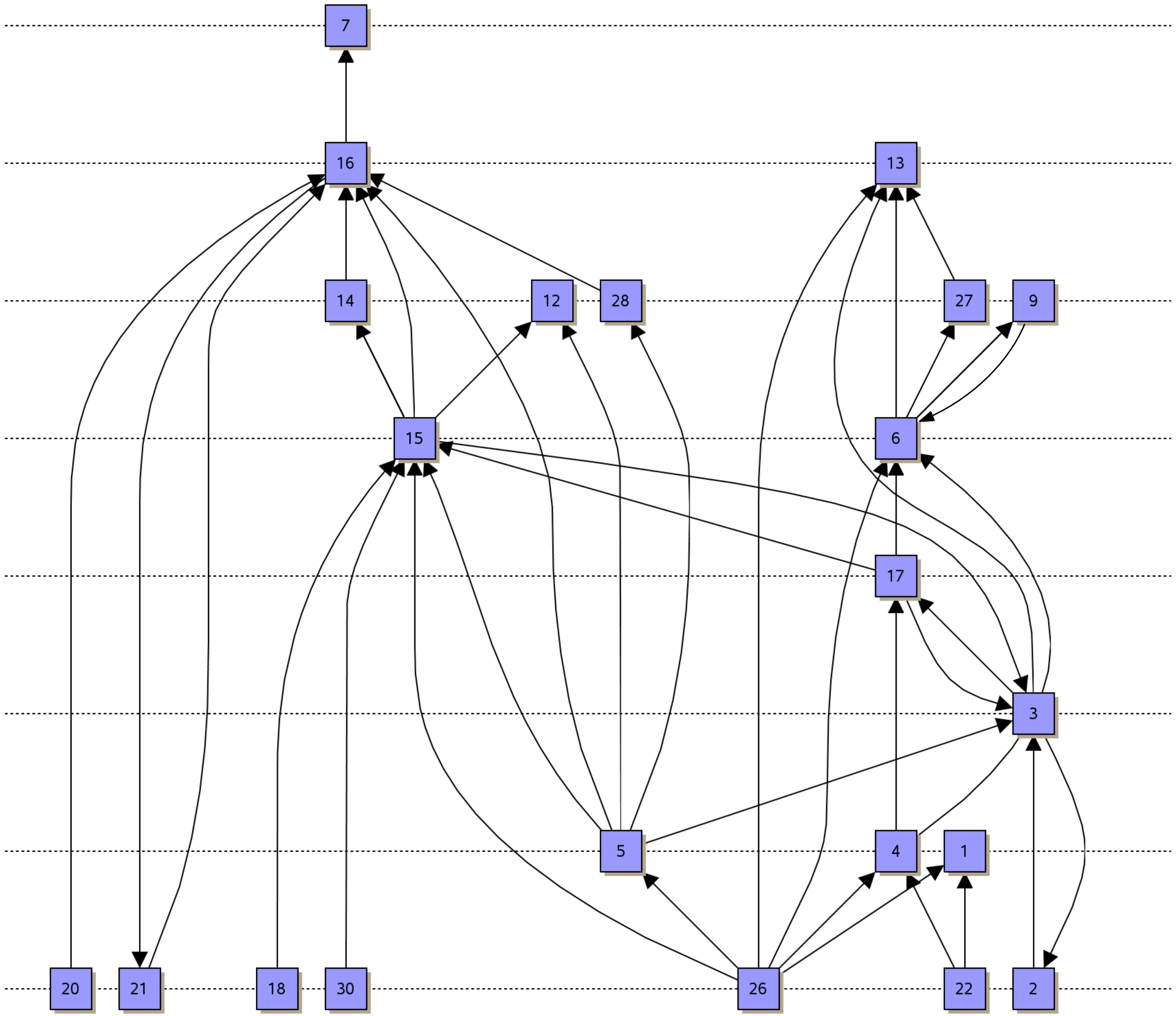
Hierarchical Drawing

Problem Statement.

- Input: digraph $G = (V, E)$
- Output: drawing of G that "closely" reproduces the hierarchical properties of G

Desirable Properties.

- edges directed upwards
- vertices occur on (few) horizontal lines
- edge crossings minimized
- edges as short as possible



Hierarchical Drawing

Problem Statement.

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Desirable Properties.

- edges directed upwards
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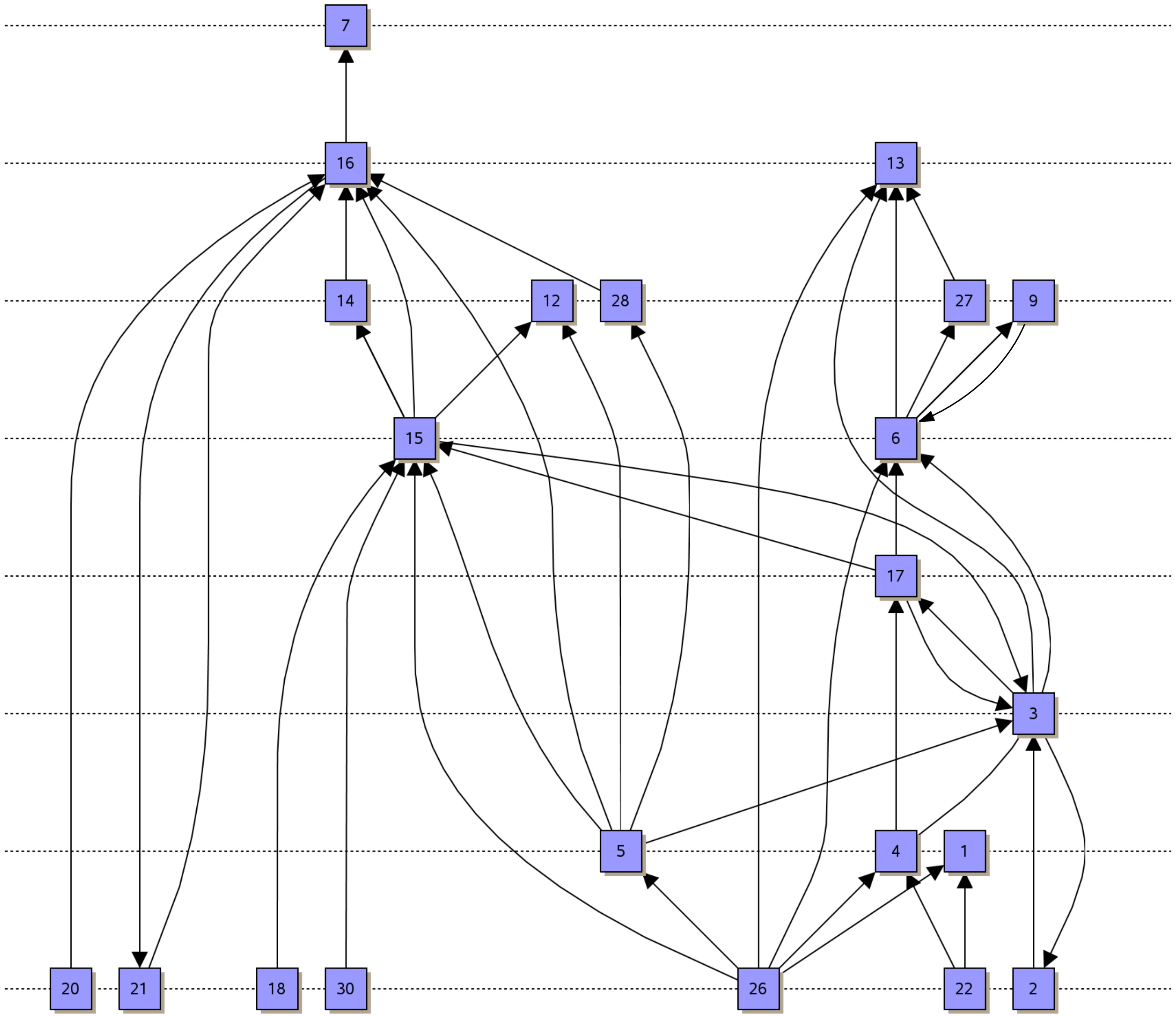
- vertices evenly spaced



Hierarchical Drawing

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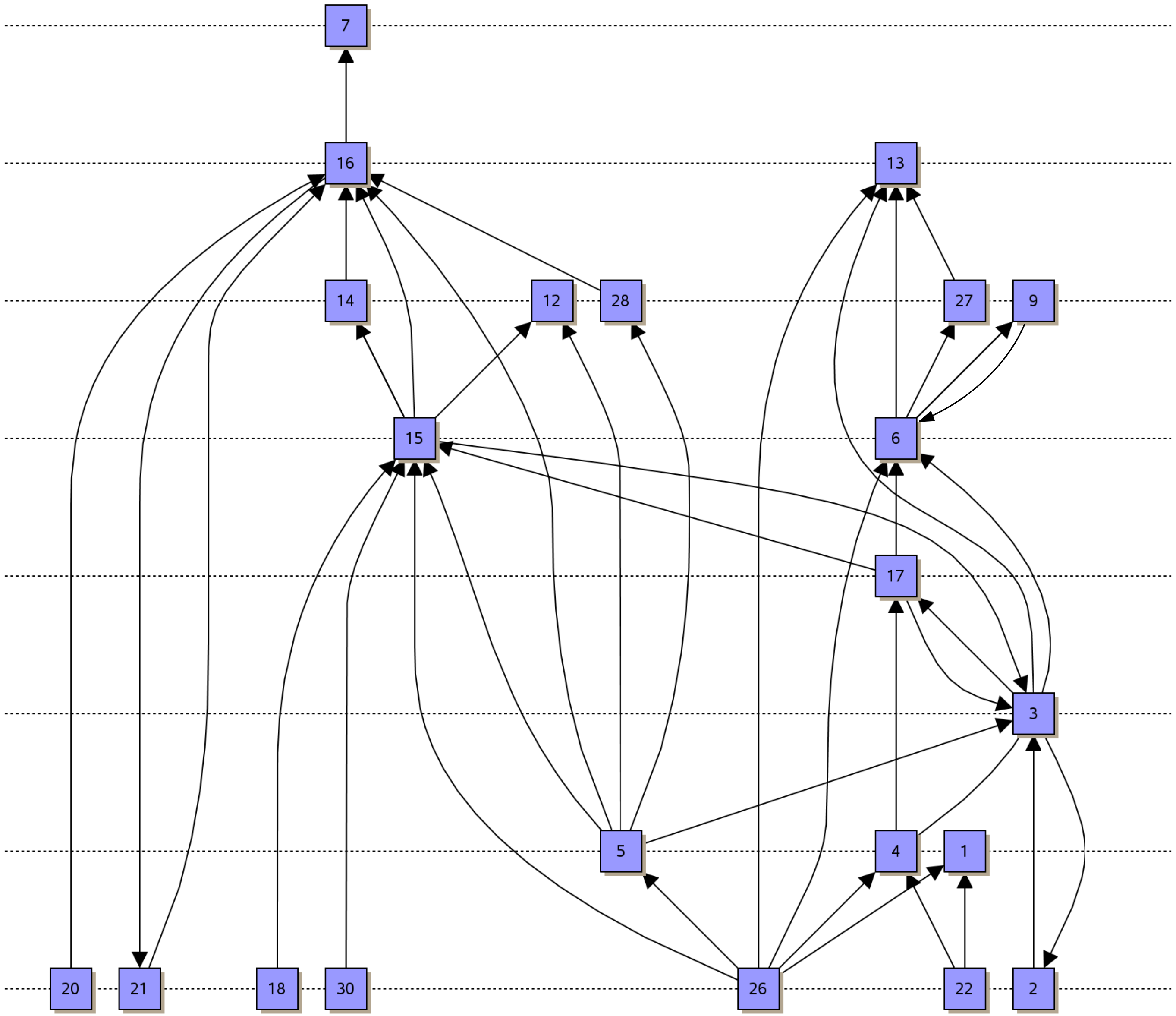


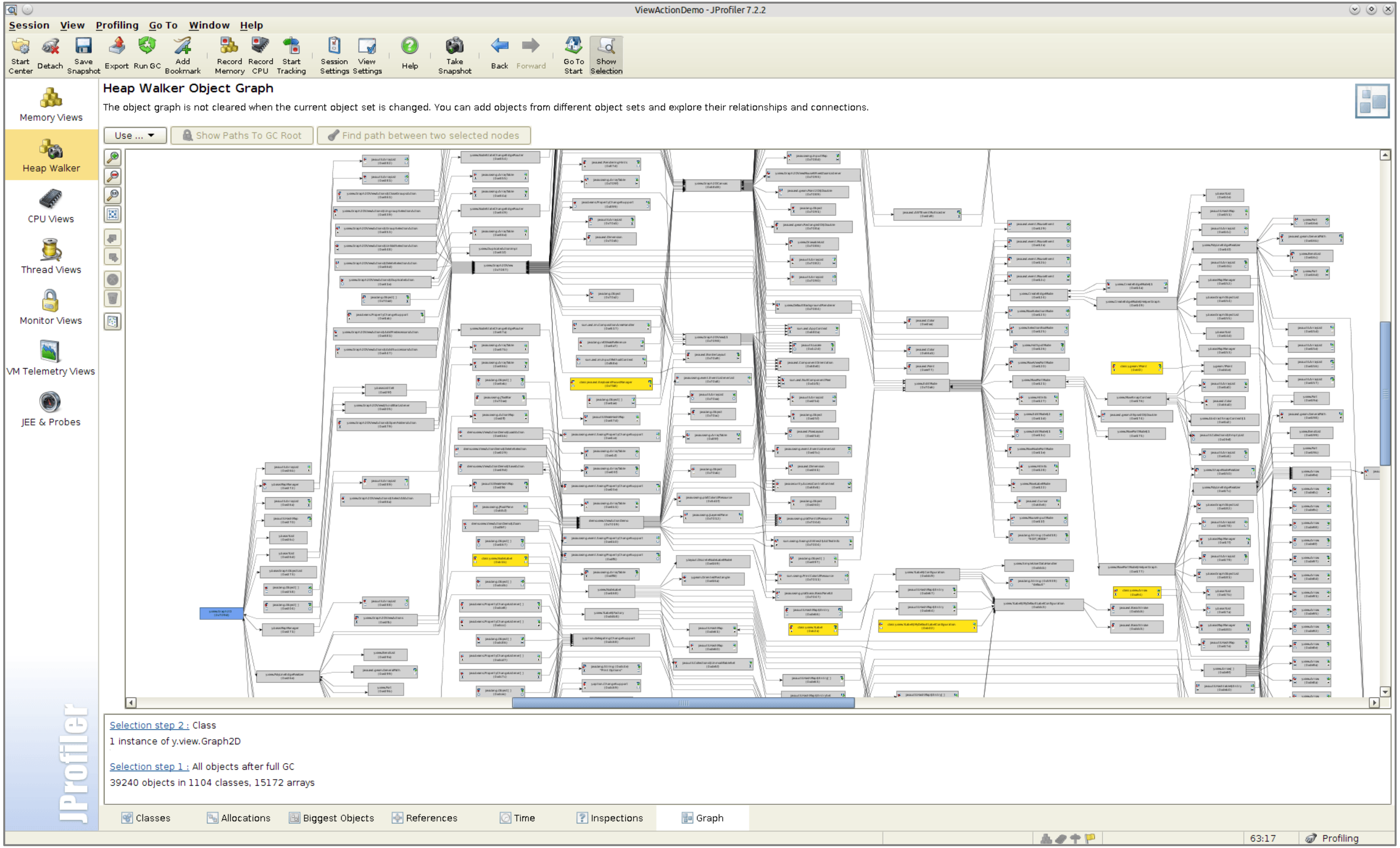
Desirable Properties.

- edges directed upwards
- vertices occur on (few) horizontal lines
- edge crossings minimized
- edges as short as possible
- vertices evenly spaced

Criteria can be contradictory!

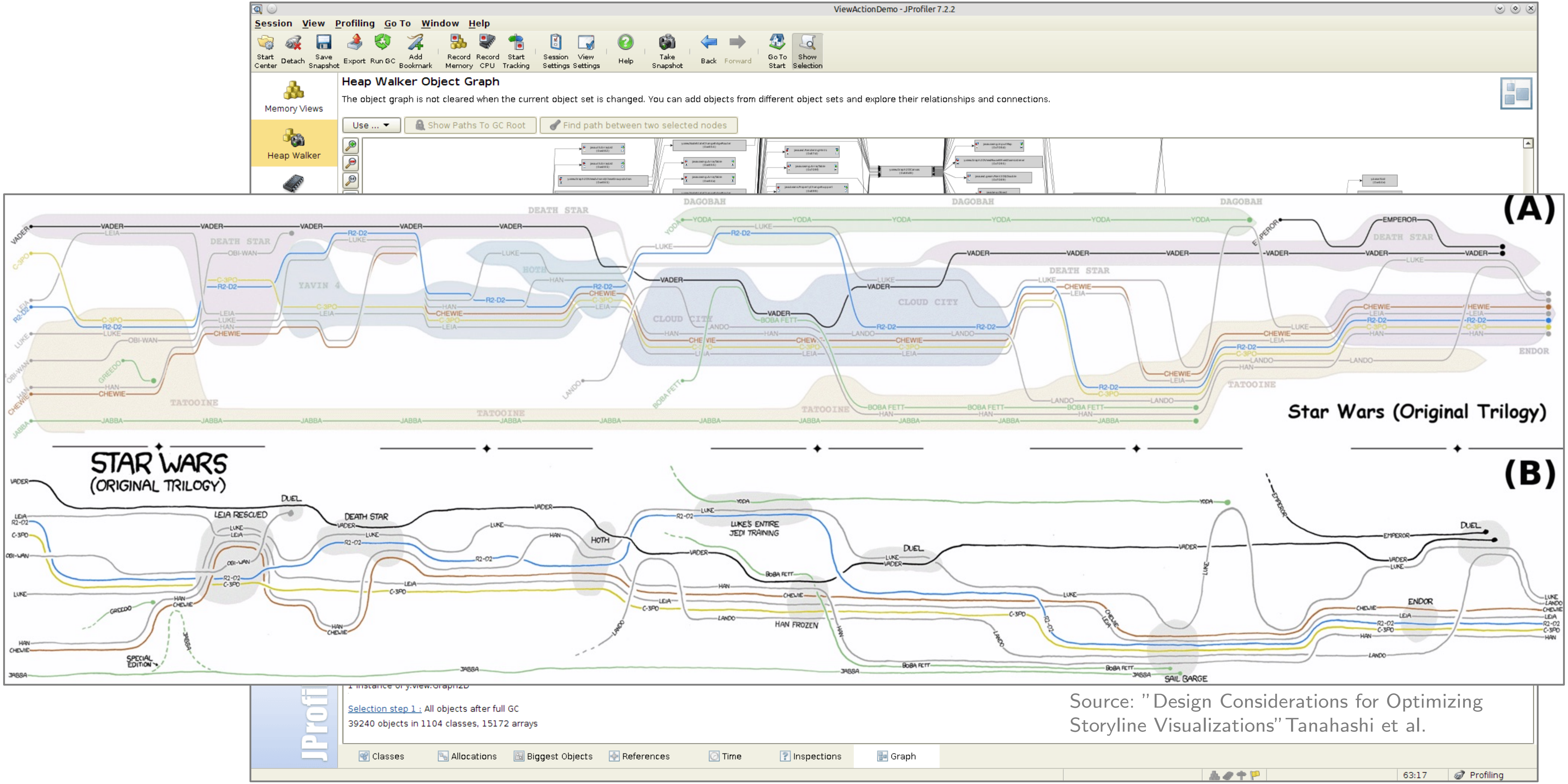
Hierarchical Drawing – Applications

yEd Gallery: Java profiler JProfiler using yFiles



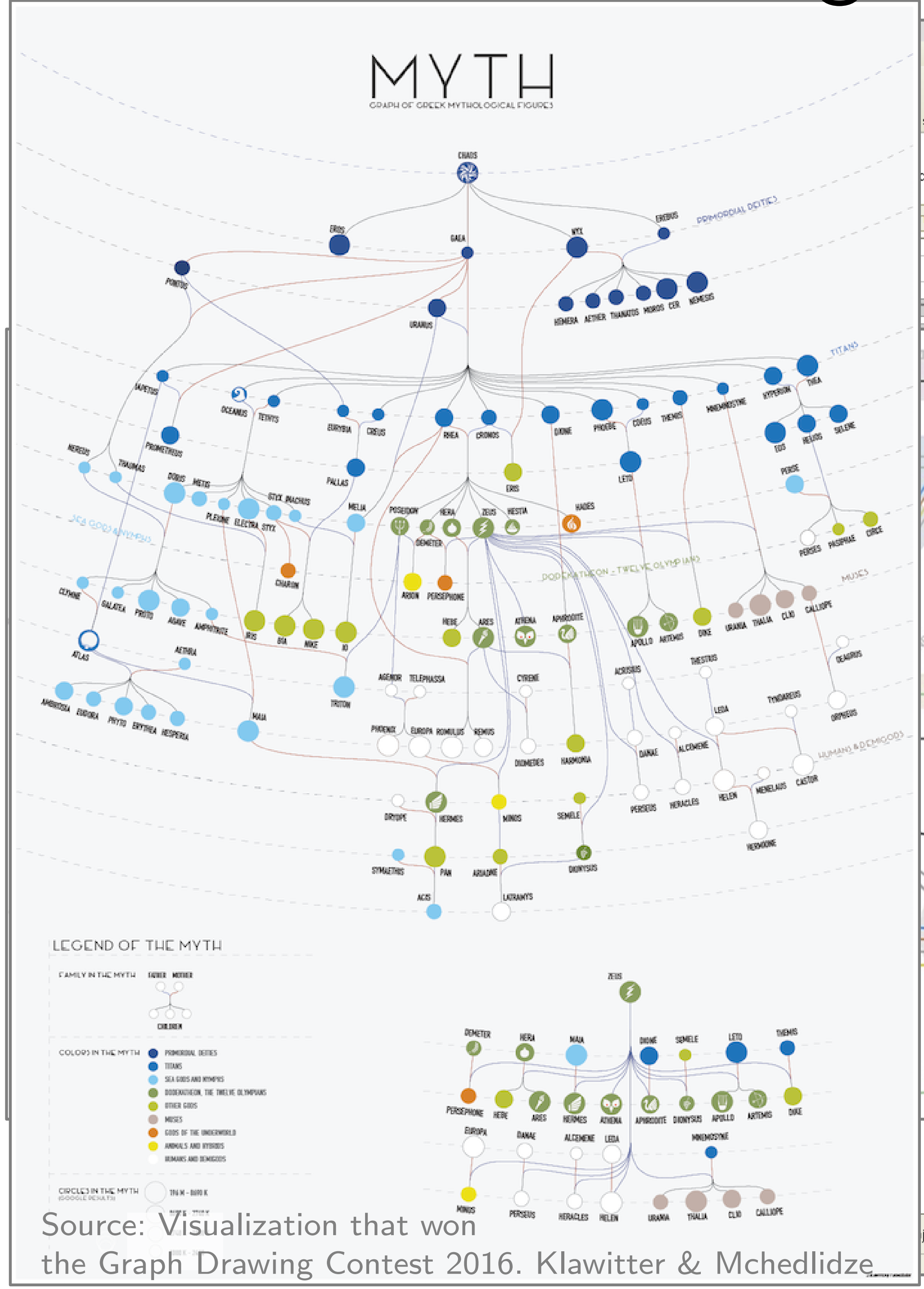
Hierarchical Drawing – Applications

yEd Gallery: Java profiler JProfiler using yFiles



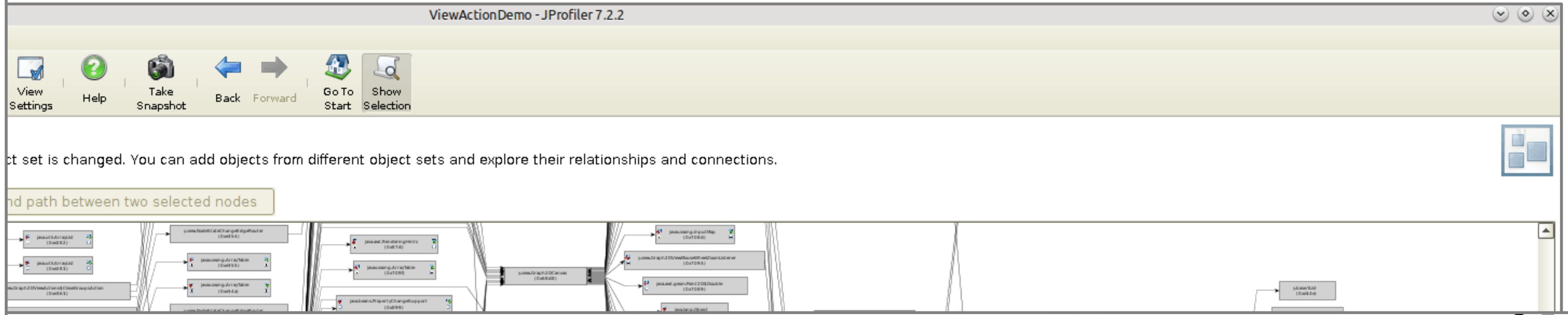
Source: "Design Considerations for Optimizing Storyline Visualizations" Tanahashi et al.

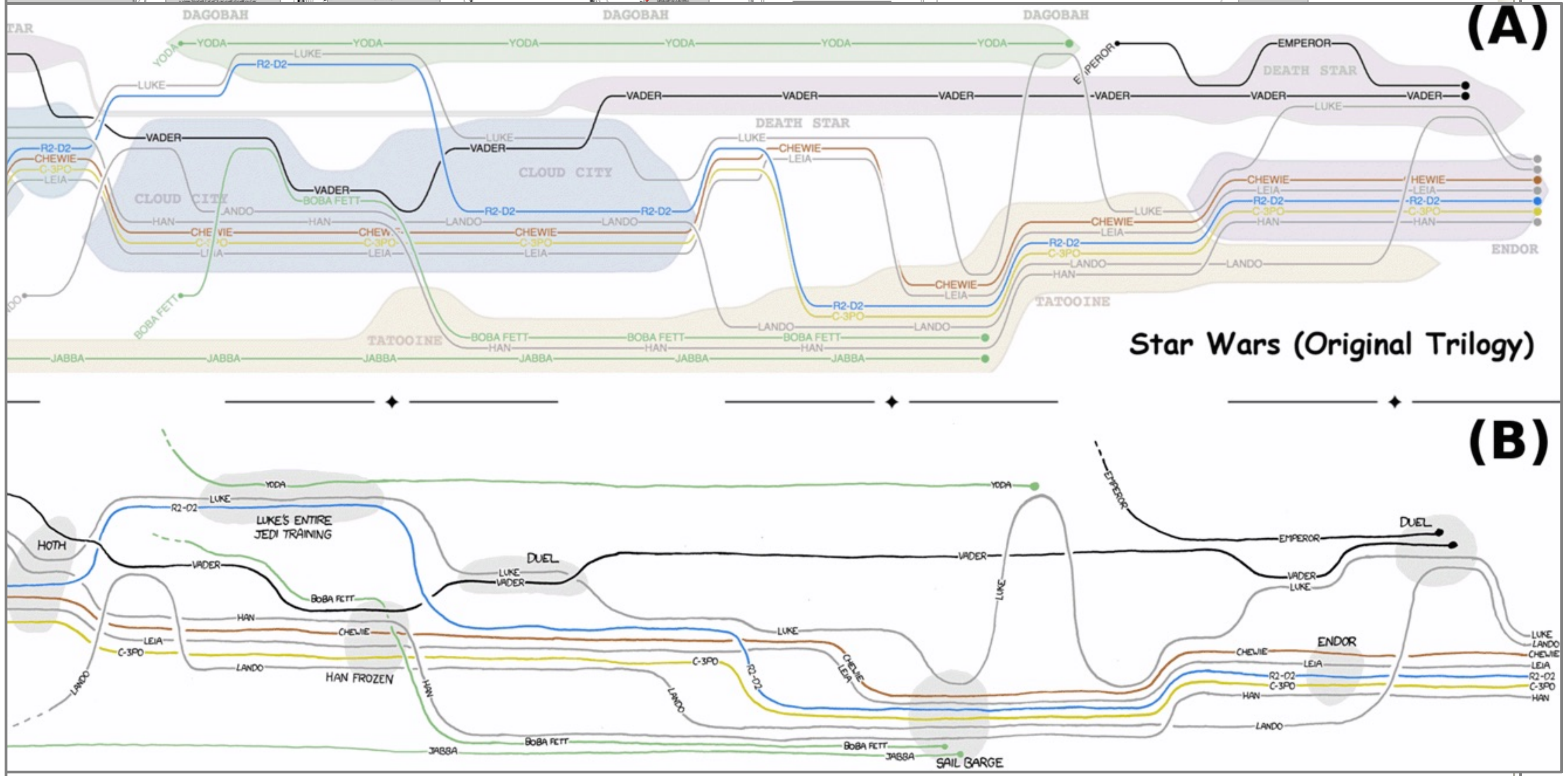
Hierarchical Drawing – Applications



Source: Visualization that won the Graph Drawing Contest 2016. Klawitter & Mchedlidze

yEd Gallery: Java profiler JProfiler using yFiles



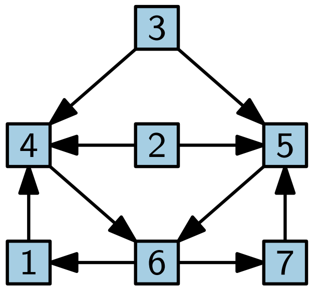


Source: "Design Considerations for Optimizing Storyline Visualizations" Tanahashi et al.

Classical Approach – Sugiyama Framework

[Sugiyama, Tagawa, Toda '81]

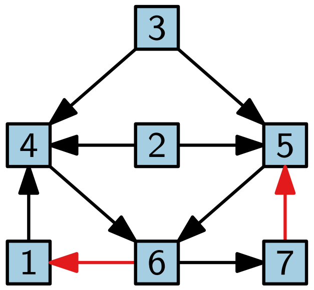
Input



Classical Approach – Sugiyama Framework

[Sugiyama, Tagawa, Toda '81]

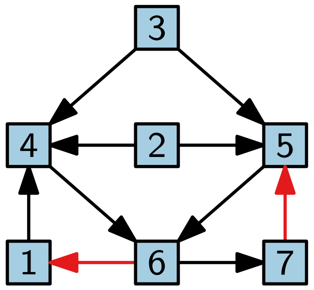
Input

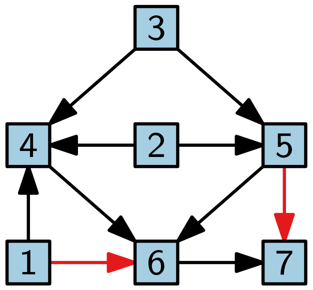


Classical Approach – Sugiyama Framework

[Sugiyama, Tagawa, Toda '81]

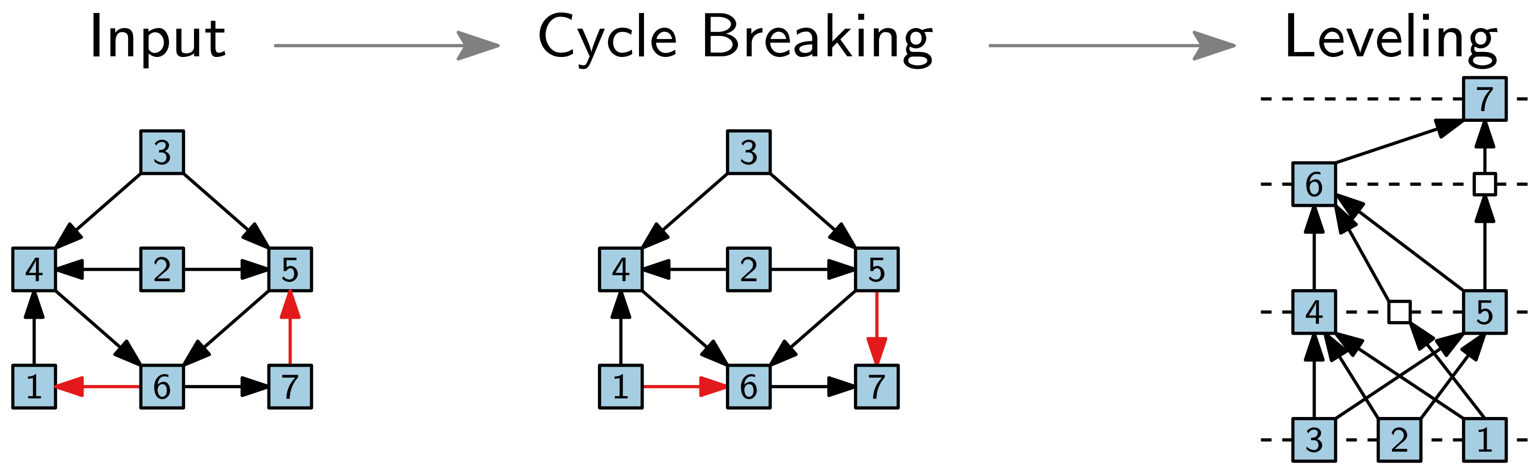
Input → Cycle Breaking





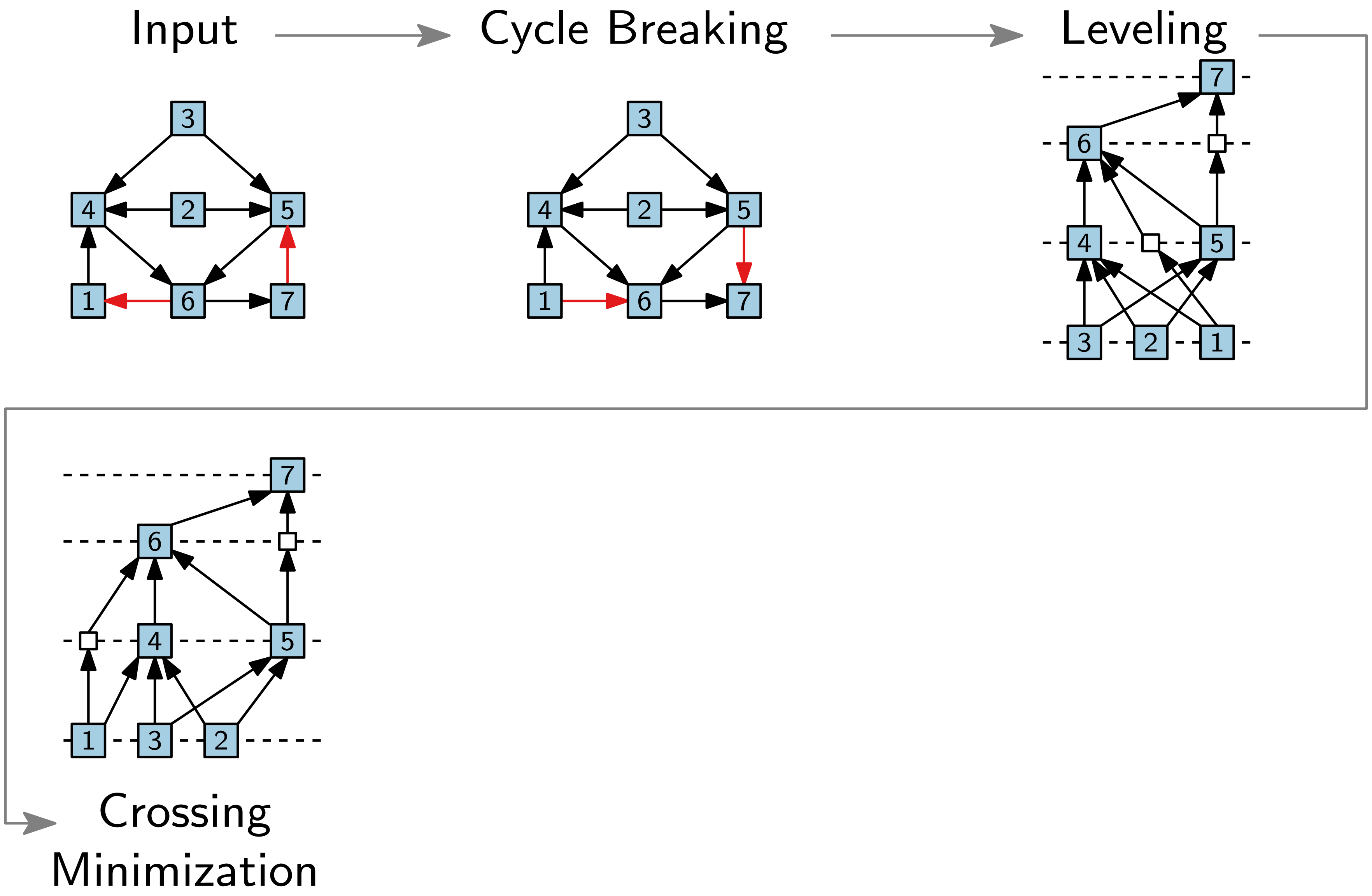
Classical Approach – Sugiyama Framework

[Sugiyama, Tagawa, Toda ’81]



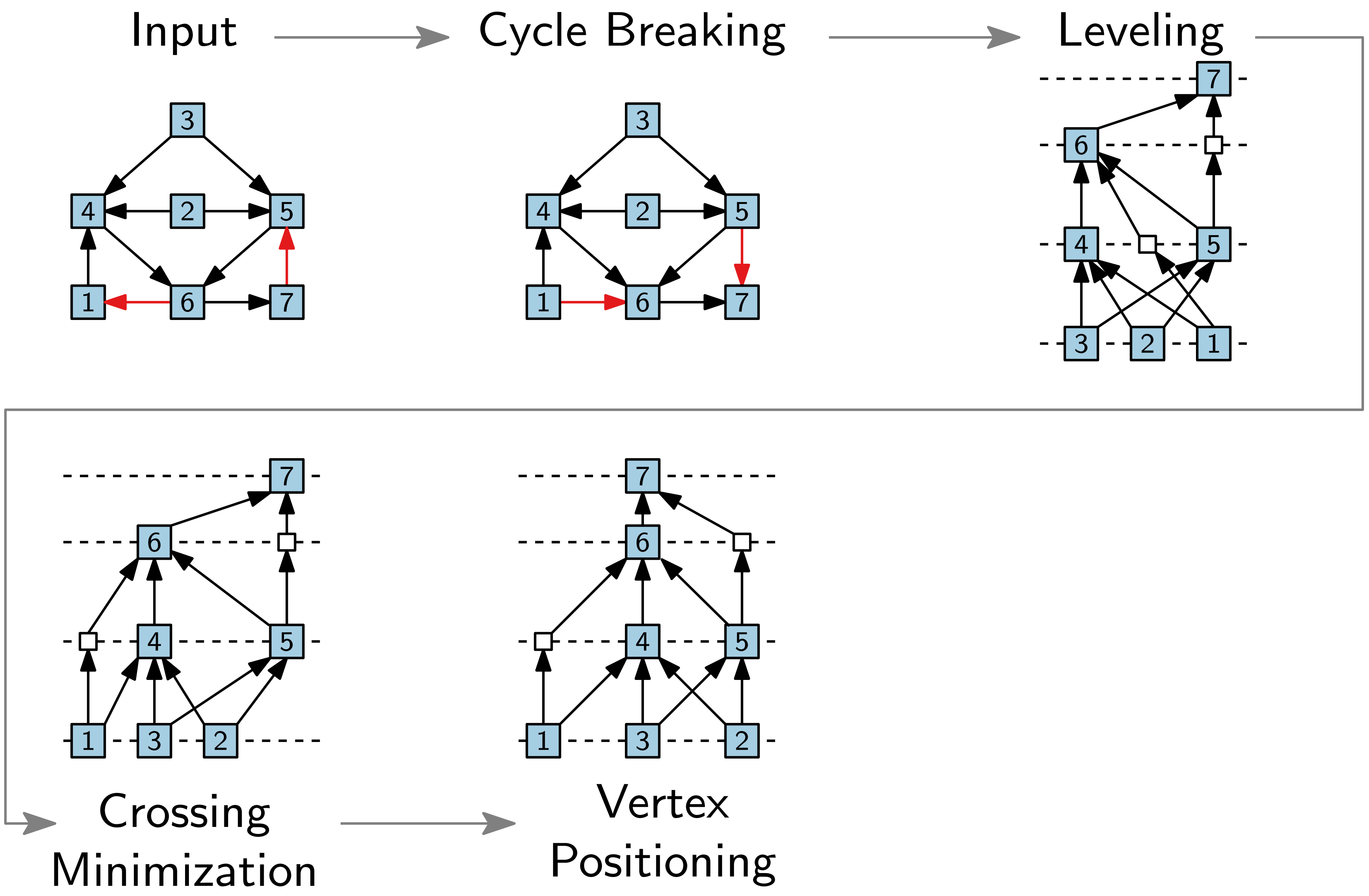
Classical Approach – Sugiyama Framework

[Sugiyama, Tagawa, Toda '81]



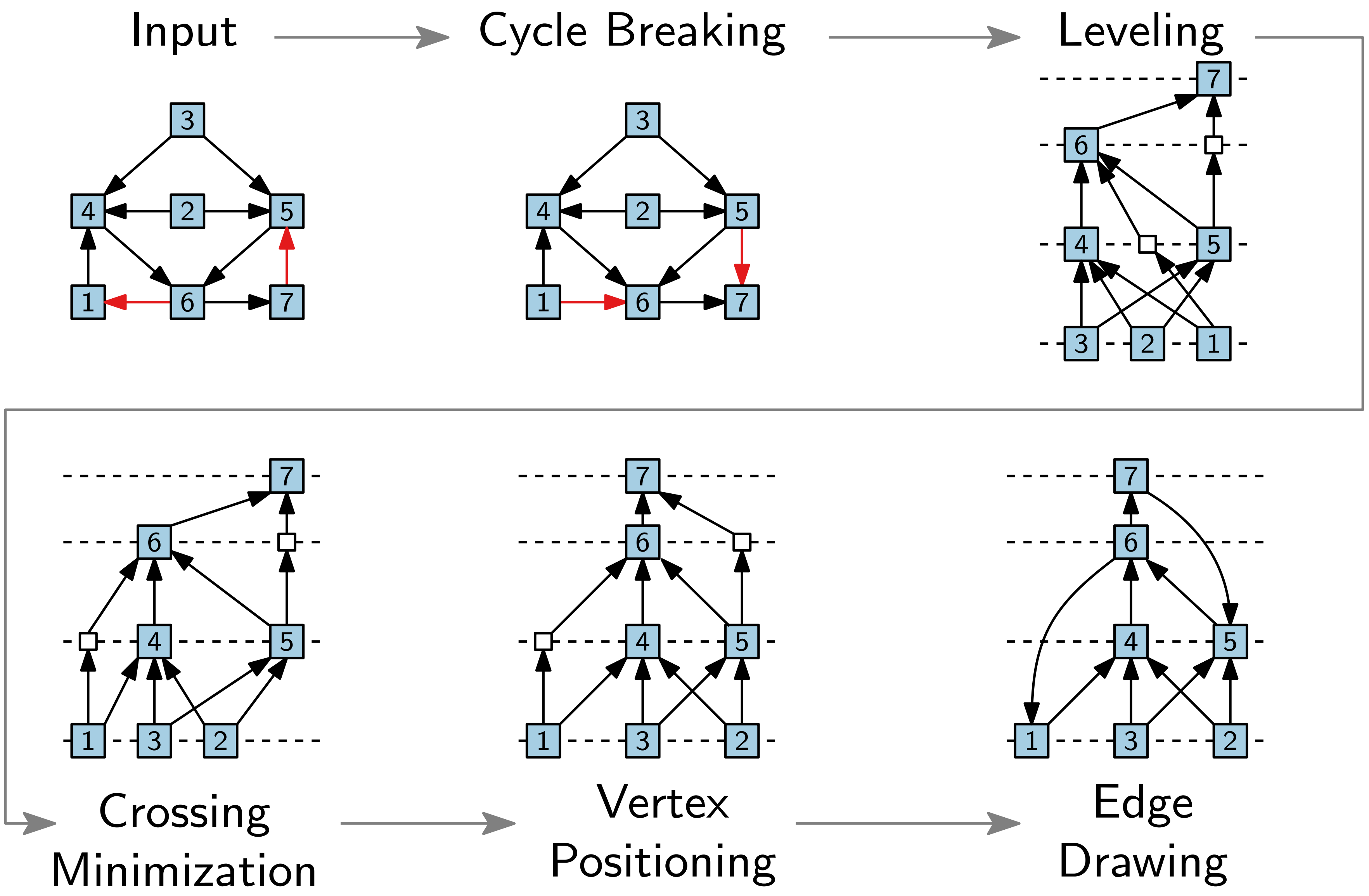
Classical Approach – Sugiyama Framework

[Sugiyama, Tagawa, Toda '81]

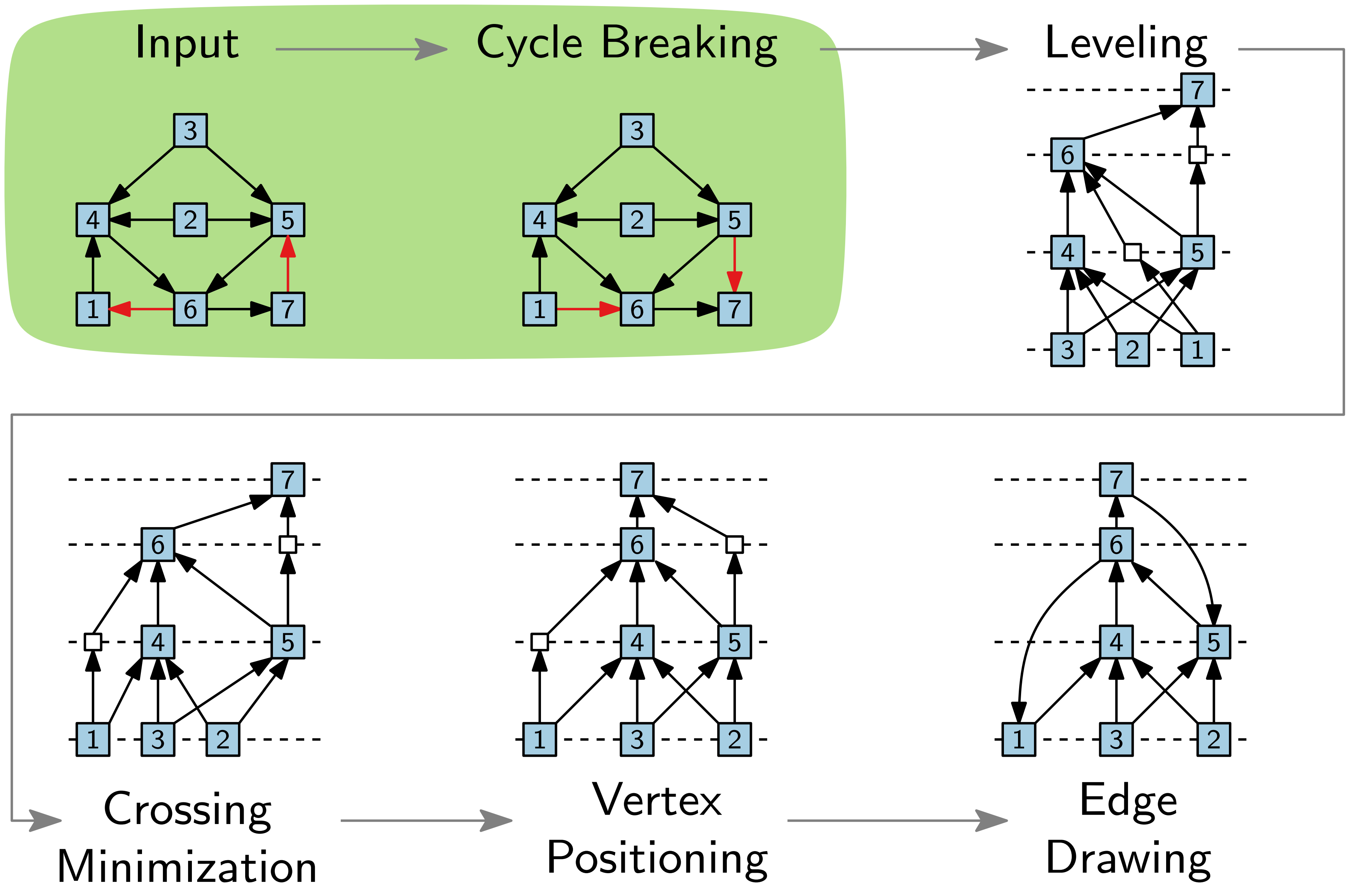


Classical Approach – Sugiyama Framework

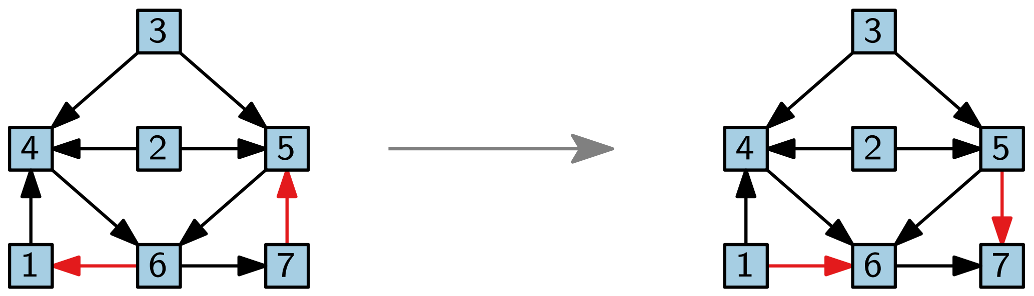
[Sugiyama, Tagawa, Toda ’81]



Step 1: Cycle breaking

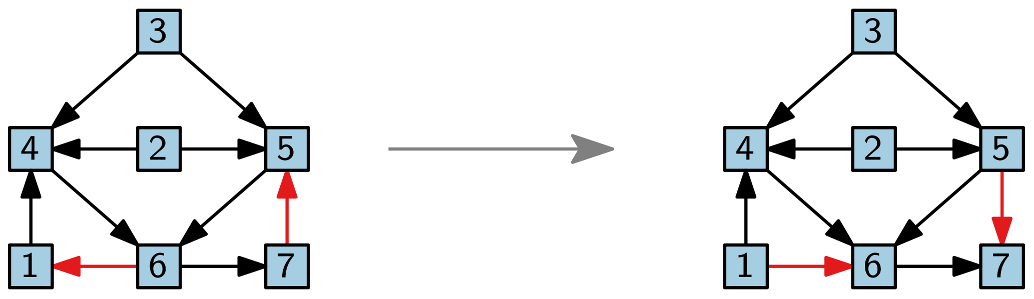


Step 1: Cycle breaking



Approach.

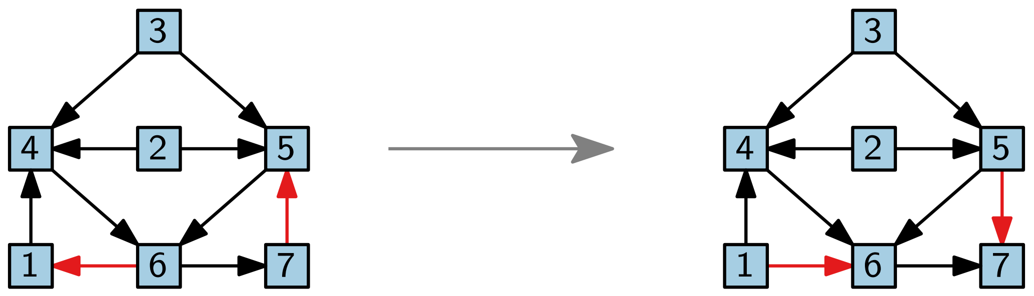
Step 1: Cycle breaking



Approach.

- Find minimum-size set $E^\star$ of edges that are not upward.

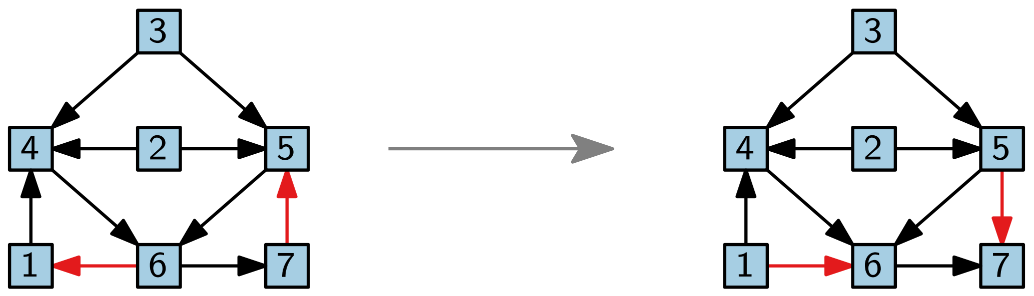
Step 1: Cycle breaking



Approach.

- Find minimum-size set $E^\star$ of edges that are not upward.
- Remove $E^\star$ and insert reversed edges.

Step 1: Cycle breaking



Approach.

- Find minimum-size set $E^\star$ of edges that are not upward.
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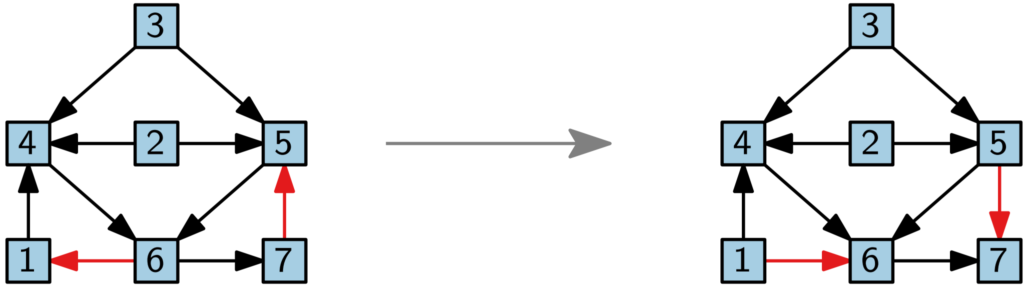
Problem MINIMUM FEEDBACK ARC SET **(FAS).**

Step 1: Cycle breaking



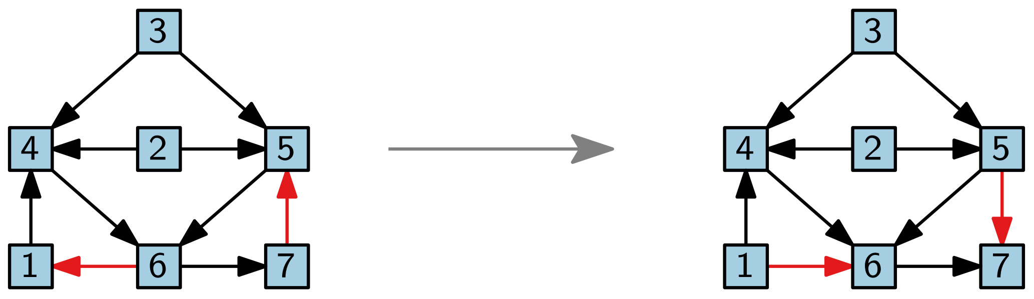
Approach.

- Find minimum-size set $E^\star$ of edges that are not upward.
- Remove $E^\star$ and insert reversed edges.

Problem MINIMUM FEEDBACK ARC SET **(FAS).**

- Input: directed graph $G = (V, E)$
- Output:

Step 1: Cycle breaking



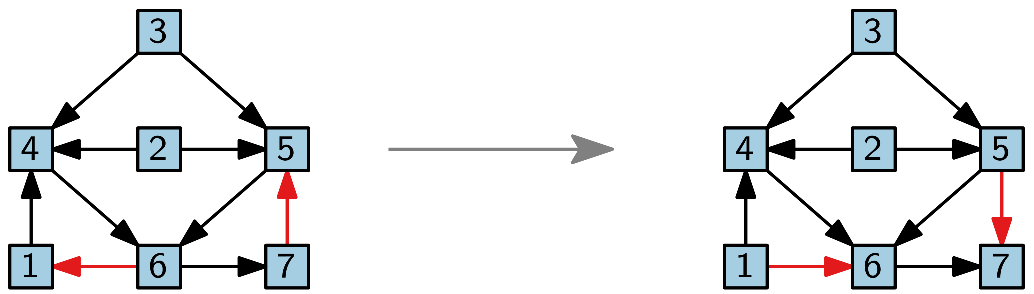
Approach.

- Find minimum-size set $E^\star$ of edges that are not upward.
- Remove $E^\star$ and insert reversed edges.

Problem MINIMUM FEEDBACK ARC SET (FAS).

- Input: directed graph $G = (V, E)$
- Output: min.-size set $E^\star \subseteq E$, such that $G^\star = (V, E \setminus E^\star)$ acyclic

Step 1: Cycle breaking



Approach.

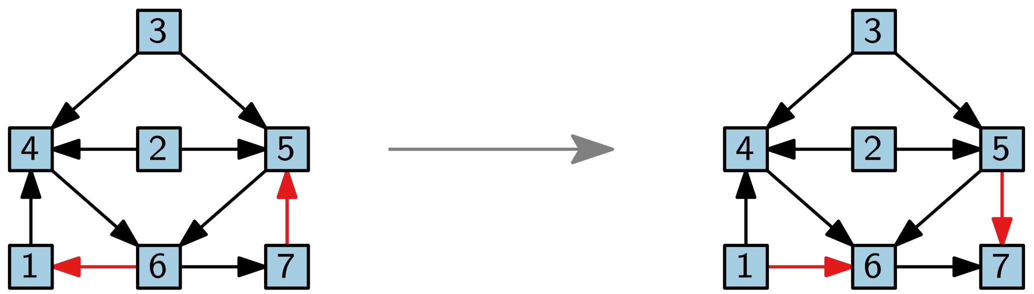
- Find minimum-size set $E^\star$ of edges that are not upward.
- Remove $E^\star$ and insert reversed edges.

Problem Minimum Feedback ~~Arc~~ Set (F~~A~~S).

- Input: directed graph $G = (V, E)$
- Output: min.-size set $E^\star \subseteq E$, such that $G^\star = (V, \cancel{E \setminus E^\star})$ acyclic

$$(E \setminus E^\star) \cup E_r^\star$$

Step 1: Cycle breaking



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- Input: directed graph $G = (V, E)$
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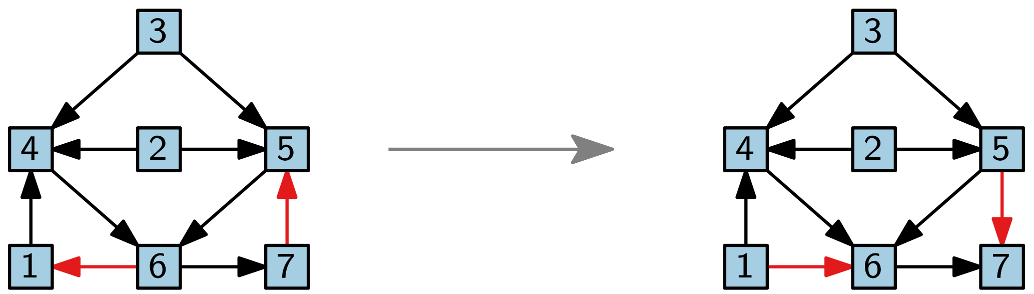
edges in $E^\star$ but reversed

$$(E \setminus E^\star) \cup \boxed{E_r^\star}$$

Step 1: Cycle breaking



Approach.

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Problem Minimum Feedback ~~Arc~~ Set **(F~~A~~S).**

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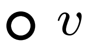
edges in $E^\star$ but reversed

$$(E \setminus E^\star) \cup E_r^\star$$

. . . NP-hard ☹

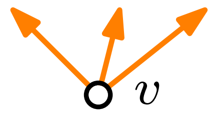
Heuristic 1

[Berger, Shor '90]



Heuristic 1

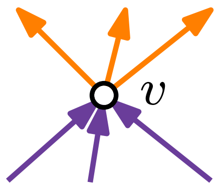
[Berger, Shor '90]



$$N^{\rightarrow}(v) \quad := \quad \{(v,u) | (v,u) \in E\}$$

Heuristic 1

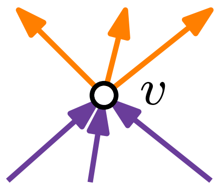
[Berger, Shor '90]



$$
\begin{aligned}
N^{\rightarrow}(v) &:= \{(v,u) | (v,u) \in E\} \\
N^{\leftarrow}(v) &:= \{(u,v) | (u,v) \in E\}
\end{aligned}
$$

Heuristic 1

[Berger, Shor '90]

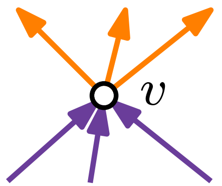


$$
\begin{aligned}
N^{\rightarrow}(v) &:= \{(v,u) \mid (v,u) \in E\} \\
N^{\leftarrow}(v) &:= \{(u,v) \mid (u,v) \in E\} \\
N(v) &:= N^{\rightarrow}(v) \cup N^{\leftarrow}(v)
\end{aligned}
$$

Heuristic 1

[Berger, Shor '90]

GreedyMakeAcyclic(Digraph $G = (V, E)$)



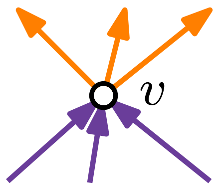
$$\begin{aligned} N^{\rightarrow}(v) &:= \{(v,u) | (v,u) \in E\} \\ N^{\leftarrow}(v) &:= \{(u,v) | (u,v) \in E\} \\ N(v) &:= N^{\rightarrow}(v) \cup N^{\leftarrow}(v) \end{aligned}$$

Heuristic 1

[Berger, Shor '90]

GreedyMakeAcyclic(Digraph $G = (V, E)$)

$E' \leftarrow \emptyset$



$$\begin{aligned} N^{\rightarrow}(v) &:= \{(v,u) | (v,u) \in E\} \\ N^{\leftarrow}(v) &:= \{(u,v) | (u,v) \in E\} \\ N(v) &:= N^{\rightarrow}(v) \cup N^{\leftarrow}(v) \end{aligned}$$

return (V, E')

Heuristic 1

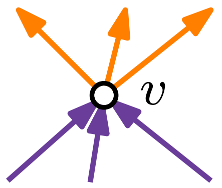
[Berger, Shor '90]

GreedyMakeAcyclic(Digraph $G = (V, E)$)

$E' \leftarrow \emptyset$

foreach $v \in V$ **do**

return (V, E')



$$
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N^{\rightarrow}(v) &:= \{(v,u) | (v,u) \in E\} \\
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\end{aligned}
$$

Heuristic 1

[Berger, Shor '90]

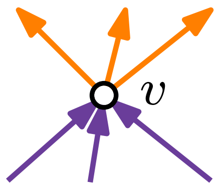
GreedyMakeAcyclic(Digraph $G = (V, E)$)

$E' \leftarrow \emptyset$

foreach $v \in V$ **do**

  if $|N^{\rightarrow}(v)| \geq |N^{\leftarrow}(v)|$ **then**

return (V, E')



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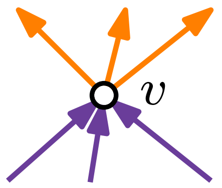
Heuristic 1

[Berger, Shor '90]

GreedyMakeAcyclic(Digraph $G = (V, E)$)

$E' \leftarrow \emptyset$
foreach $v \in V$ **do**
  if $|N^{\rightarrow}(v)| \geq |N^{\leftarrow}(v)|$ **then**
    $E' \leftarrow E' \cup N^{\rightarrow}(v)$

return (V, E')



$$N^{\rightarrow}(v) := \{(v, u) | (v, u) \in E\}$$
$$N^{\leftarrow}(v) := \{(u, v) | (u, v) \in E\}$$
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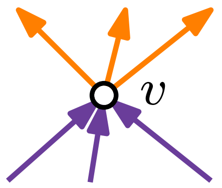
Heuristic 1

[Berger, Shor '90]



GreedyMakeAcyclic(Digraph $G = (V, E)$)

$E' \leftarrow \emptyset$

foreach $v \in V$ **do**

  if $|N^{\rightarrow}(v)| \geq |N^{\leftarrow}(v)|$ **then**

    $E' \leftarrow E' \cup N^{\rightarrow}(v)$

return (V, E')

$$N^{\rightarrow}(v) := \{(v,u) | (v,u) \in E\}$$
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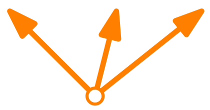

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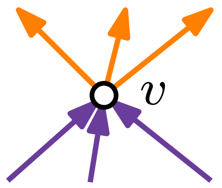

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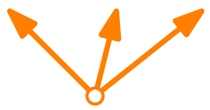

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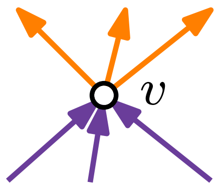

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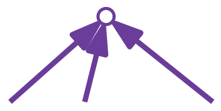

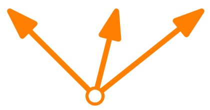

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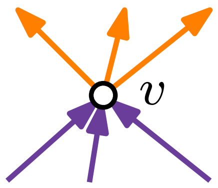

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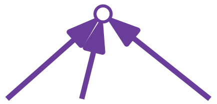

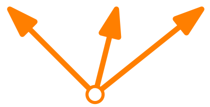

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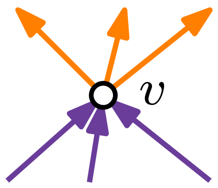

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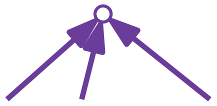

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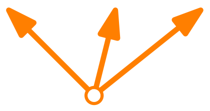

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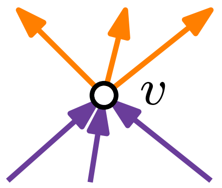

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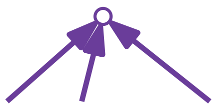

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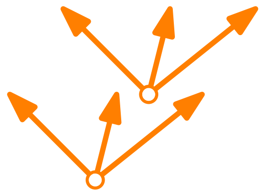

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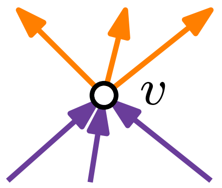

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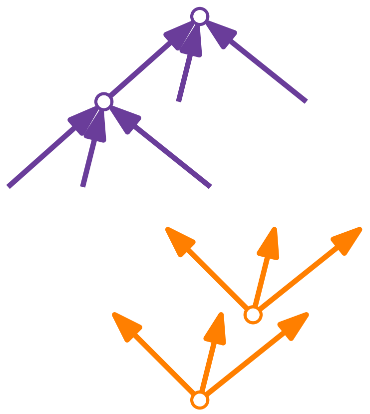

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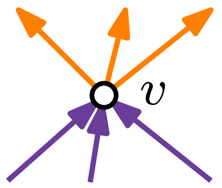

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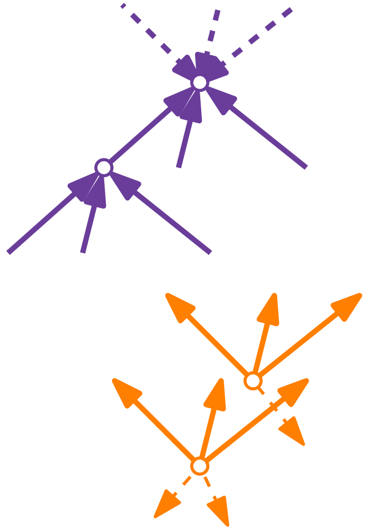

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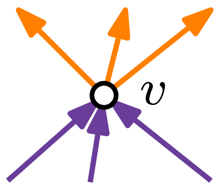

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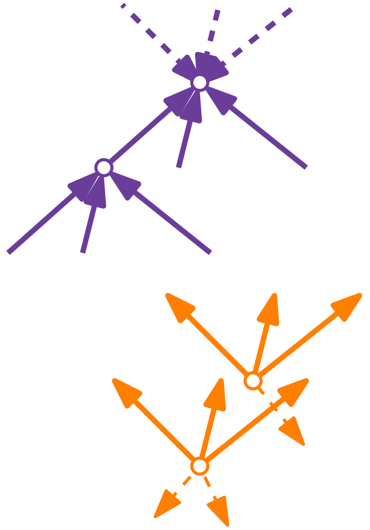

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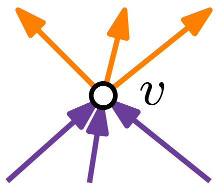

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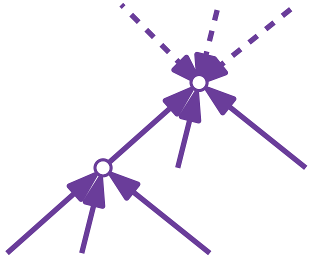

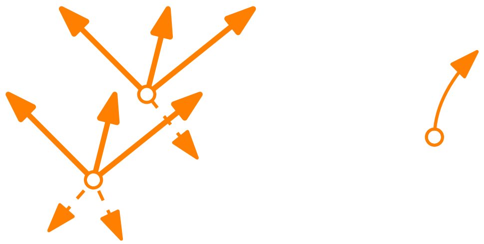

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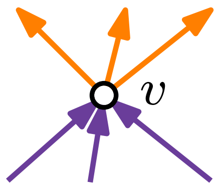

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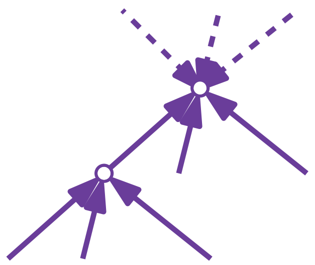

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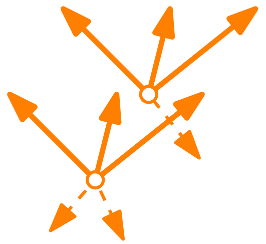

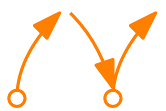

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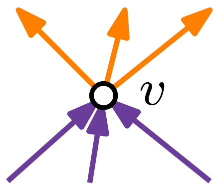

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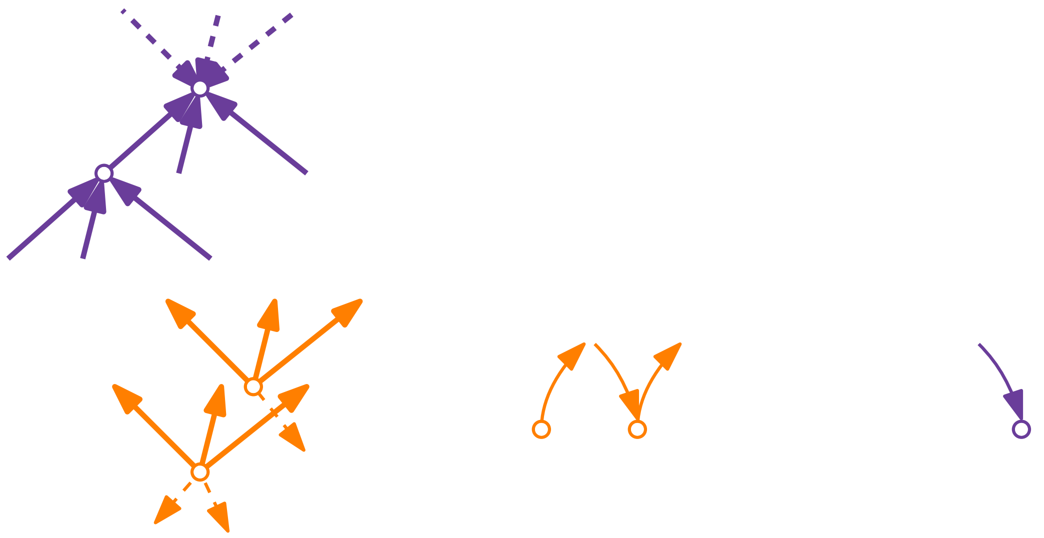

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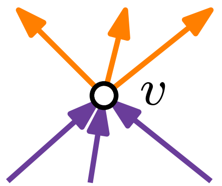

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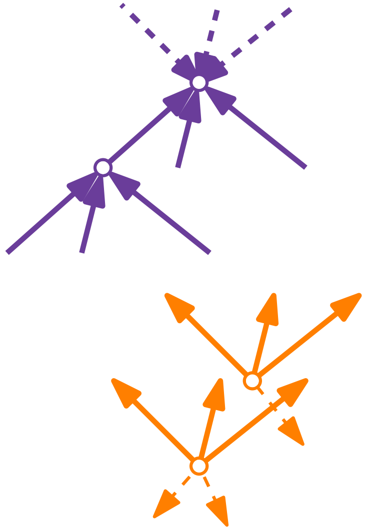

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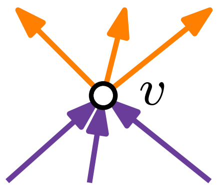

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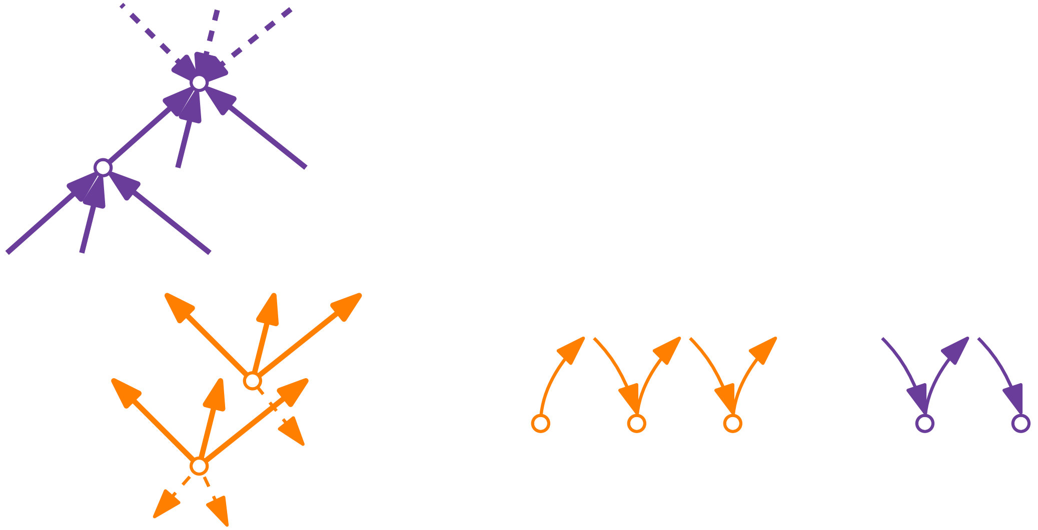

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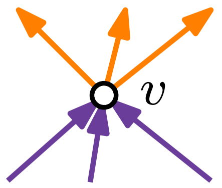

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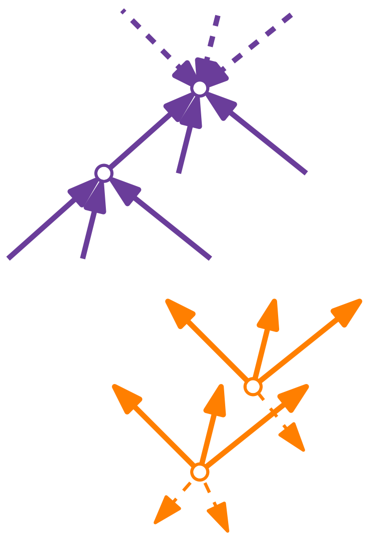

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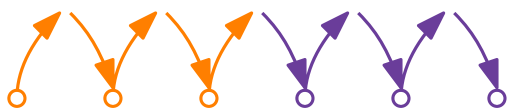

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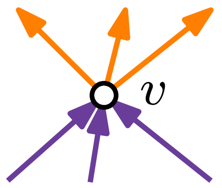

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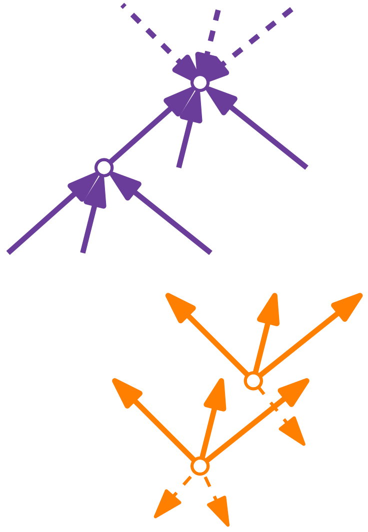

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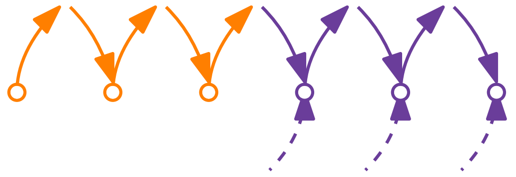

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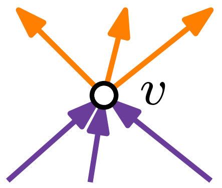

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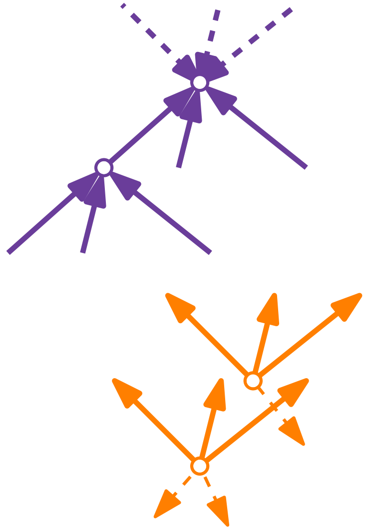

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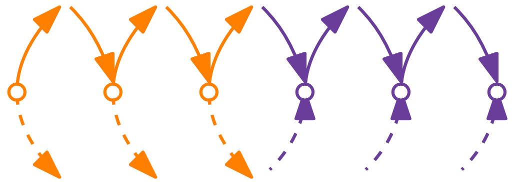
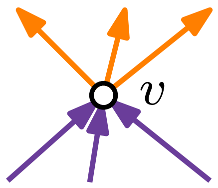


Heuristic 1

[Berger, Shor '90]



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  remove v and $N(v)$ from G.

return (V, E')

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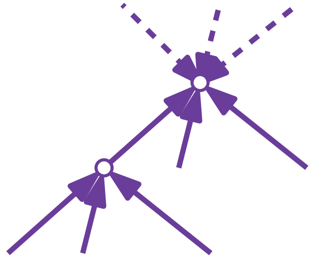

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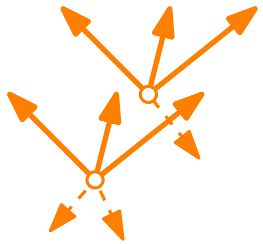

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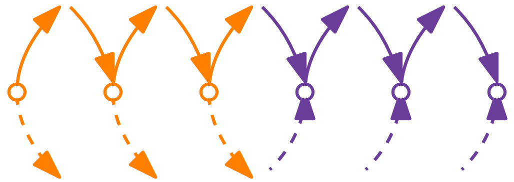

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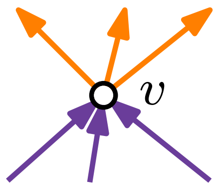

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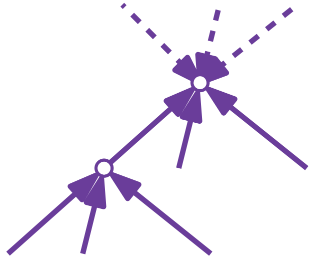

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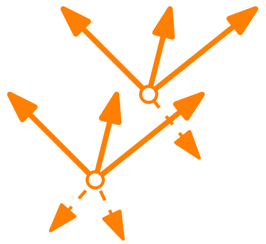

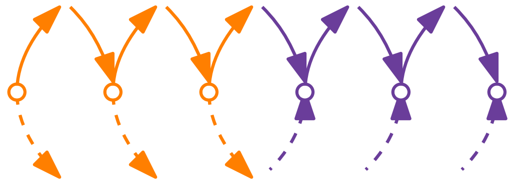

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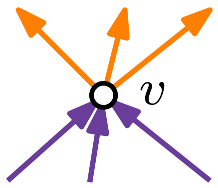

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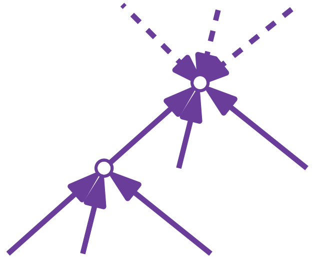
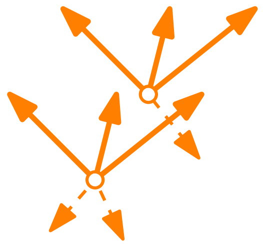
- Quality guarantee: $|E'| \geq$





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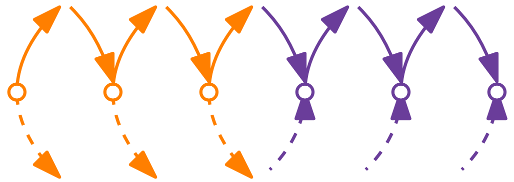

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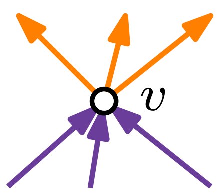

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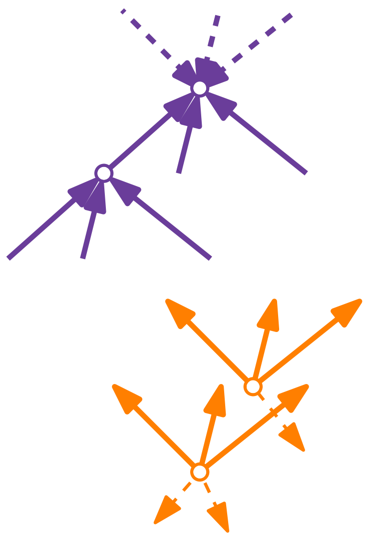
- Quality guarantee: $|E'| \geq |E|/2$



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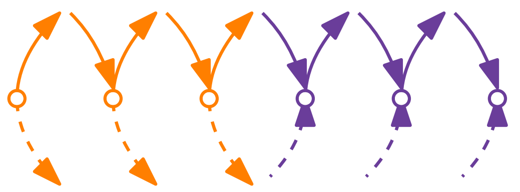
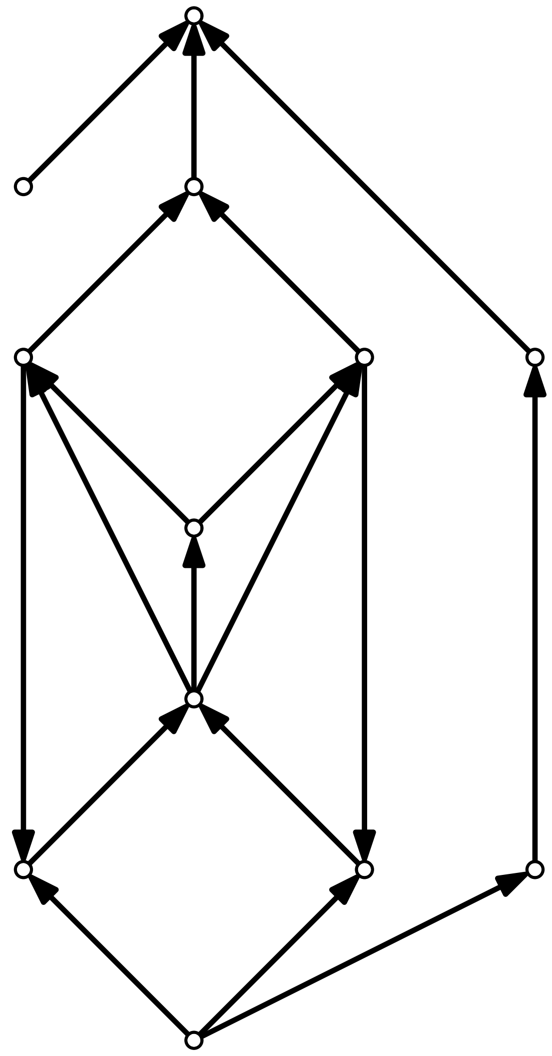


Heuristic 2

[Eades, Lin, Smyth '93]



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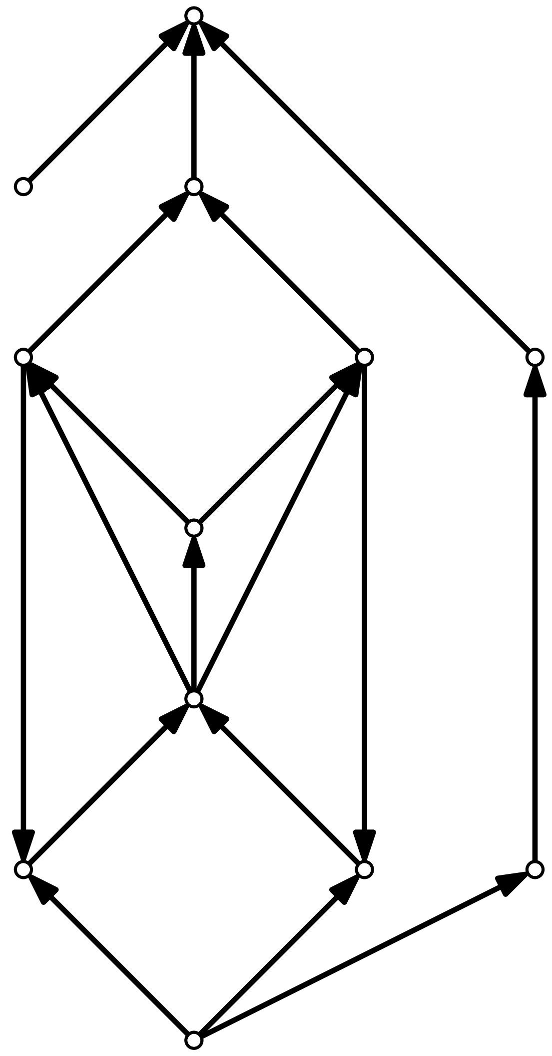

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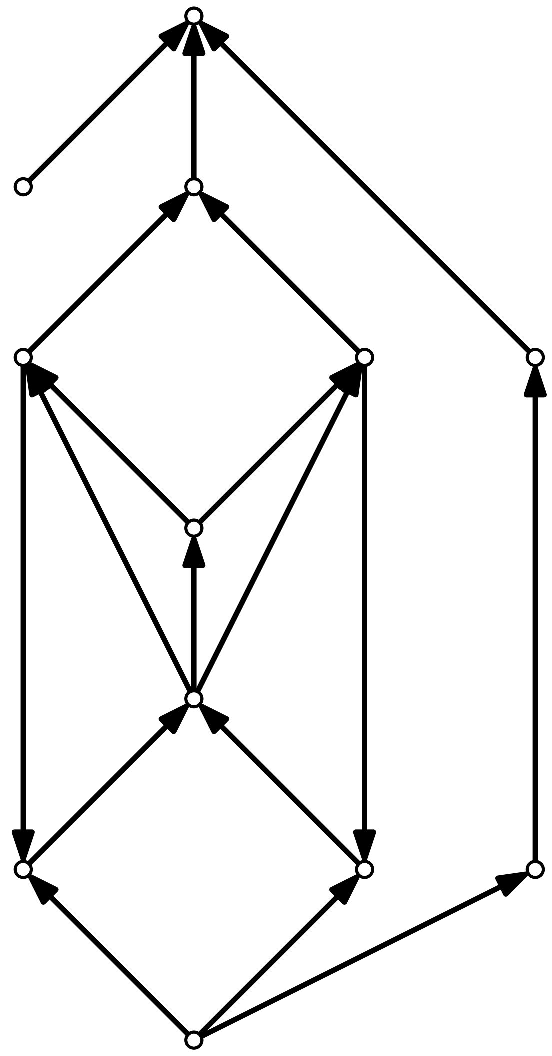

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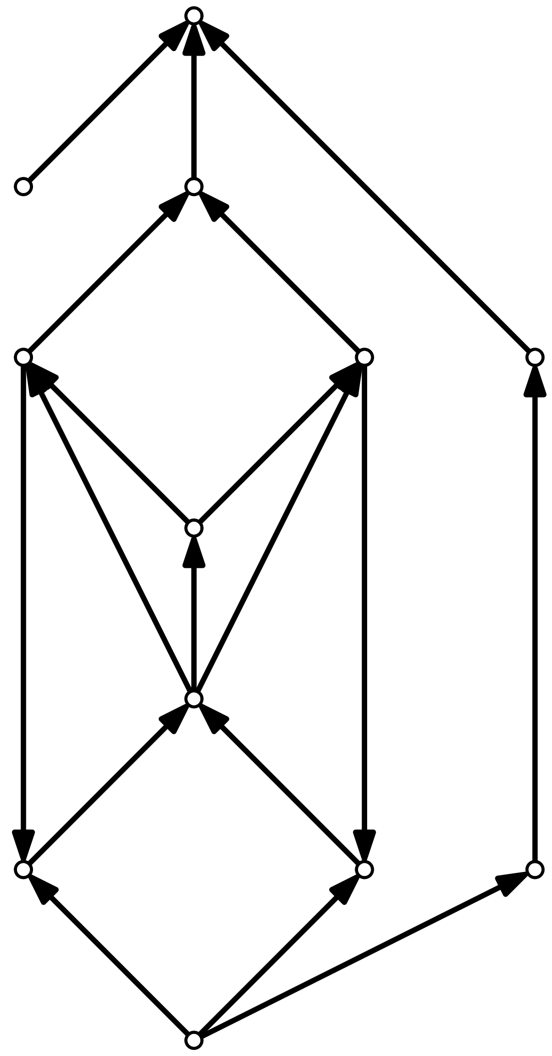

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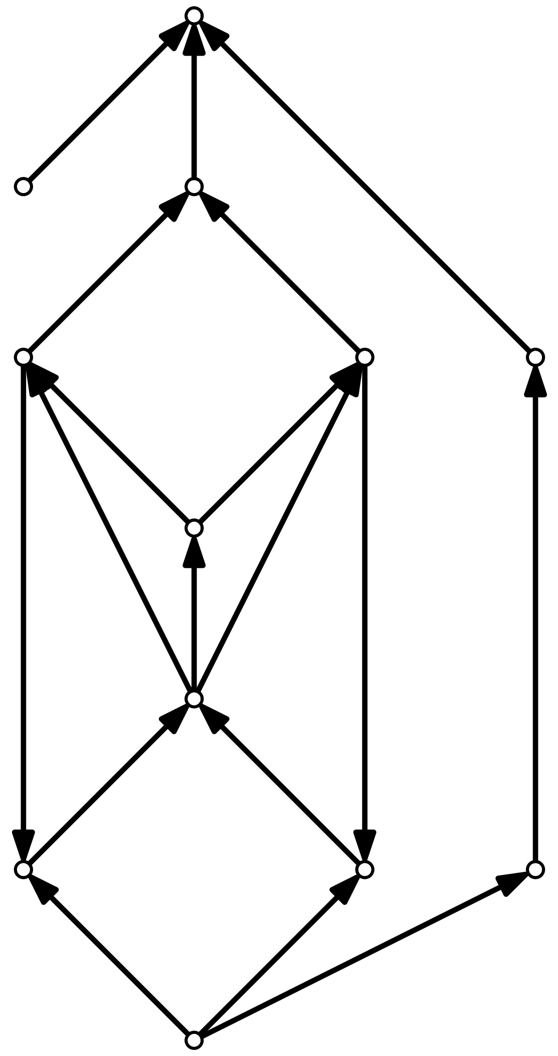

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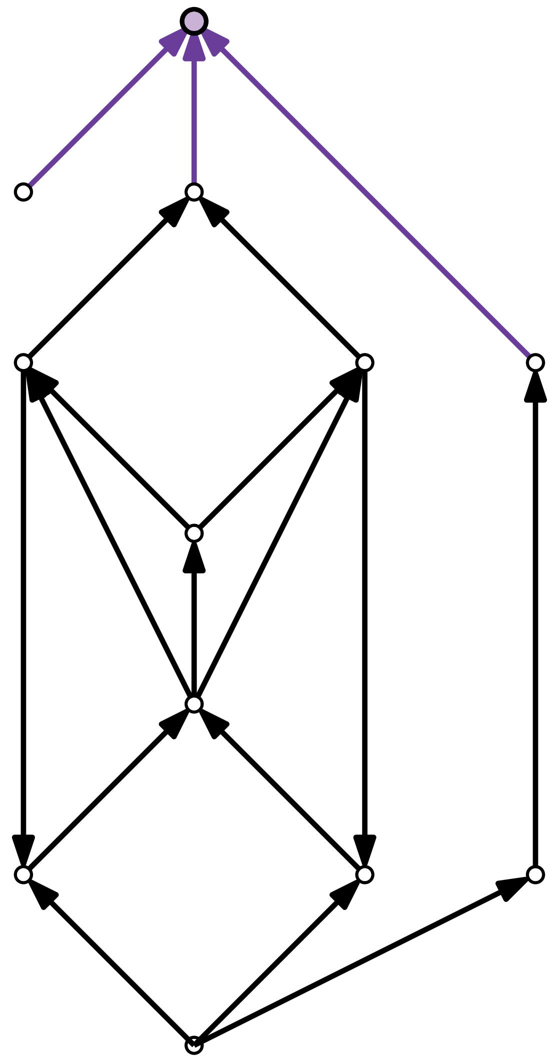

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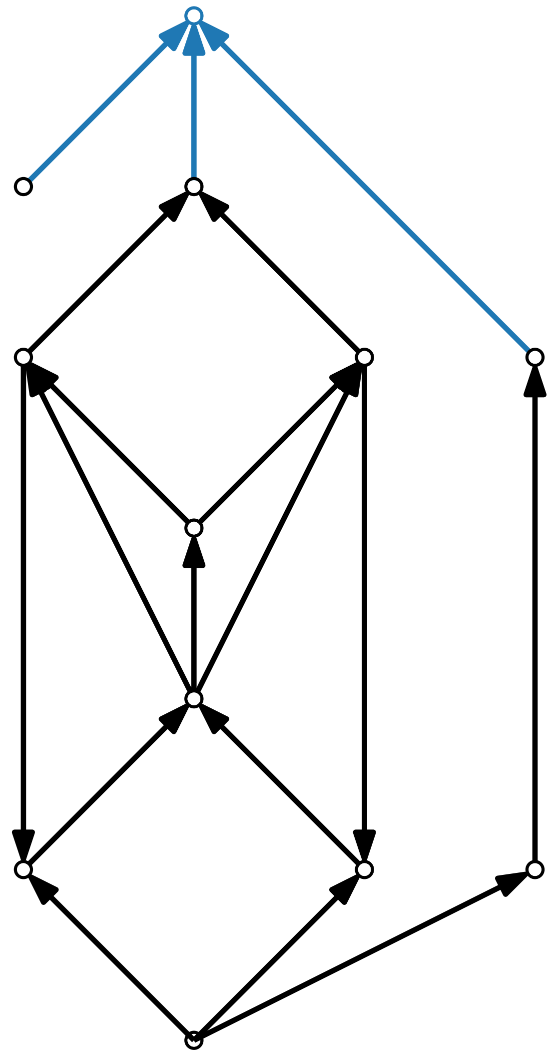

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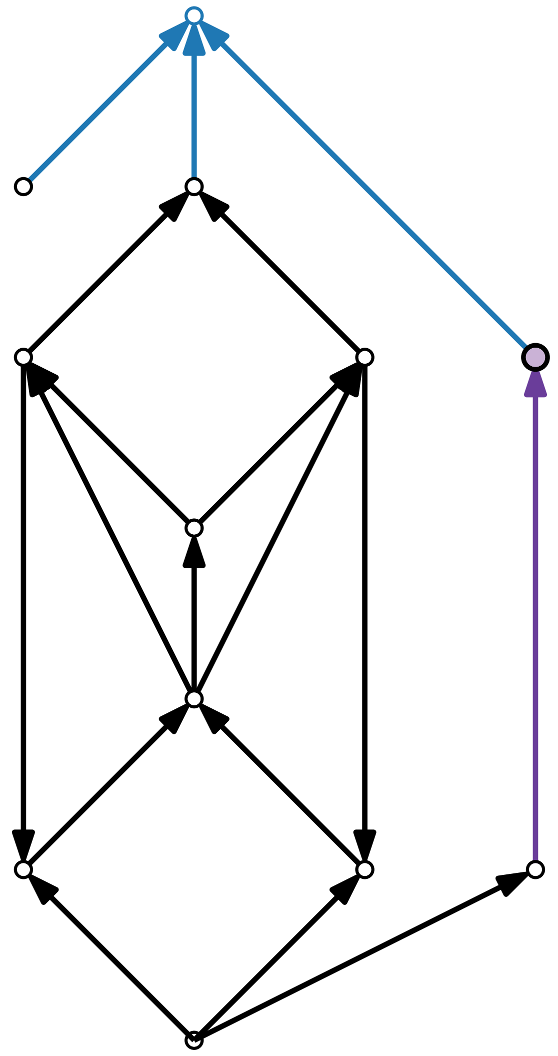

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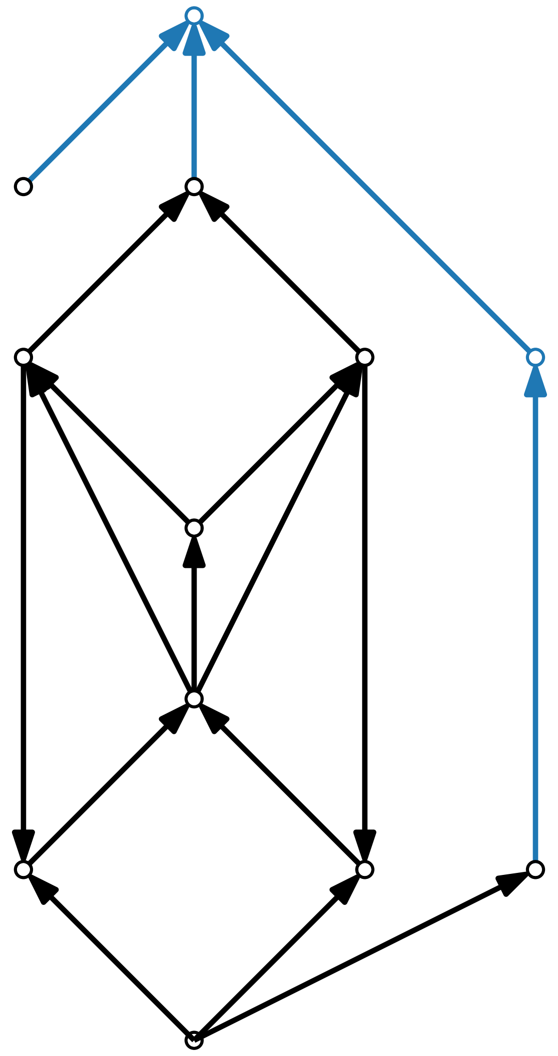

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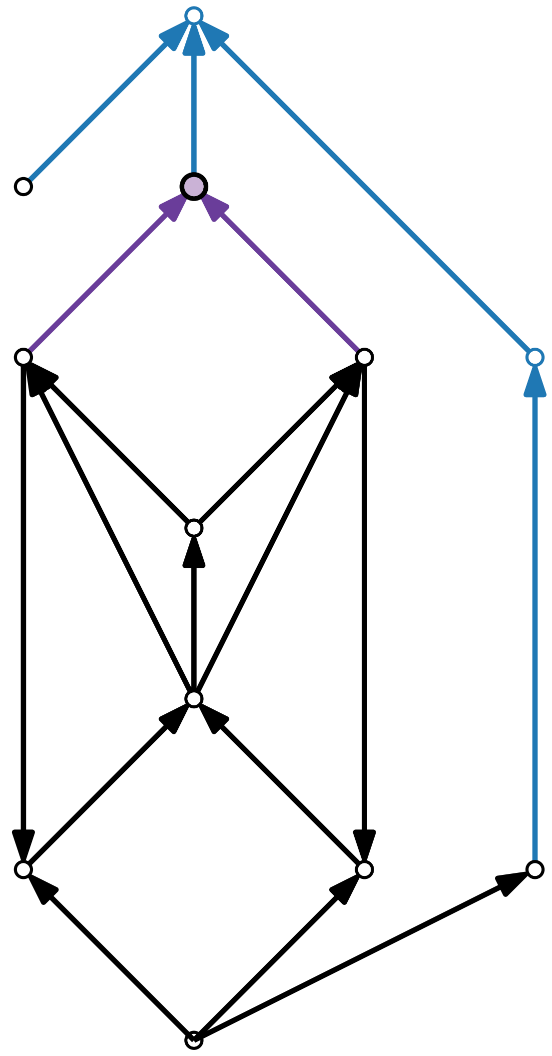

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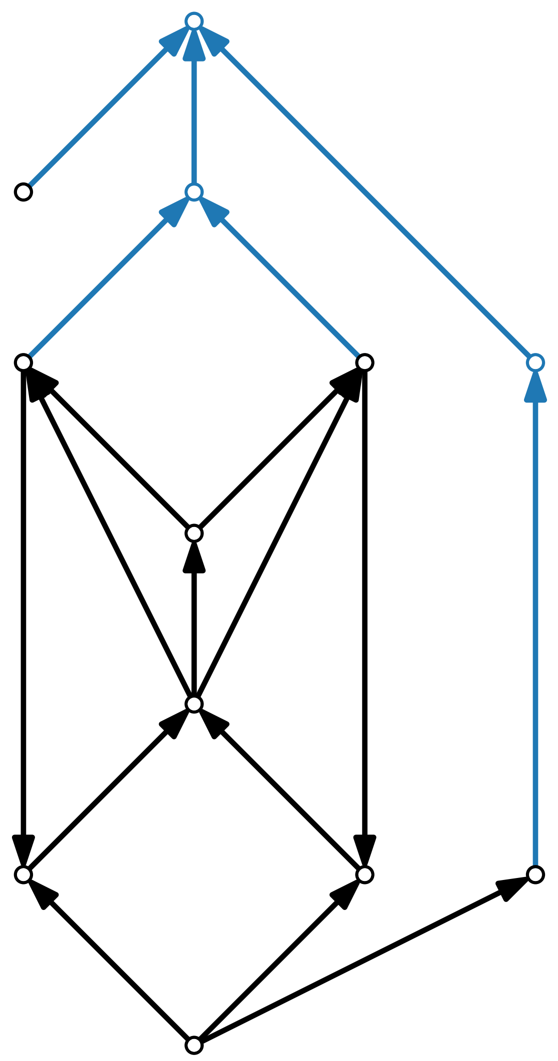

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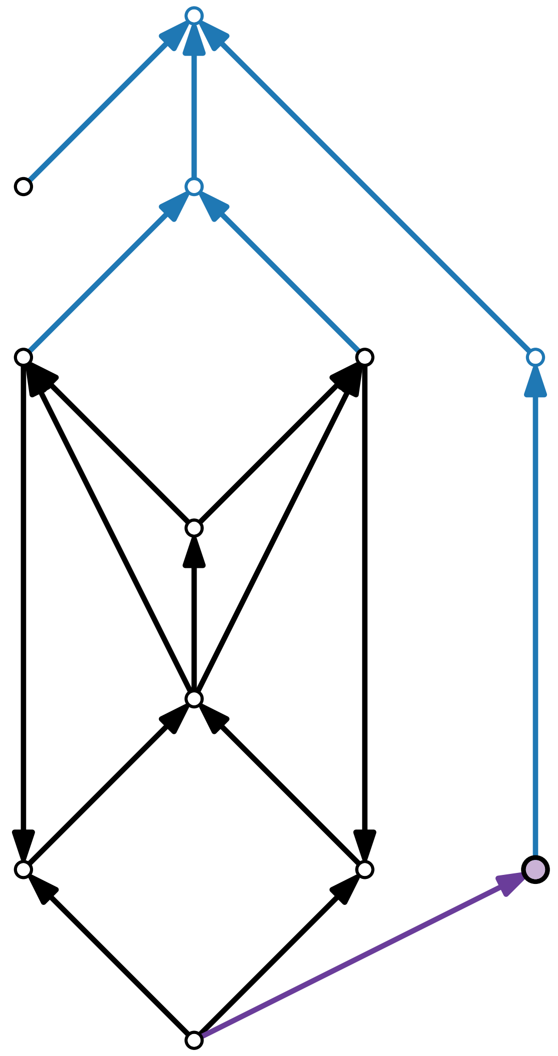

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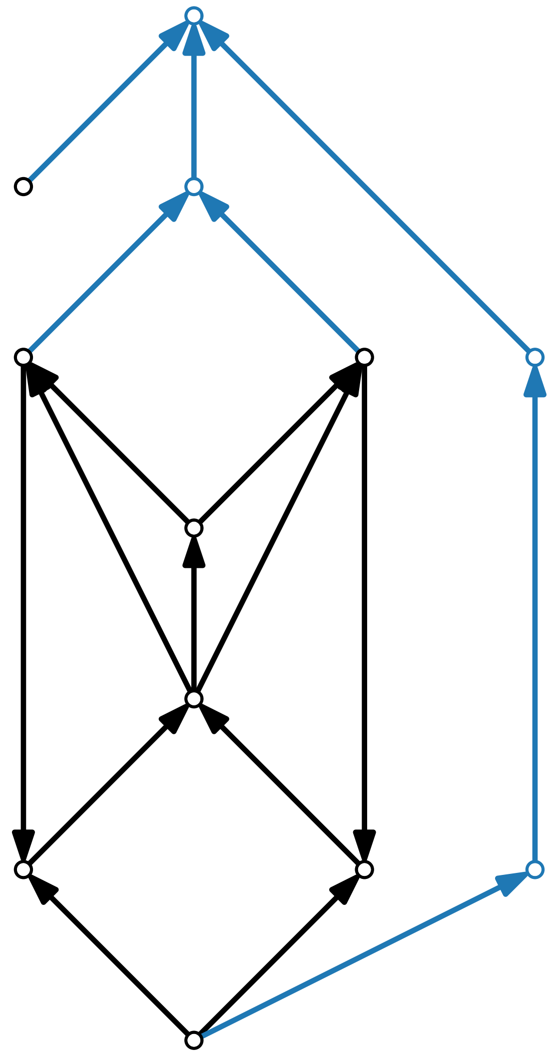

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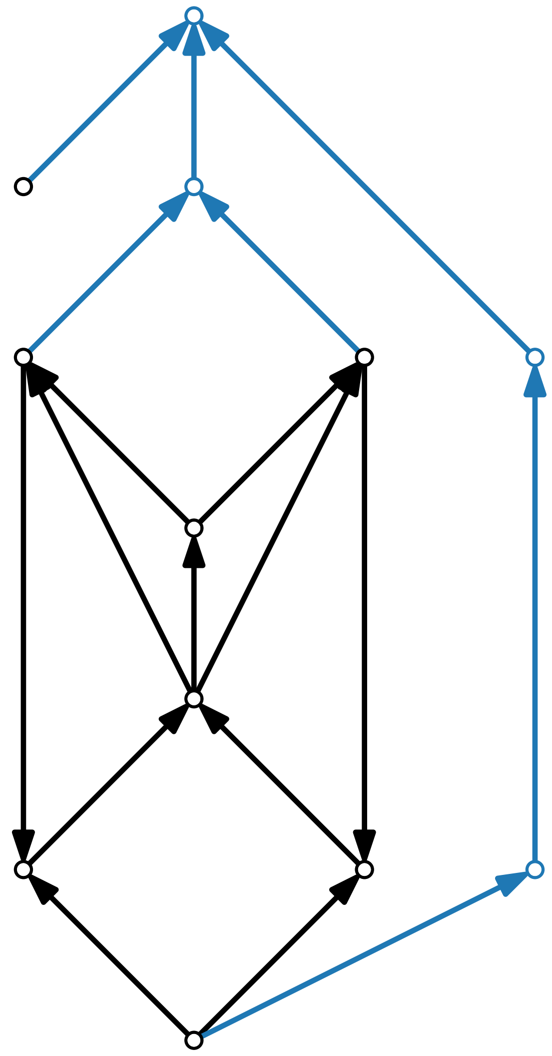

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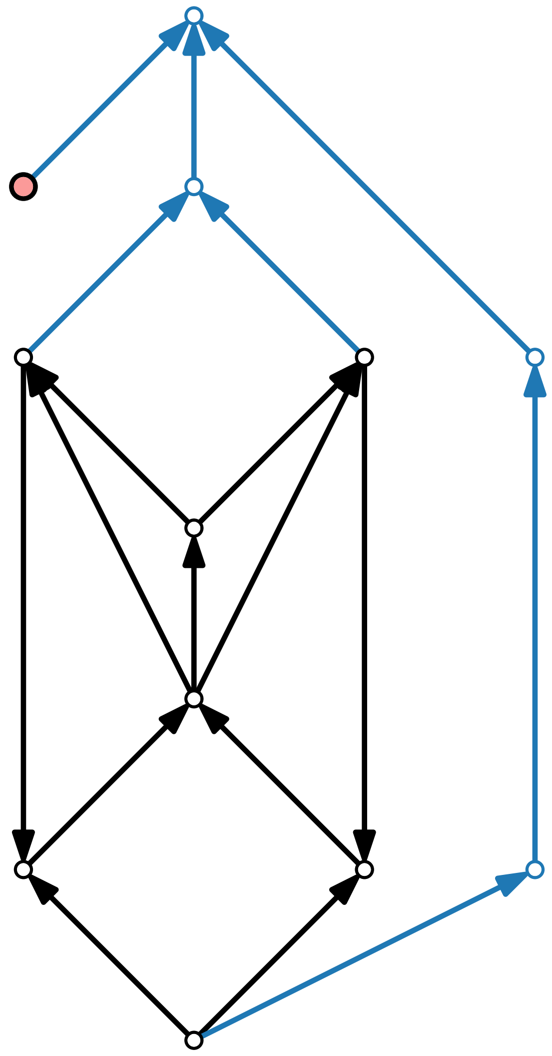

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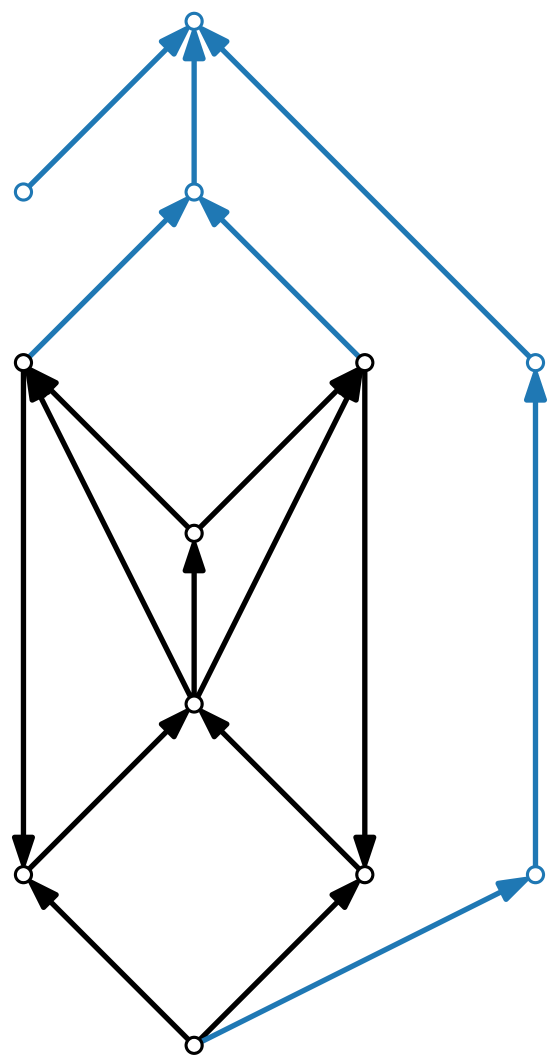

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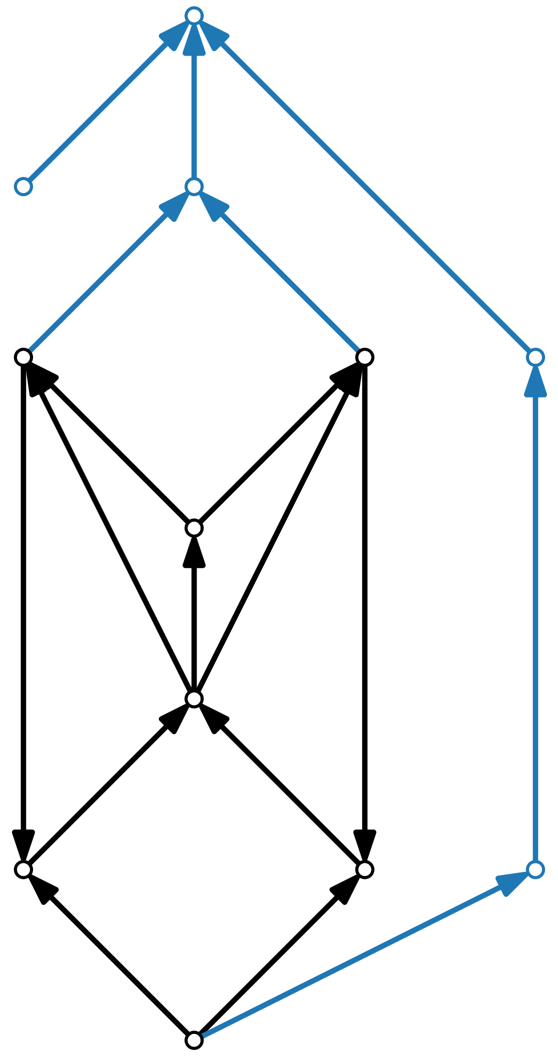

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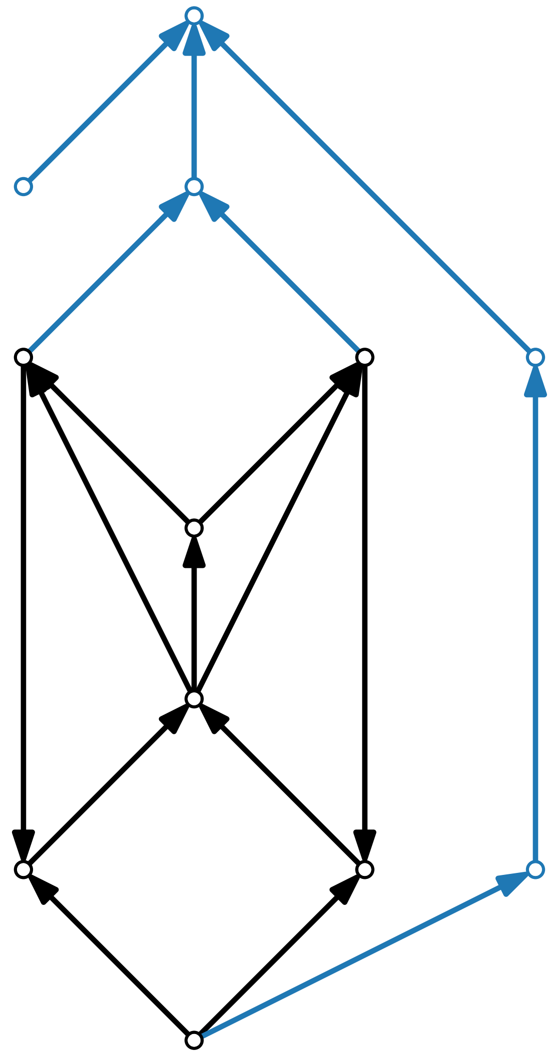

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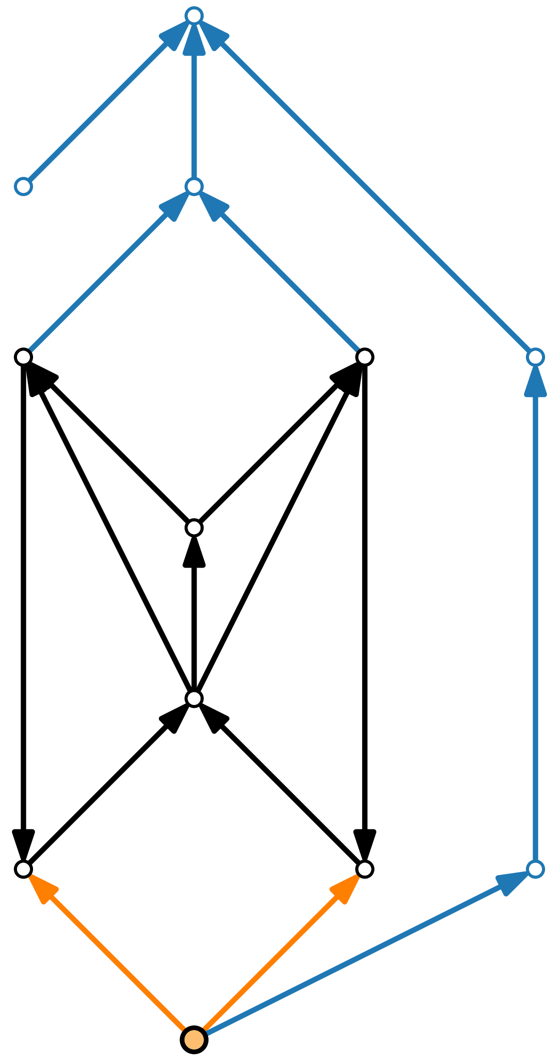

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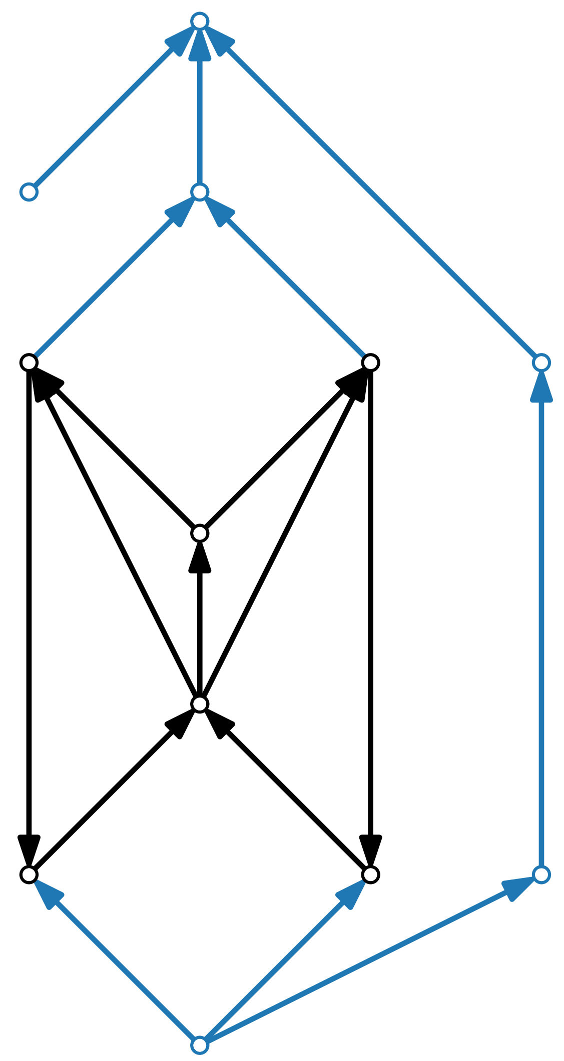

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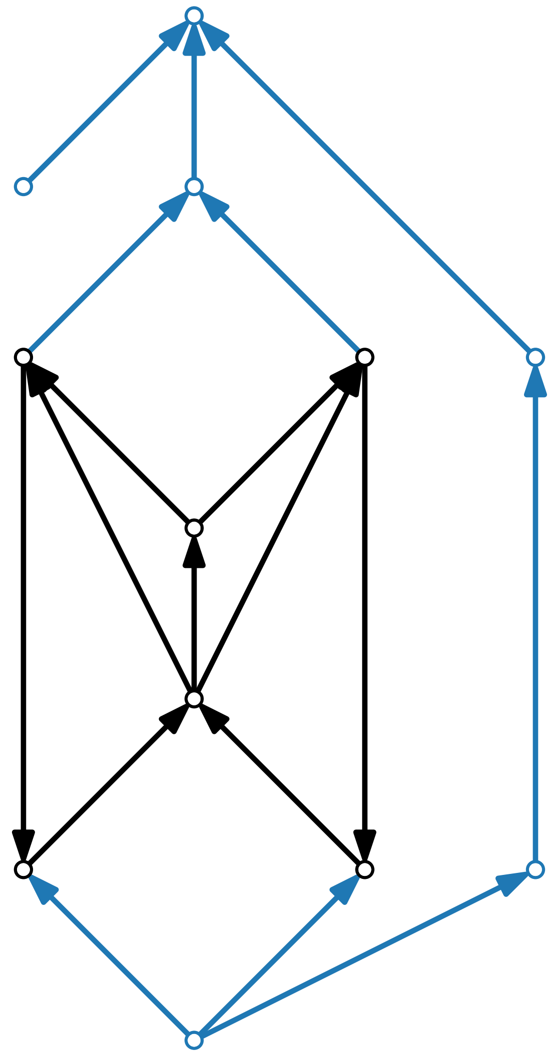

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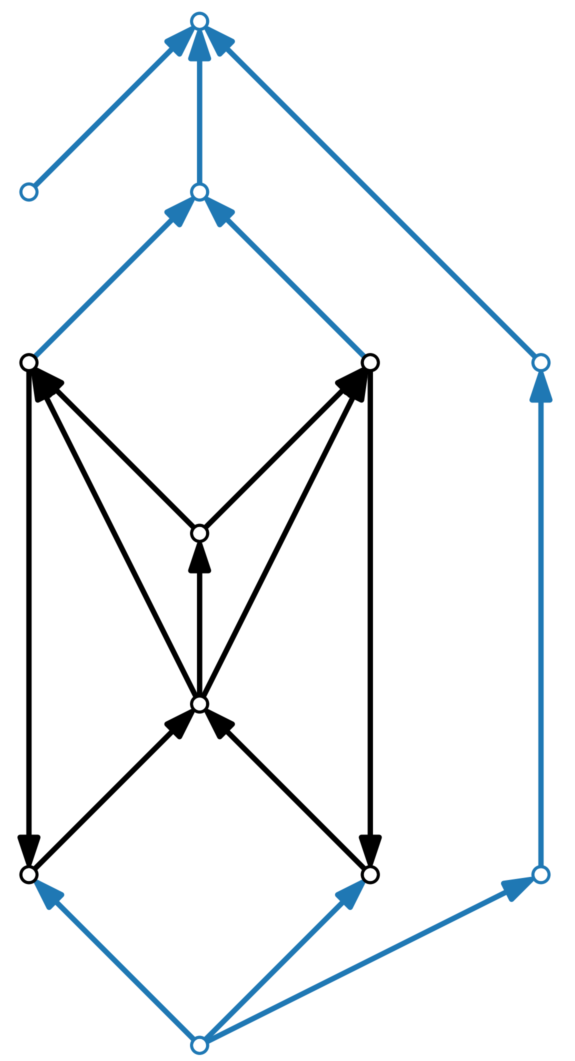

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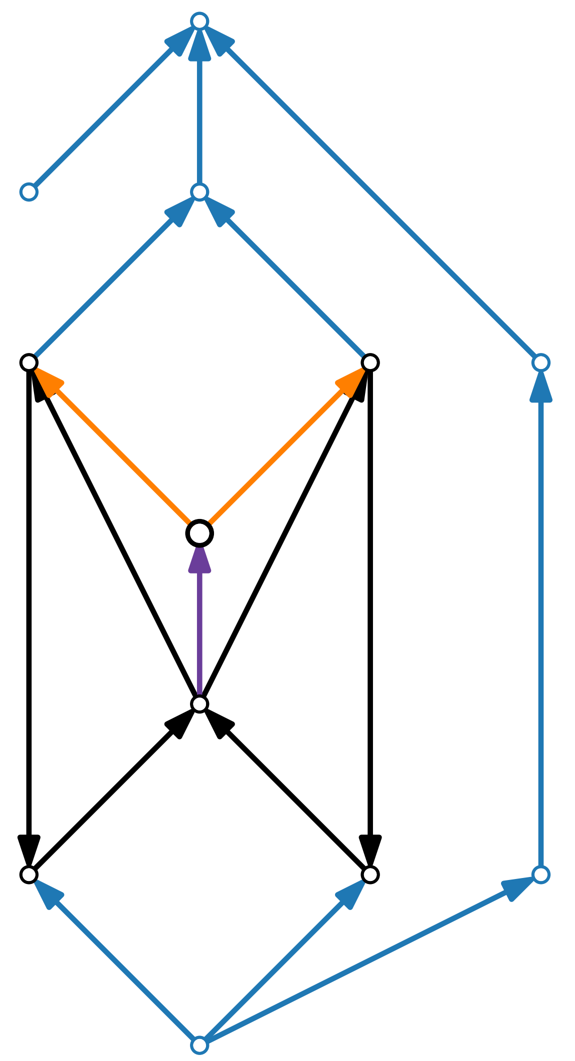

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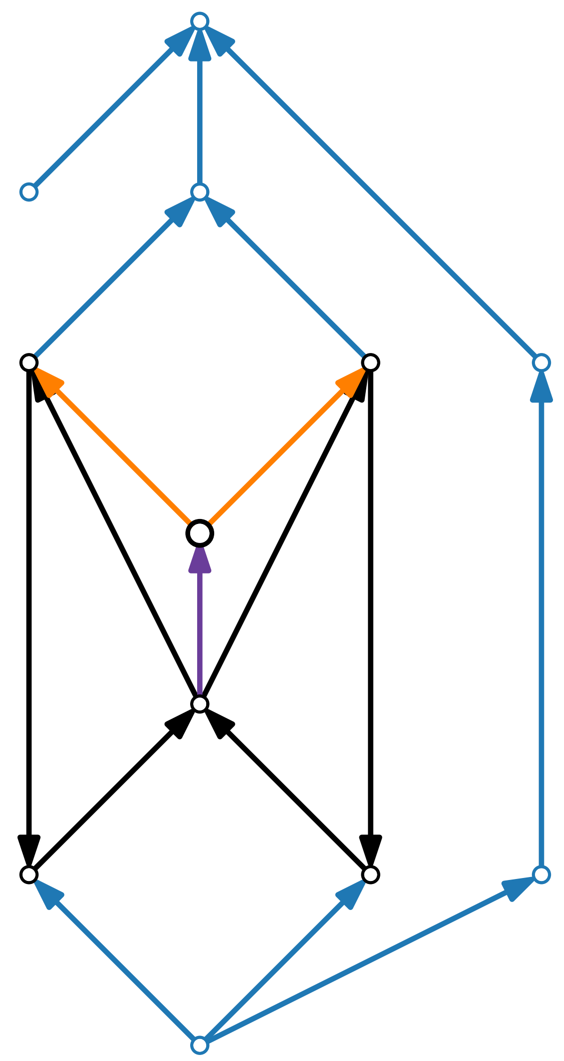

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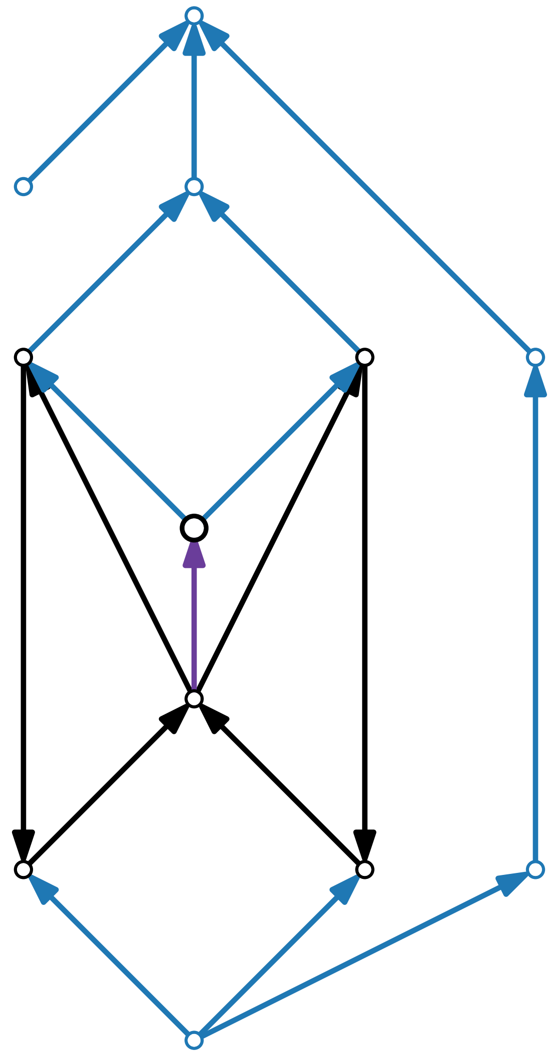

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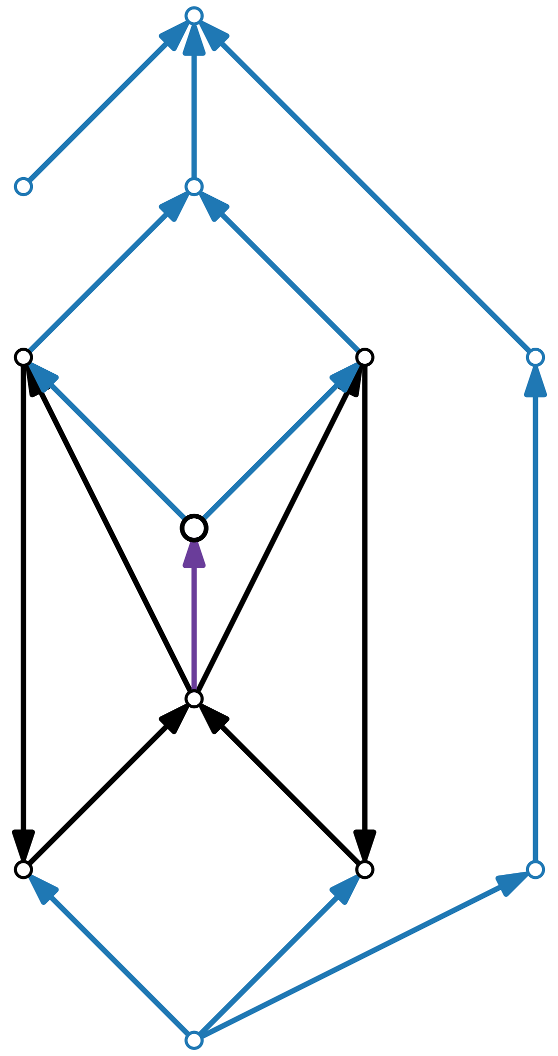

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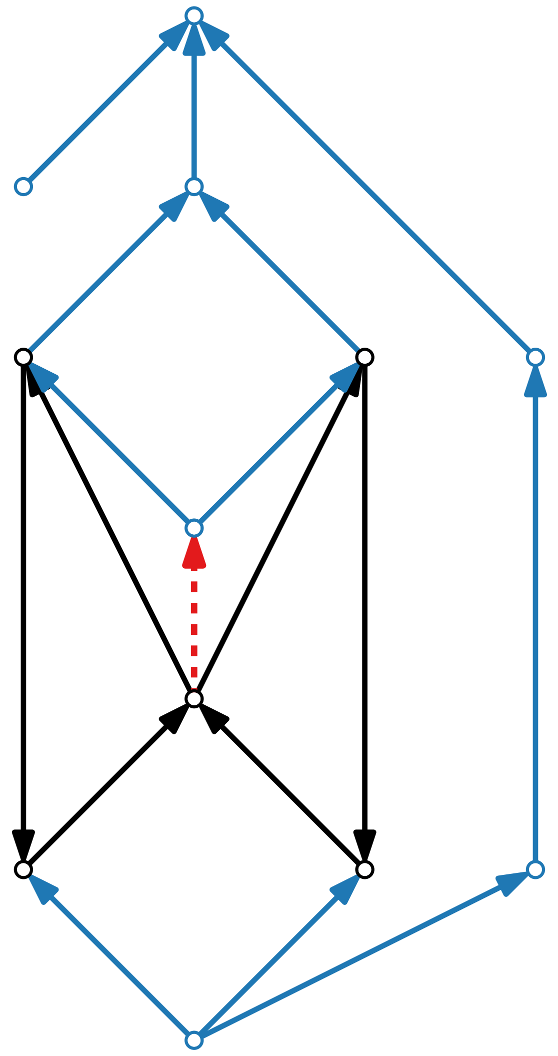

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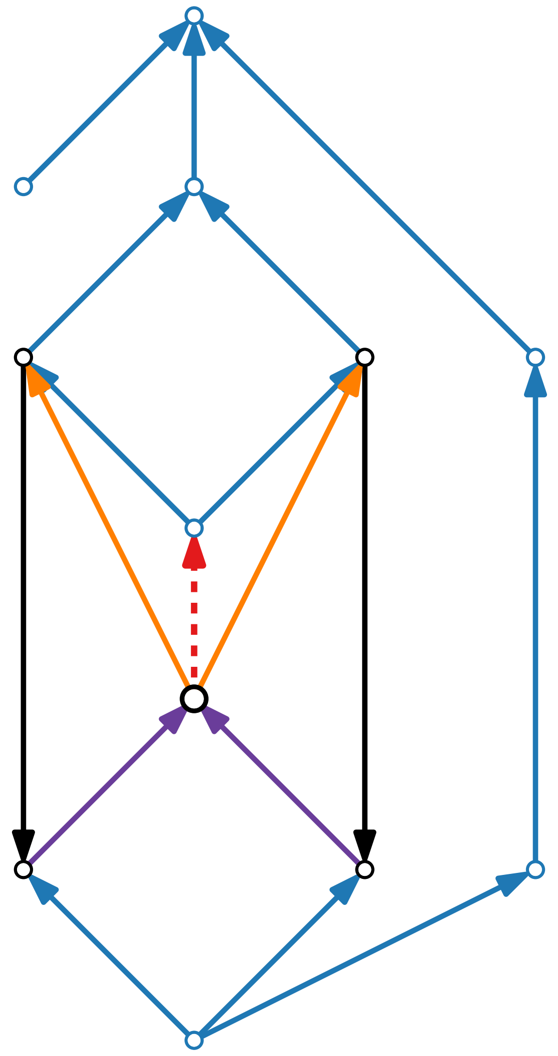

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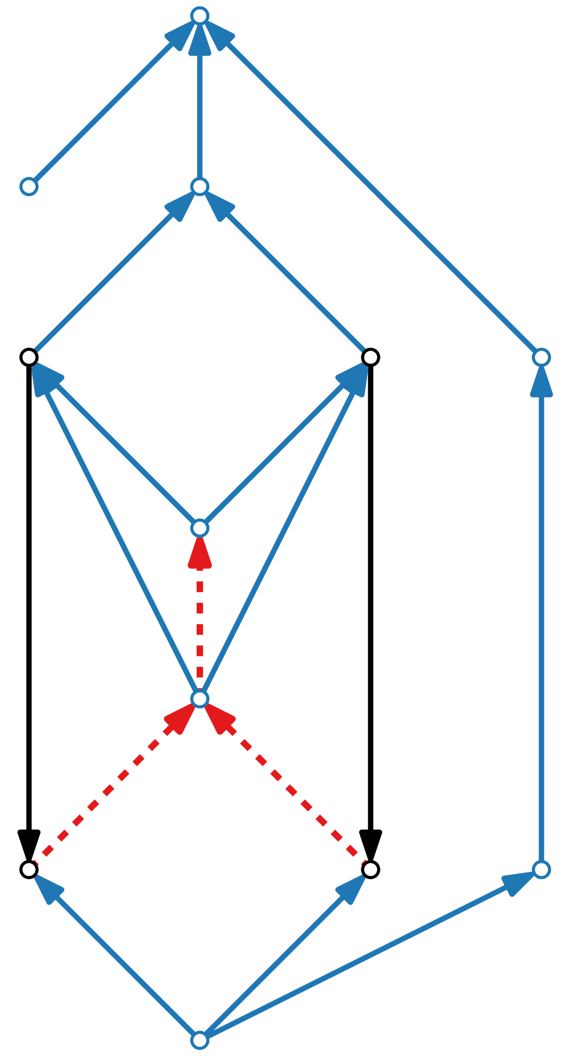

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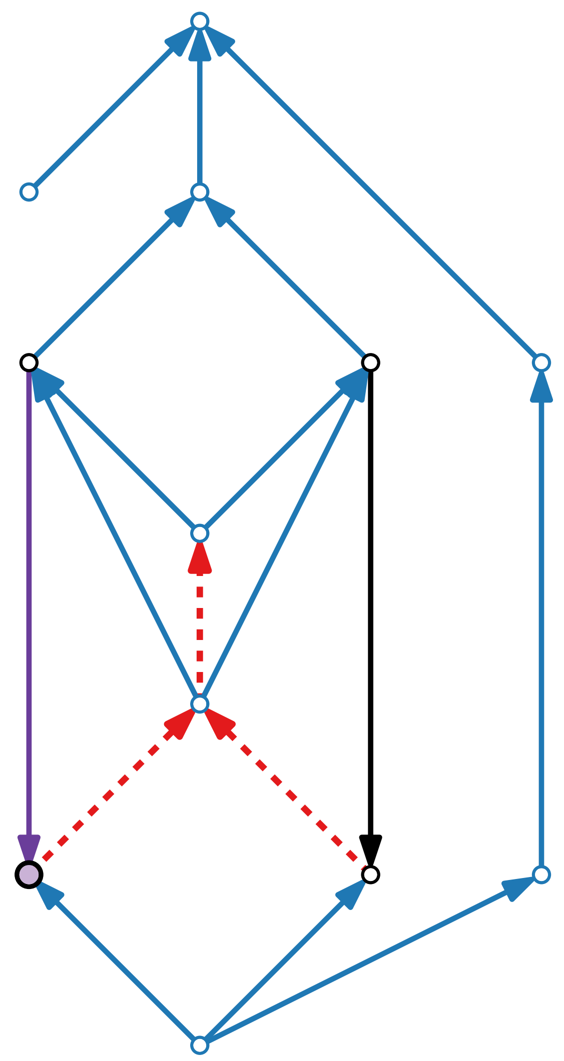

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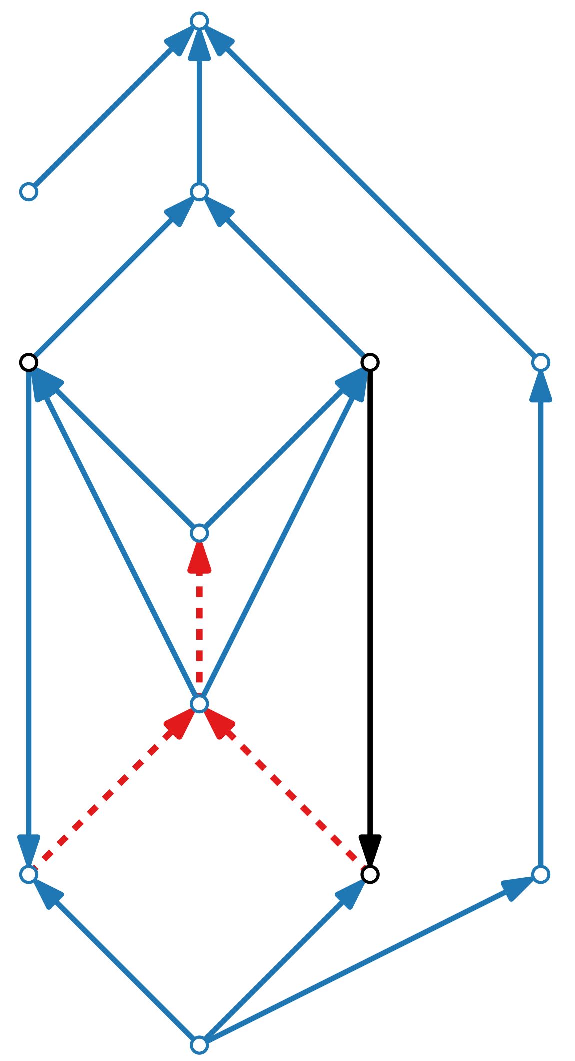

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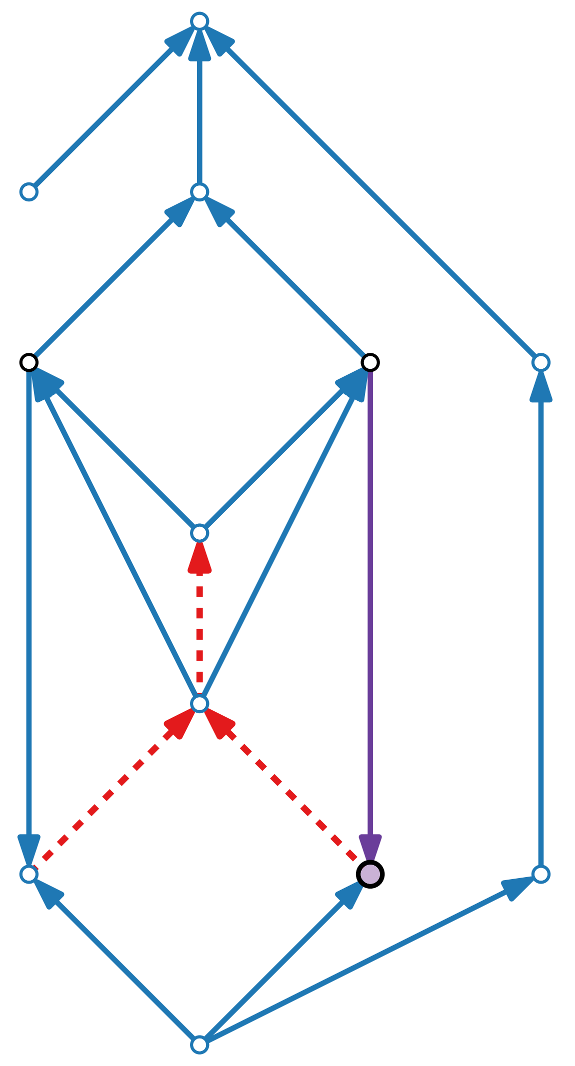

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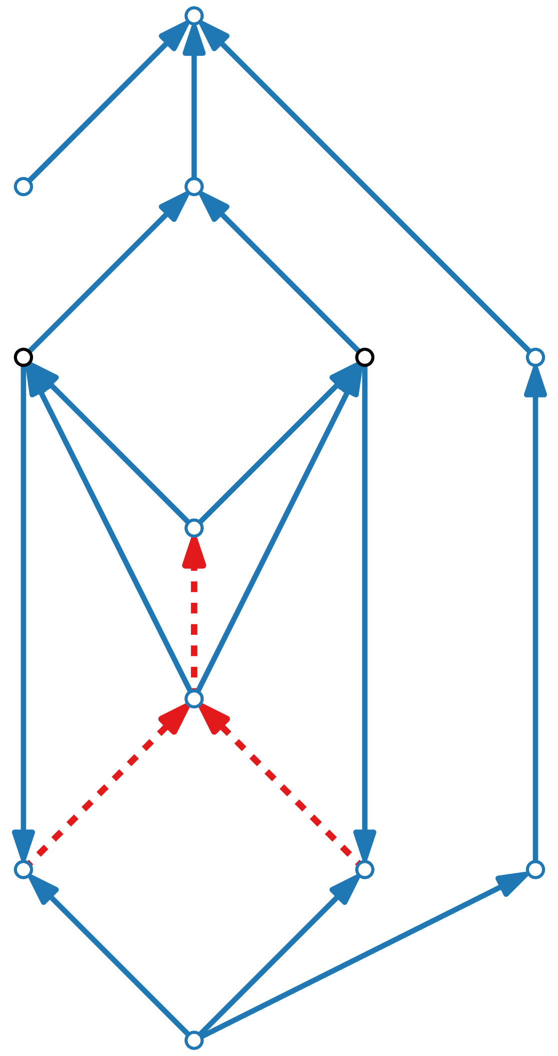

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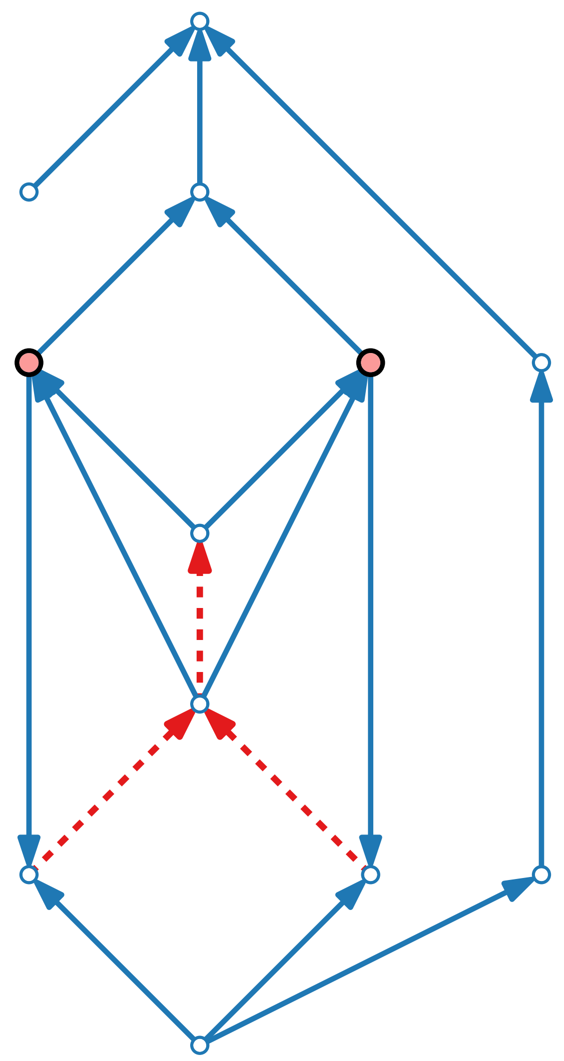

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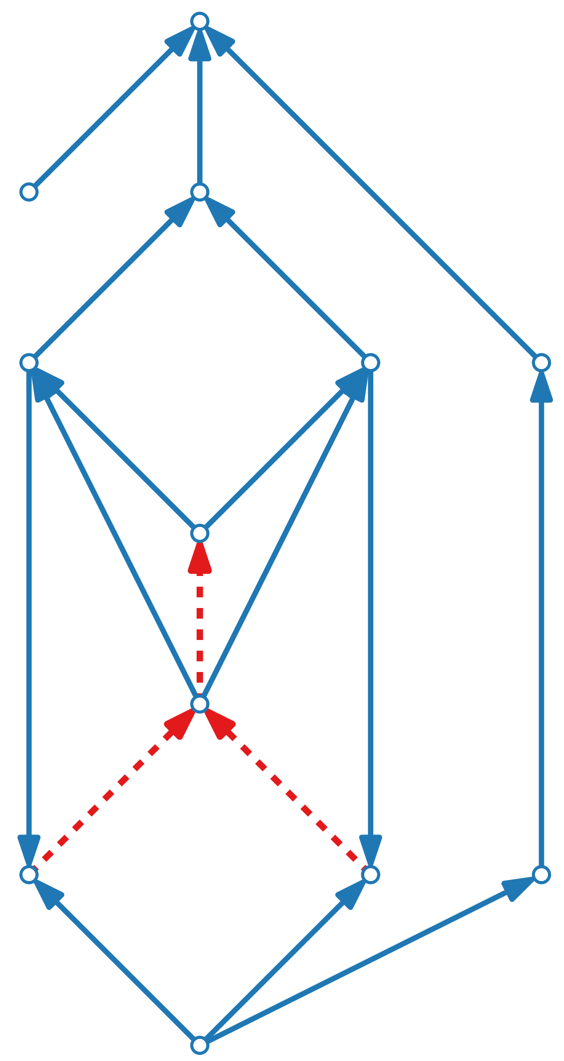

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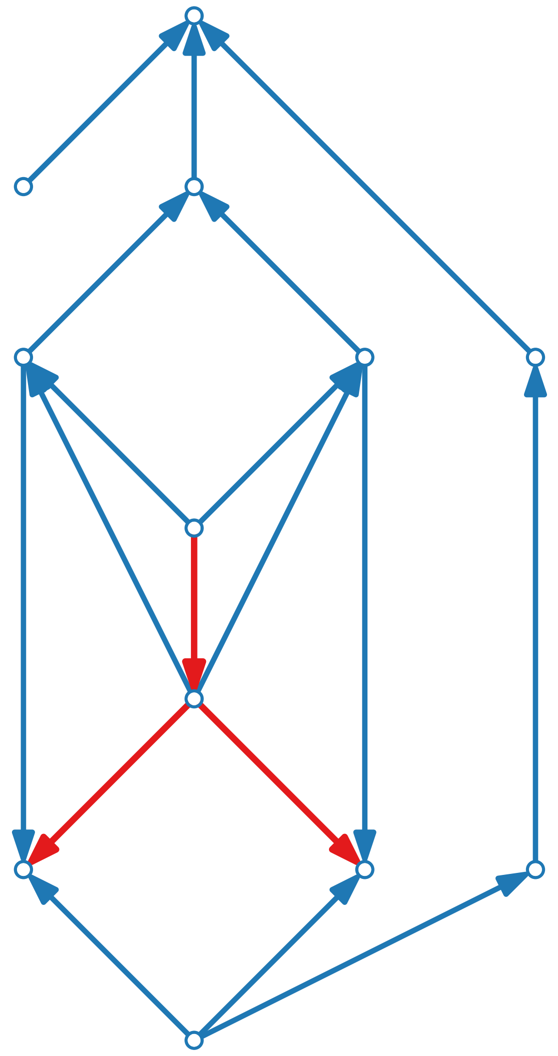

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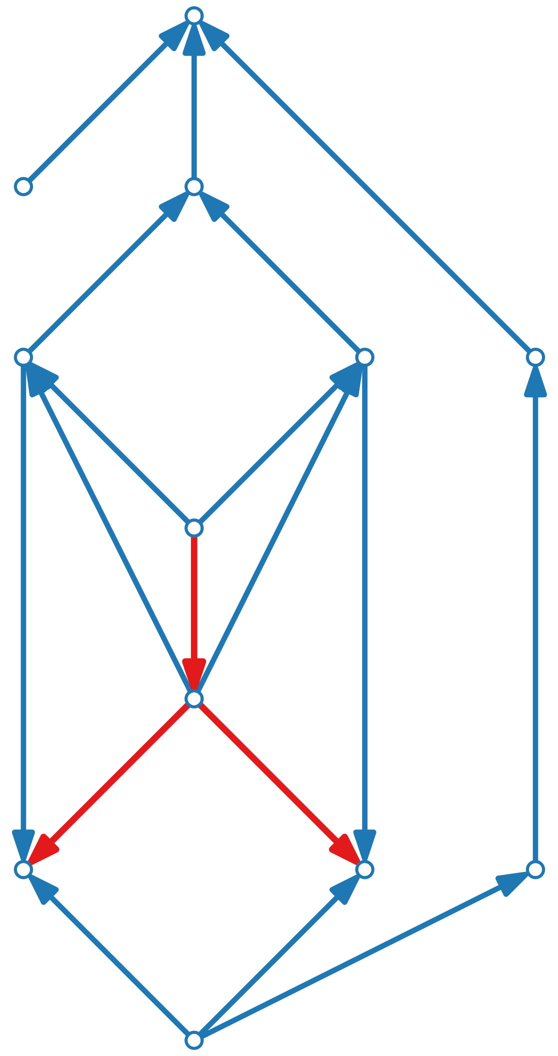

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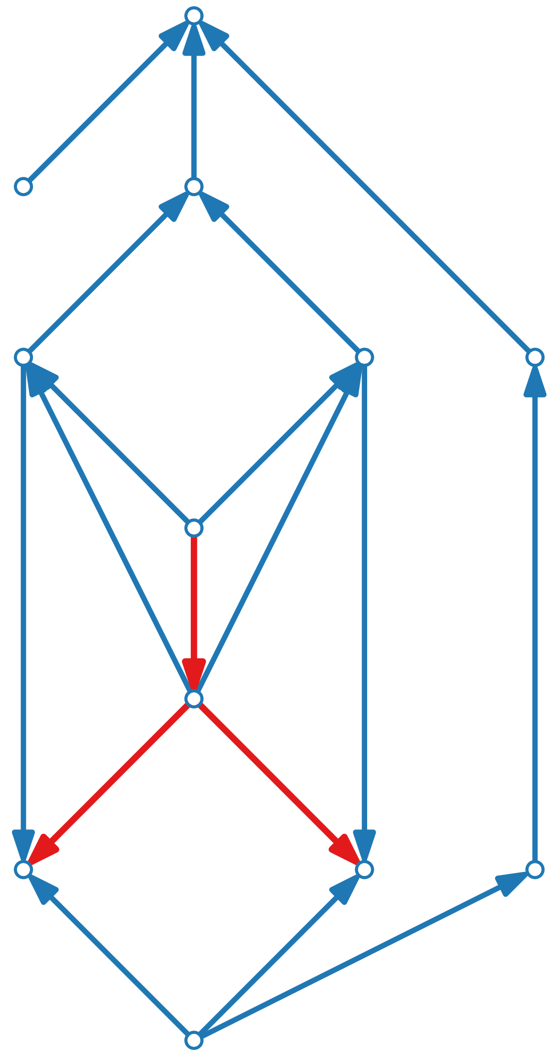

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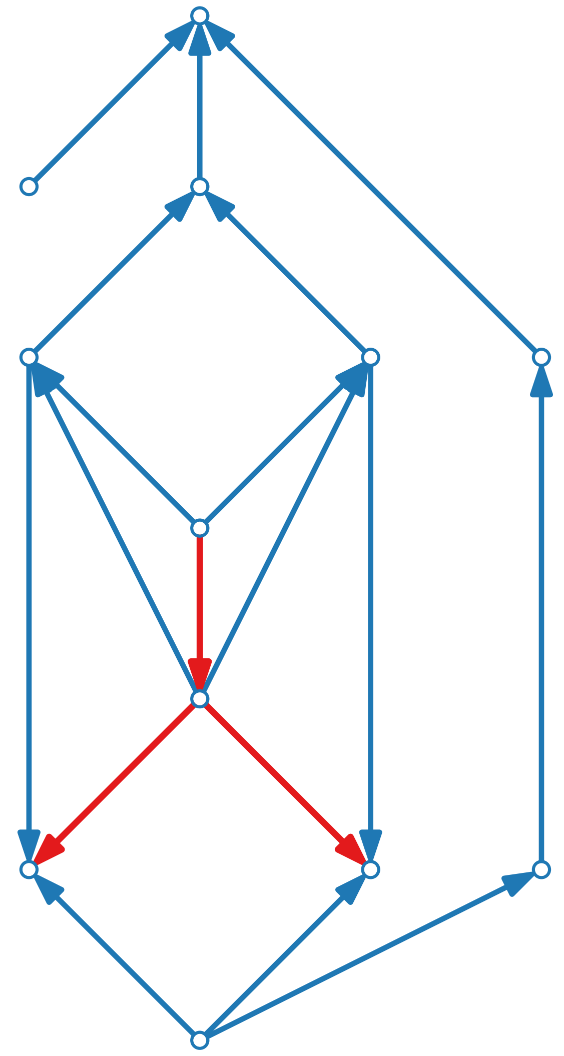

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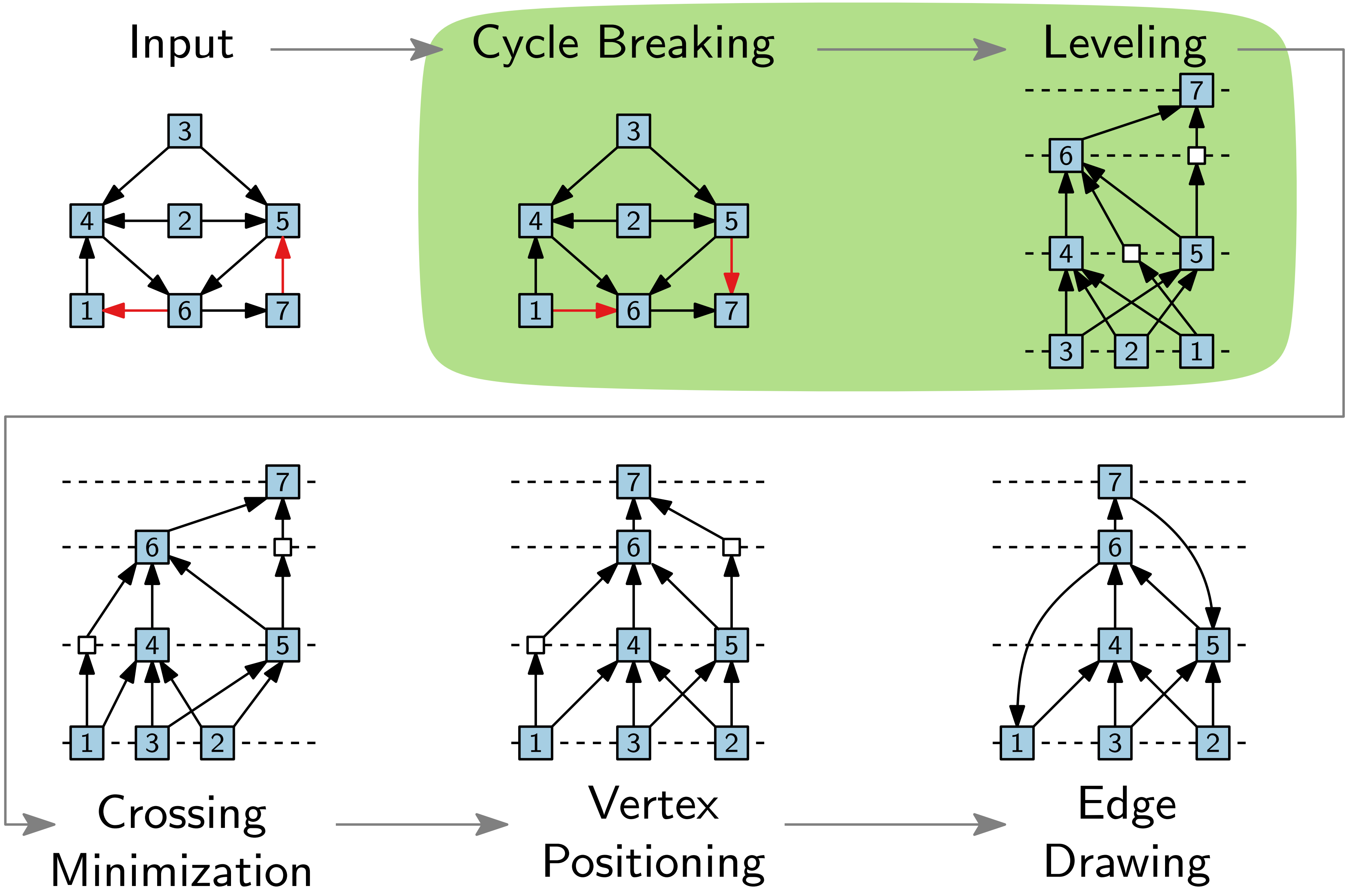
- Quality guarantee: $|E'| \geq |E|/2 + |V|/6$

Step 2: Leveling



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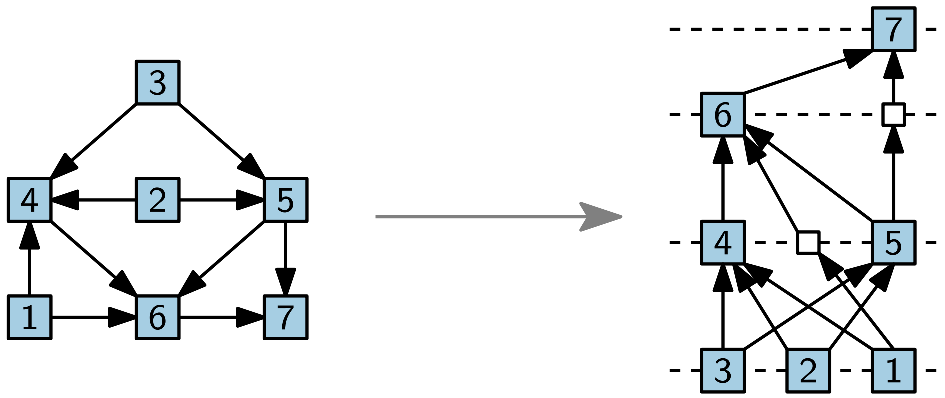

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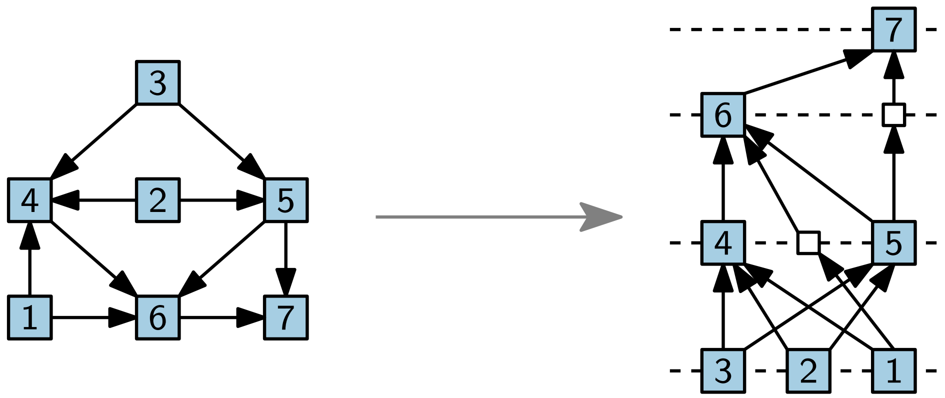

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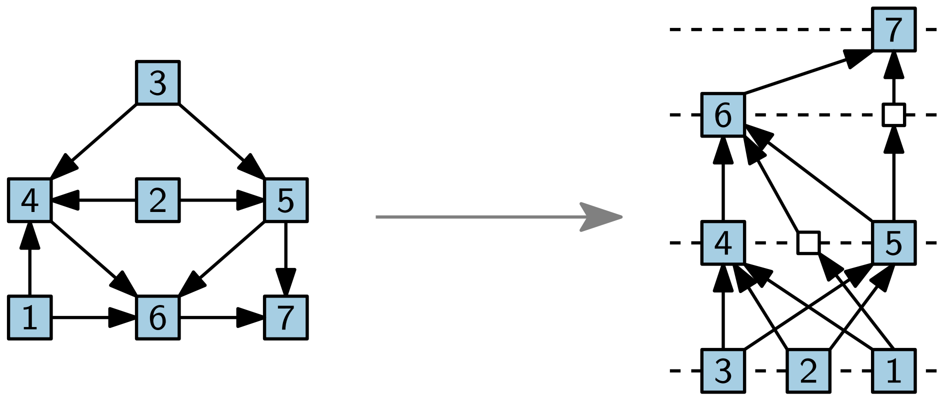

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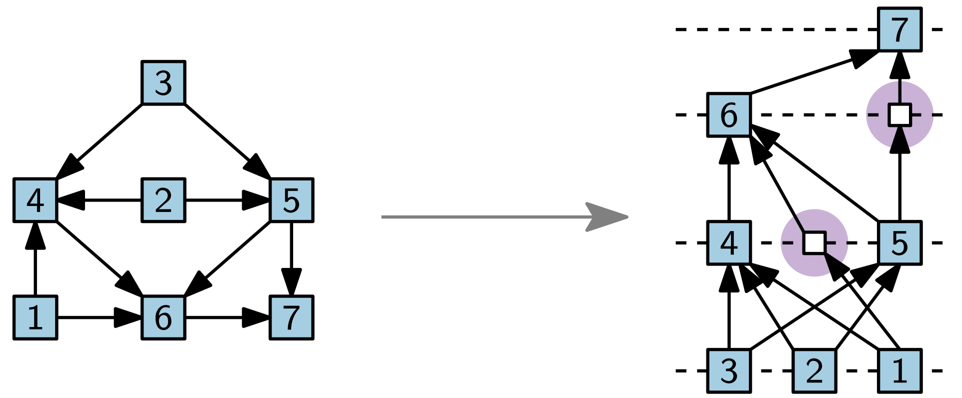

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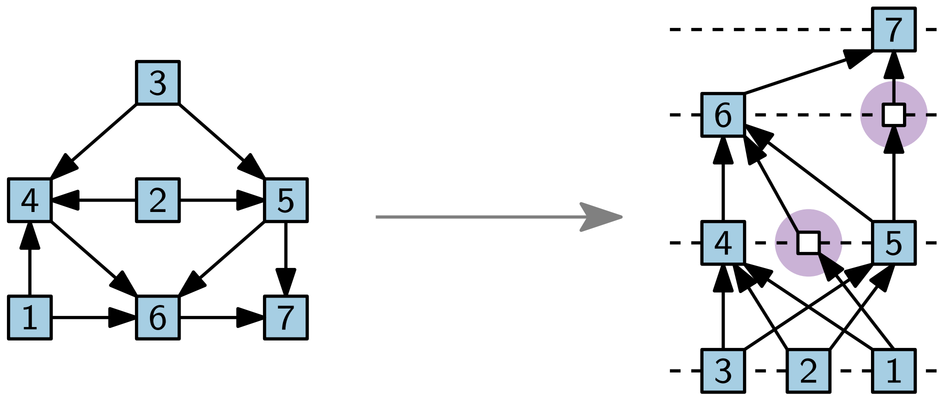

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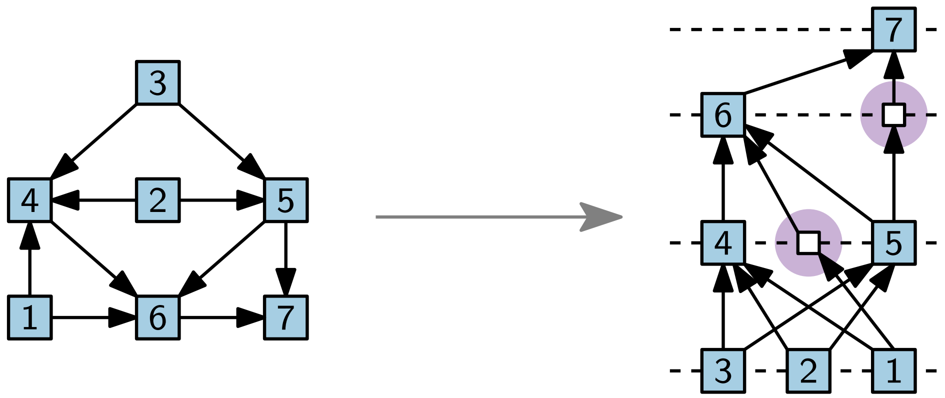

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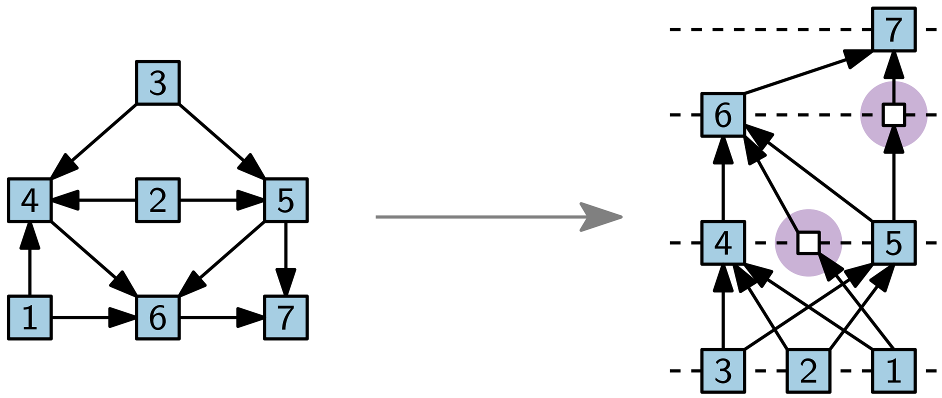

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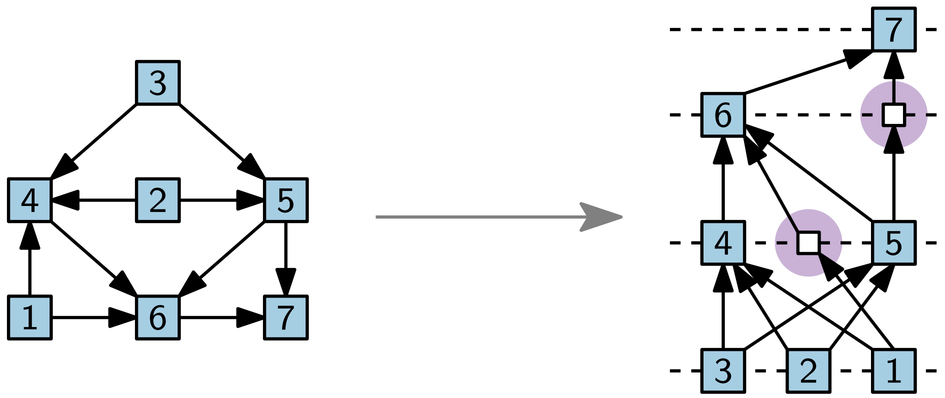

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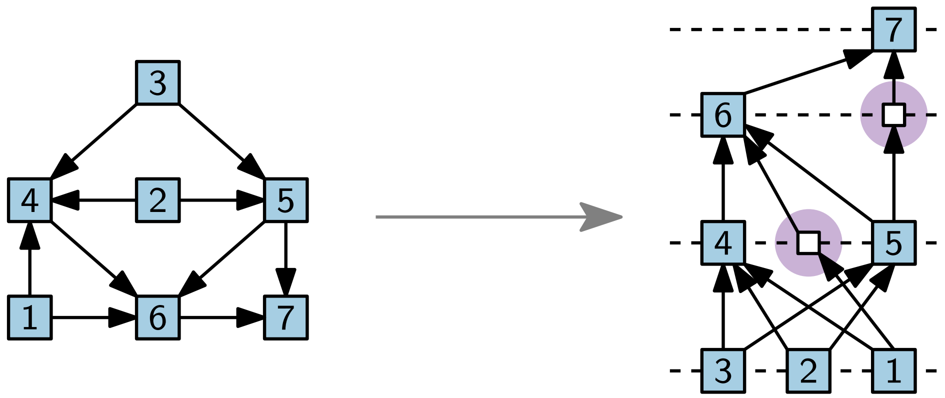

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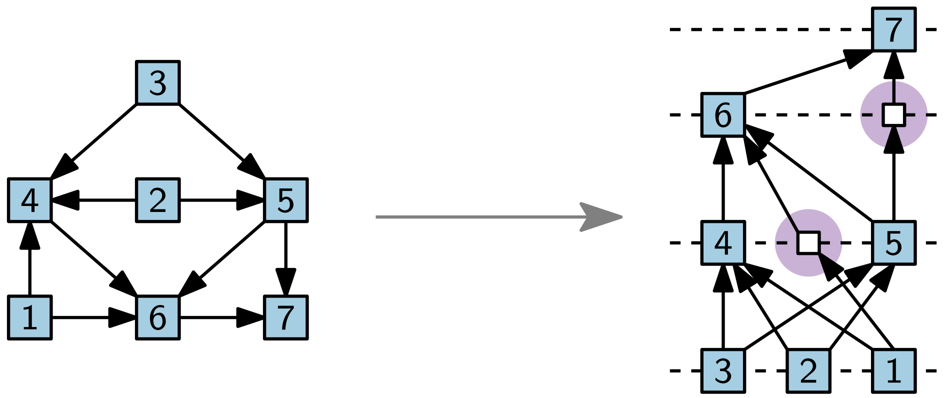

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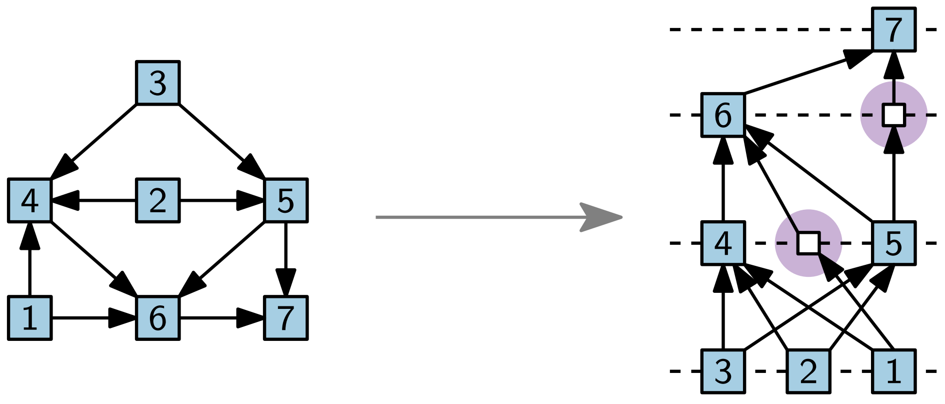

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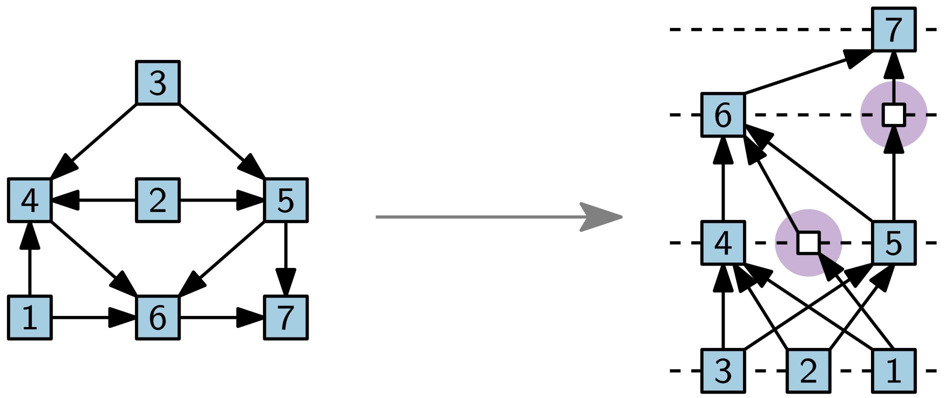

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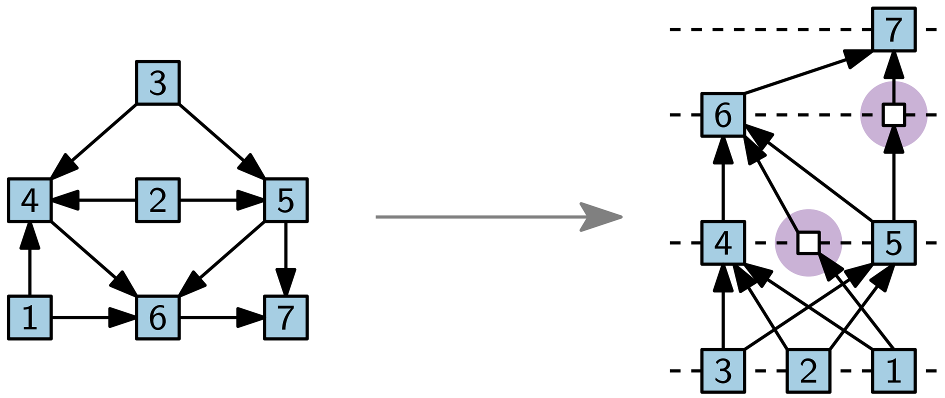

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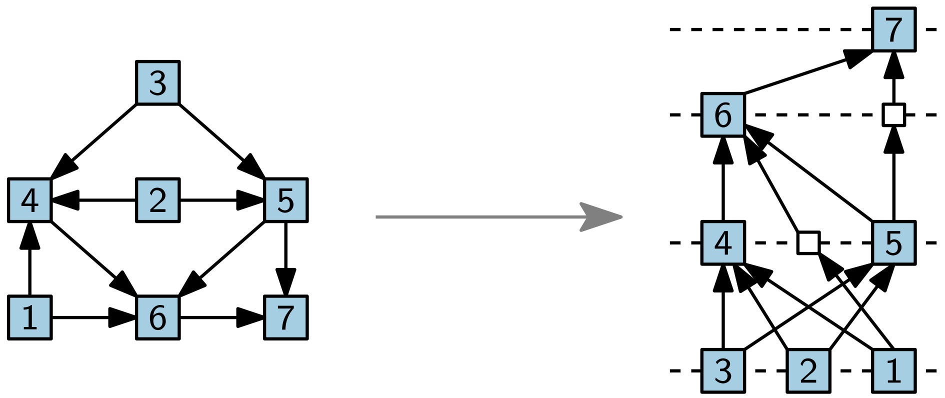
Minimize Number of Layers

Algorithm.



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- for each source q
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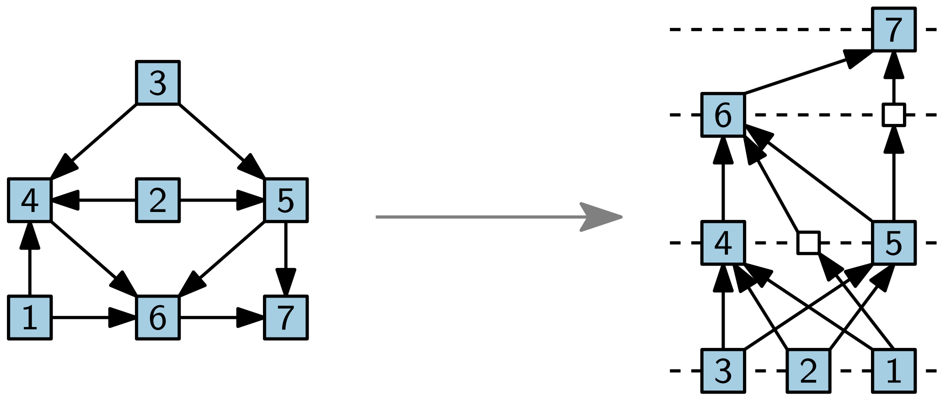

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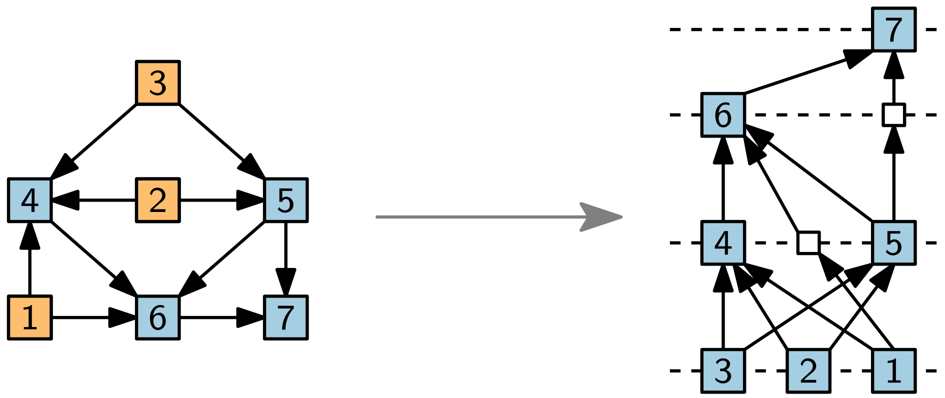

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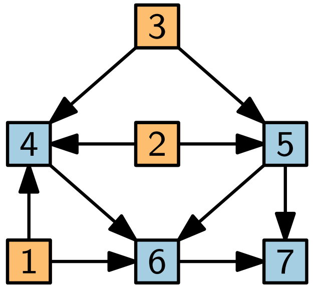

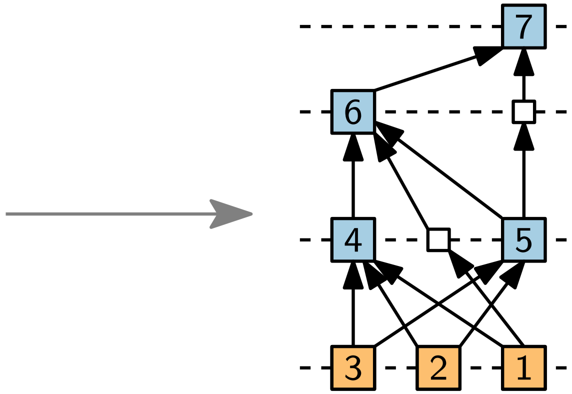

Minimize Number of Layers

Algorithm.

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- for each non-source v set
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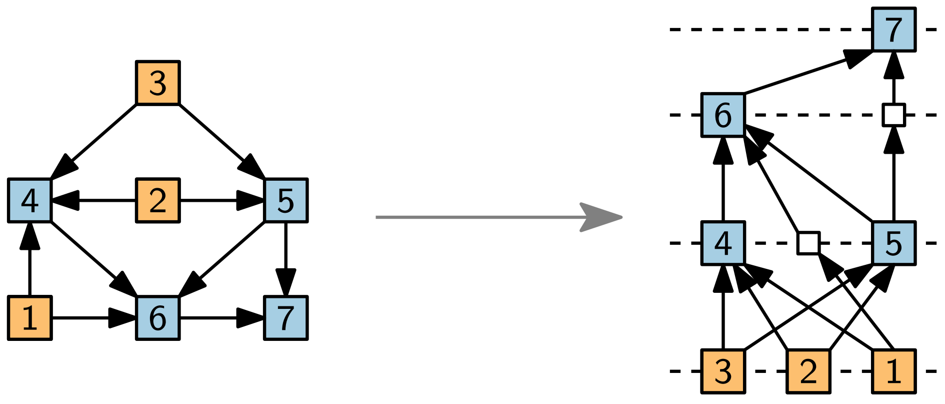

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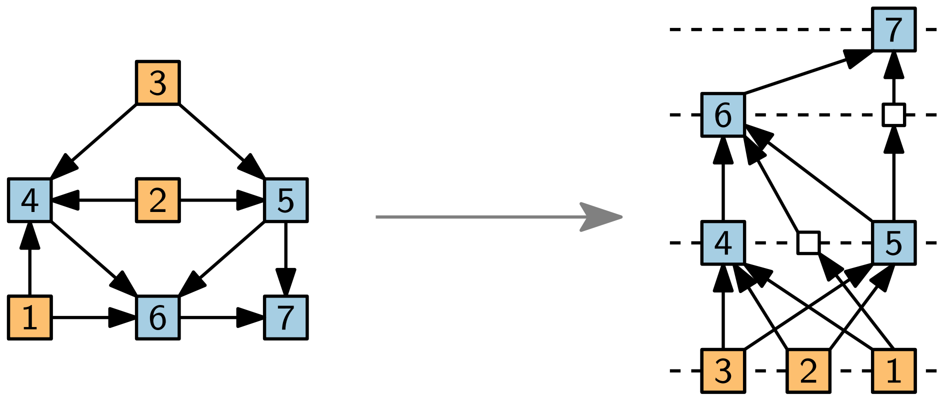

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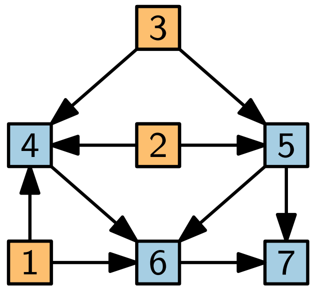

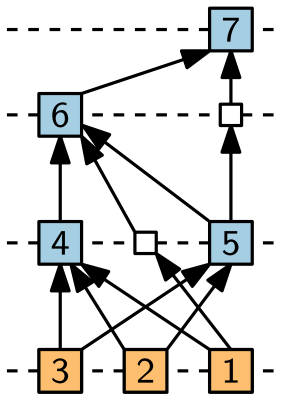

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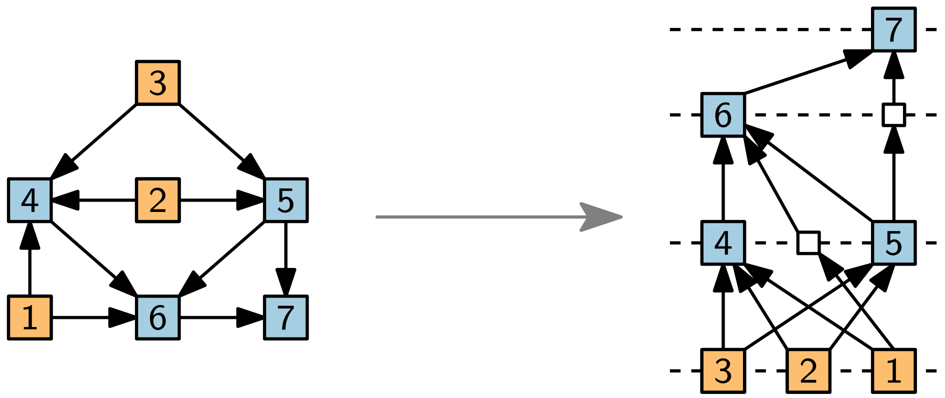

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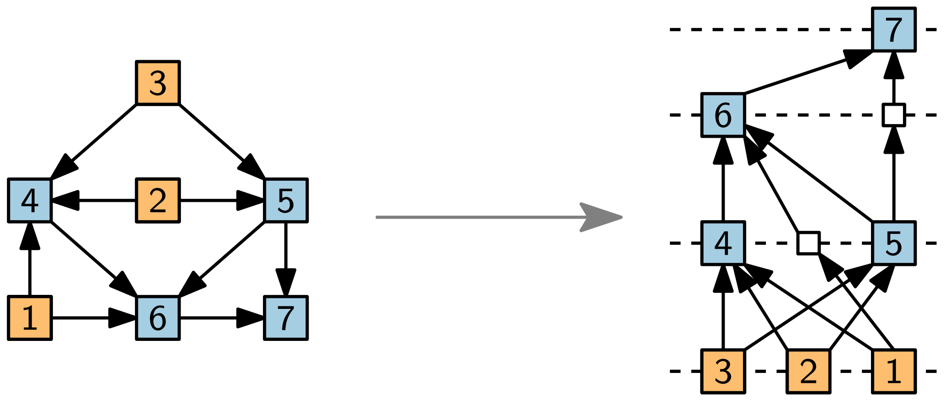

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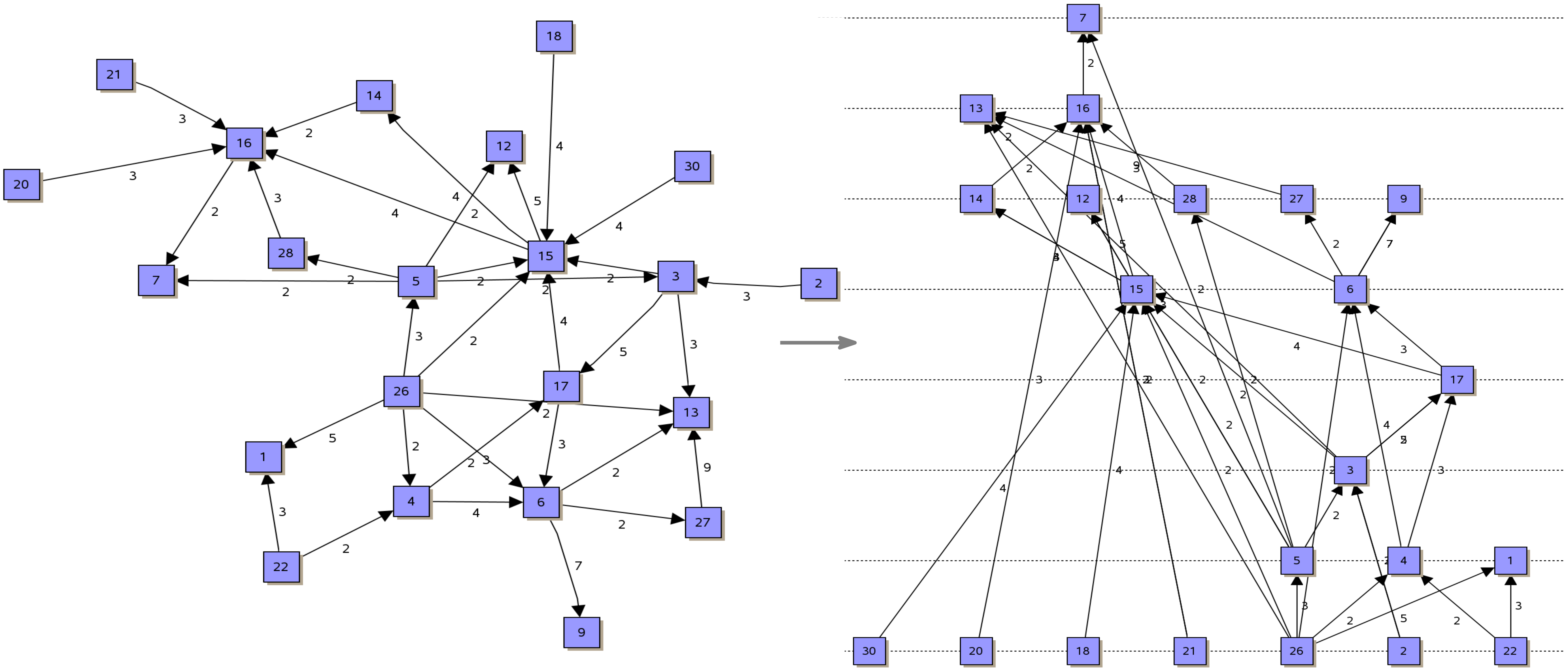
  ... which is optimal!
- Can be implemented in linear time with recursive algorithm.

Example



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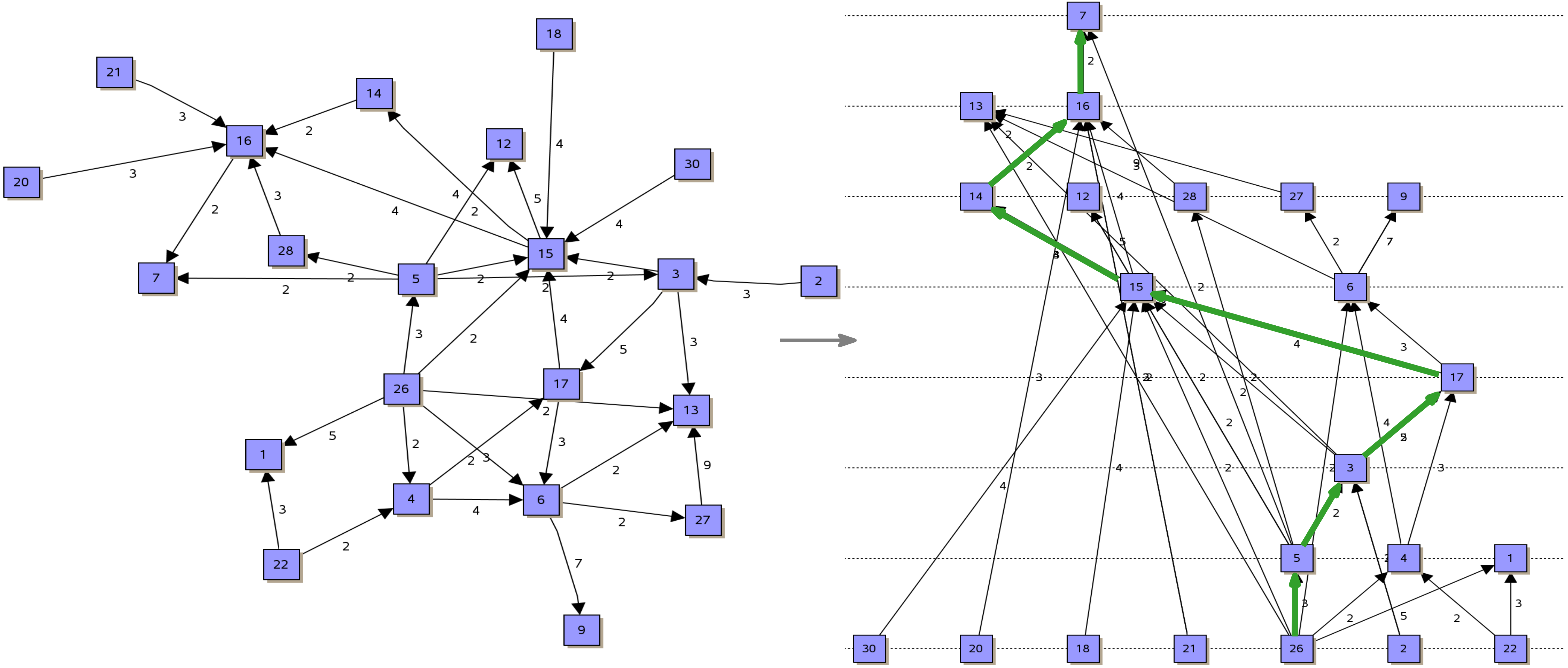

21
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Can be formulated as an integer linear program:

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One can show that:

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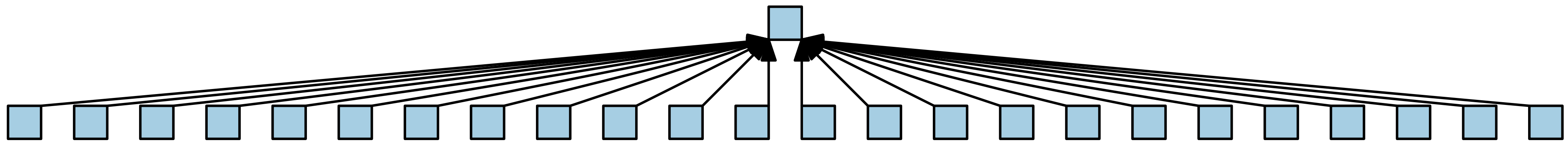
- The total edge length can be minimized in polynomial time.

Width



Drawings can be very wide.

Narrower Layer Assignment

Problem: leveling with a given maximum-width.

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- jobs stored in a list L
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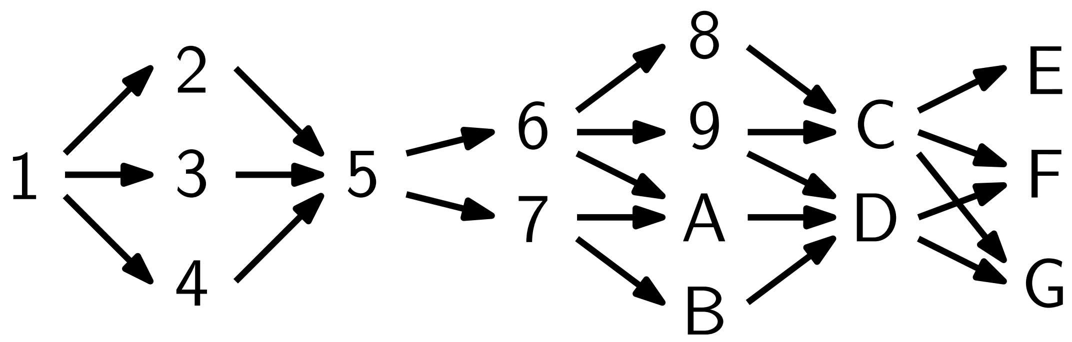
- for each time $t = 1, 2, \ldots$ we can schedule $\leq W$ available jobs
- as long as there are free machines and available jobs, take the first available job and assign it to a free machine

Approximating PCMPS

Input: Precedence graph (divided into layers of arbitrary width)



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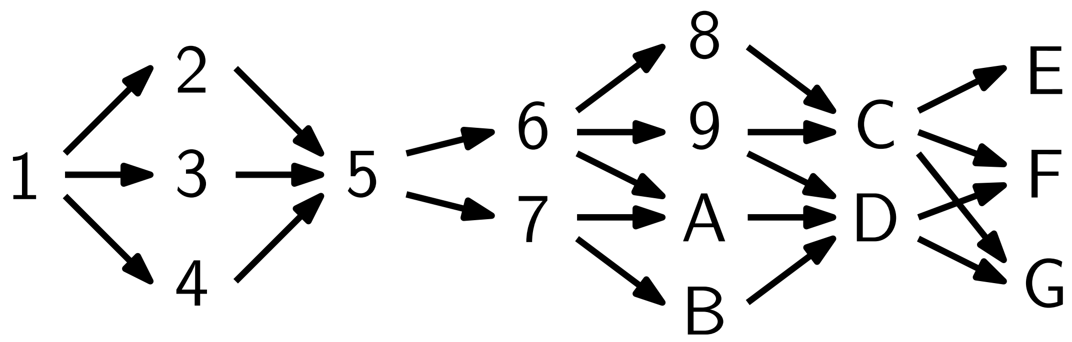

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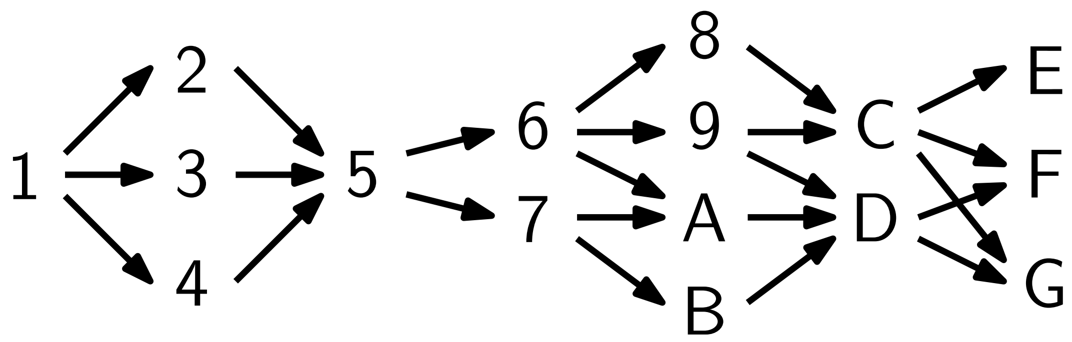

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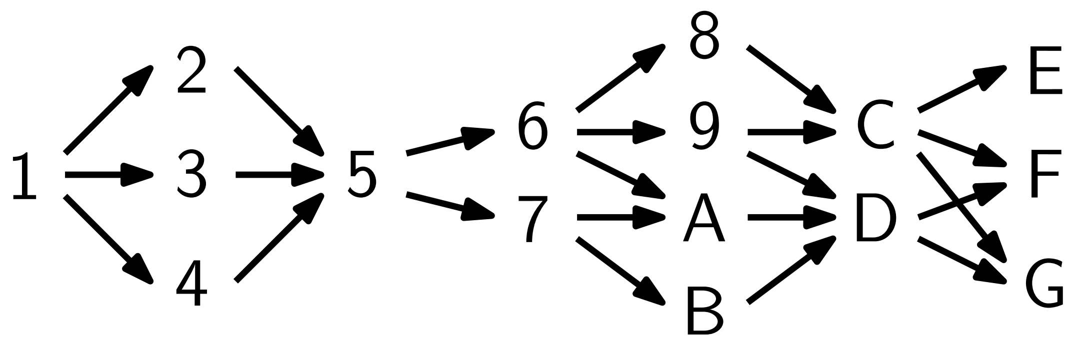

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M_1										
M_2										
t	1	2	3	4	5	6	7	8	9	10

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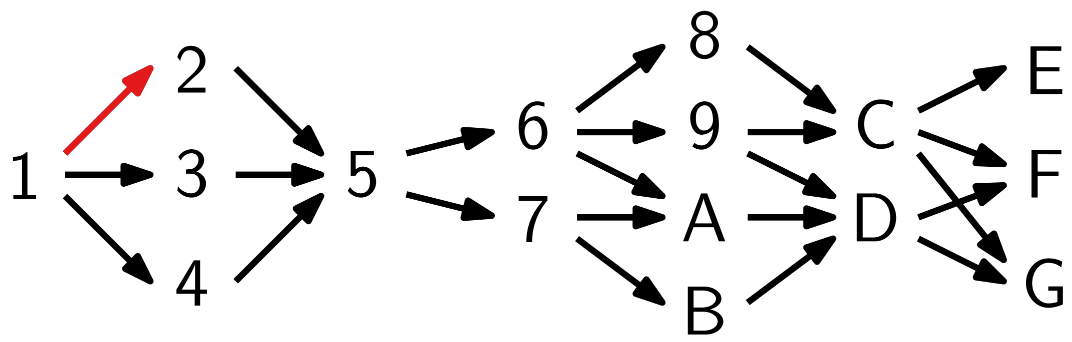

Number of machines is $W = 2$.

Output: Schedule

M_1	1									
M_2	–									
t	1	2	3	4	5	6	7	8	9	10

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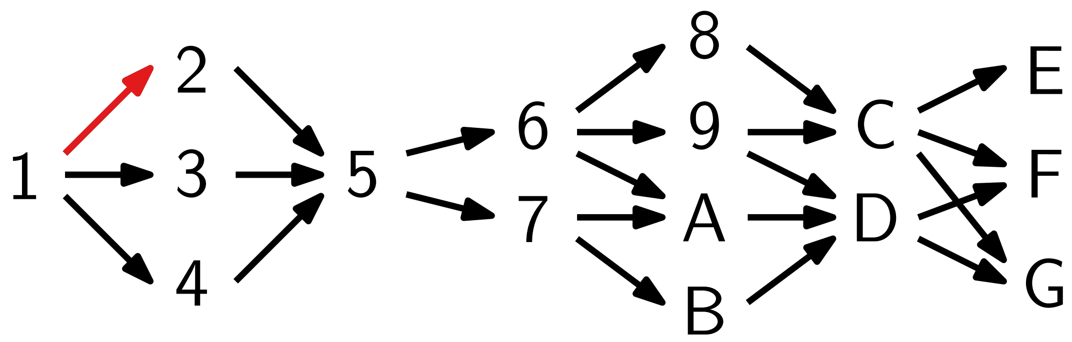

Number of machines is $W = 2$.

Output: Schedule

M_1	1	2								
M_2	–	3								
t	1	2	3	4	5	6	7	8	9	10

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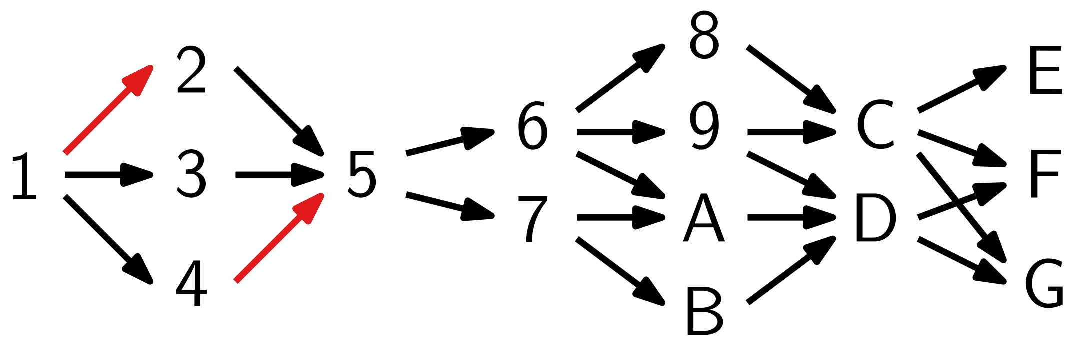

Number of machines is $W = 2$.

Output: Schedule

M_1	1	2	4							
M_2	–	3	–							
t	1	2	3	4	5	6	7	8	9	10

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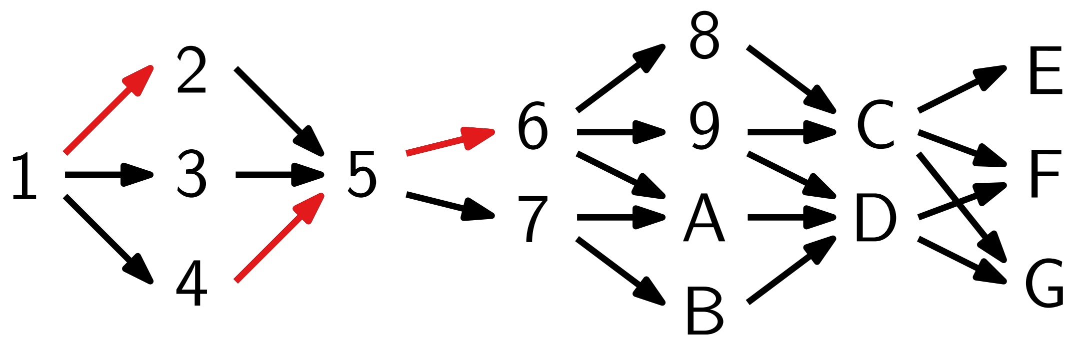

Number of machines is $W = 2$.

Output: Schedule

M_1	1	2	4	5						
M_2	–	3	–	–						
t	1	2	3	4	5	6	7	8	9	10

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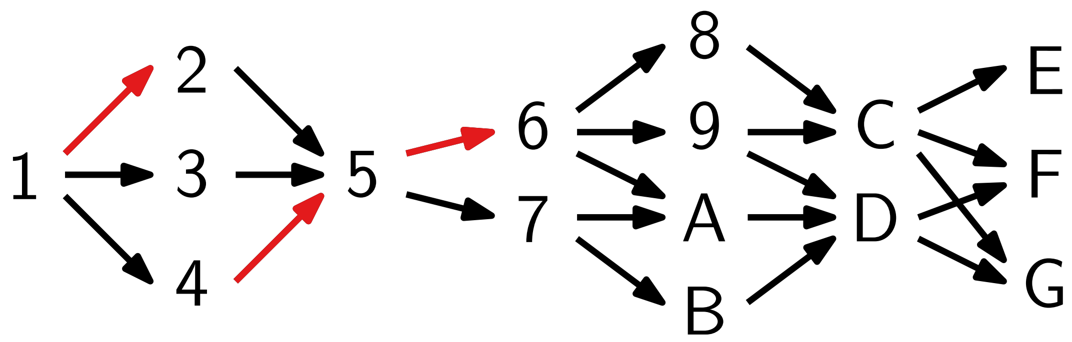

Number of machines is $W = 2$.

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M_1	1	2	4	5	6					
M_2	–	3	–	–	7					
t	1	2	3	4	5	6	7	8	9	10

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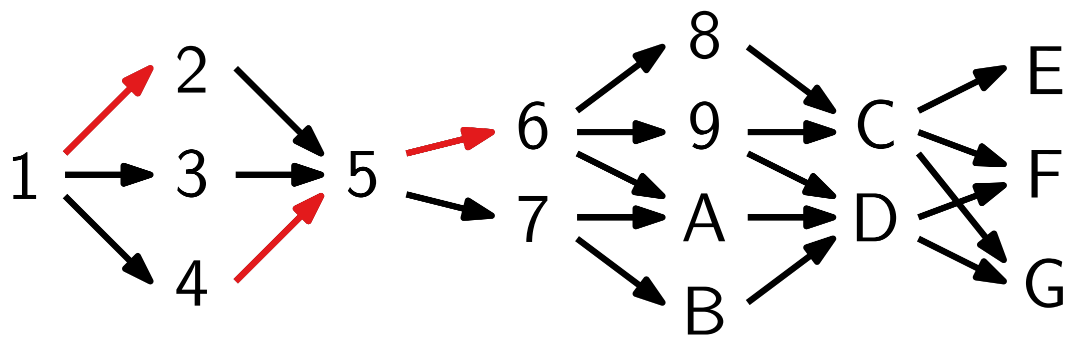

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M_1	1	2	4	5	6	8				
M_2	–	3	–	–	7	9				
t	1	2	3	4	5	6	7	8	9	10

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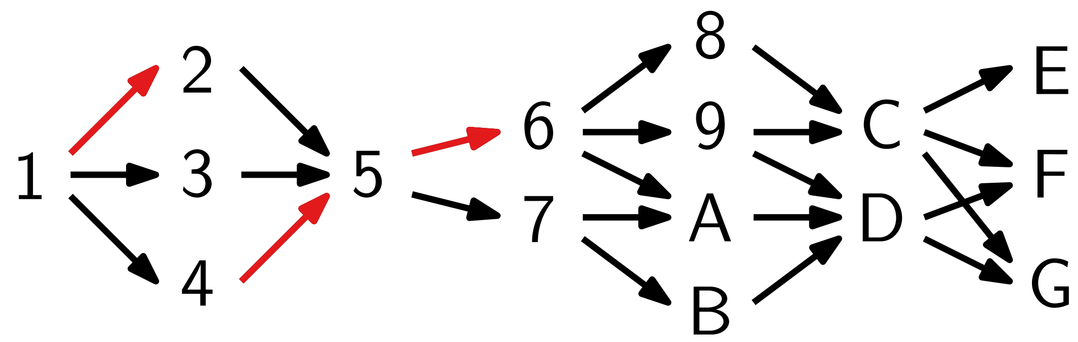

Number of machines is $W = 2$.

Output: Schedule

M_1	1	2	4	5	6	8	A			
M_2	–	3	–	–	7	9	B			
t	1	2	3	4	5	6	7	8	9	10

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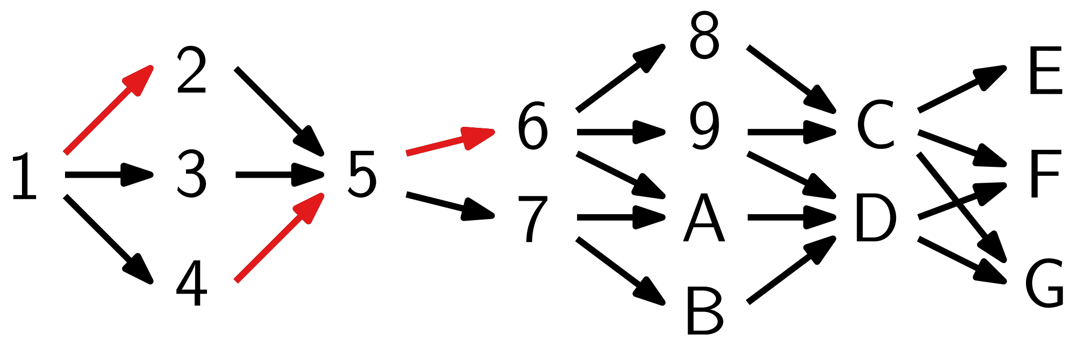

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Output: Schedule

M_1	1	2	4	5	6	8	A	C		
M_2	–	3	–	–	7	9	B	D		
t	1	2	3	4	5	6	7	8	9	10

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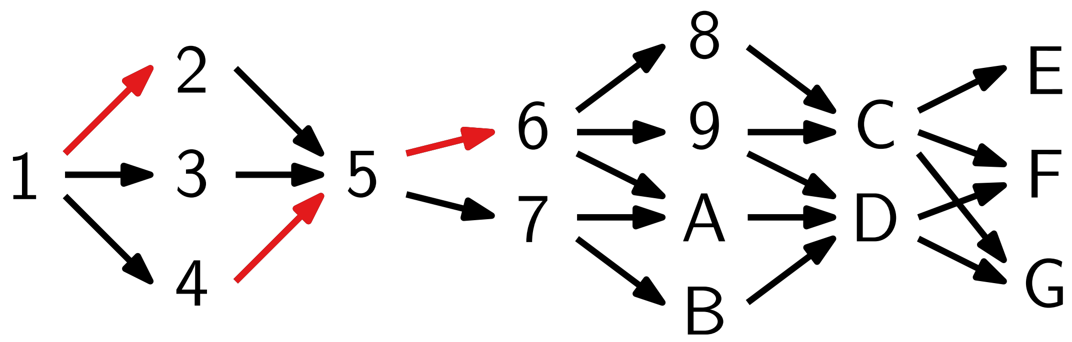

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M_2	–	3	–	–	7	9	B	D	F	
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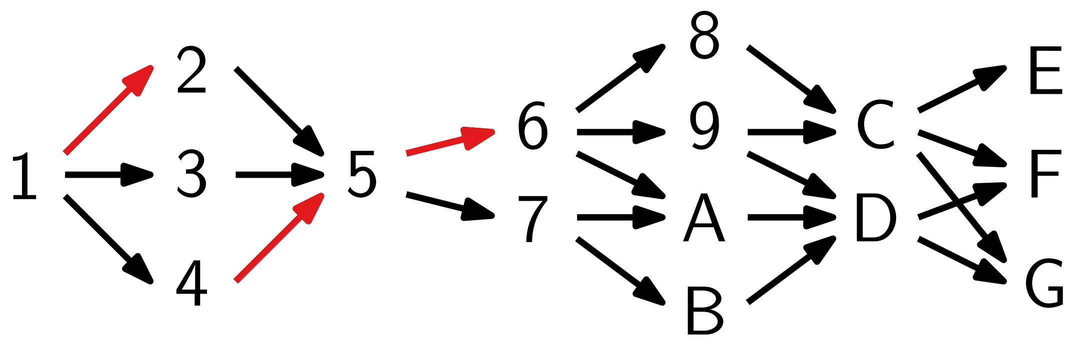

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Output: Schedule

M_1	1	2	4	5	6	8	A	C	E	G
M_2	–	3	–	–	7	9	B	D	F	–
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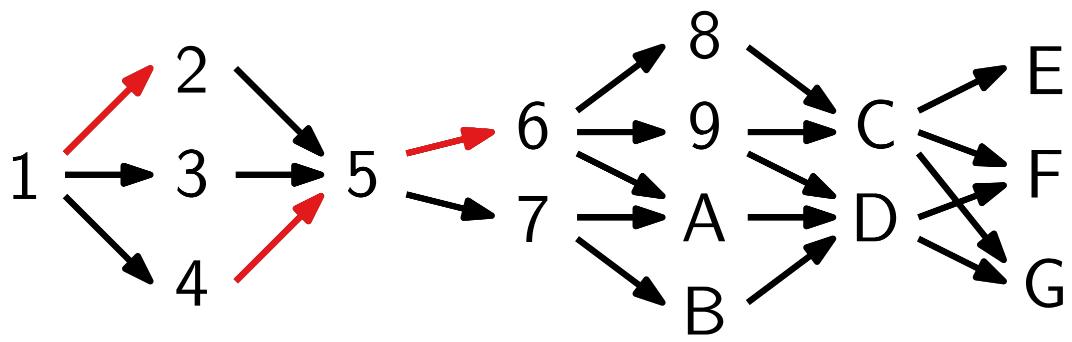

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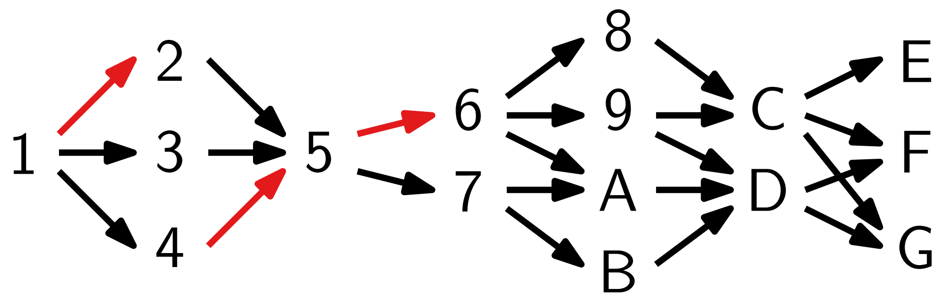
Question: Good approximation factor?

Approximating PCMPS - Analysis for $W = 2$

Precedence graph $G_<$



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M_1	1	2	4	5	6	8	A	C	E	G
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„The art of the lower bound“

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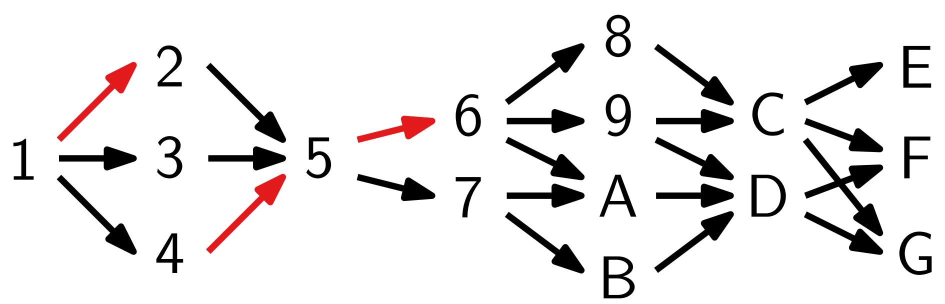

Schedule

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OPT $\geq$

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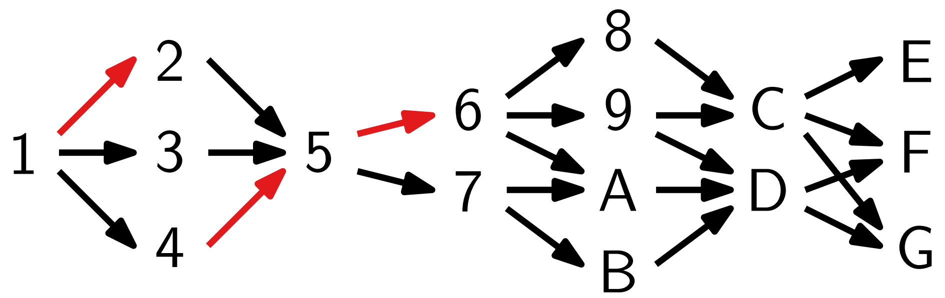

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OPT $\geq \lceil n/2 \rceil$

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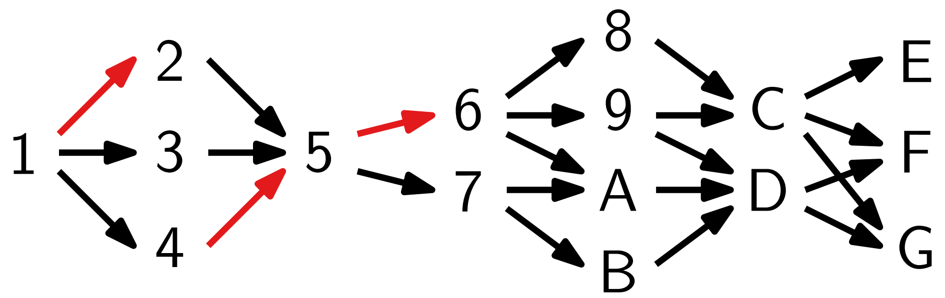

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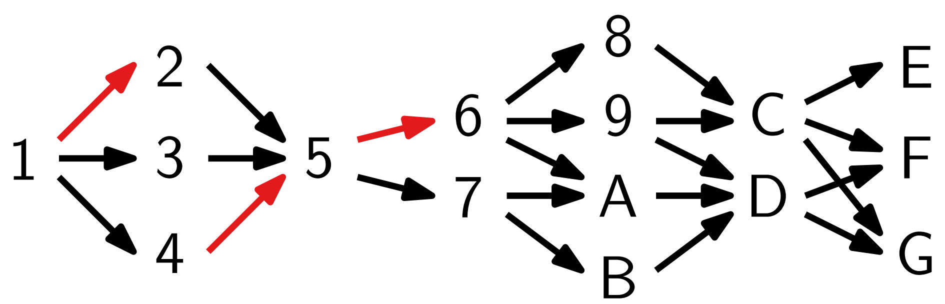
M_2	–	3	–	–	7	9	B	D	F	–
t	1	2	3	4	5	6	7	8	9	10

„The art of the lower bound“

OPT $\geq$ $\lceil n/2 \rceil$ and OPT $\geq$

Approximating PCMPS - Analysis for $W = 2$

Precedence graph $G_<$



Schedule

M_1	1	2	4	5	6	8	A	C	E	G
M_2	–	3	–	–	7	9	B	D	F	–
t	1	2	3	4	5	6	7	8	9	10

„The art of the lower bound“

OPT $\geq \lceil n/2 \rceil$ and OPT $\geq \ell :=$ Number of layers of $G_<$ ($=$ length of longest path in $G_<$)

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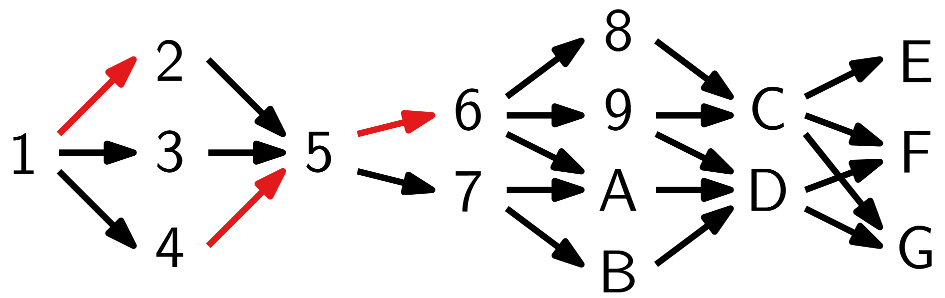

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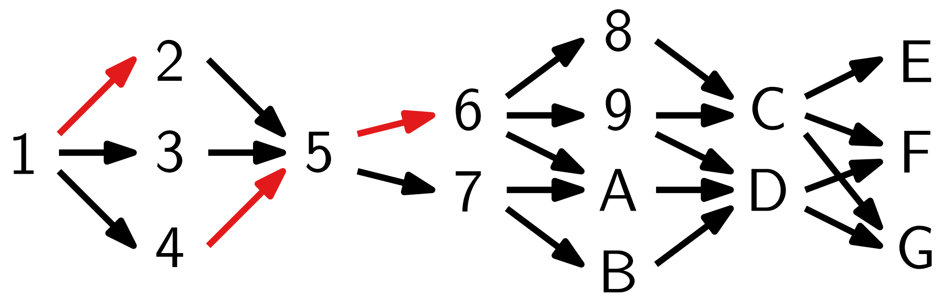

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Bound. ALG $\leq$

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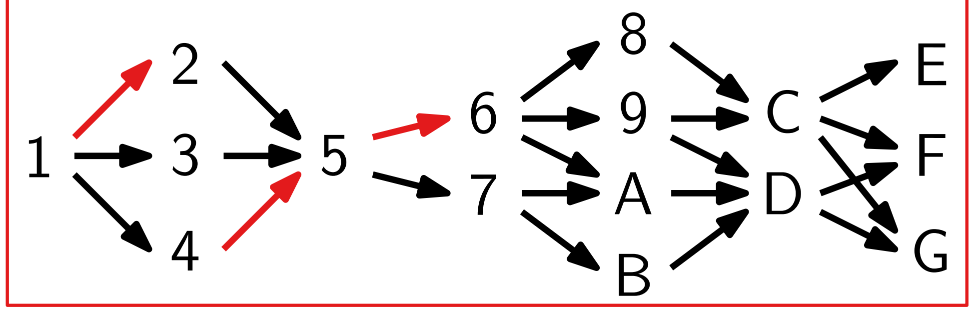

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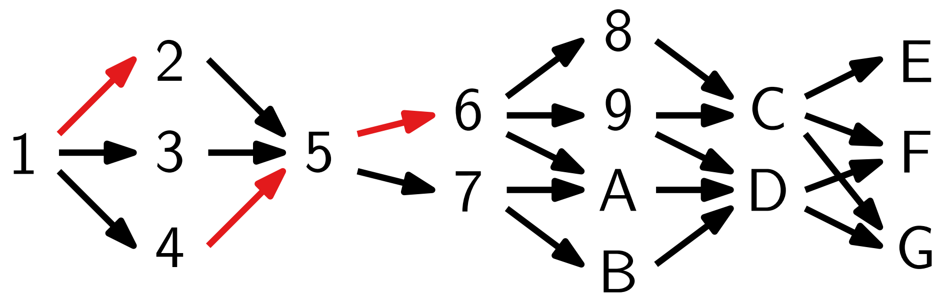

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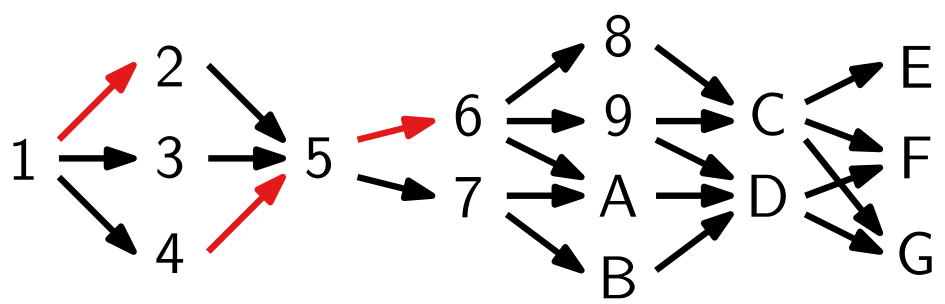

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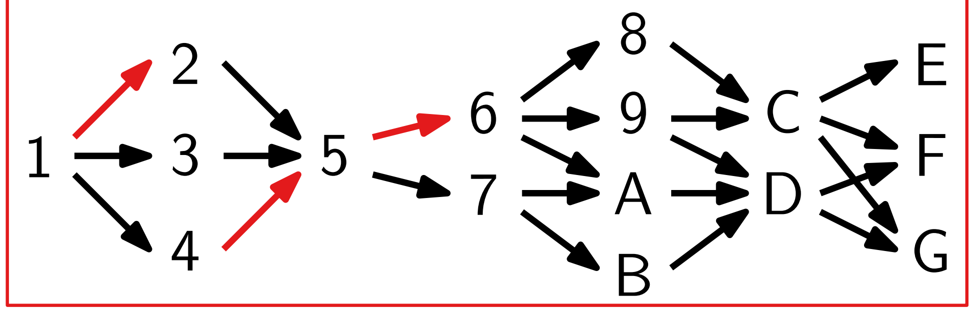

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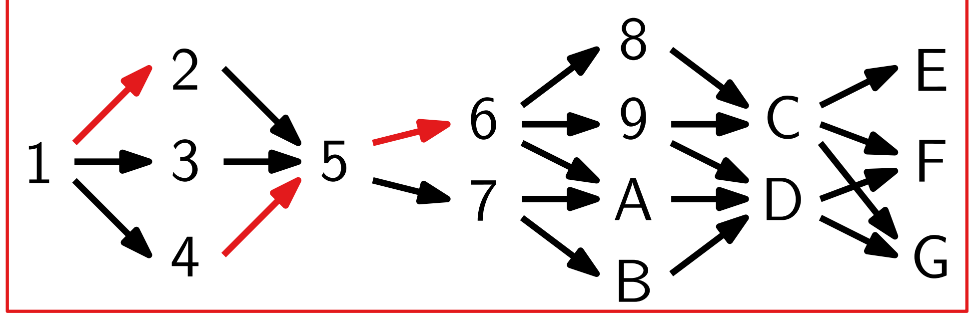

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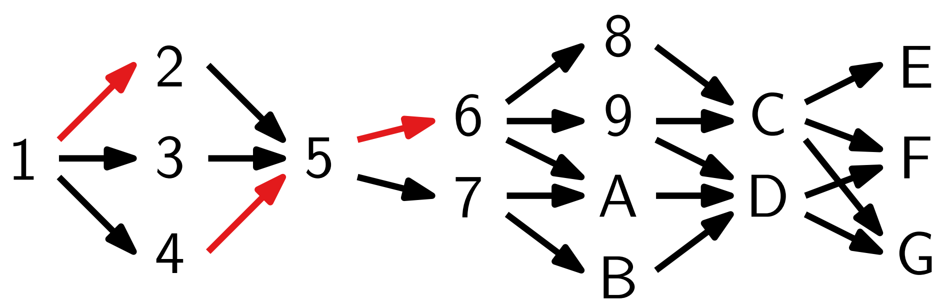

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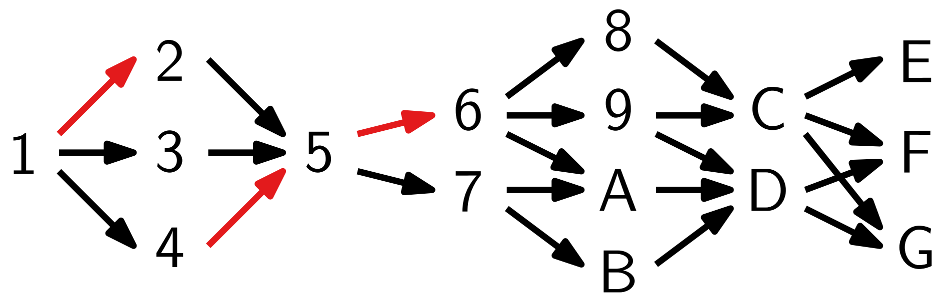

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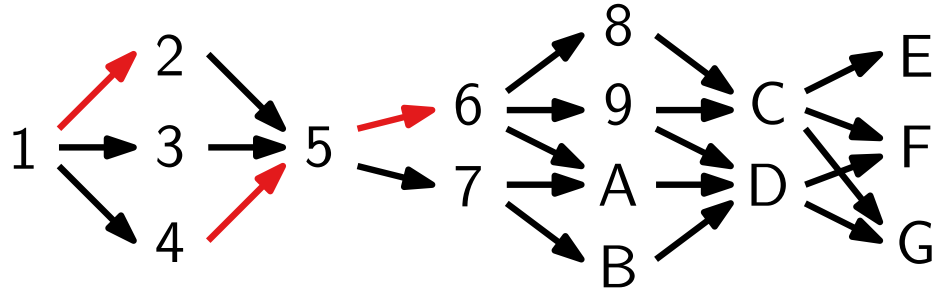

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$\leq (2 - 1/W) \cdot$ OPT in general case

Bound. ALG $\leq \left\lceil \frac{n+\ell}{2} \right\rceil \approx \lceil n/2 \rceil + \ell/2 \leq 3/2 \cdot$ OPT

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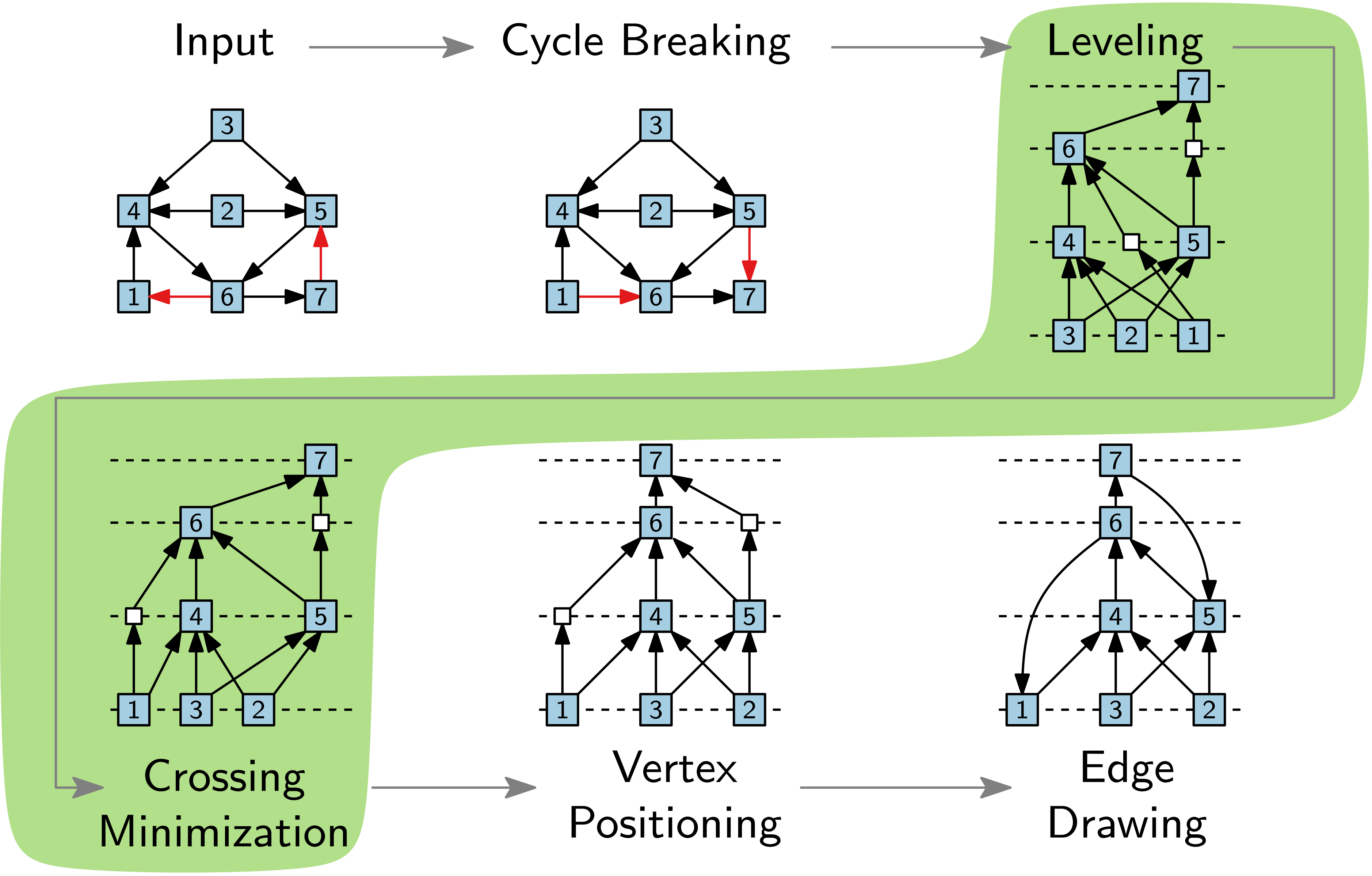
Step 3: Crossing Minimization



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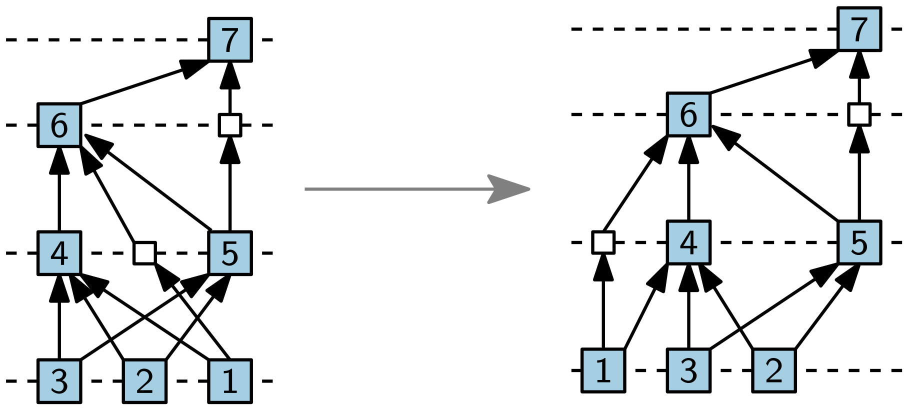
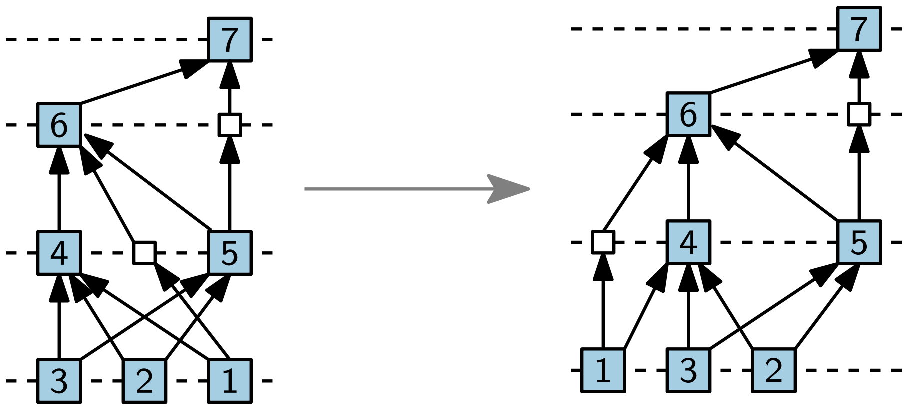


Problem.

Step 3: Crossing Minimization



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- Input: Graph G, leveling $y\colon V \to \{1, \ldots, n\}$

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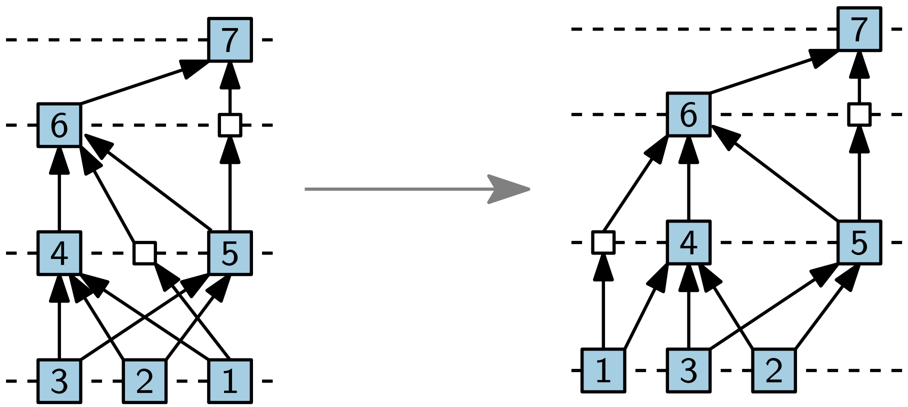

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- Input: Graph G, leveling $y\colon V \to \{1, \ldots, n\}$
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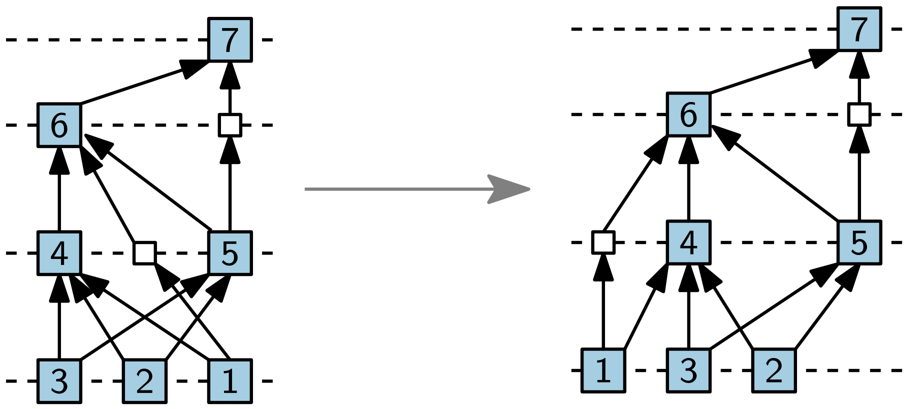

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- NP-hard, even for 2 layers [Garey & Johnson '83]

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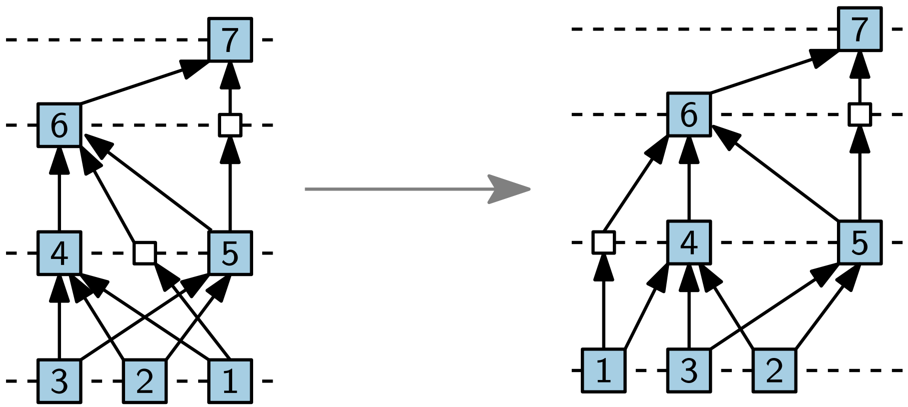

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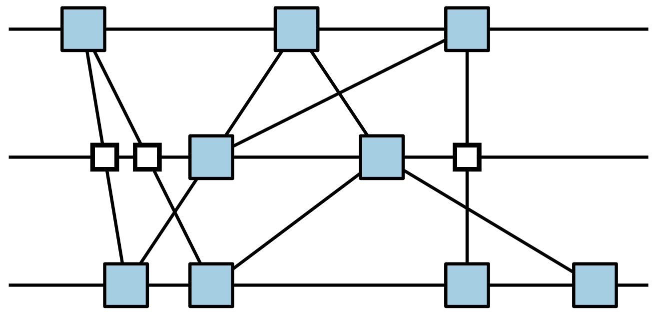
- NP-hard, even for 2 layers [Garey & Johnson '83]
- hardly any approaches optimize over multiple layers

Iterative Crossing Reduction

Observation. The number of crossings only depends on permutations of adjacent layers.

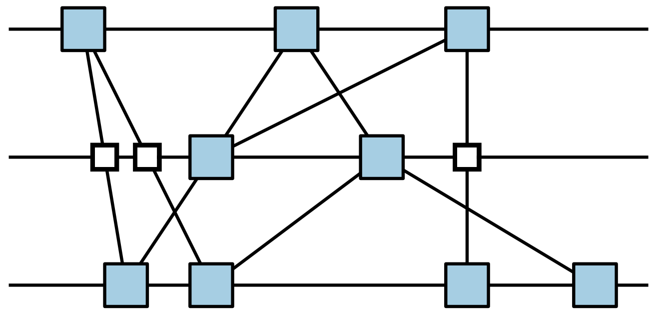


Iterative Crossing Reduction

Observation. The number of crossings only depends on permutations of adjacent layers.

Idea.

- permute one layer after the other
- treat dummy-vertices like “regular” vertices



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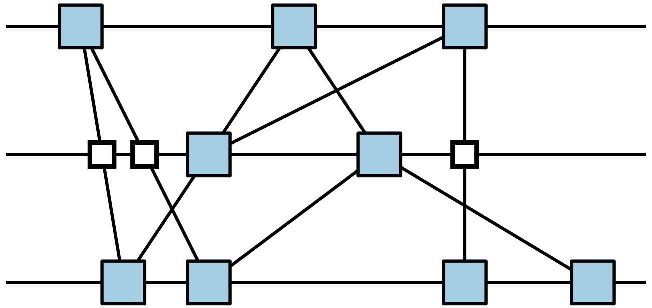

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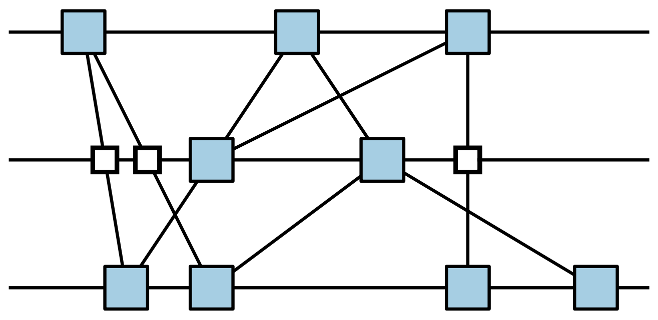

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(1) choose a random permutation of L_1

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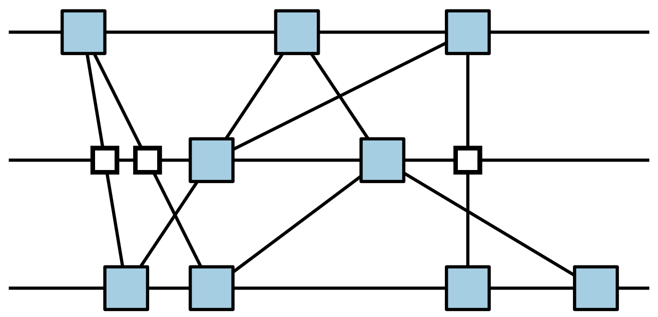

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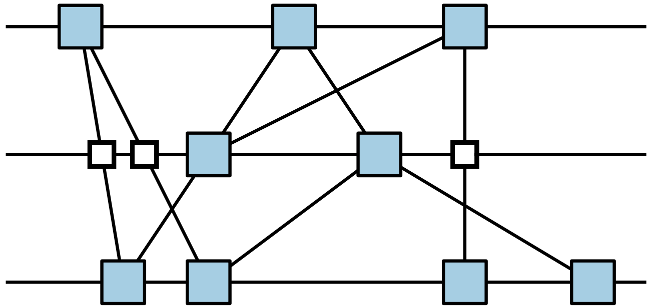

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(1) choose a random permutation of L_1

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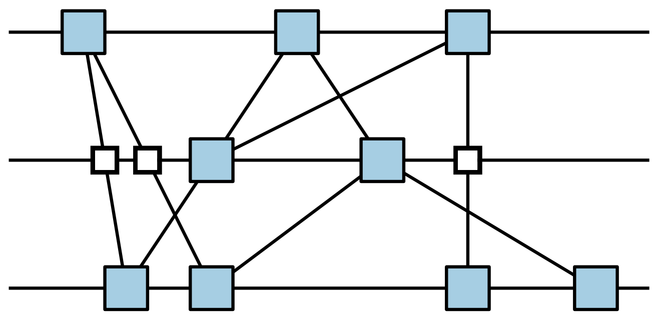

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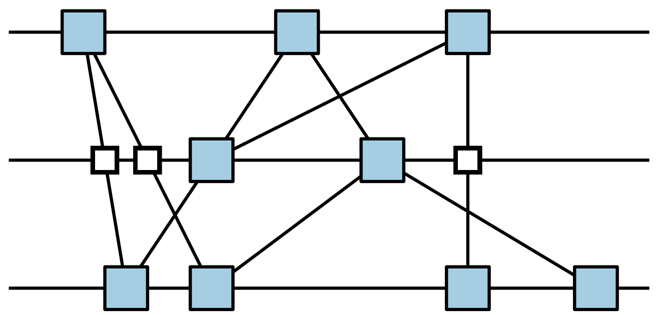

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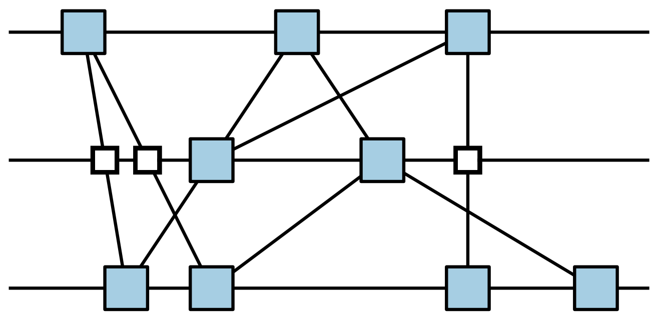

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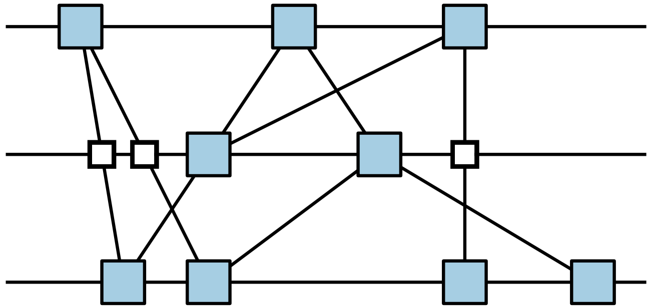

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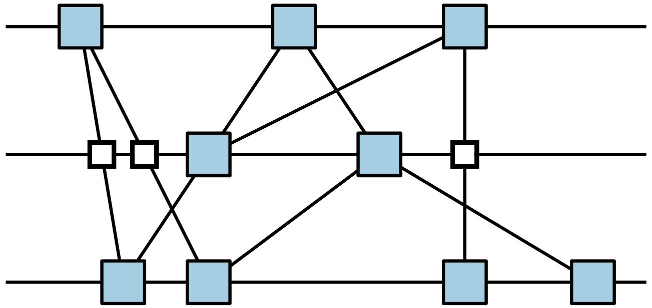

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one-sided crossing minimization

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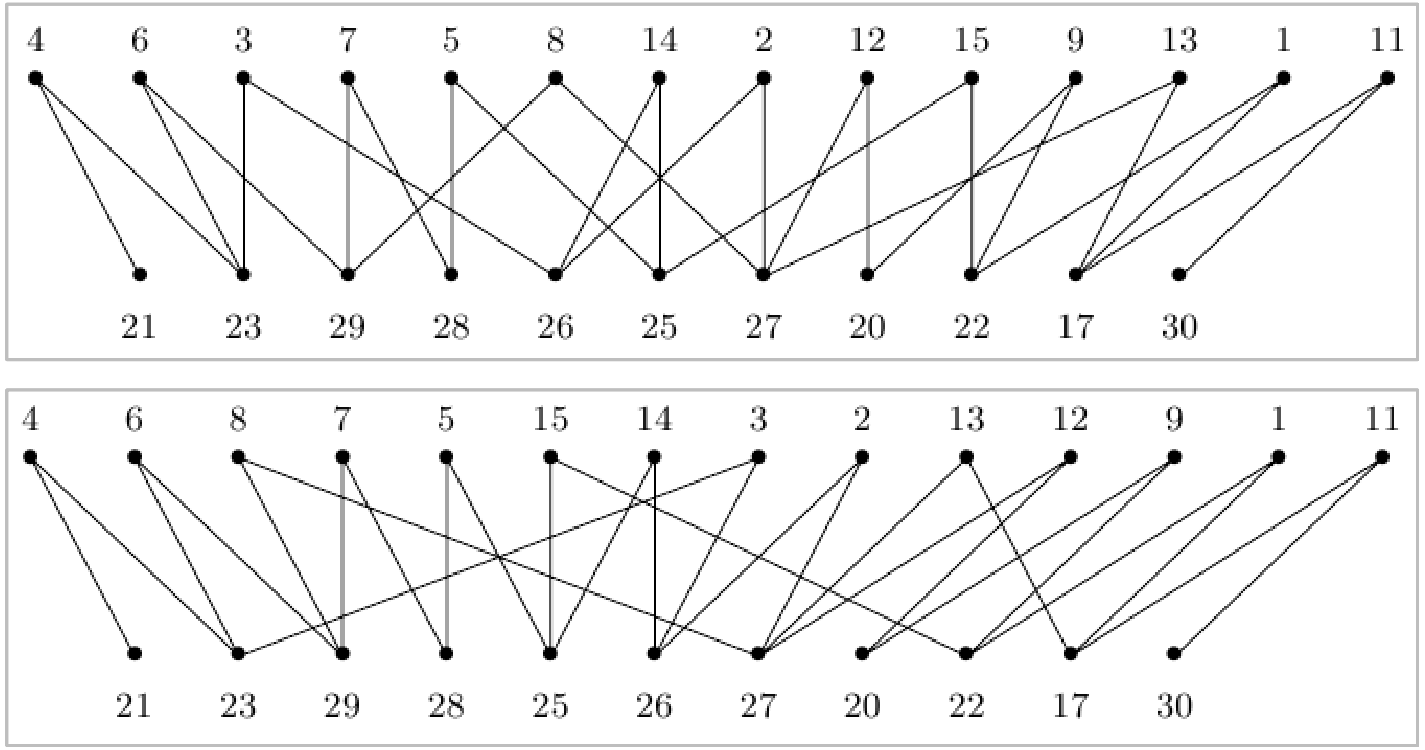
One-Sided Crossing Minimization

Problem.

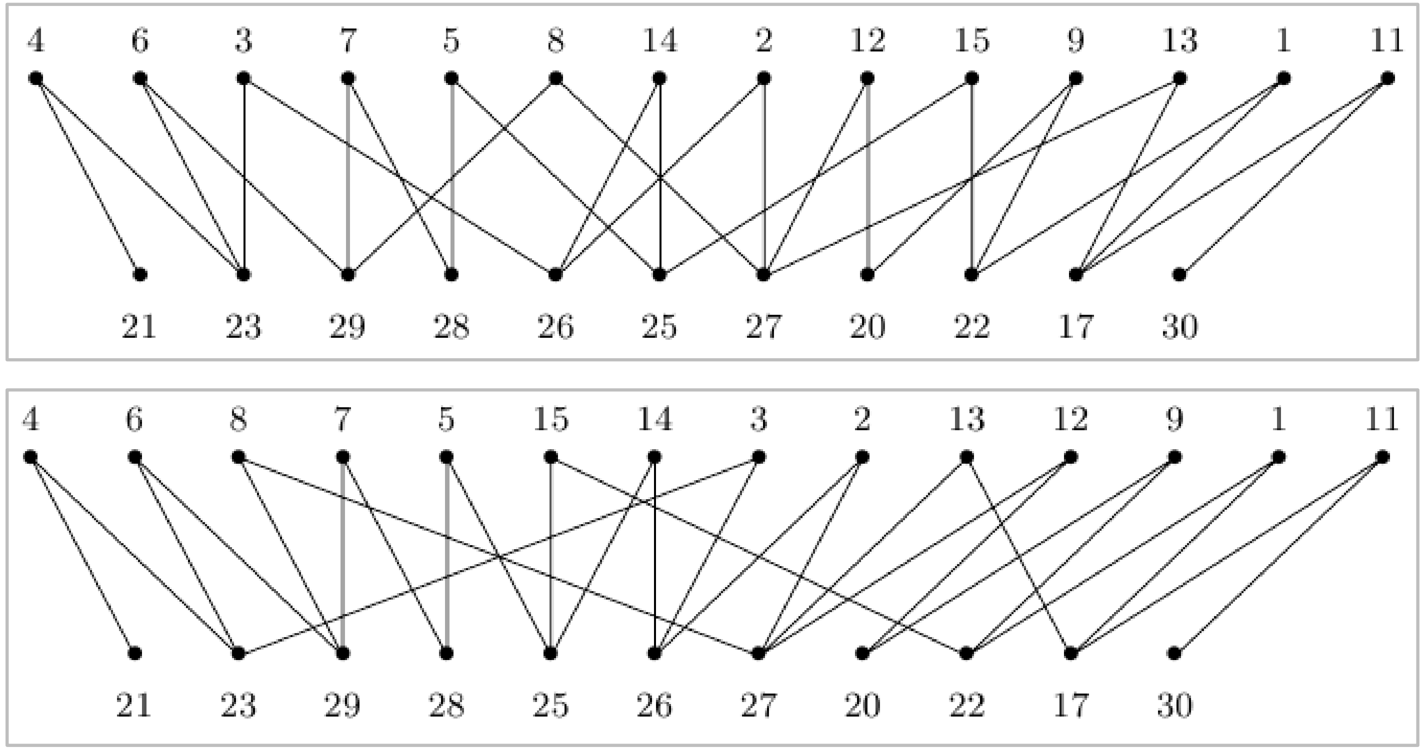


[Kaufmann & Wagner: Drawing Graphs]

One-Sided Crossing Minimization

Problem.

- Input: bipartite graph $G = (L_1 \cup L_2, E)$, permutation π_1 on L_1



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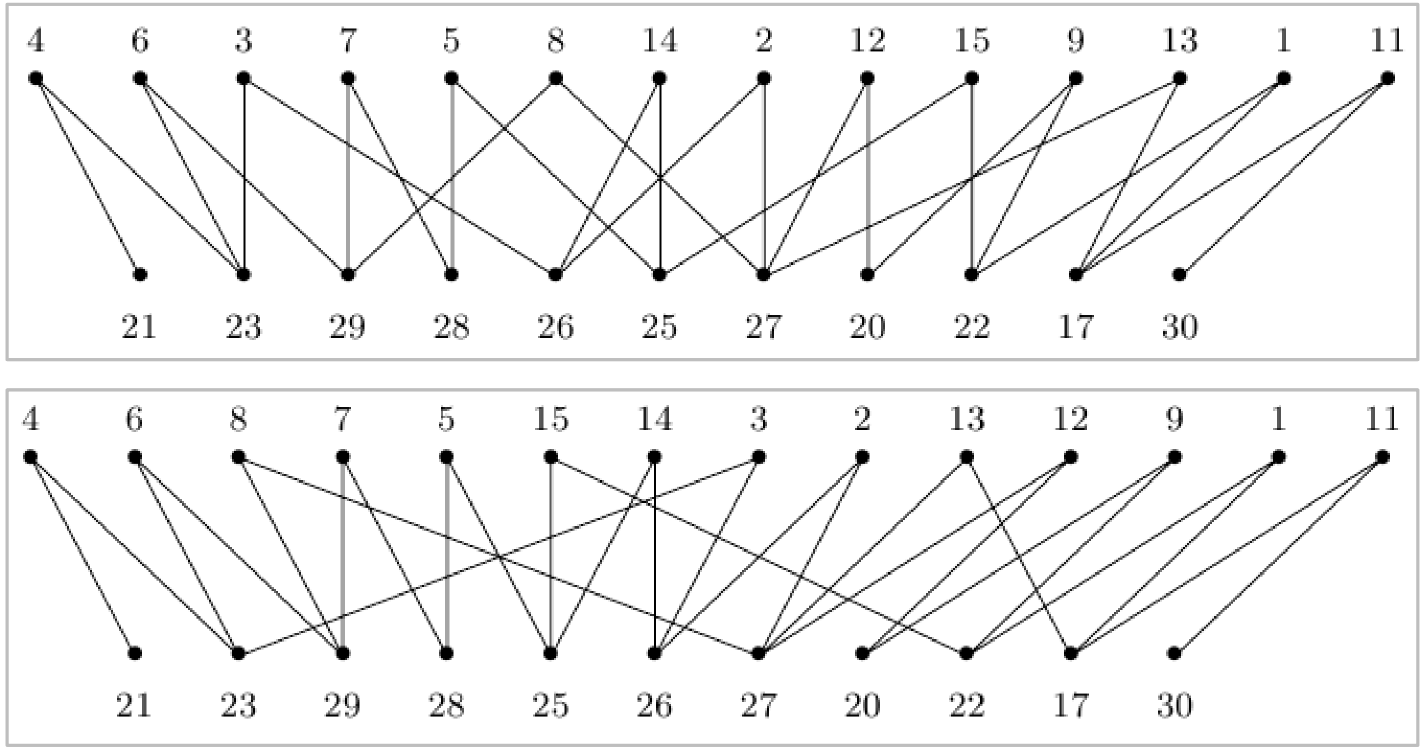

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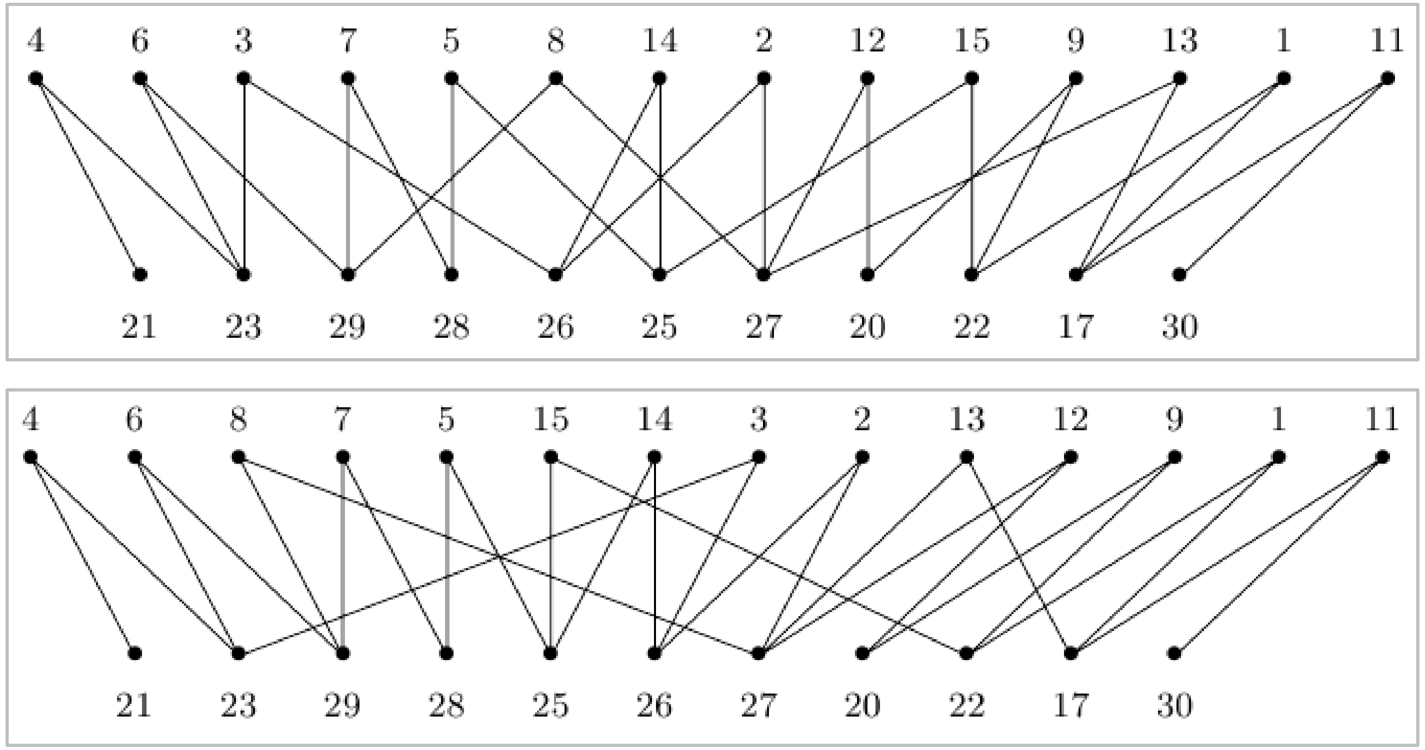
One-sided crossing minimization is NP-hard.
[Eades & Whitesides '94]



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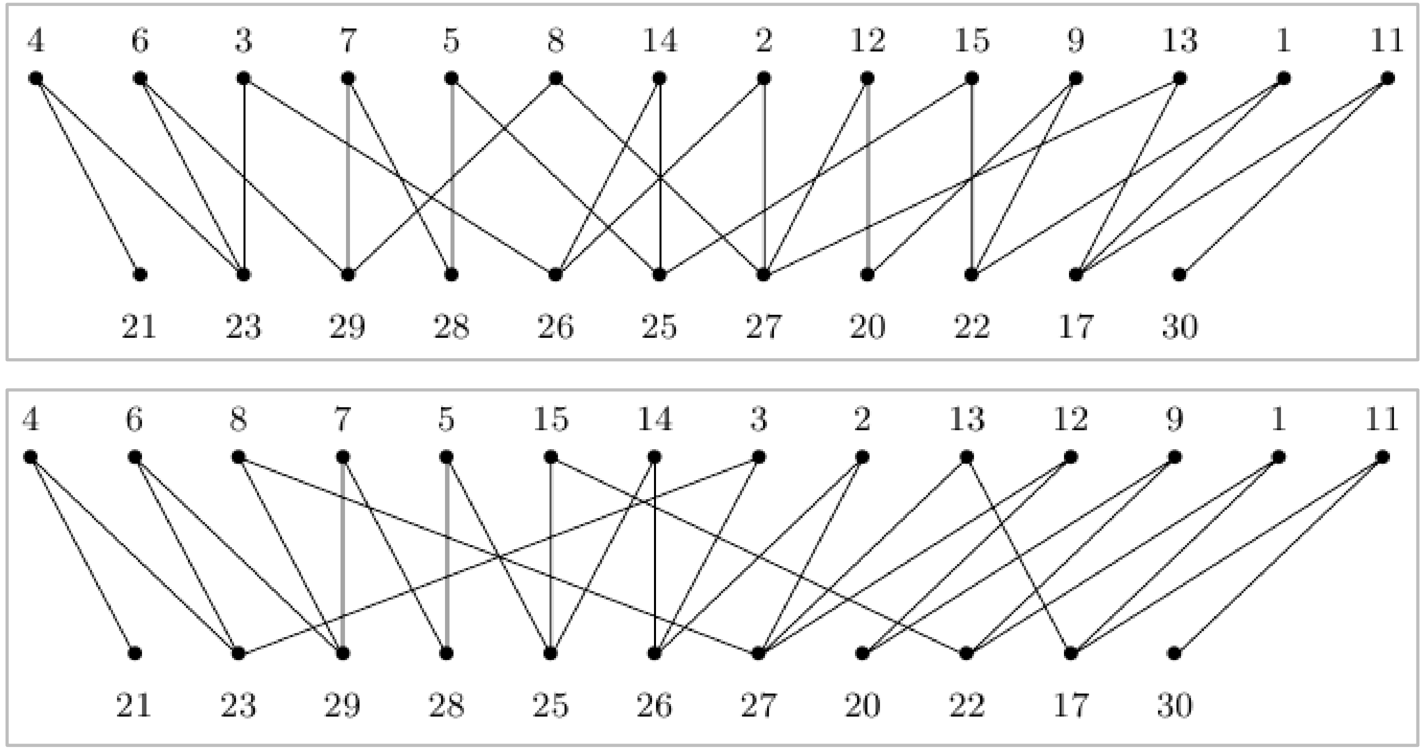

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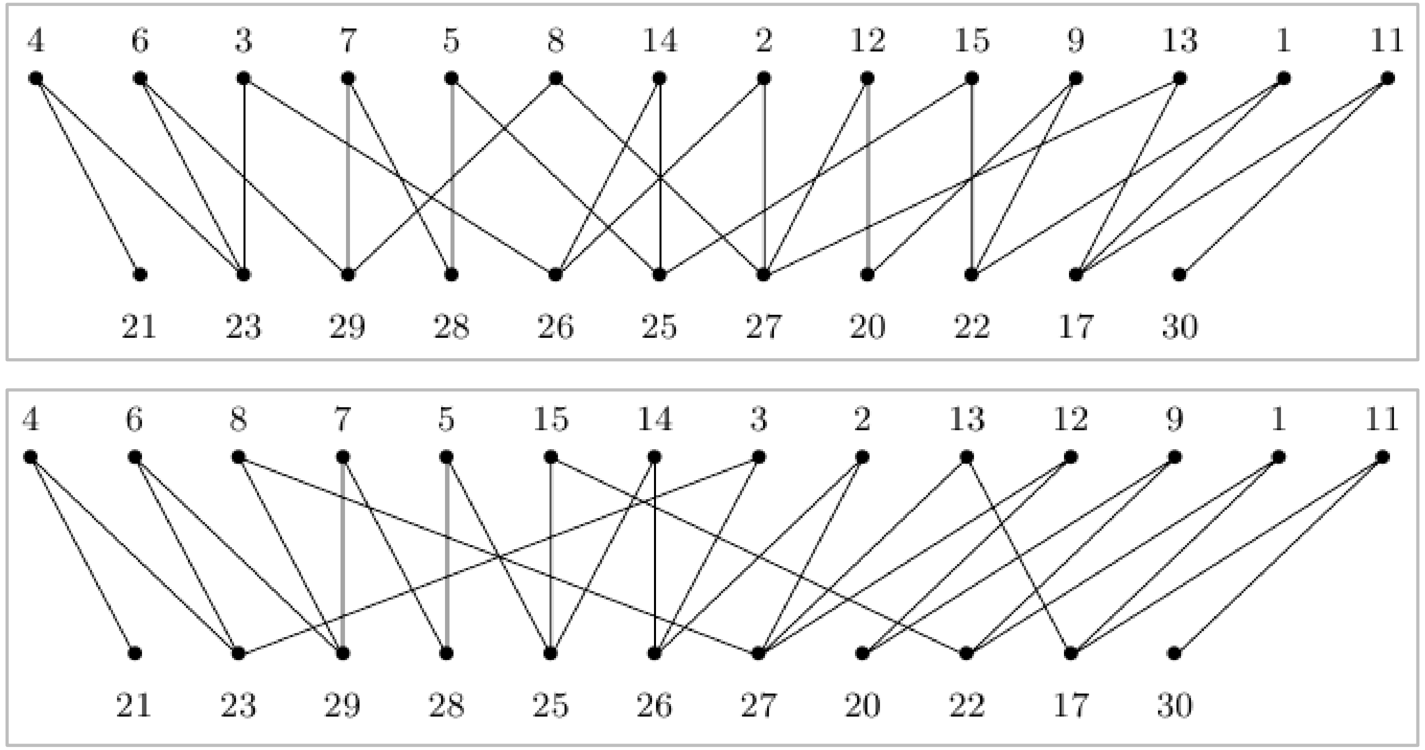
Algorithms.

- barycenter heuristic



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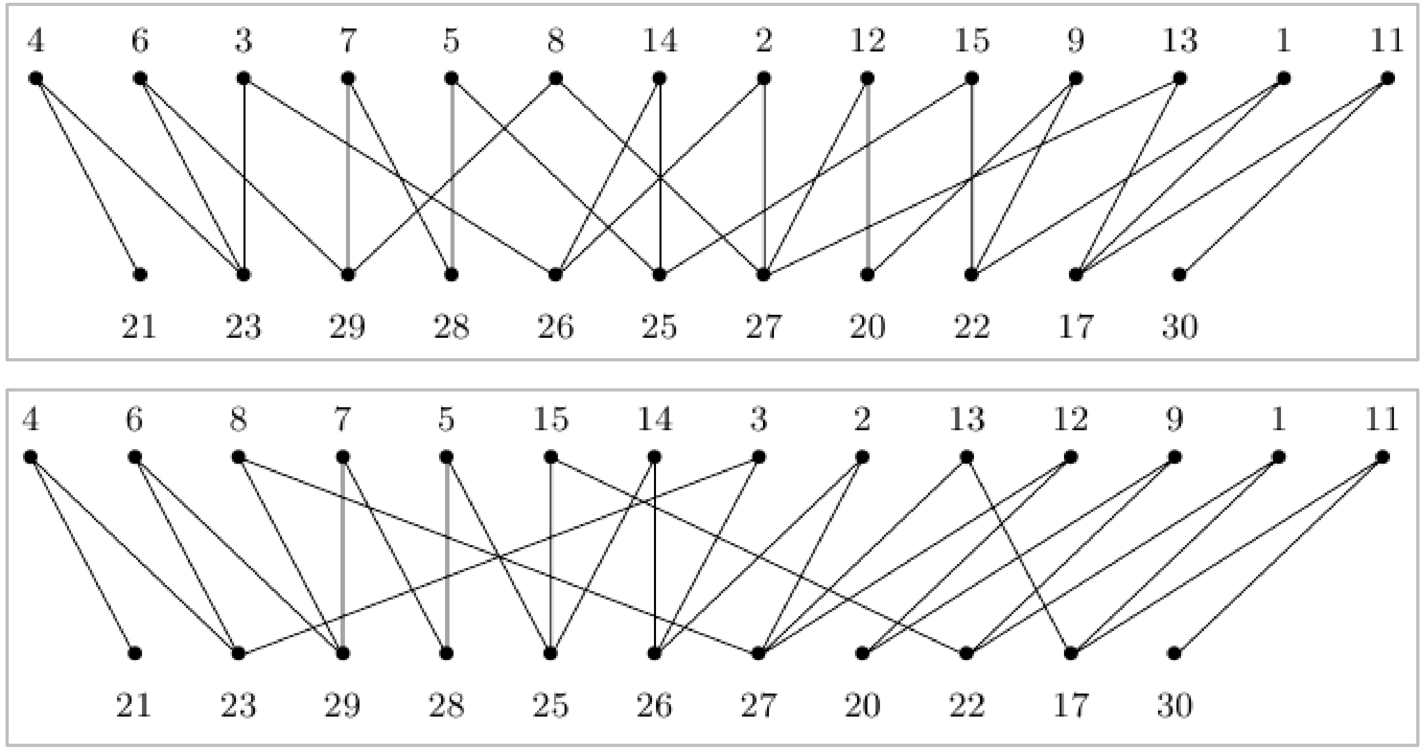
Algorithms.

- barycenter heuristic
- median heuristic



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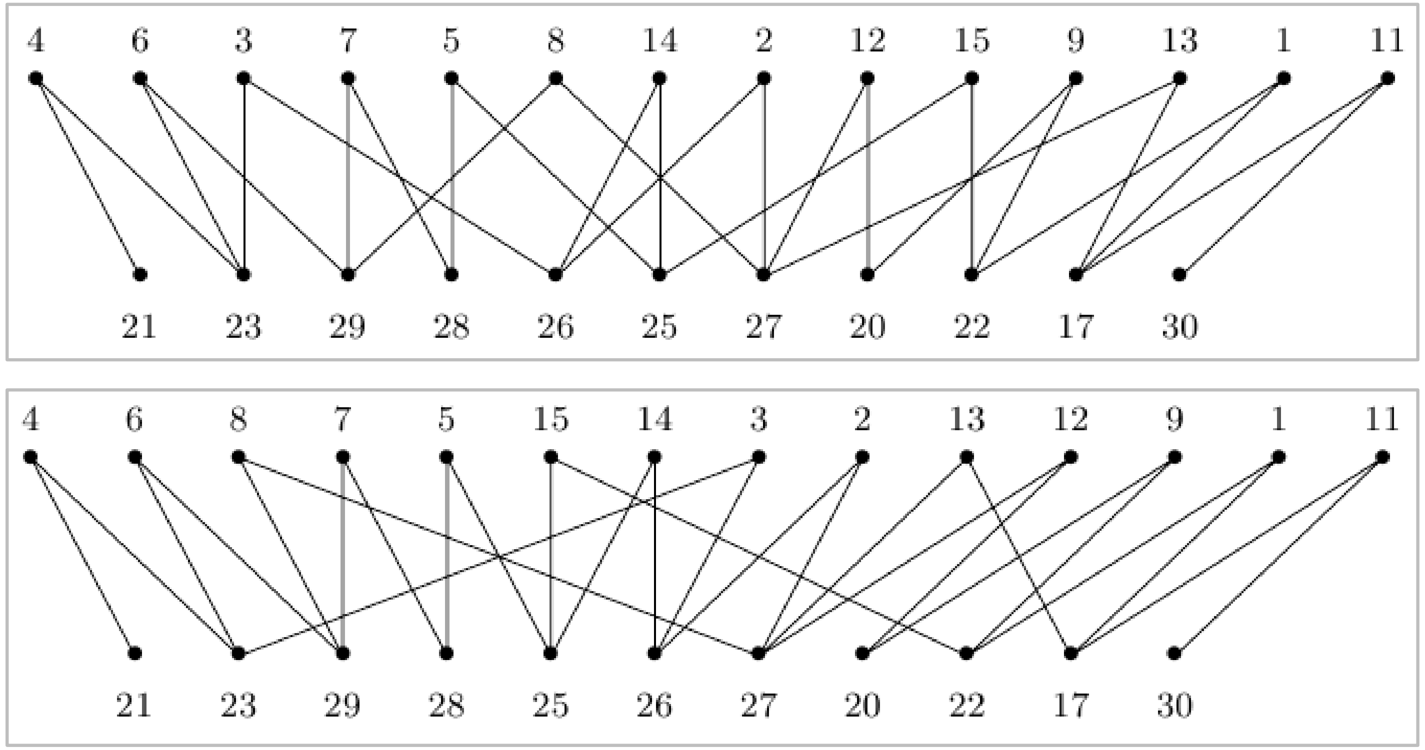

[Kaufmann & Wagner: Drawing Graphs]

One-Sided Crossing Minimization

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- Input: bipartite graph $G = (L_1 \cup L_2, E)$, permutation π_1 on L_1
- Output: permutation π_2 of L_2 minimizing the number of edge crossings.

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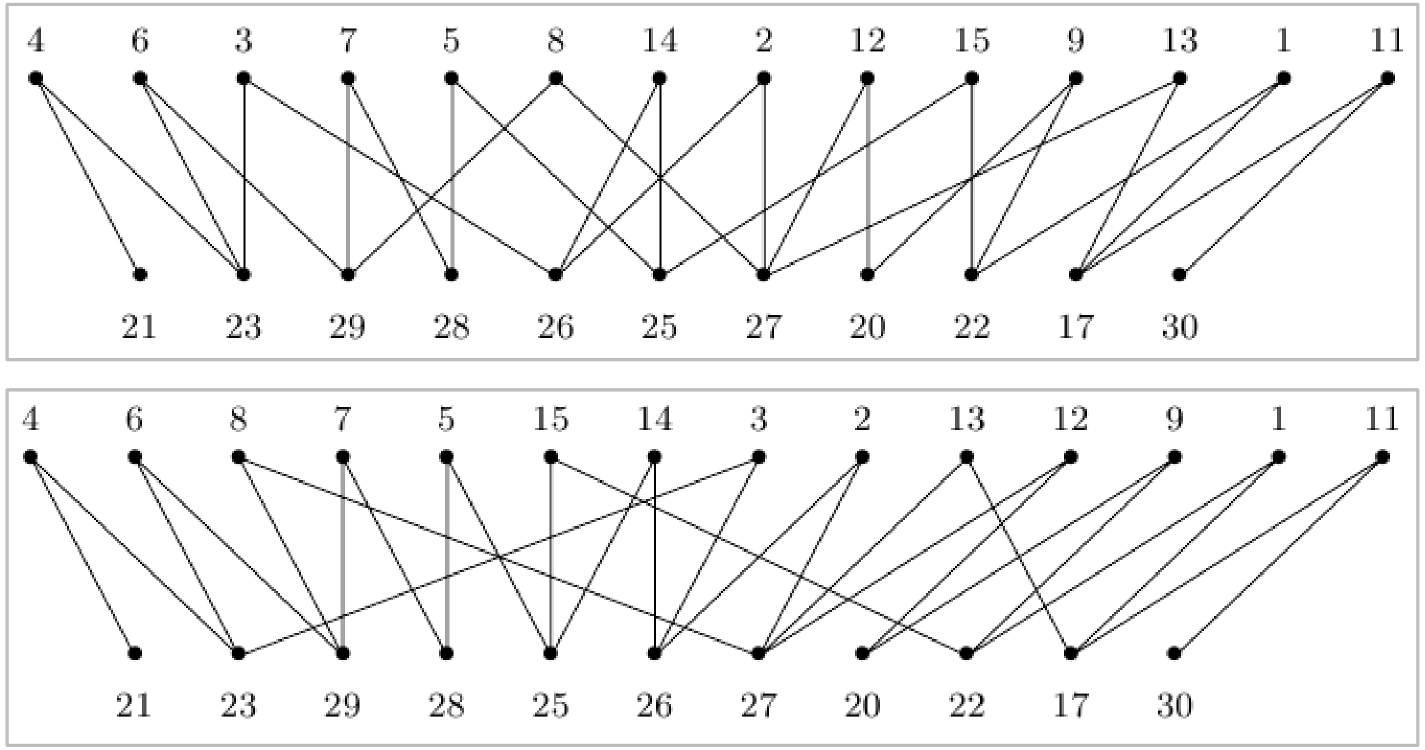

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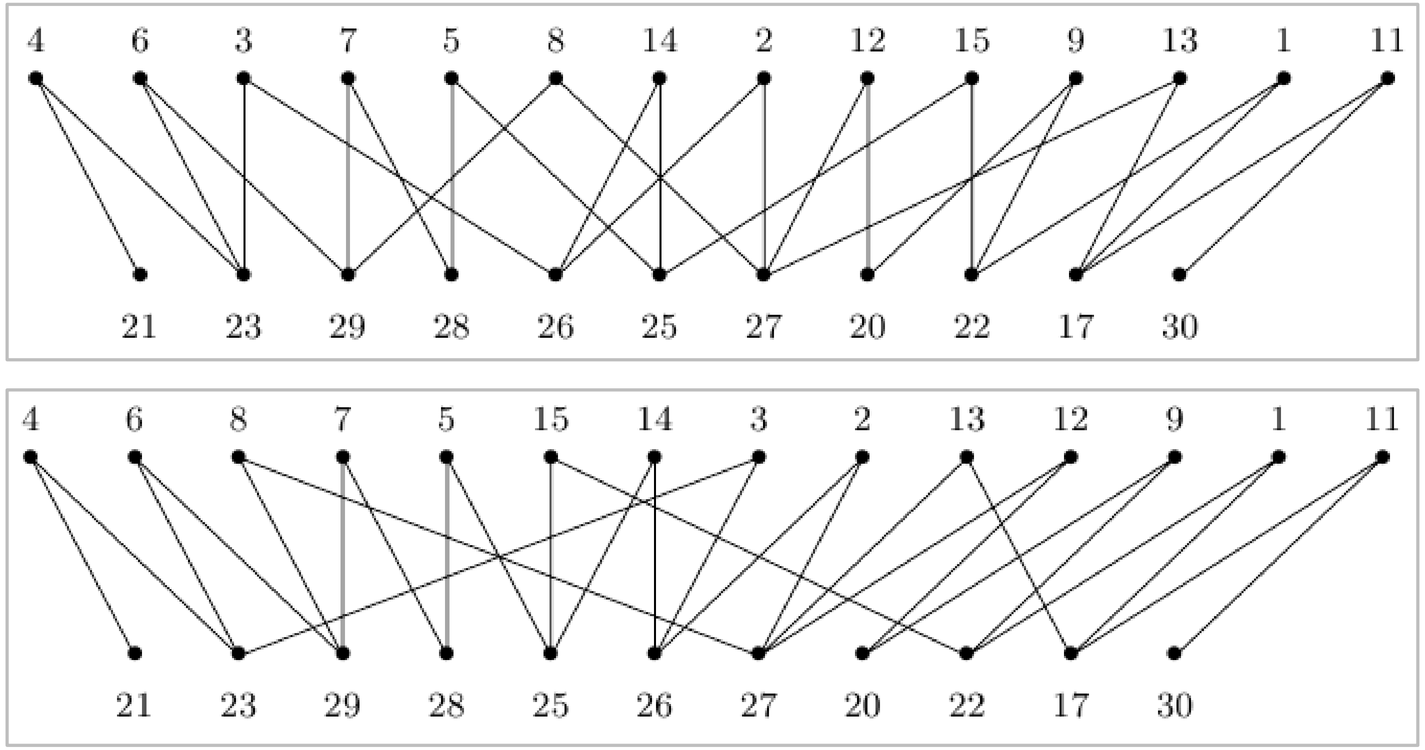

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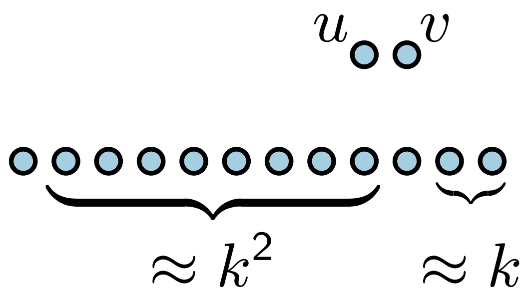

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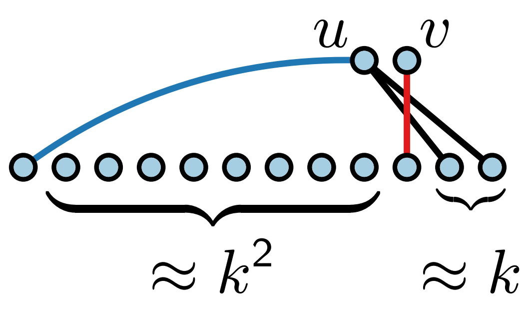

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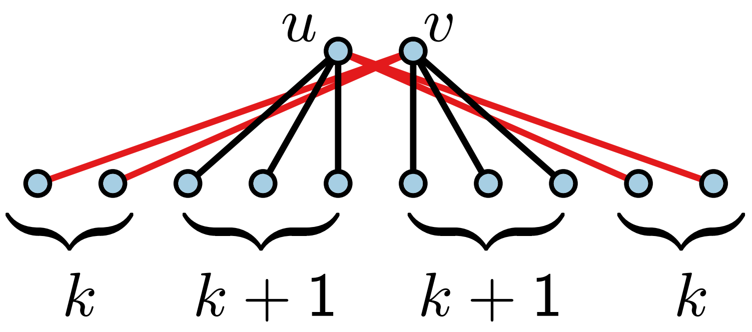

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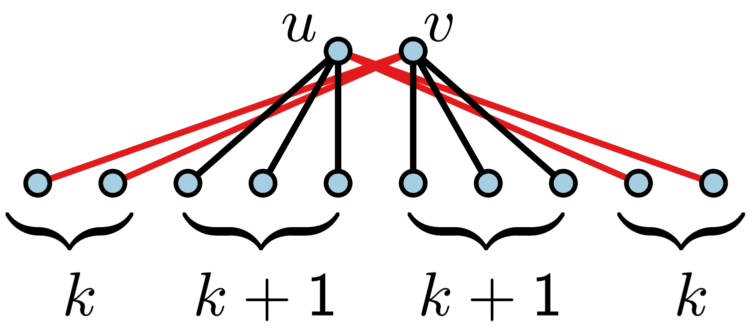

crossings: $2k(k+1) + k^2$ vs. $(k+1)^2$

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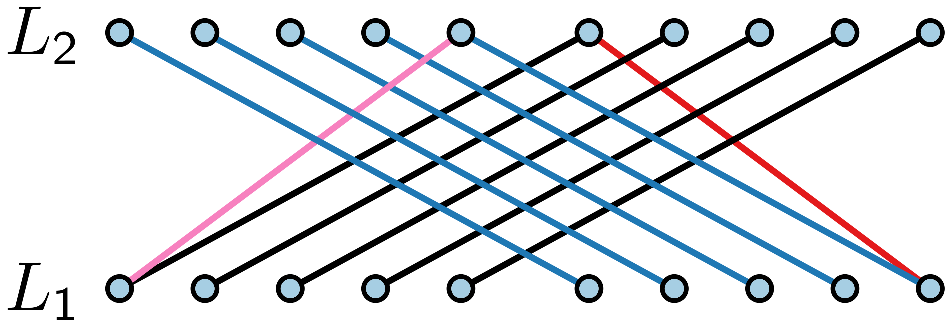

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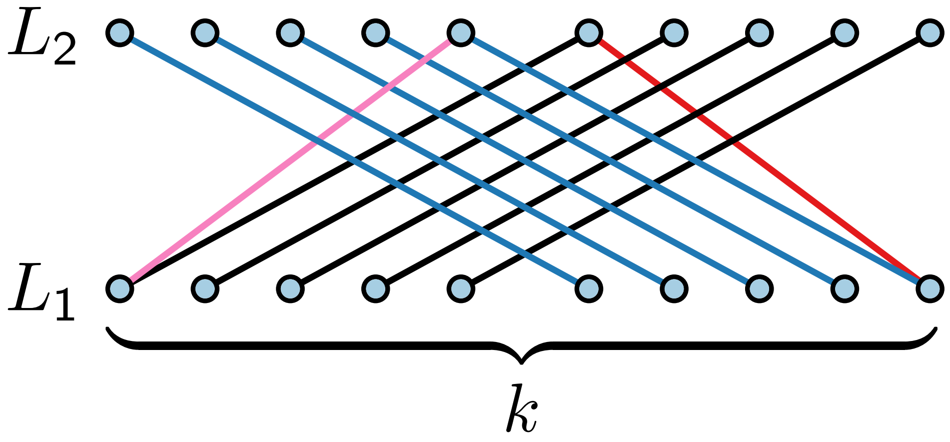

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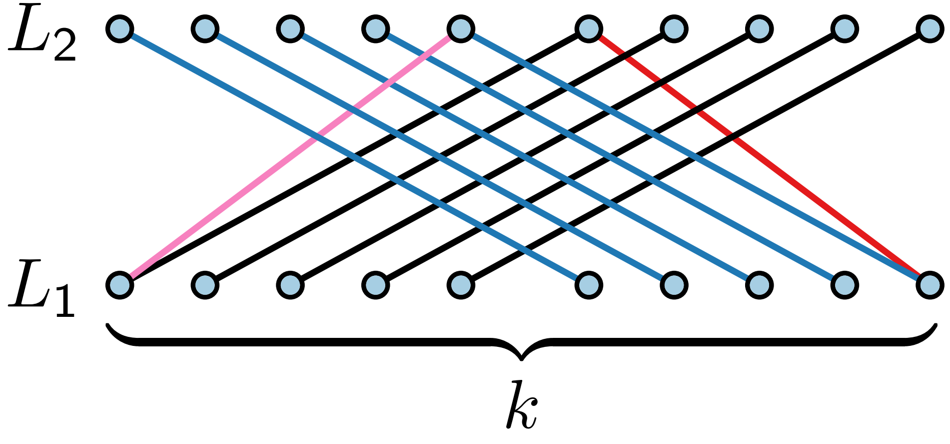

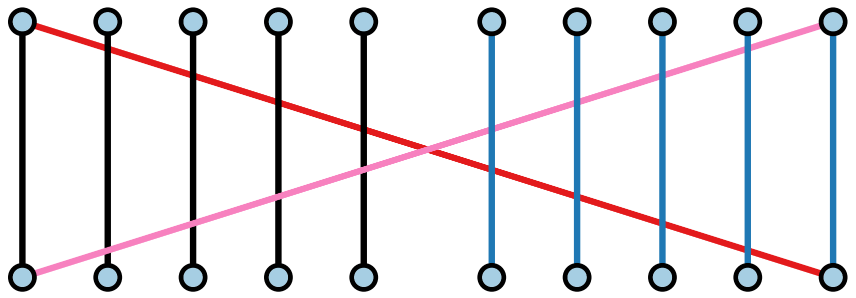

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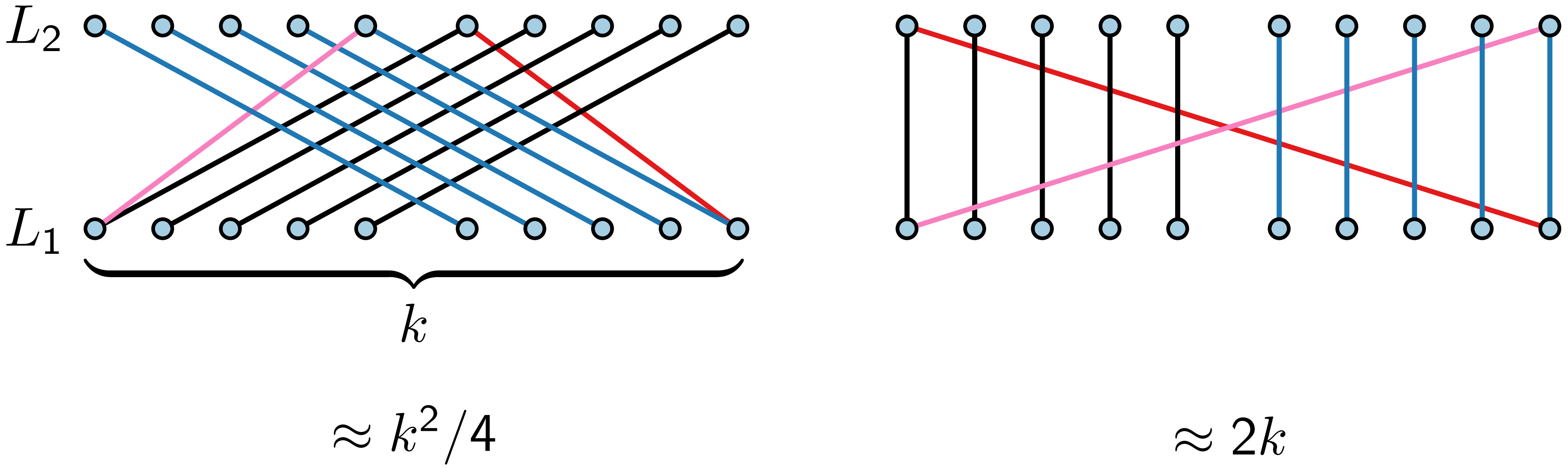

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[Jünger & Mutzel, '97]

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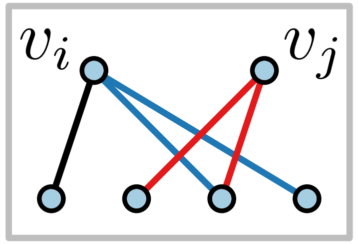

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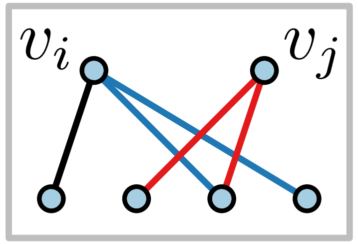

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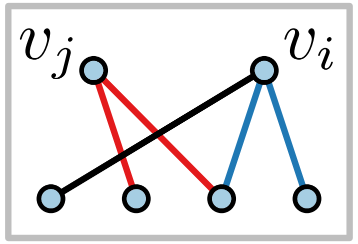

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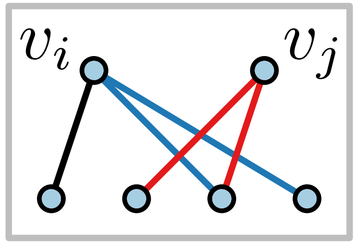

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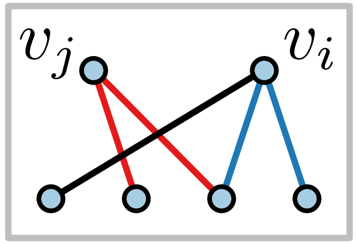

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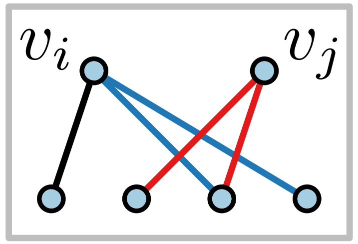

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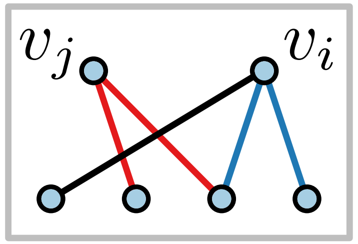

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Integer Linear Program (ILP)

[Jünger & Mutzel, '97]

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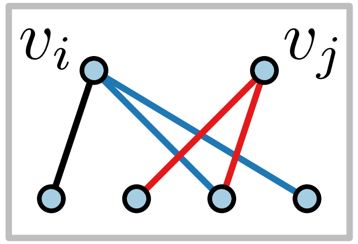

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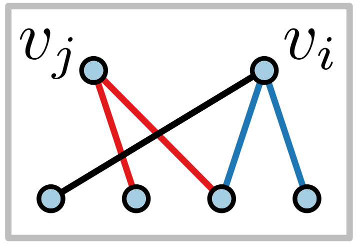

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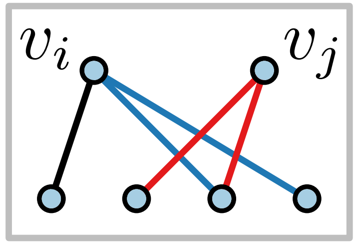

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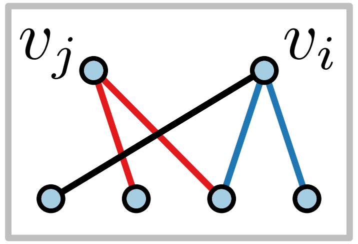

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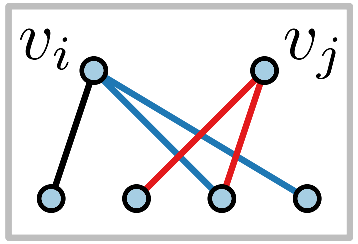

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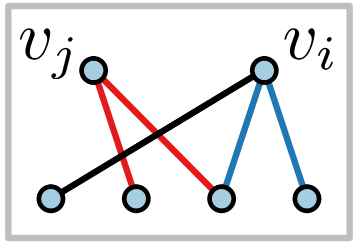

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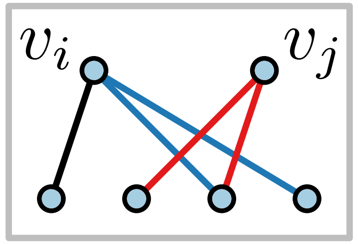

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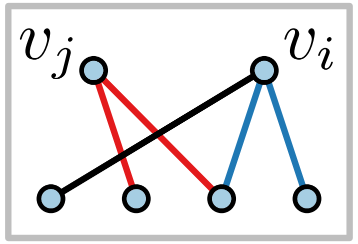

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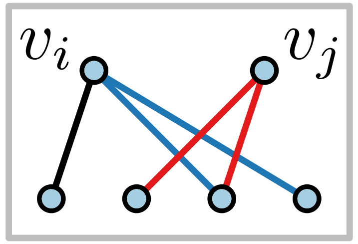

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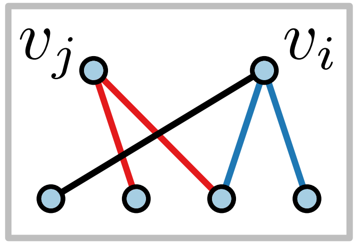

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Properties.

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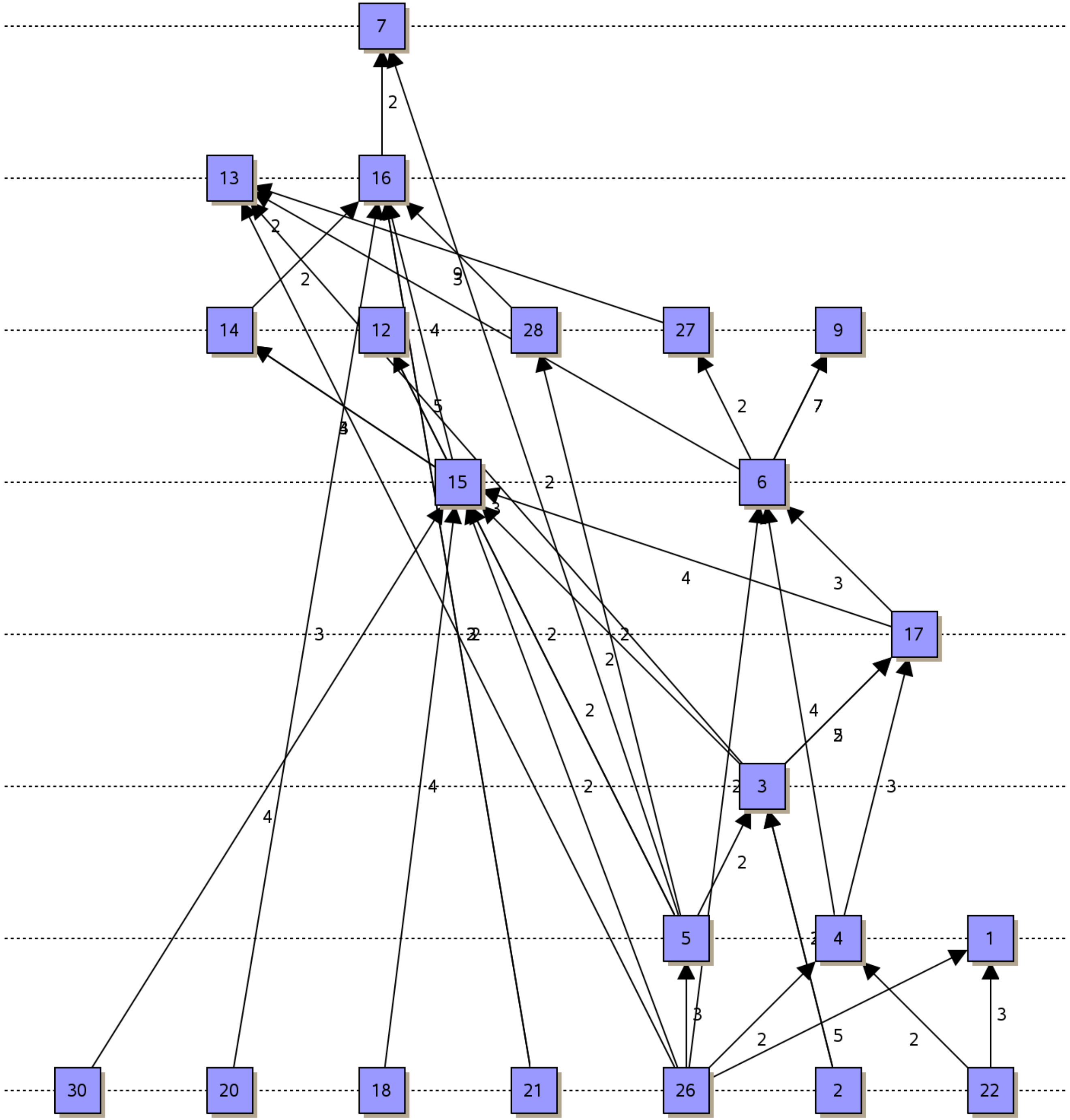
Properties.

- branch-and-cut technique applicable for this ILP
- useful for graphs of small to medium size
- finds optimal solution
- solution in polynomial time is not guaranteed

Iterations on Example



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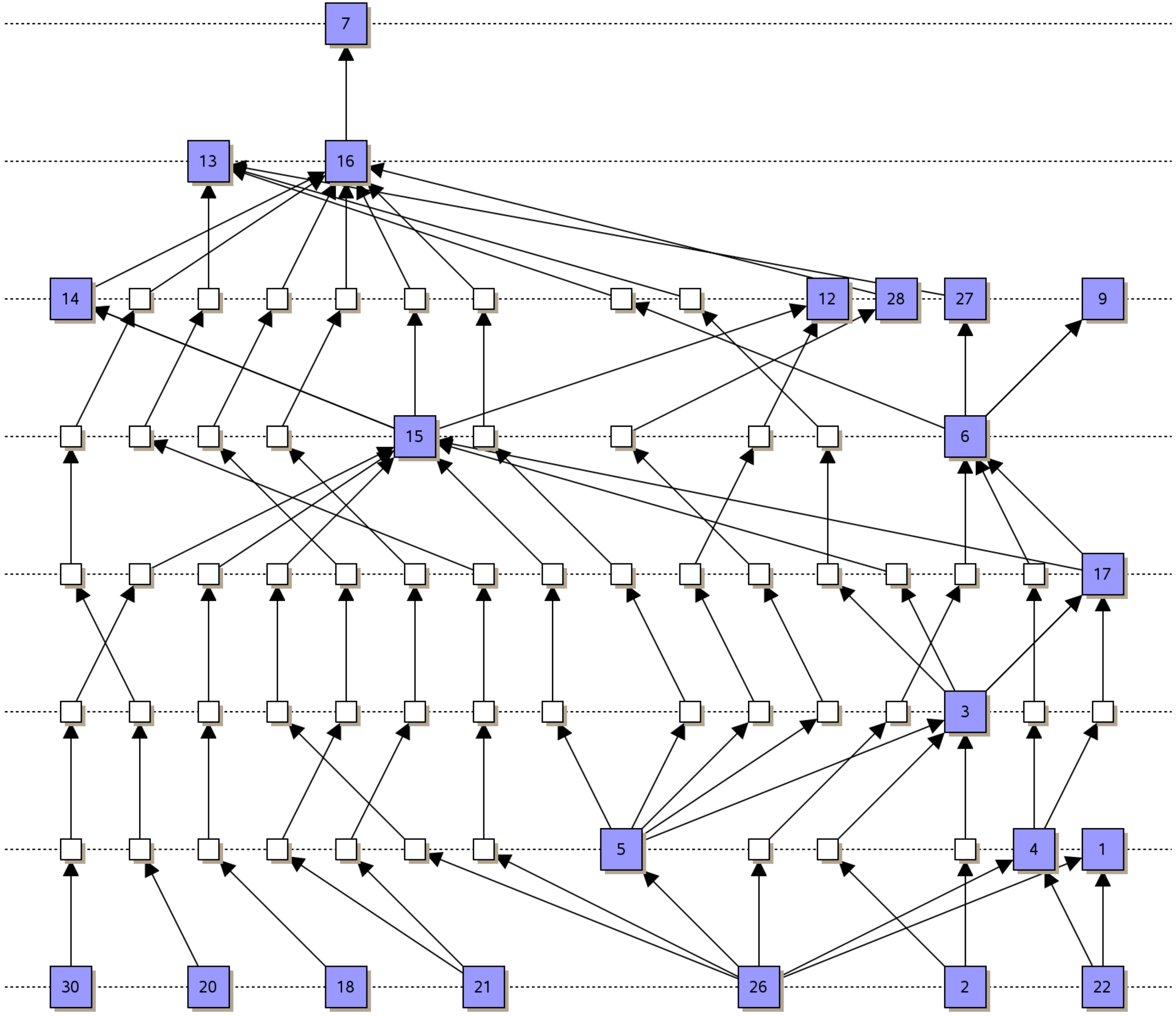

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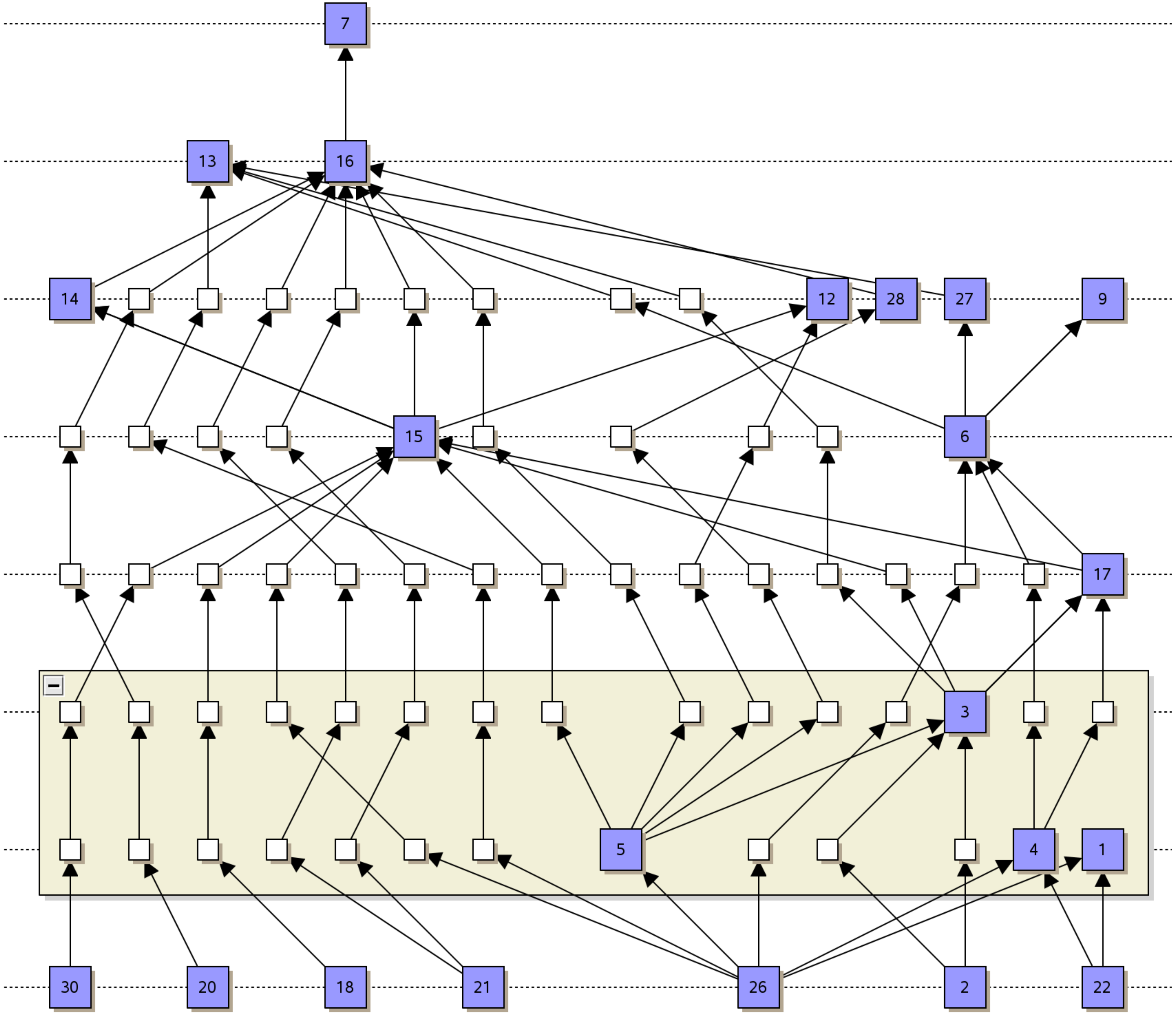

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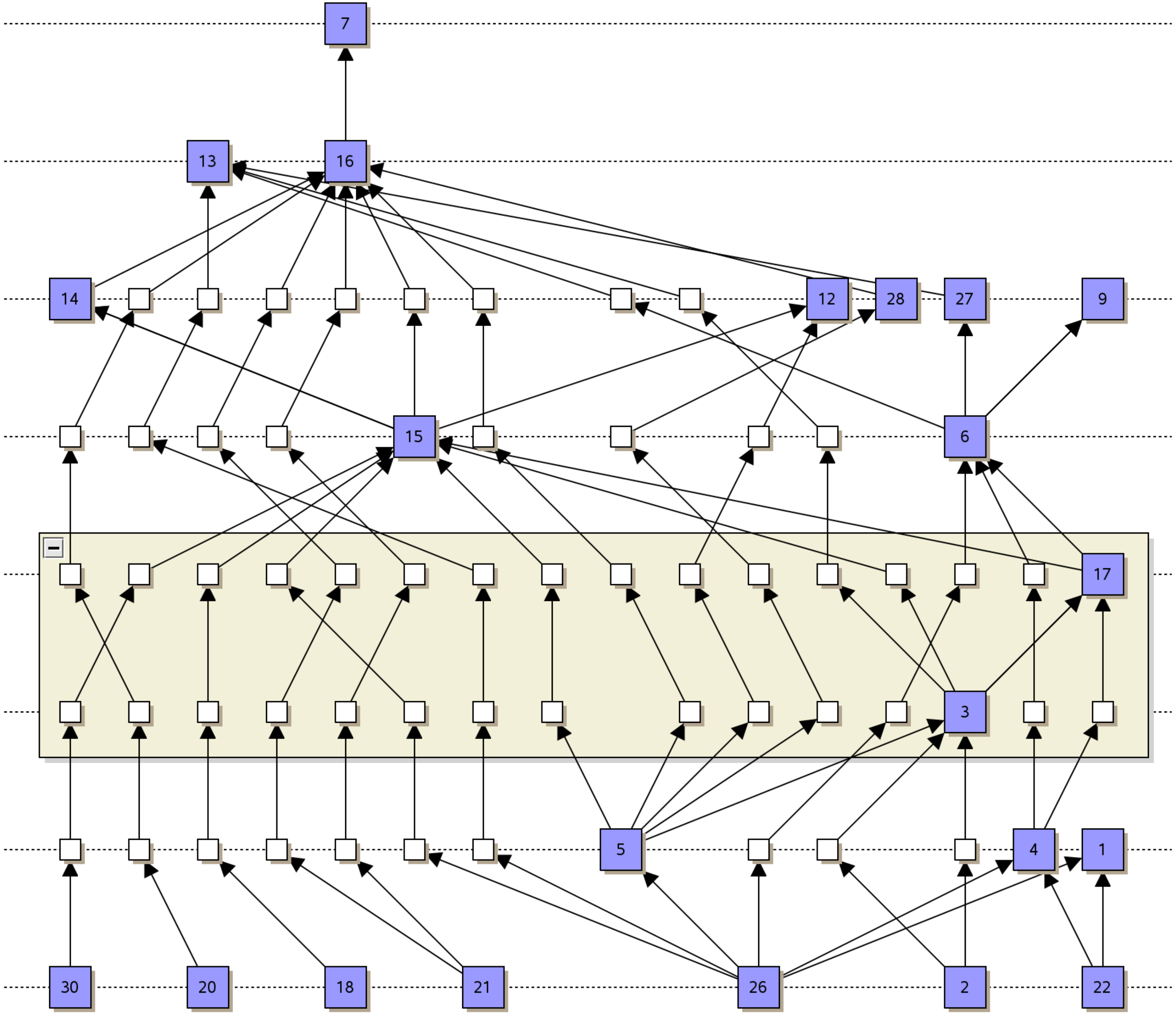

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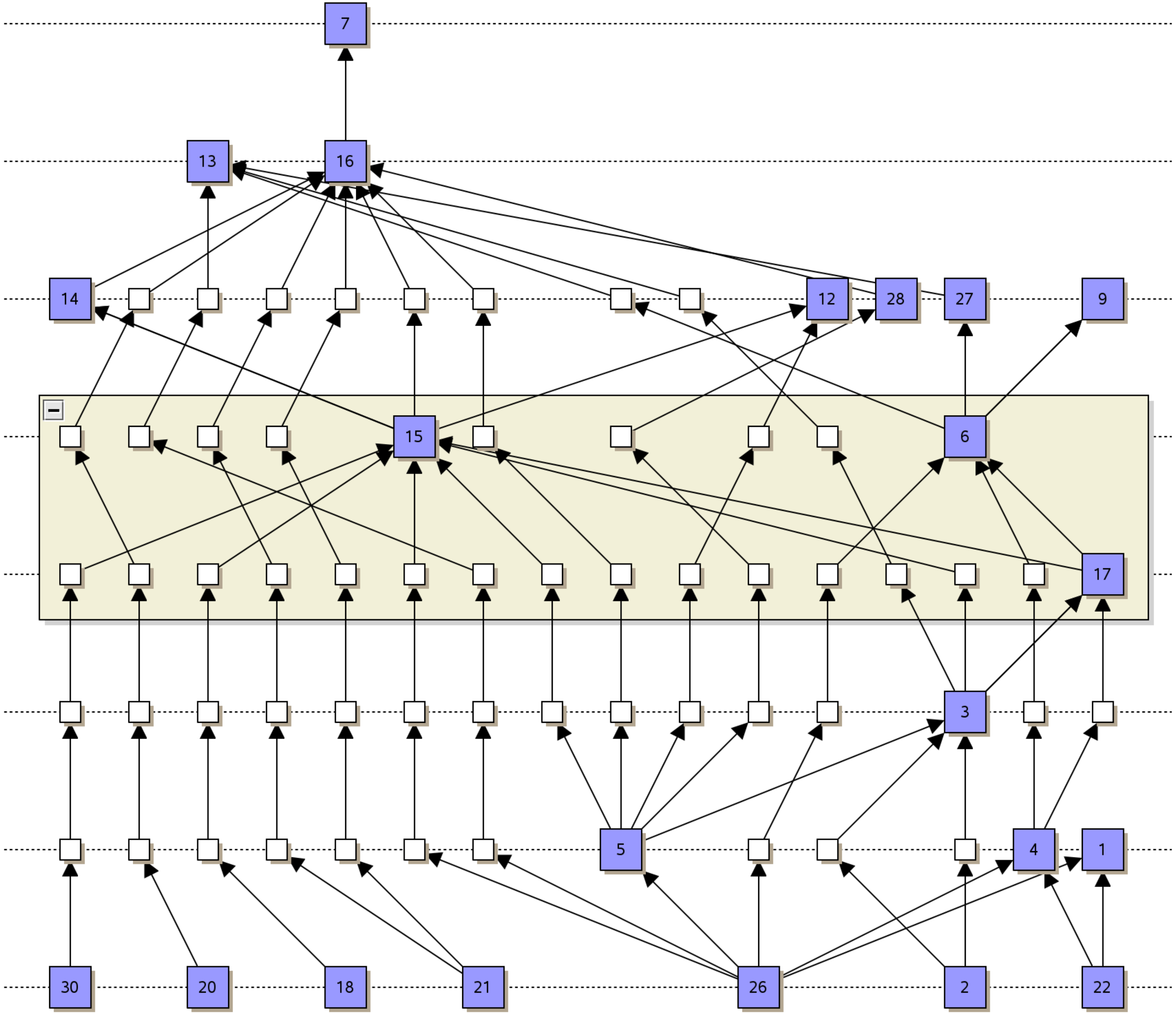

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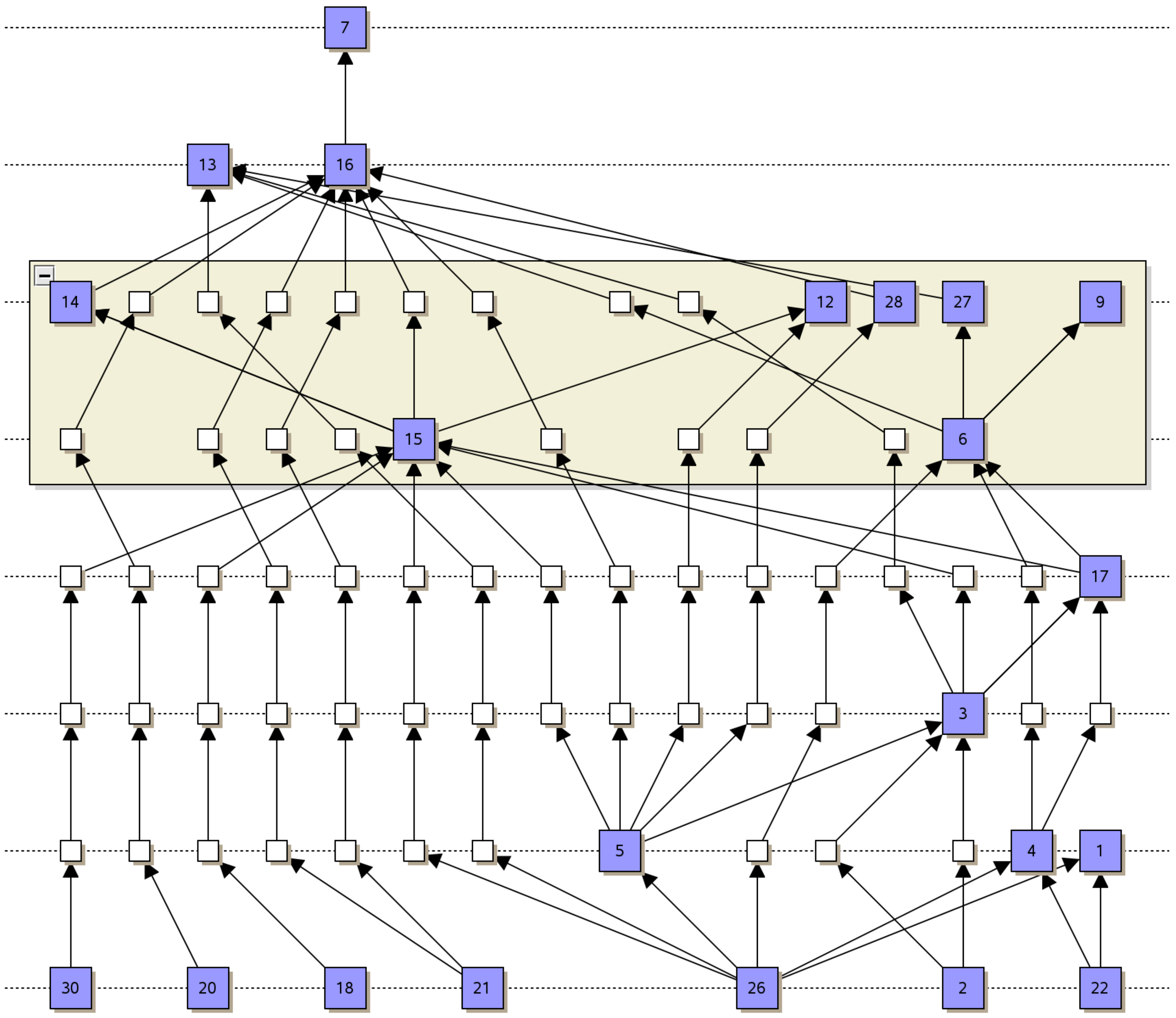

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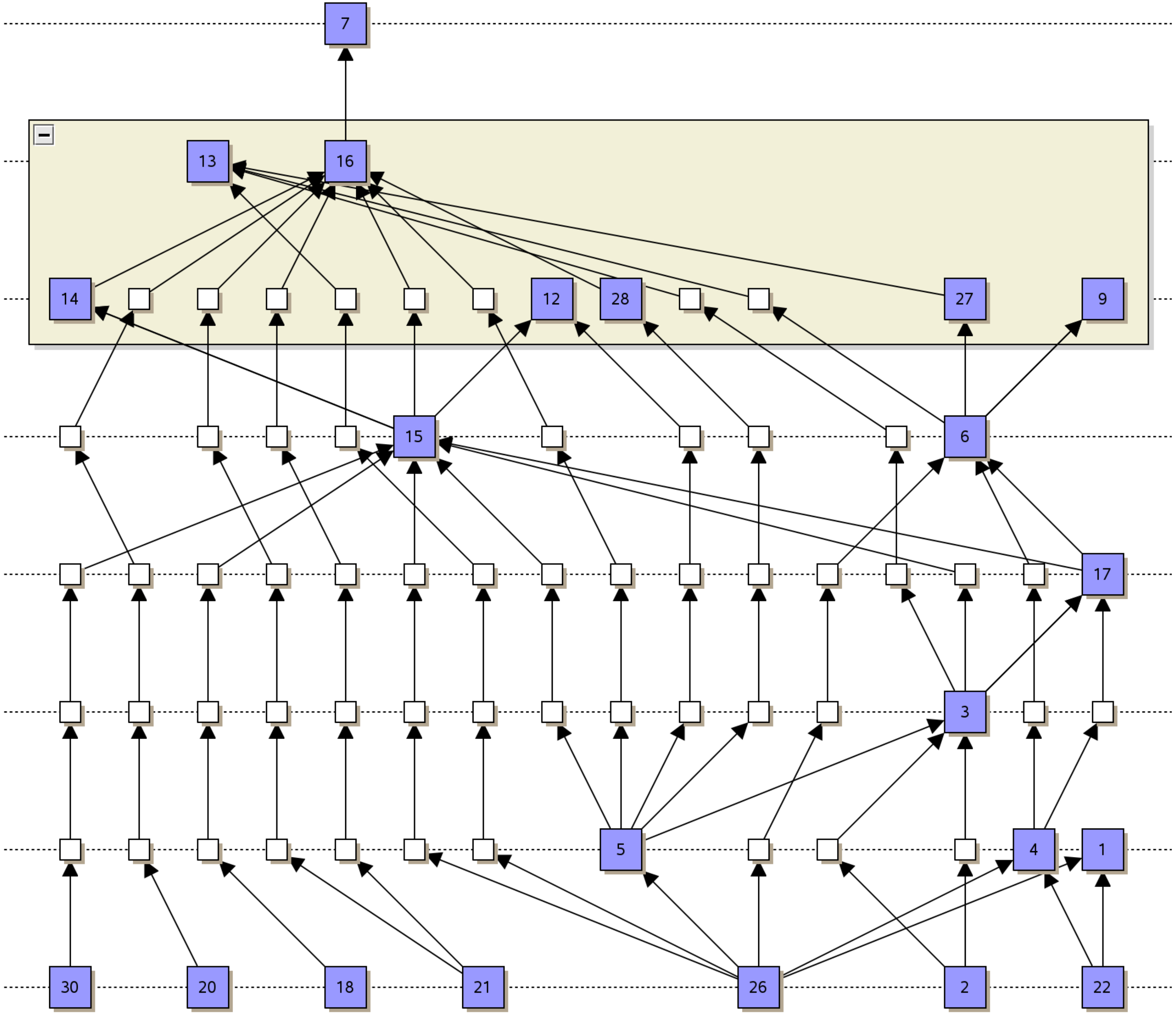

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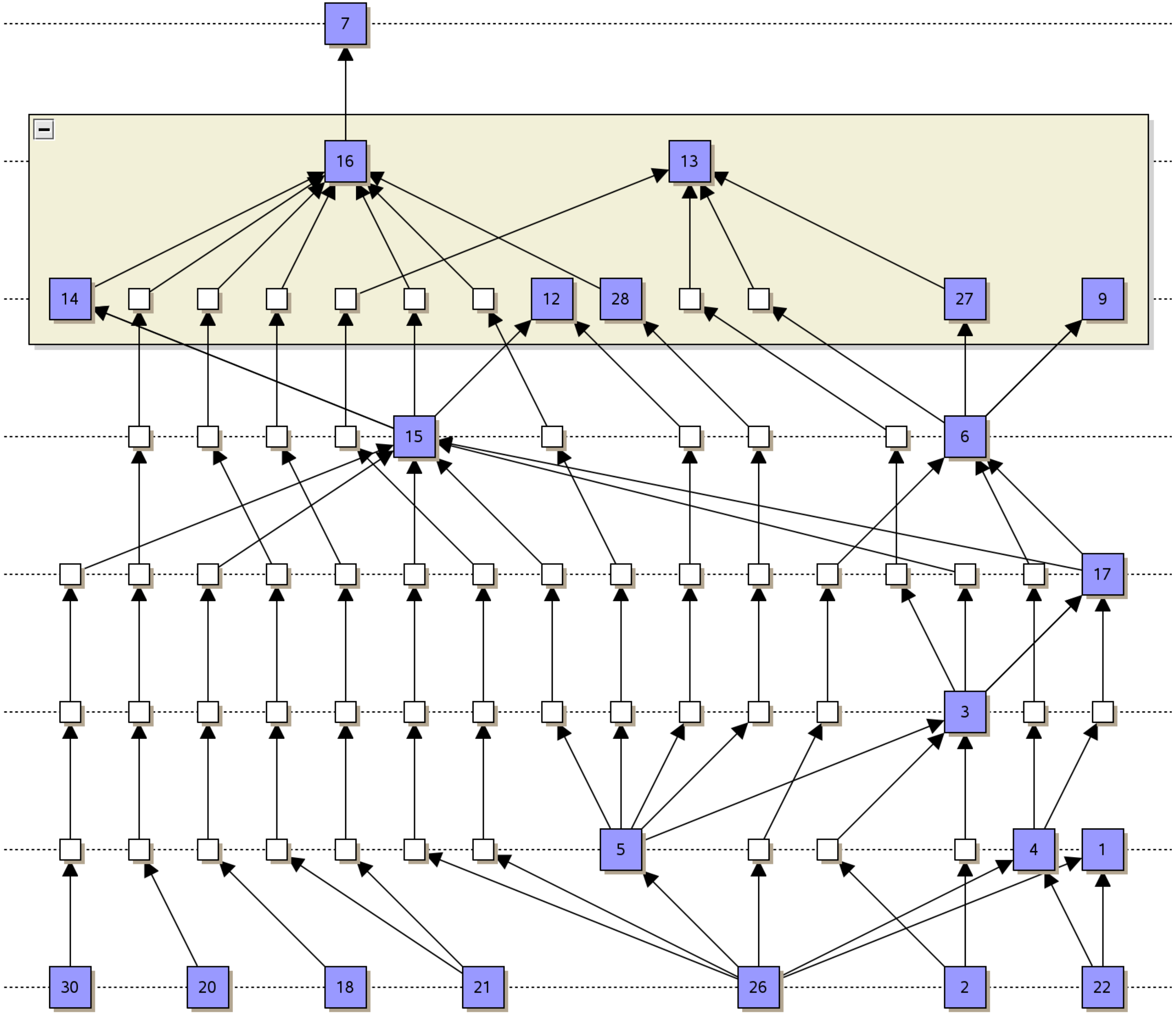

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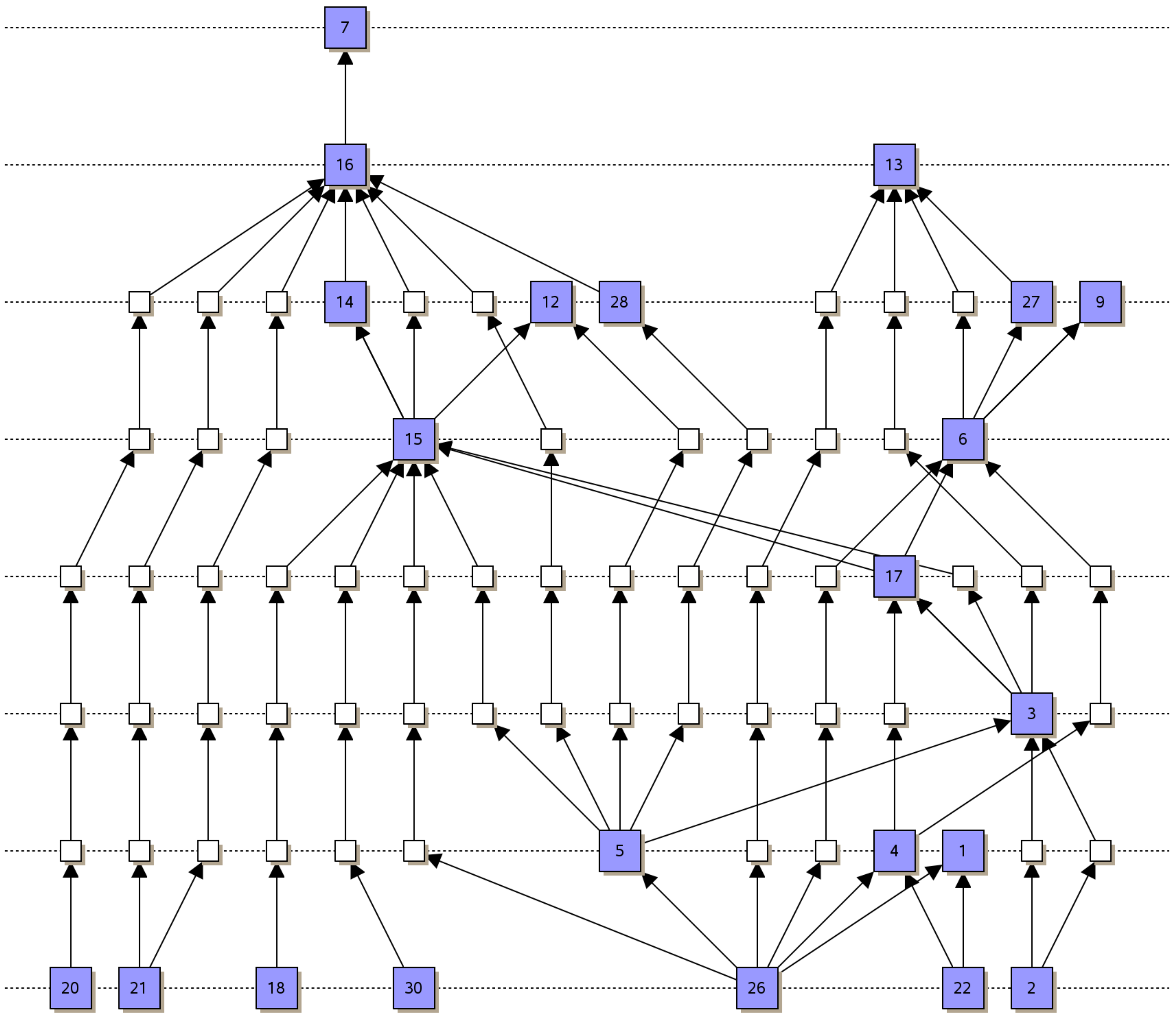
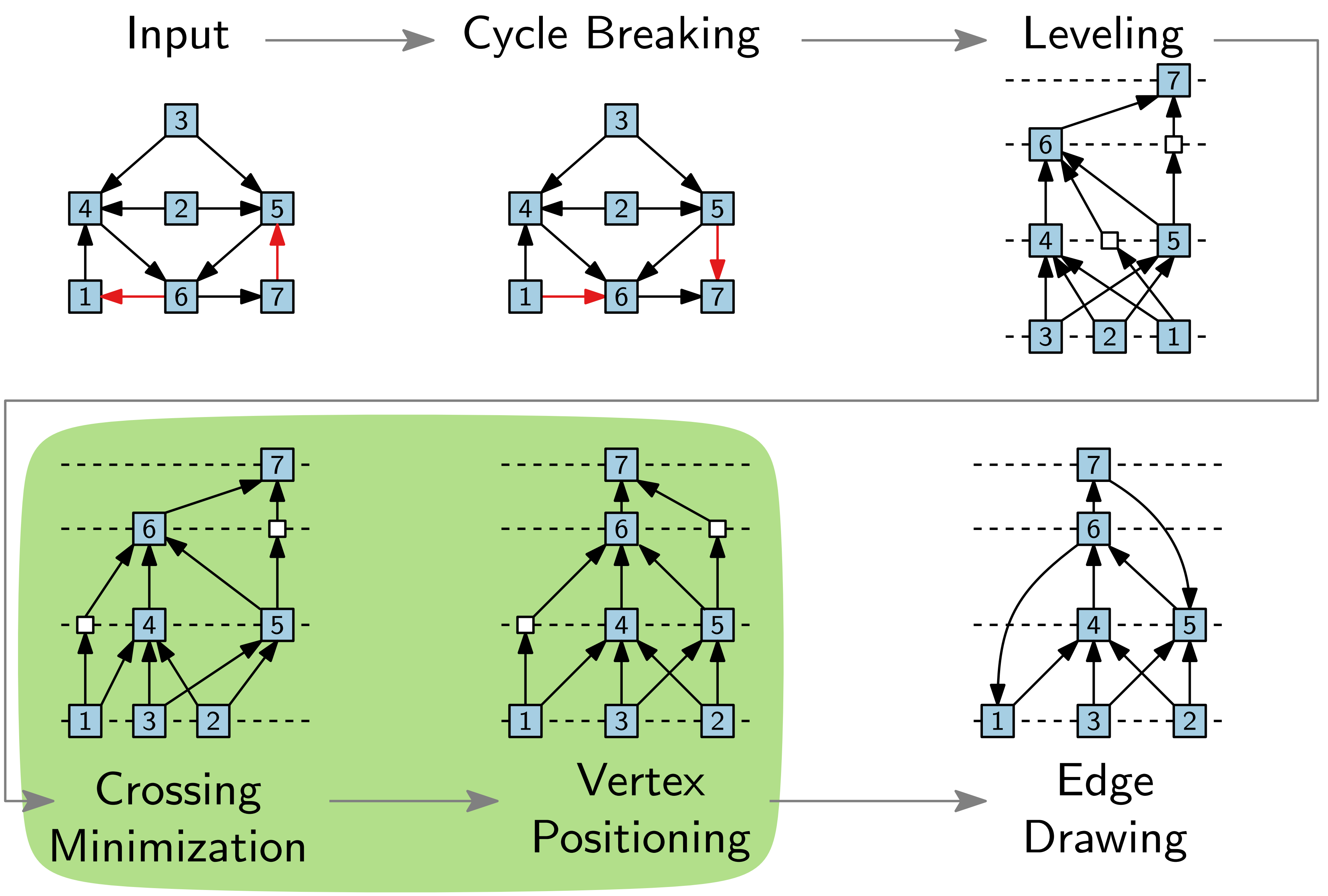


Step 4: Vertex Positioning



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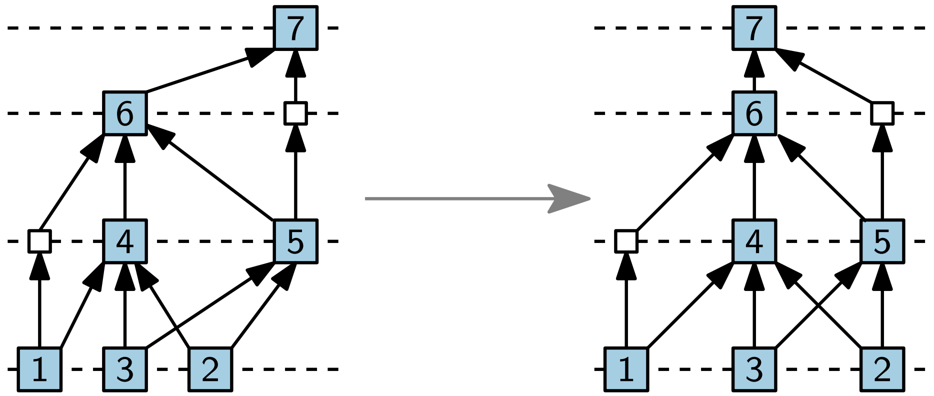
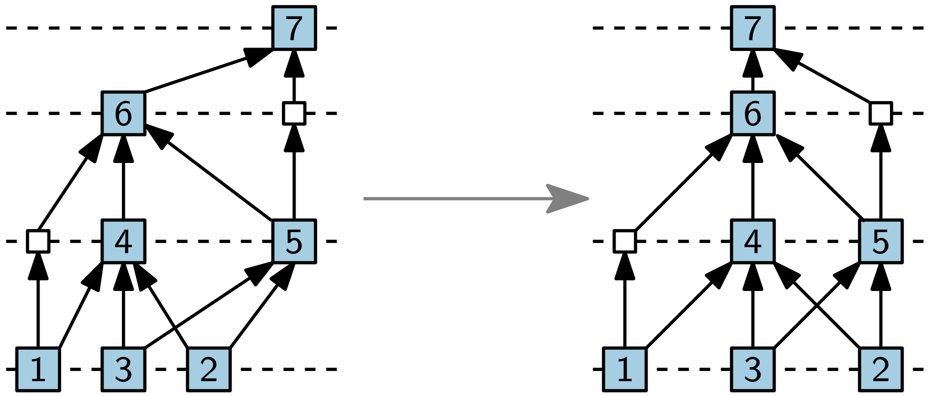


Step 4: Vertex Positioning

Goals.

- paths of a single edge should be (close to) straight
- vertices on a layer evenly spaced
- perfer vertical edges



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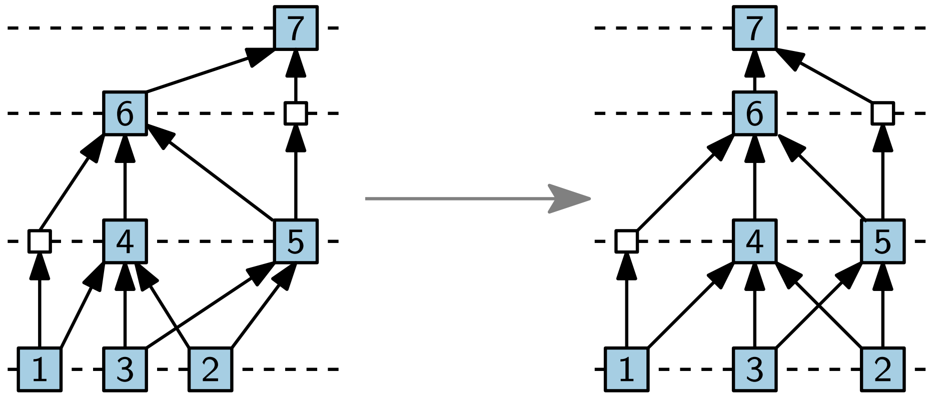

- **Exact:** Quadratic Program (QP)

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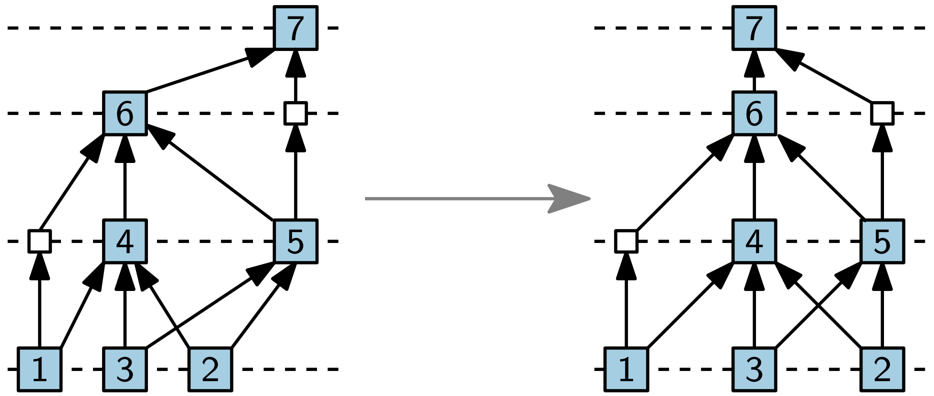
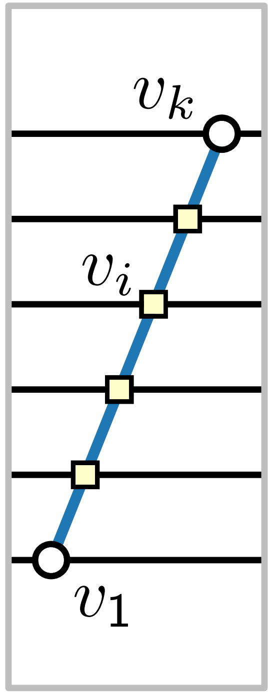


- **Exact:** Quadratic Program (QP)
- **Heuristic:** Iterative approach

Quadratic Program

- Consider the path $p_e = (v_1, \ldots, v_k)$ of an edge $e = v_1 v_k$ with dummy vertices: $v_2, \ldots, v_{k-1}$



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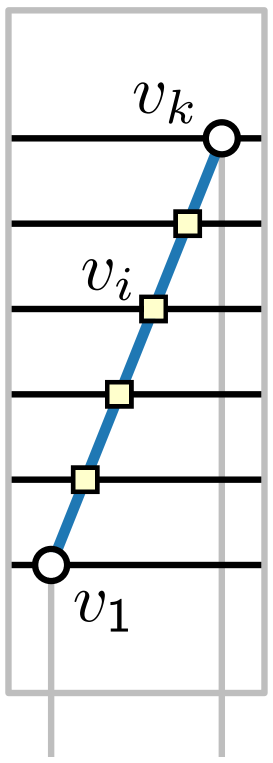

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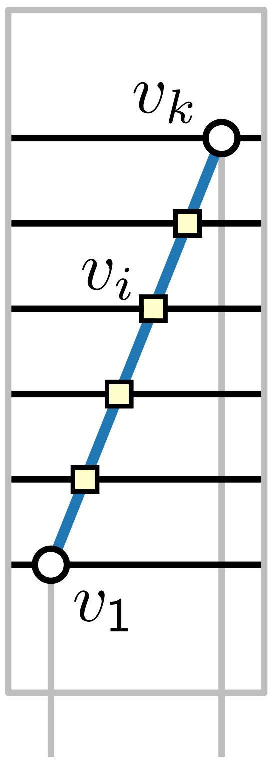

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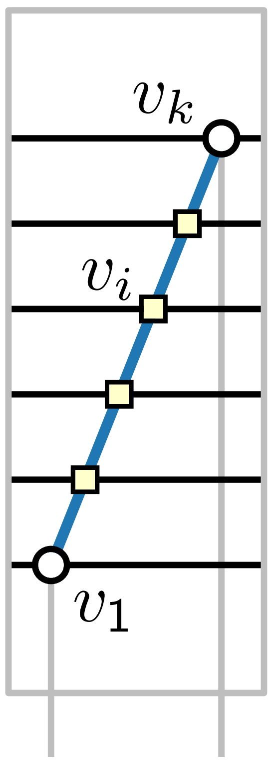

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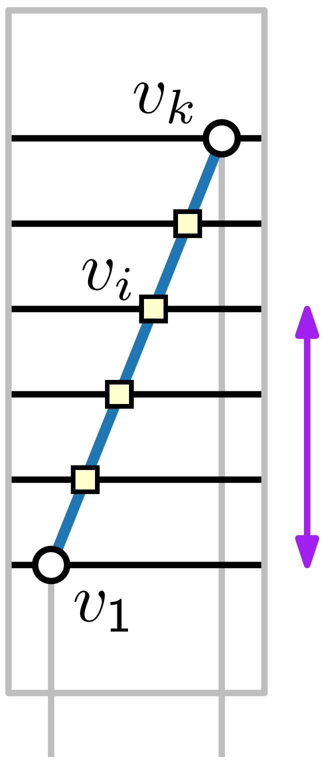

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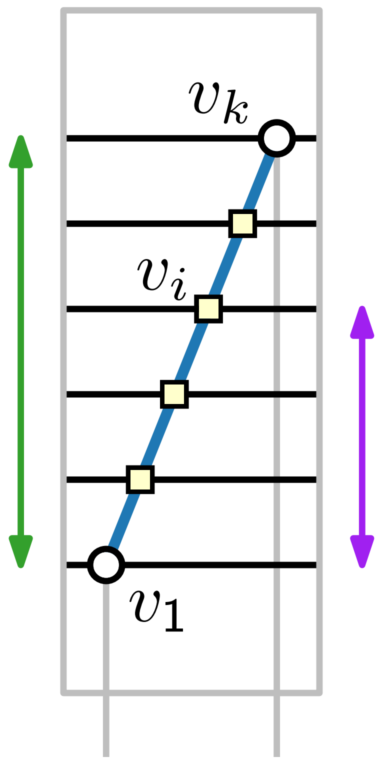

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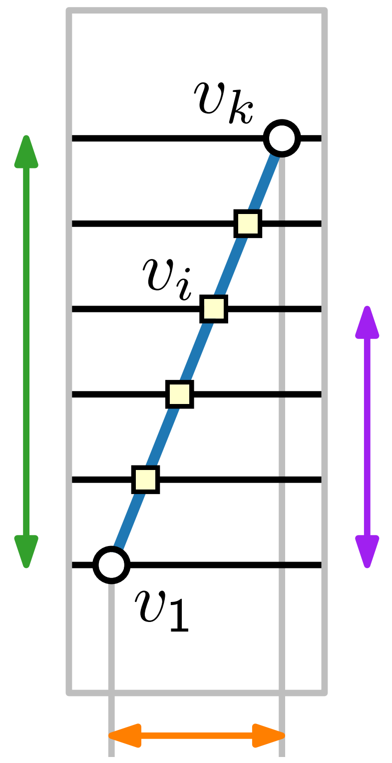

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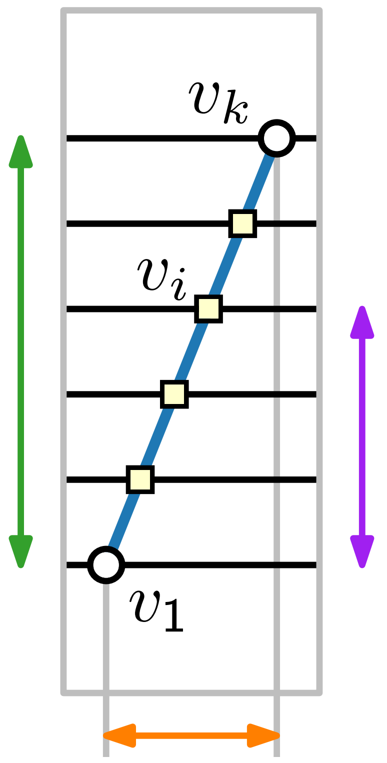

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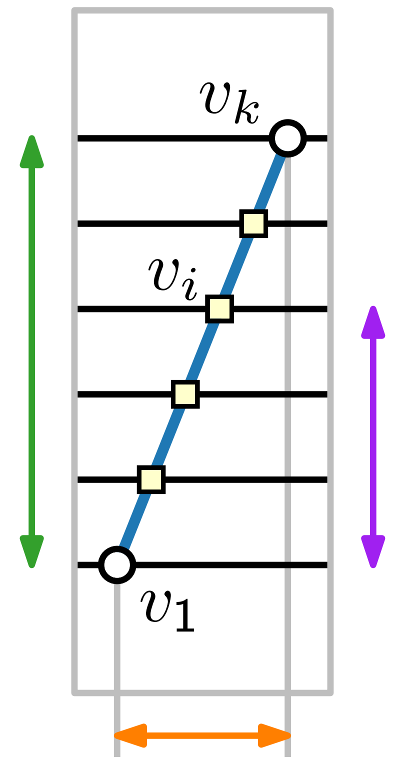

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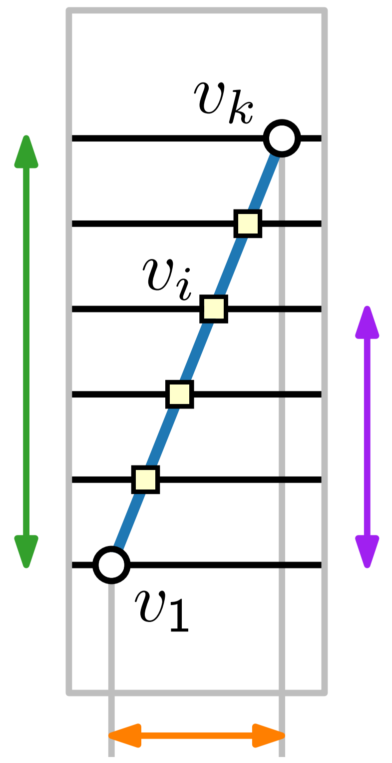

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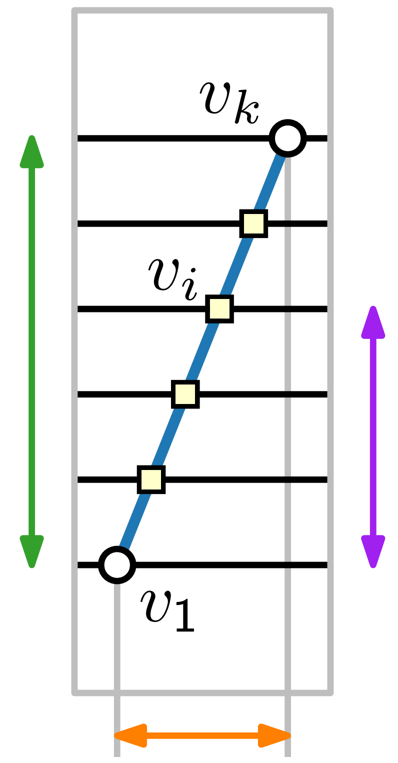

Quadratic Program

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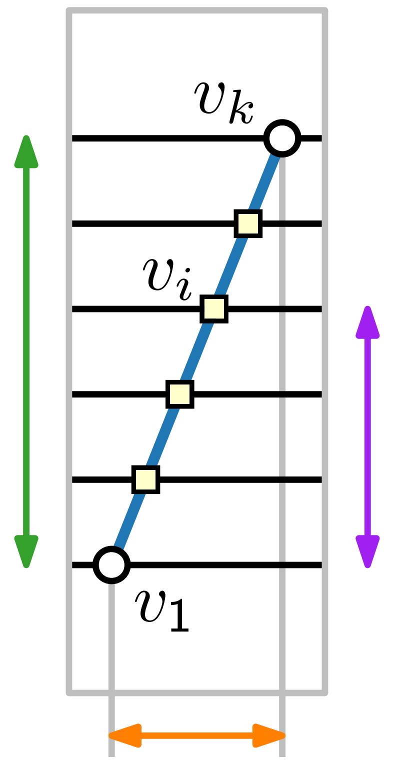

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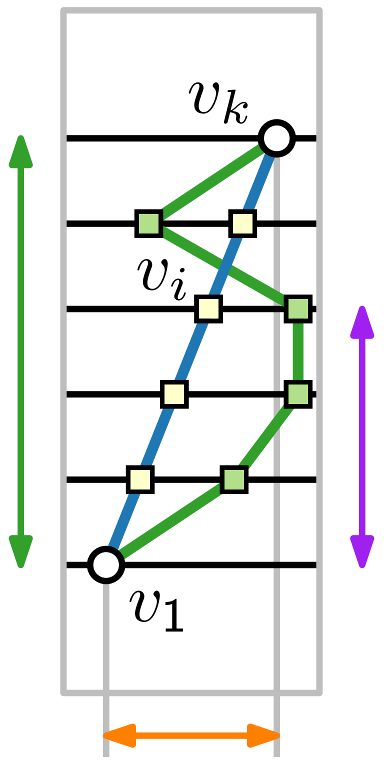

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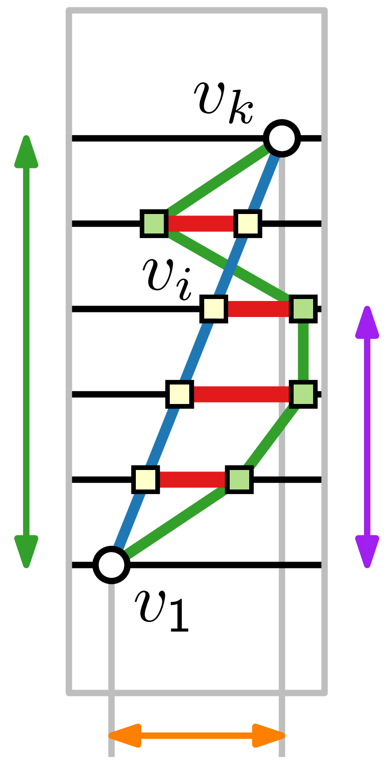

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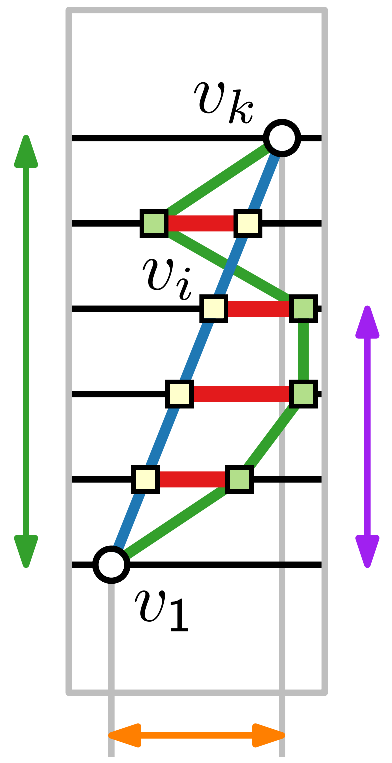

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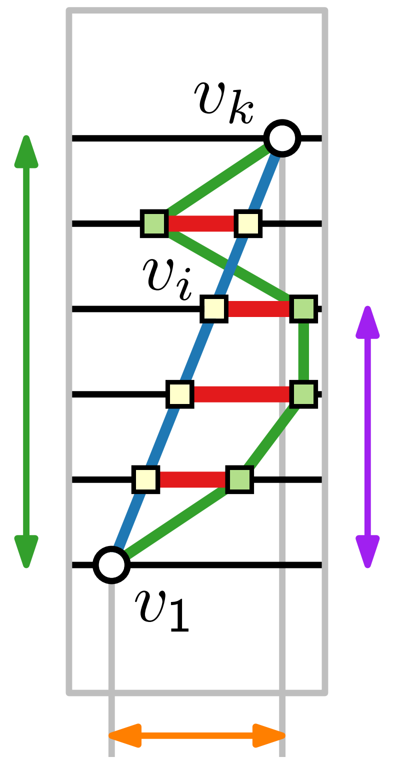

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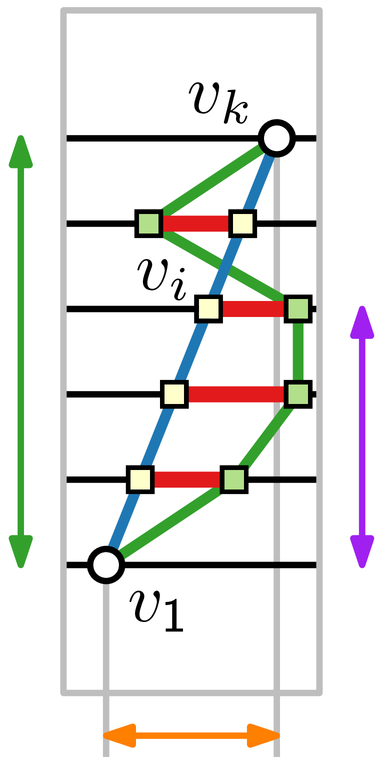

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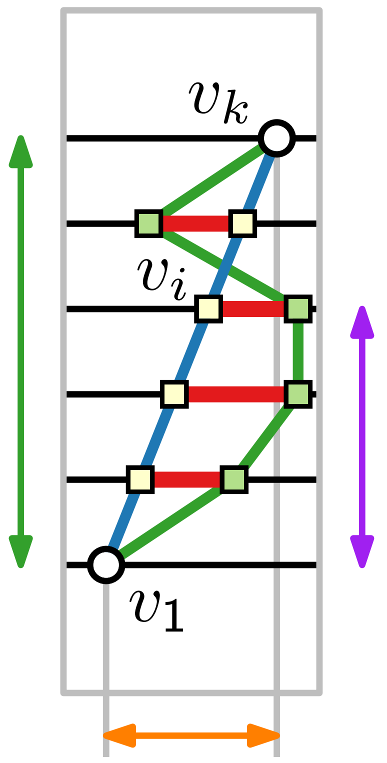

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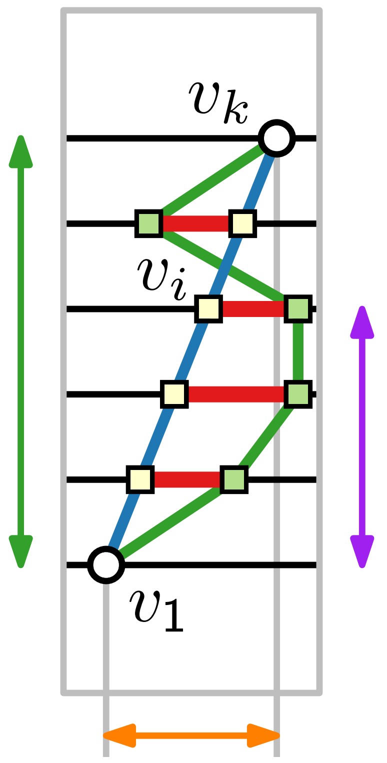

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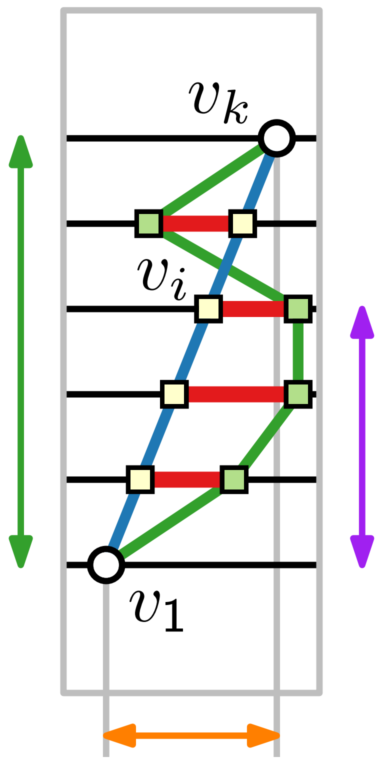

- QP is time-expensive

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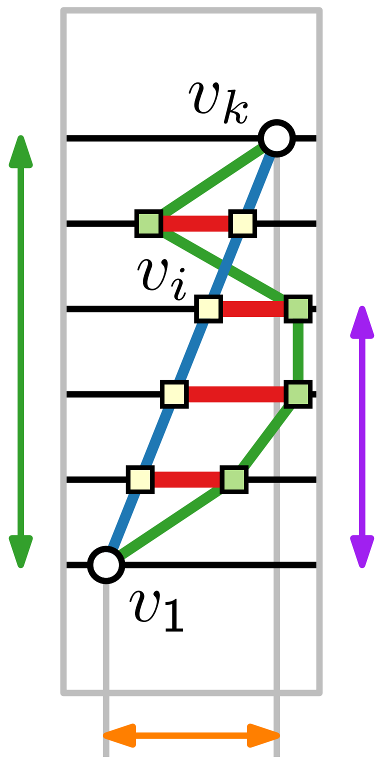

- QP is time-expensive
- width can be exponential

Iterative Heuristic

- Compute an initial layout

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- Apply the following steps as long as improvements can be made:

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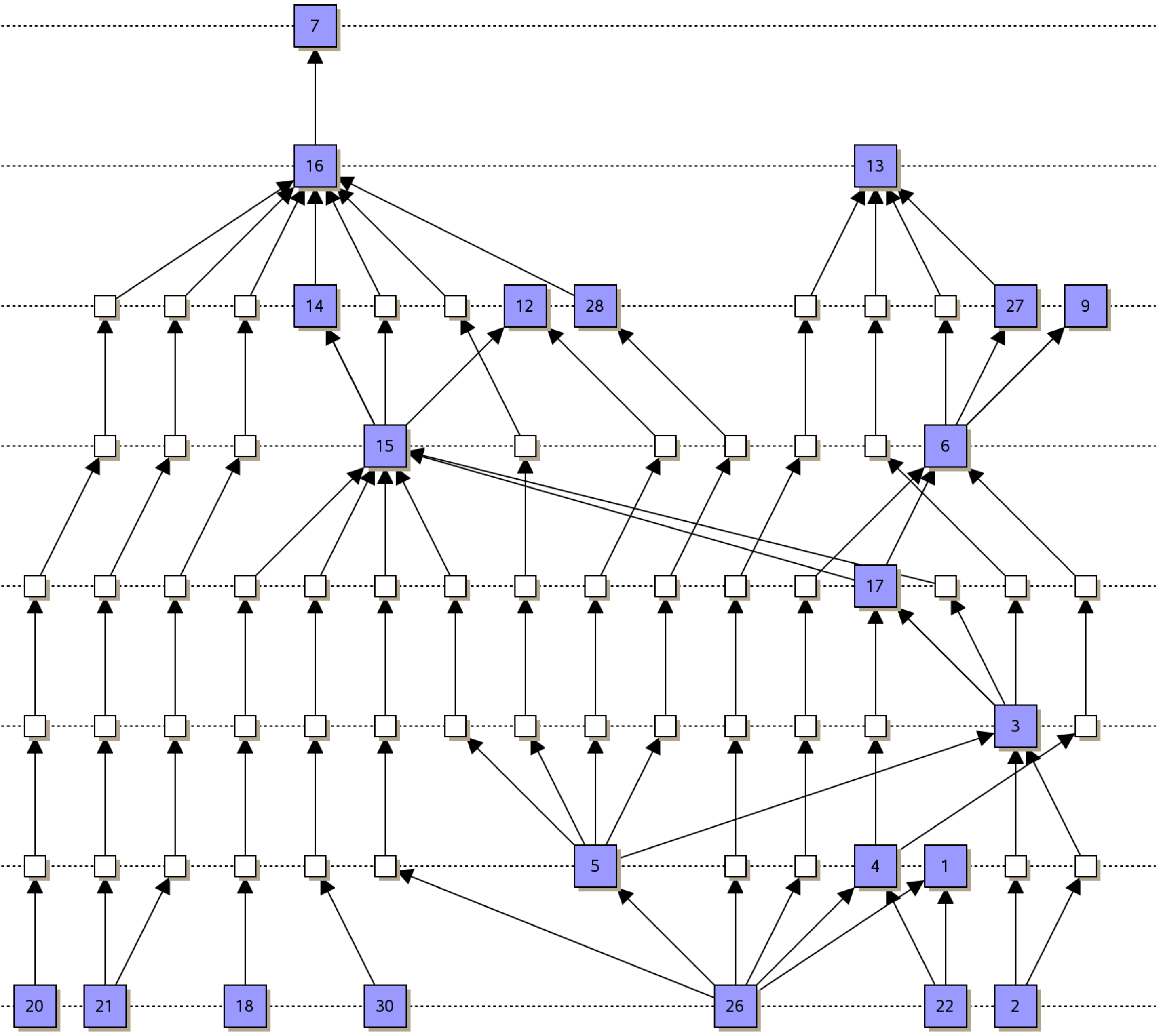
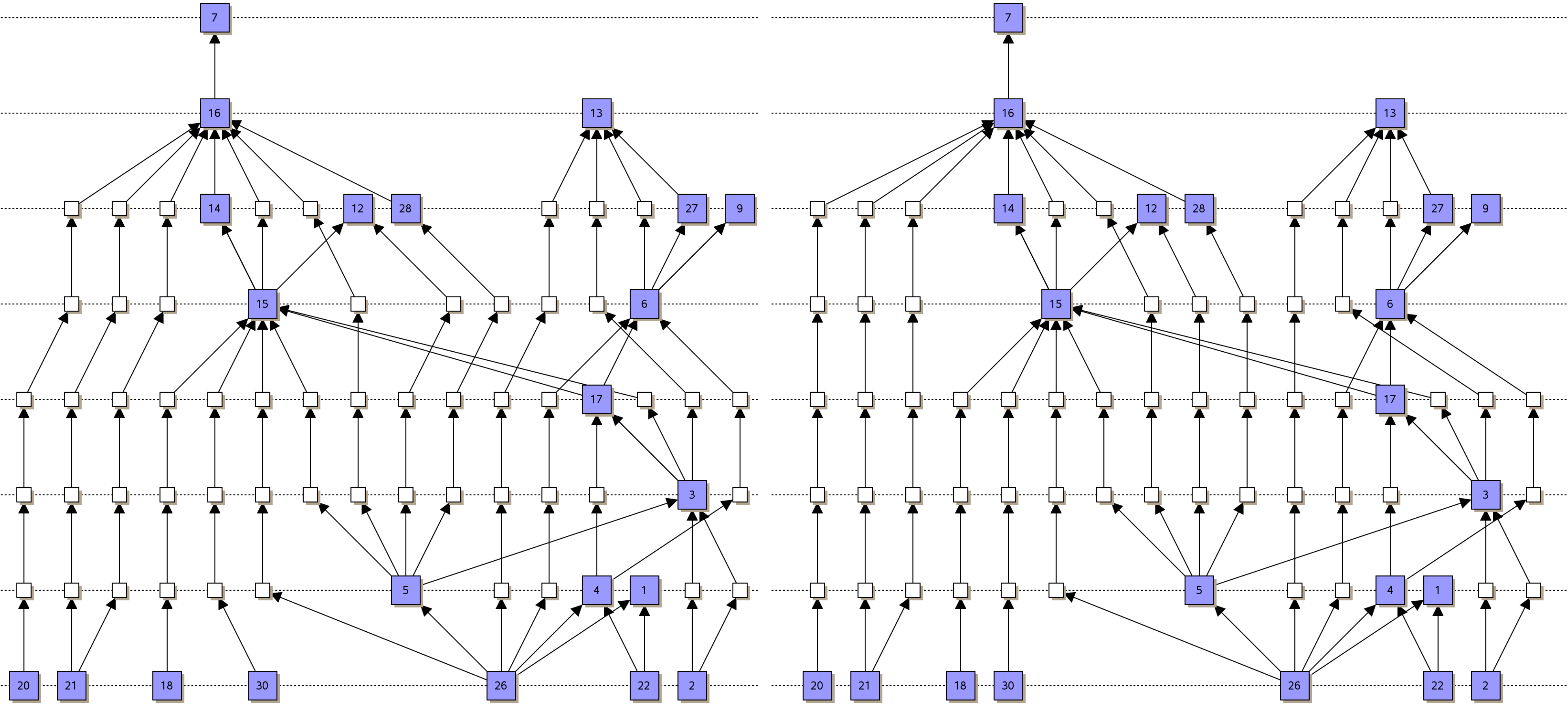
  3. compactifying the layout width

- Other algorithms include the algorithm by Brandes and Köpf ’02:
  - tries to align vertices vertically
  - does horizontal compaction afterwards
  - linear running time

Example



Example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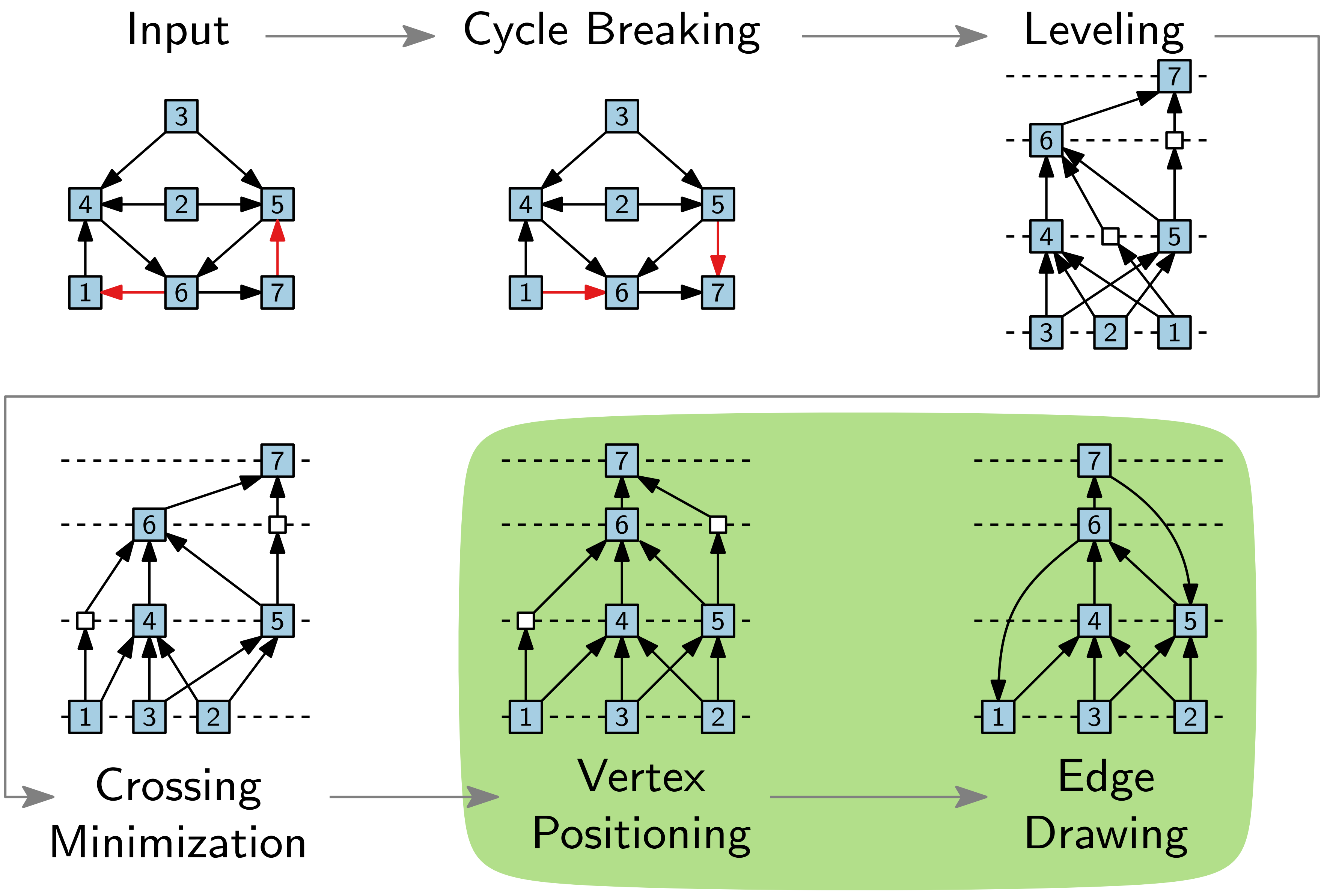
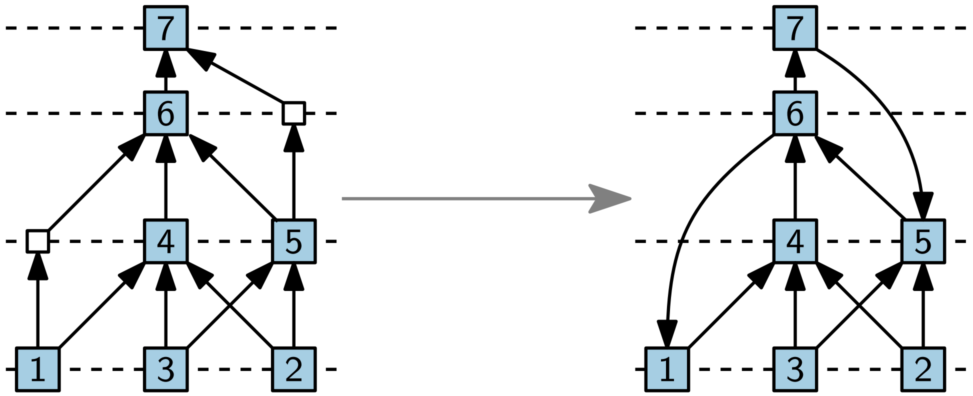
Step 5: Drawing Edges



Step 5: Drawing Edges

Possibility.
Substitute polylines by Bézier curves.



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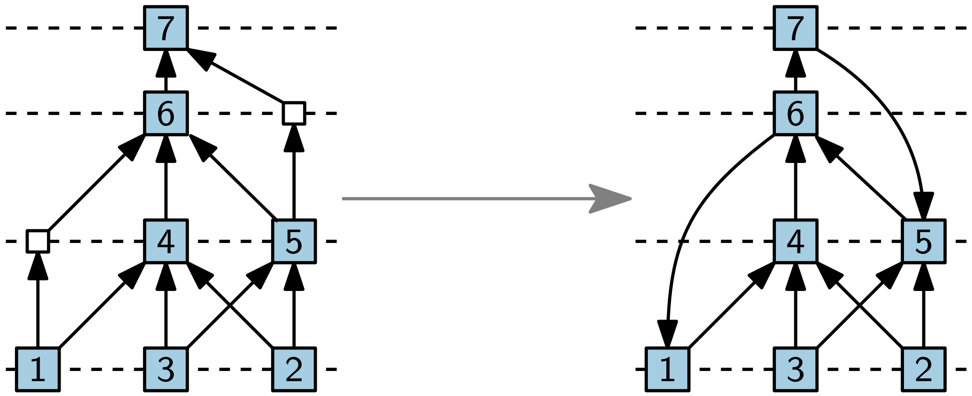
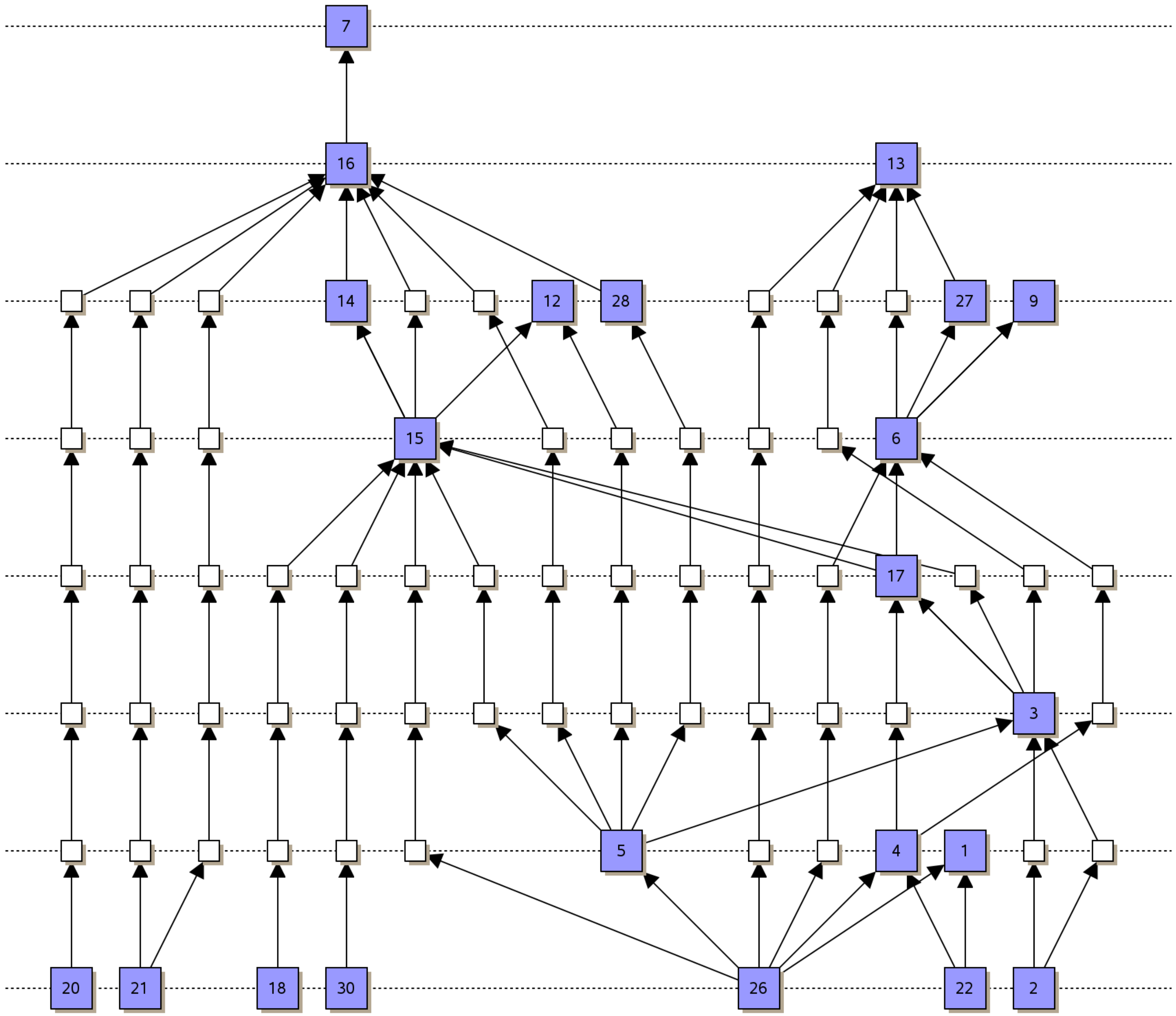
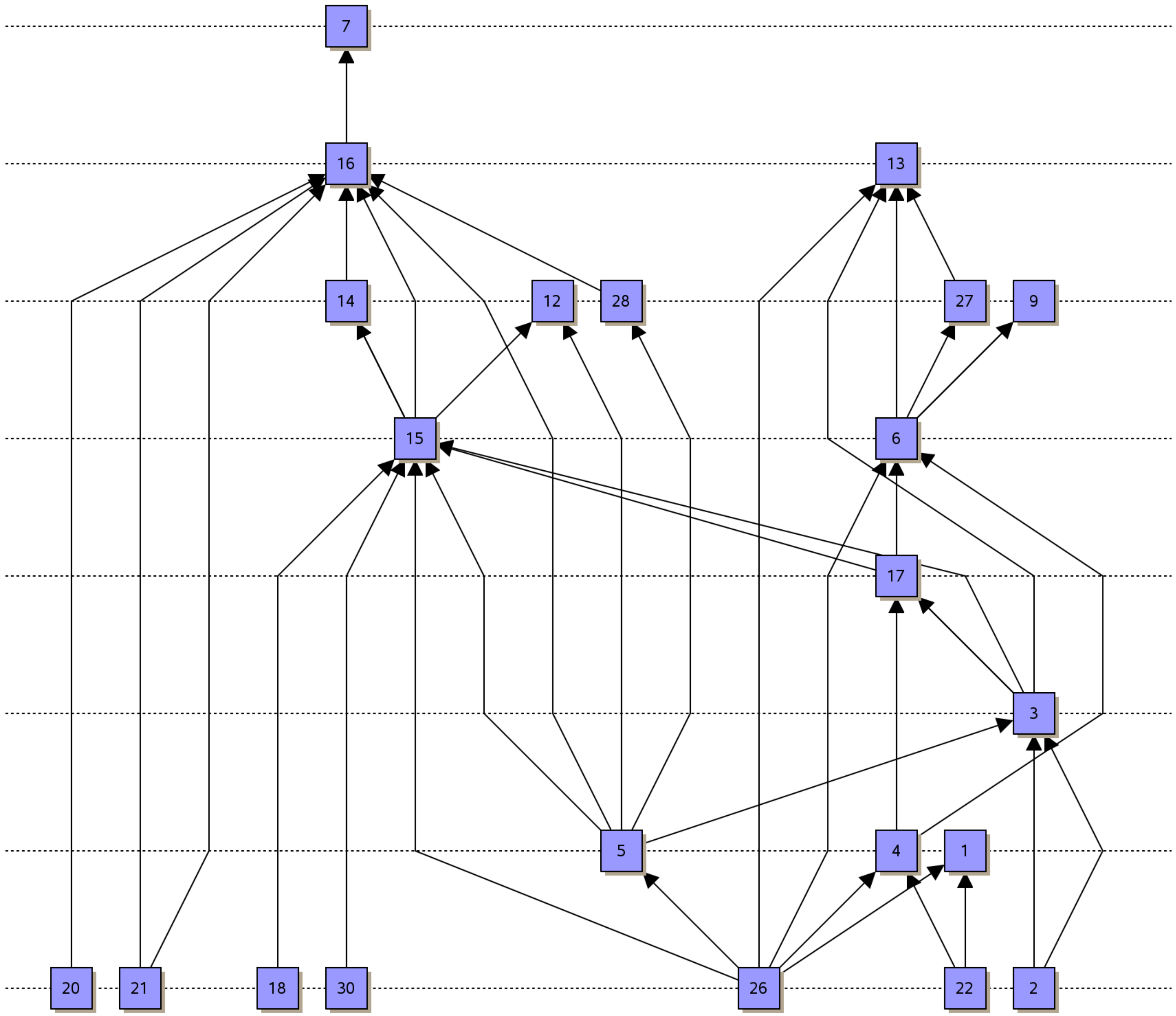


Remark.
Draw reversed edges downwards.

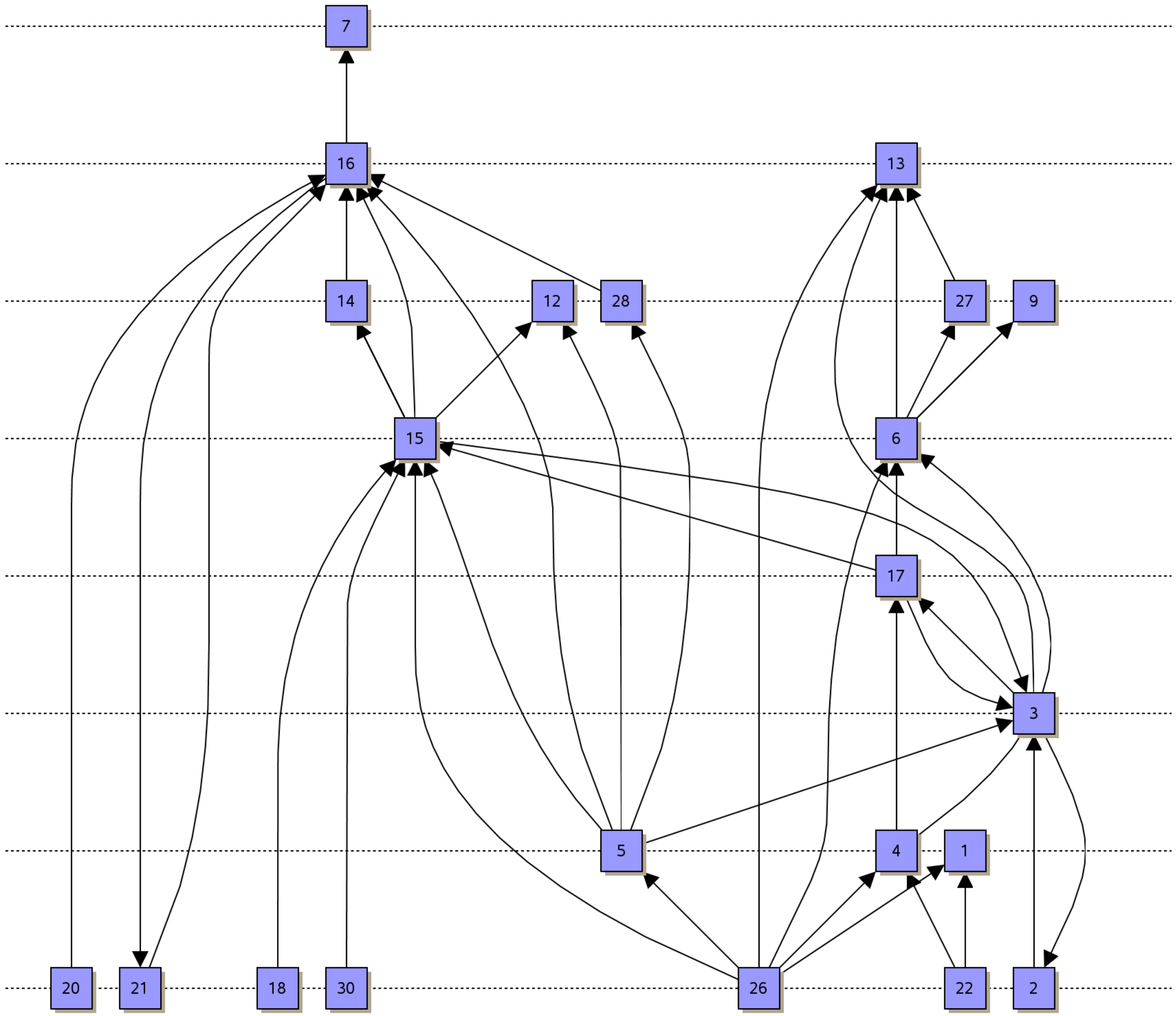
Example



Example

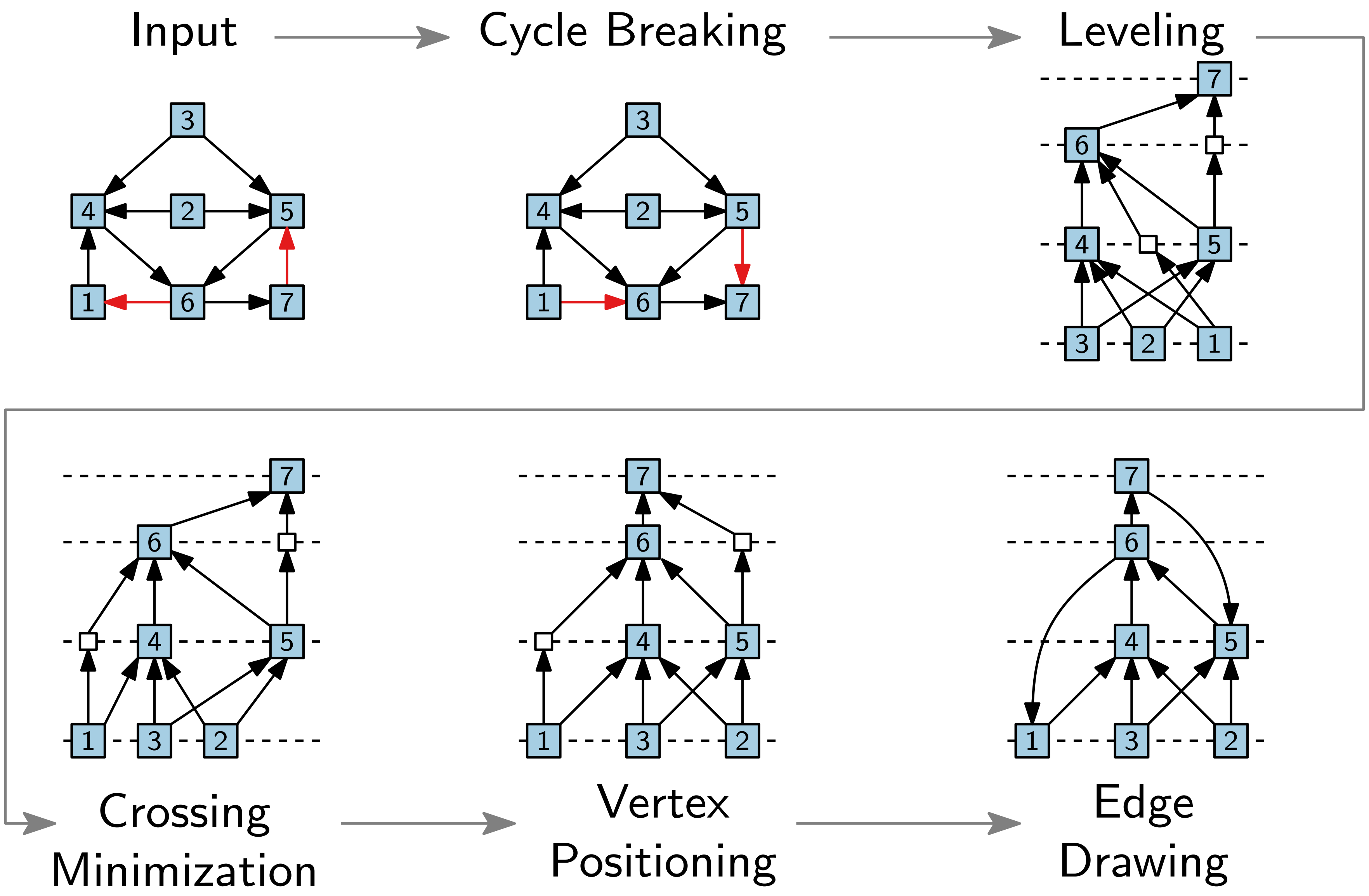


Example



Classical Approach – Sugiyama Framework

[Sugiyama, Tagawa, Toda '81]



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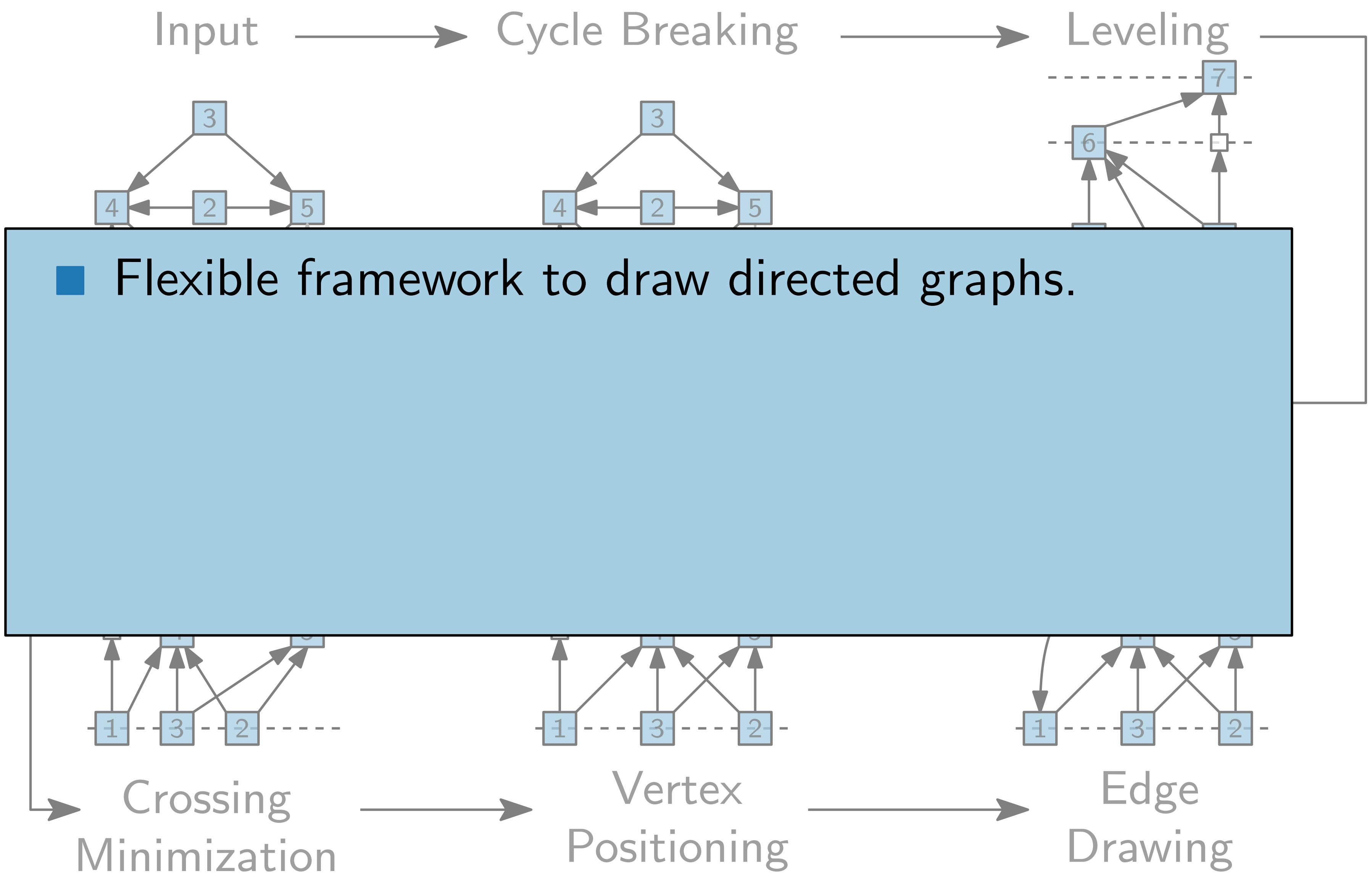

- Flexible framework to draw directed graphs.

Classical Approach – Sugiyama Framework

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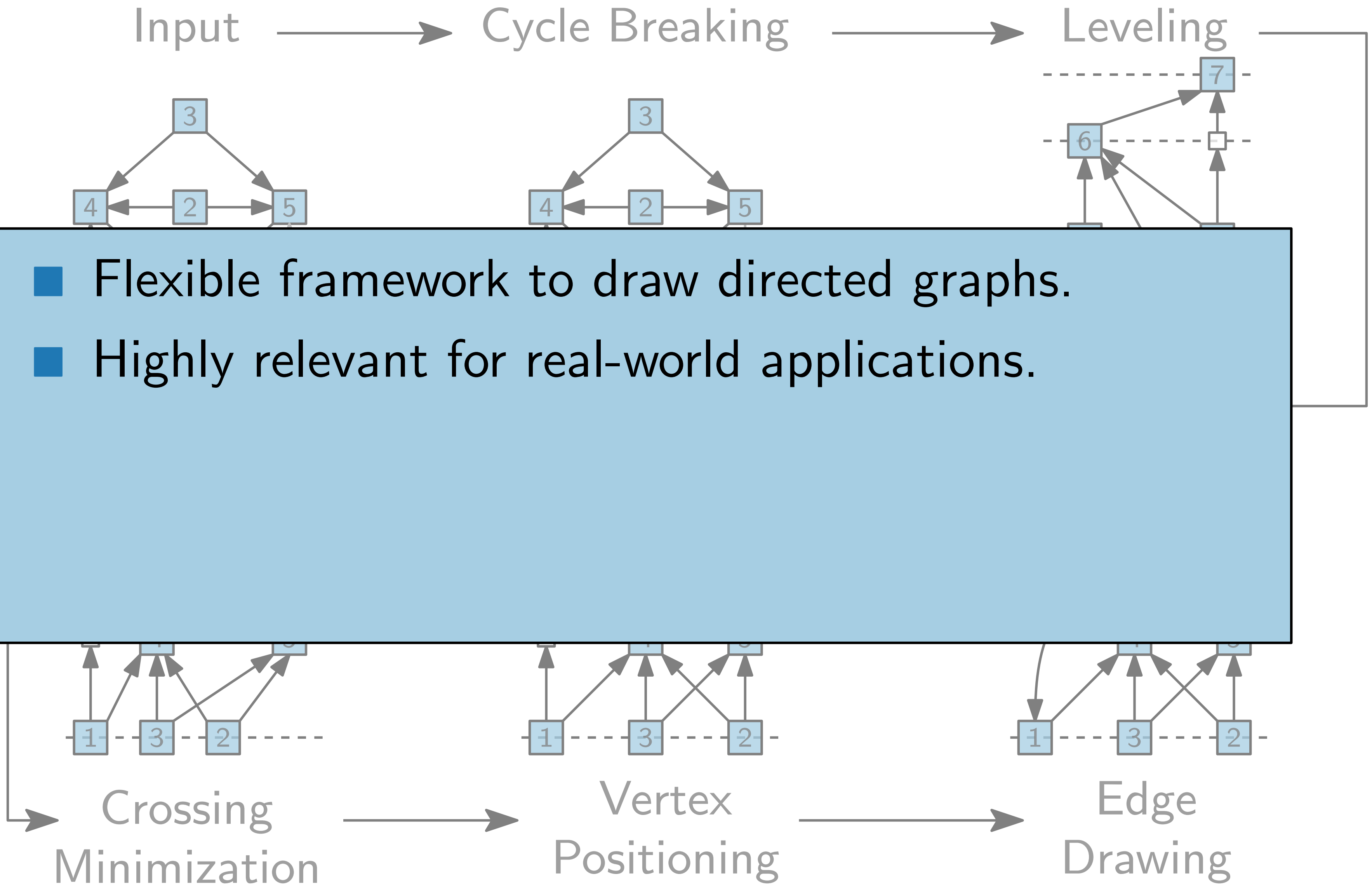

- Flexible framework to draw directed graphs.
- Highly relevant for real-world applications.

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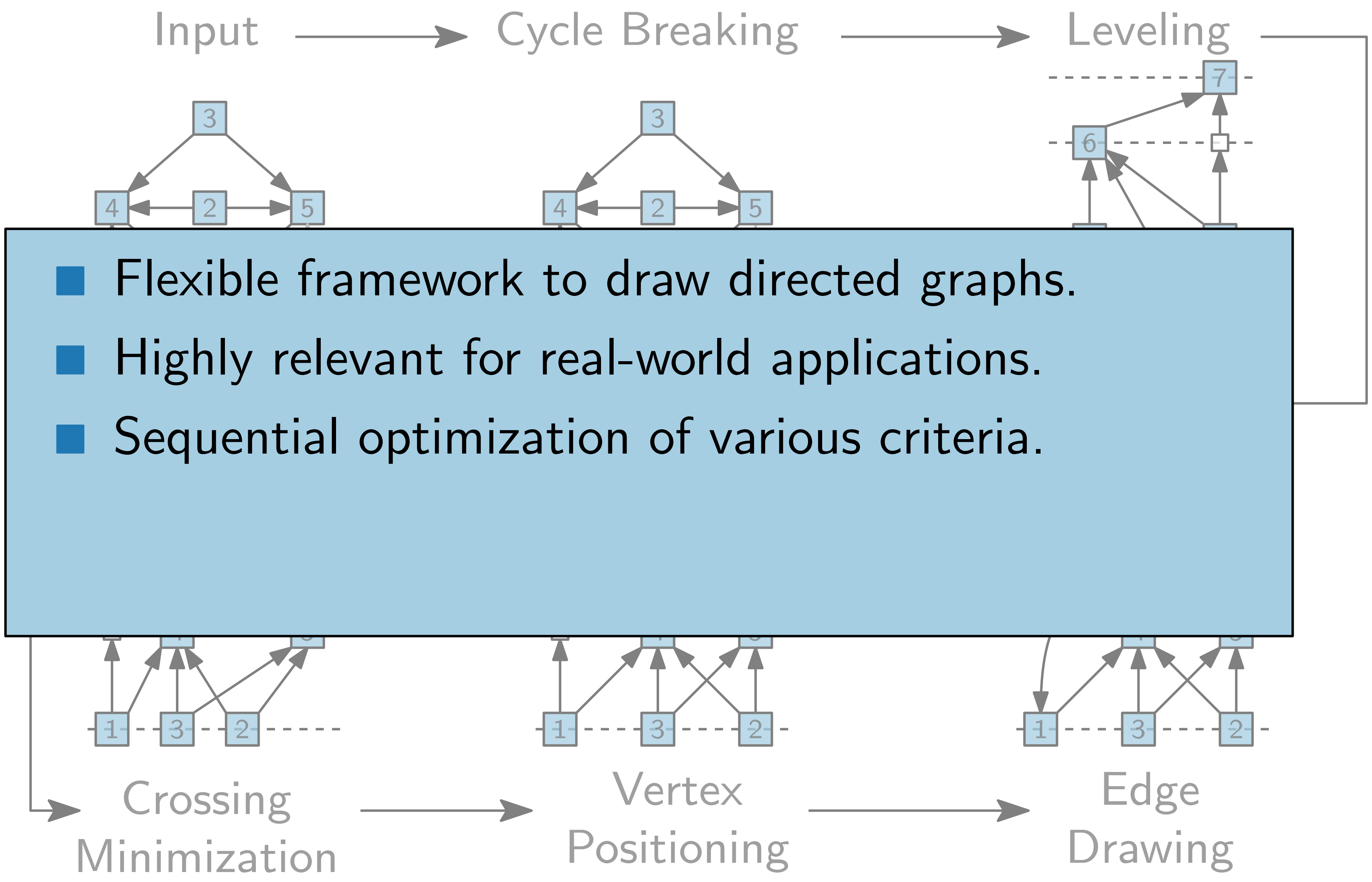

- Flexible framework to draw directed graphs.
- Highly relevant for real-world applications.
- Sequential optimization of various criteria.

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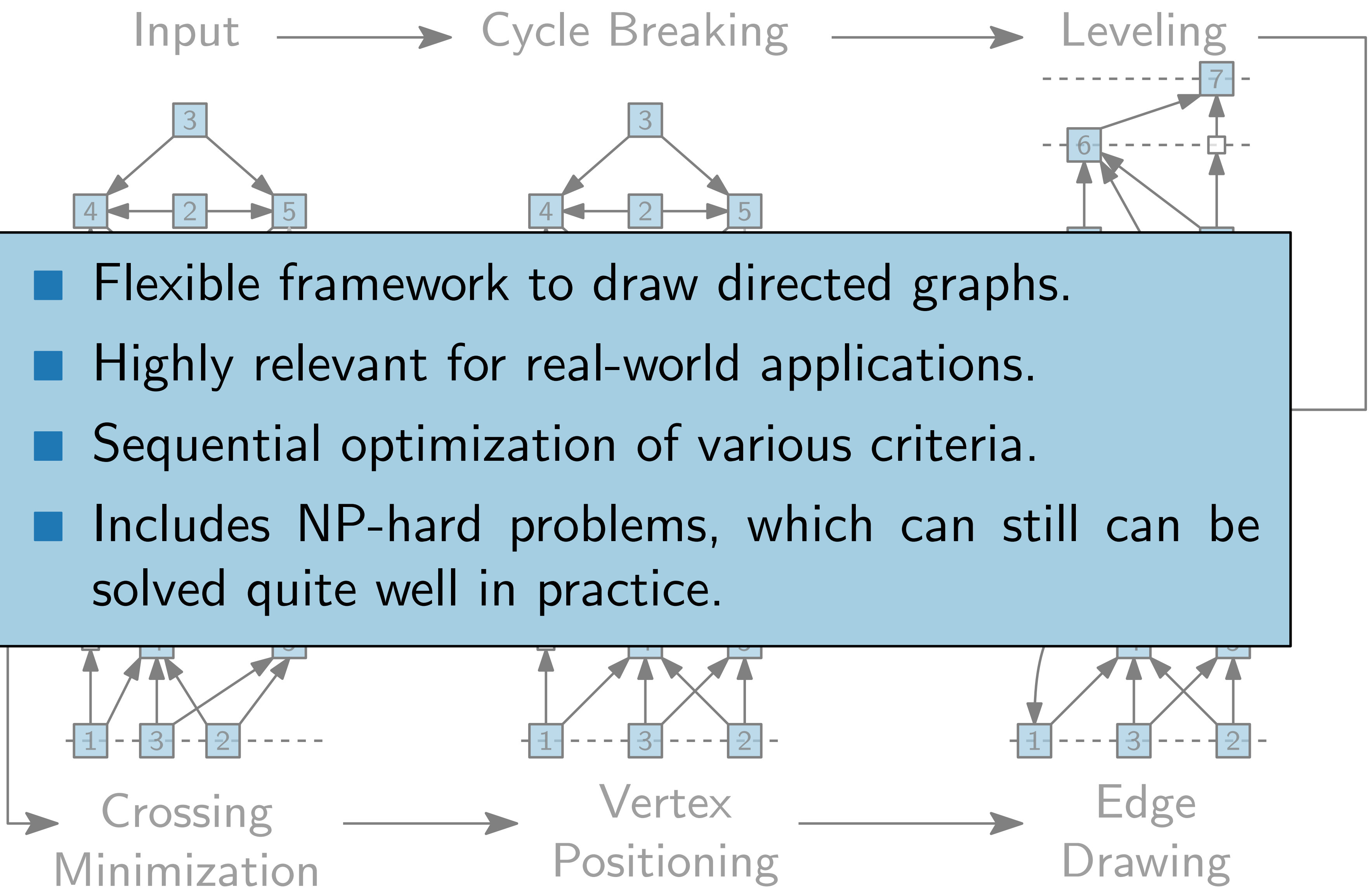

- Flexible framework to draw directed graphs.
- Highly relevant for real-world applications.
- Sequential optimization of various criteria.
- Includes NP-hard problems, which can still can be solved quite well in practice.

Literature

Detailed explanations of steps and proofs in

- [GD Ch. 11] and [DG Ch. 5]

based on

- [Sugiyama, Tagawa, Toda '81] Methods for visual understanding of hierarchical system structures

and refined with results from

- [Berger, Shor '90] Approximation algorithms for the maximum acyclic subgraph problem
- [Eades, Lin, Smith '93] A fast and effective heuristic for the feedback arc set problem
- [Garey, Johnson '83] Crossing number is NP-complete
- [Eades, Kelly '86] Heuristics for reducing crossings in 2-layered networks.
- [Eades, Whiteside '94] Drawing graphs in two layers
- [Eades, Wormland '94] Edge crossings in drawings of bipartite graphs
- [Jünger, Mutzel '97] 2-Layer Straightline Crossing Minimization: Performance of Exact and Heuristic Algorithms