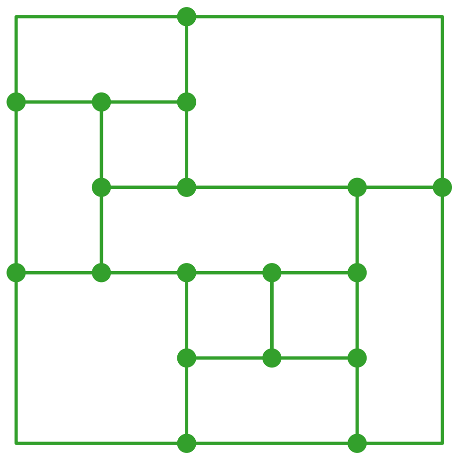
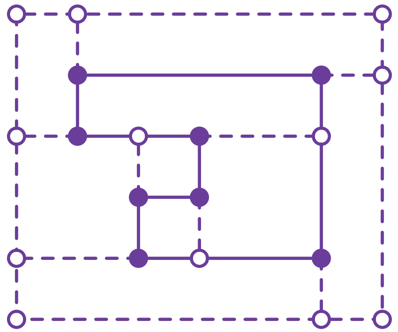
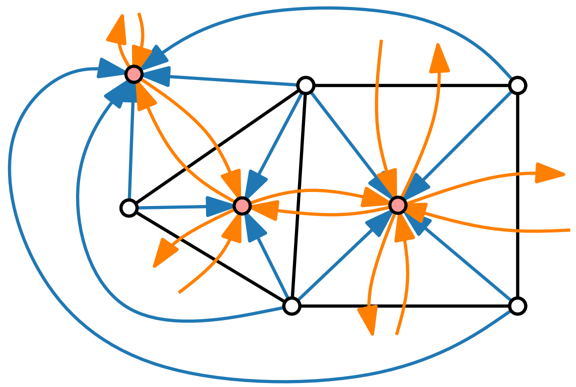
Julius-Maximilians-
UNIVERSITÄT
WÜRZBURG

Visualization of Graphs



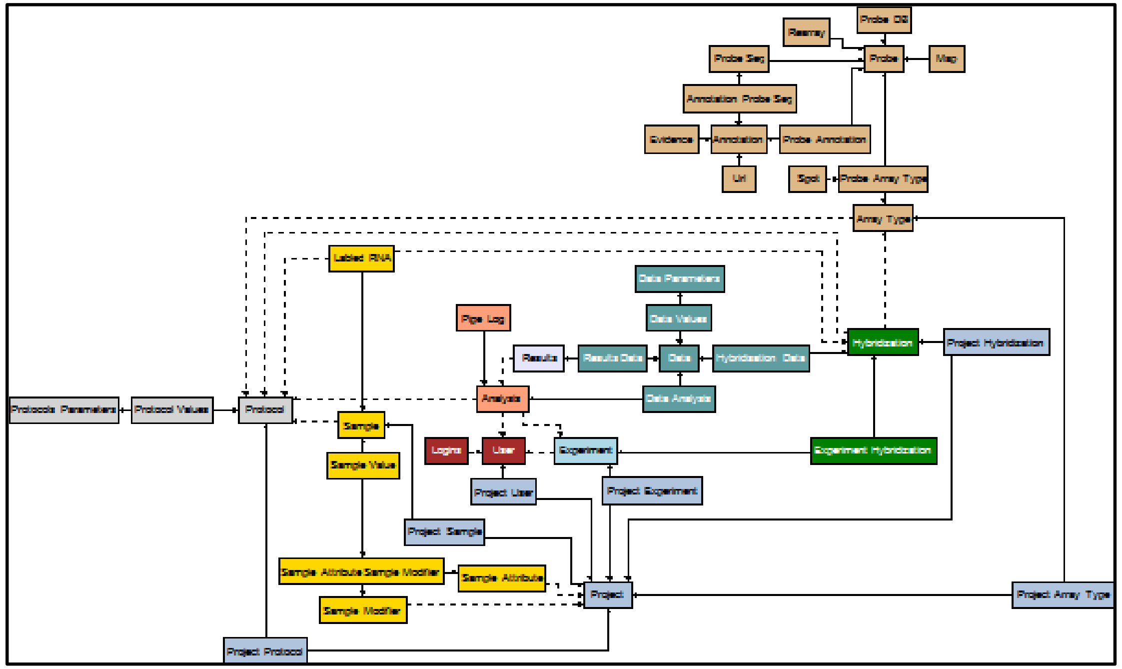
Lecture 6:
Orthogonal Layouts





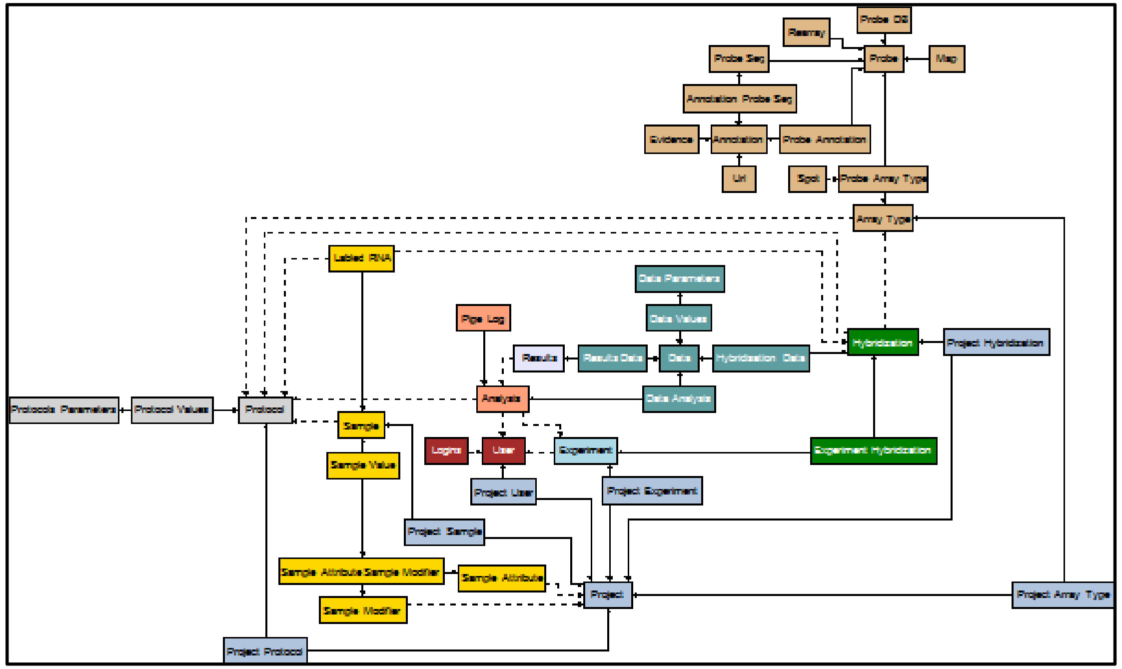
Johannes Zink

Orthogonal Layout – Applications



ER diagram in OGDF

Orthogonal Layout – Applications



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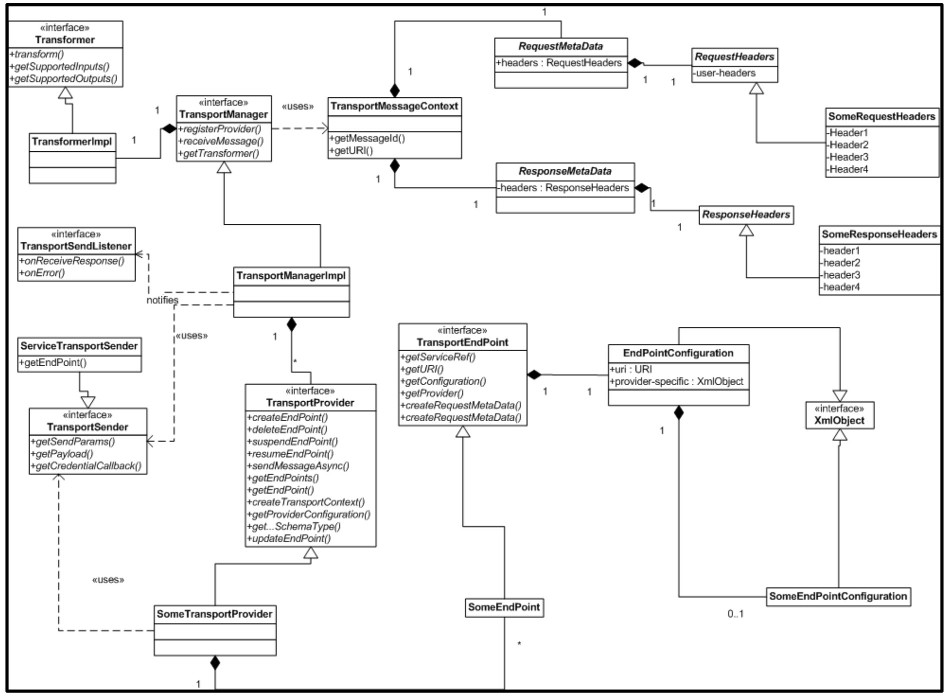
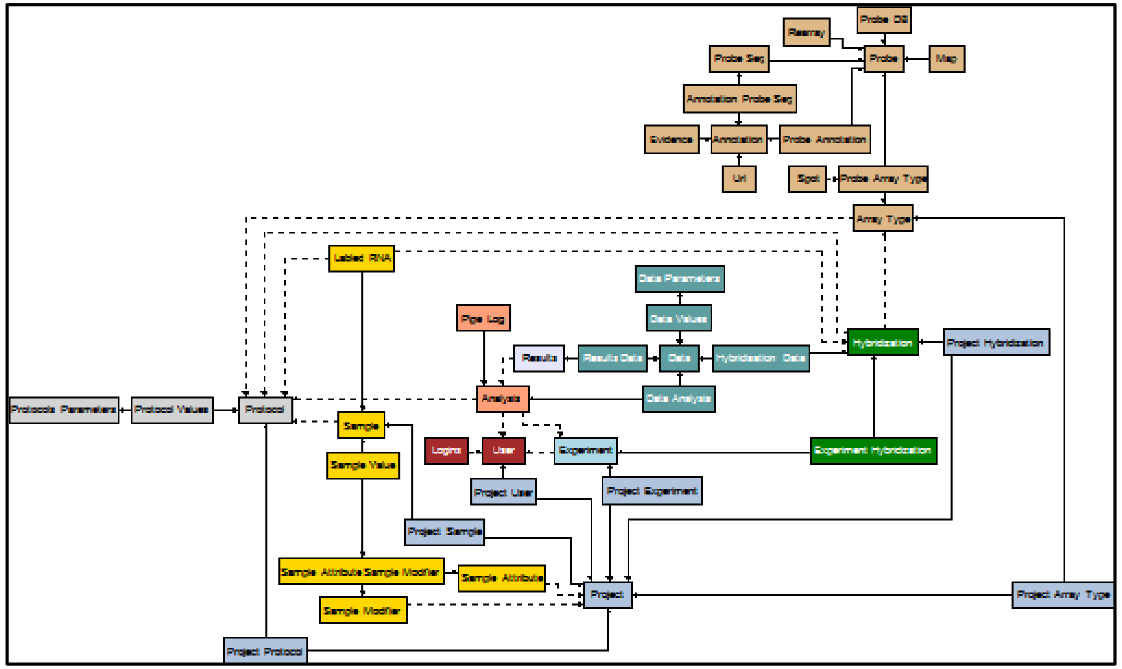


UML diagram by Oracle

Orthogonal Layout – Applications



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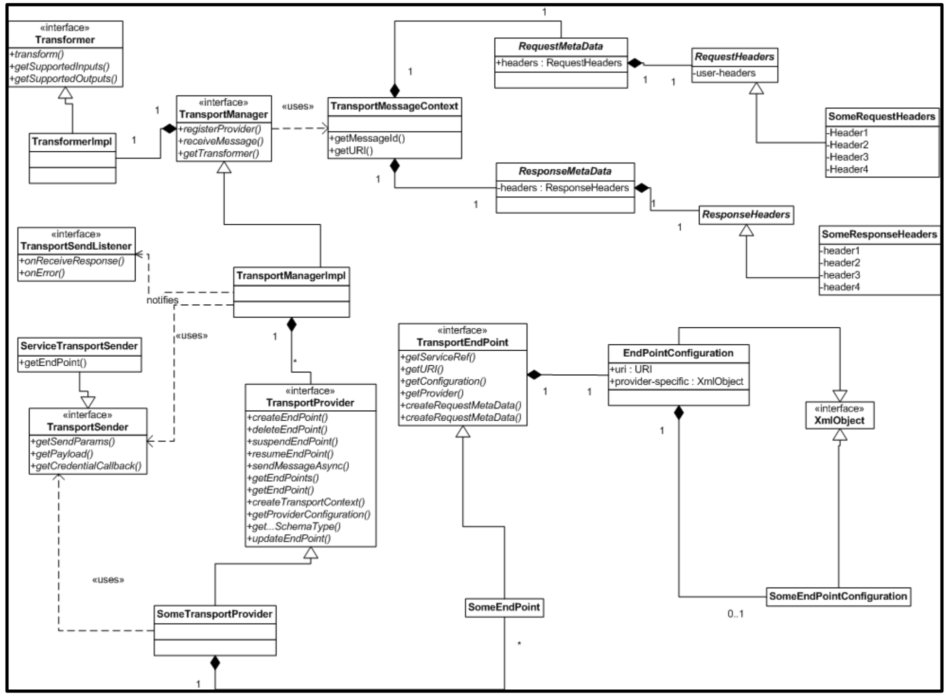

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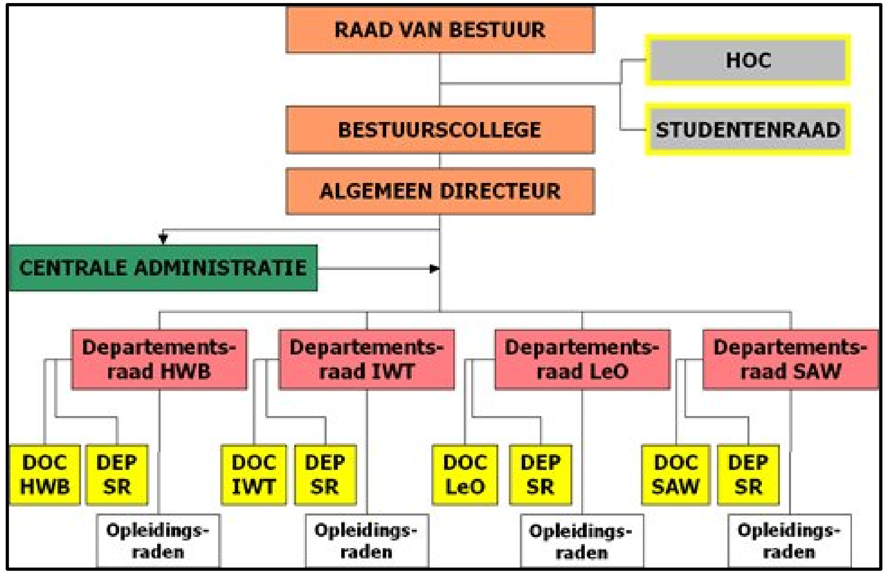

Organigram of HS Limburg

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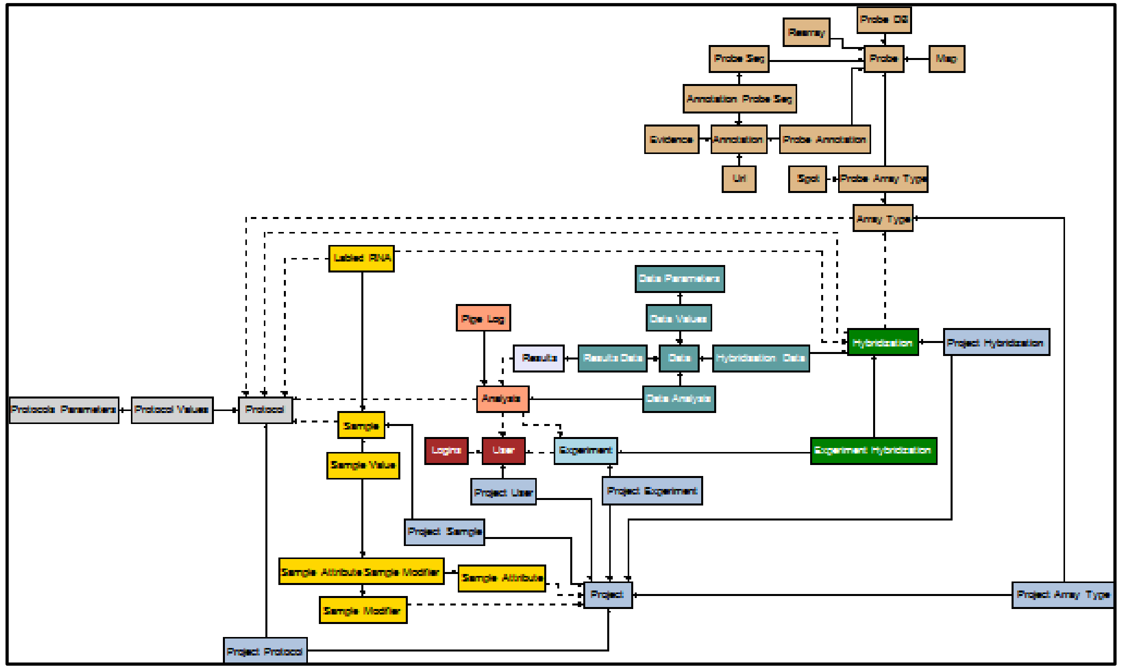

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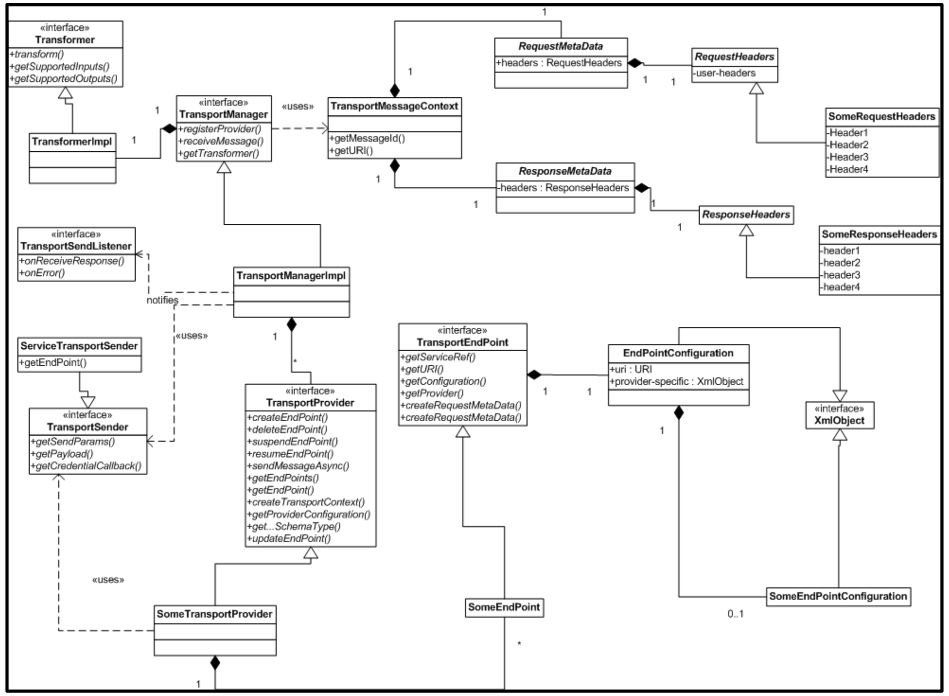

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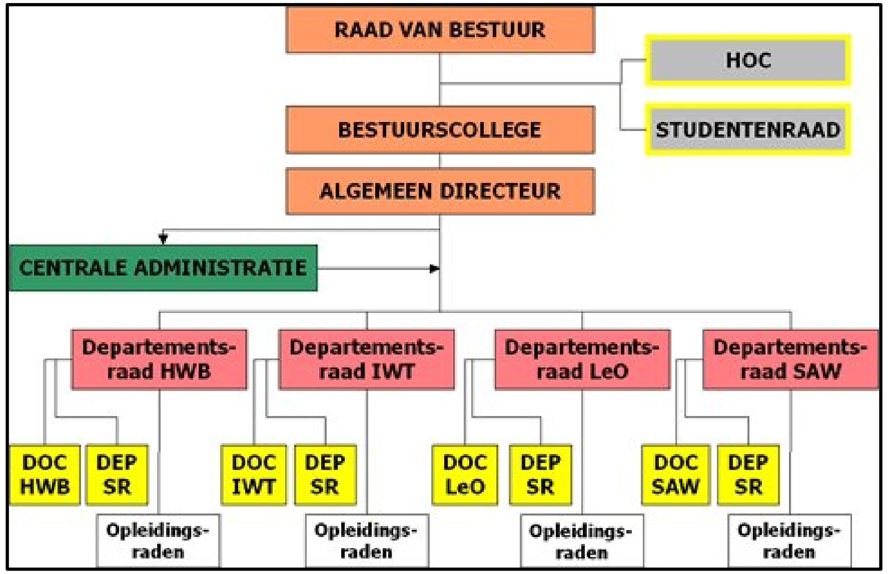

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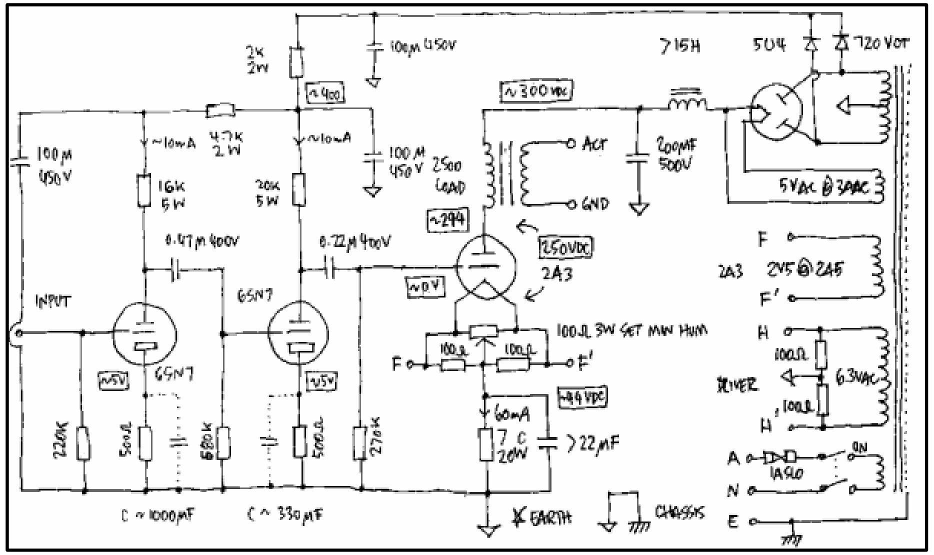

Circuit diagram by Jeff Atwood

Orthogonal Layout – Definition

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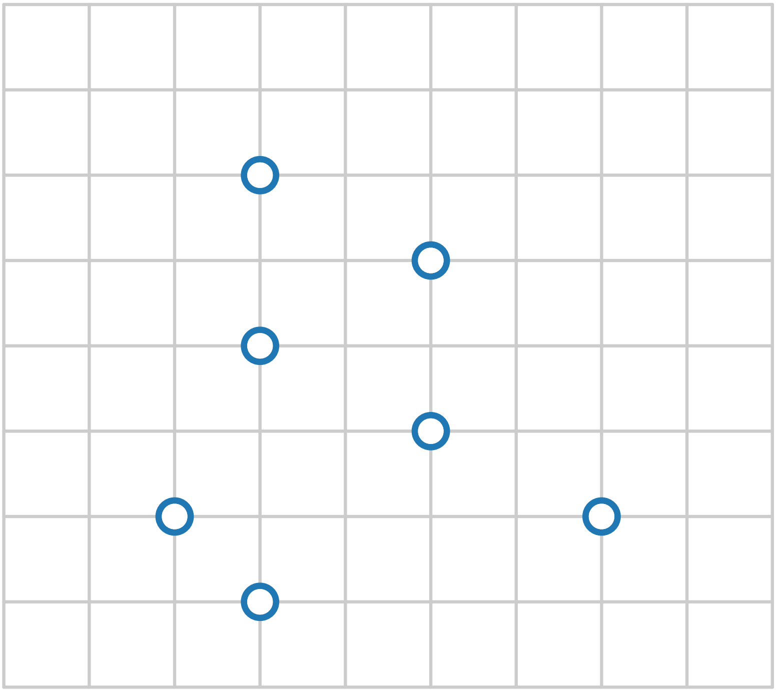

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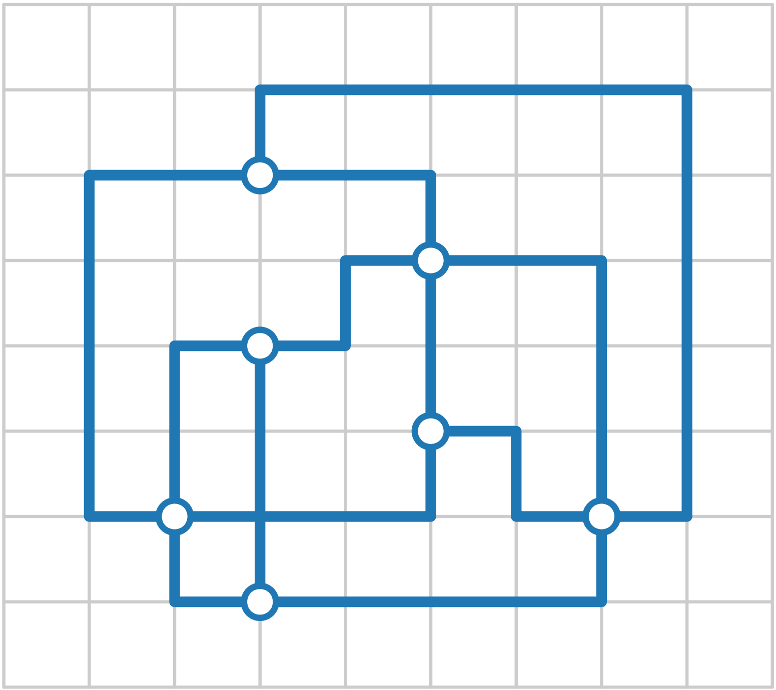

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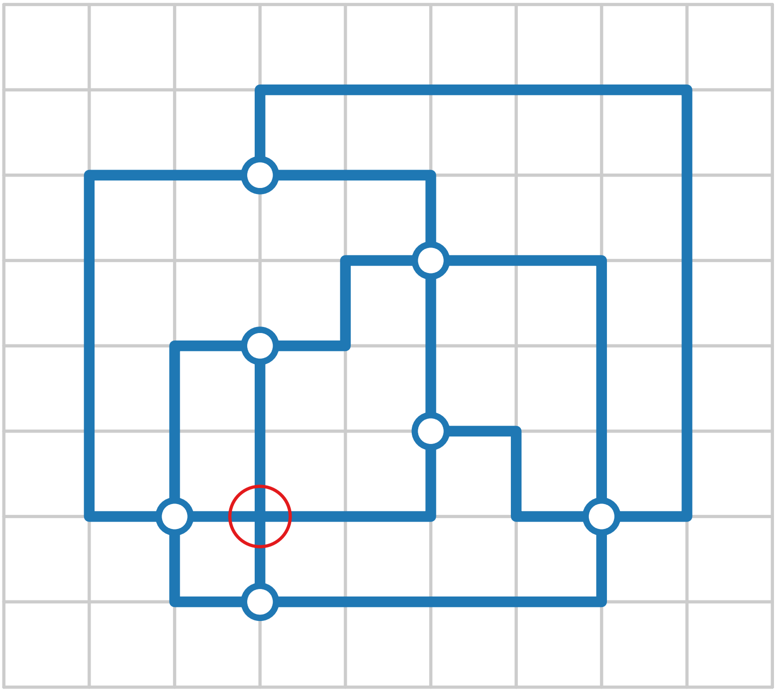

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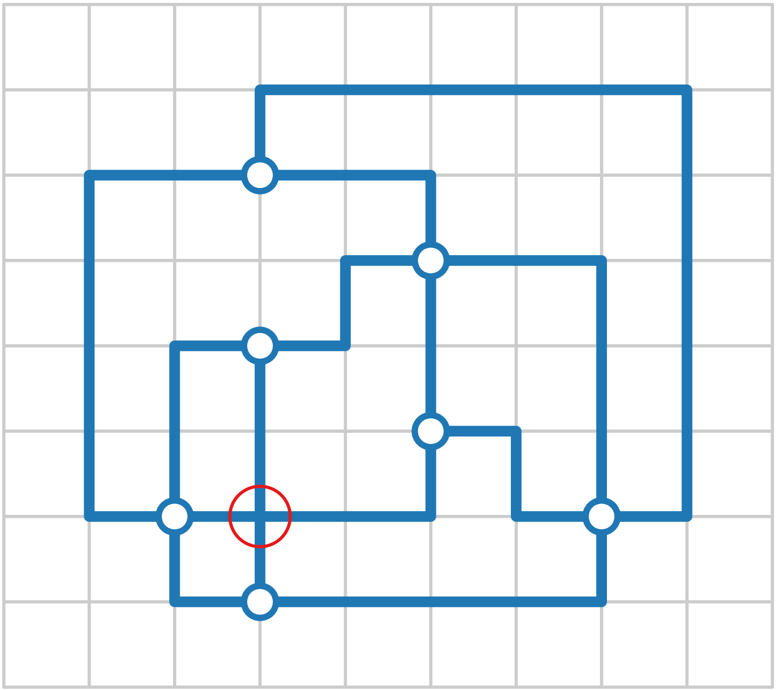

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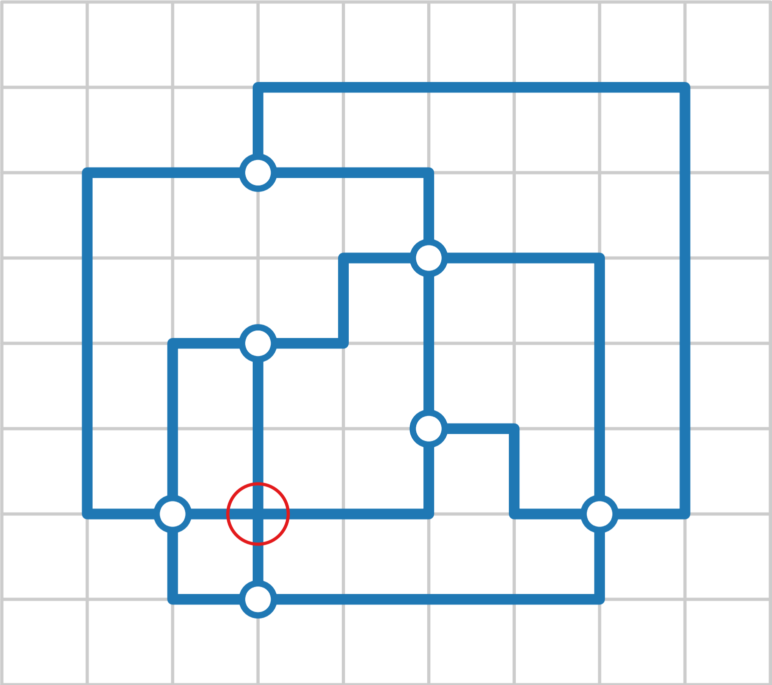

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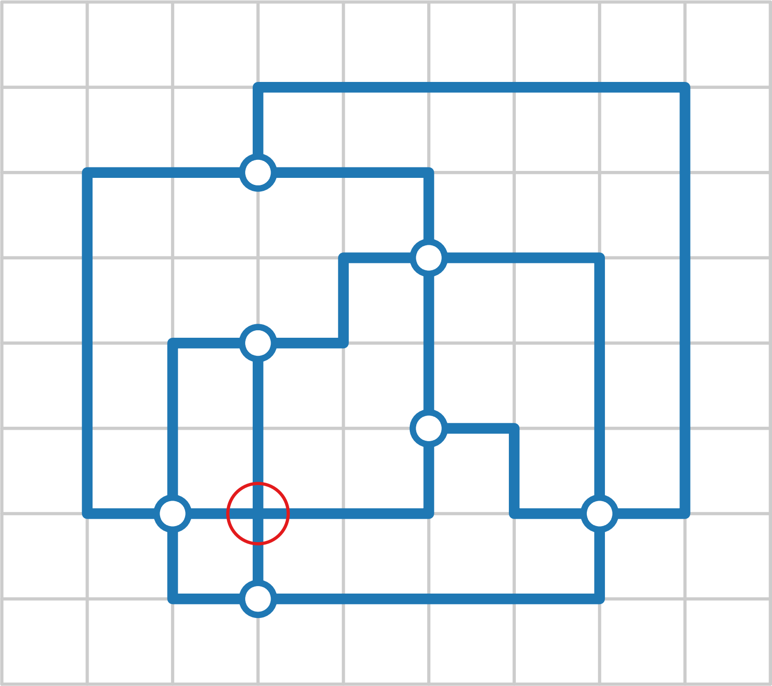

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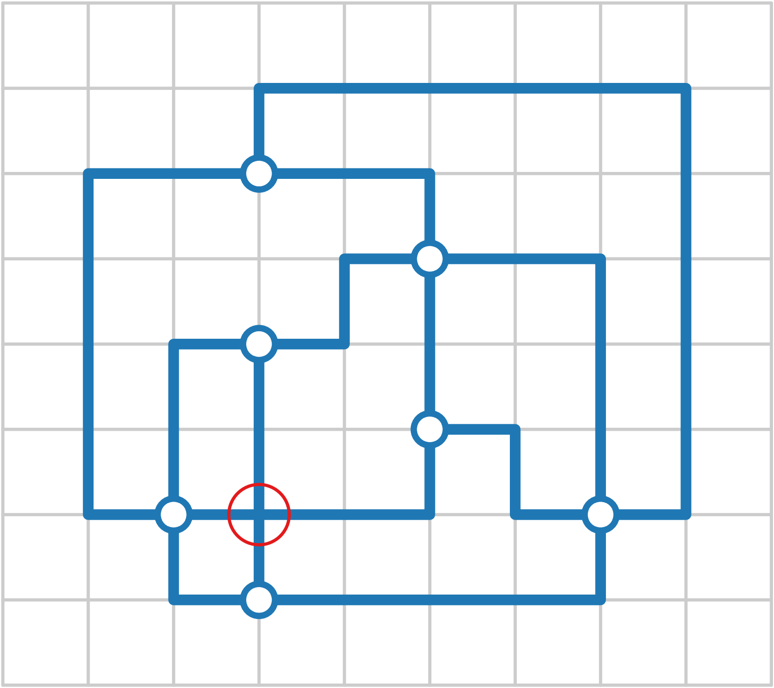

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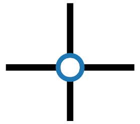
- Otherwise 

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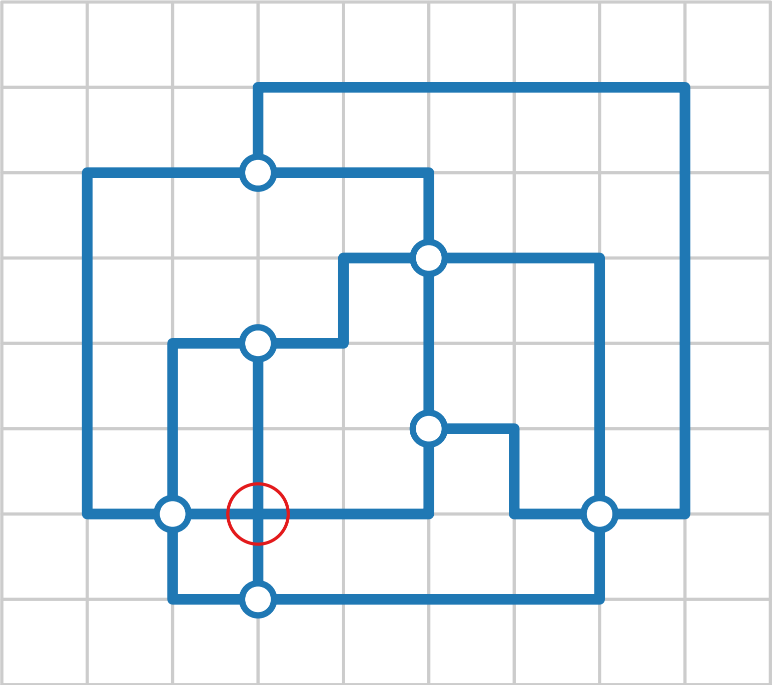

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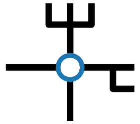
- Otherwise 

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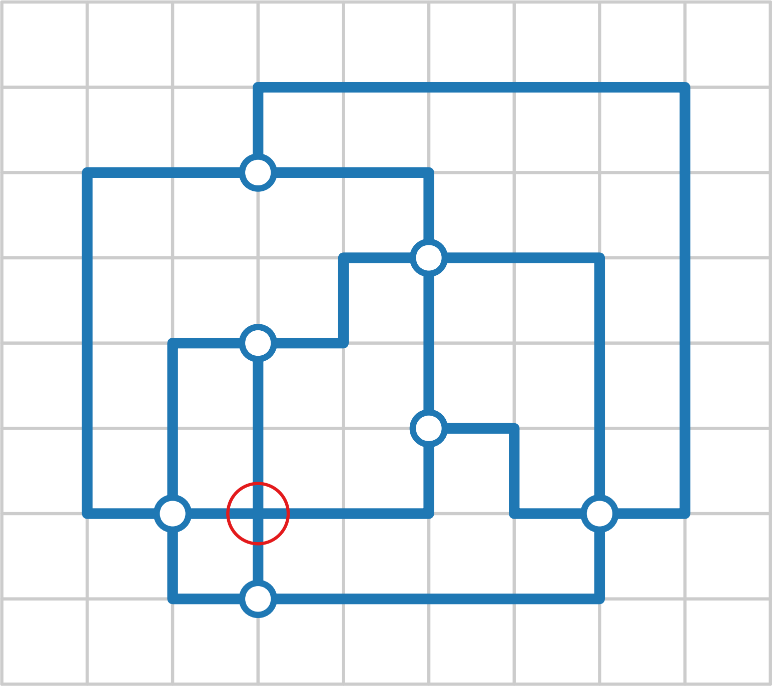

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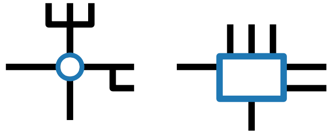
- Otherwise 

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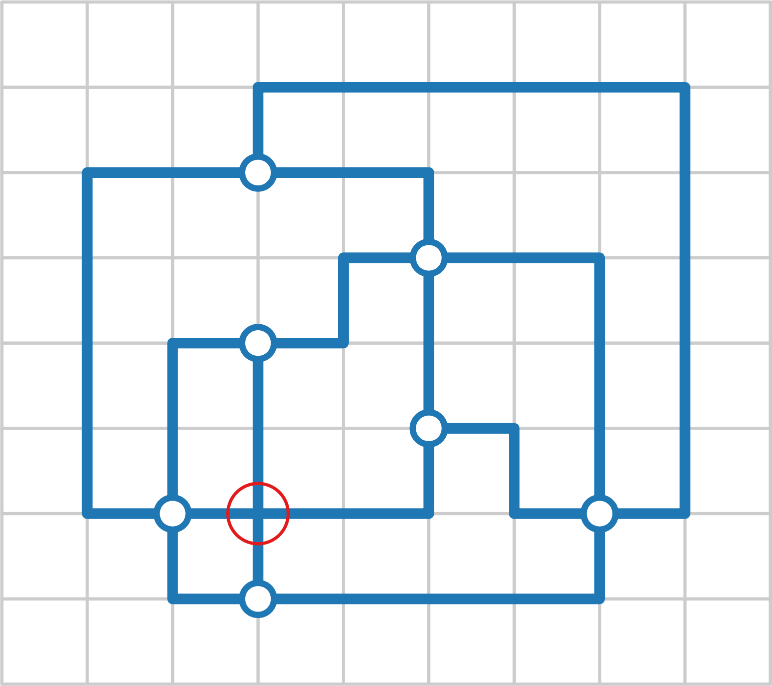

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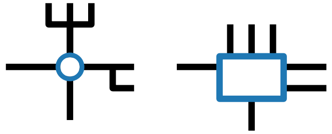
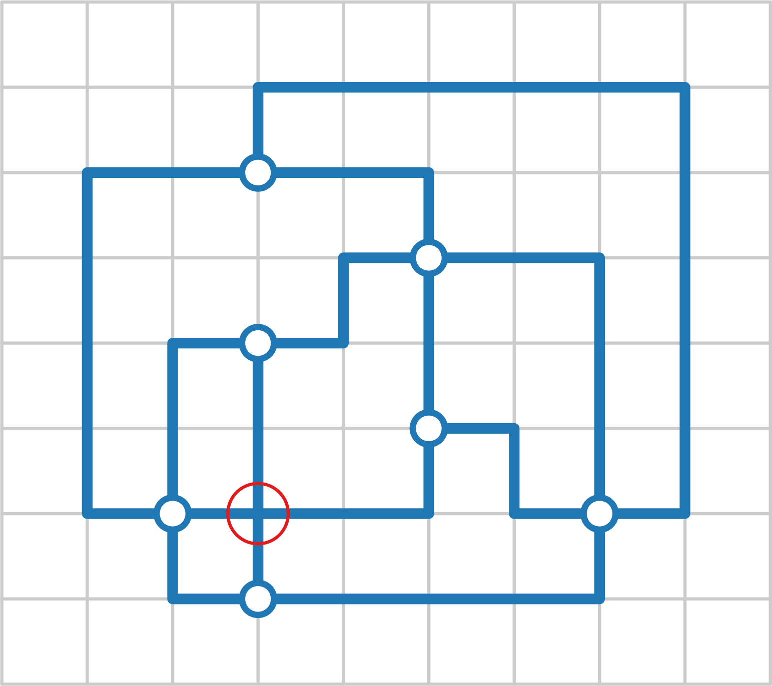


Planarization.

Orthogonal Layout – Definition



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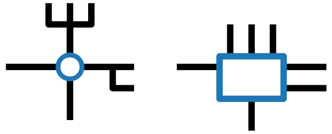

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- Fix embedding

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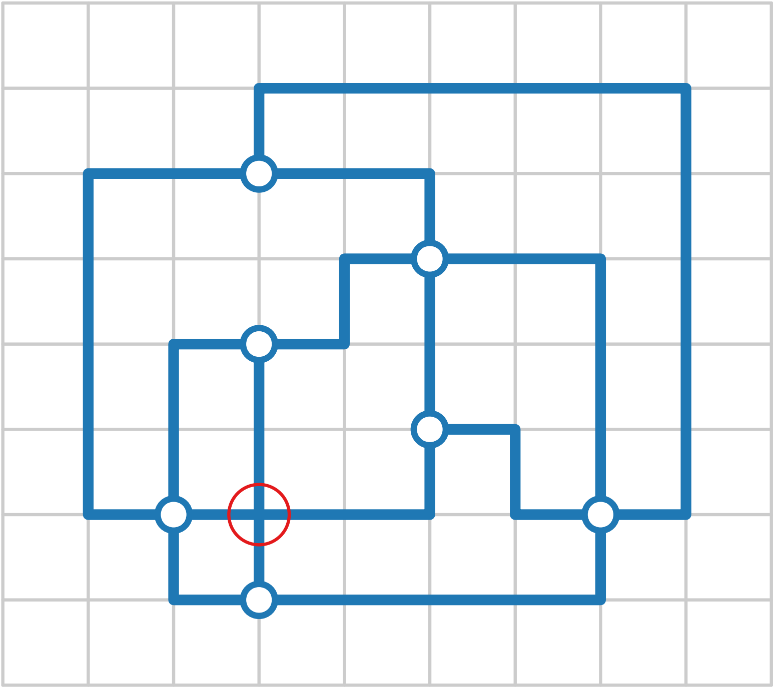

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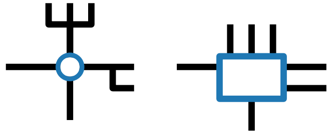

Planarization.

- Fix embedding
- Crossings become vertices

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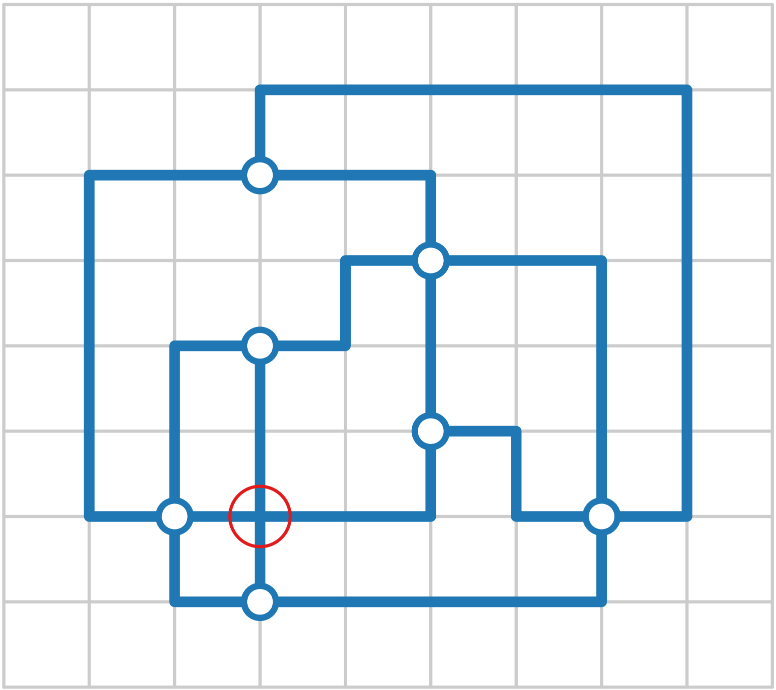

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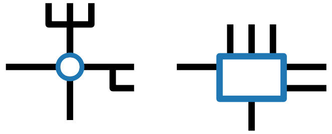
- Otherwise 

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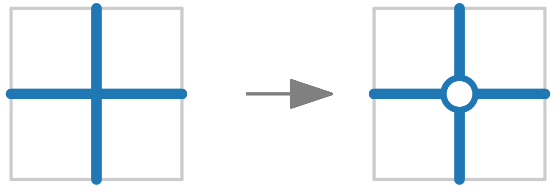

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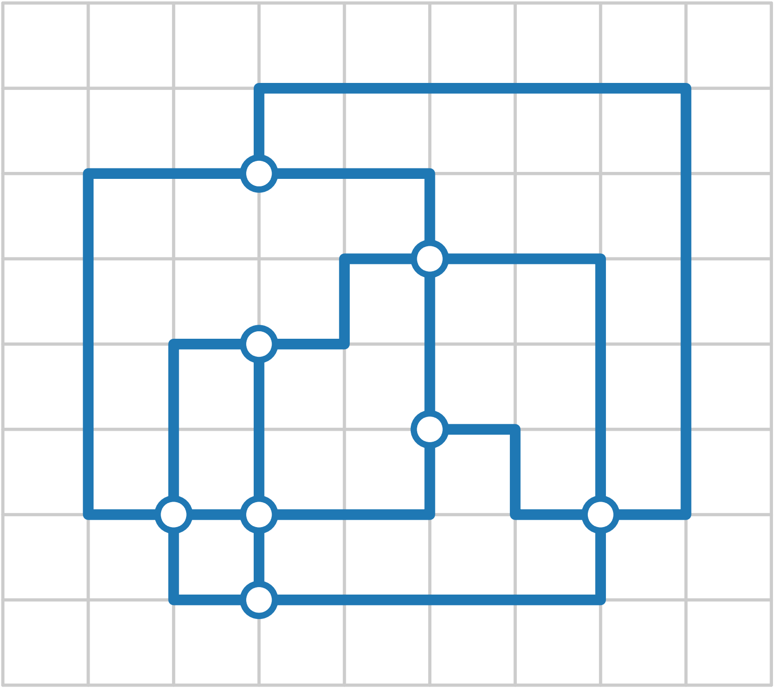

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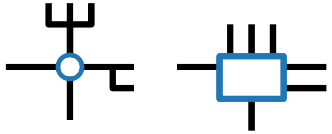

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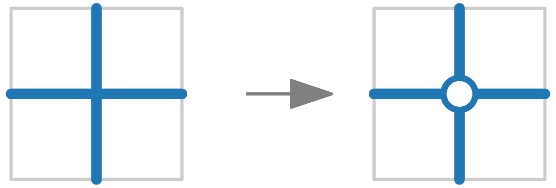

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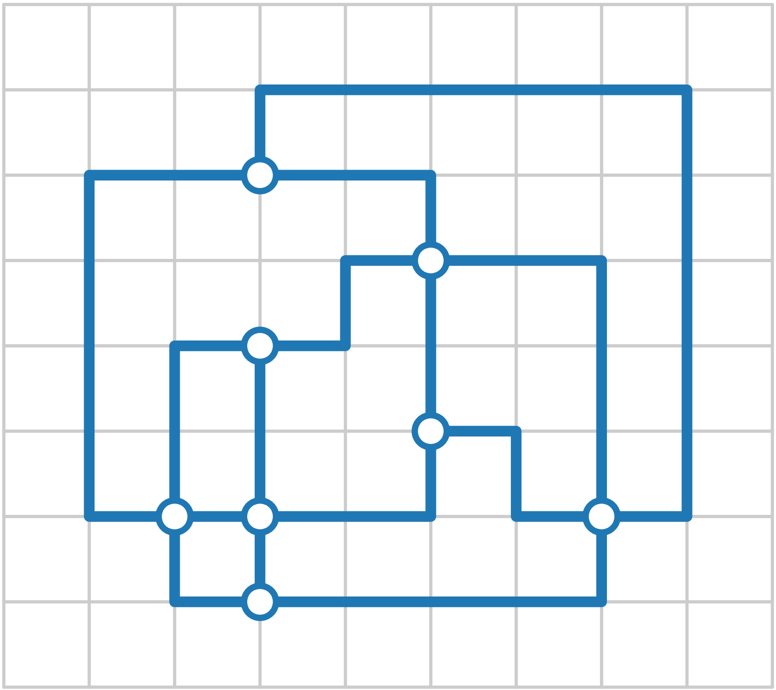

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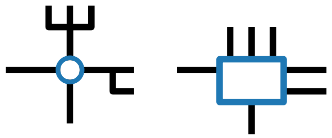

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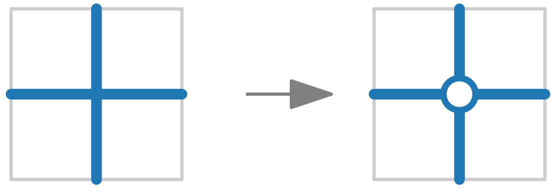
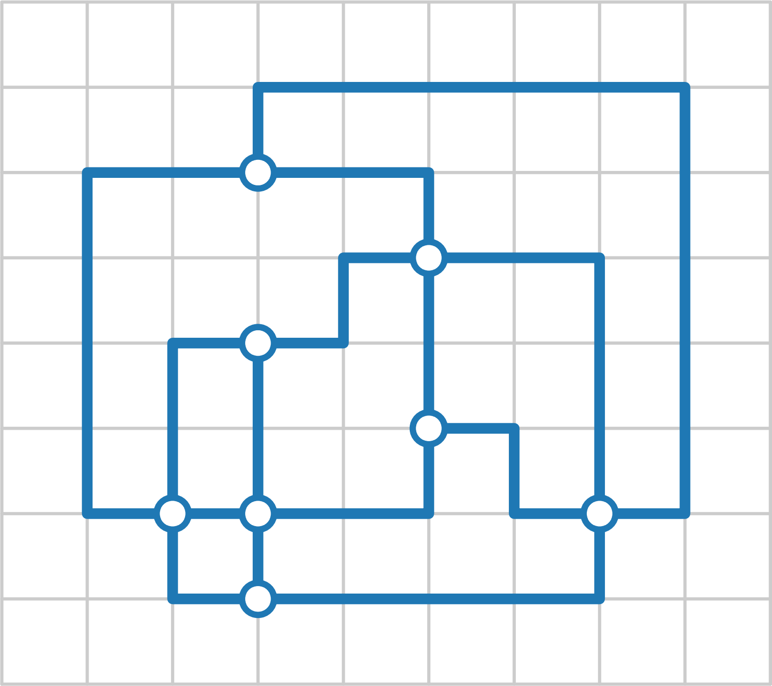


Aesthetic criteria to optimize.

Orthogonal Layout – Definition



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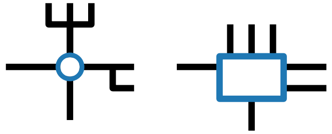

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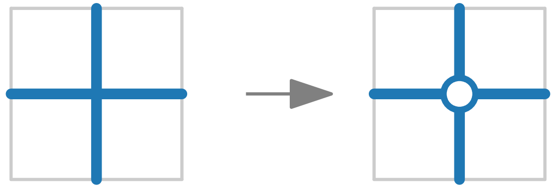
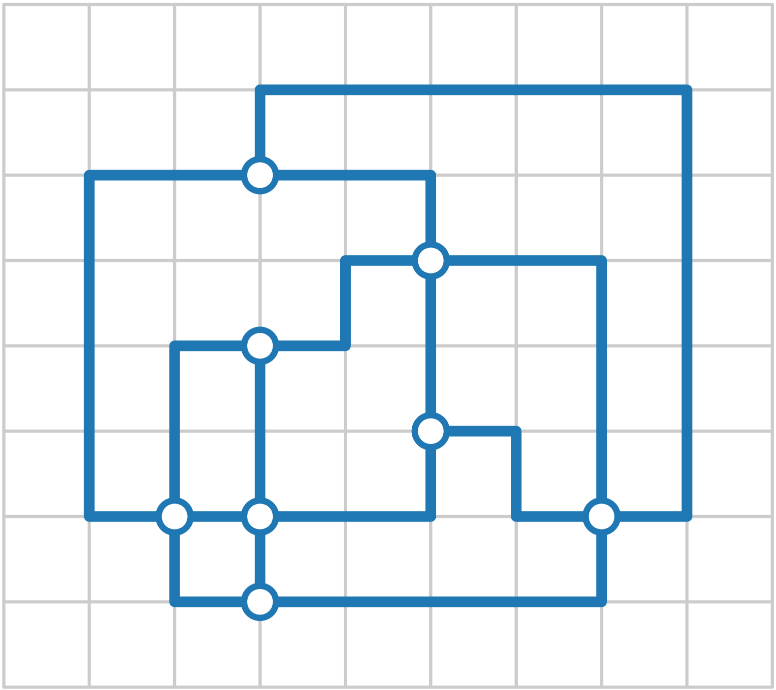


Aesthetic criteria to optimize.
- Number of bends

Orthogonal Layout – Definition



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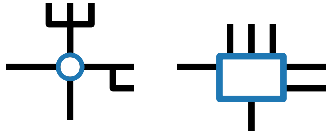

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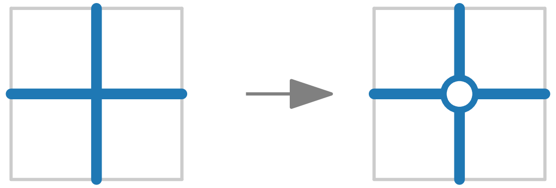
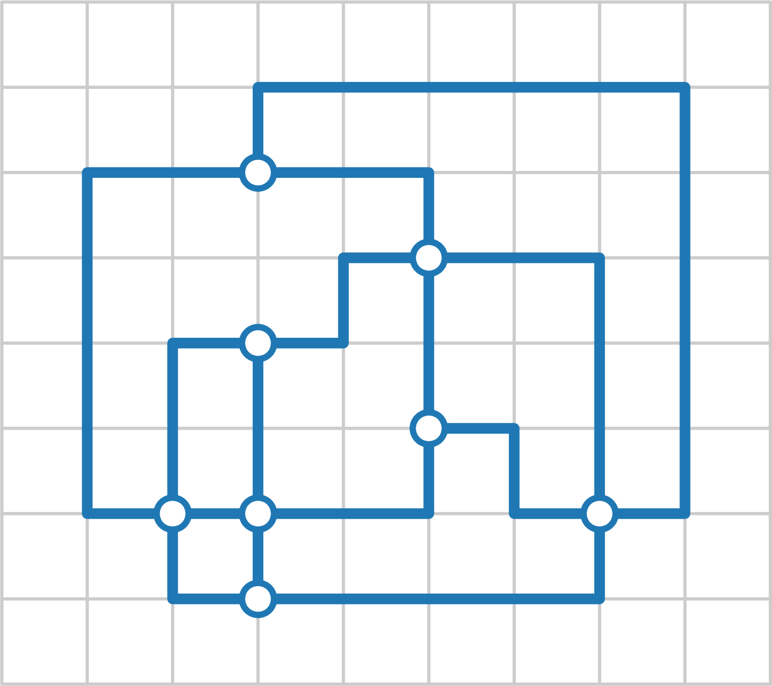


Aesthetic criteria to optimize.

- Number of bends
- Length of edges

Orthogonal Layout – Definition



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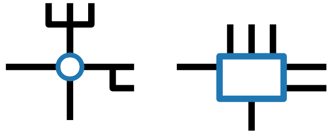

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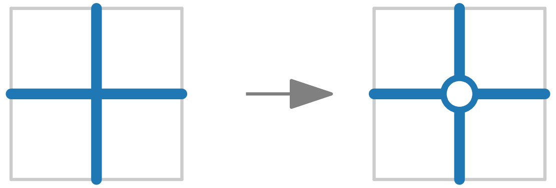
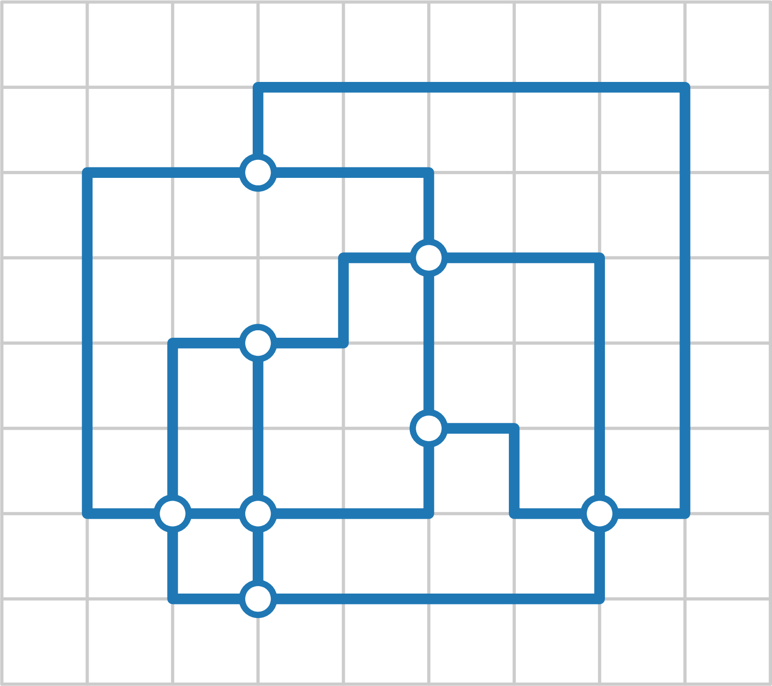


Aesthetic criteria to optimize.
- Number of bends
- Length of edges
- Width, height, area

Orthogonal Layout – Definition



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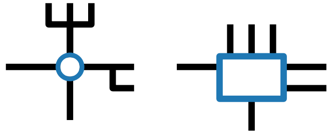

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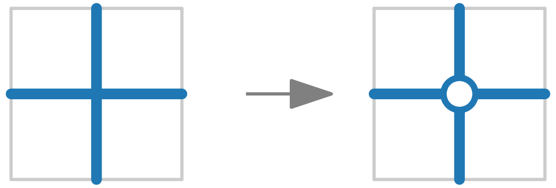

Aesthetic criteria to optimize.
- Number of bends
- Length of edges
- Width, height, area
- Monotonicity of edges
- ...

Topology – Shape – Metrics

Three-step approach: [Tamassia 1987]

TOPOLOGY — SHAPE — METRICS

Topology – Shape – Metrics

Three-step approach: [Tamassia 1987]

$V = \{v_1, v_2, v_3, v_4\}$
$E = \{v_1v_2, v_1v_3, v_1v_4, v_2v_3, v_2v_4\}$

TOPOLOGY – SHAPE – METRICS

Topology – Shape – Metrics

Three-step approach: [Tamassia 1987]

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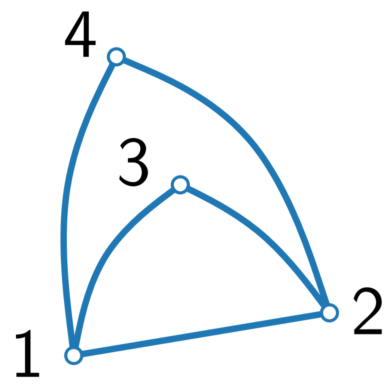
combinatorial embedding/ planarization



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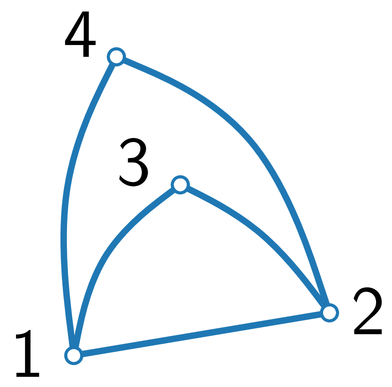

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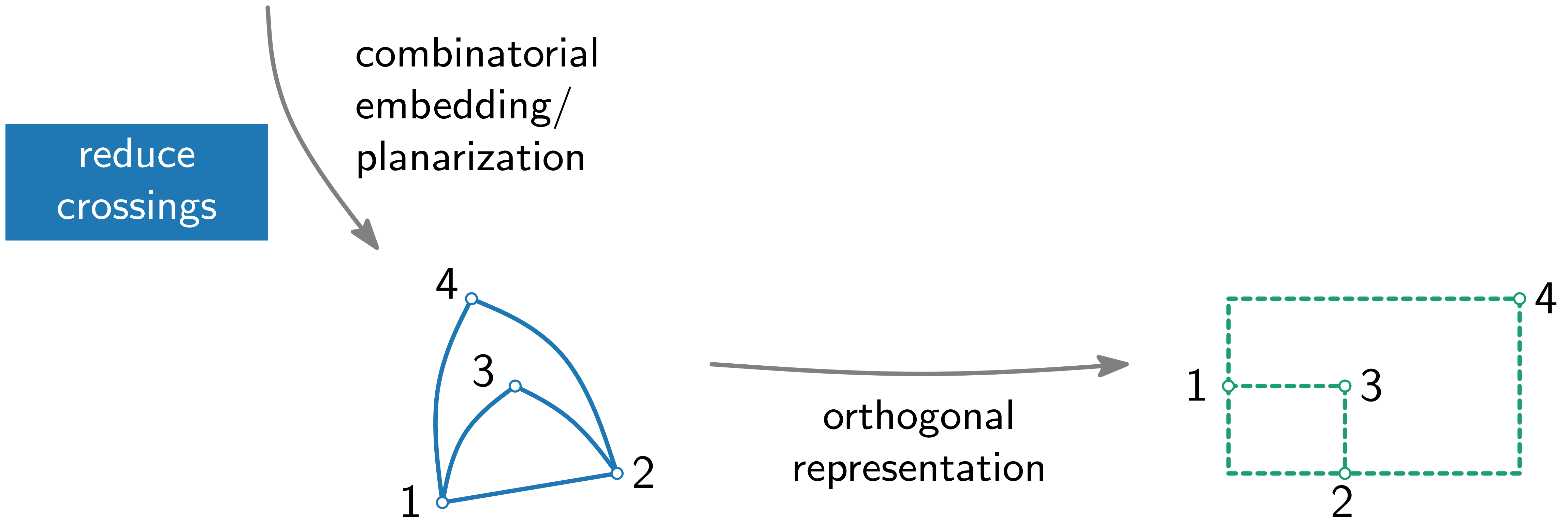

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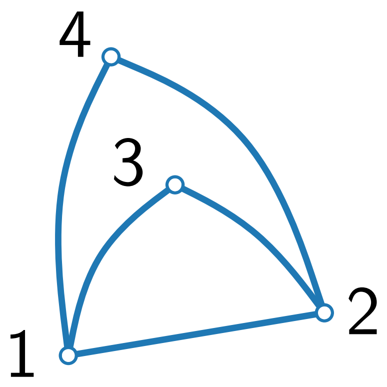

bend minimization

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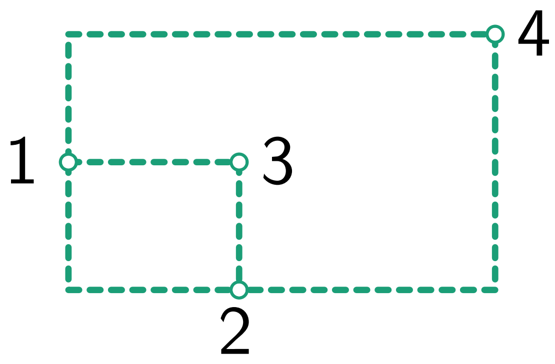

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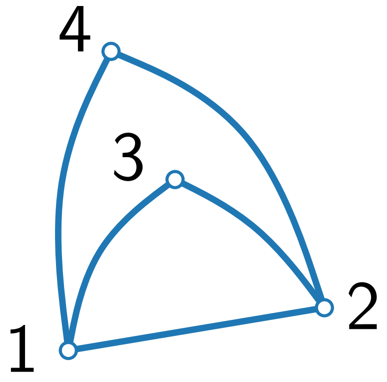

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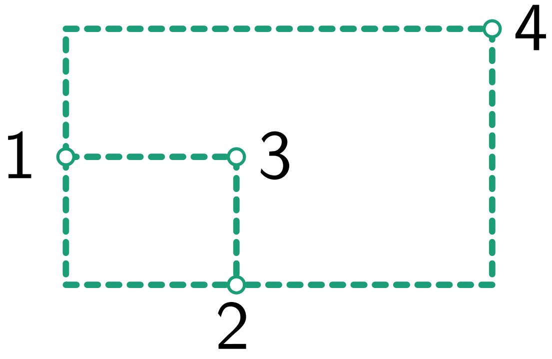
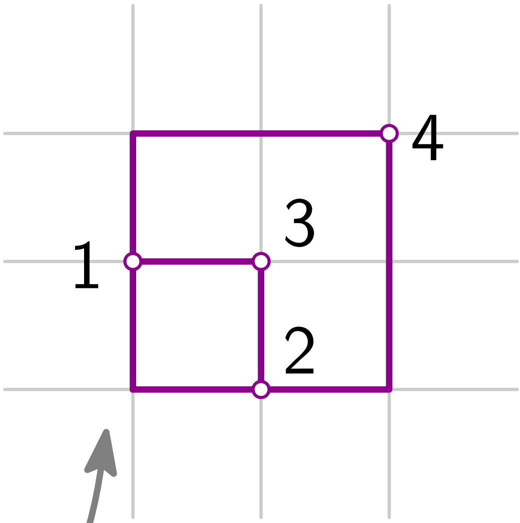


planar orthogonal drawing



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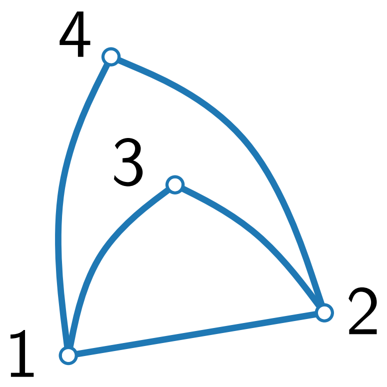

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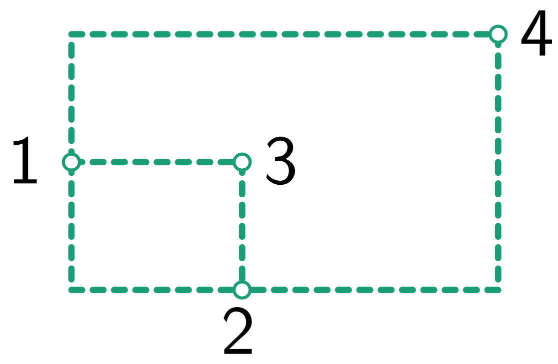

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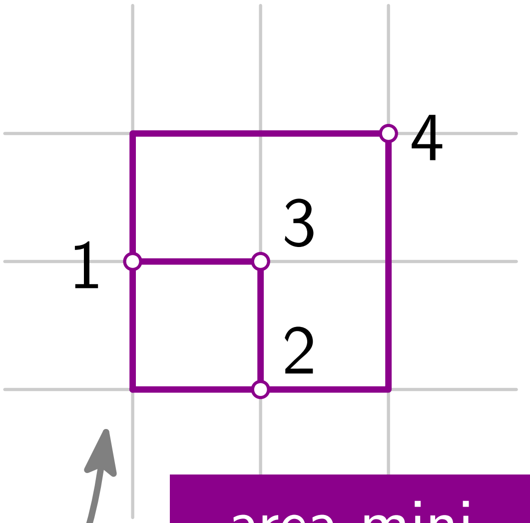

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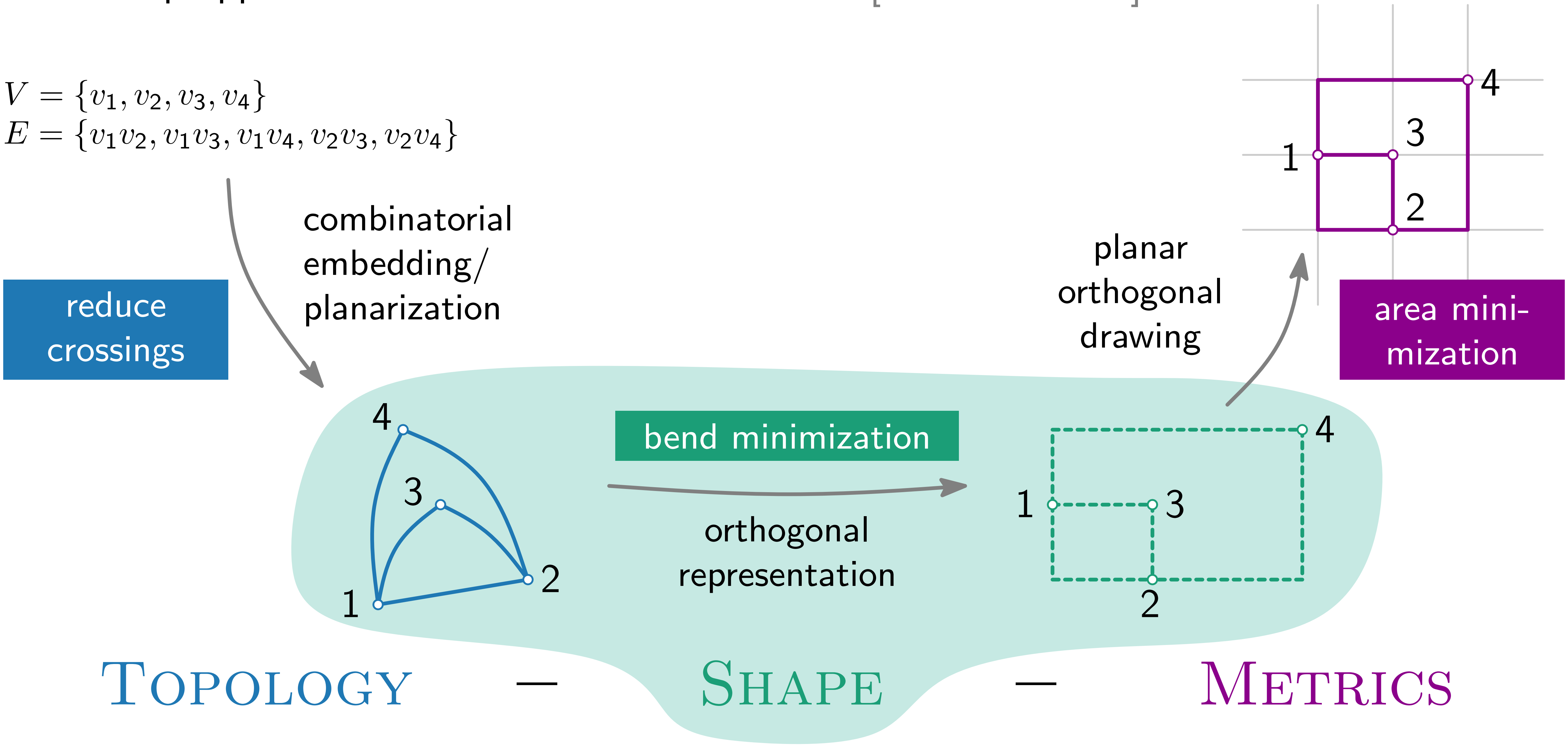

Orthogonal Representation

Idea.
Describe orthogonal drawing combinatorially.

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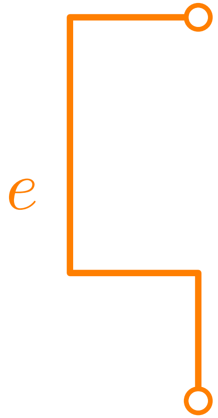

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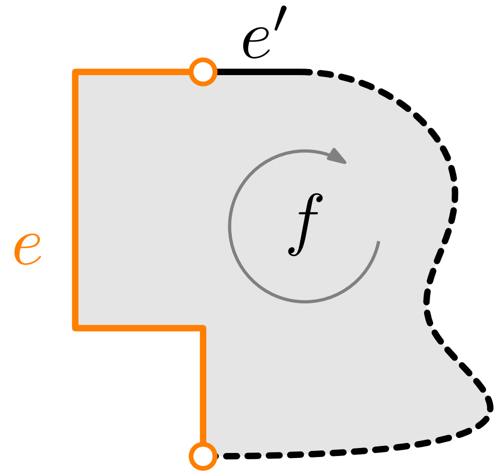

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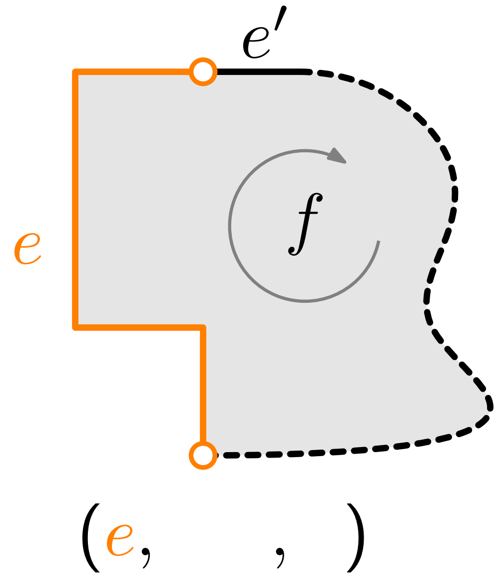

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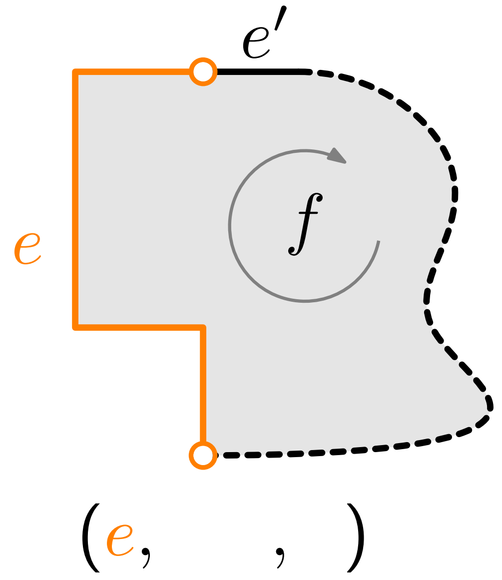

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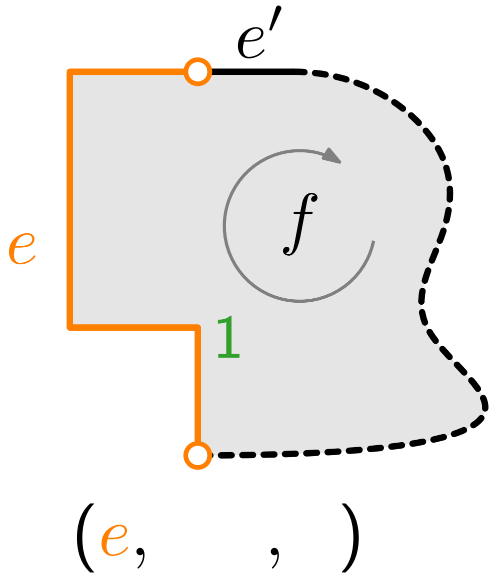

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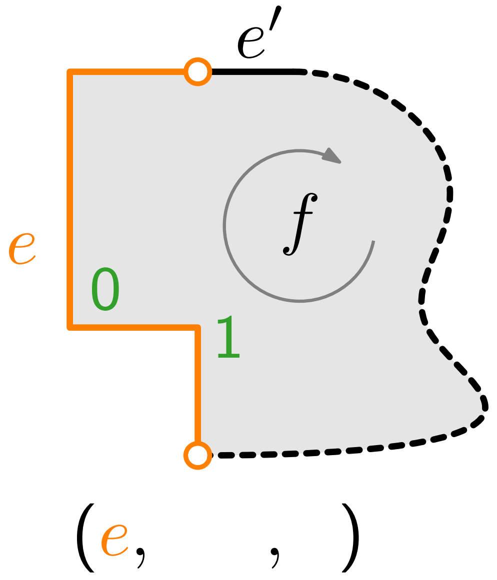

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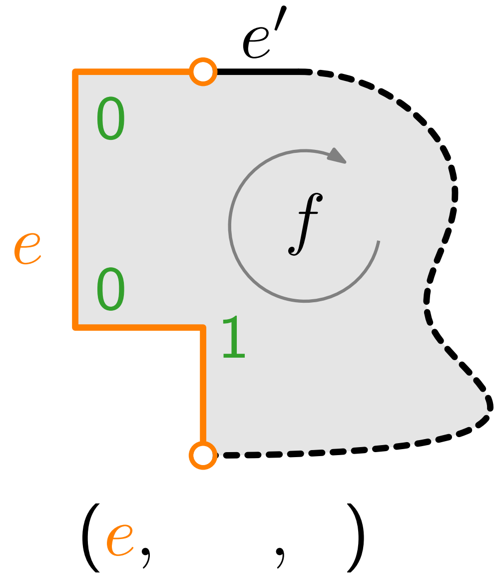

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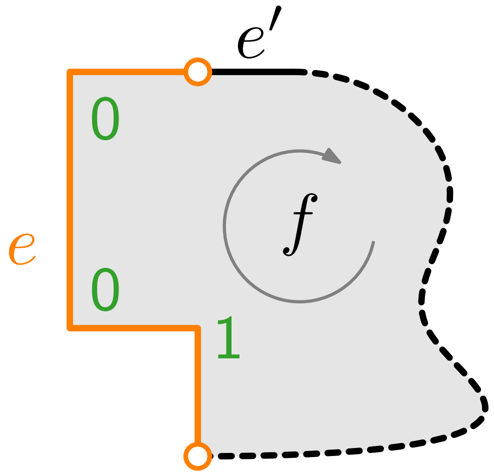

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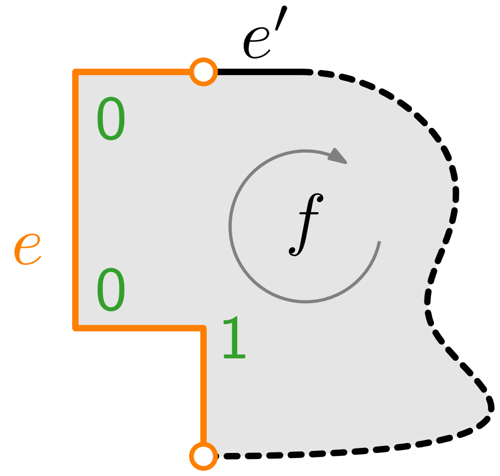

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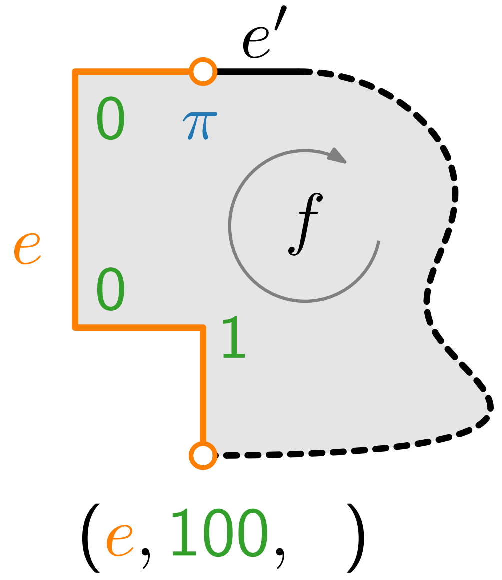

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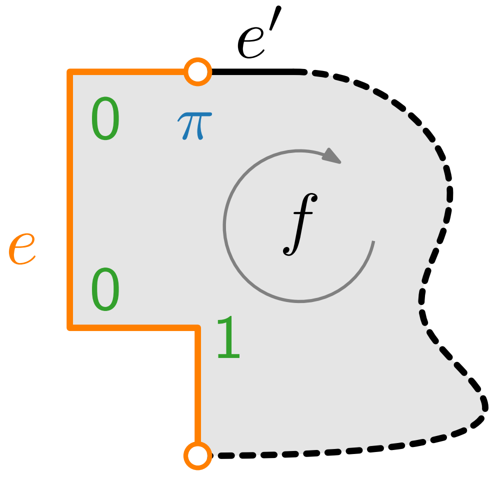

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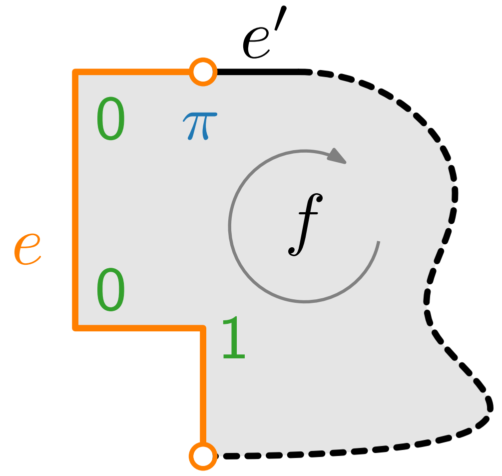

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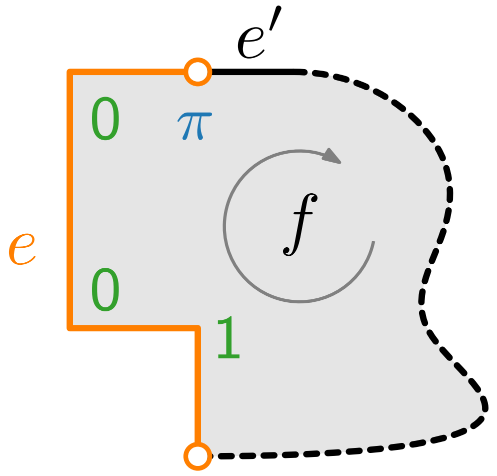

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- An **orthogonal representation** $H(G)$ of G is defined as

$$H(G) = \{H(f) \mid f \in F\}.$$

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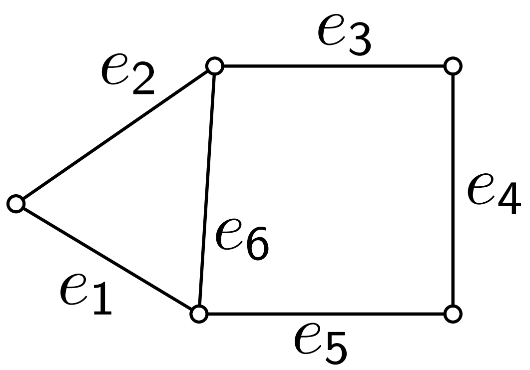

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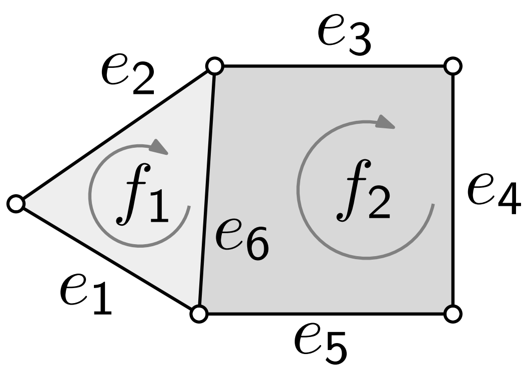

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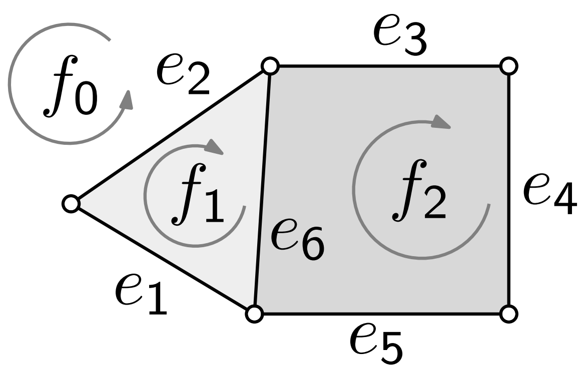

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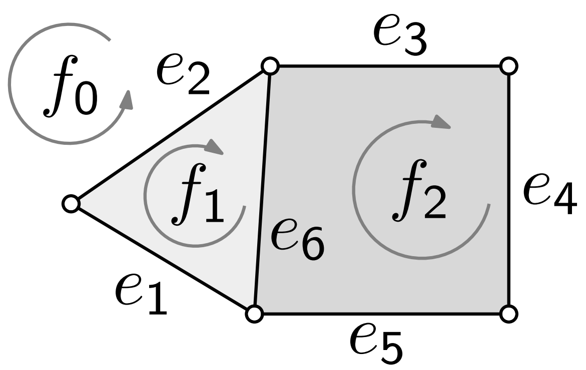

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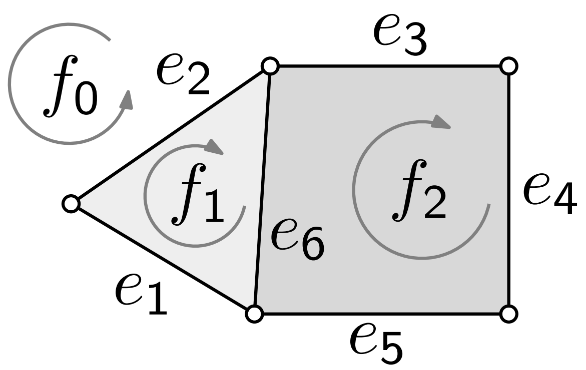

Combinatorial "drawing" of $H(G)$?

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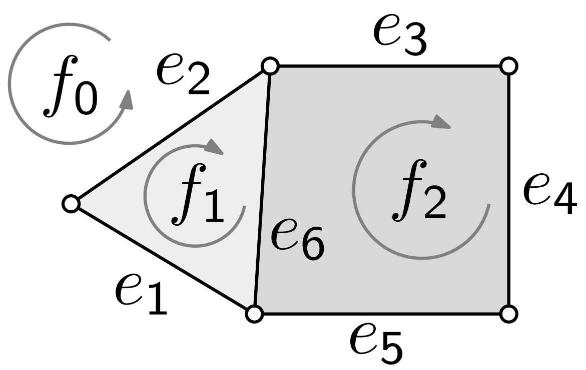

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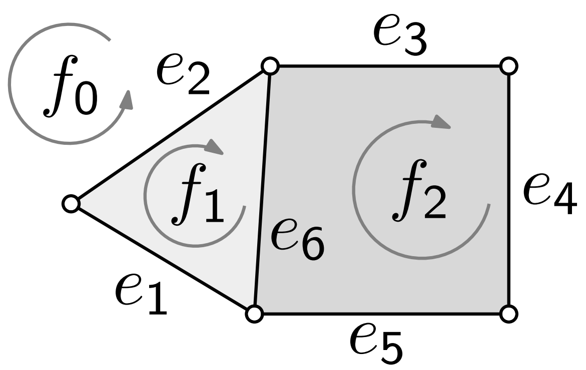

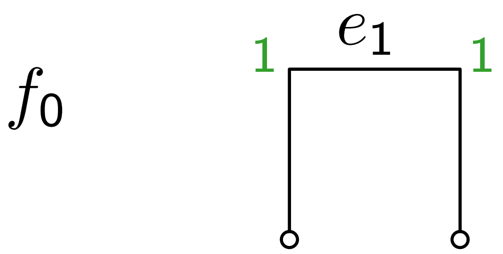

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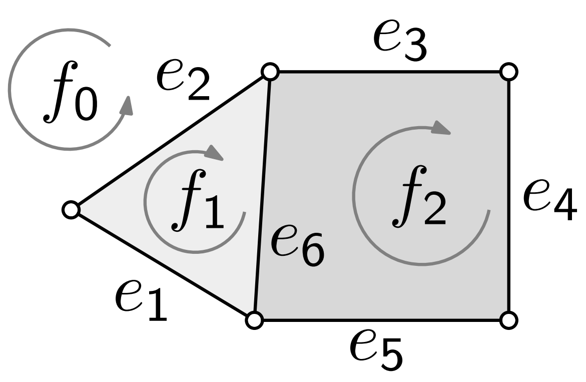

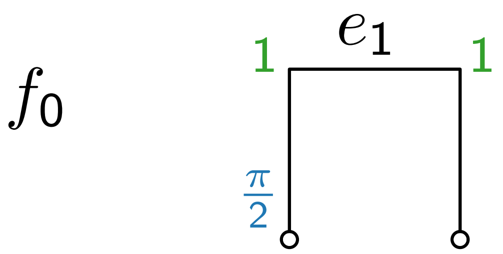

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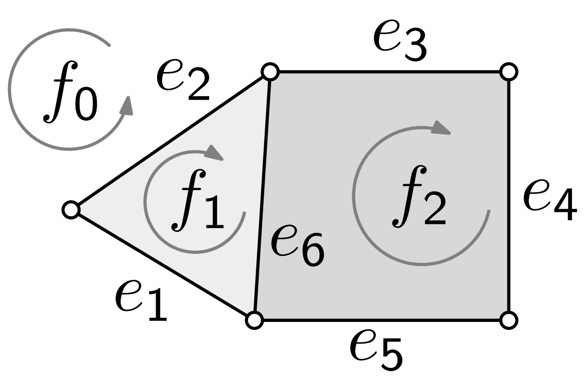

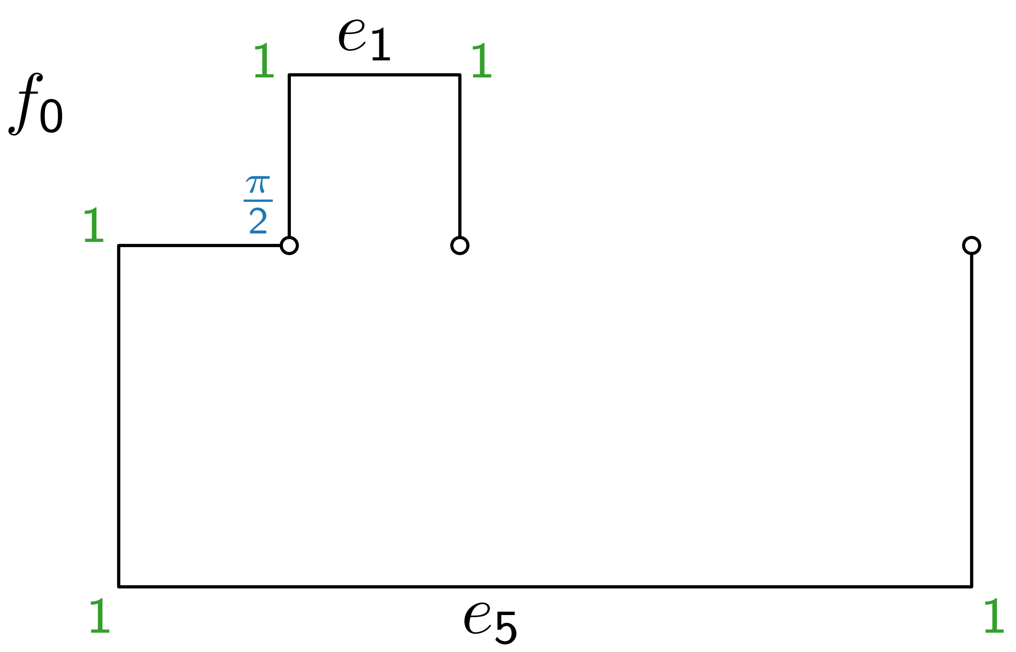

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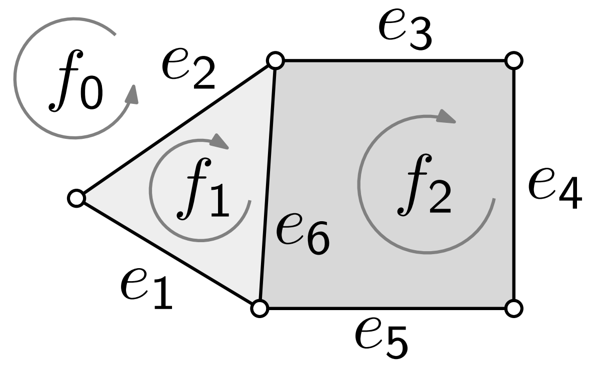
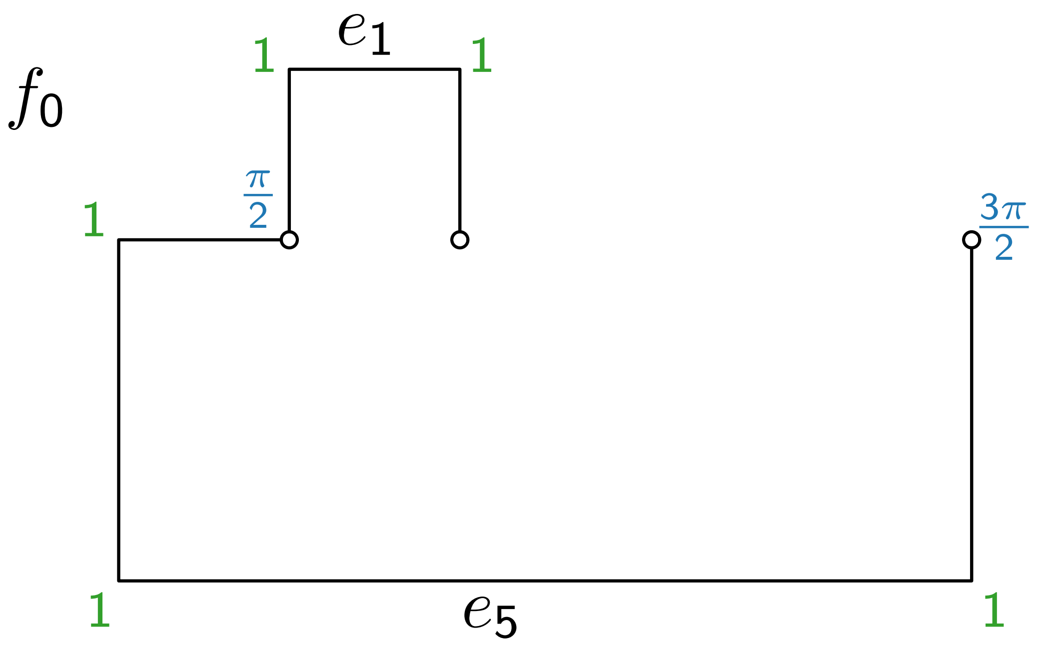
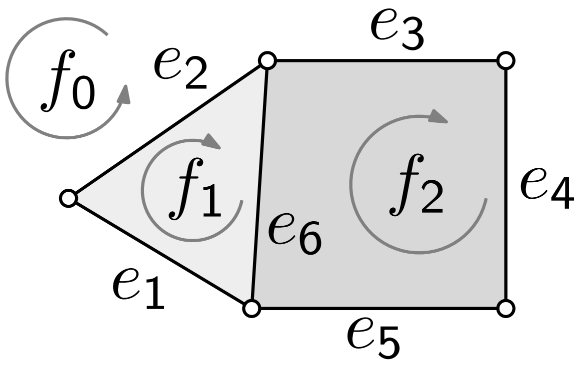
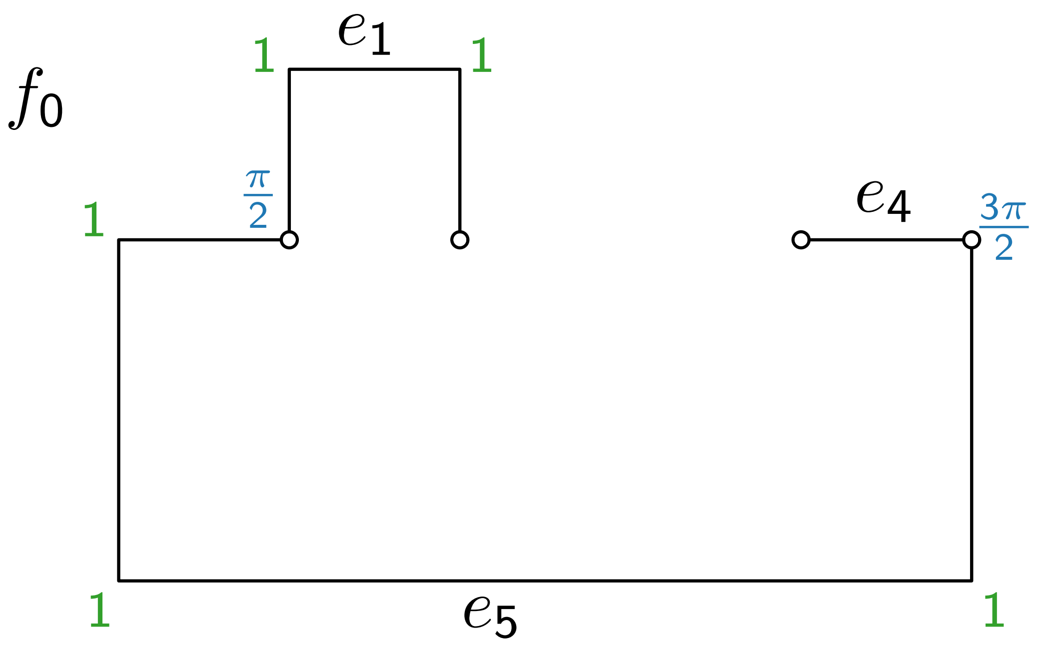




Orthogonal Representation – Example

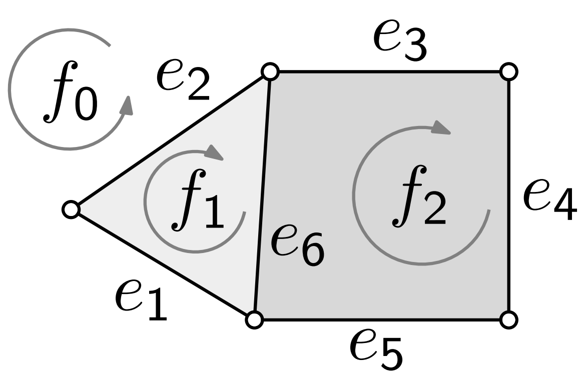
$$H(f_0) = ((e_1, 11, \tfrac{\pi}{2}), (e_5, 111, \tfrac{3\pi}{2}), (e_4, \emptyset, \pi), (e_3, \emptyset, \pi), (e_2, \emptyset, \tfrac{\pi}{2}))$$
$$H(f_1) = ((e_1, 00, \tfrac{3\pi}{2}), (e_2, \emptyset, \tfrac{\pi}{2}), (e_6, 00, \pi))$$
$$H(f_2) = ((e_5, 000, \tfrac{\pi}{2}), (e_6, 11, \tfrac{\pi}{2}), (e_3, \emptyset, \pi), (e_4, \emptyset, \tfrac{\pi}{2}))$$

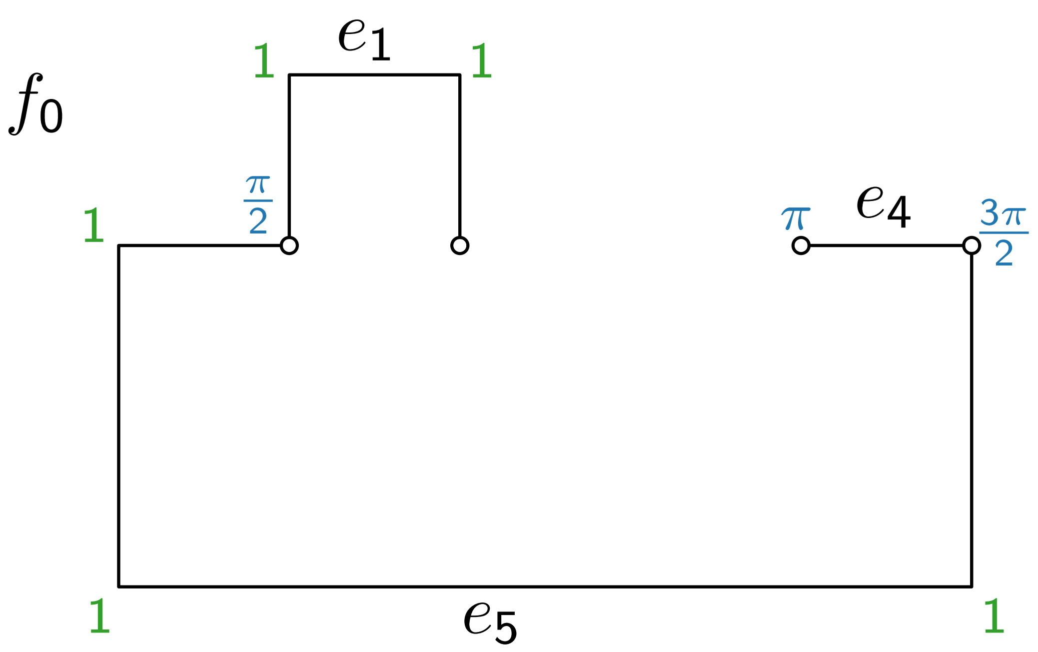




Orthogonal Representation – Example

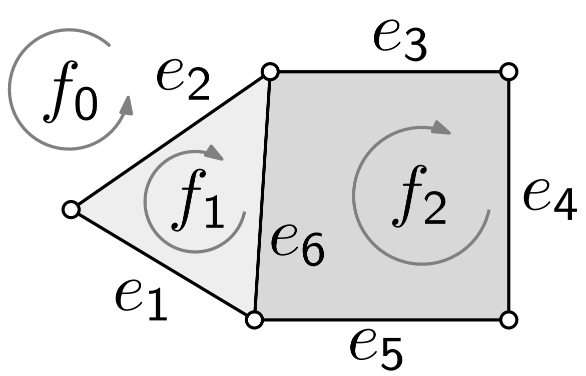
$$H(f_0) = ((e_1, 11, \tfrac{\pi}{2}), (e_5, 111, \tfrac{3\pi}{2}), (e_4, \emptyset, \pi), (e_3, \emptyset, \pi), (e_2, \emptyset, \tfrac{\pi}{2}))$$
$$H(f_1) = ((e_1, 00, \tfrac{3\pi}{2}), (e_2, \emptyset, \tfrac{\pi}{2}), (e_6, 00, \pi))$$
$$H(f_2) = ((e_5, 000, \tfrac{\pi}{2}), (e_6, 11, \tfrac{\pi}{2}), (e_3, \emptyset, \pi), (e_4, \emptyset, \tfrac{\pi}{2}))$$

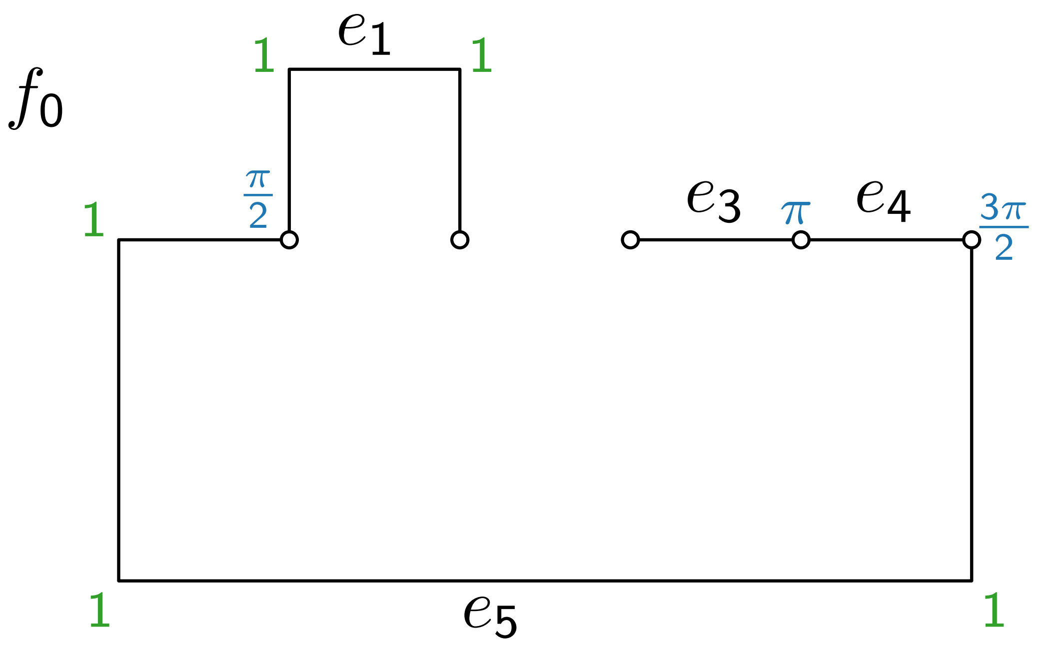




Orthogonal Representation – Example

$$H(f_0) = ((e_1, 11, \tfrac{\pi}{2}), (e_5, 111, \tfrac{3\pi}{2}), (e_4, \emptyset, \pi), (e_3, \emptyset, \pi), (e_2, \emptyset, \tfrac{\pi}{2}))$$
$$H(f_1) = ((e_1, 00, \tfrac{3\pi}{2}), (e_2, \emptyset, \tfrac{\pi}{2}), (e_6, 00, \pi))$$
$$H(f_2) = ((e_5, 000, \tfrac{\pi}{2}), (e_6, 11, \tfrac{\pi}{2}), (e_3, \emptyset, \pi), (e_4, \emptyset, \tfrac{\pi}{2}))$$



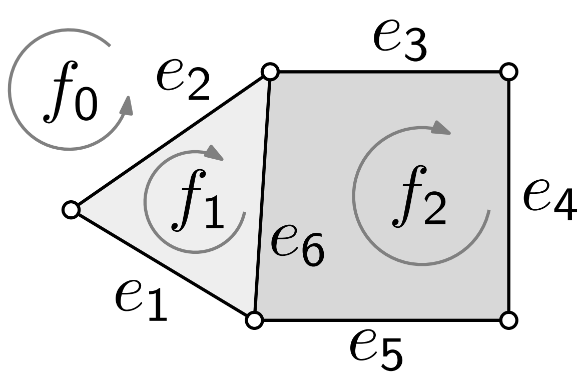


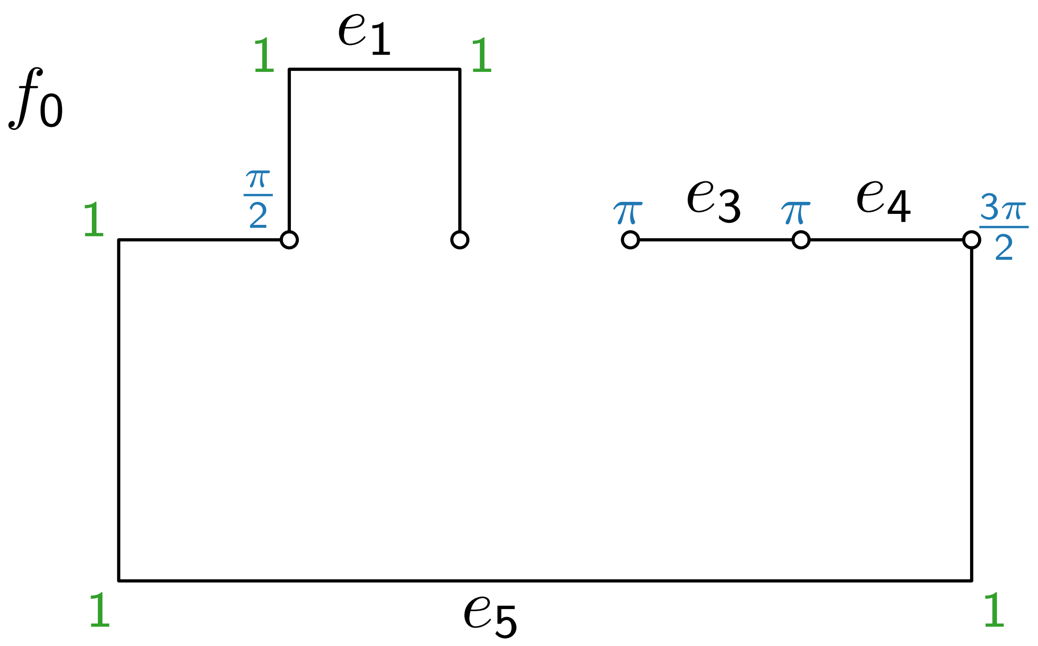
Orthogonal Representation – Example

$$H(f_0) = ((e_1, 11, \tfrac{\pi}{2}), (e_5, 111, \tfrac{3\pi}{2}), (e_4, \emptyset, \pi), (e_3, \emptyset, \pi), (e_2, \emptyset, \tfrac{\pi}{2}))$$

$$H(f_1) = ((e_1, 00, \tfrac{3\pi}{2}), (e_2, \emptyset, \tfrac{\pi}{2}), (e_6, 00, \pi))$$

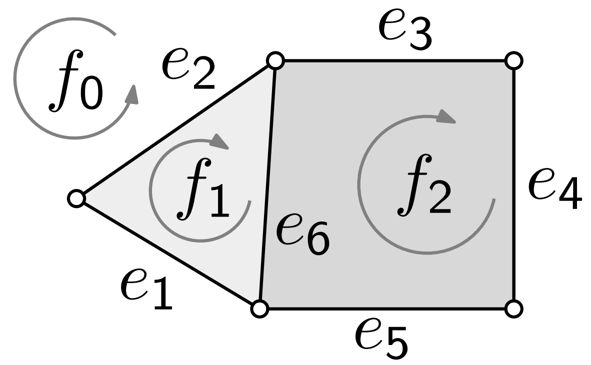
$$H(f_2) = ((e_5, 000, \tfrac{\pi}{2}), (e_6, 11, \tfrac{\pi}{2}), (e_3, \emptyset, \pi), (e_4, \emptyset, \tfrac{\pi}{2}))$$

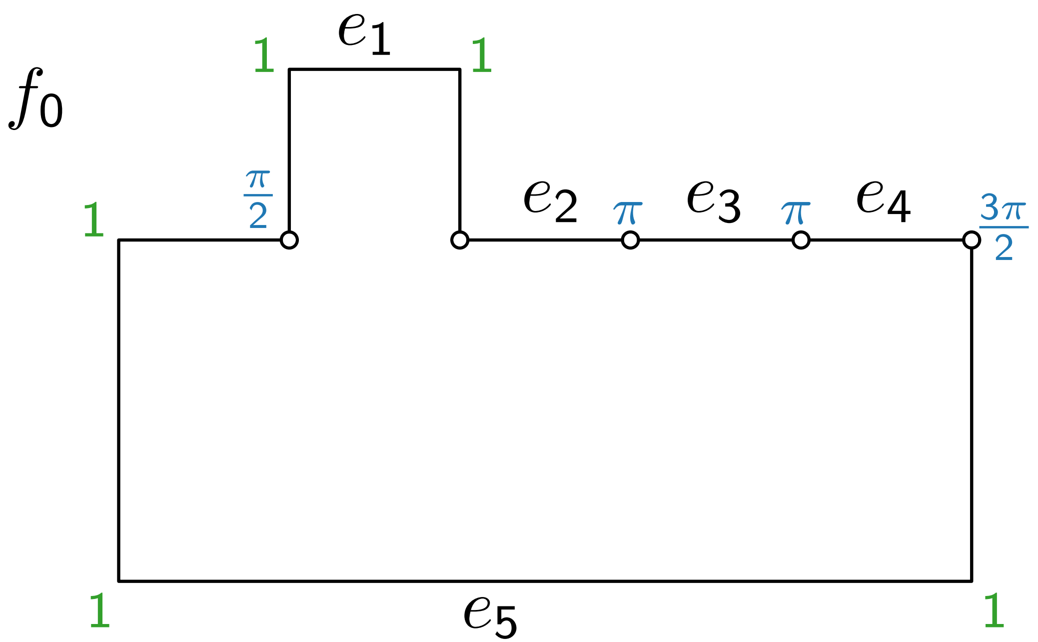




Orthogonal Representation – Example

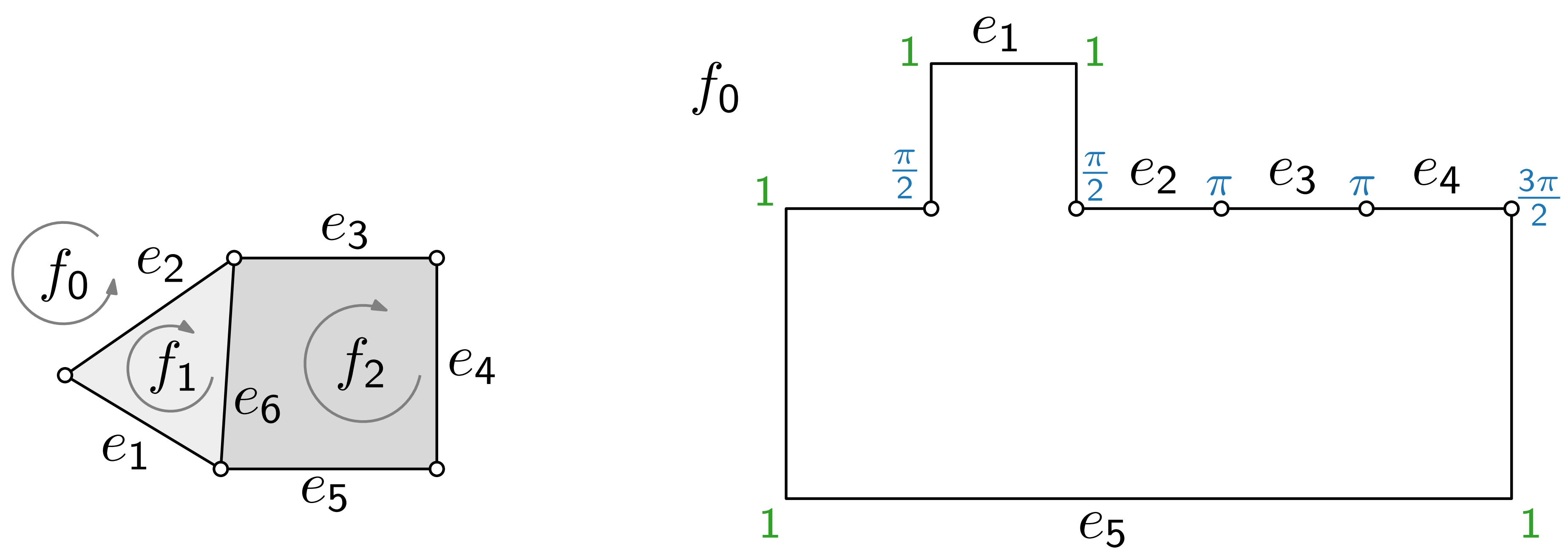
$$H(f_0) = ((e_1, 11, \tfrac{\pi}{2}), (e_5, 111, \tfrac{3\pi}{2}), (e_4, \emptyset, \pi), (e_3, \emptyset, \pi), (e_2, \emptyset, \tfrac{\pi}{2}))$$
$$H(f_1) = ((e_1, 00, \tfrac{3\pi}{2}), (e_2, \emptyset, \tfrac{\pi}{2}), (e_6, 00, \pi))$$
$$H(f_2) = ((e_5, 000, \tfrac{\pi}{2}), (e_6, 11, \tfrac{\pi}{2}), (e_3, \emptyset, \pi), (e_4, \emptyset, \tfrac{\pi}{2}))$$





Orthogonal Representation – Example

$$H(f_0) = ((e_1, 11, \tfrac{\pi}{2}), (e_5, 111, \tfrac{3\pi}{2}), (e_4, \emptyset, \pi), (e_3, \emptyset, \pi), (e_2, \emptyset, \tfrac{\pi}{2}))$$
$$H(f_1) = ((e_1, 00, \tfrac{3\pi}{2}), (e_2, \emptyset, \tfrac{\pi}{2}), (e_6, 00, \pi))$$
$$H(f_2) = ((e_5, 000, \tfrac{\pi}{2}), (e_6, 11, \tfrac{\pi}{2}), (e_3, \emptyset, \pi), (e_4, \emptyset, \tfrac{\pi}{2}))$$

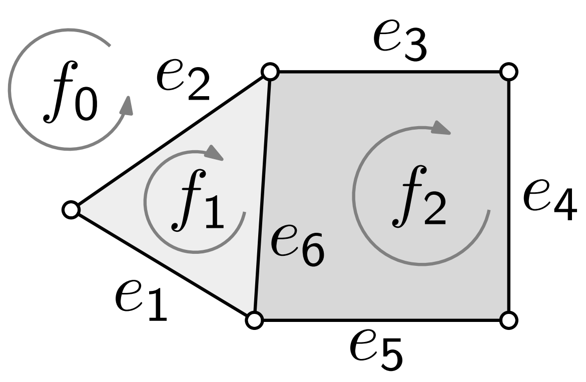


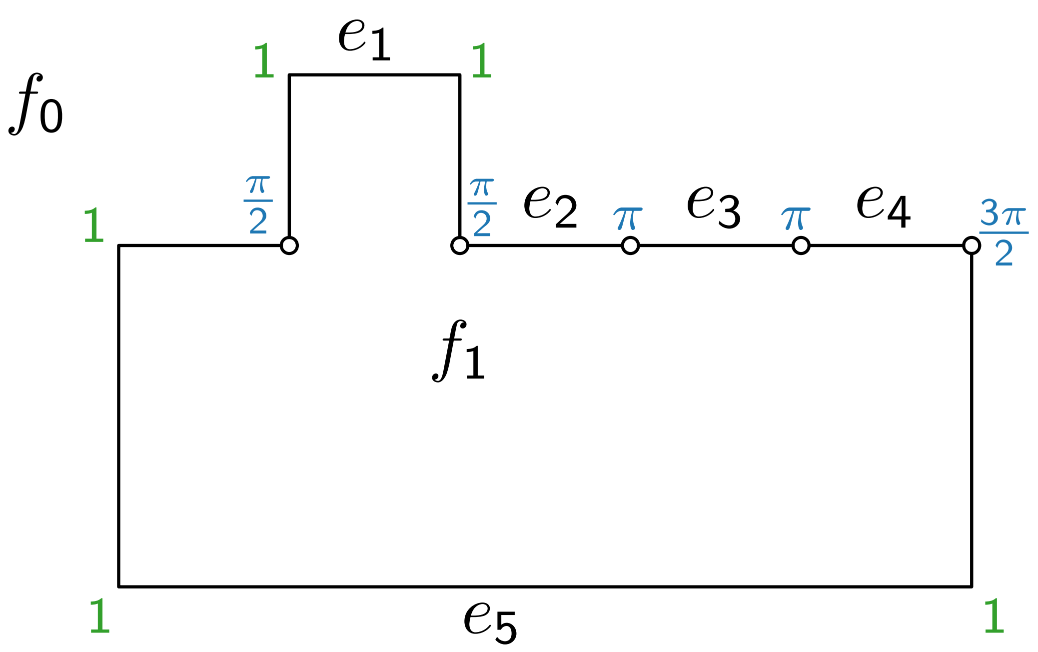
Orthogonal Representation – Example

$$H(f_0) = ((e_1, 11, \tfrac{\pi}{2}), (e_5, 111, \tfrac{3\pi}{2}), (e_4, \emptyset, \pi), (e_3, \emptyset, \pi), (e_2, \emptyset, \tfrac{\pi}{2}))$$

$$H(f_1) = ((e_1, 00, \tfrac{3\pi}{2}), (e_2, \emptyset, \tfrac{\pi}{2}), (e_6, 00, \pi))$$

$$H(f_2) = ((e_5, 000, \tfrac{\pi}{2}), (e_6, 11, \tfrac{\pi}{2}), (e_3, \emptyset, \pi), (e_4, \emptyset, \tfrac{\pi}{2}))$$





Orthogonal Representation – Example

$$H(f_0) = ((e_1, 11, \tfrac{\pi}{2}), (e_5, 111, \tfrac{3\pi}{2}), (e_4, \emptyset, \pi), (e_3, \emptyset, \pi), (e_2, \emptyset, \tfrac{\pi}{2}))$$
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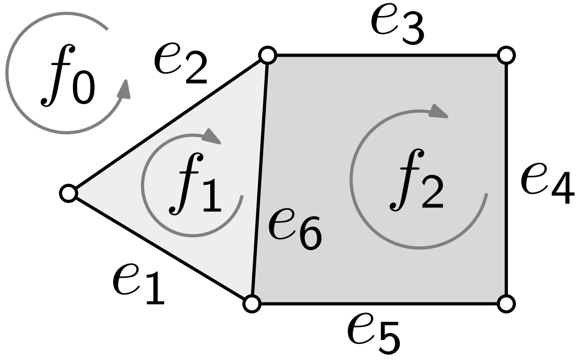
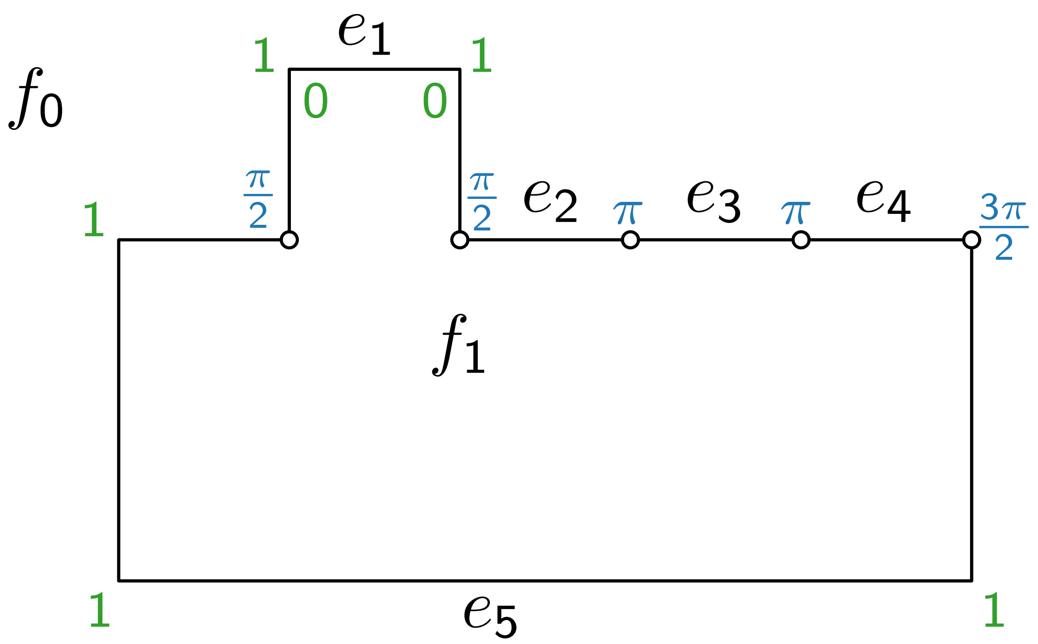
$$H(f_2) = ((e_5, 000, \tfrac{\pi}{2}), (e_6, 11, \tfrac{\pi}{2}), (e_3, \emptyset, \pi), (e_4, \emptyset, \tfrac{\pi}{2}))$$





Orthogonal Representation – Example

$$H(f_0) = ((e_1, 11, \tfrac{\pi}{2}), (e_5, 111, \tfrac{3\pi}{2}), (e_4, \emptyset, \pi), (e_3, \emptyset, \pi), (e_2, \emptyset, \tfrac{\pi}{2}))$$
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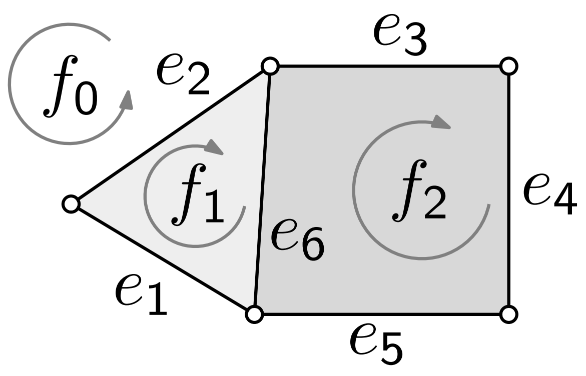

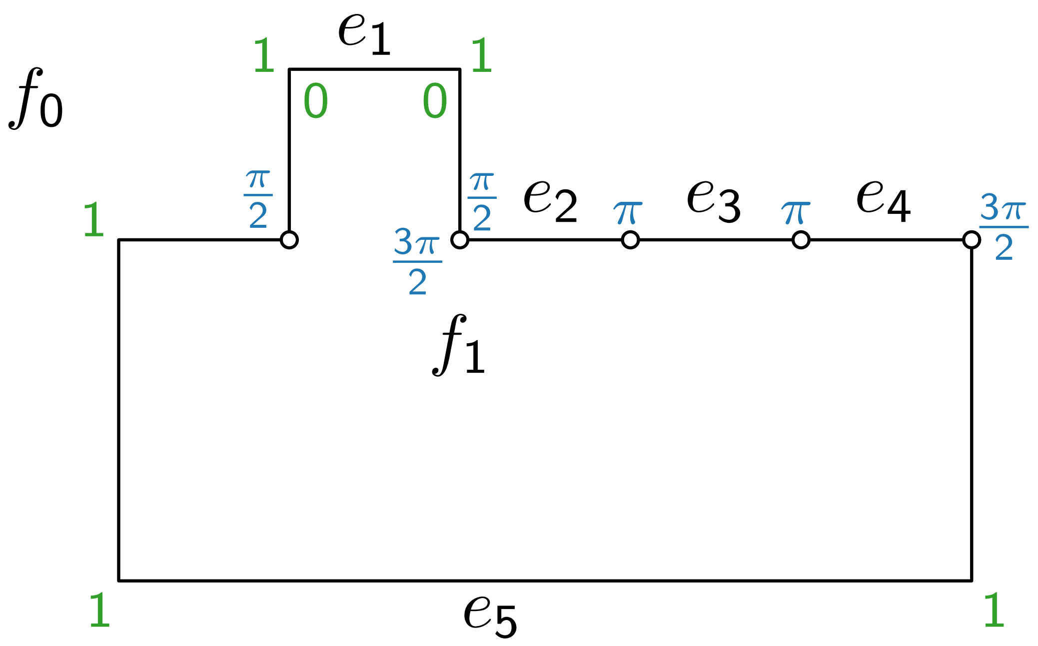

Orthogonal Representation – Example

$$H(f_0) = ((e_1, 11, \tfrac{\pi}{2}), (e_5, 111, \tfrac{3\pi}{2}), (e_4, \emptyset, \pi), (e_3, \emptyset, \pi), (e_2, \emptyset, \tfrac{\pi}{2}))$$
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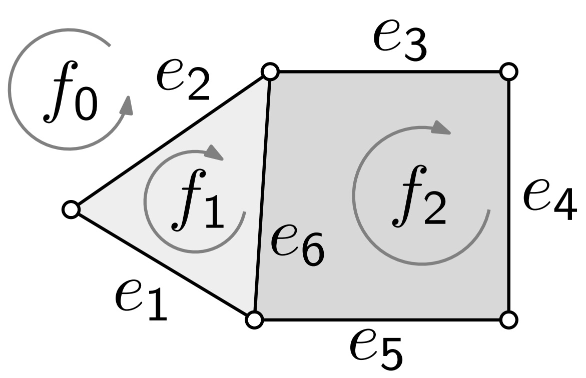

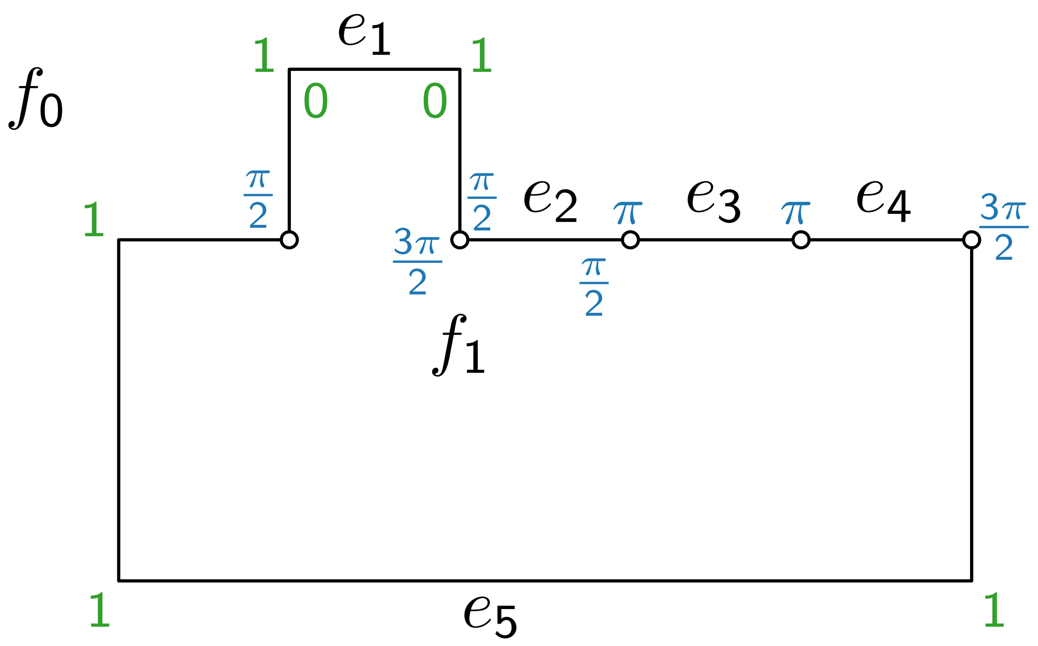

Orthogonal Representation – Example

$$H(f_0) = ((e_1, 11, \tfrac{\pi}{2}), (e_5, 111, \tfrac{3\pi}{2}), (e_4, \emptyset, \pi), (e_3, \emptyset, \pi), (e_2, \emptyset, \tfrac{\pi}{2}))$$
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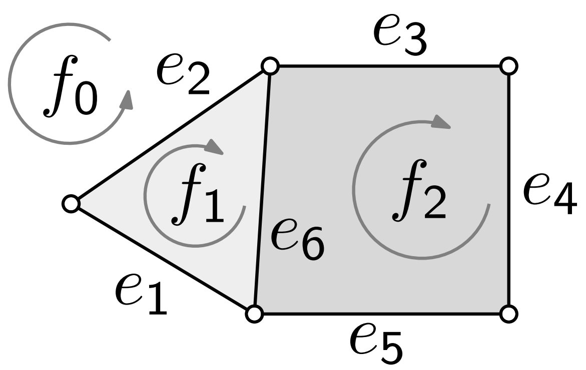
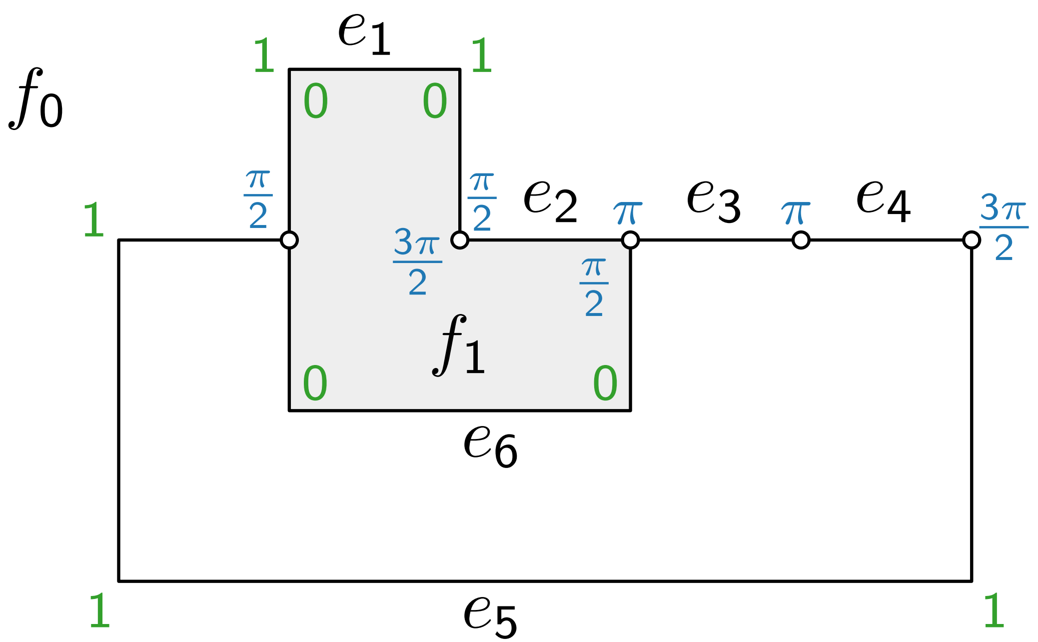
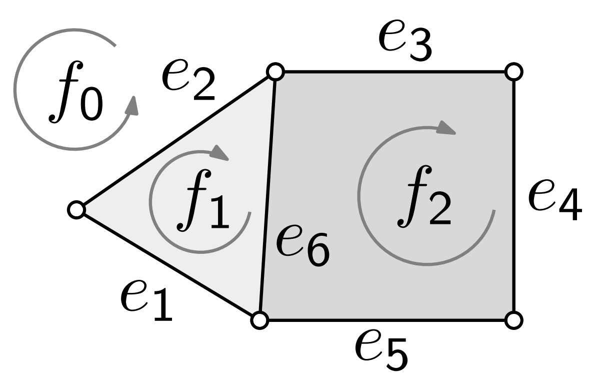
$$H(f_2) = ((e_5, 000, \tfrac{\pi}{2}), (e_6, 11, \tfrac{\pi}{2}), (e_3, \emptyset, \pi), (e_4, \emptyset, \tfrac{\pi}{2}))$$

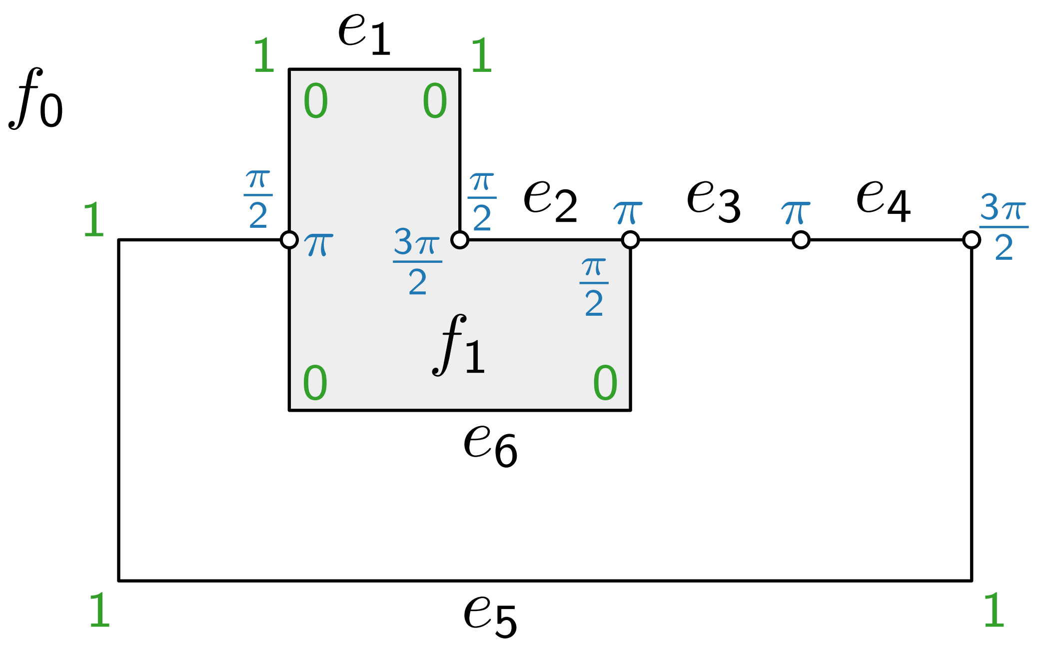




Orthogonal Representation – Example

$$H(f_0) = ((e_1, 11, \tfrac{\pi}{2}), (e_5, 111, \tfrac{3\pi}{2}), (e_4, \emptyset, \pi), (e_3, \emptyset, \pi), (e_2, \emptyset, \tfrac{\pi}{2}))$$
$$H(f_1) = ((e_1, 00, \tfrac{3\pi}{2}), (e_2, \emptyset, \tfrac{\pi}{2}), (e_6, 00, \pi))$$
$$H(f_2) = ((e_5, 000, \tfrac{\pi}{2}), (e_6, 11, \tfrac{\pi}{2}), (e_3, \emptyset, \pi), (e_4, \emptyset, \tfrac{\pi}{2}))$$



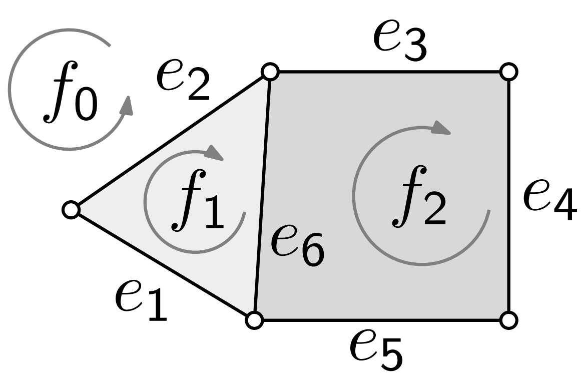


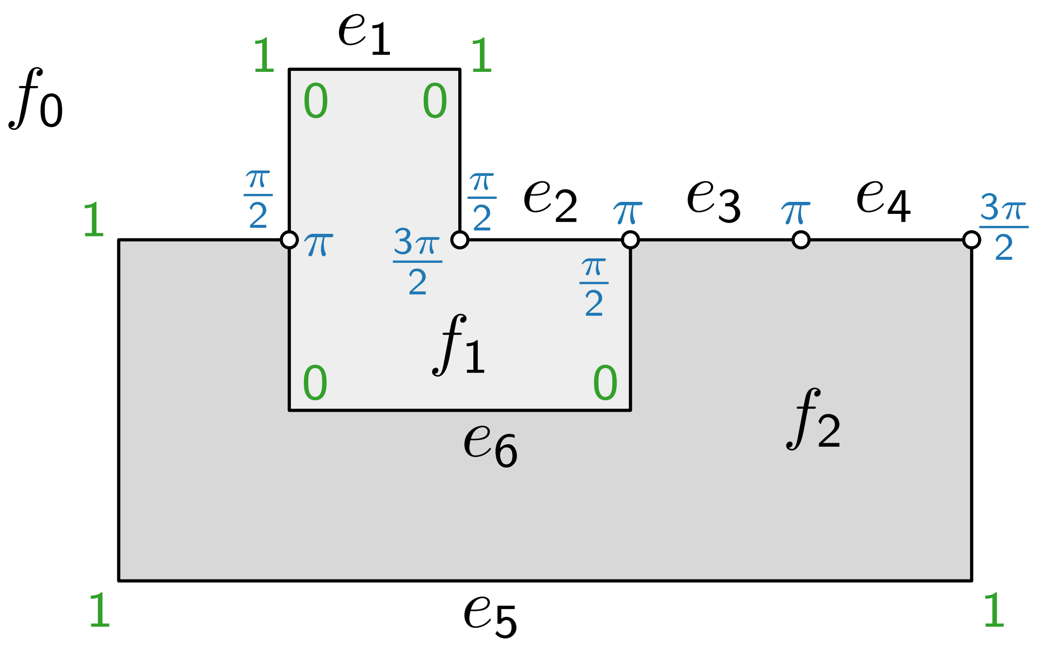
Orthogonal Representation – Example

$$H(f_0) = ((e_1, 11, \tfrac{\pi}{2}), (e_5, 111, \tfrac{3\pi}{2}), (e_4, \emptyset, \pi), (e_3, \emptyset, \pi), (e_2, \emptyset, \tfrac{\pi}{2}))$$

$$H(f_1) = ((e_1, 00, \tfrac{3\pi}{2}), (e_2, \emptyset, \tfrac{\pi}{2}), (e_6, 00, \pi))$$

$$H(f_2) = ((e_5, 000, \tfrac{\pi}{2}), (e_6, 11, \tfrac{\pi}{2}), (e_3, \emptyset, \pi), (e_4, \emptyset, \tfrac{\pi}{2}))$$





Orthogonal Representation – Example

$$H(f_0) = ((e_1, 11, \tfrac{\pi}{2}), (e_5, 111, \tfrac{3\pi}{2}), (e_4, \emptyset, \pi), (e_3, \emptyset, \pi), (e_2, \emptyset, \tfrac{\pi}{2}))$$
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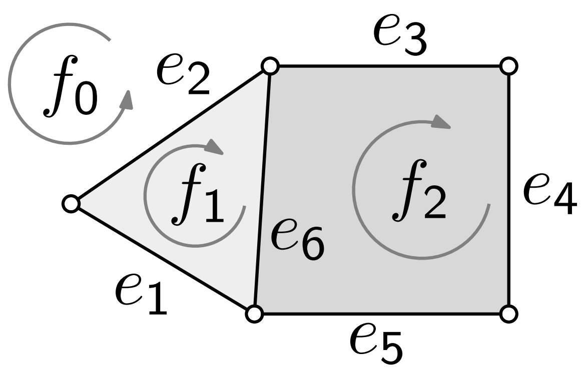

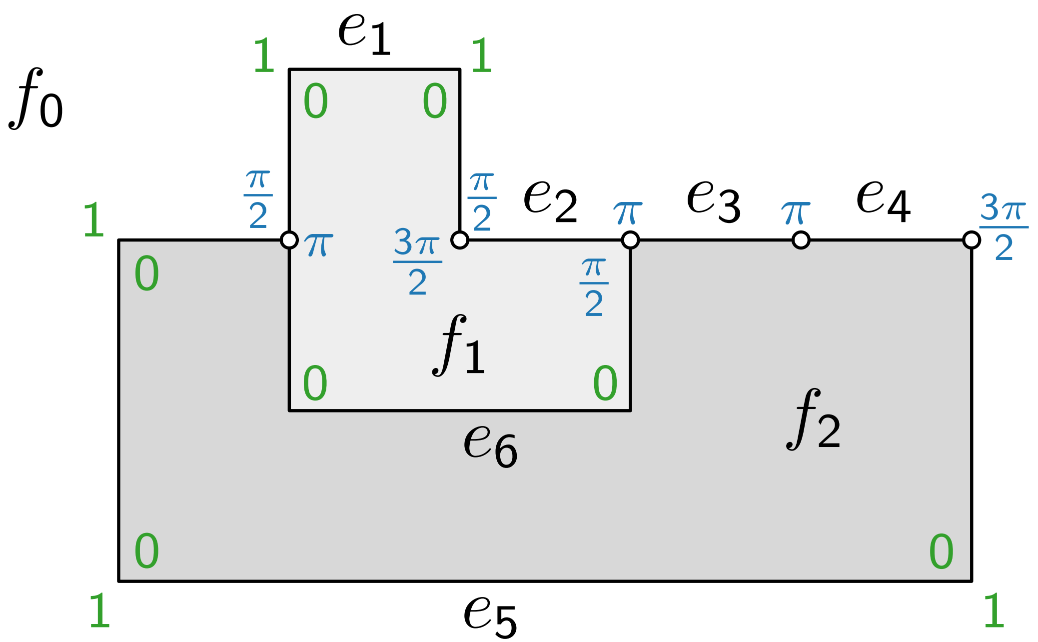

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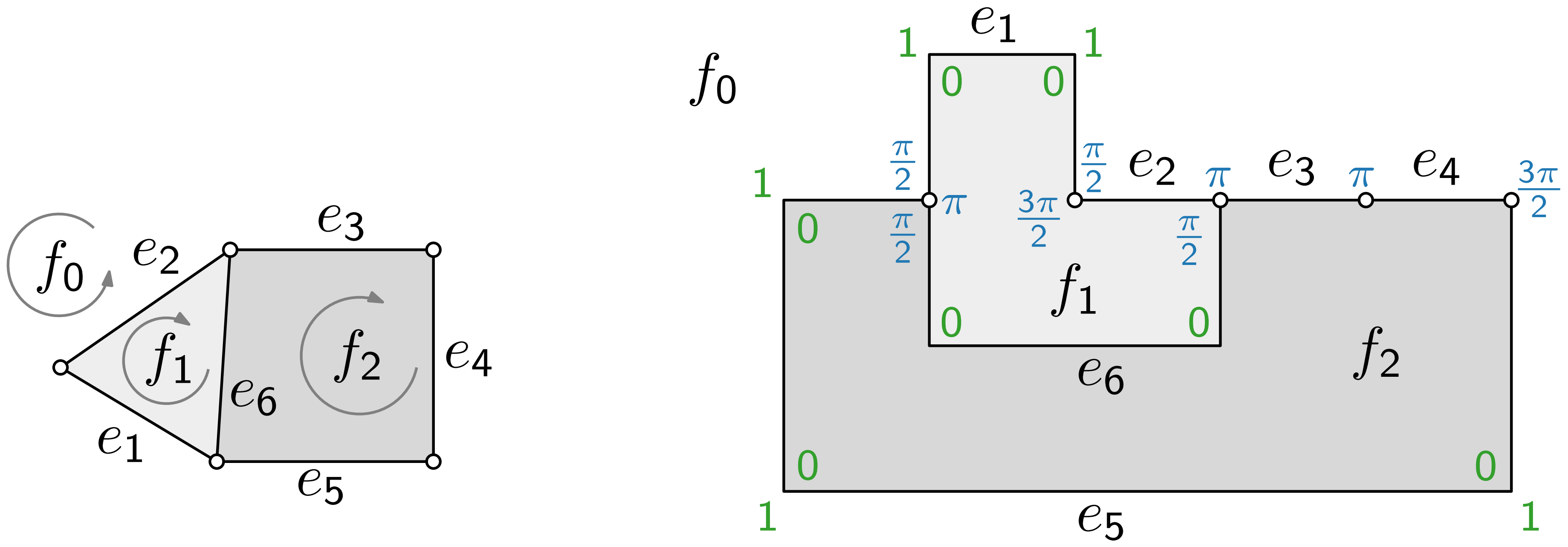

Orthogonal Representation – Example

$$H(f_0) = ((e_1, 11, \tfrac{\pi}{2}), (e_5, 111, \tfrac{3\pi}{2}), (e_4, \emptyset, \pi), (e_3, \emptyset, \pi), (e_2, \emptyset, \tfrac{\pi}{2}))$$
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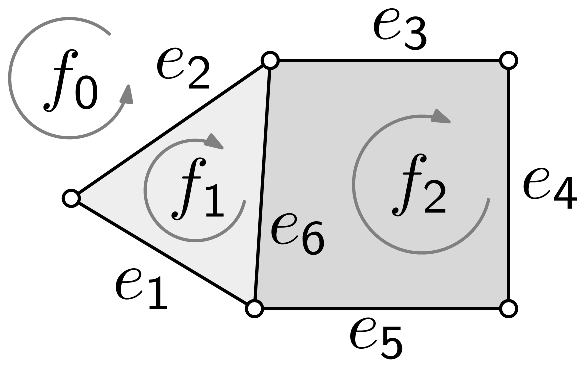

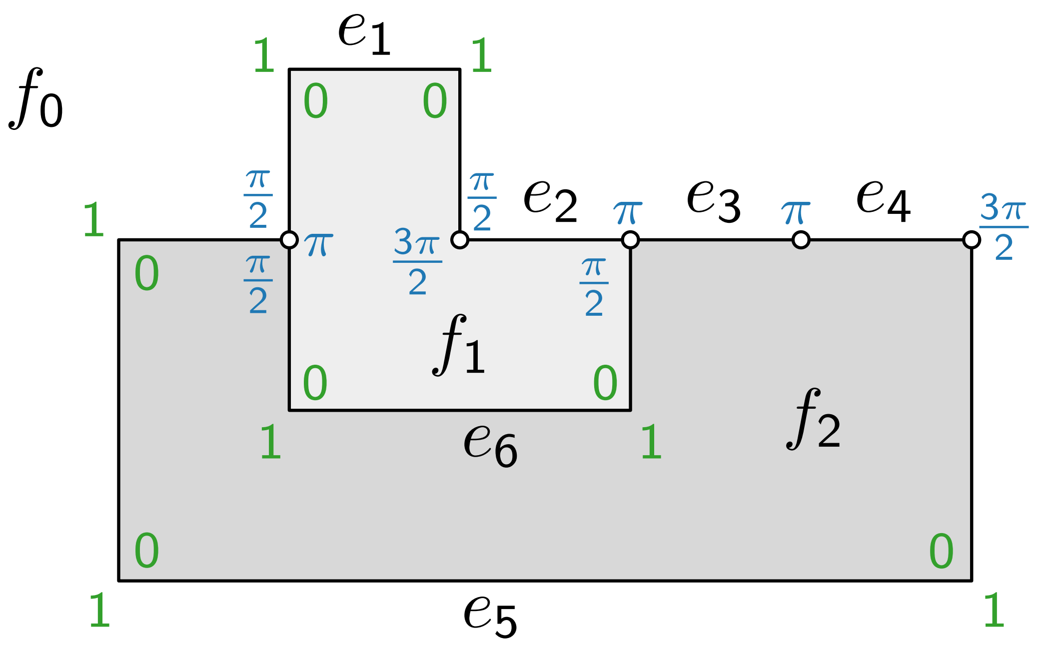

Orthogonal Representation – Example

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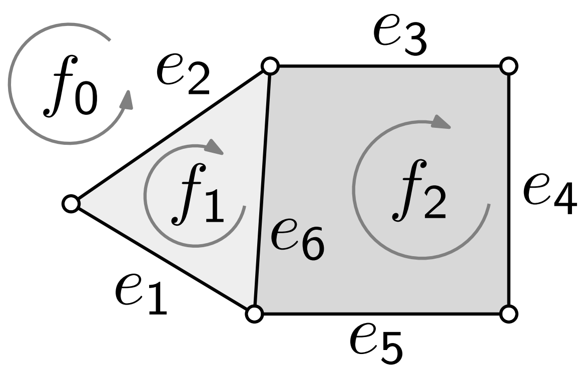

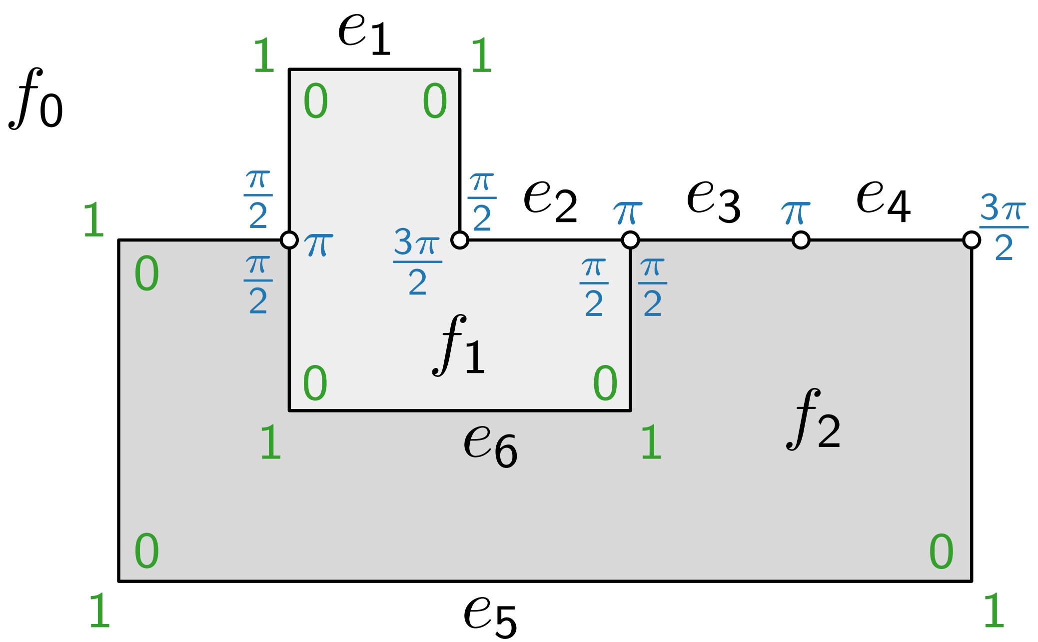

Orthogonal Representation – Example

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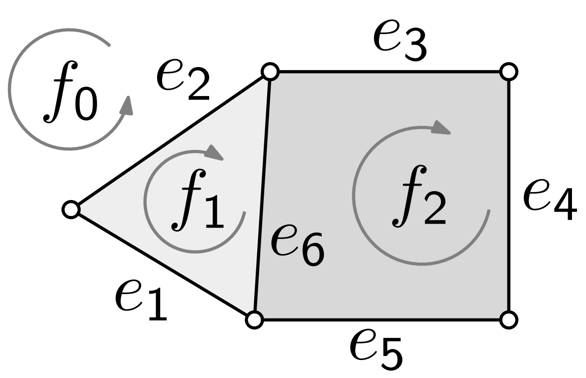

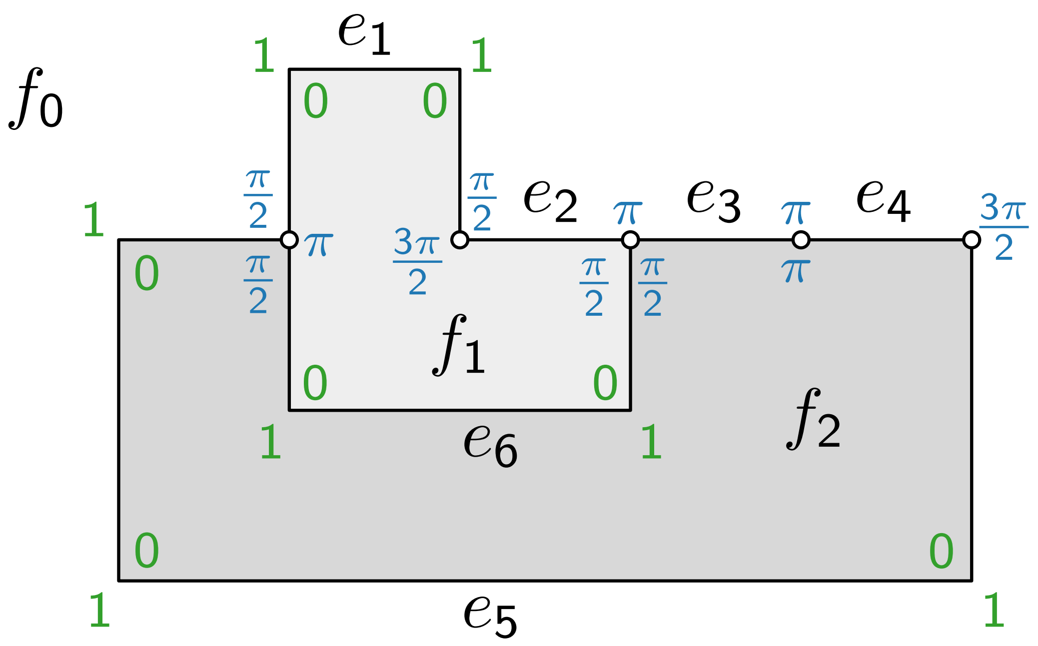

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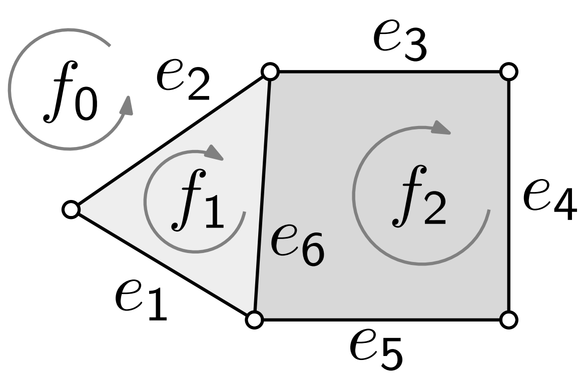

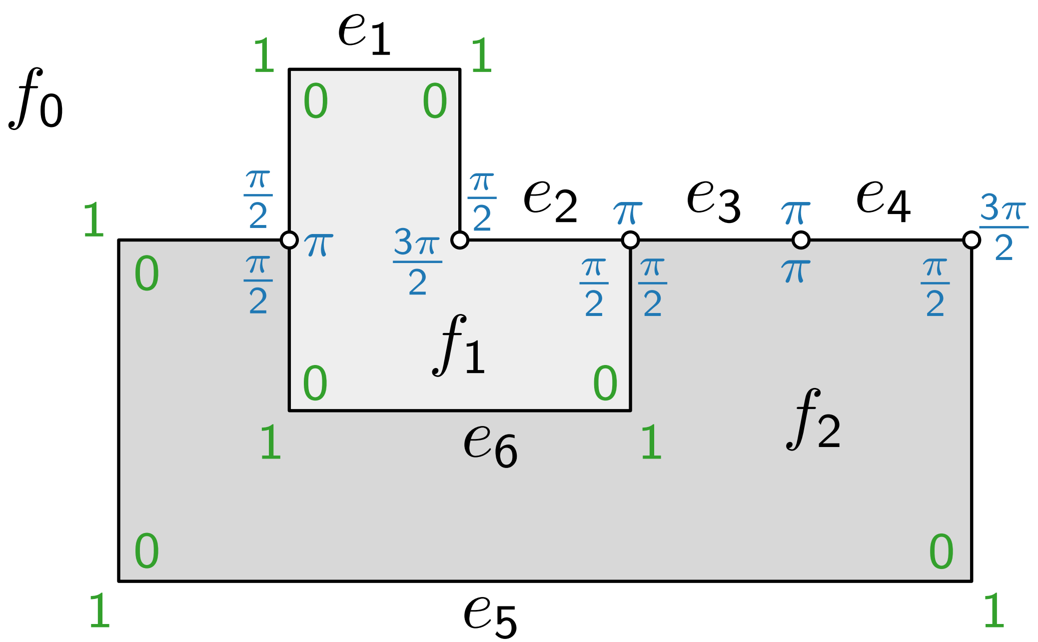

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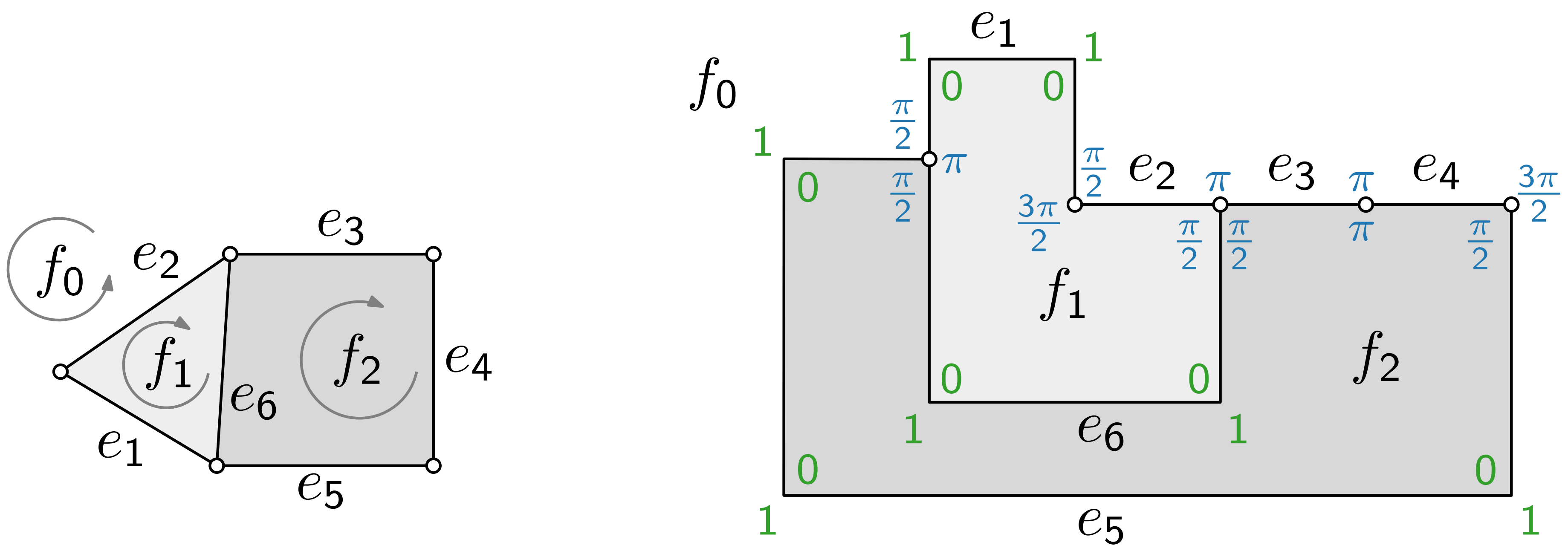
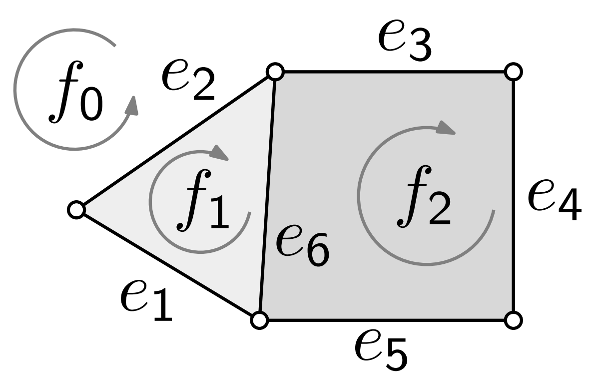
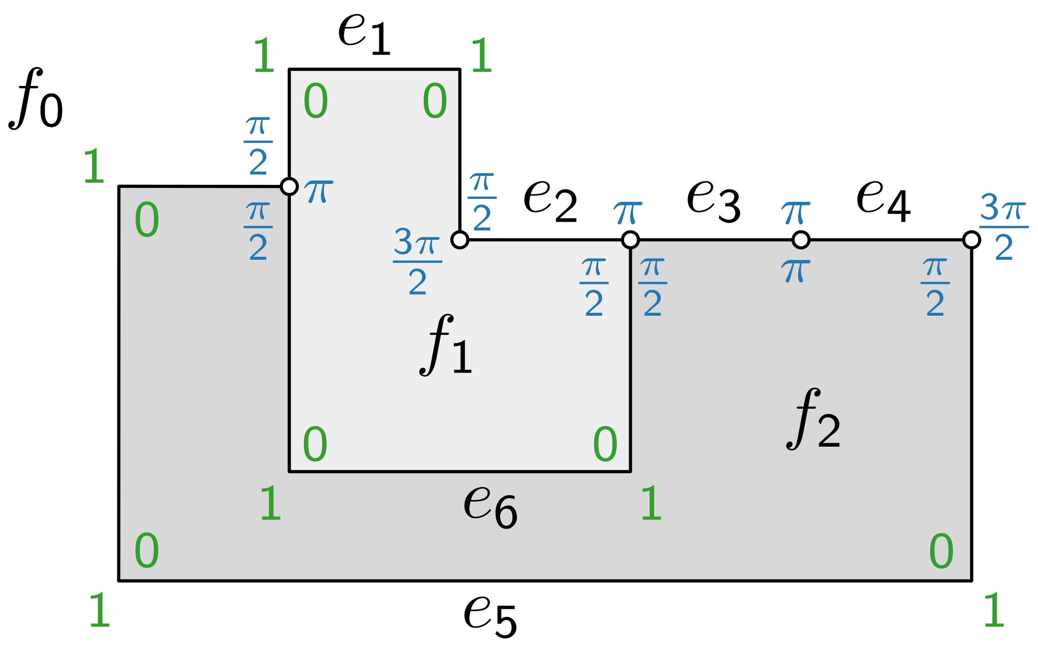
$$H(f_2) = ((e_5, 000, \tfrac{\pi}{2}), (e_6, 11, \tfrac{\pi}{2}), (e_3, \emptyset, \pi), (e_4, \emptyset, \tfrac{\pi}{2}))$$



Orthogonal Representation – Example

$$H(f_0) = ((e_1, 11, \tfrac{\pi}{2}), (e_5, 111, \tfrac{3\pi}{2}), (e_4, \emptyset, \pi), (e_3, \emptyset, \pi), (e_2, \emptyset, \tfrac{\pi}{2}))$$
$$H(f_1) = ((e_1, 00, \tfrac{3\pi}{2}), (e_2, \emptyset, \tfrac{\pi}{2}), (e_6, 00, \pi))$$
$$H(f_2) = ((e_5, 000, \tfrac{\pi}{2}), (e_6, 11, \tfrac{\pi}{2}), (e_3, \emptyset, \pi), (e_4, \emptyset, \tfrac{\pi}{2}))$$

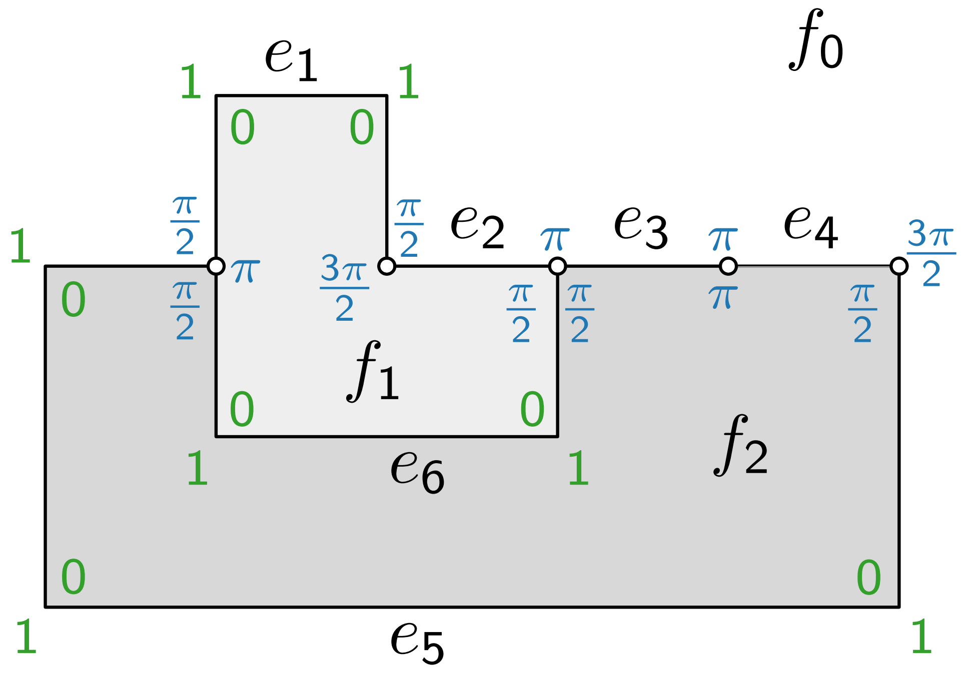




Concrete coordinates are not fixed yet!

Correctness of an Orthogonal Representation

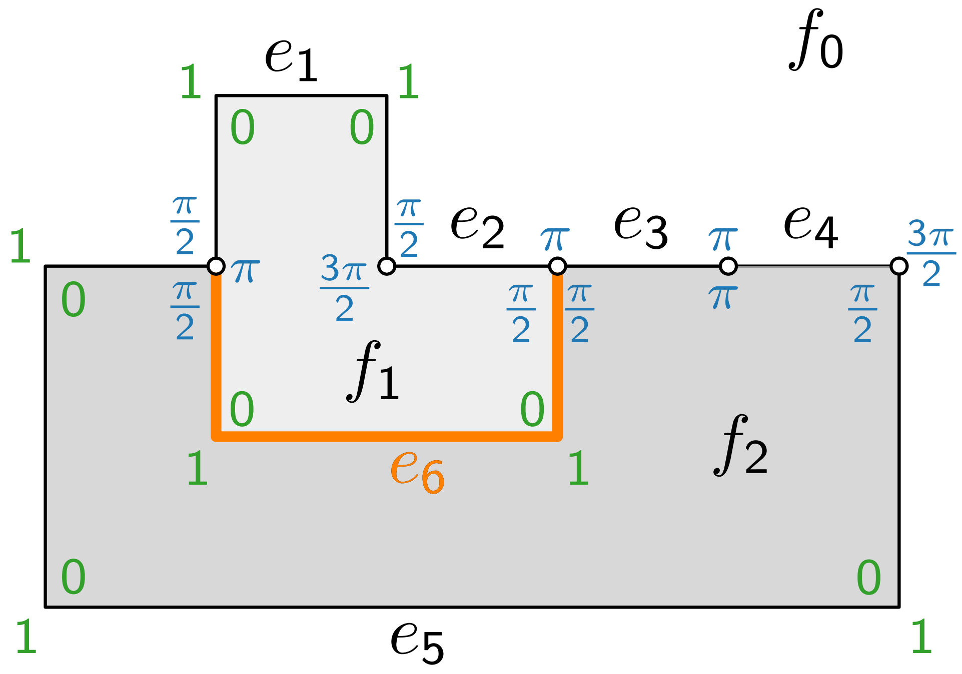
(H1) $H(G)$ corresponds to F, f_0.



Correctness of an Orthogonal Representation

(H1) $H(G)$ corresponds to F, f_0.

(H2) For each **edge** $\{u, v\}$ shared by faces f and g with $((u,v), \delta_1, \alpha_1) \in H(f)$ and $((v,u), \delta_2, \alpha_2) \in H(g)$



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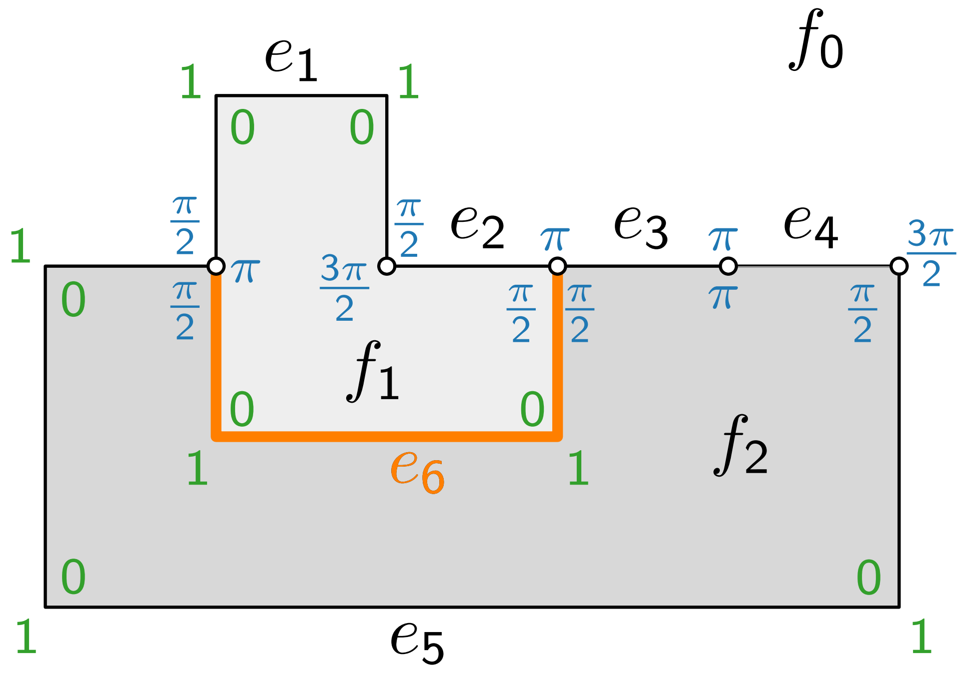

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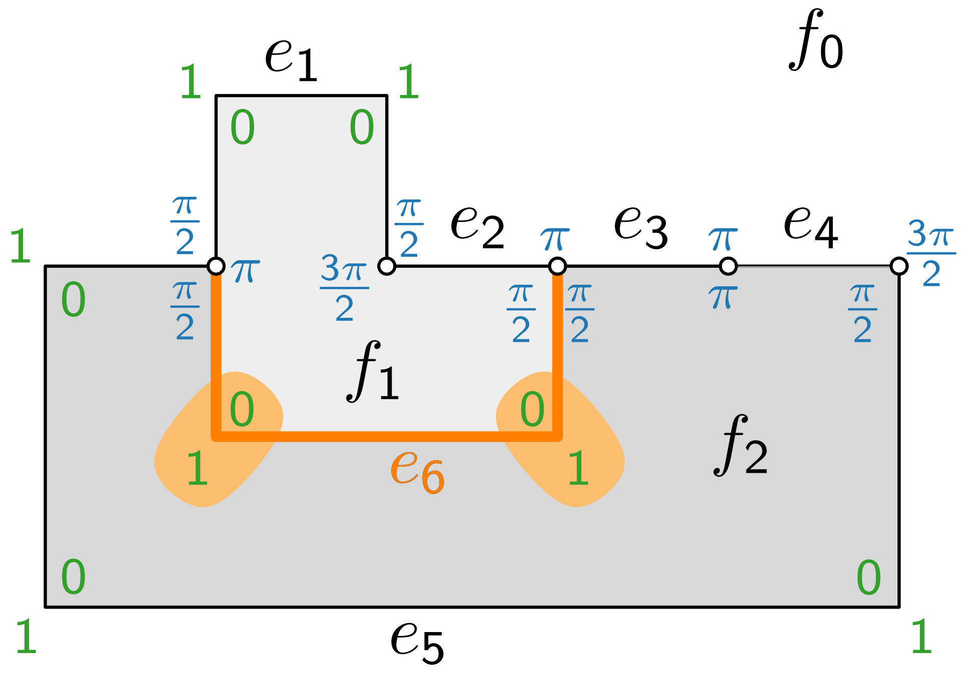

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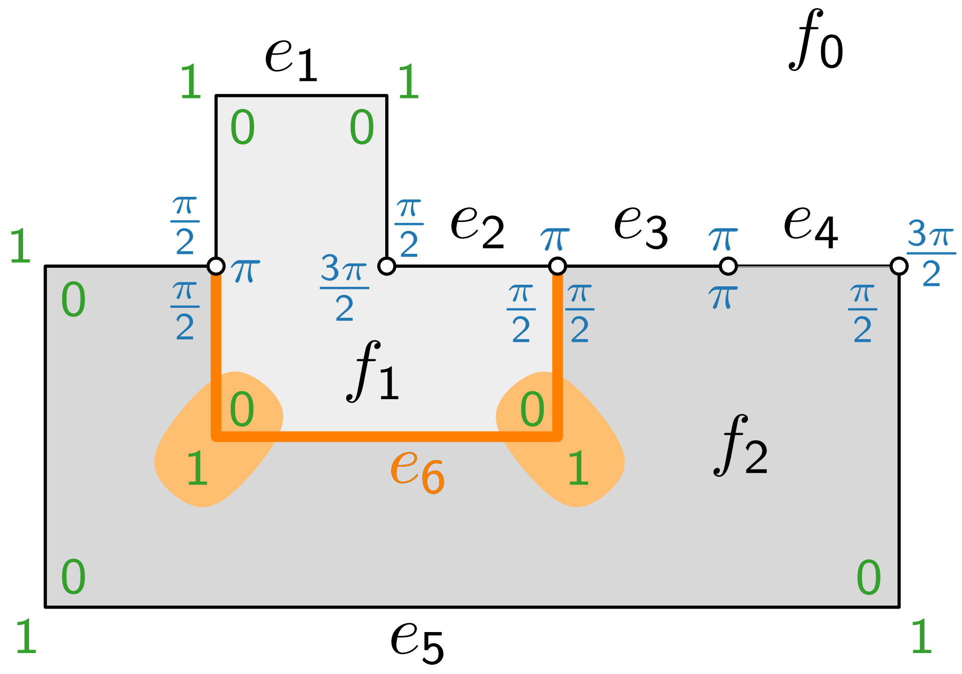
(H3) Let $|\delta|_0$ (resp. $|\delta|_1$) be the number of zeros (resp. ones) in δ, and let $r = (e, \delta, \alpha)$.
Let $C(r) := |\delta|_0 - |\delta|_1 - \alpha/\frac{\pi}{2} + 2$.



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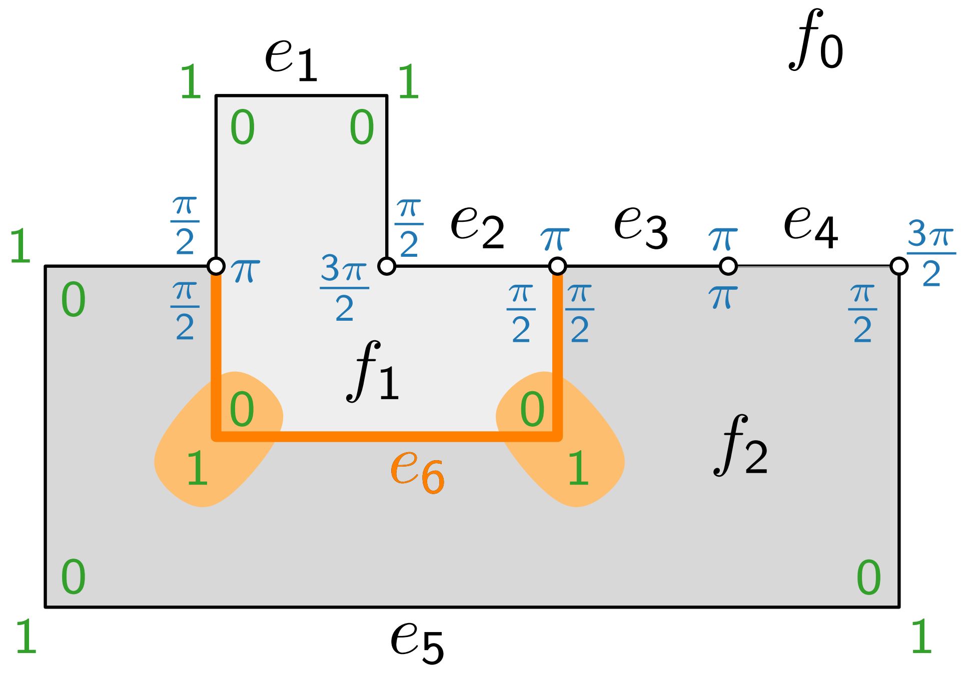
For each **face** f, it holds that:



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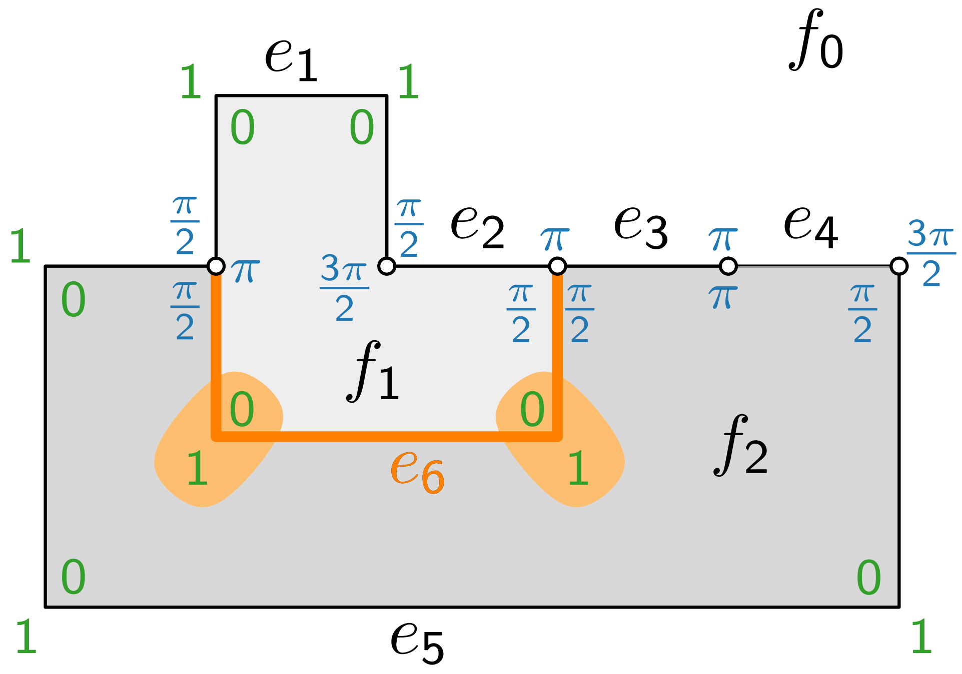
$$\sum_{r \in H(f)} C(r) = \begin{cases} -4 & \text{if } f = f_0 \\ +4 & \text{otherwise.} \end{cases}$$



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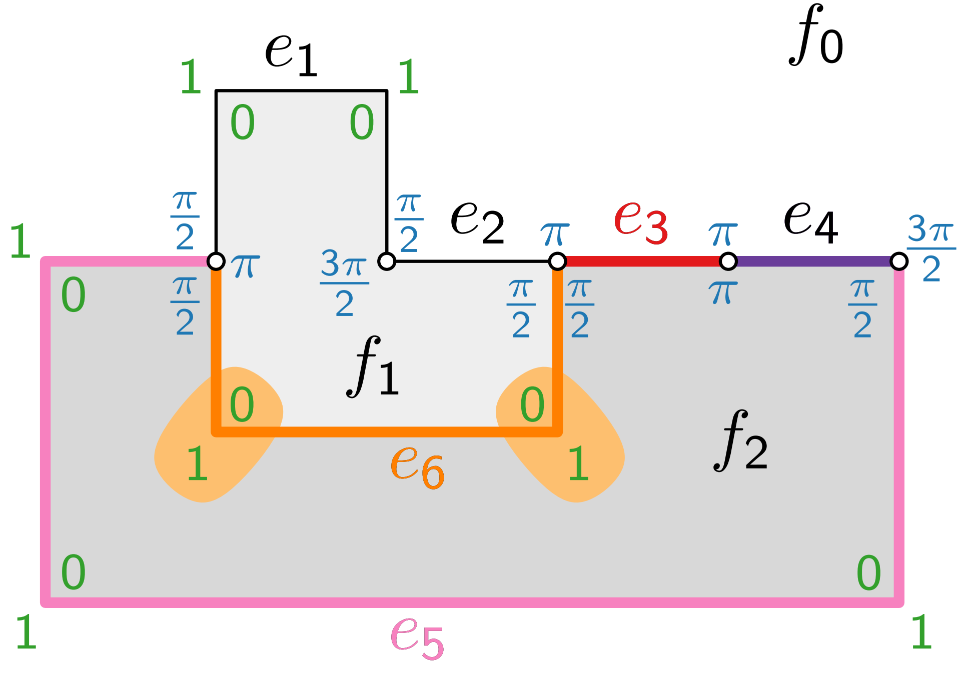

$C(e_3) = \quad - \quad - \quad + 2 =$

$C(e_4) = \quad - \quad - \quad + 2 =$

$C(e_5) = \quad - \quad - \quad + 2 =$

$C(e_6) = \quad - \quad - \quad + 2 =$

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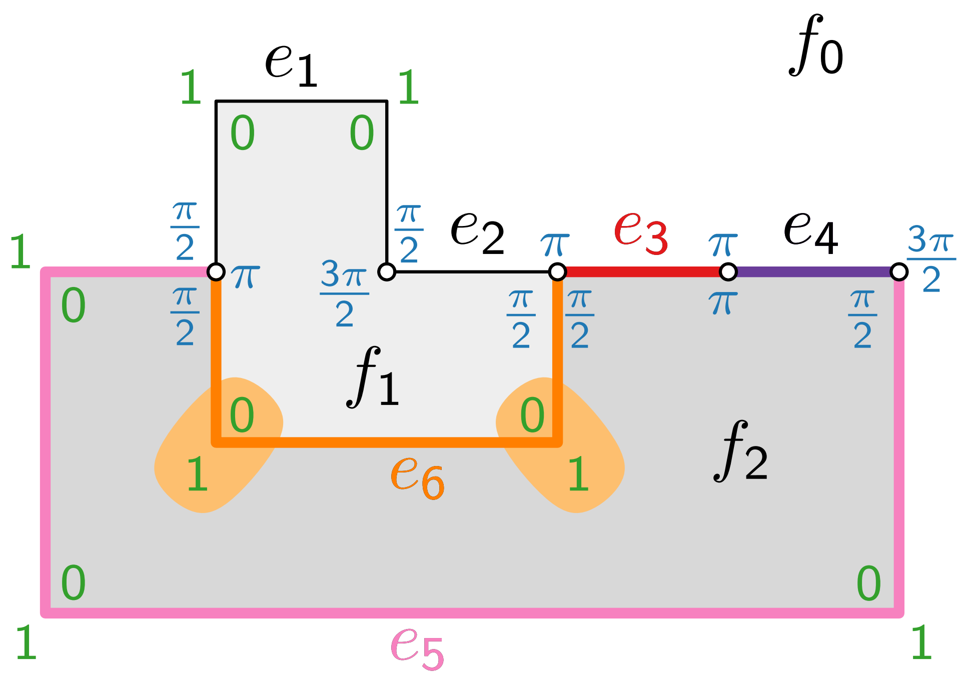

$C(e_3) = 0 - \quad - \quad + 2 =$

$C(e_4) = \quad - \quad - \quad + 2 =$

$C(e_5) = \quad - \quad - \quad + 2 =$

$C(e_6) = \quad - \quad - \quad + 2 =$

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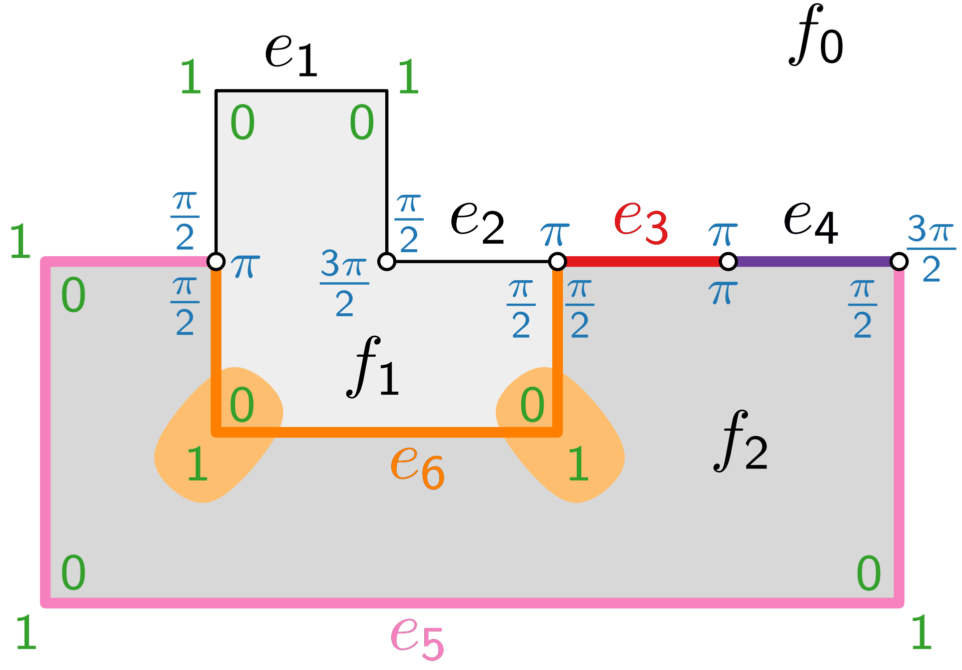

$C(e_3) = 0 - 0 - \quad + 2 =$

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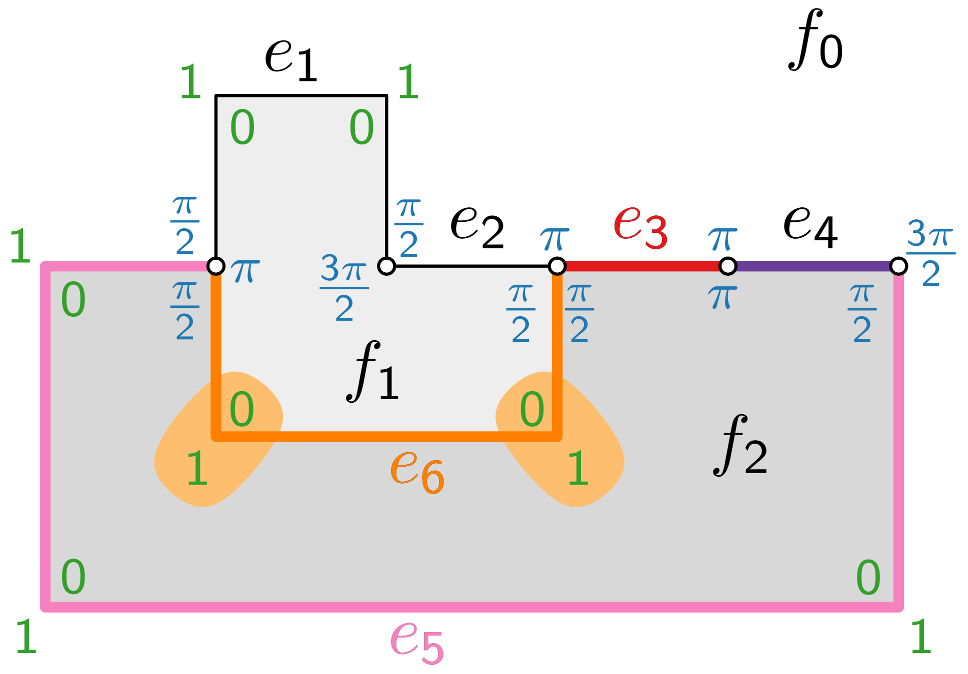

$C(e_3) = 0 - 0 - 2 + 2 =$

$C(e_4) = \quad - \quad - \quad + 2 =$

$C(e_5) = \quad - \quad - \quad + 2 =$

$C(e_6) = \quad - \quad - \quad + 2 =$

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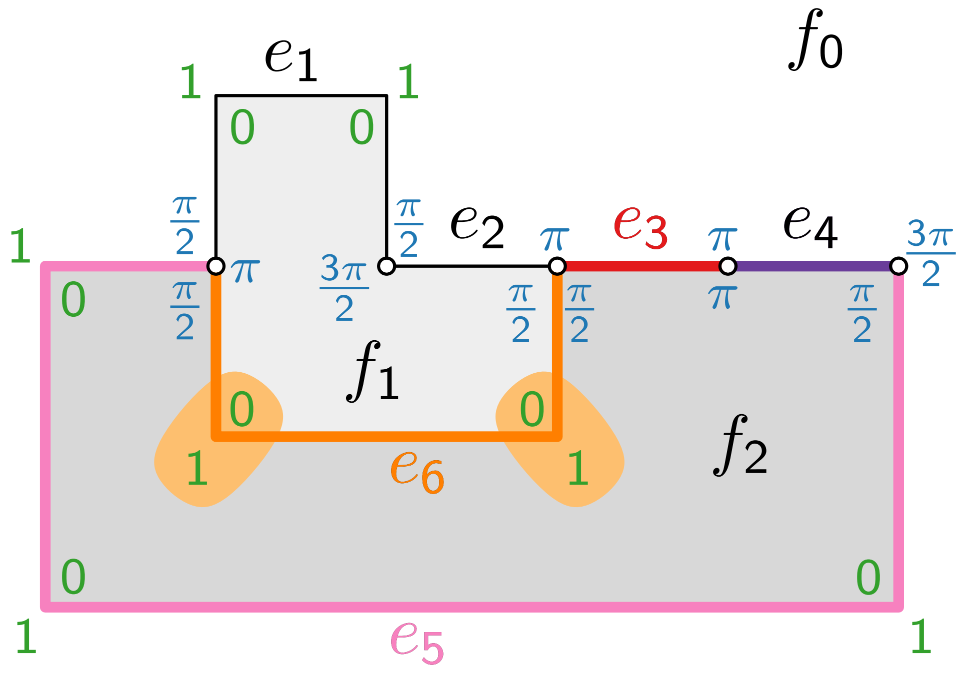

$C(e_3) = 0 - 0 - 2 + 2 = 0$

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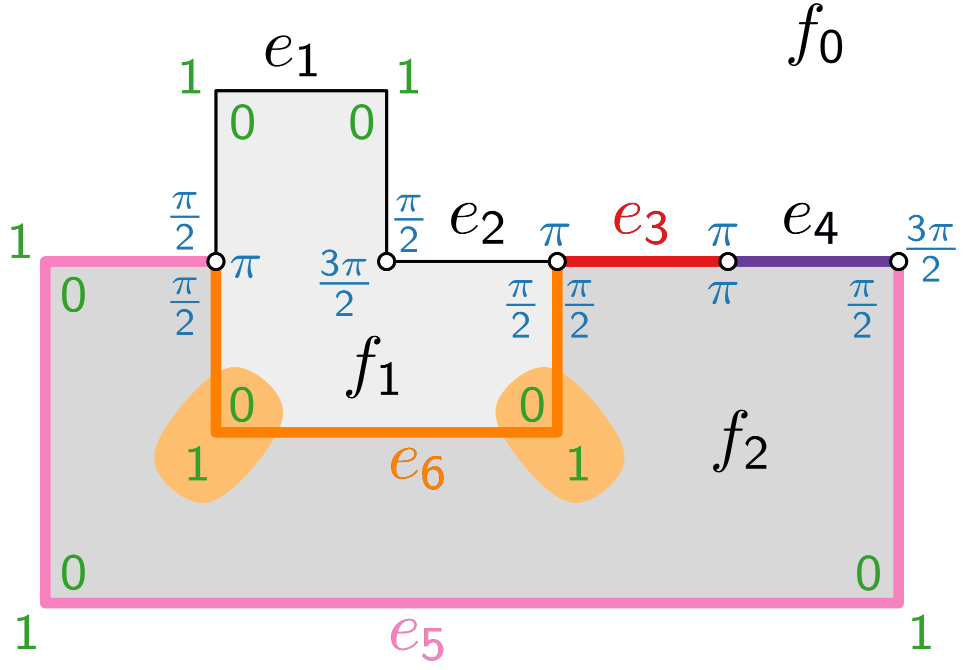

$C(e_3) = 0 - 0 - 2 + 2 = 0$

$C(e_4) = 0 - 0 - 1 + 2 =$

$C(e_5) = \quad - \quad - \quad + 2 =$

$C(e_6) = \quad - \quad - \quad + 2 =$

Correctness of an Orthogonal Representation

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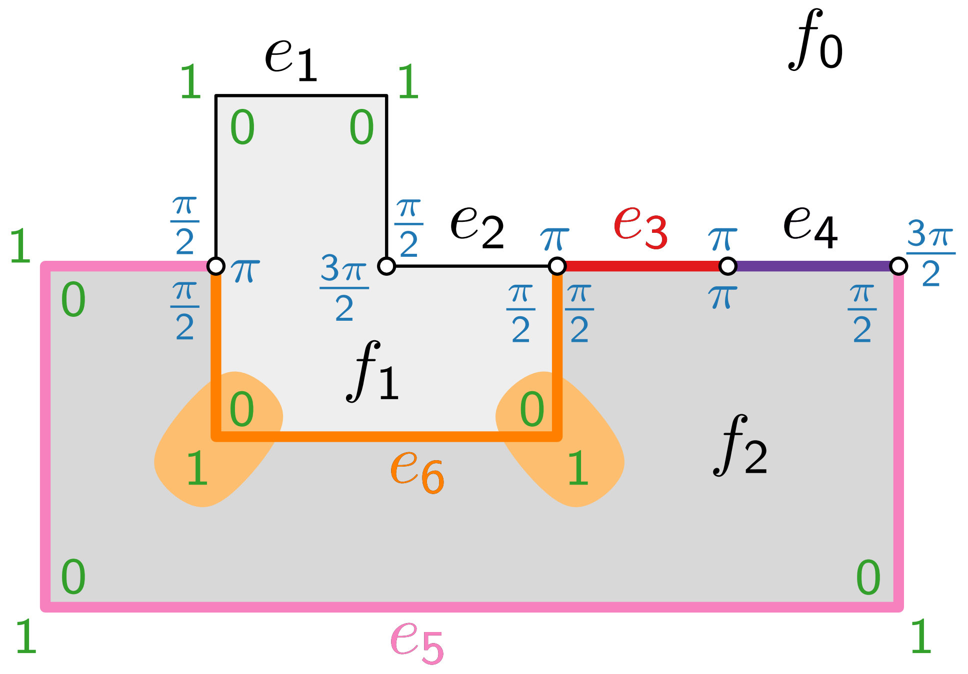

$C(e_3) = 0 - 0 - 2 + 2 = 0$

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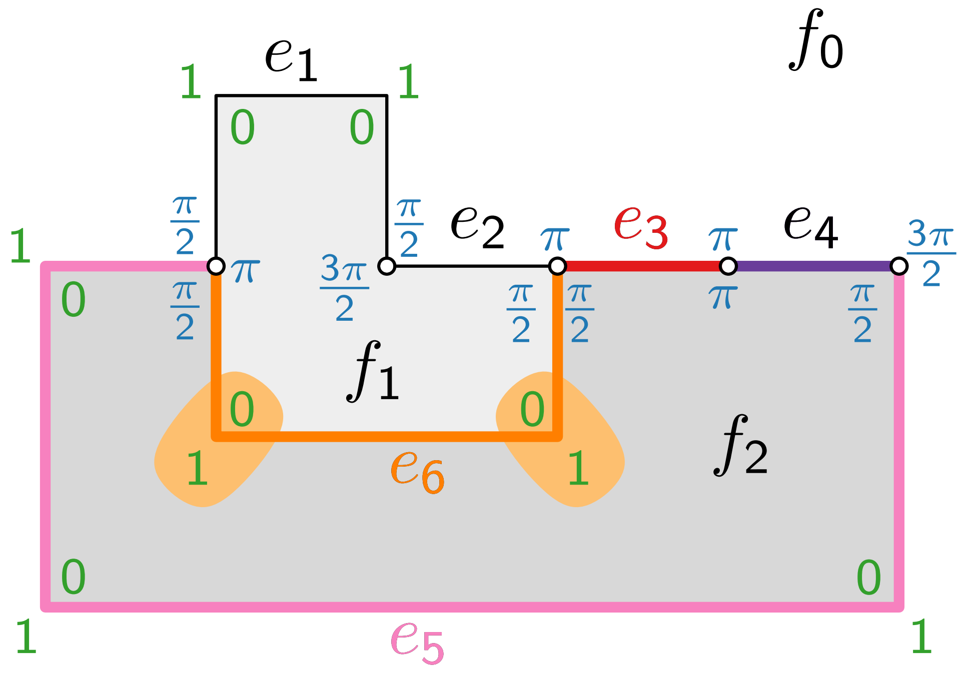

$C(e_3) = 0 - 0 - 2 + 2 = 0$

$C(e_4) = 0 - 0 - 1 + 2 = 1$

$C(e_5) = 3 - 0 - \quad + 2 =$

$C(e_6) = \quad - \quad - \quad + 2 =$

Correctness of an Orthogonal Representation

(H1) $H(G)$ corresponds to F, f_0.

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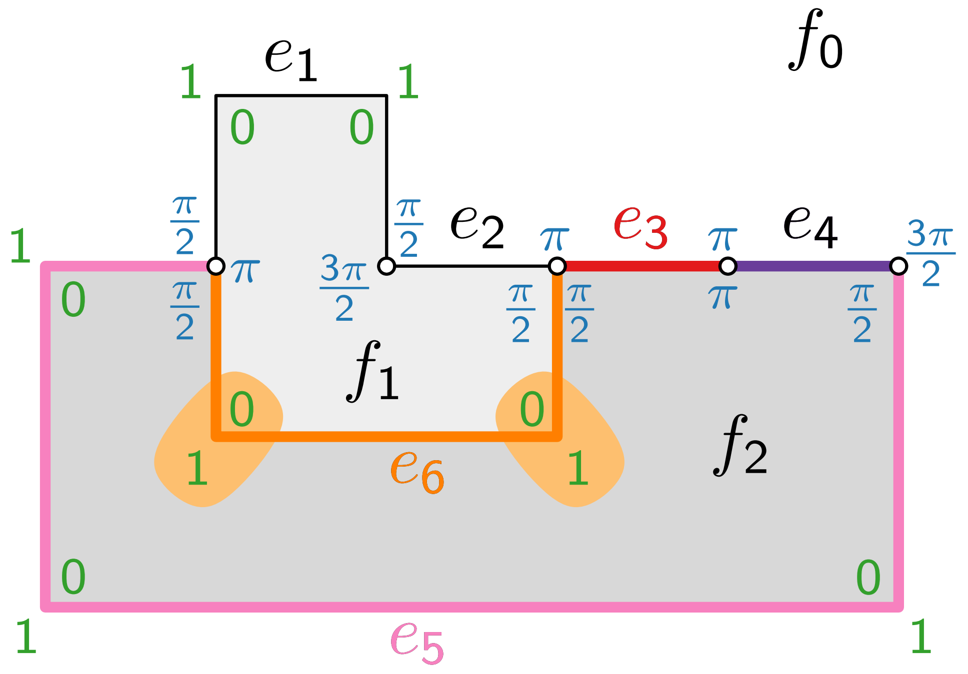

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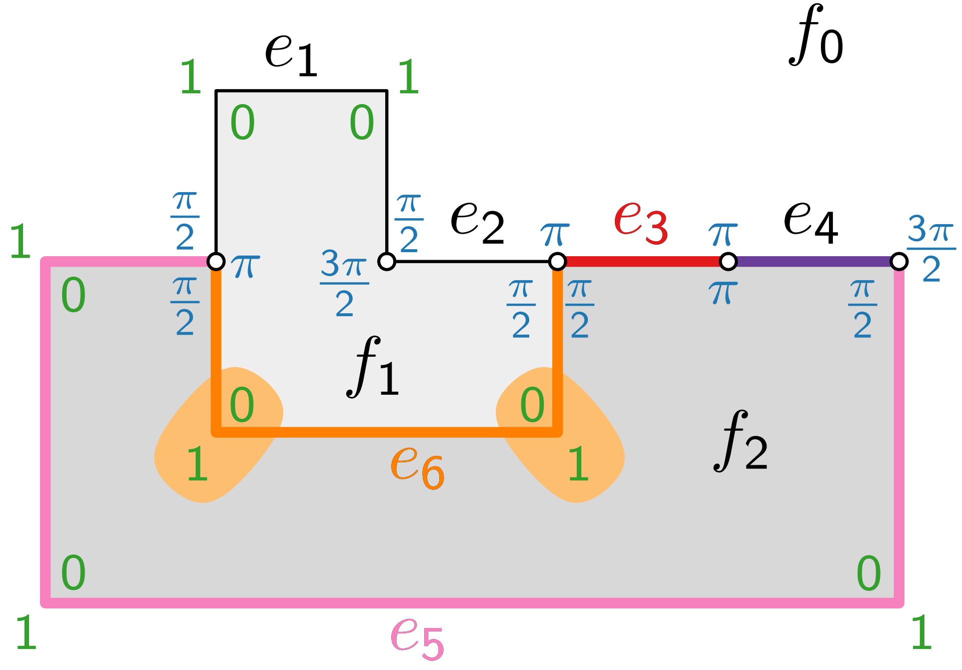

$C(e_3) = 0 - 0 - 2 + 2 = 0$

$C(e_4) = 0 - 0 - 1 + 2 = 1$

$C(e_5) = 3 - 0 - 1 + 2 = 4$

$C(e_6) = \quad - \quad - \quad + 2 =$

Correctness of an Orthogonal Representation

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(H2) For each **edge** $\{u, v\}$ shared by faces f and g with $((u, v), \delta_1, \alpha_1) \in H(f)$ and $((v, u), \delta_2, \alpha_2) \in H(g)$, the sequence δ_1 is like δ_2, but reversed and inverted.

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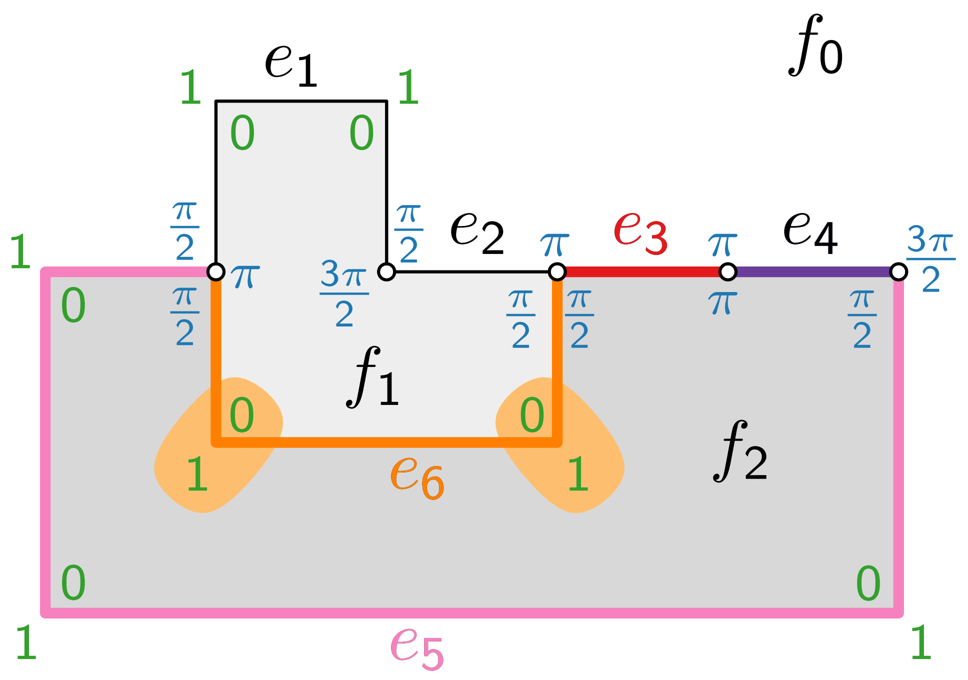

$C(e_3) = 0 - 0 - 2 + 2 = 0$

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$C(e_6) = 0 - 2 - 1 + 2 =$

Correctness of an Orthogonal Representation

(H1) $H(G)$ corresponds to F, f_0.

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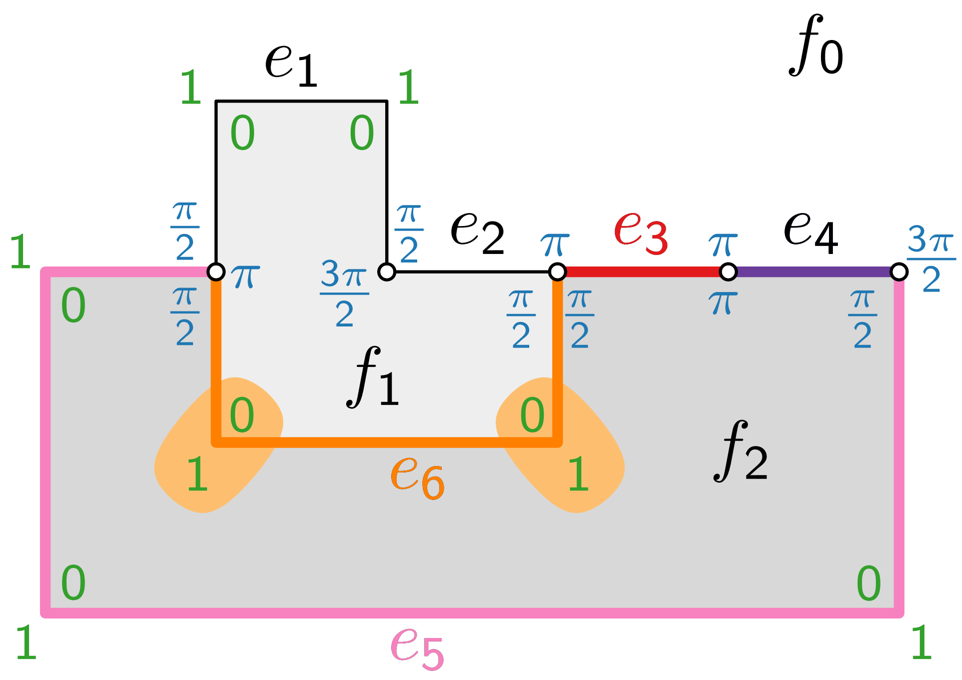

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$C(e_6) = 0 - 2 - 1 + 2 = -1$

Correctness of an Orthogonal Representation

(H1) $H(G)$ corresponds to F, f_0.

(H2) For each **edge** $\{u, v\}$ shared by faces f and g with $((u, v), \delta_1, \alpha_1) \in H(f)$ and $((v, u), \delta_2, \alpha_2) \in H(g)$, the sequence δ_1 is like δ_2, but reversed and inverted.

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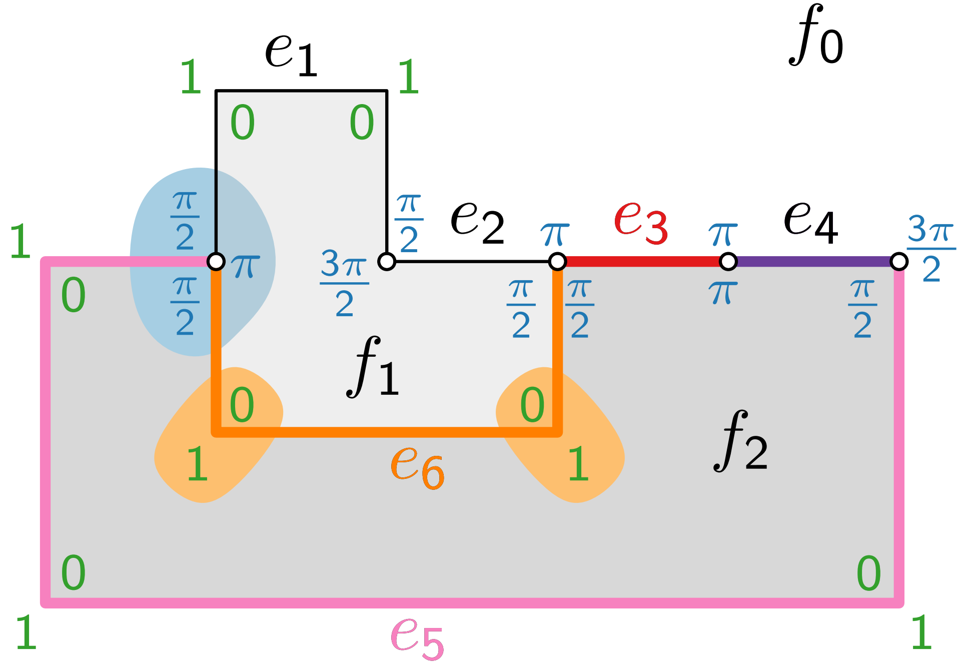
(H4) For each **vertex** v, the sum of incident angles is 2π.



$C(e_3) = 0 - 0 - 2 + 2 = 0$

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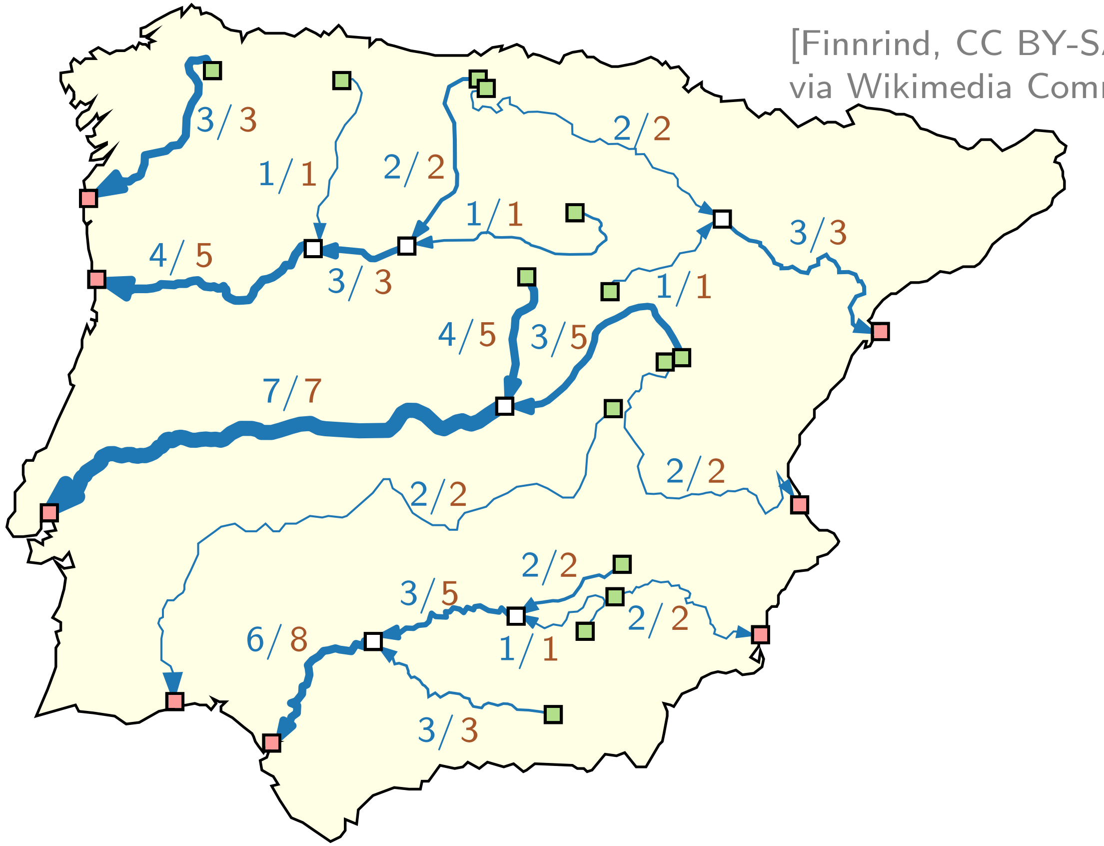
Reminder: s-t-Flow Networks



Flow network $(G = (V, E); S, T; u)$ with

- directed graph $G = (V, E)$
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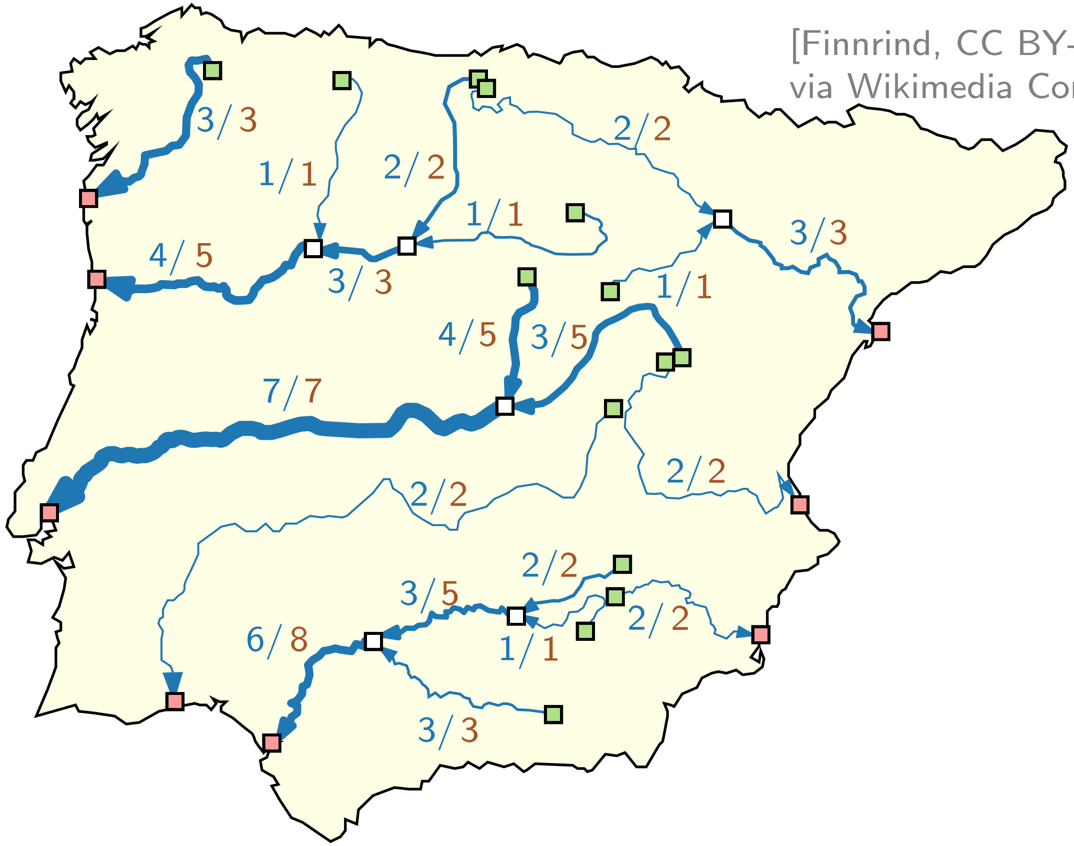
Reminder: s-t-Flow Networks



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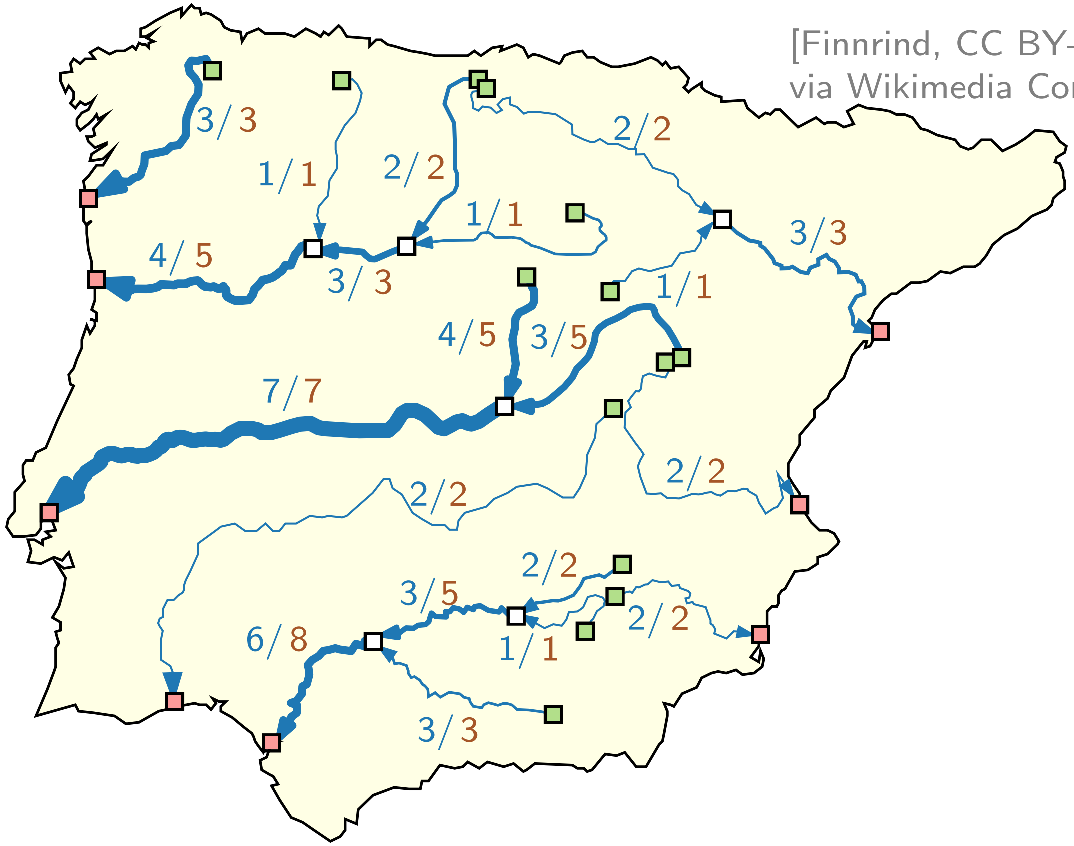

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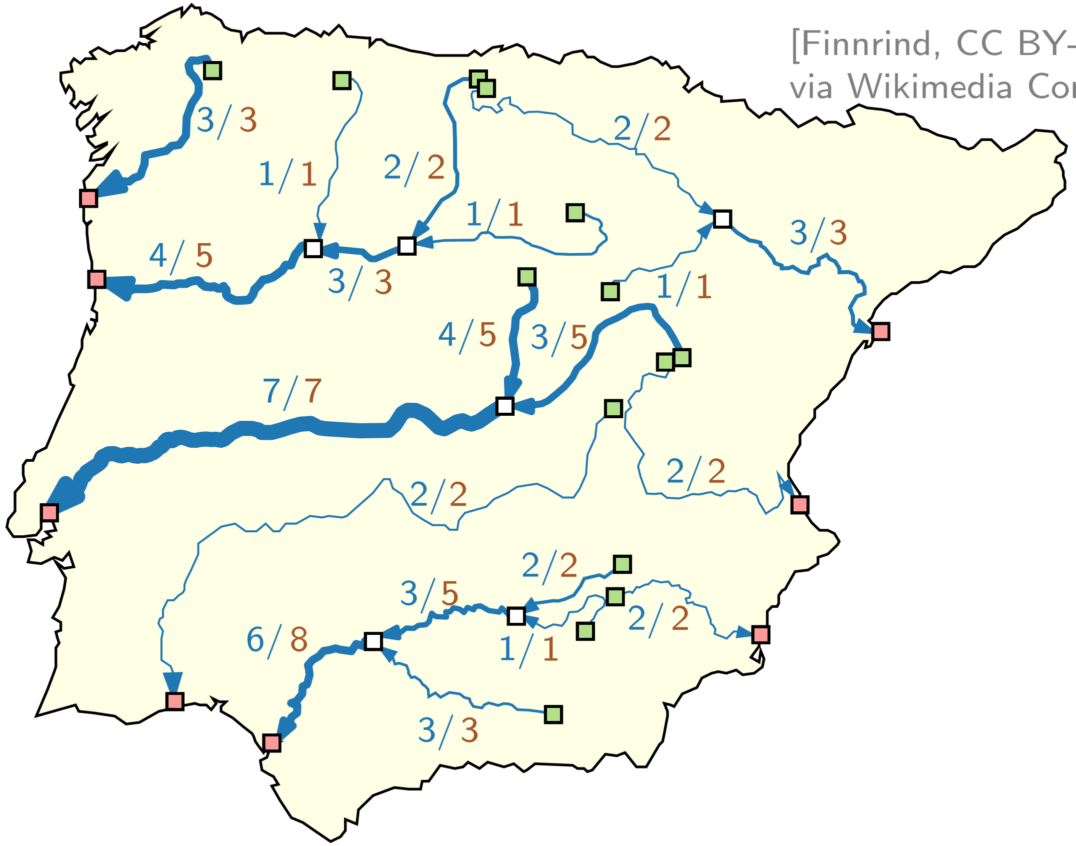

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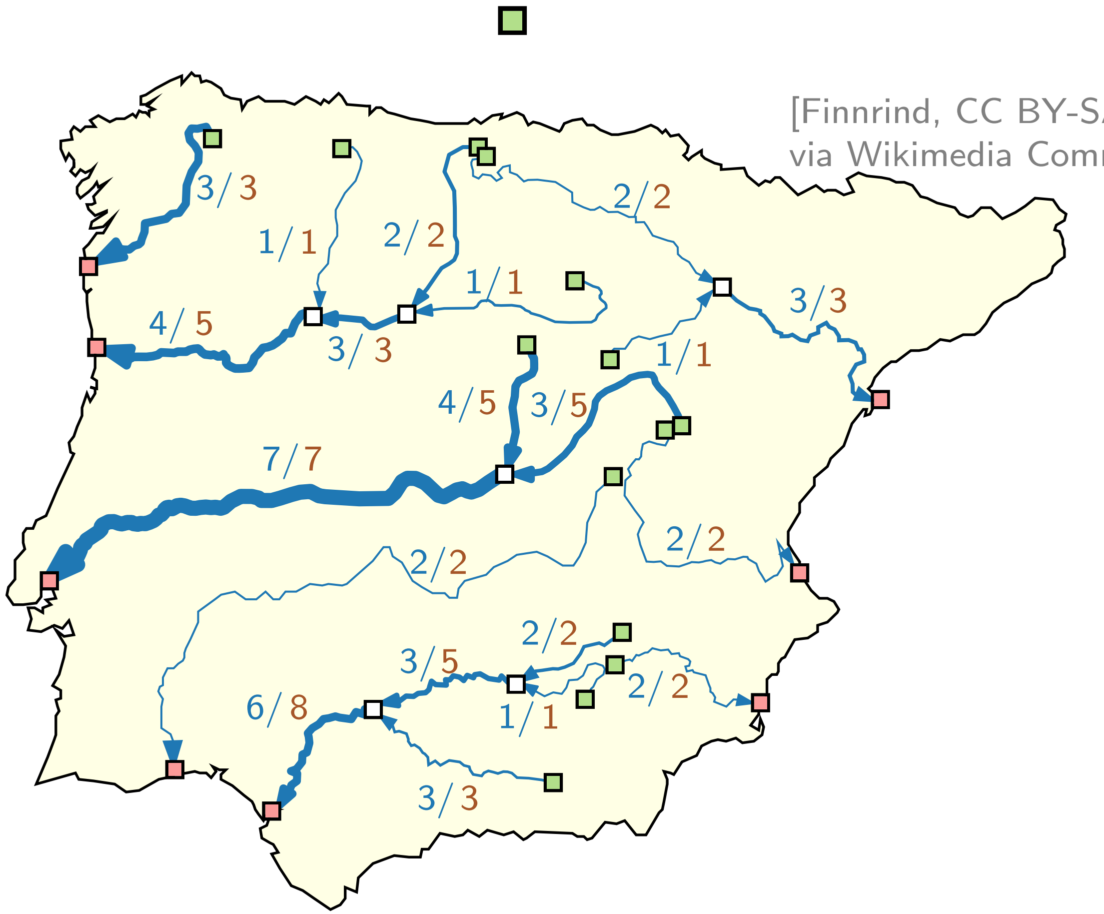

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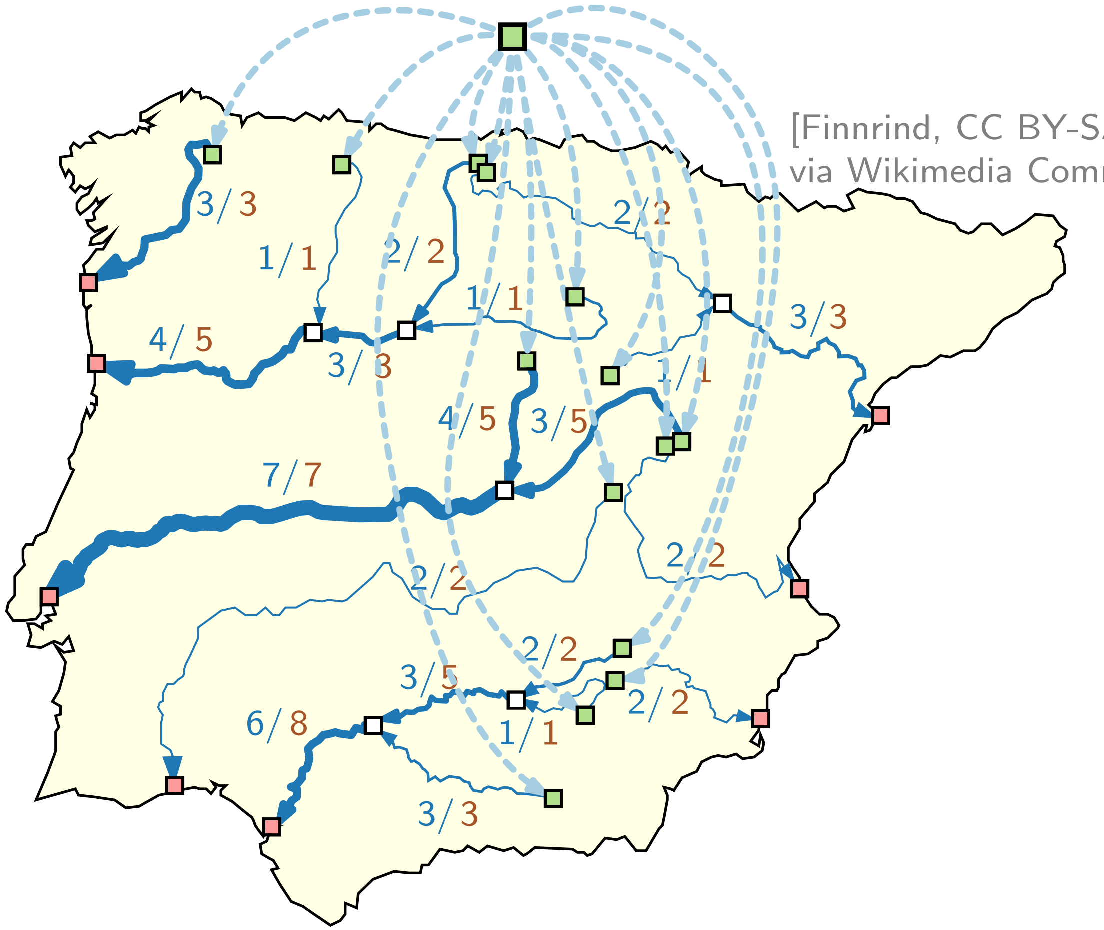

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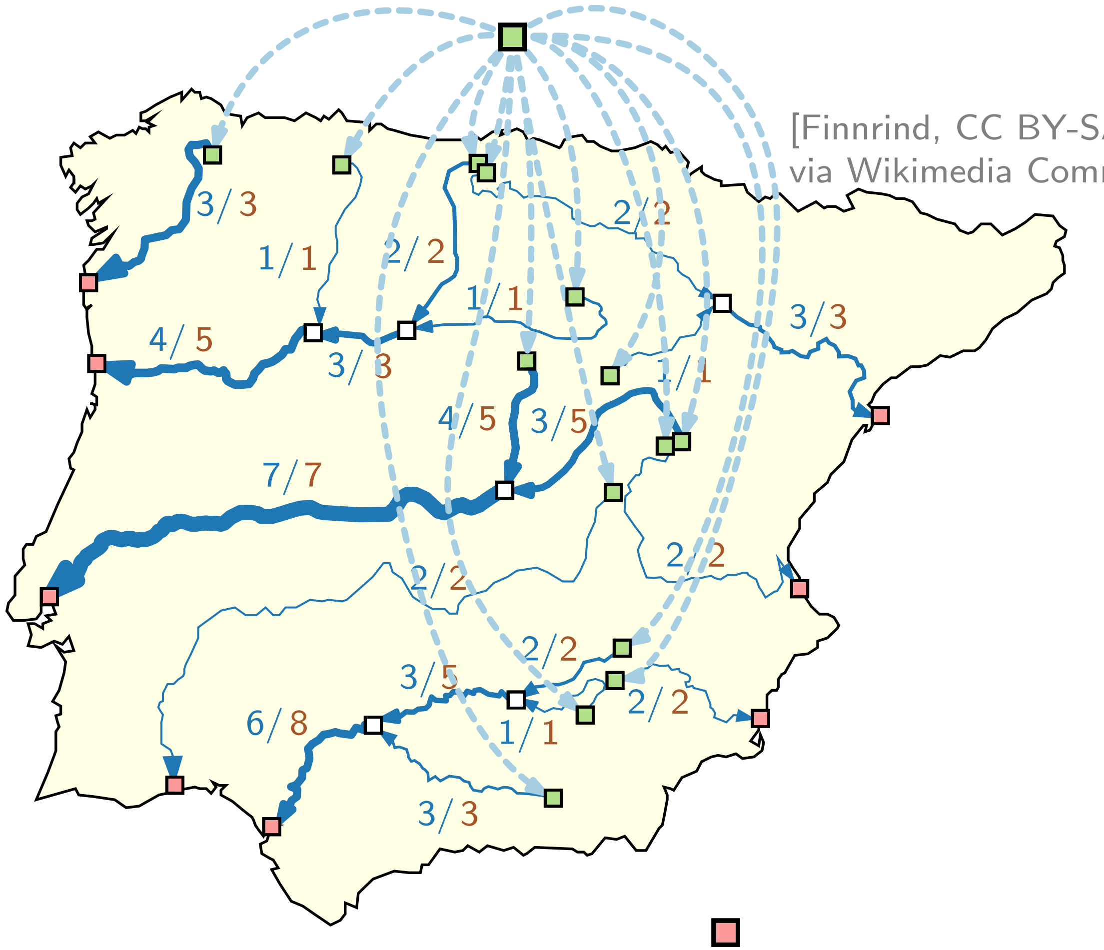

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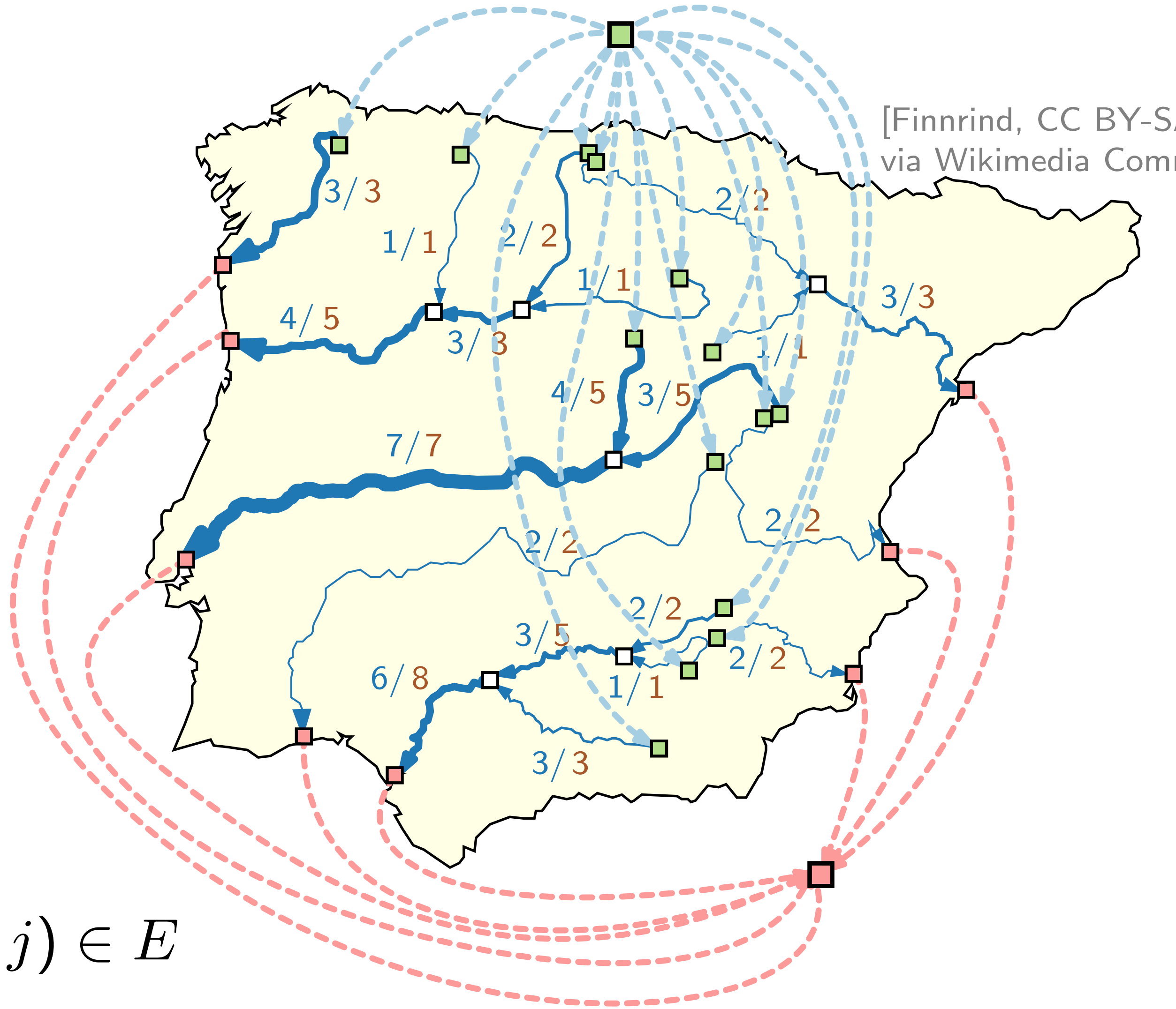

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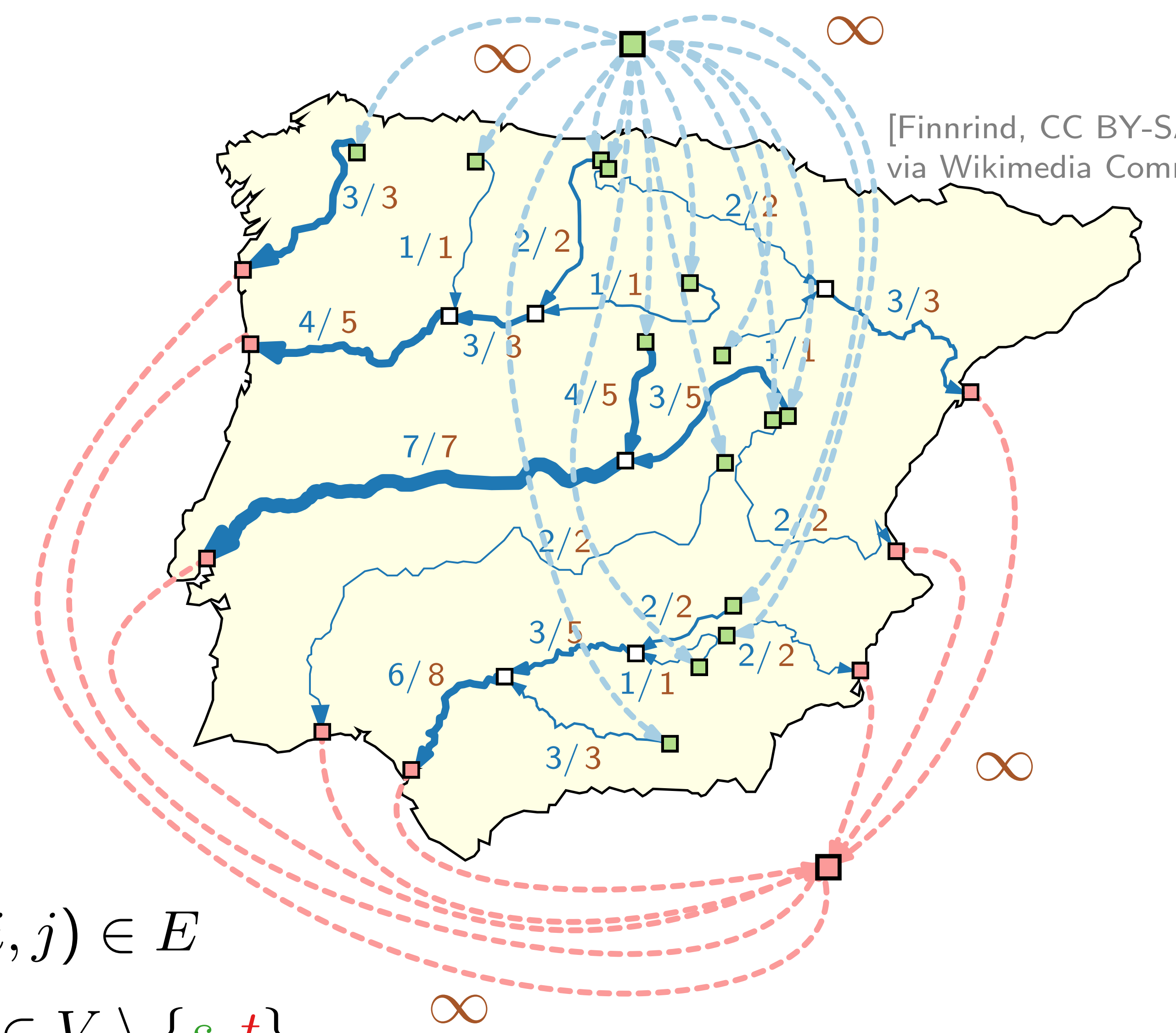

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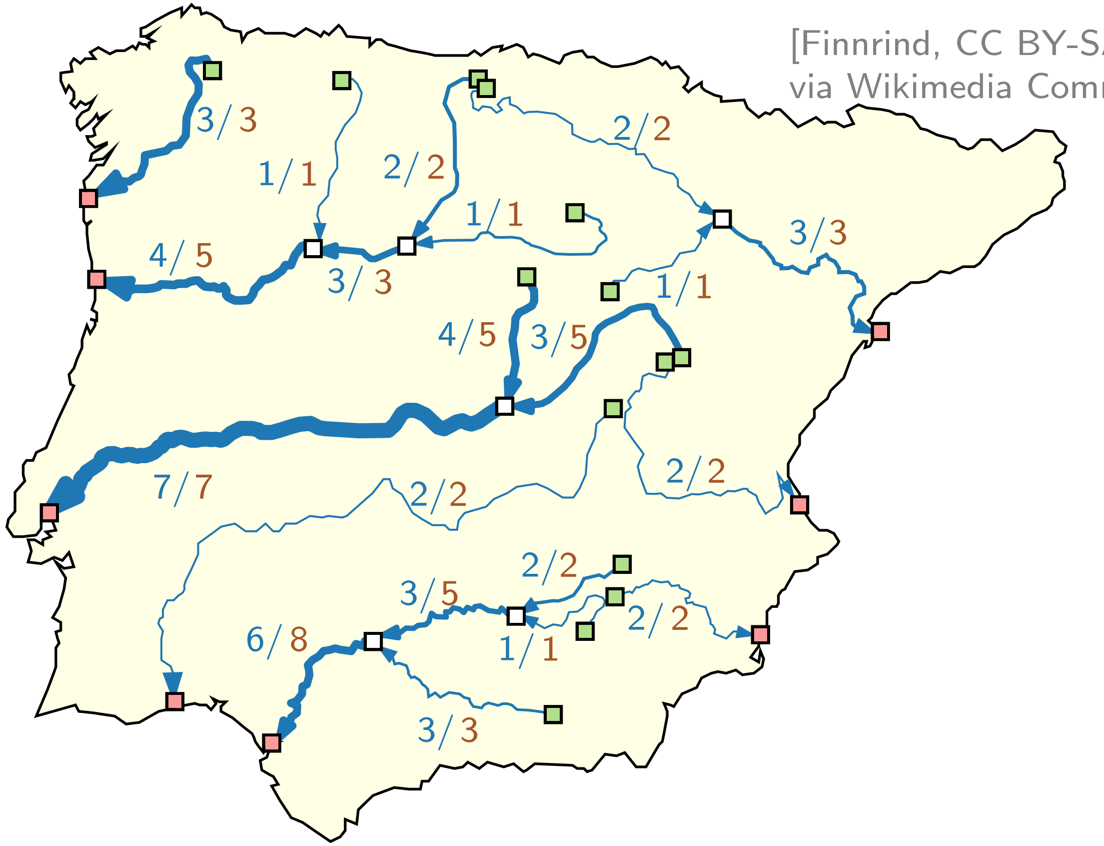

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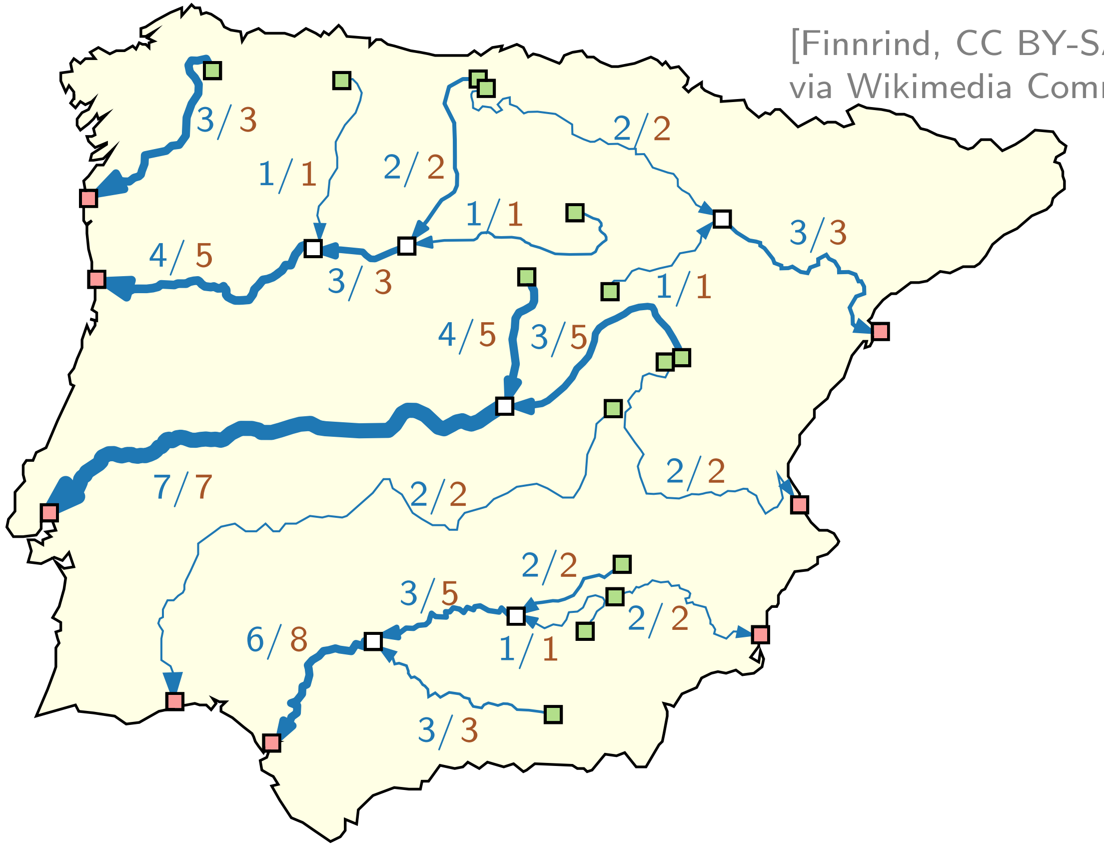

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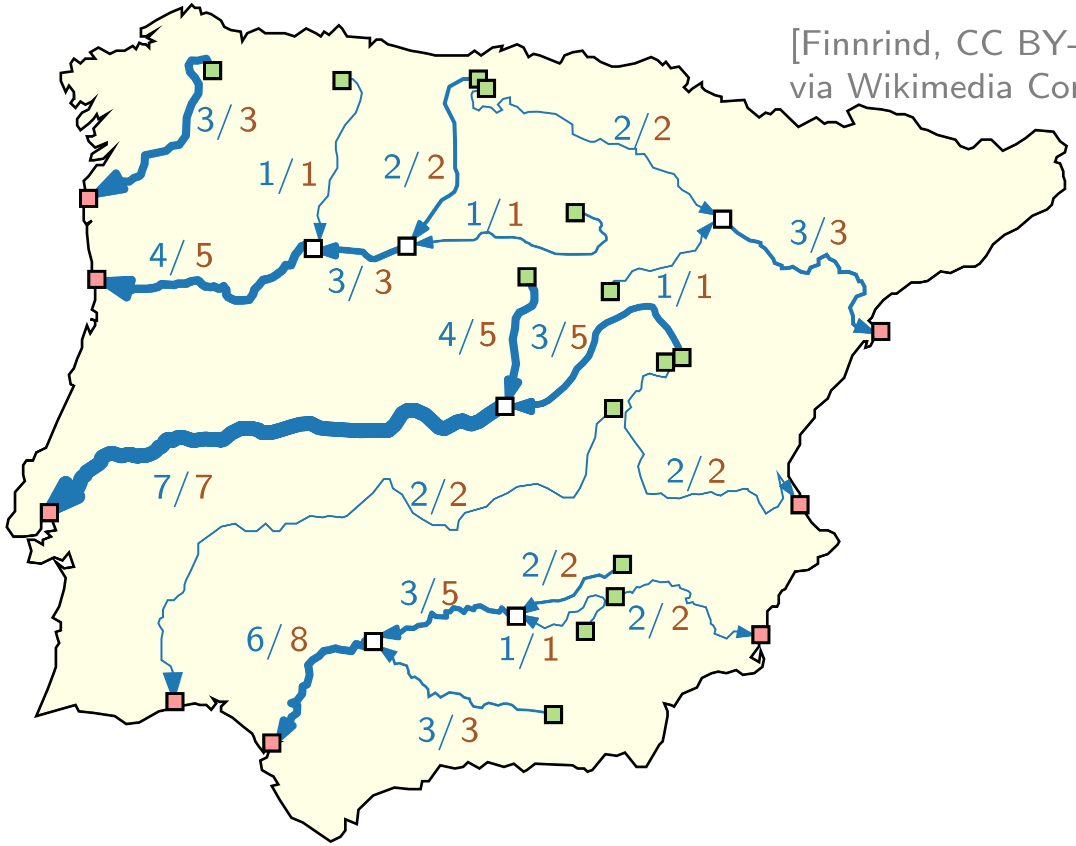

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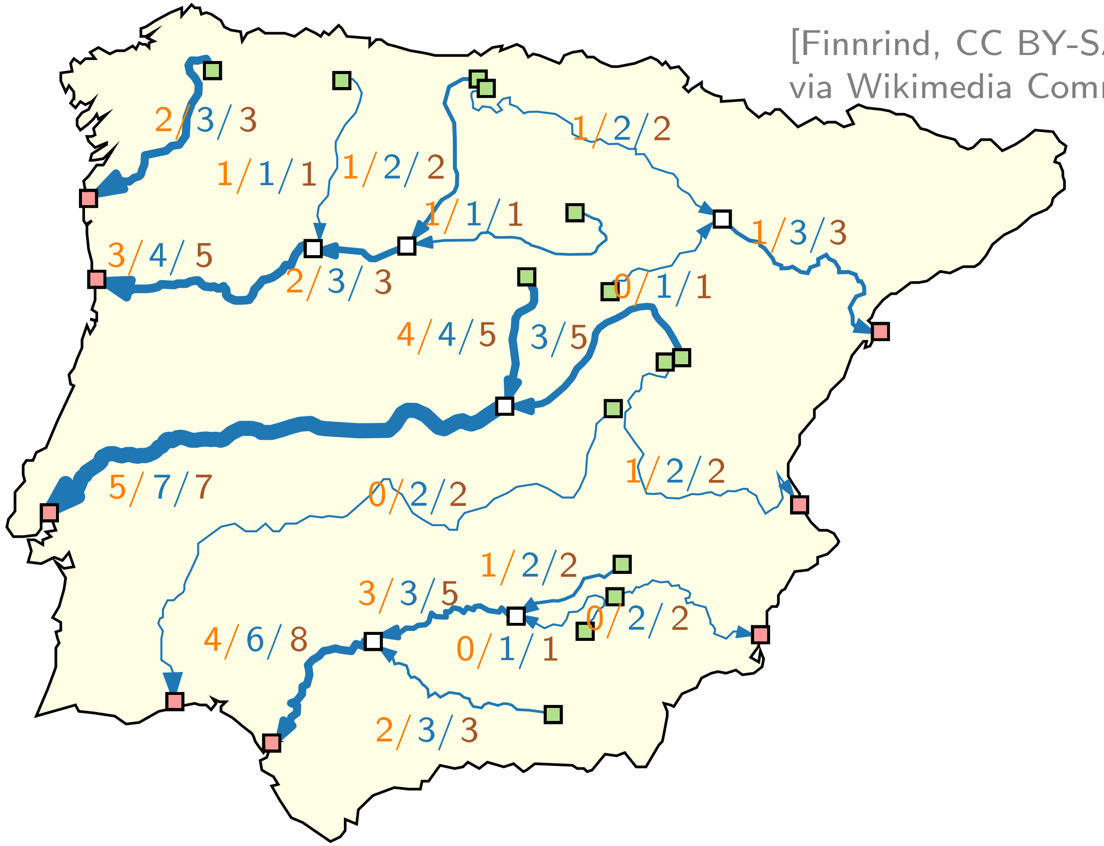

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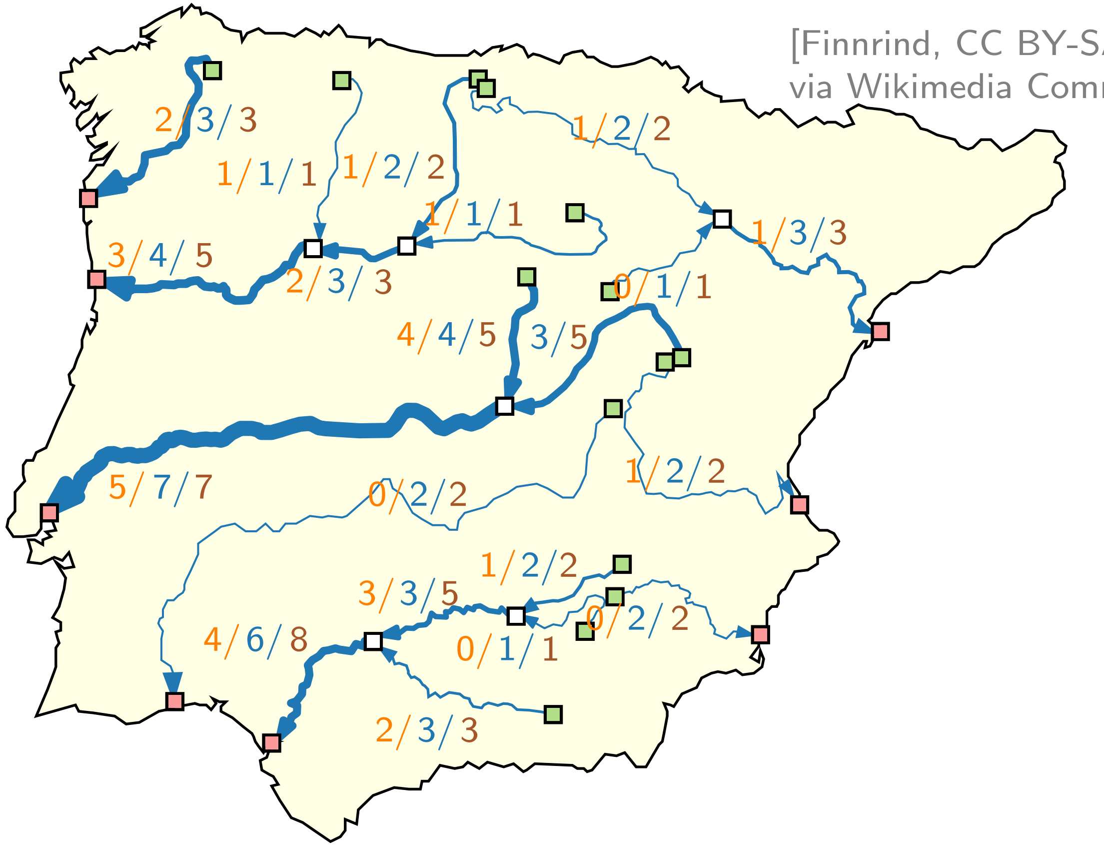

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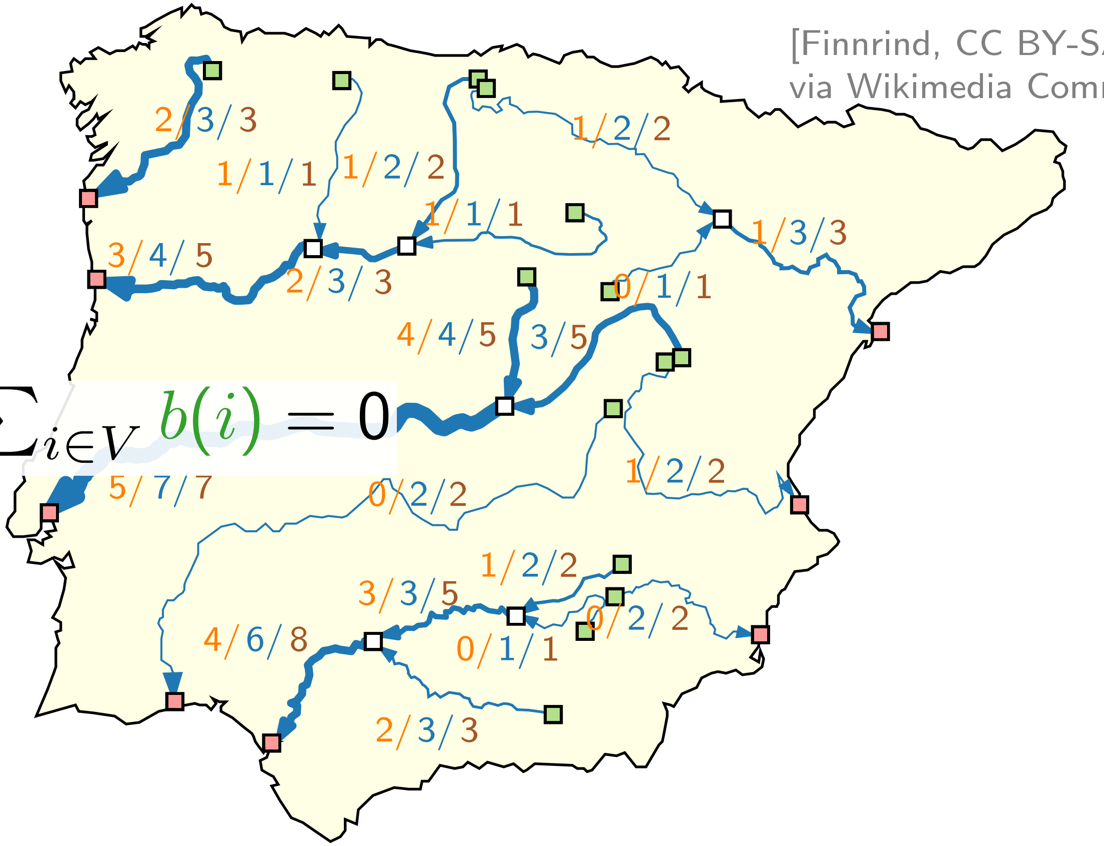

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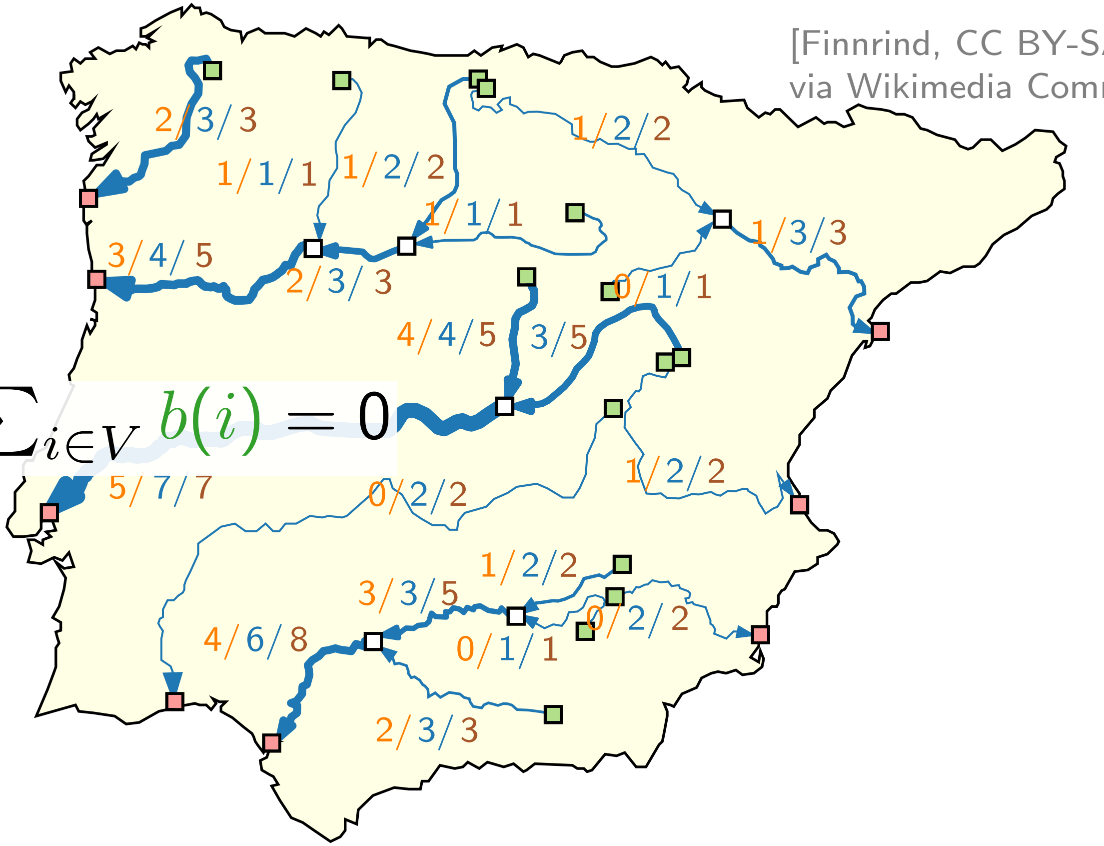

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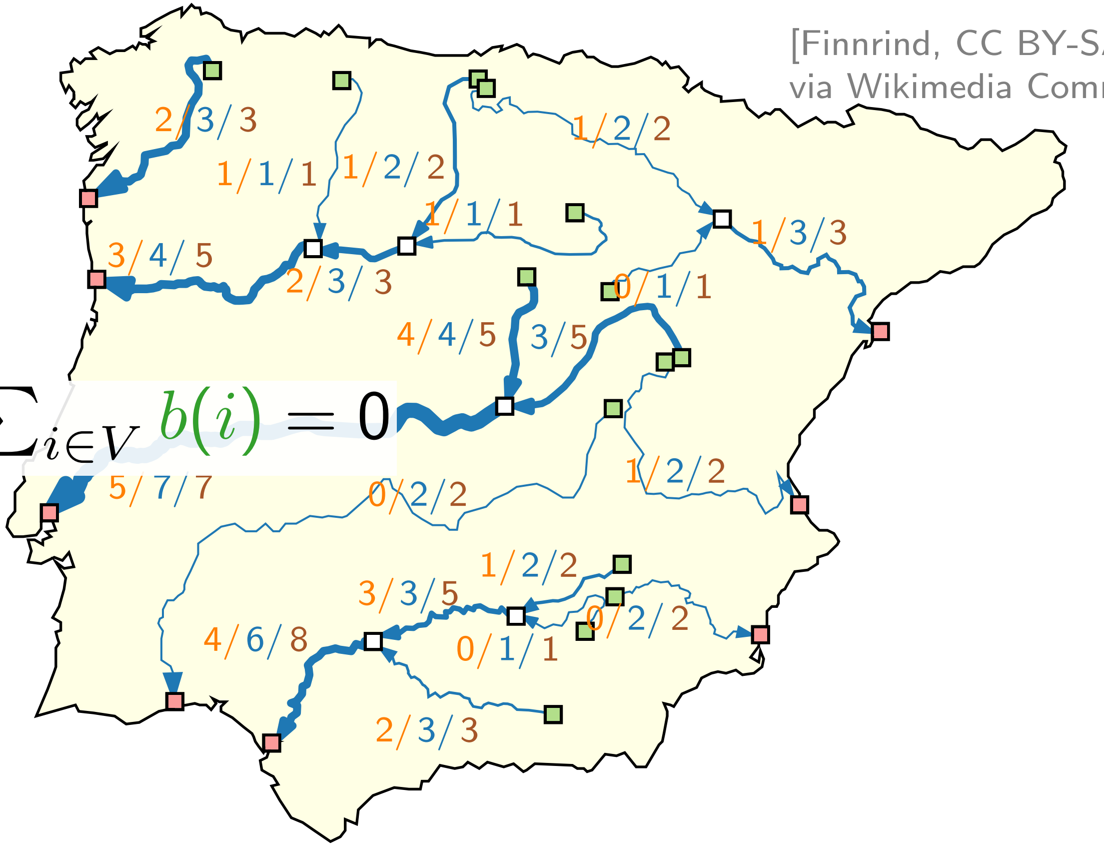

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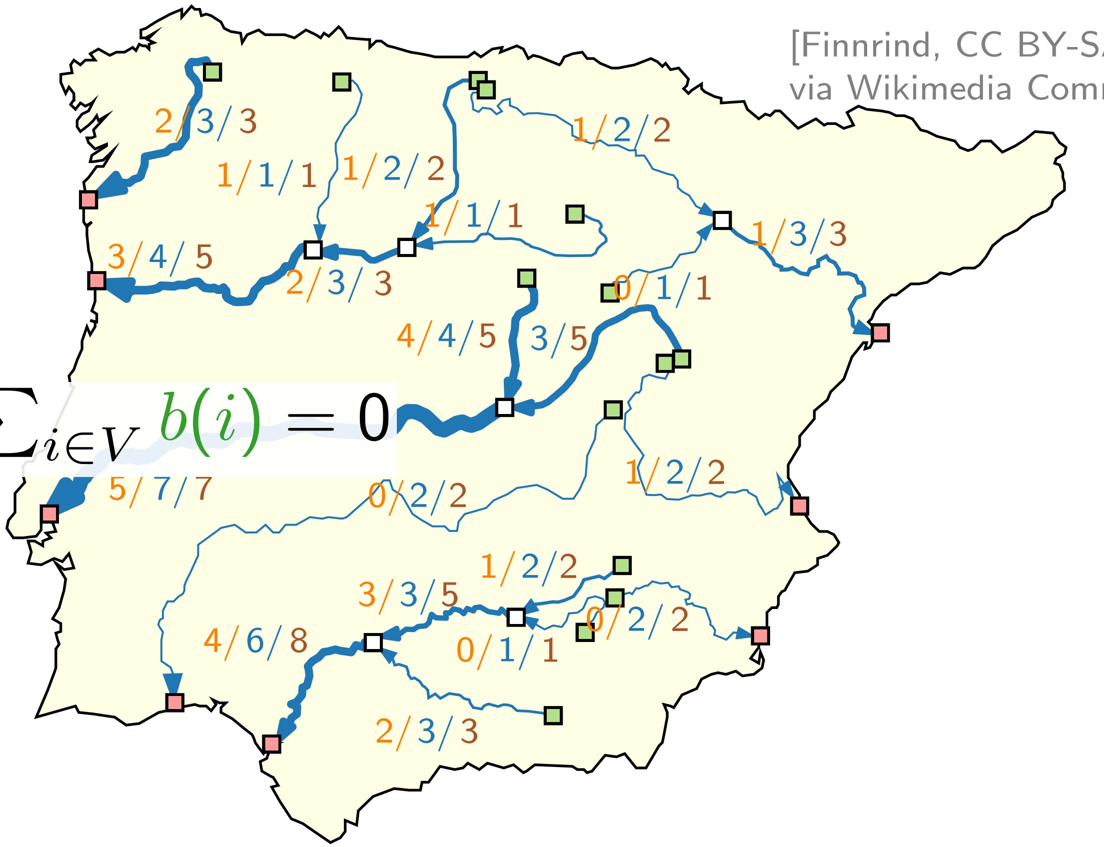

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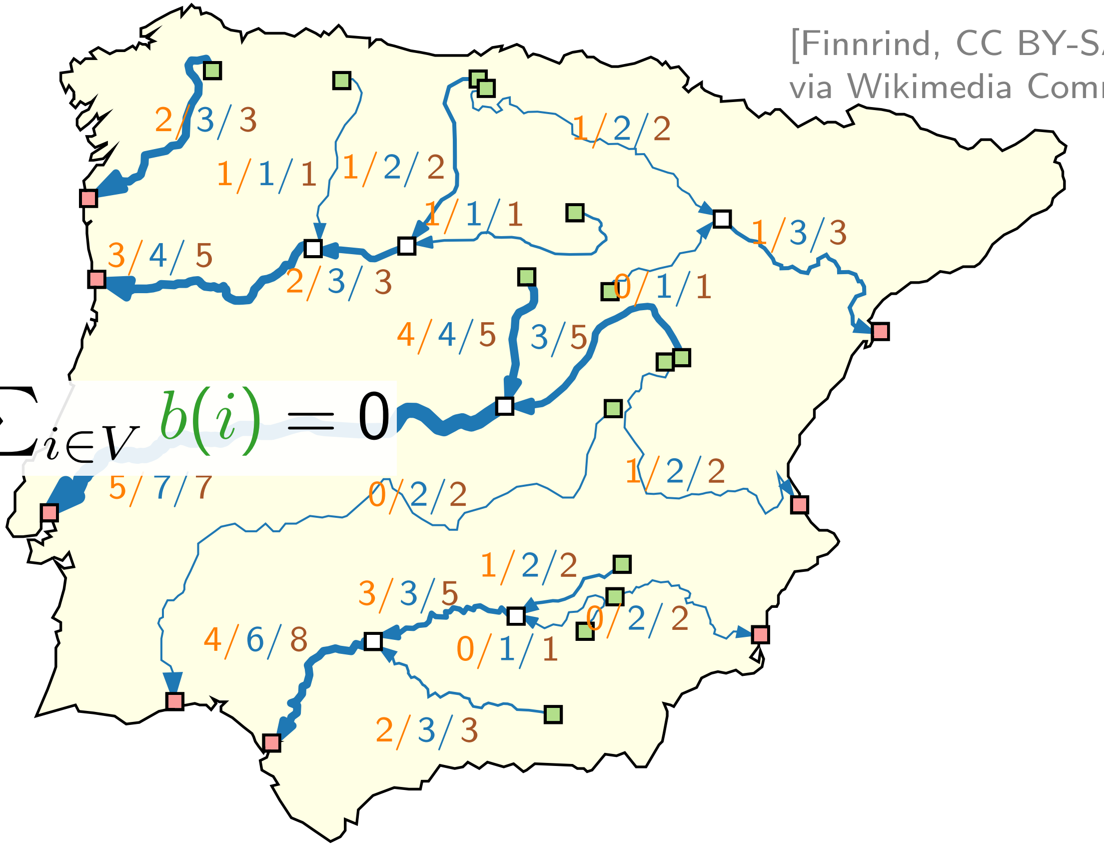

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A **minimum cost flow** is a valid flow where cost(X) is minimized.

General Flow Network – Algorithms

Polynomial Algorithms

#	Due to	Year	Running Time
1	Edmonds and Karp	1972	$O((n + m') \log U \, S(n, m, nC))$
2	Rock	1980	$O((n + m') \log U \, S(n, m, nC))$
3	Rock	1980	$O(n \log C \, M(n, m, U))$
4	Bland and Jensen	1985	$O(m \log C \, M(n, m, U))$
5	Goldberg and Tarjan	1987	$O(nm \log (n^2/m) \log (nC))$
6	Goldberg and Tarjan	1988	$O(nm \log n \log (nC))$
7	Ahuja, Goldberg, Orlin and Tarjan	1988	$O(nm \log \log U \log (nC))$

Strongly Polynomial Algorithms

#	Due to	Year	Running Time
1	Tardos	1985	$O(m^4)$
2	Orlin	1984	$O((n + m')^2 \log n \, S(n, m))$
3	Fujishige	1986	$O((n + m')^2 \log n \, S(n, m))$
4	Galil and Tardos	1986	$O(n^2 \log n \, S(n, m))$
5	Goldberg and Tarjan	1987	$O(nm^2 \log n \log(n^2/m))$
6	Goldberg and Tarjan	1988	$O(nm^2 \log^2 n)$
7	Orlin (this paper)	1988	$O((n + m') \log n \, S(n, m))$

$S(n, m)$	$= O(m + n \log n)$	Fredman and Tarjan [1984]
$S(n, m, C)$	$= O(\text{Min}\,(m + n\sqrt{\log C}),$ $(m \log \log C))$	Ahuja, Mehlhorn, Orlin and Tarjan [1990]; Van Emde Boas, Kaas and Zijlstra[1977]
$M(n, m)$	$= O(\min\,(nm + n^{2+\varepsilon}, nm \log n)$ where ε is any fixed constant.	King, Rao, and Tarjan [1991]
$M(n, m, U)$	$= O(nm \log (\frac{n}{m}\sqrt{\log U} + 2))$	Ahuja, Orlin and Tarjan [1989]

[Orlin 1991]

General Flow Network – Algorithms

Polynomial Algorithms

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3	Rock	1980	O(n log C M(n, m, U))
4	Bland and Jensen	1985	O(m log C M(n, m, U))
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Strongly Polynomial Algorithms

#	Due to	Year	Running Time
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4	Galil and Tardos	1986	O(n^2 log n S(n, m))
5	Goldberg and Tarjan	1987	O(nm^2 log n log(n^2/m))
6	Goldberg and Tarjan	1988	O(nm^2 $\log^2$ n)
7	Orlin (this paper)	1988	O((n + m') log n S(n, m))

S(n, m)	=	O(m + n log n)	Fredman and Tarjan [1984]
S(n, m, C)	=	O(Min (m + n$\sqrt{\log C}$), (m log log C))	Ahuja, Mehlhorn, Orlin and Tarjan [1990] Van Emde Boas, Kaas and Zijlstra[1977]
M(n, m)	=	O(min (nm + $n^{2+\varepsilon}$, nm log n) where ε is any fixed constant.	King, Rao, and Tarjan [1991]
M(n, m, U)	=	O(nm log ($\frac{n}{m}\sqrt{\log U}$ + 2))	Ahuja, Orlin and Tarjan [1989]

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Theorem. [Orlin 1991]
The minimum cost flow problem can be solved in $O(n^2 \log^2 n + m^2 \log n)$ time.

General Flow Network – Algorithms

Polynomial Algorithms

#	Due to	Year	Running Time
1	Edmonds and Karp	1972	O((n + m') log U S(n, m, nC))
2	Rock	1980	O((n + m') log U S(n, m, nC))
3	Rock	1980	O(n log C M(n, m, U))
4	Bland and Jensen	1985	O(m log C M(n, m, U))
5	Goldberg and Tarjan	1987	O(nm log (n^2/m) log (nC))
6	Goldberg and Tarjan	1988	O(nm log n log (nC))
7	Ahuja, Goldberg, Orlin and Tarjan	1988	O(nm log log U log (nC))

Strongly Polynomial Algorithms

#	Due to	Year	Running Time
1	Tardos	1985	O(m^4)
2	Orlin	1984	O($(n + m')^2$ log n S(n, m))
3	Fujishige	1986	O($(n + m')^2$ log n S(n, m))
4	Galil and Tardos	1986	O(n^2 log n S(n, m))
5	Goldberg and Tarjan	1987	O(nm^2 log n log(n^2/m))
6	Goldberg and Tarjan	1988	O(nm^2 $\log^2$ n)
7	Orlin (this paper)	1988	O((n + m') log n S(n, m))

S(n, m)	=	O(m + n log n)	Fredman and Tarjan [1984]
S(n, m, C)	=	O(Min (m + n$\sqrt{\log C}$), (m log log C))	Ahuja, Mehlhorn, Orlin and Tarjan [1990] Van Emde Boas, Kaas and Zijlstra[1977]
M(n, m)	=	O(min (nm + $n^{2+\varepsilon}$, nm log n) where ε is any fixed constant.	King, Rao, and Tarjan [1991]
M(n, m, U)	=	O(nm log ($\frac{n}{m}\sqrt{\log U}$ + 2))	Ahuja, Orlin and Tarjan [1989]

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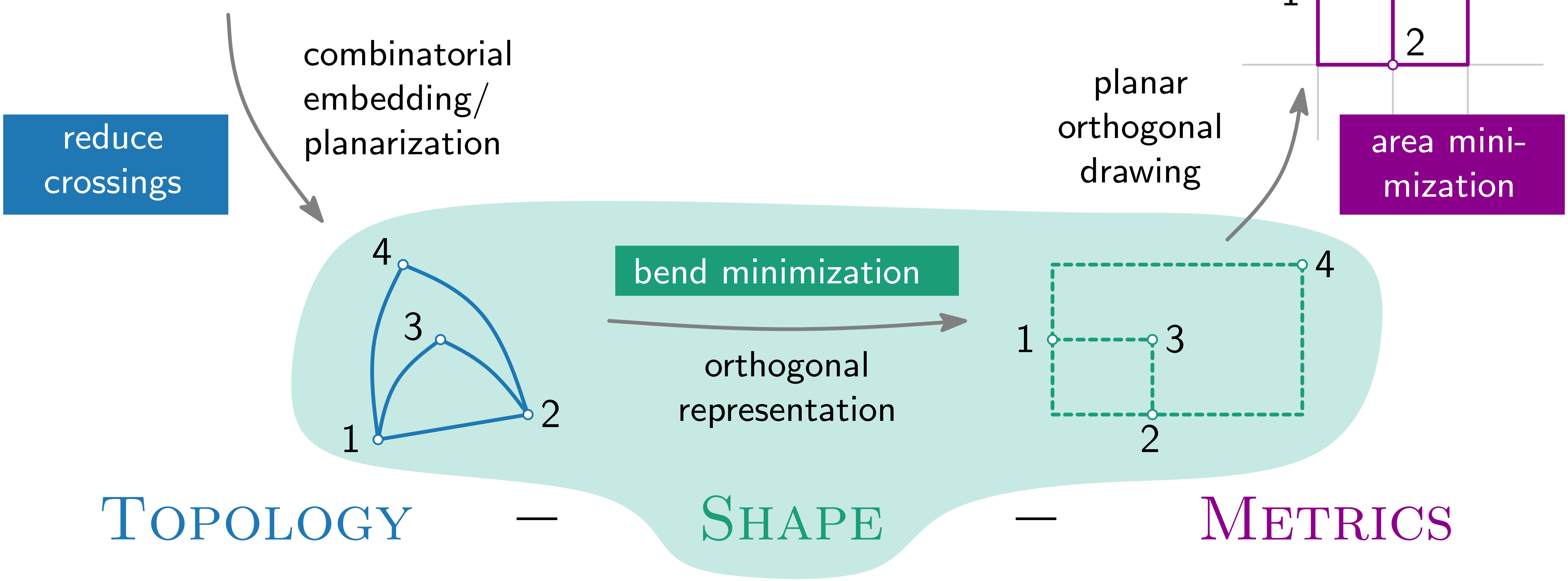
The minimum cost flow problem can be solved in $O(n^2 \log^2 n + m^2 \log n)$ time.

Theorem. [Cornelsen & Karrenbauer 2011]
The minimum cost flow problem for planar graphs with bounded costs and face sizes can be solved in $O(n^{3/2})$ time.

Topology – Shape – Metrics

Three-step approach: [Tamassia 1987]

$V = \{v_1, v_2, v_3, v_4\}$
$E = \{v_1v_2, v_1v_3, v_1v_4, v_2v_3, v_2v_4\}$



Bend Minimization with Given Embedding

Geometric orthogonal bend minimization.

Given:

Find:

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Given:
- Plane graph $G = (V, E)$ with maximum degree 4

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Given:
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Compare with the following variation.

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- faces $\xrightarrow{\measuredangle}$ neighbouring faces ($\#$ bends toward the neighbour)

Flow Network for Bend Minimization

(H1) $H(G)$ corresponds to F, f_0.

(H2) For each **edge** $\{u, v\}$ shared by faces f and g, sequence δ_1 is reversed and inverted δ_2.

(H3) For each **face** f it holds that:

$$\sum_{r \in H(f)} C(r) = \begin{cases} -4 & \text{if } f = f_0 \\ +4 & \text{otherwise.} \end{cases}$$

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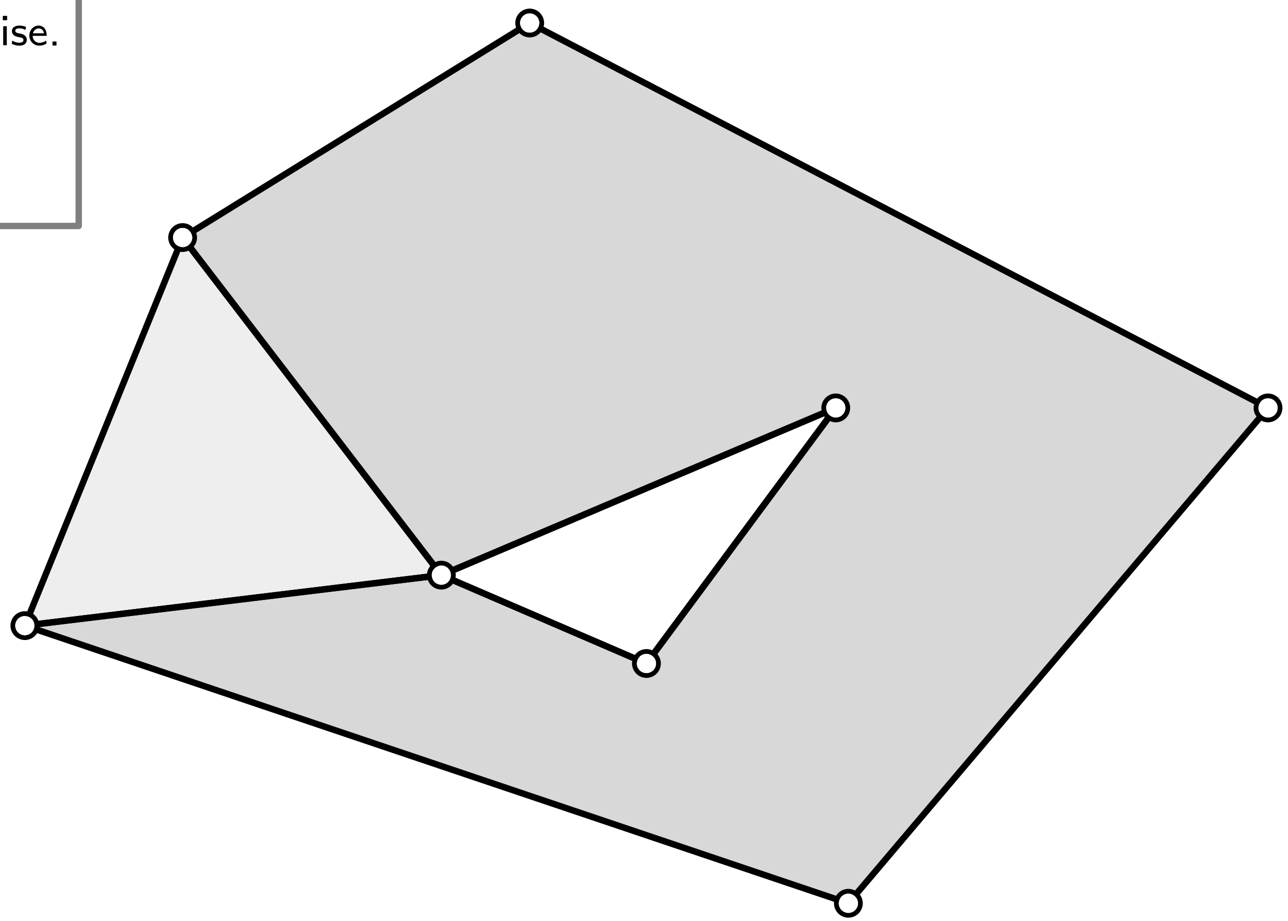

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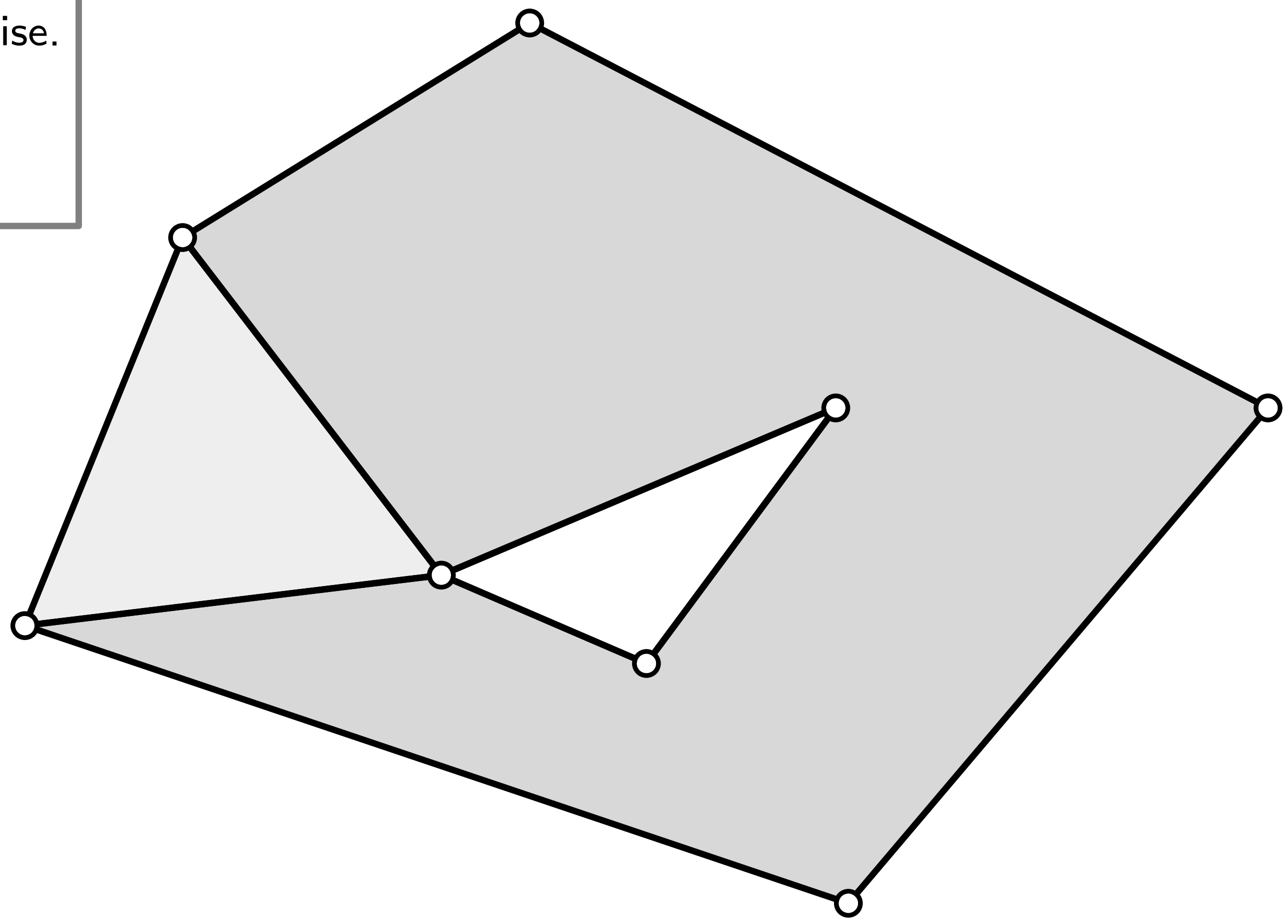

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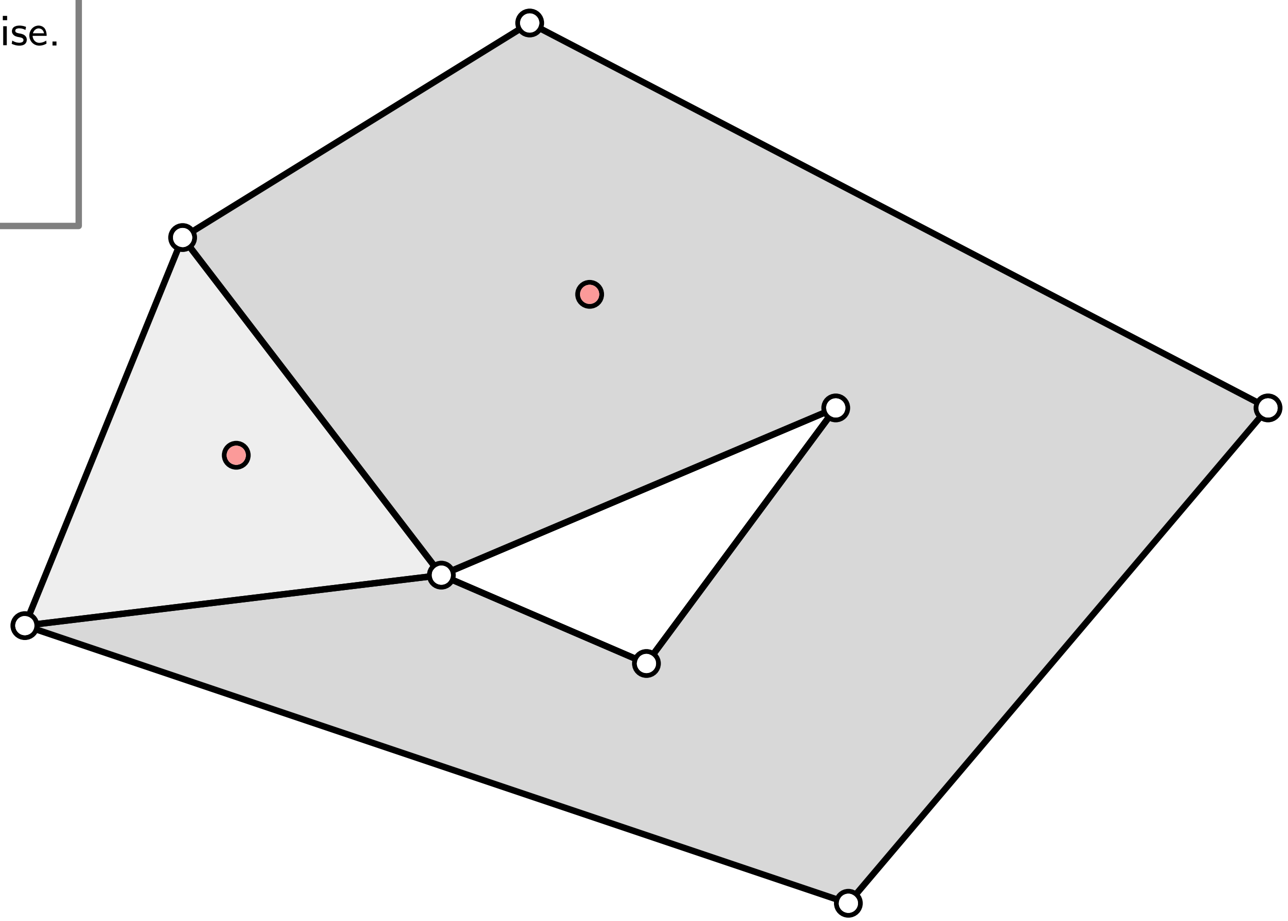

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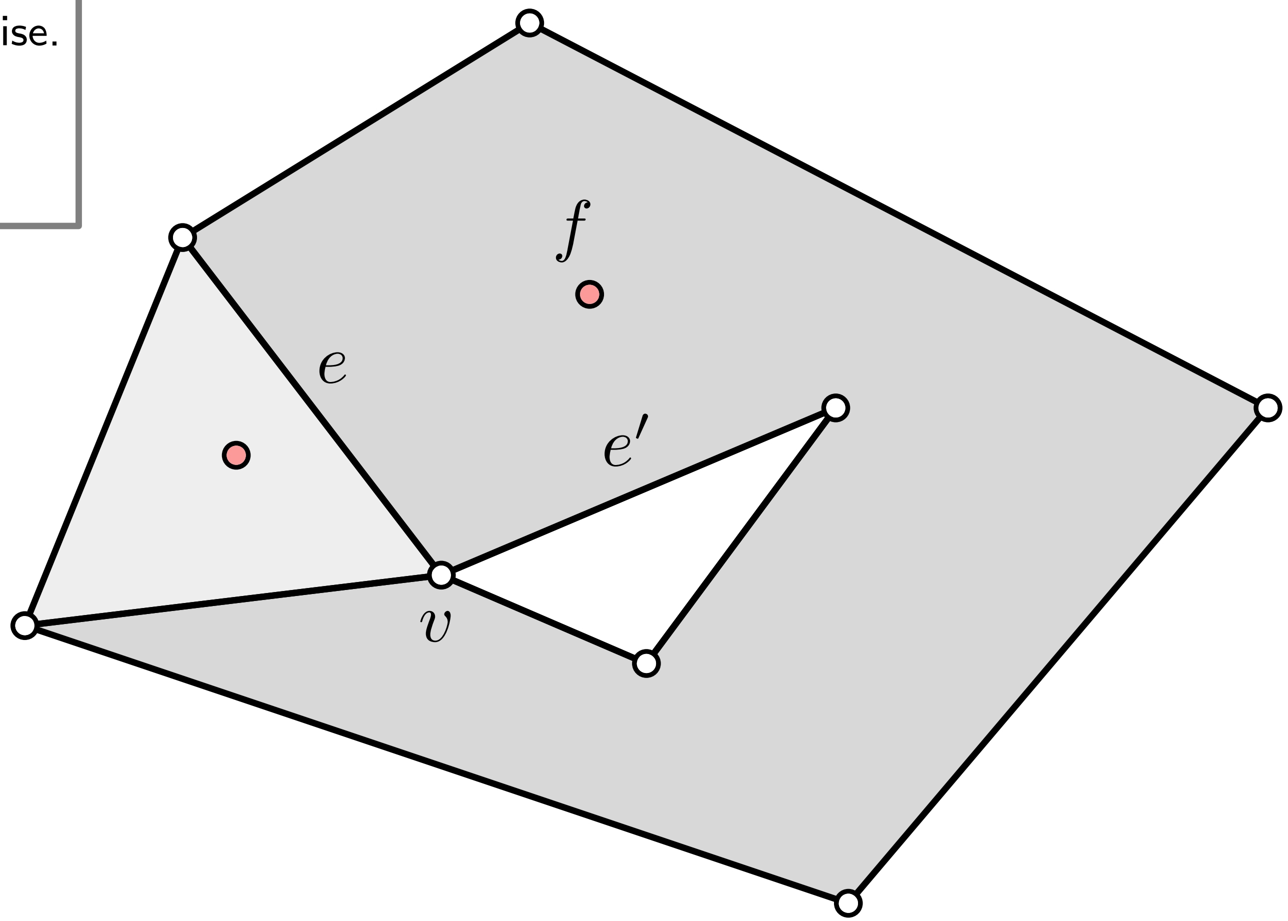

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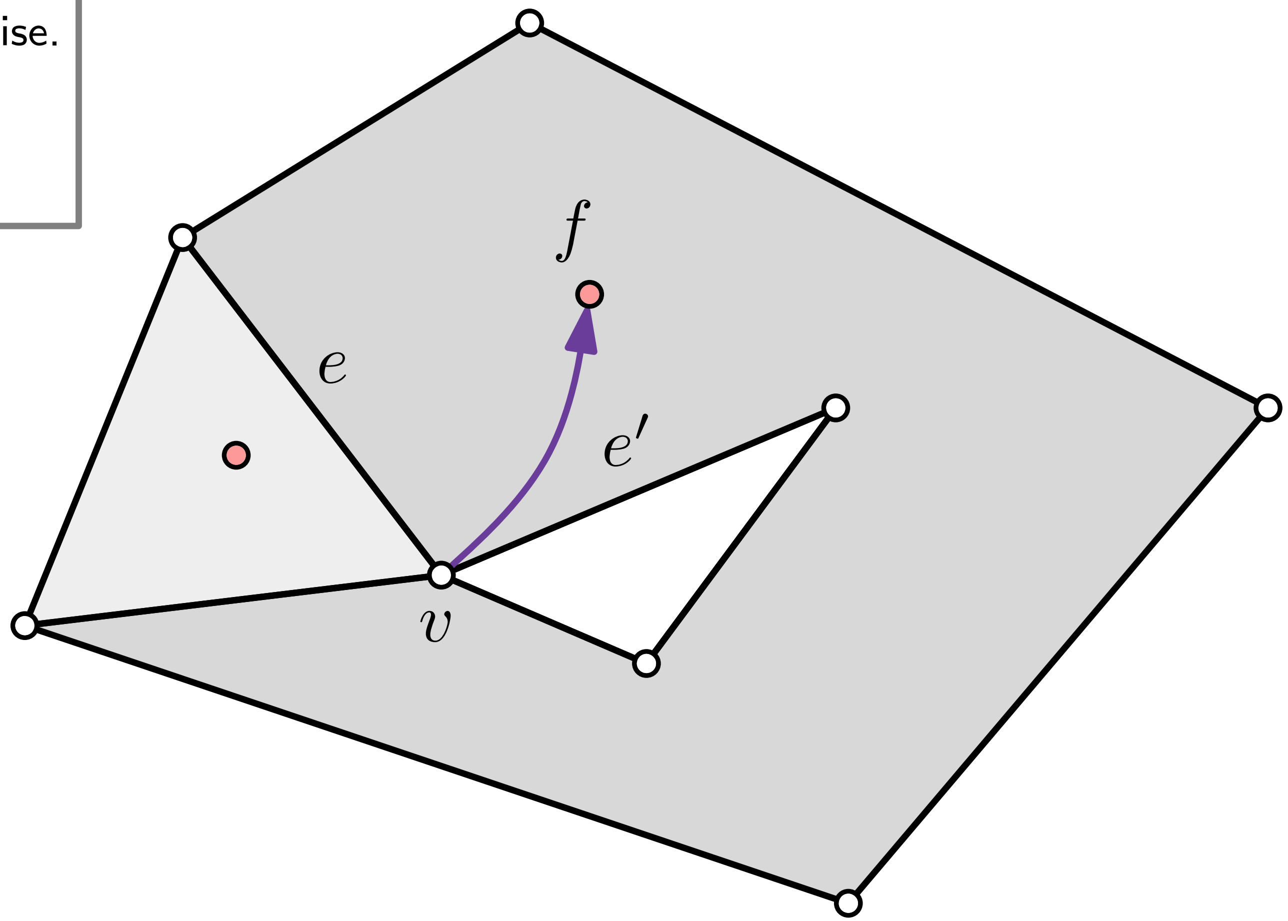

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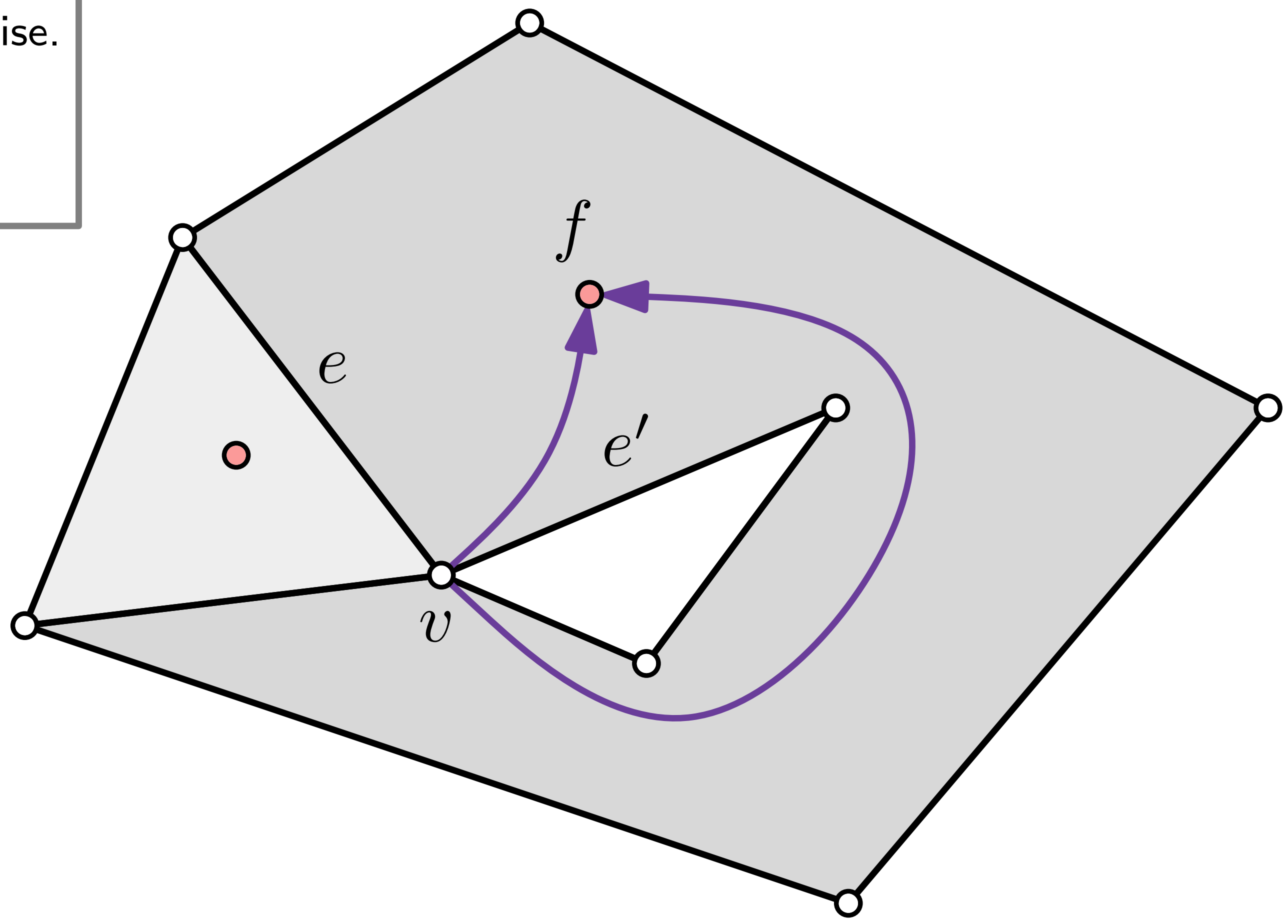

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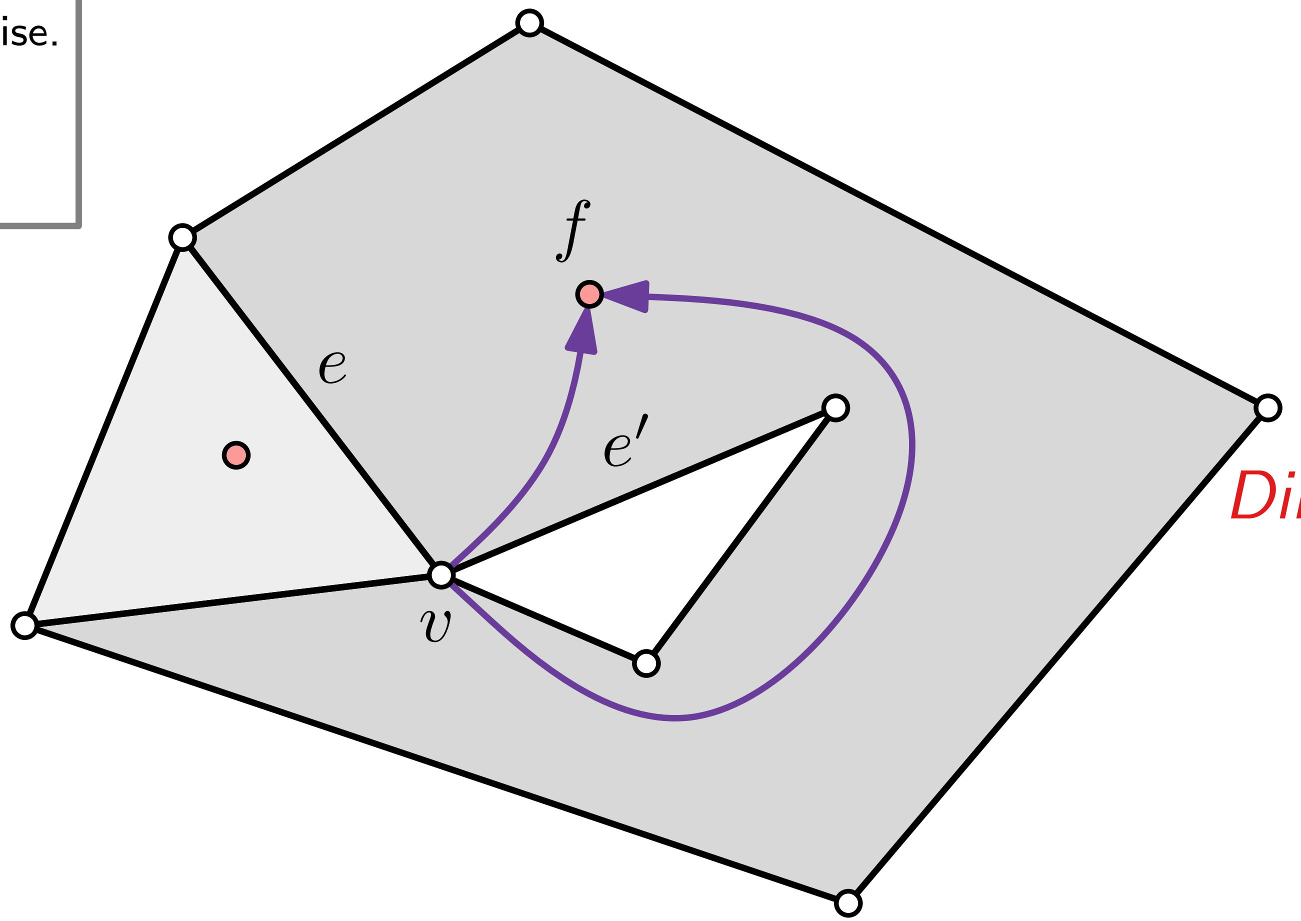

Directed multigraph!

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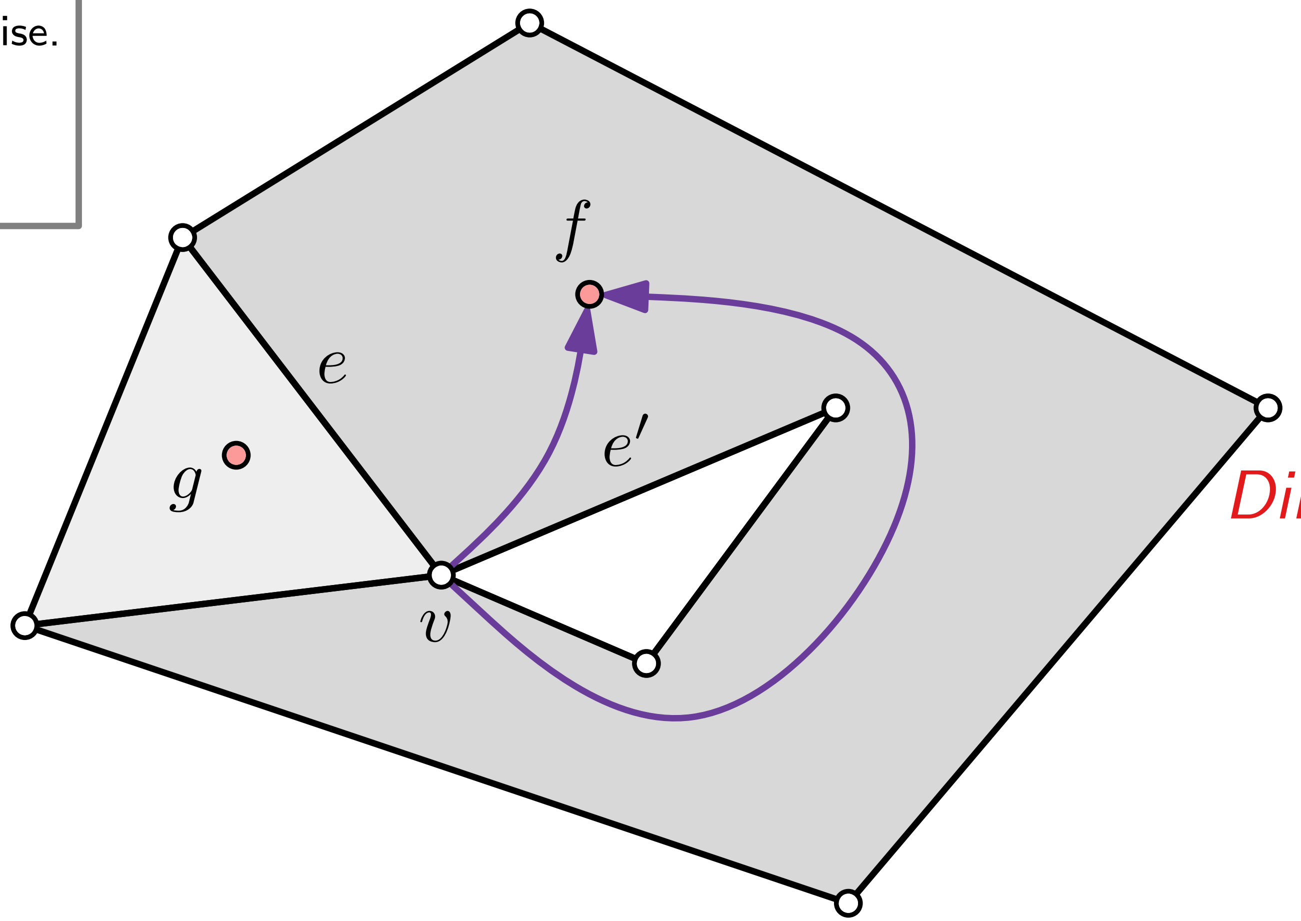

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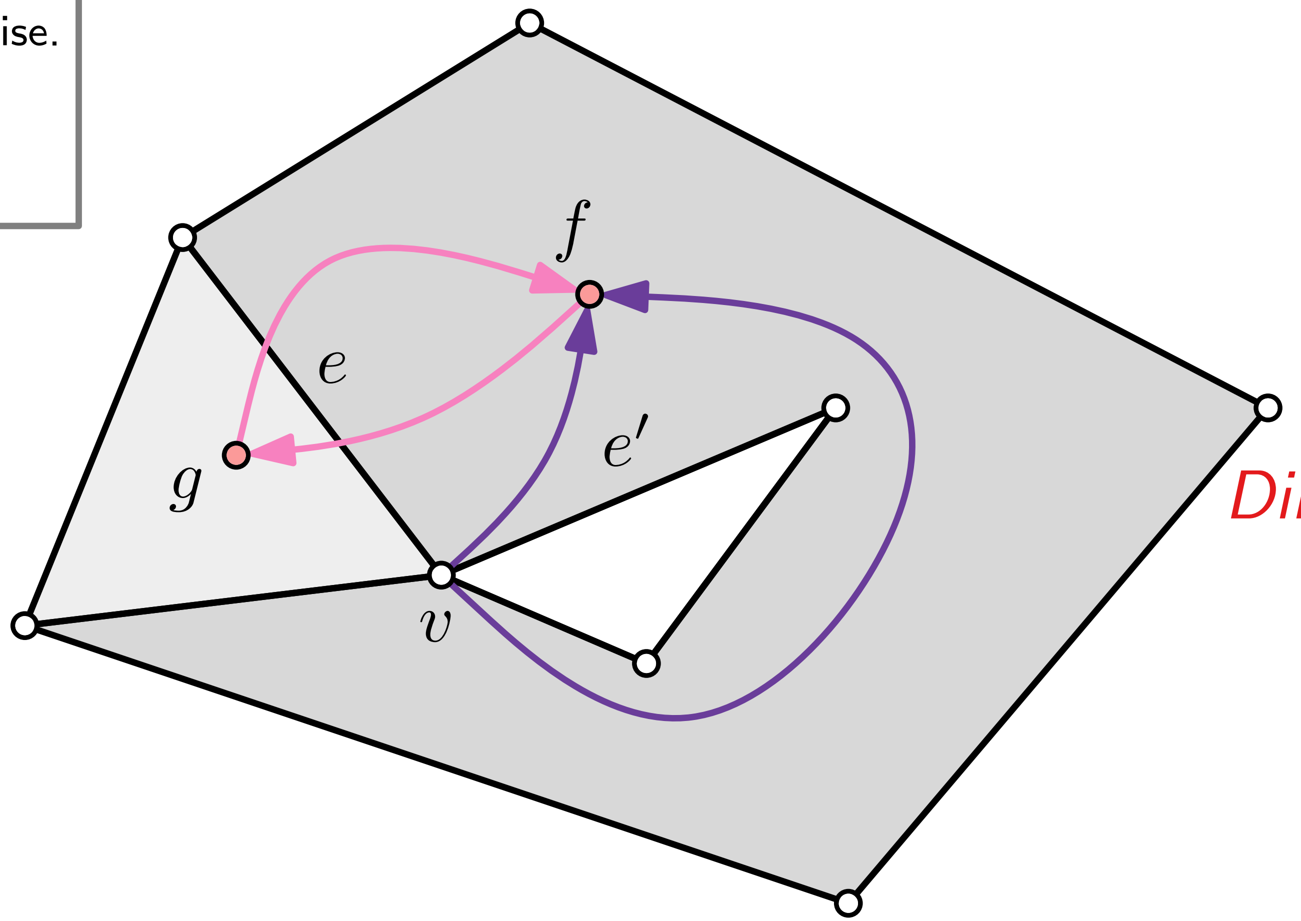

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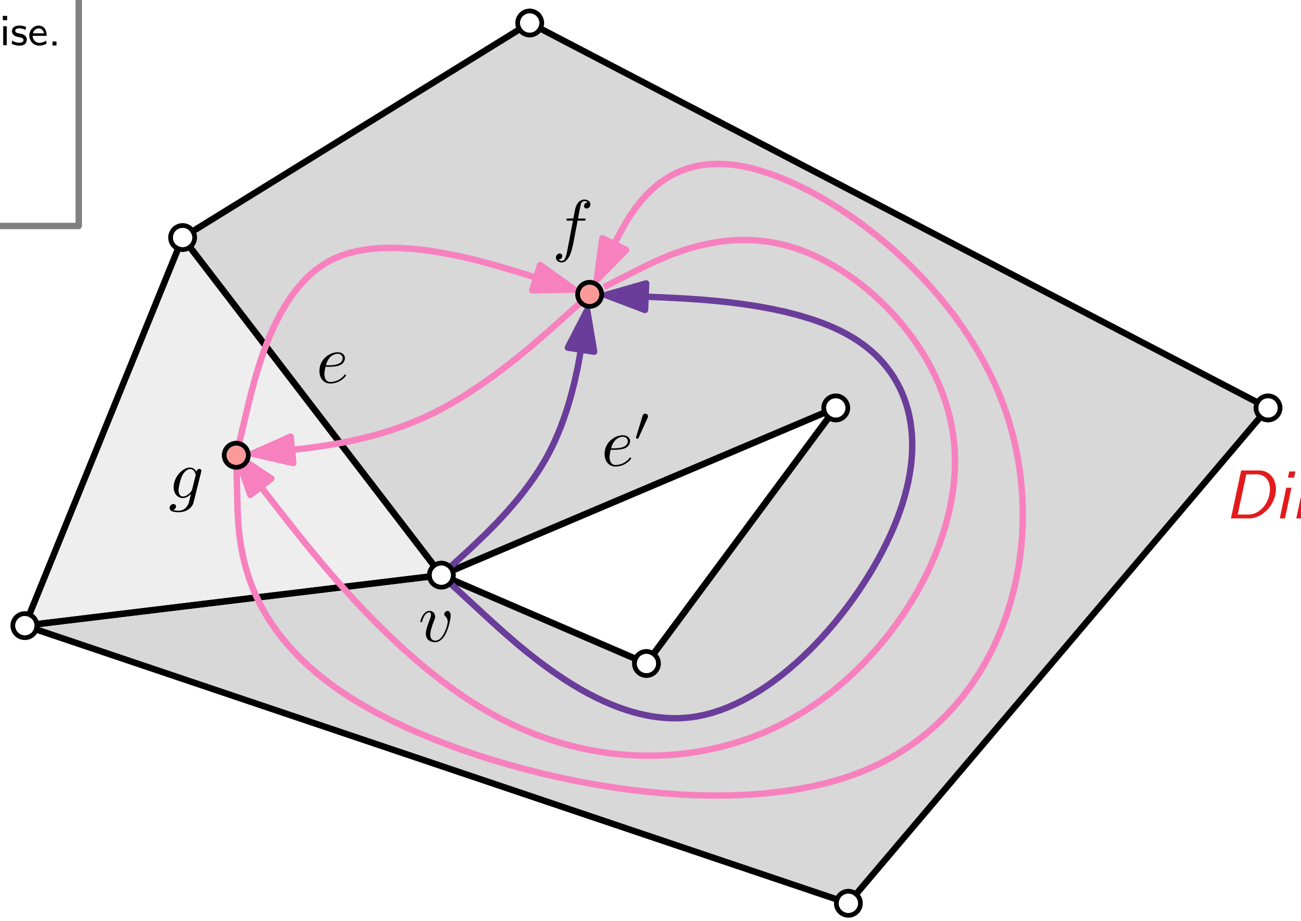

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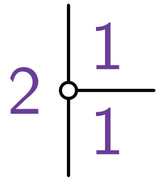

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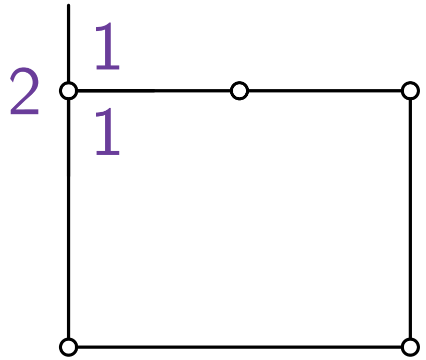

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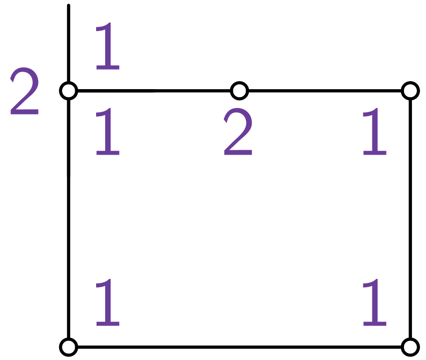

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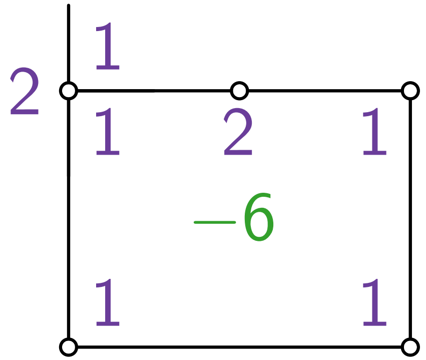

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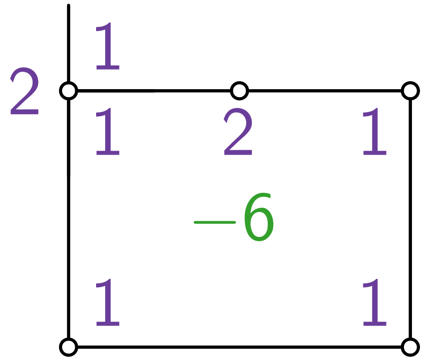

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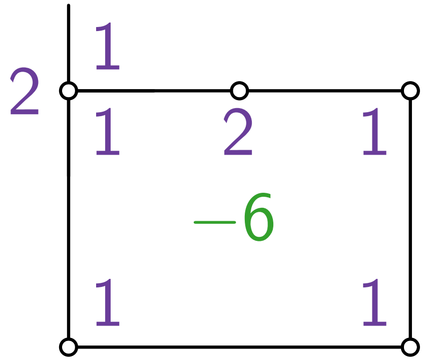

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Define flow network $N(G) = ((V \cup F, E'); b; \ell; u; \text{cost})$:

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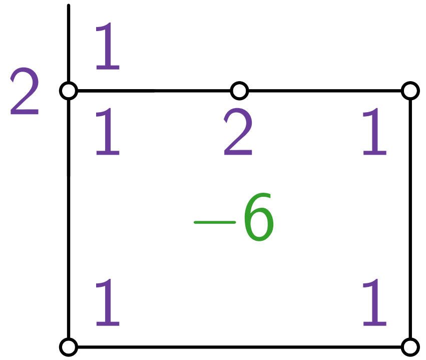

Flow Network for Bend Minimization

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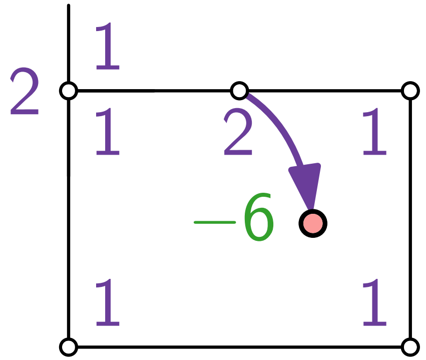

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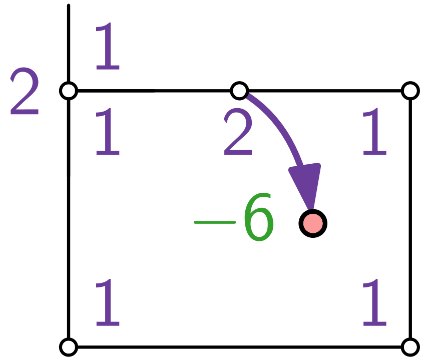

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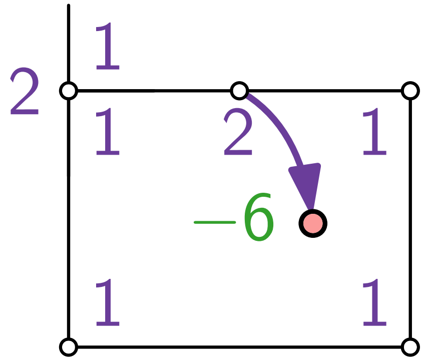

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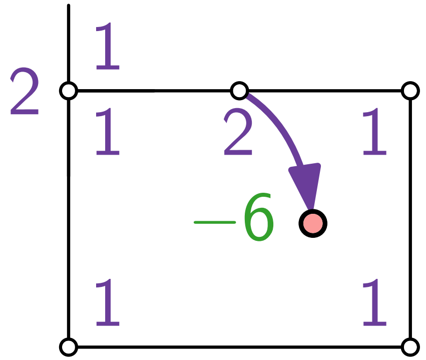

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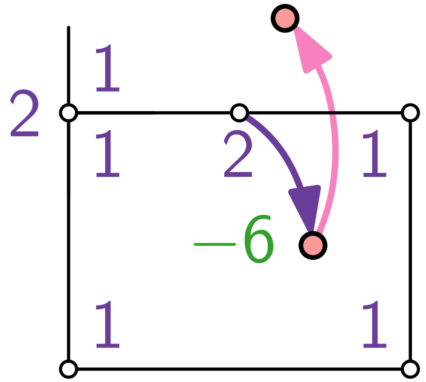

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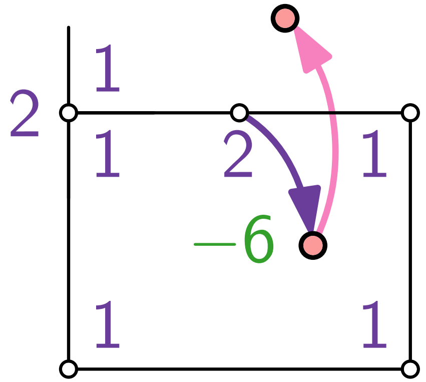

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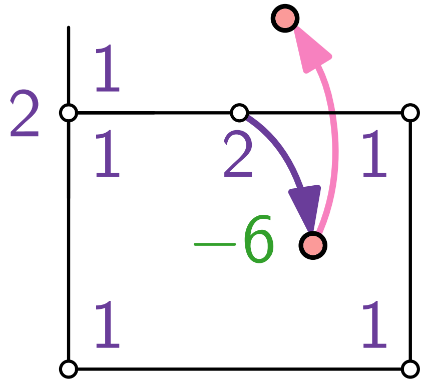

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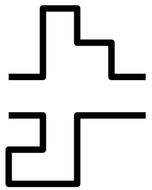

We model only the *number* of bends. Why is it enough?

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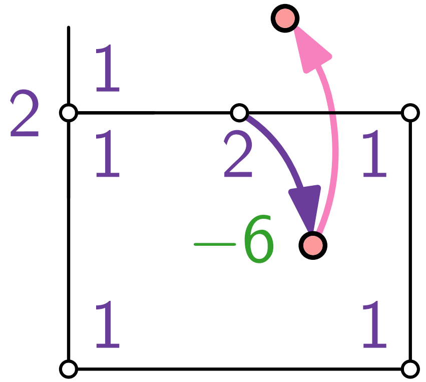

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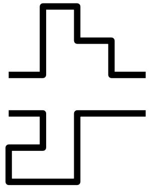

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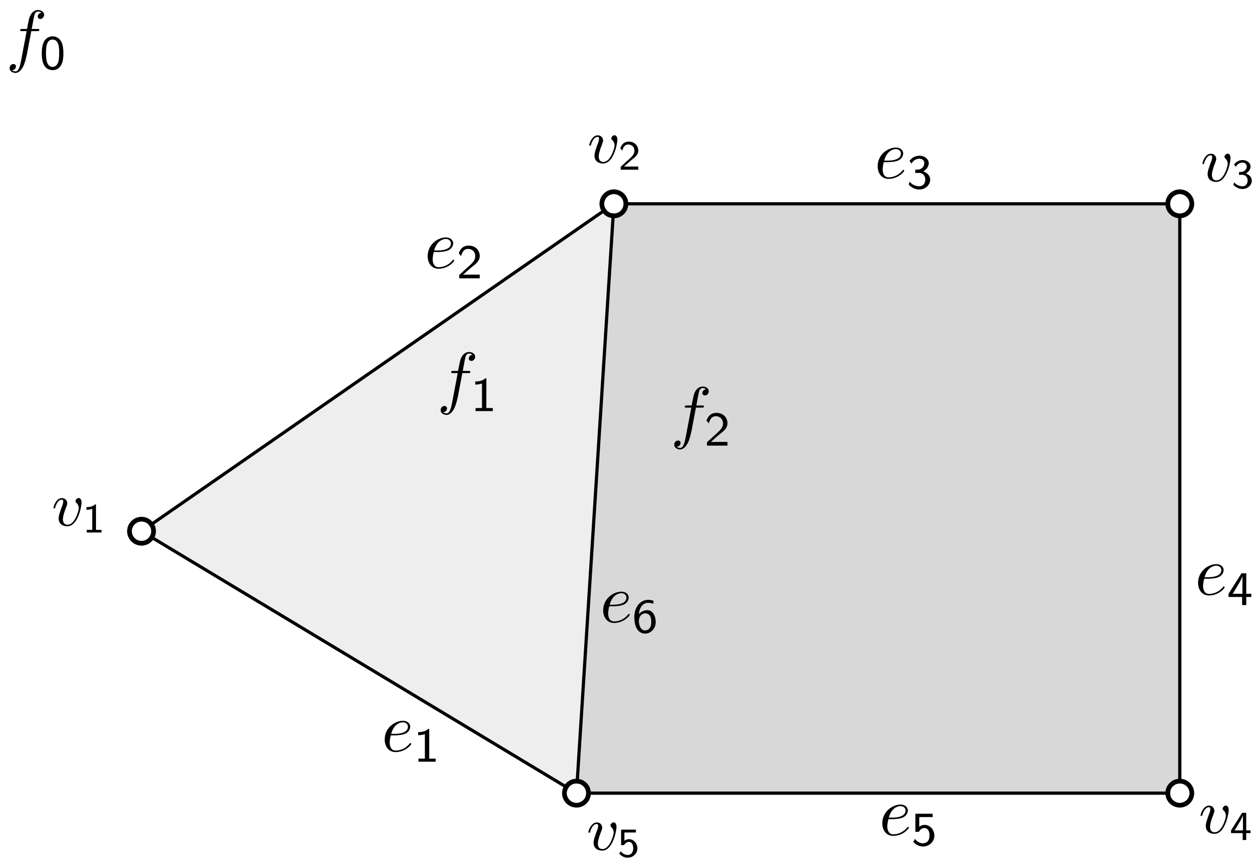
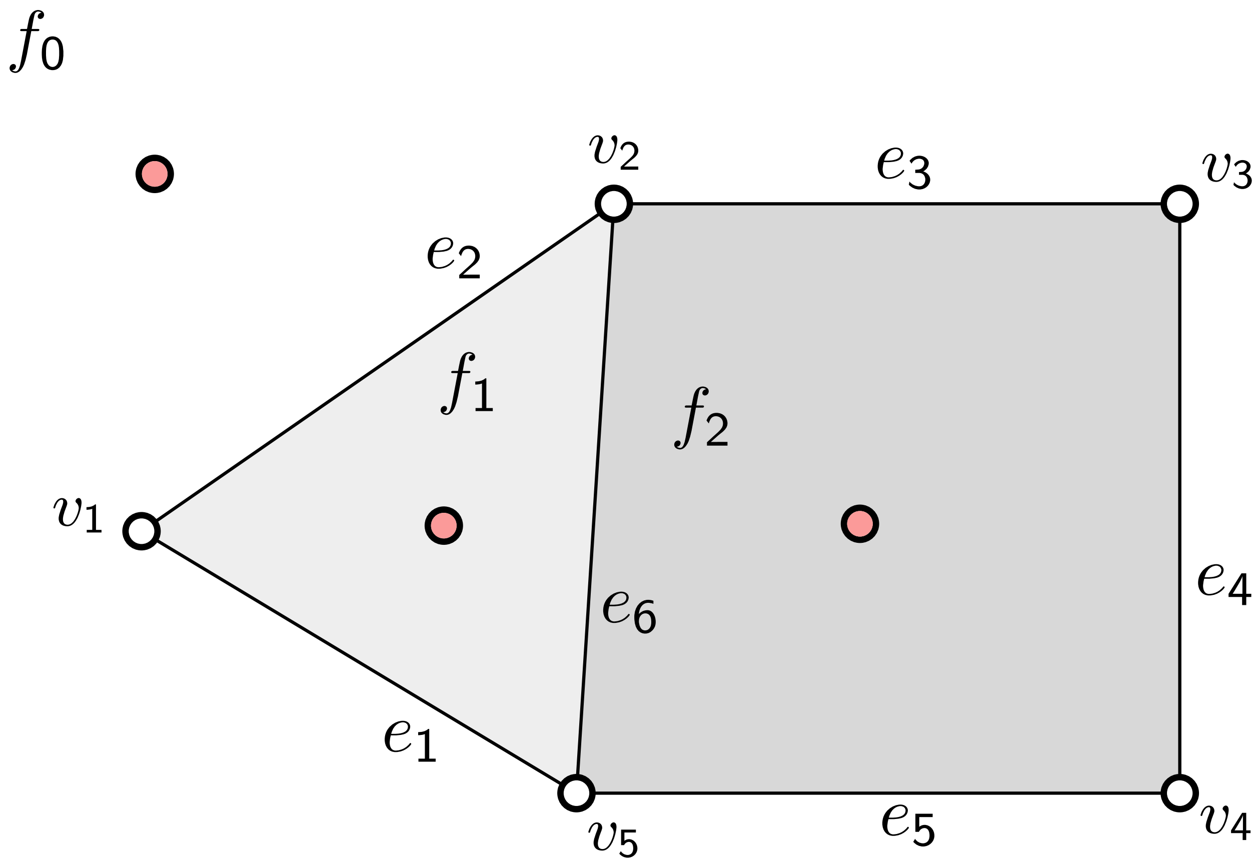
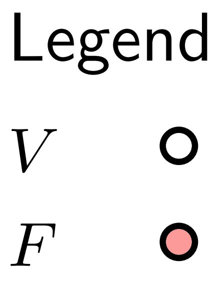
→ *Exercise!*

Flow Network Example



Flow Network Example





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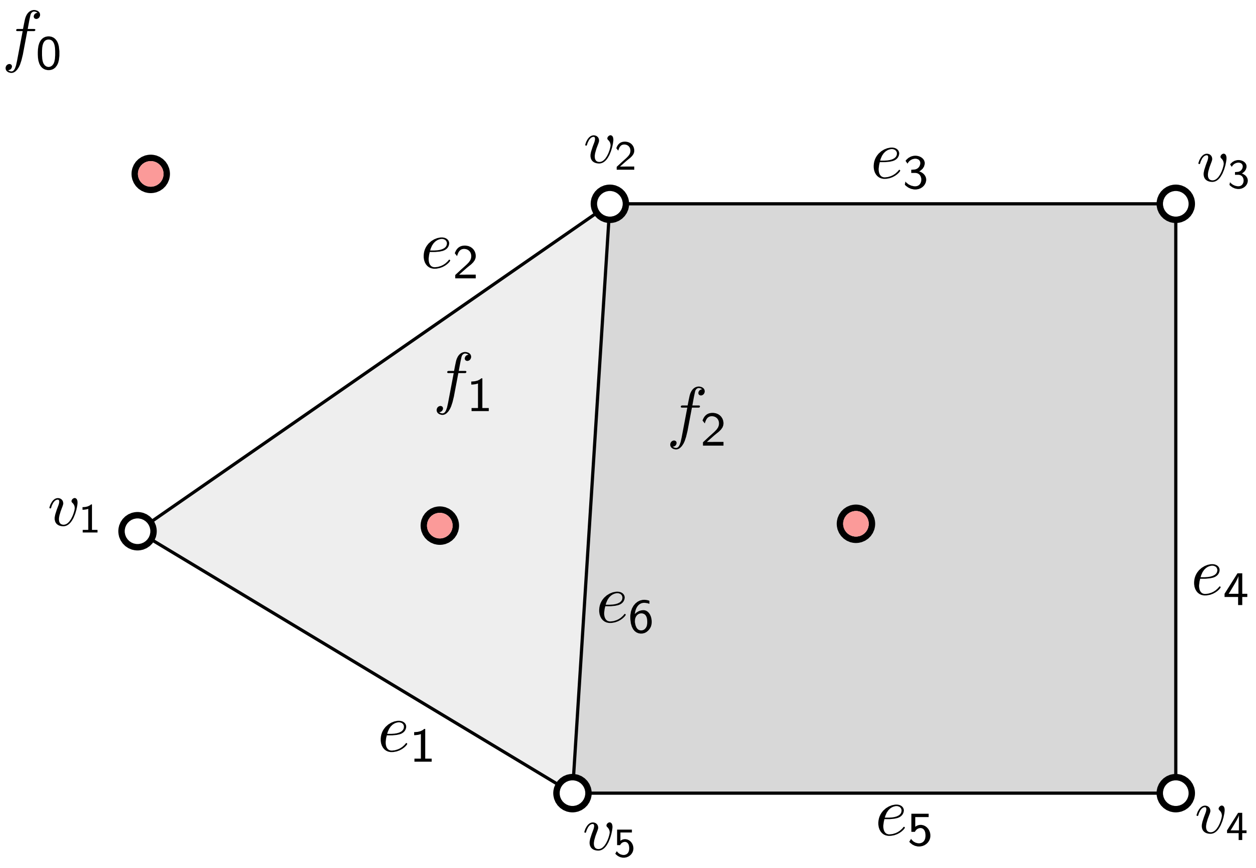

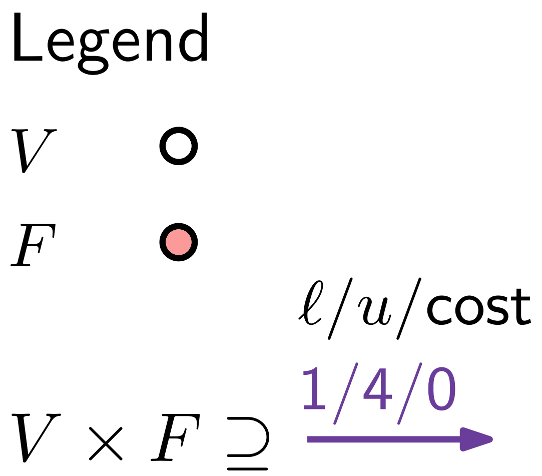

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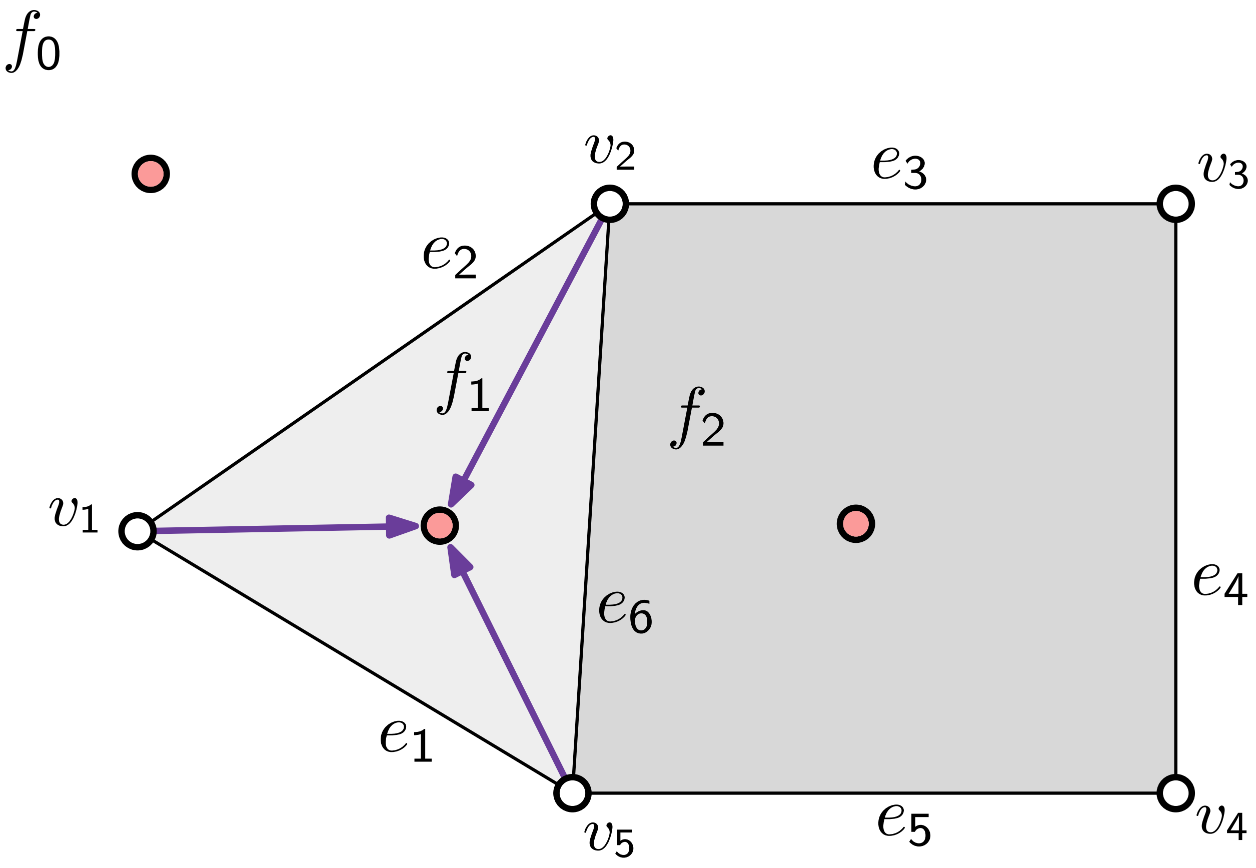


Legend

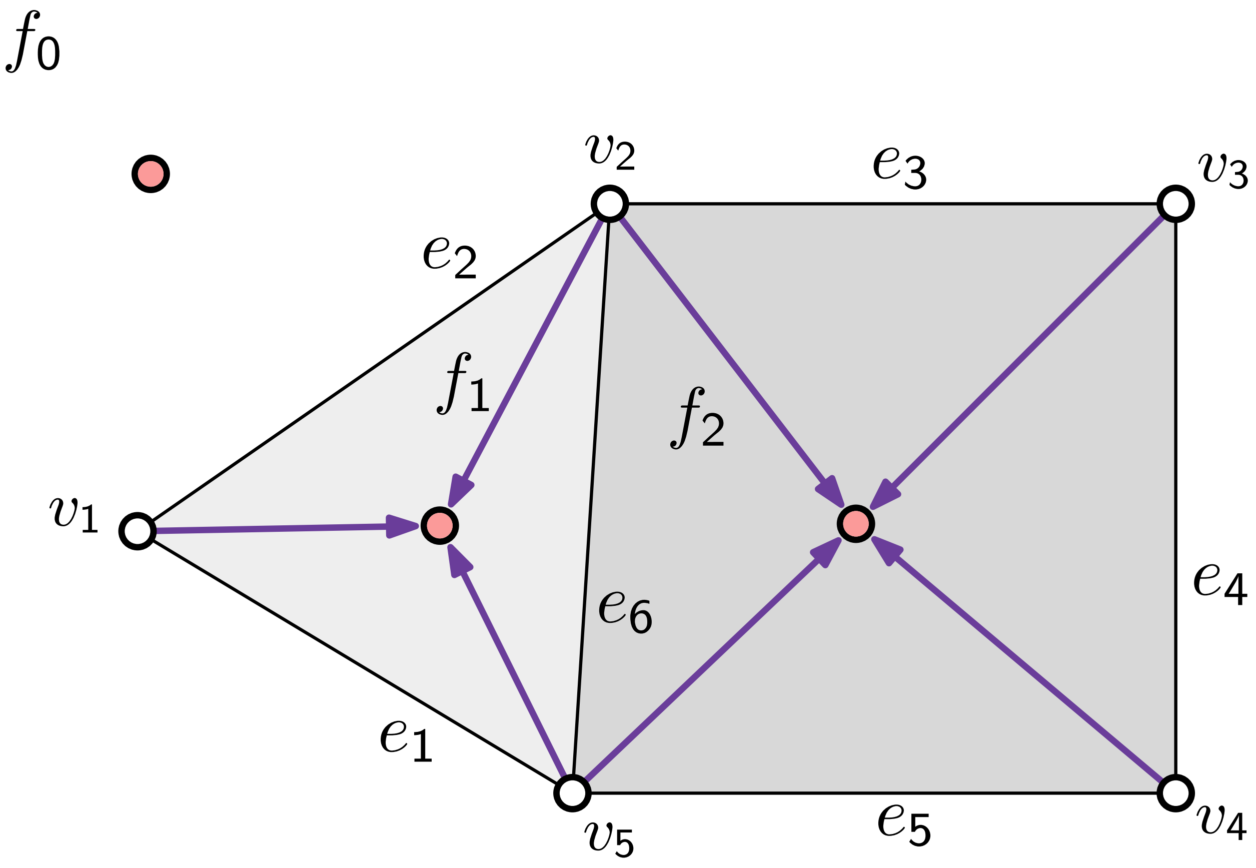
V ○

F ●

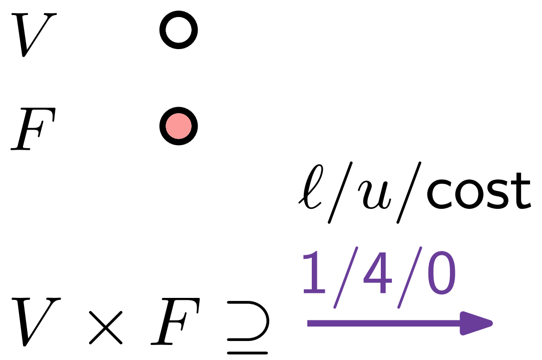
$\ell/u/\text{cost}$

$V \times F \supseteq$ 1/4/0 →

Flow Network Example



Legend



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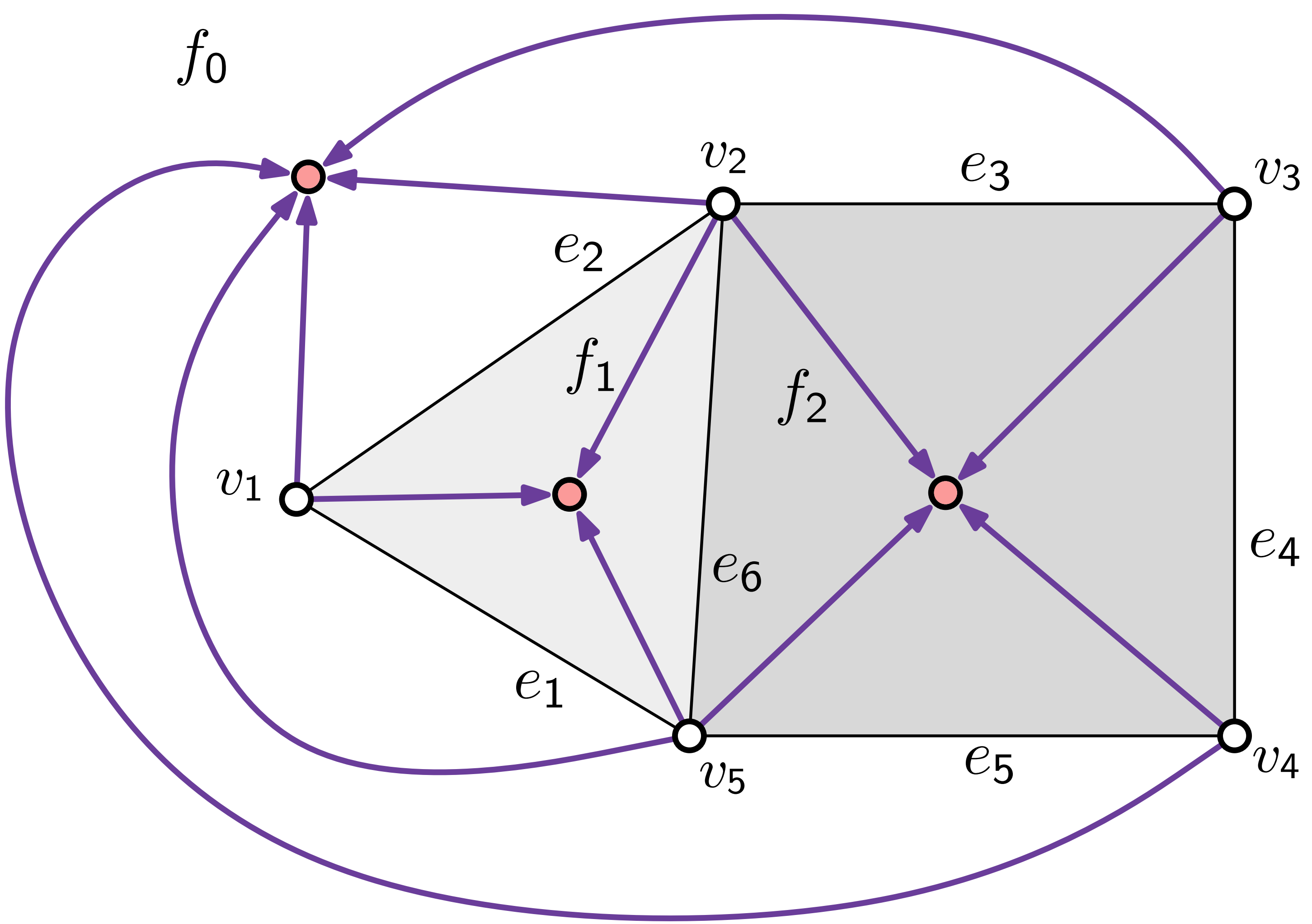

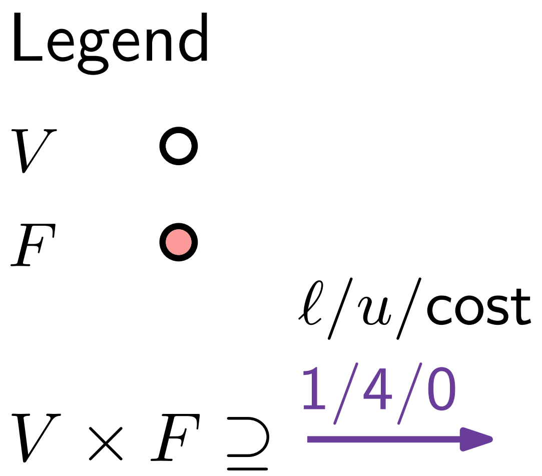

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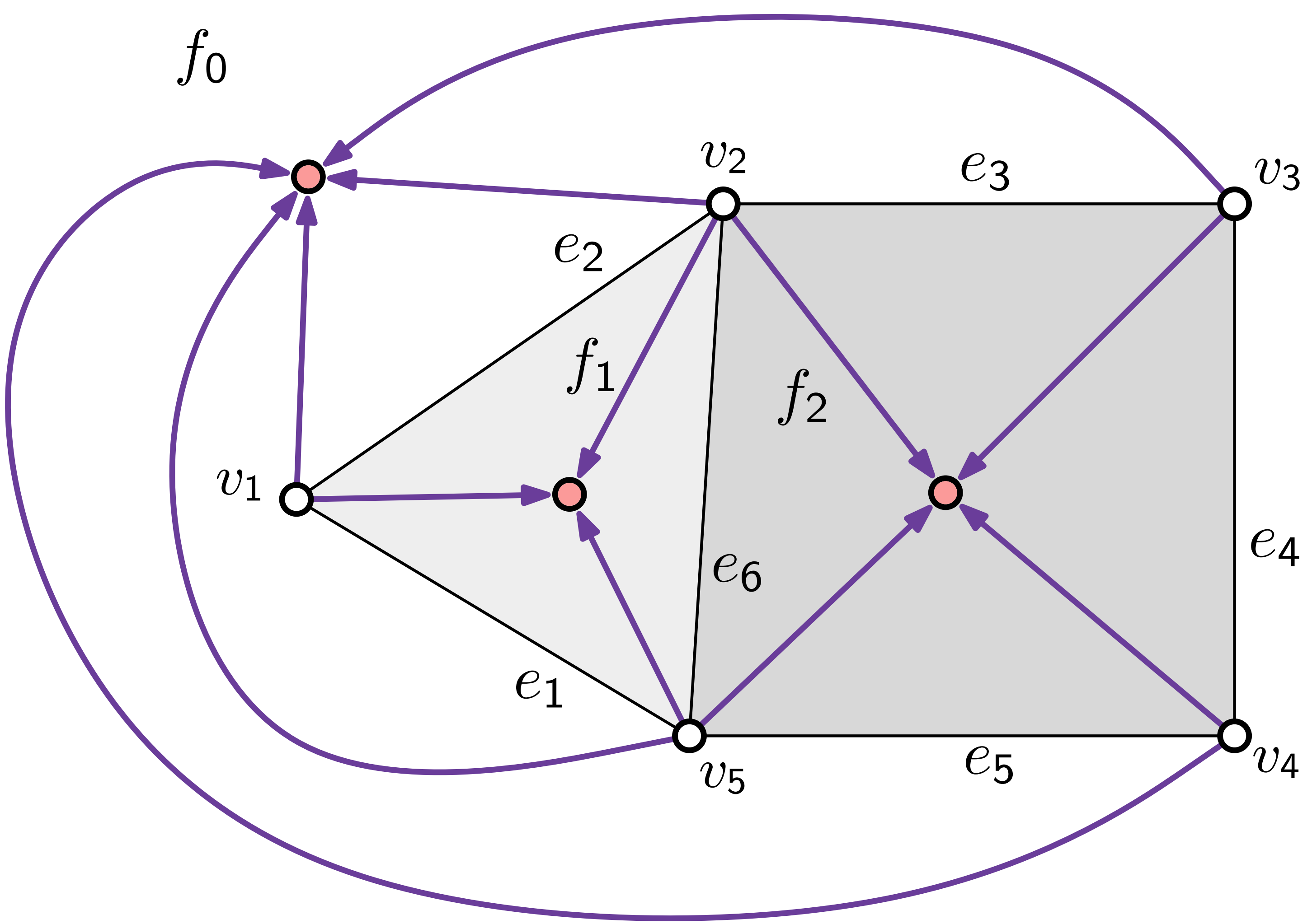

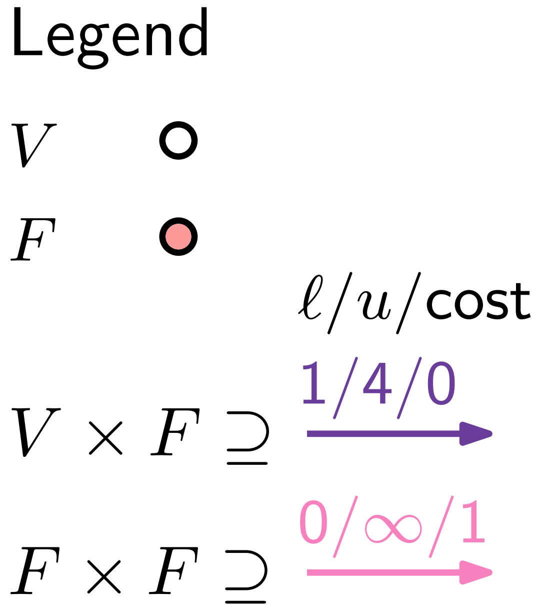

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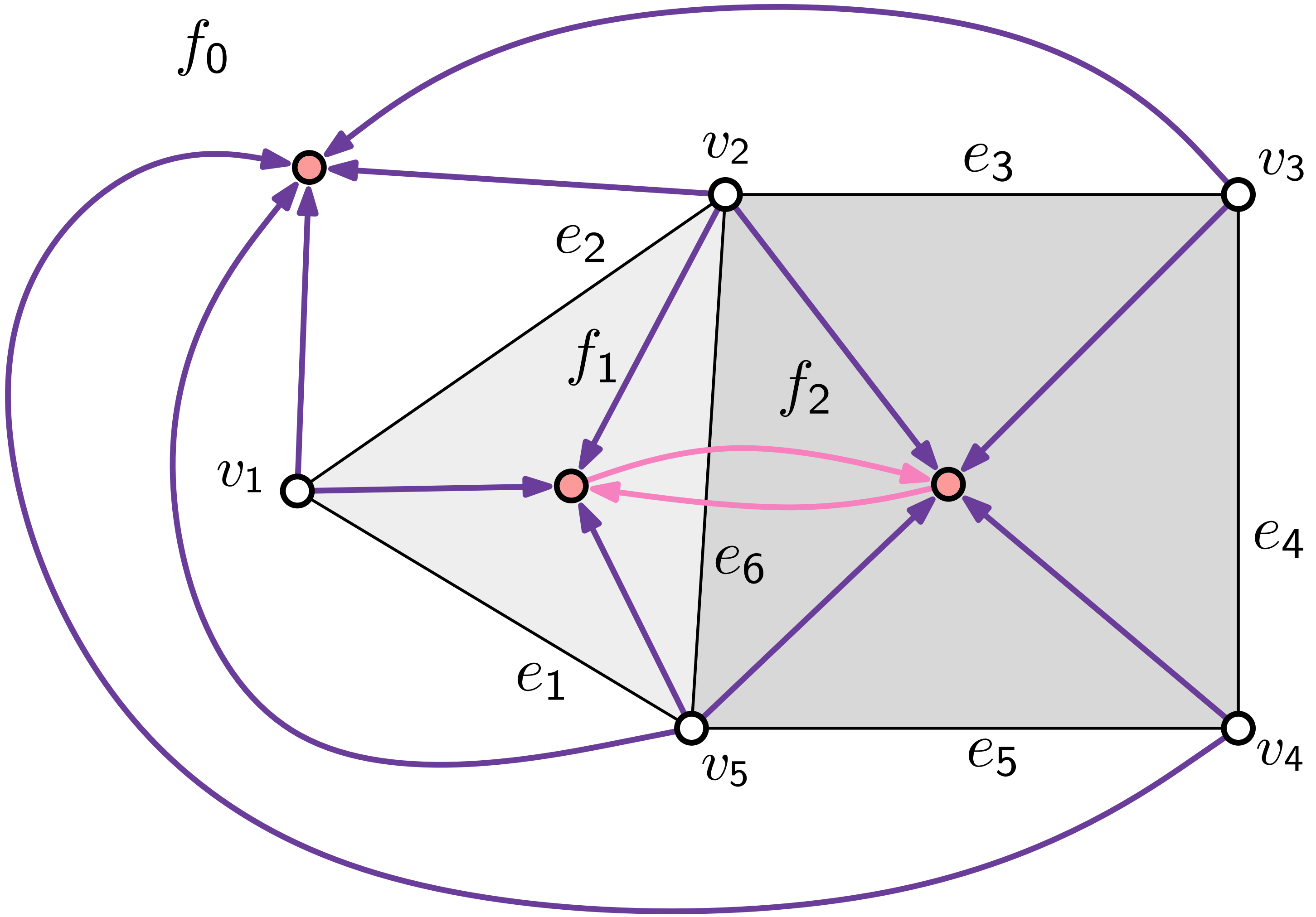

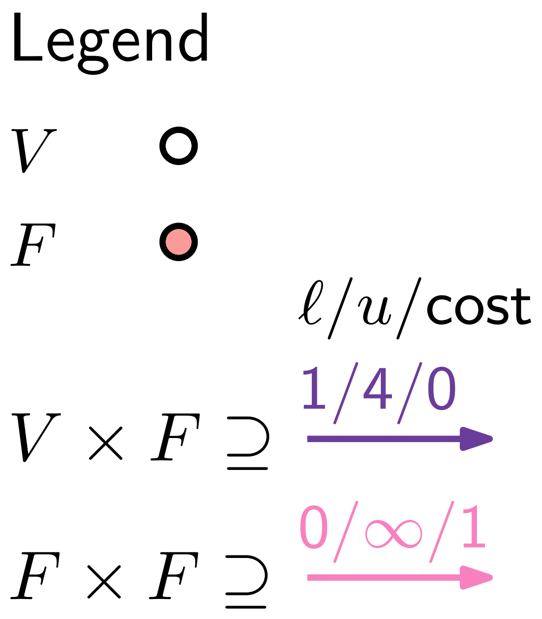

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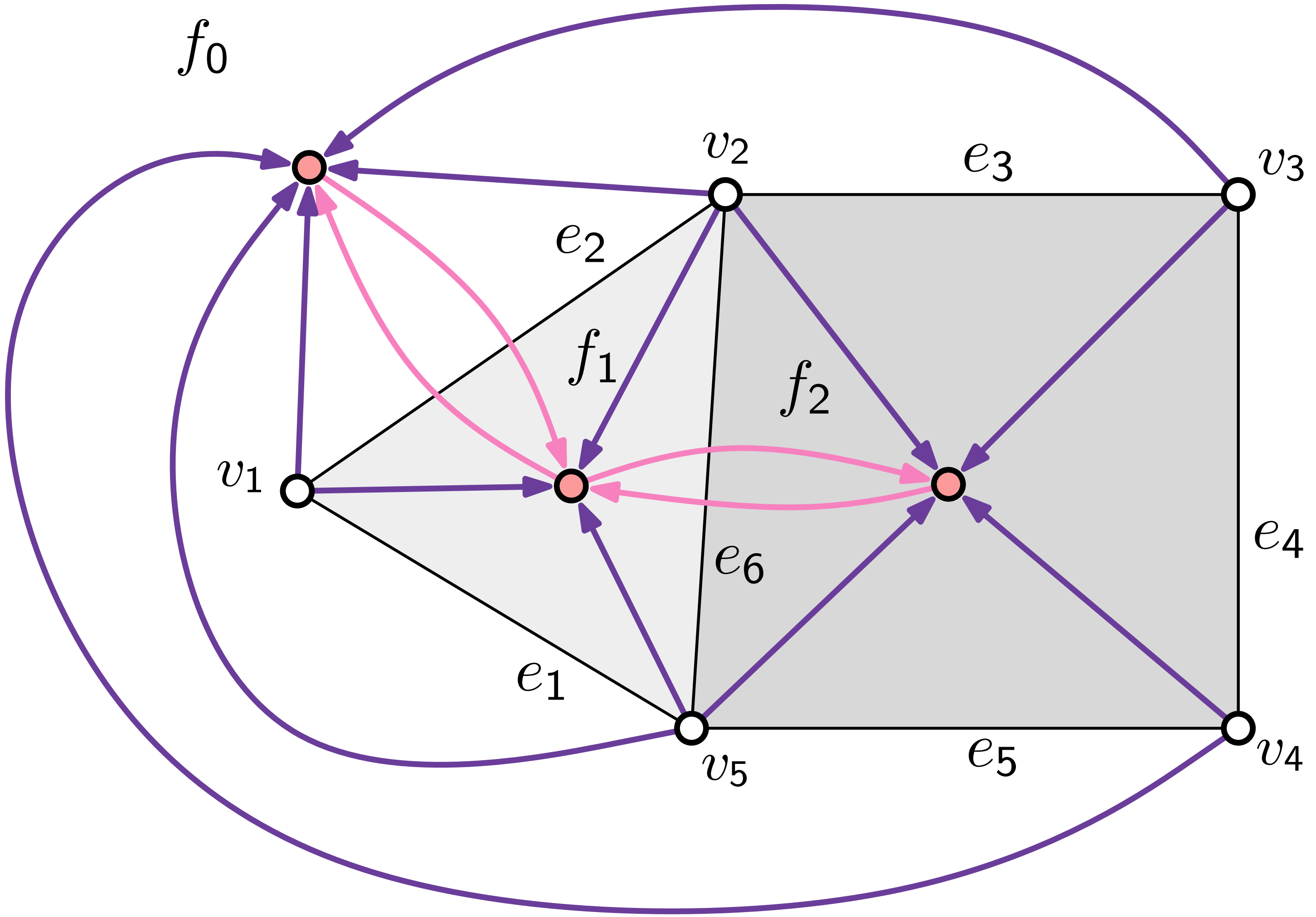

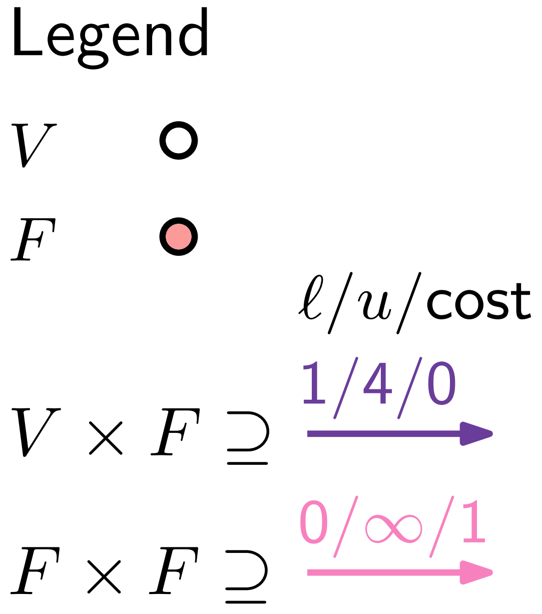

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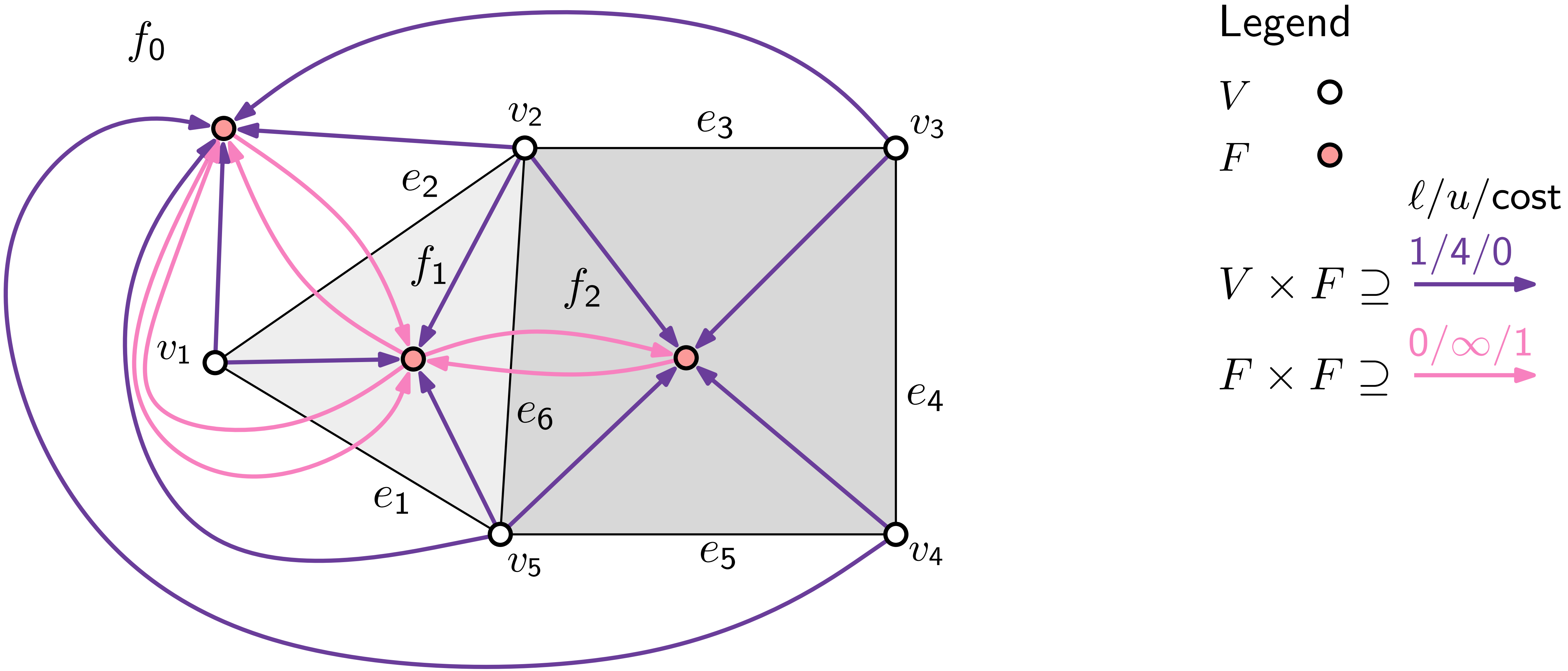

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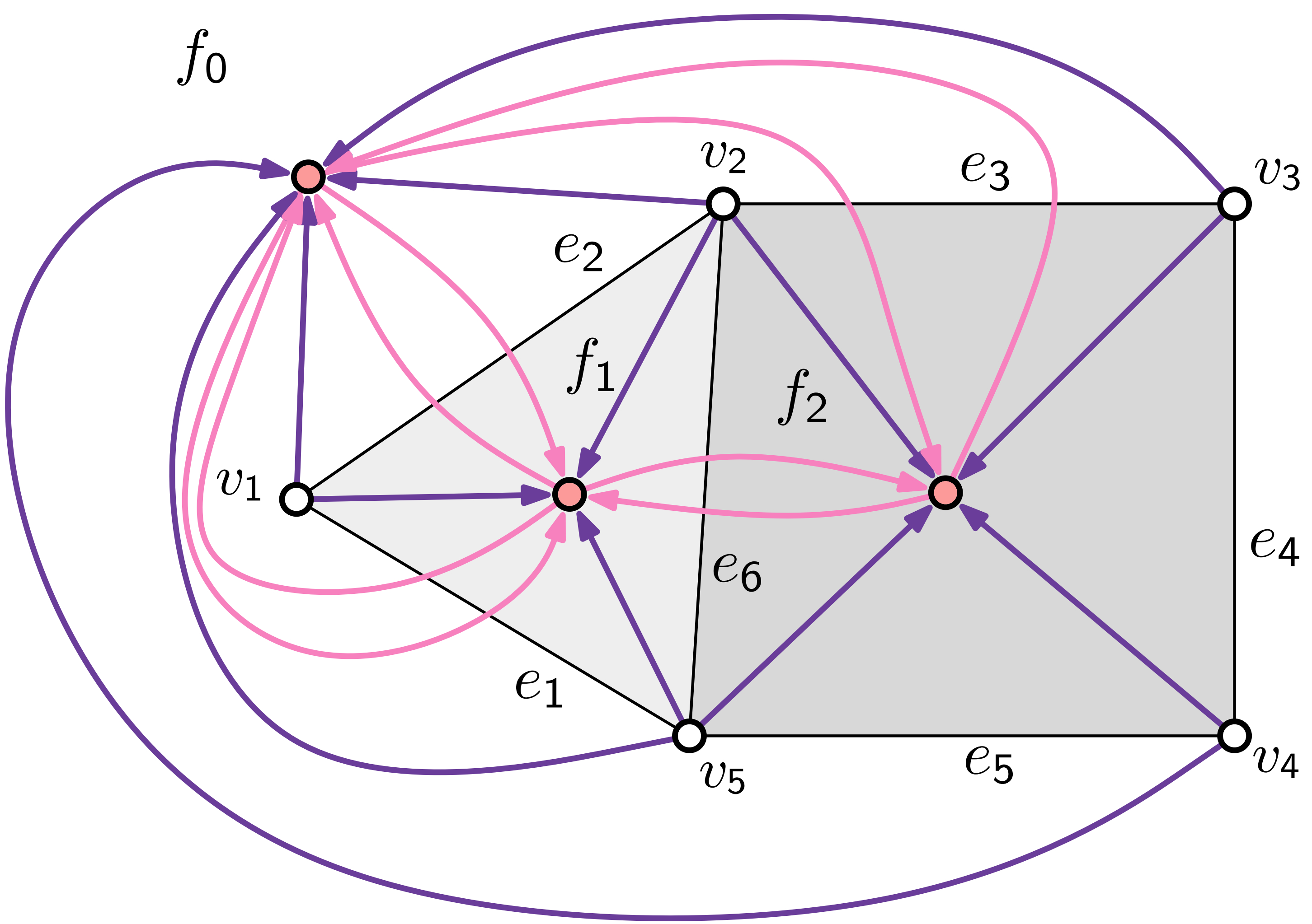

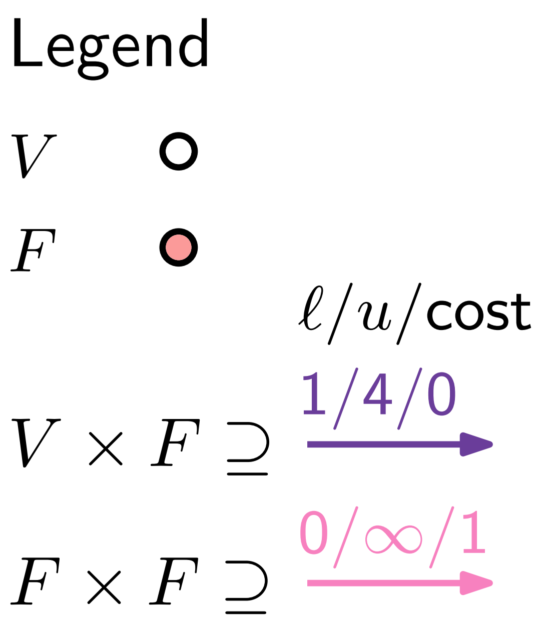

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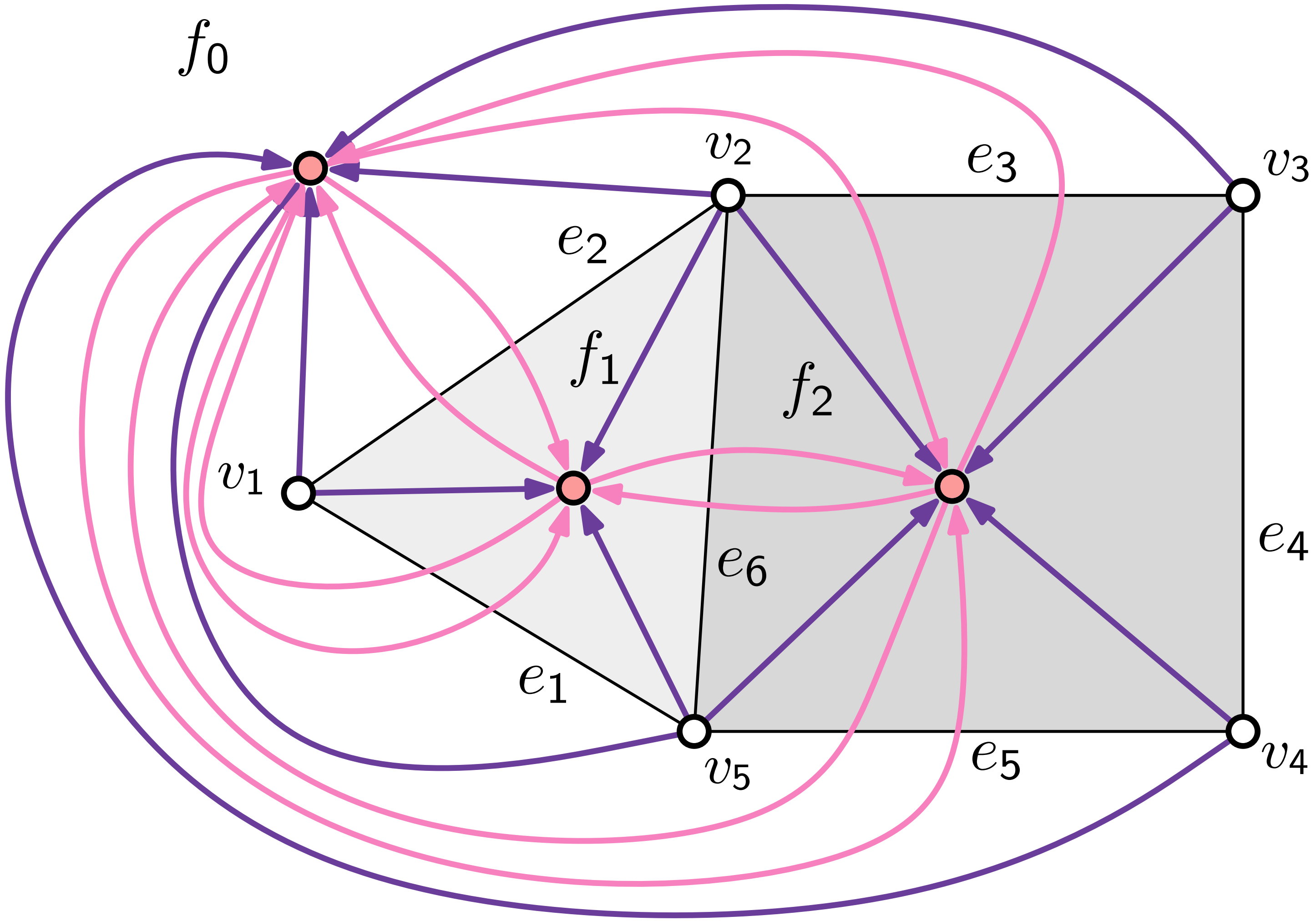


Legend

V ○

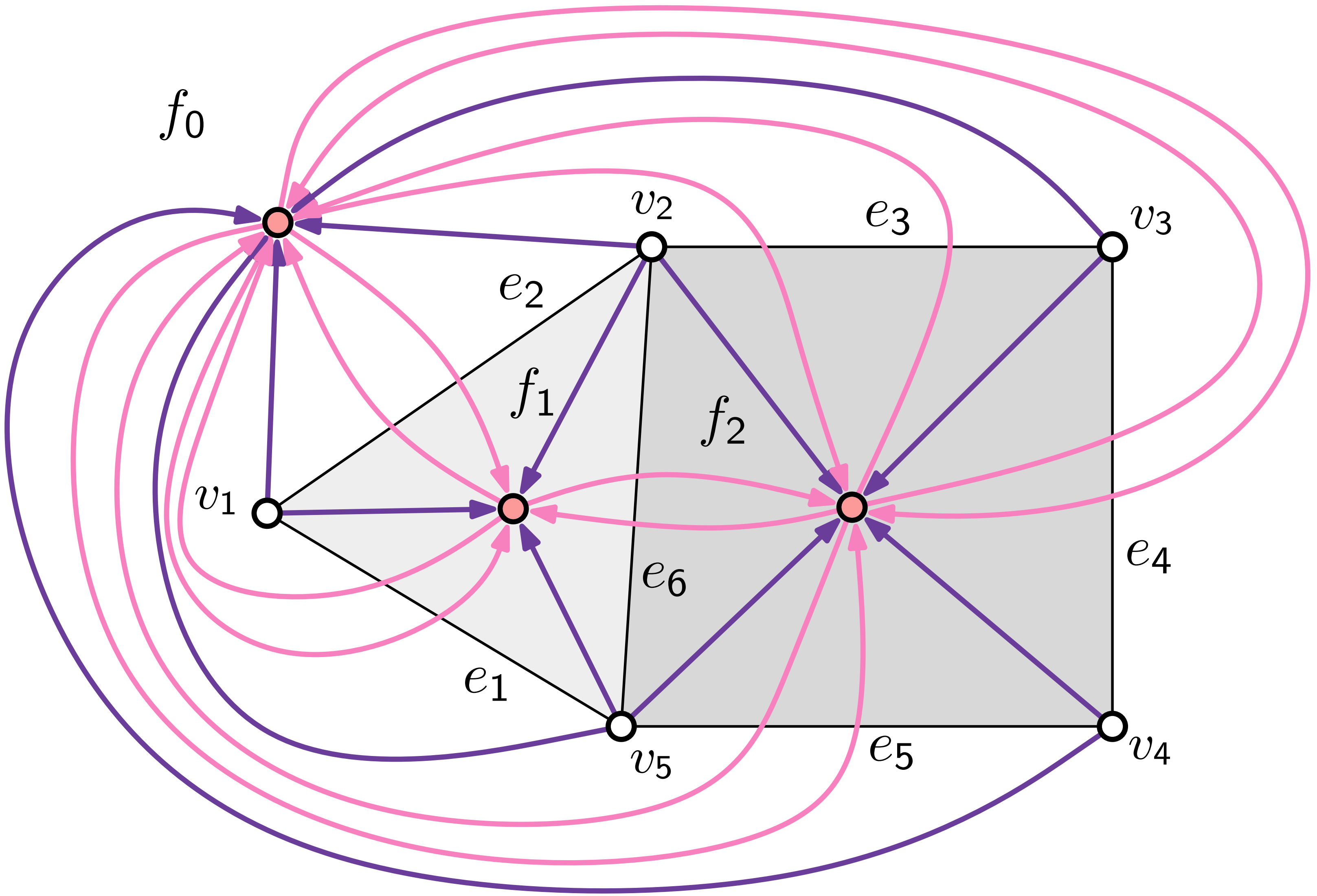
F ●

$\ell/u/\text{cost}$

$V \times F \supseteq$ 1/4/0

$F \times F \supseteq$ $0/\infty/1$

Flow Network Example



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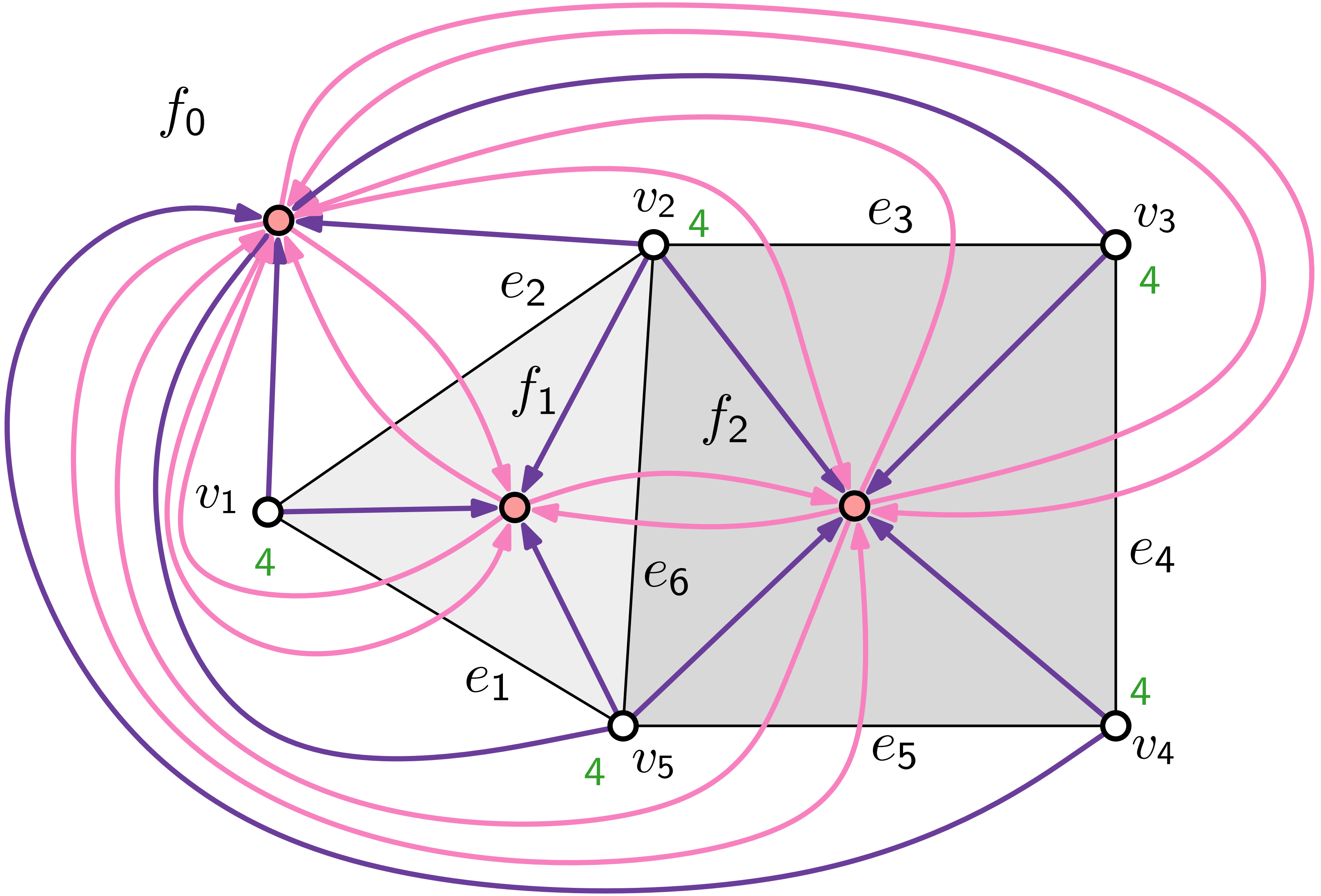

Legend

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F

$\ell/u/$cost

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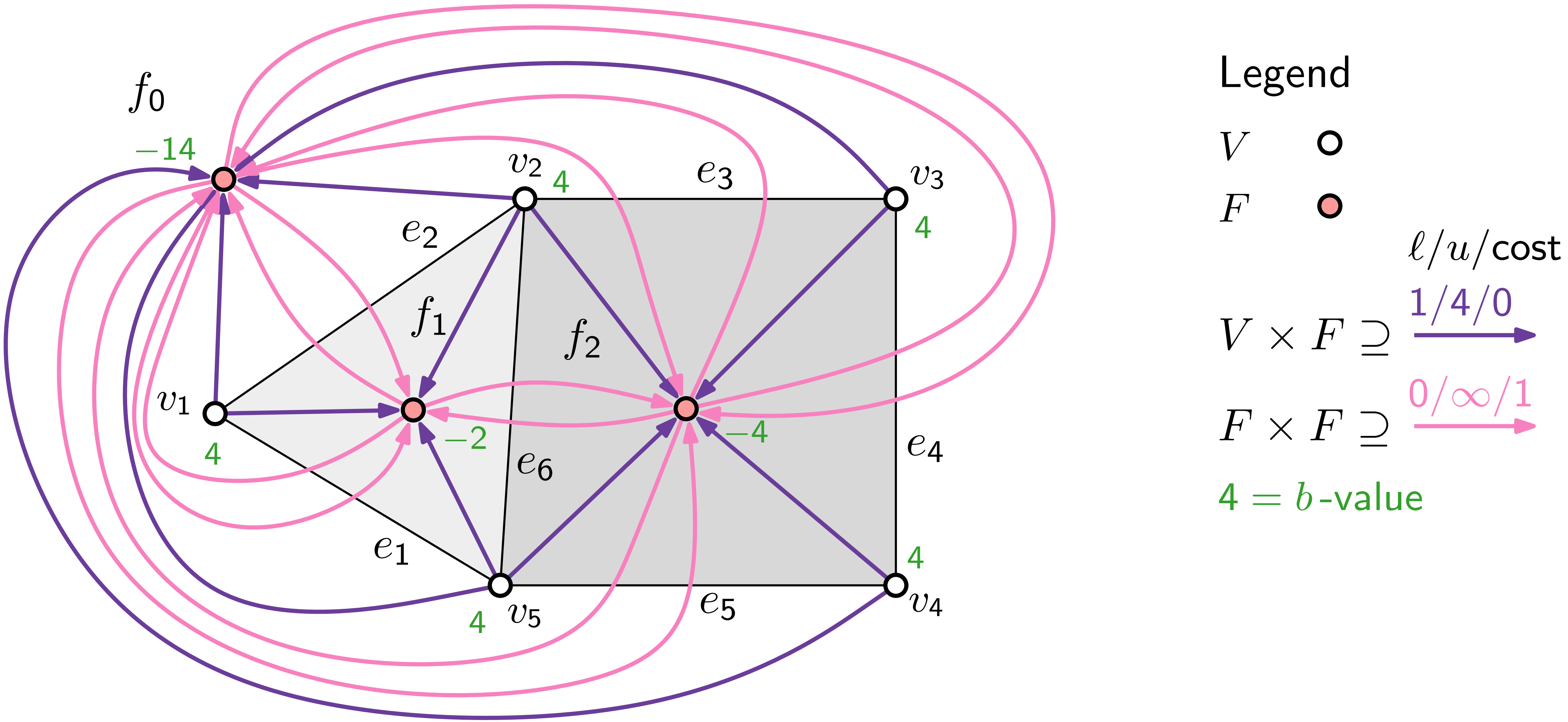
$4 = b$-value

Flow Network Example



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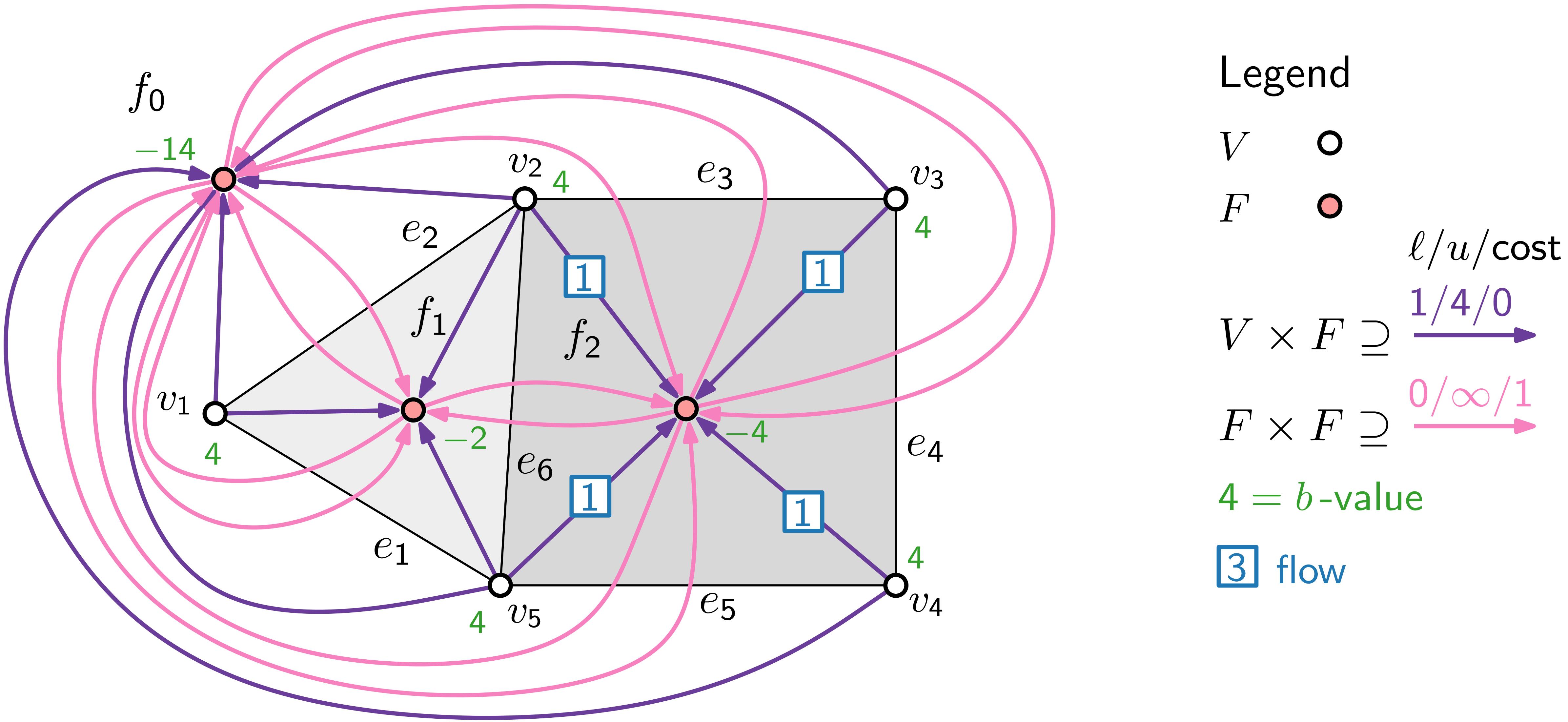

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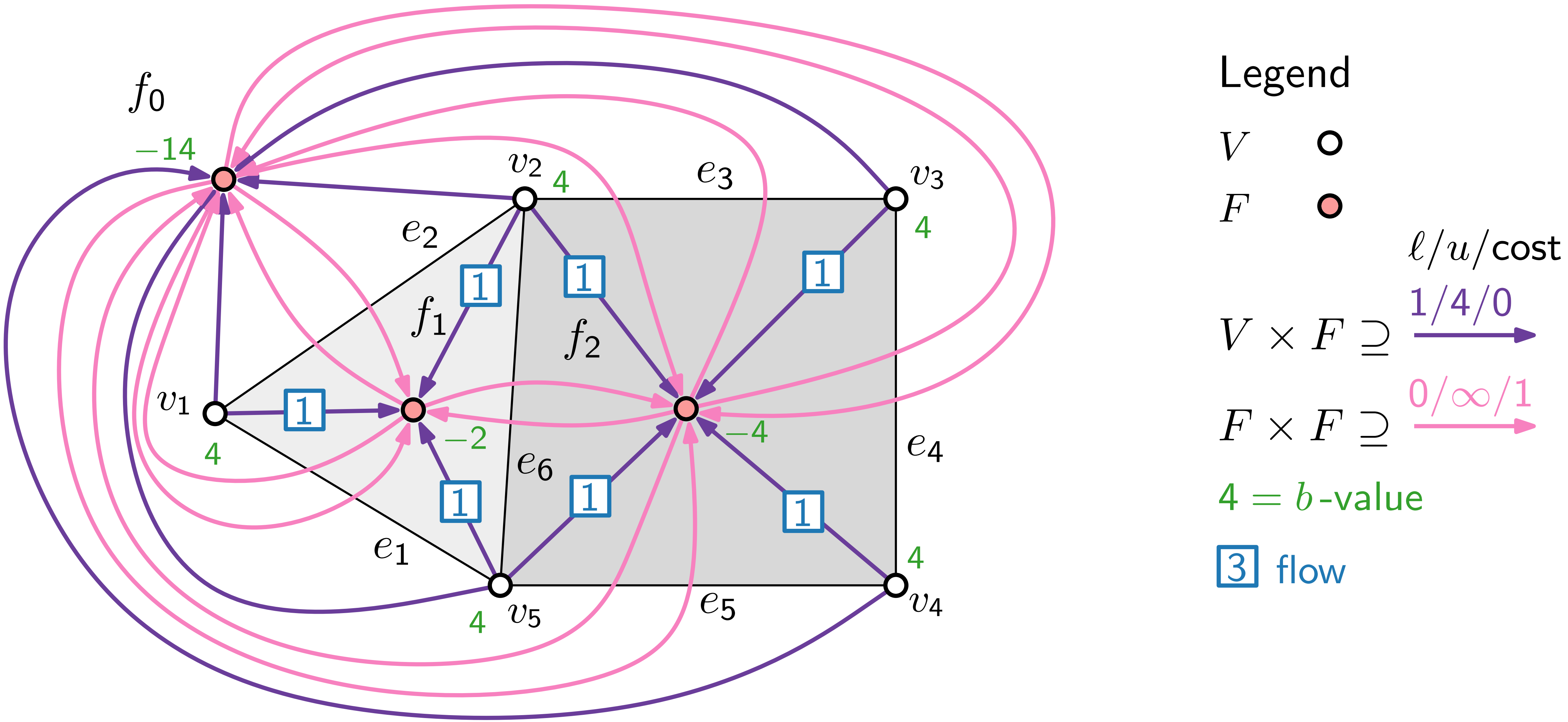

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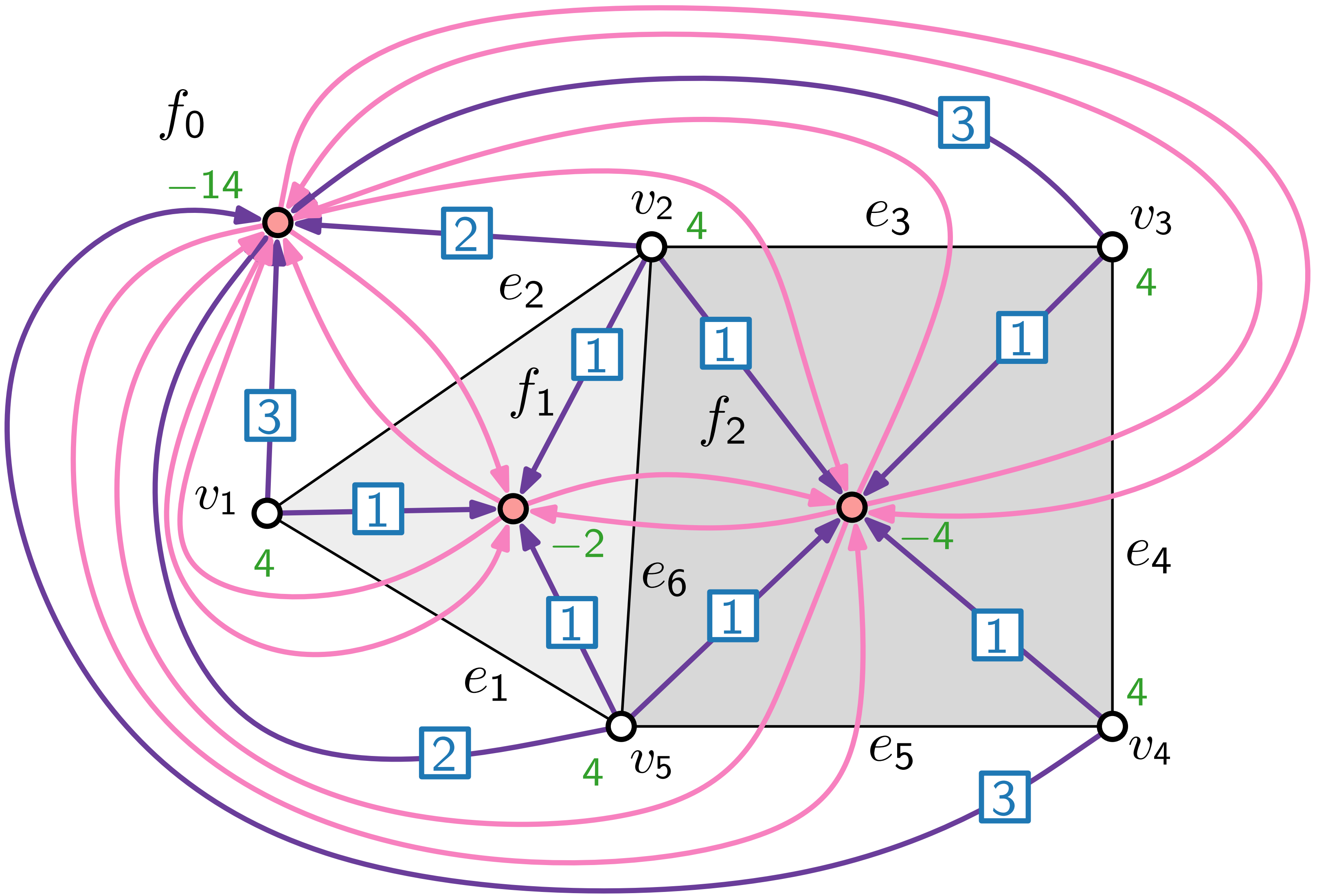

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$\ell/u/$cost

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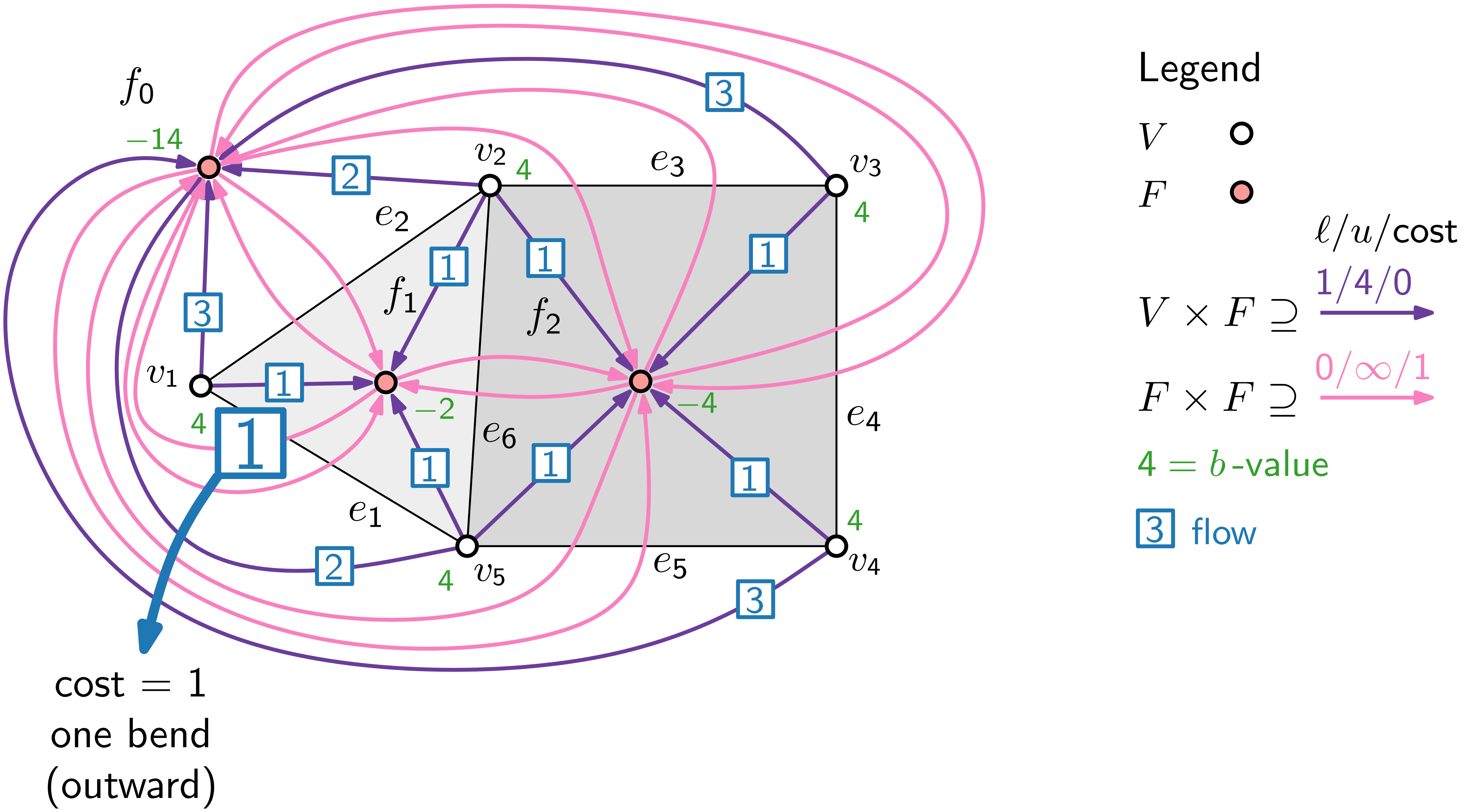
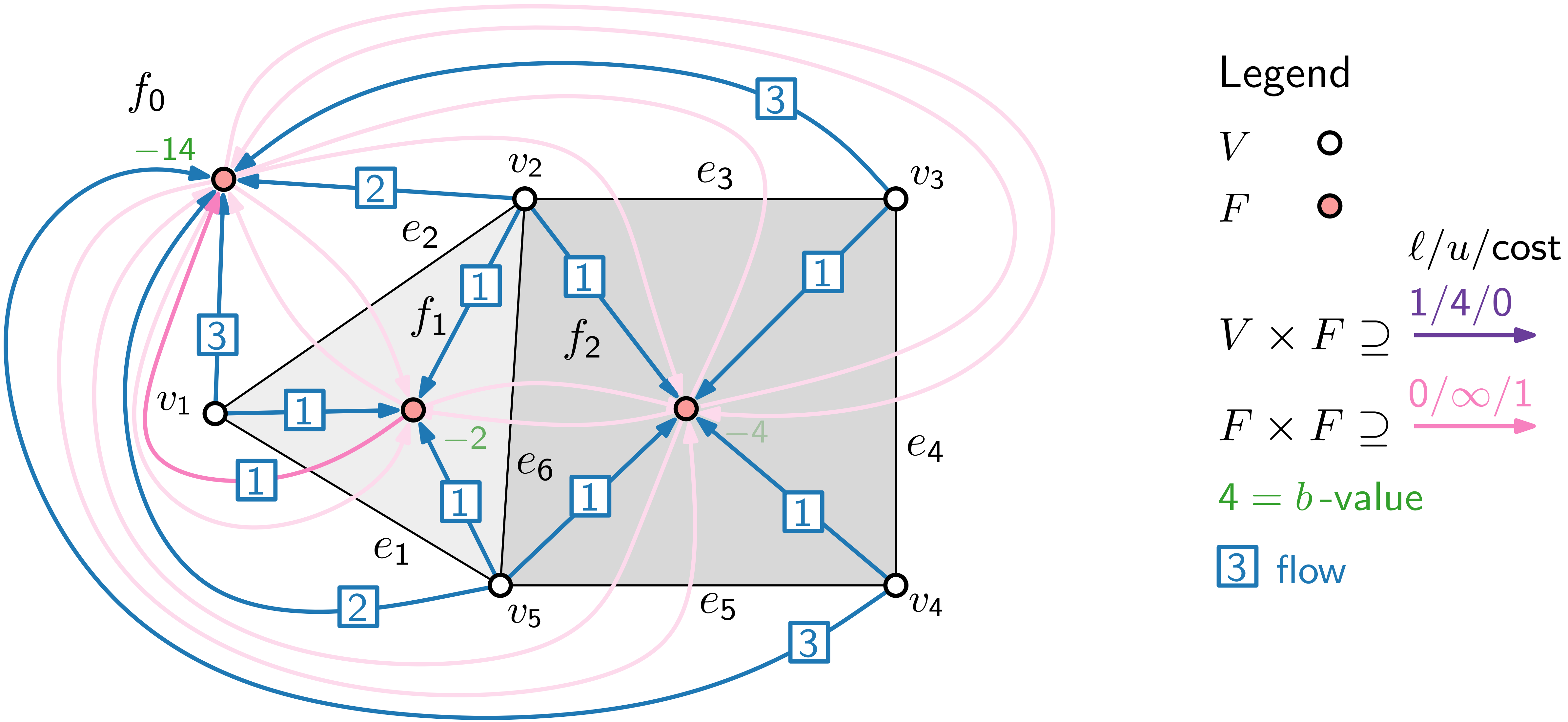
$4 = b$-value

3 flow

Flow Network Example



Flow Network Example



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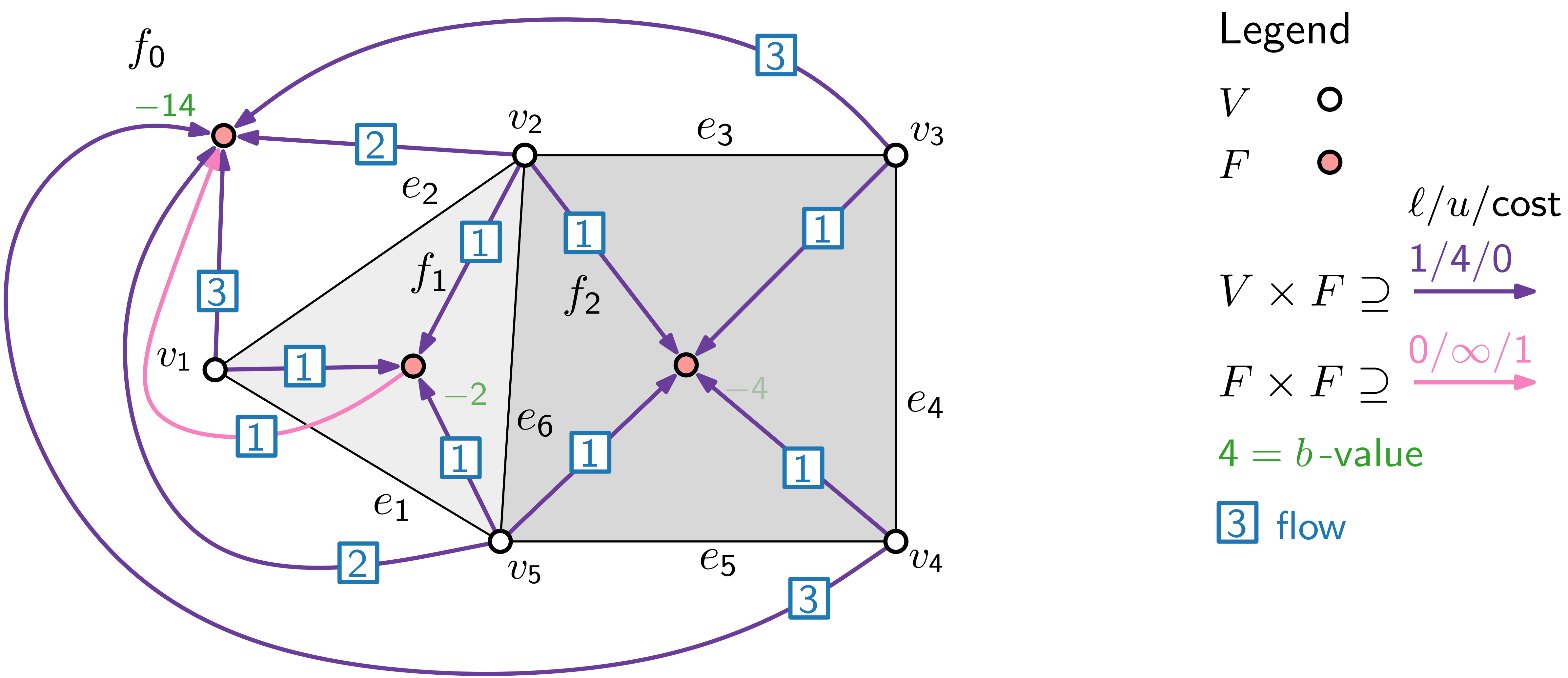

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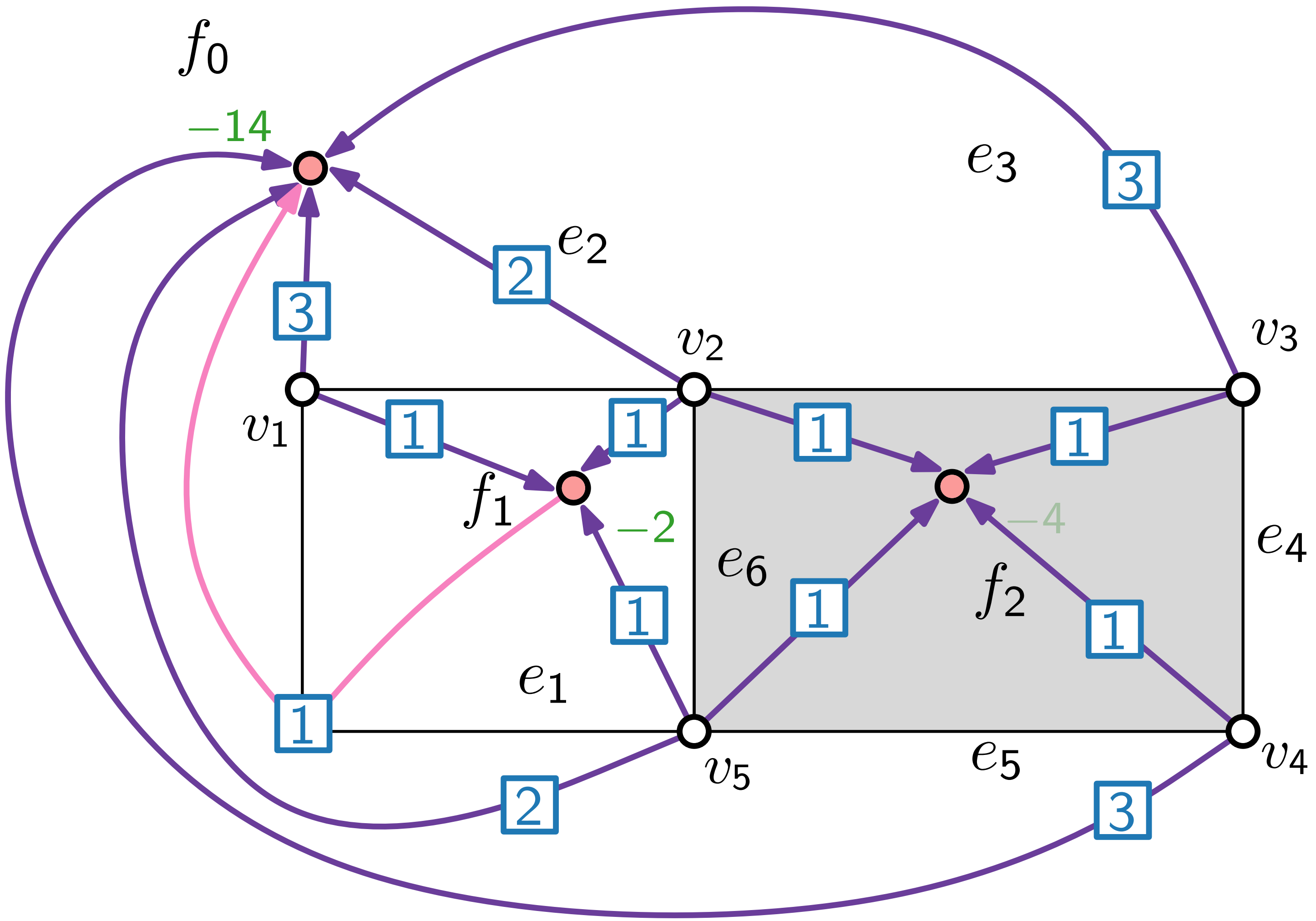

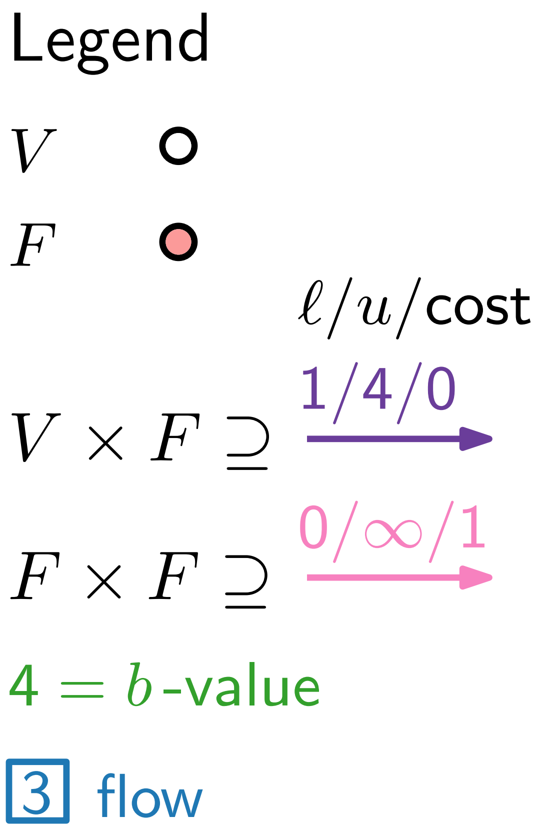

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Theorem. [Tamassia '87]

A plane graph (G, F, f_0) has a valid orthogonal representation $H(G)$ with k bends. $\Leftrightarrow$

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(H1) $H(G)$ corresponds to F, f_0.

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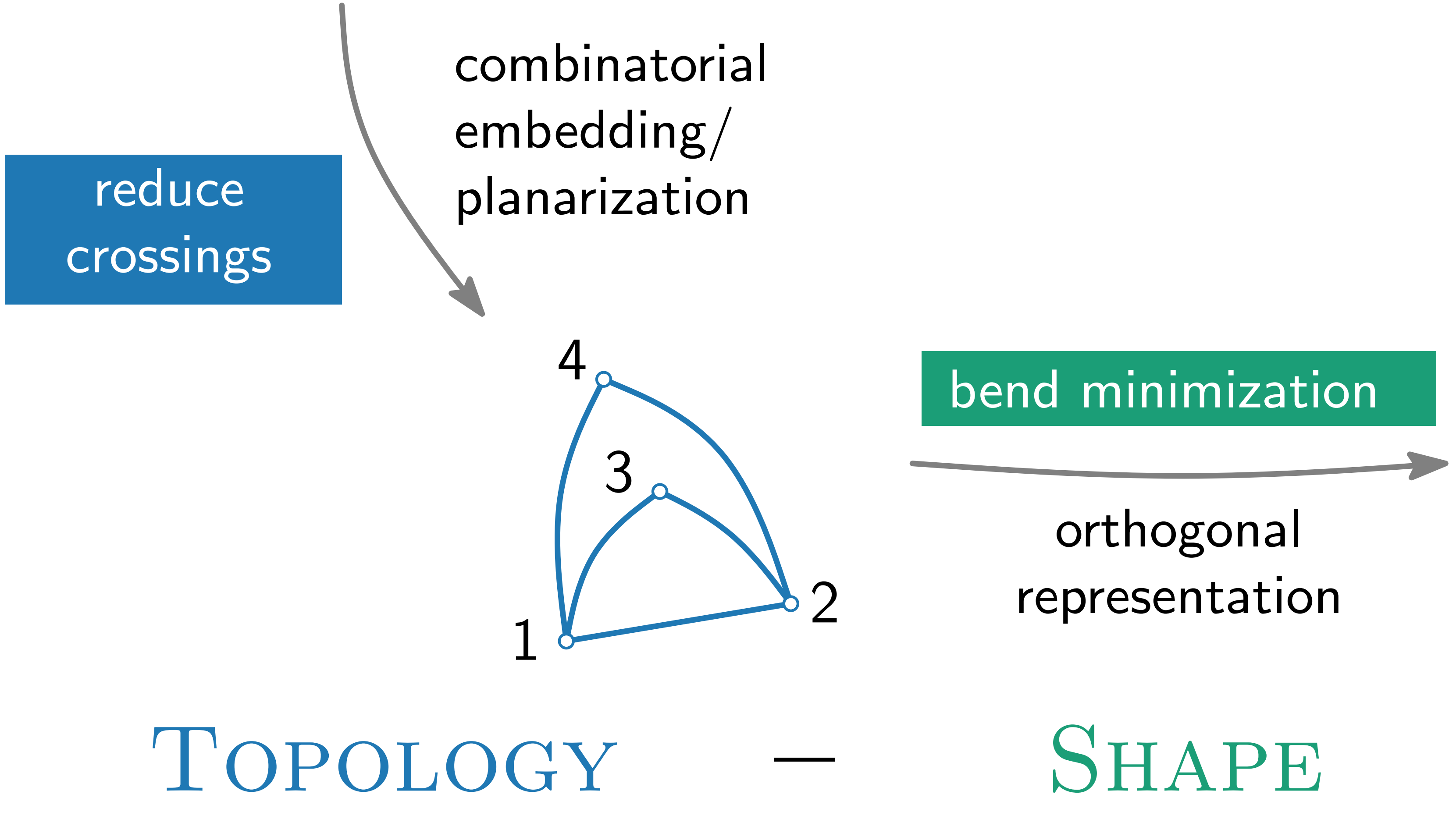
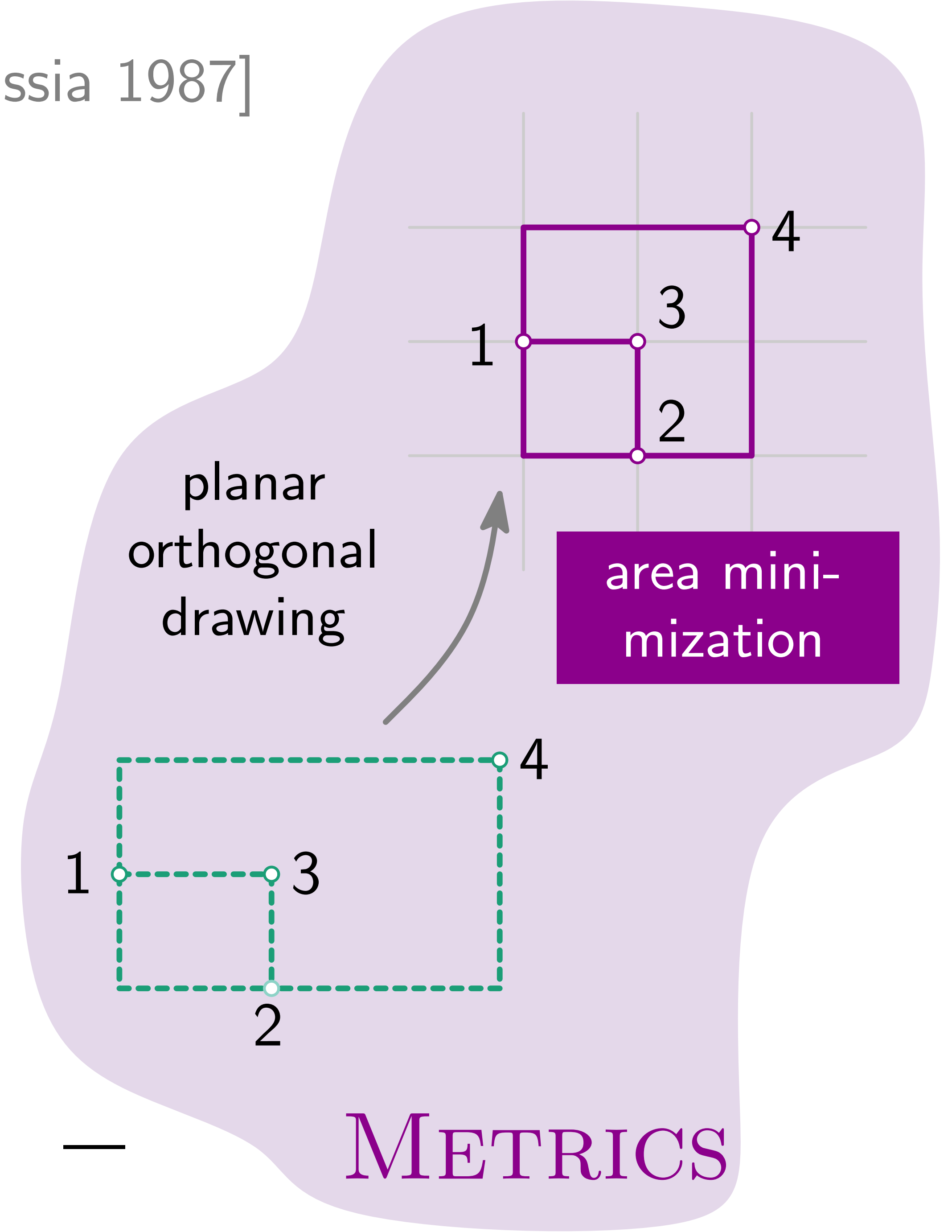
Bend minimization without given combinatorial embedding is NP-hard.

Topology – Shape – Metrics

Three-step approach: [Tamassia 1987]

$$V = \{v_1, v_2, v_3, v_4\}$$
$$E = \{v_1v_2, v_1v_3, v_1v_4, v_2v_3, v_2v_4\}$$





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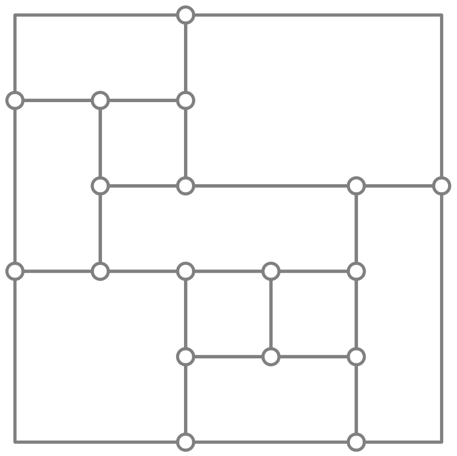
Idea.

- Formulate flow network for horizontal/vertical compaction

Flow Network for Edge-Length Assignment



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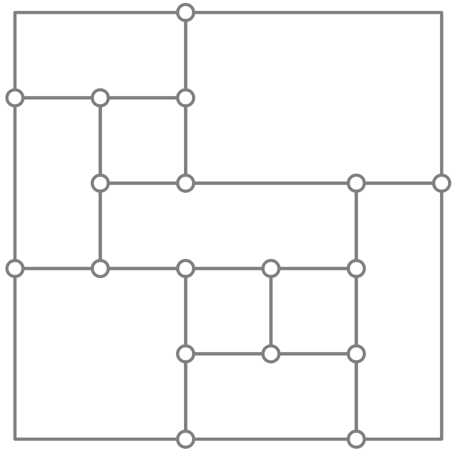

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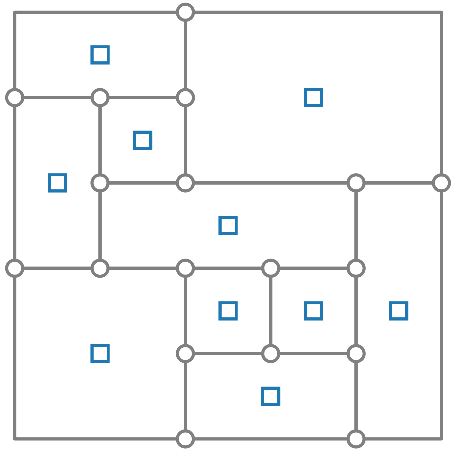

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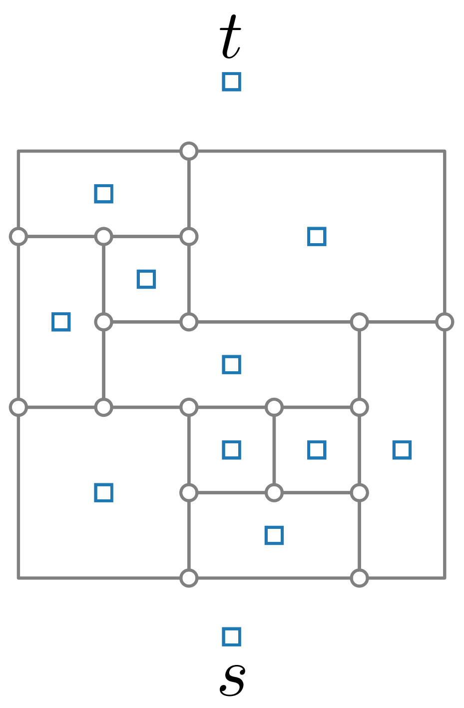

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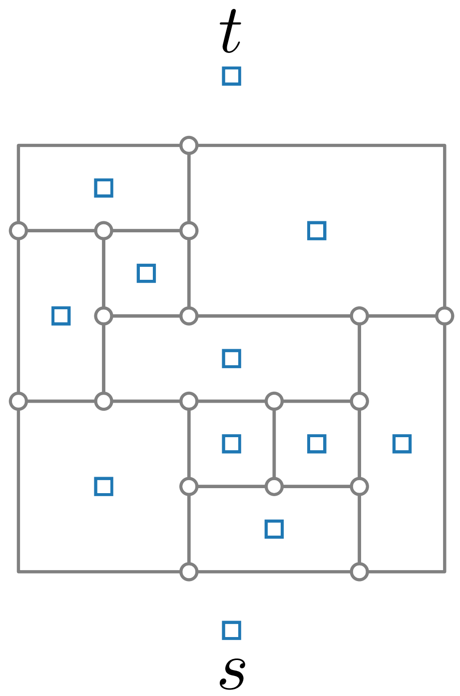

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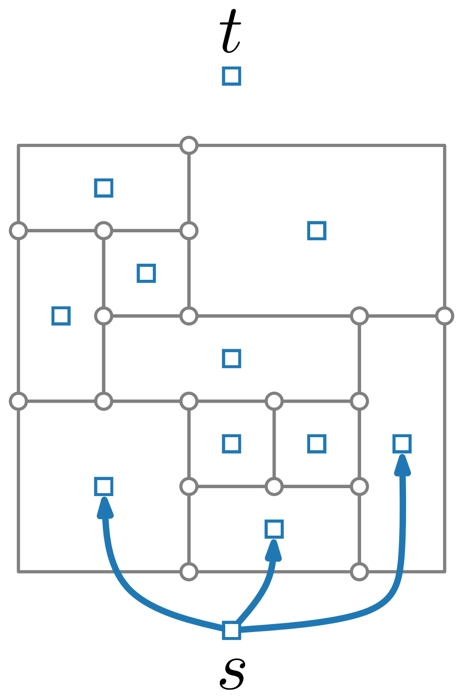

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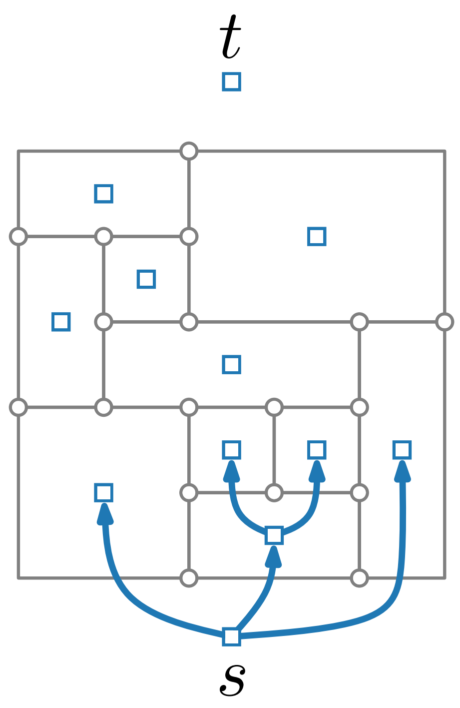

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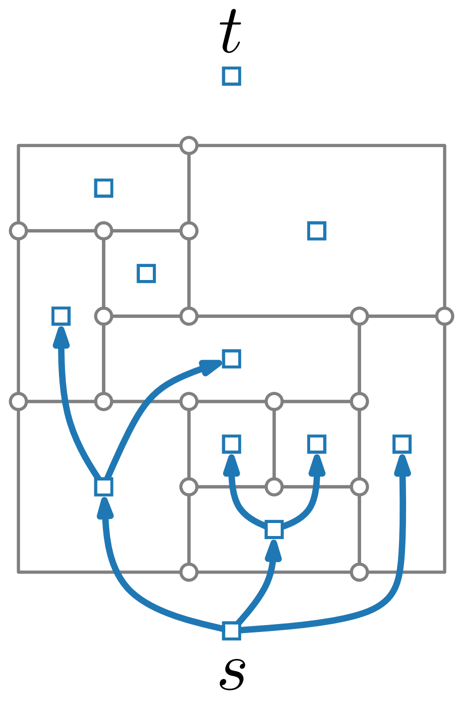

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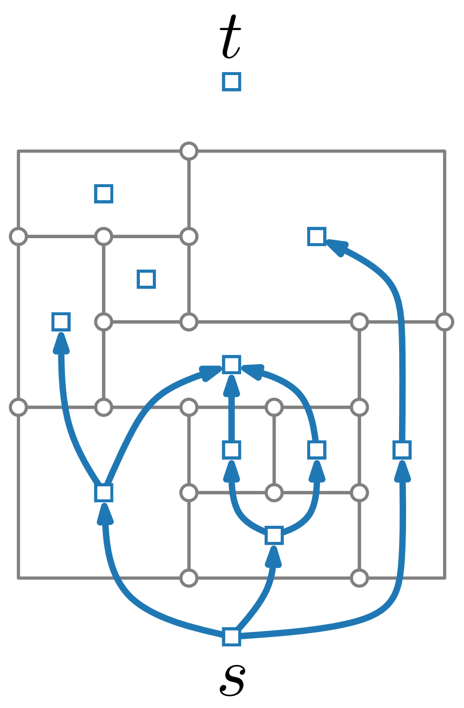

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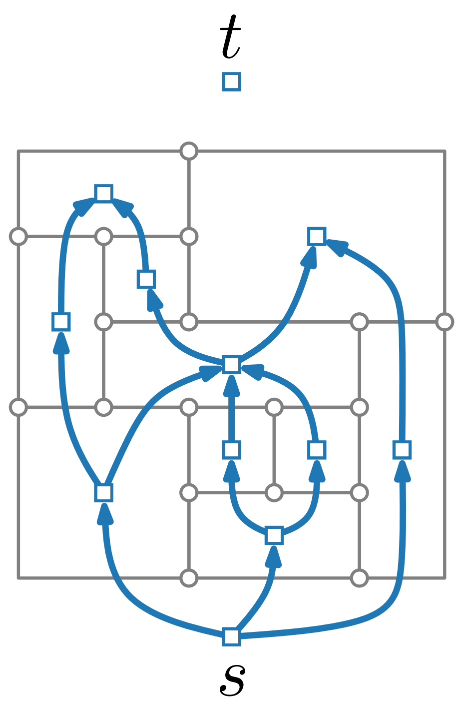
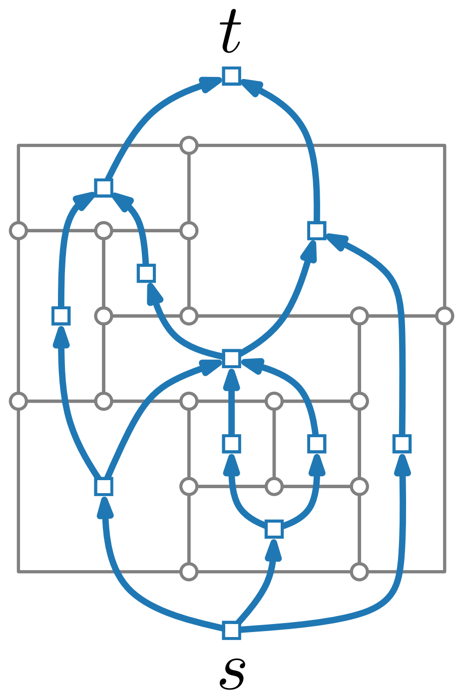


Flow Network for Edge-Length Assignment

Definition.

Flow Network $N_{\text{hor}} = ((W_{\text{hor}}, E_{\text{hor}}); b; \ell; u; \text{cost})$

- $W_{\text{hor}} = F \setminus \{f_0\} \cup \{s, t\}$
- $E_{\text{hor}} = \{(f, g) \mid f, g$ share a *horizontal* segment and f lies *below* $g\}$



Flow Network for Edge-Length Assignment

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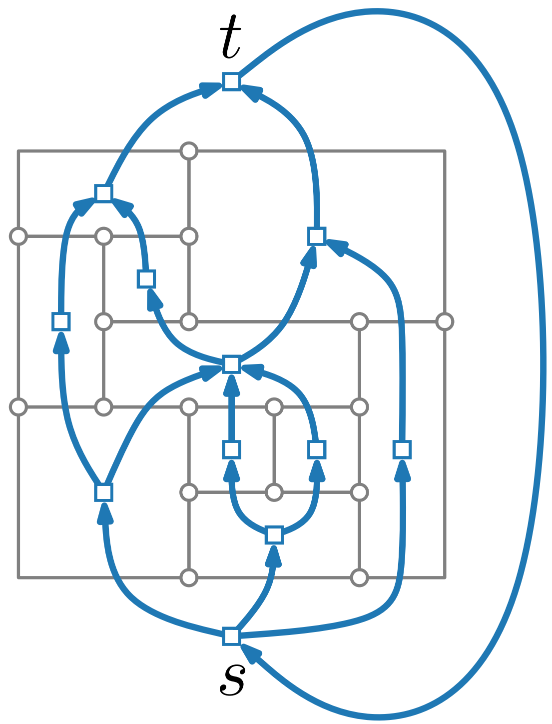
- $E_{\text{hor}} = \{(f, g) \mid f, g$ share a *horizontal* segment and f lies *below* $g\} \cup \{(t, s)\}$



Flow Network for Edge-Length Assignment

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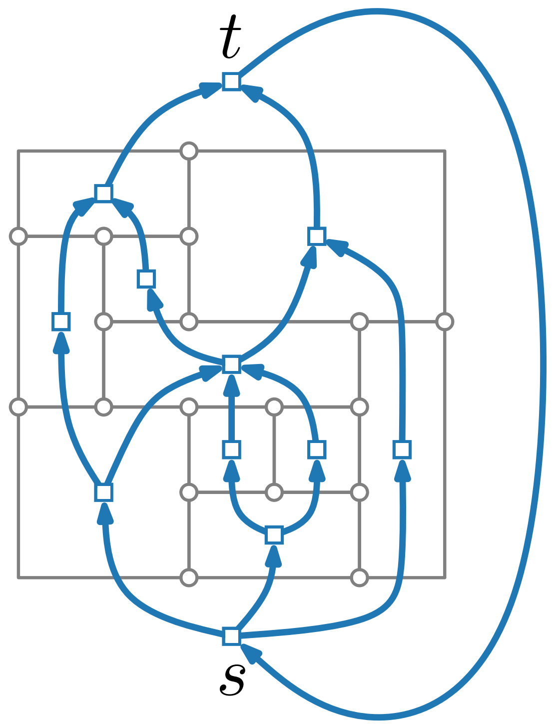
- $W_{\text{hor}} = F \setminus \{f_0\} \cup \{s, t\}$ □
- $E_{\text{hor}} = \{(f, g) \mid f, g$ share a *horizontal* segment and f lies *below* $g\} \cup \{(t, s)\}$
- $\ell(a) = 1 \quad \forall a \in E_{\text{hor}}$



Flow Network for Edge-Length Assignment

Definition.

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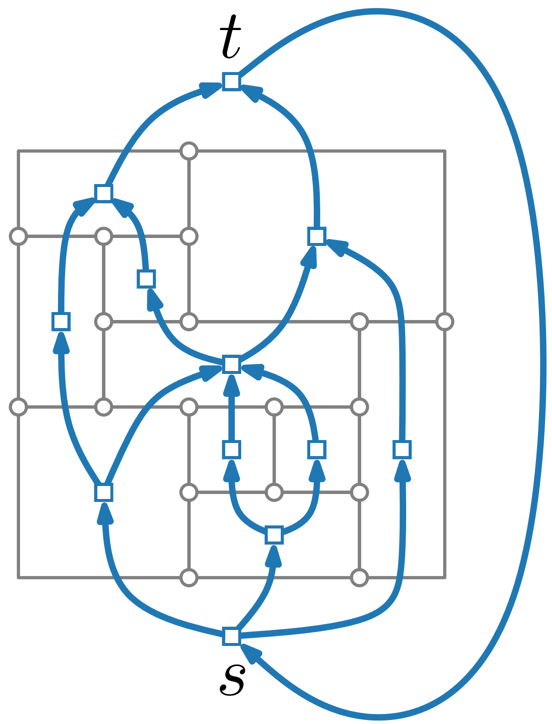
- $W_{\text{hor}} = F \setminus \{f_0\} \cup \{s, t\}$
- $E_{\text{hor}} = \{(f, g) \mid f, g$ share a *horizontal* segment and f lies *below* $g\} \cup \{(t, s)\}$
- $\ell(a) = 1 \quad \forall a \in E_{\text{hor}}$
- $u(a) = \infty \quad \forall a \in E_{\text{hor}}$



Flow Network for Edge-Length Assignment

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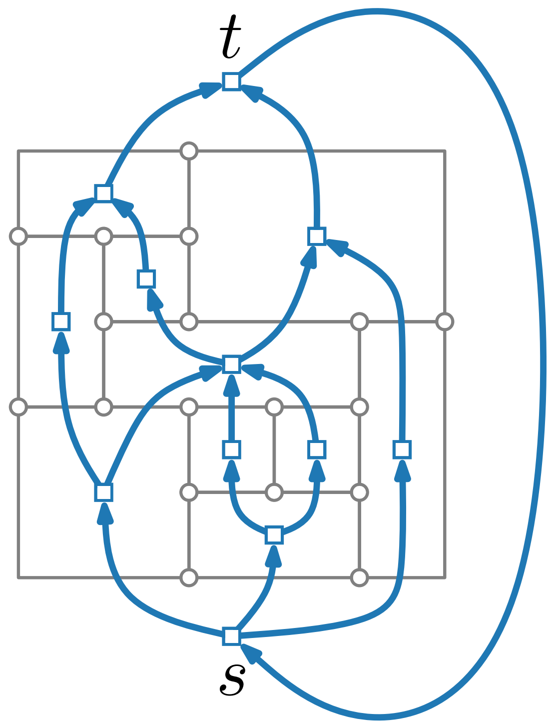
- $\ell(a) = 1 \quad \forall a \in E_{\text{hor}}$
- $u(a) = \infty \quad \forall a \in E_{\text{hor}}$
- $\text{cost}(a) = 1 \quad \forall a \in E_{\text{hor}}$



Flow Network for Edge-Length Assignment

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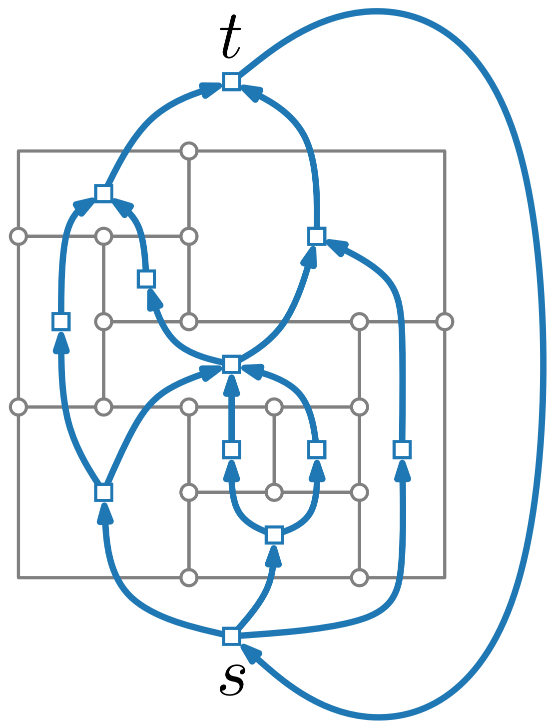
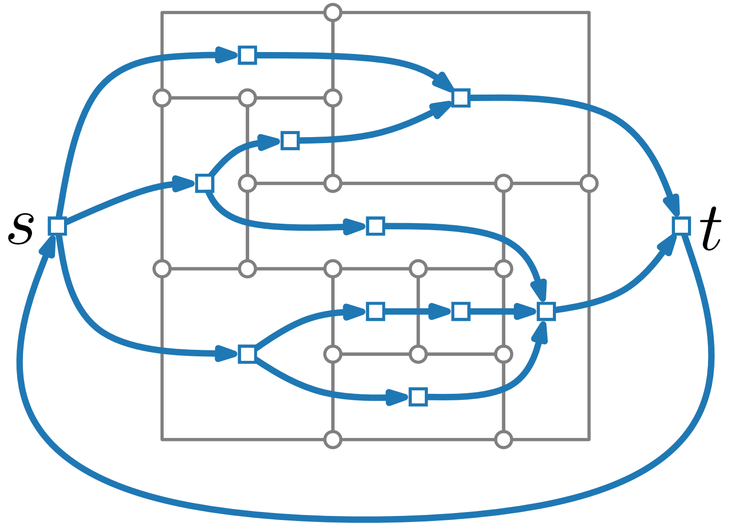
- $\ell(a) = 1 \quad \forall a \in E_{\text{hor}}$
- $u(a) = \infty \quad \forall a \in E_{\text{hor}}$
- $\text{cost}(a) = 1 \quad \forall a \in E_{\text{hor}}$
- $b(f) = 0 \quad \forall f \in W_{\text{hor}}$



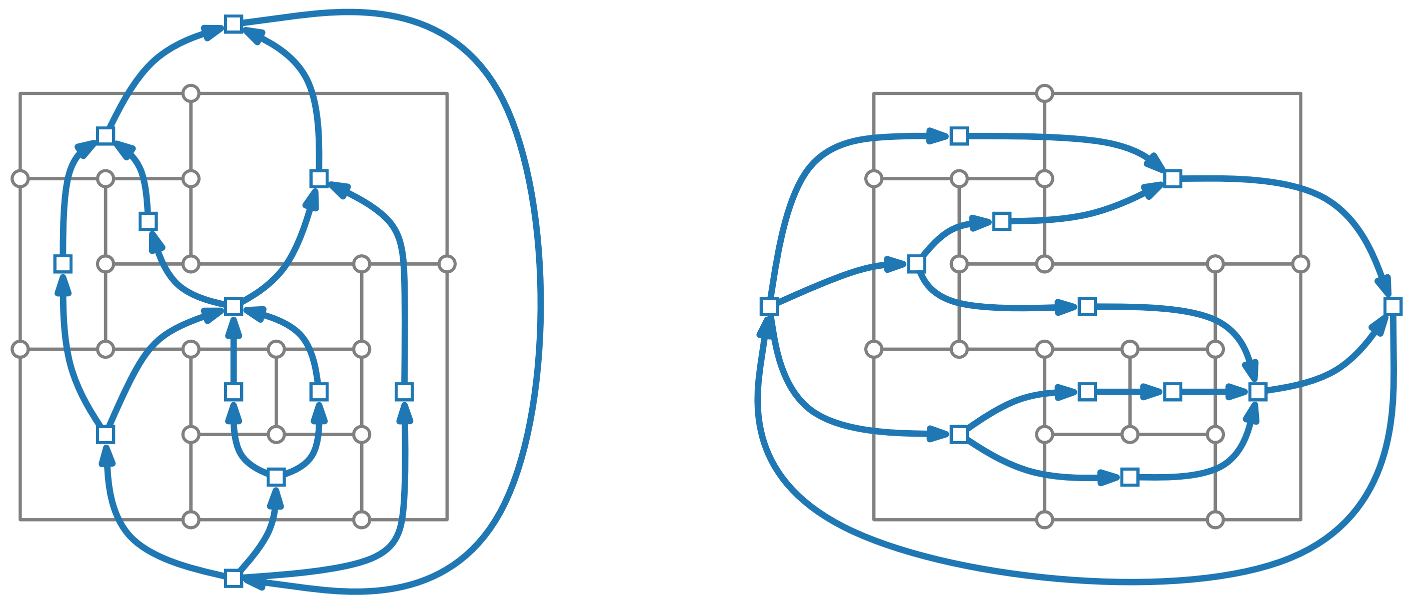
Flow Network for Edge-Length Assignment

Definition.
Flow Network $N_{\text{ver}} = ((W_{\text{ver}}, E_{\text{ver}}); b; \ell; u; \text{cost})$

- $W_{\text{ver}} = F \setminus \{f_0\} \cup \{s, t\}$
- $E_{\text{ver}} = \{(f, g) \mid f, g$ share a *vertical* segment and f lies to the *left* of $g\} \cup \{(t, s)\}$
- $\ell(a) = 1 \quad \forall a \in E_{\text{ver}}$
- $u(a) = \infty \quad \forall a \in E_{\text{ver}}$
- $\text{cost}(a) = 1 \quad \forall a \in E_{\text{ver}}$
- $b(f) = 0 \quad \forall f \in W_{\text{ver}}$

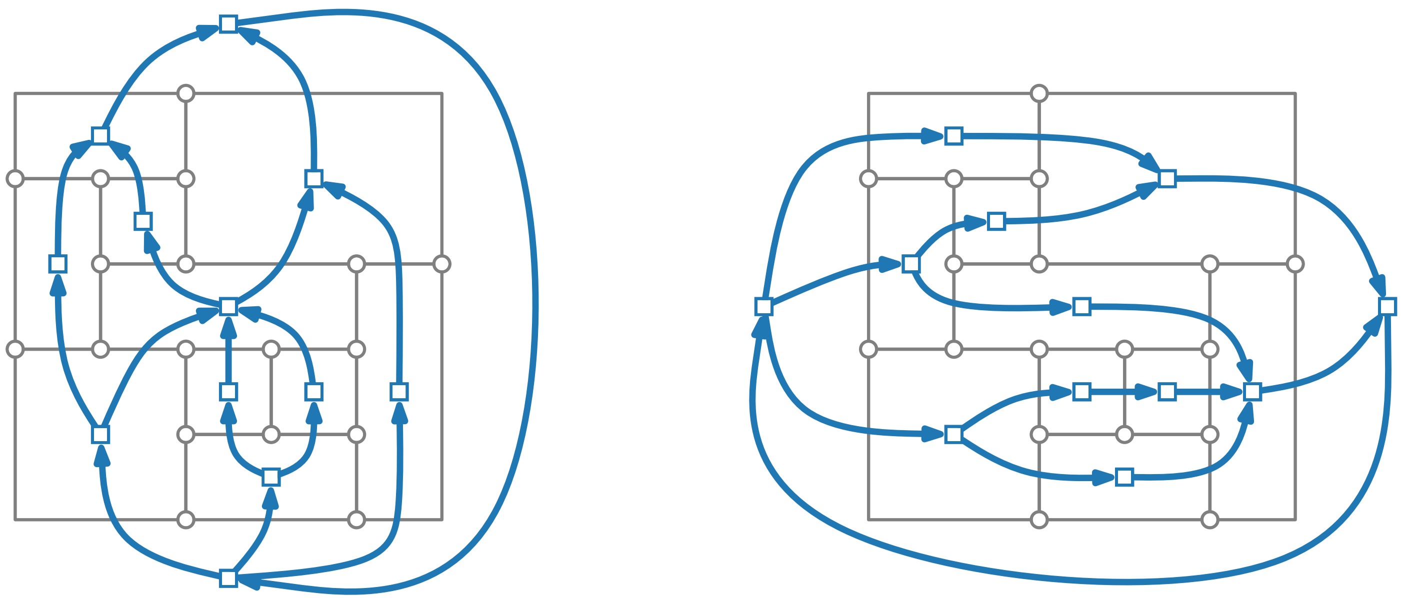


Compaction – Result



Theorem.
A valid flow for N_{hor} and N_{ver} exists $\Leftrightarrow$ corresponding edge lengths induce an orthogonal drawing.

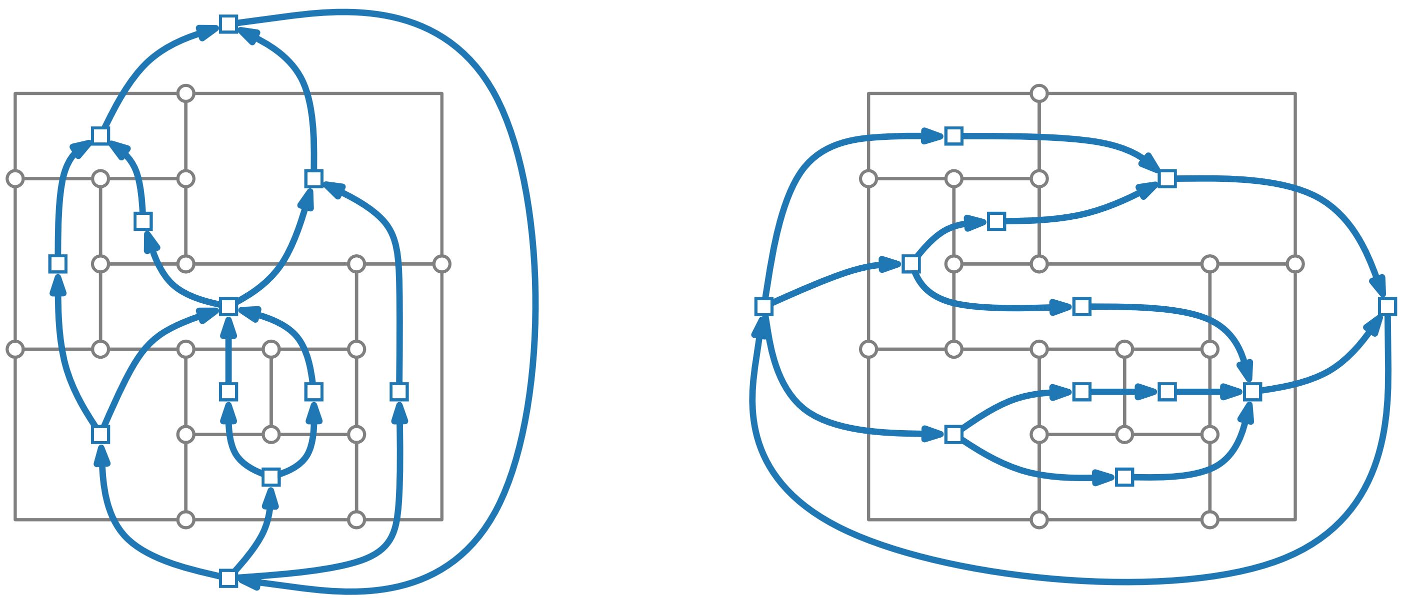
Compaction – Result



Theorem.
A valid flow for N_{hor} and N_{ver} exists $\Leftrightarrow$
corresponding edge lengths induce an orthogonal drawing.

What values of the drawing do the following quantities represent?

Compaction – Result

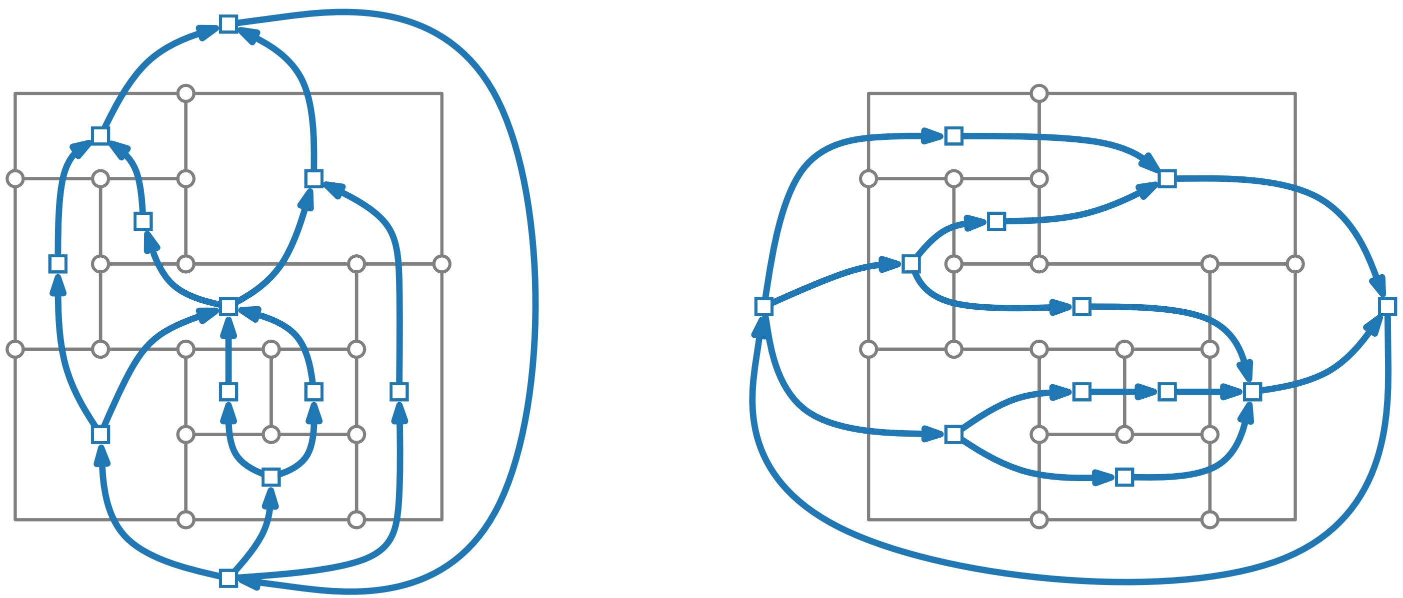


Theorem.
A valid flow for N_{hor} and N_{ver} exists $\Leftrightarrow$
corresponding edge lengths induce an orthogonal drawing.

What values of the drawing do the following quantities represent?

- $|X_{\text{hor}}(t, s)|$ and $|X_{\text{ver}}(t, s)|$?

Compaction – Result

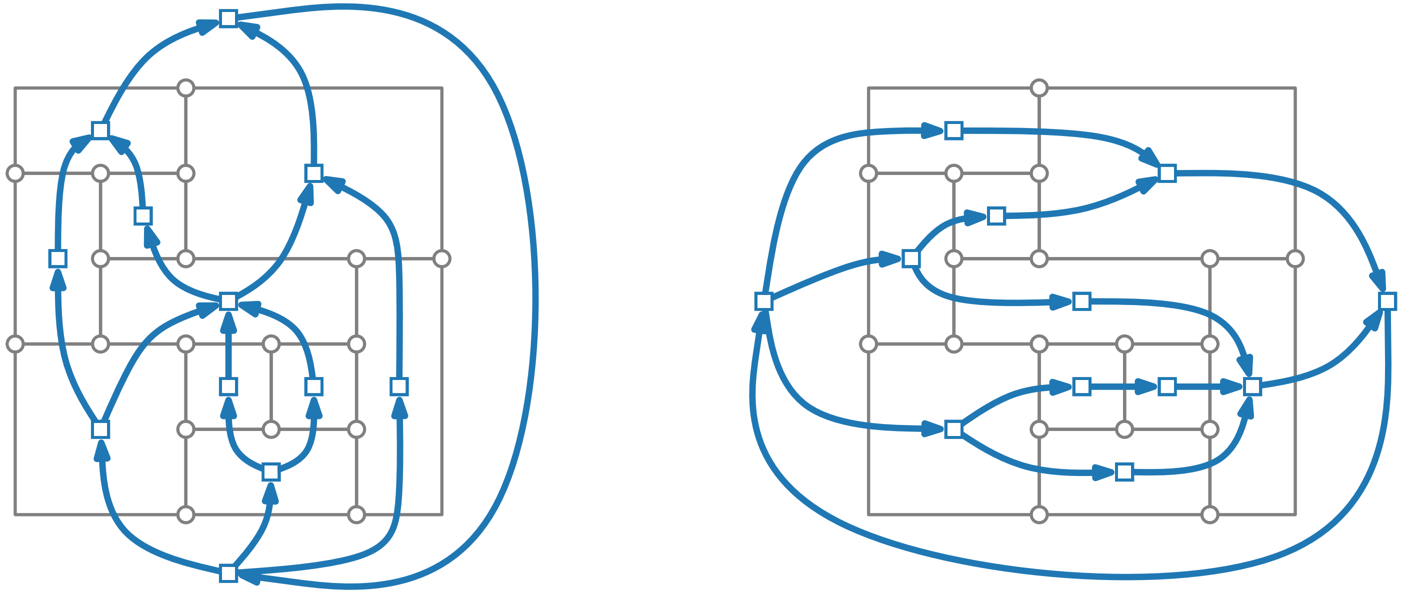


Theorem.
A valid flow for N_{hor} and N_{ver} exists $\Leftrightarrow$
corresponding edge lengths induce an orthogonal drawing.

What values of the drawing do the following quantities represent?

- $|X_{\text{hor}}(t,s)|$ and $|X_{\text{ver}}(t,s)|$? width and height of the drawing

Compaction – Result

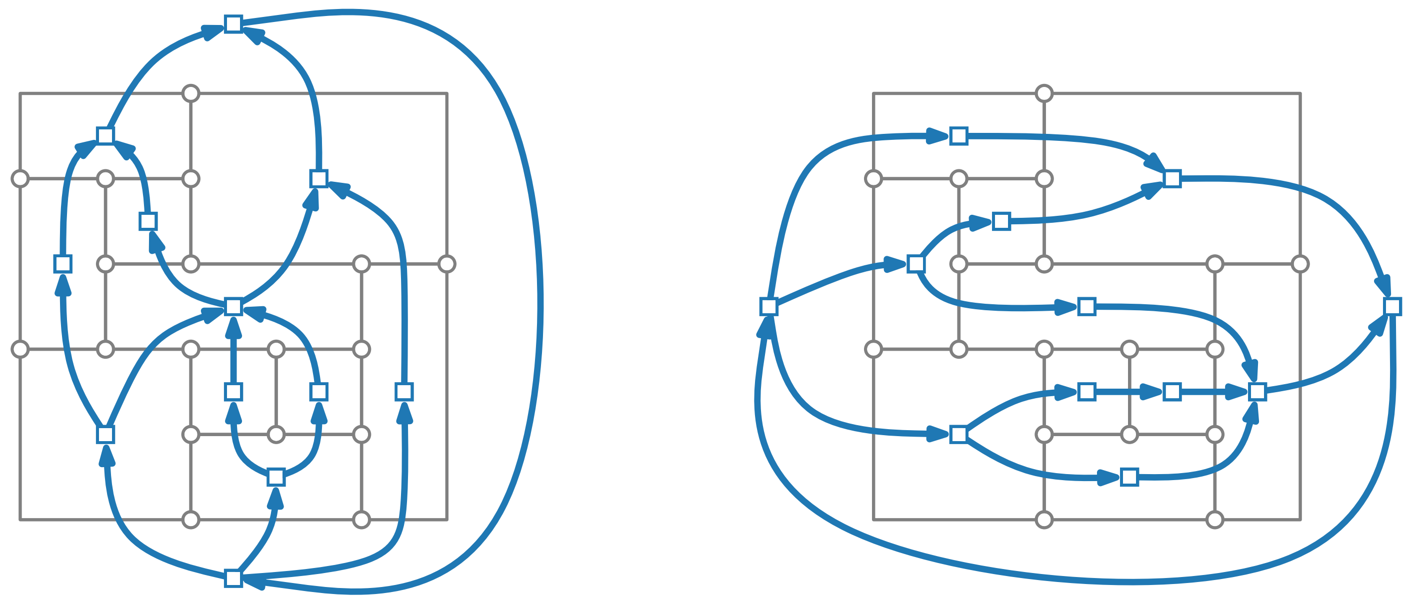


Theorem.
A valid flow for N_{hor} and N_{ver} exists $\Leftrightarrow$
corresponding edge lengths induce an orthogonal drawing.

What values of the drawing do the following quantities represent?

- $|X_{\text{hor}}(t, s)|$ and $|X_{\text{ver}}(t, s)|$? width and height of the drawing
- $\sum_{e \in E_{\text{hor}}} X_{\text{hor}}(e) + \sum_{e \in E_{\text{ver}}} X_{\text{ver}}(e)$

Compaction – Result

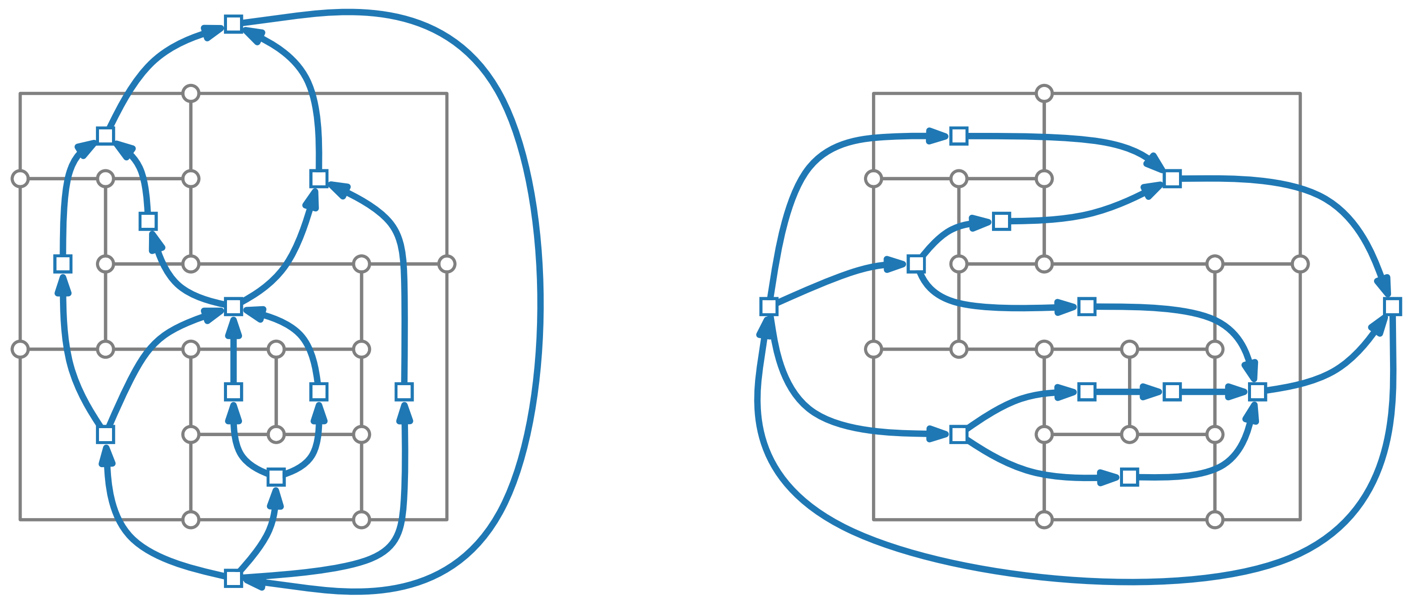


Theorem.
A valid flow for N_{hor} and N_{ver} exists $\Leftrightarrow$
corresponding edge lengths induce an orthogonal drawing.

What values of the drawing do the following quantities represent?

- $|X_{\text{hor}}(t,s)|$ and $|X_{\text{ver}}(t,s)|$? width and height of the drawing
- $\sum_{e \in E_{\text{hor}}} X_{\text{hor}}(e) + \sum_{e \in E_{\text{ver}}} X_{\text{ver}}(e)$ total edge length

Compaction – Result



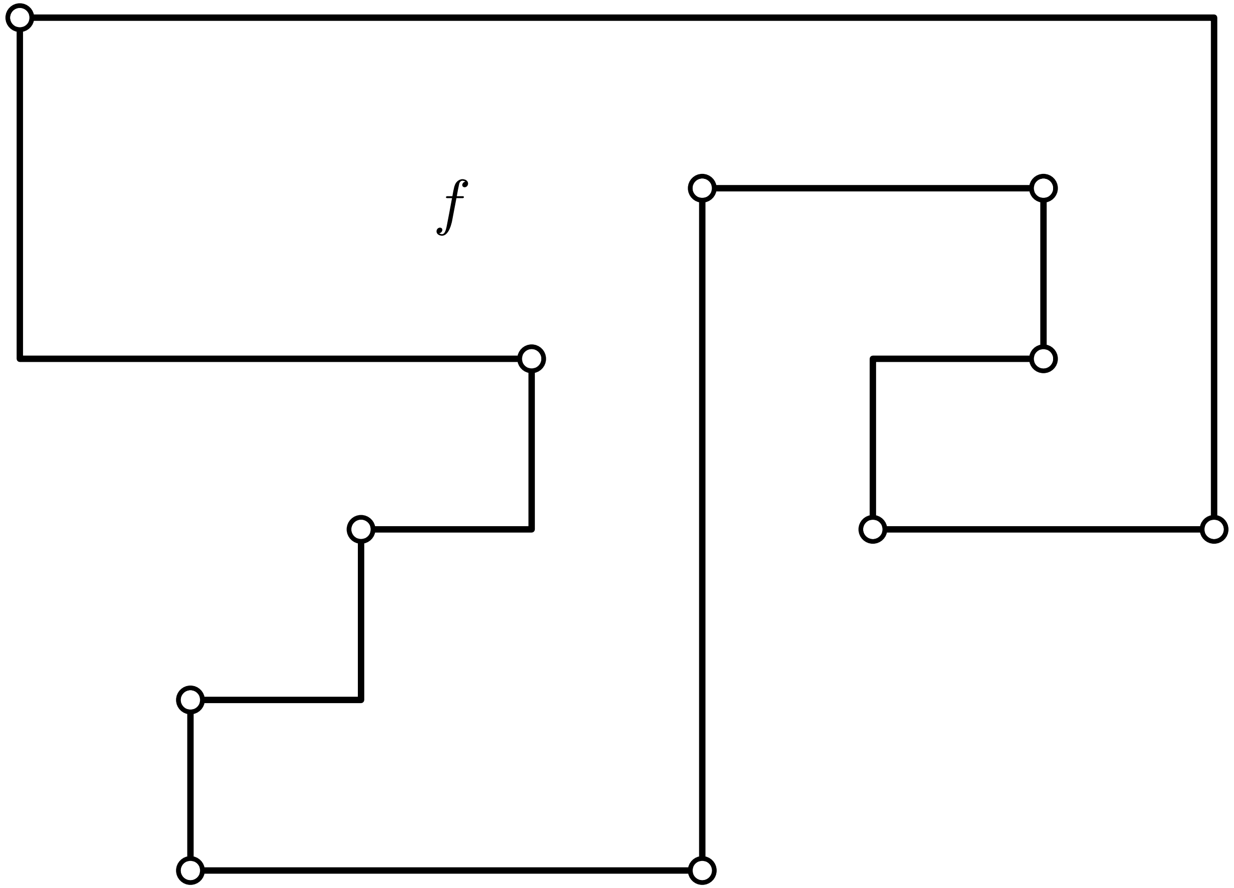
What if not all faces are rectangular?

Theorem.
A valid flow for N_{hor} and N_{ver} exists $\Leftrightarrow$ corresponding edge lengths induce an orthogonal drawing.

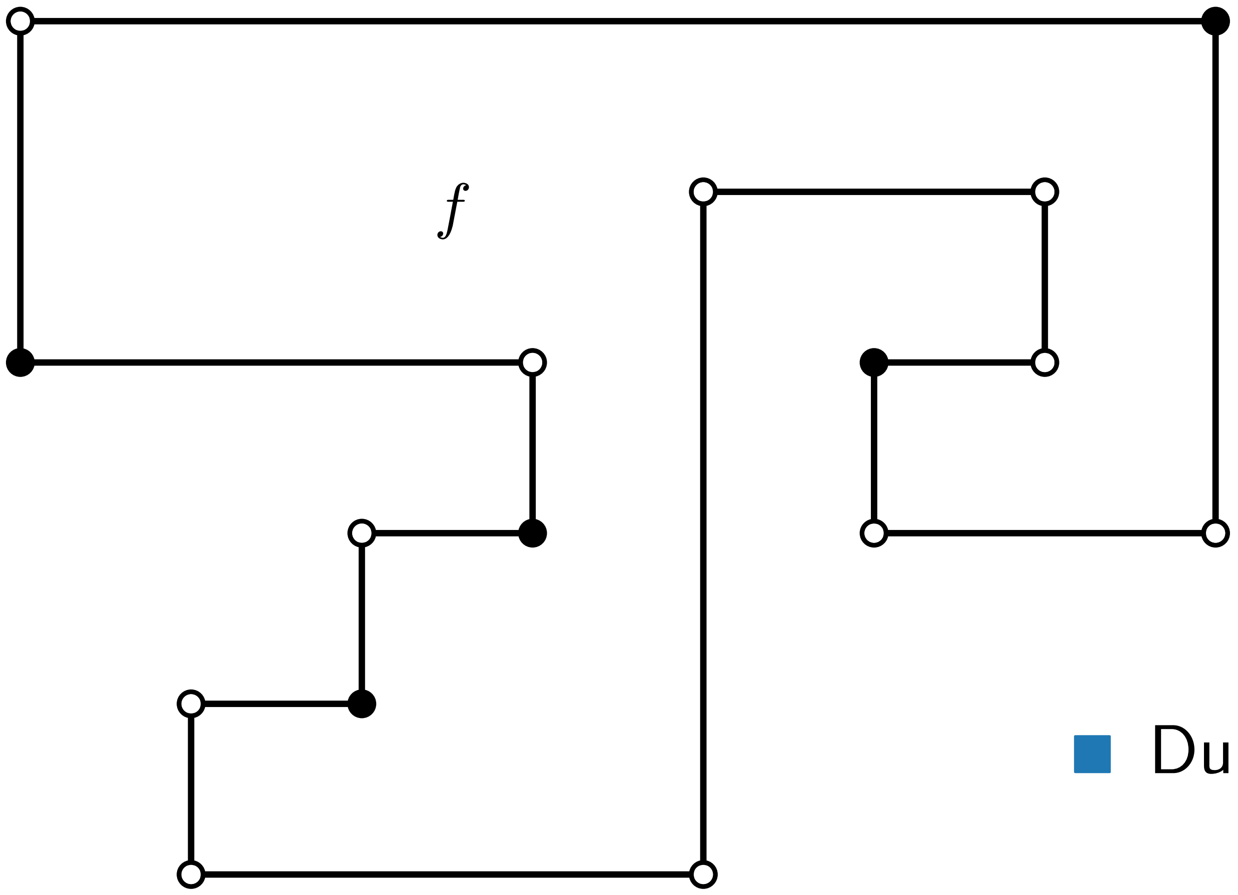
What values of the drawing do the following quantities represent?

- $|X_{\text{hor}}(t,s)|$ and $|X_{\text{ver}}(t,s)|$? width and height of the drawing
- $\sum_{e \in E_{\text{hor}}} X_{\text{hor}}(e) + \sum_{e \in E_{\text{ver}}} X_{\text{ver}}(e)$ total edge length

Refinement of G and $H(G)$ – Inner Face

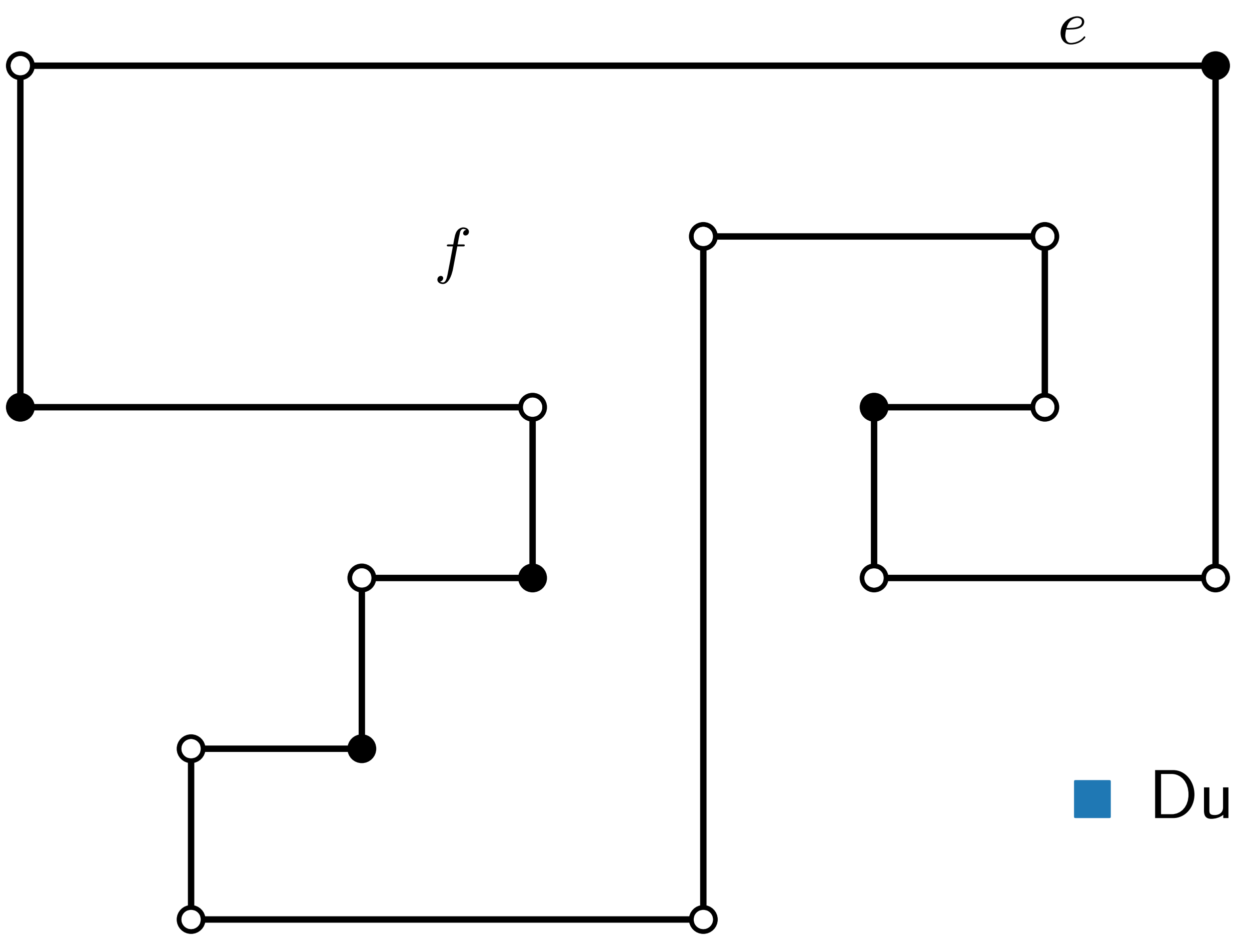


Refinement of G and $H(G)$ – Inner Face



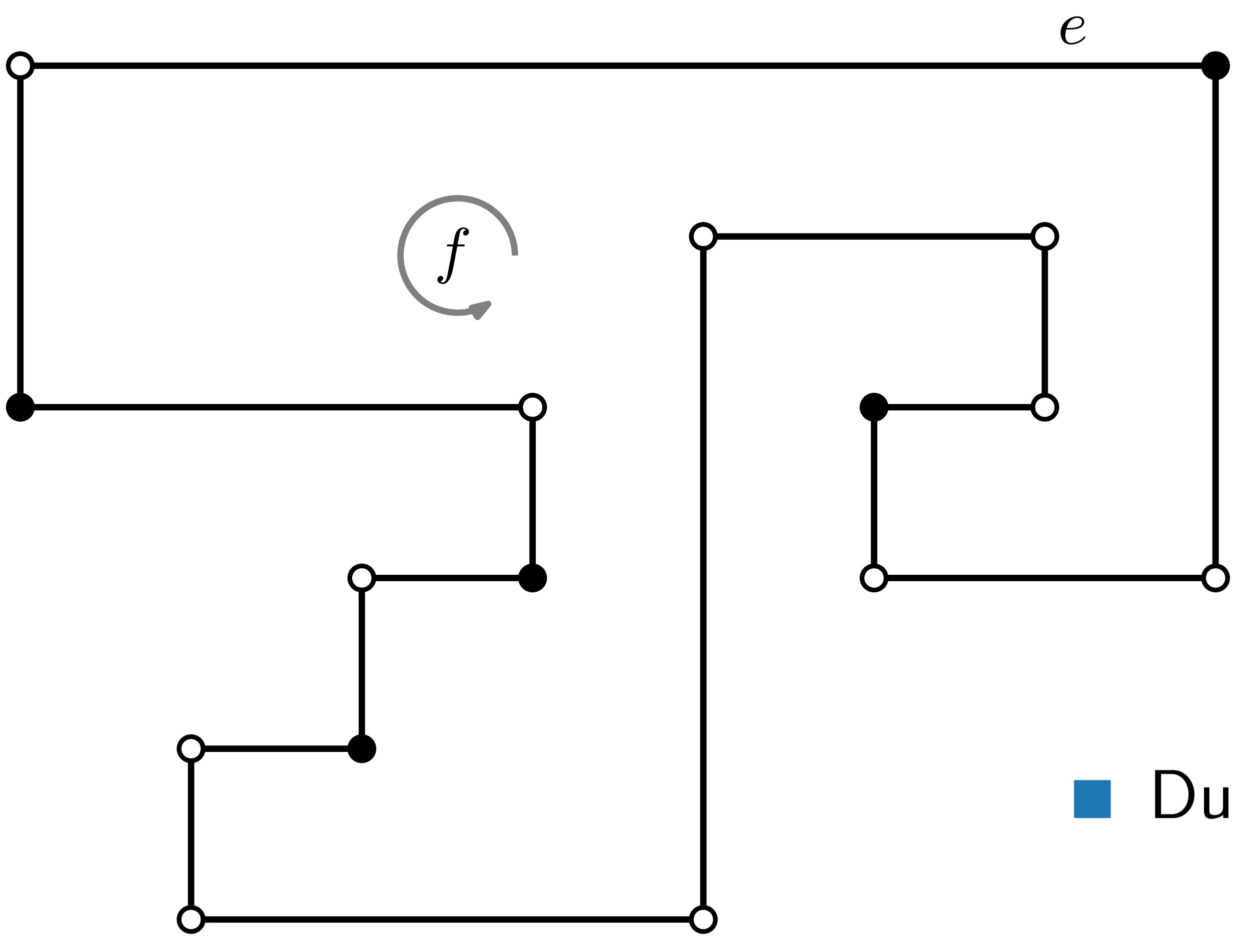
- Dummy vertices for bends

Refinement of G and $H(G)$ – Inner Face



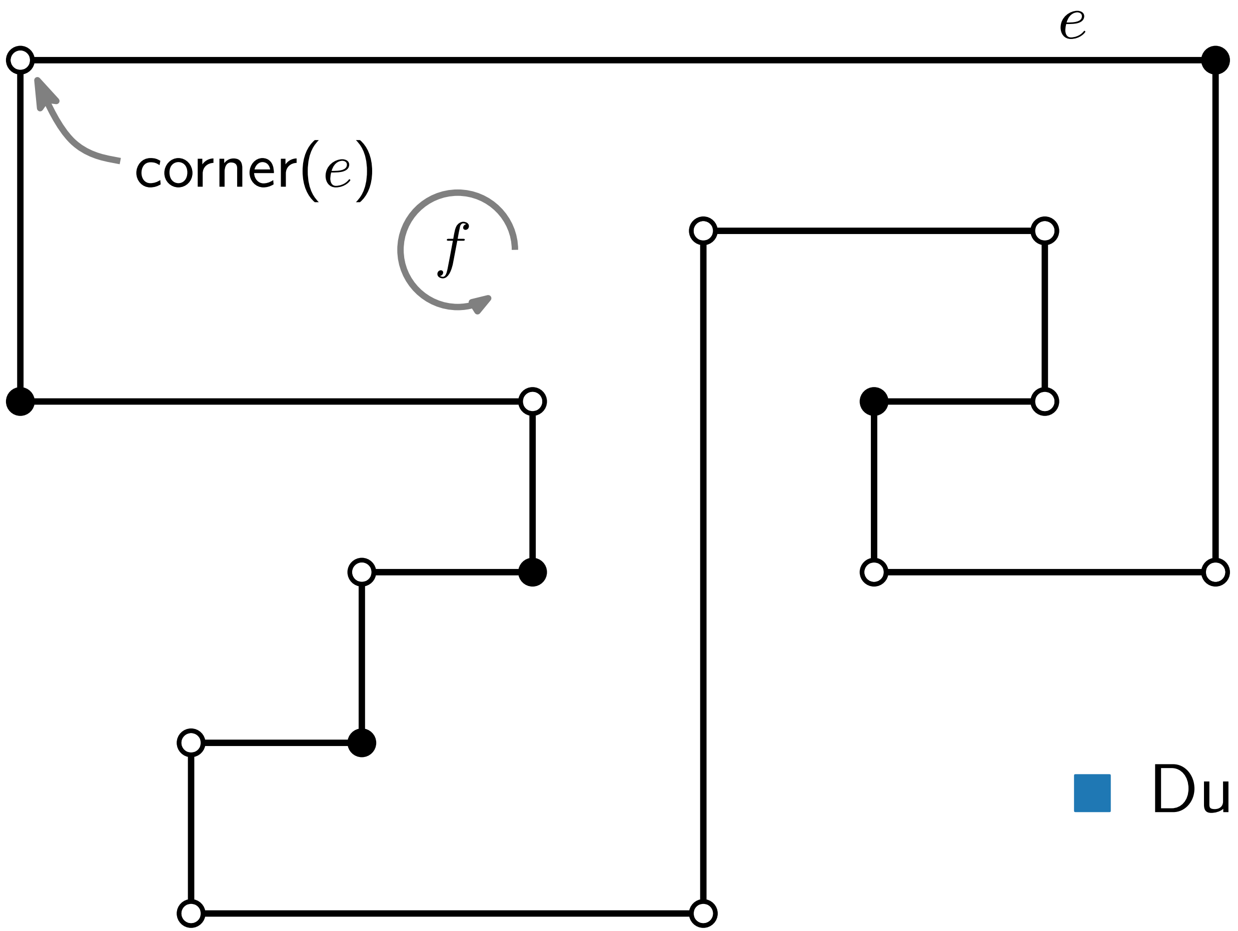
- Dummy vertices for bends

Refinement of G and $H(G)$ – Inner Face



- Dummy vertices for bends

Refinement of G and $H(G)$ – Inner Face



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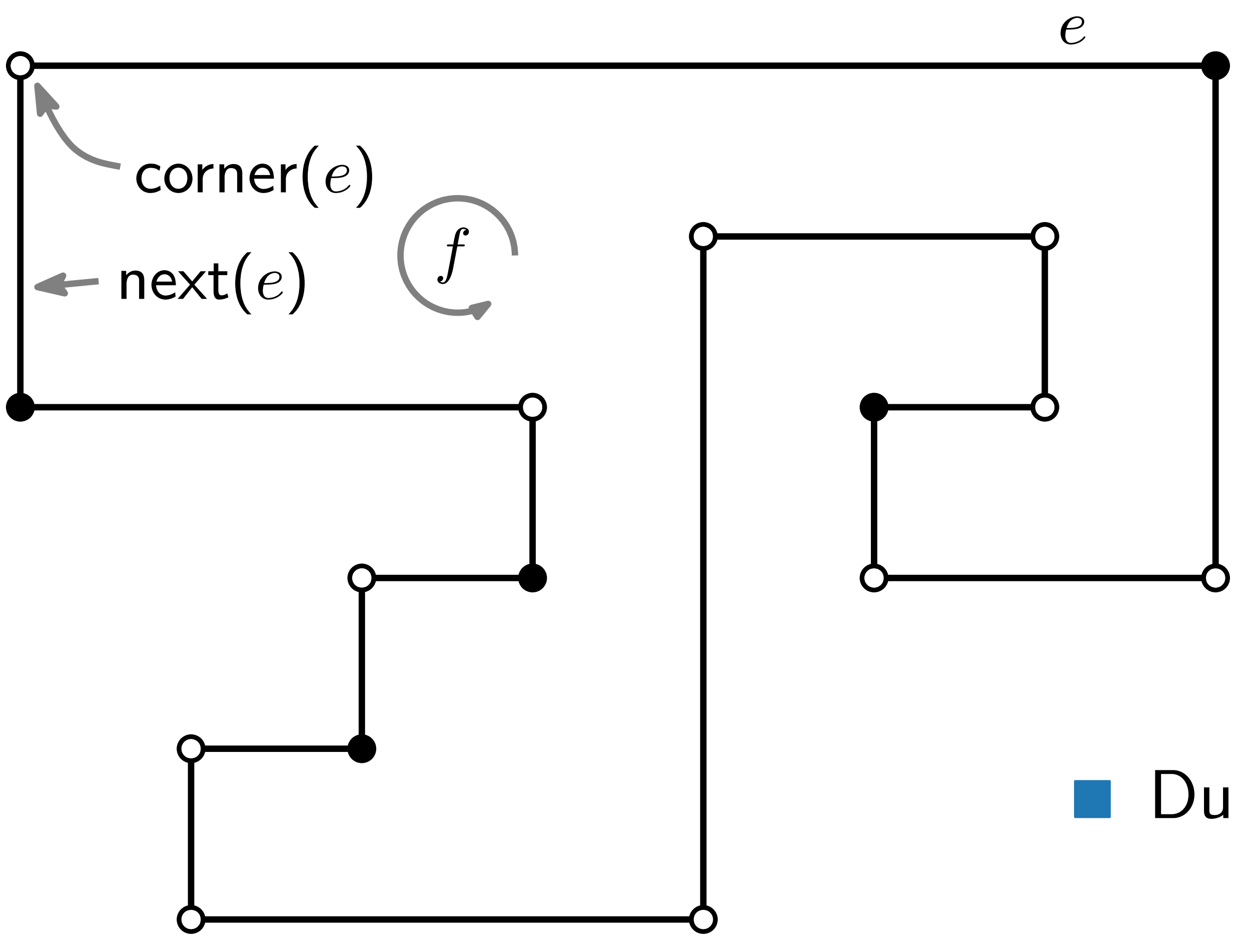
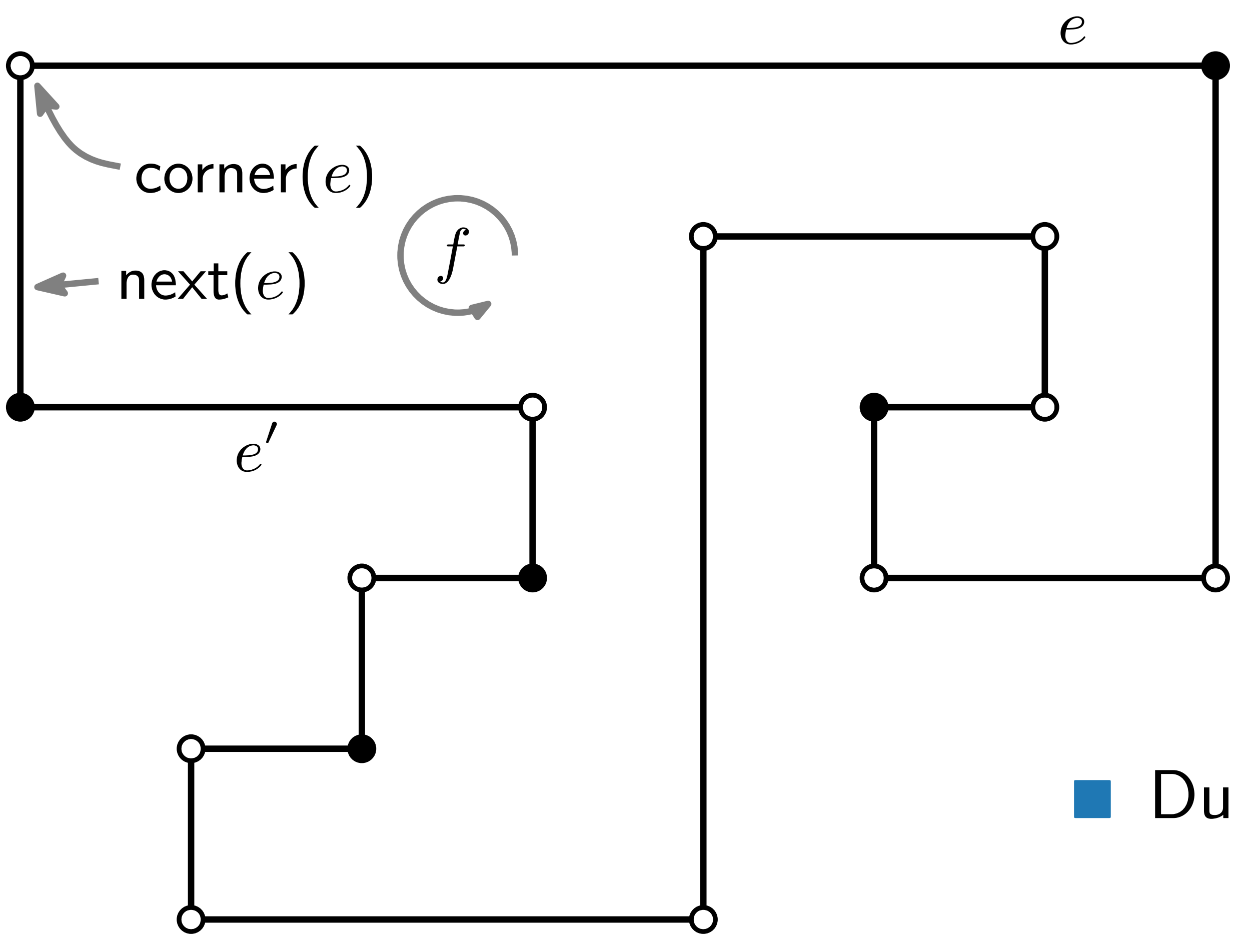
Refinement of G and $H(G)$ – Inner Face



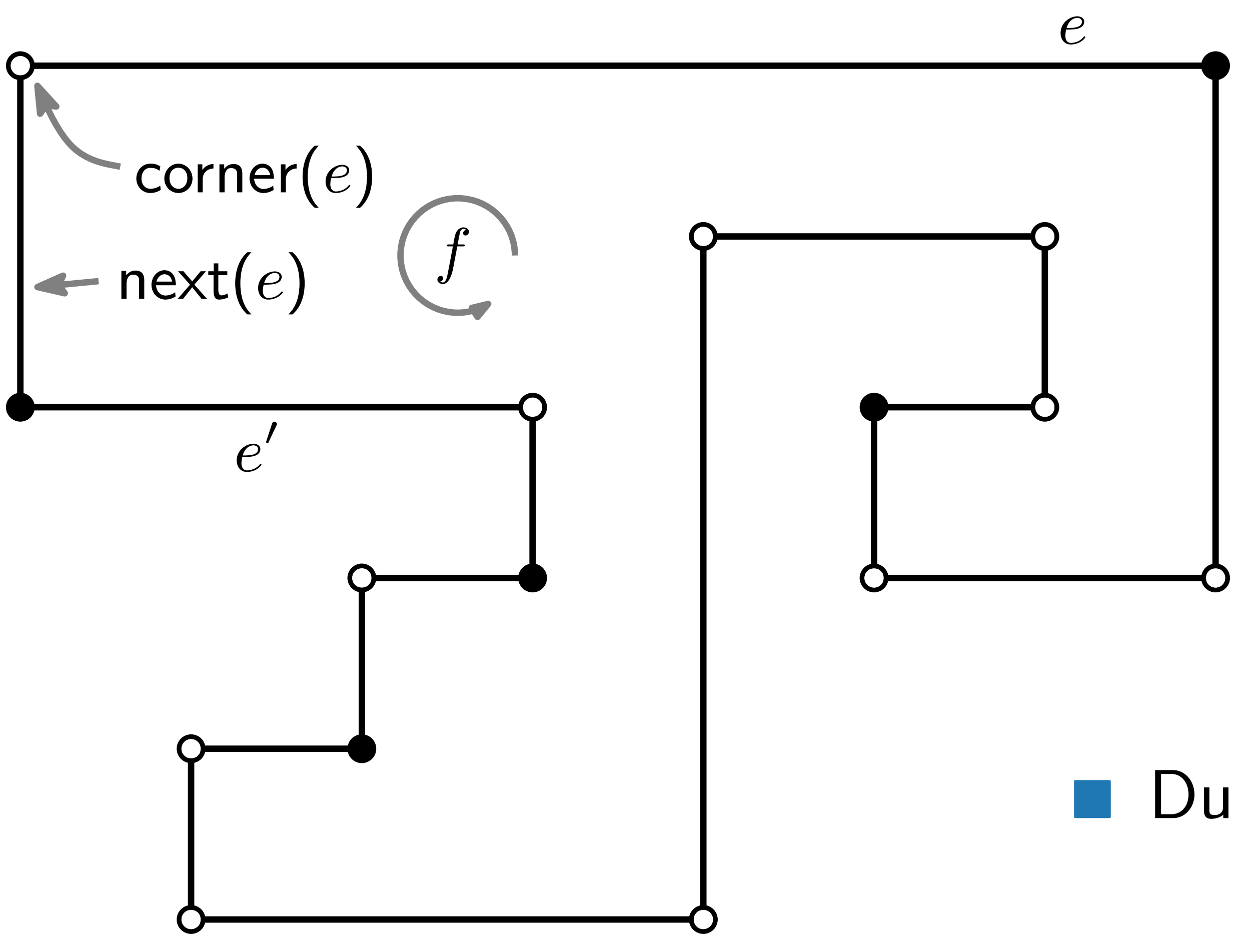
- Dummy vertices for bends

Refinement of G and $H(G)$ – Inner Face



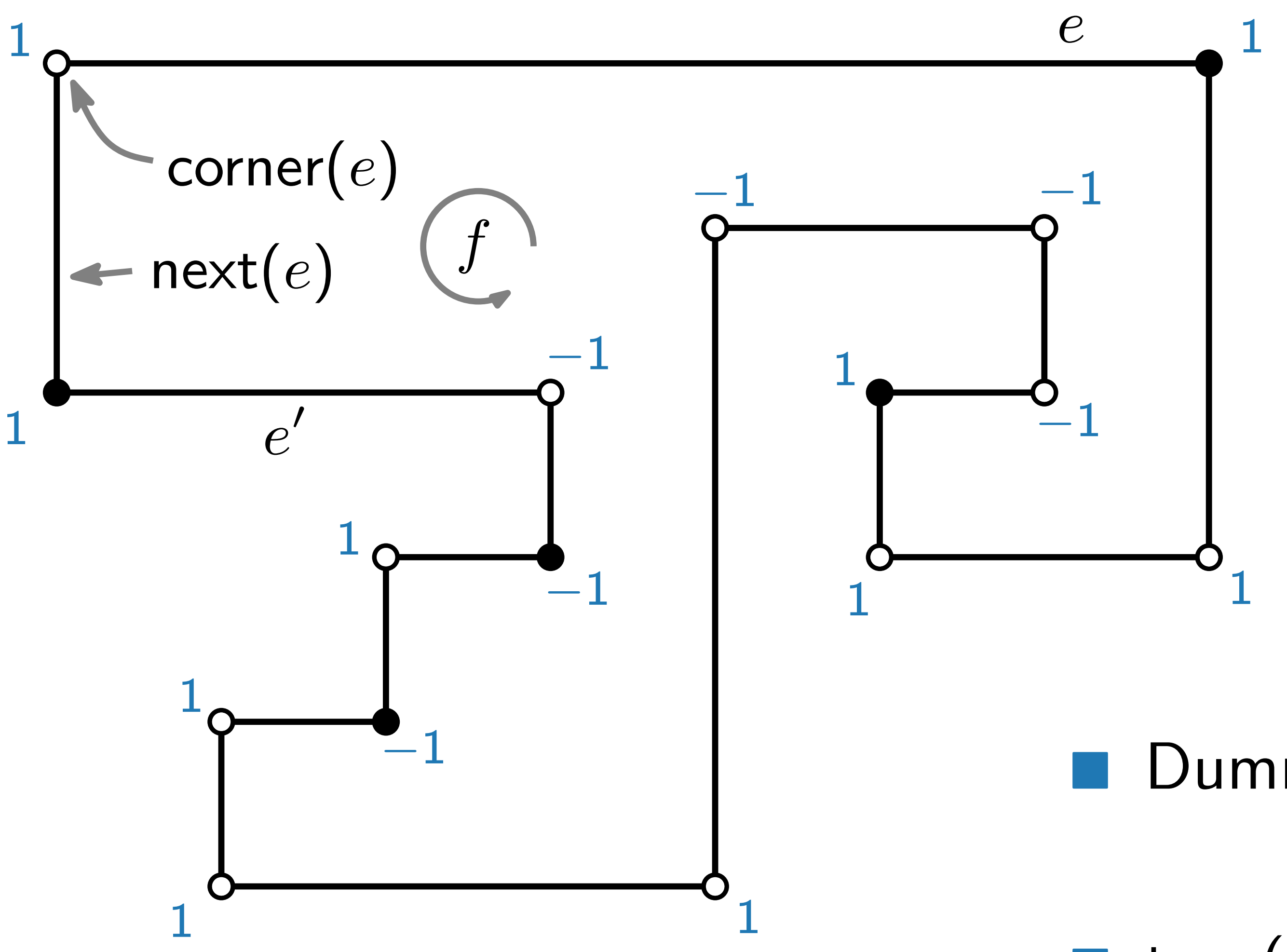
- Dummy vertices for bends

Refinement of G and $H(G)$ – Inner Face



- Dummy vertices for bends
- $\text{turn}(e) = \begin{cases} 1 & \text{left turn} \\ 0 & \text{no turn} \\ -1 & \text{right turn} \end{cases}$

Refinement of G and $H(G)$ – Inner Face



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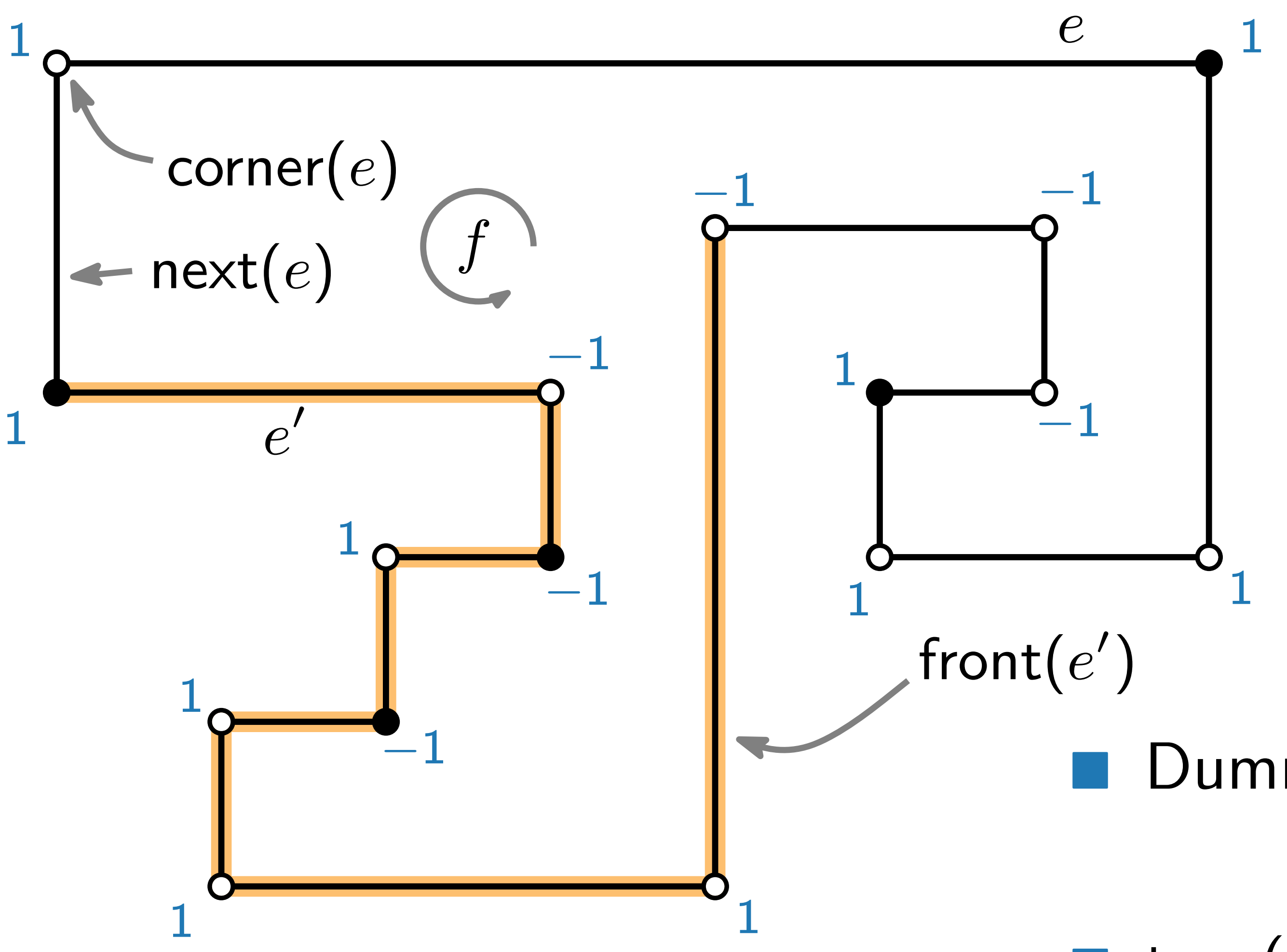

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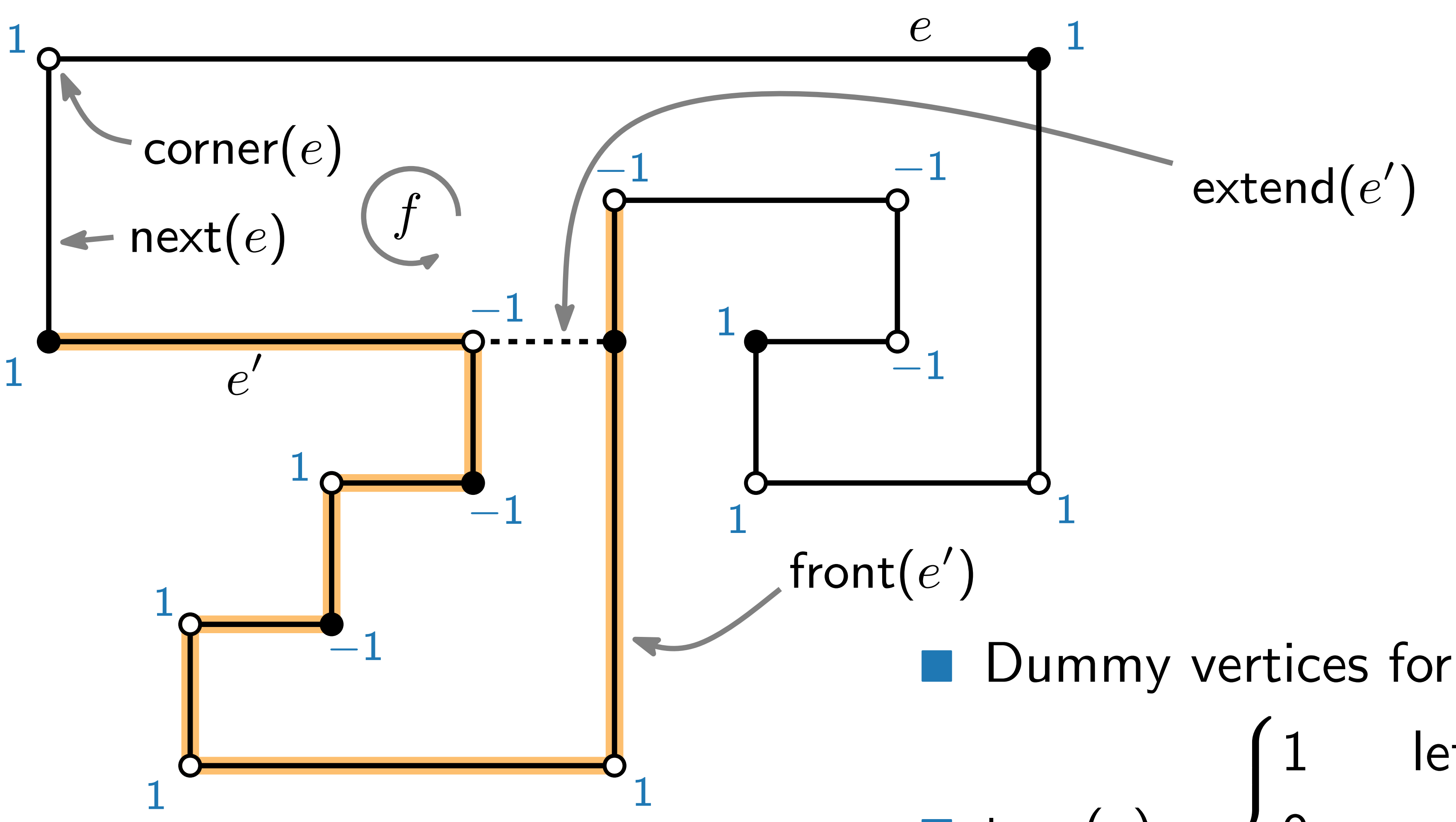

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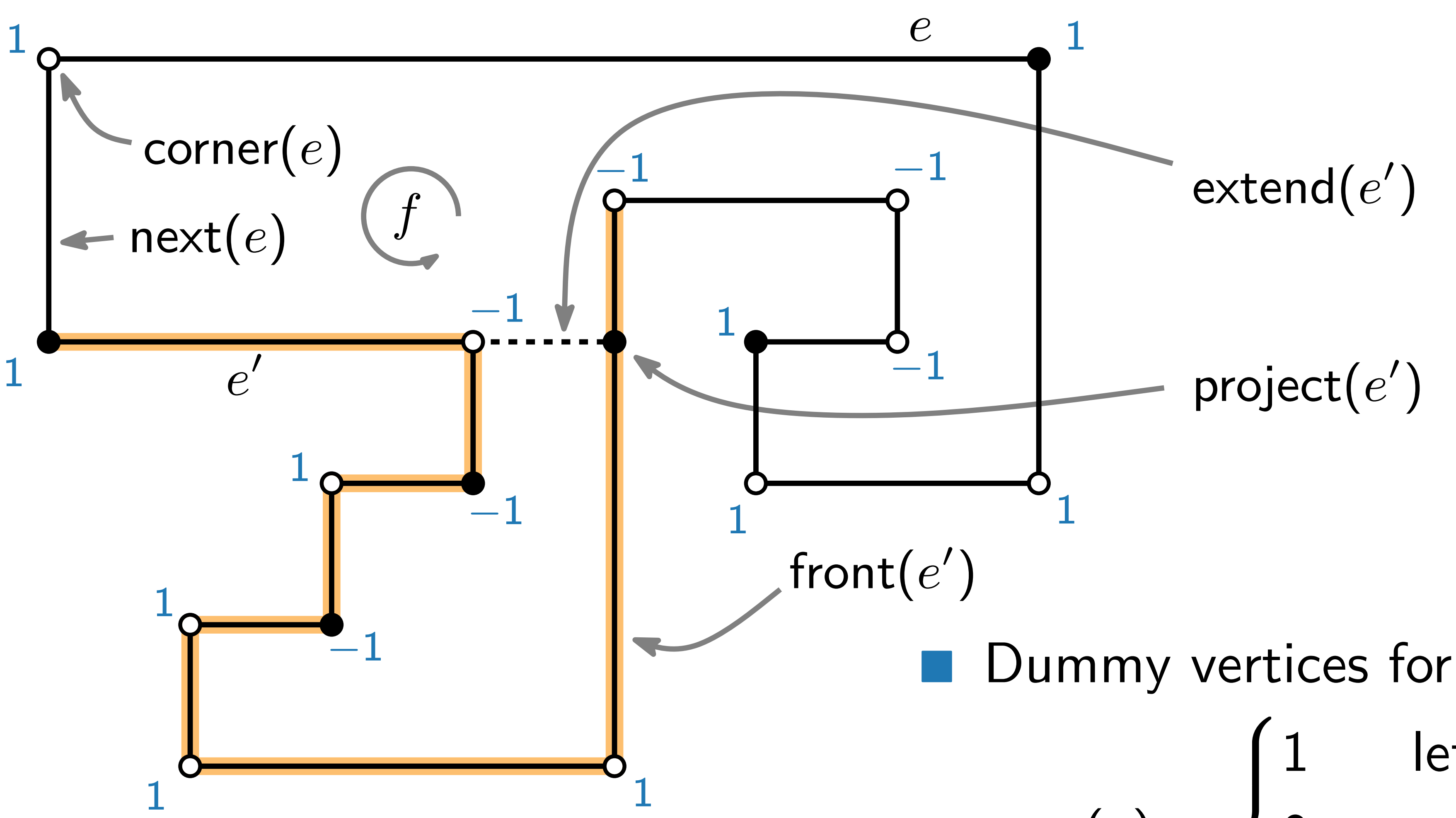
Refinement of G and $H(G)$ – Inner Face



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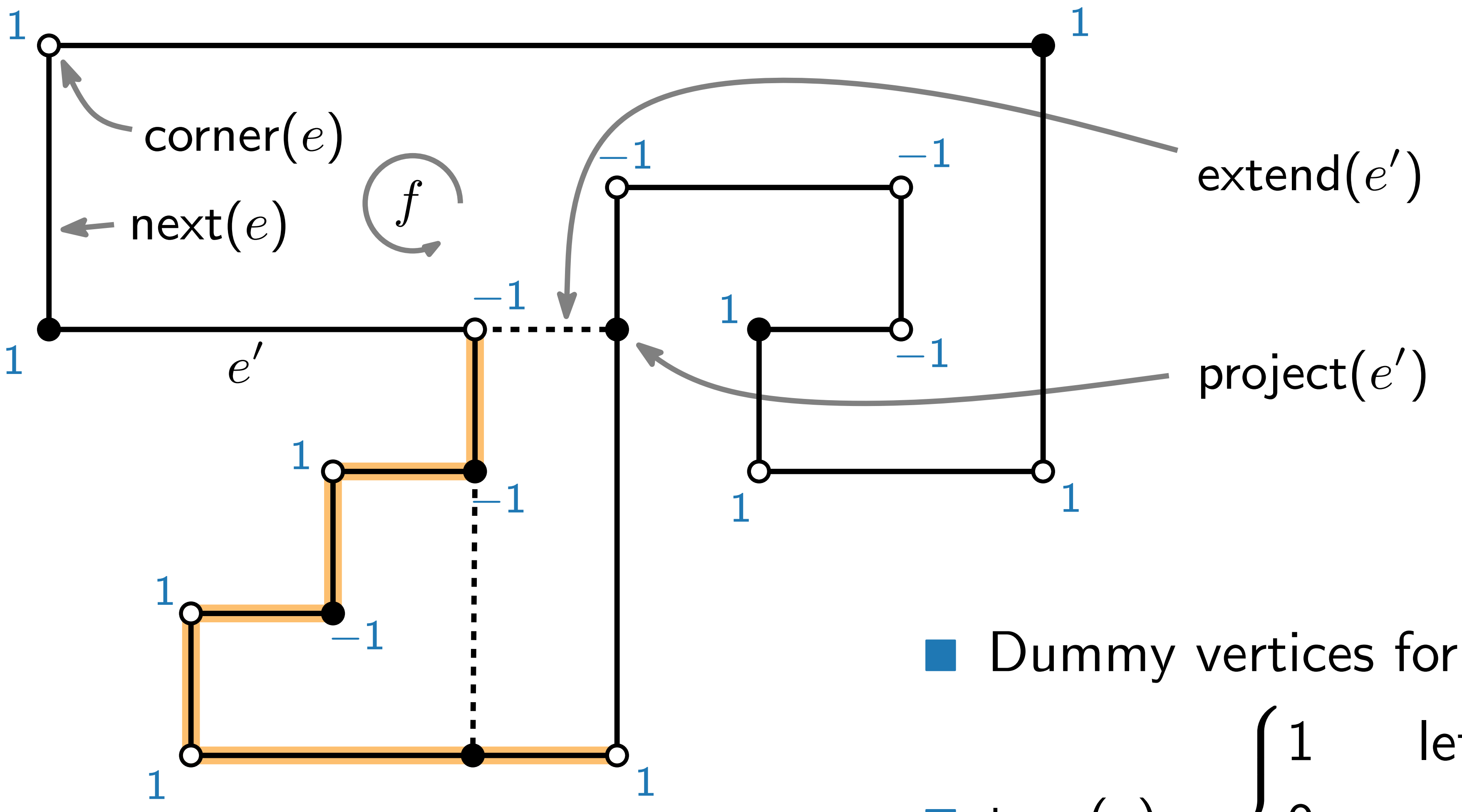

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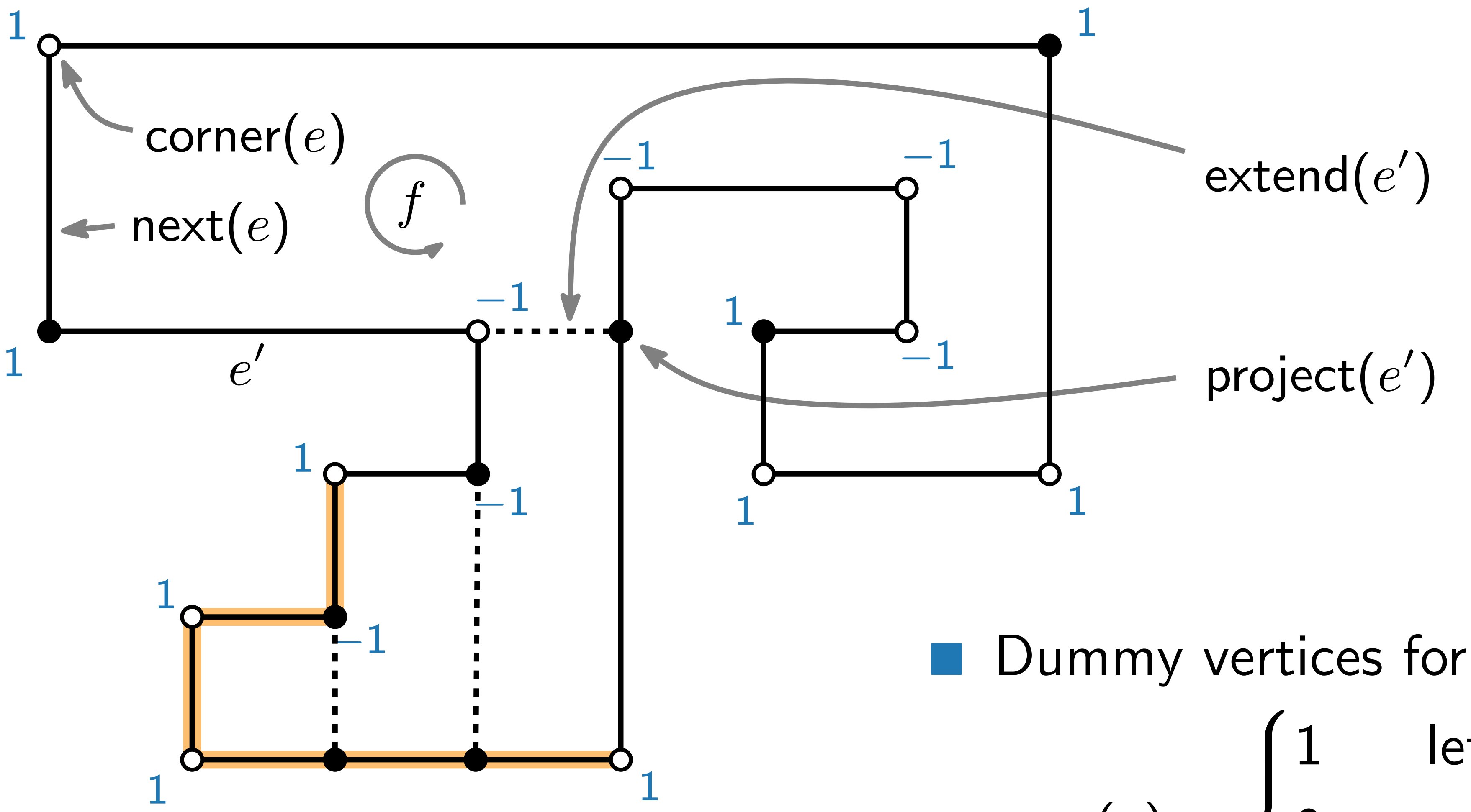

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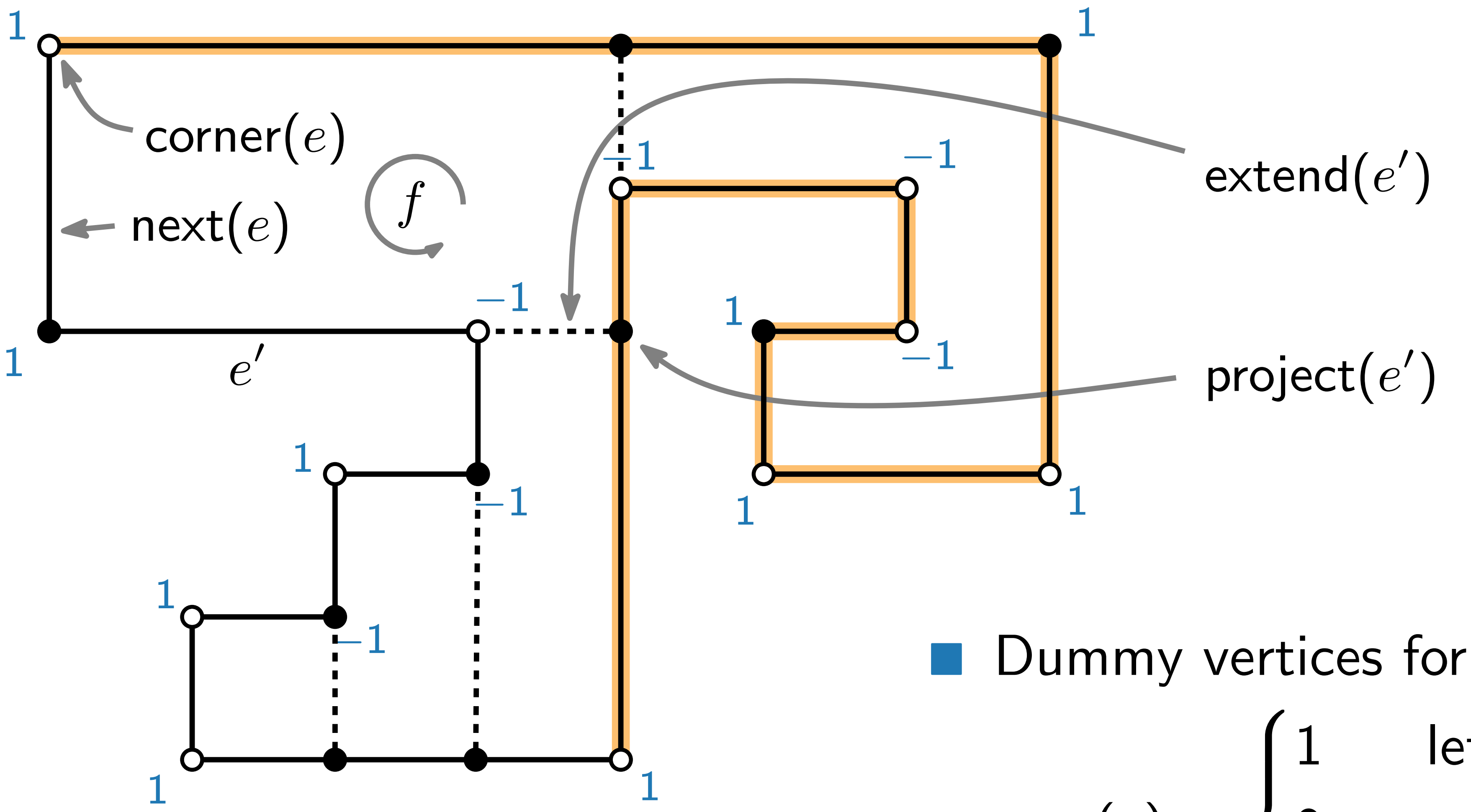

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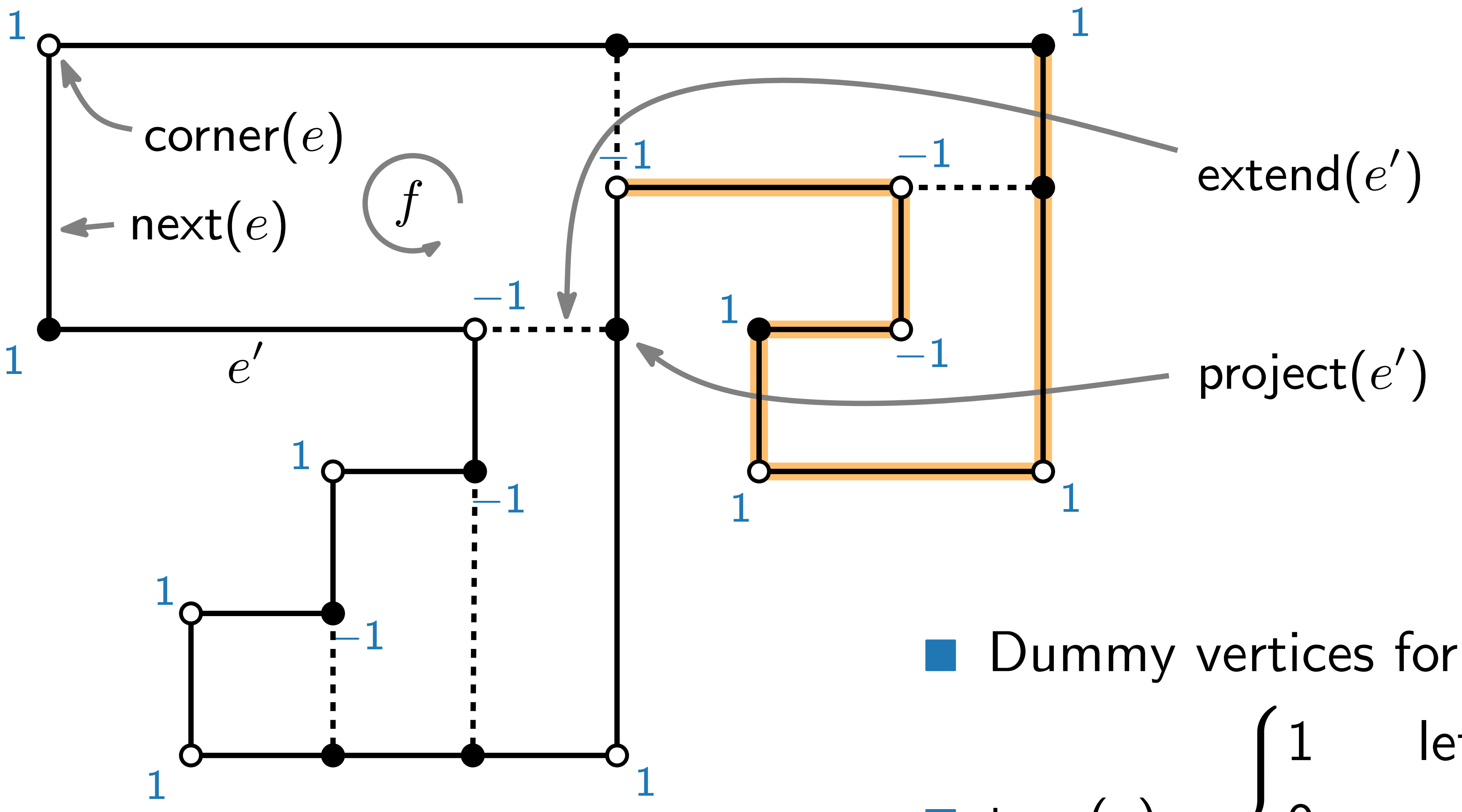

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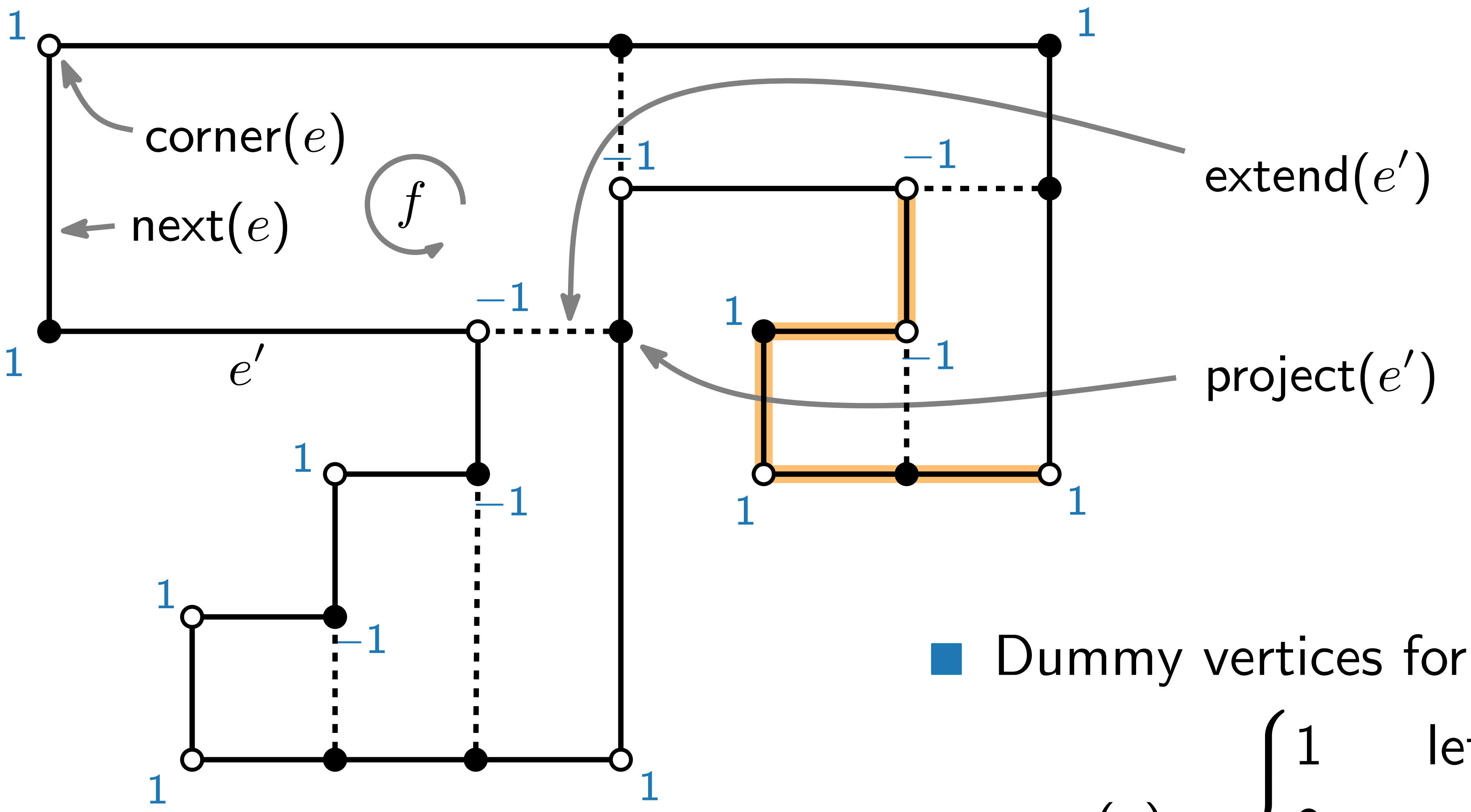

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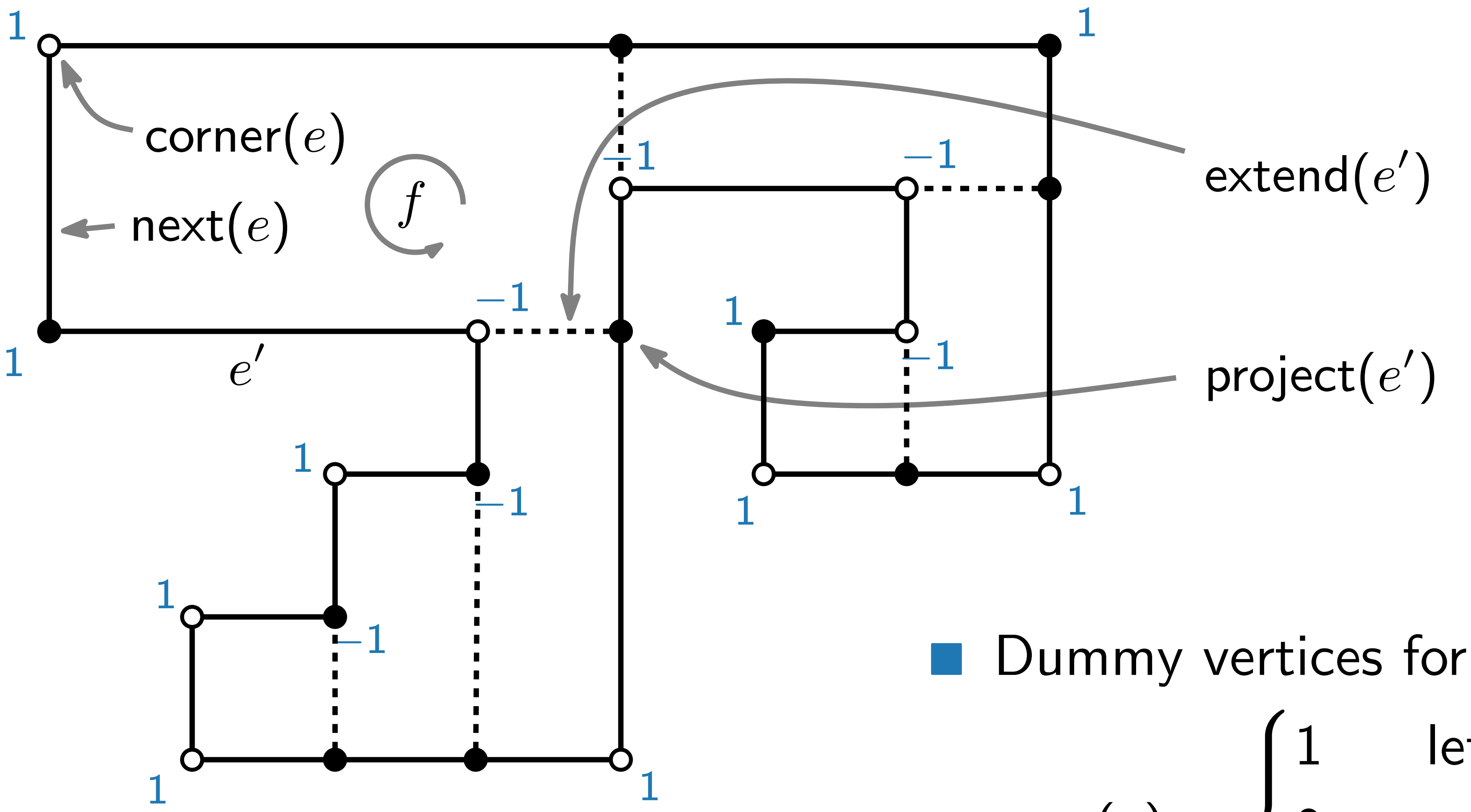

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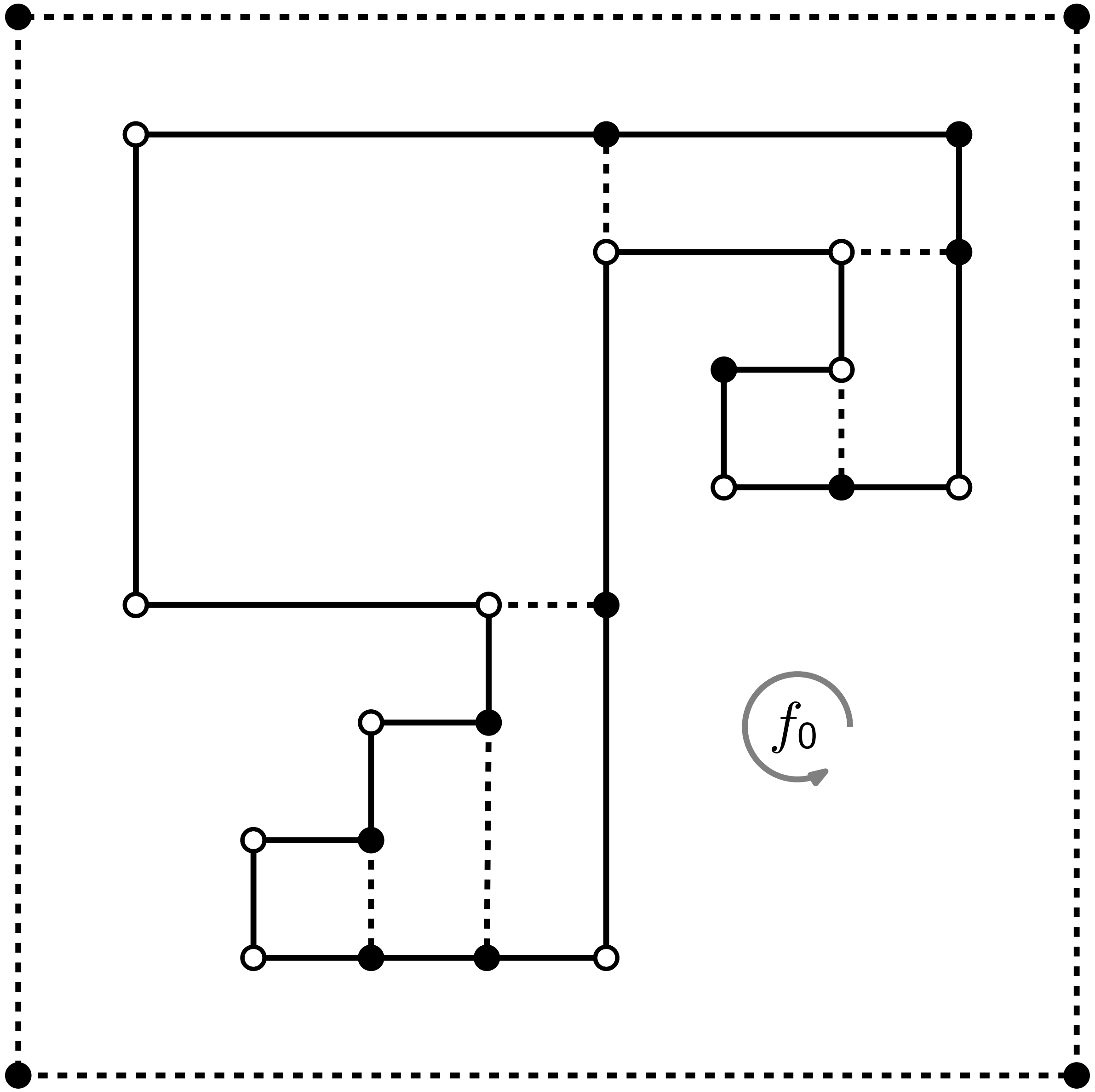
Refinement of G and $H(G)$ – Outer Face



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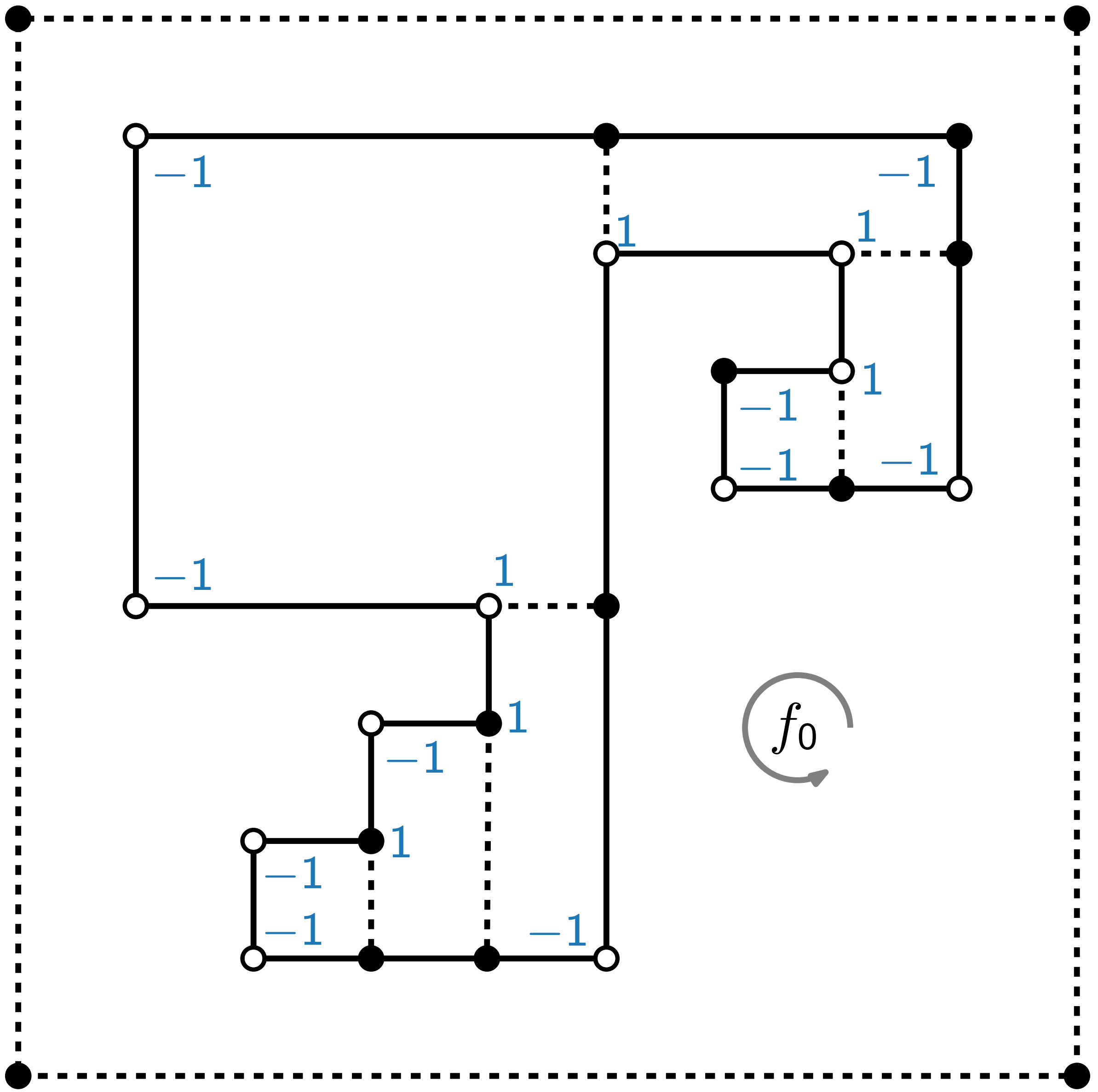

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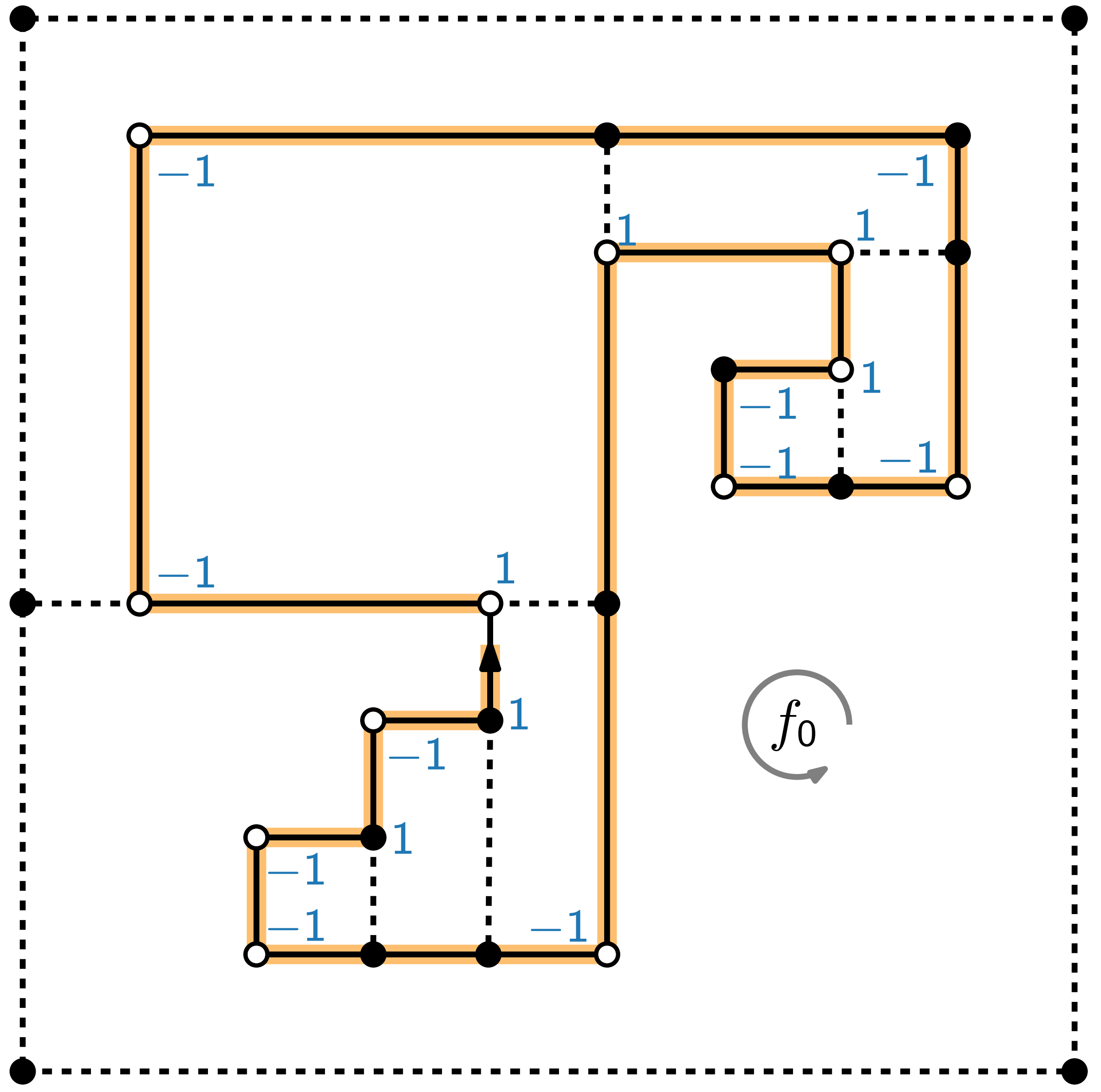

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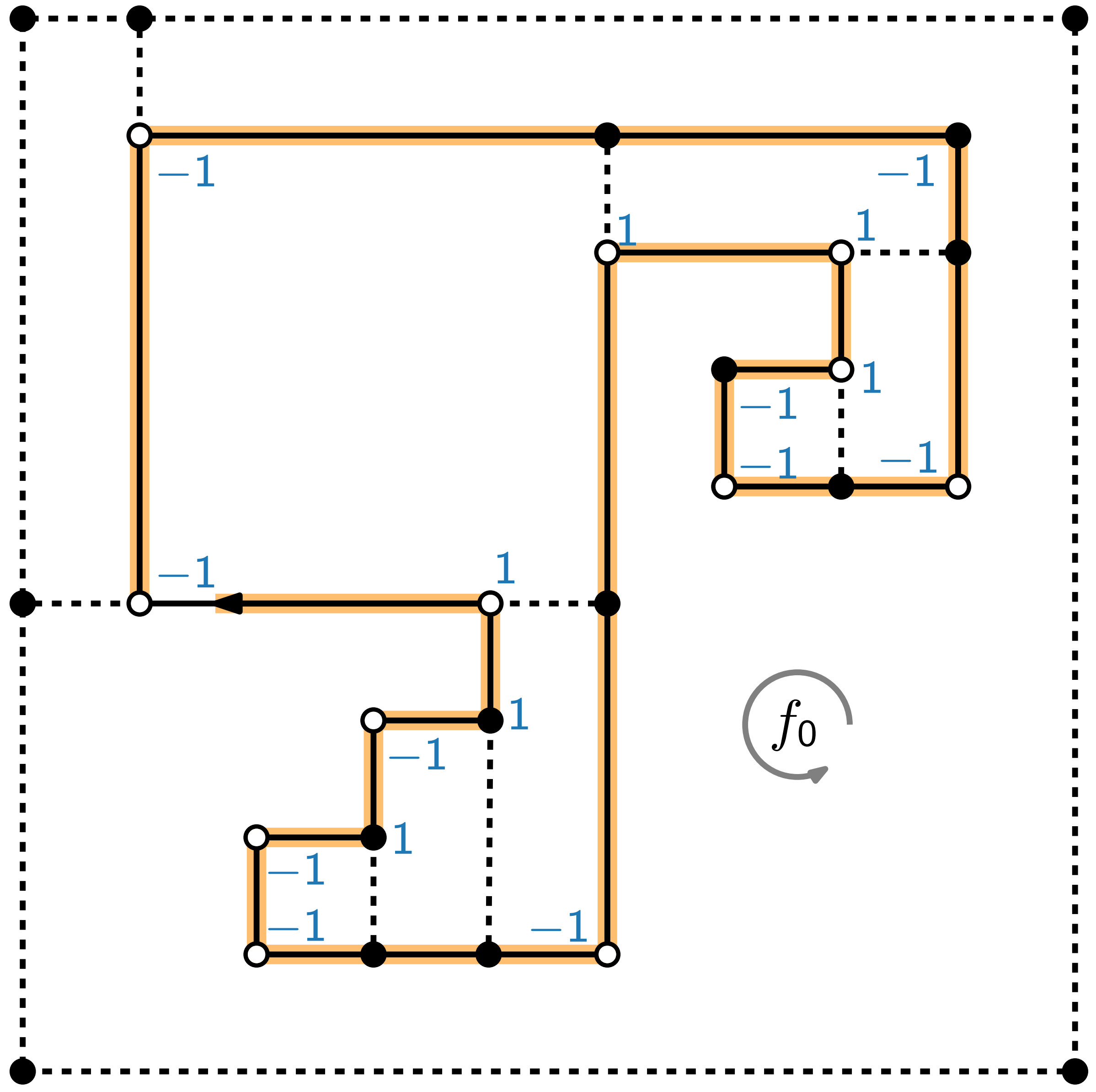
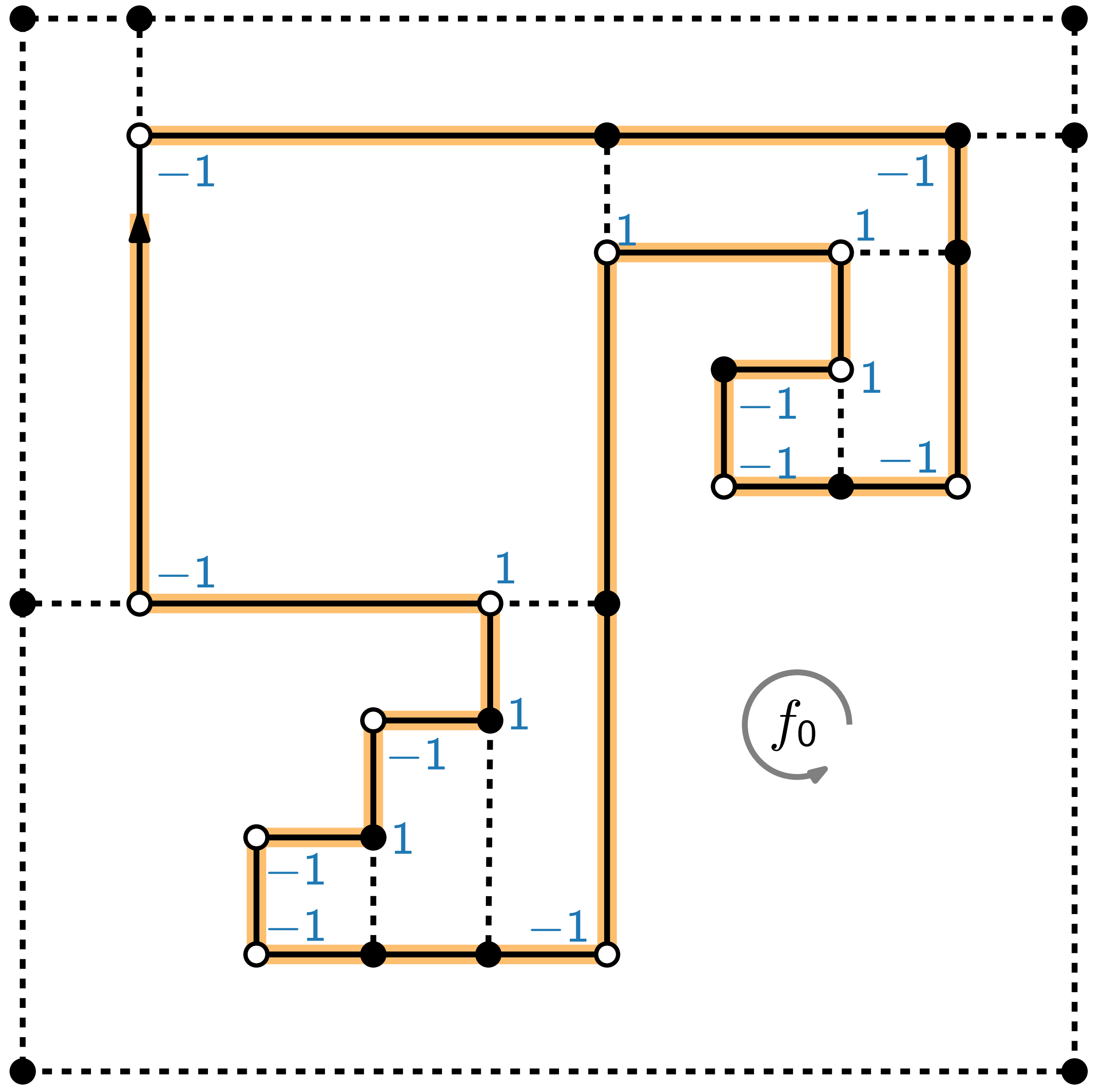
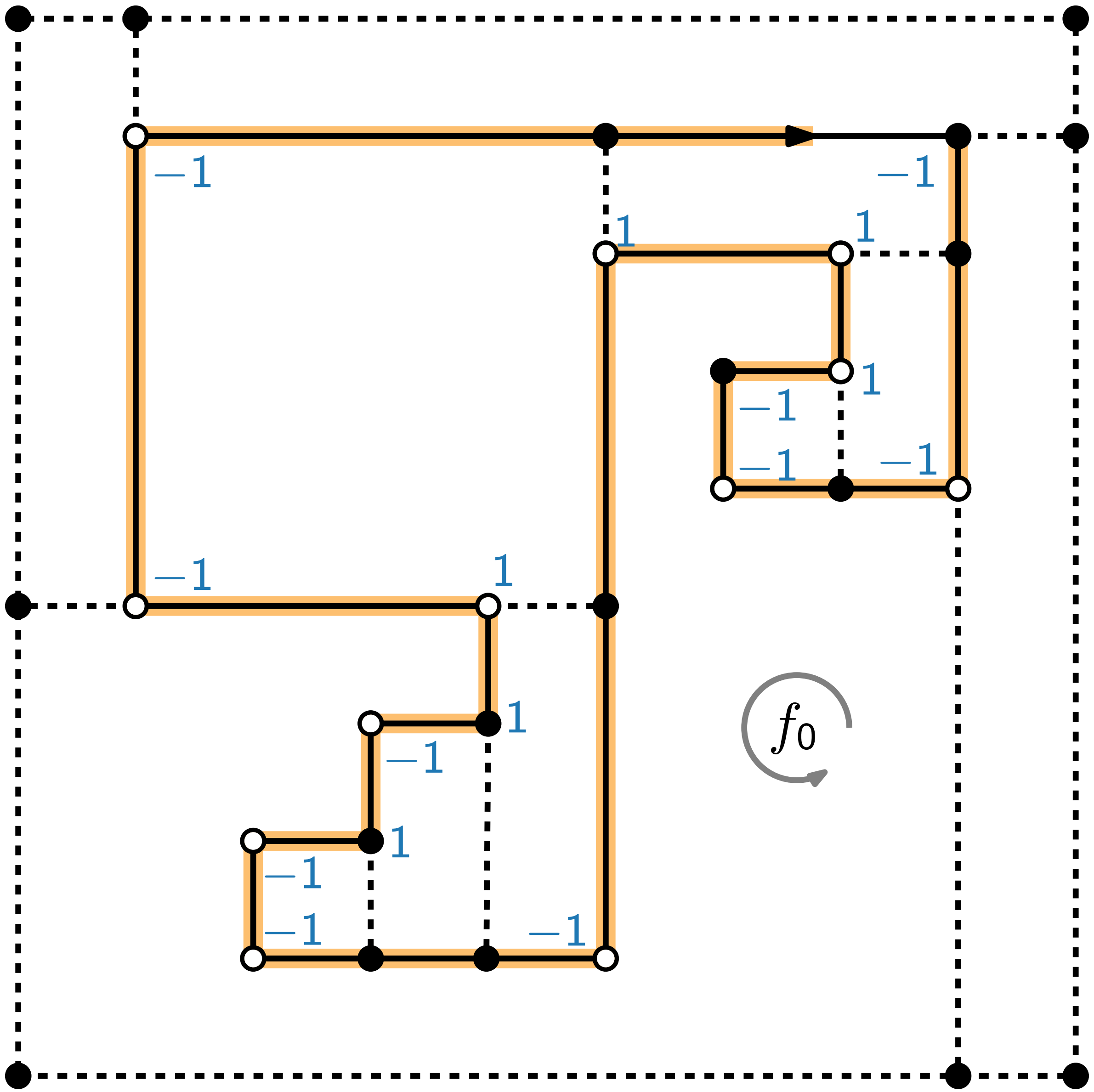


Refinement of G and $H(G)$ – Outer Face



Refinement of G and $H(G)$ – Outer Face



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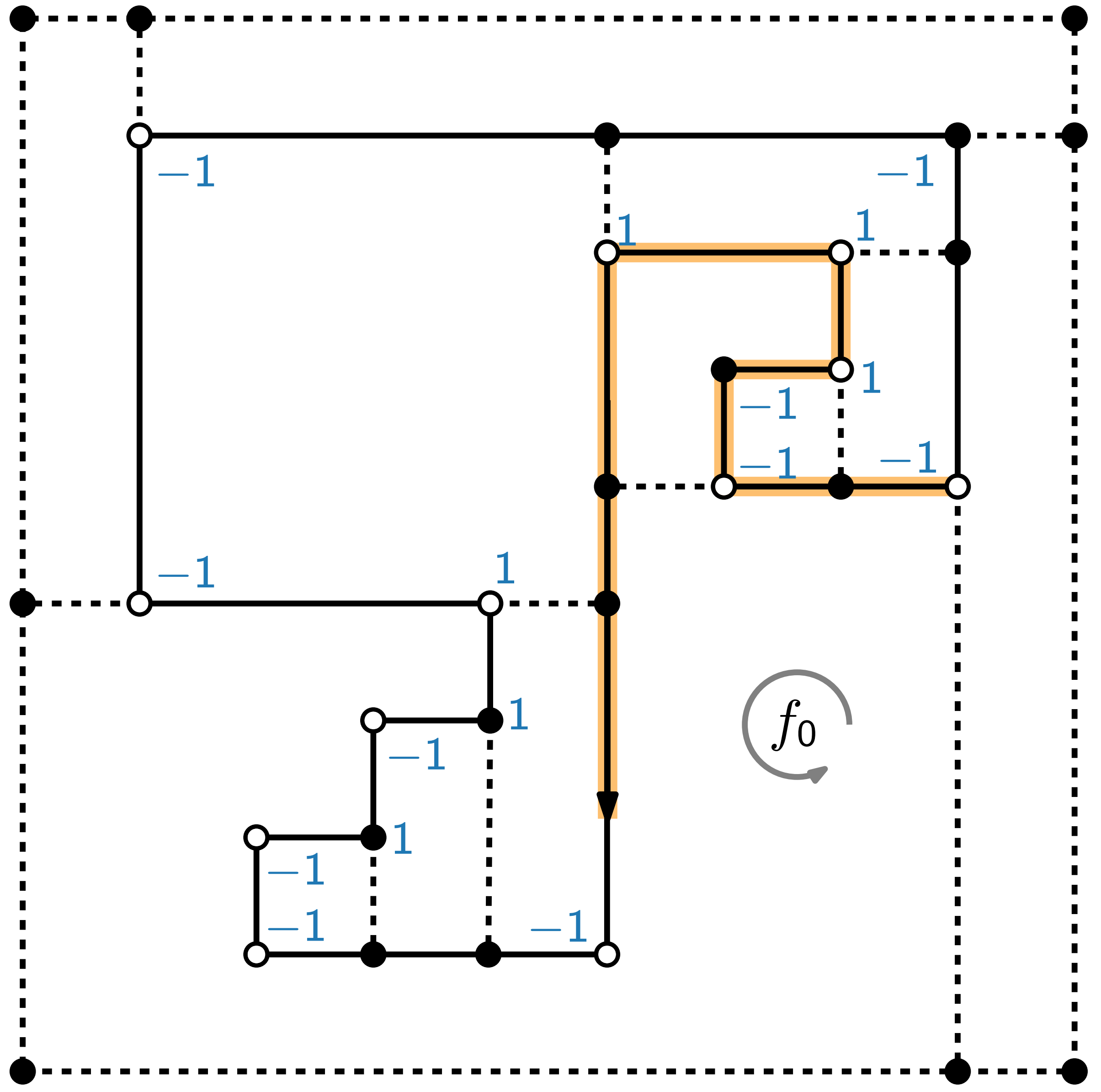

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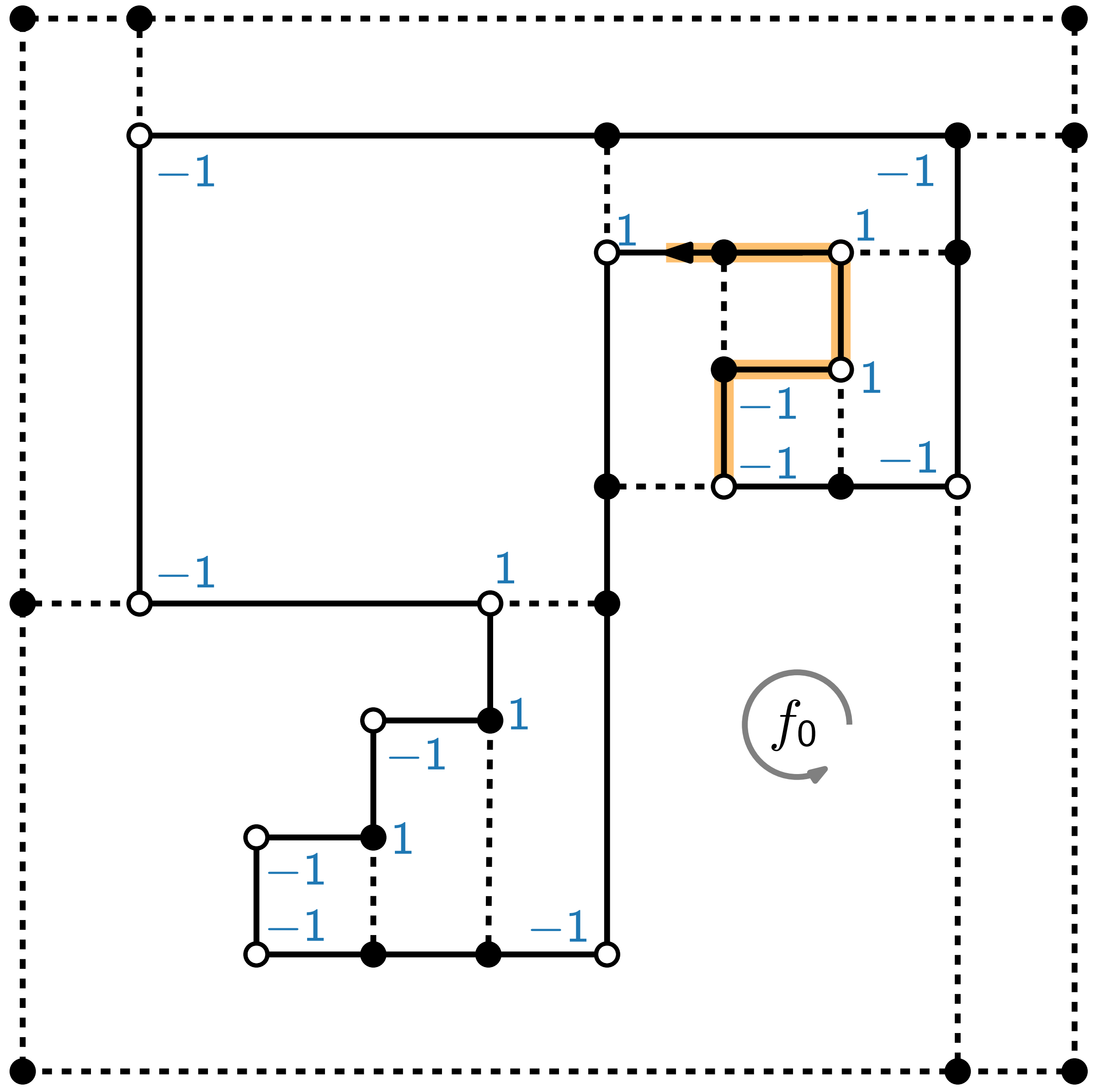
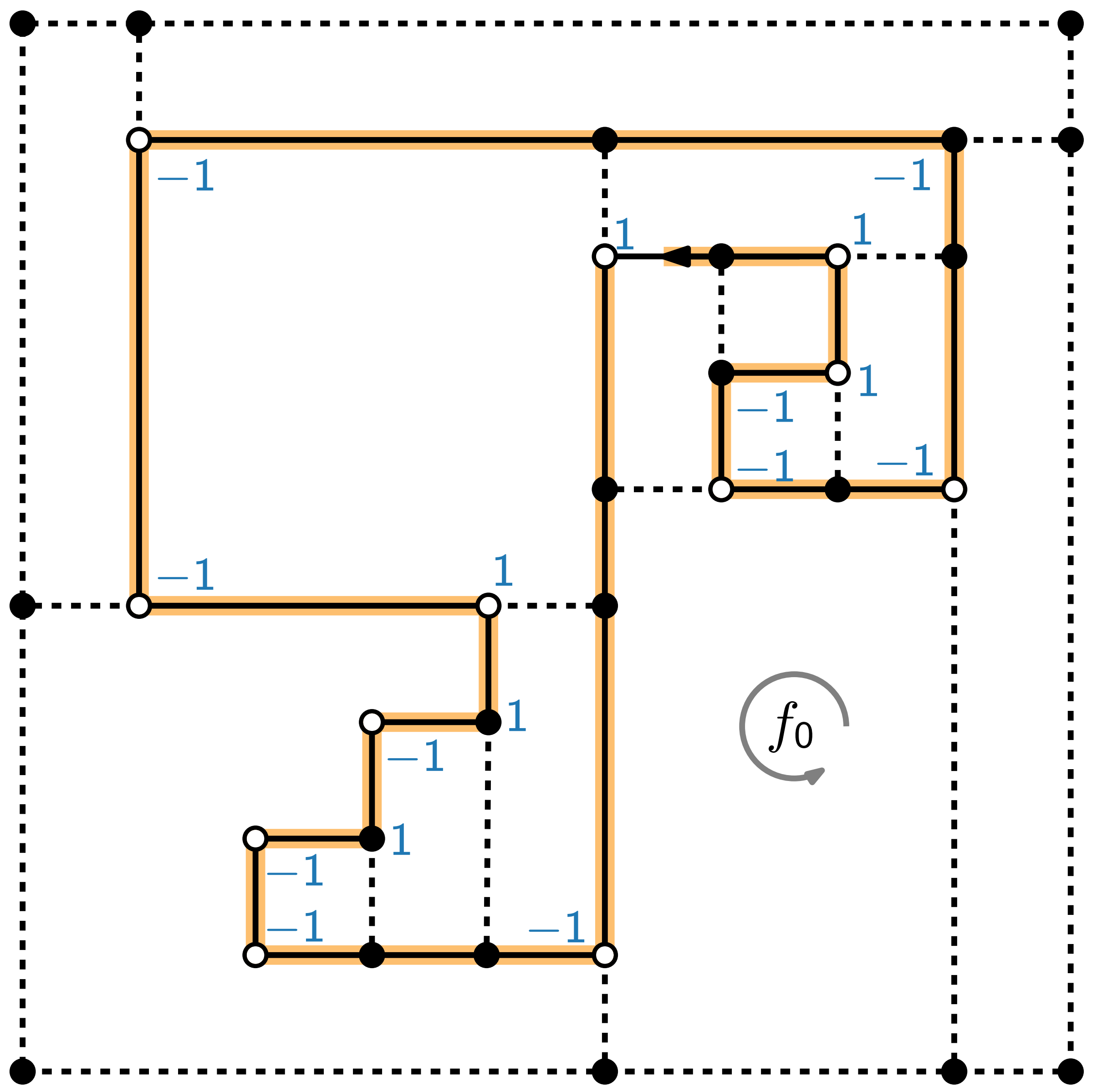
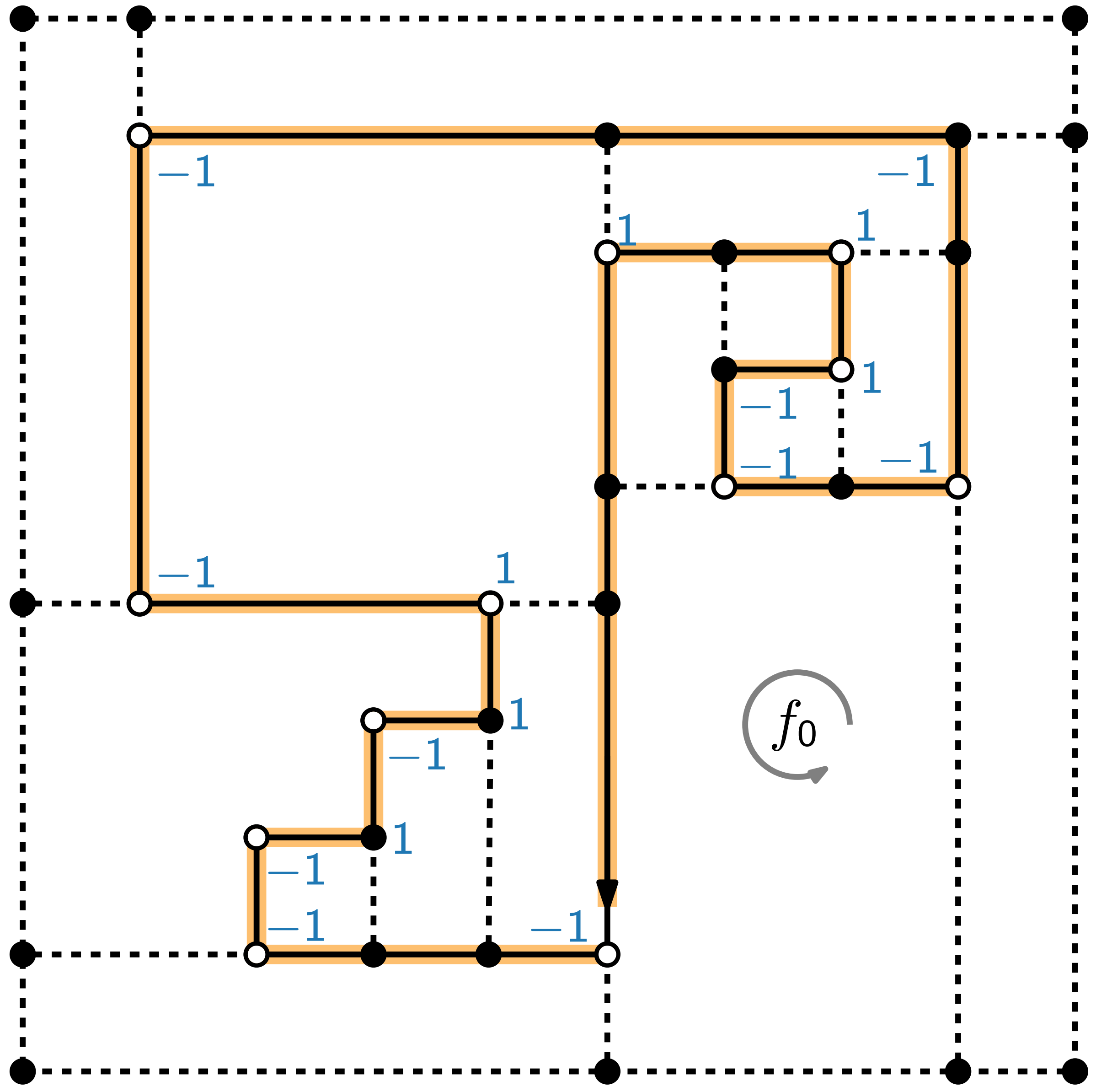


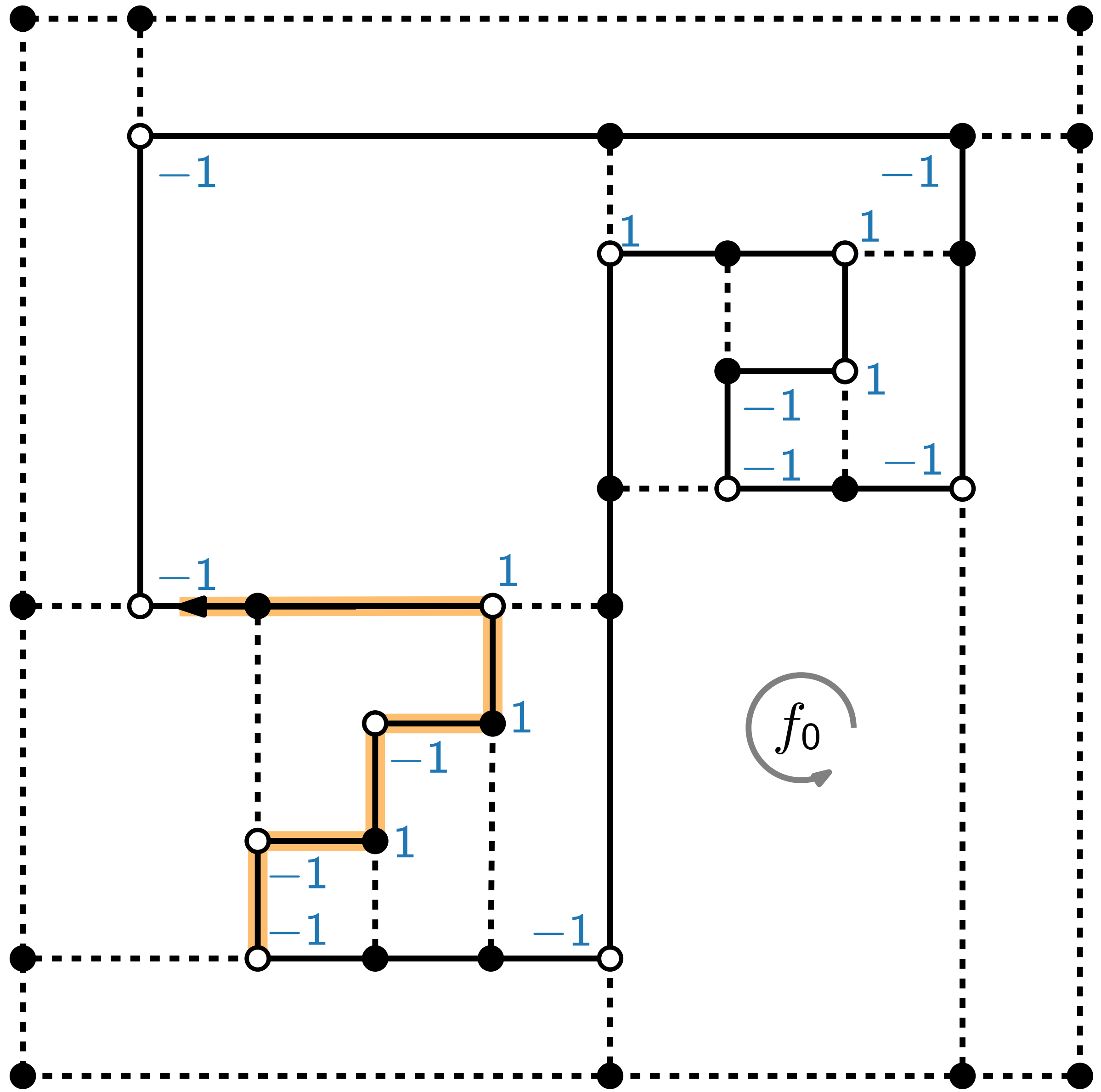
Refinement of G and $H(G)$ – Outer Face



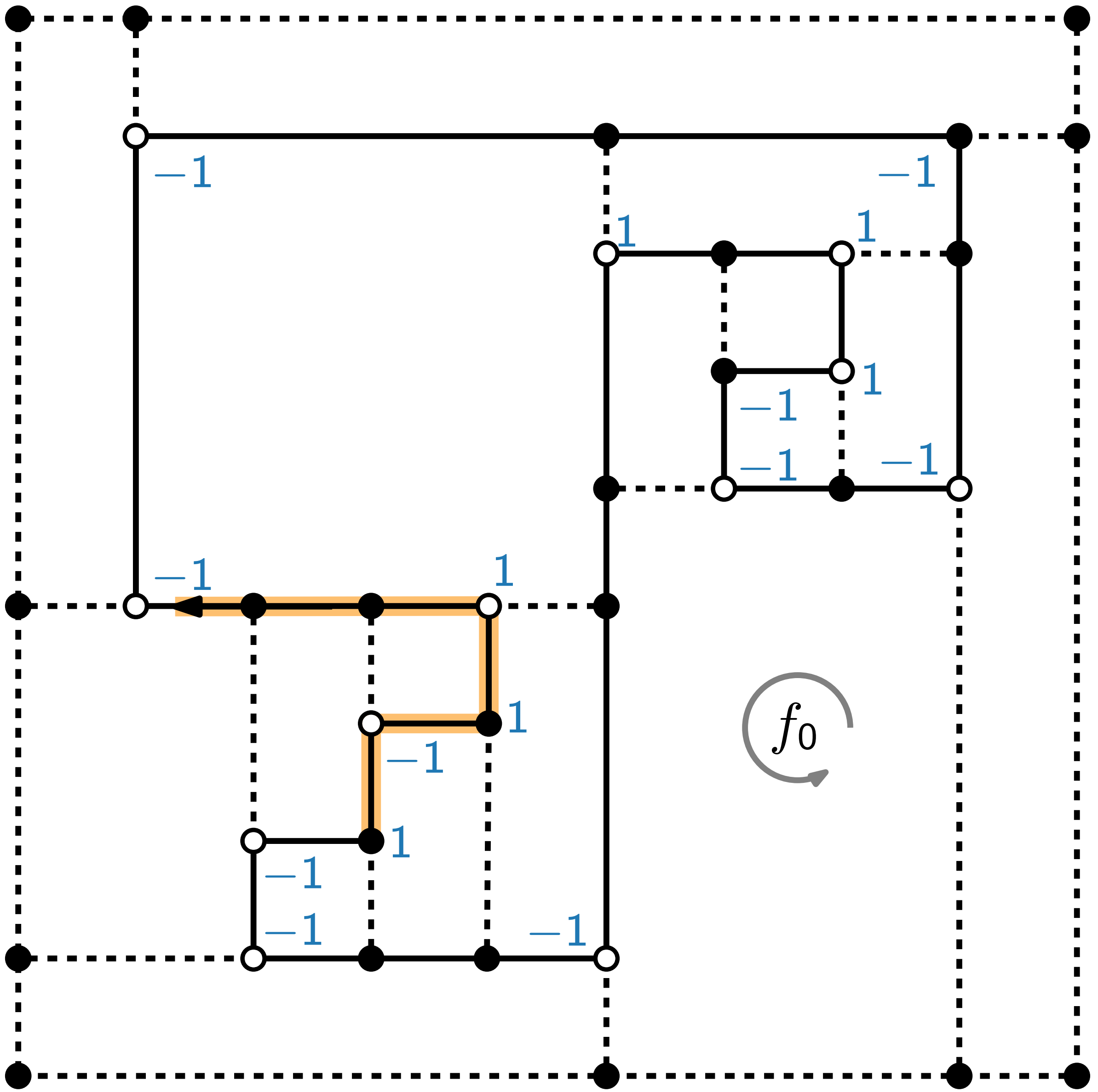
Refinement of G and $H(G)$ – Outer Face



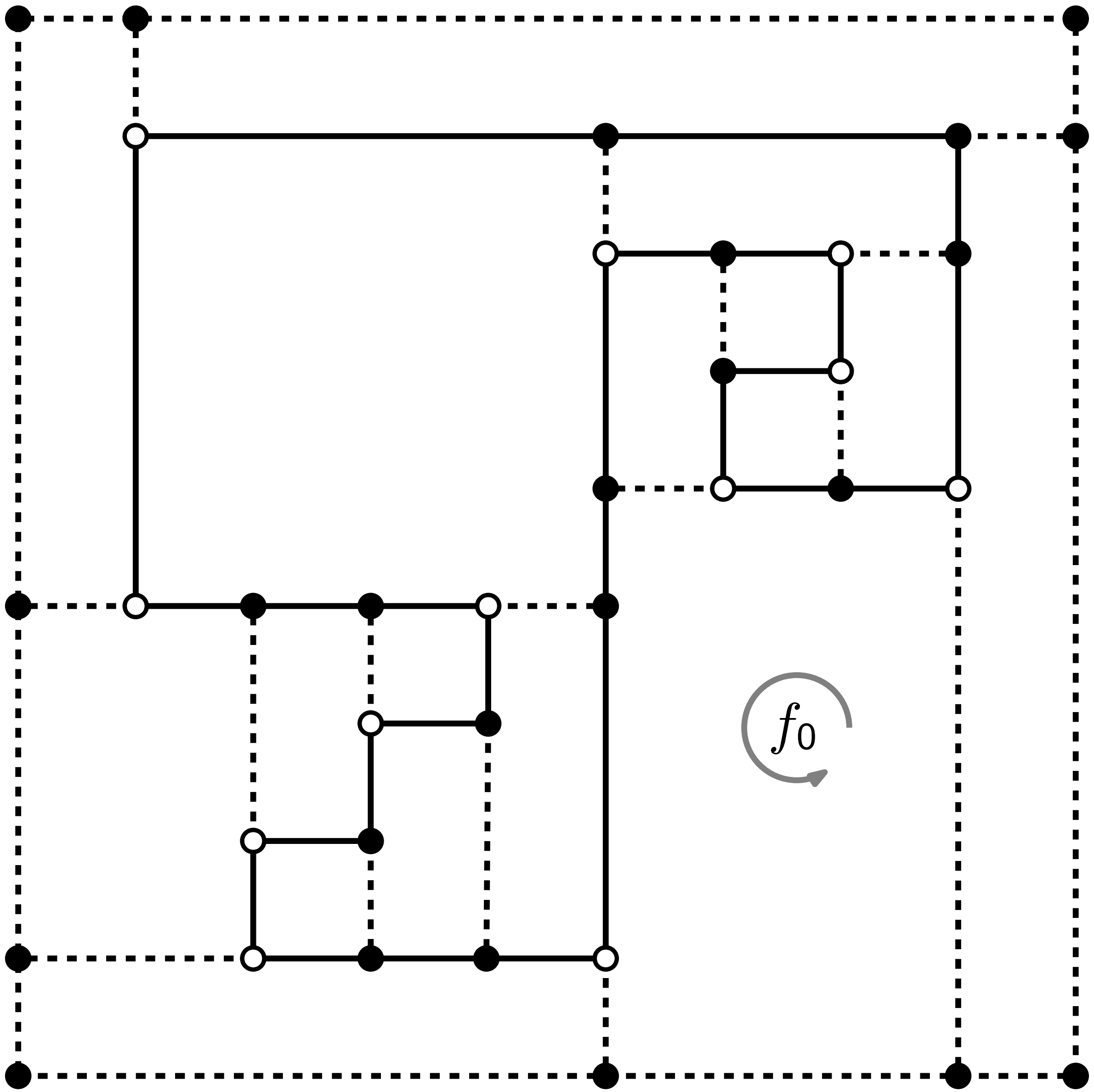
Refinement of G and $H(G)$ – Outer Face



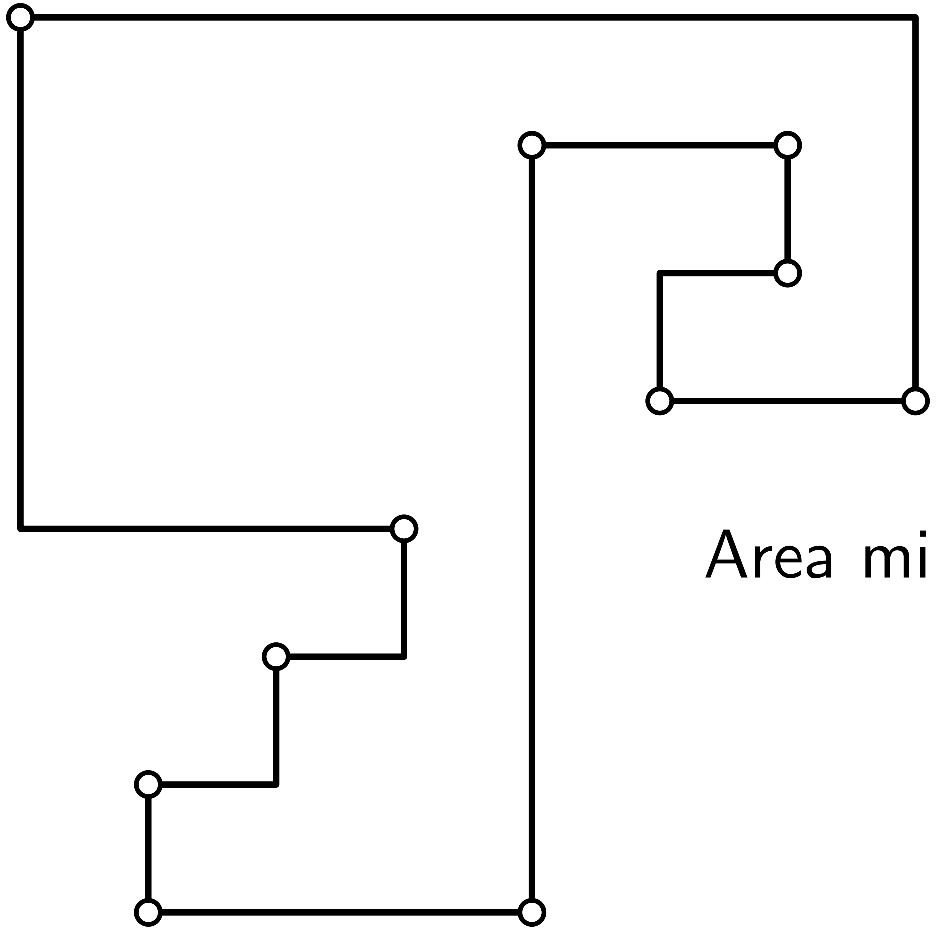
Refinement of G and $H(G)$ – Outer Face



Refinement of G and $H(G)$ – Outer Face

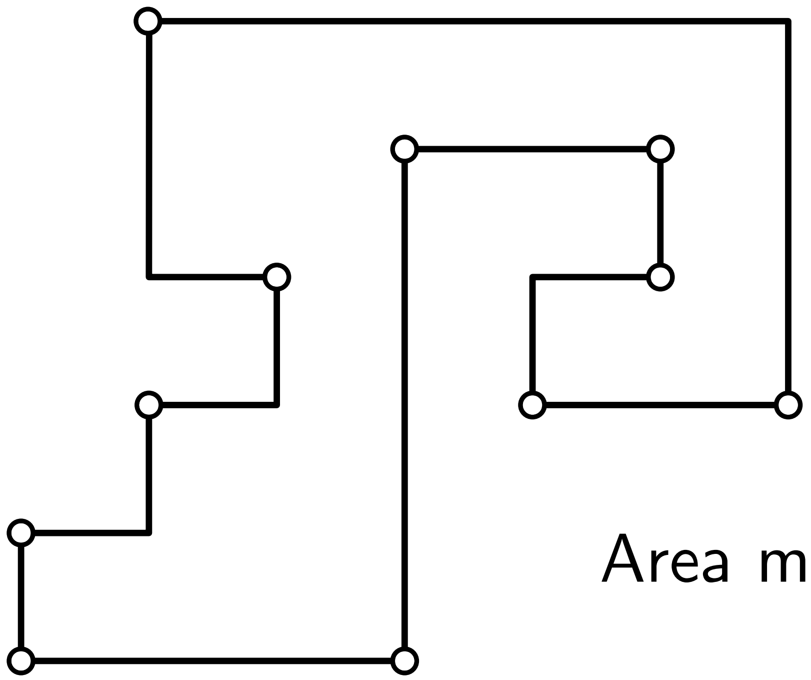


Refinement of G and $H(G)$ – Outer Face



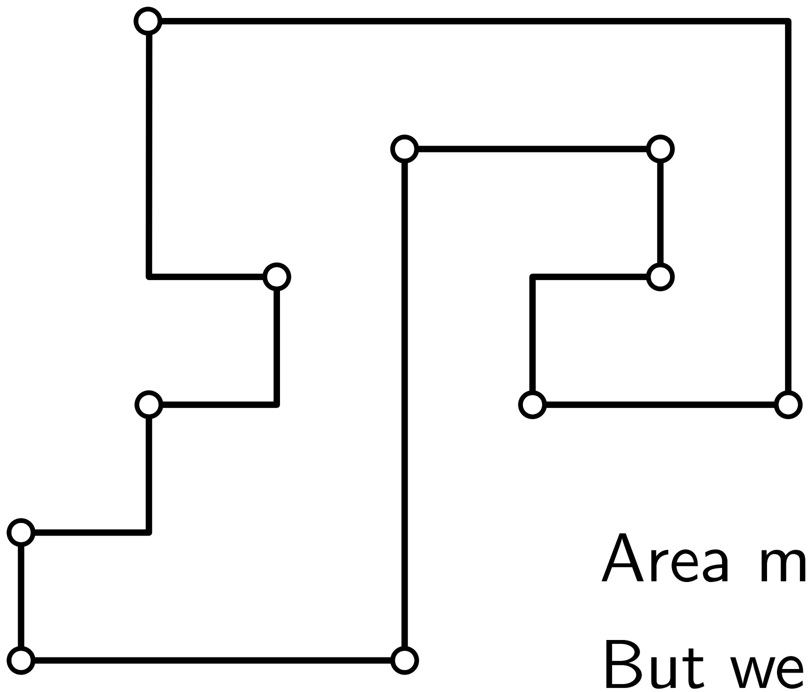
Area minimized?

Refinement of G and $H(G)$ – Outer Face



Area minimized? No!

Refinement of G and $H(G)$ – Outer Face



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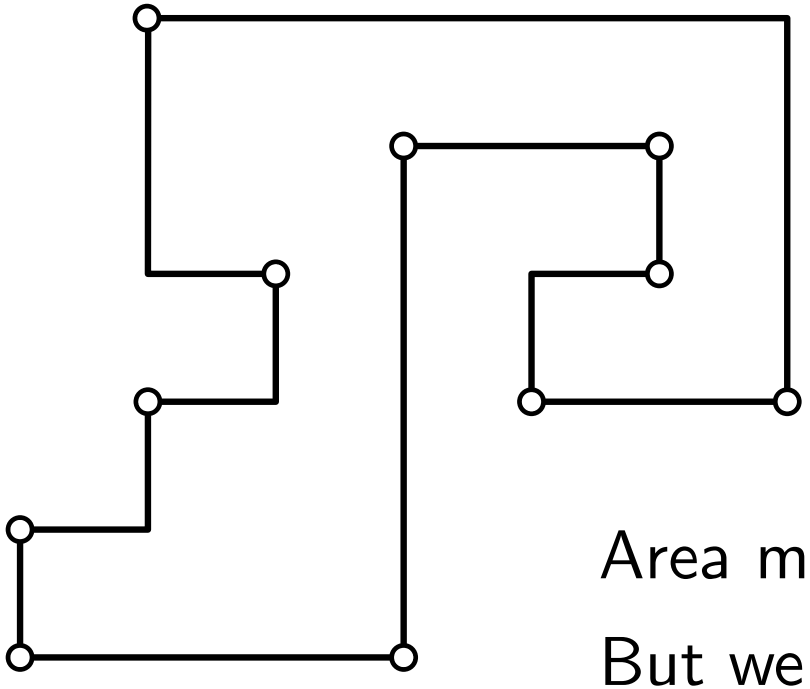

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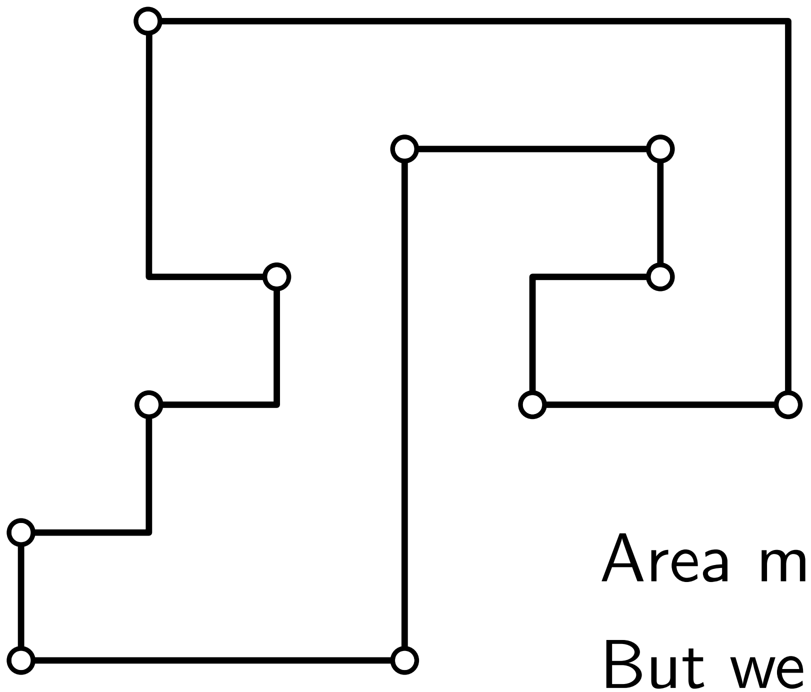

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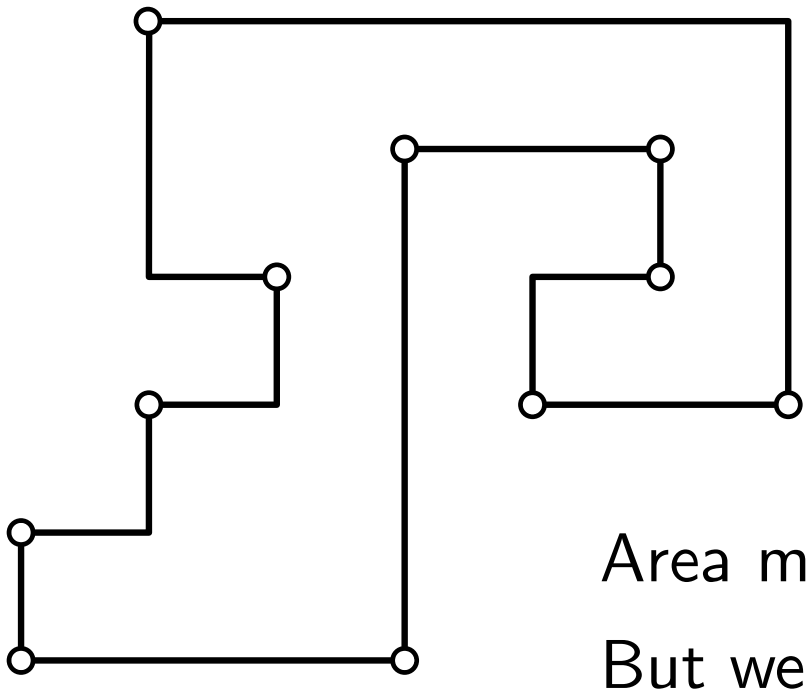

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Theorem. [EFKSSW 2022]
Compaction is NP-hard even for orthogonal representations of *cycles*.

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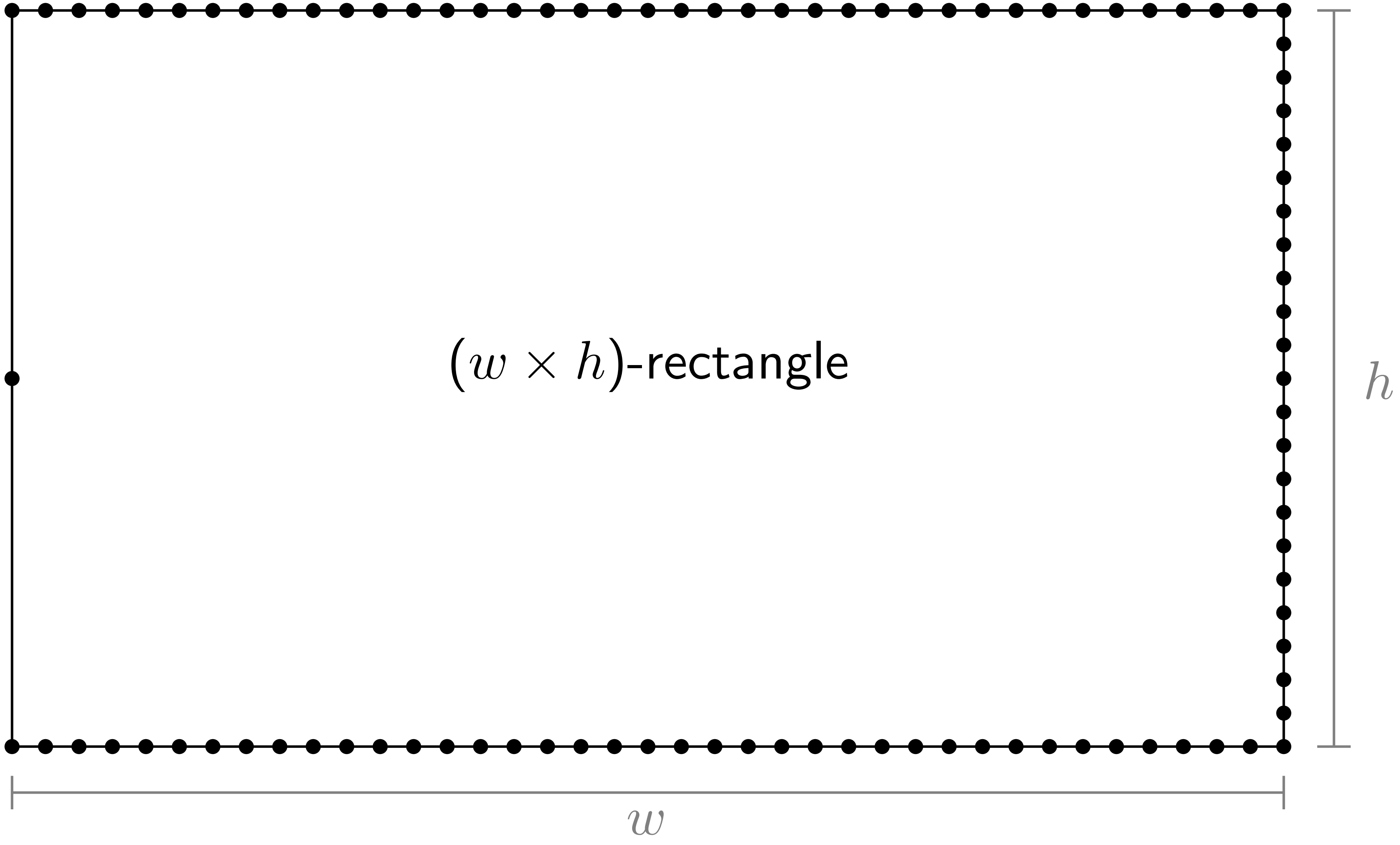
Idea of the reduction:

- Given SAT instance $\Phi \Rightarrow$ construct a plane graph G and a orthogonal description $H(G)$
- Φ is satisfiable $\Leftrightarrow$ G can be drawn w.r.t. $H(G)$ in area K for some specific number K

Boundary, Belt, and "Piston" Gadget



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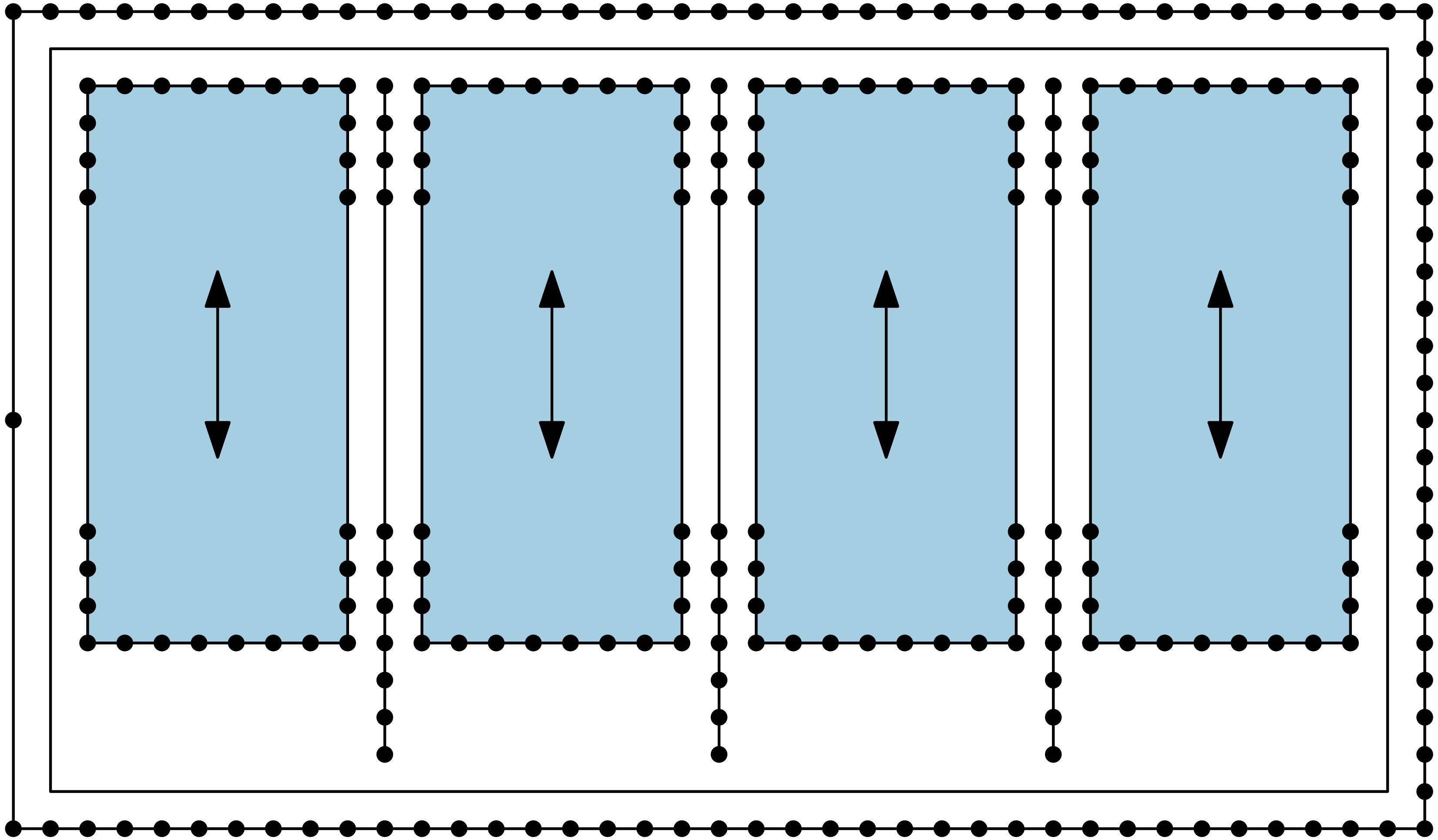

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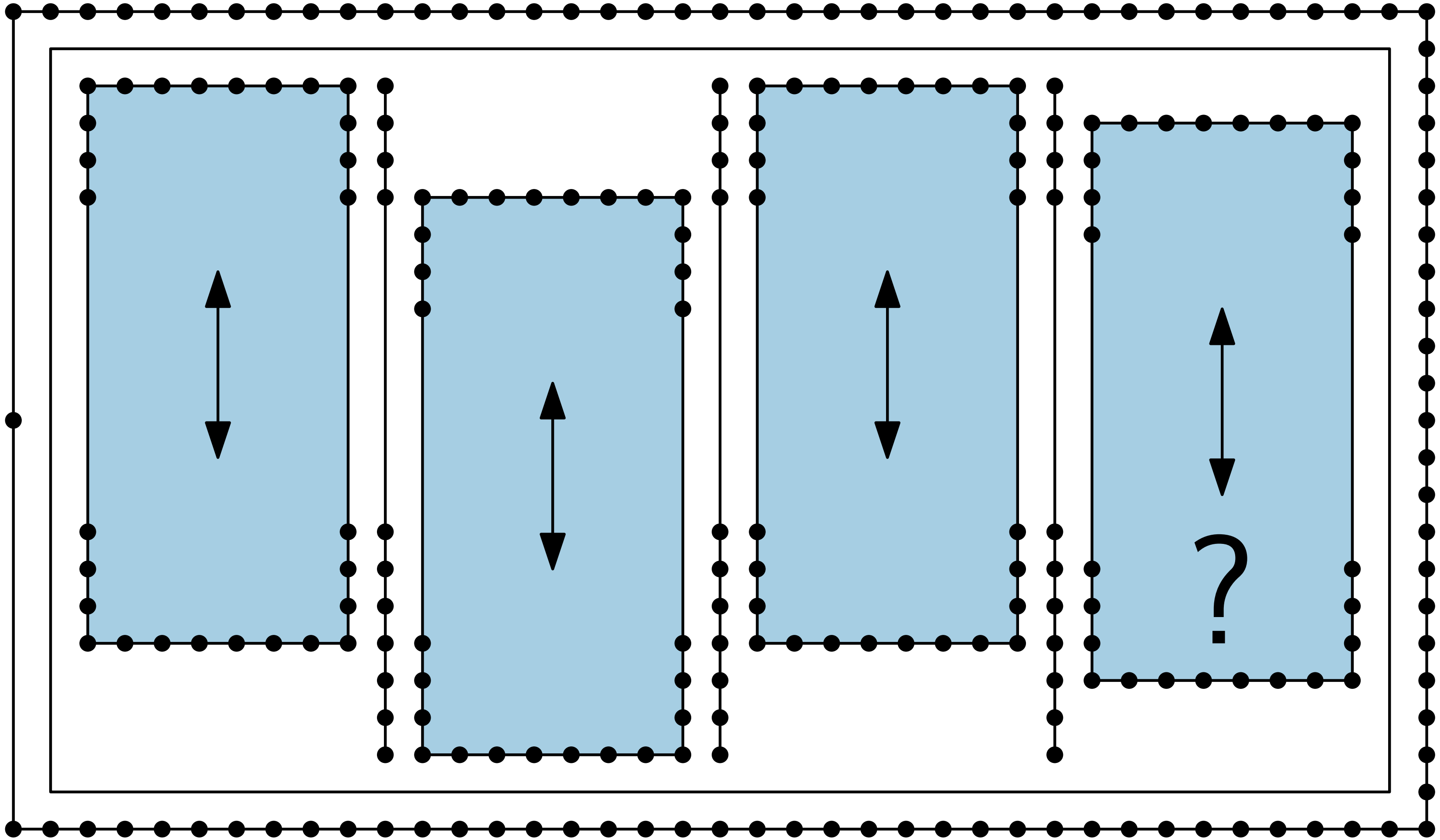

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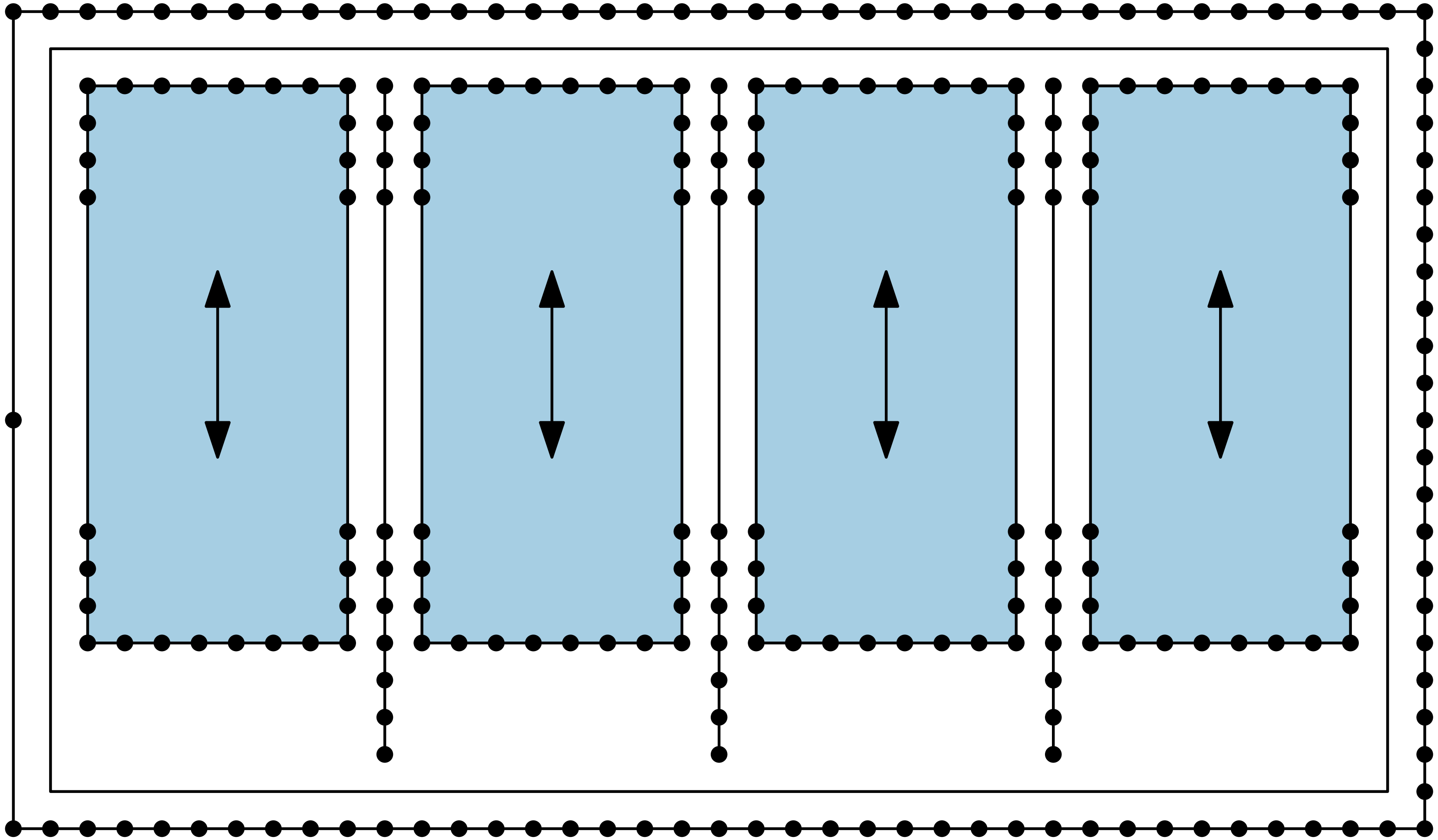

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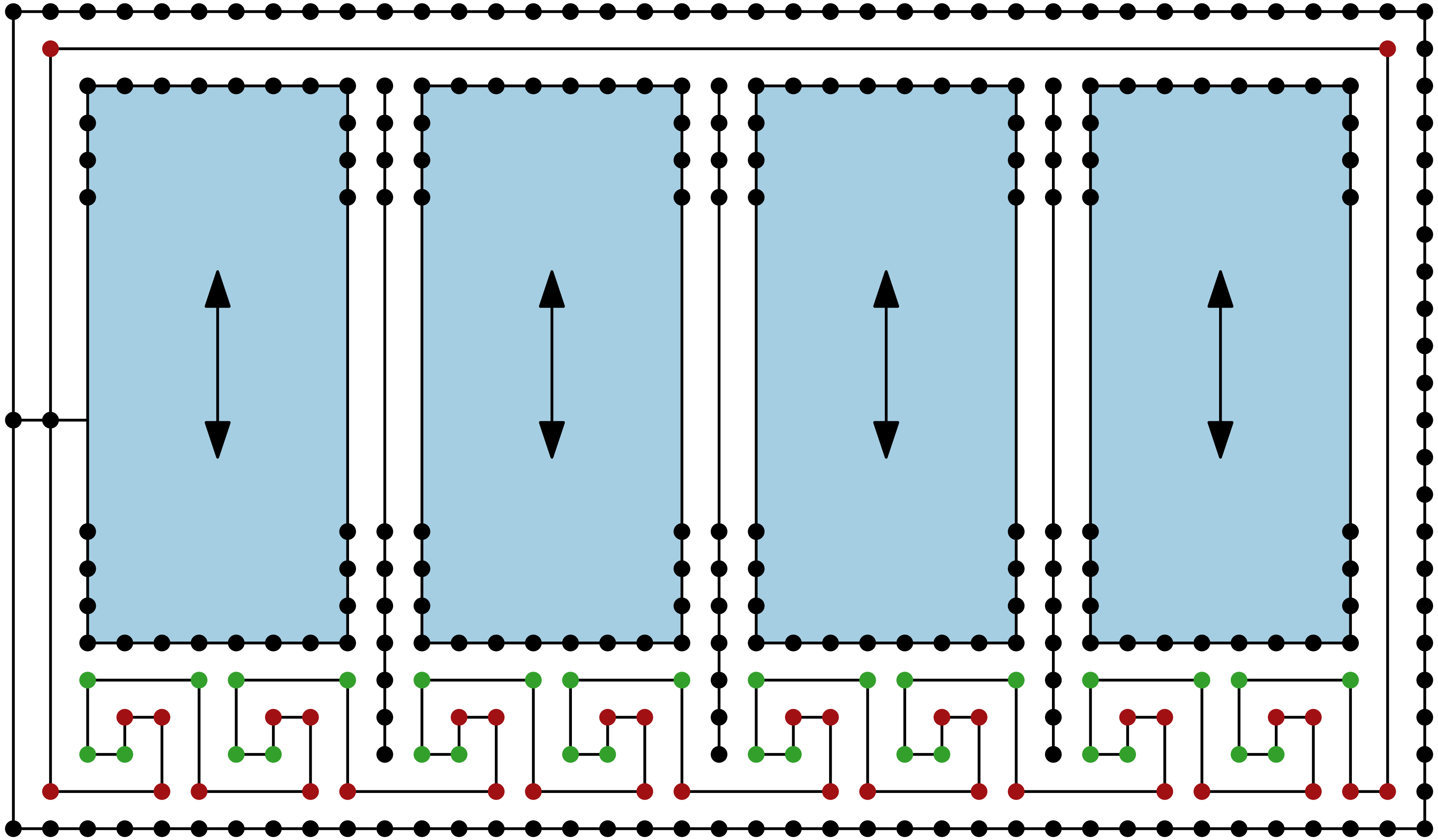

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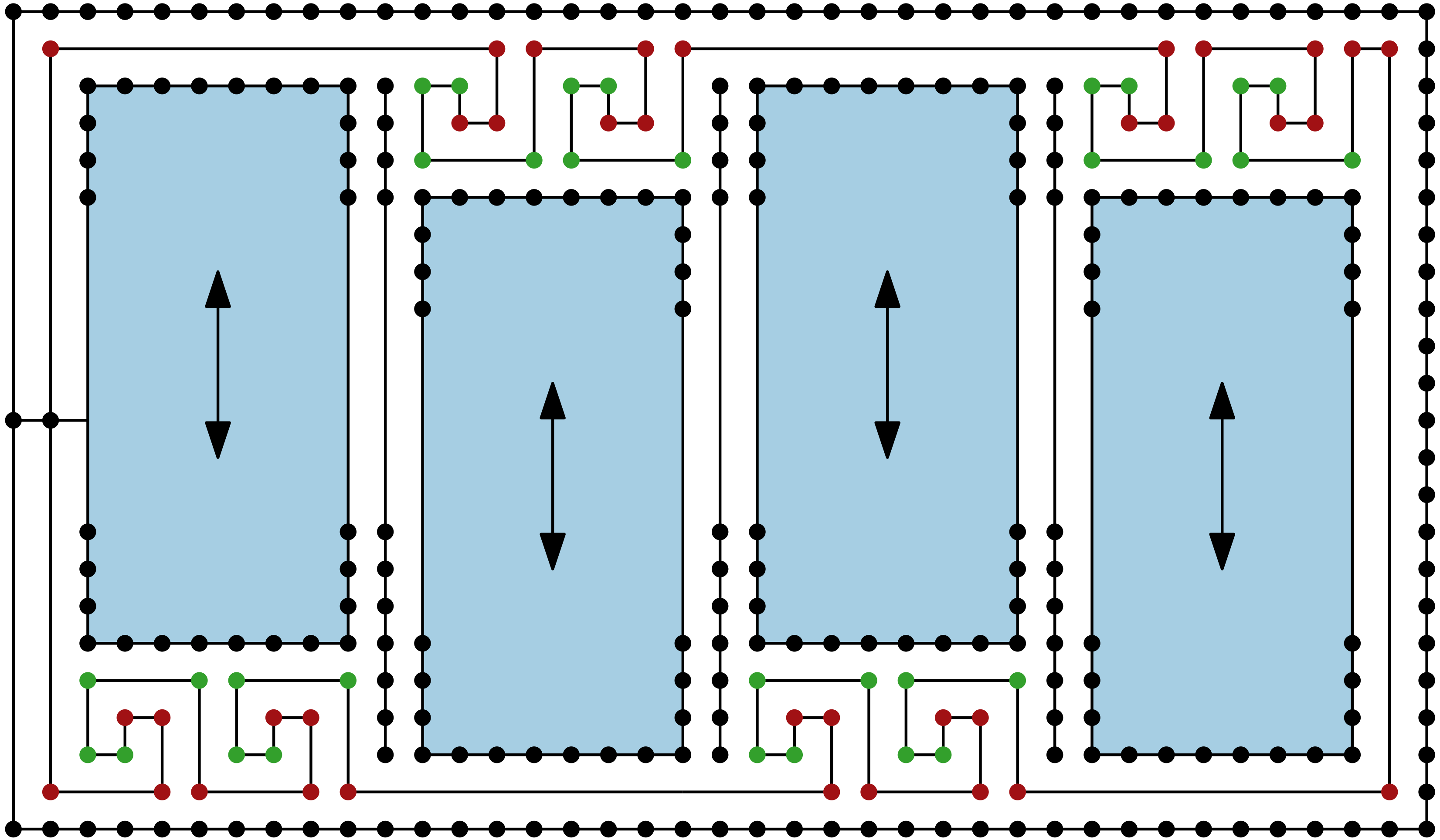

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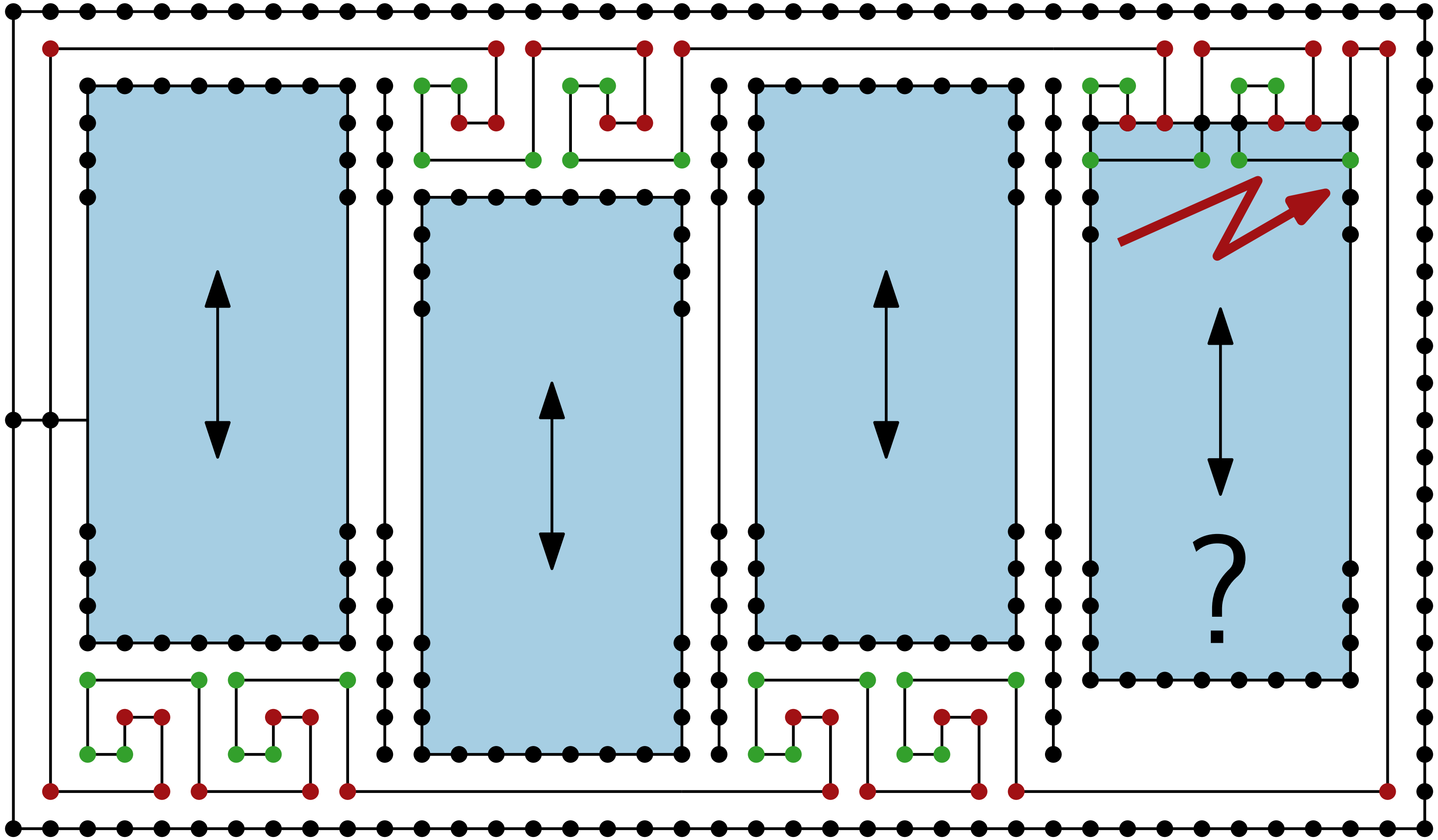

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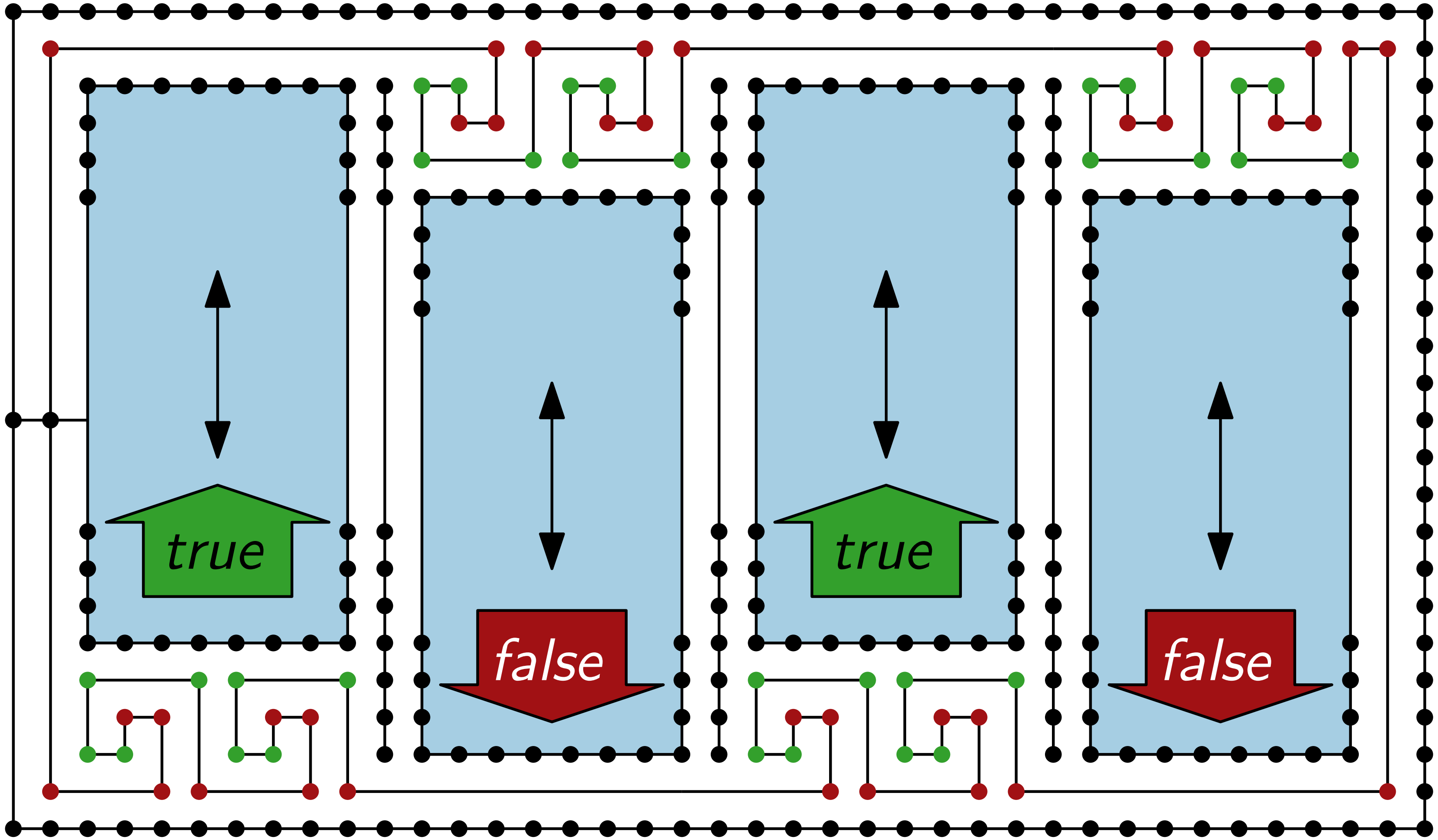
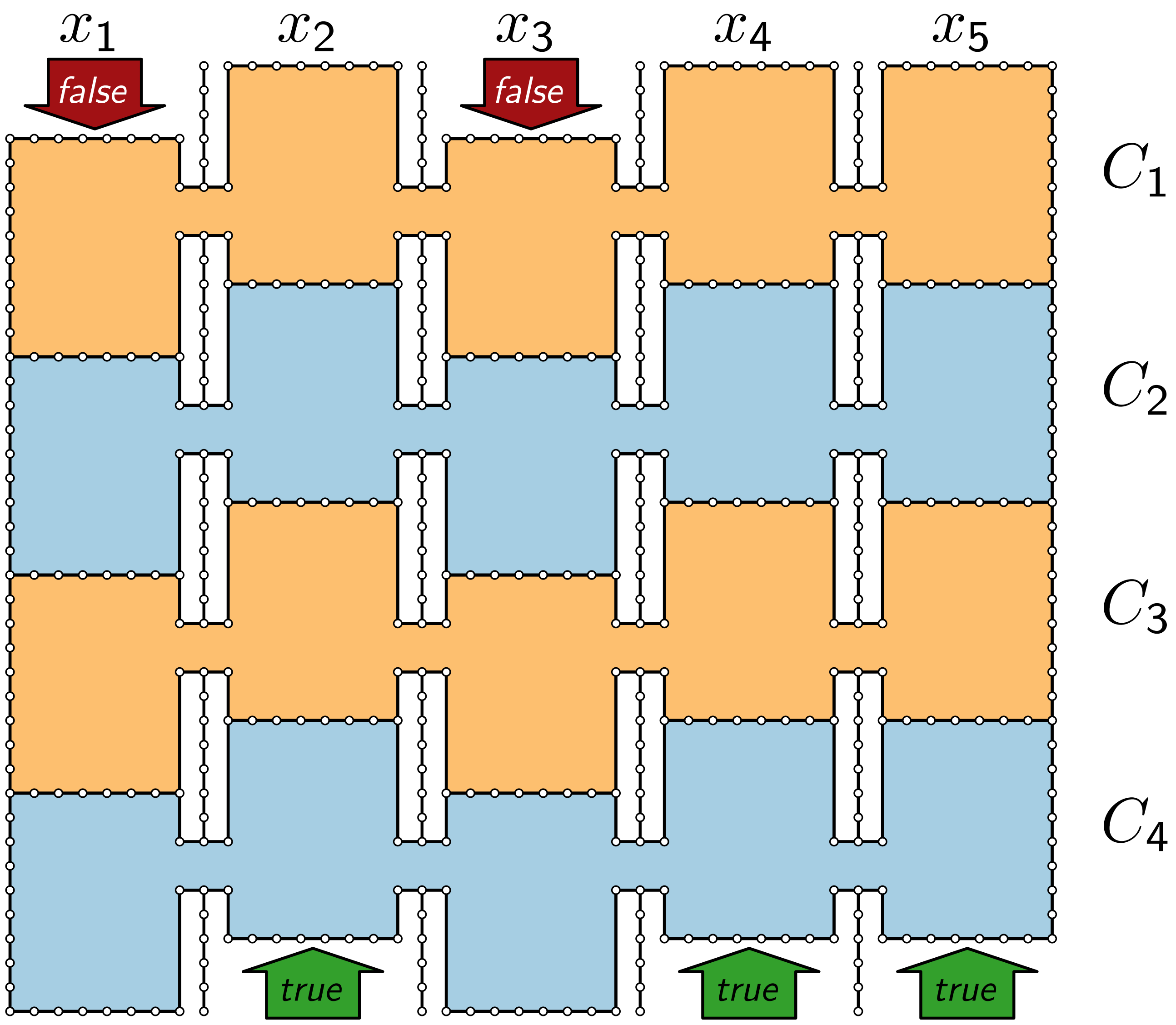


Clause Gadgets



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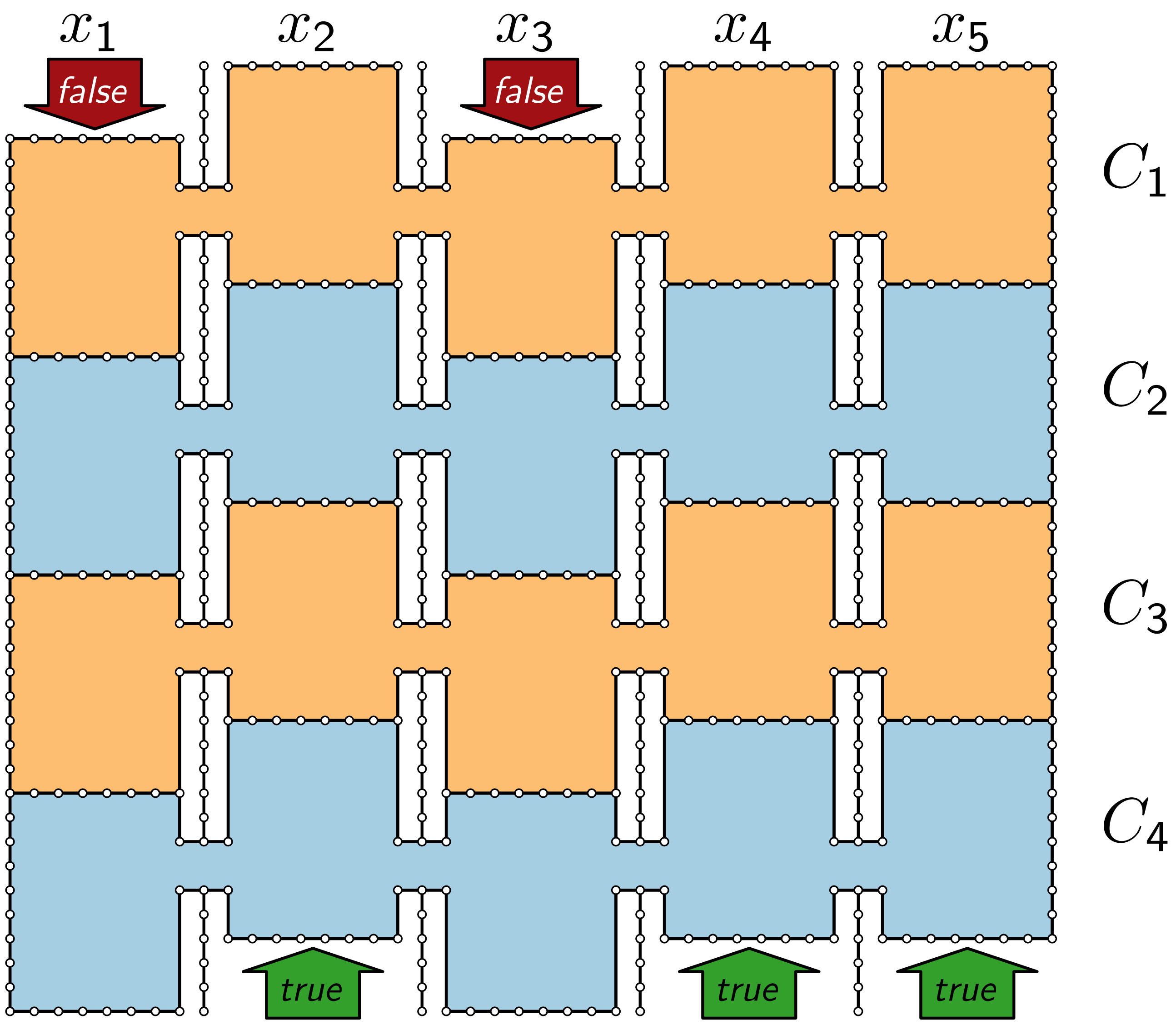

Example:

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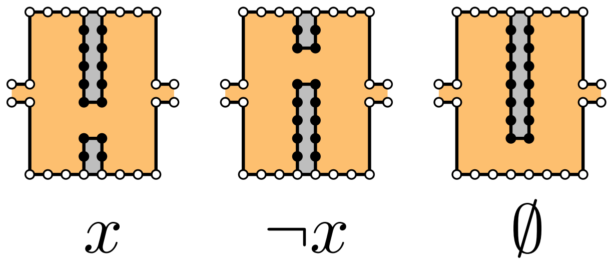

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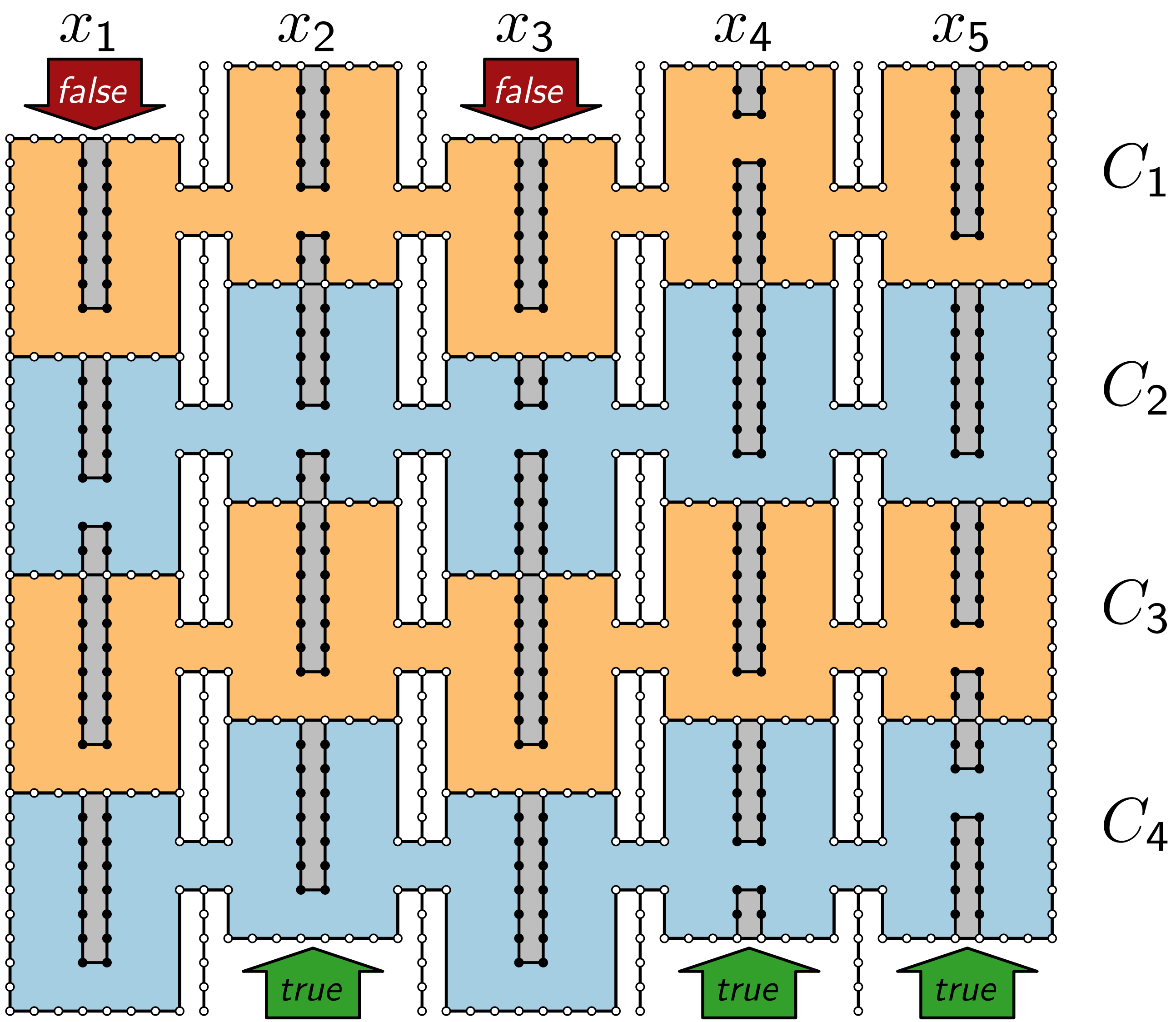

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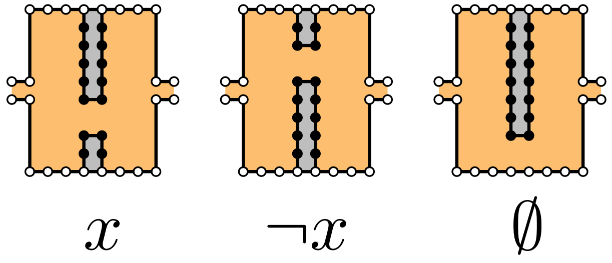

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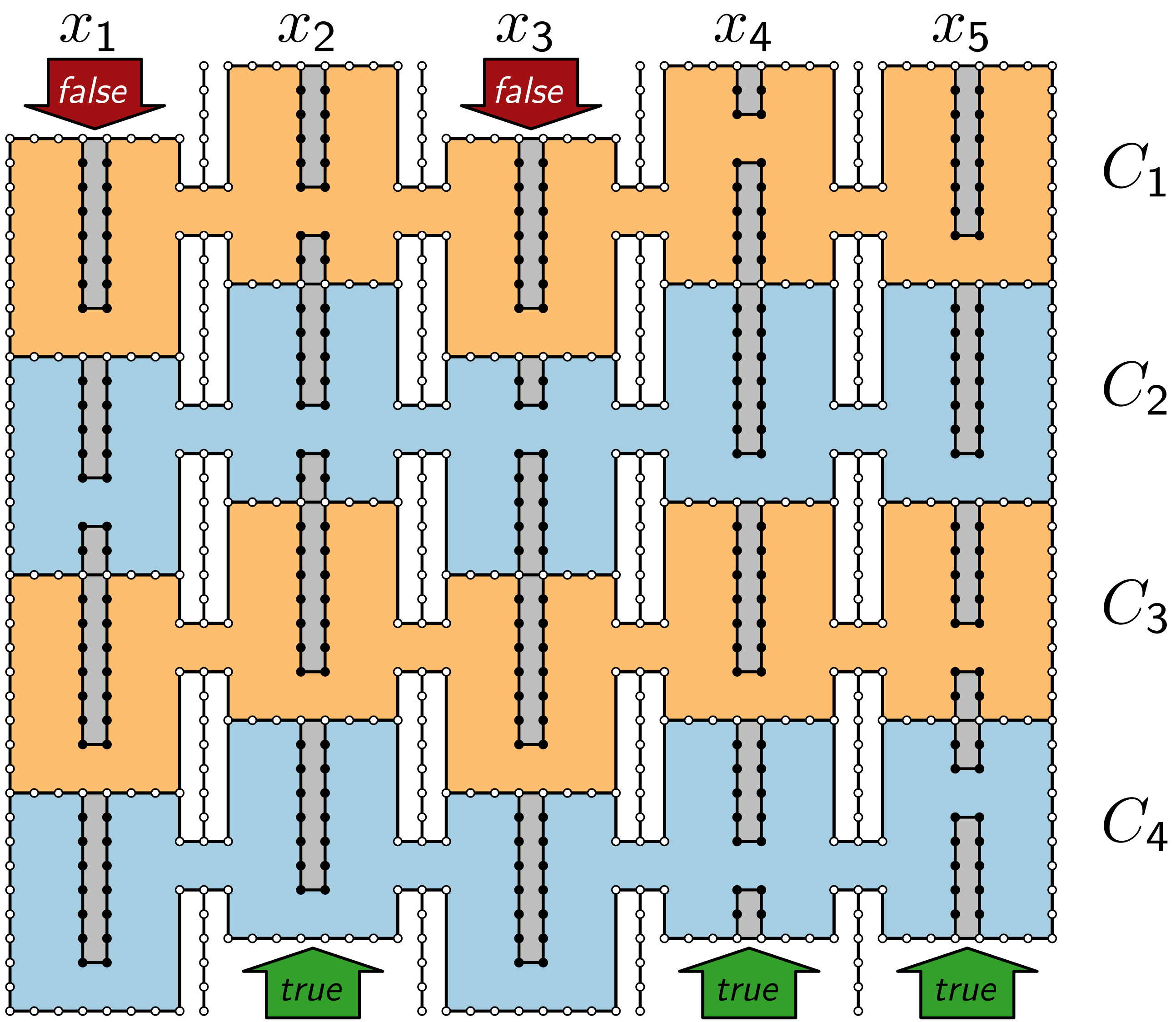

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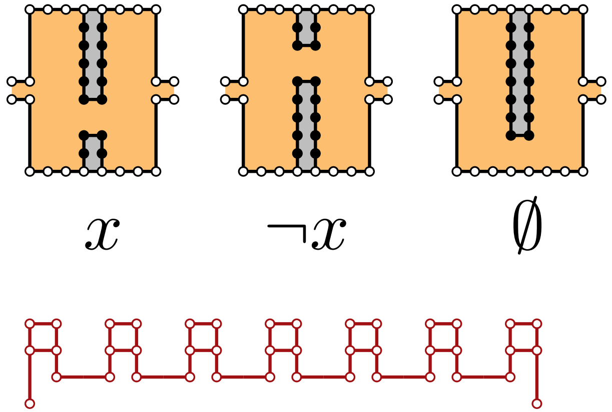

insert $(2n-1)$-chain through each clause

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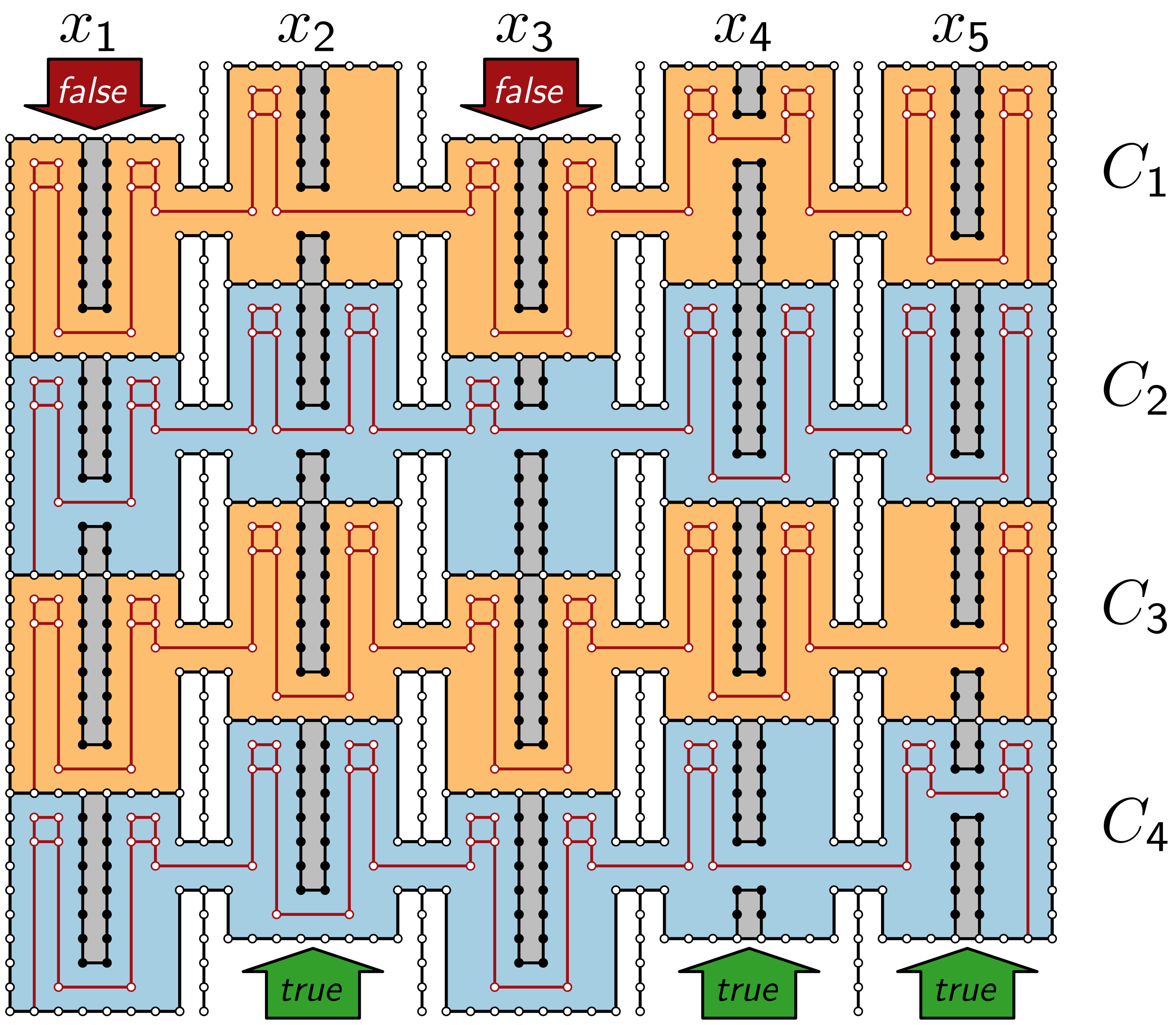

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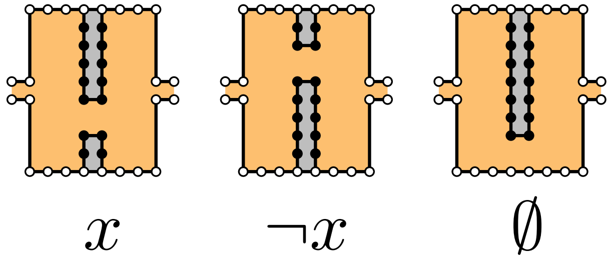

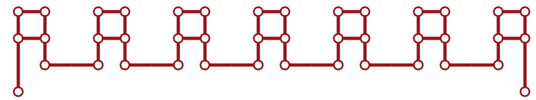

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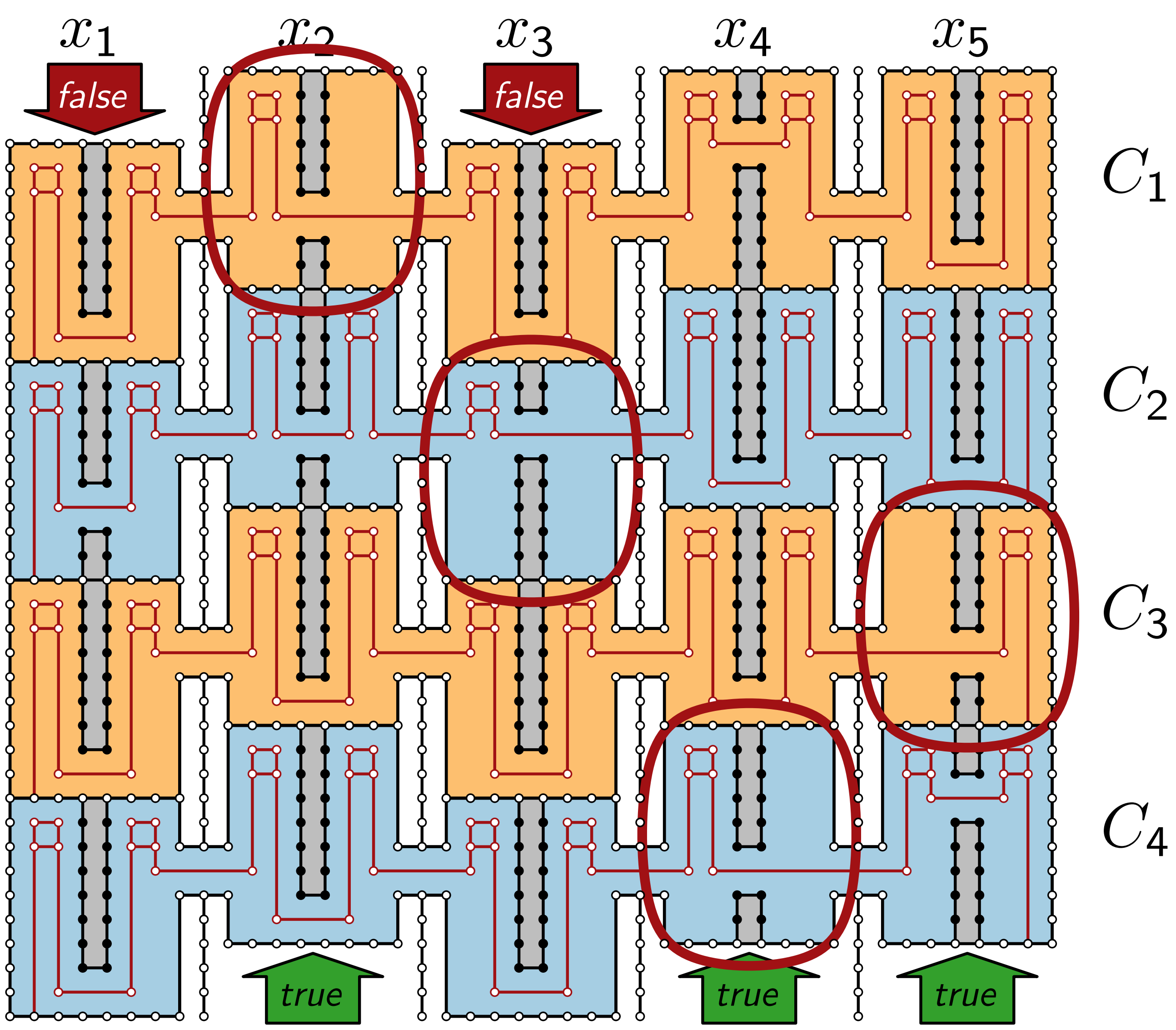

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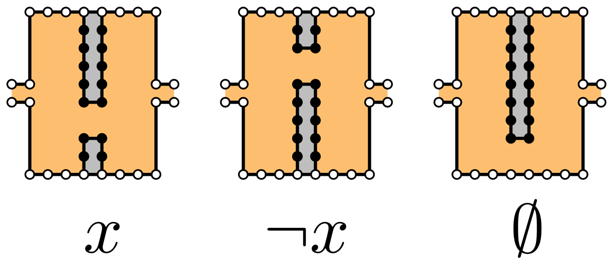

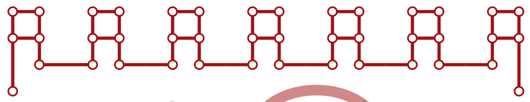

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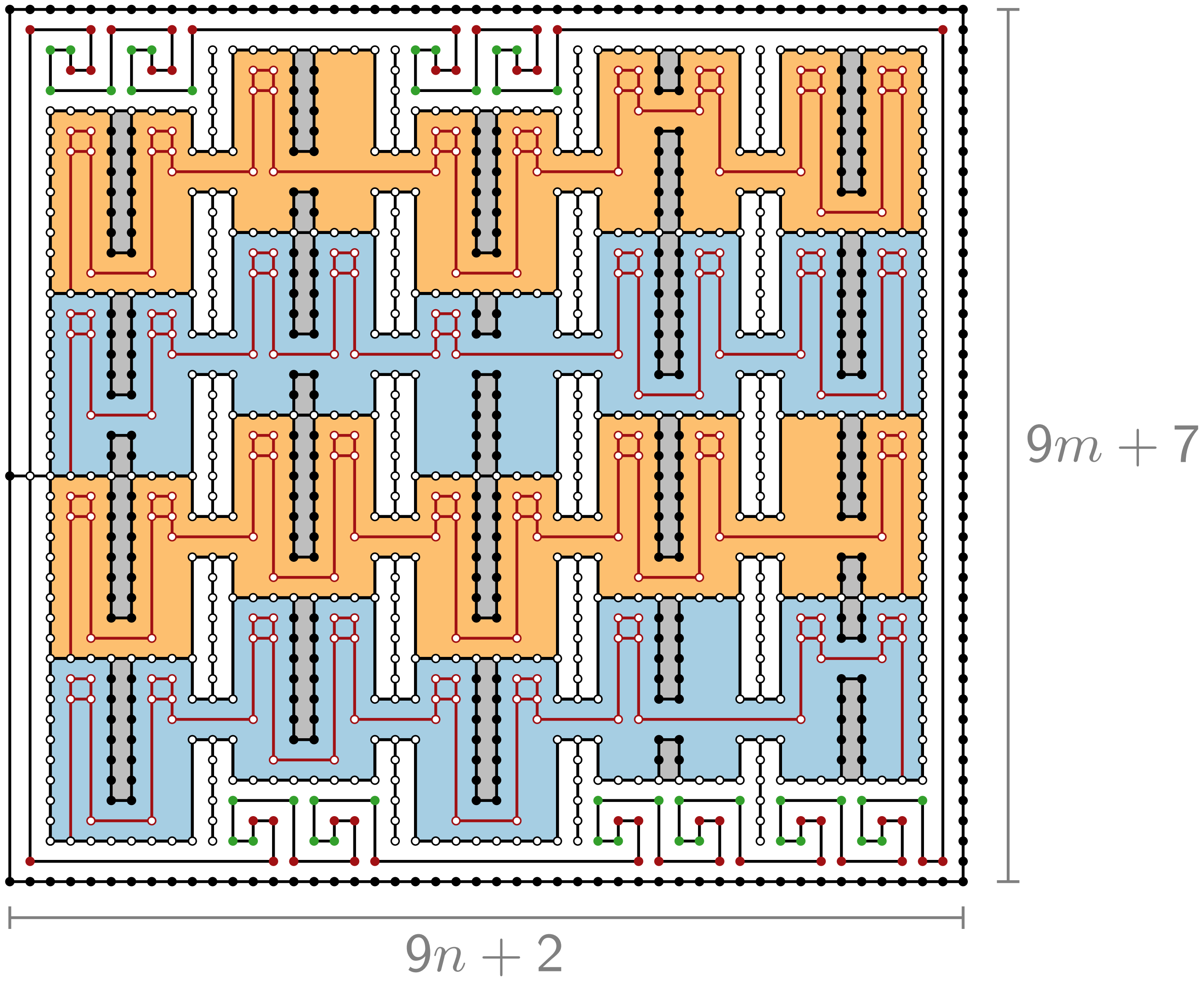
$\rightarrow$ for every clause, there needs to be ≥ 1 "gap of a literal" to be on the same height as the "tunnel" to the next literal

Complete Reduction



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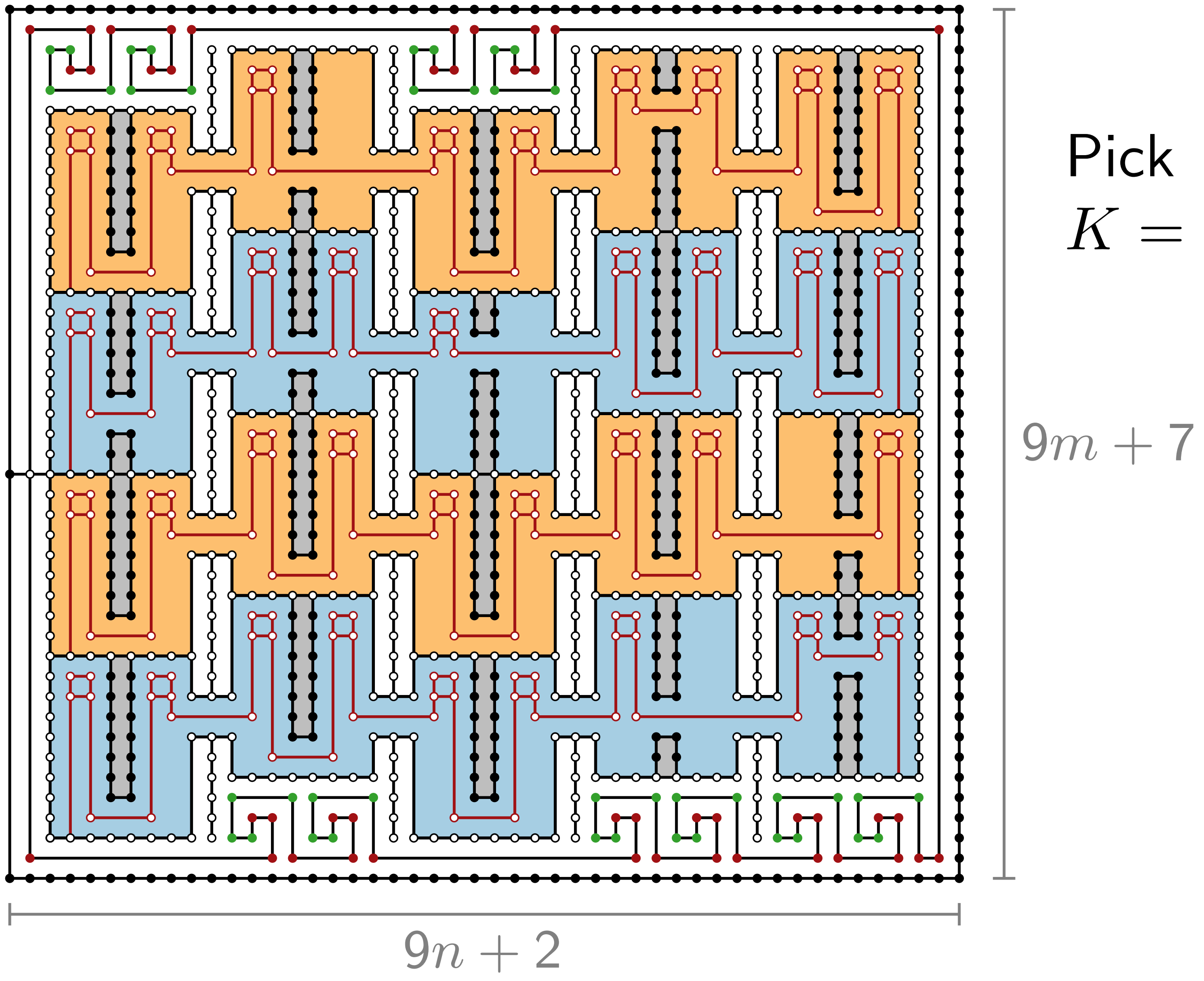

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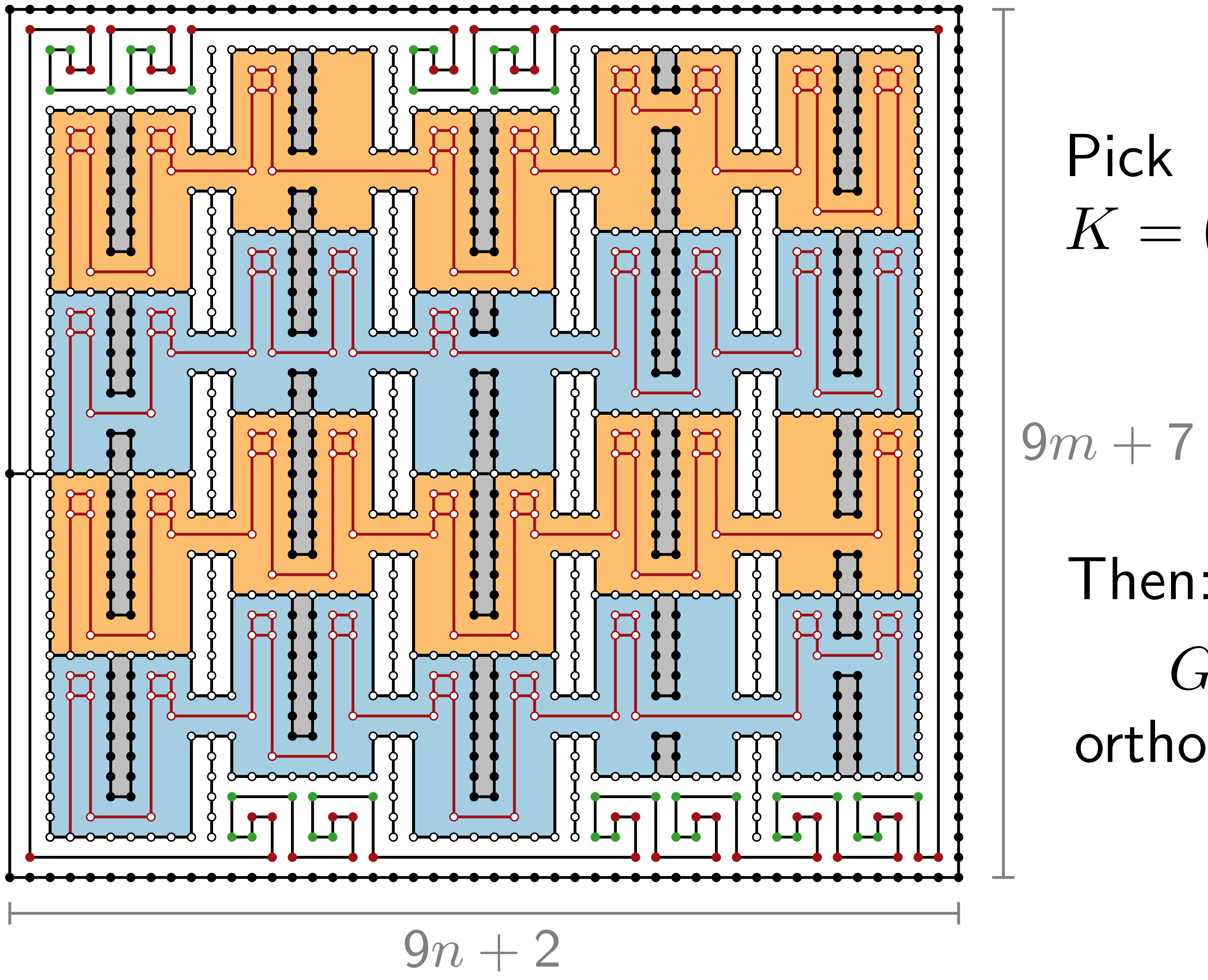

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Then:

G under $H(G)$ has an orthogonal drawing in area K

$\Leftrightarrow$

Φ satisfiable

□

Literature

- [GD Ch. 5] for detailed explanation
- [Tamassia 1987] “On embedding a graph in the grid with the minmum number of bends”
  Original paper on flow for bend minimization.
- [Patrignani 2001] “On the complexity of orthogonal compaction”
  NP-hardness proof for orthogonal representation of planar max-degree-4 graphs.
- [Evans, Fleszar, Kindermann, Saeedi, Shin, Wolff 2022]
  “Minimum rectilinear polygons for given angle sequences”
  NP-hardness proof for compaction of cycles.