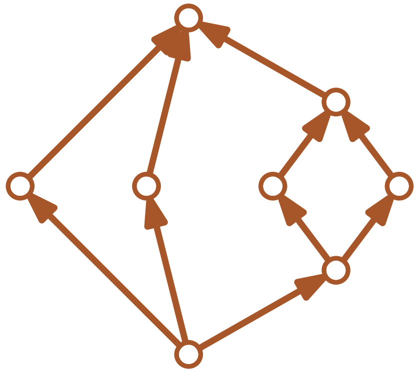
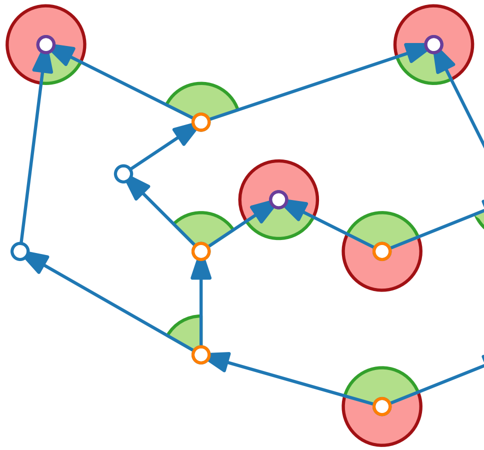
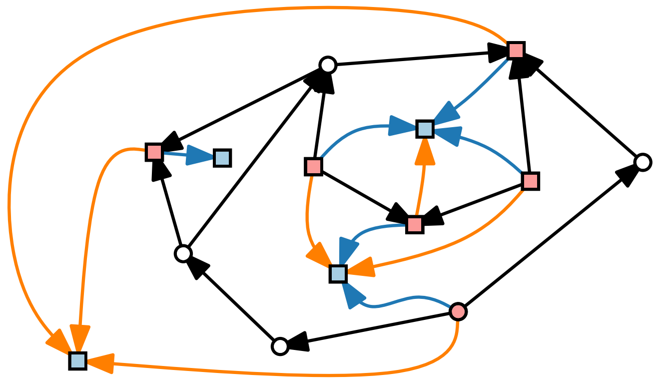
Visualization of Graphs

Lecture 5:
Upward Planar Drawings



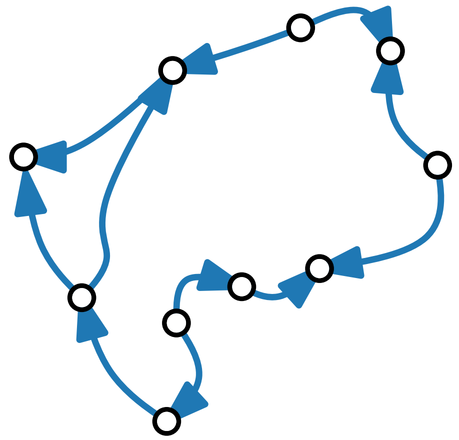
Part I:
Recognition



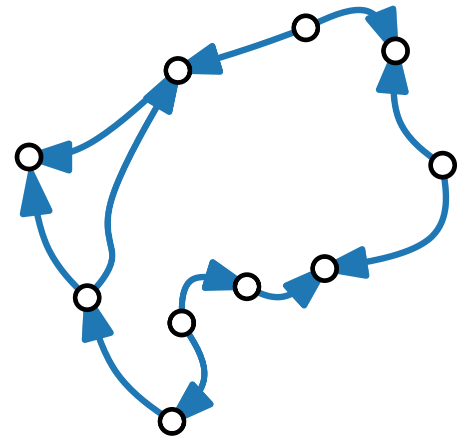


Johannes Zink

Upward Planar Drawings – Motivation



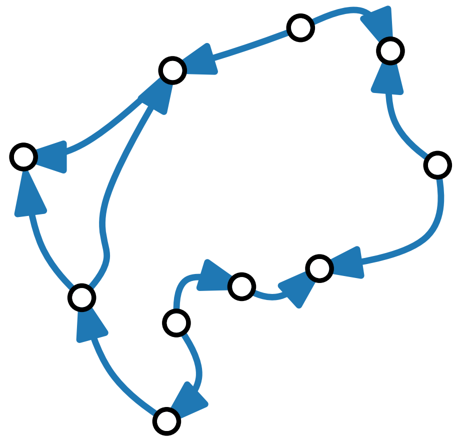
Upward Planar Drawings – Motivation

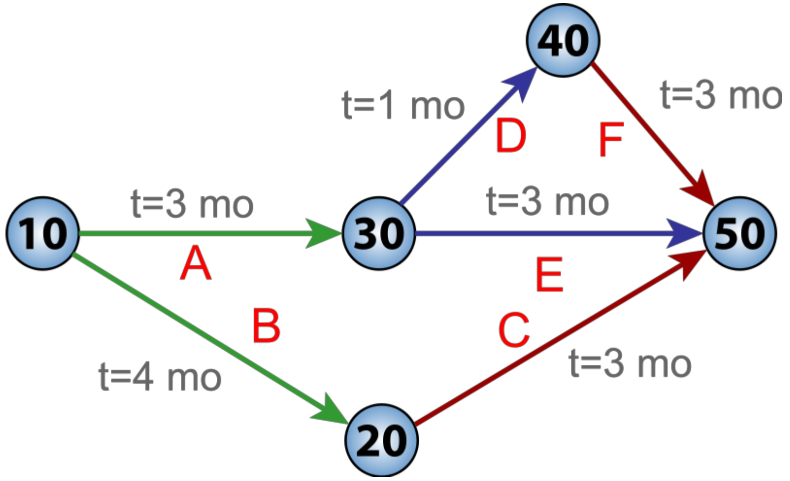


- What may the direction of edges in a directed graph represent?

Upward Planar Drawings – Motivation

- What may the direction of edges in a directed graph represent?
  - Time



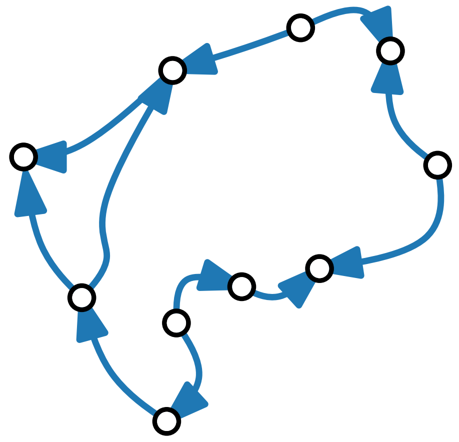


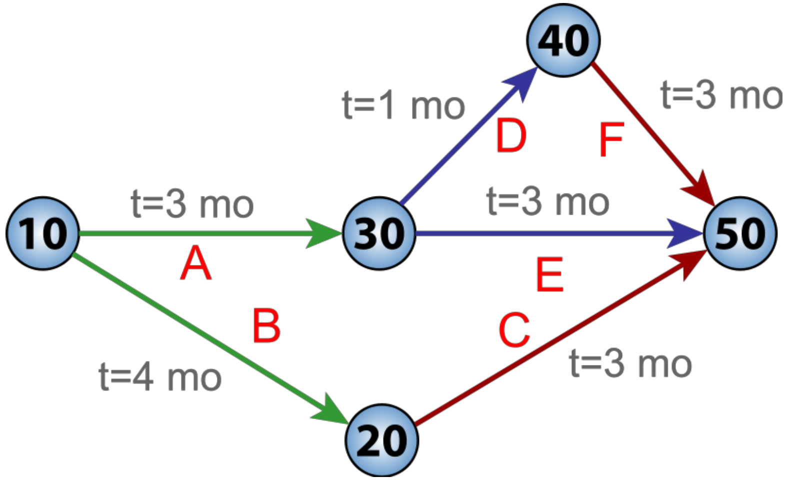
PERT diagram

Program Evaluation and Review Technique
(Project management)

Upward Planar Drawings – Motivation

- What may the direction of edges in a directed graph represent?
  - Time
  - Flow





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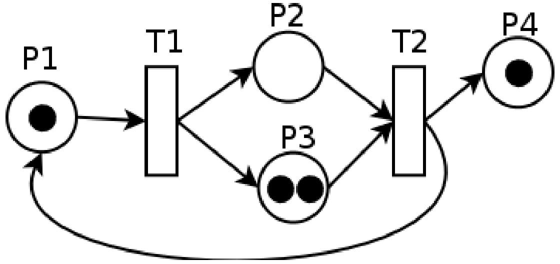
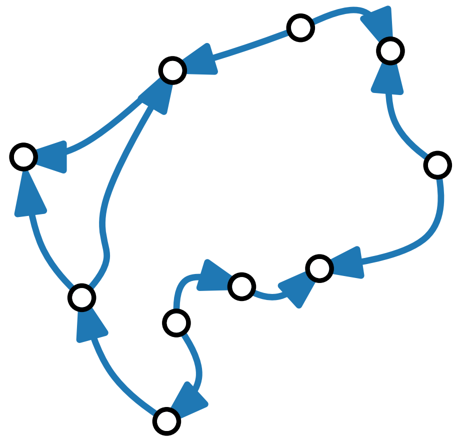


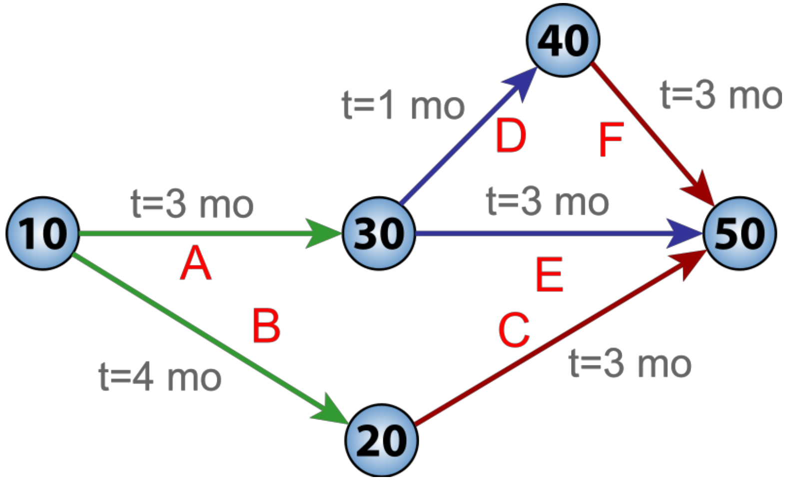
Petri net

Place/Transition net
(Modeling languages for distributed systems)

Upward Planar Drawings – Motivation

- What may the direction of edges in a directed graph represent?
  - Time
  - Flow
  - Hierarchy





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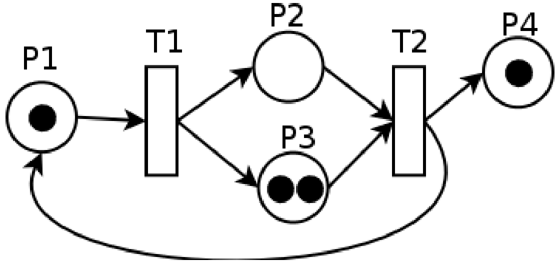

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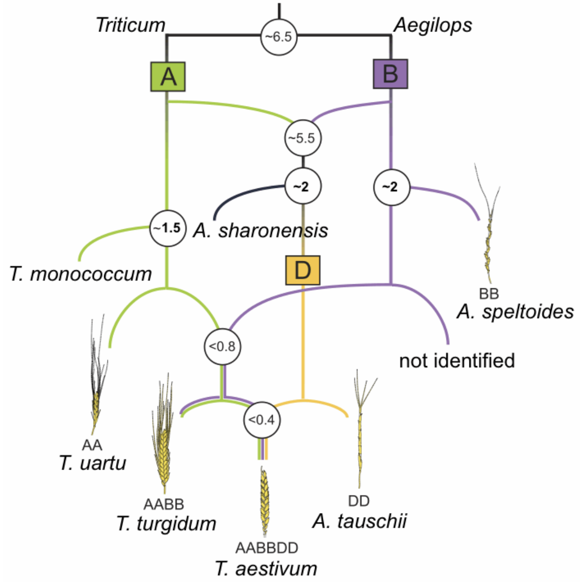
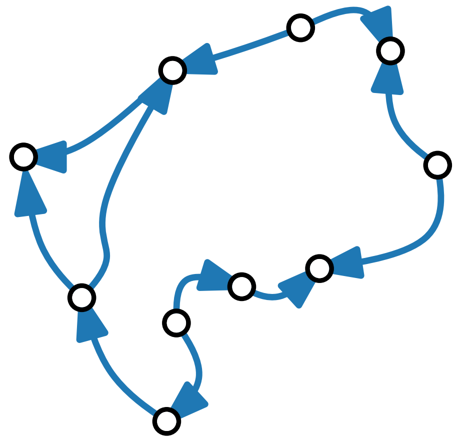


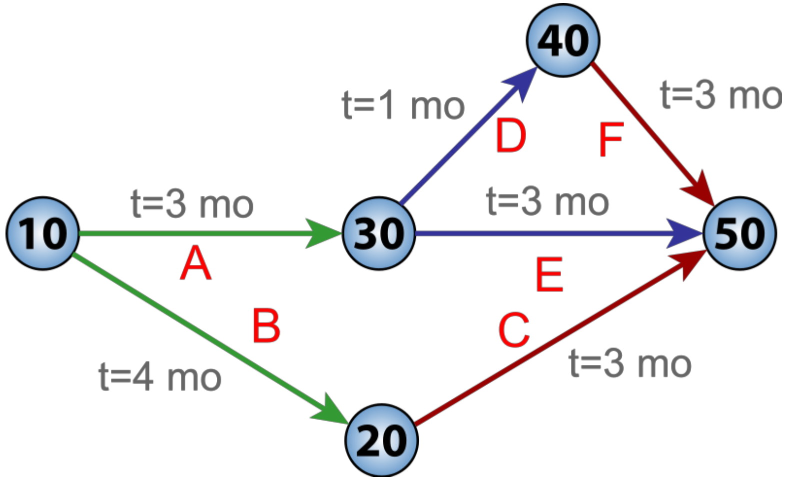
Phylogenetic network

Ancestral trees / networks
(Biology)

Upward Planar Drawings – Motivation



- What may the direction of edges in a directed graph represent?
  - Time
  - Flow
  - Hierarchy
  - ...



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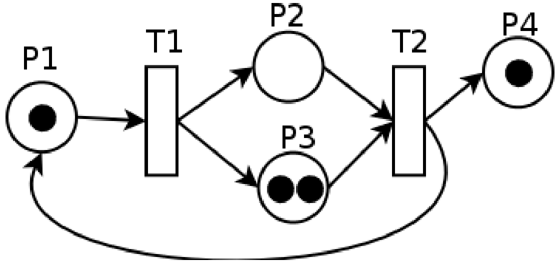

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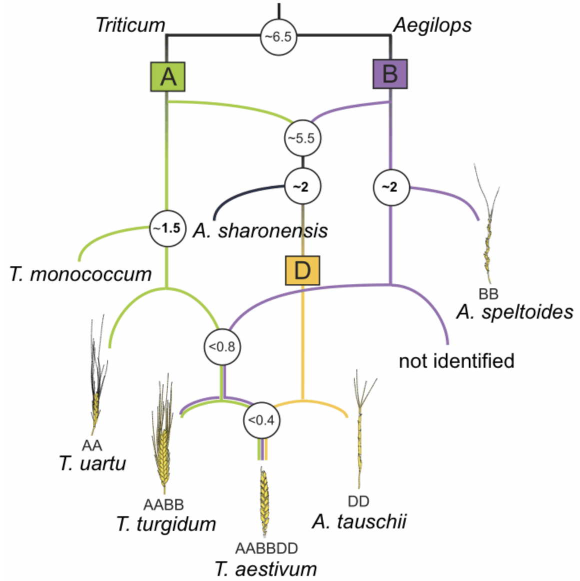

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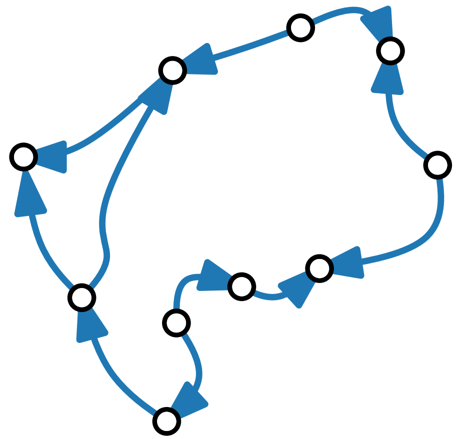
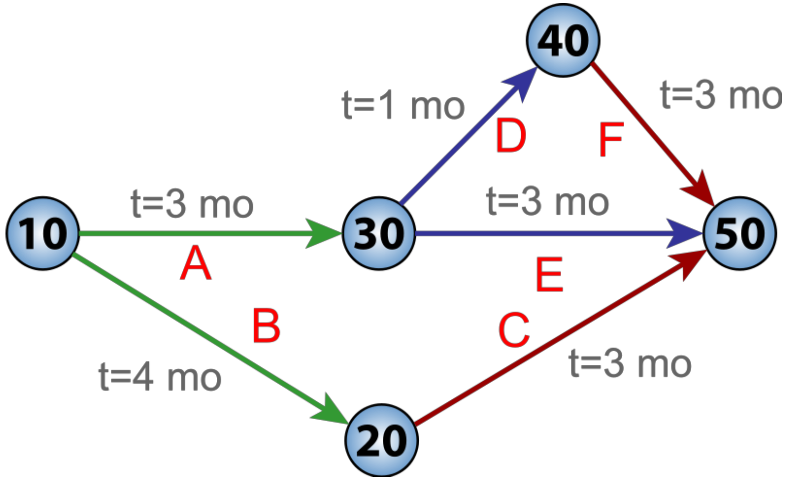
Upward Planar Drawings – Motivation



- What may the direction of edges in a directed graph represent?
  - Time
  - Flow
  - Hierarchy
  - ...
- We aim for drawings where the general direction is preserved.



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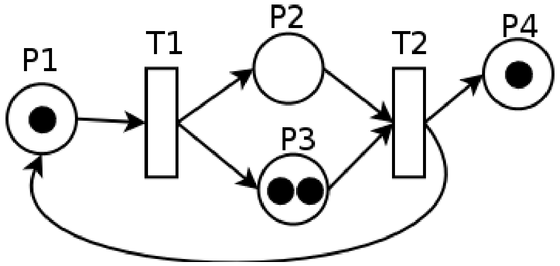

Petri net

Place/Transition net
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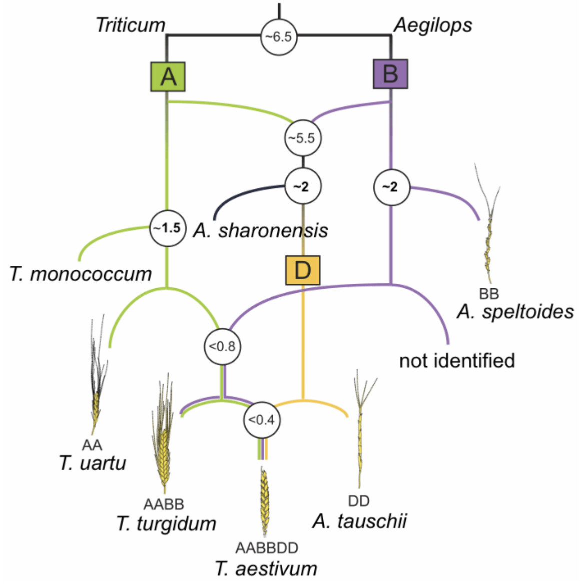

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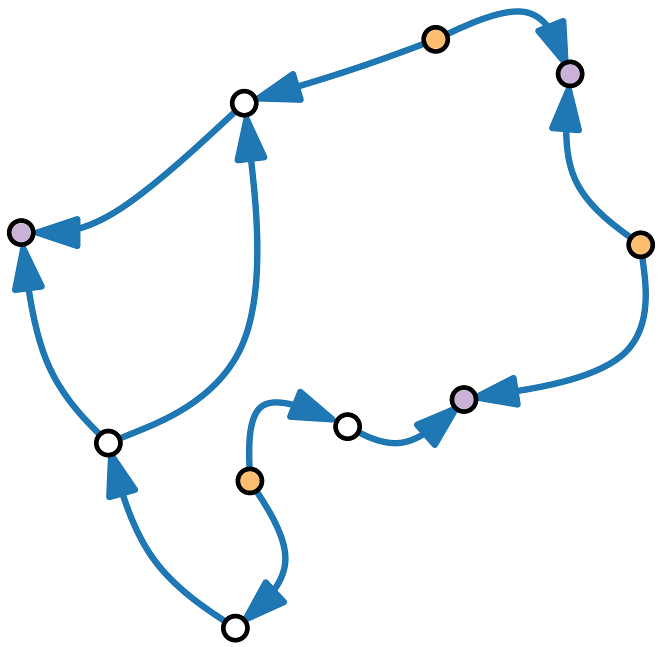
Upward Planar Drawings – Definition

A directed graph (*digraph*) is **upward planar** when it admits a drawing that is



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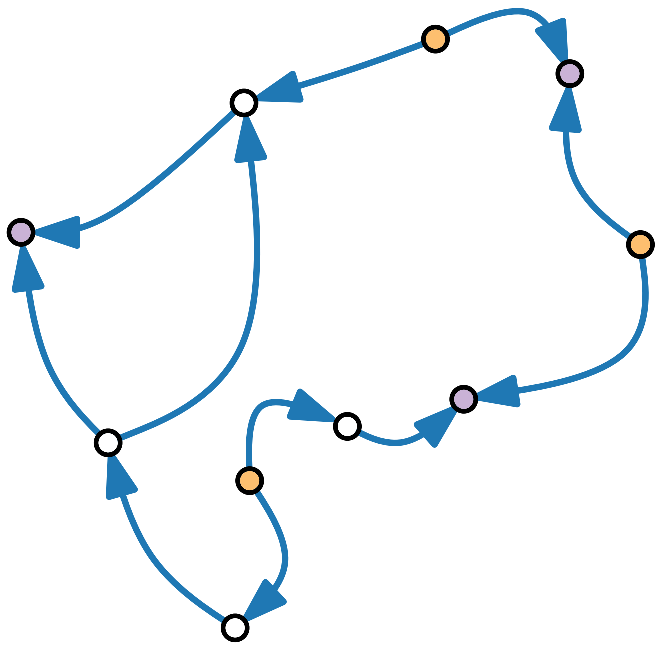

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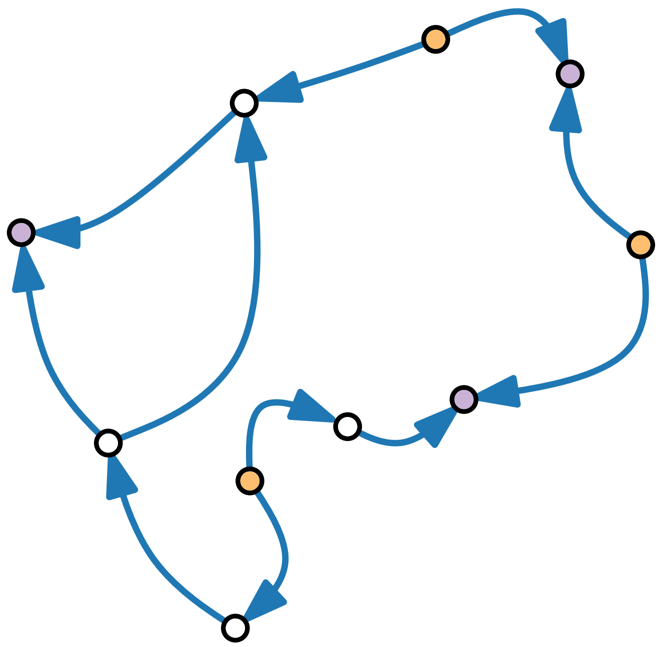

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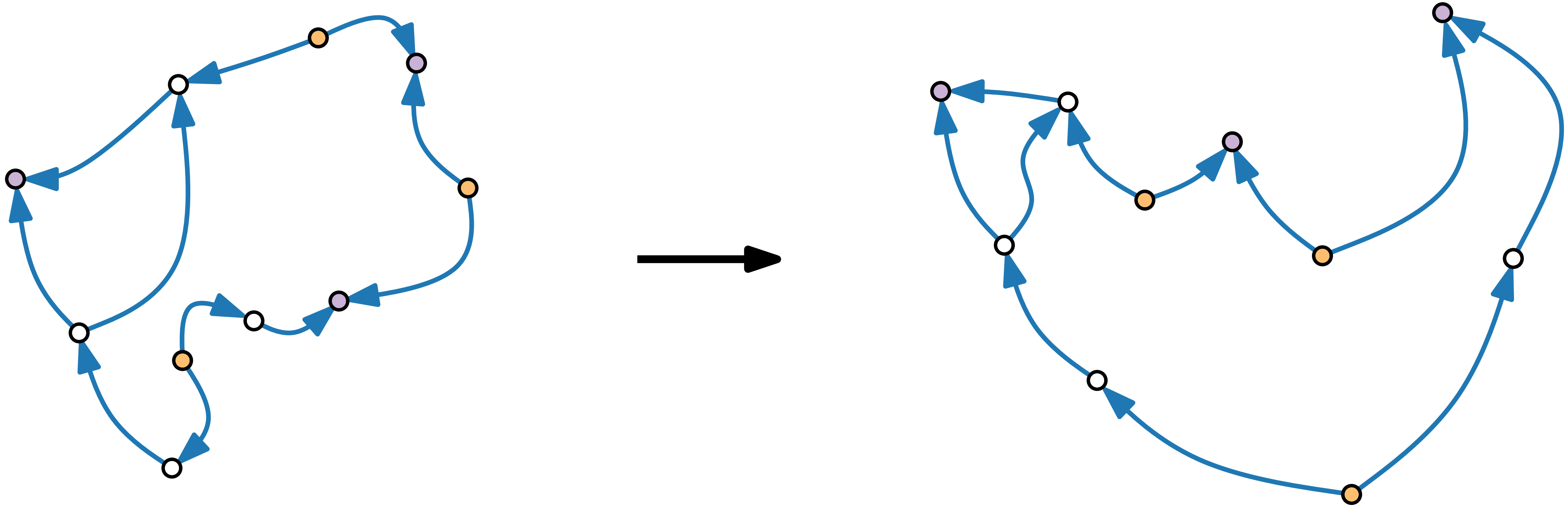

Upward Planarity – Necessary Conditions

- For an (embedded) digraph to be upward planar, it needs to . . .

Upward Planarity – Necessary Conditions

- For an (embedded) digraph to be upward planar, it needs to ...
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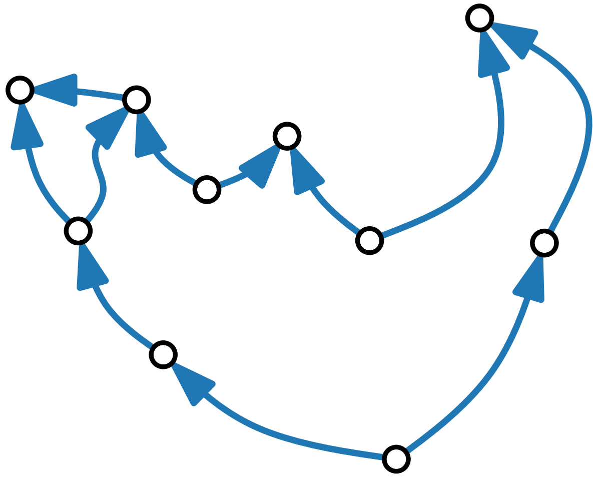

Upward Planarity – Necessary Conditions

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  - be planar
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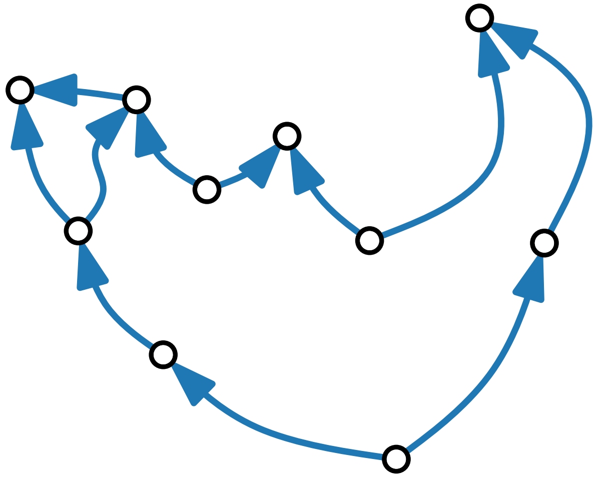

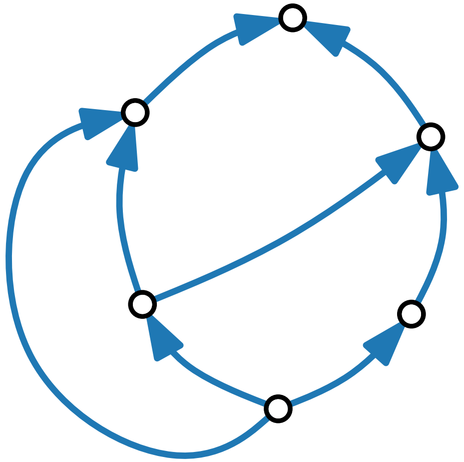

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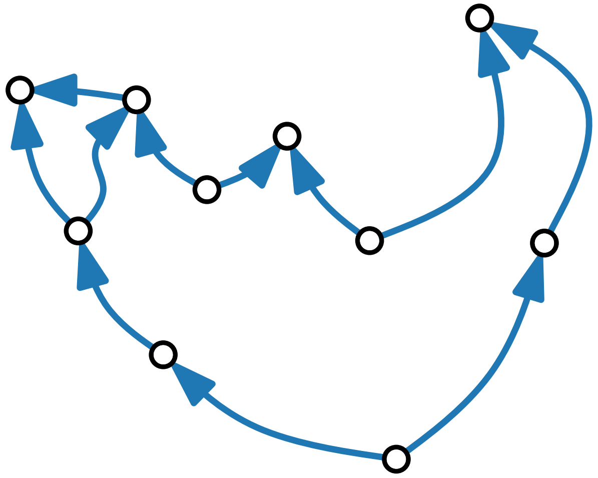

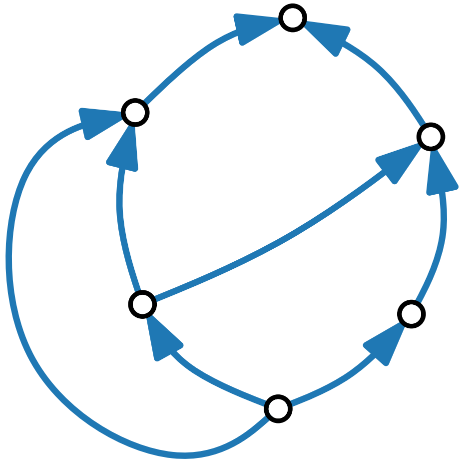

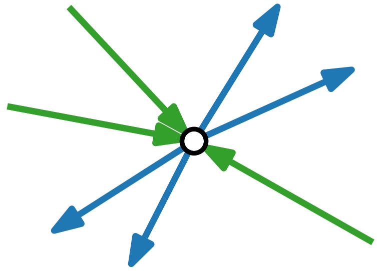

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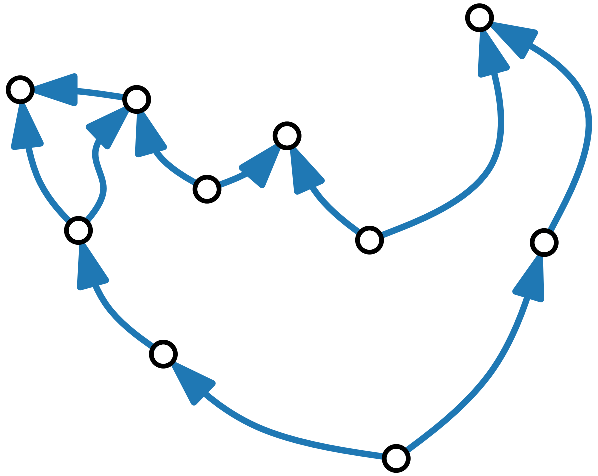

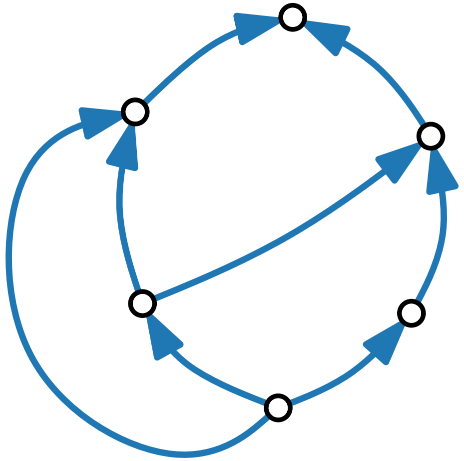

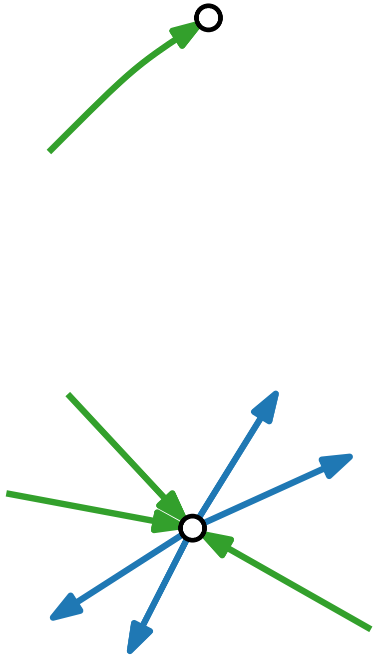

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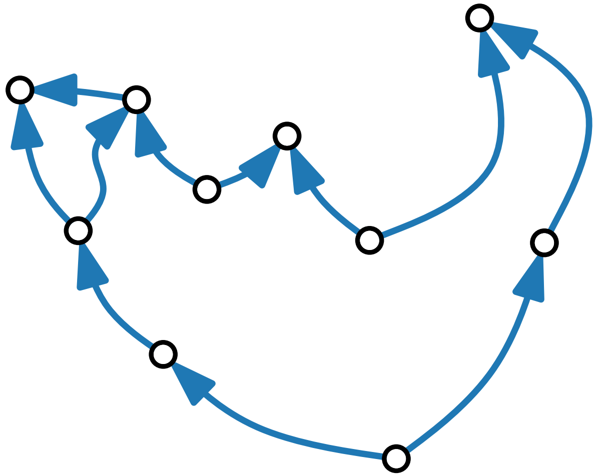

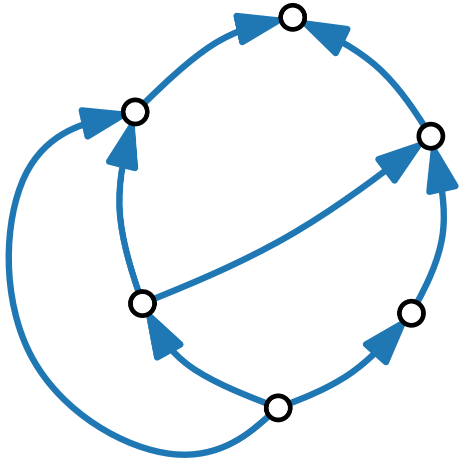

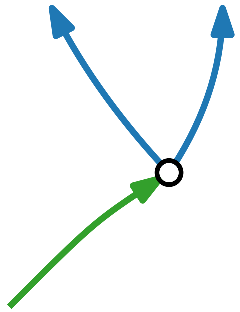

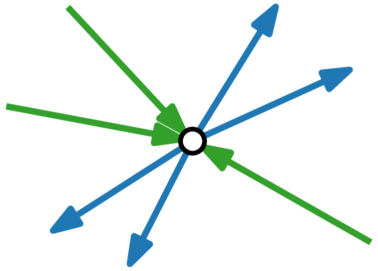

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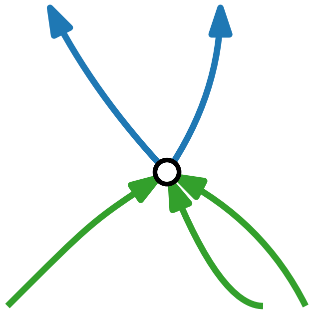

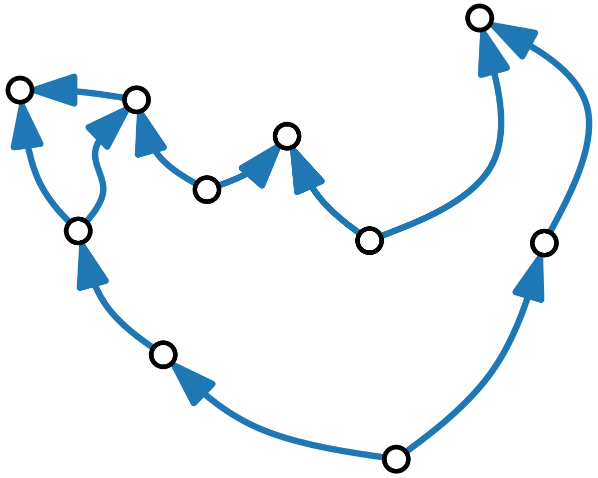

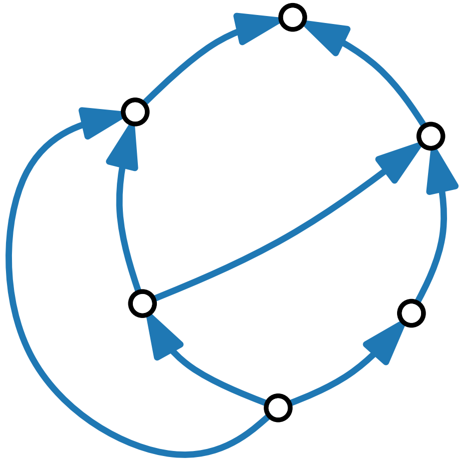

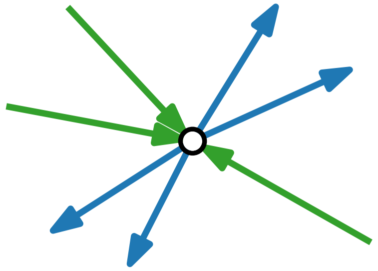

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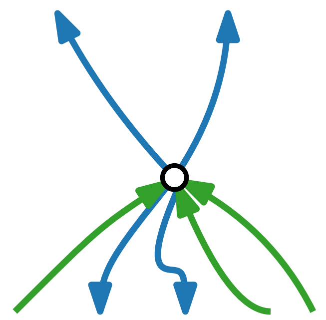

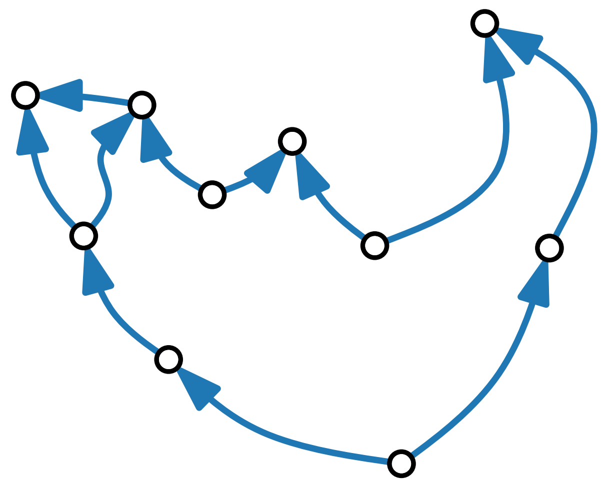

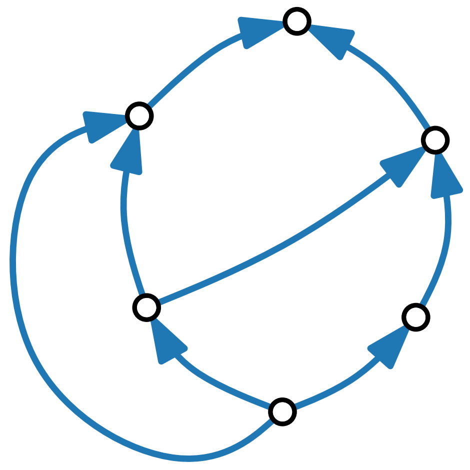

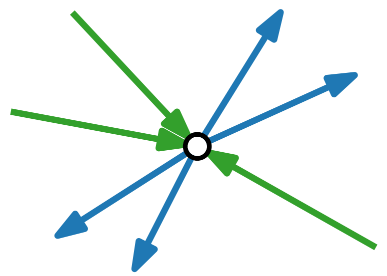

Upward Planarity – Necessary Conditions

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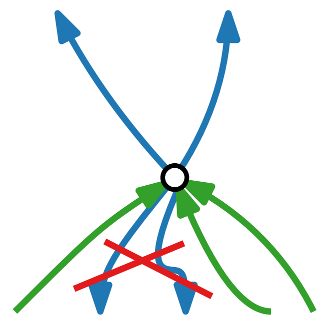

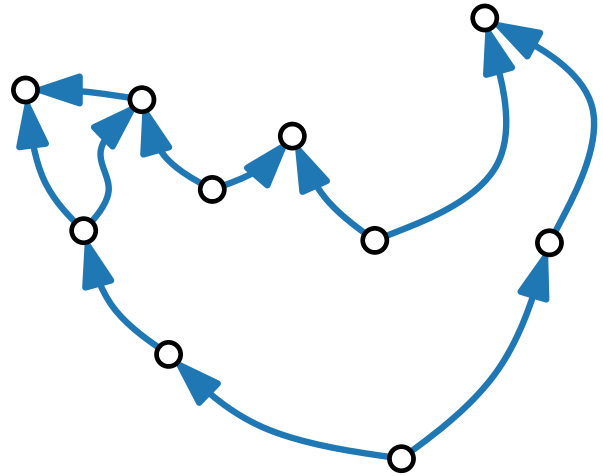

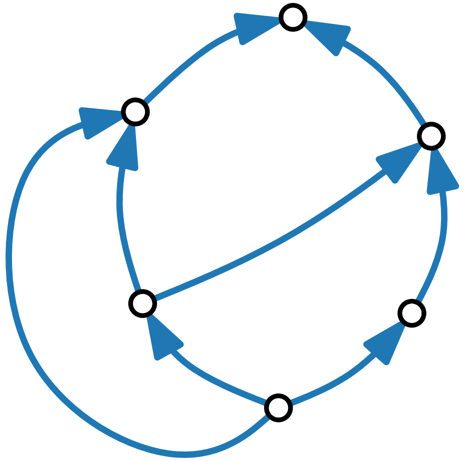

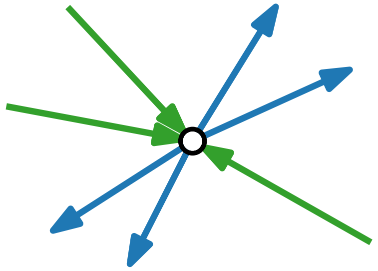

Upward Planarity – Necessary Conditions

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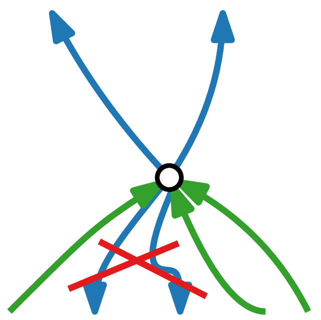

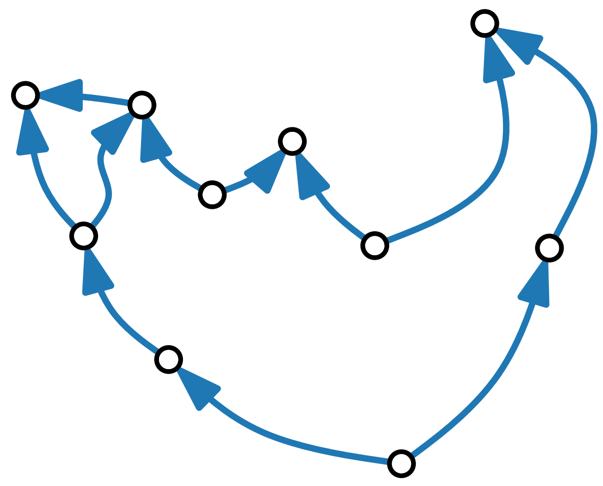

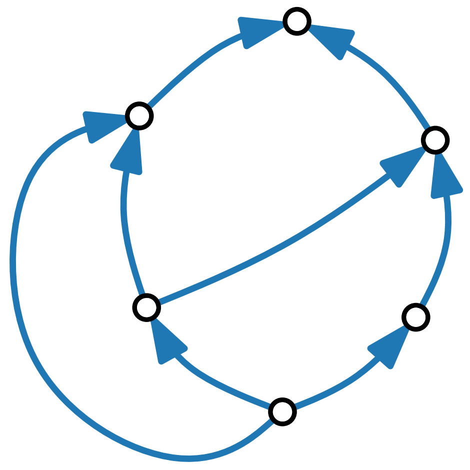

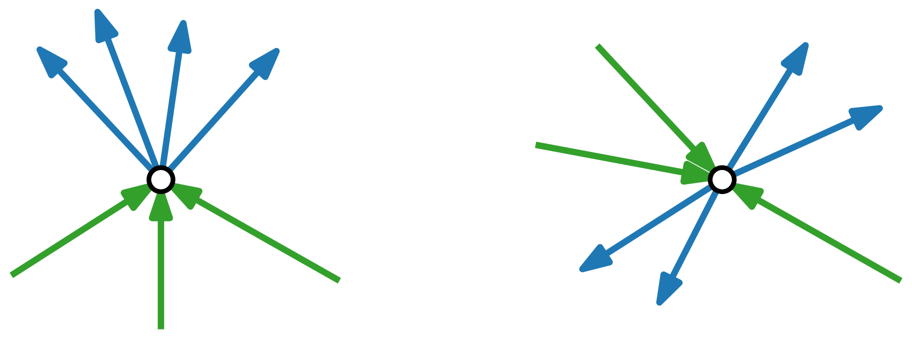

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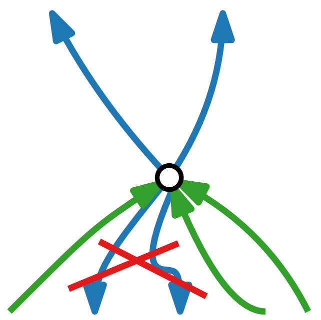
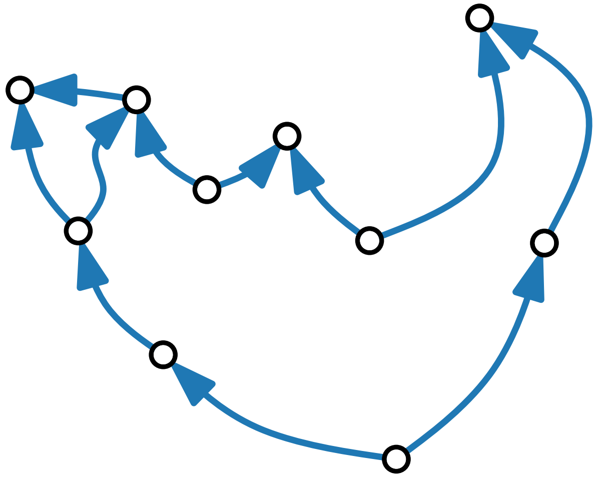
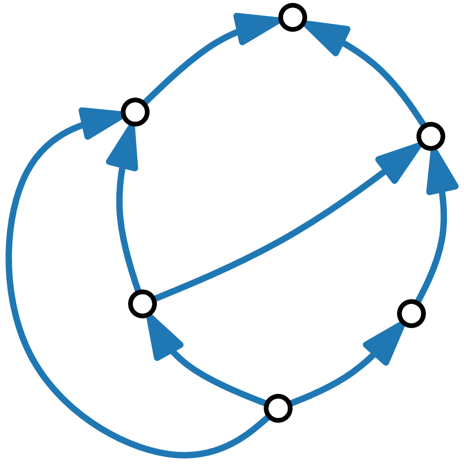
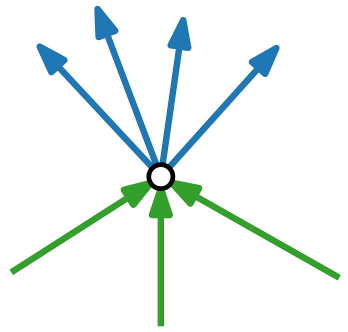
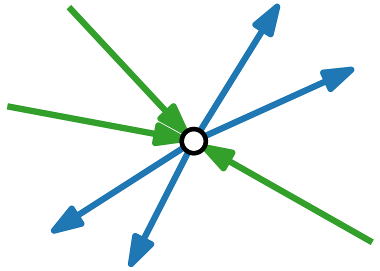








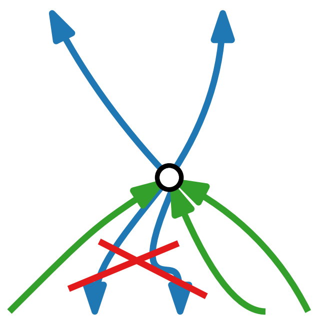
bimodal vertex

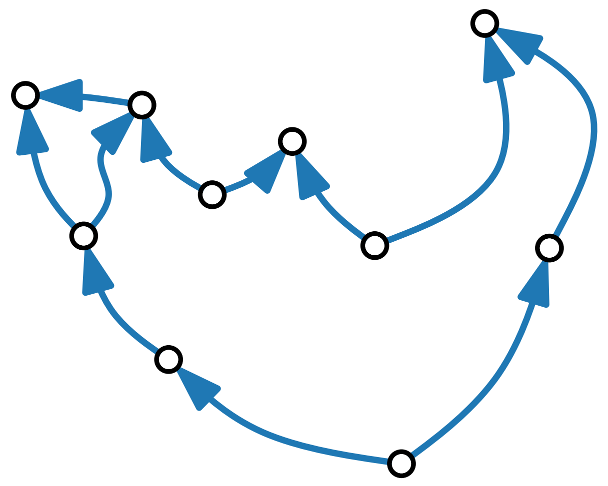


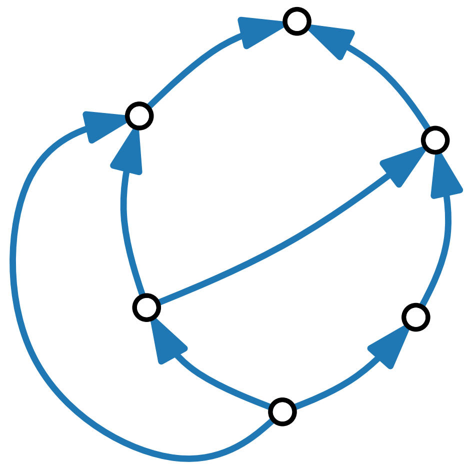
not bimodal

Upward Planarity – Necessary Conditions

- For an (embedded) digraph to be upward planar, it needs to . . .
  - be planar
  - be acyclic
  - have a bimodal embedding







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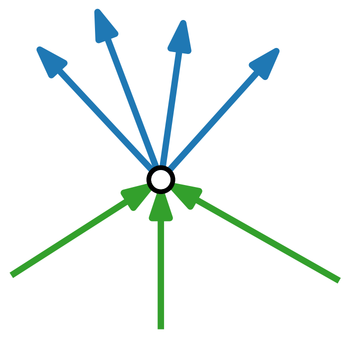

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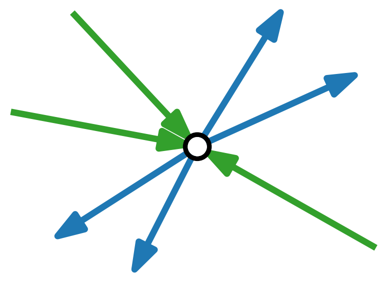

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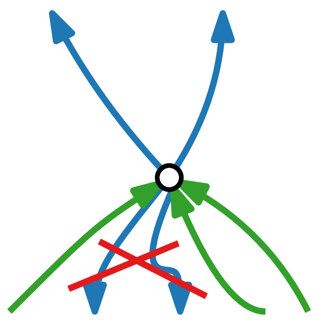
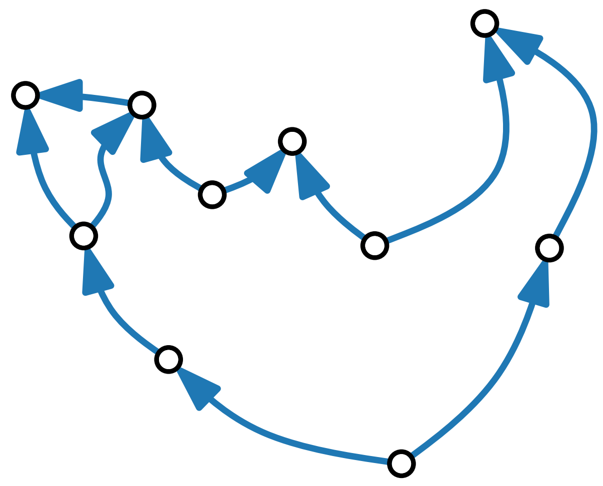
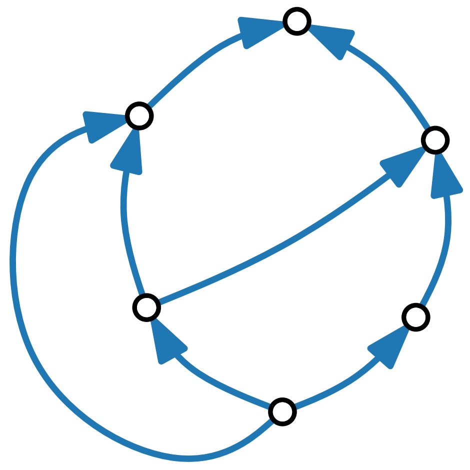
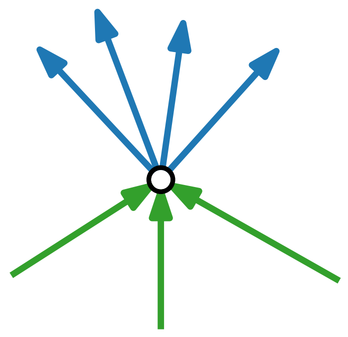


- ... but these conditions are *not sufficient*.

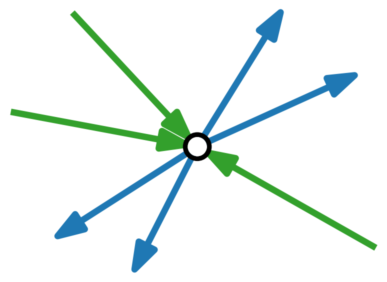




bimodal vertex

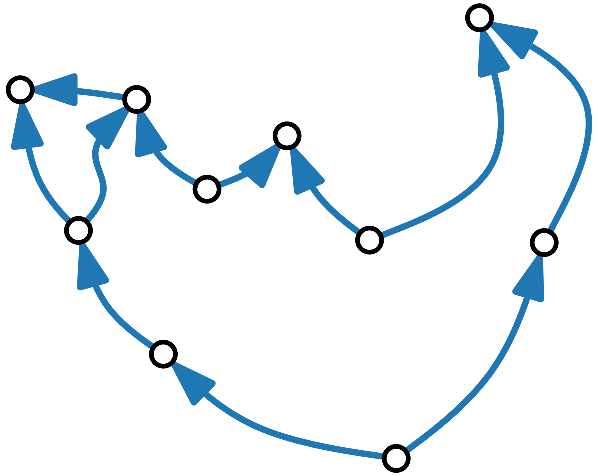


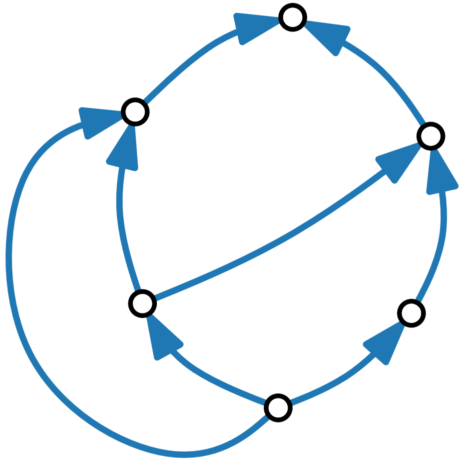
not bimodal

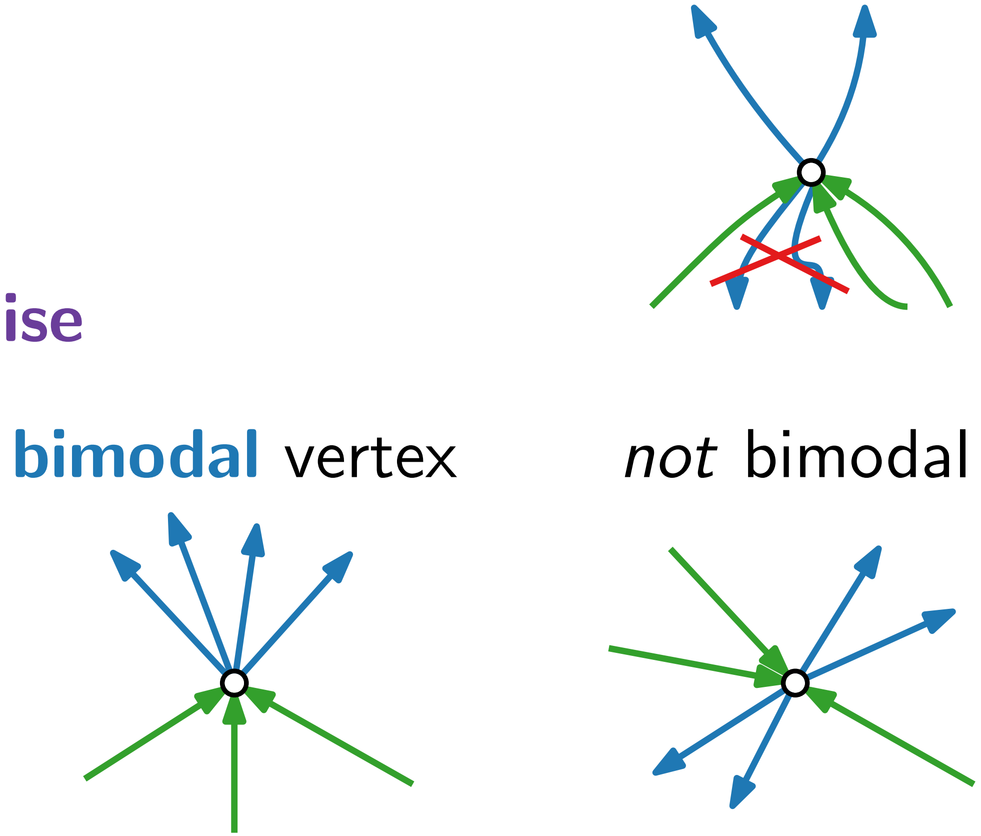


Upward Planarity – Necessary Conditions

- For an (embedded) digraph to be upward planar, it needs to . . .
  - be planar
  - be acyclic
  - have a bimodal embedding
- . . . but these conditions are *not sufficient*. → **Exercise**







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acyclic digraph with
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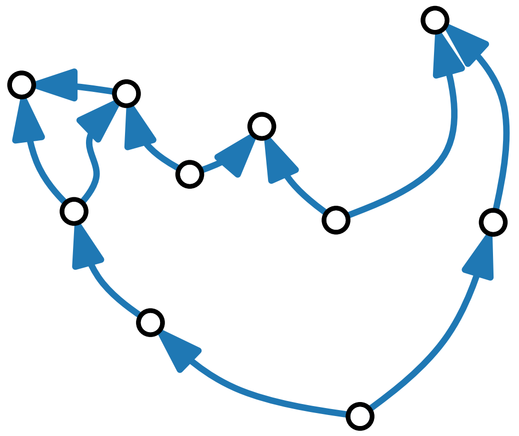

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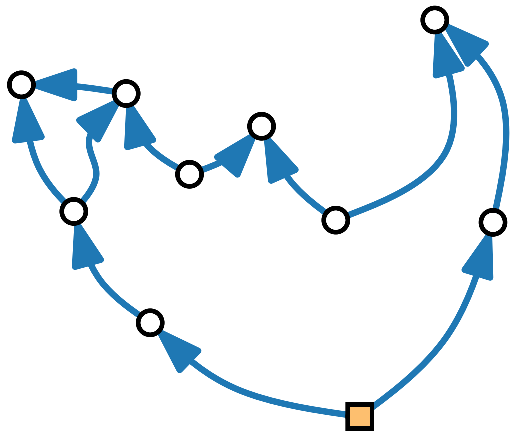

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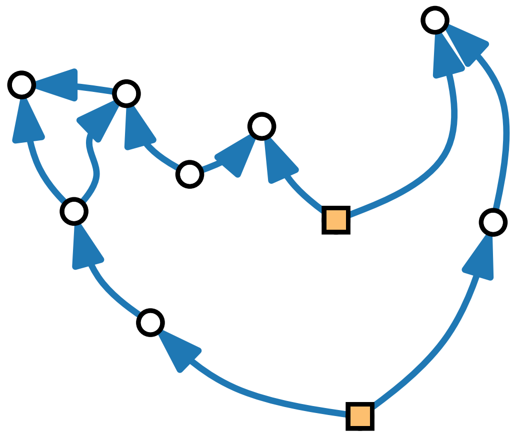

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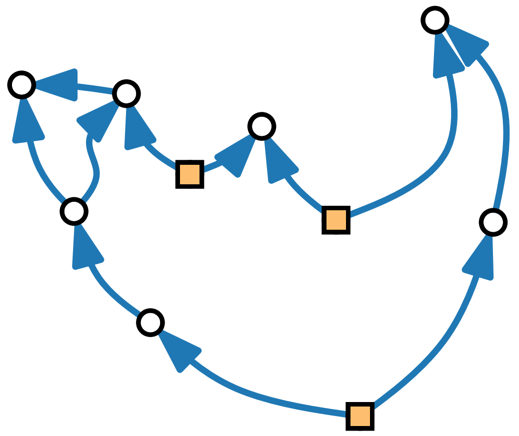

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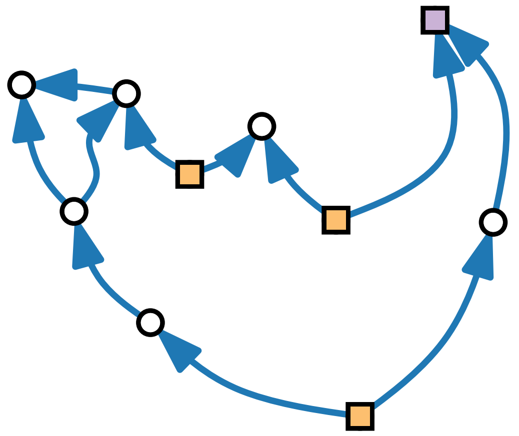

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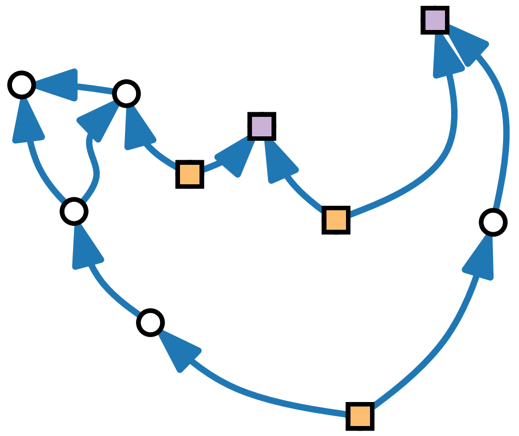

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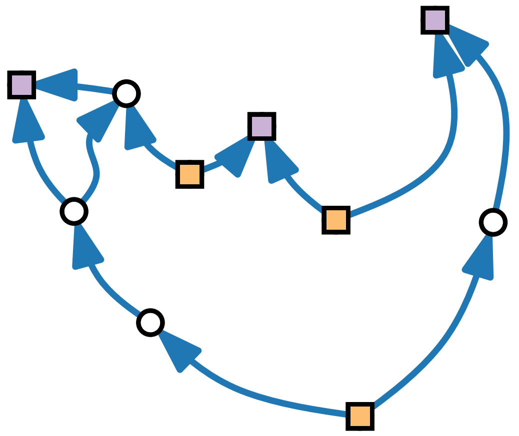

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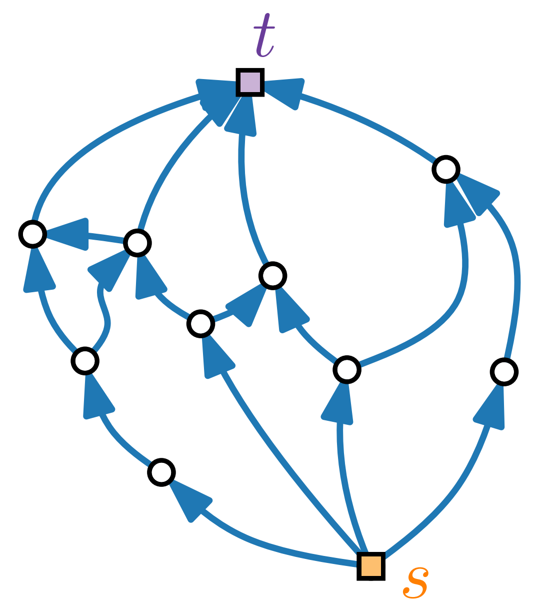

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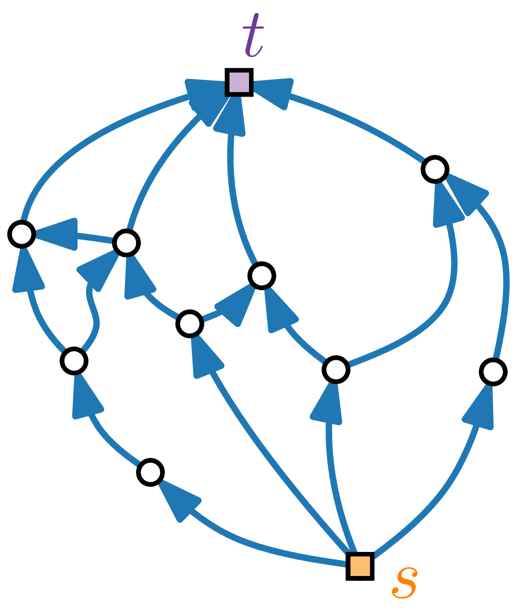

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Additionally:
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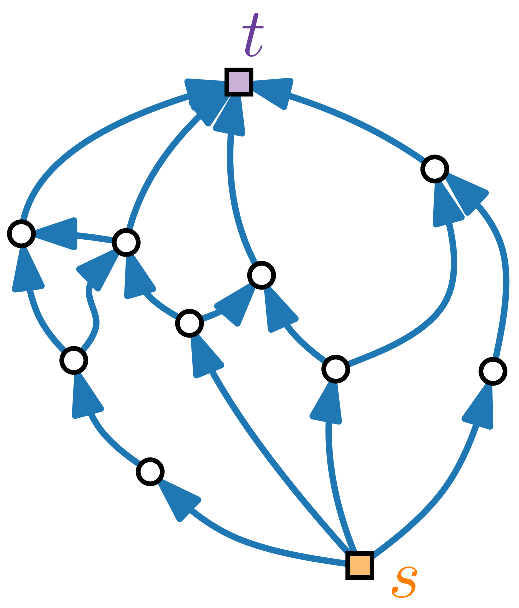

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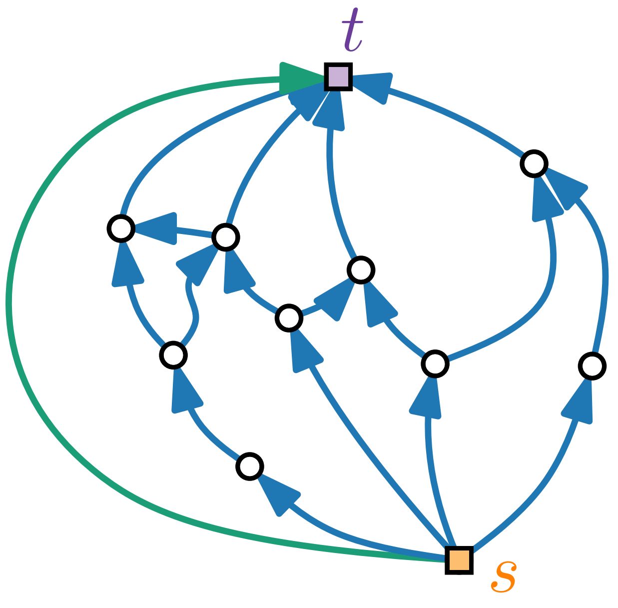

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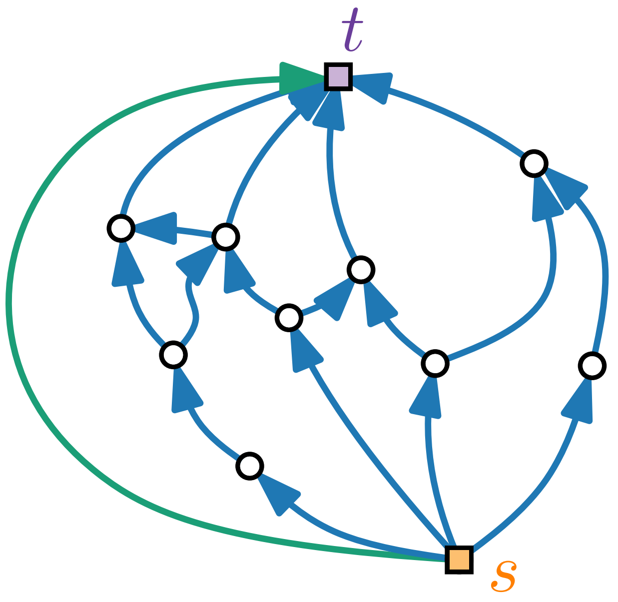

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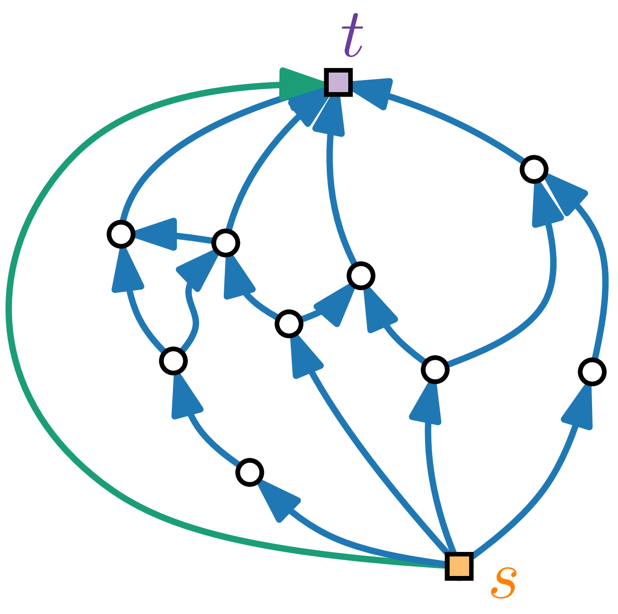

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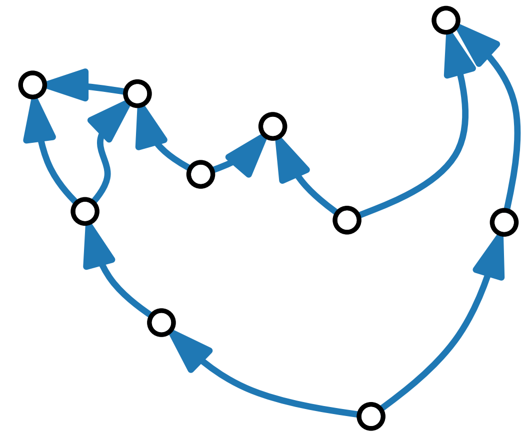

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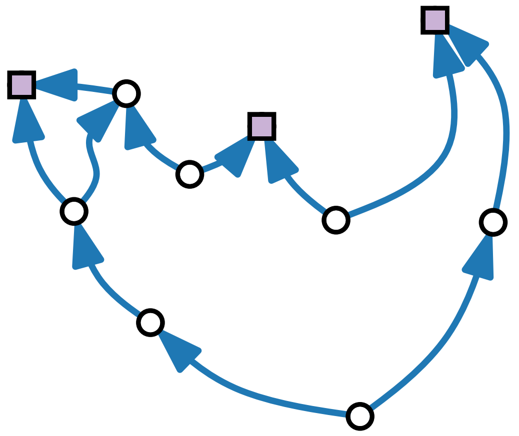

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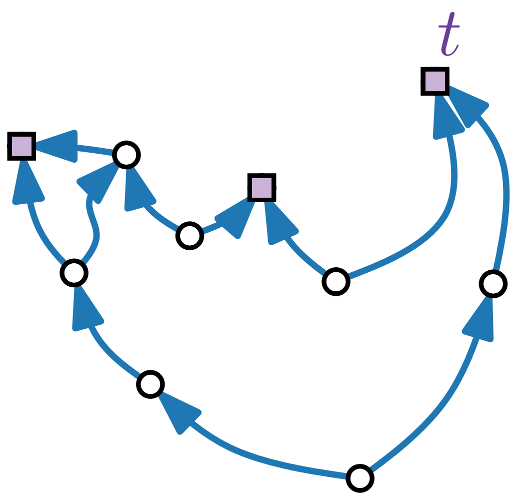

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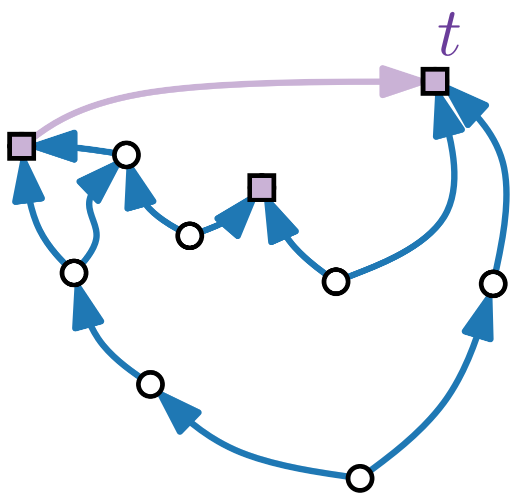

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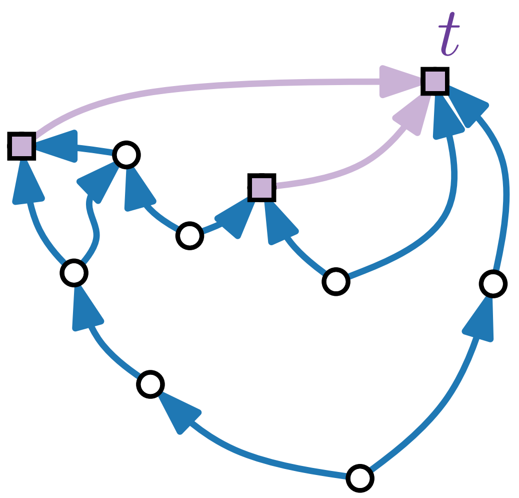

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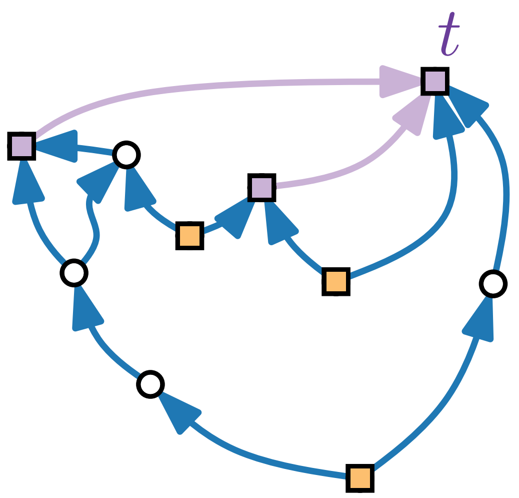

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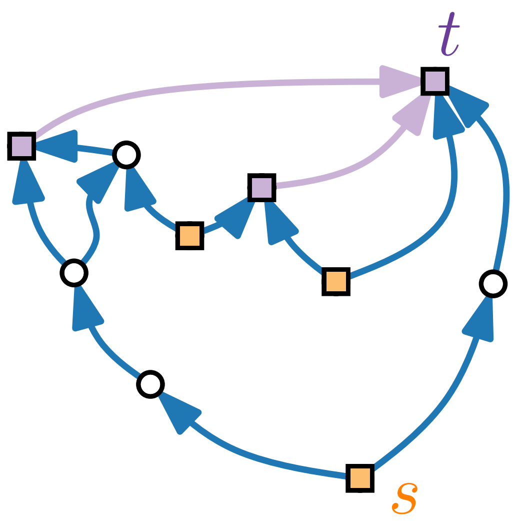

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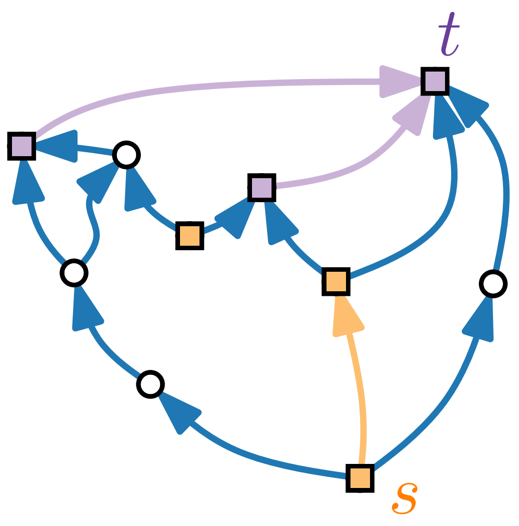

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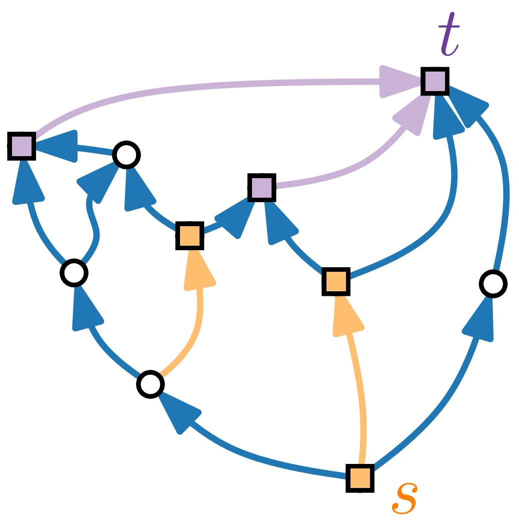

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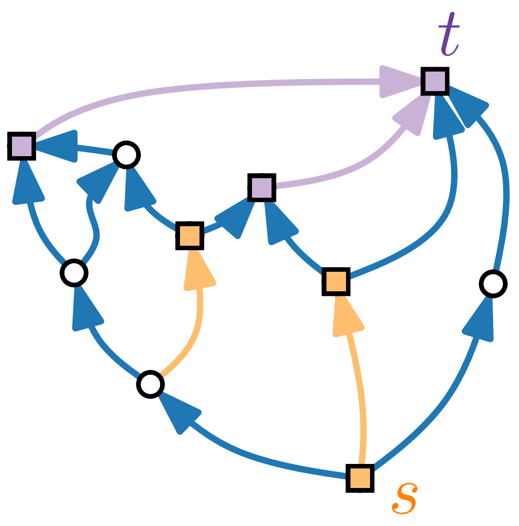

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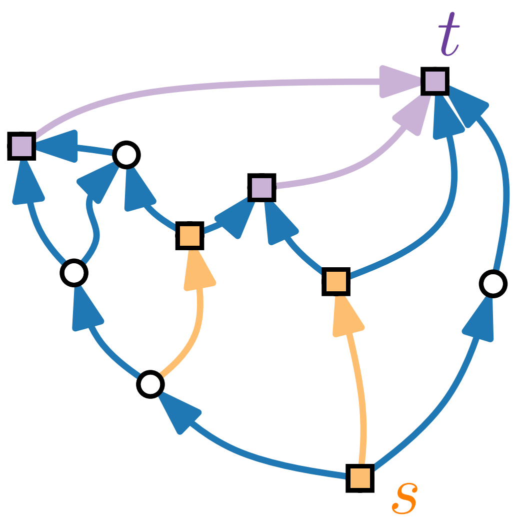

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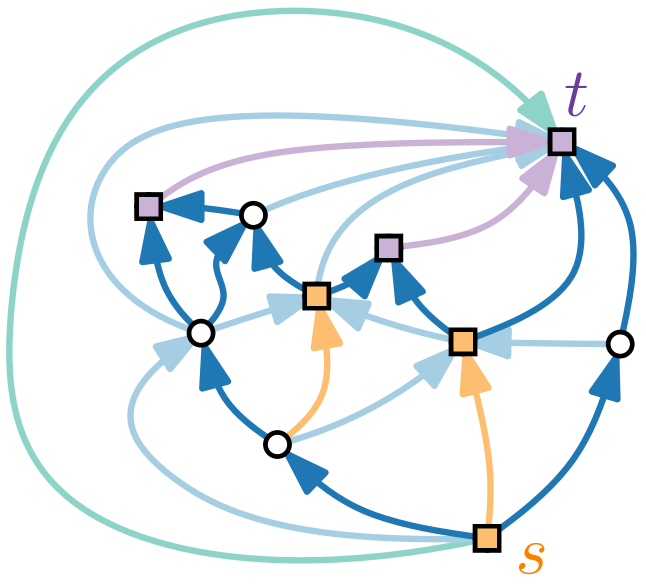

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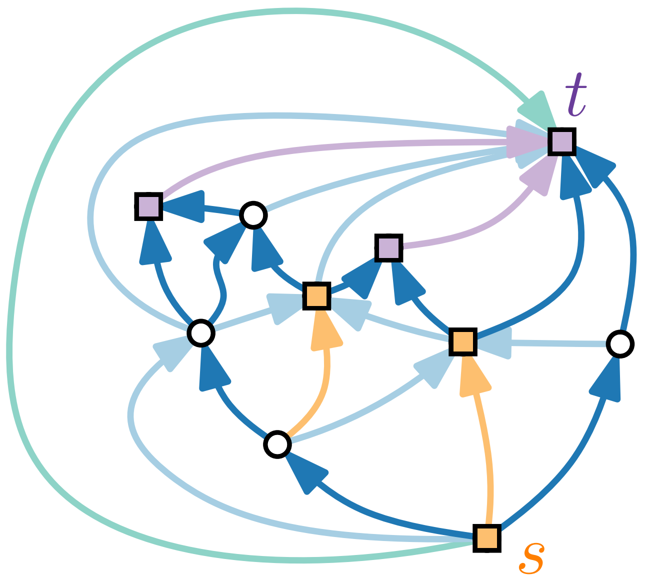

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Claim.
Can be drawn
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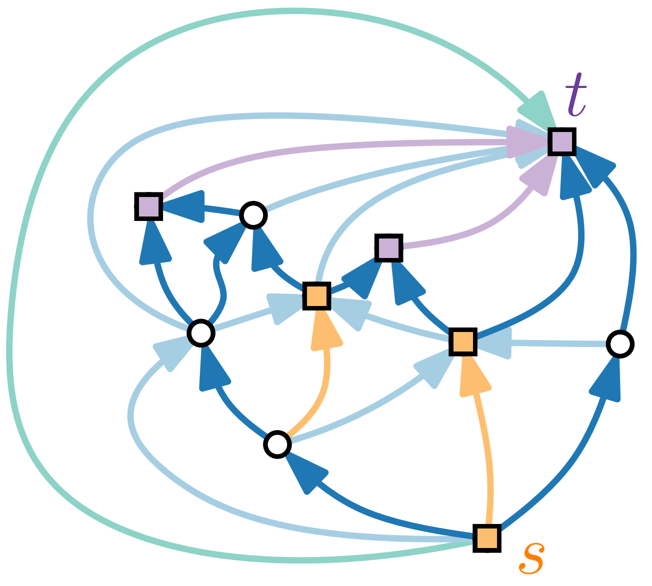

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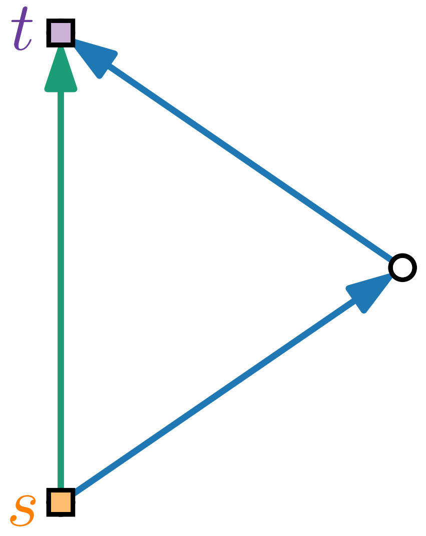

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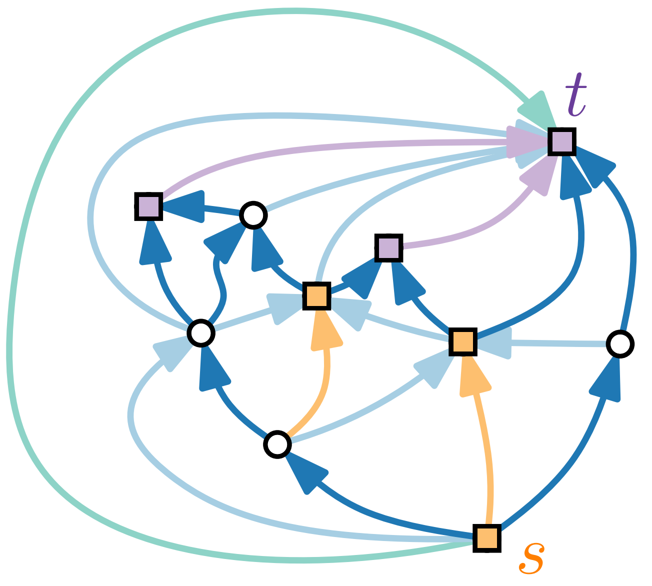

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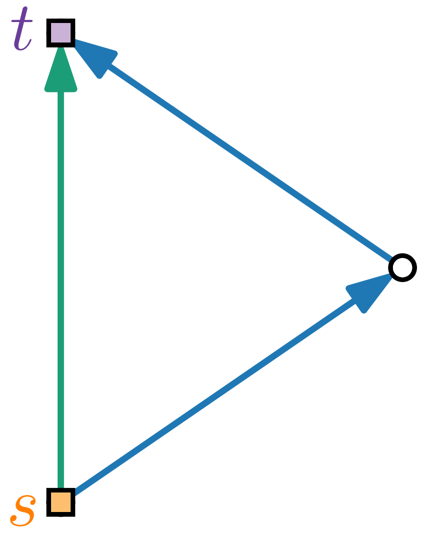
Induction on the number of vertices n.



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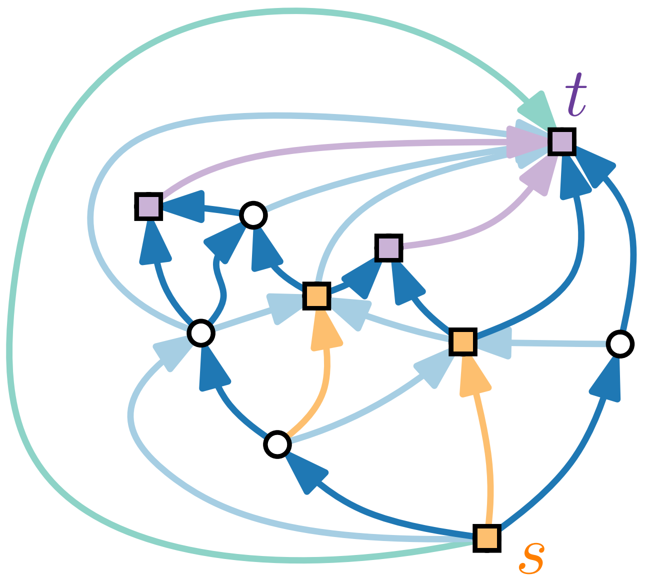

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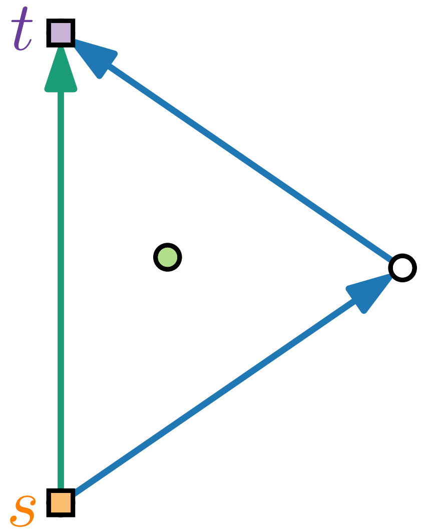

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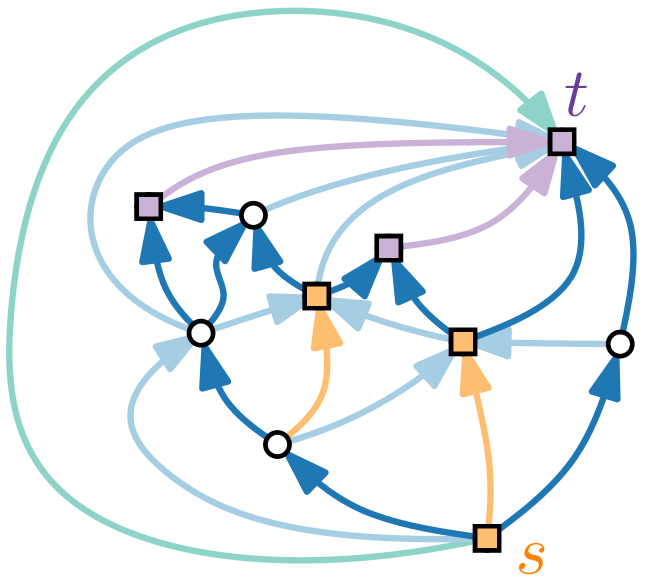

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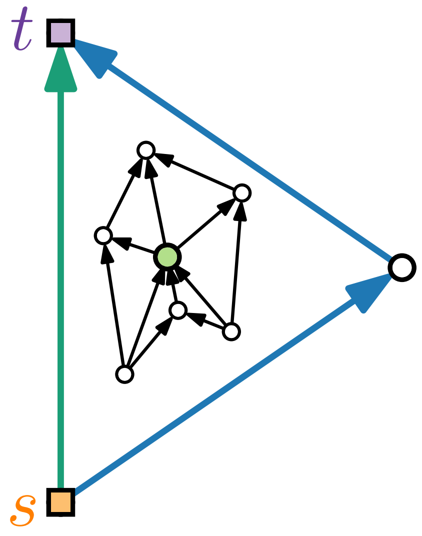

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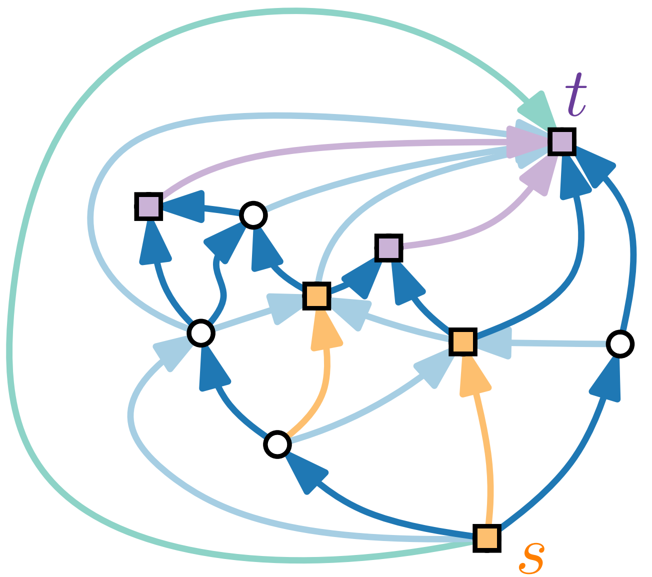

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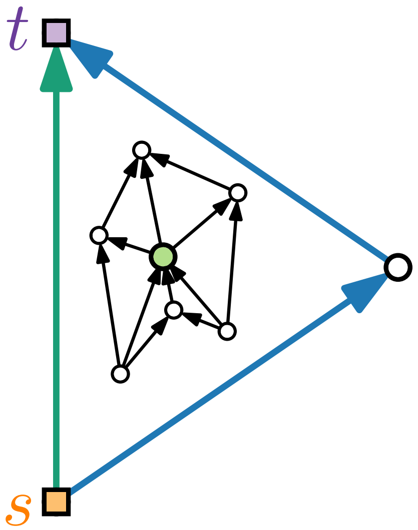

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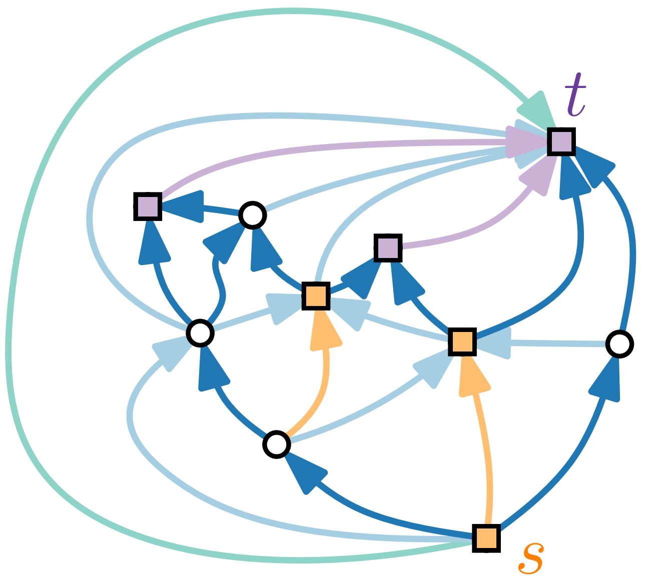

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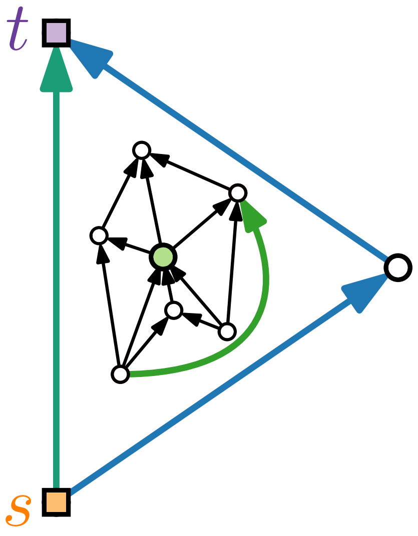

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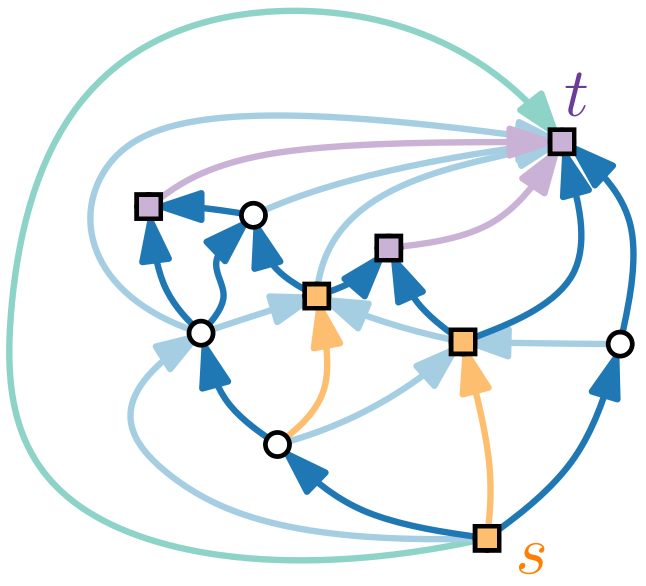

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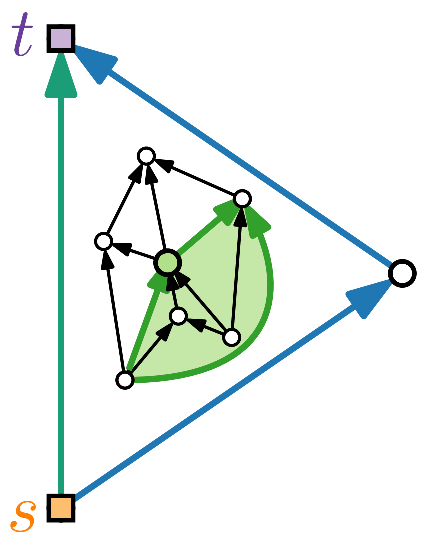

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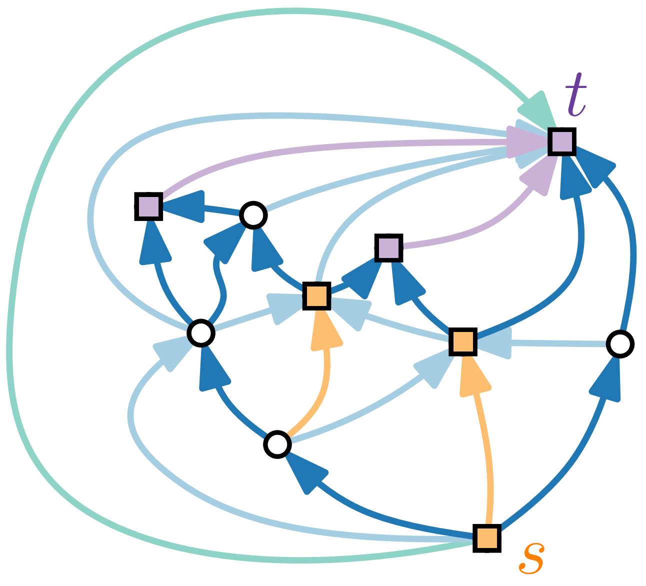

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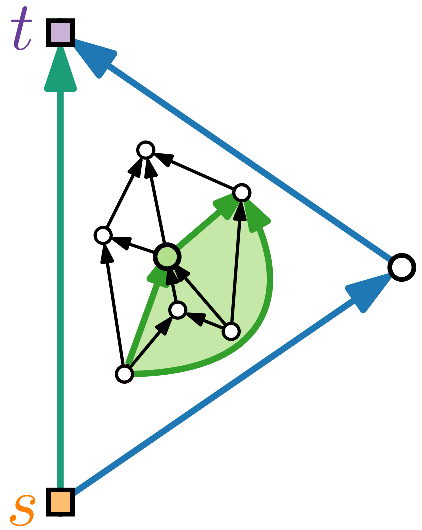

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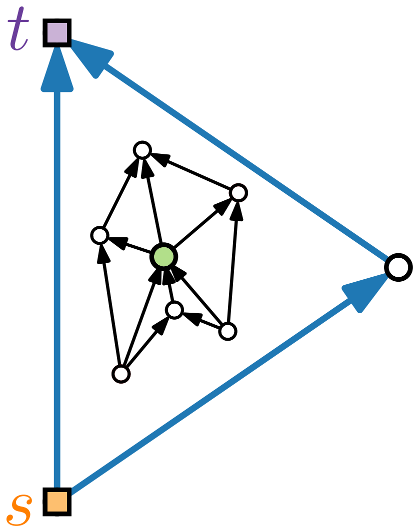

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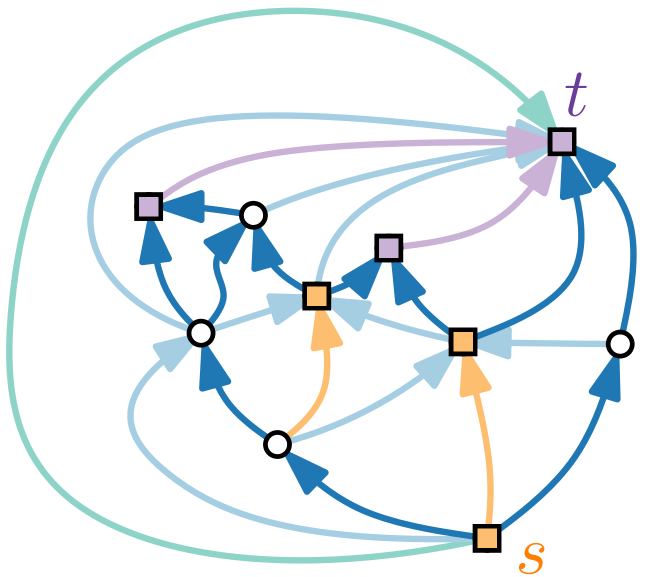

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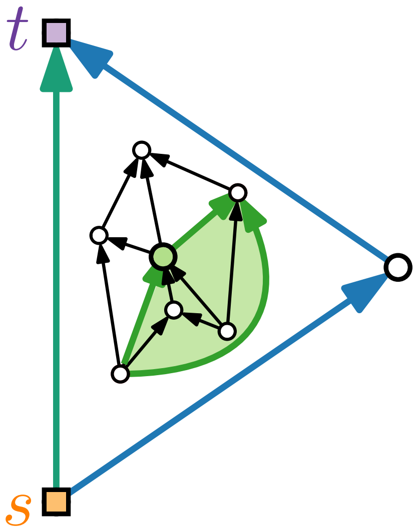

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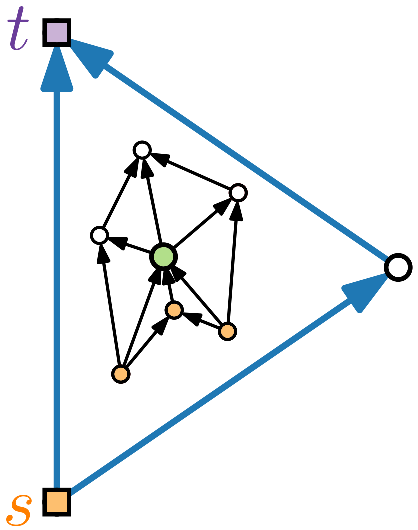

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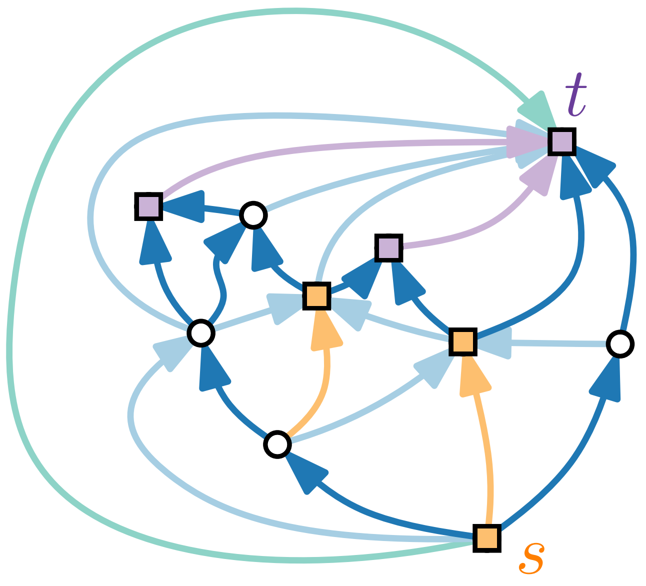

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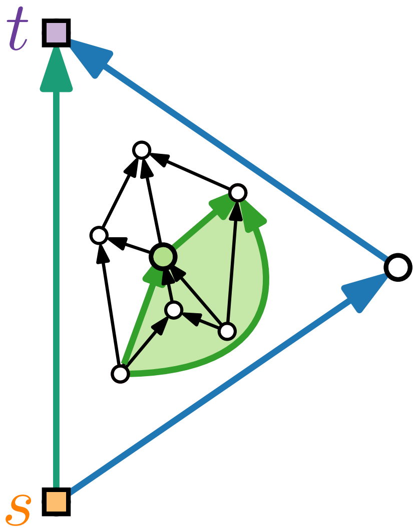

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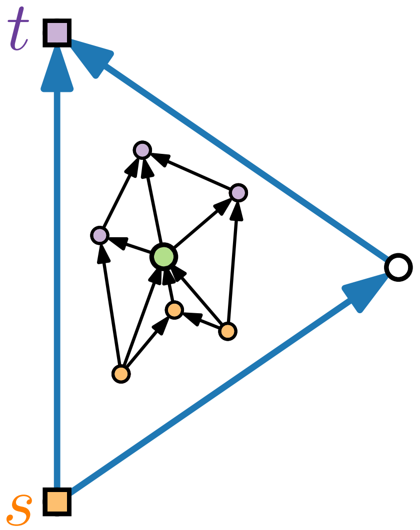

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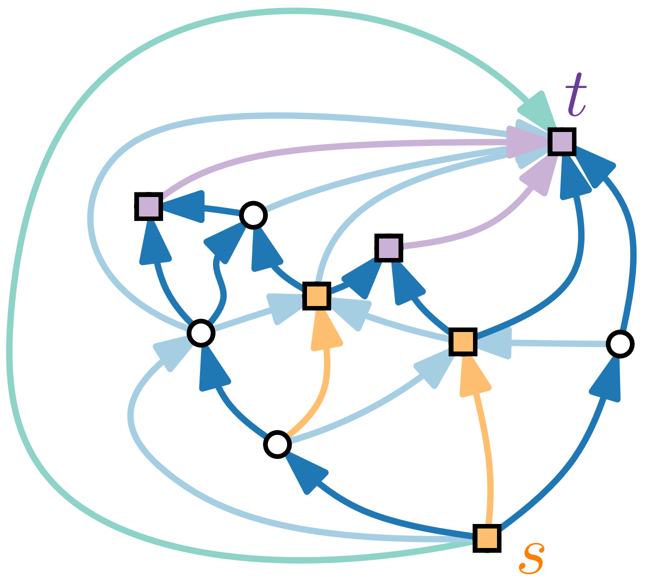

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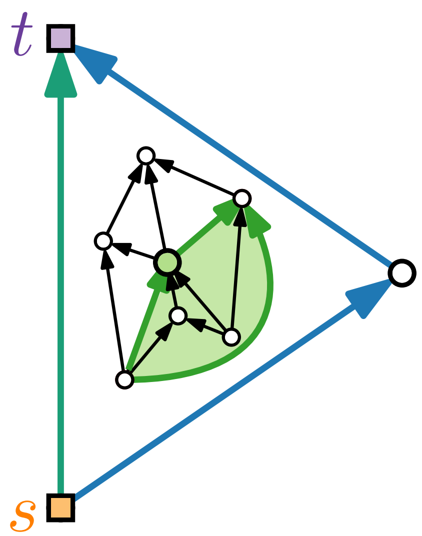

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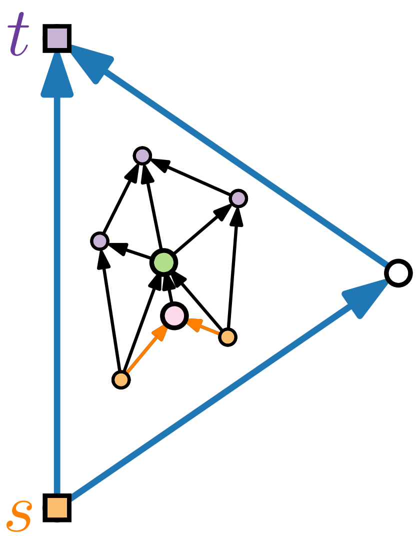

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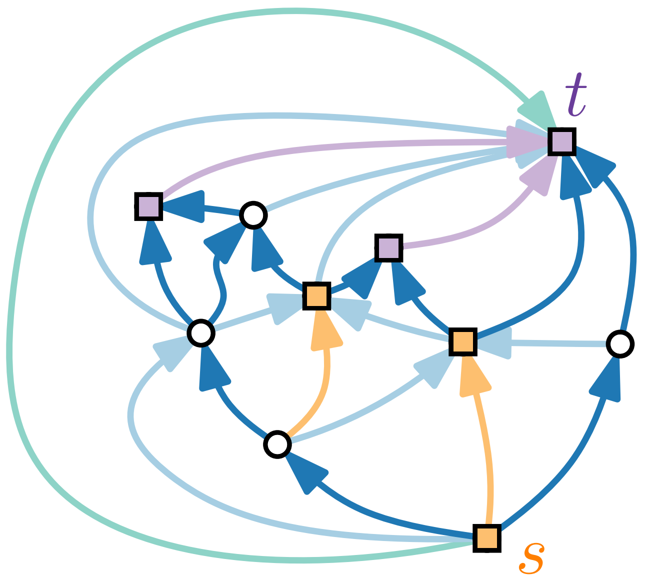

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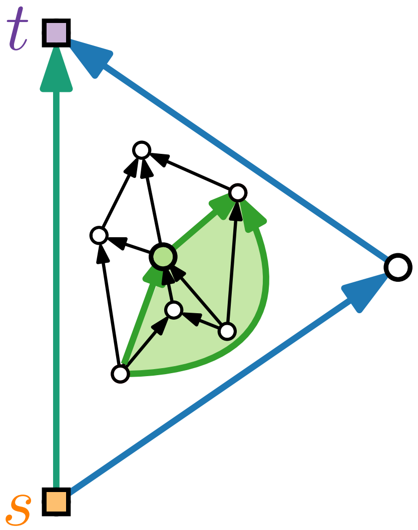

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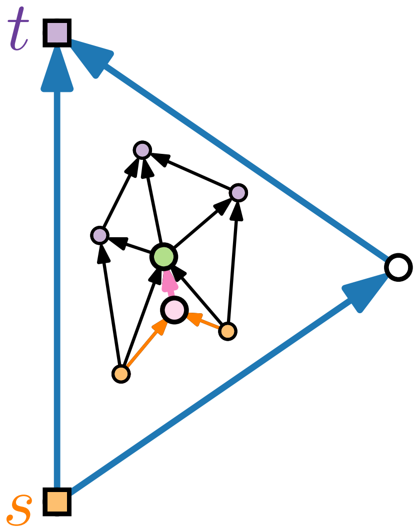

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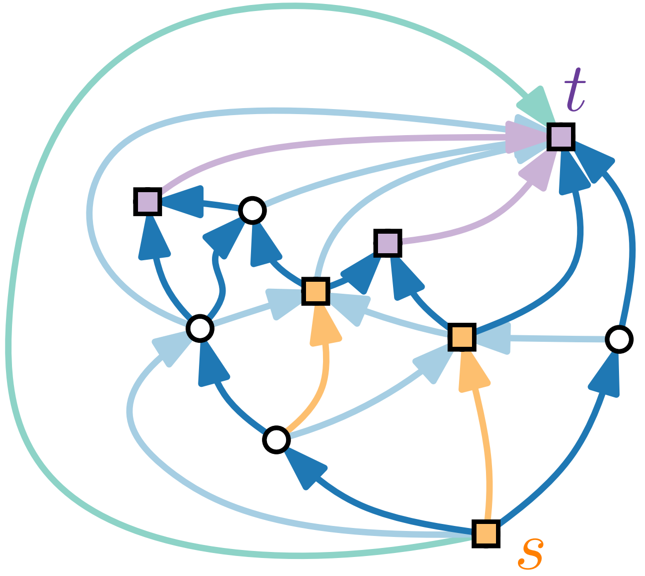

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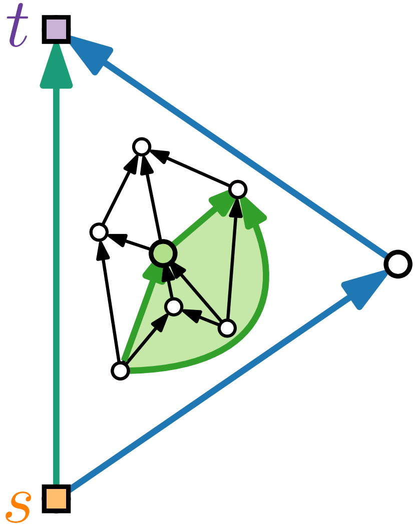

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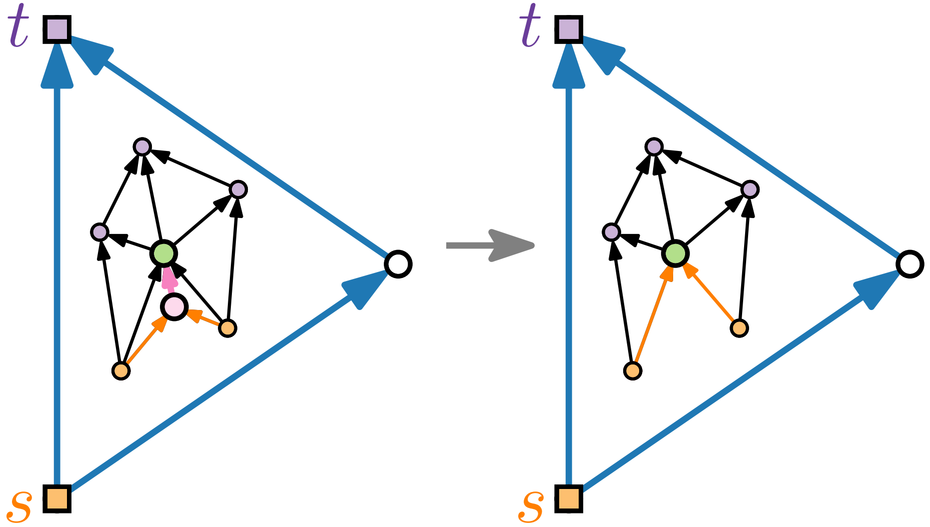

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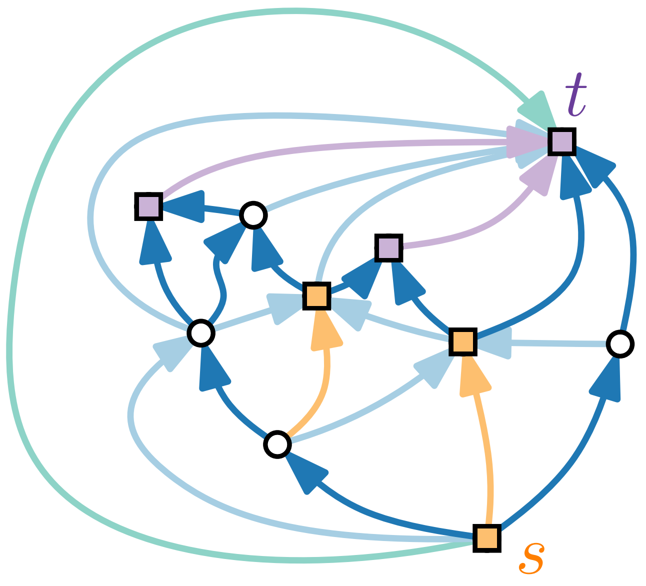

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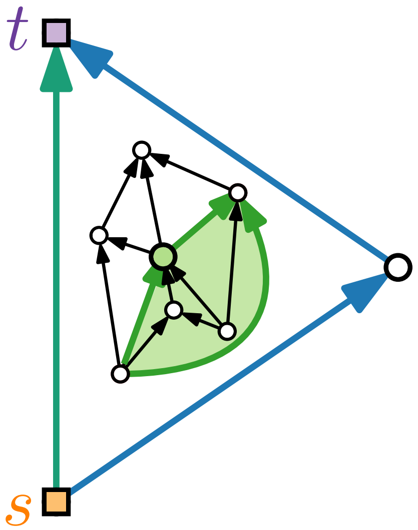

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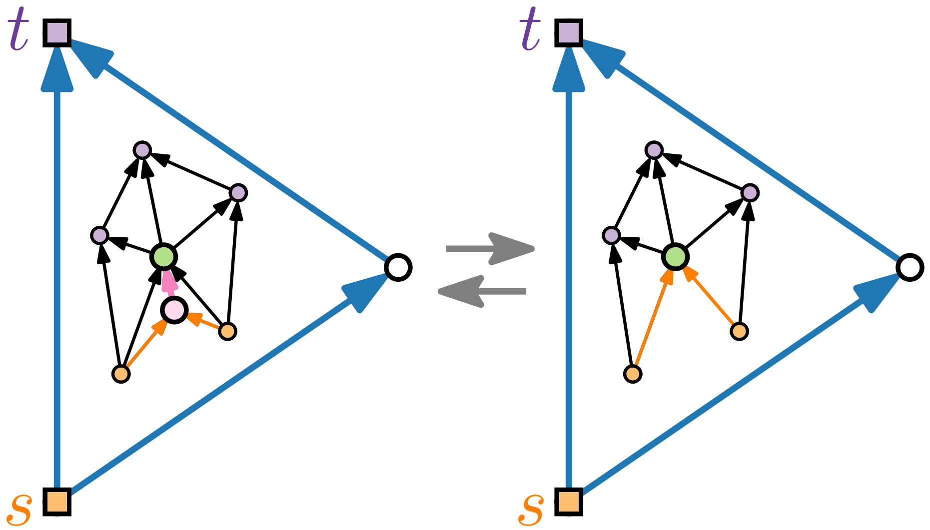

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Given a *planar acyclic* digraph G,
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Test whether G is upward planar (w.r.t. to F and f_0).

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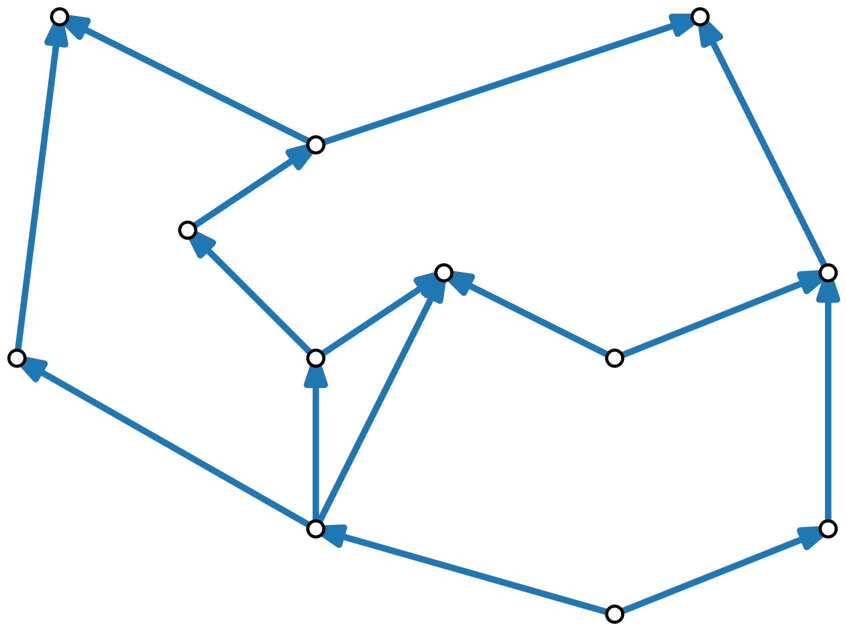
Plan.

- Find a property that any upward planar drawing of G satisfies.
- Formalize this property.
- Specify an algorithm to test this property.

Angles, Local Sources & Sinks

Definitions.



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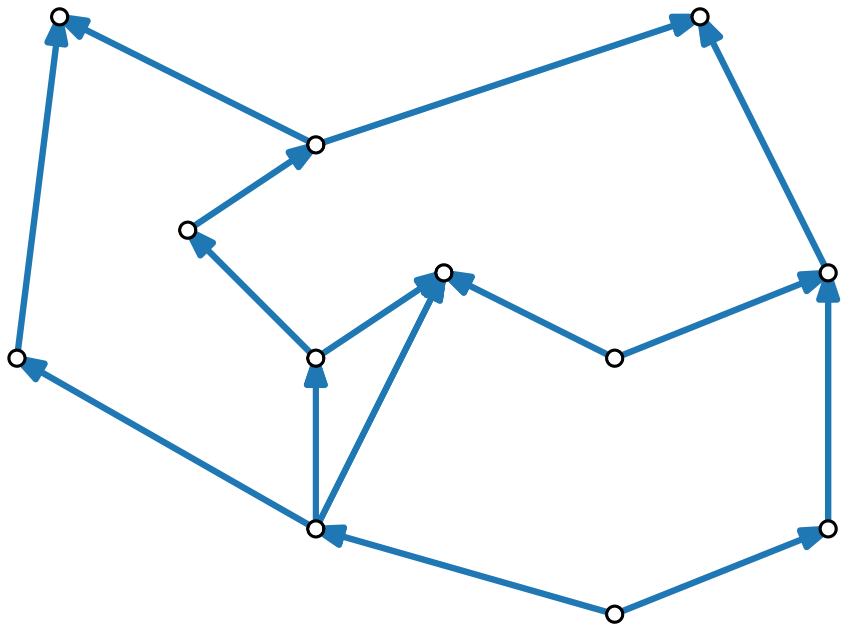

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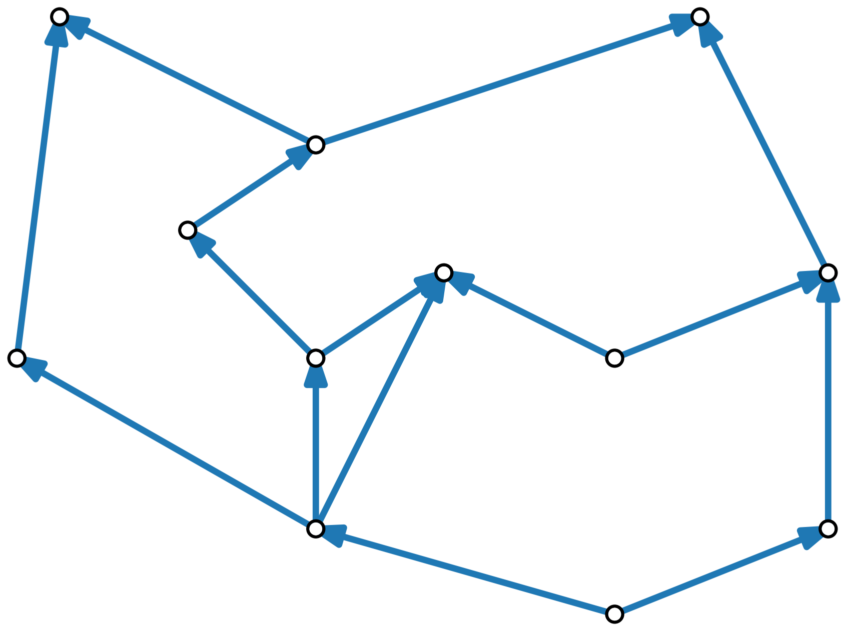

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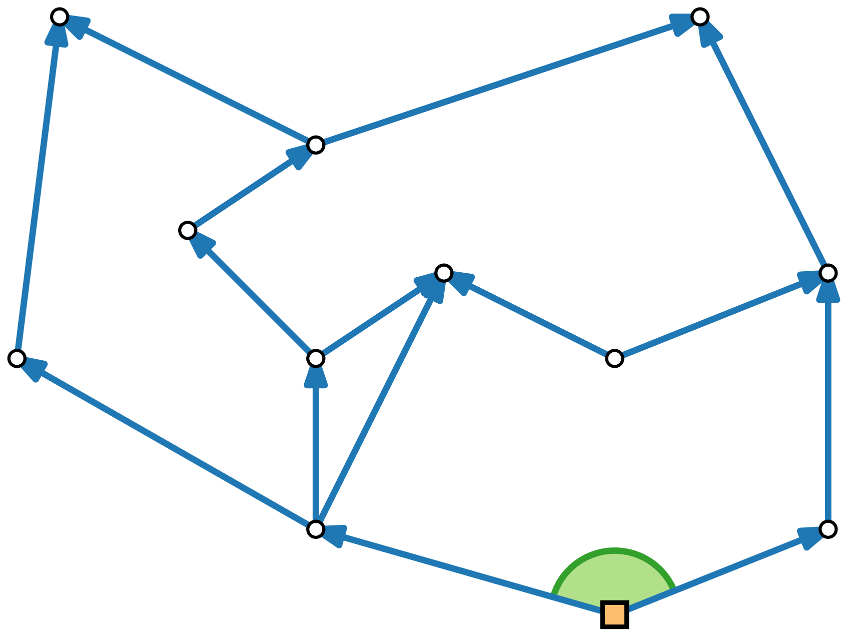

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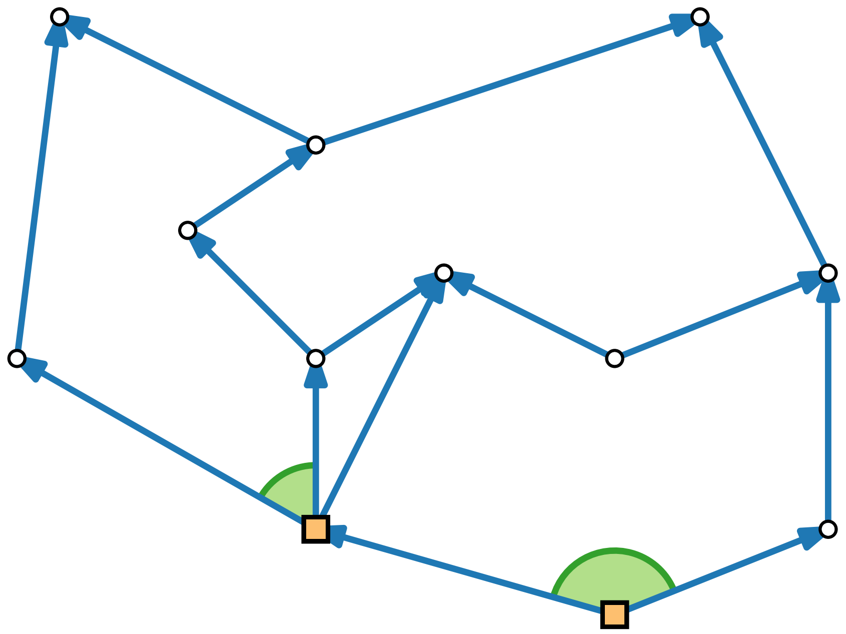

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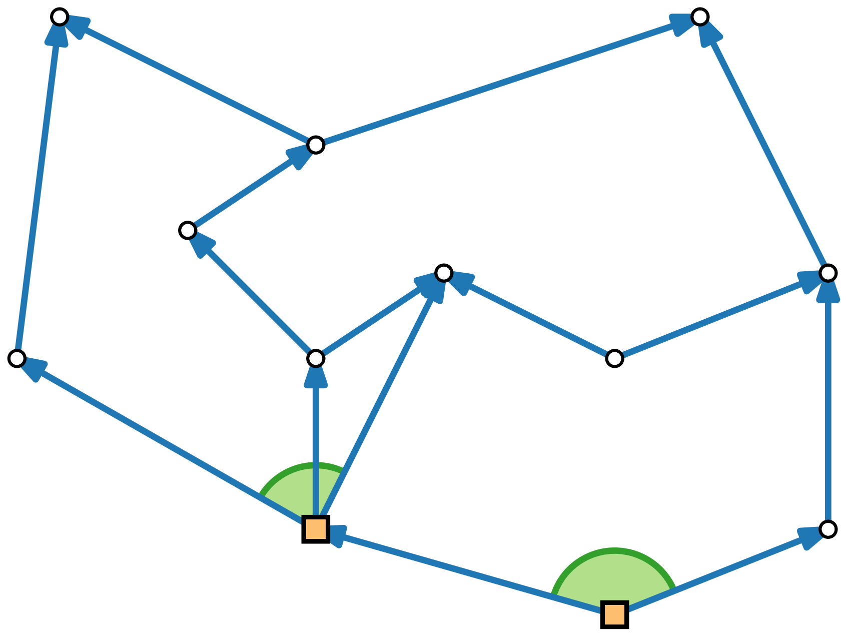

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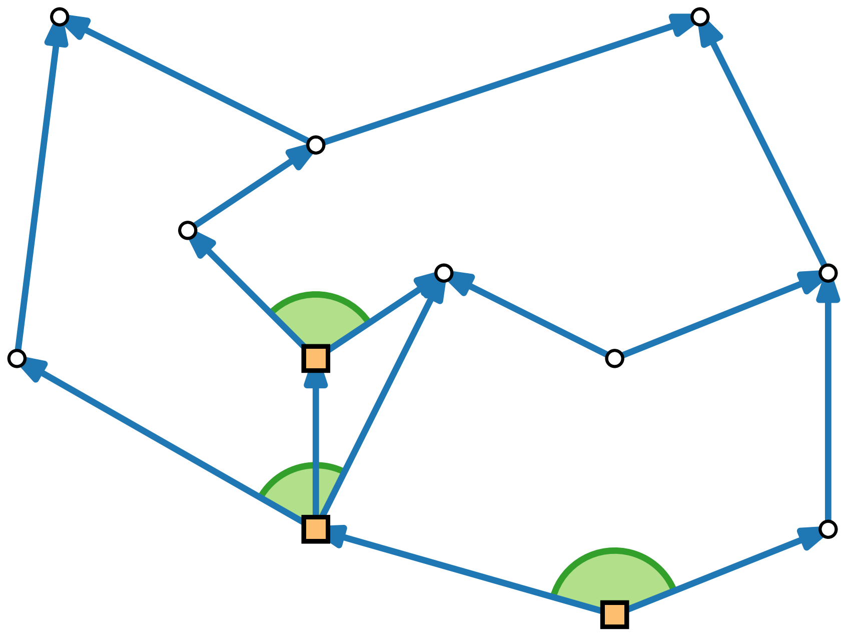

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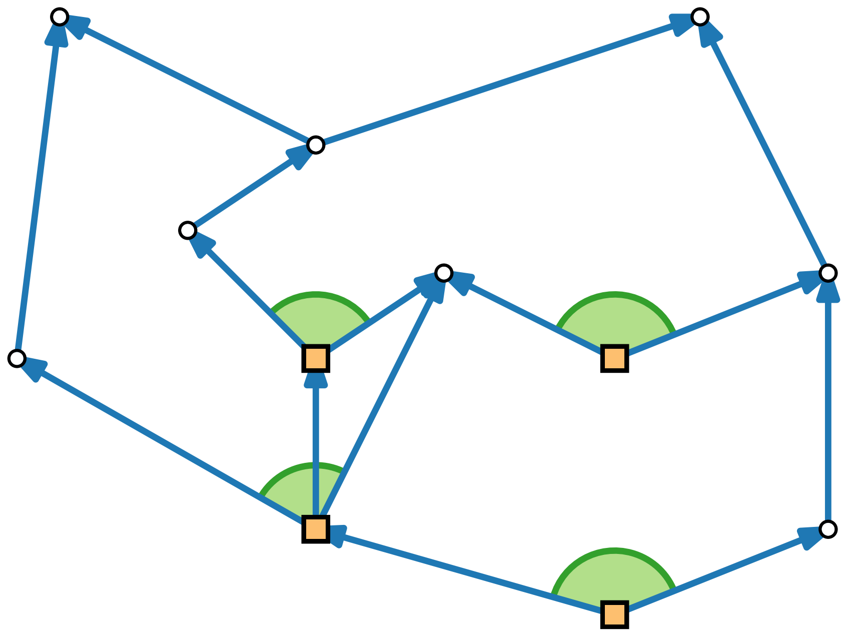

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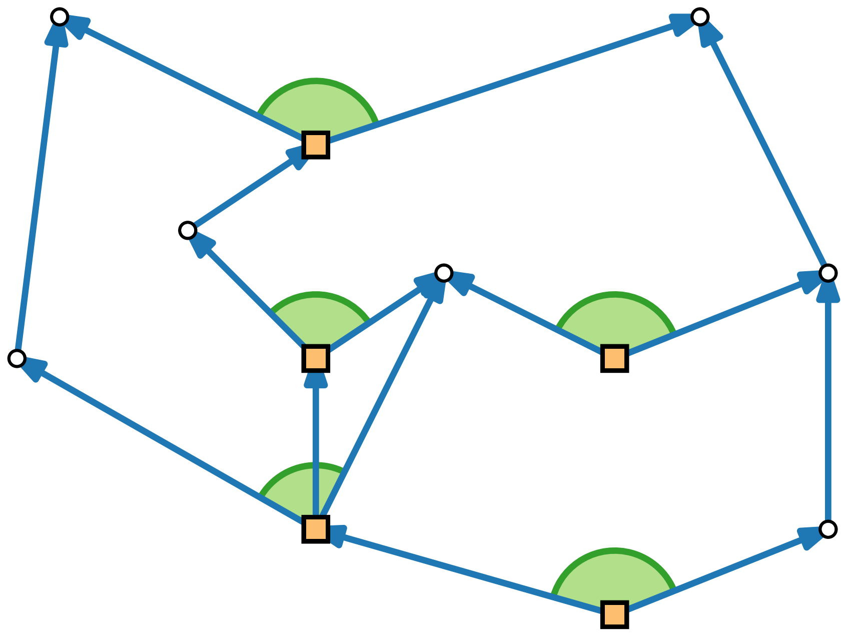

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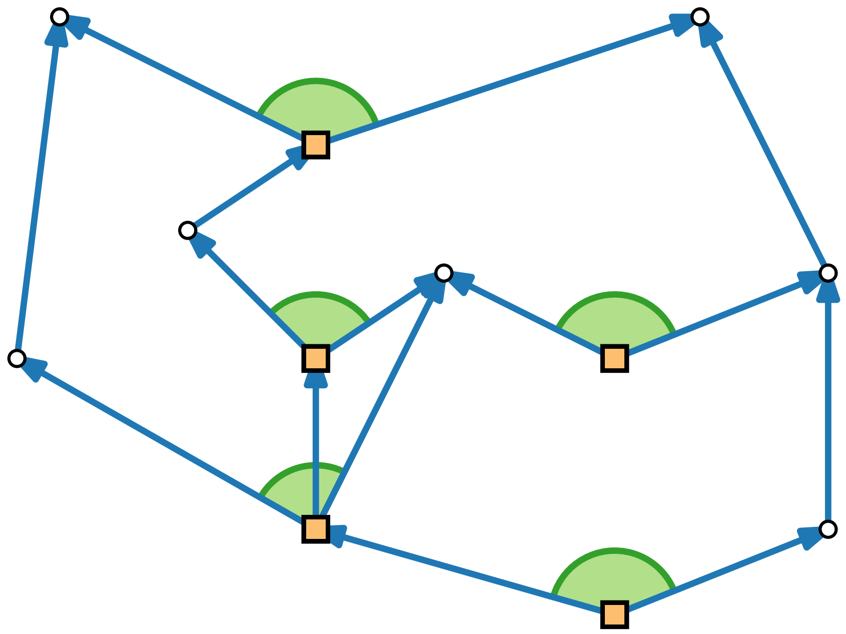

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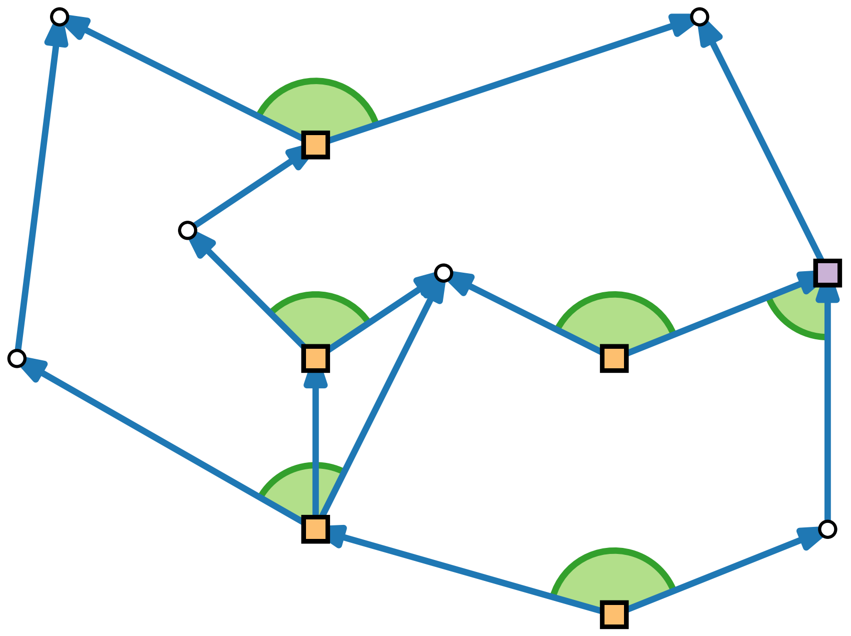

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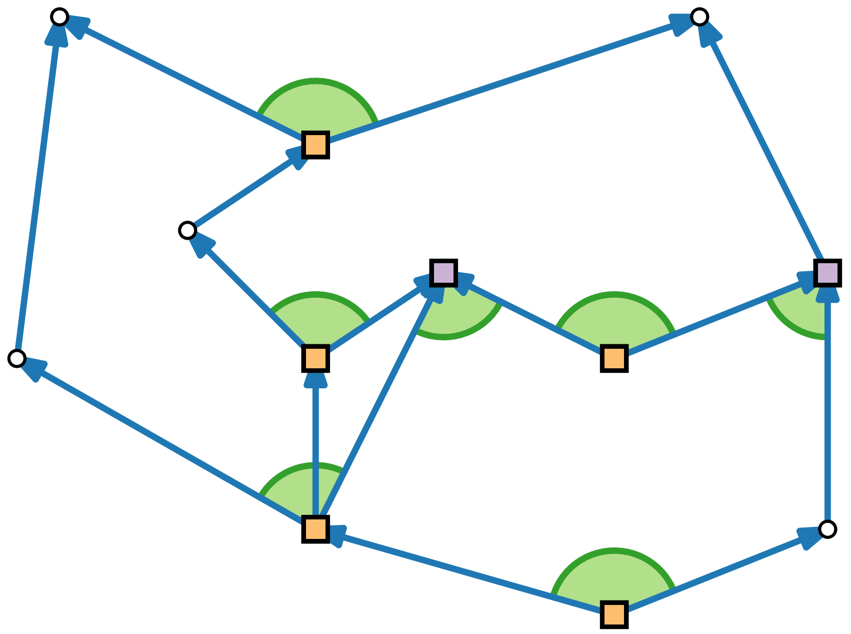

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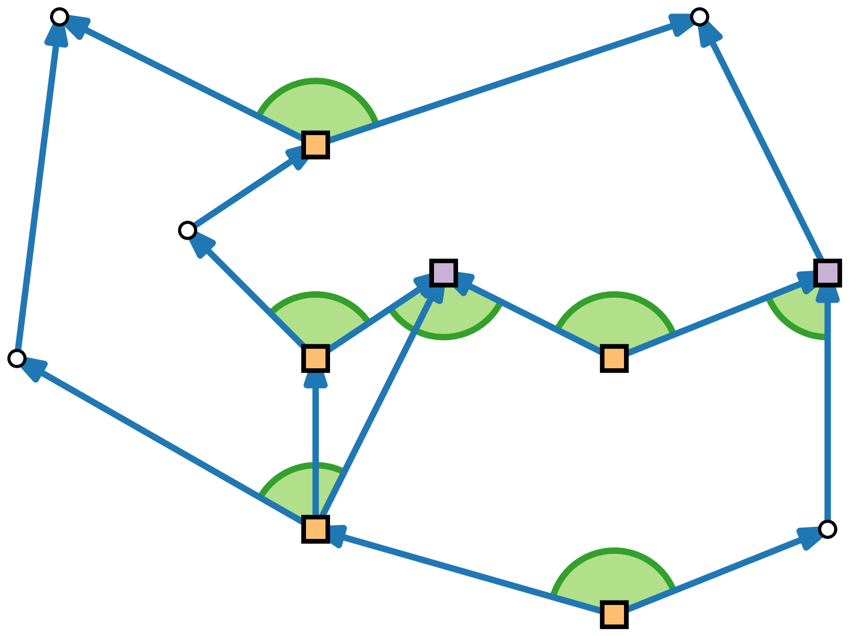

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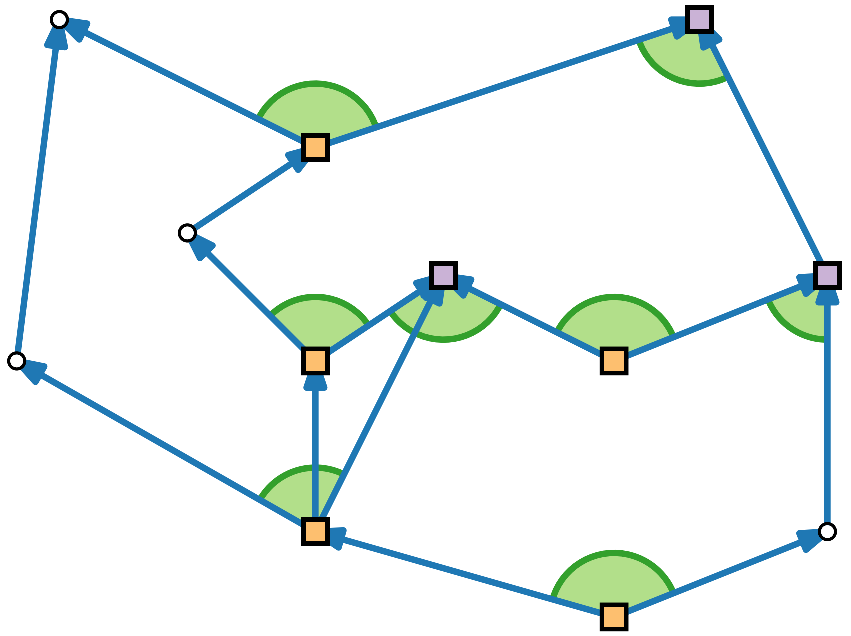

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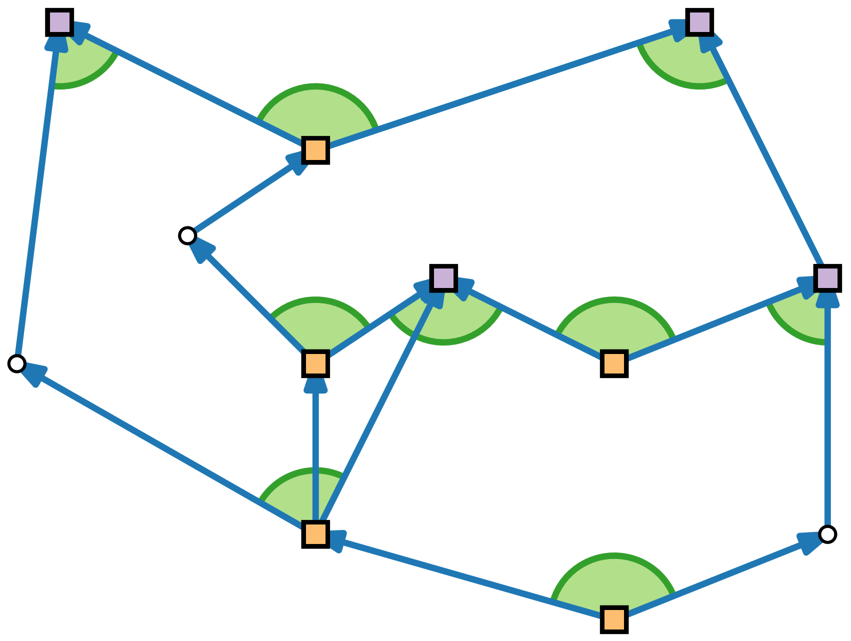

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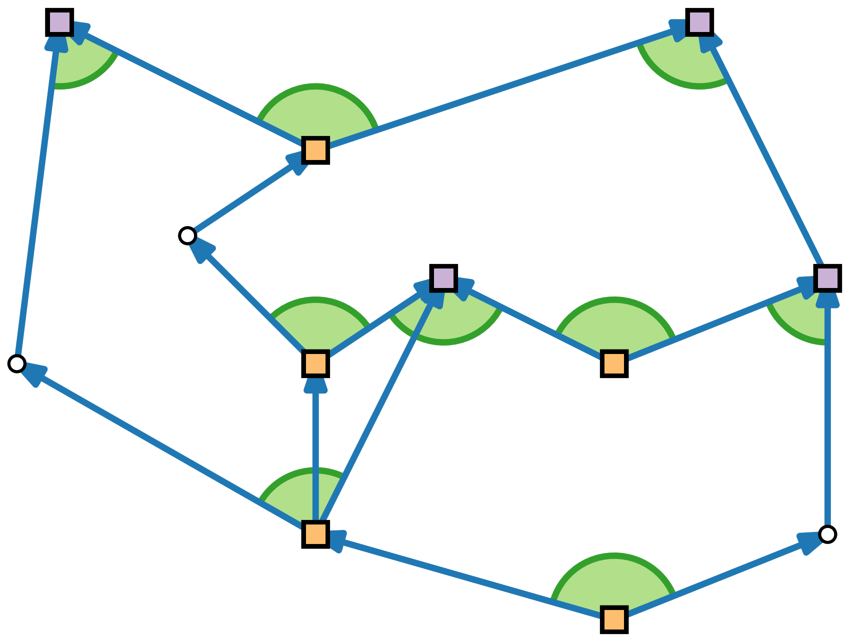

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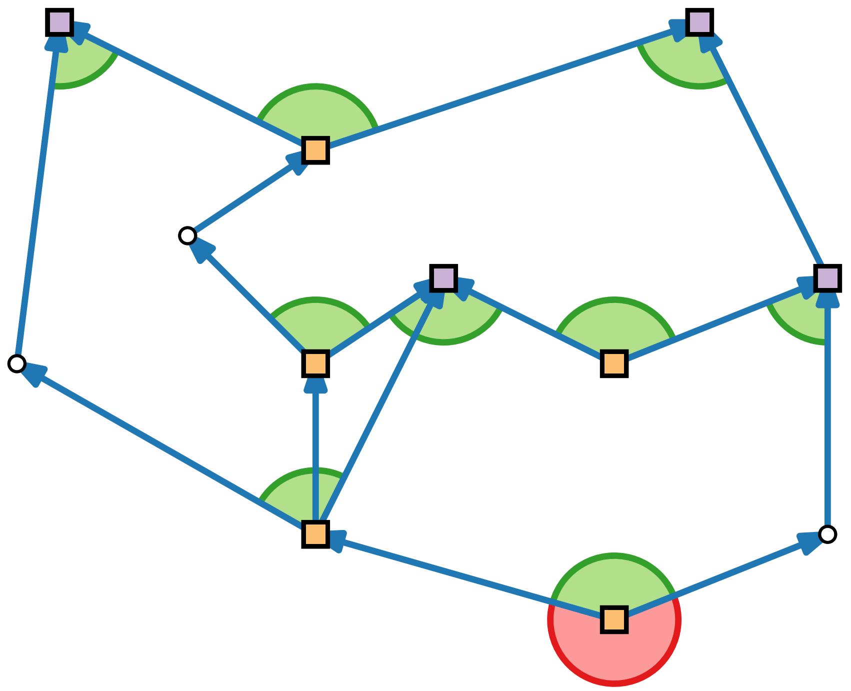

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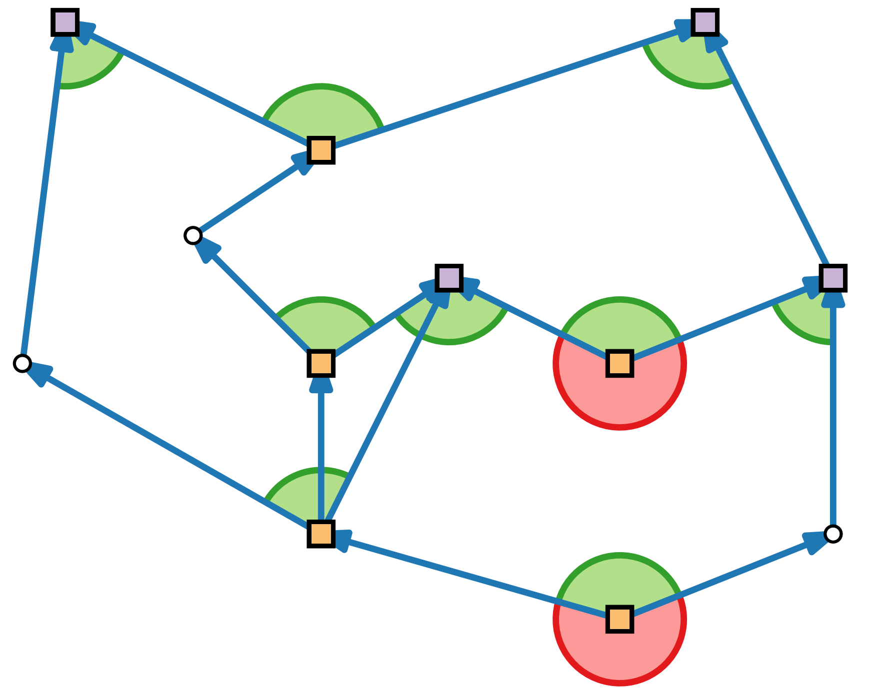

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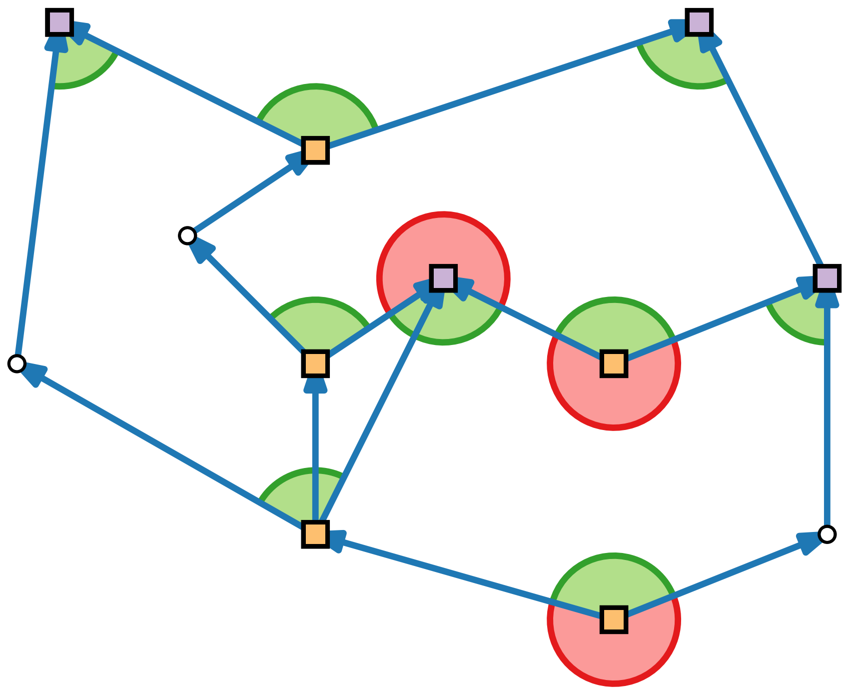

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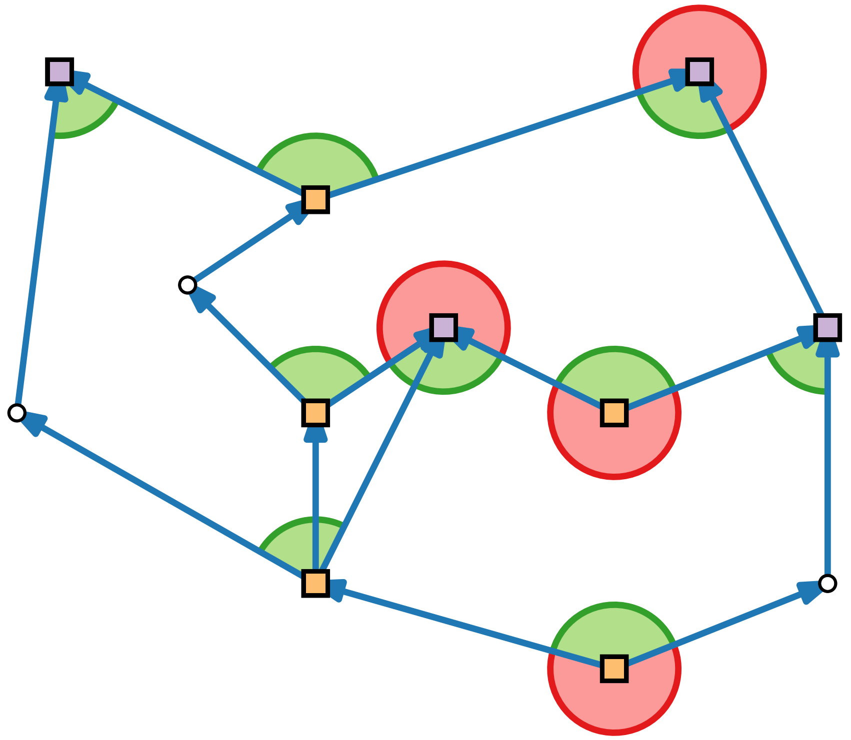

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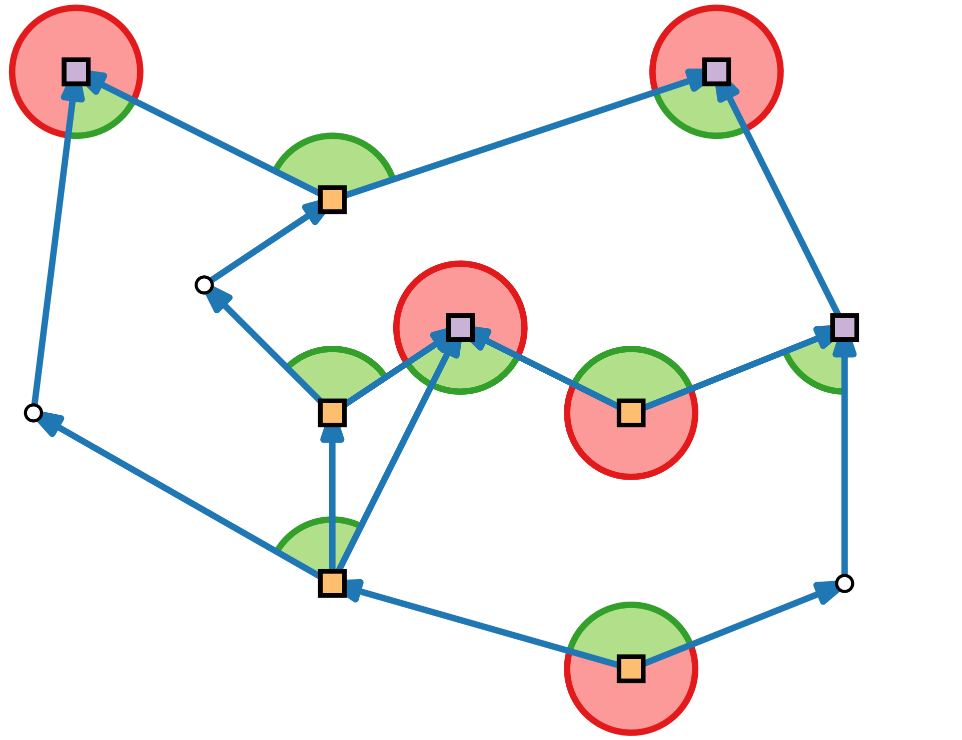

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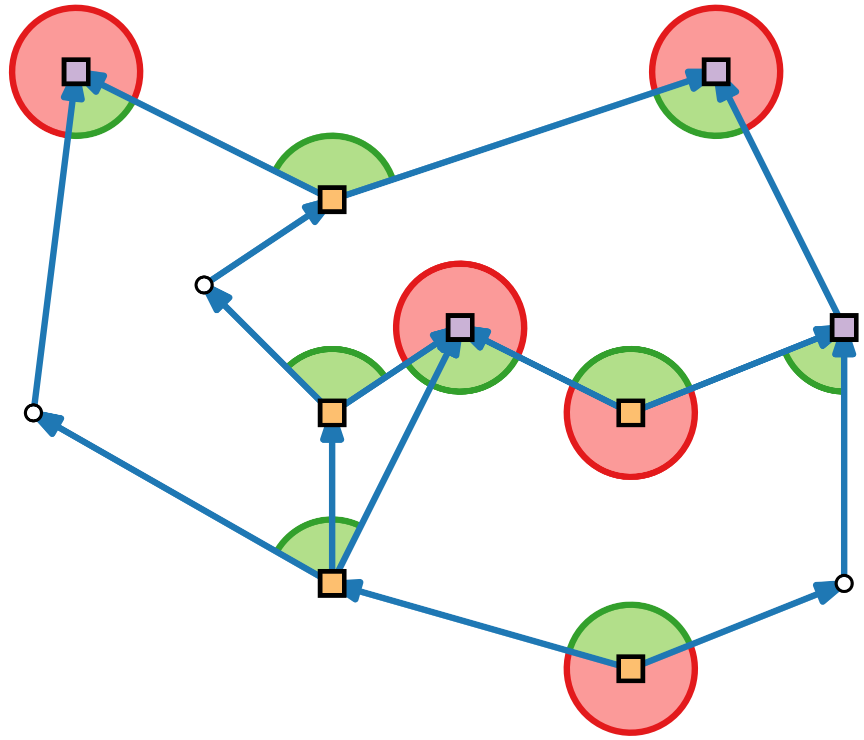

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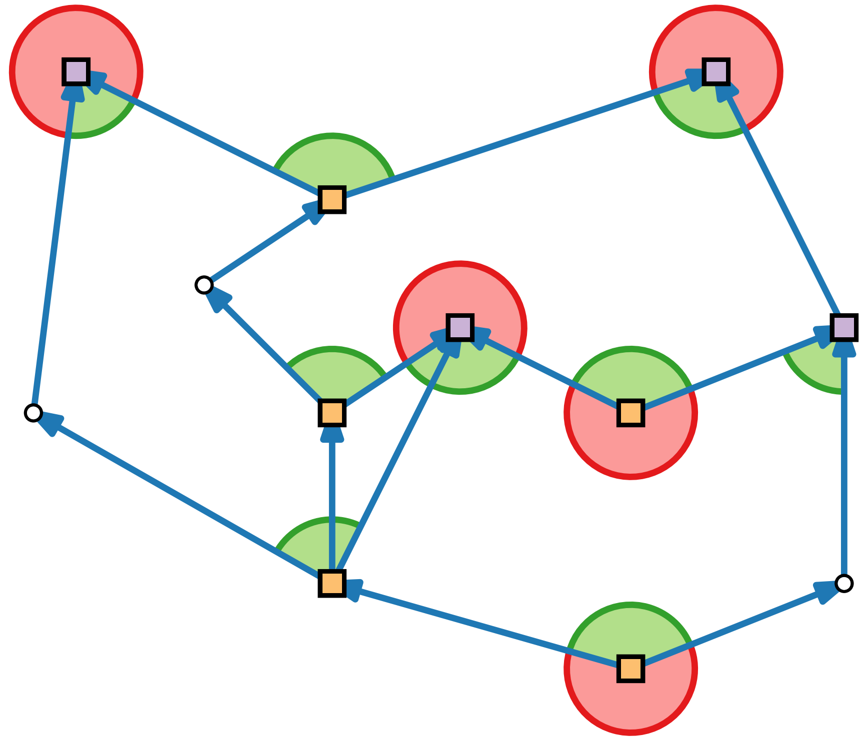

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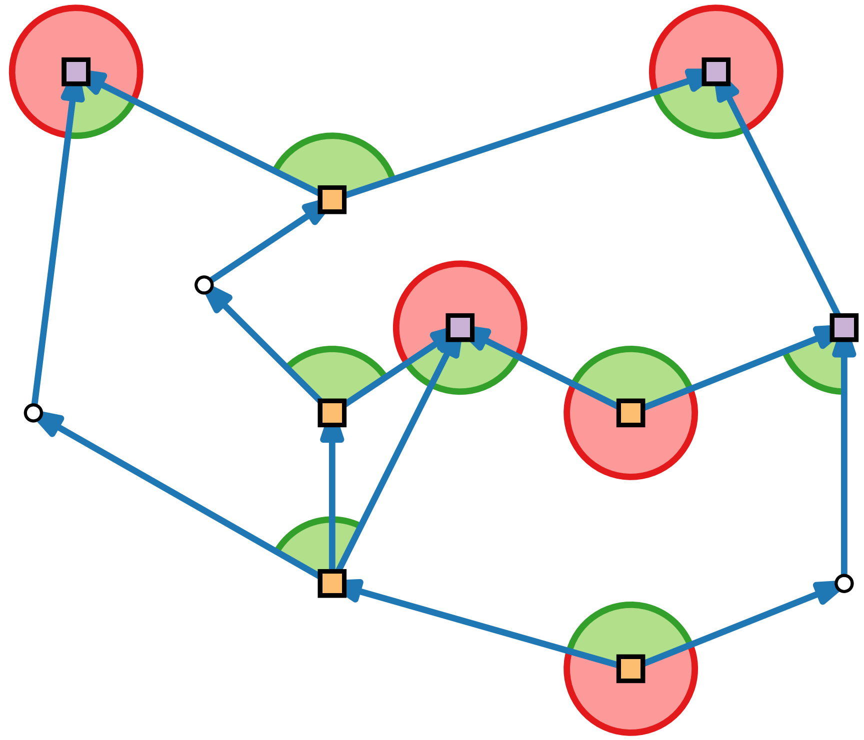

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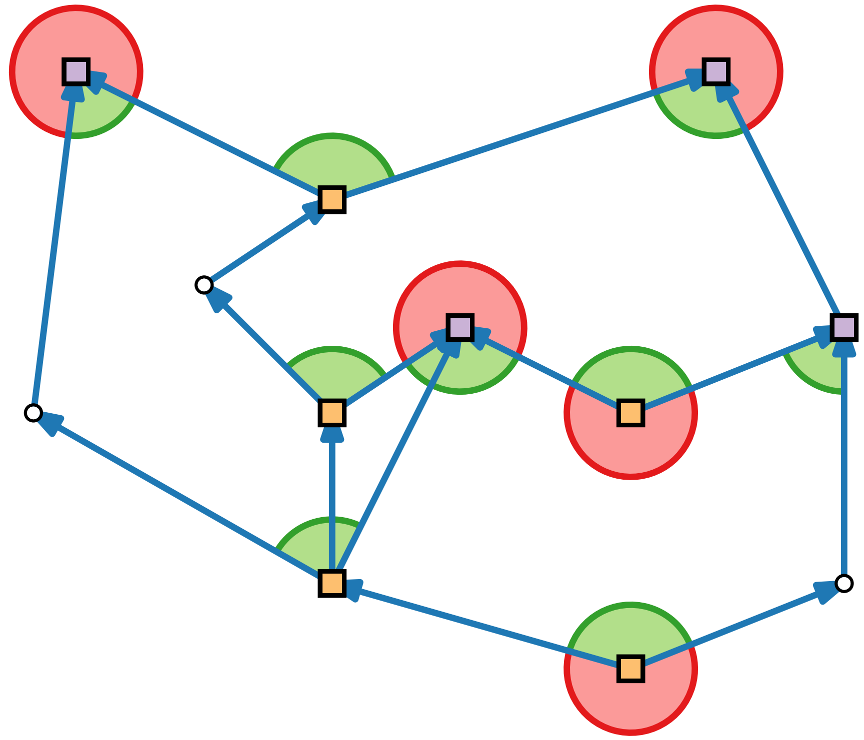

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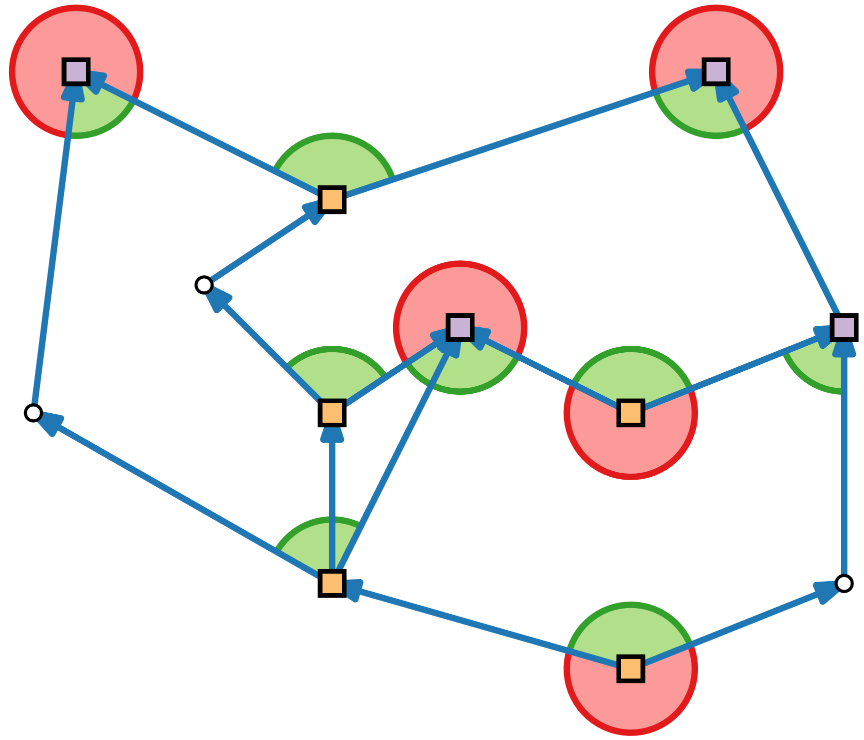

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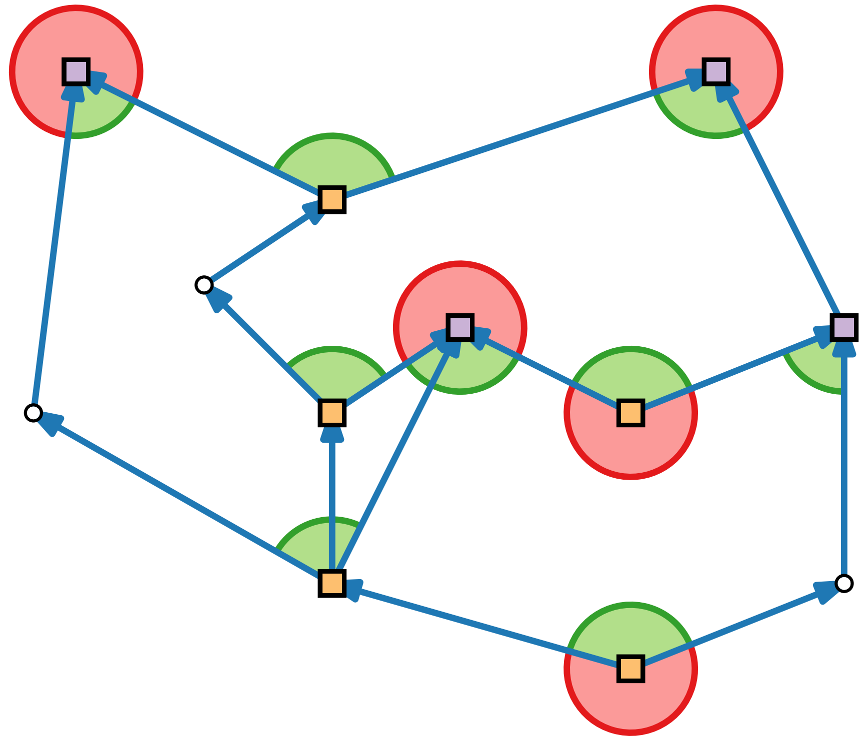
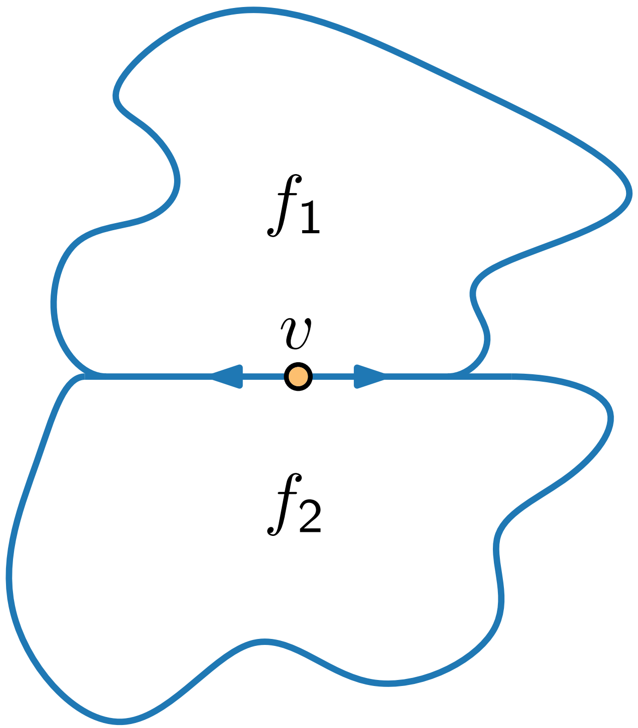


Lemma 1.
$L(f) + S(f) = 2A(f)$

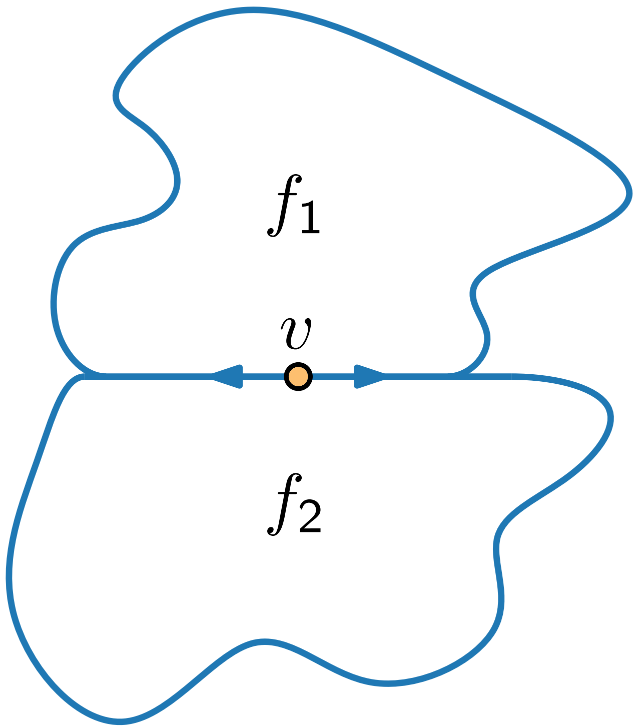
Assignment Problem

- Vertex v is a global source at faces f_1 and f_2.



Assignment Problem

- Vertex v is a global source at faces f_1 and f_2.
- Does v have a **large** angle in f_1 or f_2?



Angle Relations

Lemma 2.

$$L(f) - S(f) = \begin{cases} -2, & f \neq f_0 \\ +2, & f = f_0 \end{cases}$$

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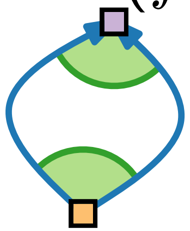

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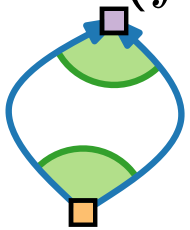
- $L(f) = 0$  $\Rightarrow S(f) = 2$ ✓

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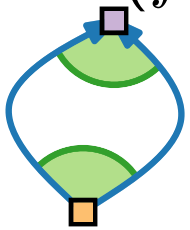
- $L(f) = 0$  $\Rightarrow S(f) = 2$ ✓

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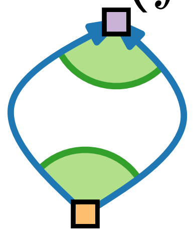

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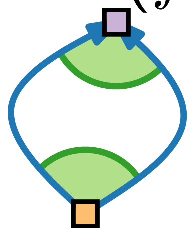

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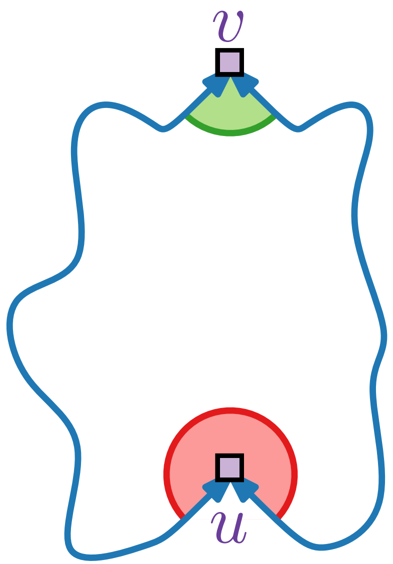

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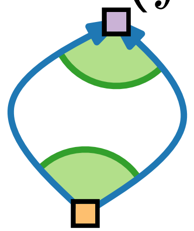
- $L(f) = 0$  $\Rightarrow S(f) = 2$ ✓

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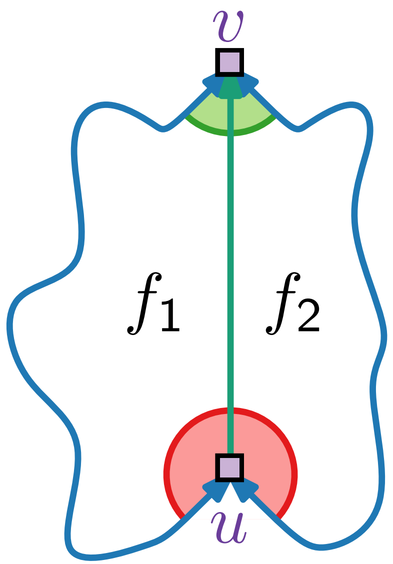

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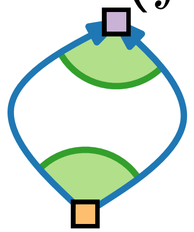
- $L(f) = 0$  $\Rightarrow S(f) = 2$ ✓

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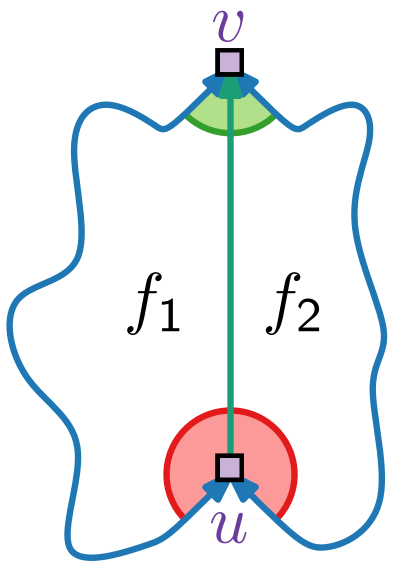

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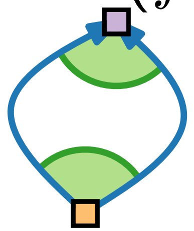
- $L(f) = 0$  $\Rightarrow S(f) = 2$ ✓

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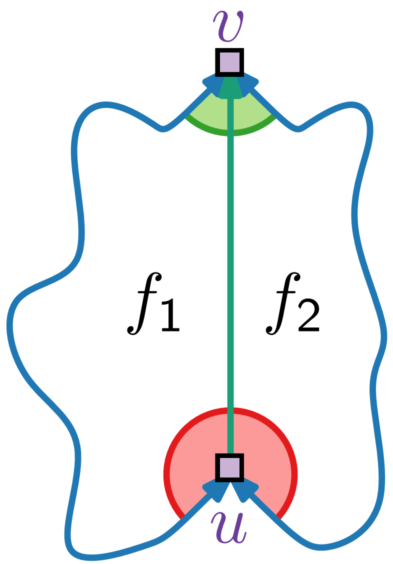

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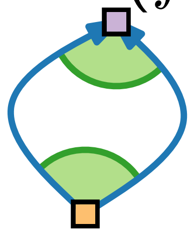

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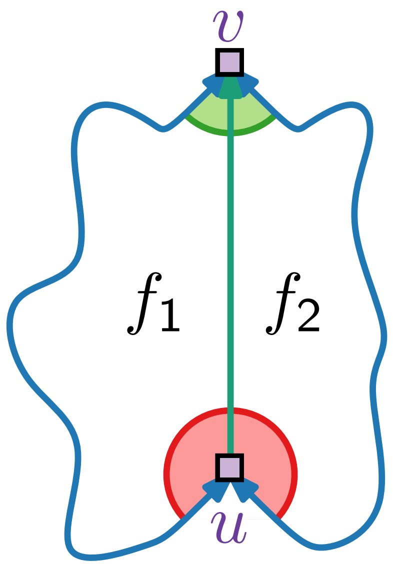

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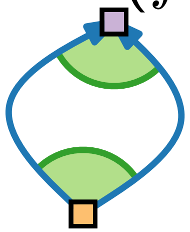

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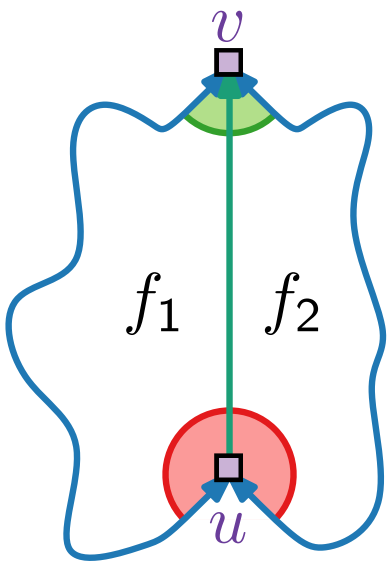

$-2 \qquad -2$

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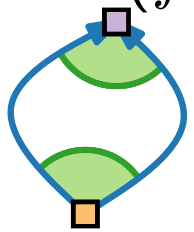
- $L(f) = 0$  $\Rightarrow S(f) = 2$ ✓

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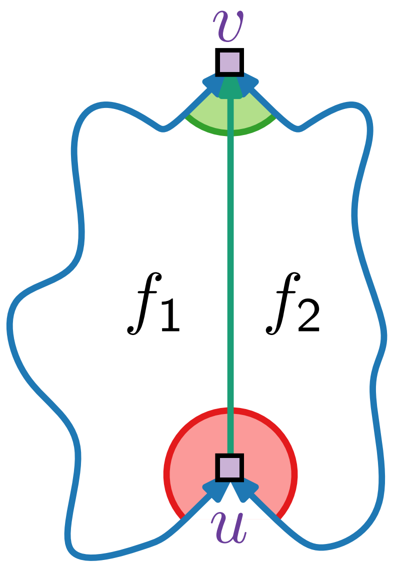

$$\begin{aligned} L(f) - S(f) &= \overbrace{L(f_1)}^{-2} + \overbrace{L(f_2)}^{-2} + 1 \\ &\quad - (S(f_1) + S(f_2) - 1) \\ &= -2 \end{aligned}$$

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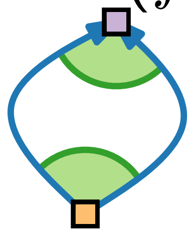

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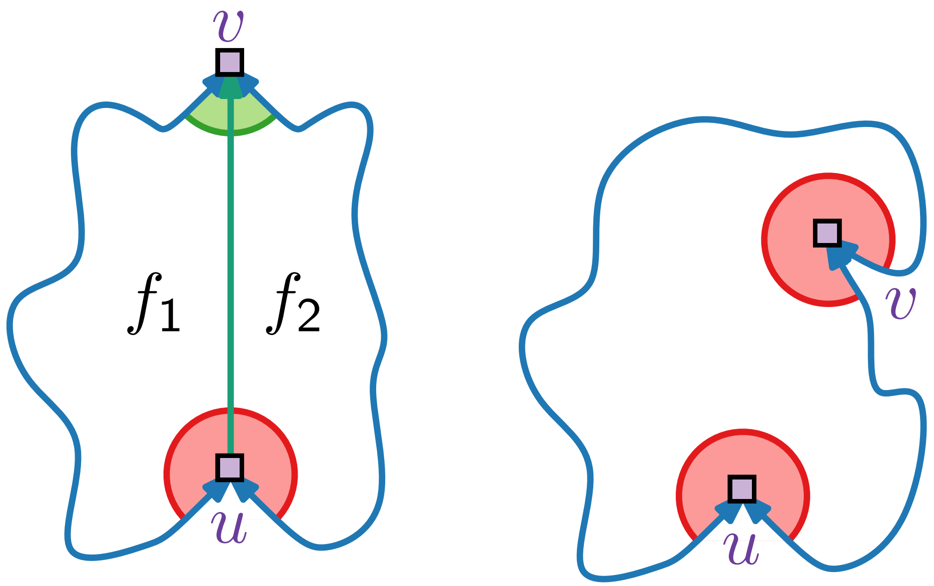

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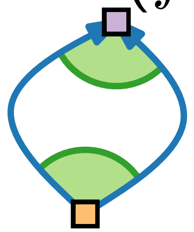
- $L(f) = 0$  $\Rightarrow S(f) = 2$ ✓

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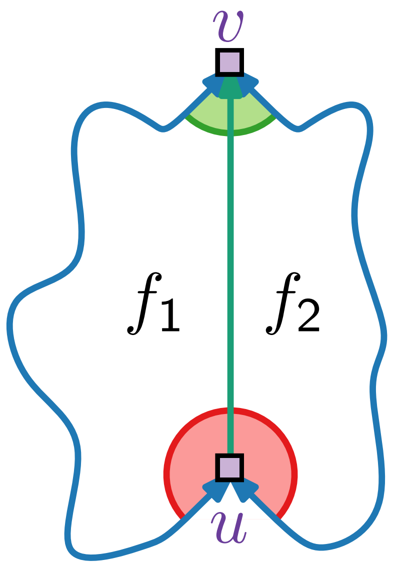

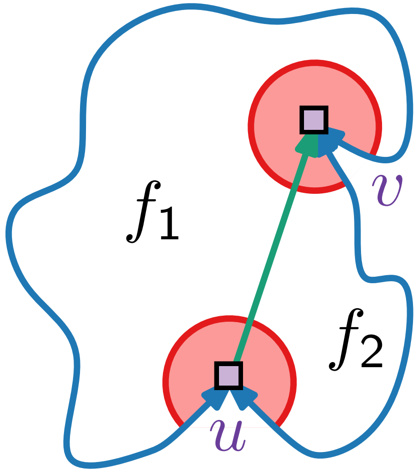

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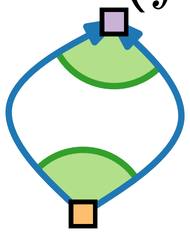
- $L(f) = 0$  $\Rightarrow S(f) = 2$ ✓

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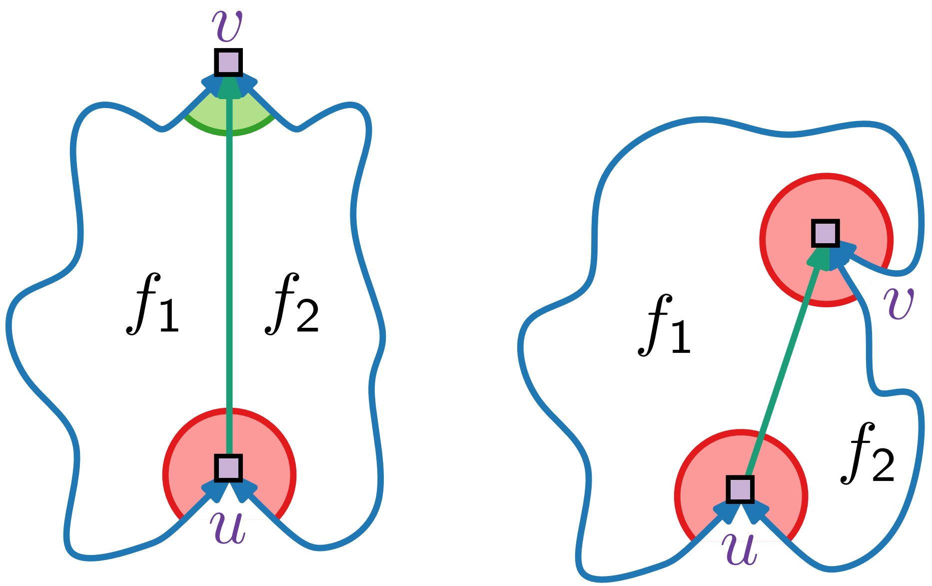

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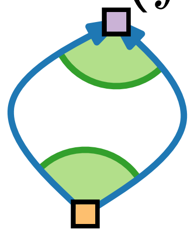
- $L(f) = 0$  $\Rightarrow S(f) = 2$ ✓

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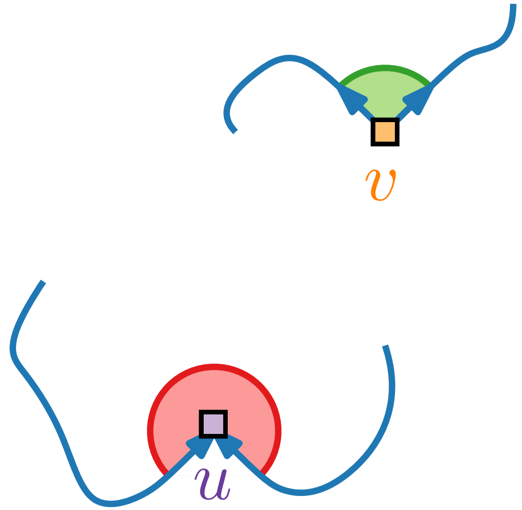

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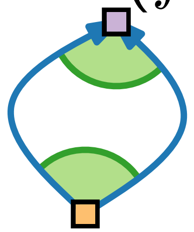
- $L(f) = 0$  $\Rightarrow S(f) = 2$ ✓

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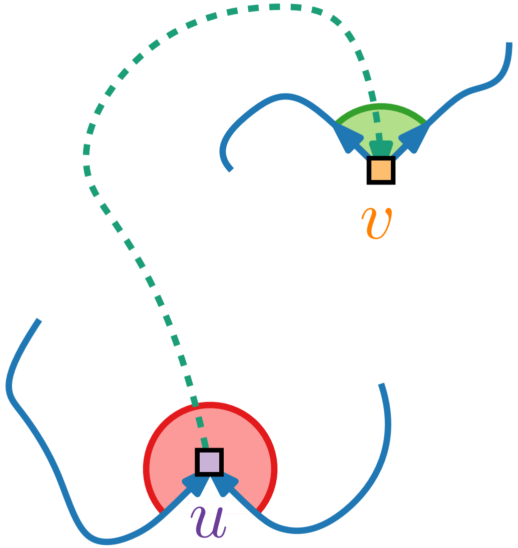

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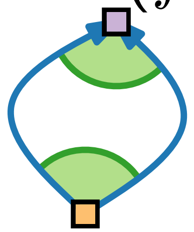
- $L(f) = 0$  $\Rightarrow S(f) = 2$ ✓

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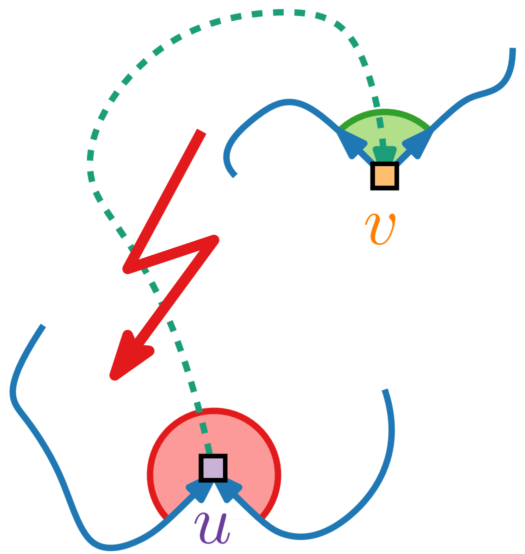

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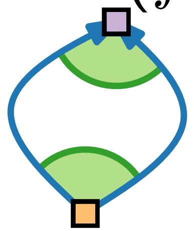
- $L(f) = 0$  $\Rightarrow S(f) = 2$ ✓

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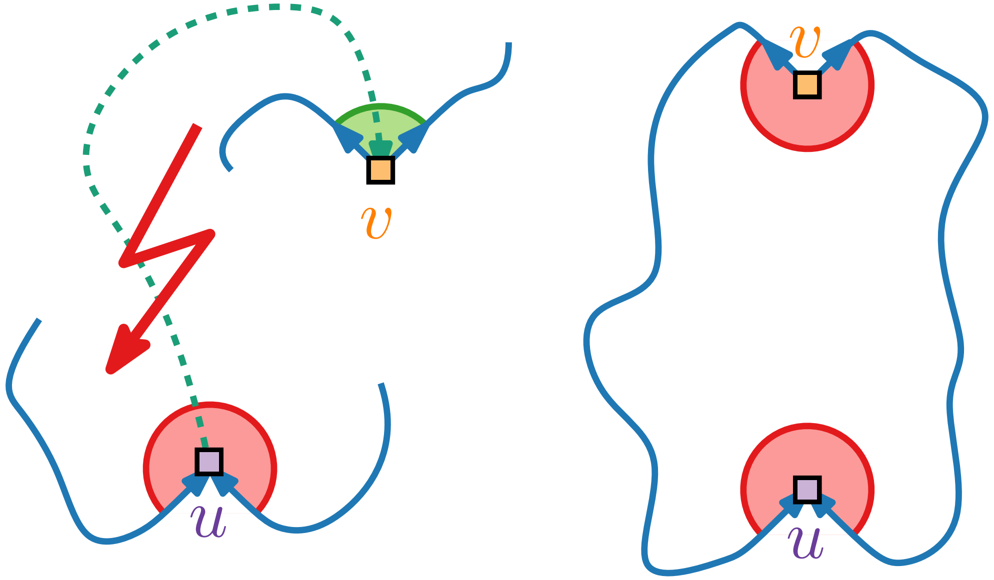

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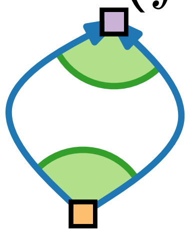

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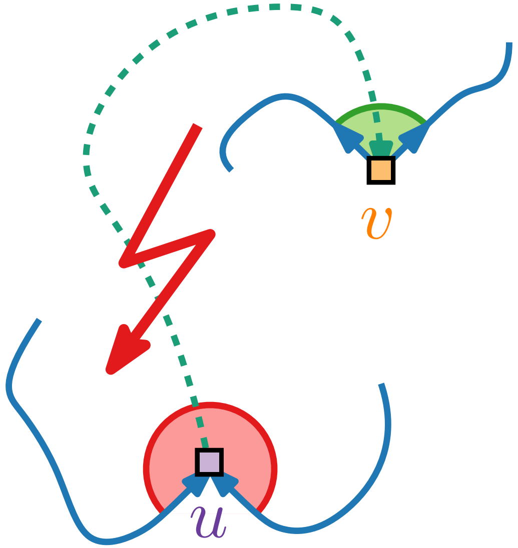

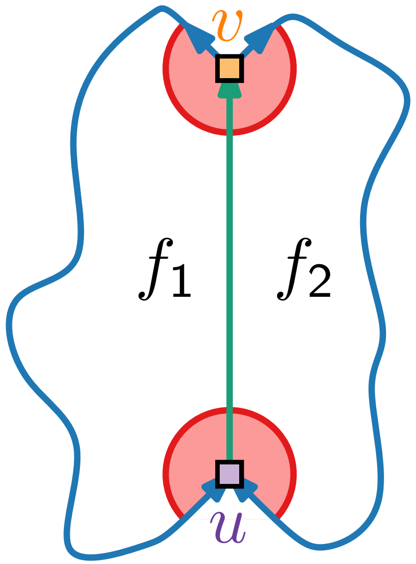

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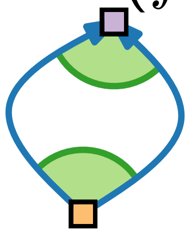
- $L(f) = 0$  $\Rightarrow S(f) = 2$ ✓

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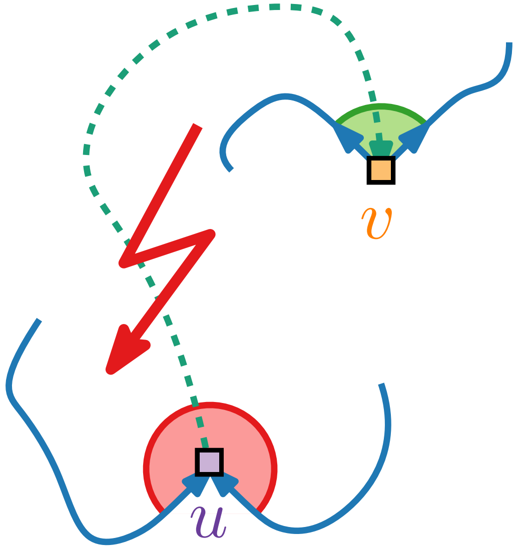

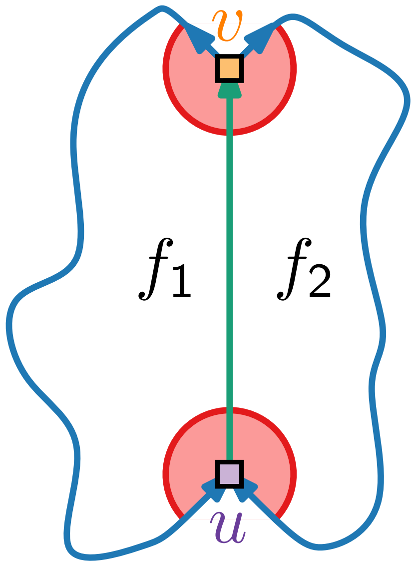

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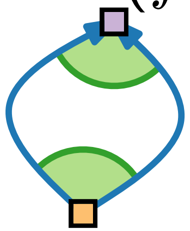

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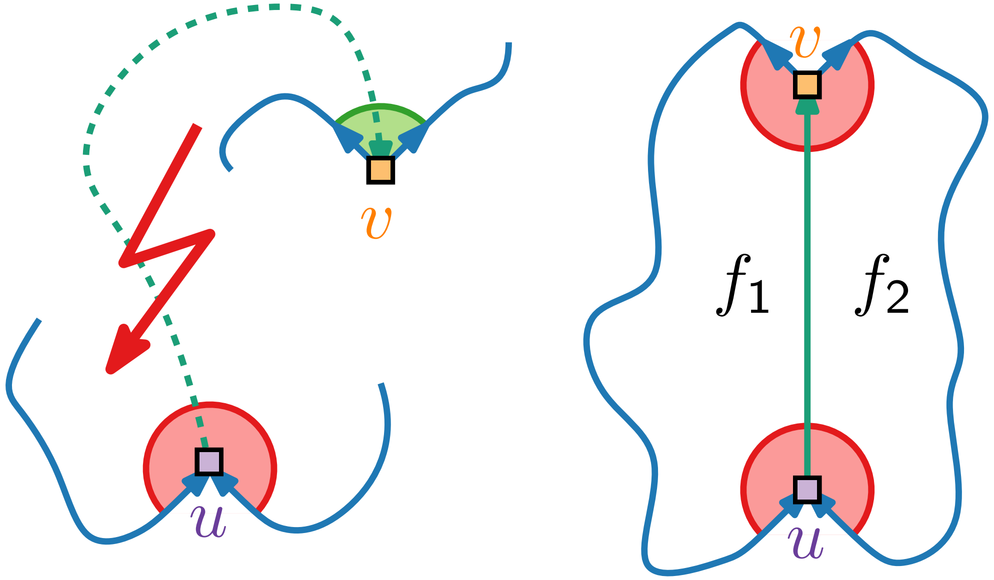

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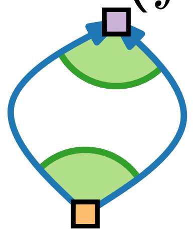

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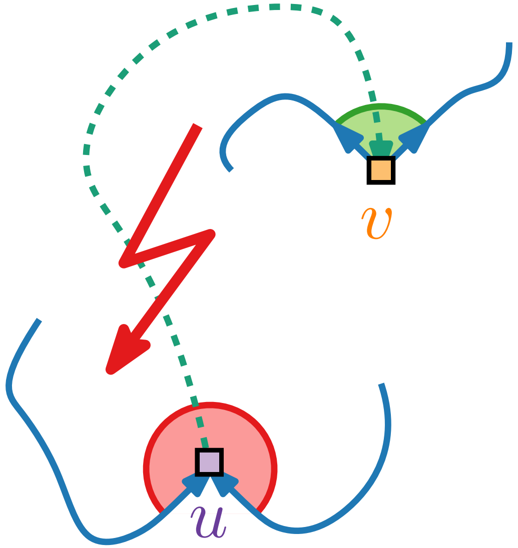

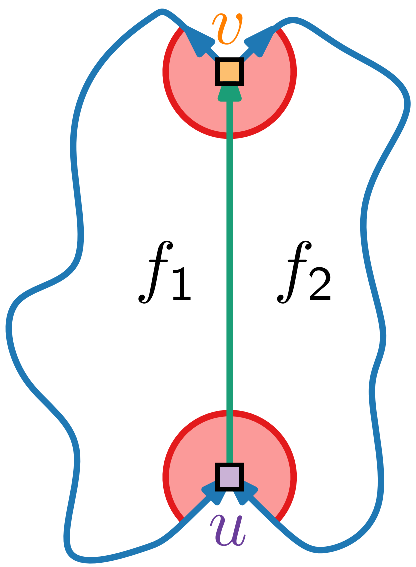

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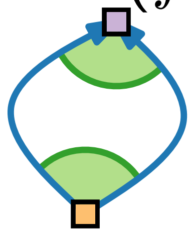

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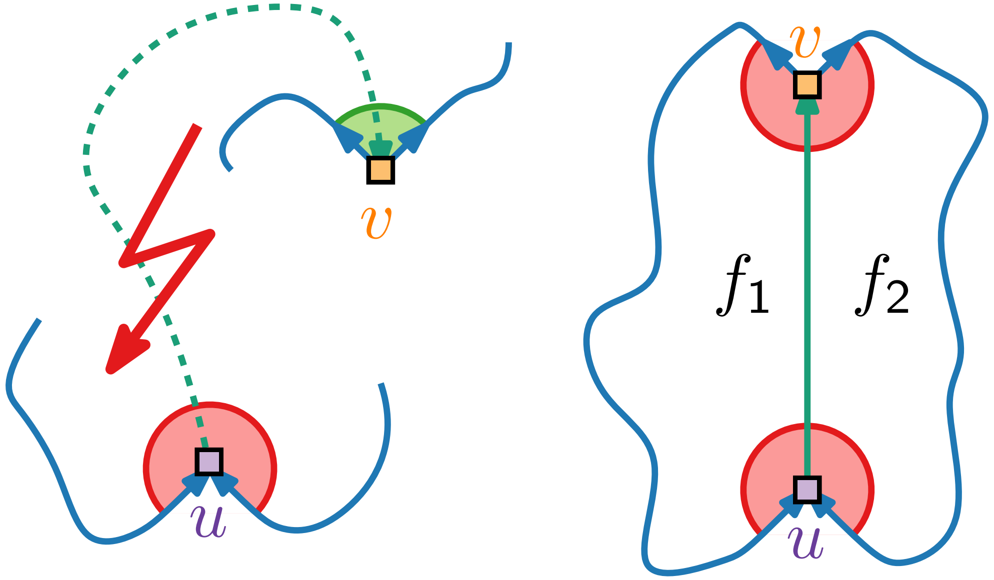

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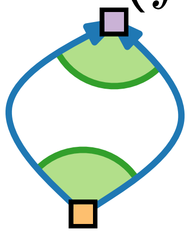

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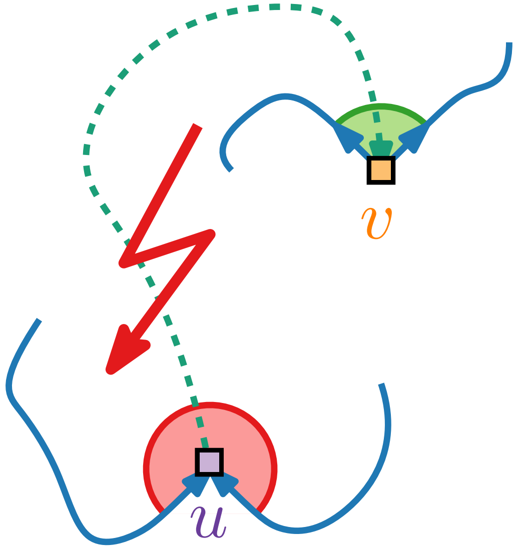

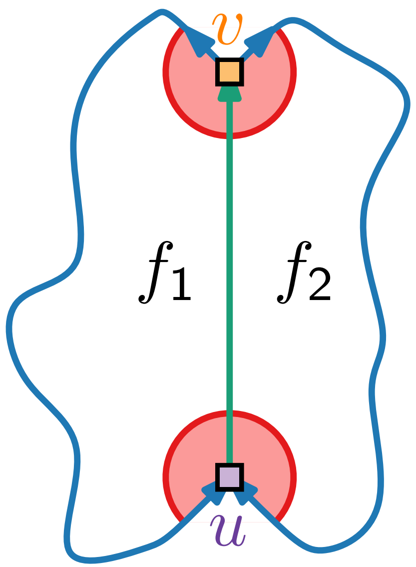

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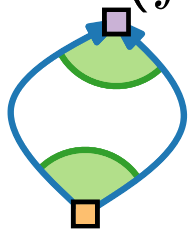
- $L(f) = 0$  $\Rightarrow S(f) = 2$ ✓

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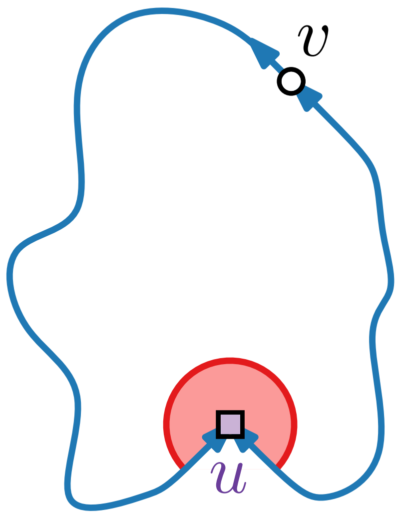

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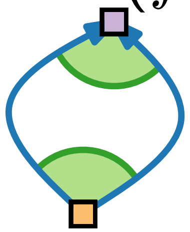
- $L(f) = 0$  $\Rightarrow S(f) = 2$ ✓

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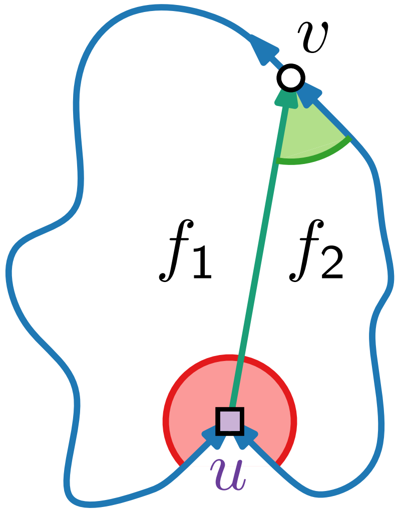

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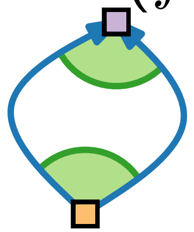
- $L(f) = 0$  $\Rightarrow S(f) = 2$ ✓

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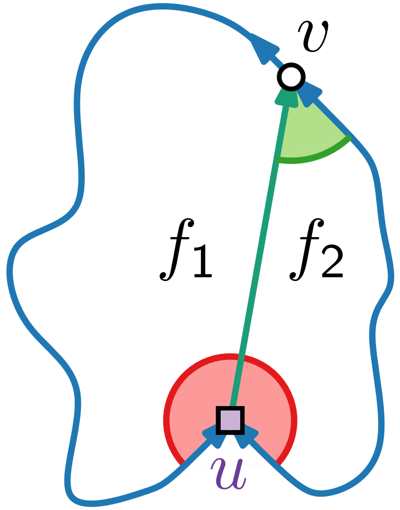

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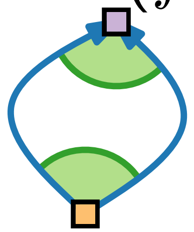

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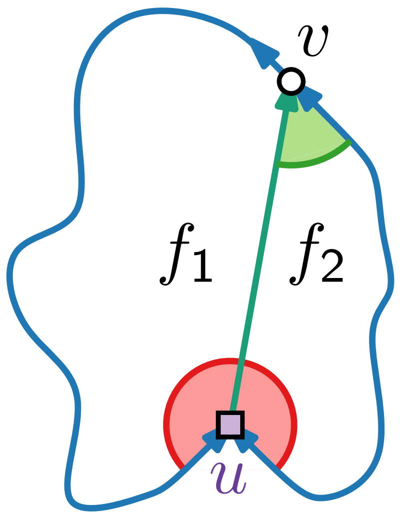

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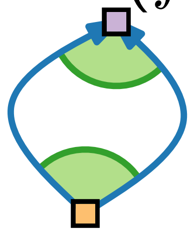
- $L(f) = 0$  $\Rightarrow S(f) = 2$ ✓

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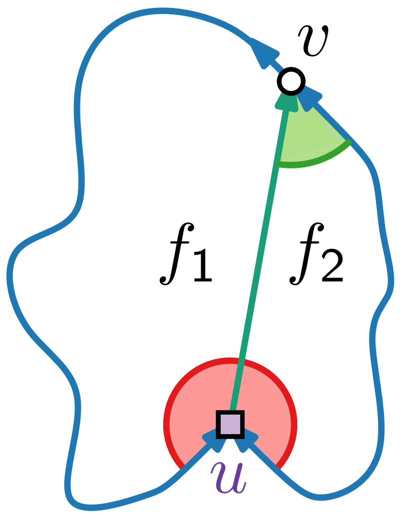

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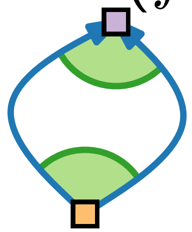
- $L(f) = 0$  $\Rightarrow S(f) = 2$ ✓

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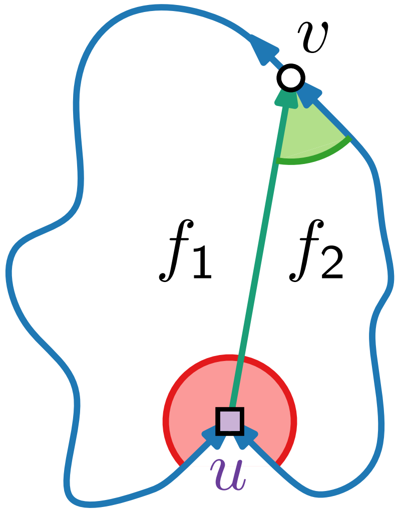

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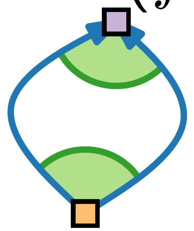
- $L(f) = 0$  $\Rightarrow S(f) = 2$ ✓

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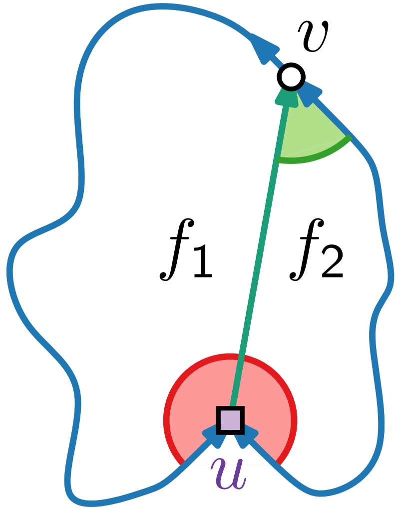

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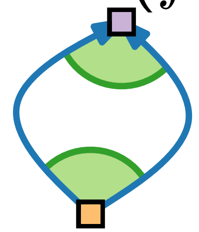
- $L(f) = 0$  $\Rightarrow S(f) = 2$ ✓

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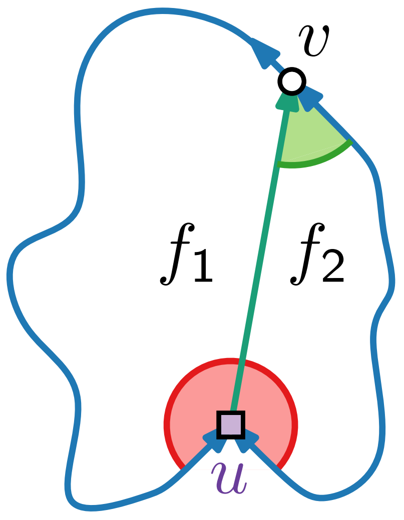

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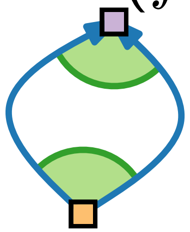

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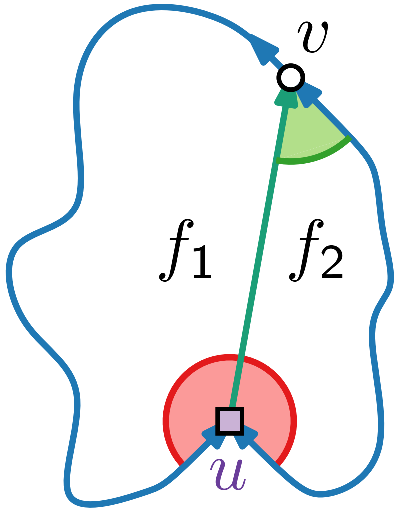

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- Similar argumentation for the outer face f_0

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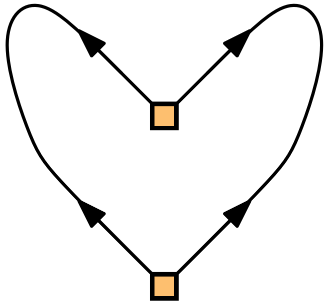

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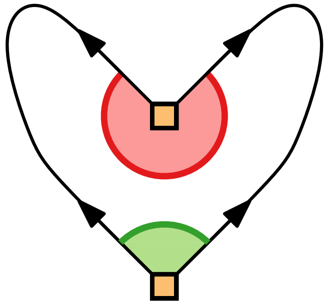

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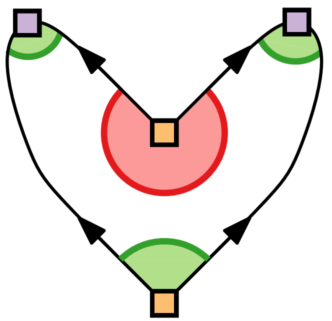

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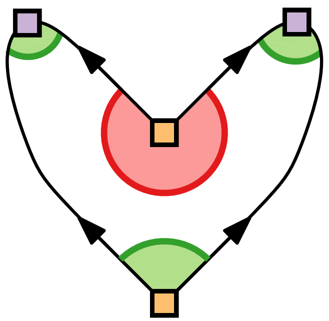

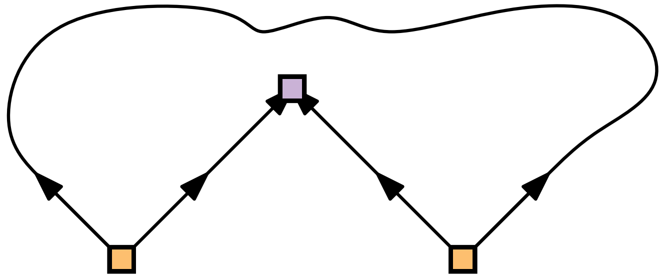

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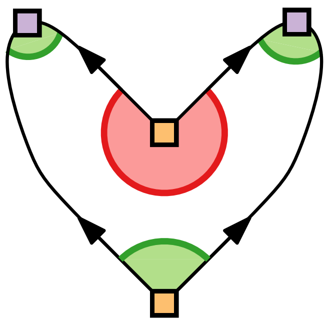

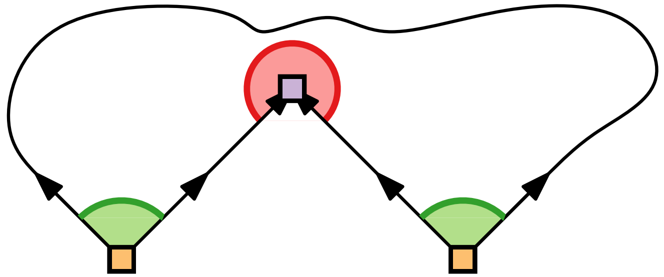

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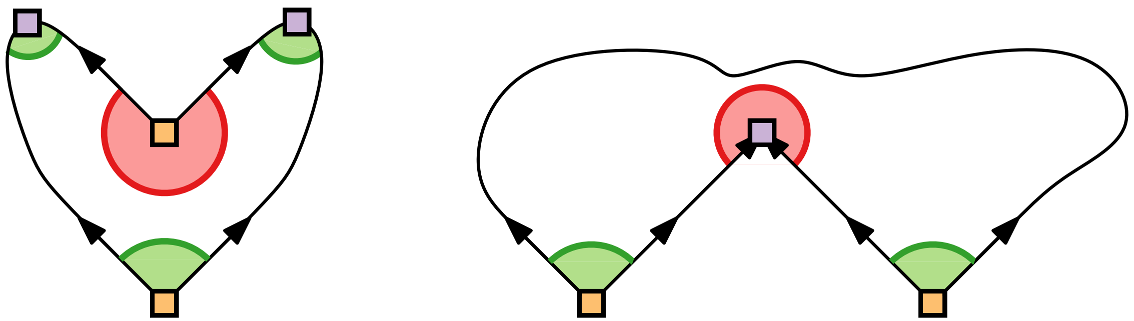

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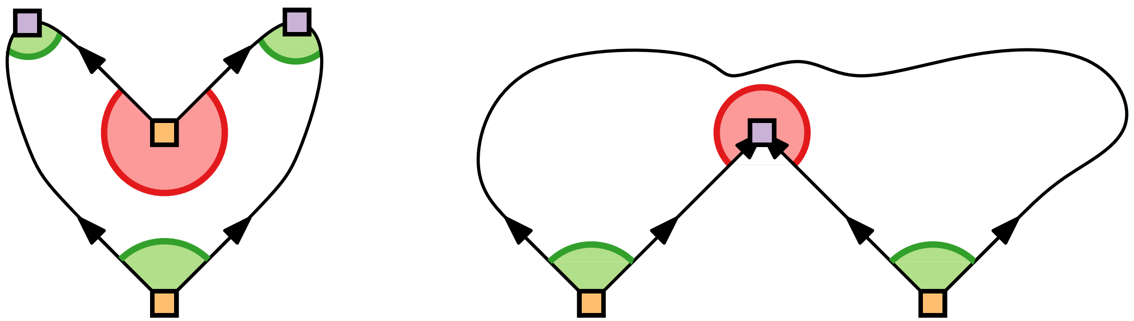

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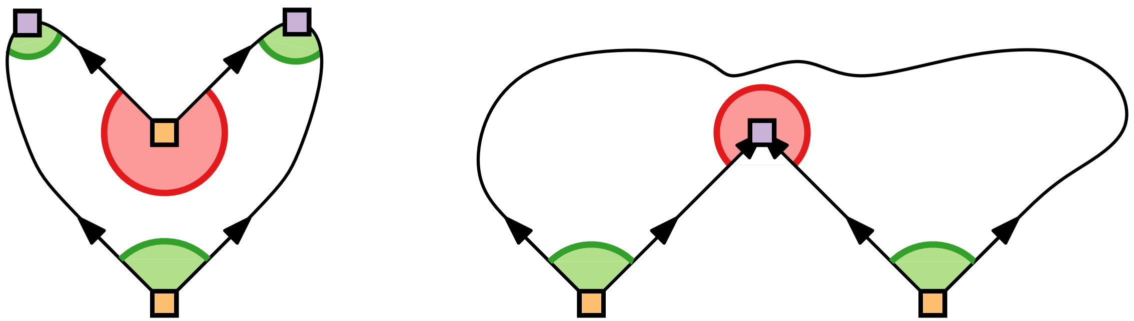

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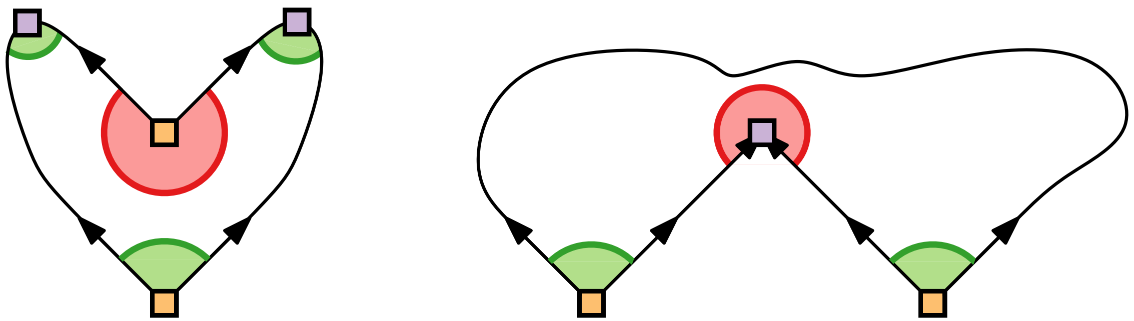

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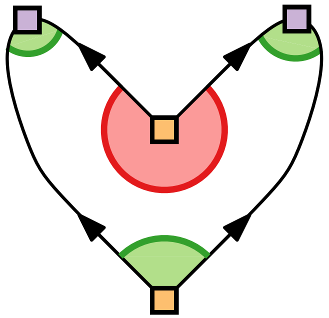
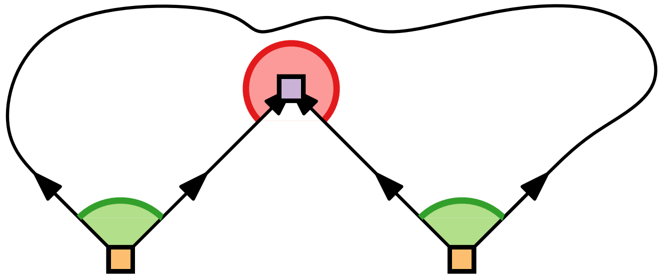

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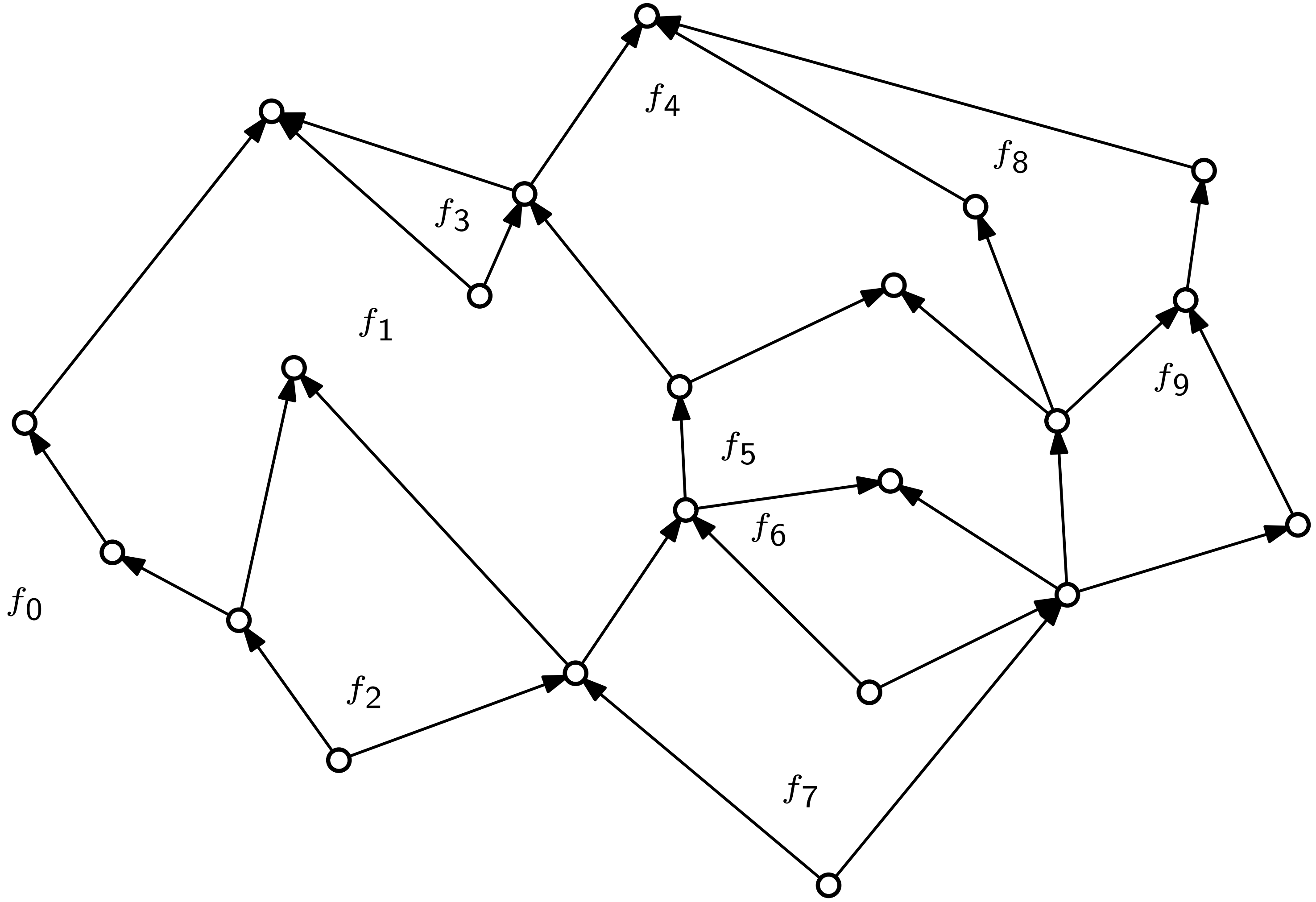

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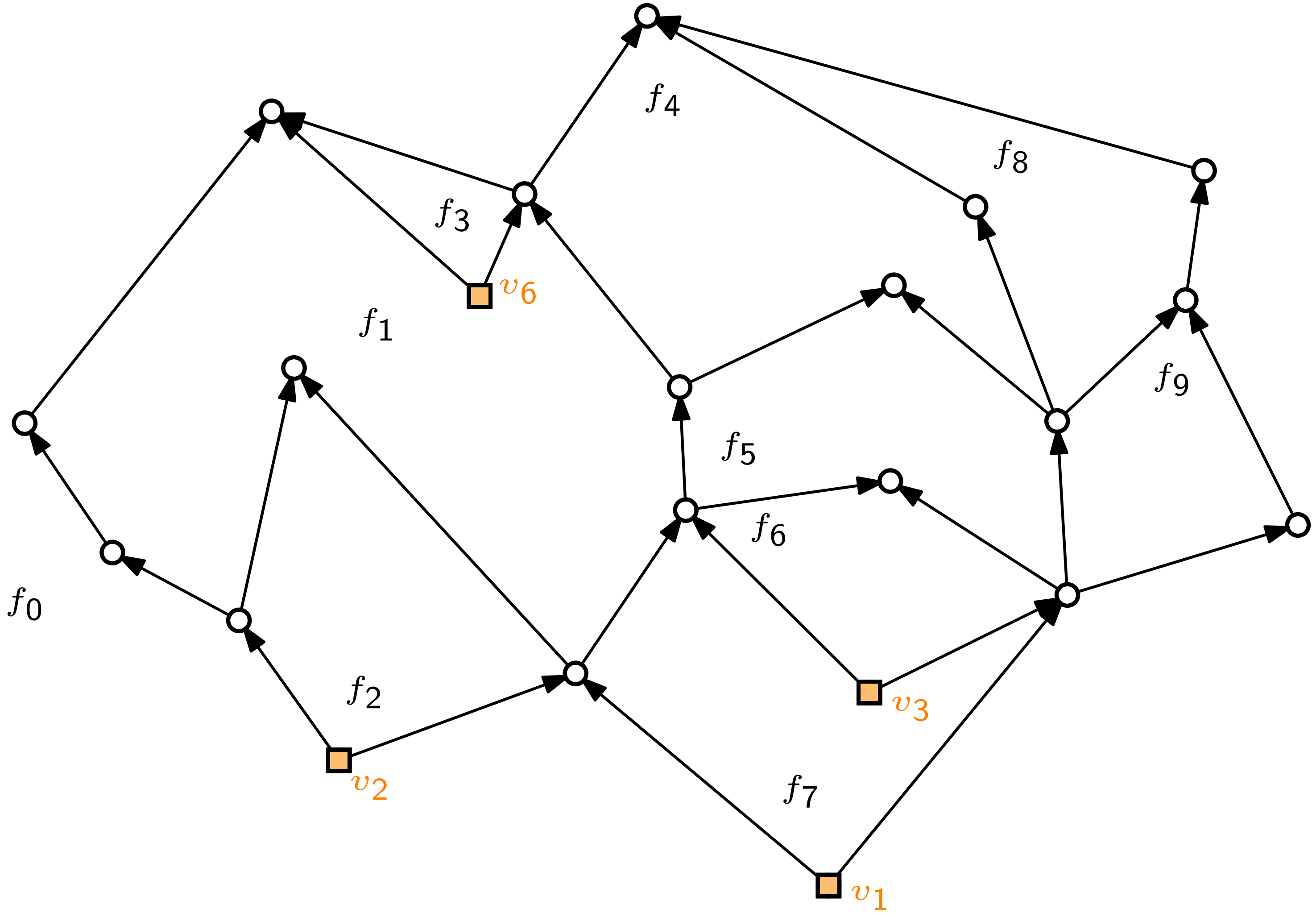

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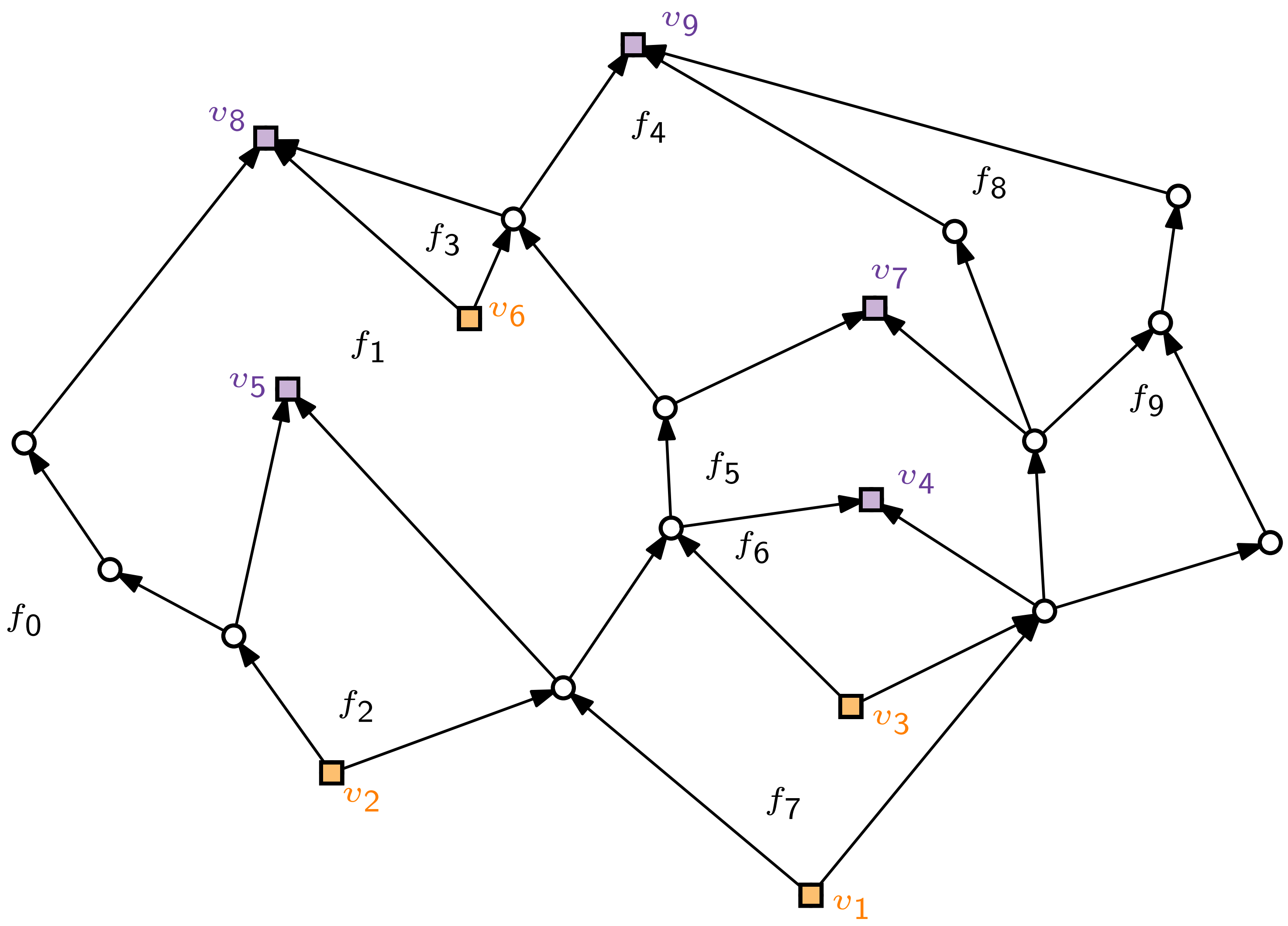

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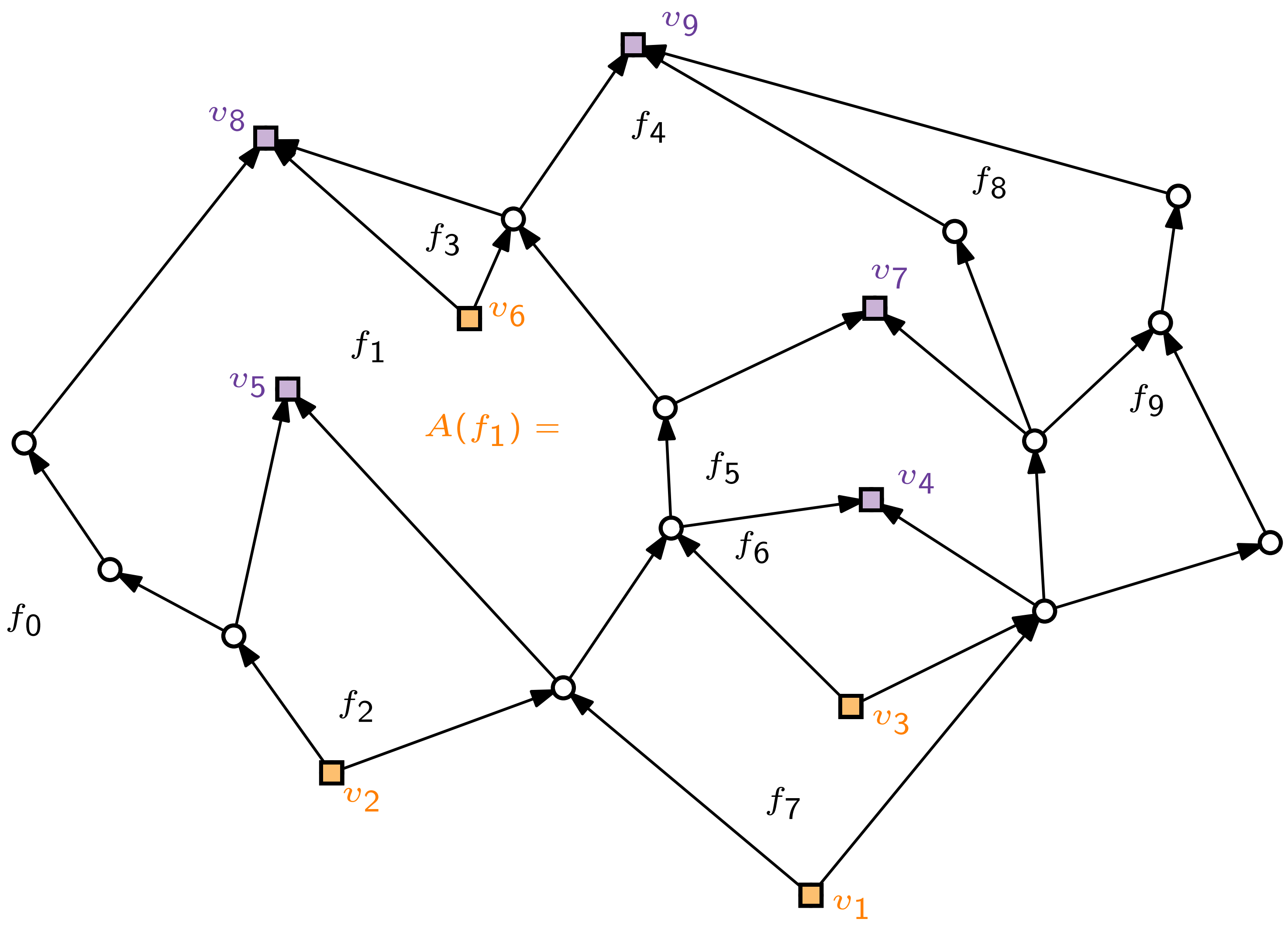

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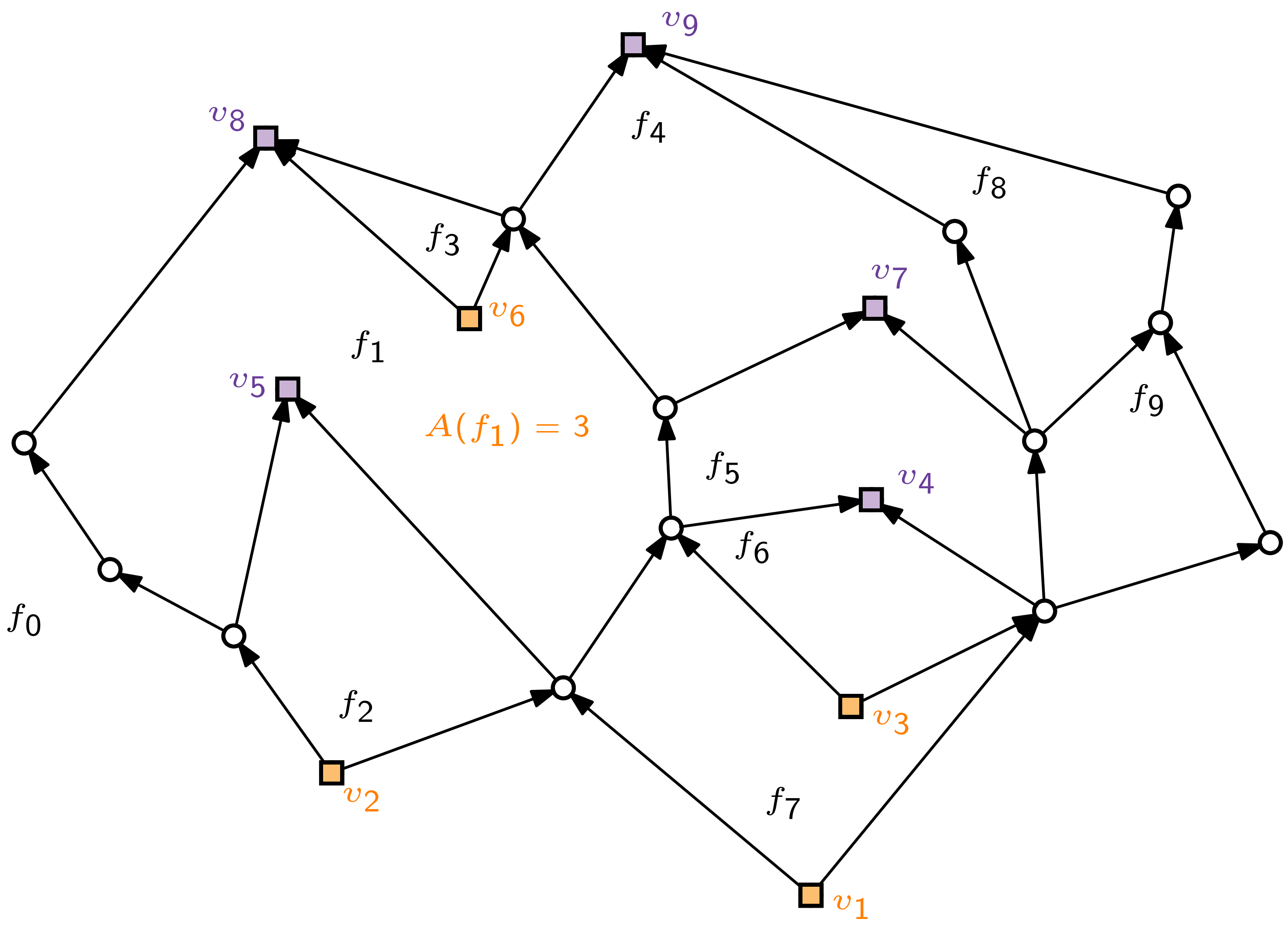

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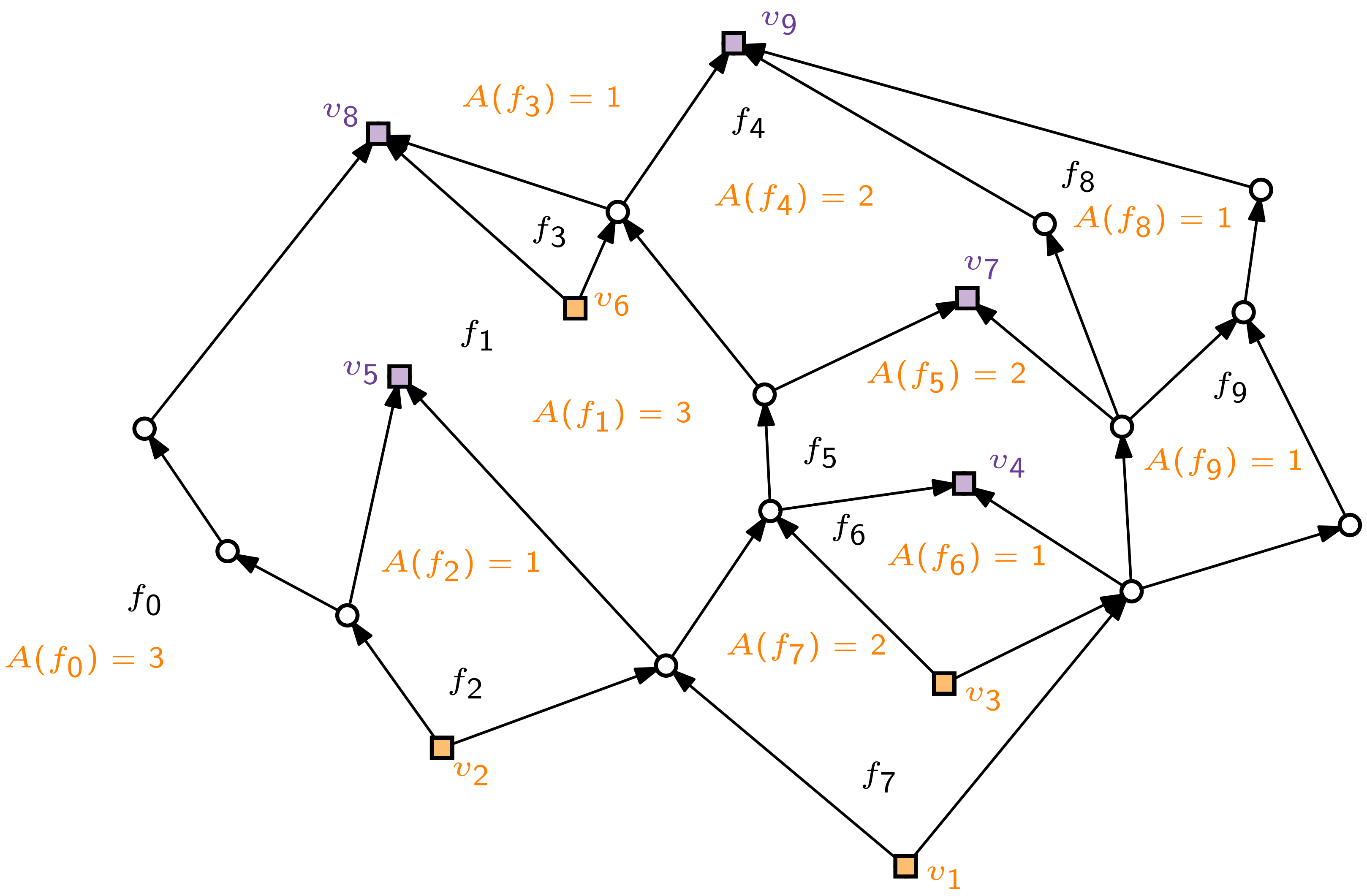

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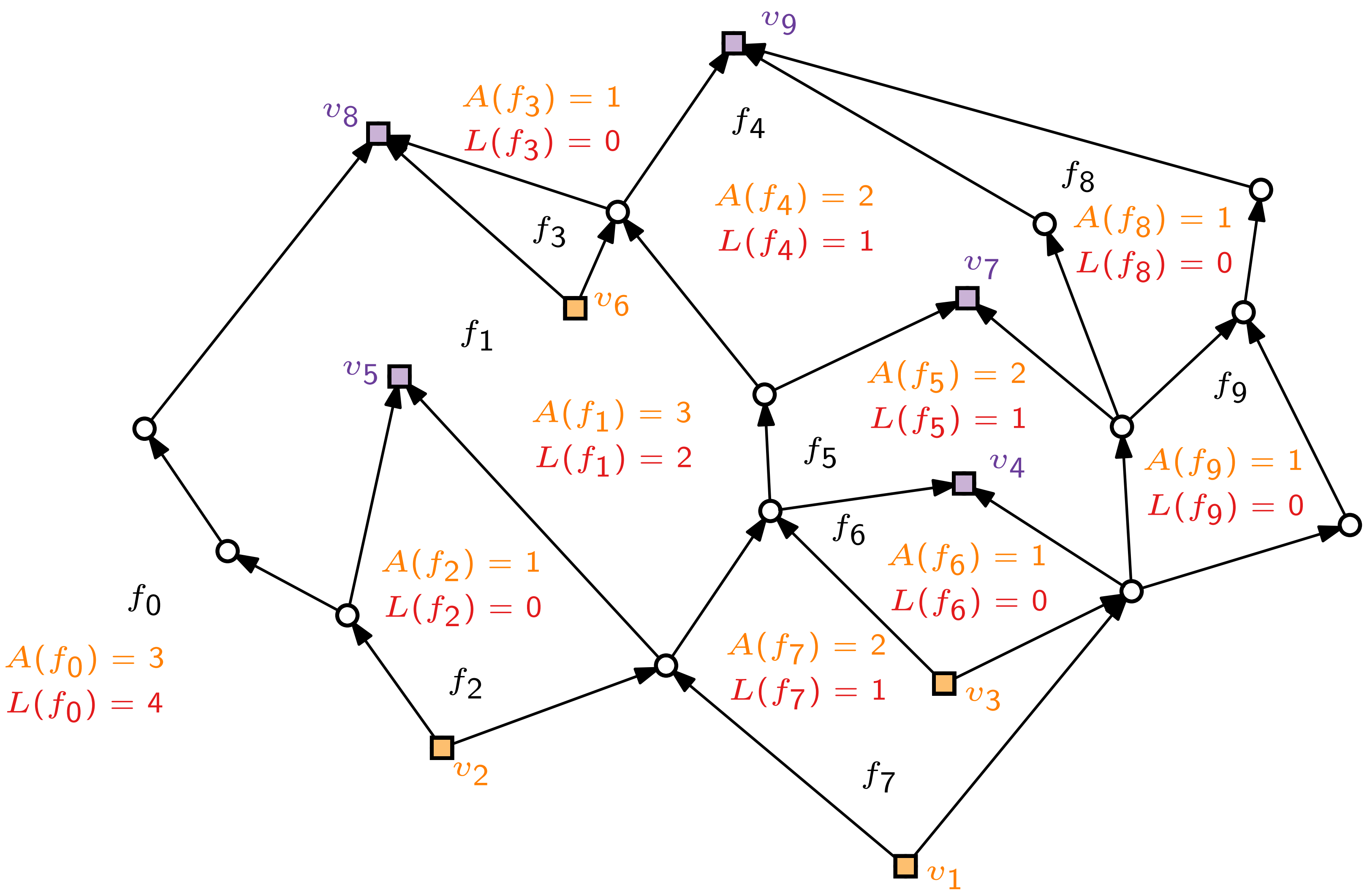

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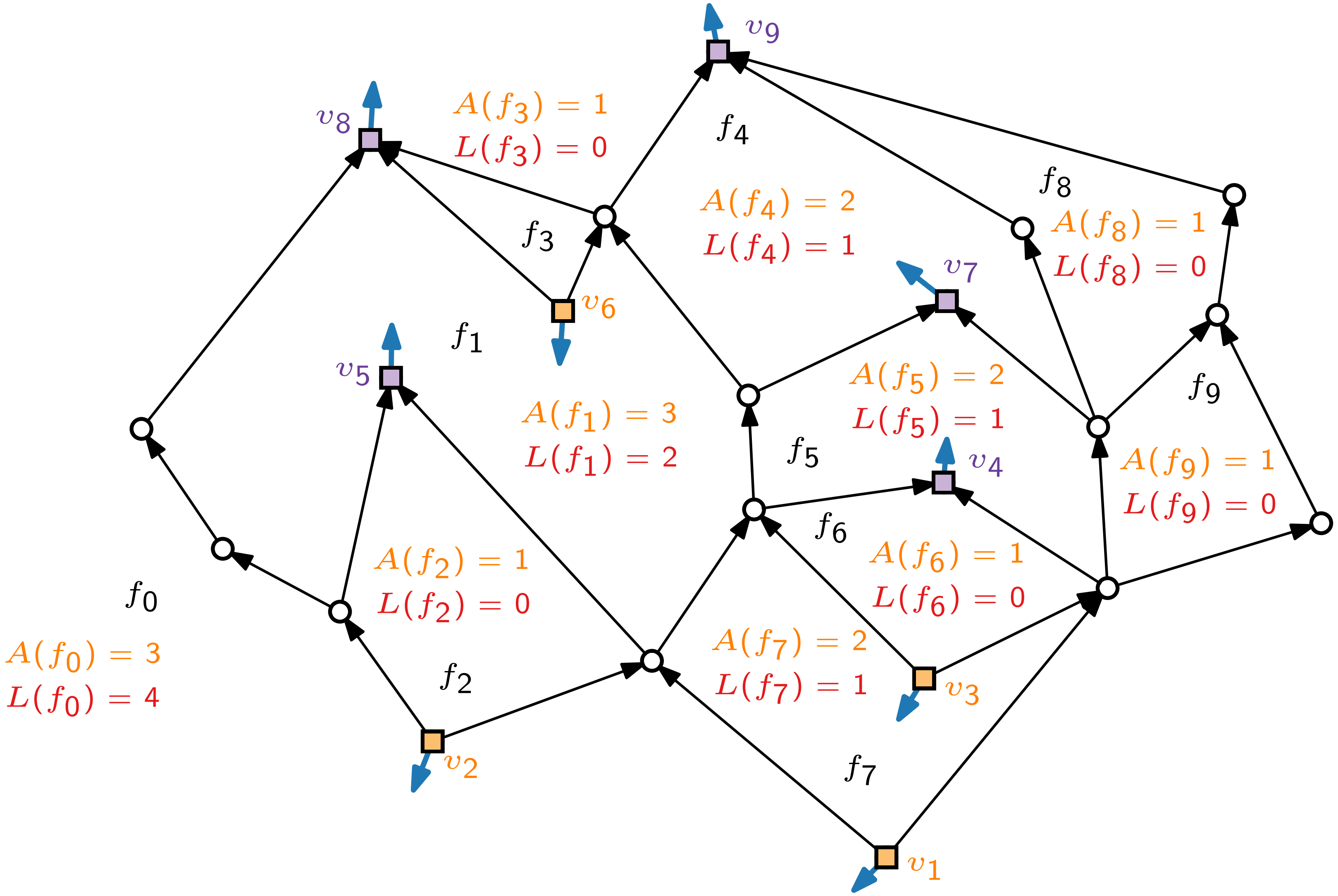

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- Construct planar st-digraph that is supergraph of G.
- Apply equivalence from Theorem 1.

G is upward planar. $\Leftrightarrow$ G is a spanning subgraph of a planar st-digraph.

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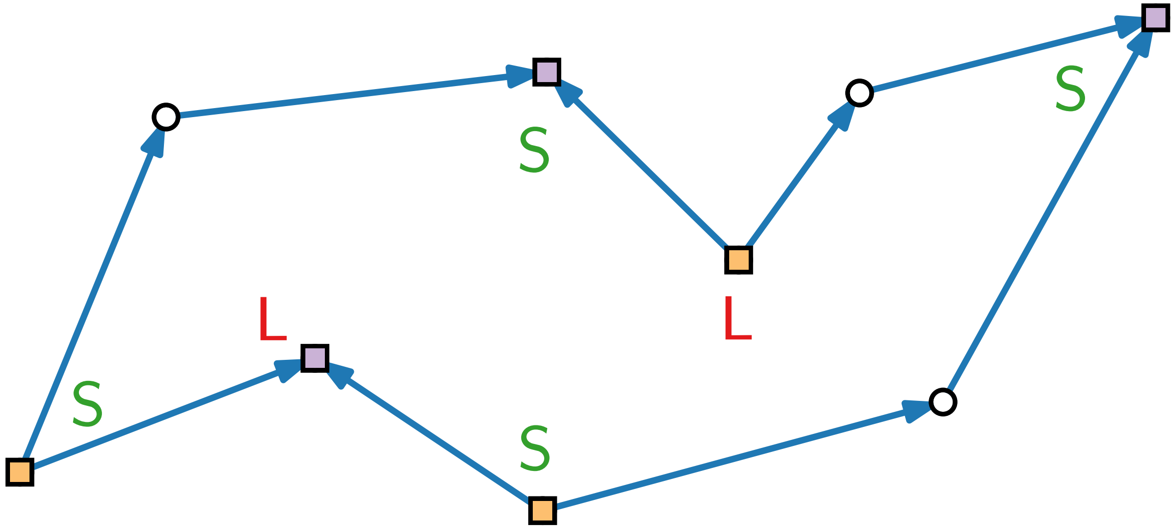

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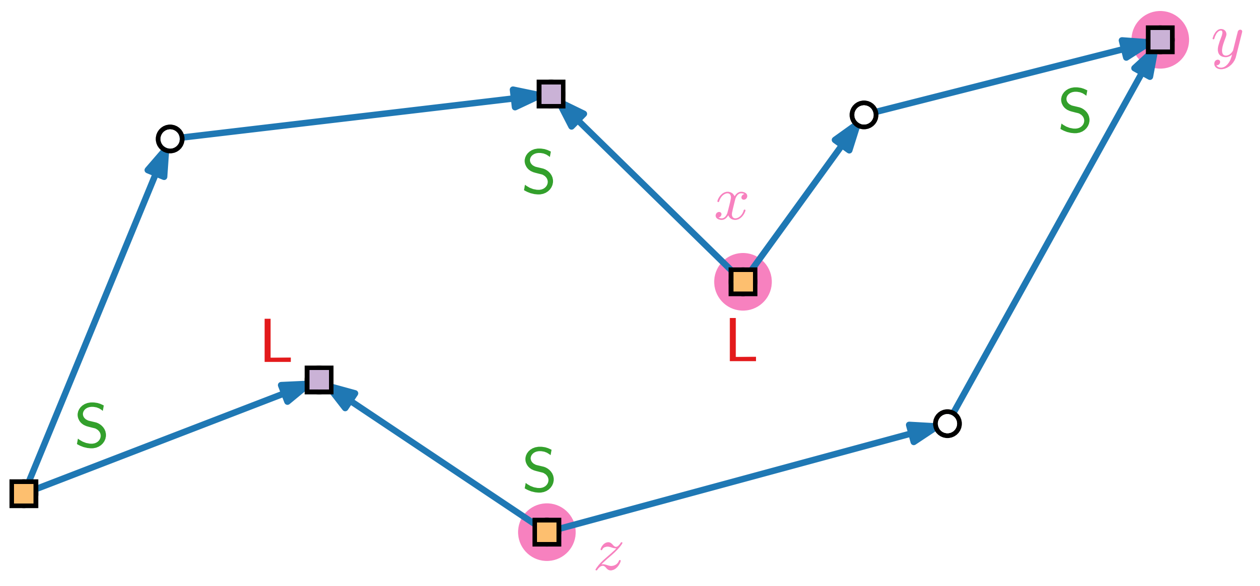

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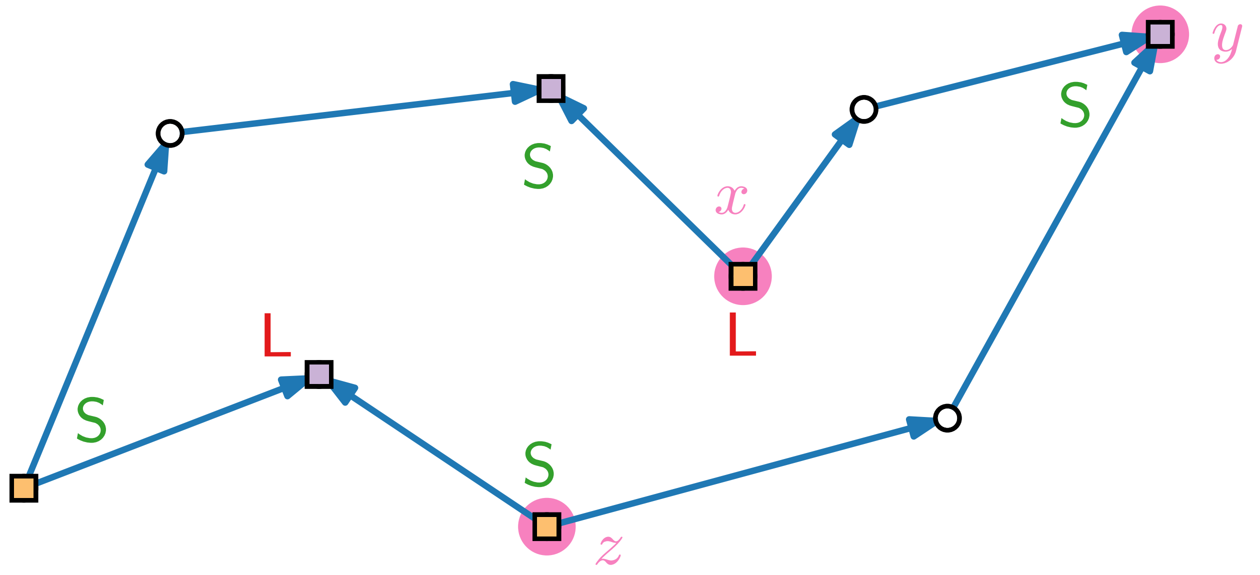

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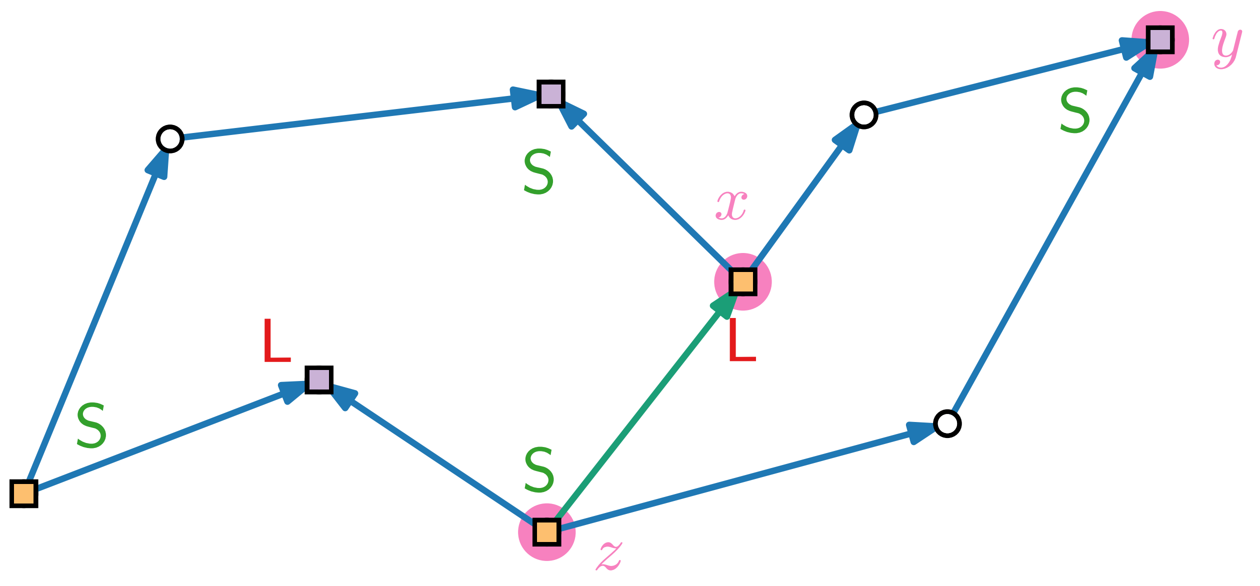

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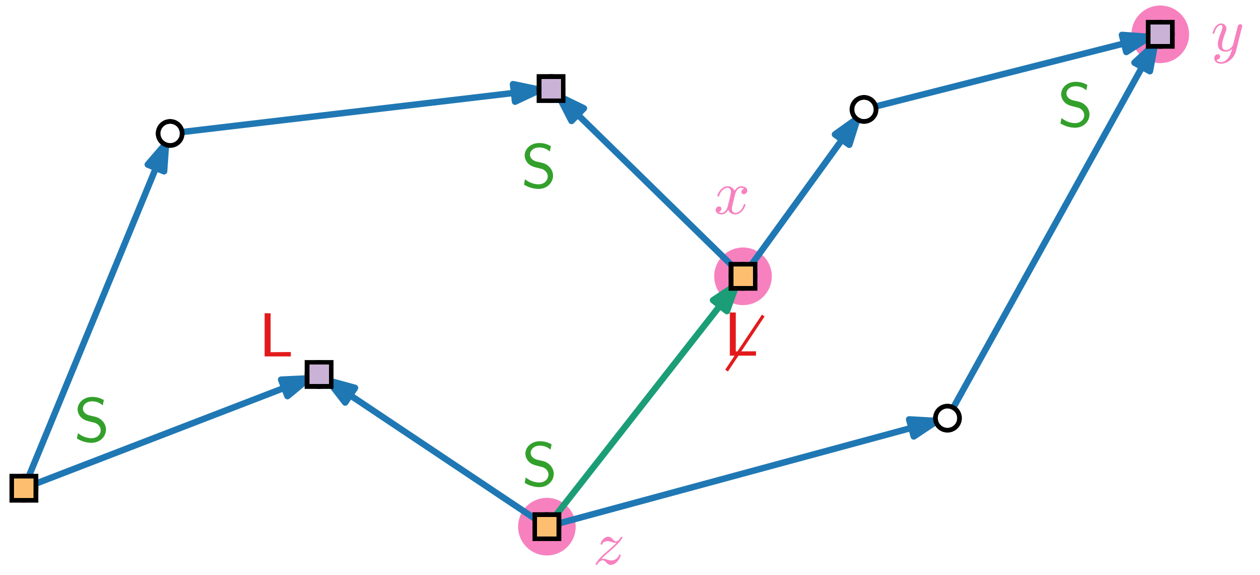

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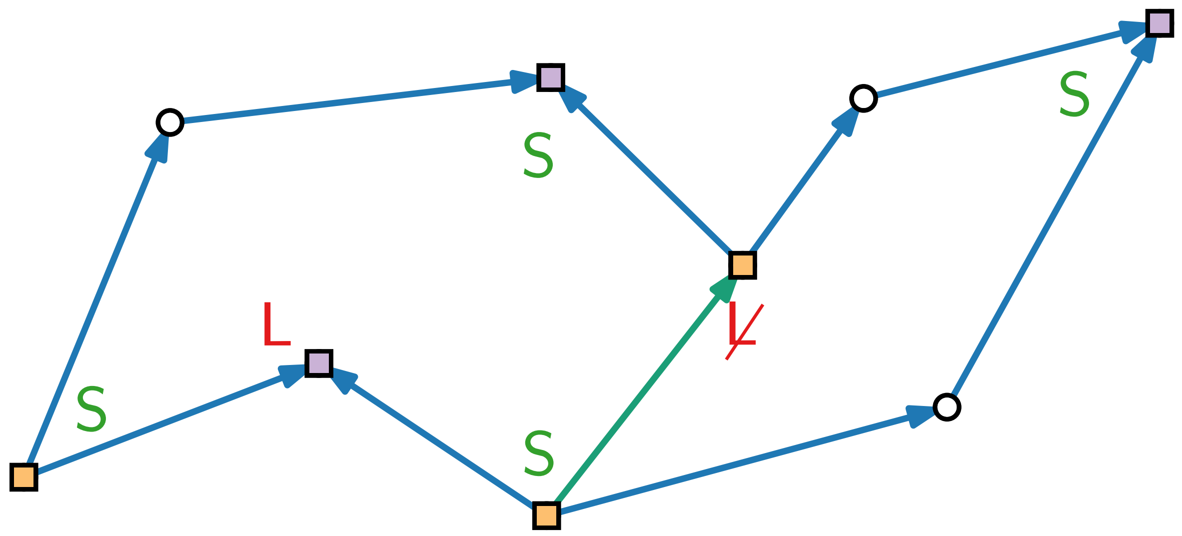

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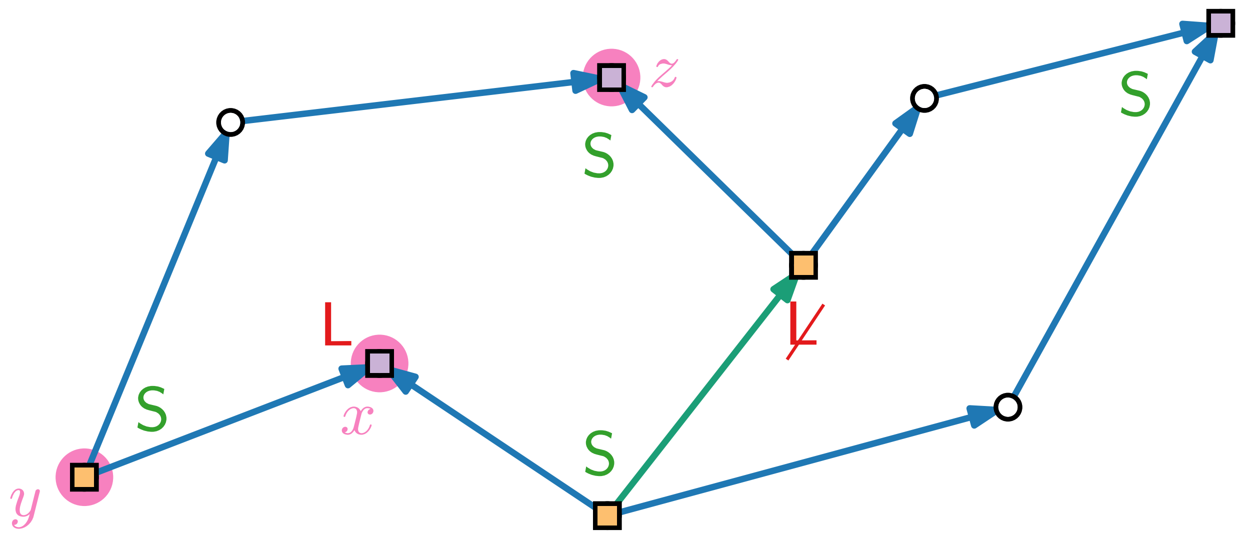

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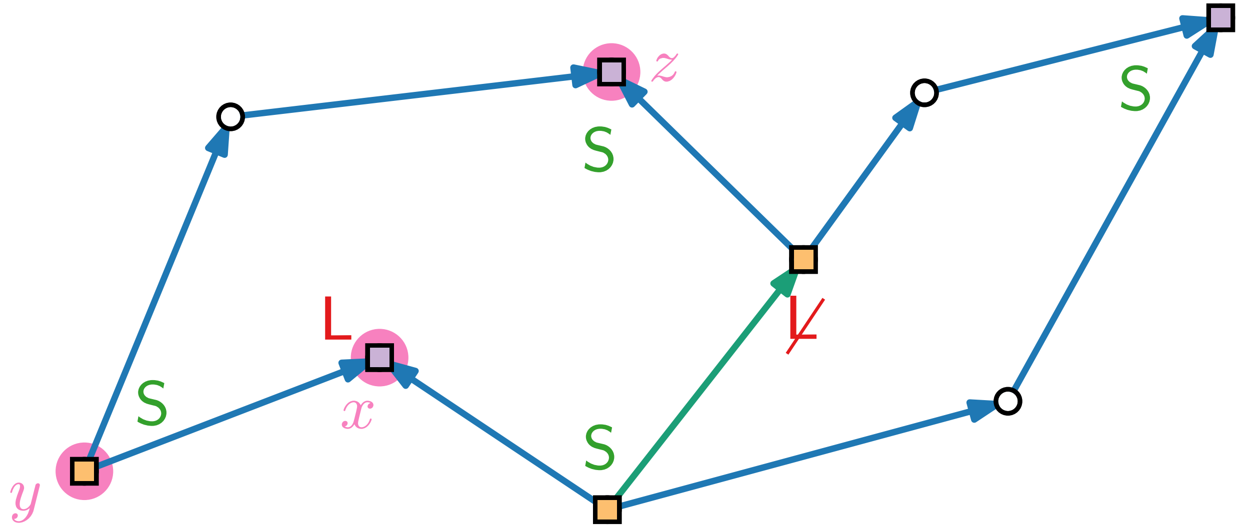

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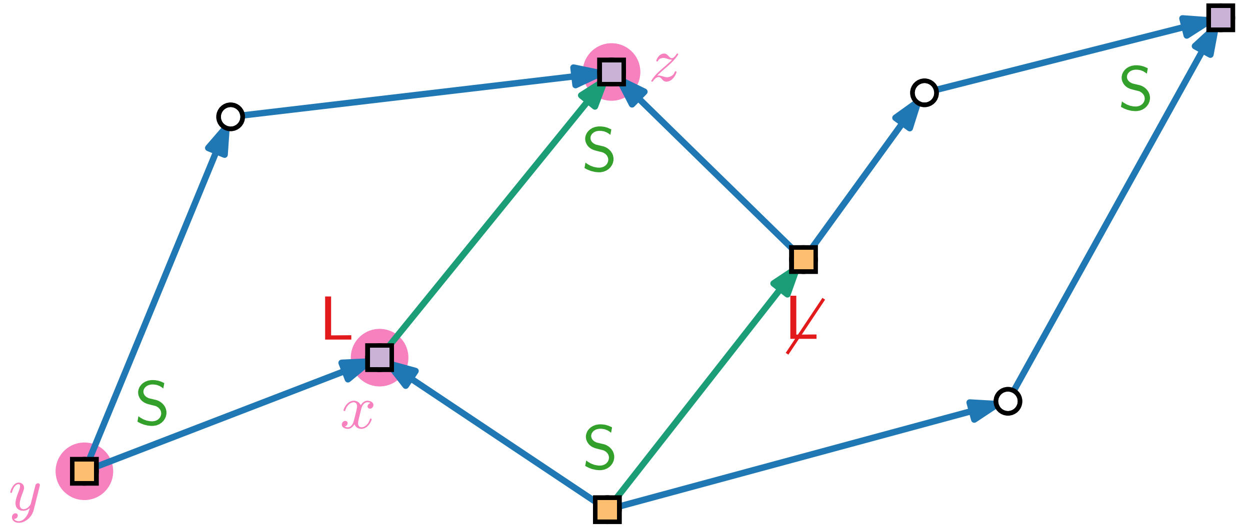

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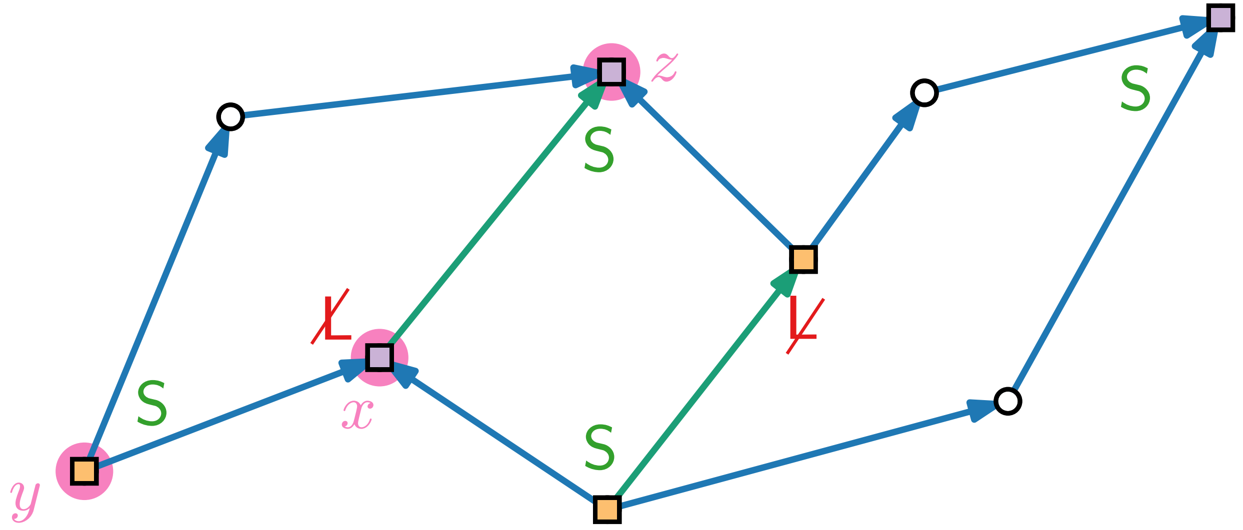

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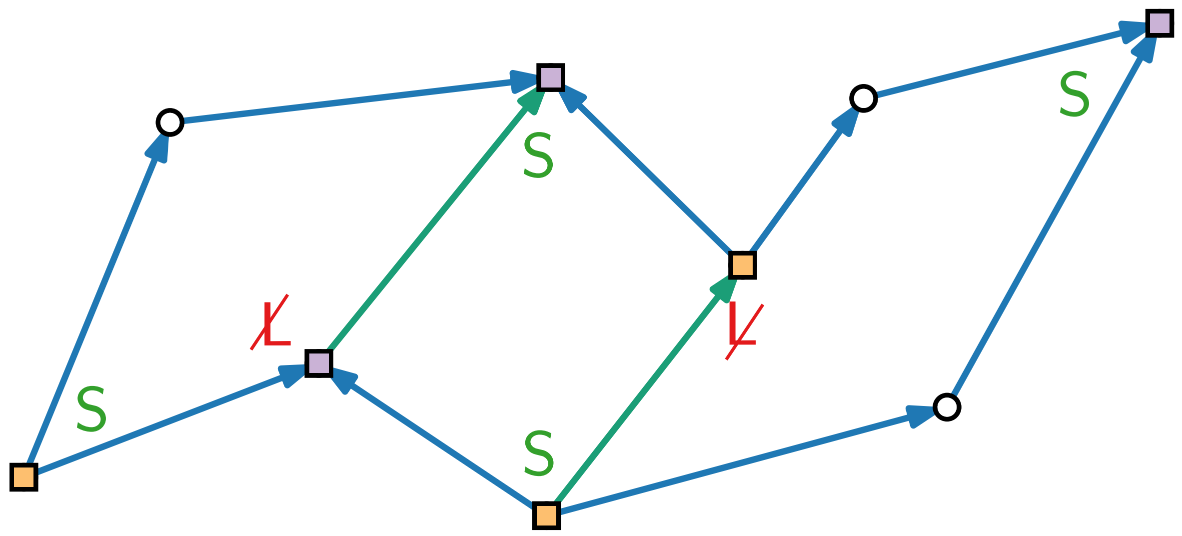

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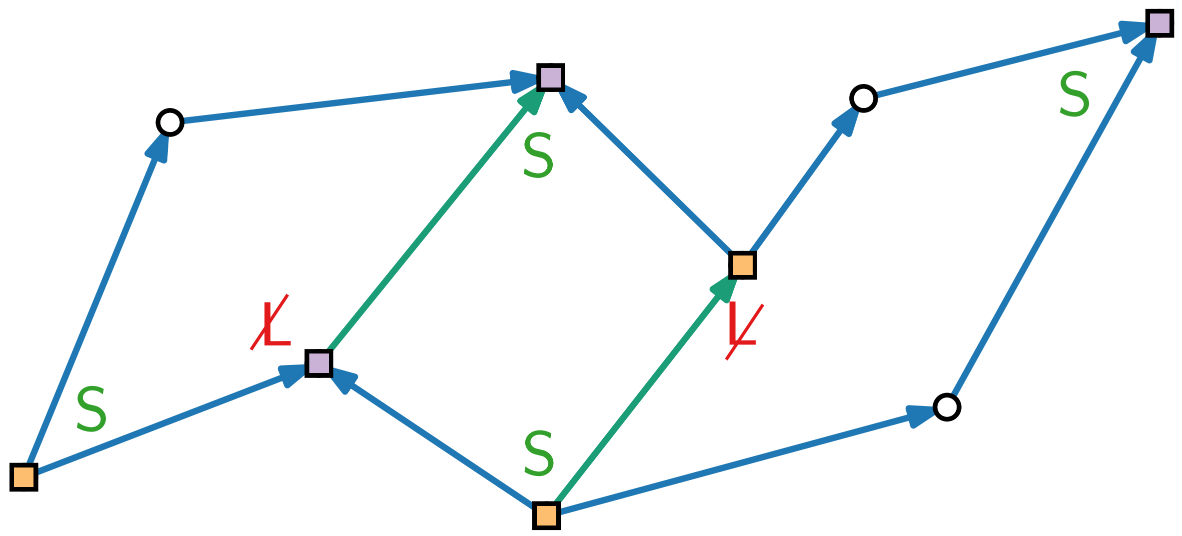

Refinement Algorithm: $\Phi, F, f_0 \to$ st-digraph

Let f be a face.
Consider the clockwise angle sequence σ_f of L / S on local sources and sinks of f.

- Goal: Add edges to break large angles (sources and sinks).
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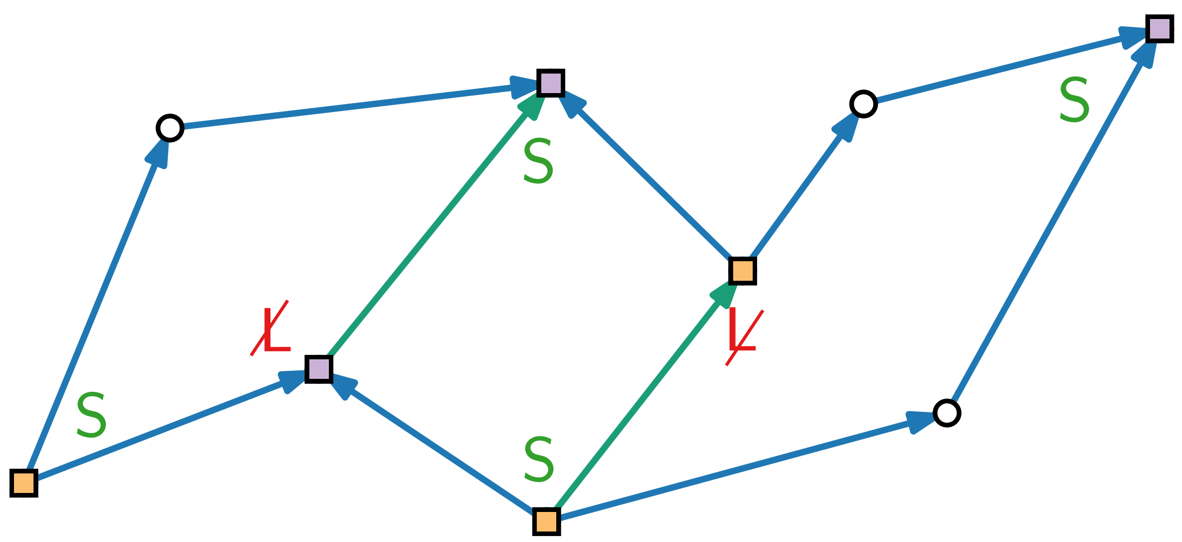

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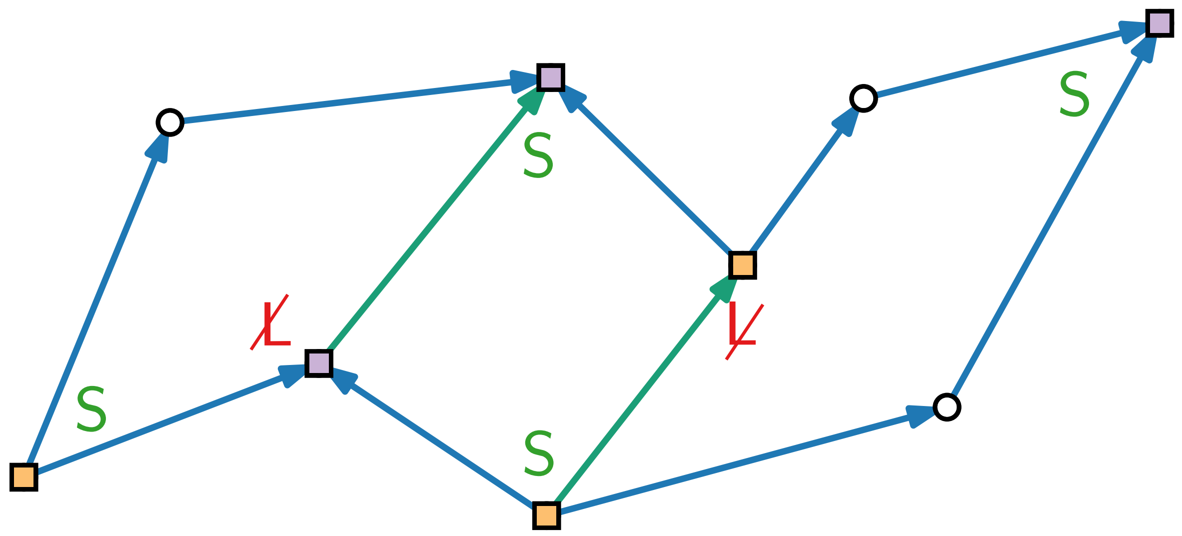

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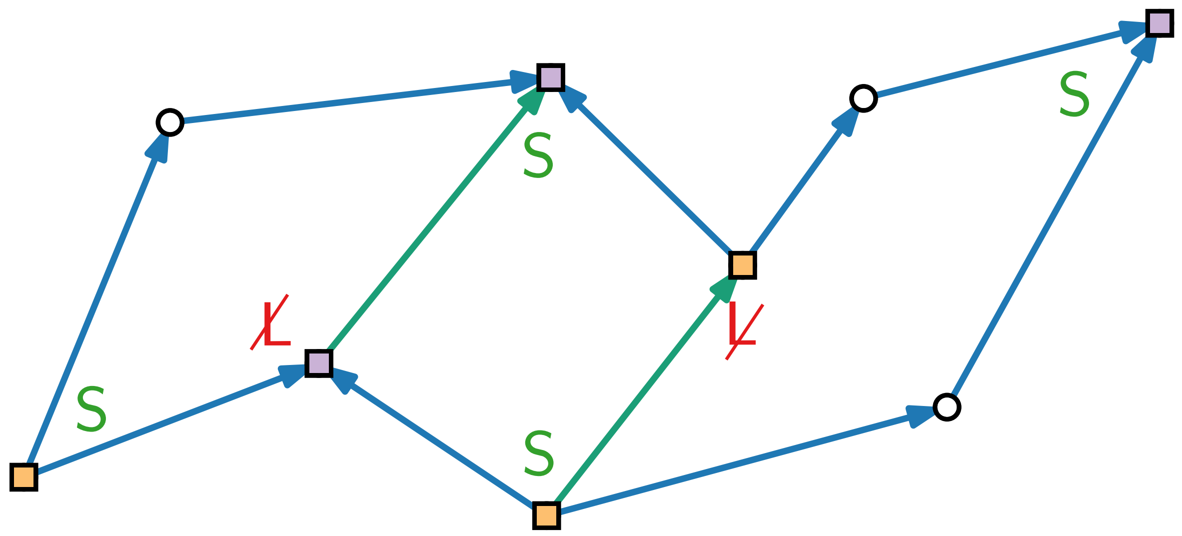

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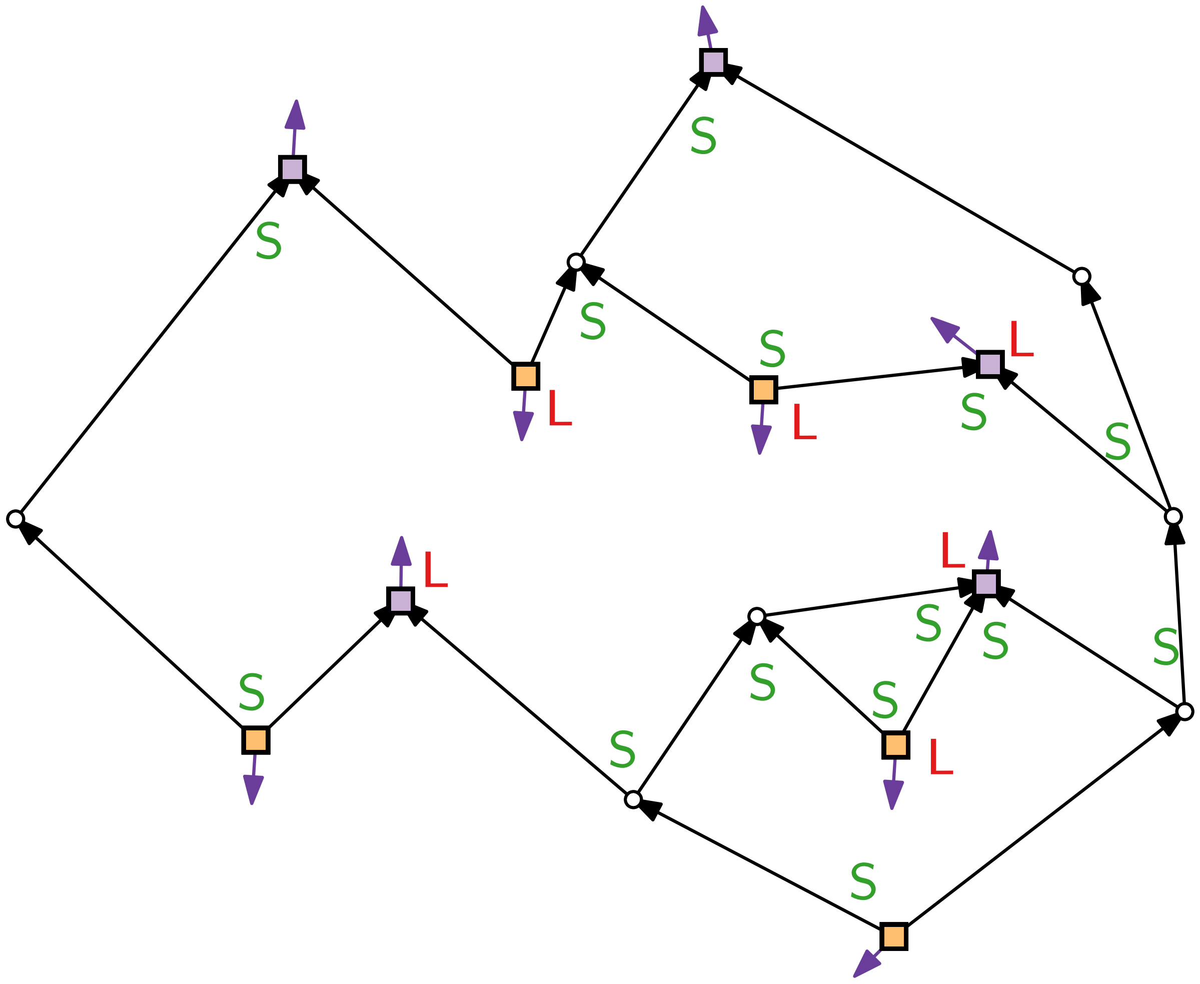

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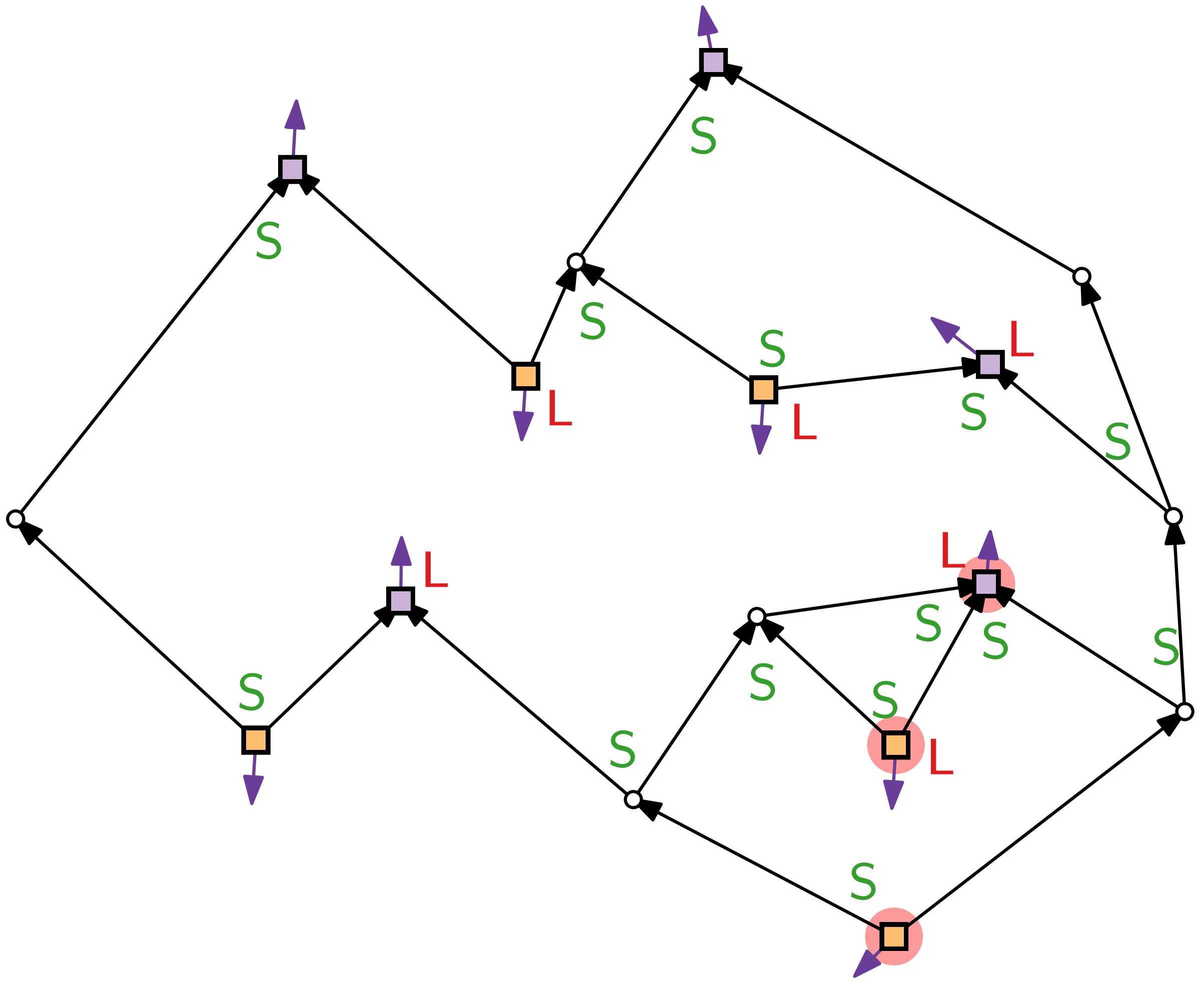

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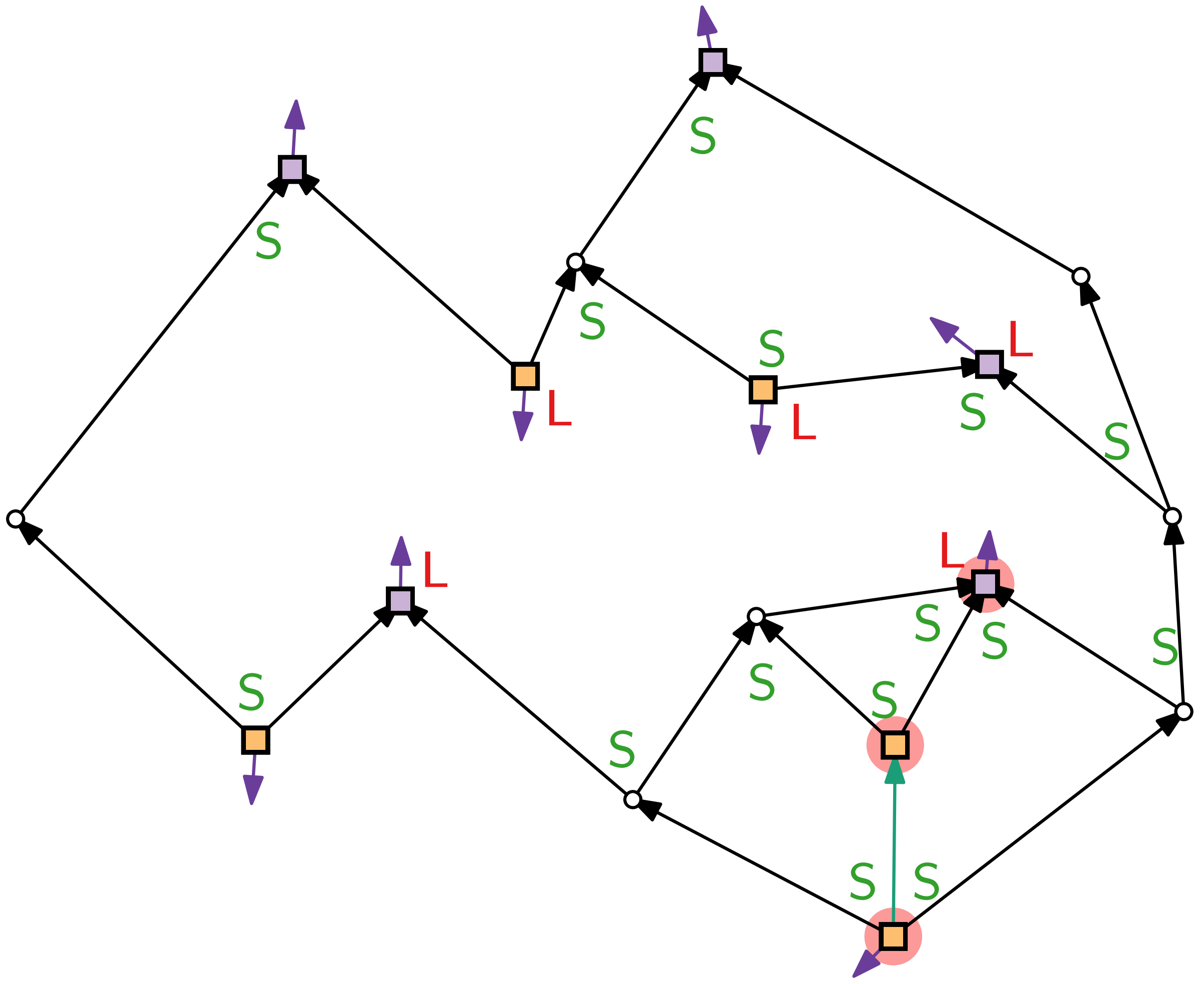

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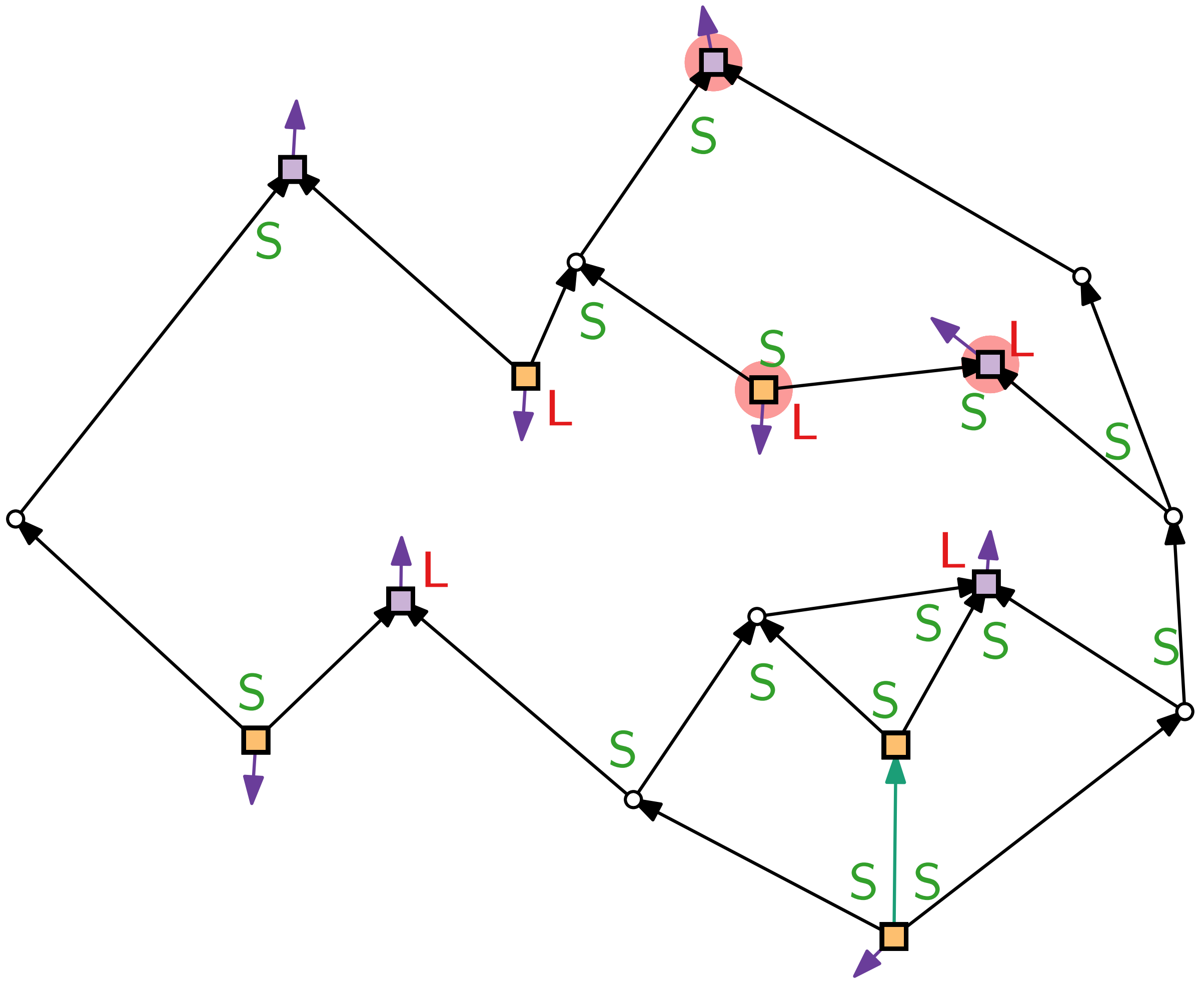

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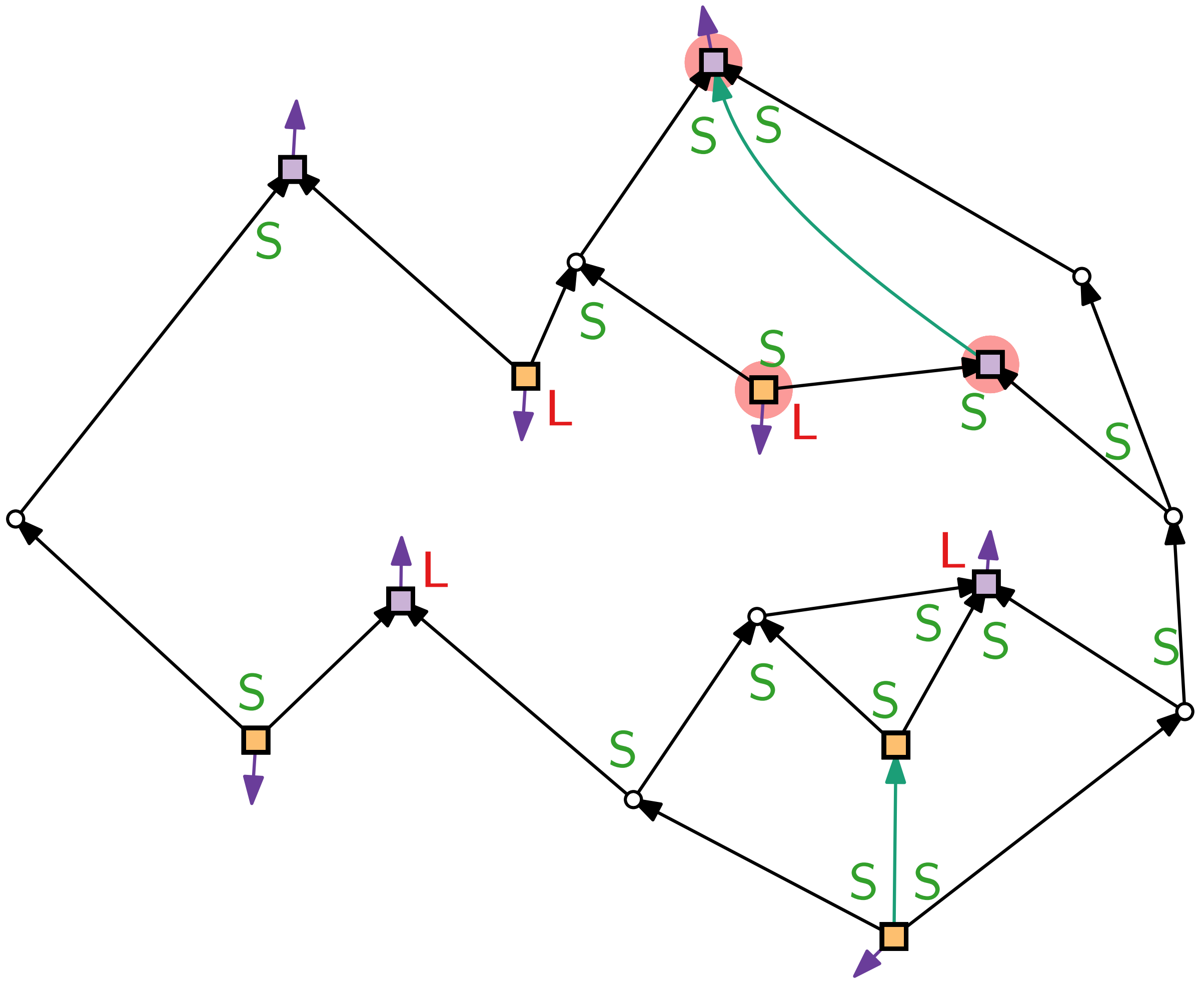

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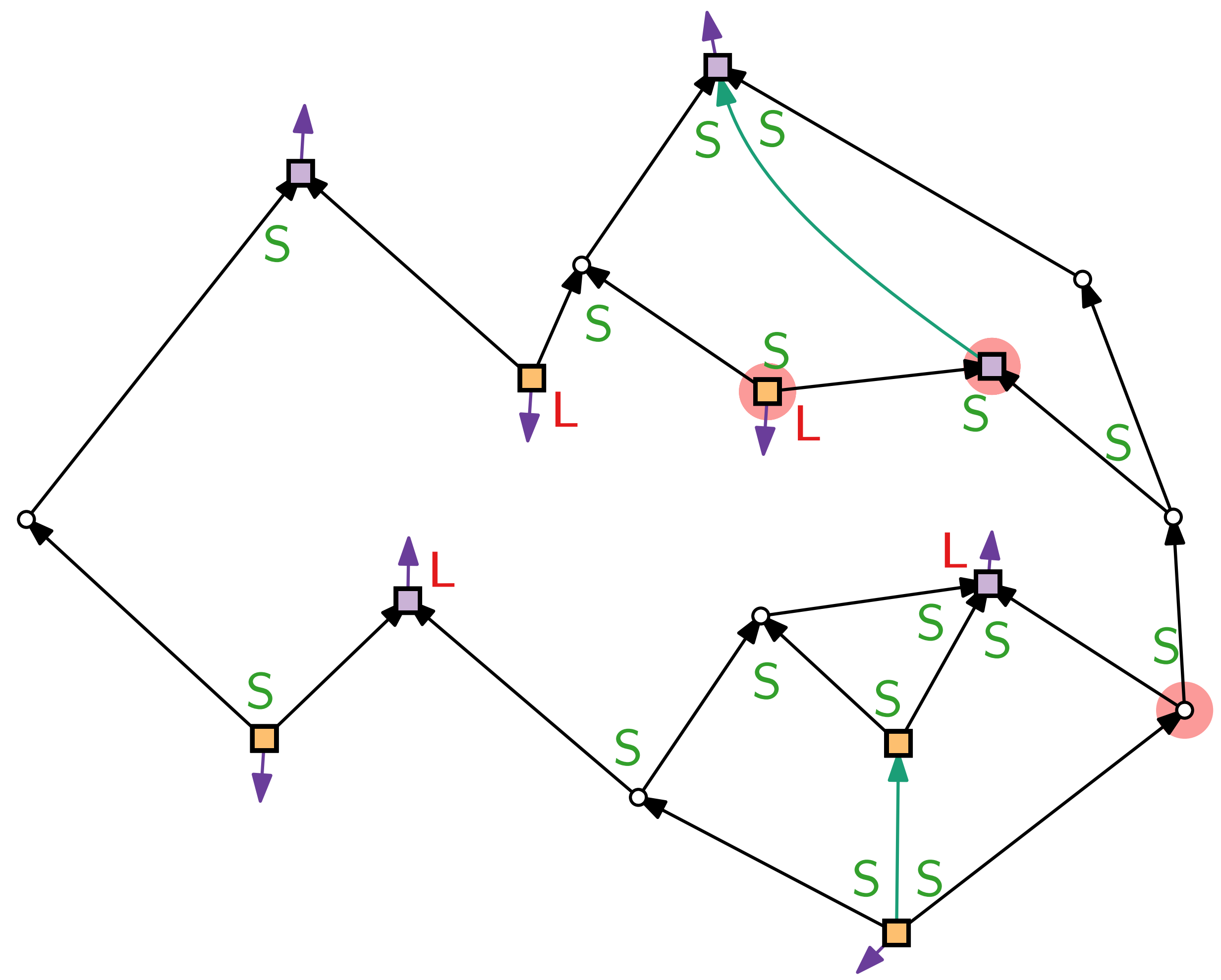

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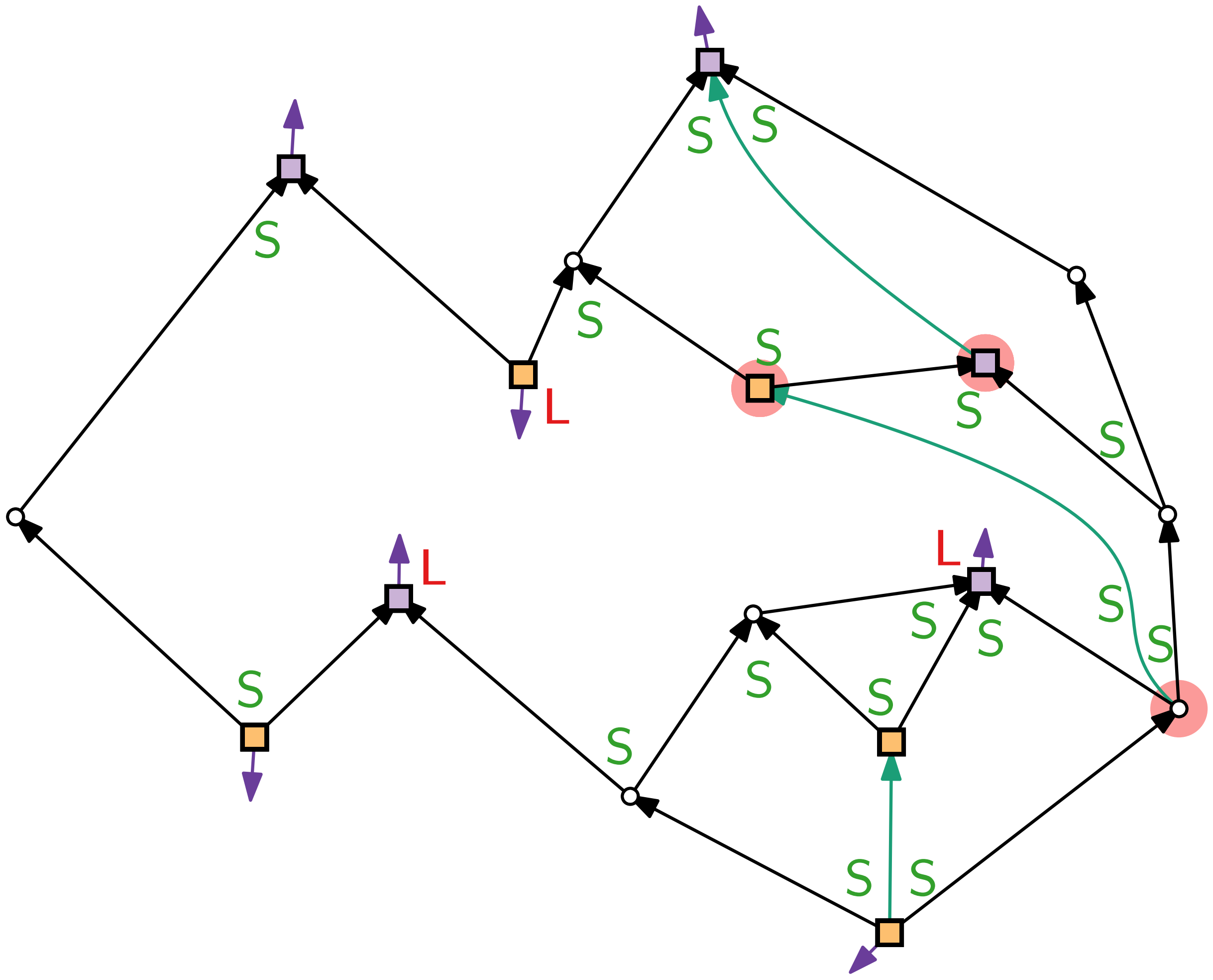

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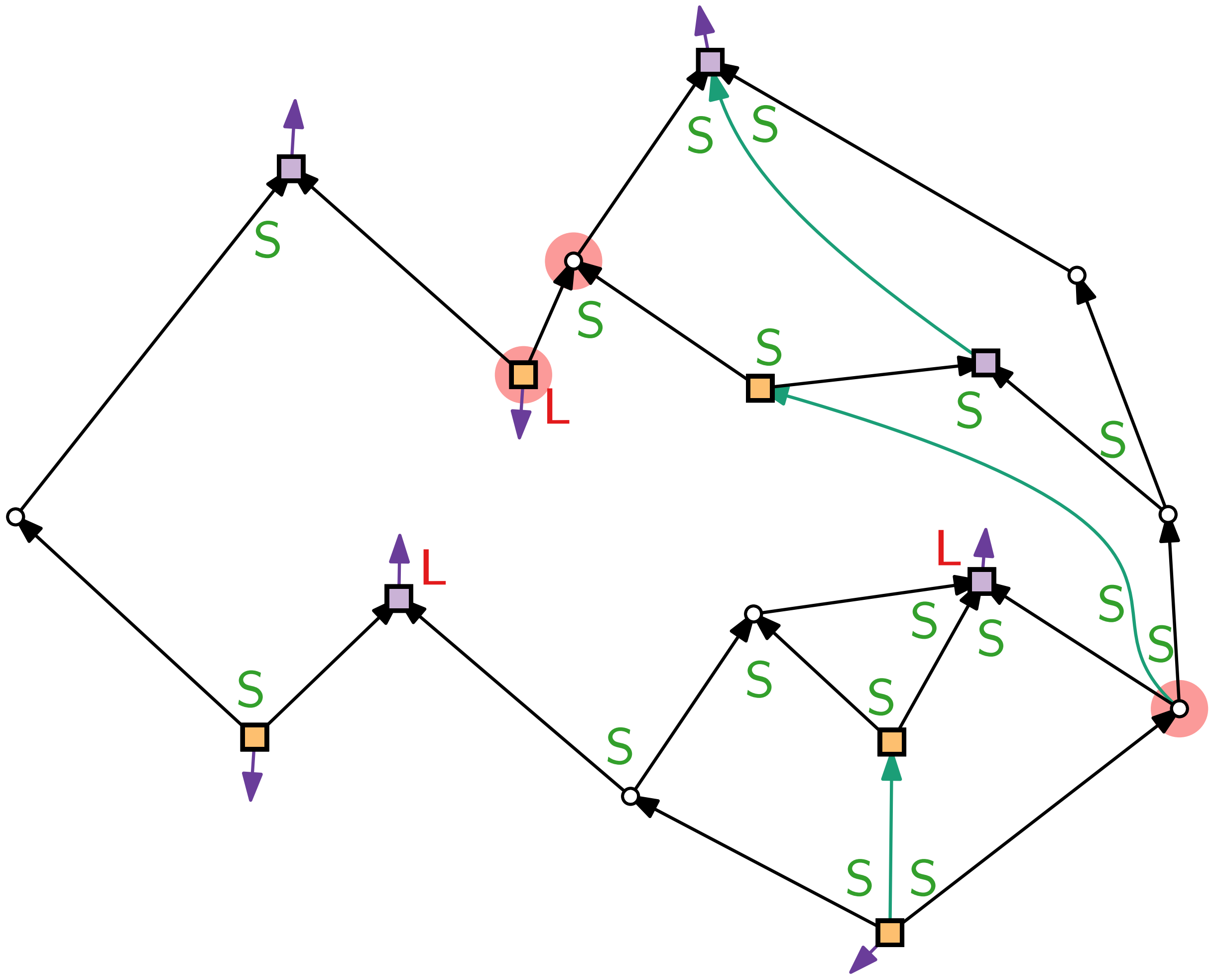

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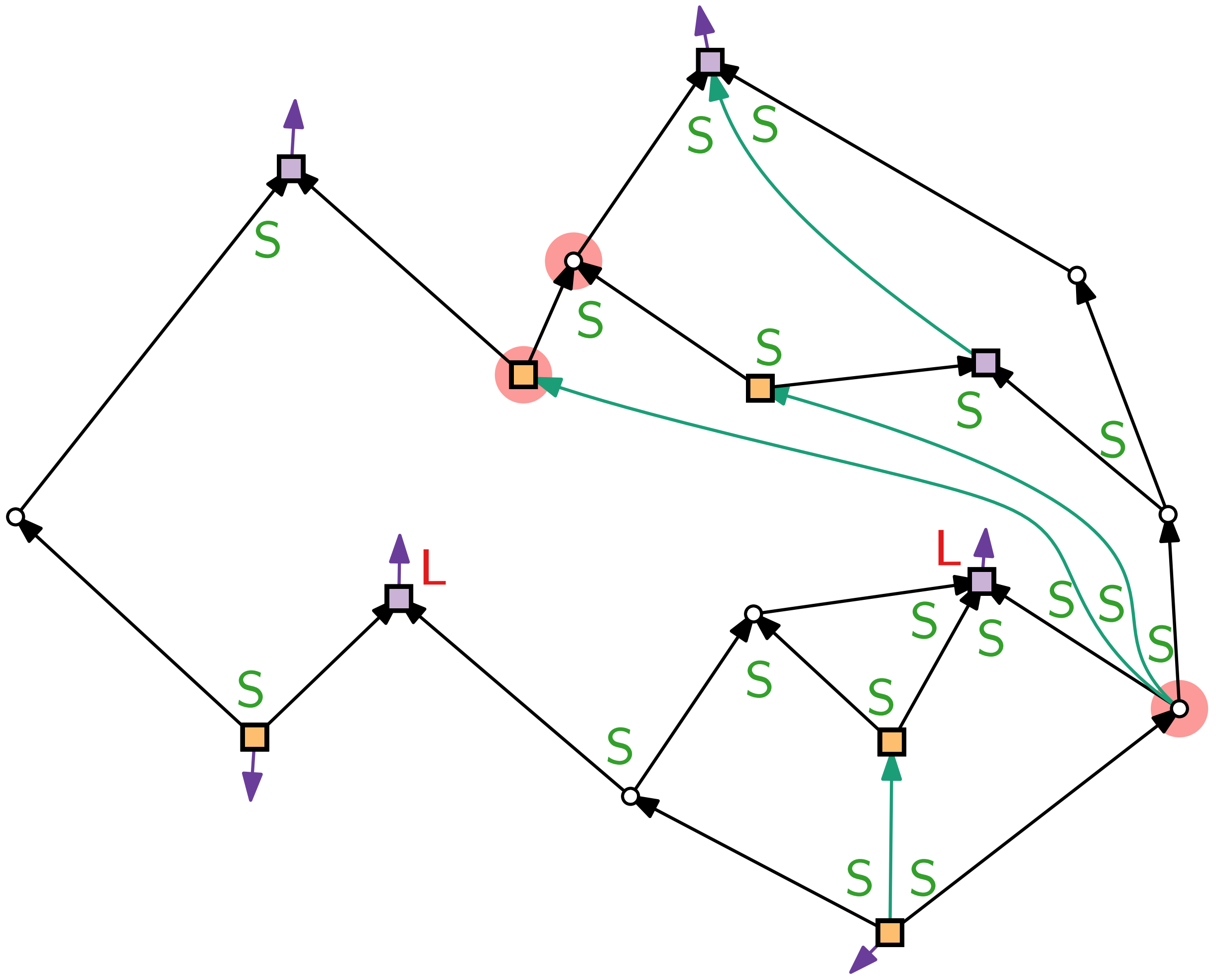

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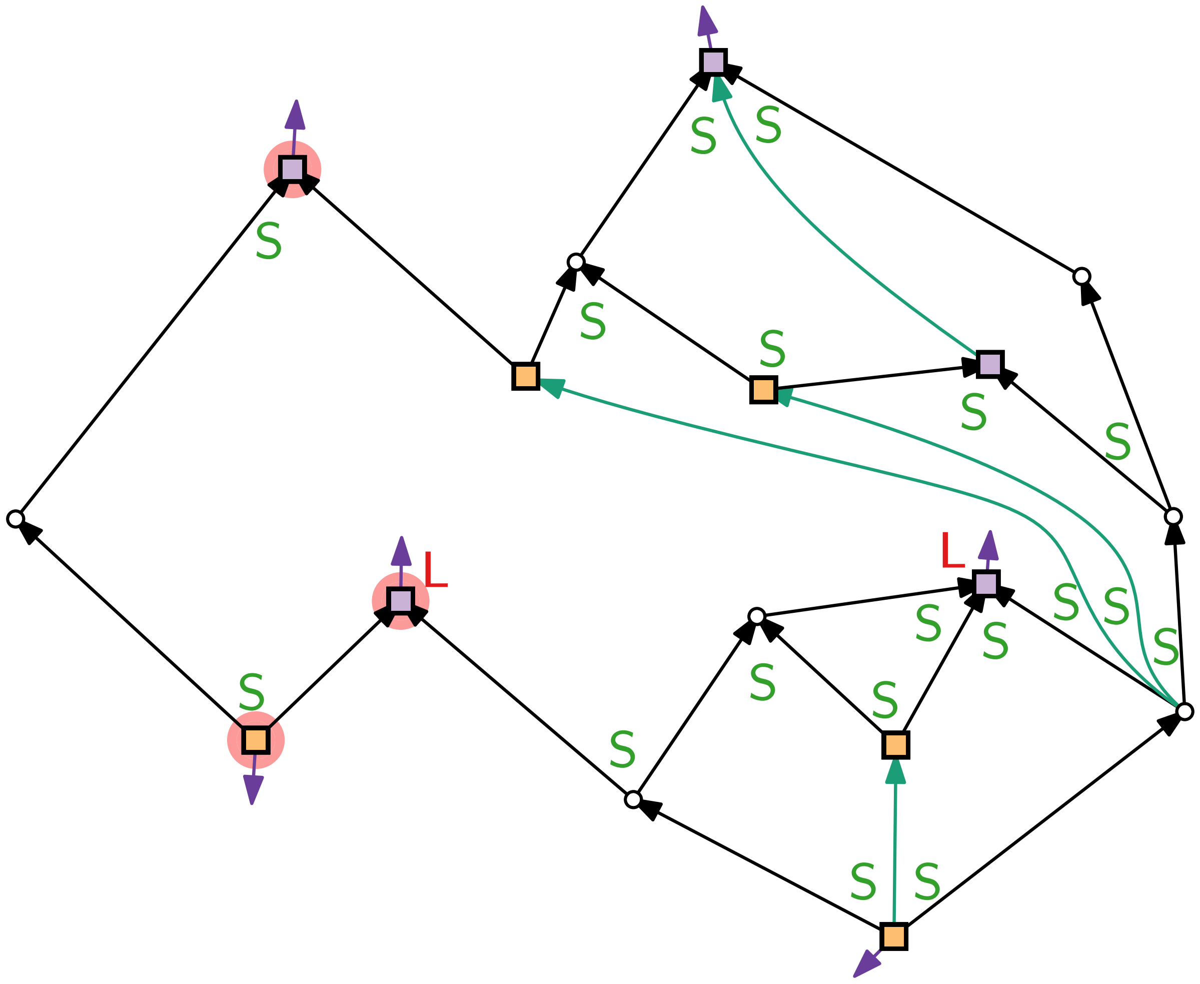

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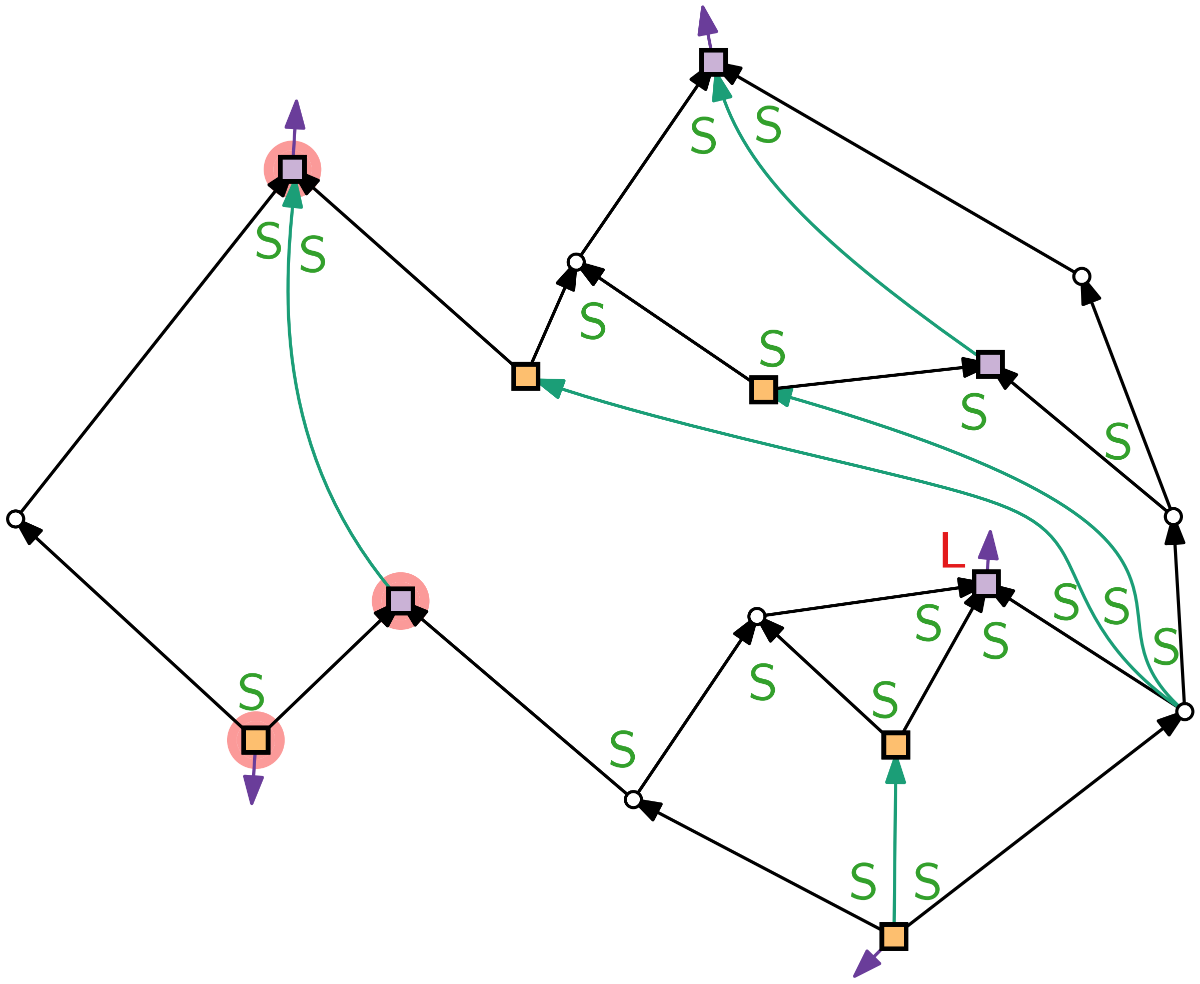

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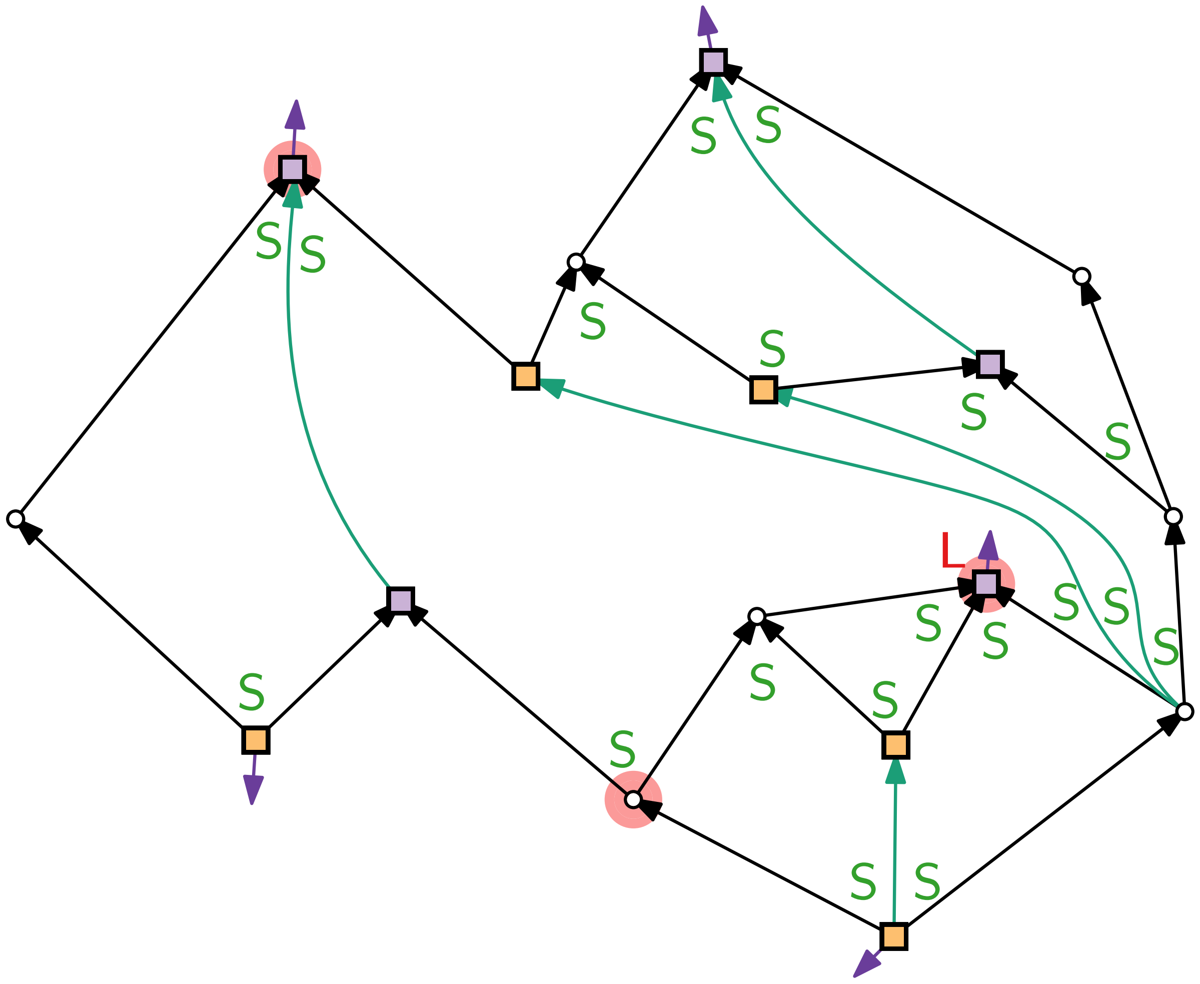

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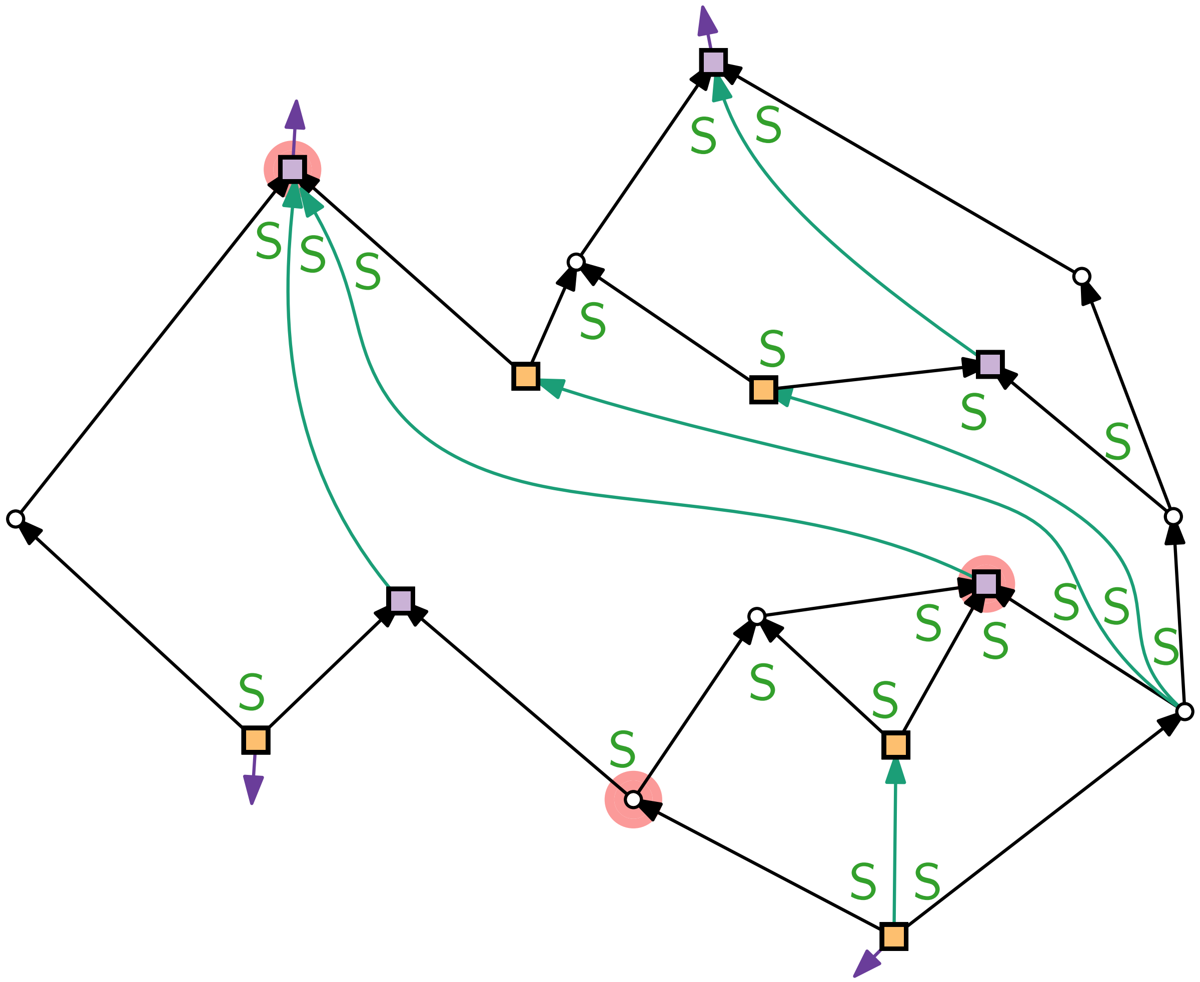

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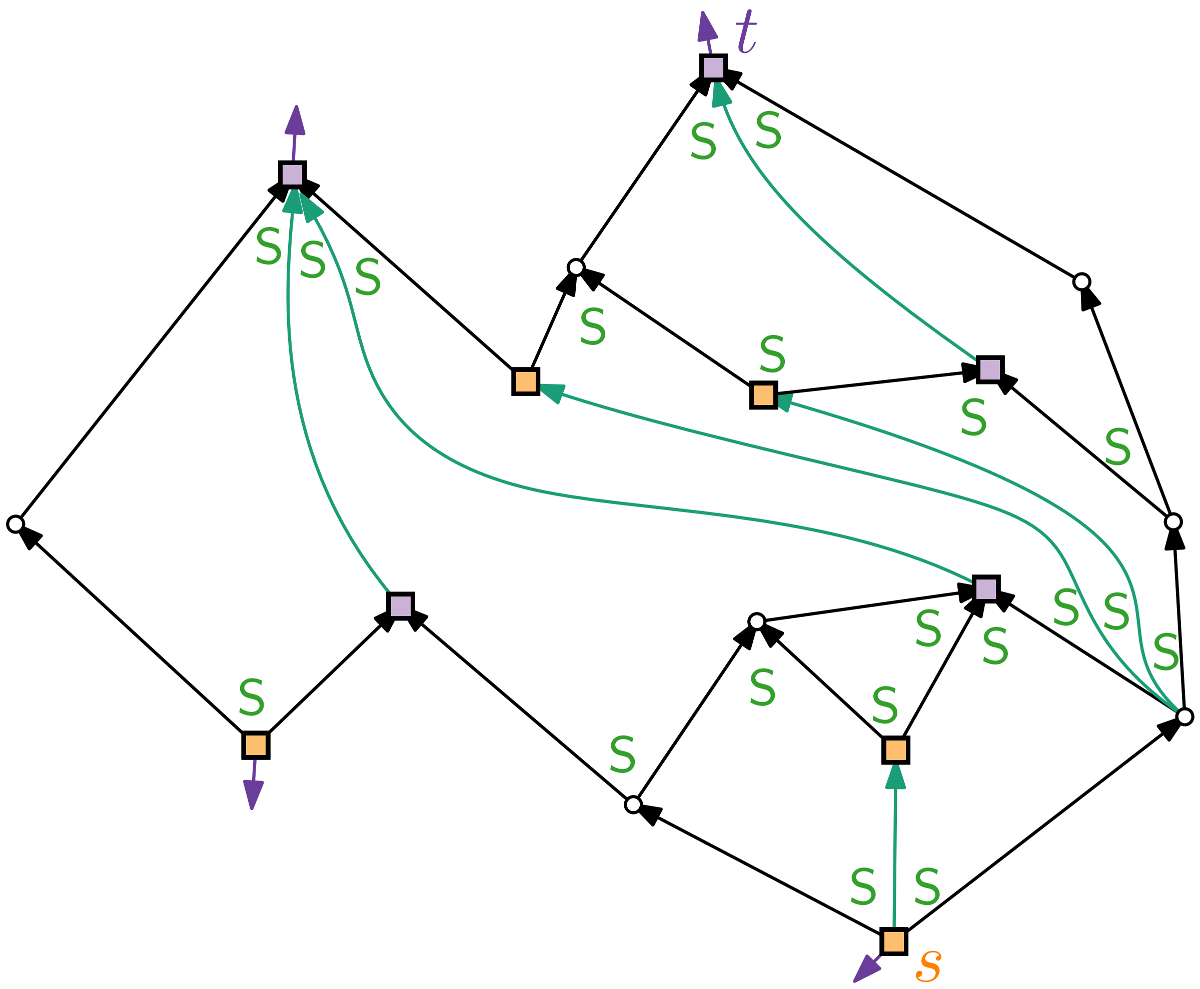

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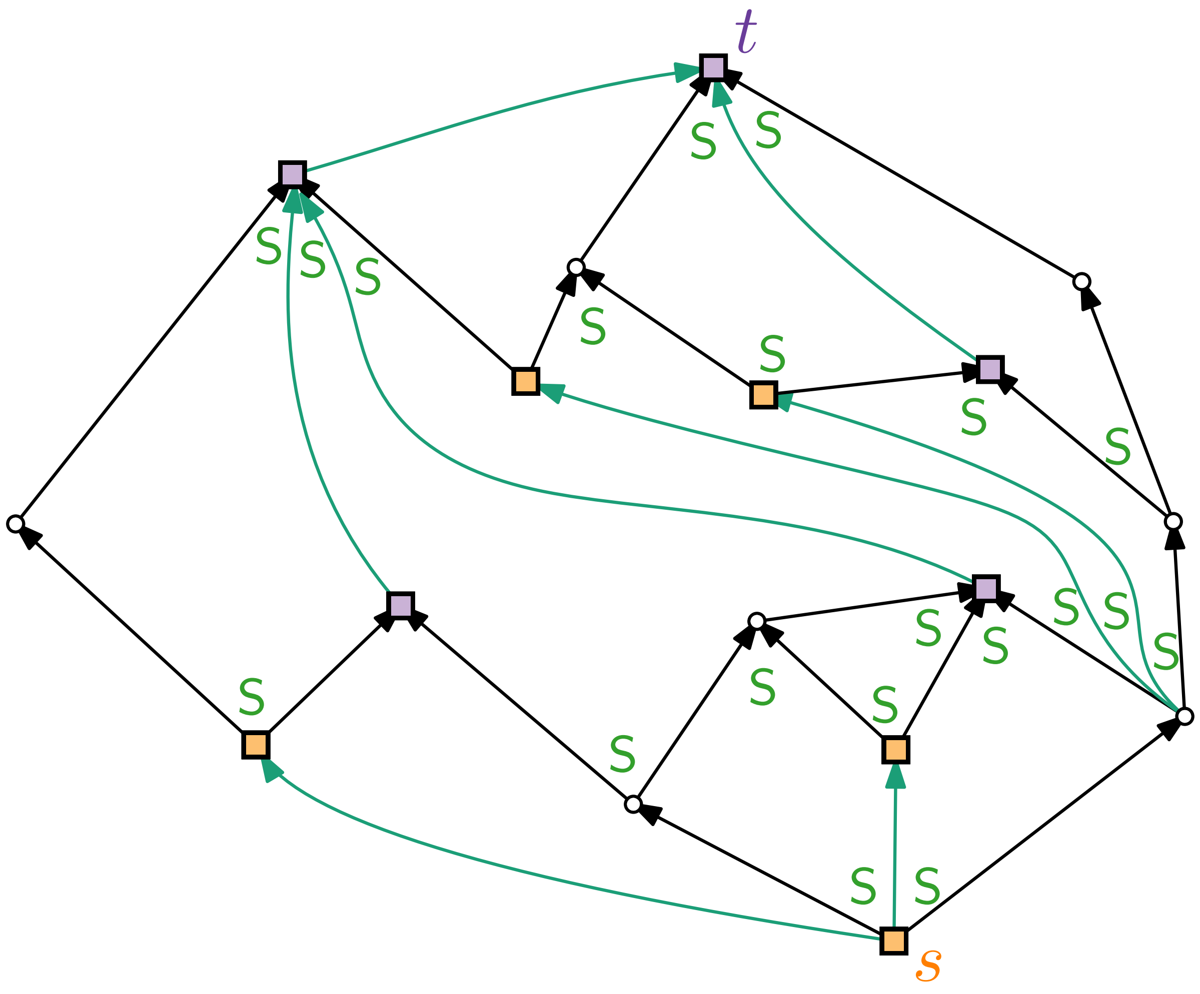

Result Upward Planarity Test

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- Deleted edges added in refinement step.

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Flow $(v, f) = 1$ from global source / sink v to the incident face f its large angle gets assigned to.

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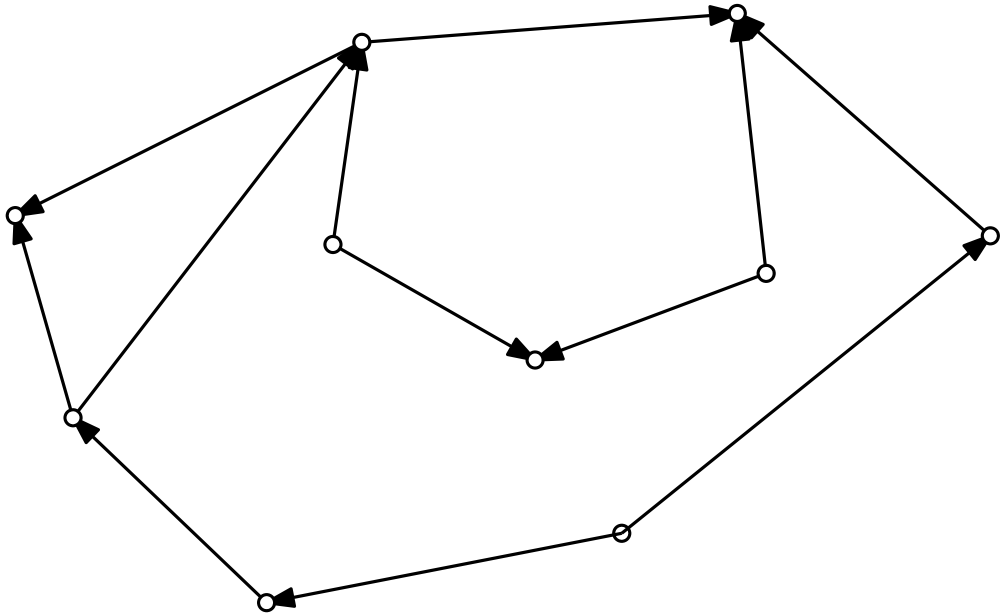

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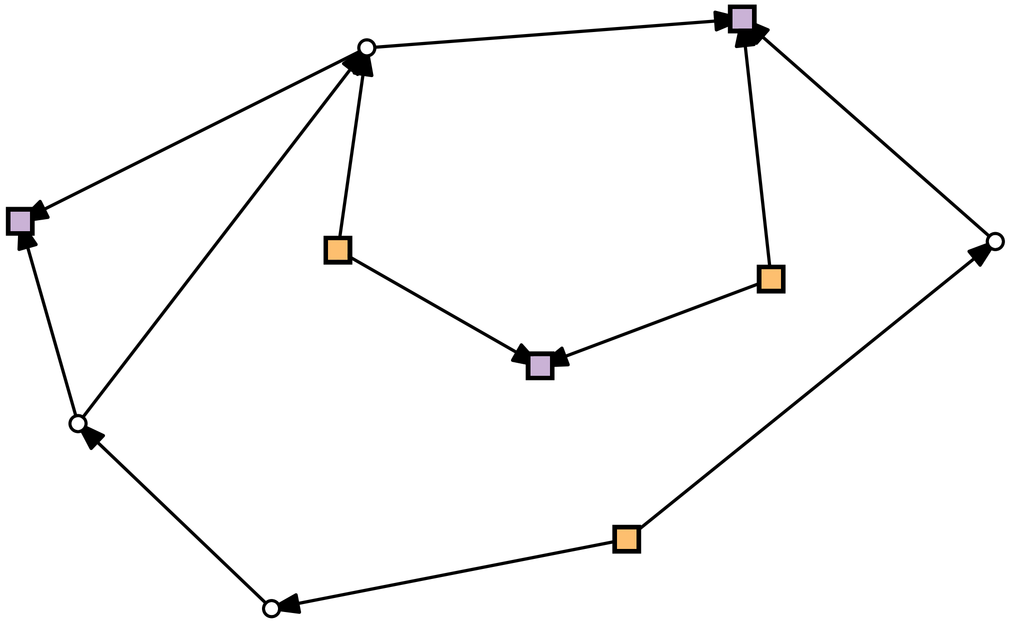

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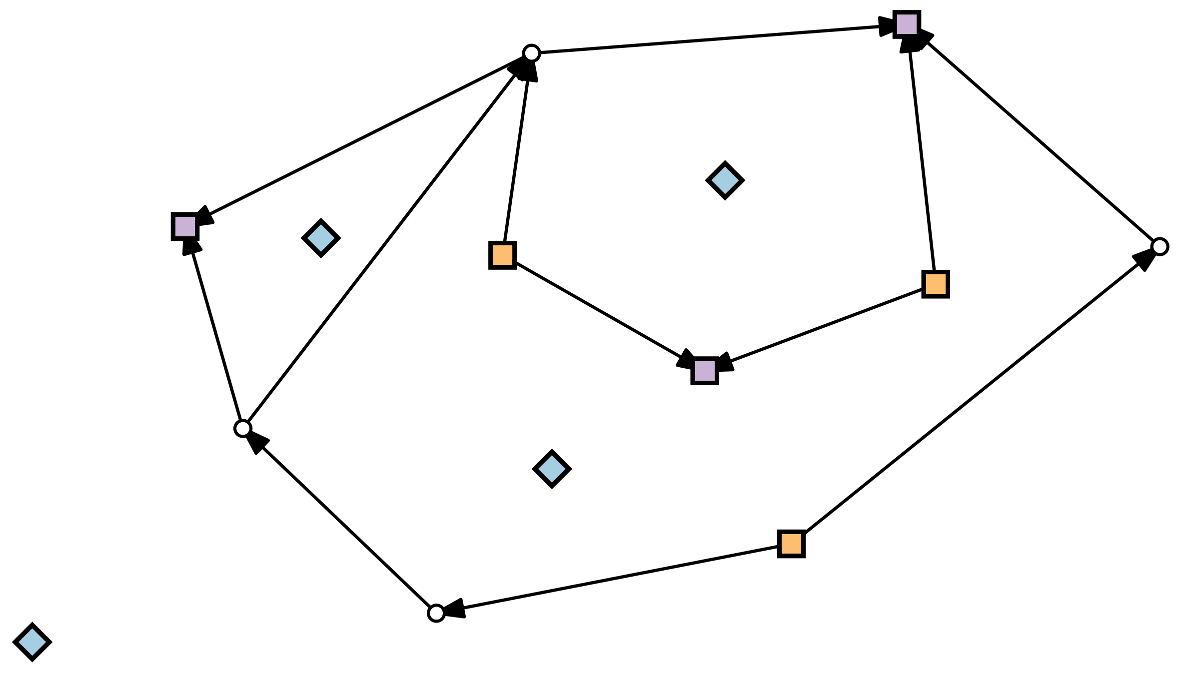

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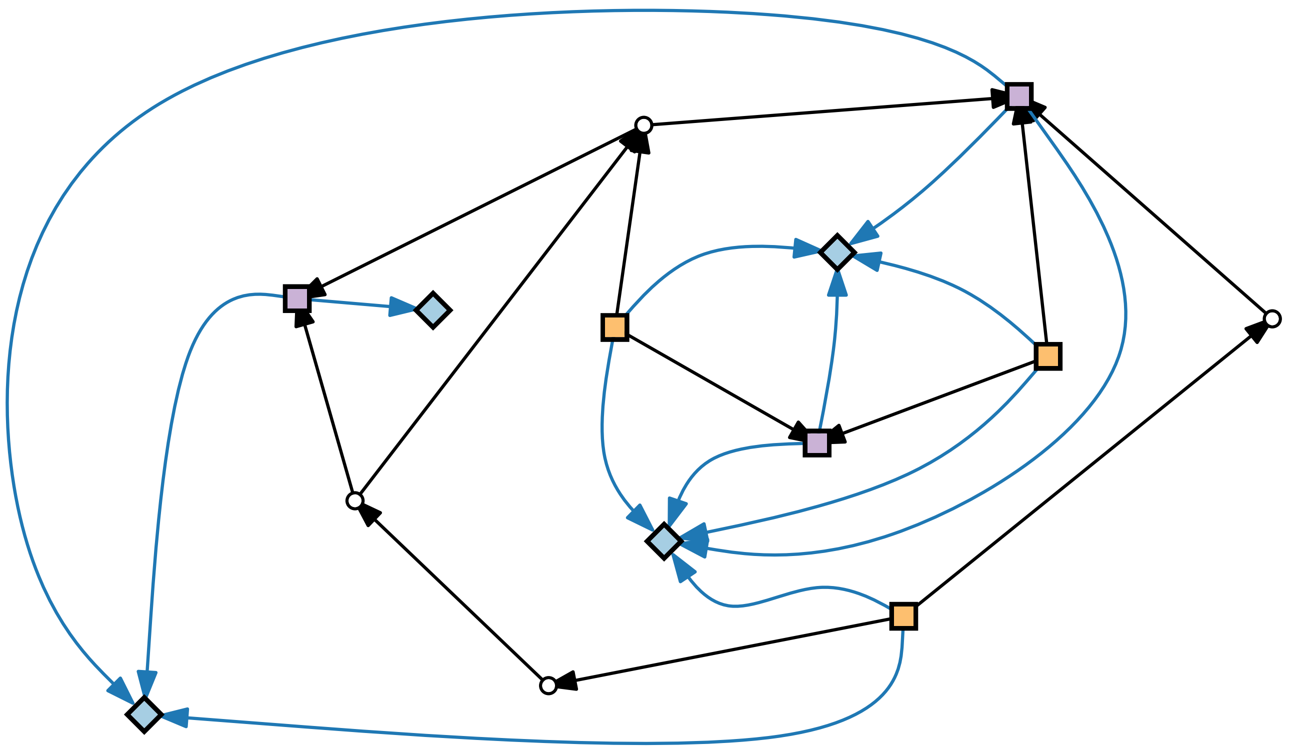

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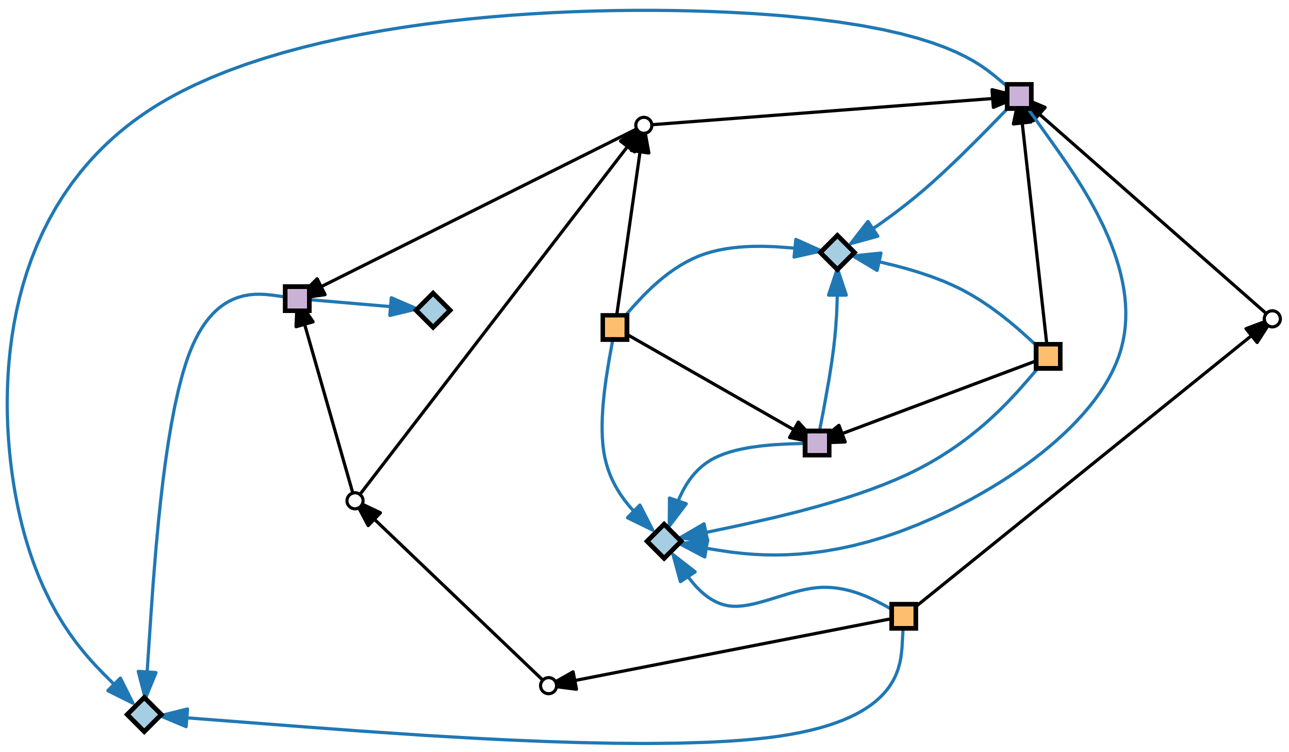

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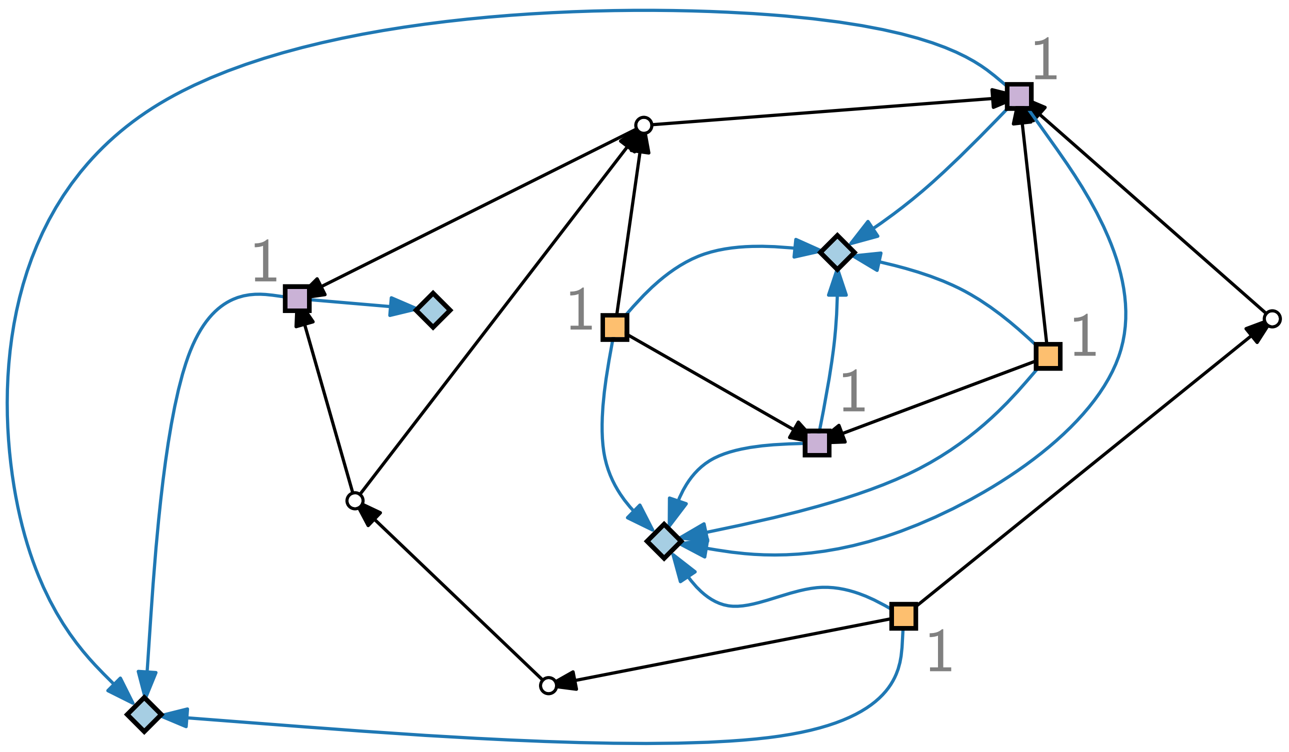

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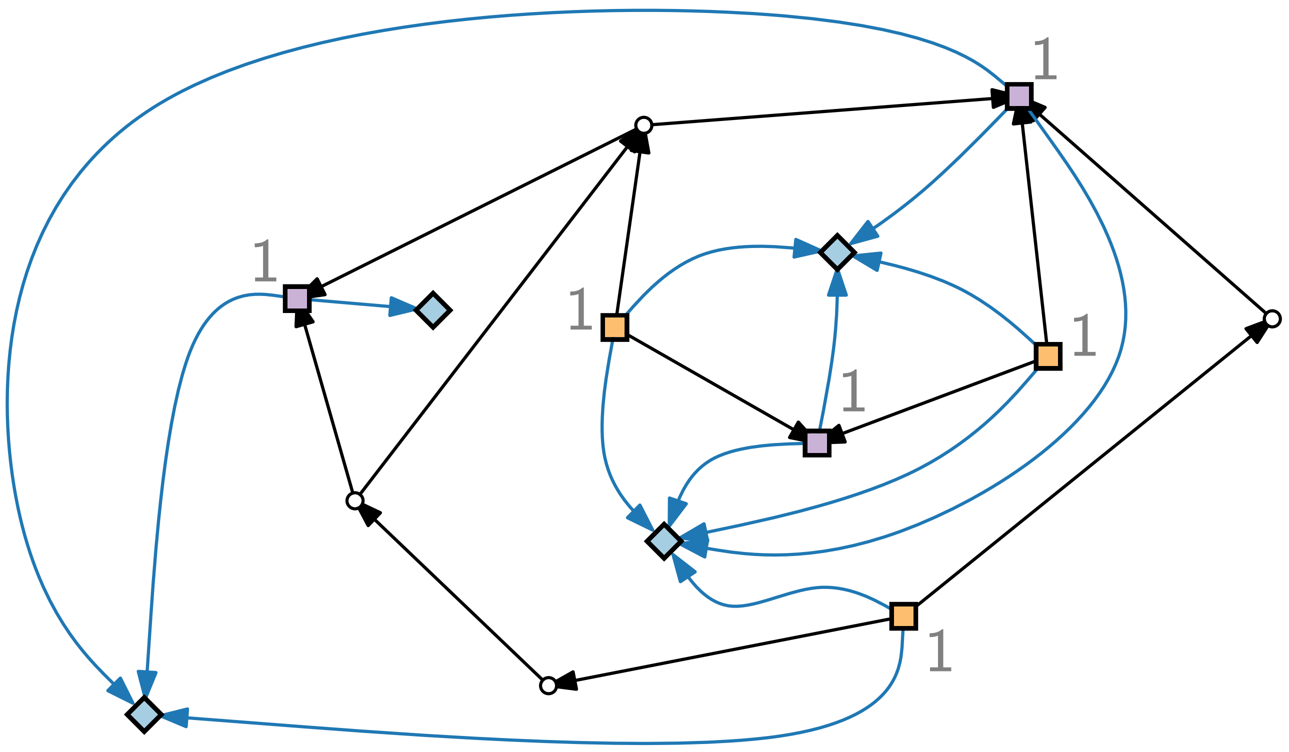

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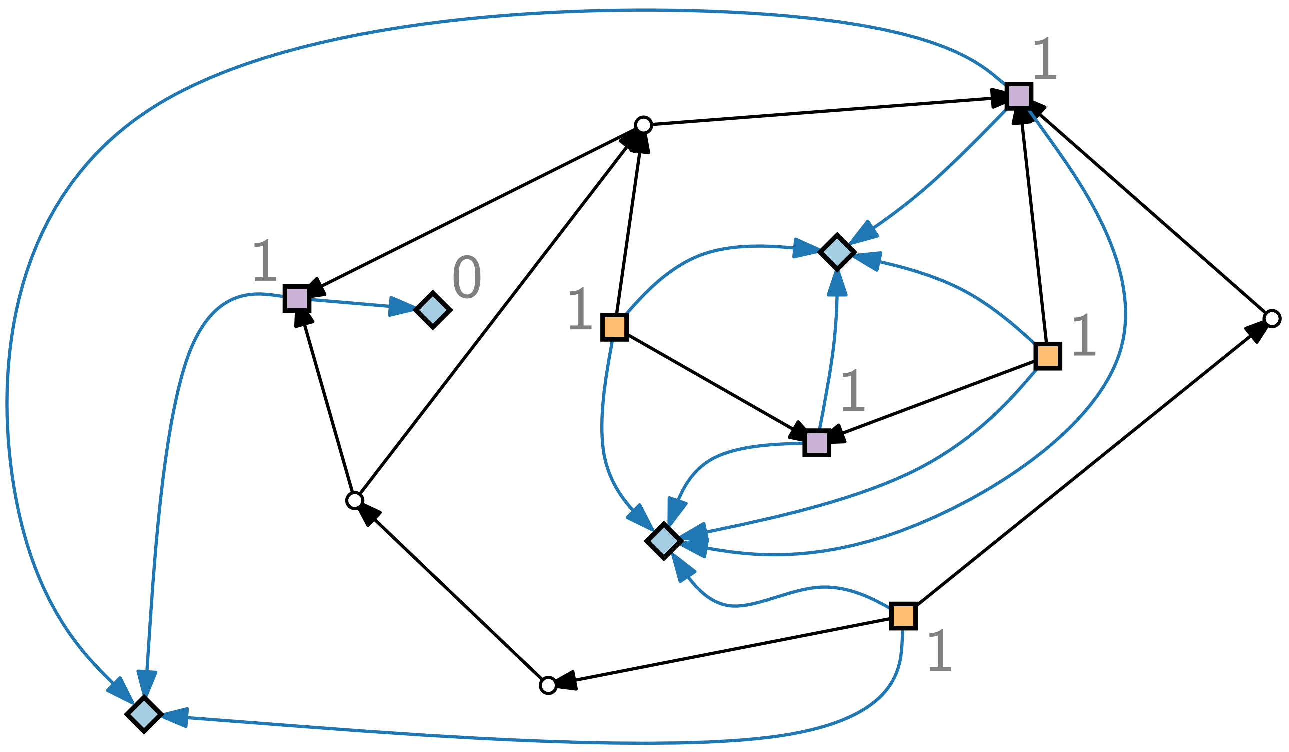

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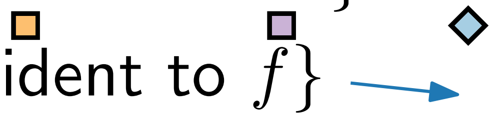

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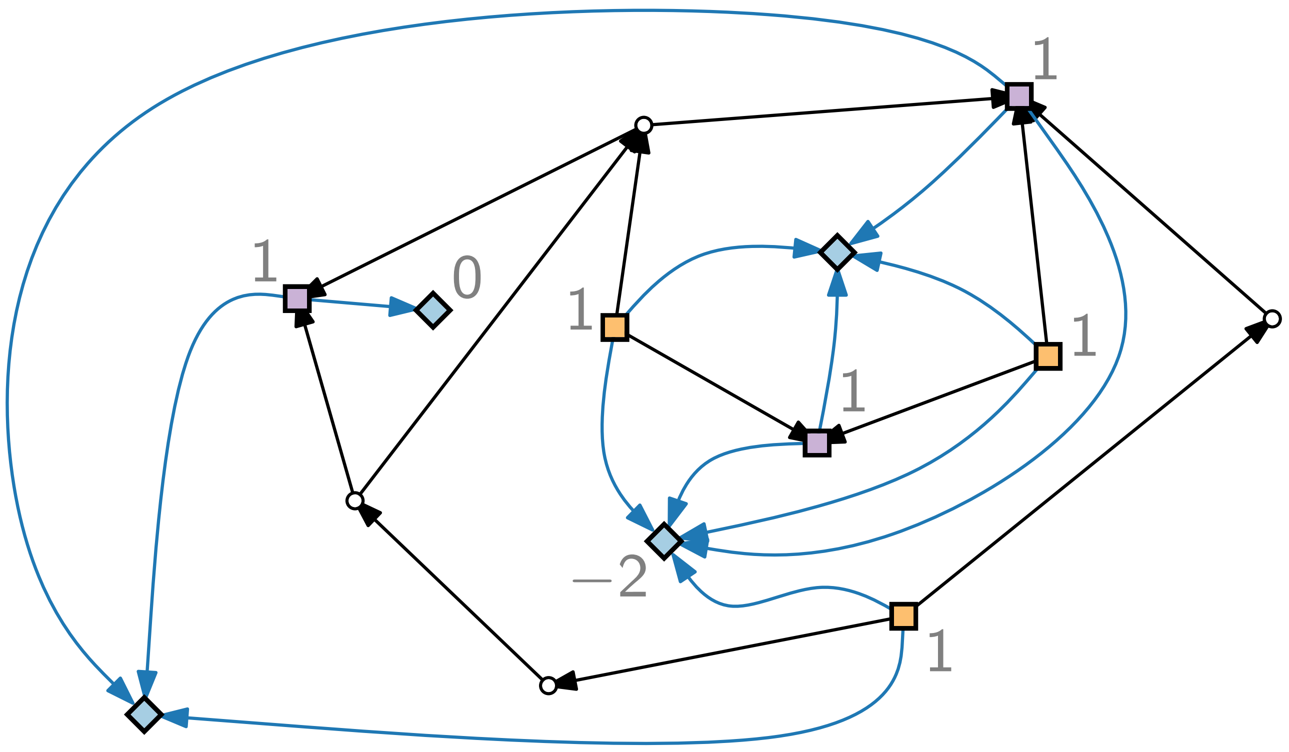

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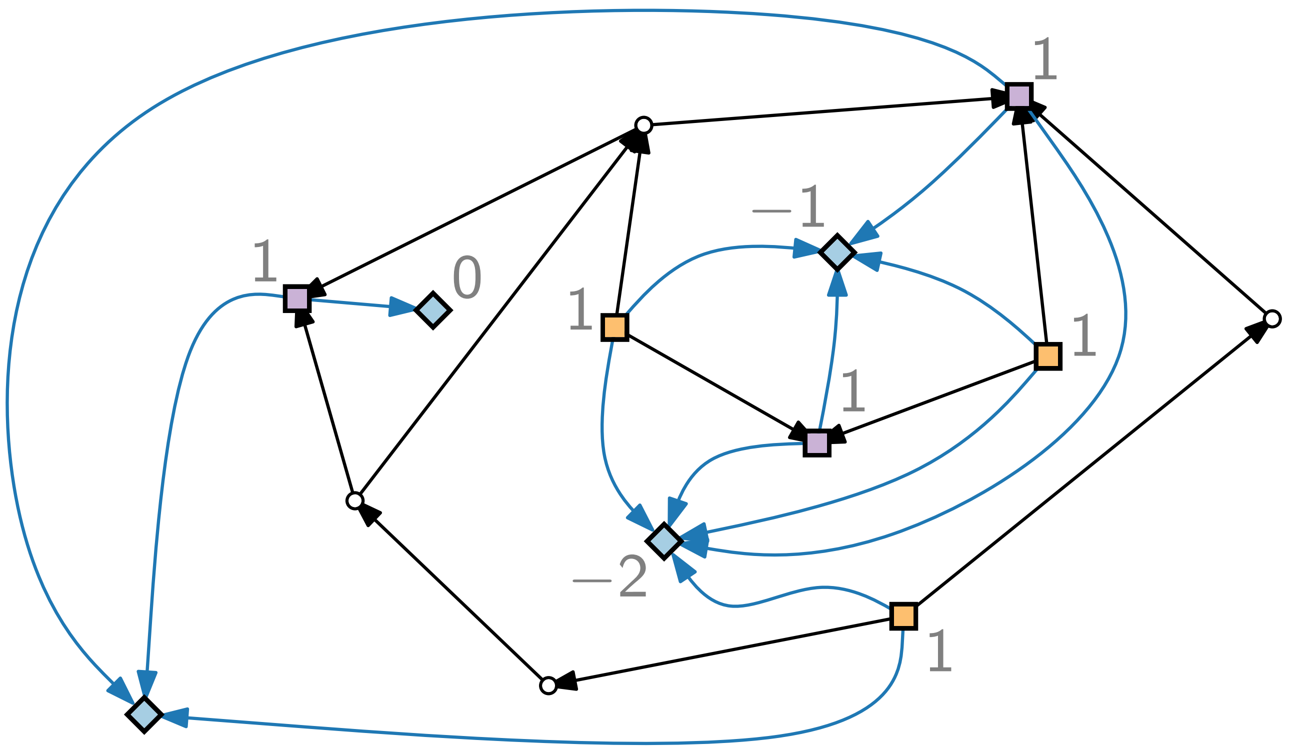

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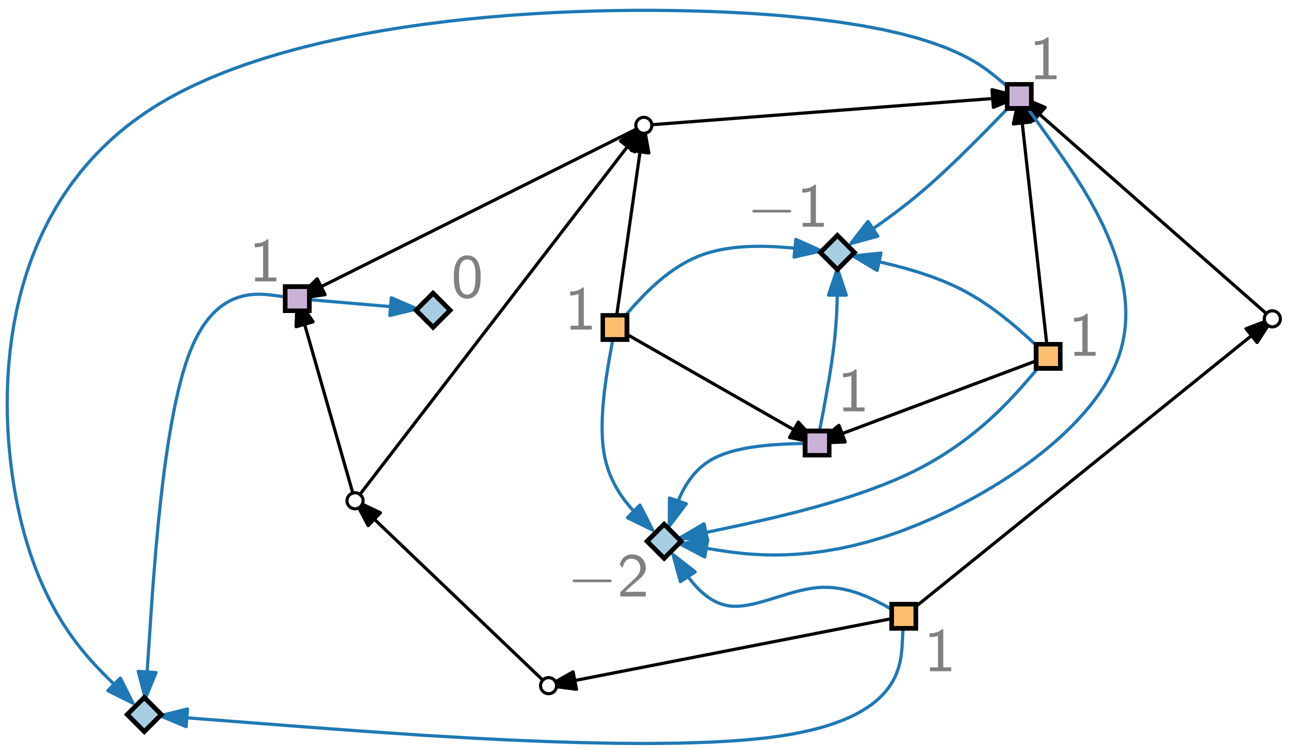

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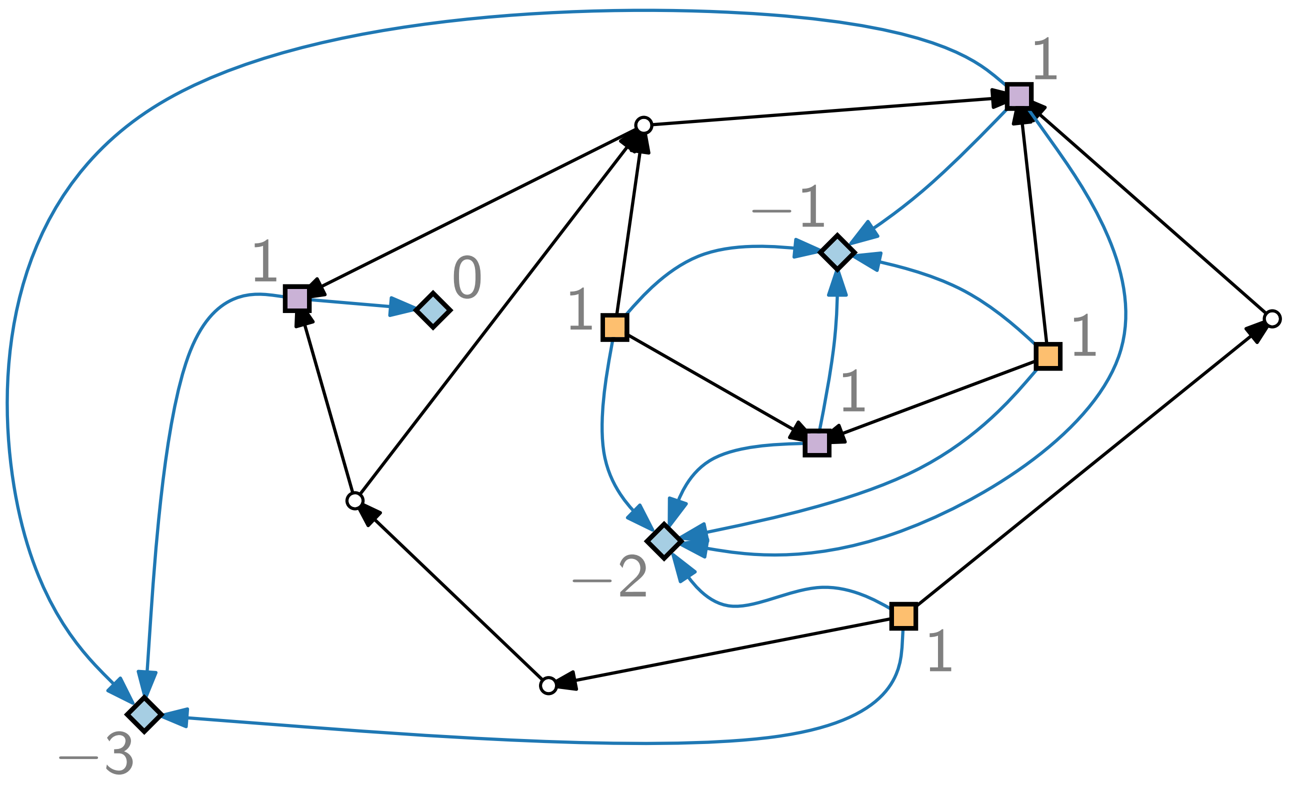

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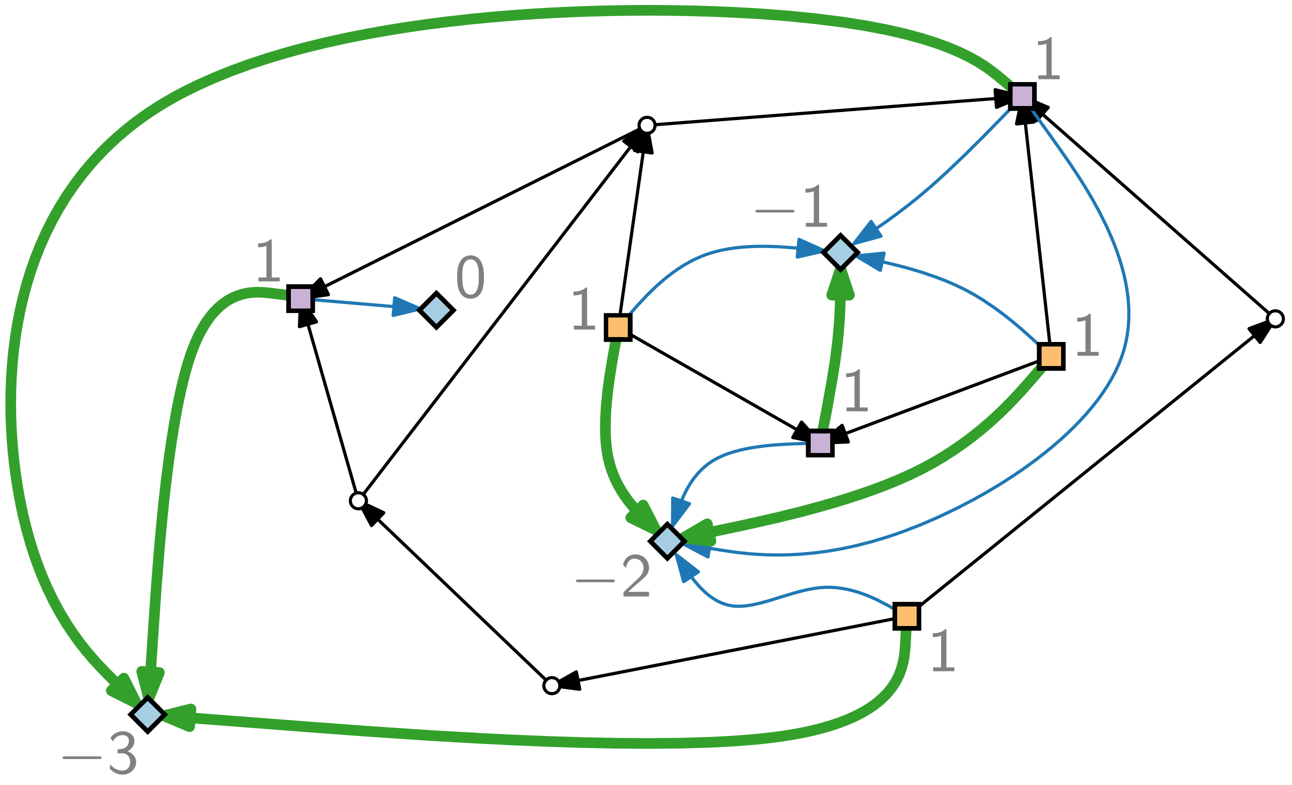
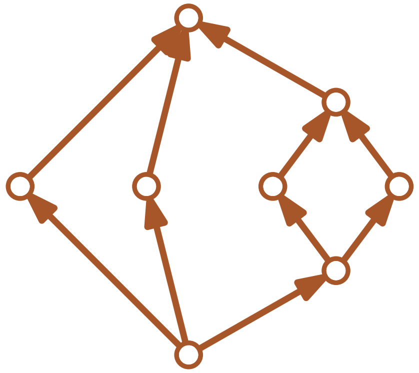
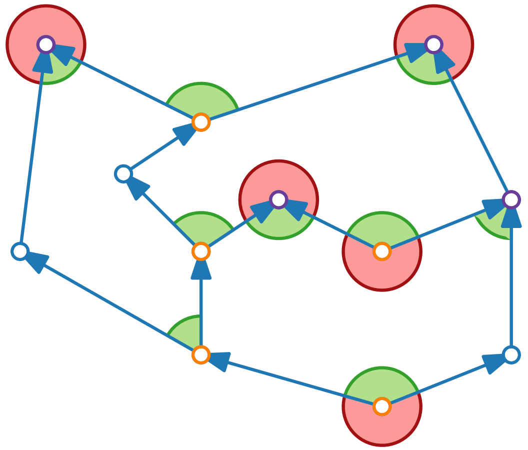


Julius-Maximilians-
UNIVERSITÄT
WÜRZBURG

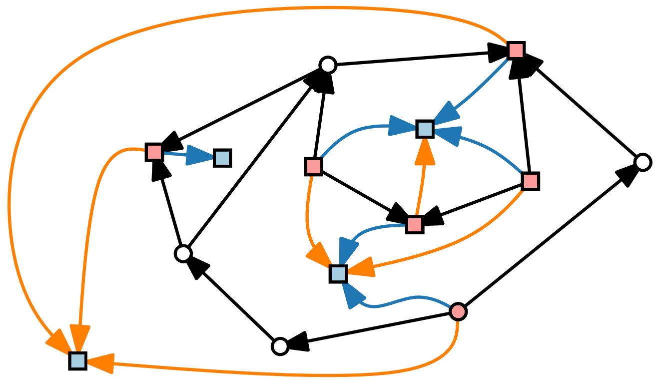
Visualization of Graphs



Lecture 5:
Upward Planar Drawings



Part II:
Series-Parallel Graphs



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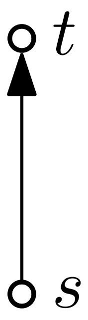

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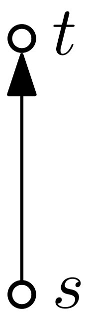

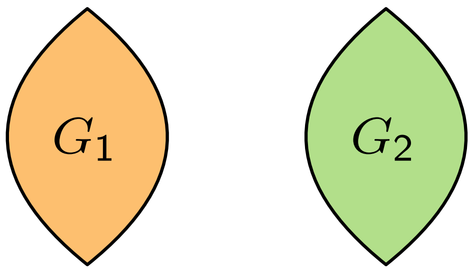

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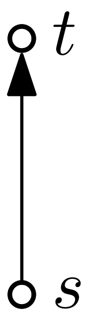

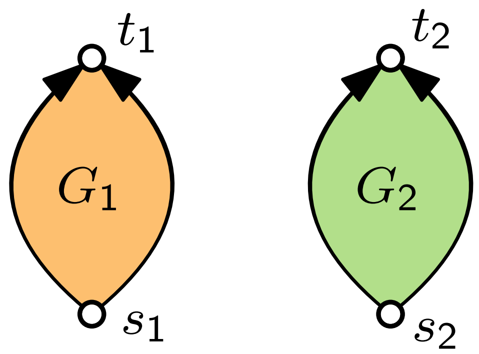

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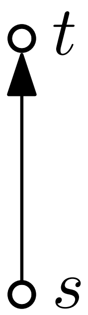

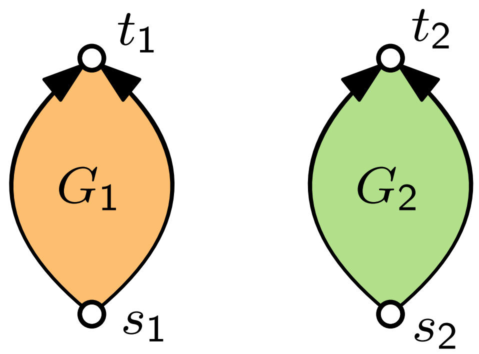

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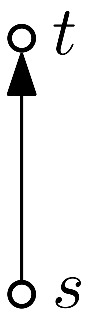
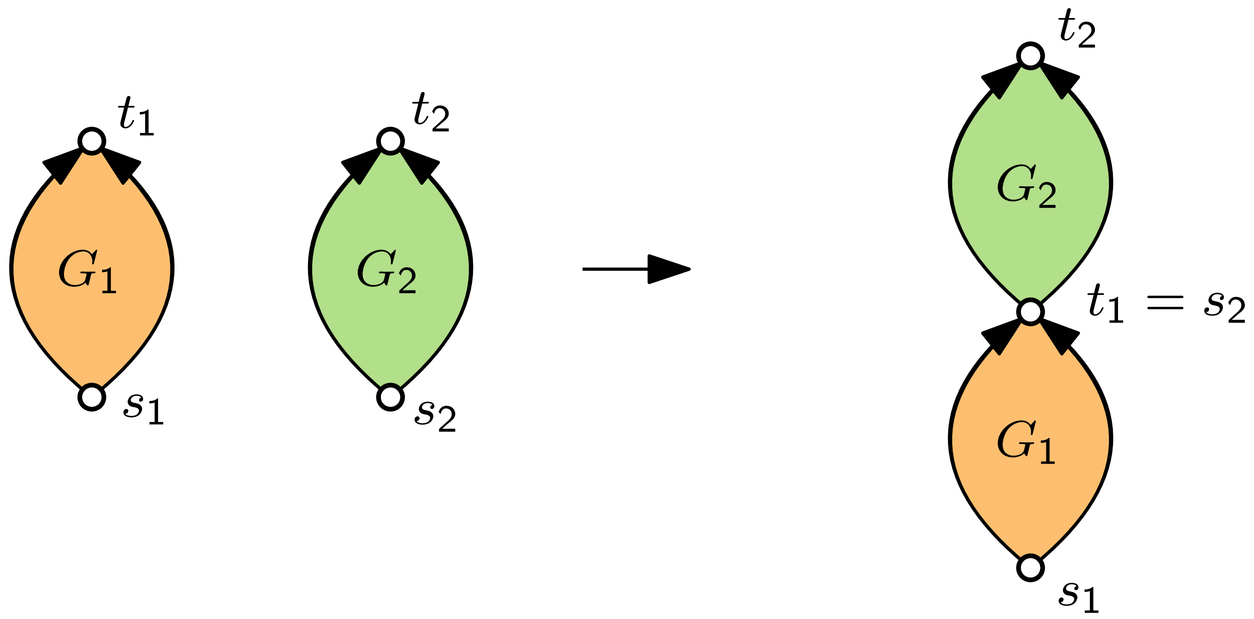


Series composition



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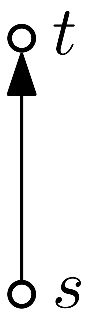

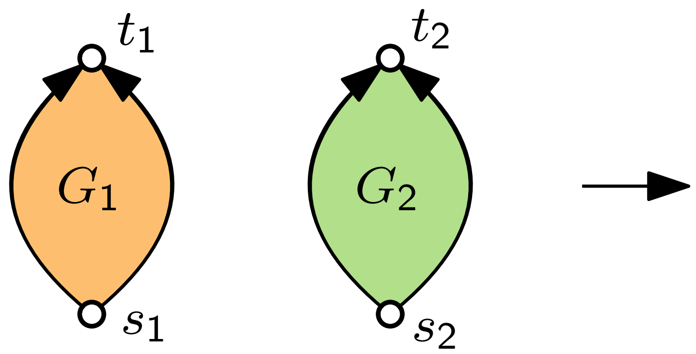

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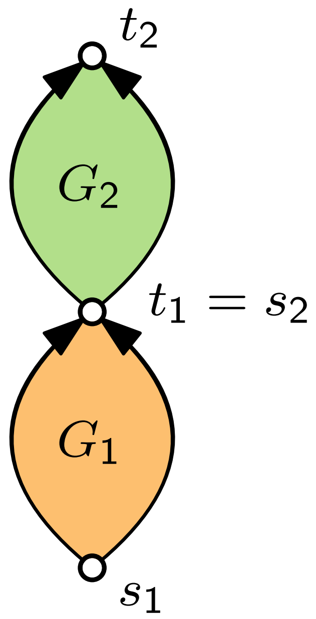
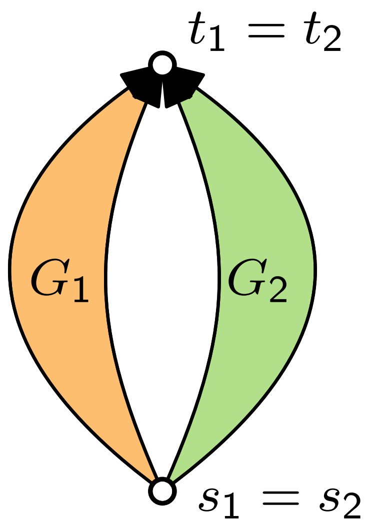


Parallel composition



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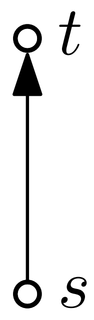
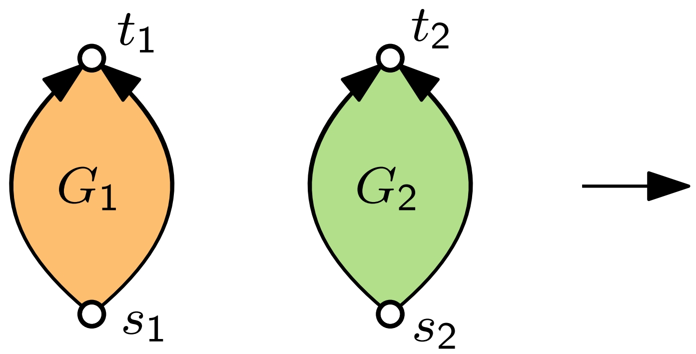


Convince yourself that series-parallel graphs are planar!



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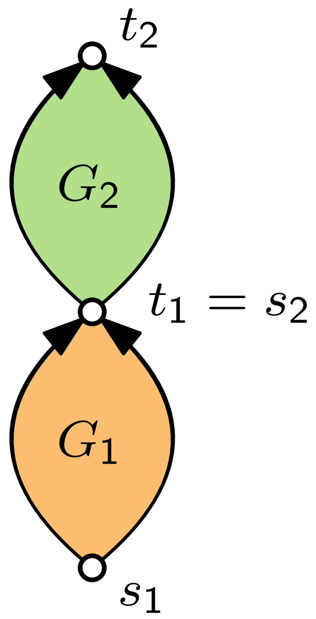

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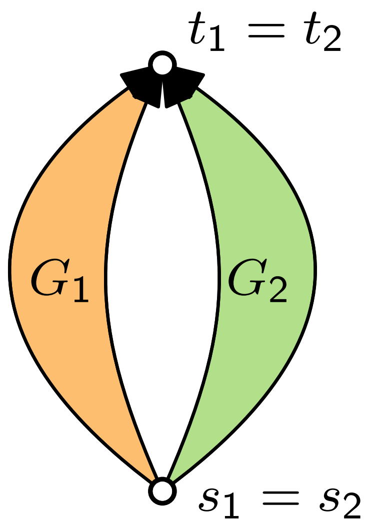

Series-Parallel Graphs – Decomposition Tree

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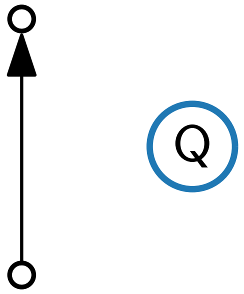

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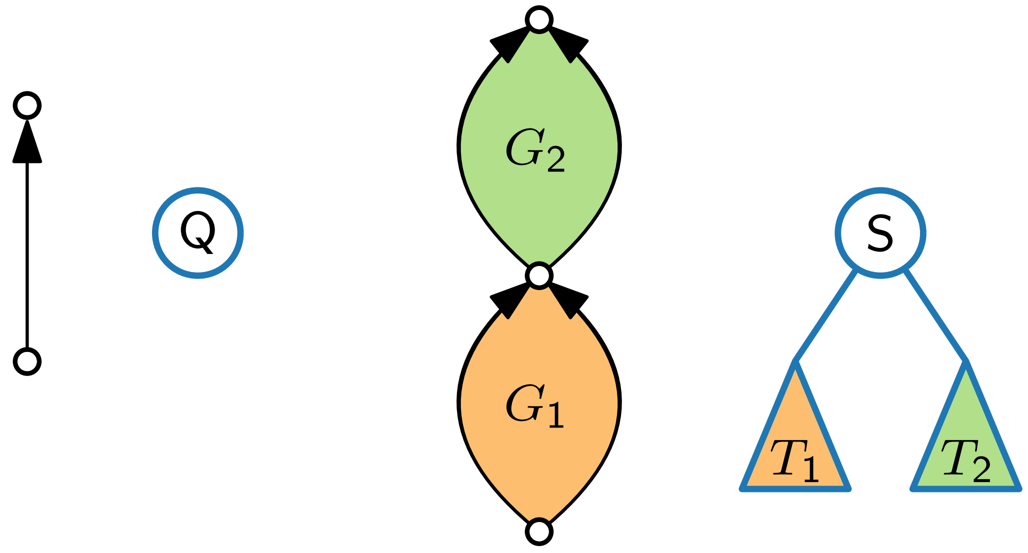

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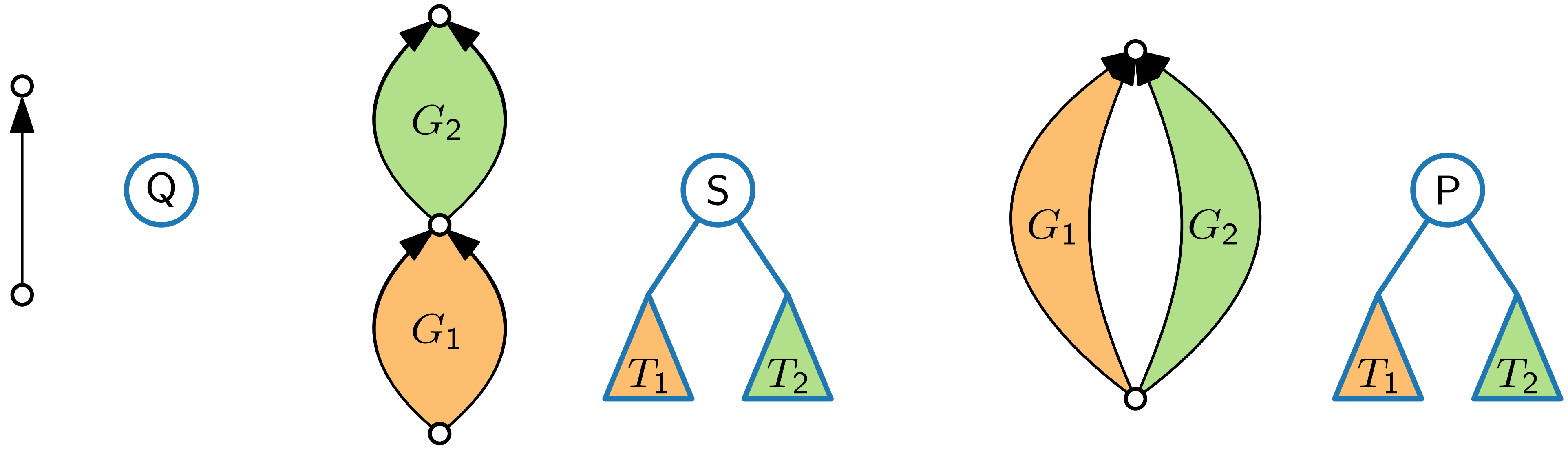
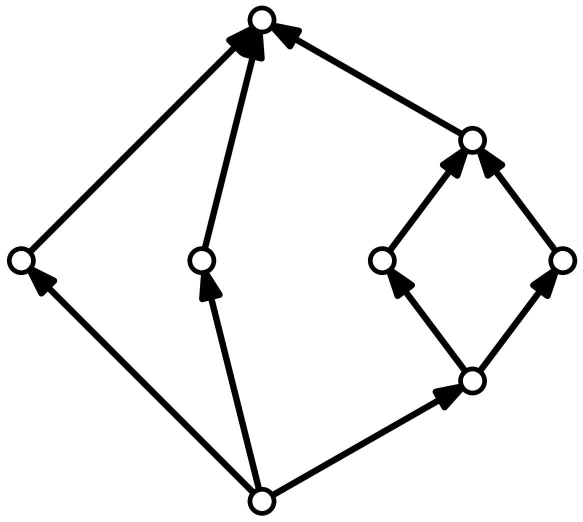


Series-Parallel Graphs – Decomposition Example



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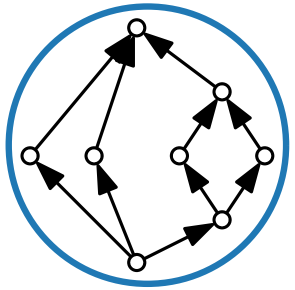

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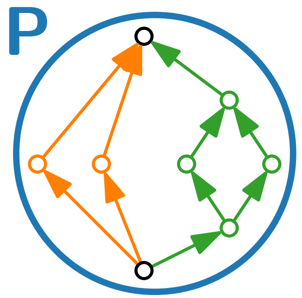

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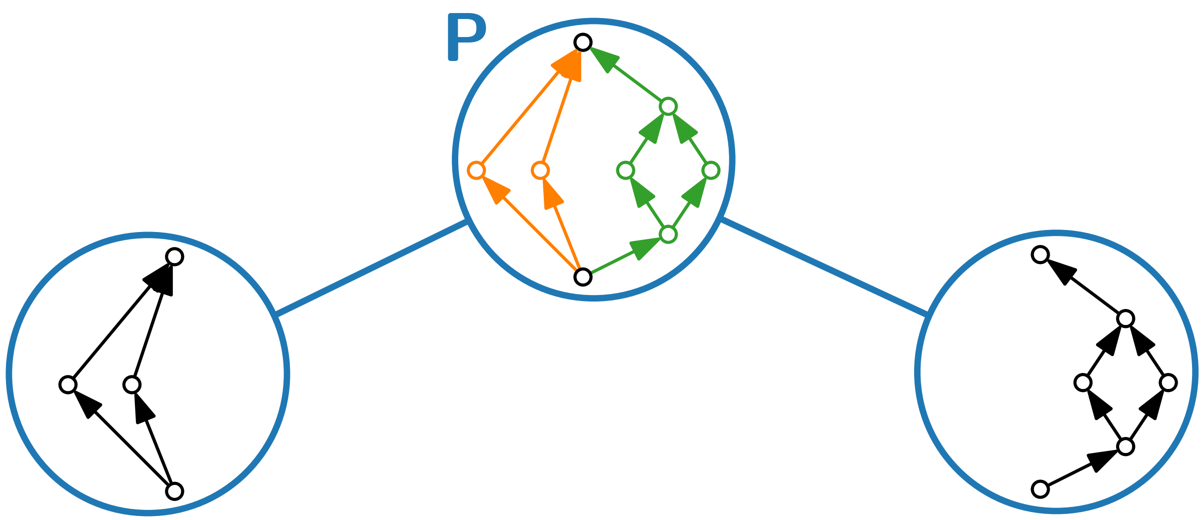

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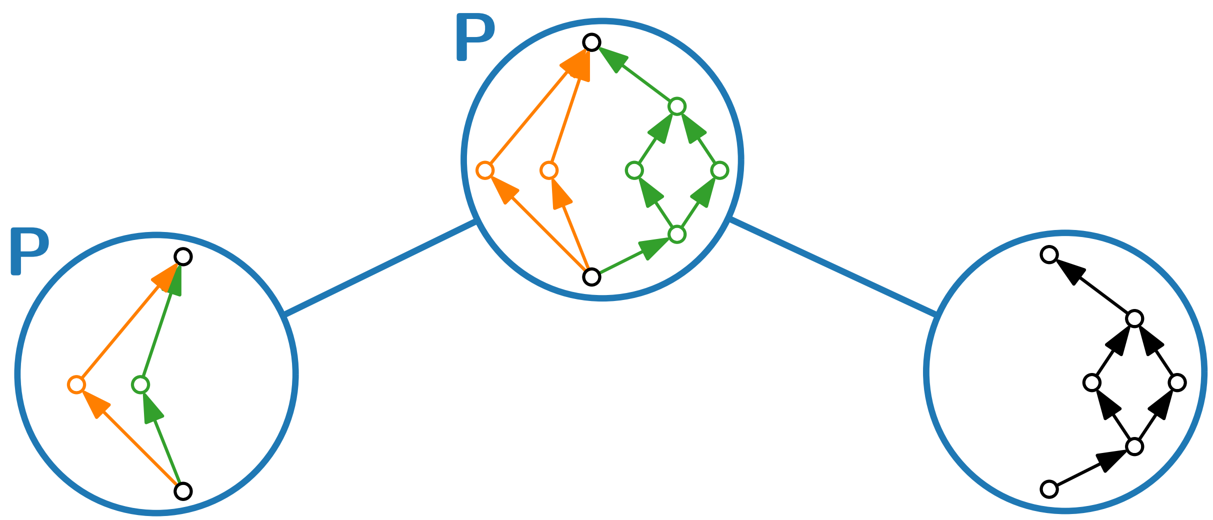

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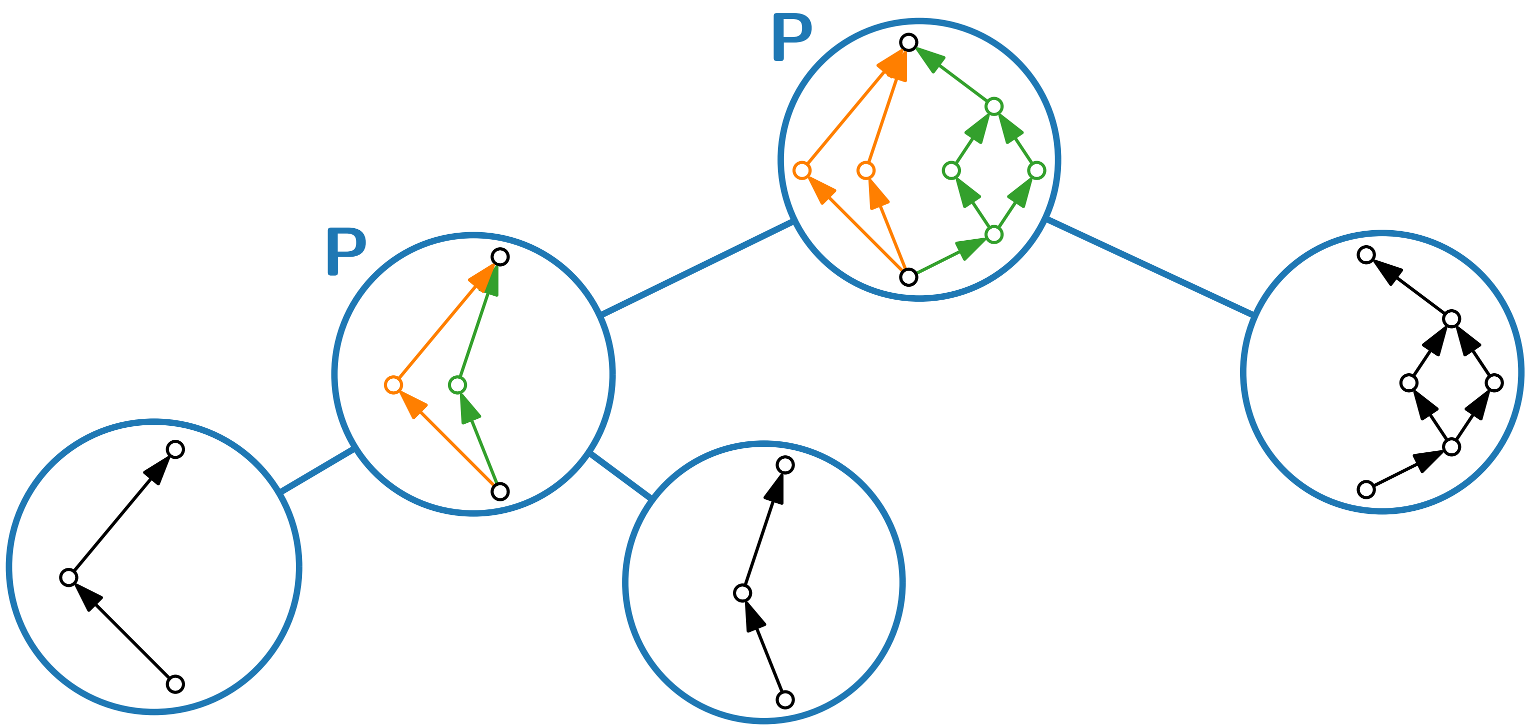

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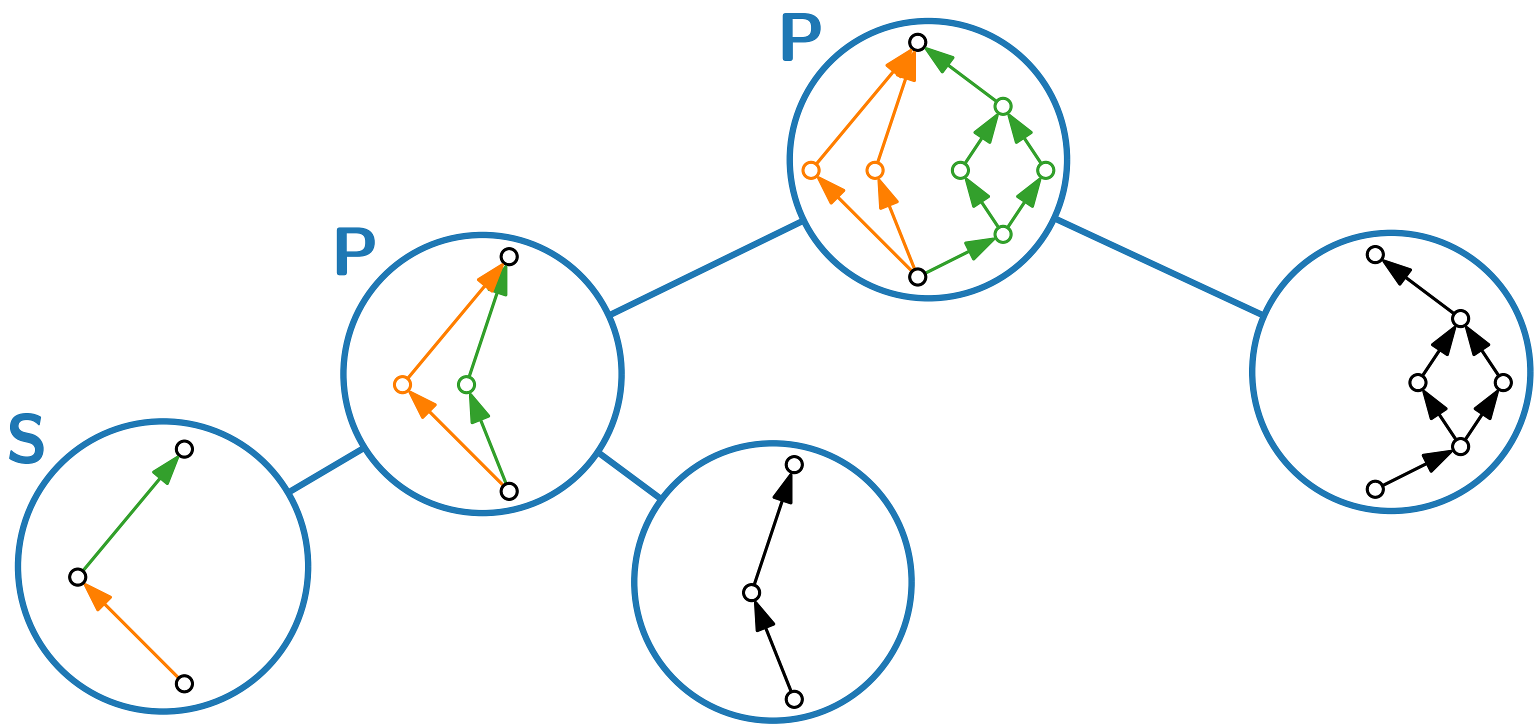

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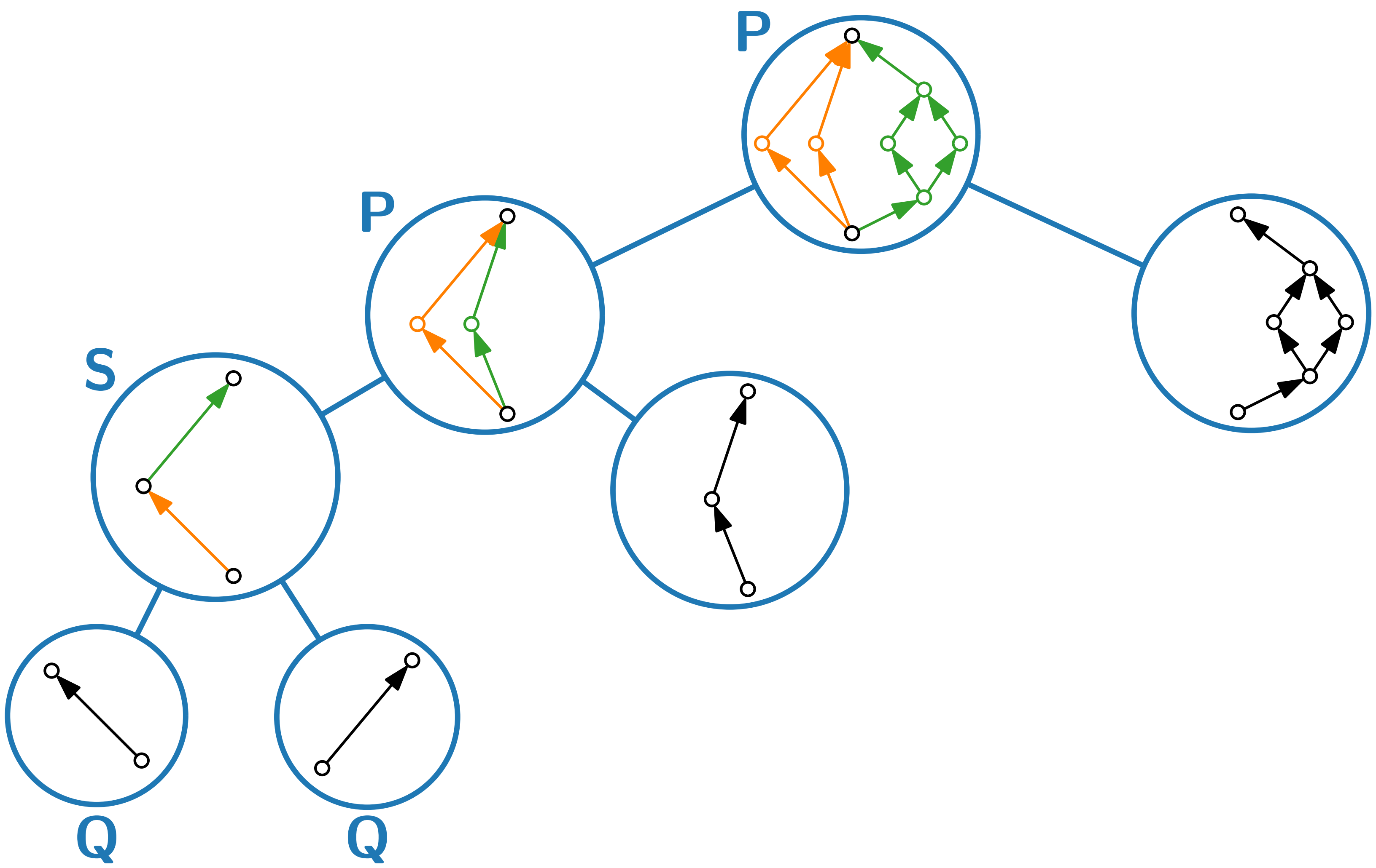

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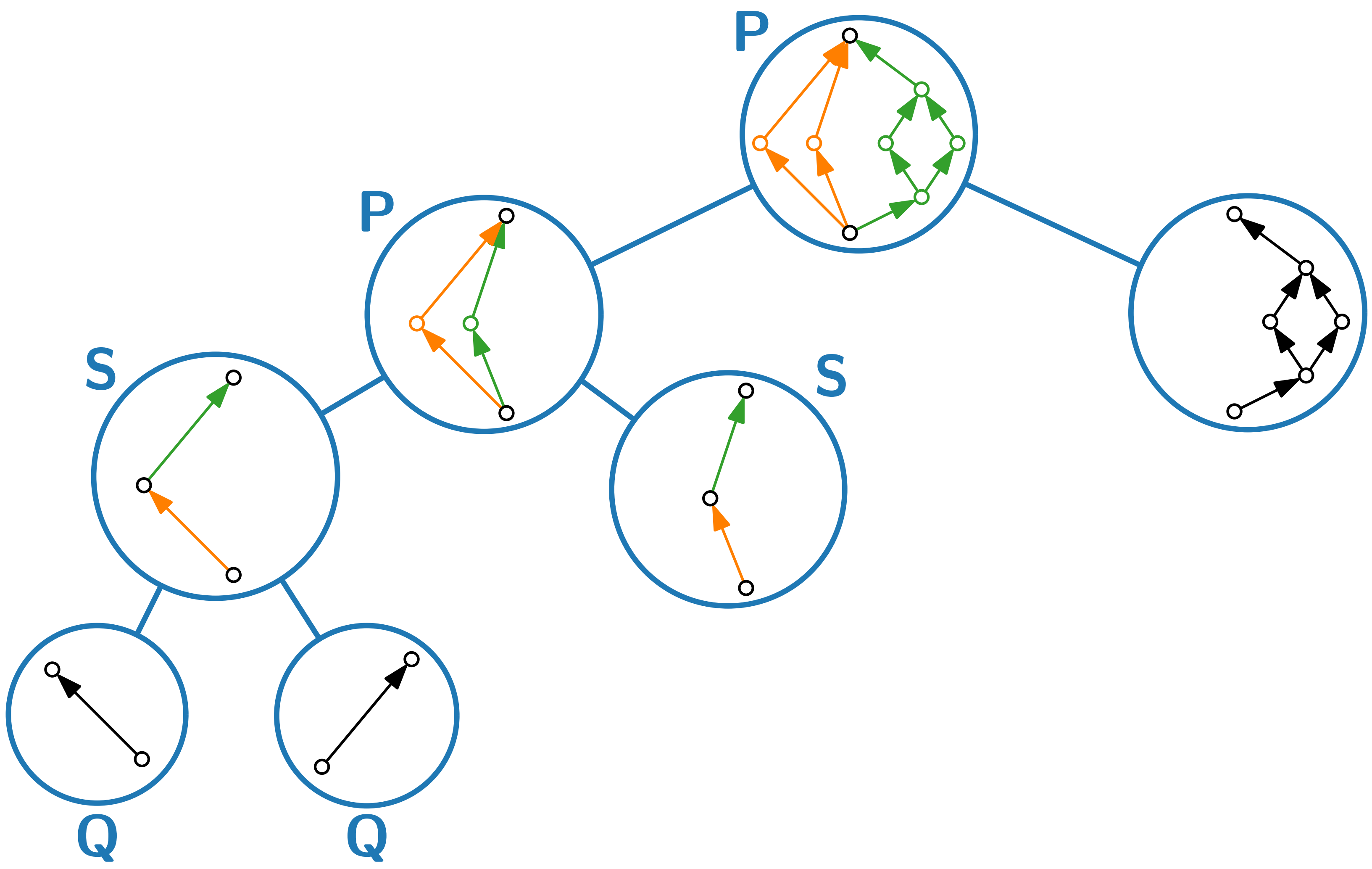

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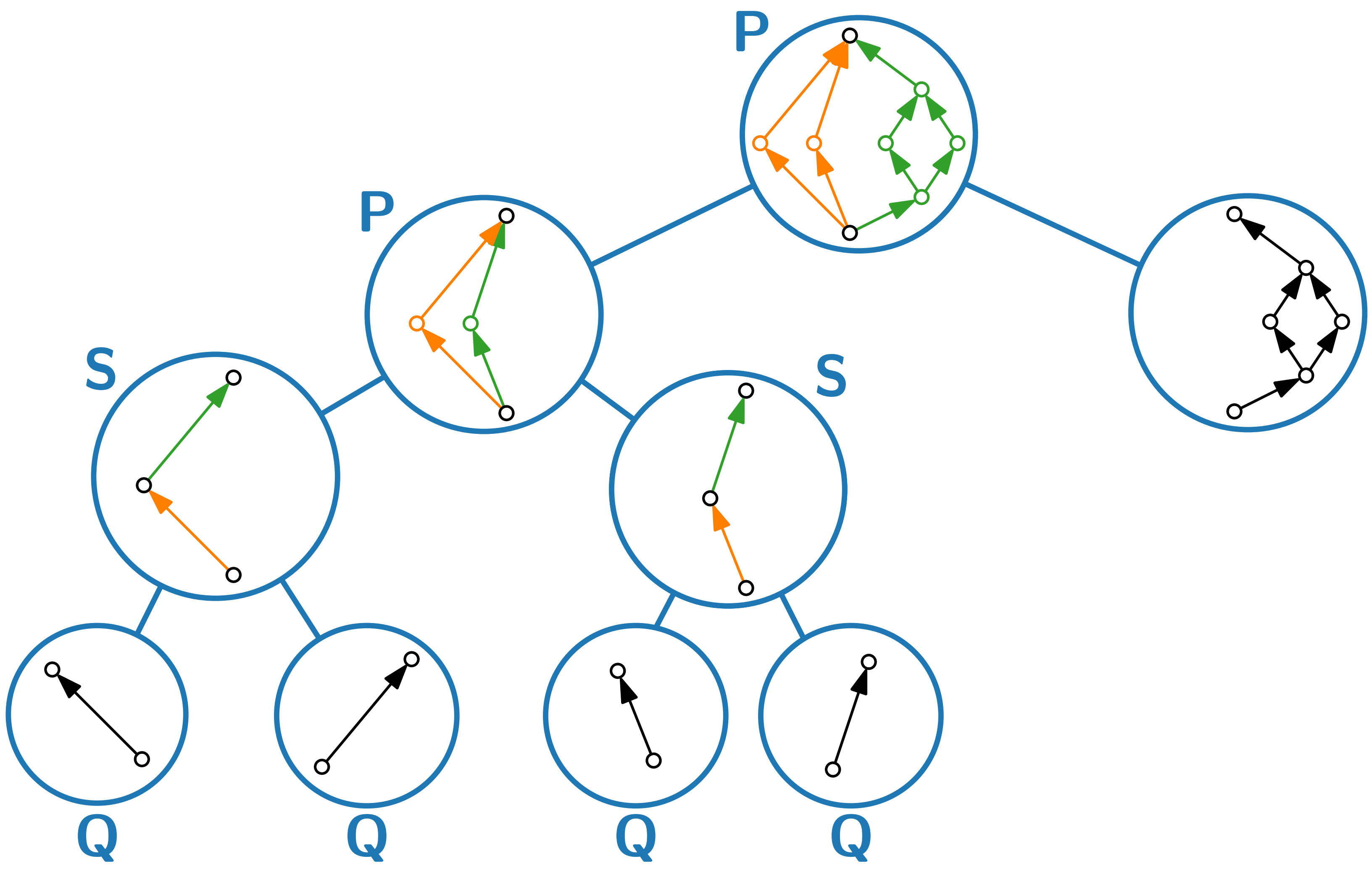

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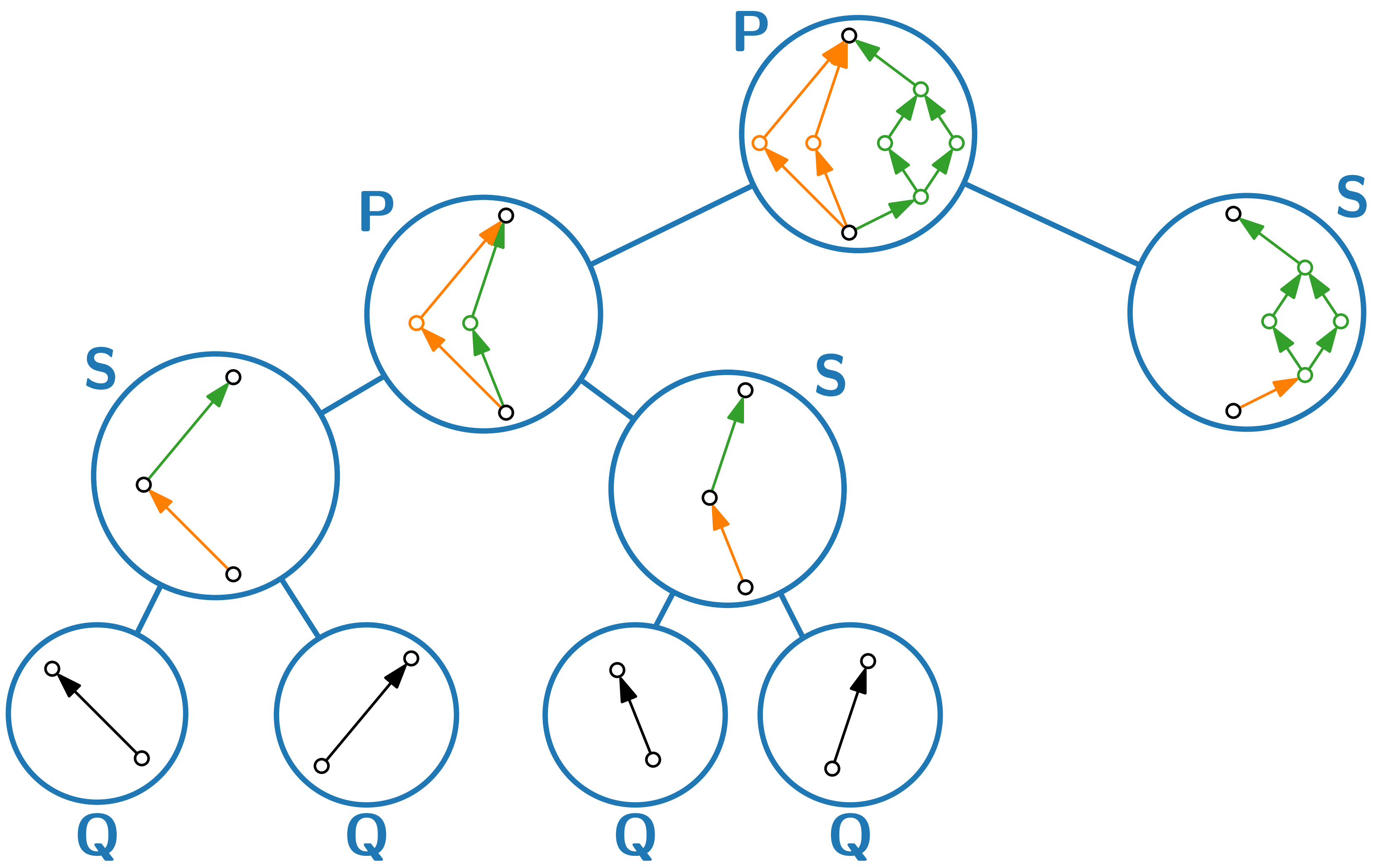

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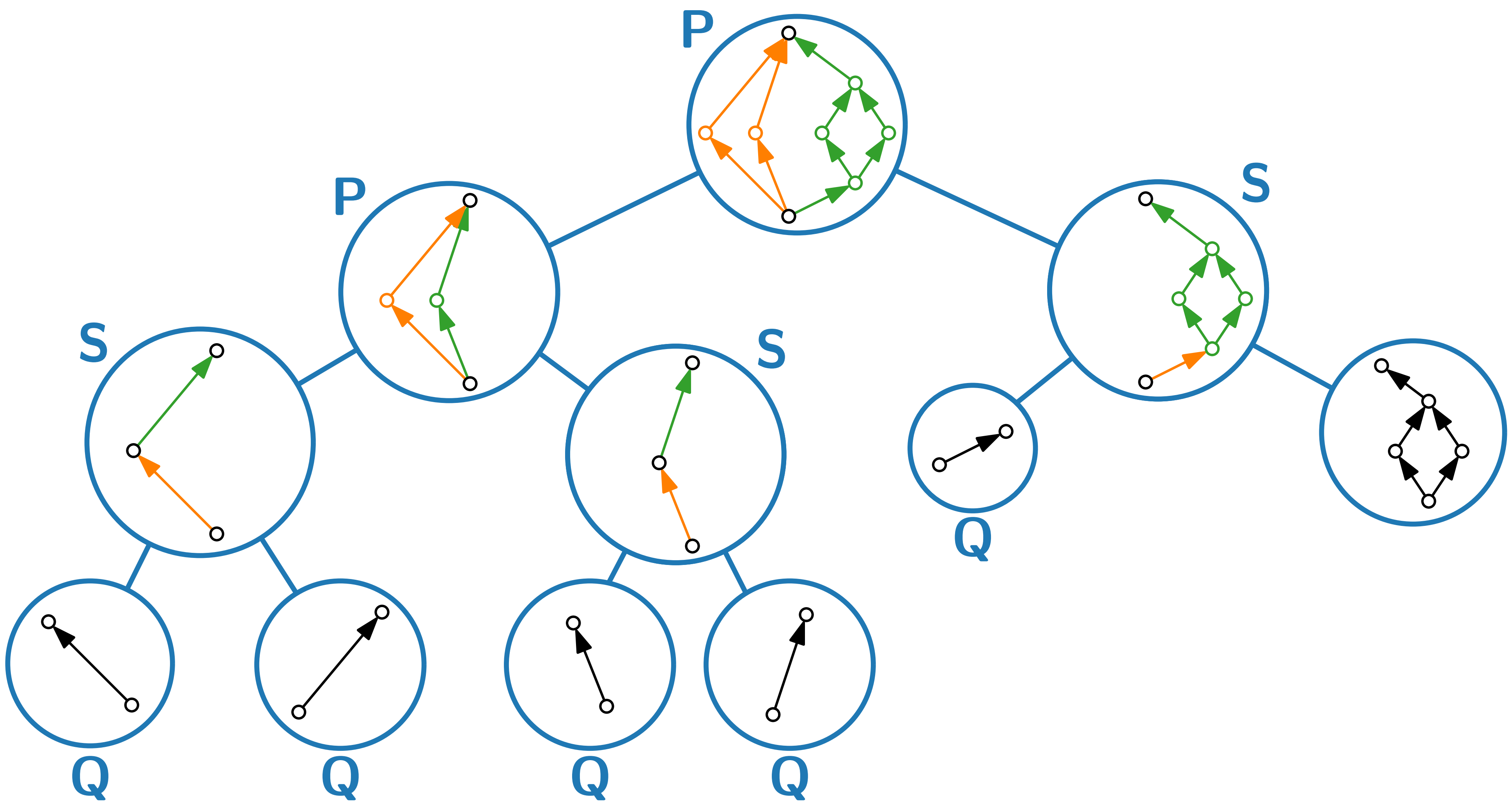

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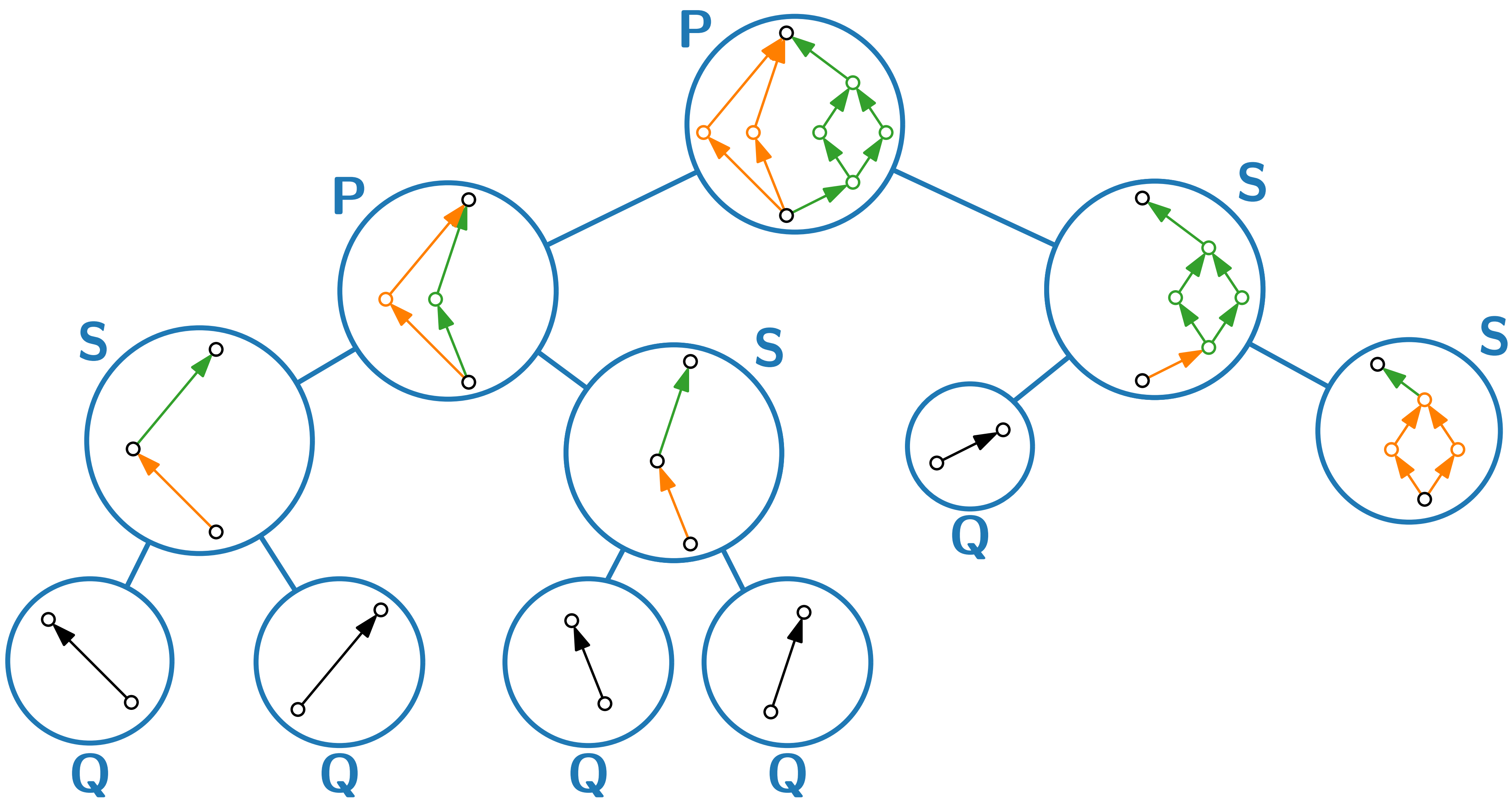

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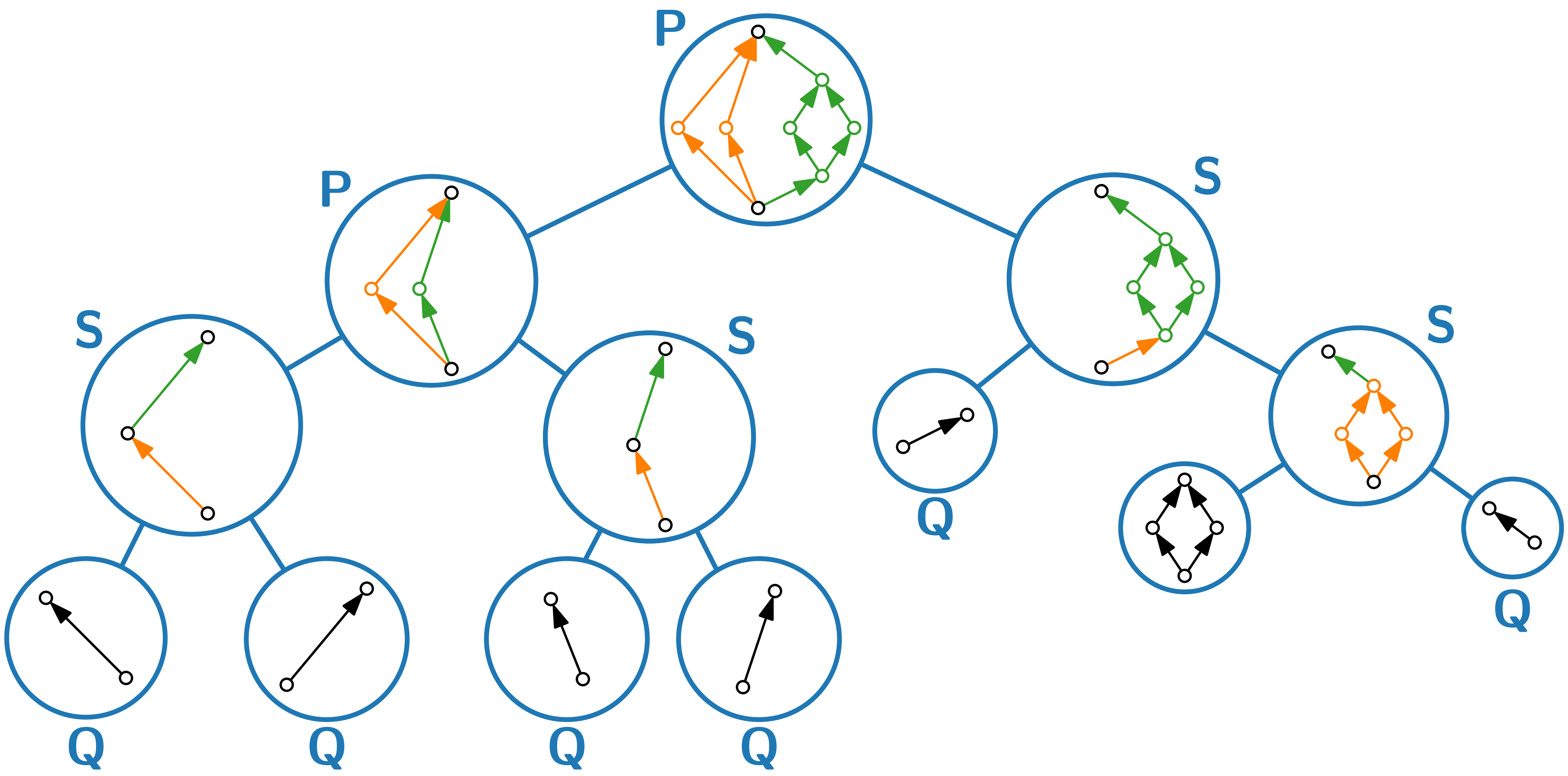

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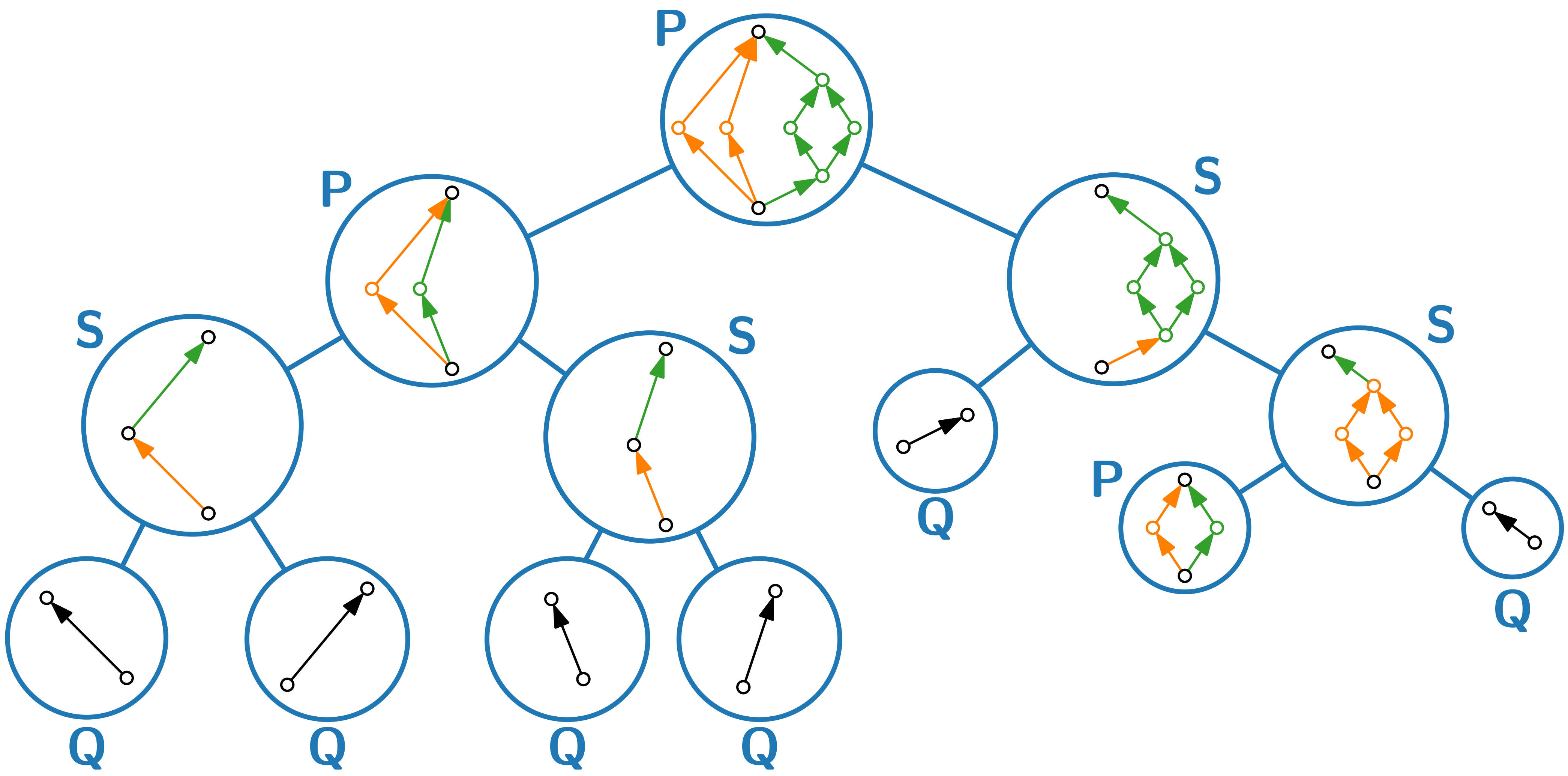

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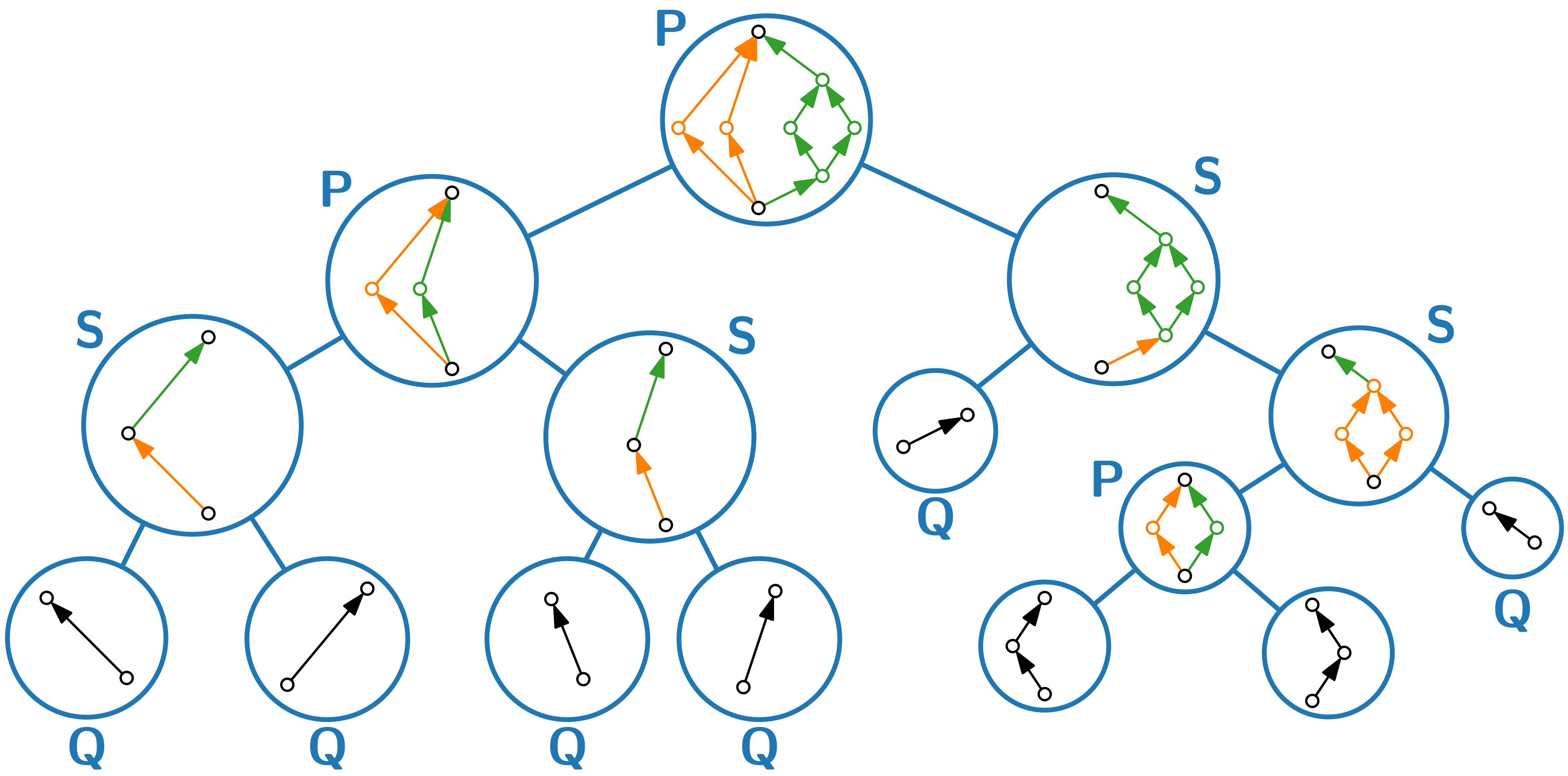

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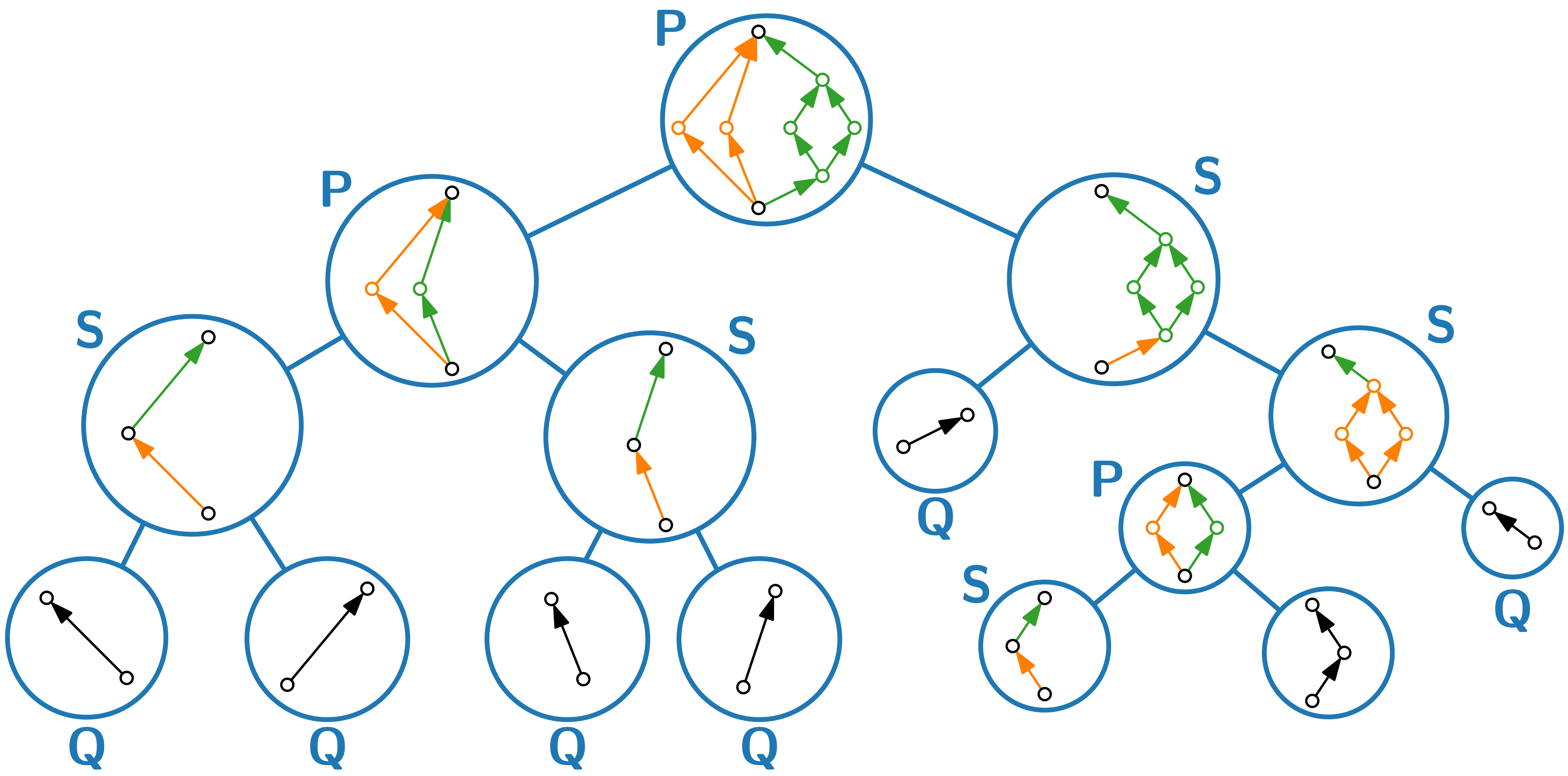

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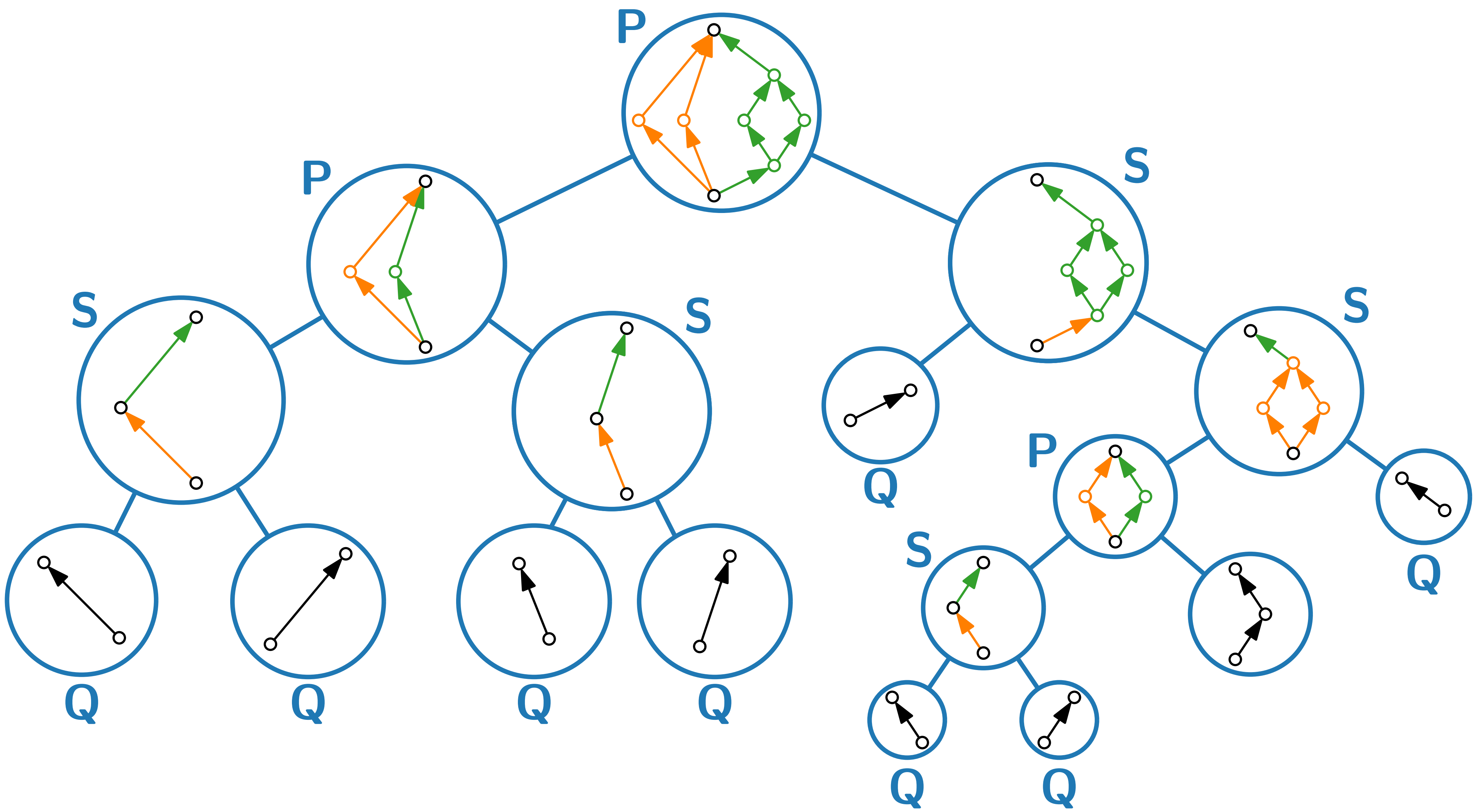

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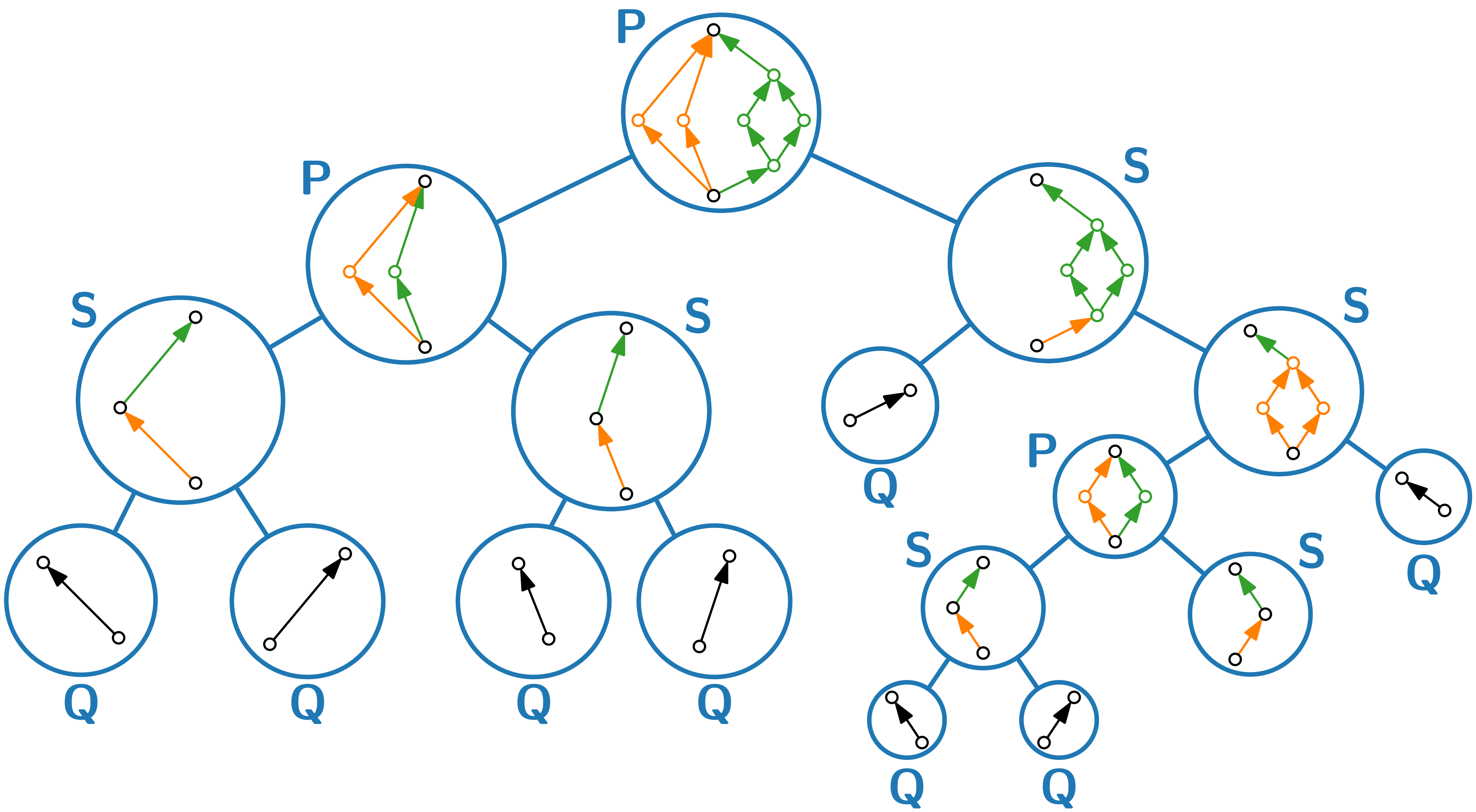

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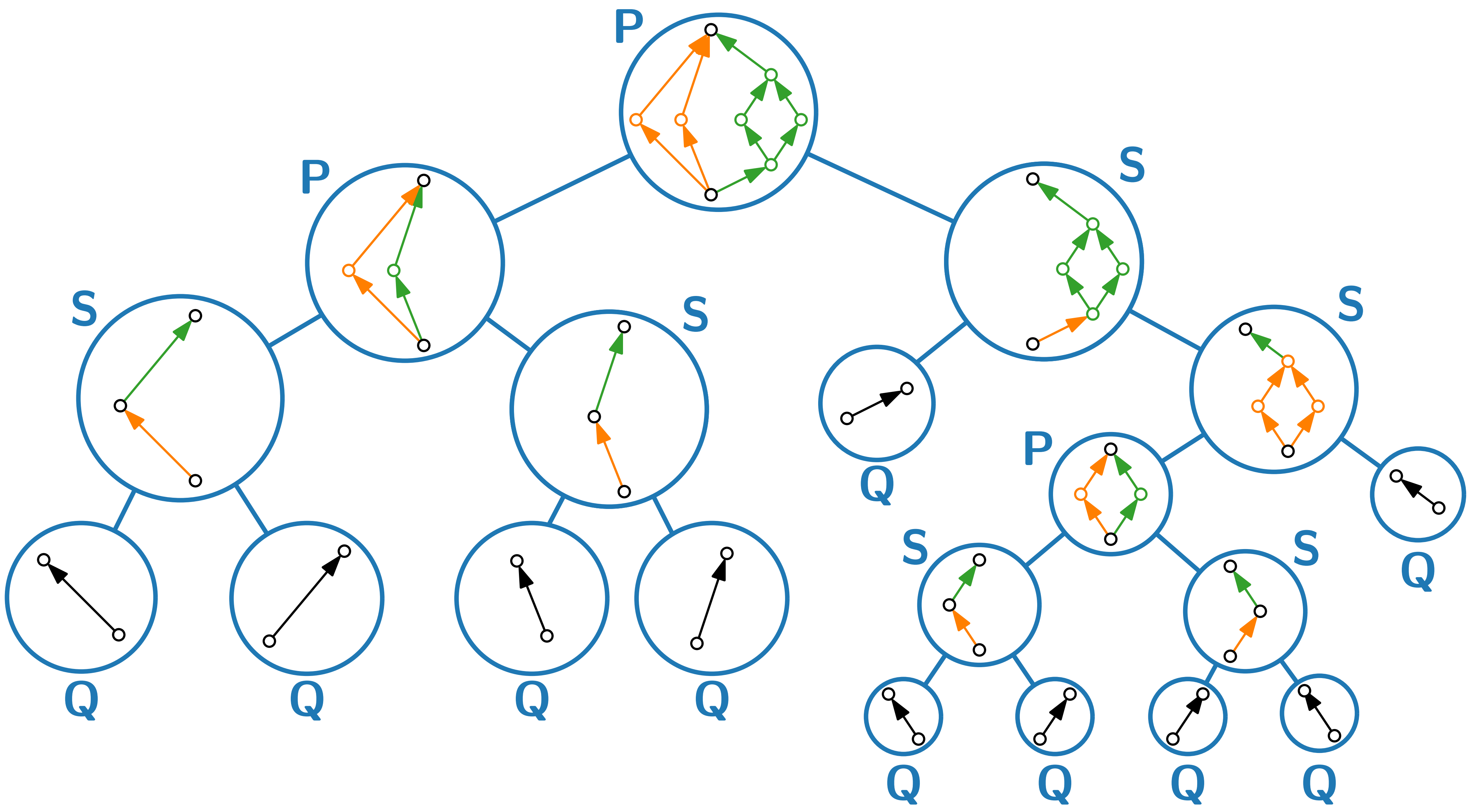
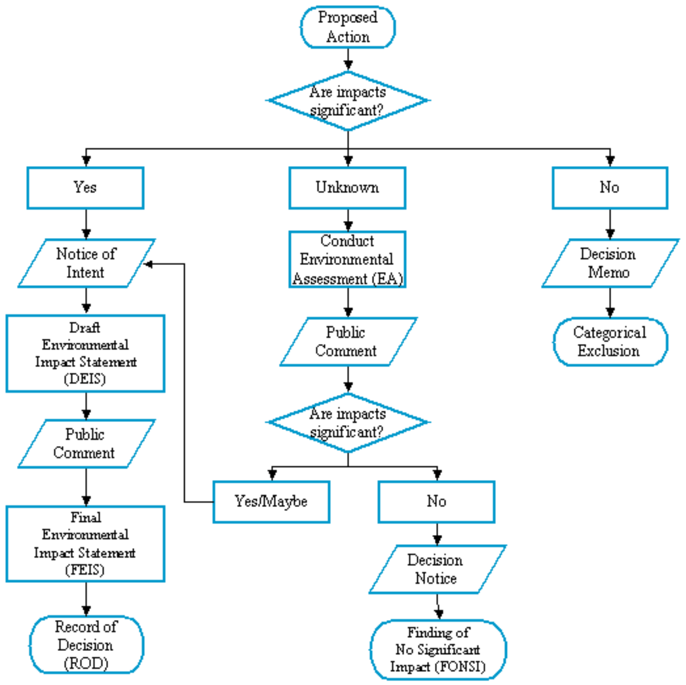
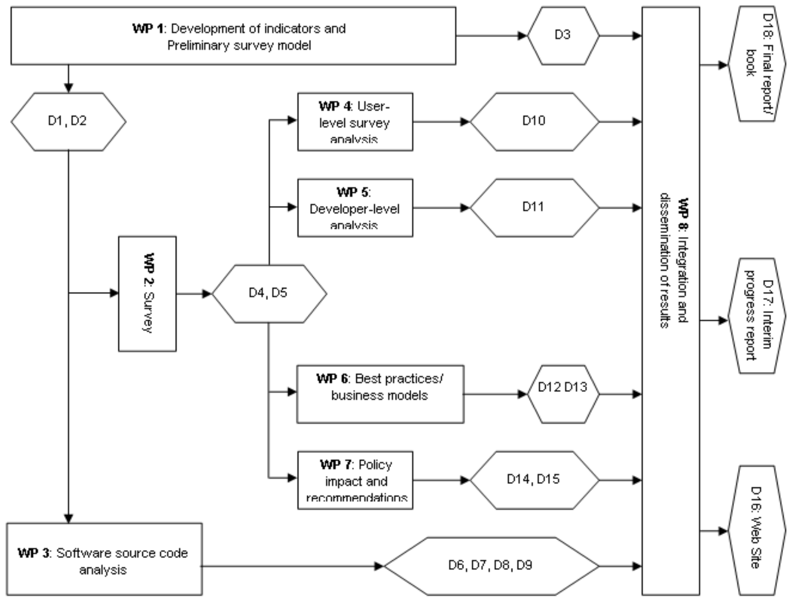


Series-Parallel Graphs – Applications



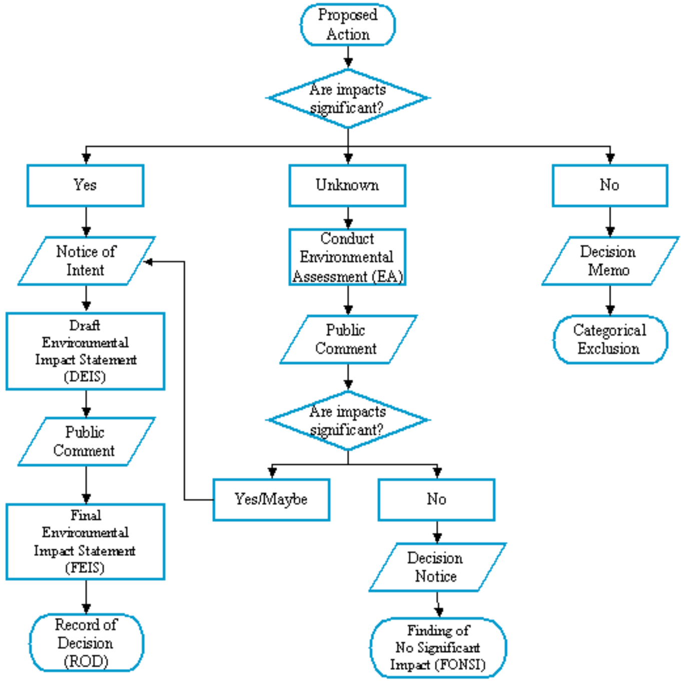
Flowcharts



PERT-Diagrams

(Program Evaluation and Review Technique)

Series-Parallel Graphs – Applications



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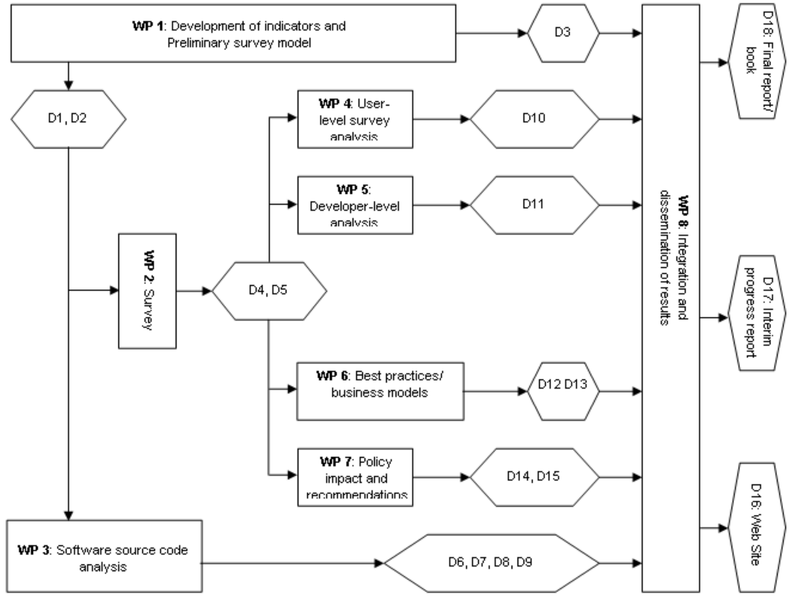


PERT-Diagrams

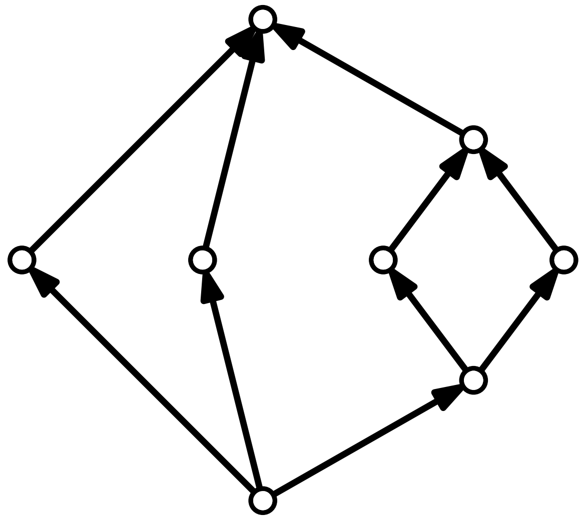
(Program Evaluation and Review Technique)

Computational complexity:

Series-parallel graphs often admit linear-time algorithms for NP-hard problems, e.g., minimum maximal matching, maximum independent set, Hamiltonian completion.

Series-Parallel Graphs – Drawing Style

Drawing conventions

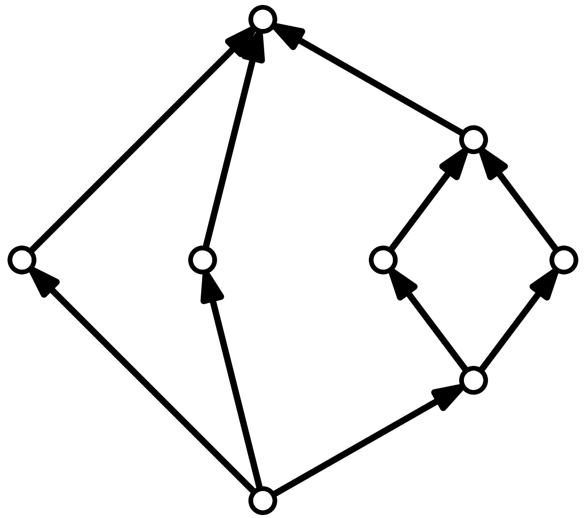


Drawing aesthetics to optimize

Series-Parallel Graphs – Drawing Style

Drawing conventions

- Planarity



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Series-Parallel Graphs – Drawing Style

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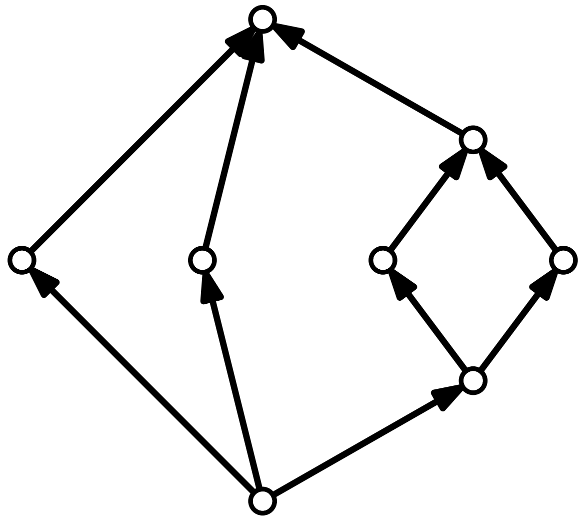
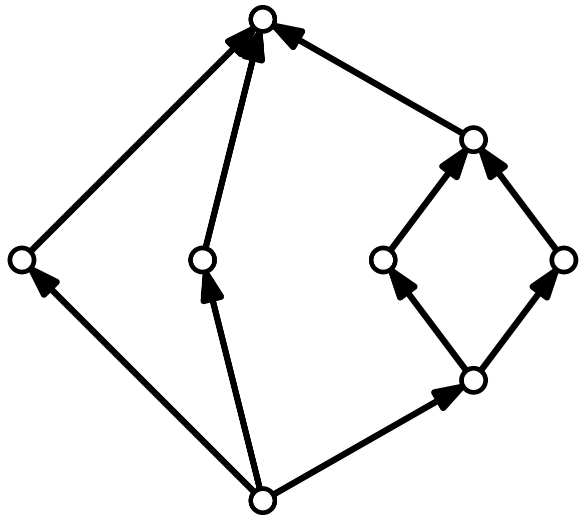


Series-Parallel Graphs – Drawing Style

Drawing conventions

- Planarity
- Straight-line edges
- Upward

Drawing aesthetics to optimize



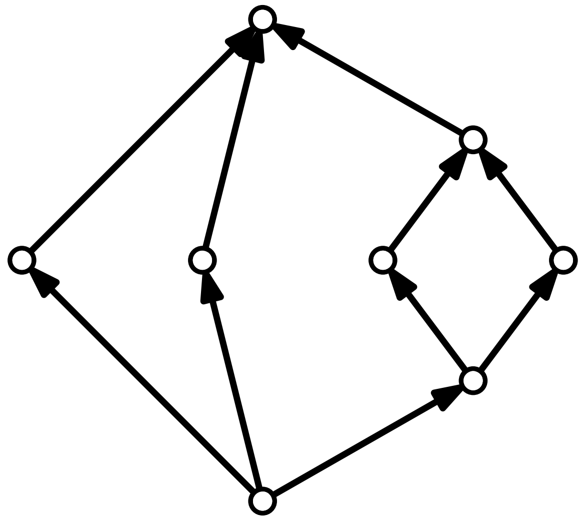
Series-Parallel Graphs – Drawing Style

Drawing conventions

- Planarity
- Straight-line edges
- Upward

Drawing aesthetics to optimize

- Area



Series-Parallel Graphs – Drawing Style

Drawing conventions
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- Straight-line edges
- Upward

Drawing aesthetics to optimize
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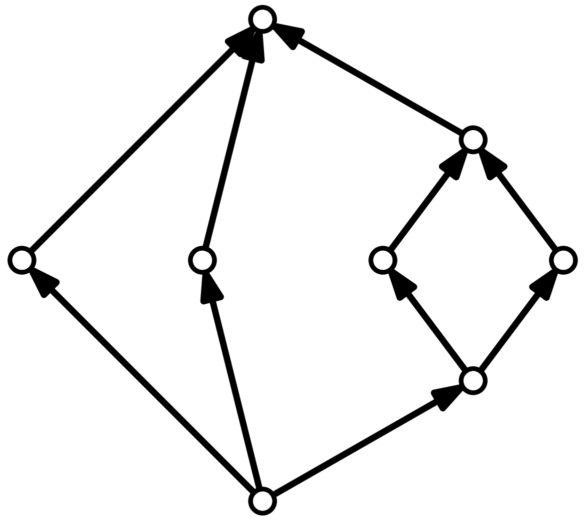
- Symmetry



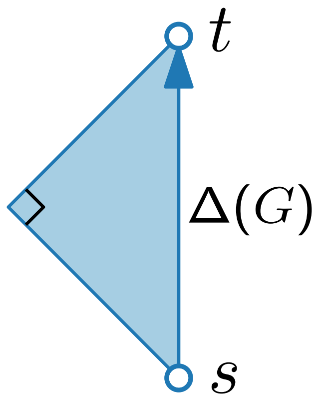
Series-Parallel Graphs – Straight-Line Drawings

Divide & conquer algorithm using the decomposition tree

Series-Parallel Graphs – Straight-Line Drawings

Divide & conquer algorithm using the decomposition tree

- Invariant: draw G inside a right-angled isosceles bounding triangle $\Delta(G)$ with s at the bottom and t at the top



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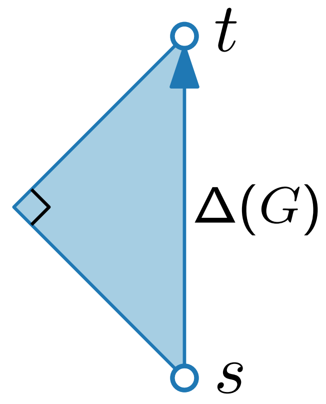

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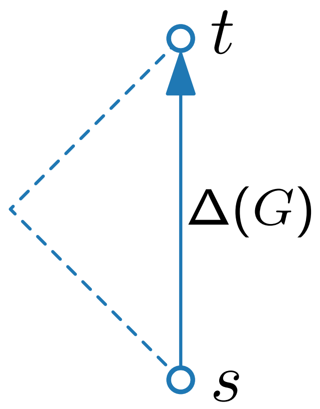

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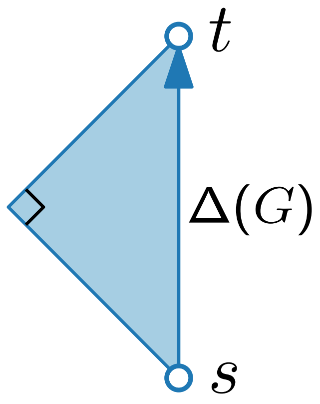

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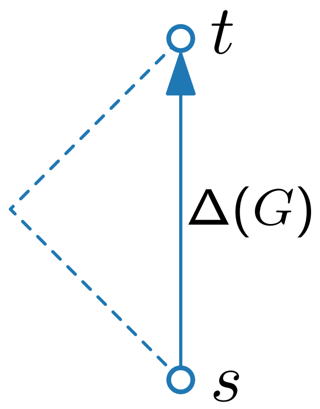

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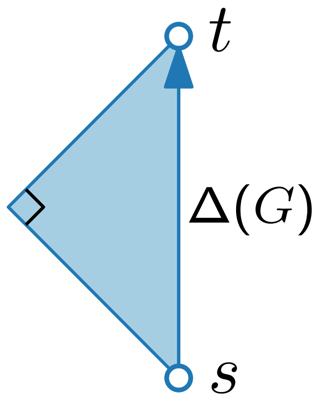
Series-Parallel Graphs – Straight-Line Drawings



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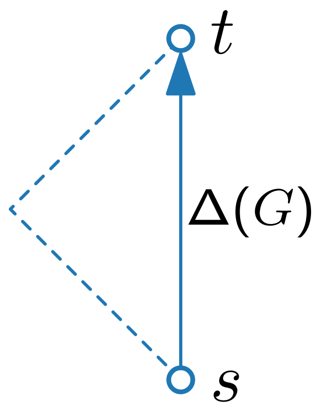

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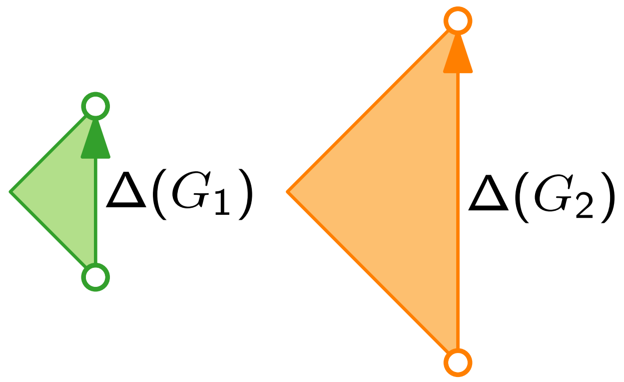

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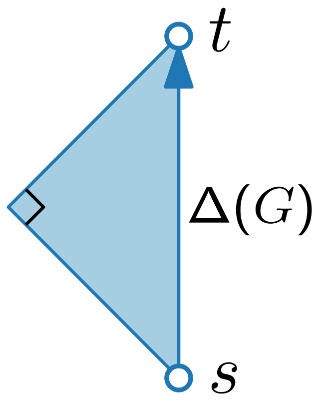

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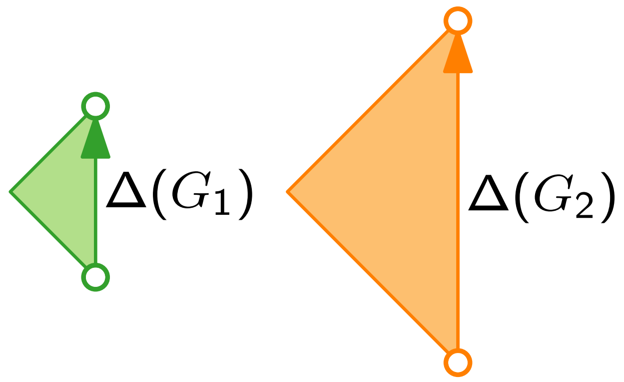
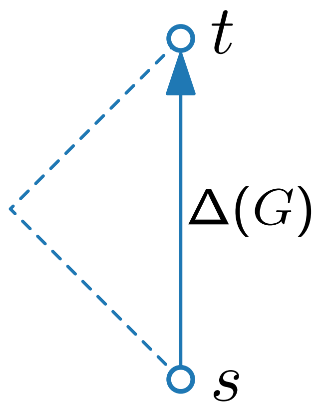
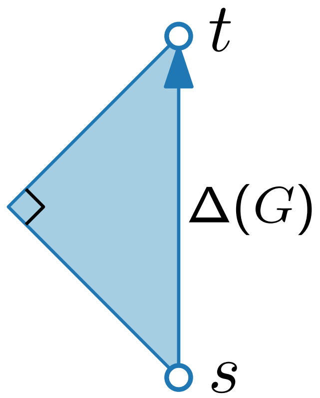


Conquer:



Series-Parallel Graphs – Straight-Line Drawings

Divide & conquer algorithm using the decomposition tree



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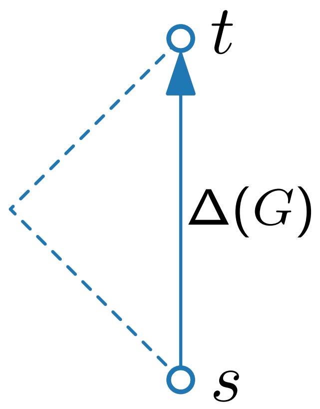

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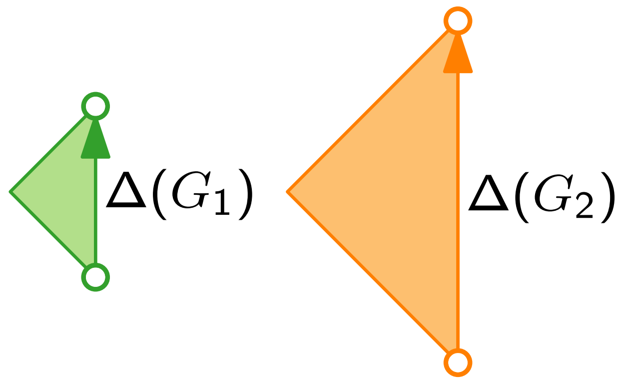
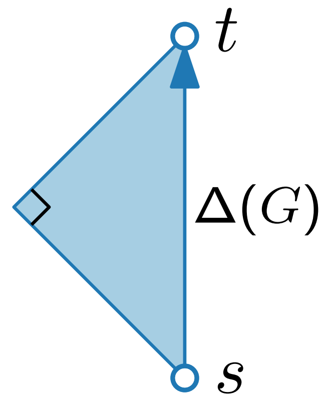


Conquer:

- S-nodes: series compositions

Series-Parallel Graphs – Straight-Line Drawings

Divide & conquer algorithm using the decomposition tree



- Invariant: draw G inside a right-angled isosceles bounding triangle $\Delta(G)$ with s at the bottom and t at the top

Base case: Q-nodes

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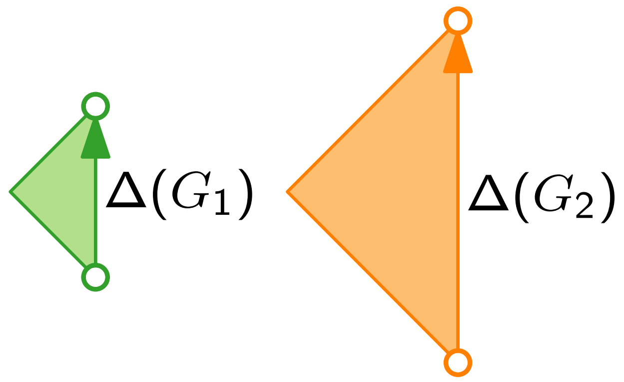
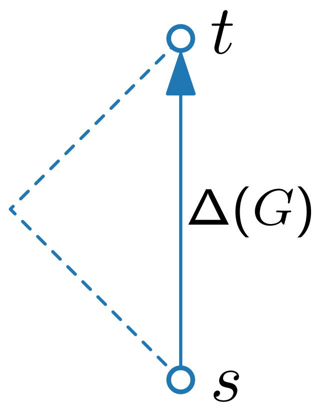
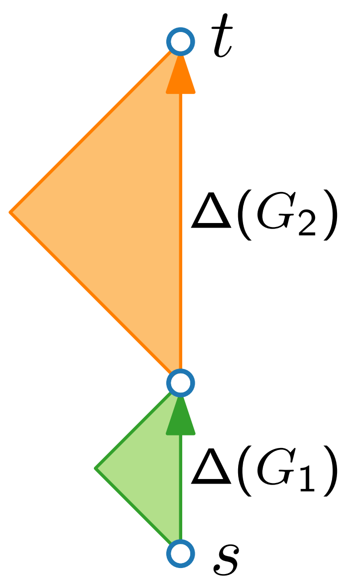


Conquer:

- S-nodes: series compositions





Series-Parallel Graphs – Straight-Line Drawings

Divide & conquer algorithm using the decomposition tree

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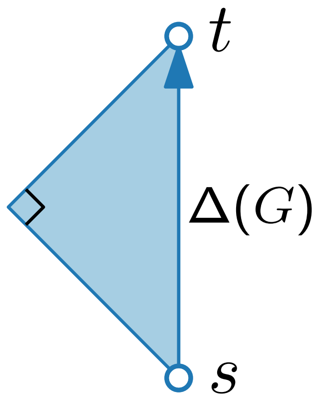

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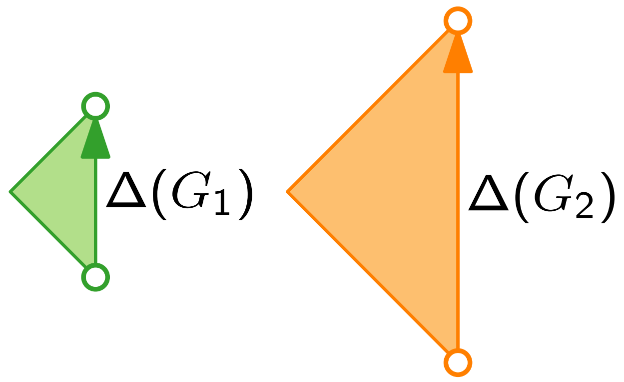

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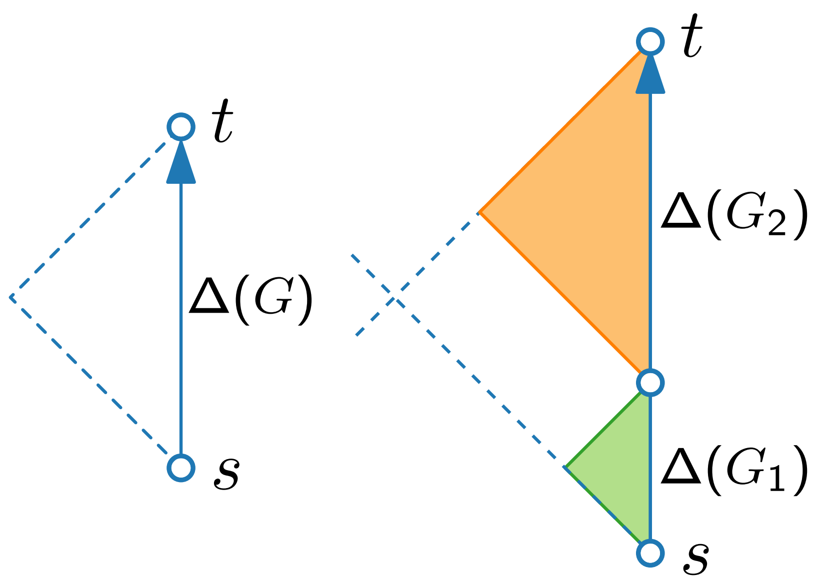

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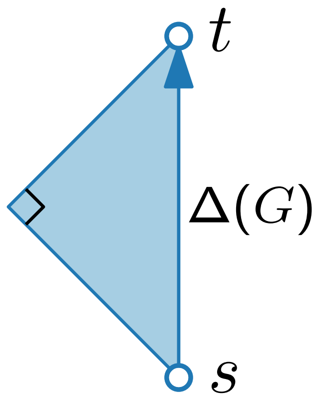

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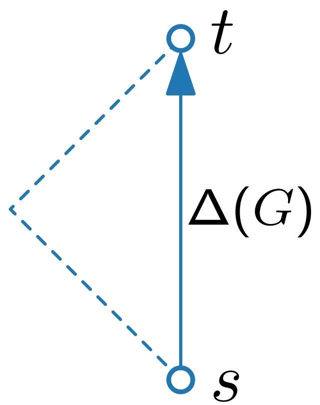

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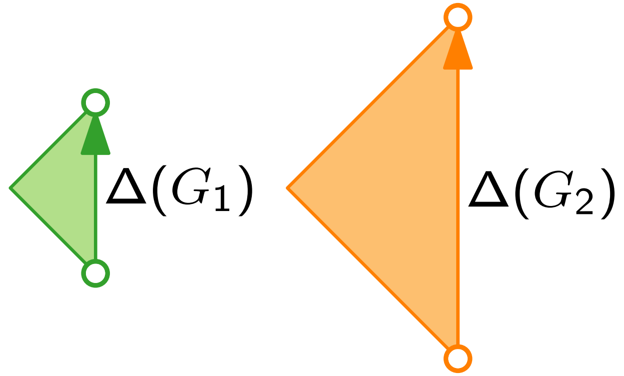

Conquer:

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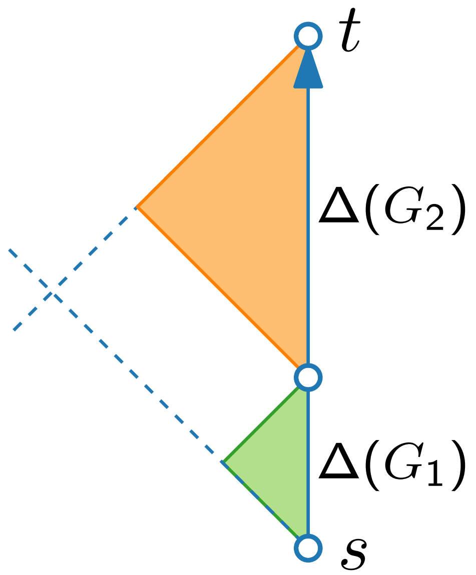
- P-nodes: parallel compositions



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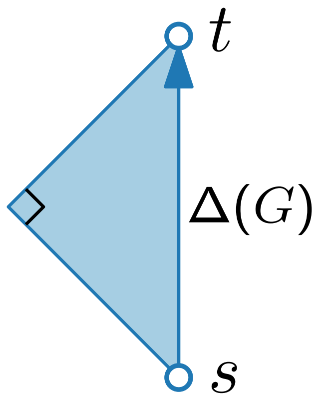

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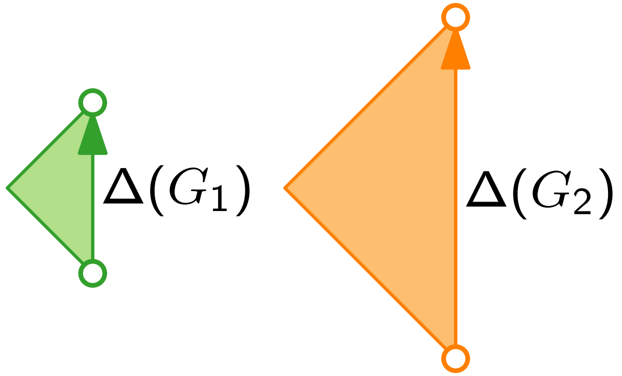

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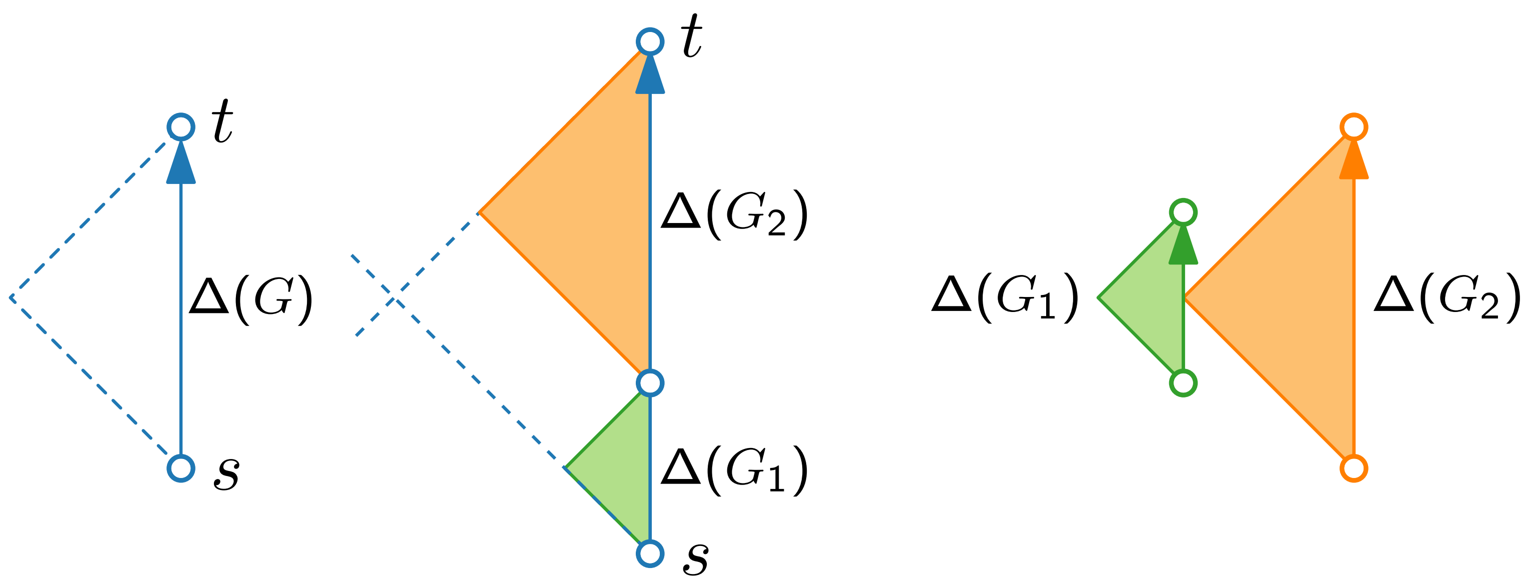

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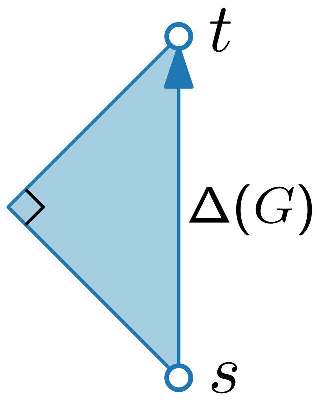

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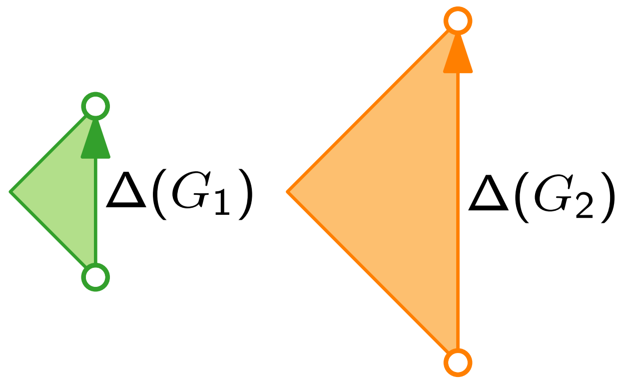

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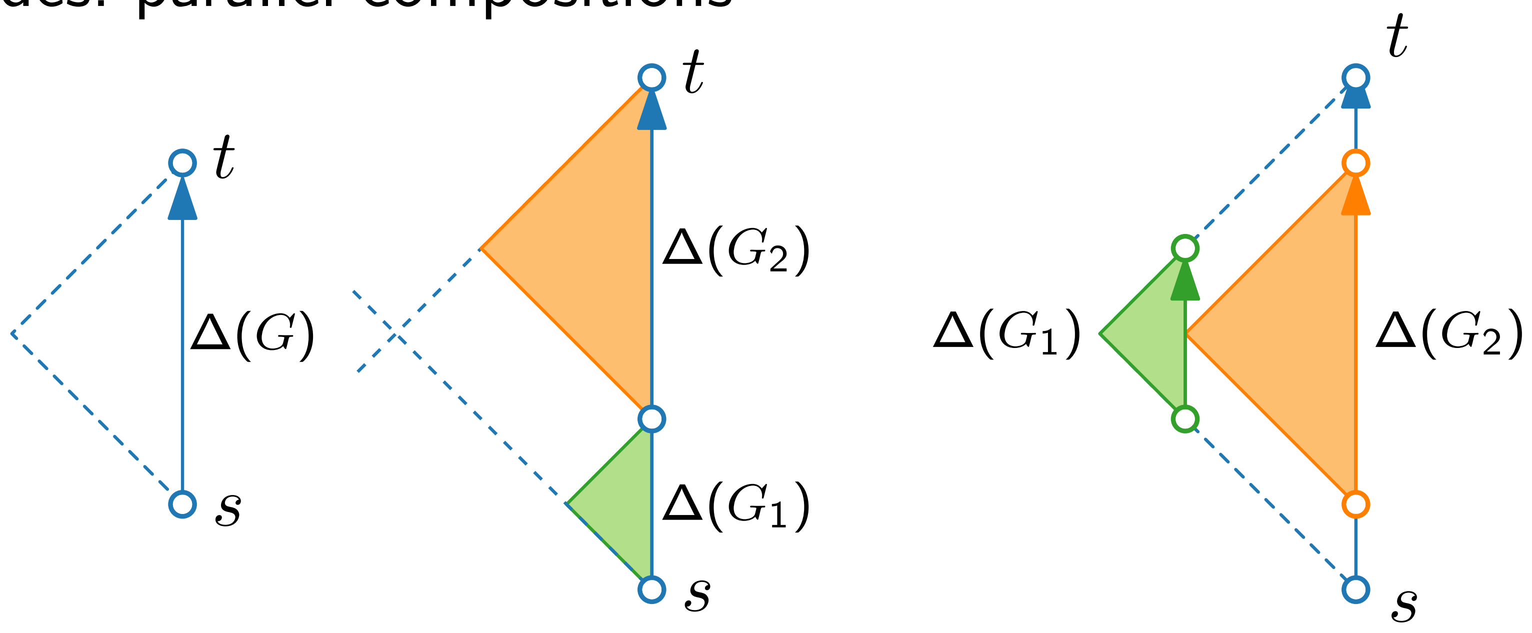

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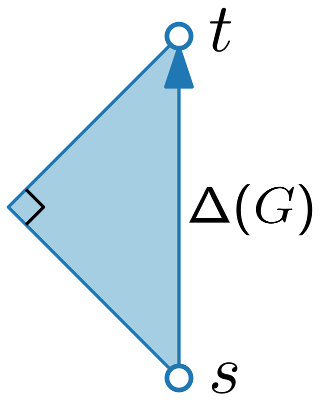

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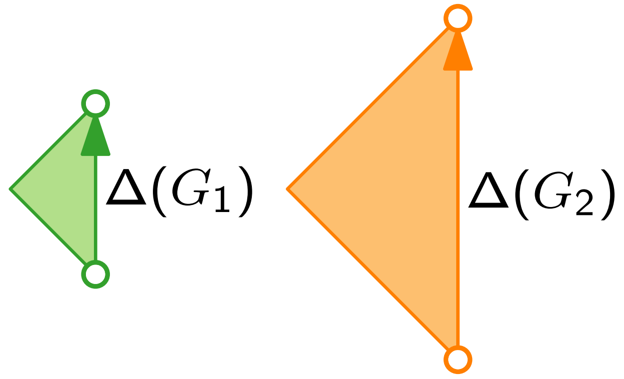

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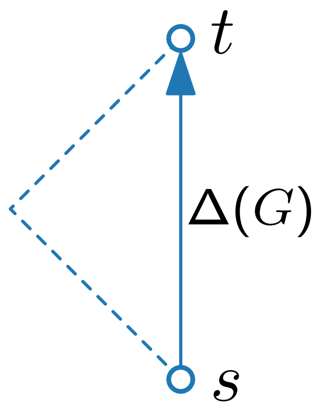

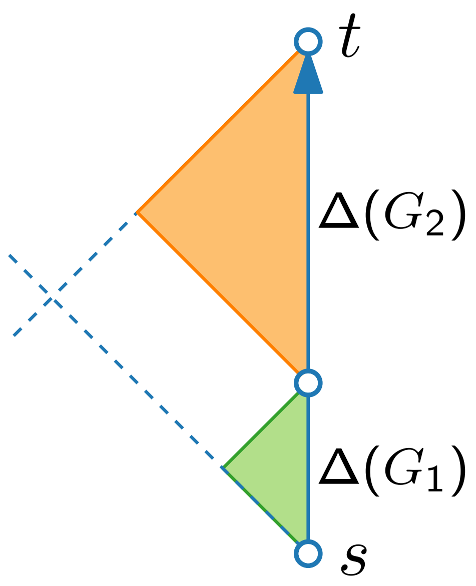

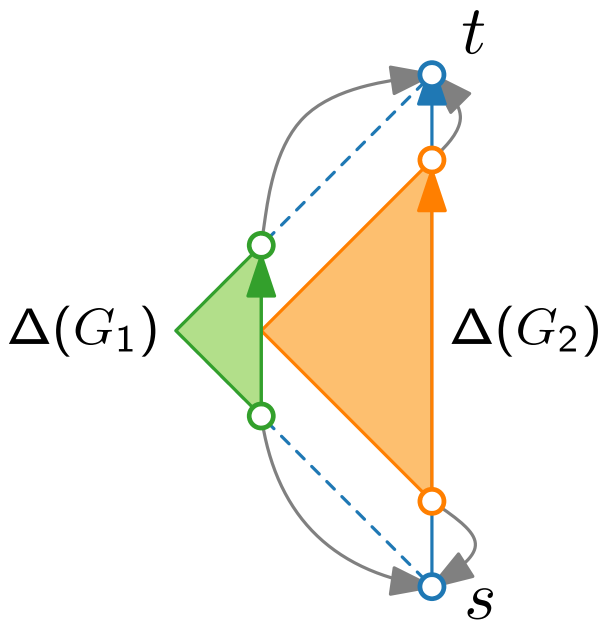

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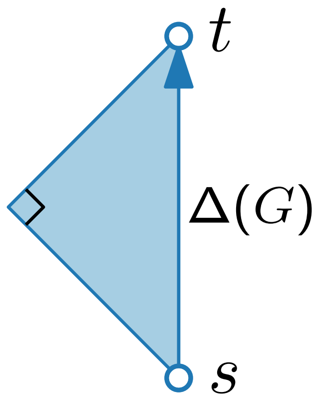

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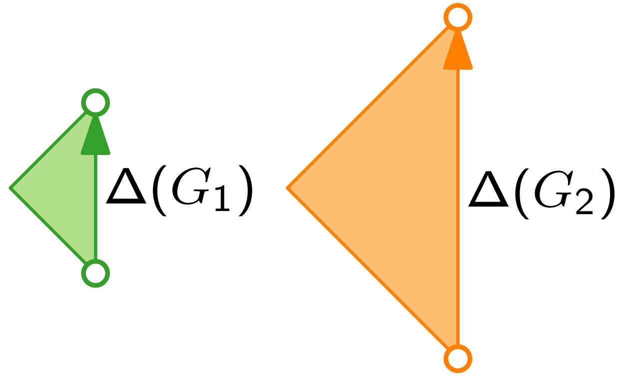

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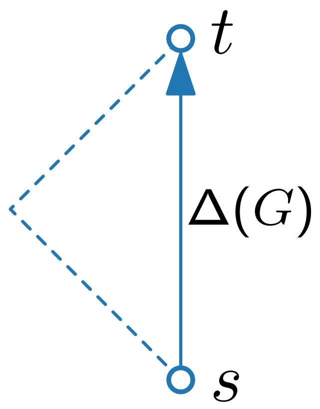
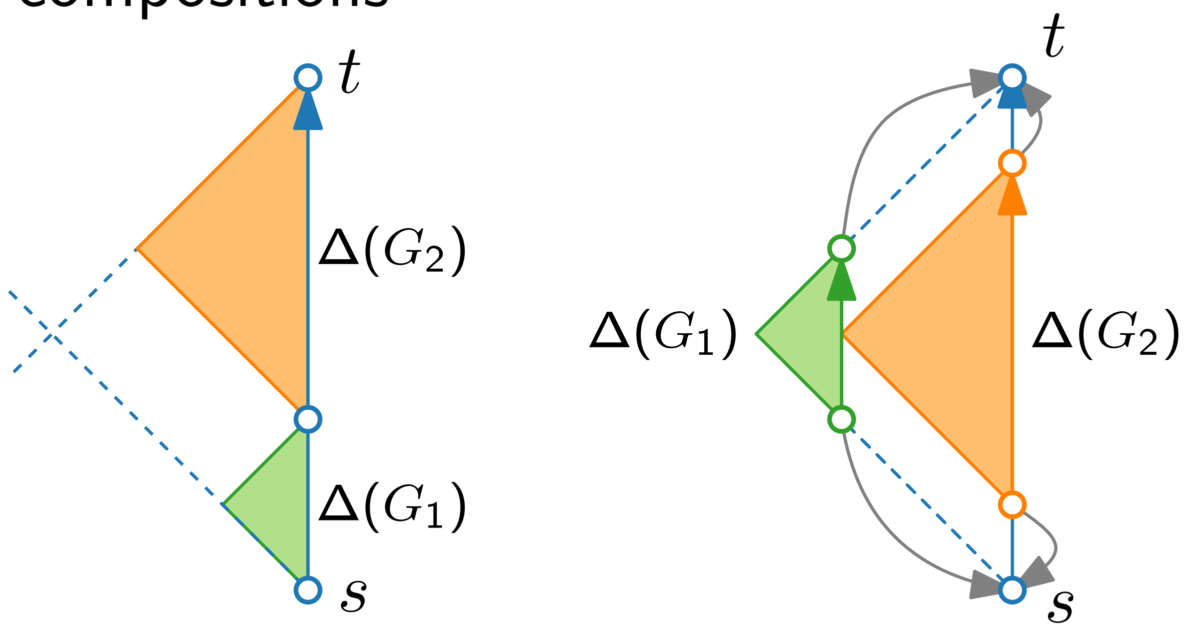
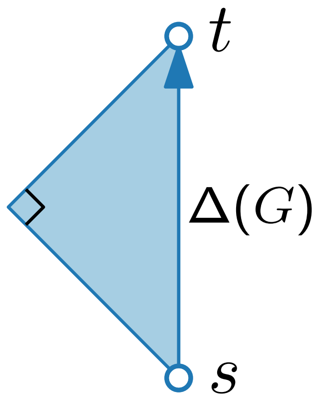




Do you see any problem?

Series-Parallel Graphs – Straight-Line Drawings

Divide & conquer algorithm using the decomposition tree



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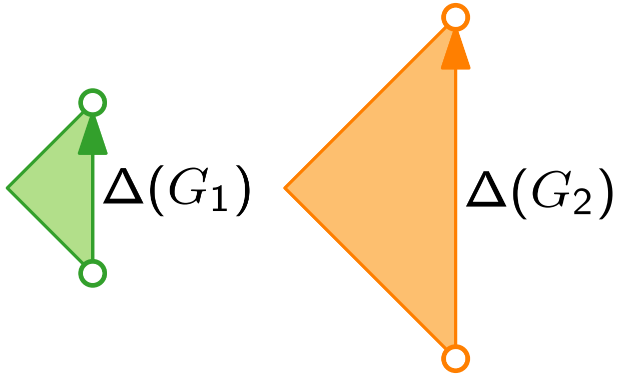

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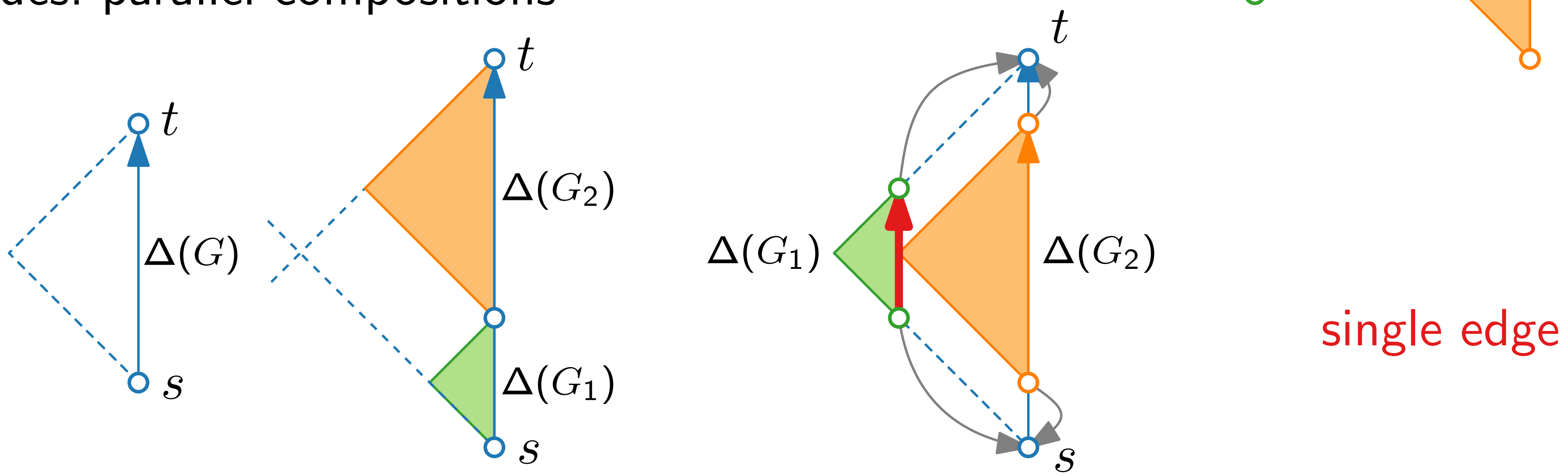
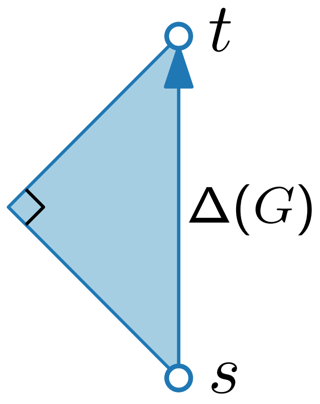


Series-Parallel Graphs – Straight-Line Drawings

Divide & conquer algorithm using the decomposition tree



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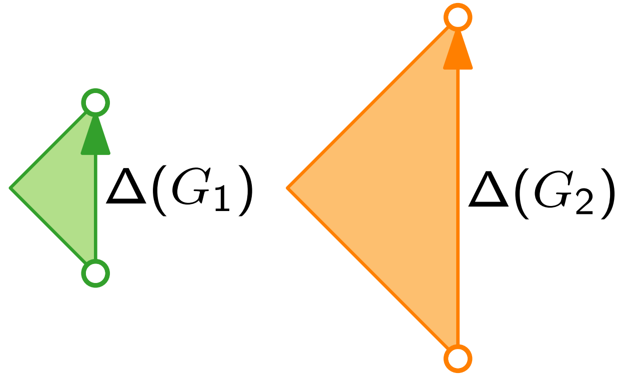

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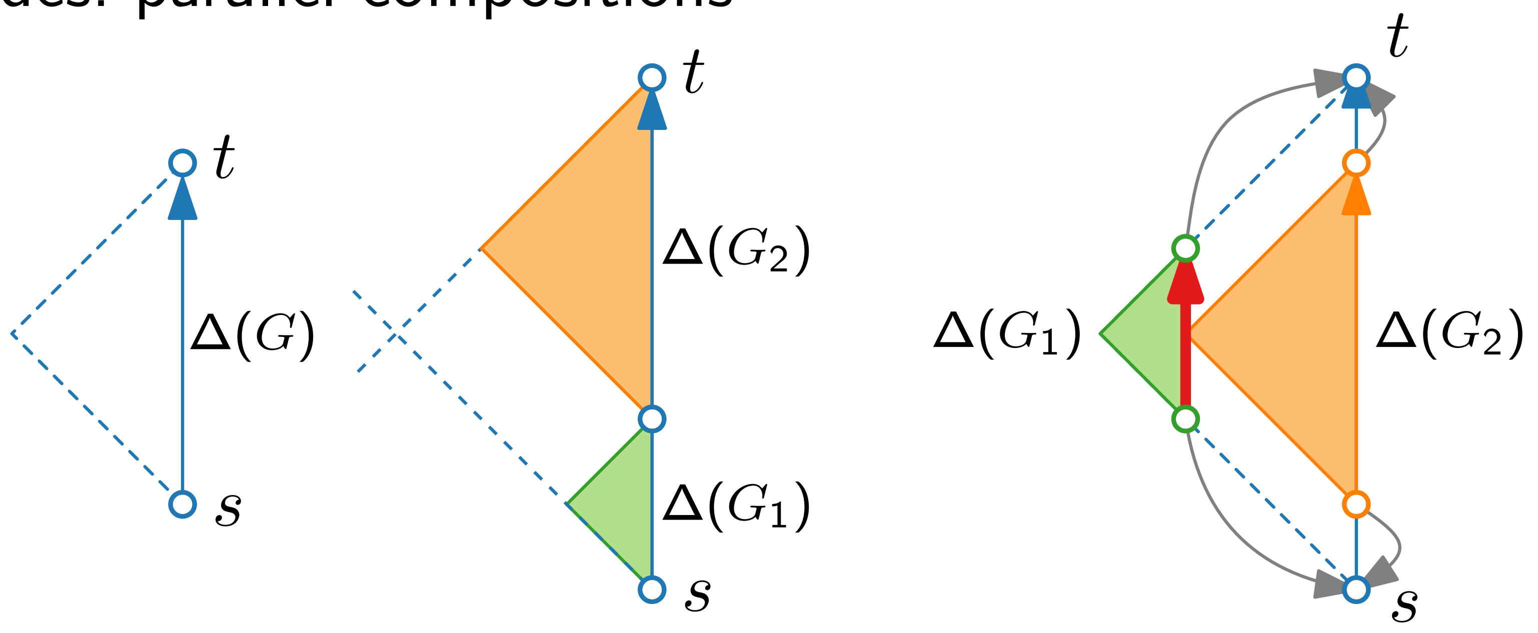

change embedding!

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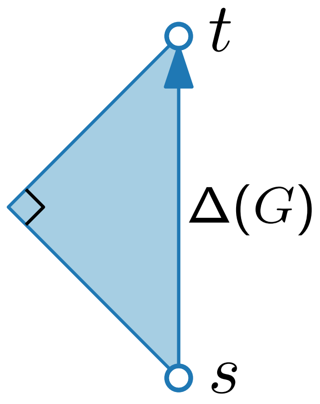

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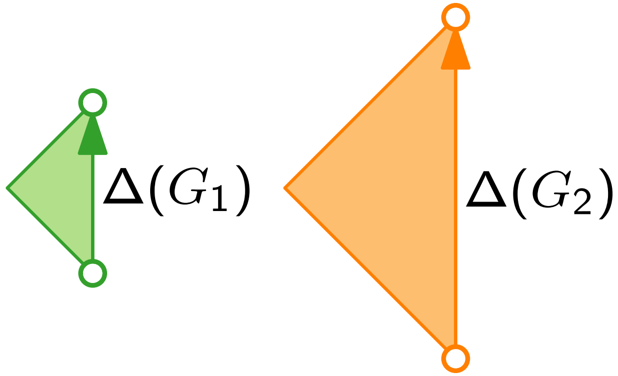

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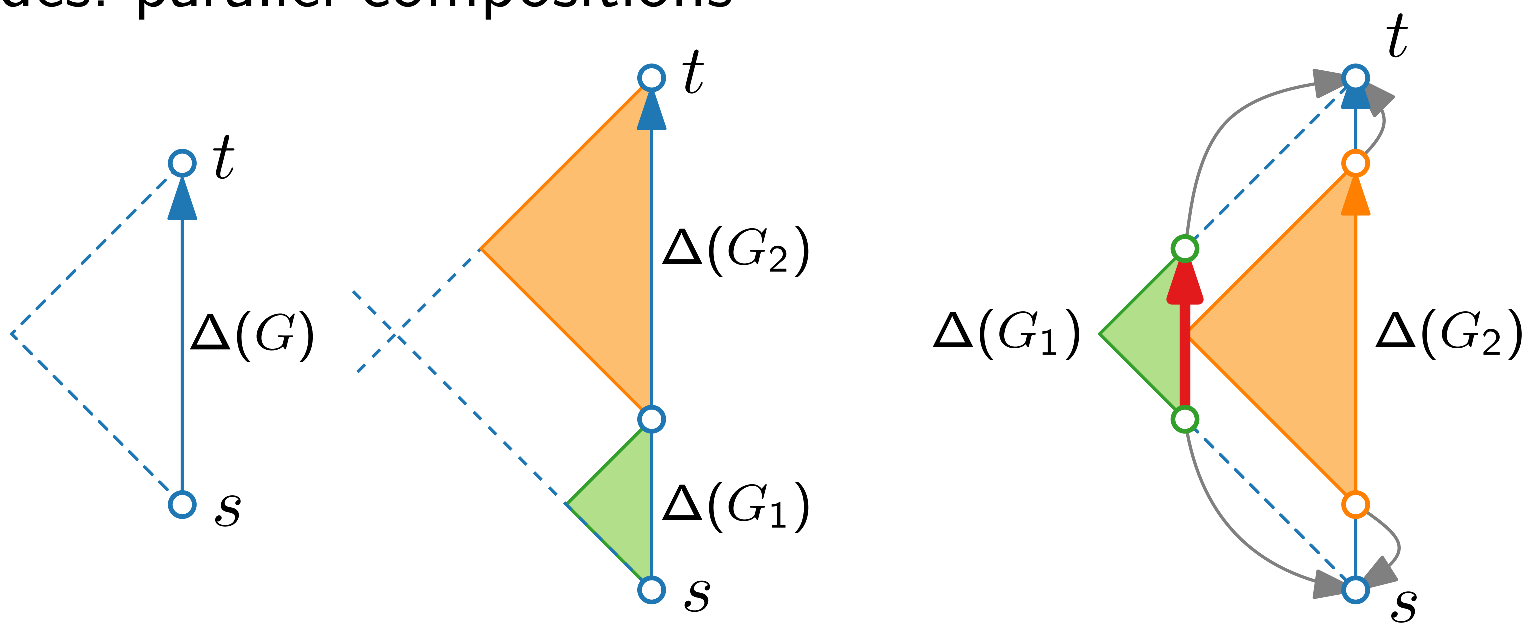

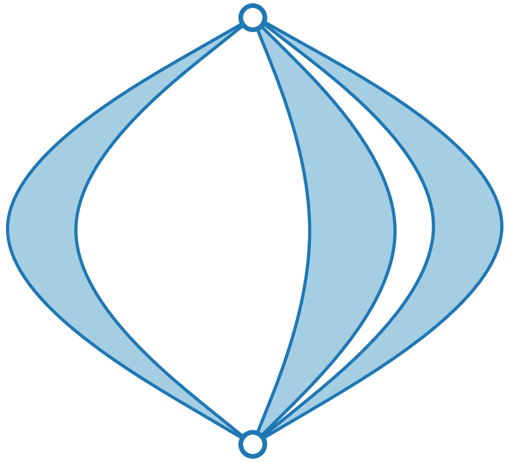

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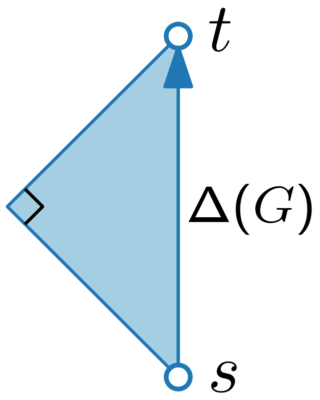

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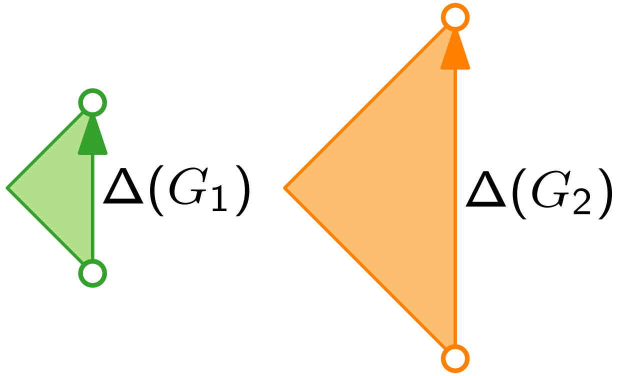

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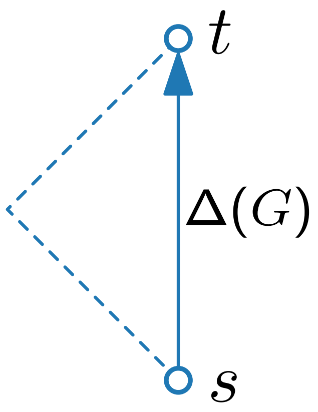

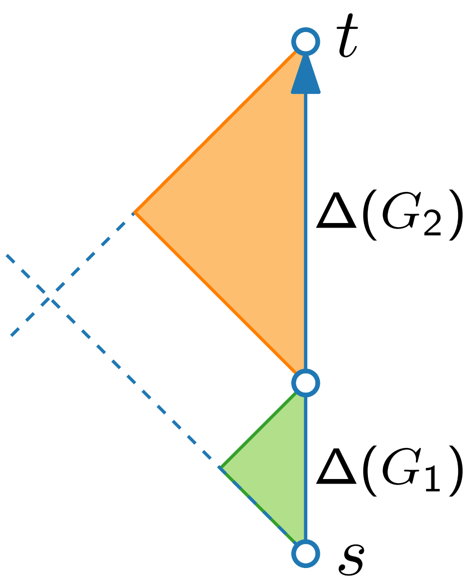

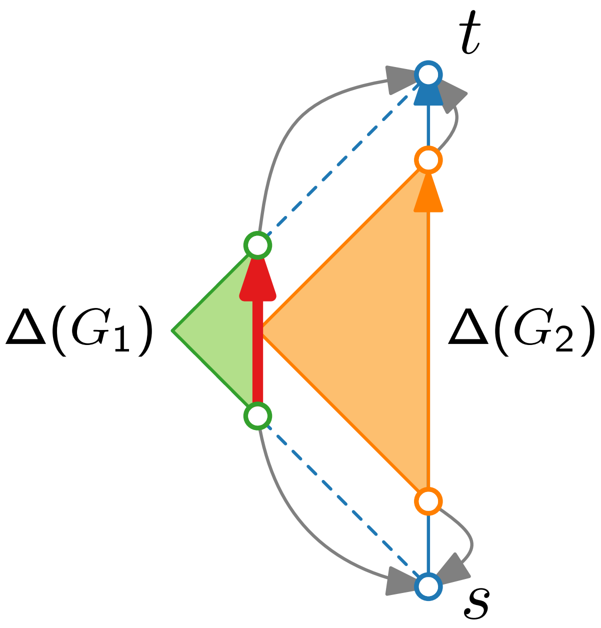

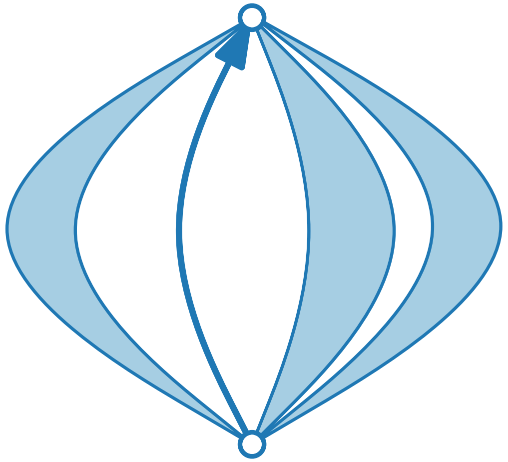

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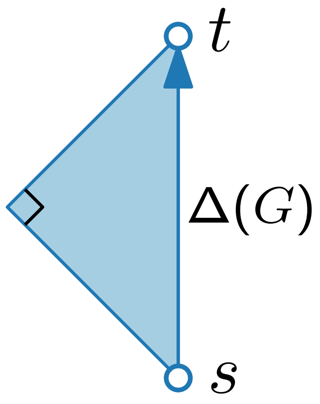

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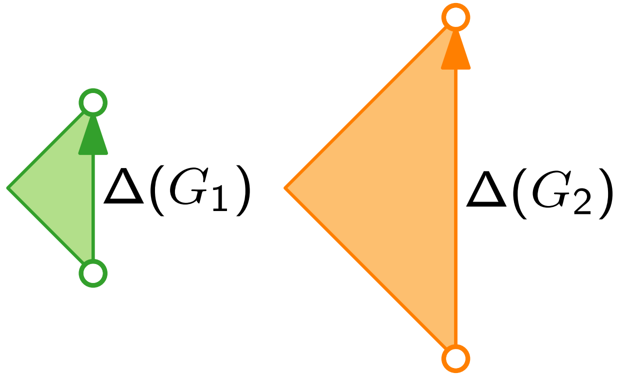

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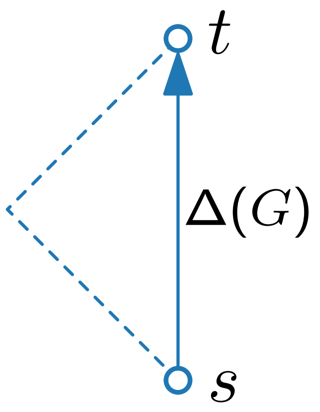

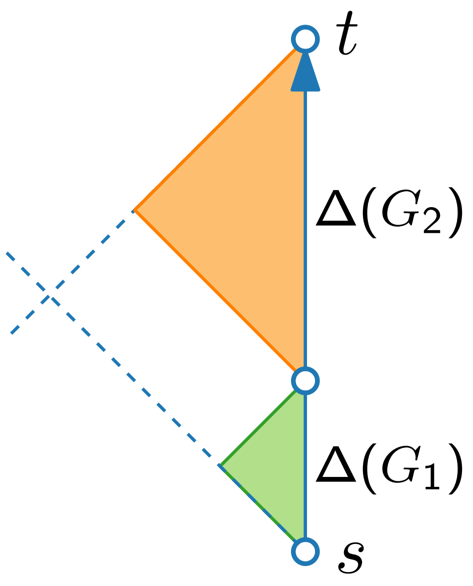

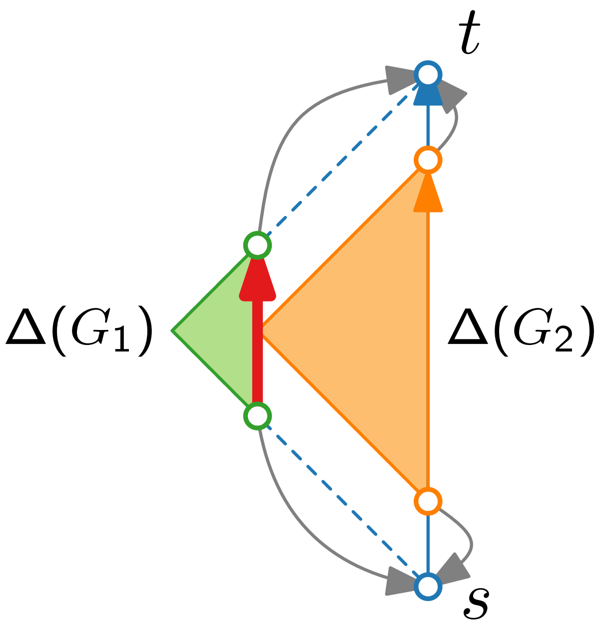

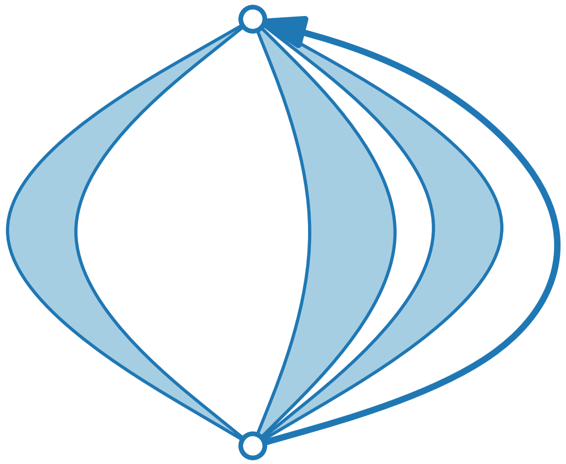

Series-Parallel Graphs – Straight-Line Drawings

- What makes parallel composition possible without creating crossings?

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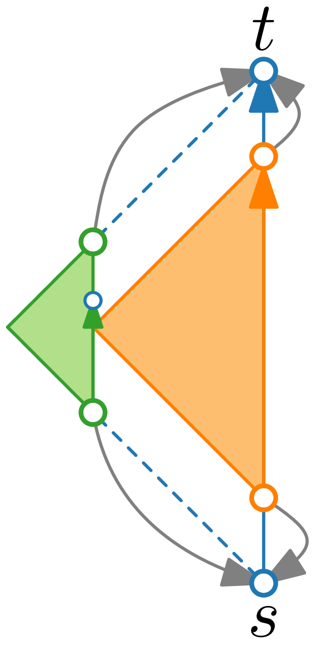

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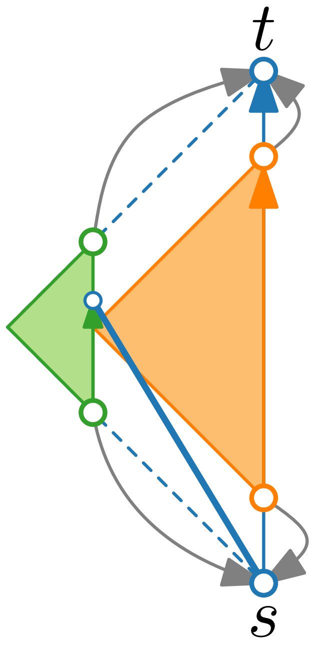

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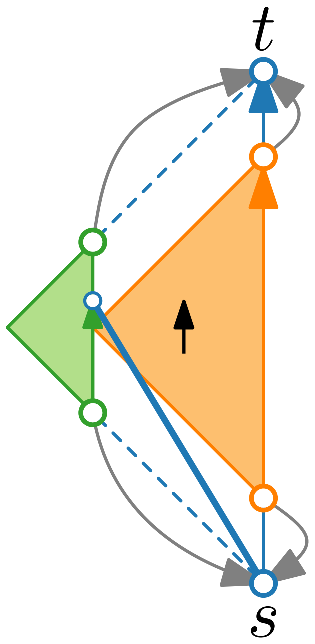

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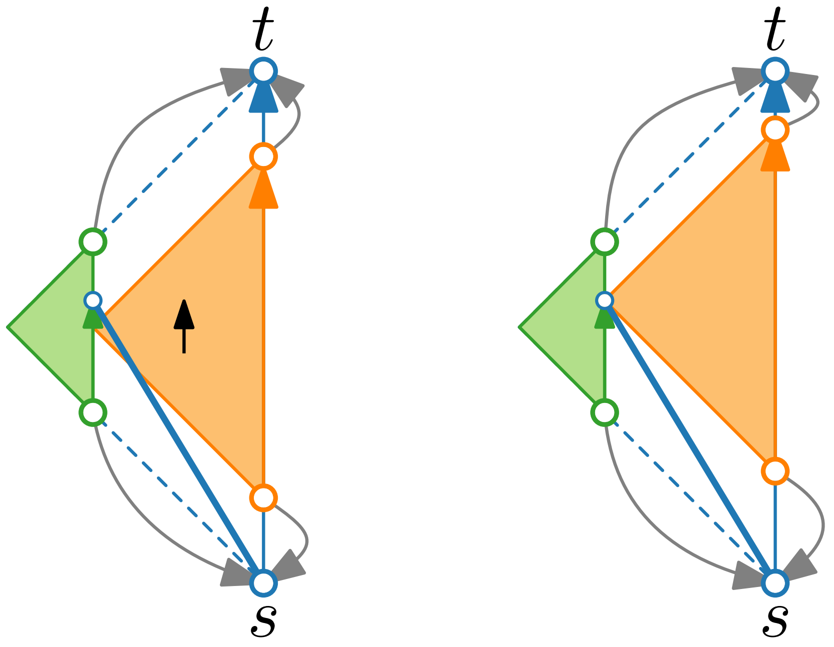

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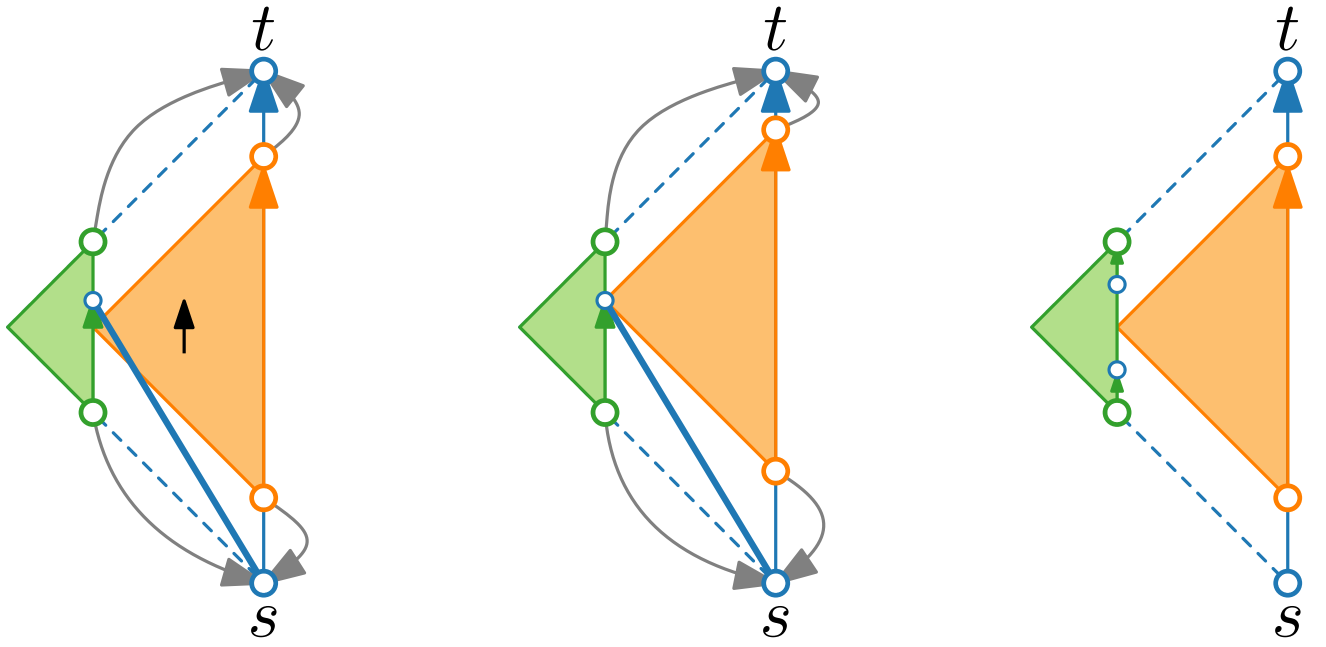

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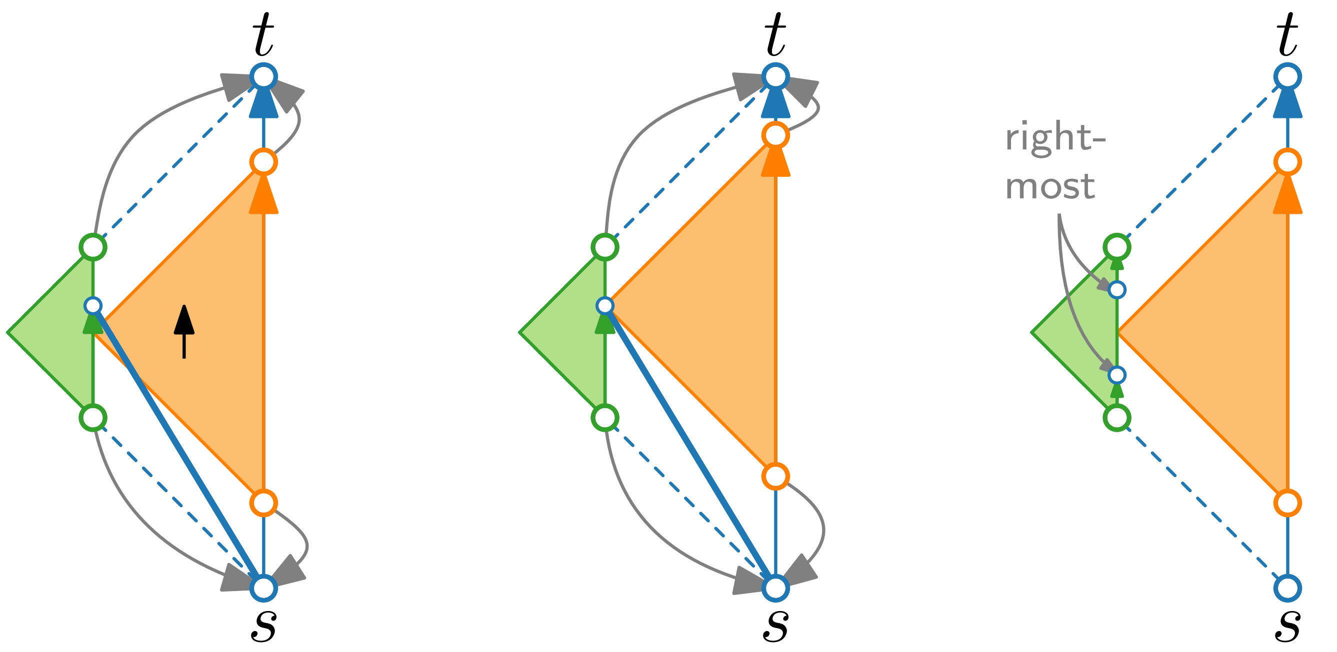

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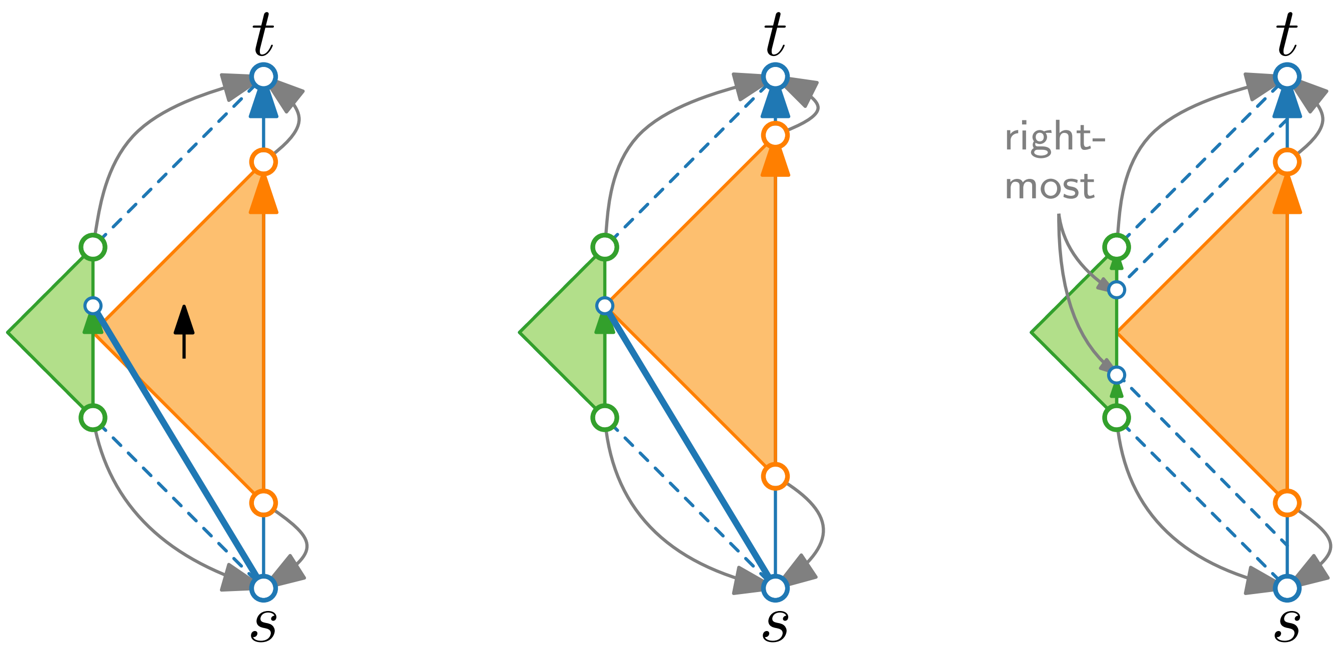

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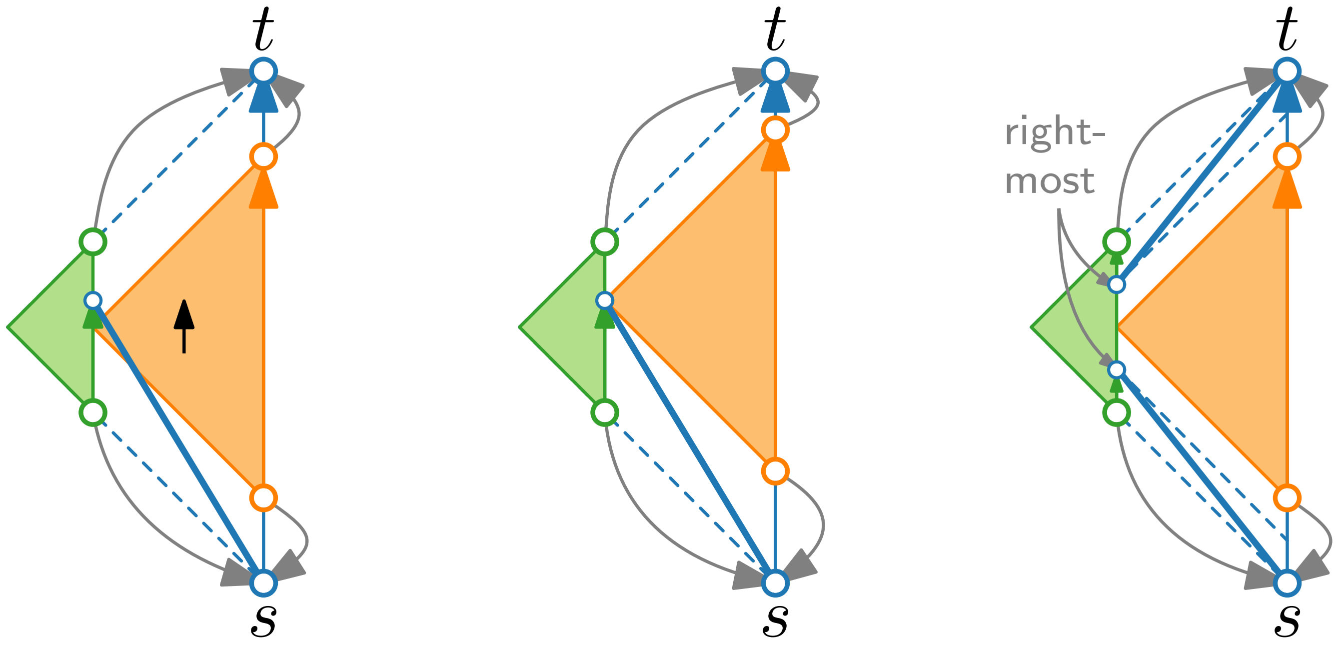

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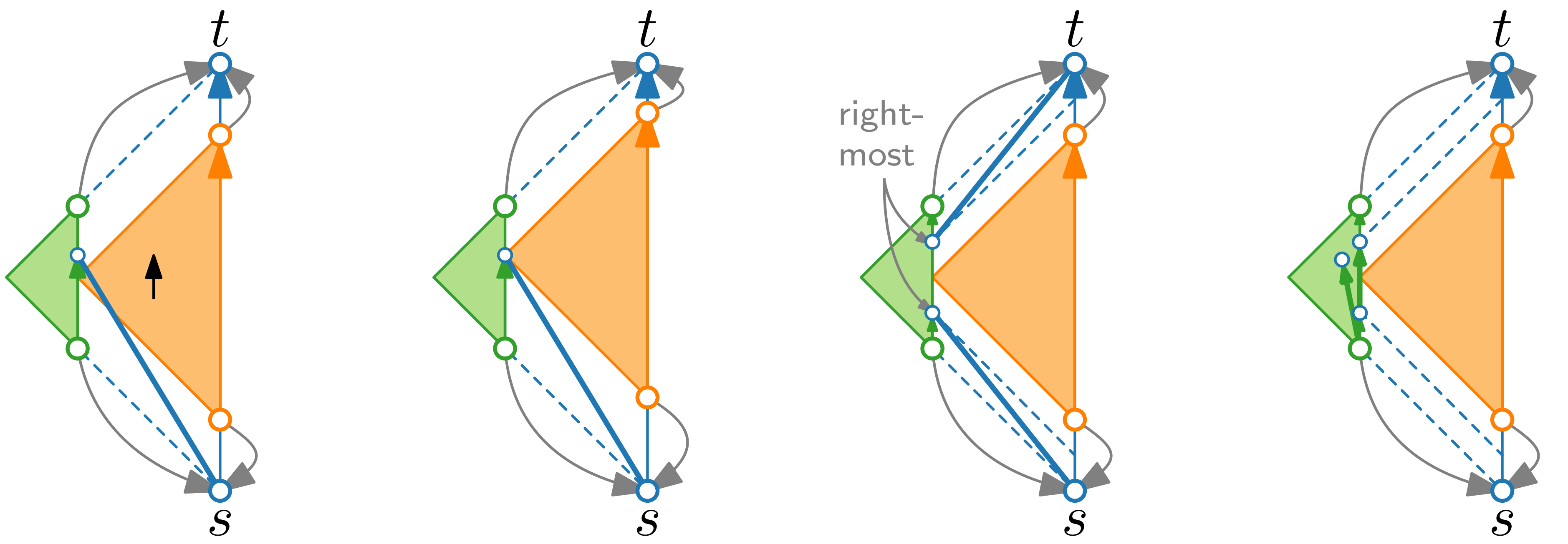

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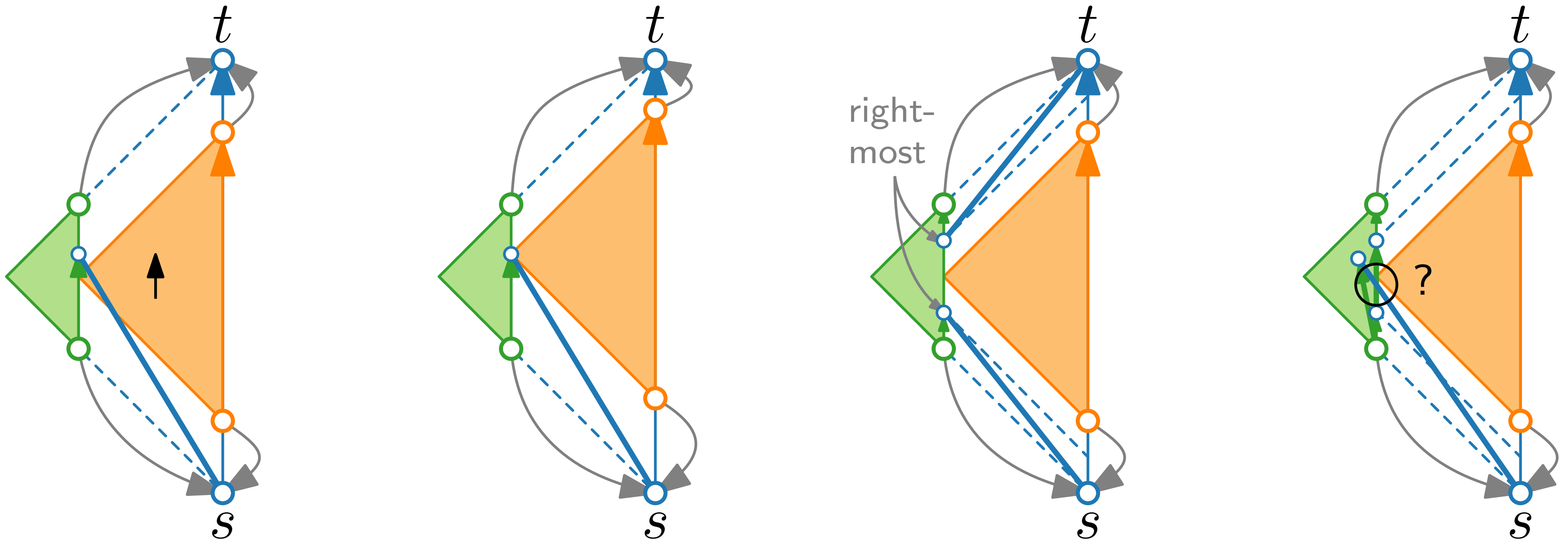

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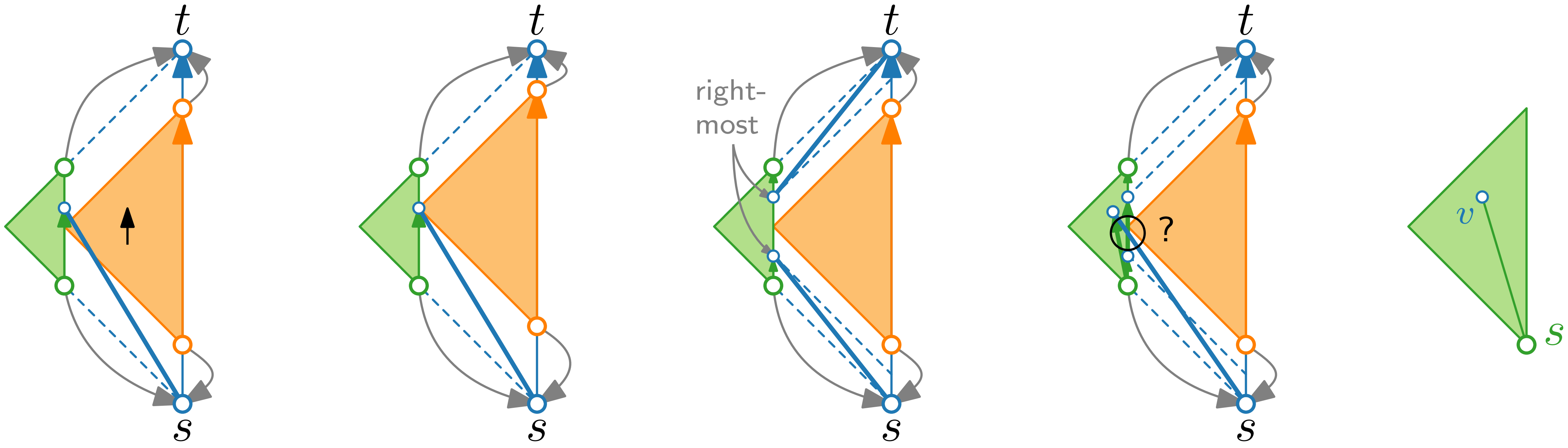

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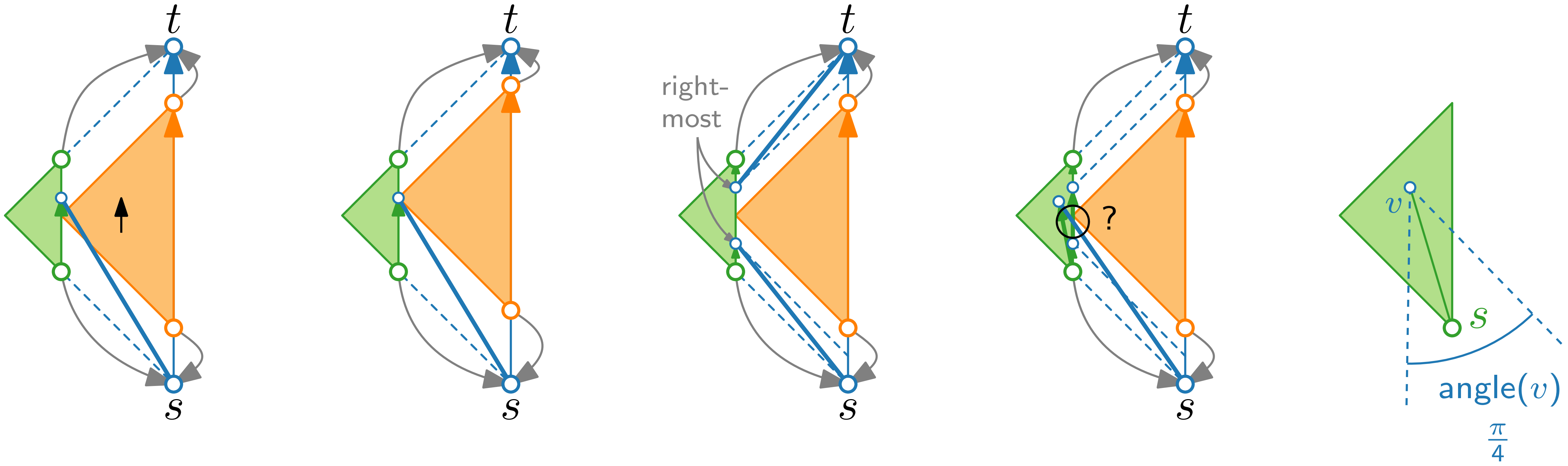

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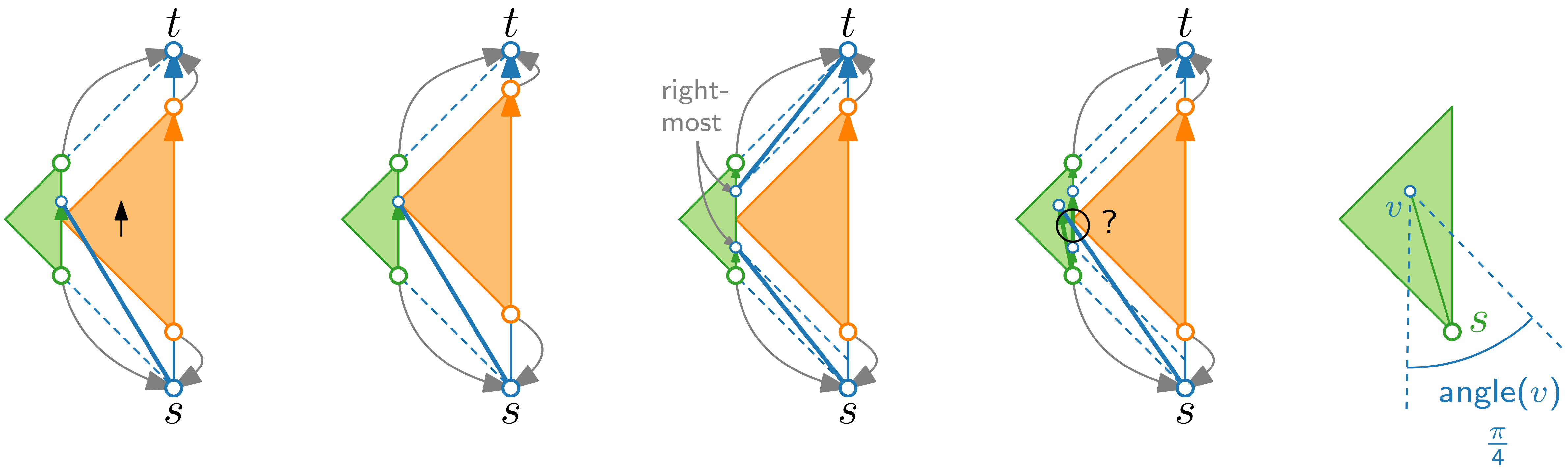

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the only vertex in angle(v) is s

Series-Parallel Graphs – Straight-Line Drawings

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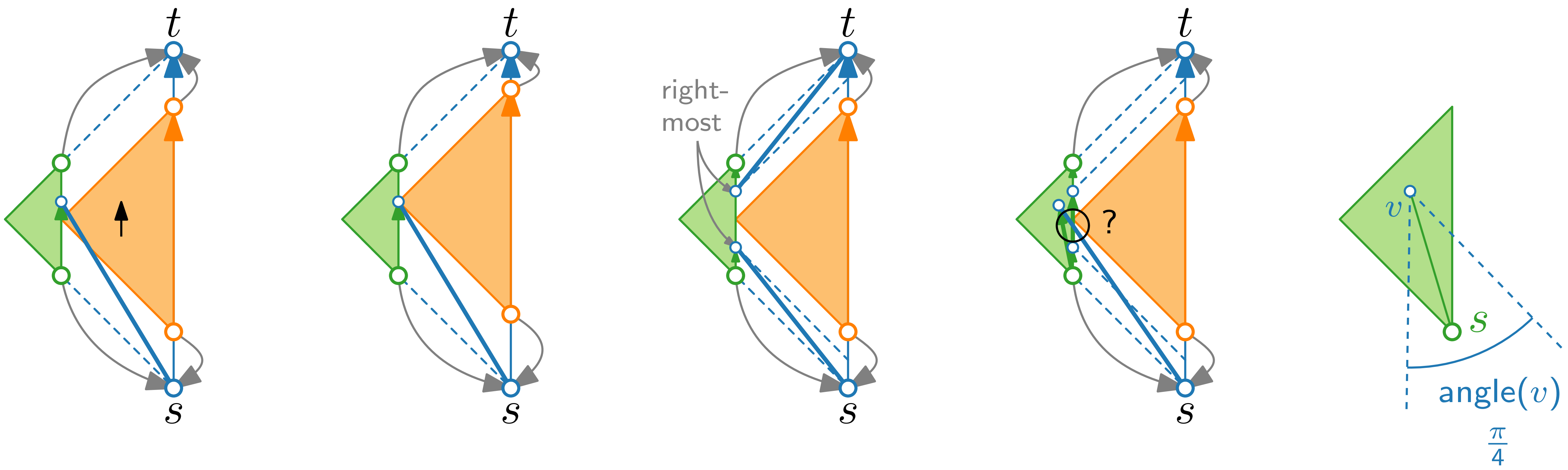

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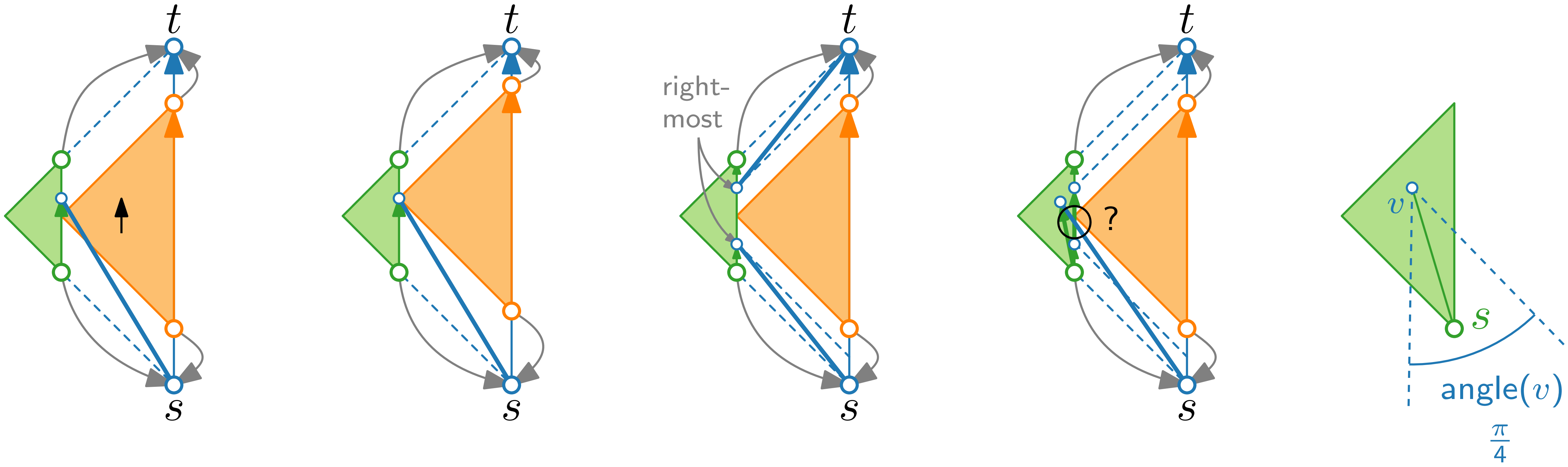
- This condition **is** preserved during the induction step.

Series-Parallel Graphs – Straight-Line Drawings

- What makes parallel composition possible without creating crossings?



Assume the following holds:
the only vertex in angle(v) is s

- This condition **is** preserved during the induction step.

Lemma.
The drawing produced by the algorithm is planar.

Series-Parallel Graphs – Result

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Γ can be computed in $\mathcal{O}(n)$ time.

Series-Parallel Graphs – Fixed Embedding

Theorem. [Bertolazzi et al. 94]

For any $n \geq 1$, there exists a $2n$-vertex series-parallel graph G_n in an embedding such that any upward planar straight-line drawing of G_n that respects the given embedding requires $\Omega(4^n)$ area.

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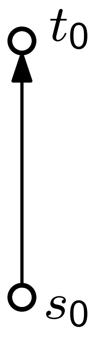

G_1

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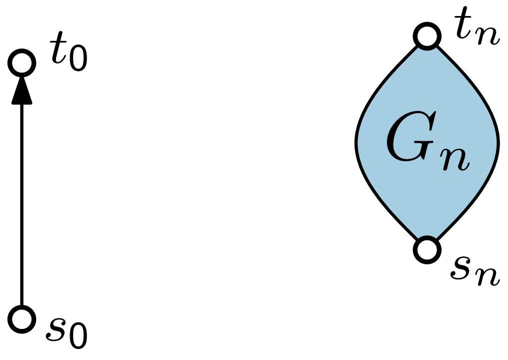

G_1 G_{n+1}

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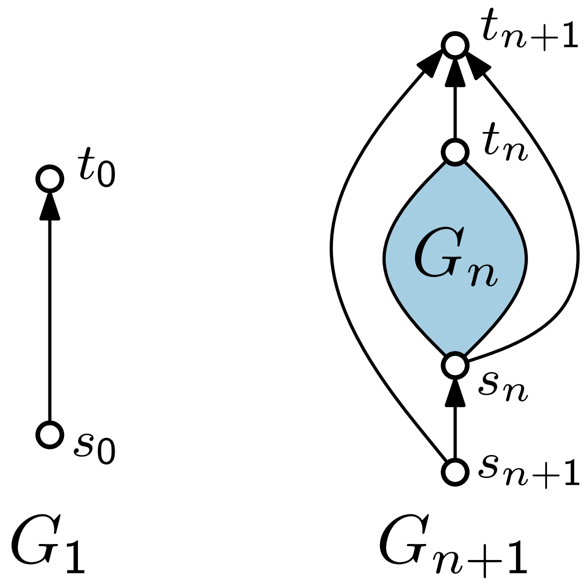

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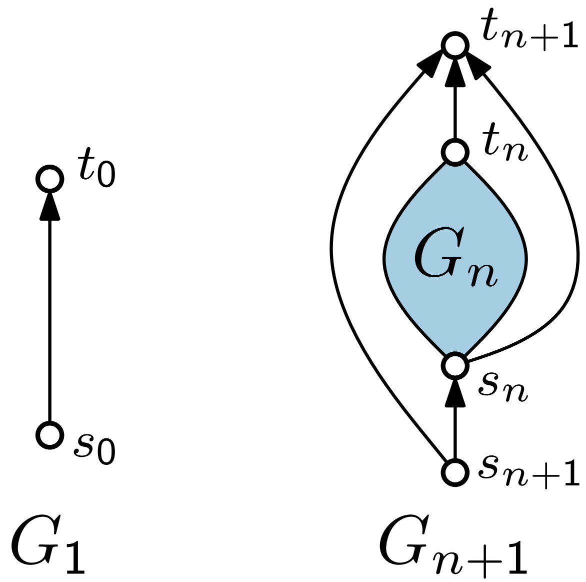

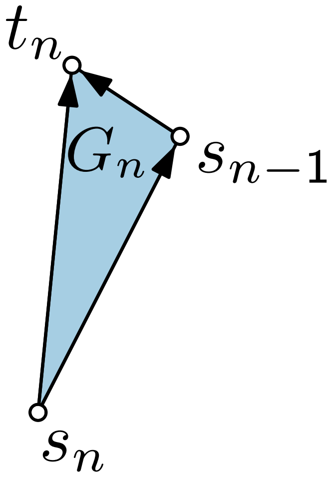

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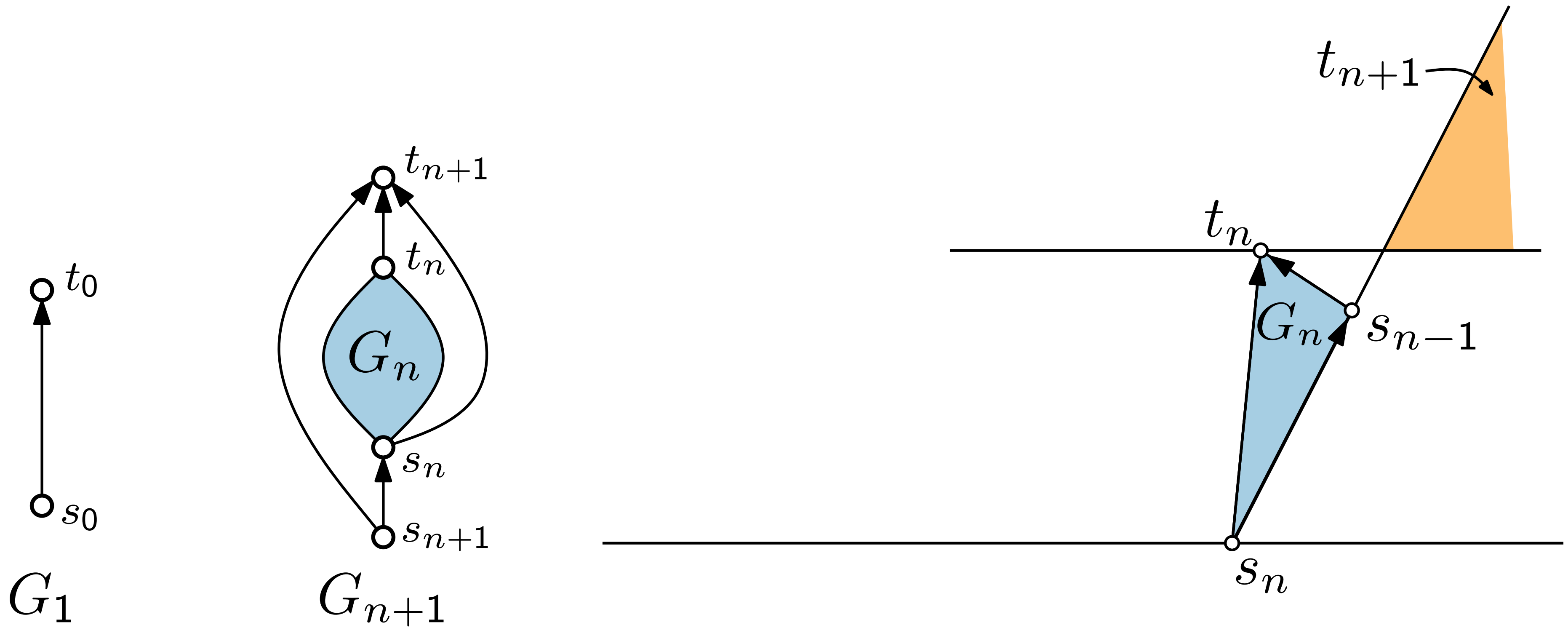

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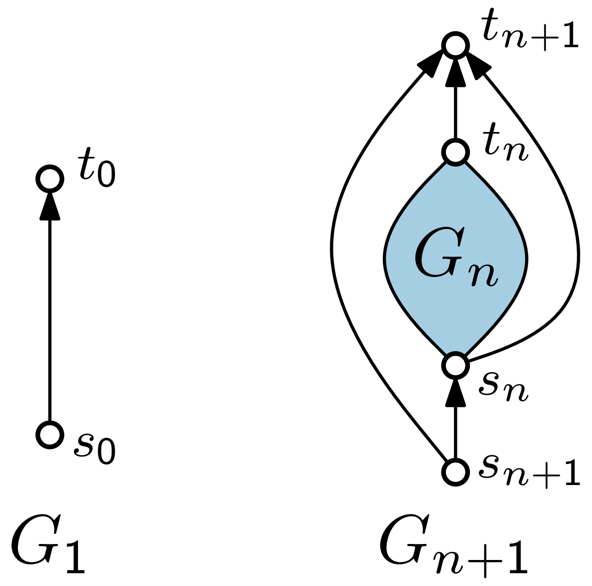

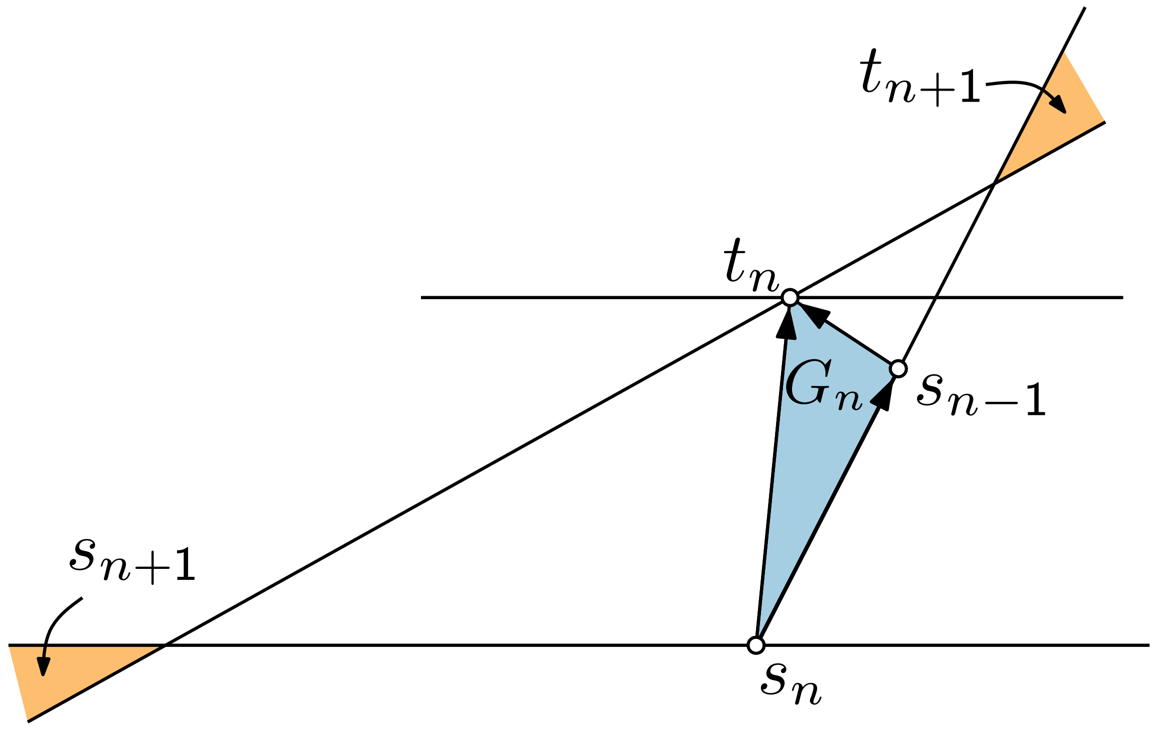

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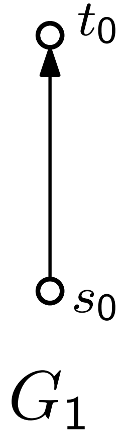

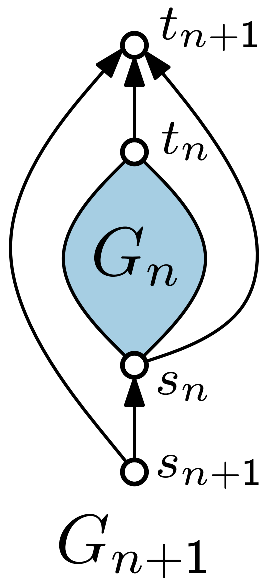

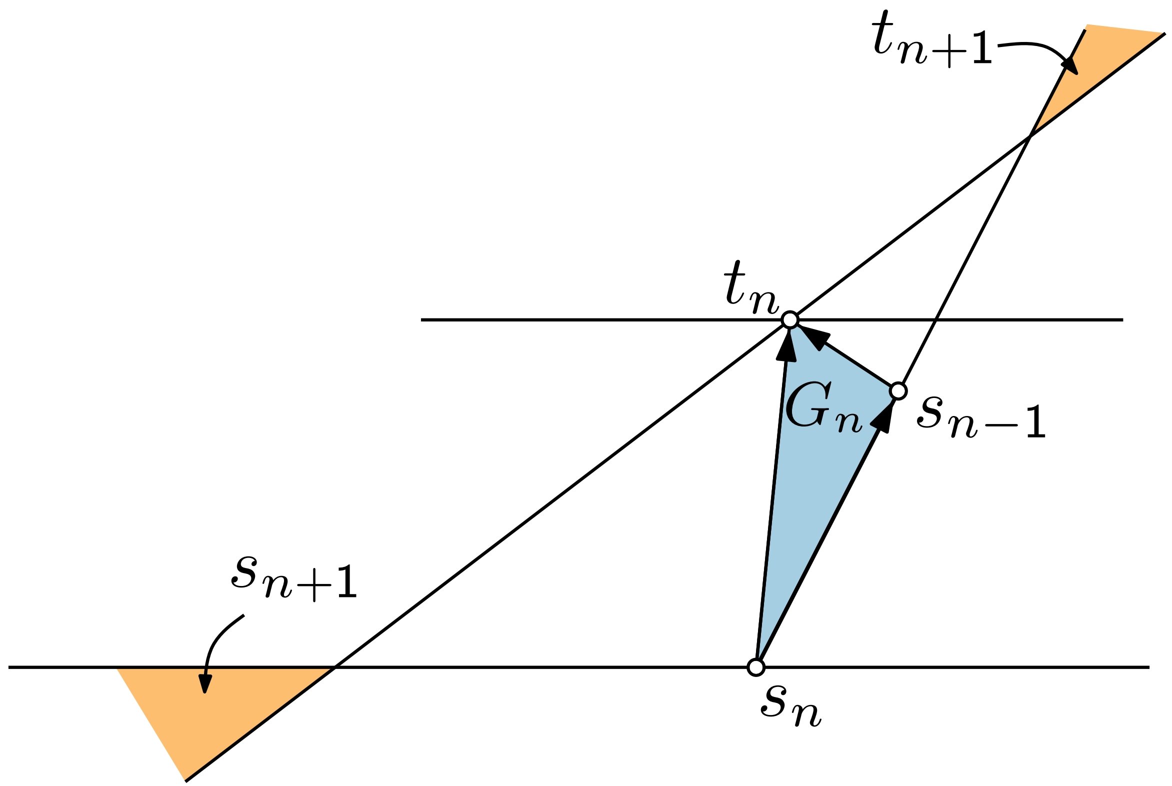

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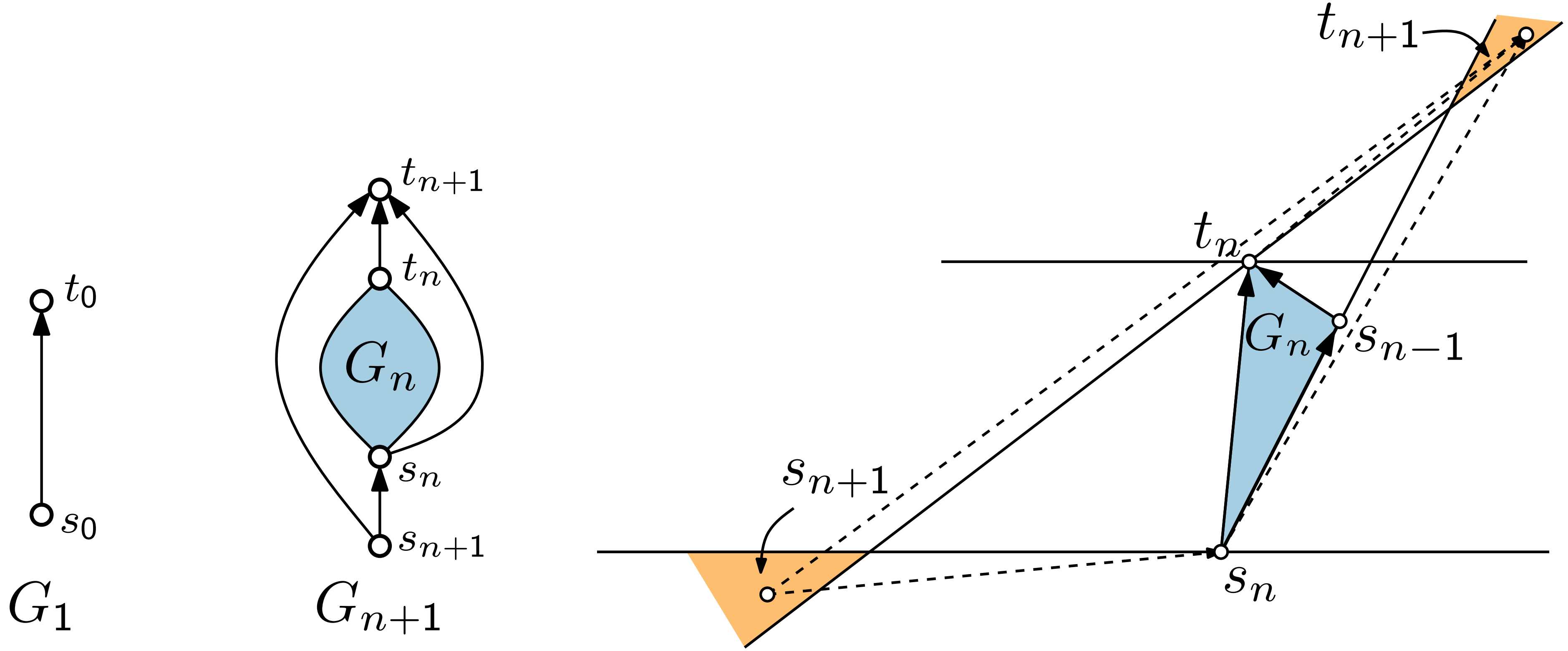

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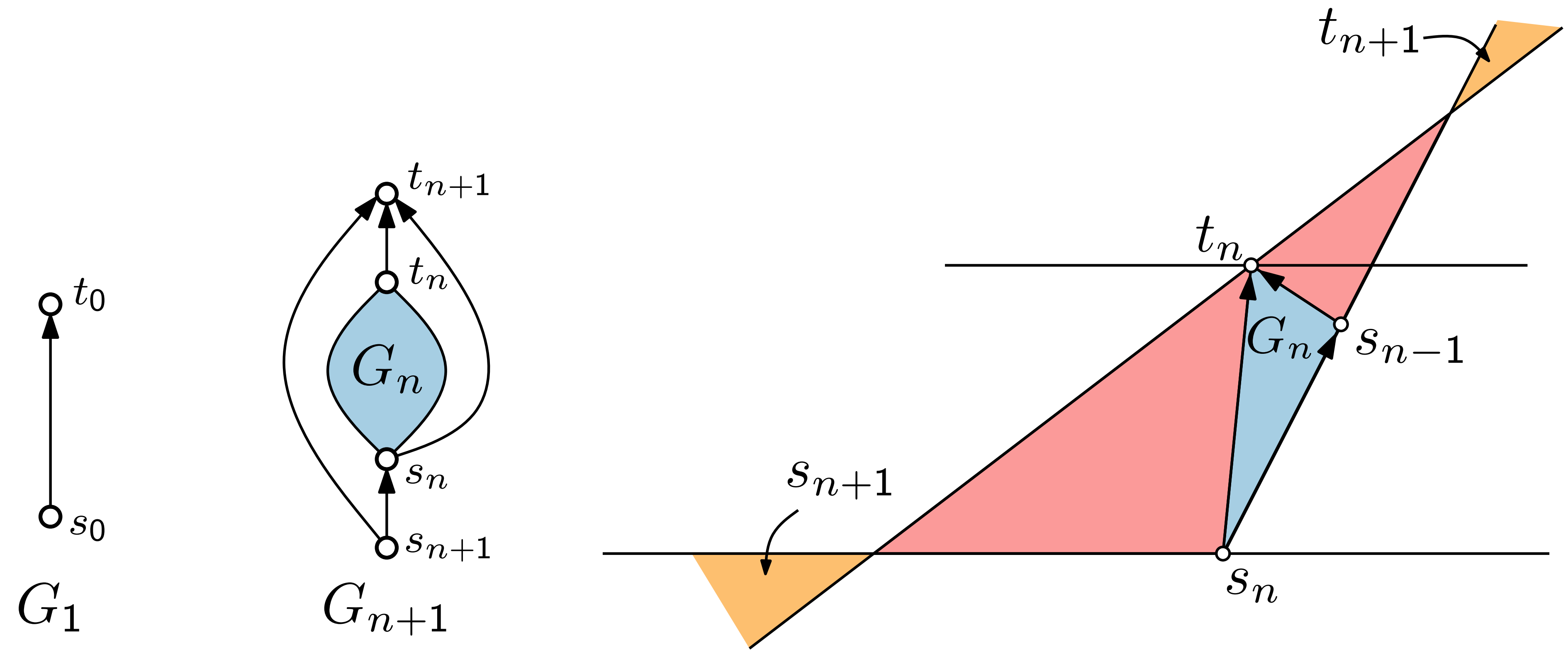

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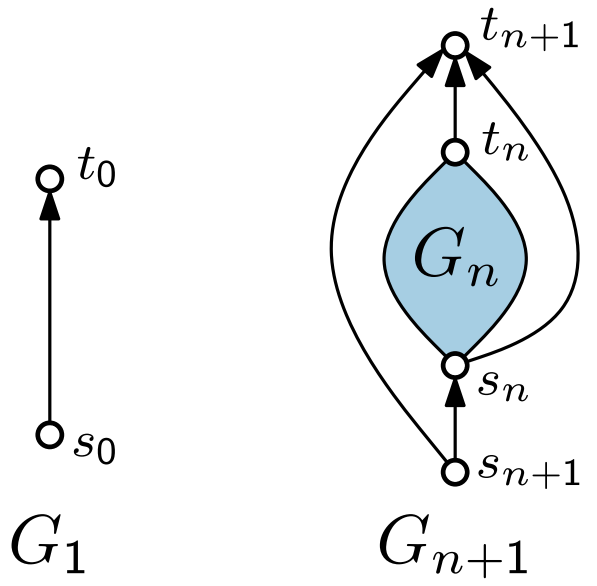

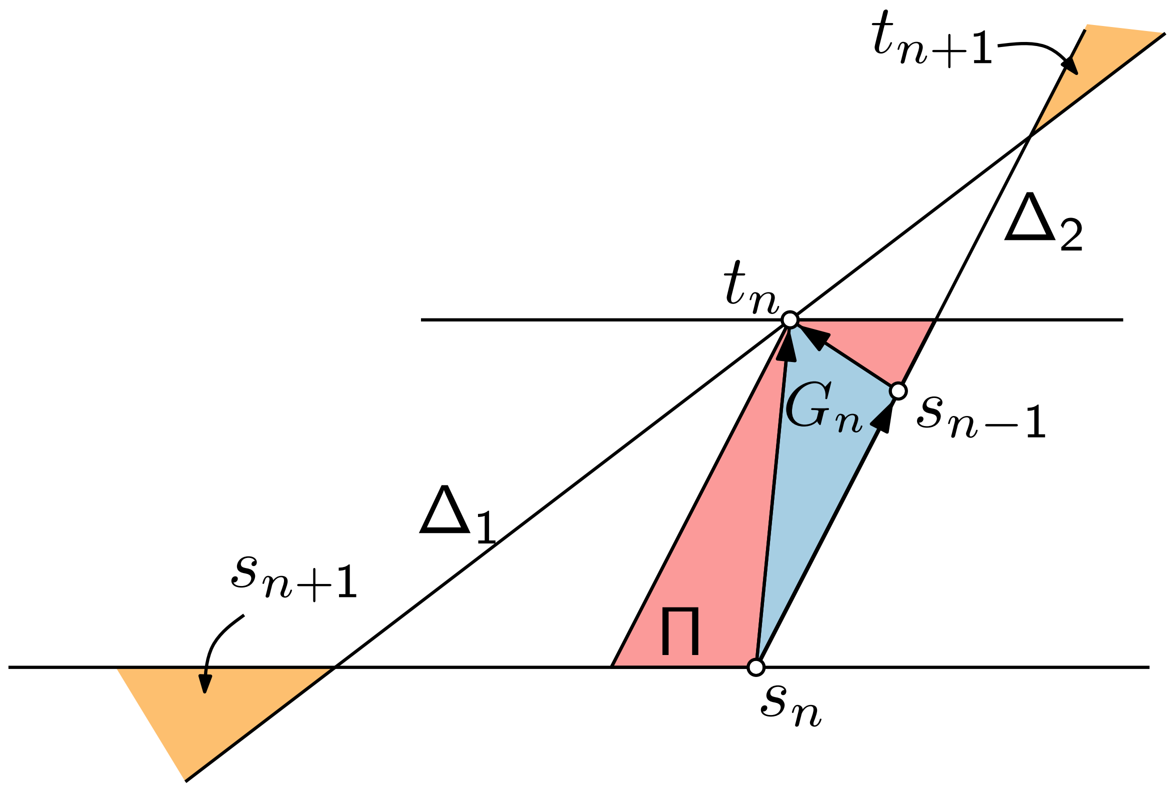

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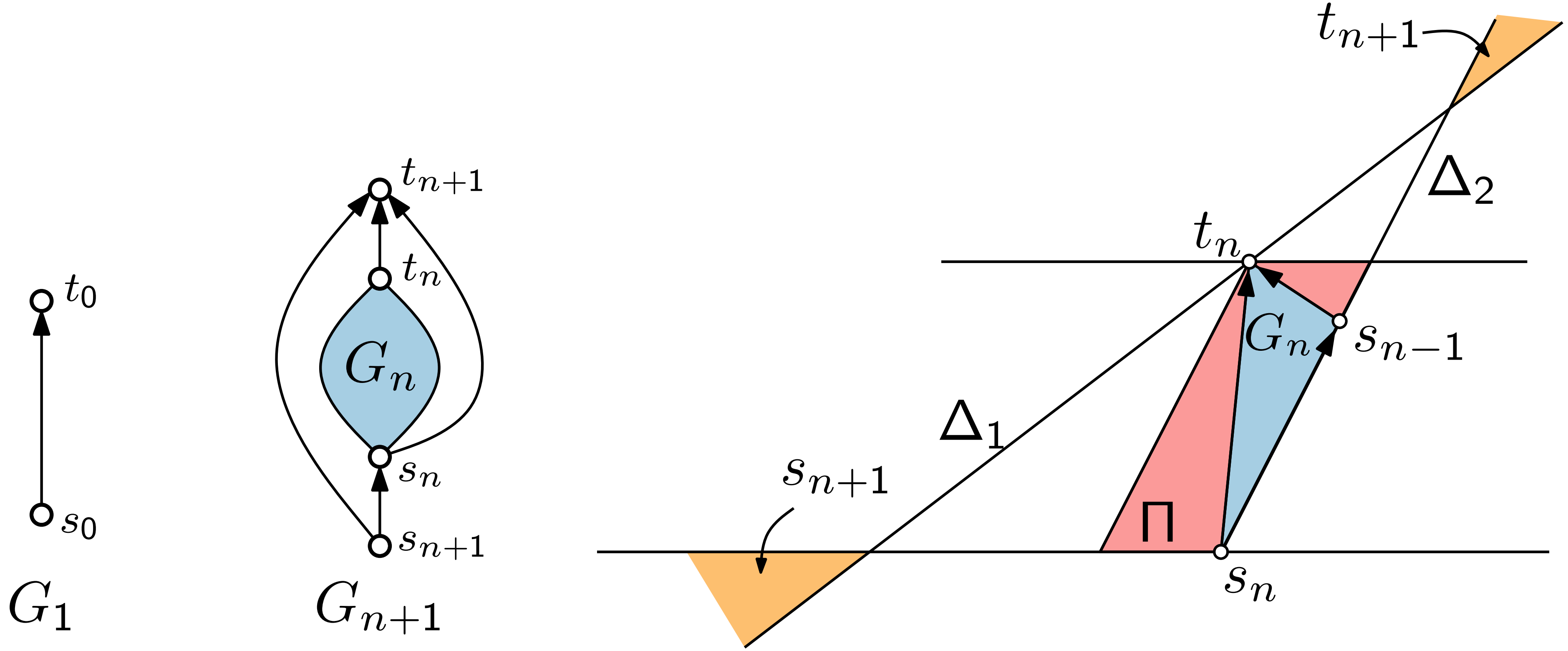
- $2 \cdot \text{Area}(G_n) < \text{Area}(\Pi)$



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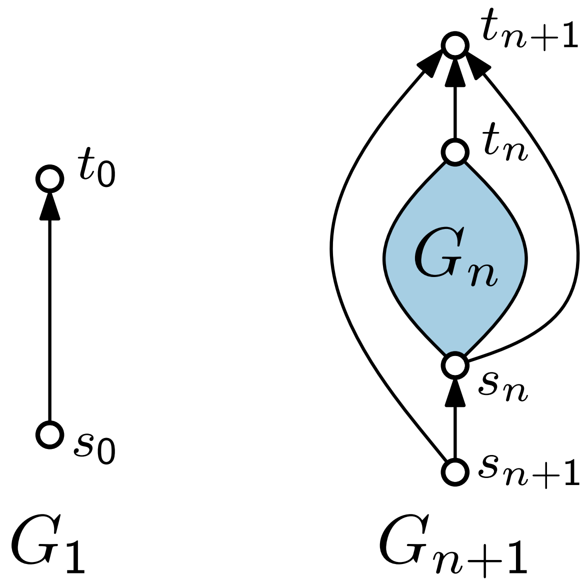

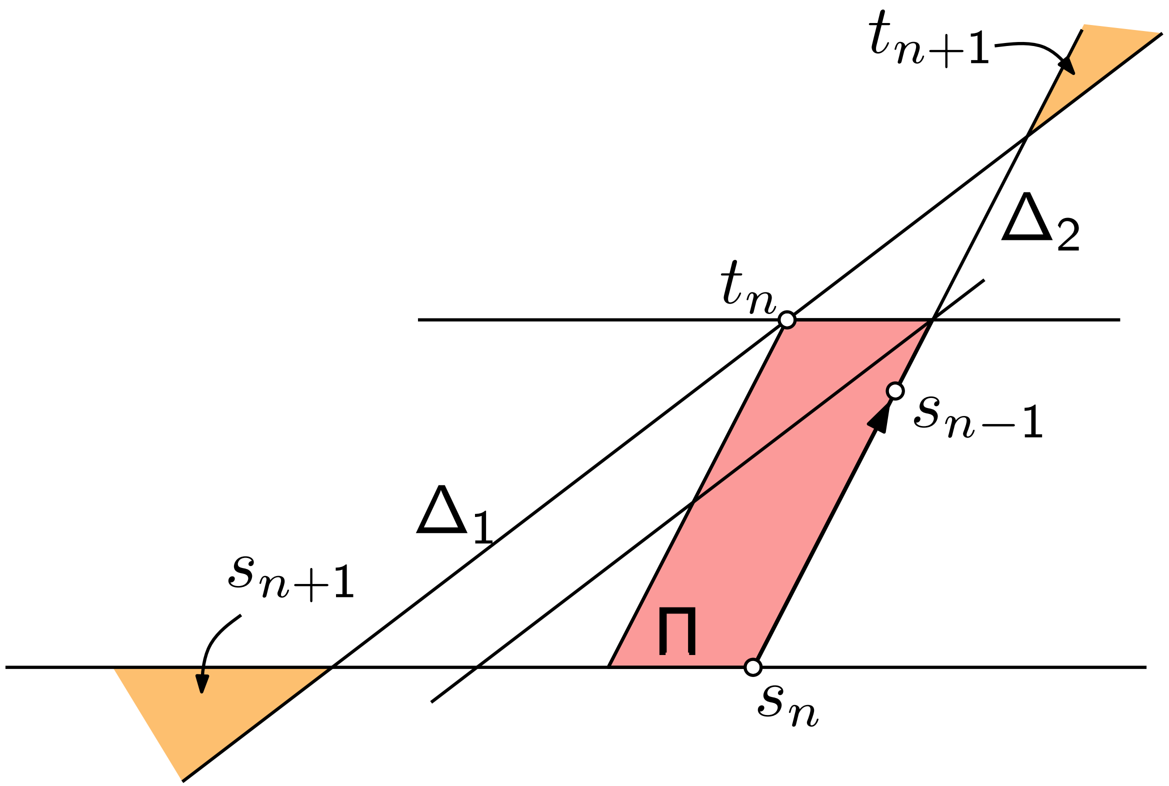

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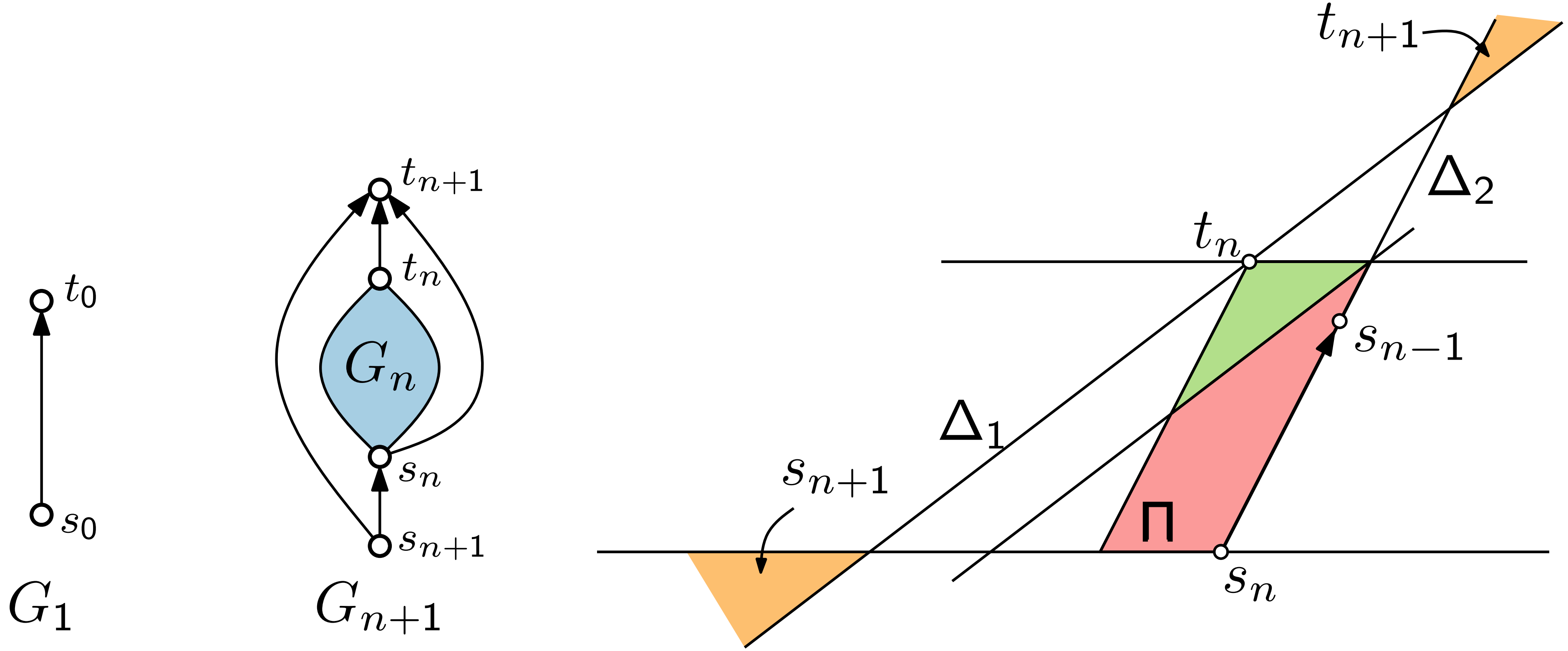

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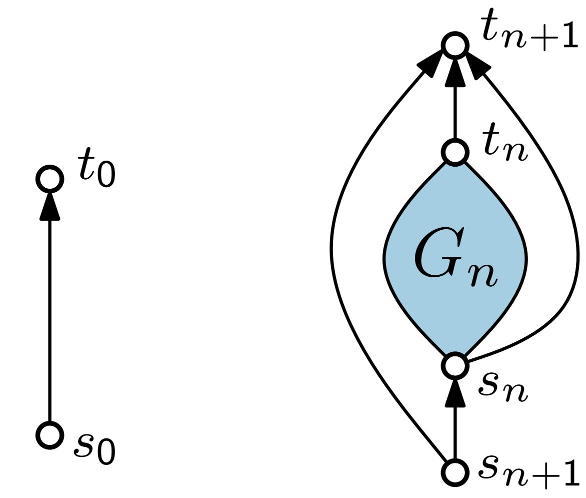
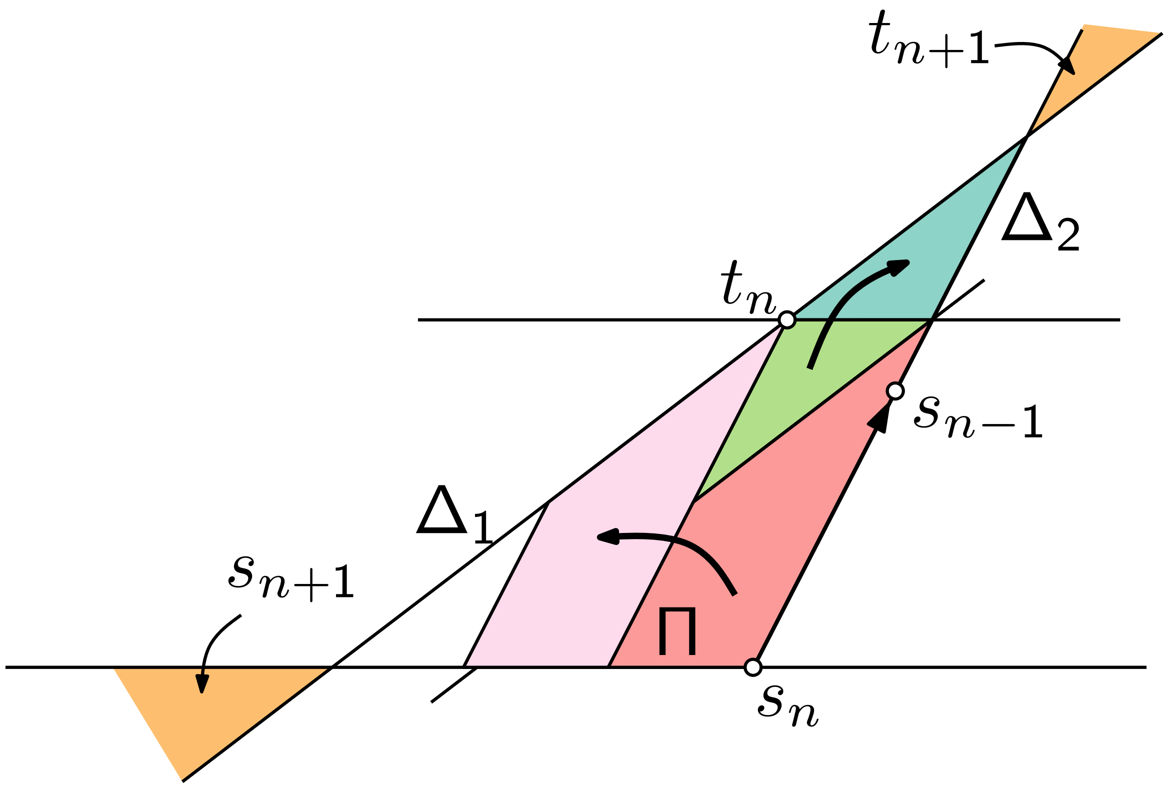


G_1 G_{n+1}



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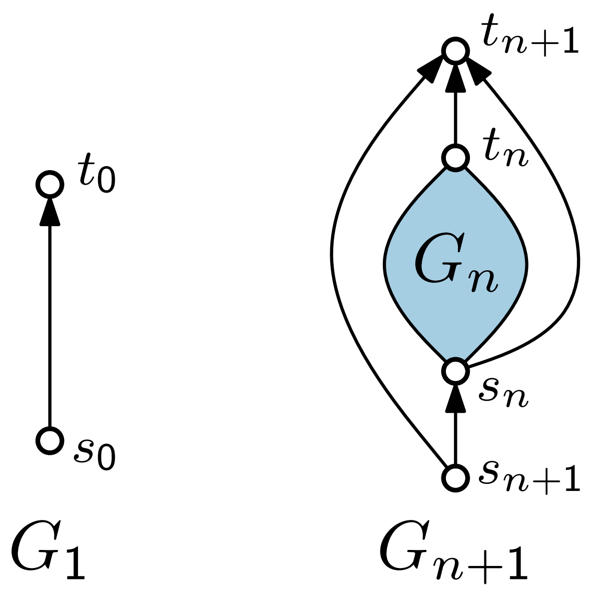

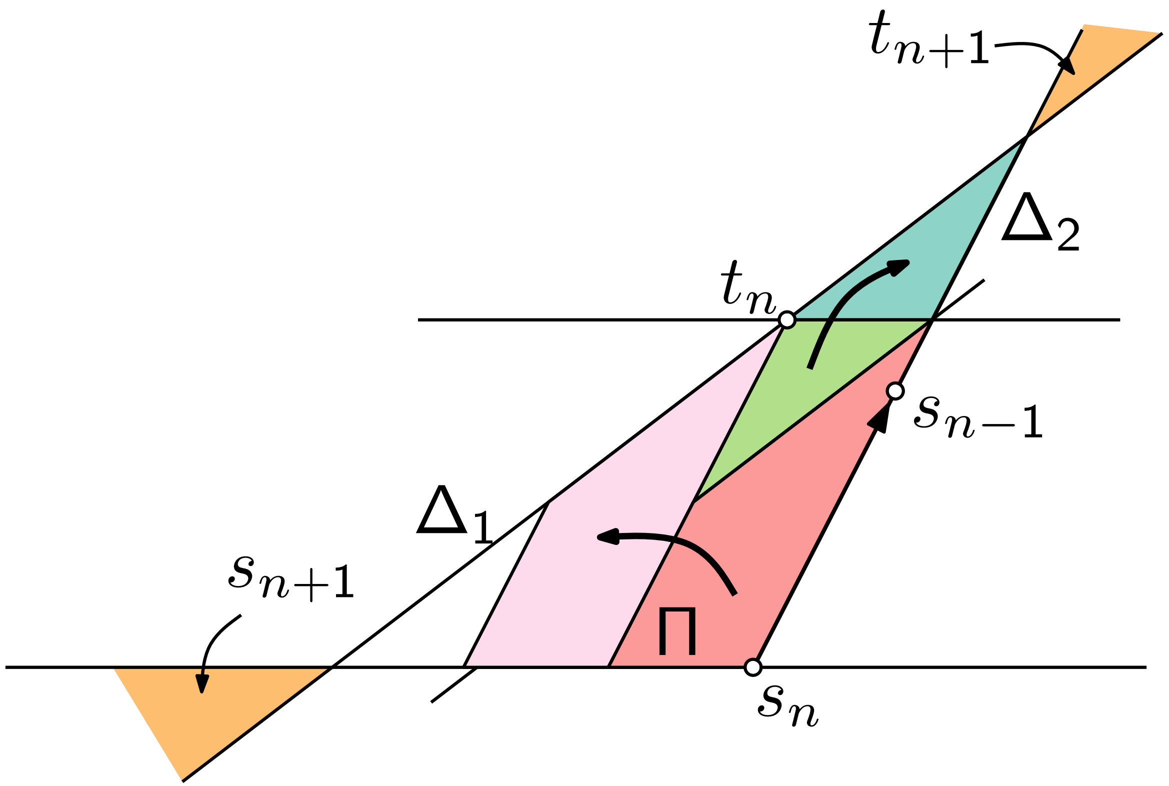

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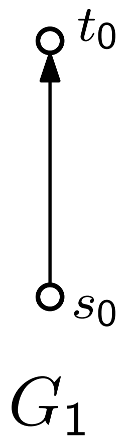
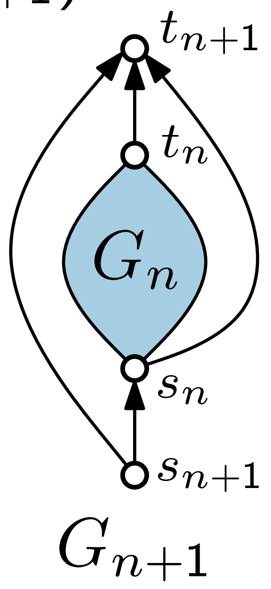
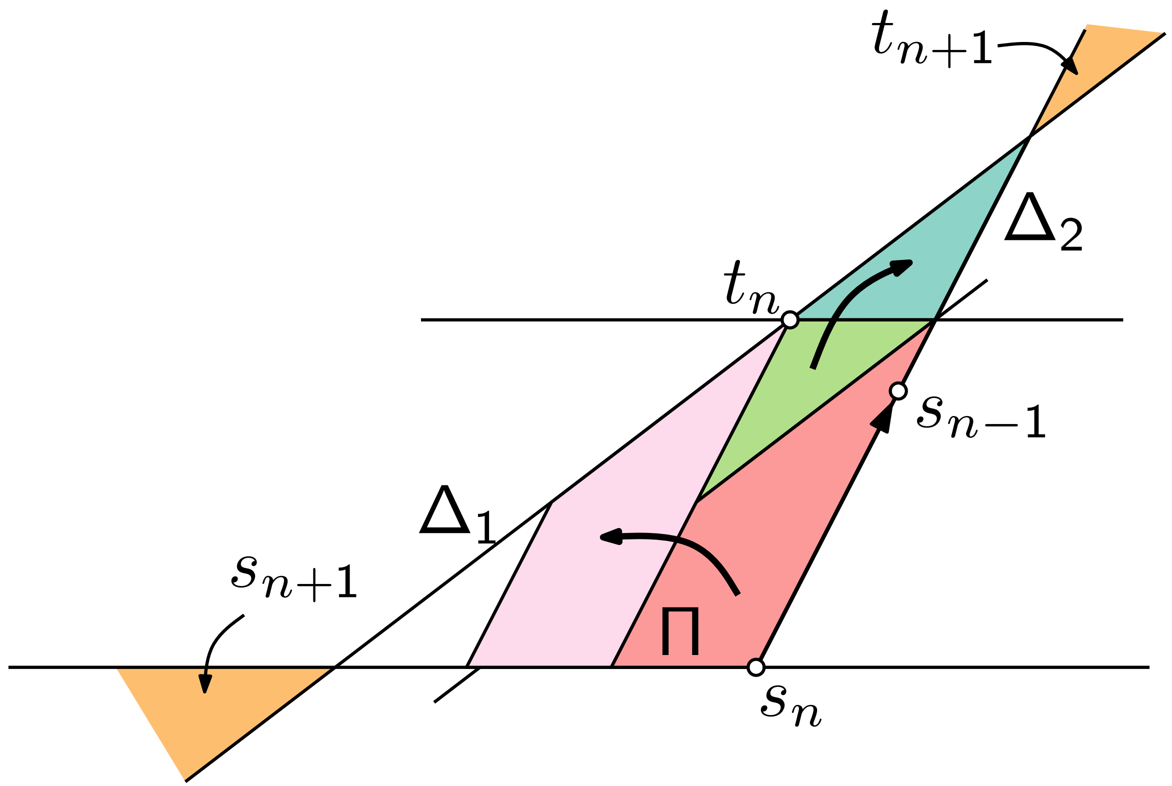
$\Rightarrow 4 \cdot \text{Area}(G_n) < \text{Area}(G_{n+1})$







Discussion

- There exist fixed-parameter (FPT) algorithms to test upward planarity of general digraphs with the parameter being the number of triconnected components.

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- Many related concepts have been studied:
  upward drawings of mixed graphs, upward drawings with layers for the vertices, upward planarity on cylinder/torus, ...

Literature

- See [GD Ch. 6] for detailed explanation on upward planarity.
- See [GD Ch. 3] for divide and conquer methods of series-parallel graphs

Orginal papers referenced:

- [Kelly '87] Fundamentals of Planar Ordered Sets
- [Di Battista &Tamassia '88] Algorithms for Plane Representations of Acyclic Digraphs
- [Garg &Tamassia '95]
  On the Computational Complexity of Upward and Rectilinear Planarity Testing
- [Hutton & Lubiw '96] Upward Planar Drawing of Single-Source Acyclic Digraphs
- [Bertolazzi, Di Battista, Mannino, Tamassia '94]
  Upward Drawings of Triconnected Digraphs
- [Healy & Lynch '05] Building Blocks of Upward Planar Digraphs
- [Didimo, Giardano, Liotta '09] Upward Spirality and Upward Planarity Testing
- [Abbasi, Healy, Rextin '10]
  Improving the running time of embedded upward planarity testing