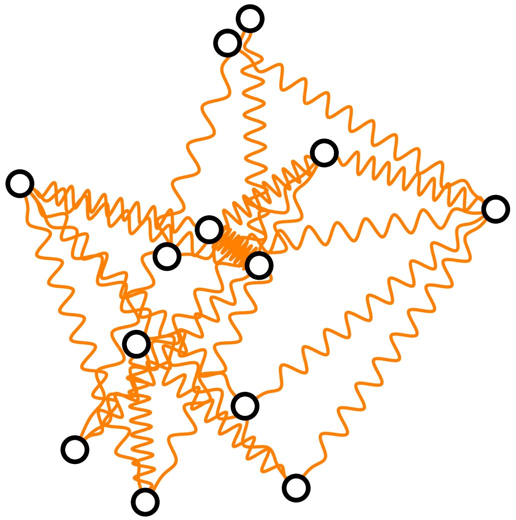
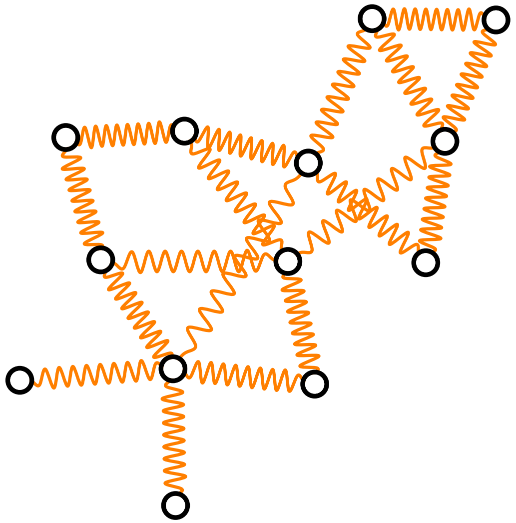
Julius-Maximilians-
UNIVERSITÄT
WÜRZBURG

Visualization of Graphs

Lecture 2:
Force-Directed Drawing Algorithms



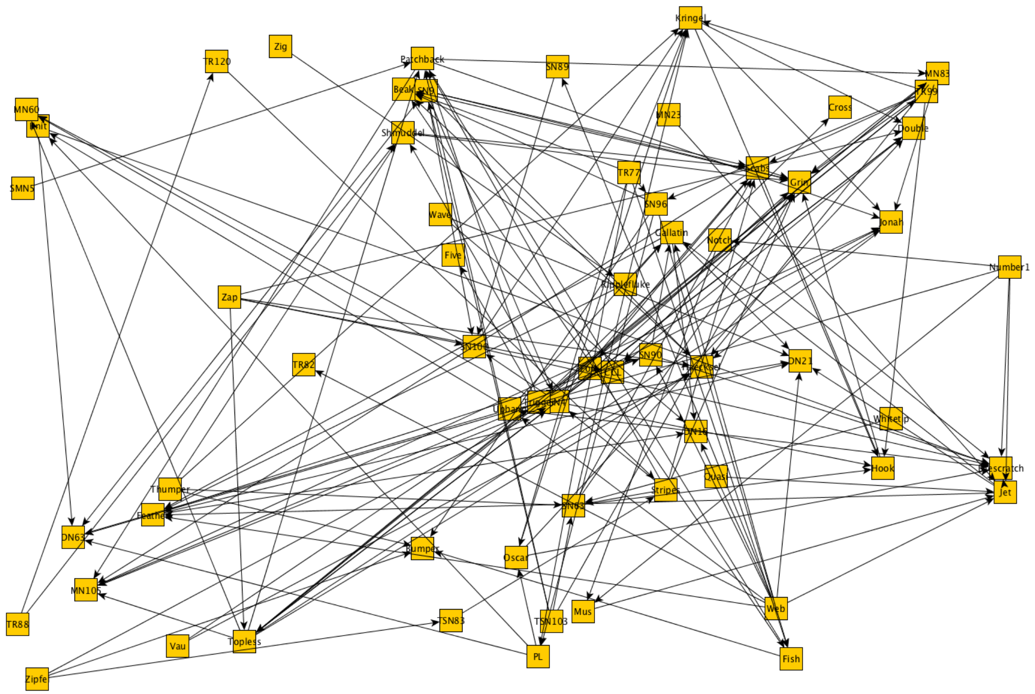
Part I:
Spring Embedders



Johannes Zink

General Layout Problem

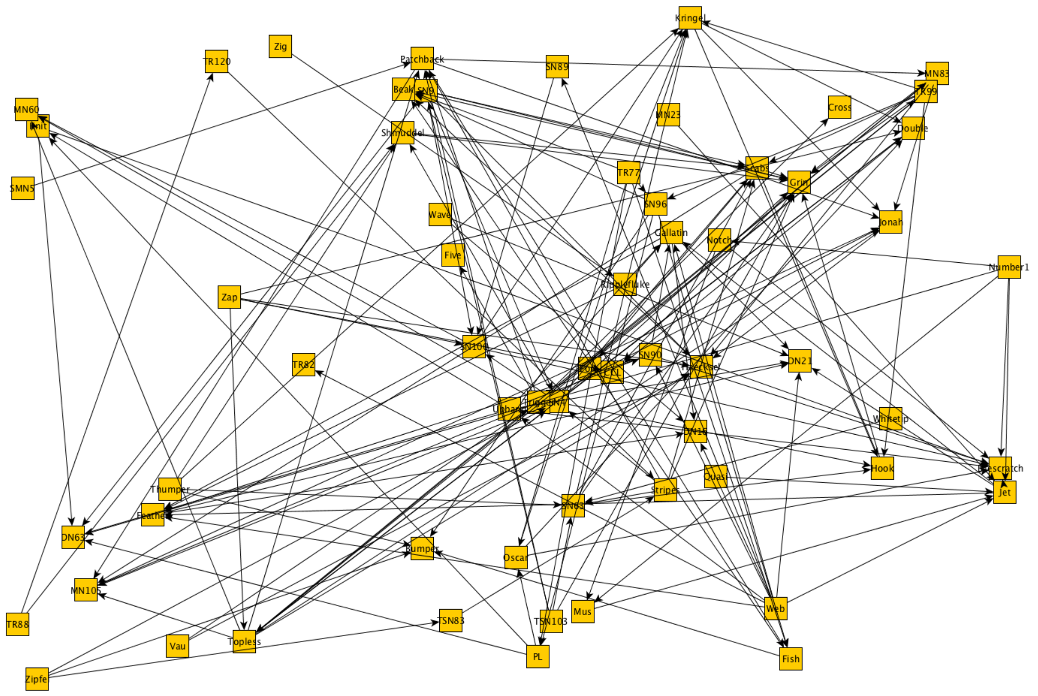
Input: Graph G

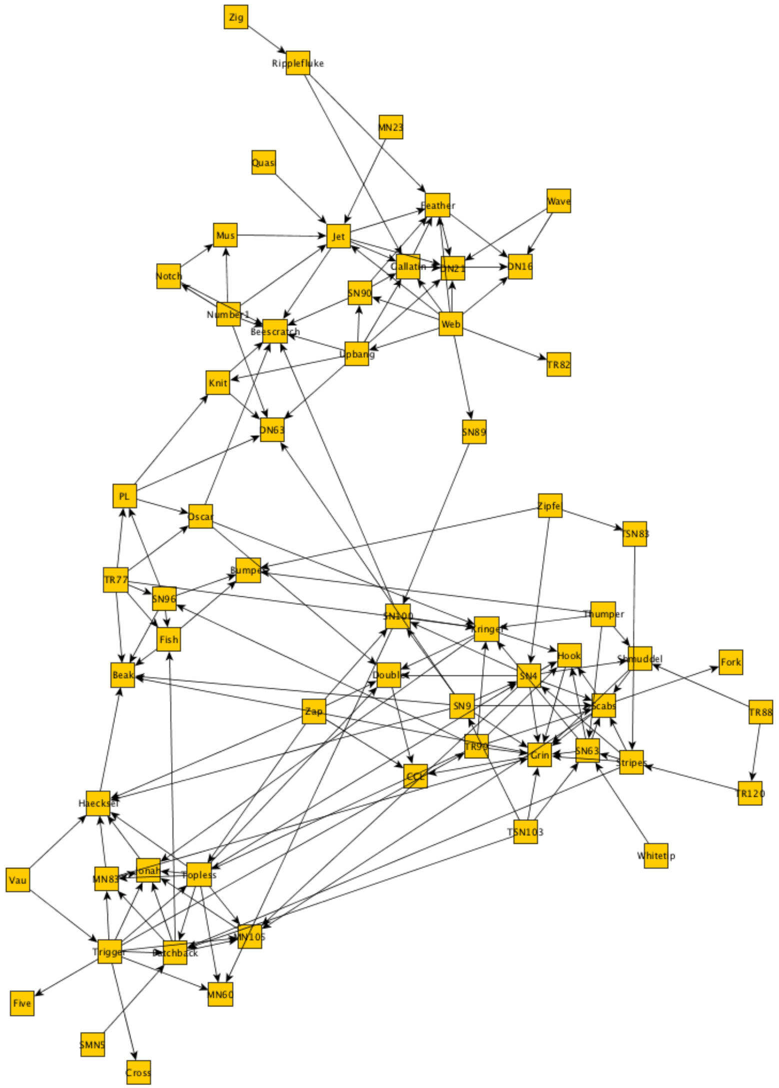


General Layout Problem

Input: Graph G

Output: Clear and readable straight-line drawing of G





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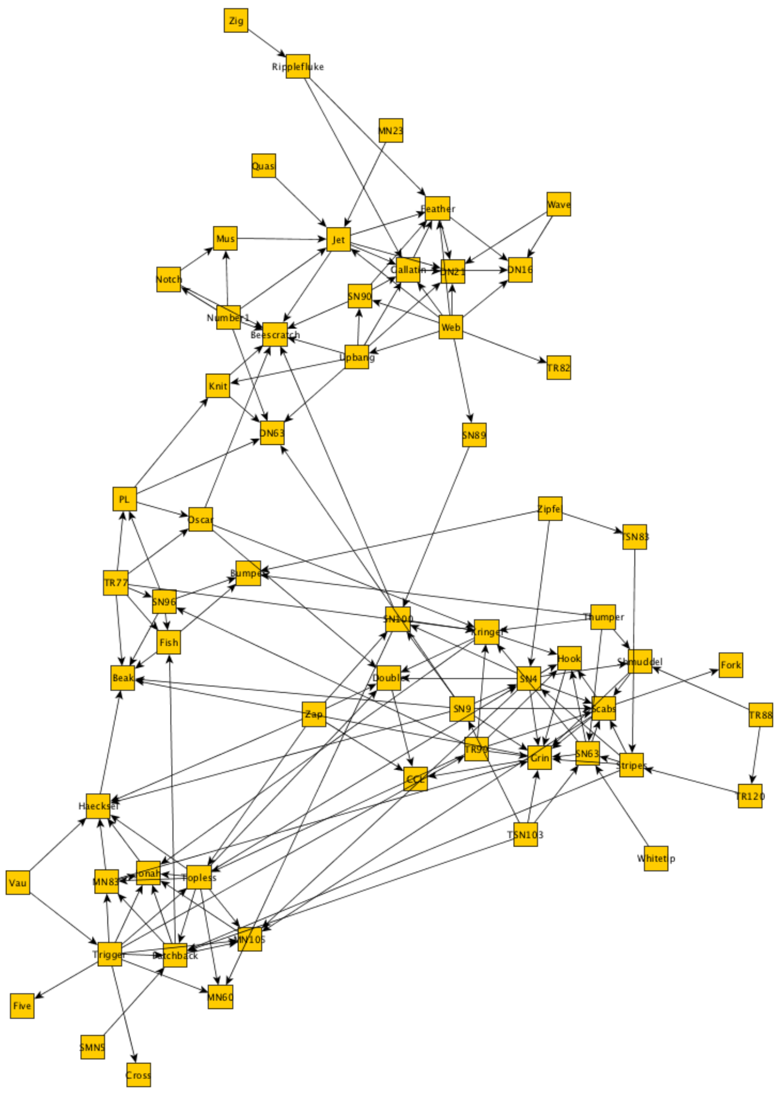
Drawing aesthetics to optimize:



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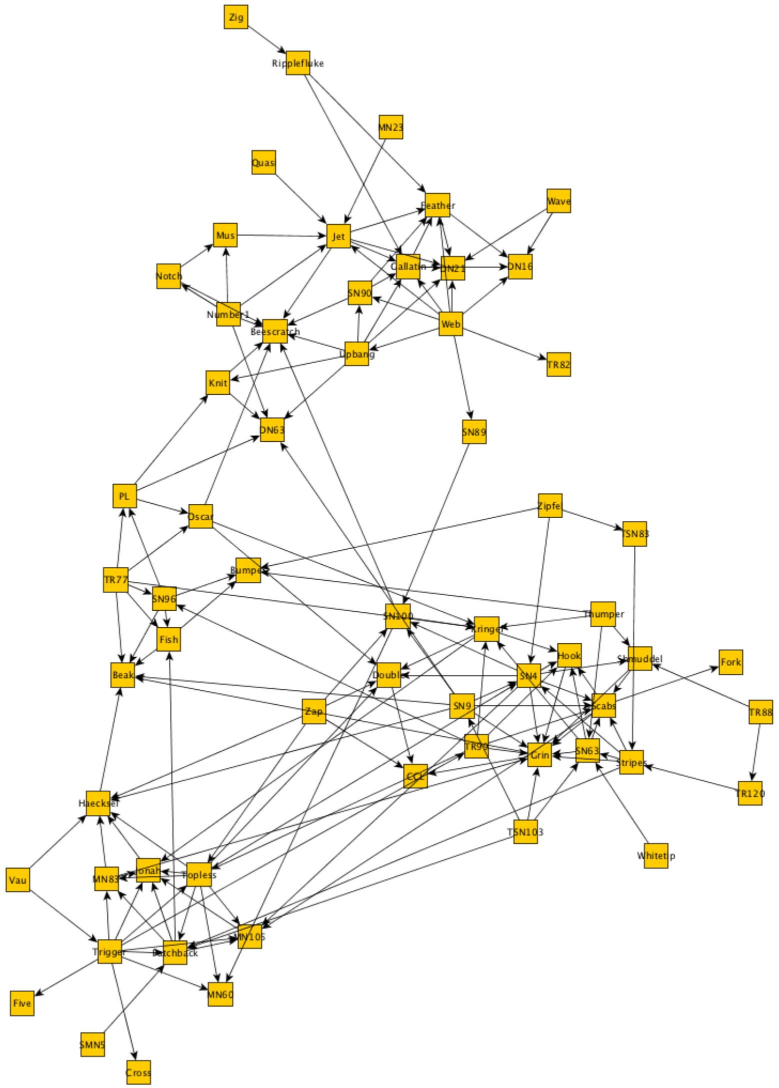
Drawing aesthetics to optimize:

- adjacent vertices are close



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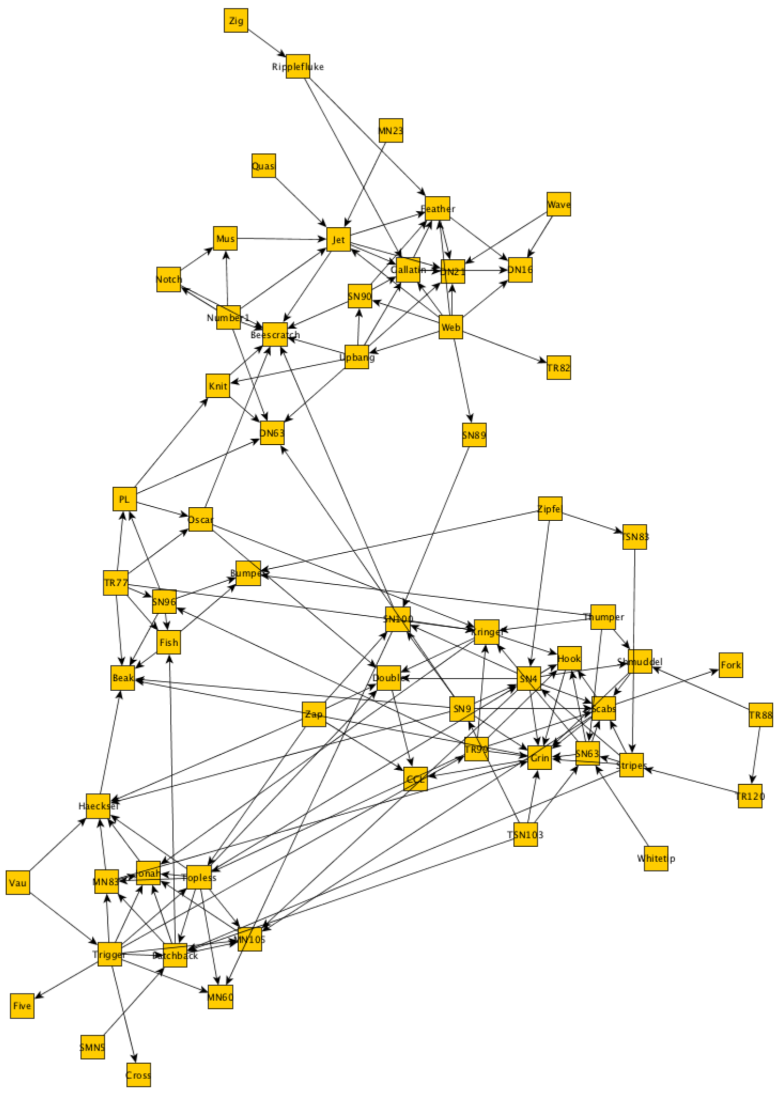

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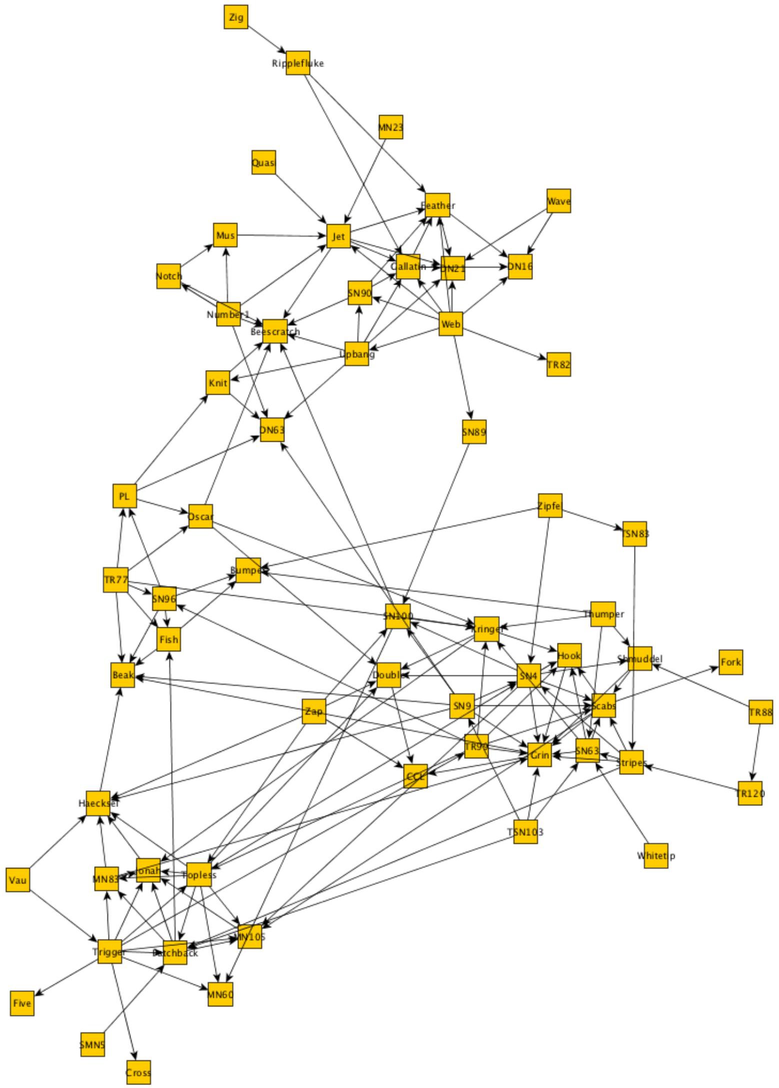
Drawing aesthetics to optimize:

- adjacent vertices are close
- non-adjacent vertices are far apart
- edges short, straight-line, **similar length**



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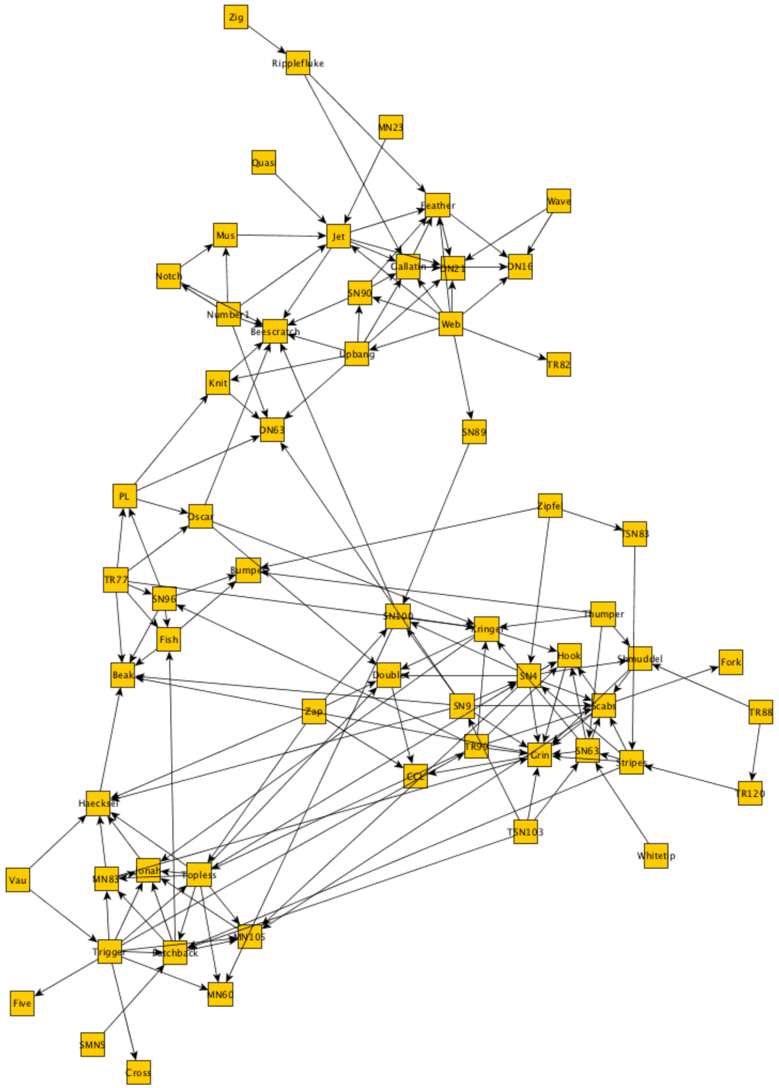

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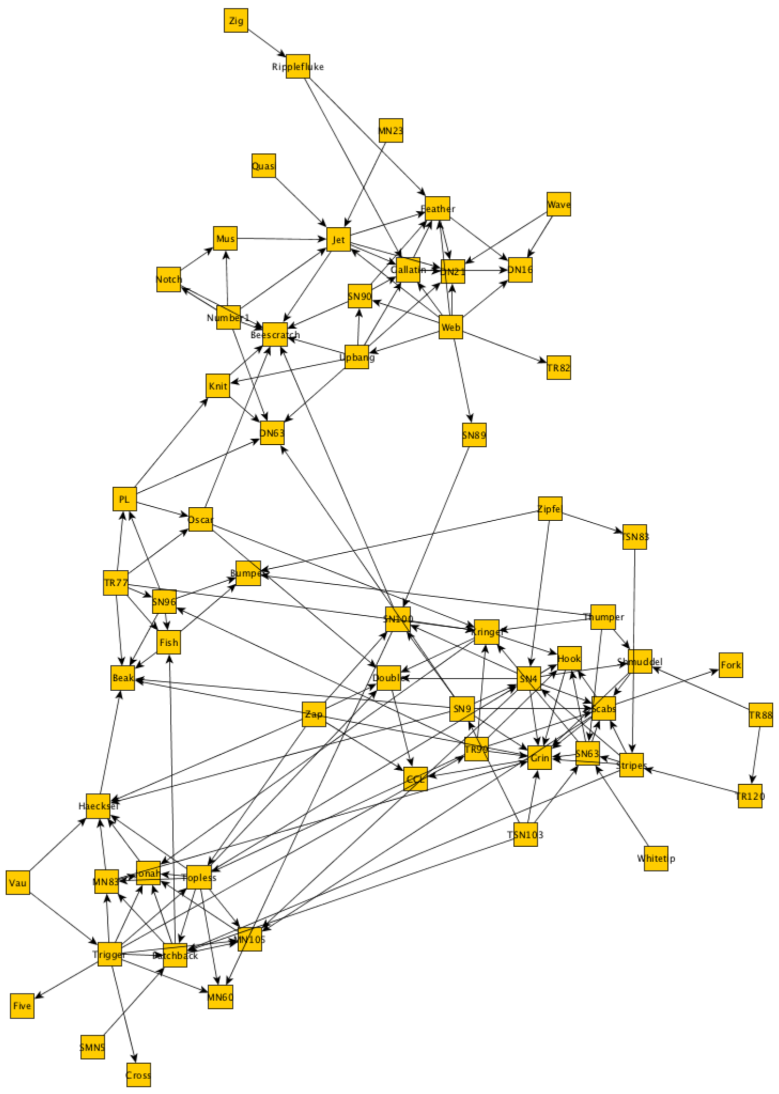

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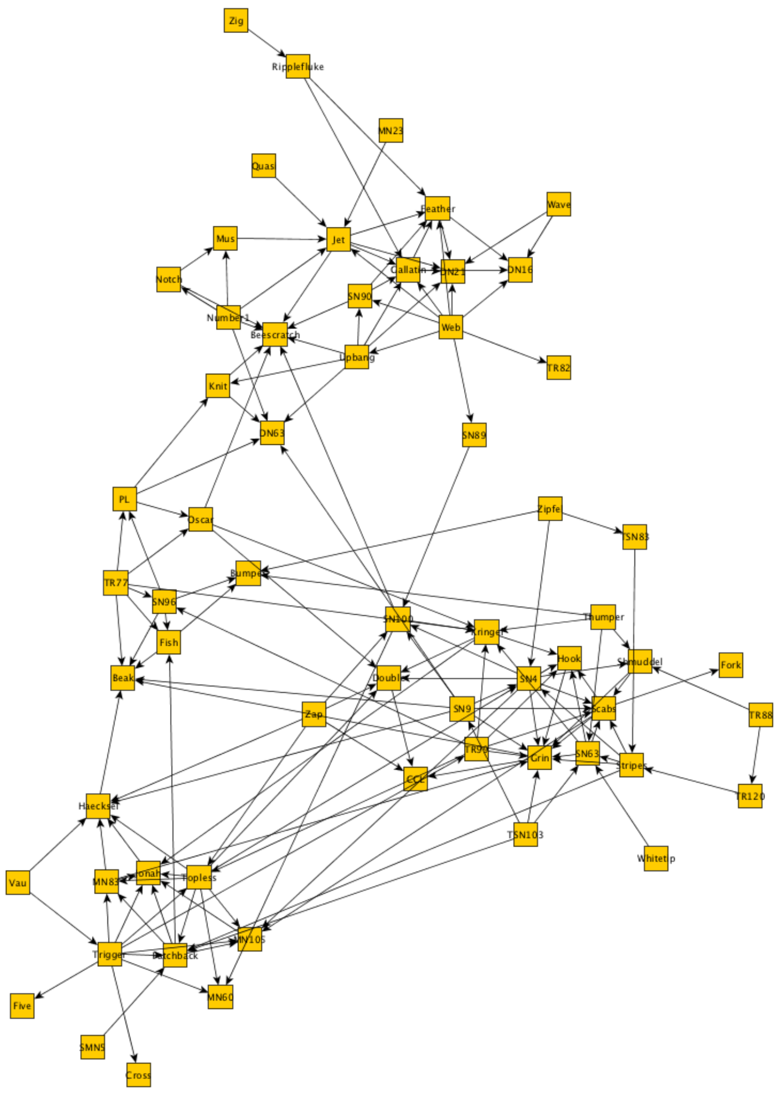

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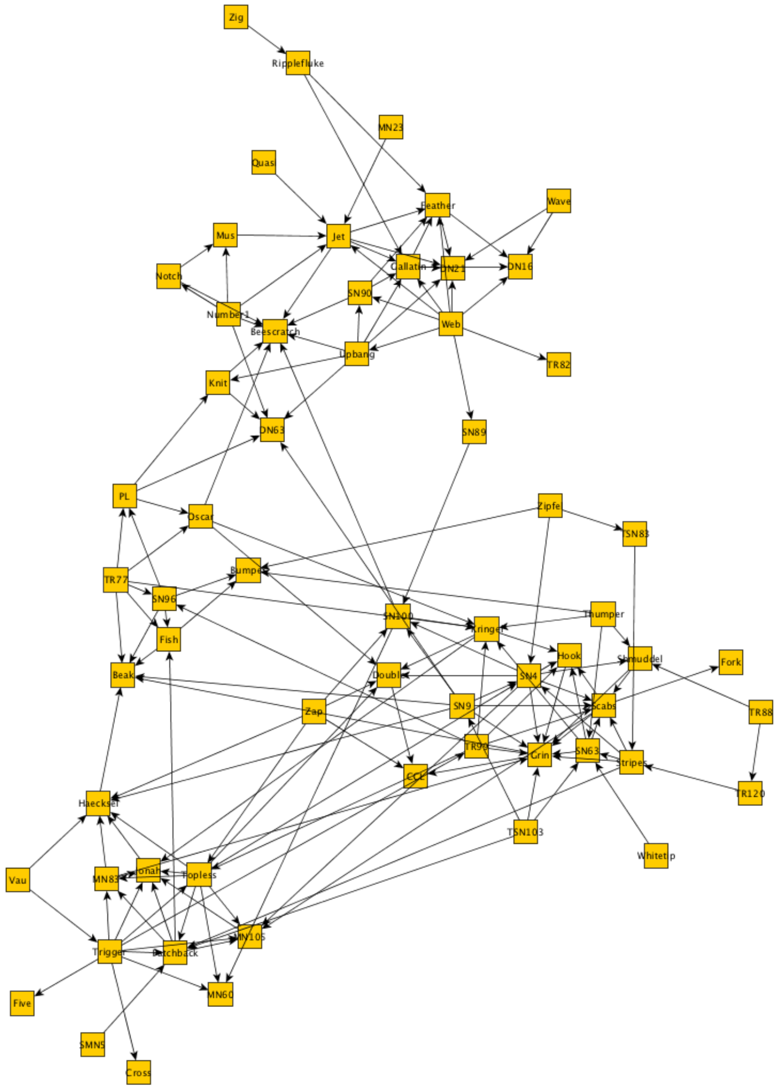

Optimization criteria partially contradict each other.

Fixed Edge Lengths?

Input: Graph $G = (V, E)$, required edge length $\ell(e)$ for each $e \in E$.

Output: Drawing of G that realizes the given edge lengths.

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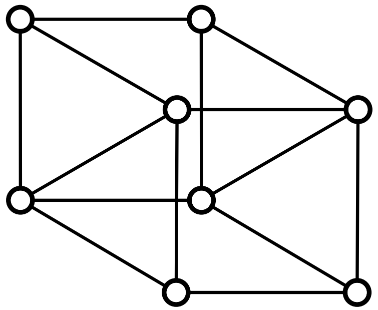

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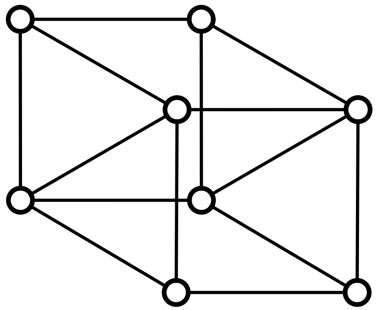

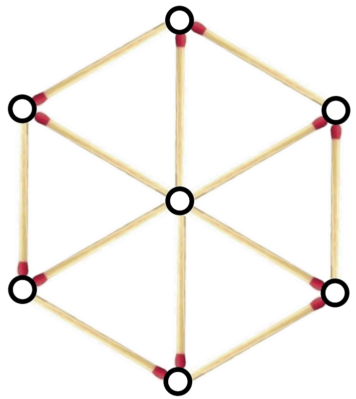

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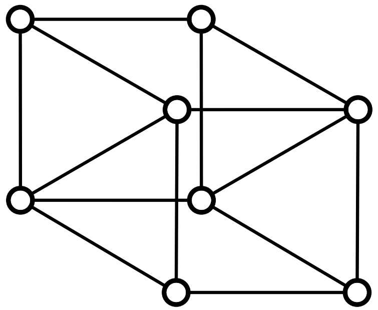

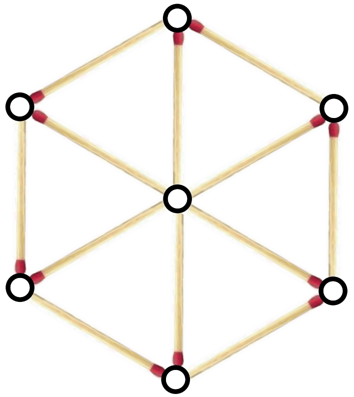

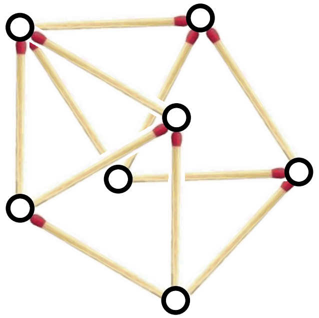

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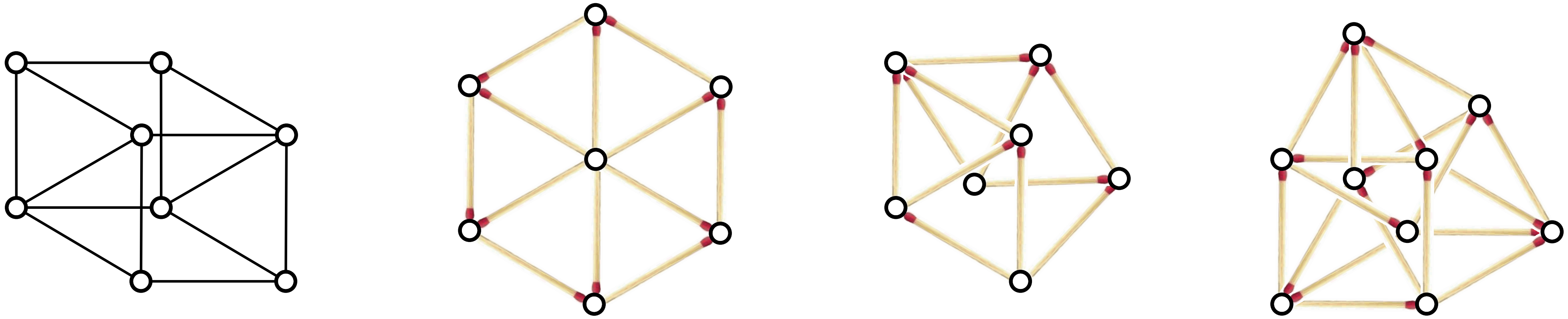

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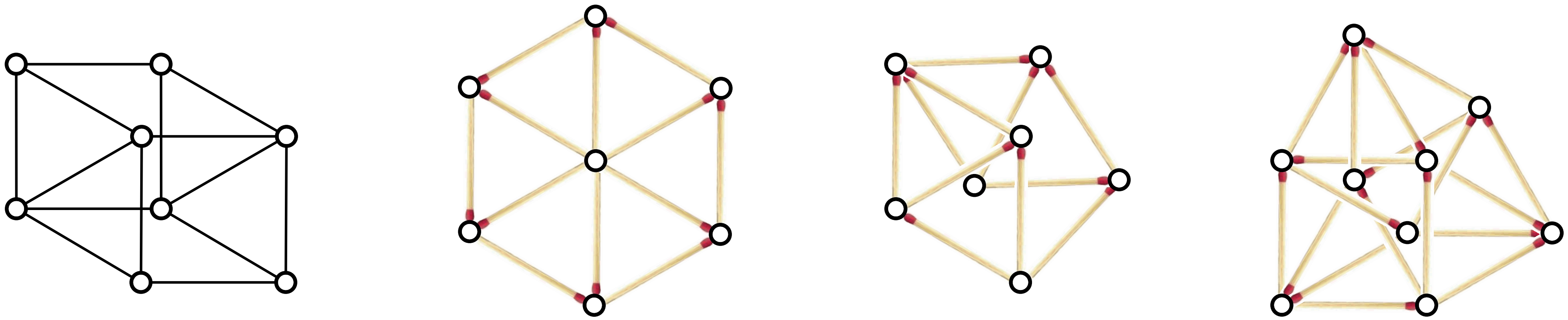

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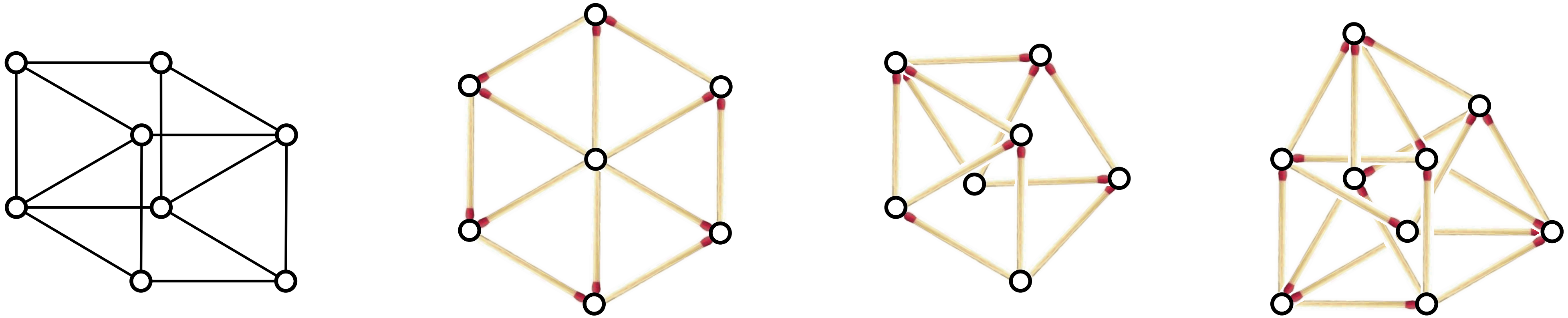

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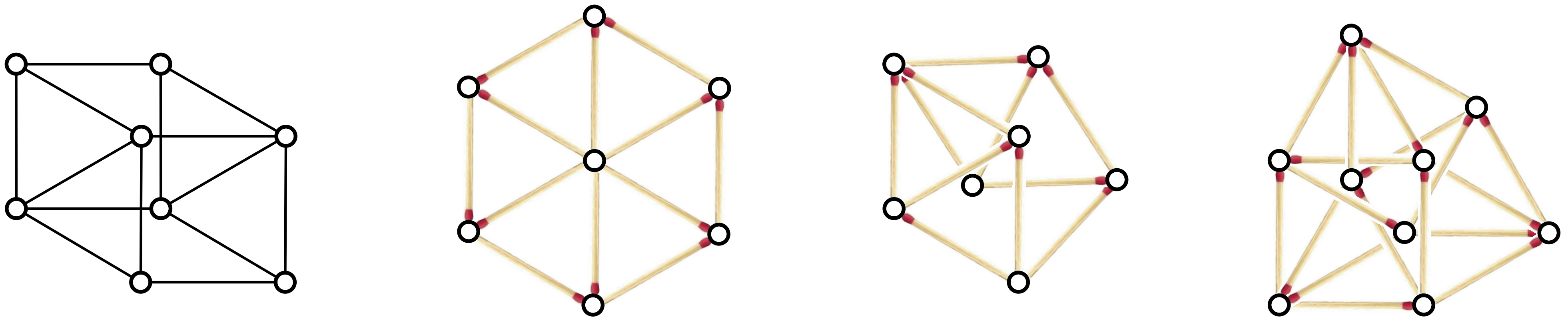

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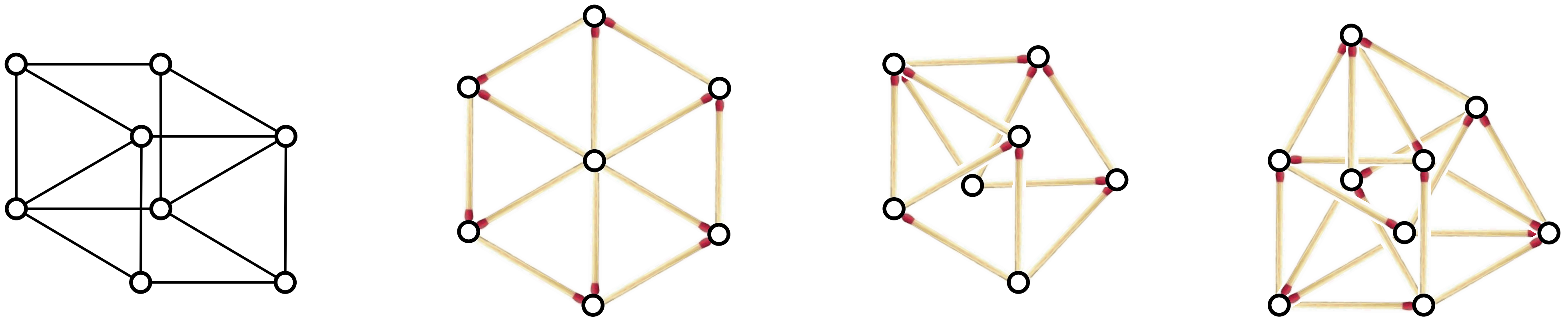

NP-hard for

- uniform edge lengths in any dimension [Johnson '82]
- uniform edge lengths in planar drawings [Eades, Wormald '90]
- edge lengths $\{1, 2\}$ [Saxe '80]

Physical Analogy

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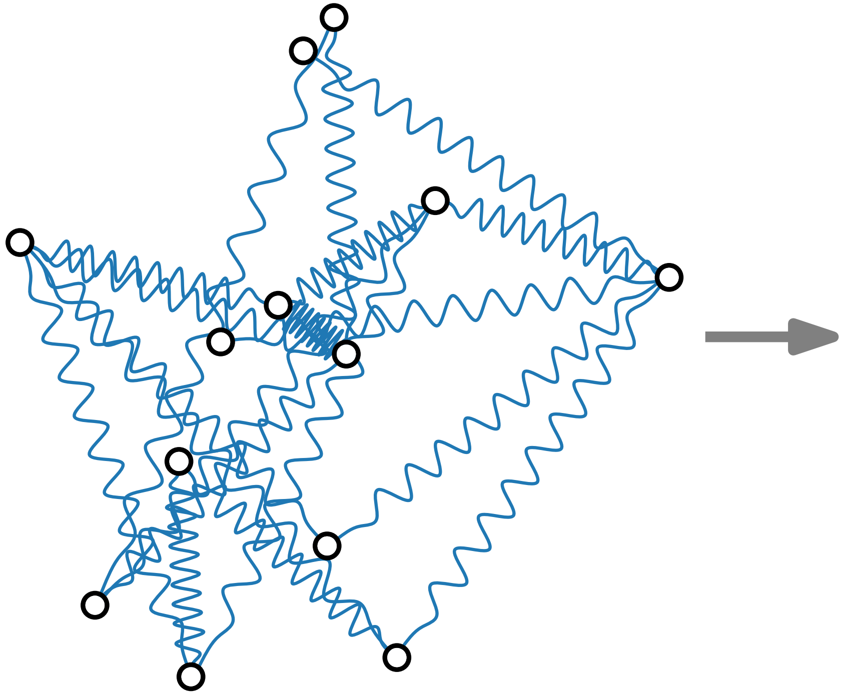

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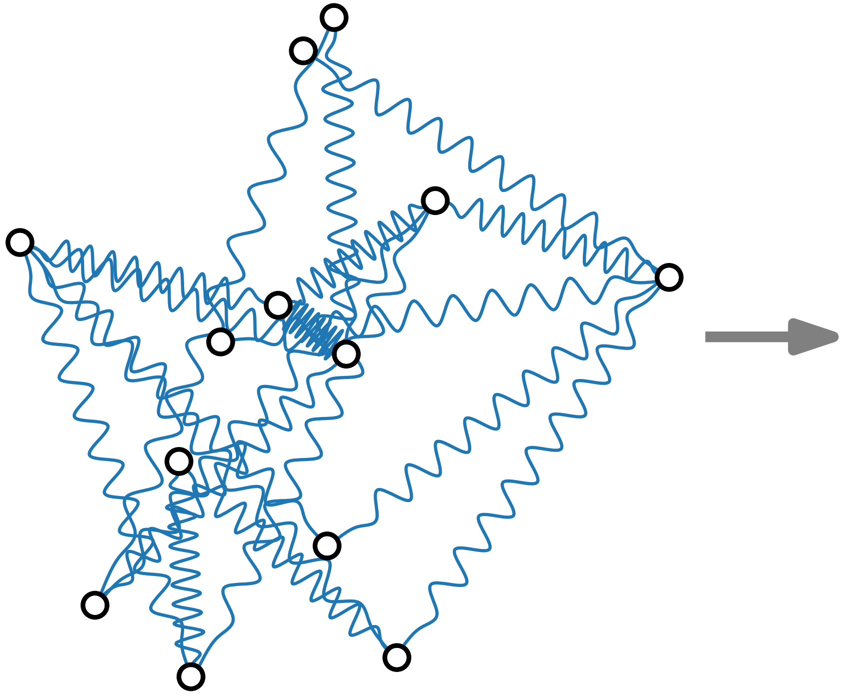

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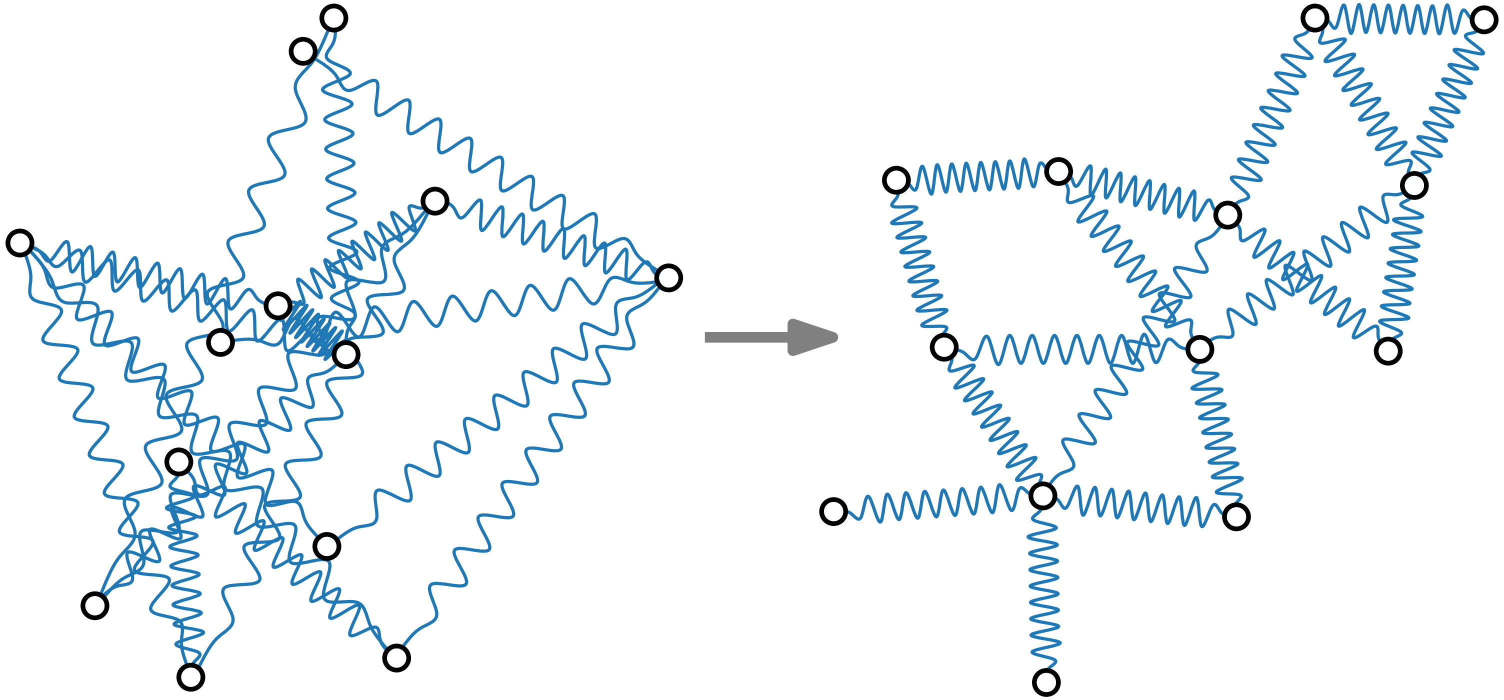

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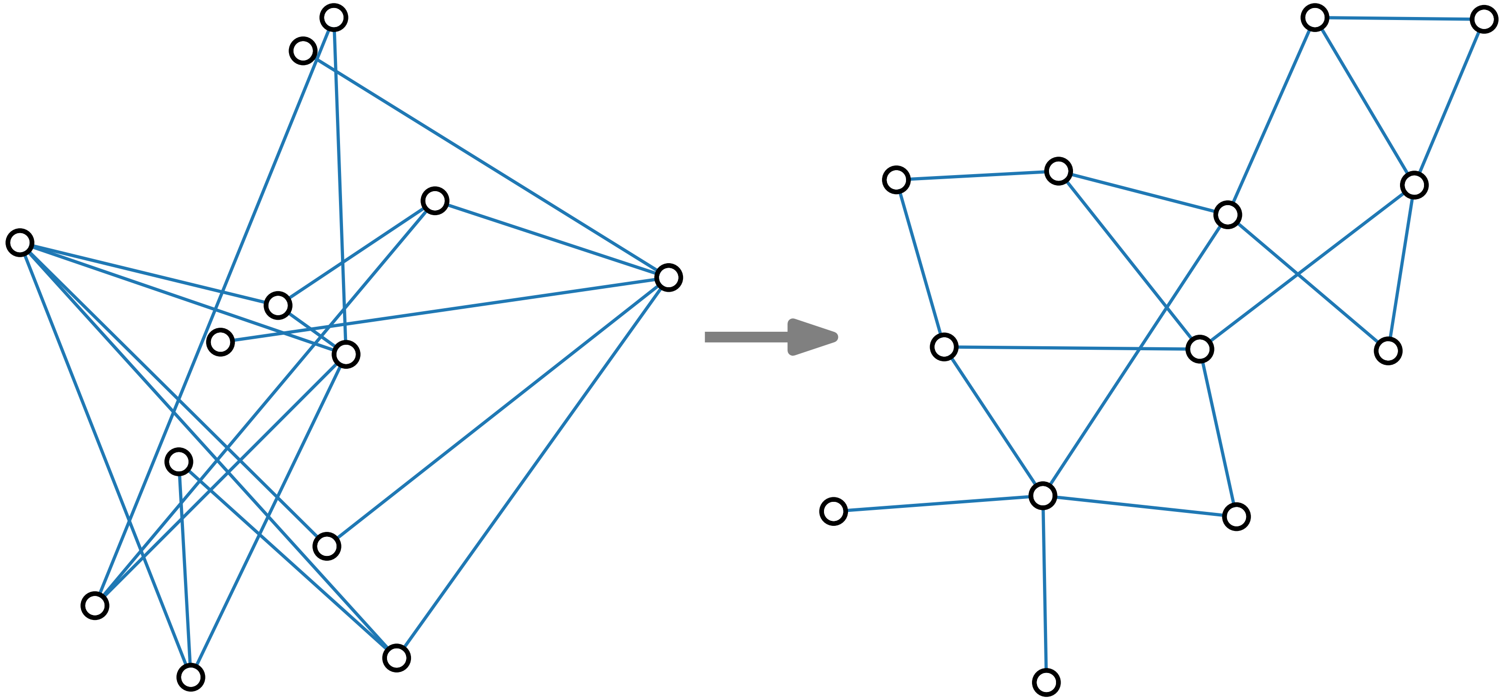

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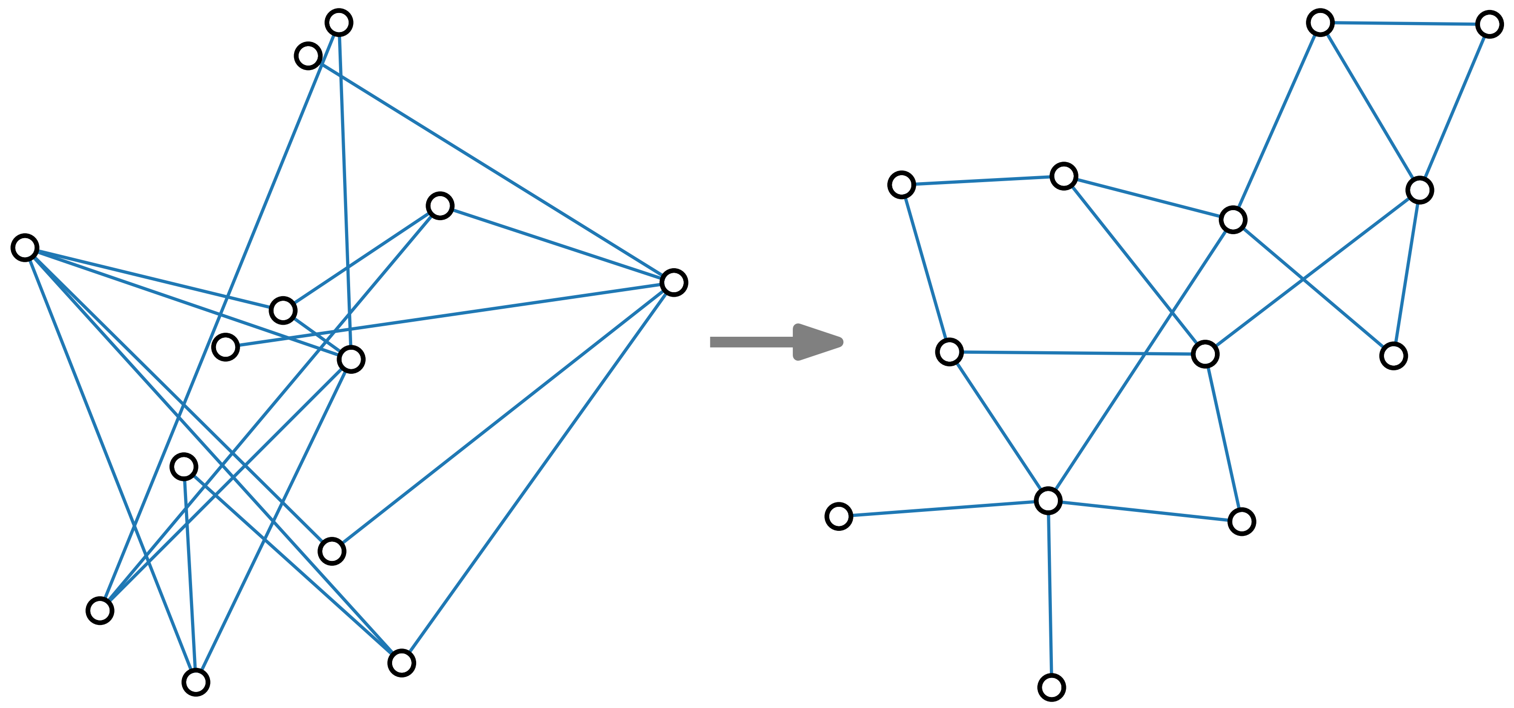
Attractive forces.



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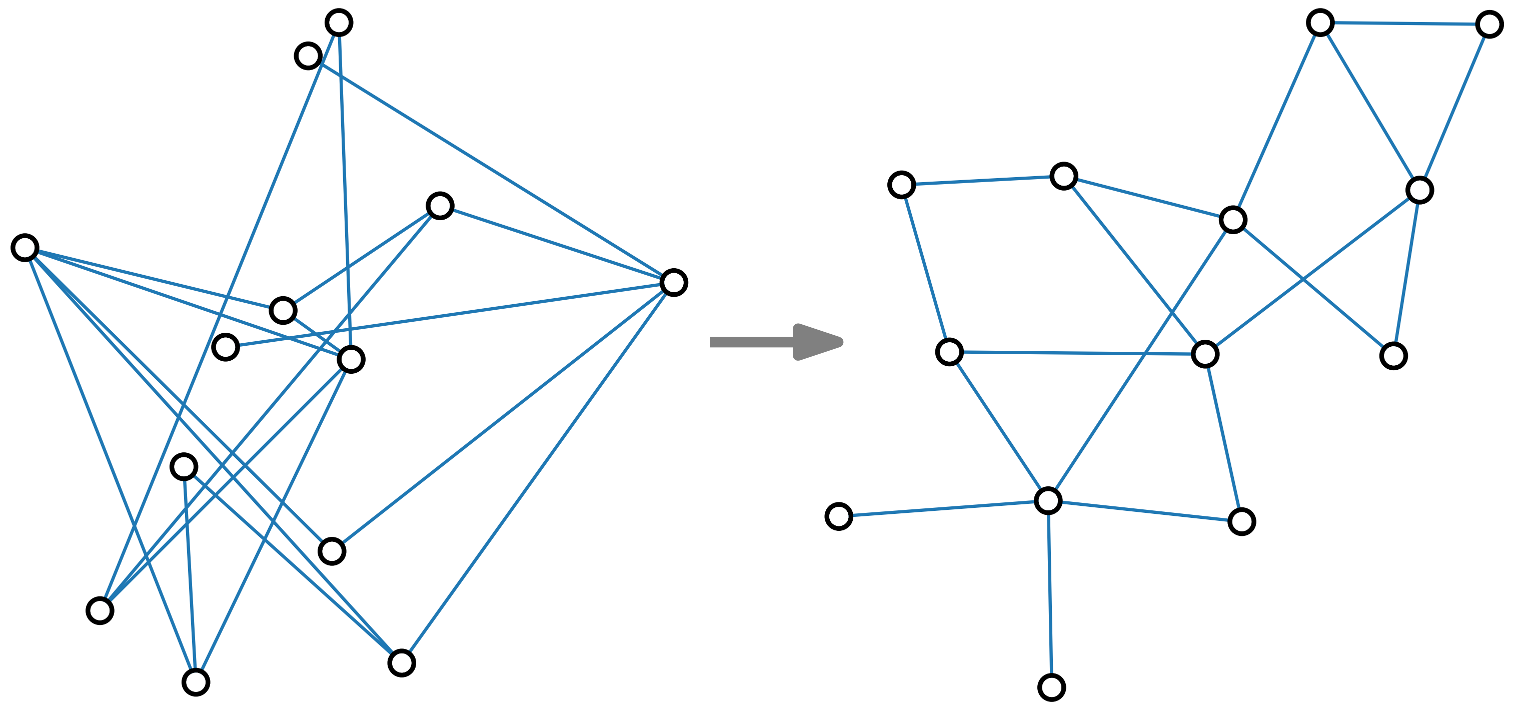

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pairs $\{u, v\}$ of adjacent vertices:

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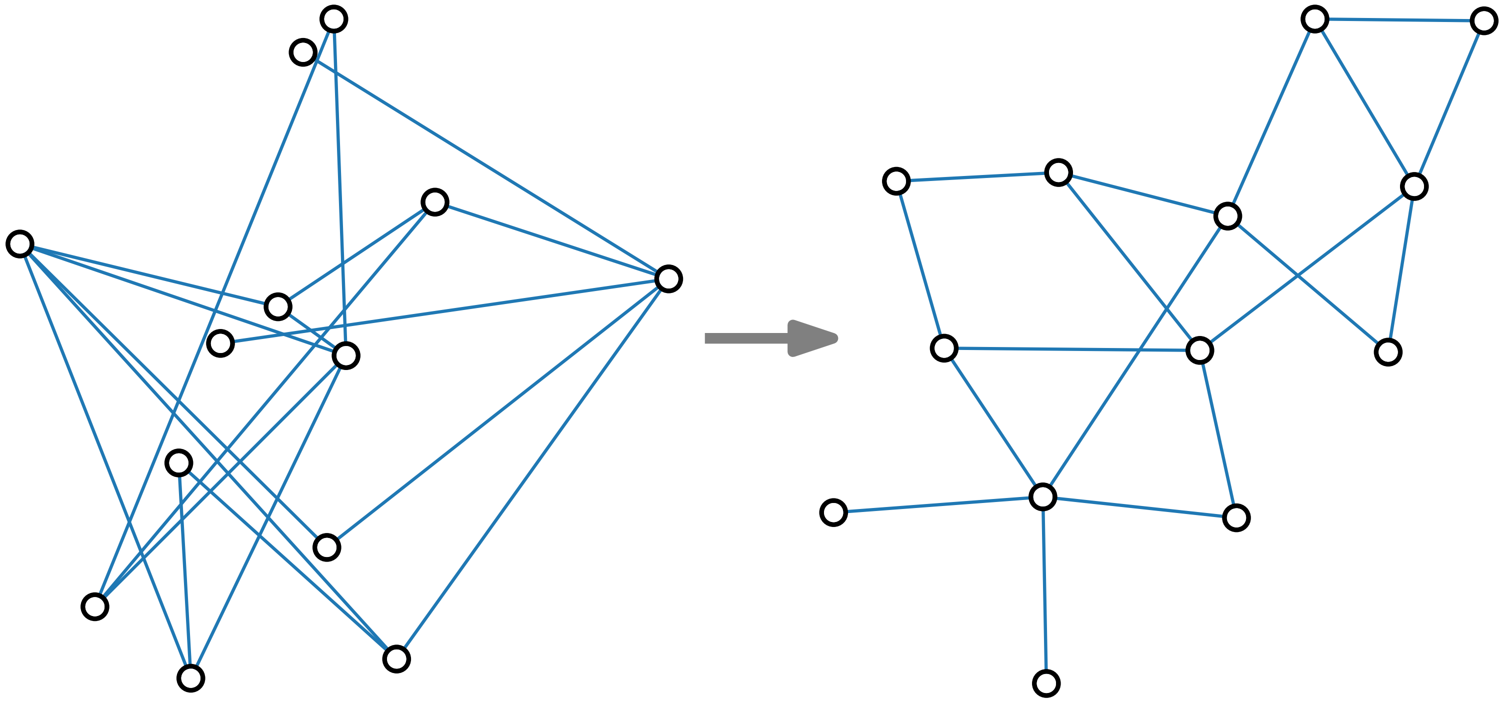

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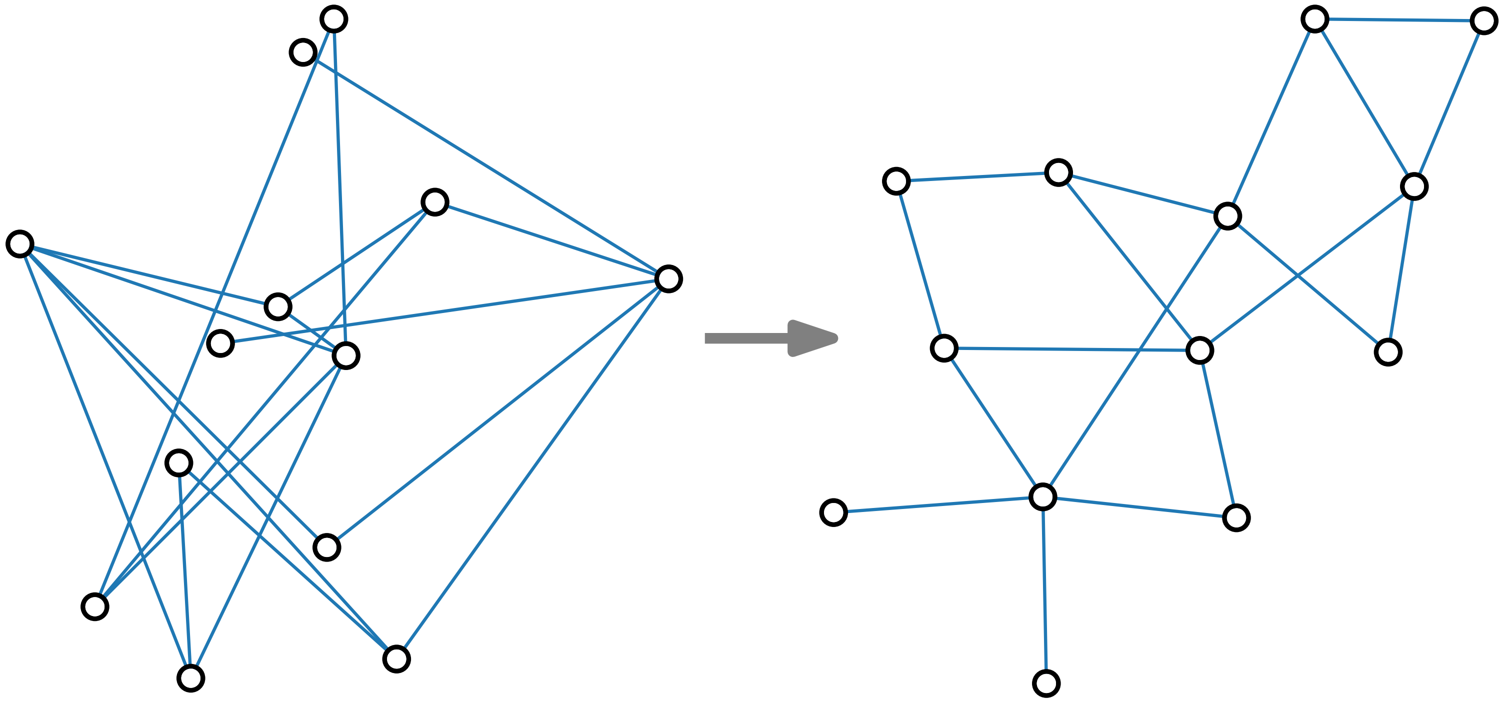

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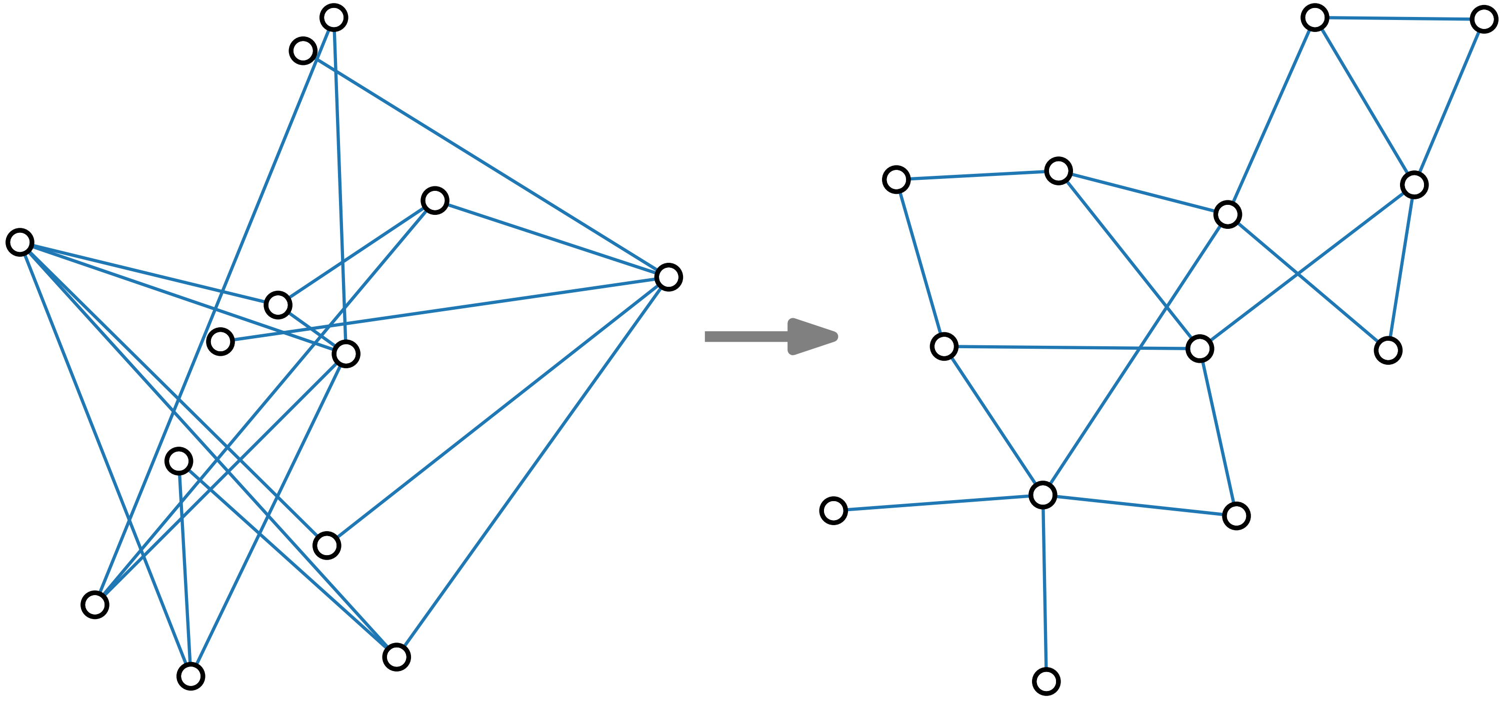

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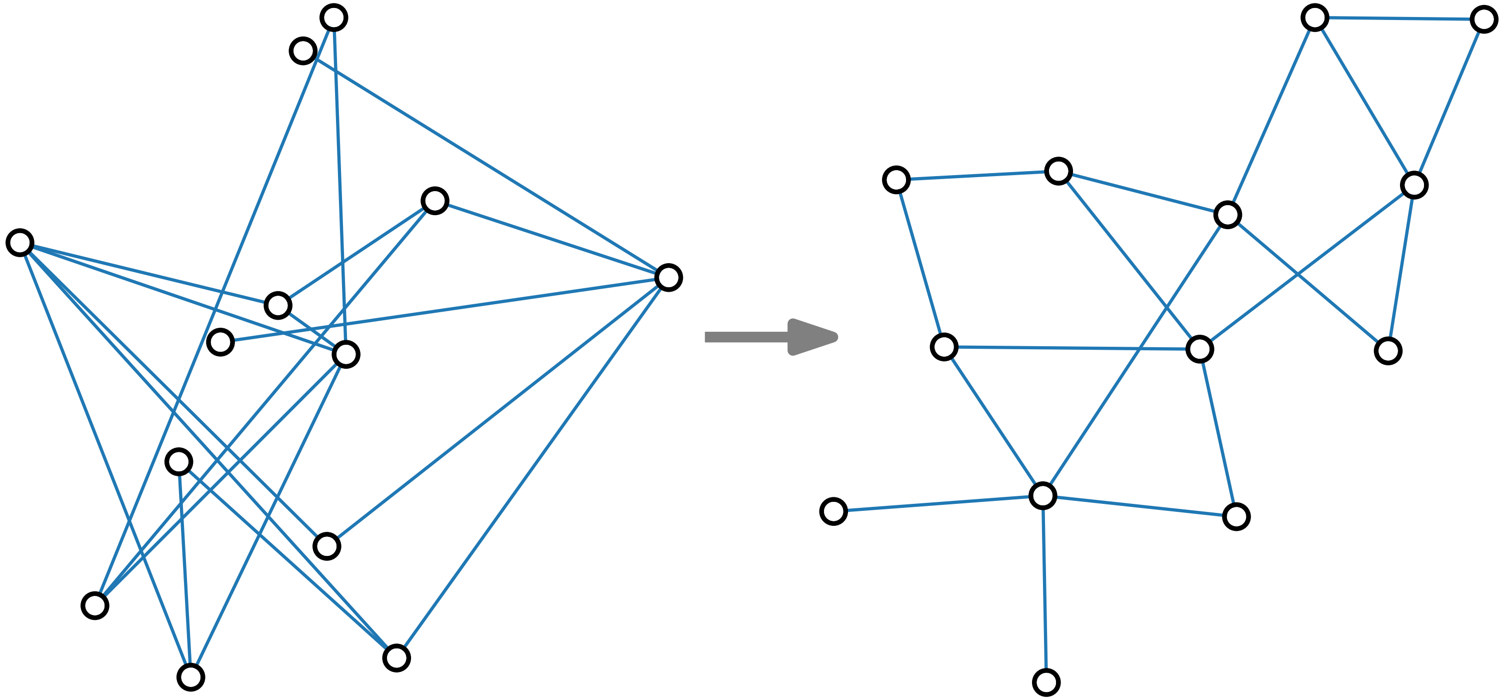

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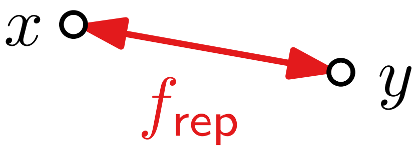

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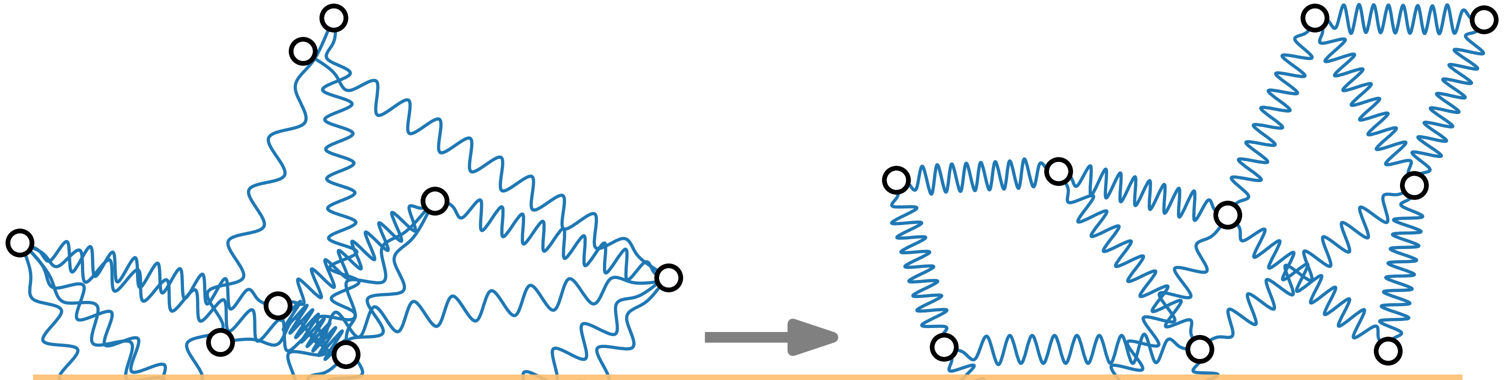

So-called **spring-embedder** algorithms that work according to this or similar principles are among the most frequently used graph-drawing methods in practice.

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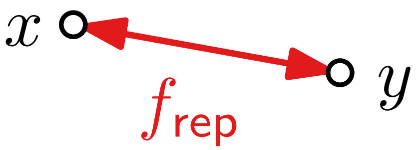

Force-Directed Algorithms

ForceDirected($G = (V, E)$, $p = (p_v)_{v \in V}$, $\varepsilon > 0$, $K \in \mathbb{N}$)

return p

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initial layout; may be randomly chosen positions

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ForceDirected($G = (V, E)$, $p = (p_v)_{v \in V}$, $\varepsilon > 0$, $K \in \mathbb{N}$)

$t \leftarrow 1$

while $t < K$ **and** $\max_{v \in V} \|F_v(t)\| > \varepsilon$ **do**

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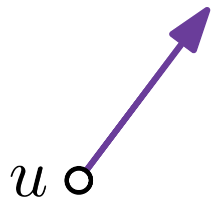

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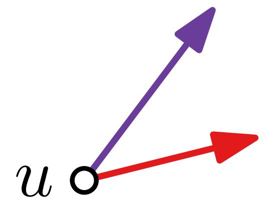

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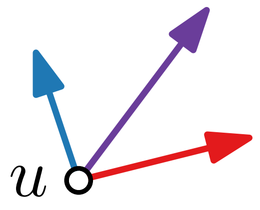

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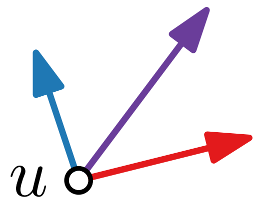

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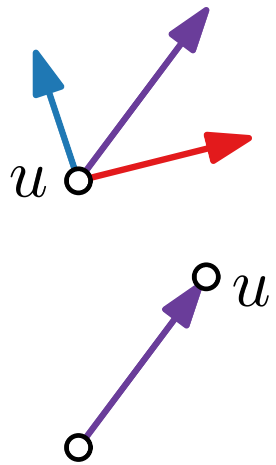

Force-Directed Algorithms

initial layout; may be randomly chosen positions

threshold

max # iterations

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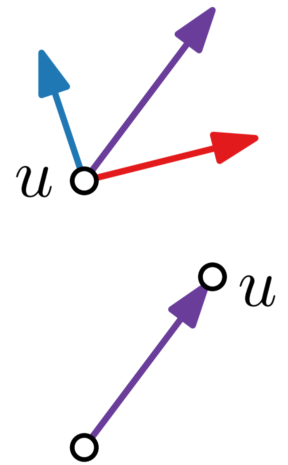

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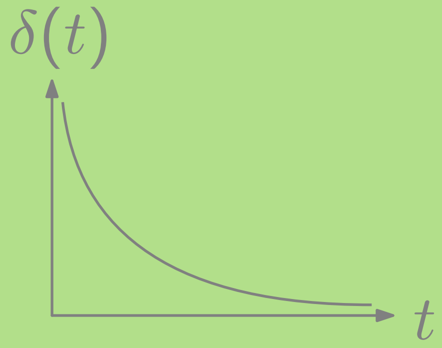

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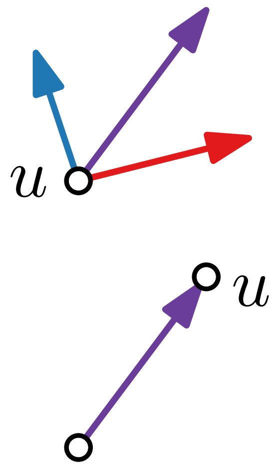

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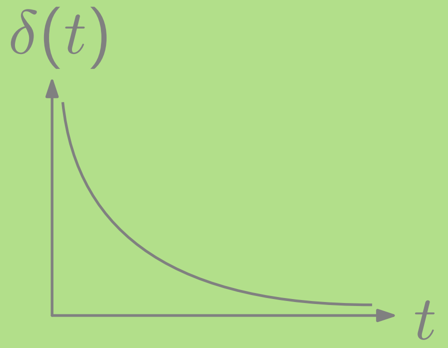

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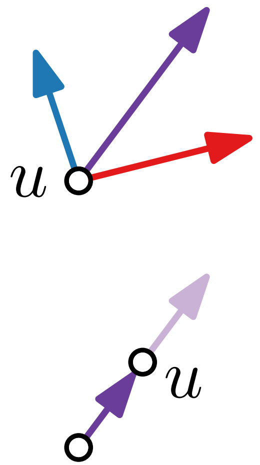

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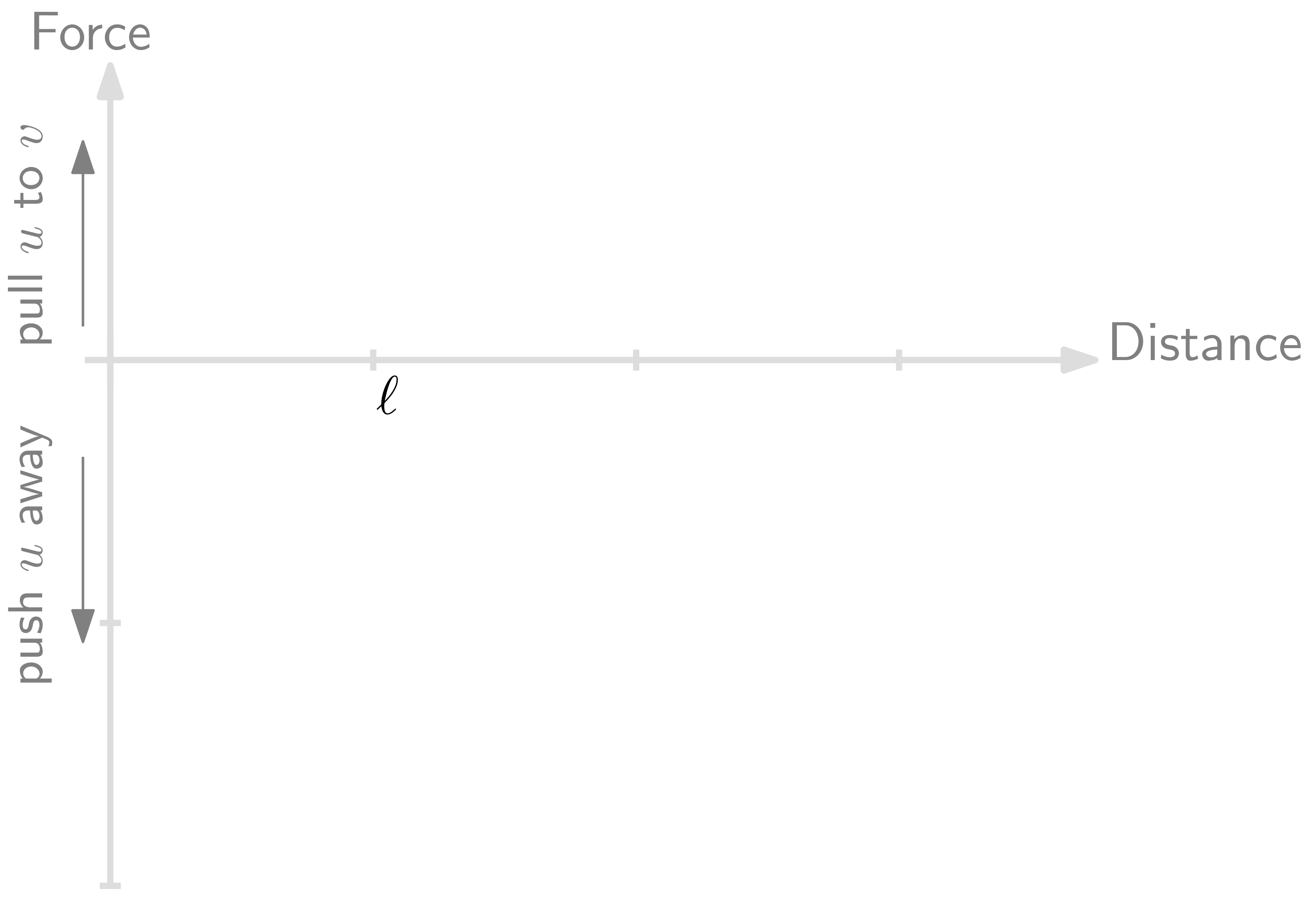

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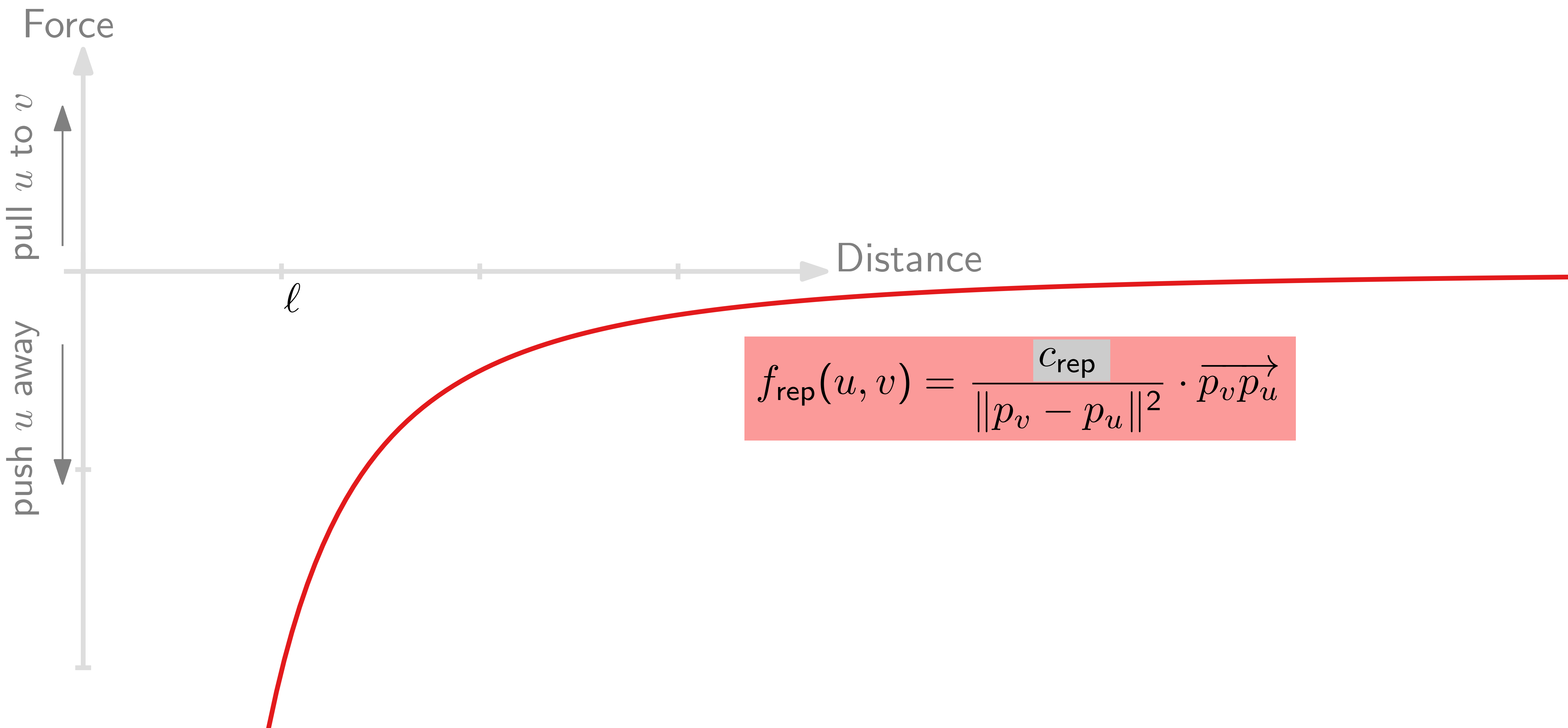

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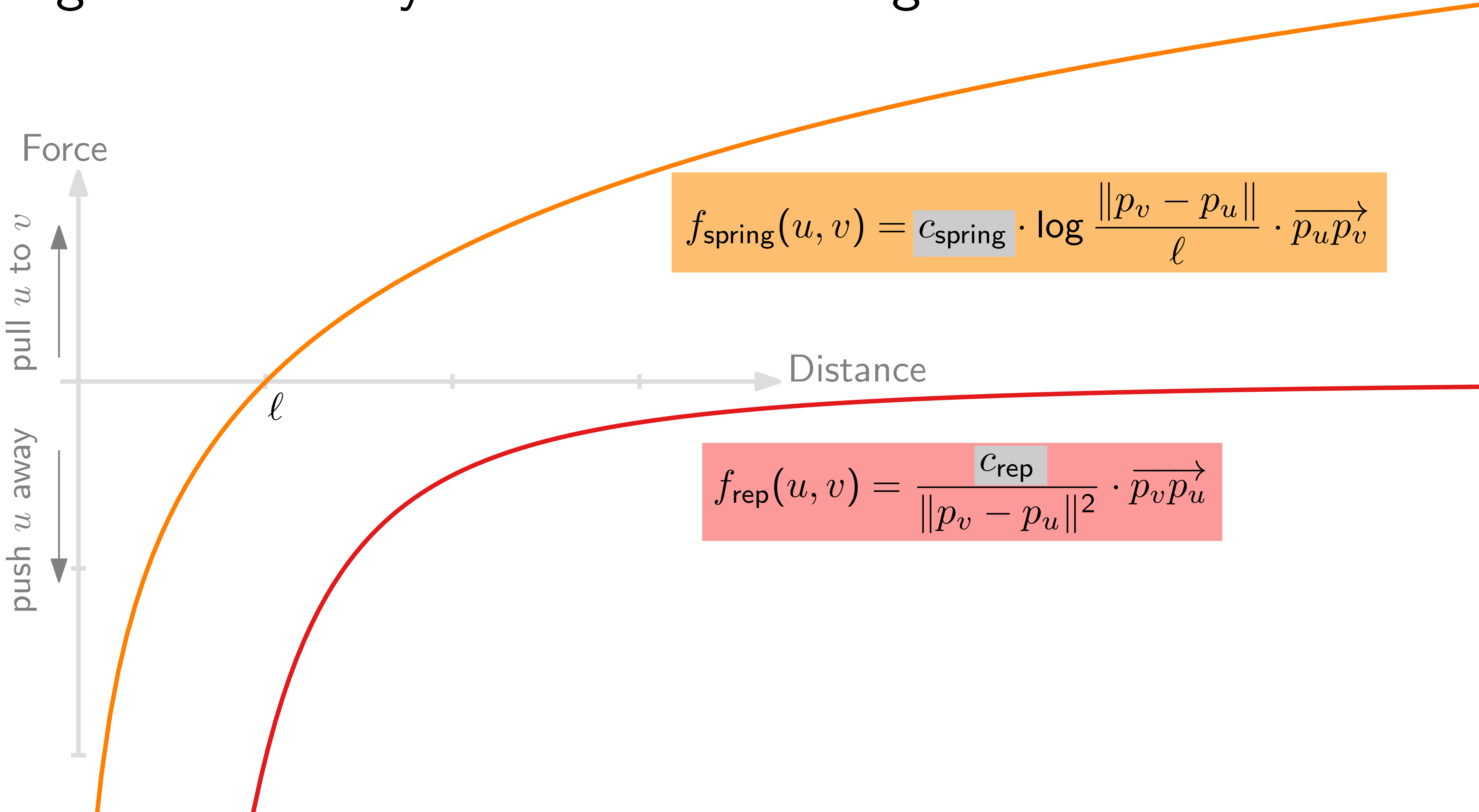

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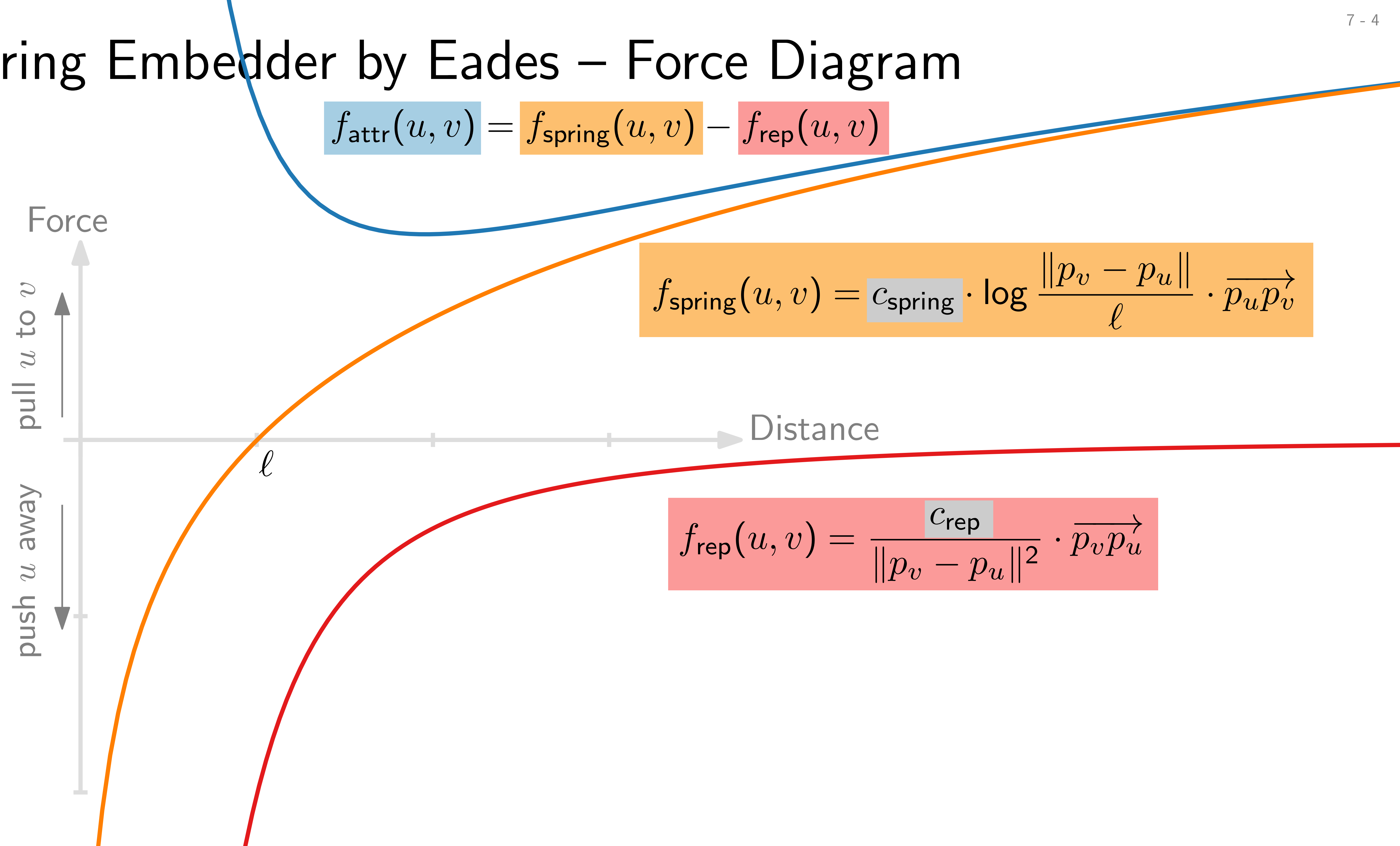

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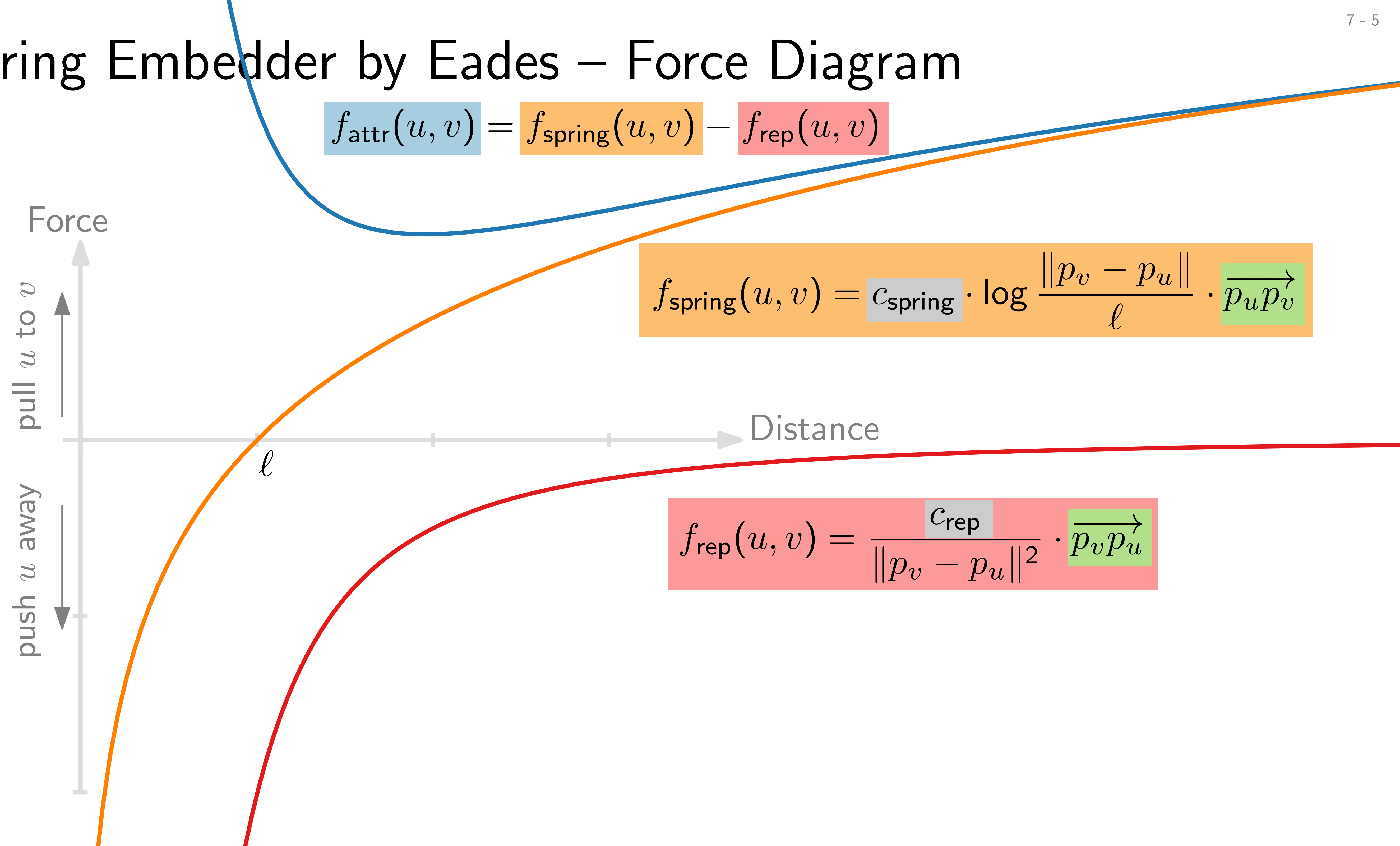

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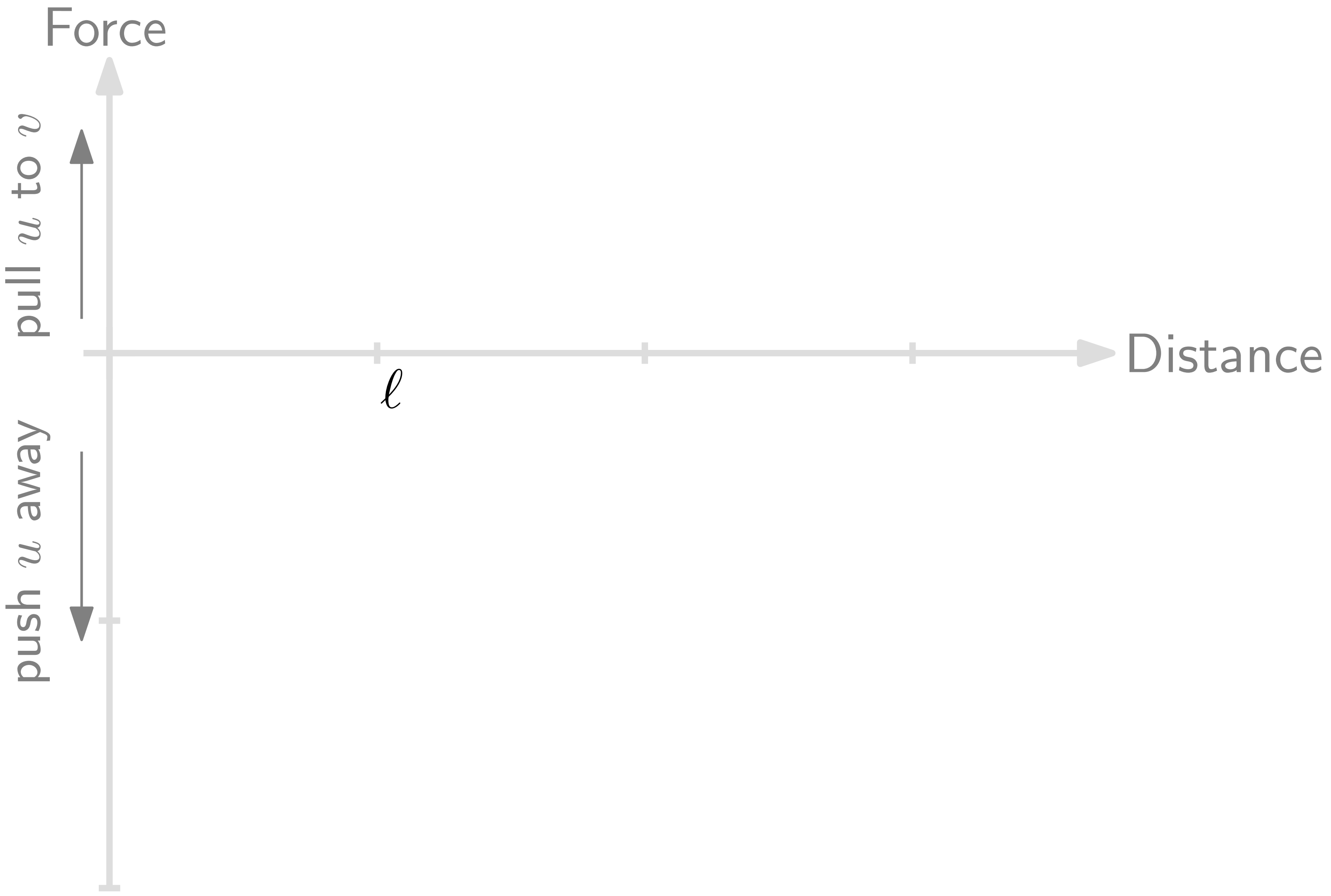
  foreach $u \in V$ **do**
    $p_u \leftarrow p_u + \delta(t) \cdot F_u(t)$
  $t \leftarrow t + 1$
return p

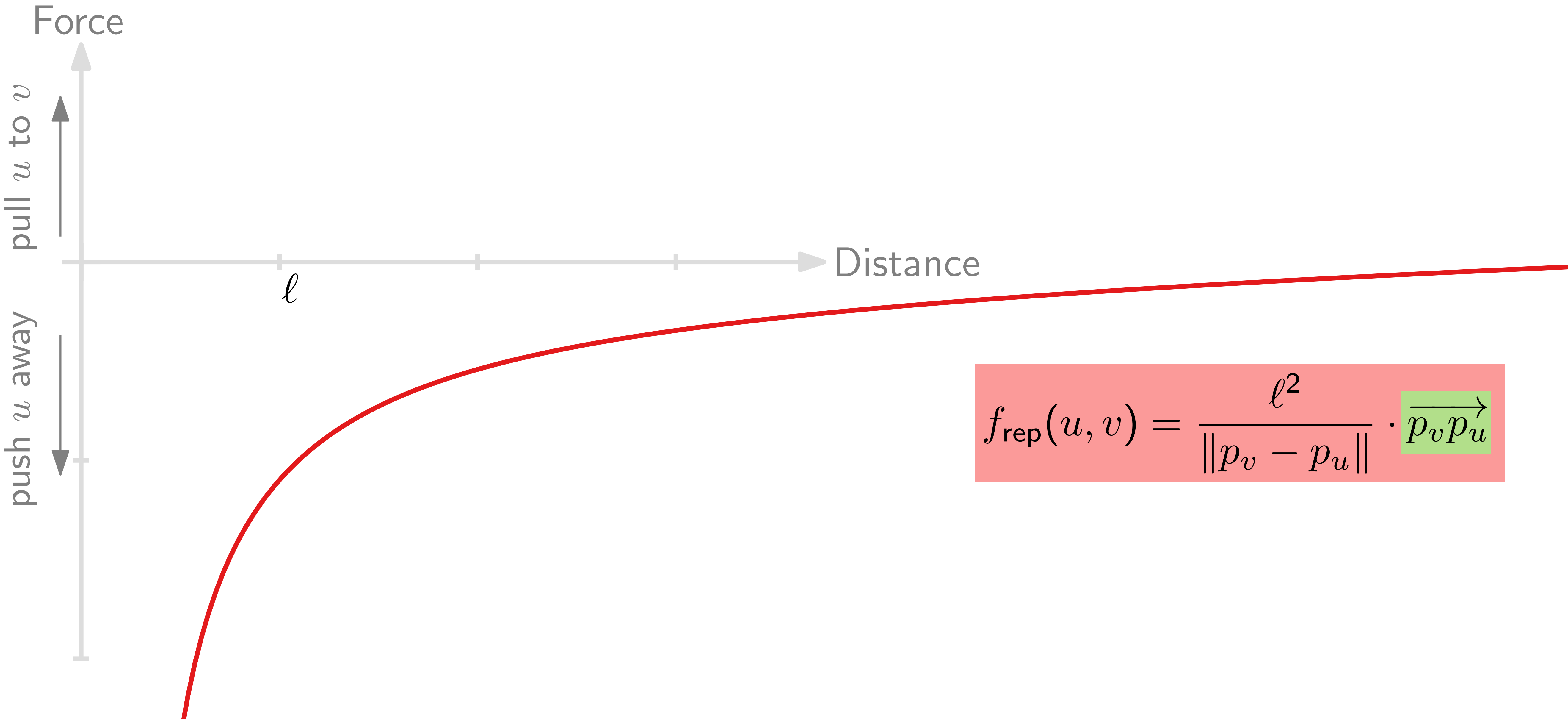
Notation.

- $\|p_u - p_v\|$ = Euclidean distance between u and v
- $\overrightarrow{p_u p_v}$ = unit vector pointing from u to v
- ℓ = ideal spring length for edges

Fruchterman & Reingold – Force Diagram



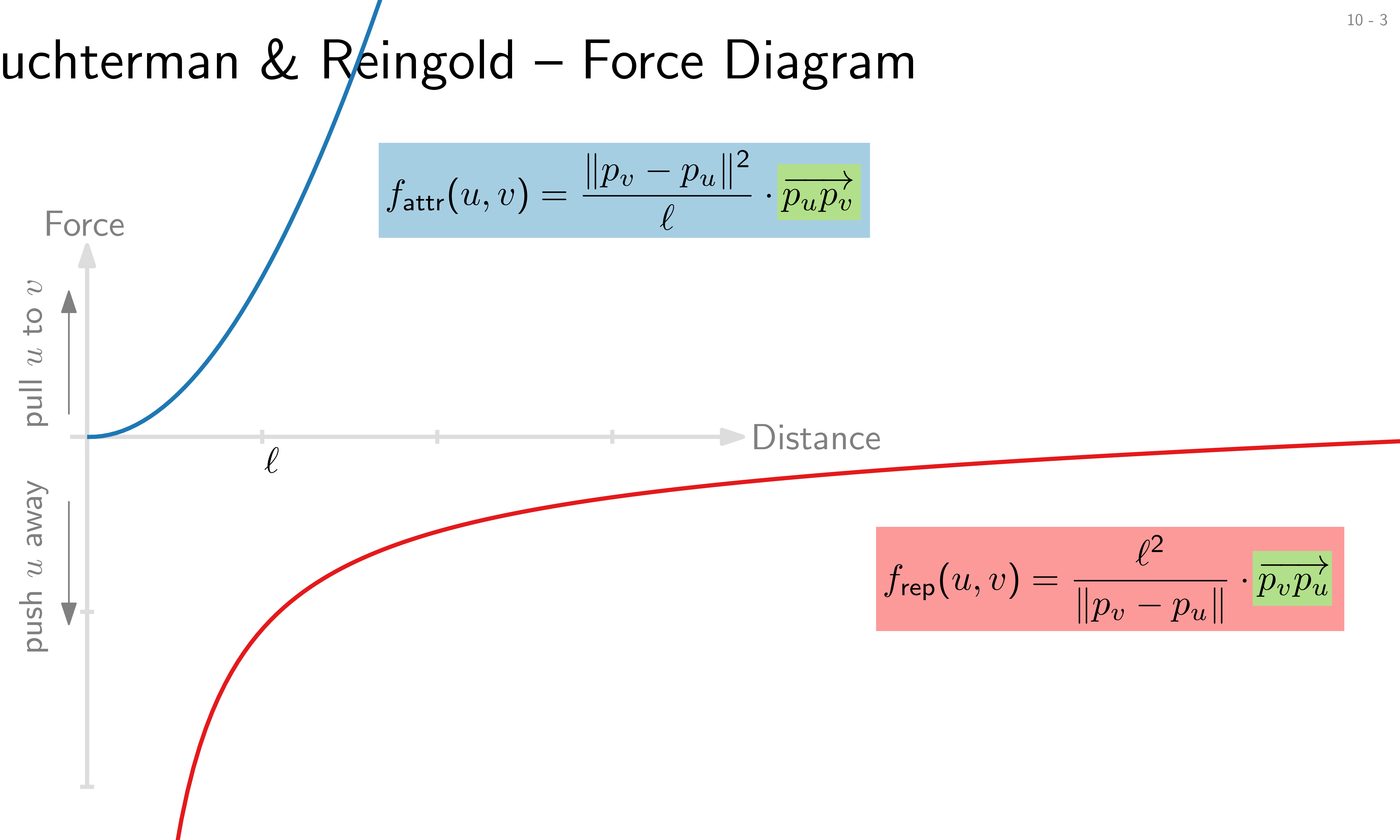
Fruchterman & Reingold – Force Diagram



$$f_{\mathsf{rep}}(u,v) = \frac{\ell^2}{\|p_v - p_u\|} \cdot \overrightarrow{p_v p_u}$$

Fruchterman & Reingold – Force Diagram

$$f_{\mathsf{attr}}(u,v) = \frac{\|p_v - p_u\|^2}{\ell} \cdot \overrightarrow{p_u p_v}$$



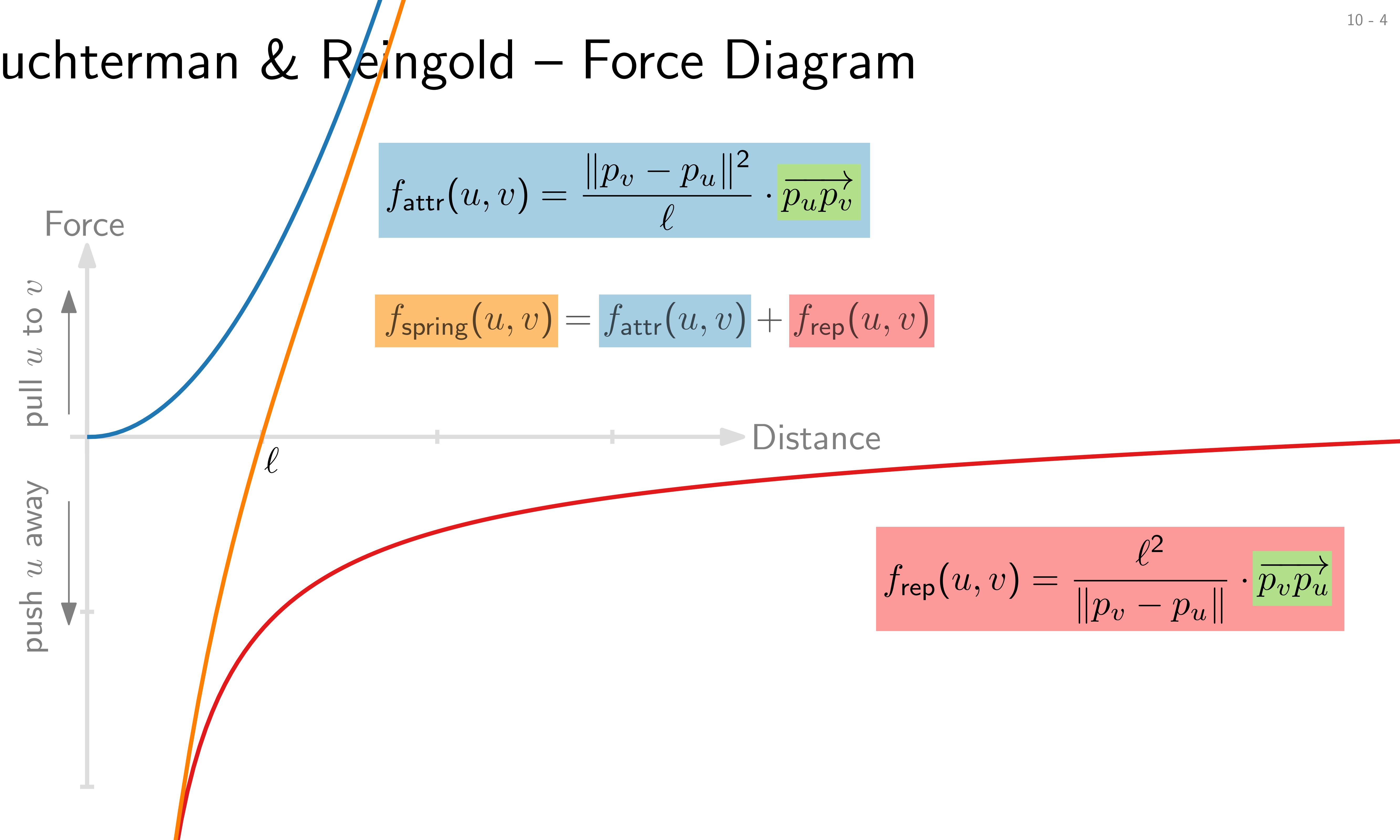
$$f_{\mathsf{rep}}(u,v) = \frac{\ell^2}{\|p_v - p_u\|} \cdot \overrightarrow{p_v p_u}$$

Fruchterman & Reingold – Force Diagram

$$f_{\text{attr}}(u,v) = \frac{\|p_v - p_u\|^2}{\ell} \cdot \overrightarrow{p_u p_v}$$

$$f_{\text{spring}}(u,v) = f_{\text{attr}}(u,v) + f_{\text{rep}}(u,v)$$

$$f_{\text{rep}}(u,v) = \frac{\ell^2}{\|p_v - p_u\|} \cdot \overrightarrow{p_v p_u}$$



Adaptability

Inertia. (“Trägheit”)

- Define vertex mass $\Phi(v) = 1 + \deg(v)/2$
- Set $f_{\text{attr}}(p_u, p_v) \leftarrow f_{\text{attr}}(p_u, p_v) \cdot 1/\Phi(v)$

Adaptability

Inertia. ("Trägheit")

- Define vertex mass $\Phi(v) = 1 + \mathrm{deg}(v)/2$
- Set $f_{\mathrm{attr}}(p_u, p_v) \leftarrow f_{\mathrm{attr}}(p_u, p_v) \cdot 1/\Phi(v)$

Gravitation.

- Define centroid $\sigma_V = 1/|V| \cdot \sum_{v \in V} p_v$
- Add force $f_{\mathrm{grav}}(p_v) = c_{\mathrm{grav}} \cdot \Phi(v) \cdot \overrightarrow{p_v \sigma_V}$

Adaptability

Inertia. ("Trägheit")

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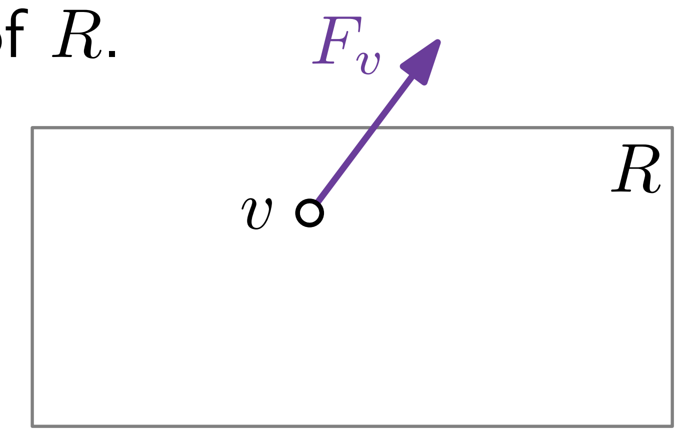
- Add force $f_{\mathrm{grav}}(p_v) = c_{\mathrm{grav}} \cdot \Phi(v) \cdot \overrightarrow{p_v \sigma_V}$

Restricted drawing area.

If F_v points beyond area R, clip vector appropriately at the border of R.



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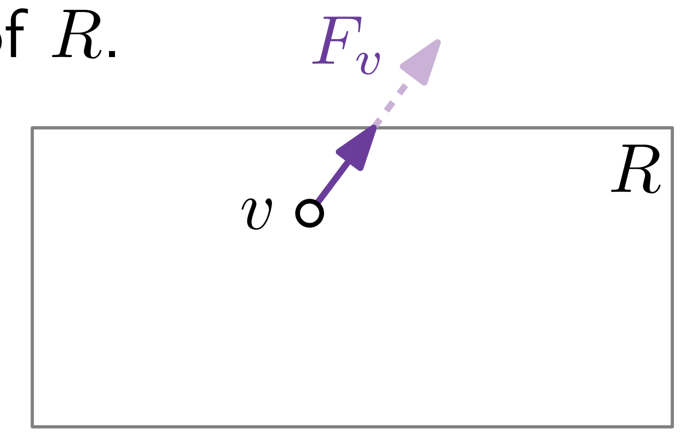

Adaptability

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- Define vertex mass $\Phi(v) = 1 + \deg(v)/2$
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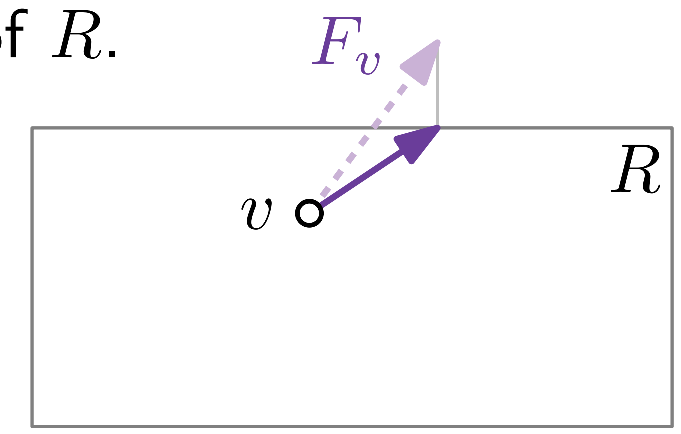

Adaptability

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Gravitation.

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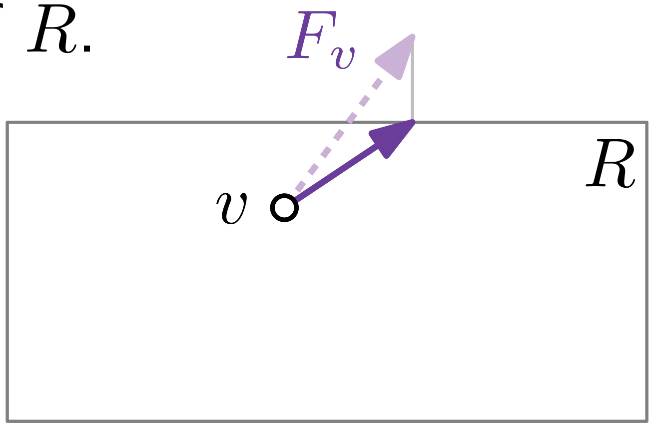
Restricted drawing area.

If F_v points beyond area R, clip vector appropriately at the border of R.



And many more...

- magnetic orientation of edges [GD Ch. 10.4]
- other energy models
- planarity preserving
- speed-ups

Speeding up "Convergence" by Adaptive Displacement $\delta_v(t)$

ForceDirected($G = (V, E)$, $p = (p_v)_{v \in V}$, $\varepsilon > 0$, $K \in \mathbb{N}$)
  $t \leftarrow 1$
  while $t < K$ **and** $\max_{v \in V} \|F_v(t)\| > \varepsilon$ **do**
    foreach $u \in V$ **do**
      $F_u(t) \leftarrow \sum_{v \in V} f_{\mathsf{rep}}(u, v) + \sum_{v \in \mathsf{Adj}[u]} f_{\mathsf{attr}}(u, v)$
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    $t \leftarrow t + 1$
  return p

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  foreach $u \in V$ **do**

    $p_u \leftarrow p_u + \cancel{\delta(t)} \cdot F_u(t)$

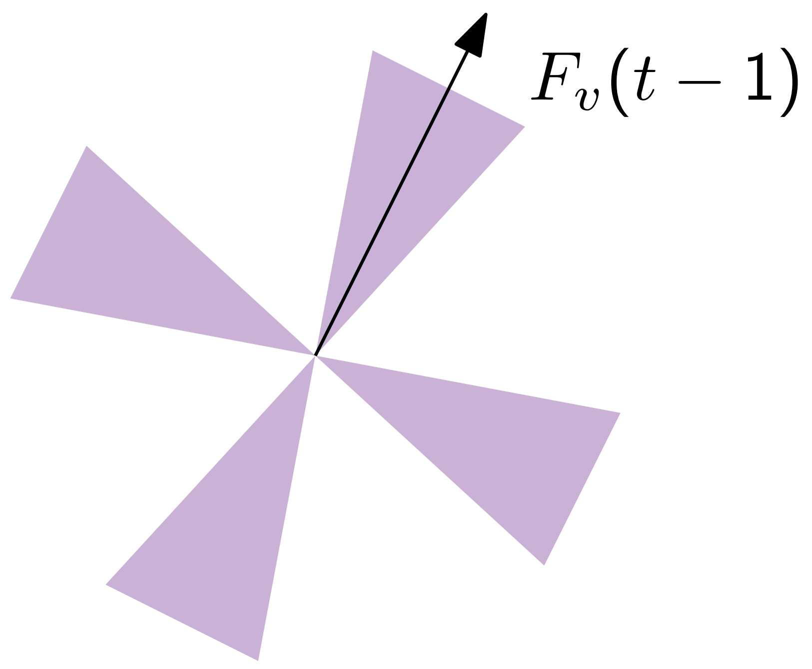
    $\delta_v(t)$

  $t \leftarrow t + 1$

return p

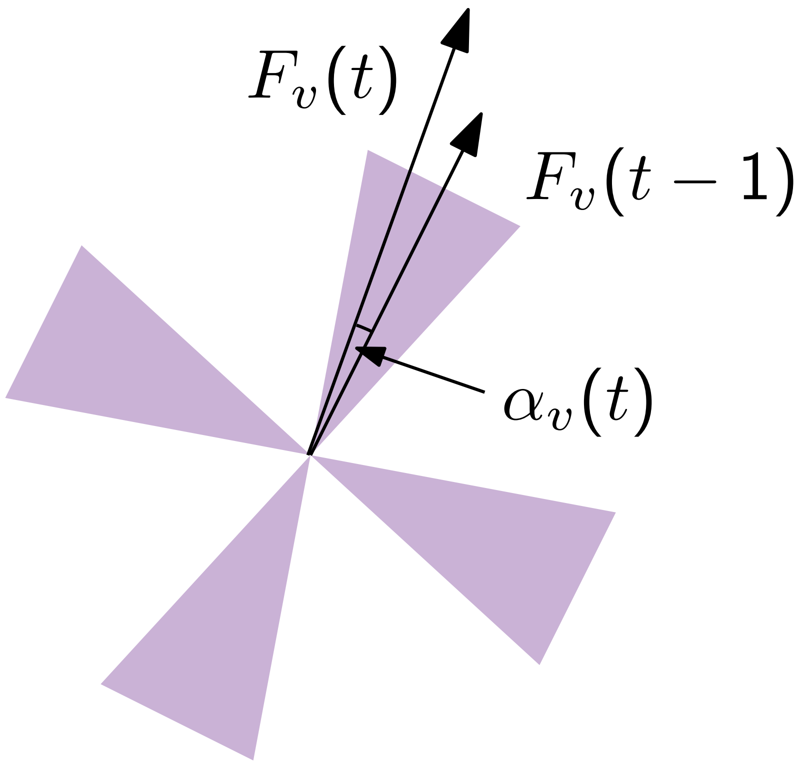
Speeding up "Convergence" by Adaptive Displacement $\delta_v(t)$

[Frick, Ludwig, Mehldau '95]



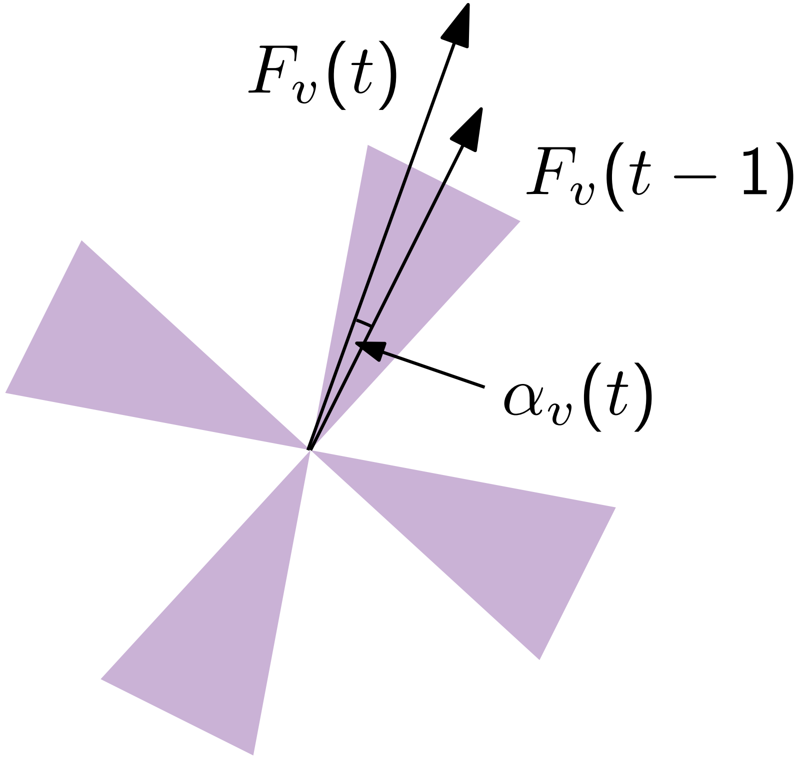
Speeding up "Convergence" by Adaptive Displacement $\delta_v(t)$

[Frick, Ludwig, Mehldau '95]



Speeding up "Convergence" by Adaptive Displacement $\delta_v(t)$

[Frick, Ludwig, Mehldau '95]

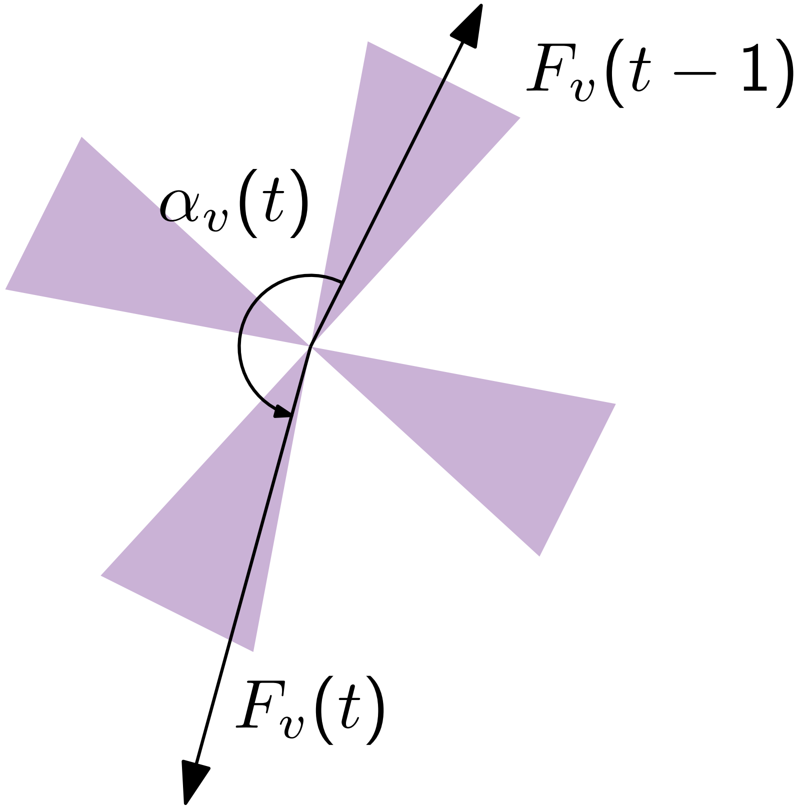


Same direction.

$\rightarrow$ increase temperature $\delta_v(t)$

Speeding up "Convergence" by Adaptive Displacement $\delta_v(t)$

[Frick, Ludwig, Mehldau '95]

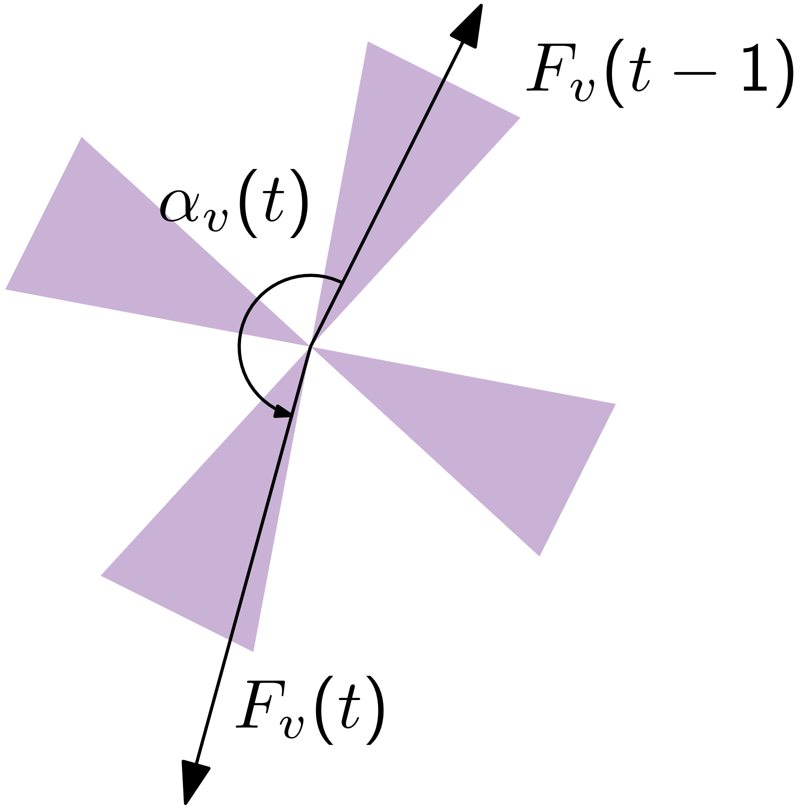


Same direction.

$\rightarrow$ increase temperature $\delta_v(t)$

Speeding up "Convergence" by Adaptive Displacement $\delta_v(t)$

[Frick, Ludwig, Mehldau '95]

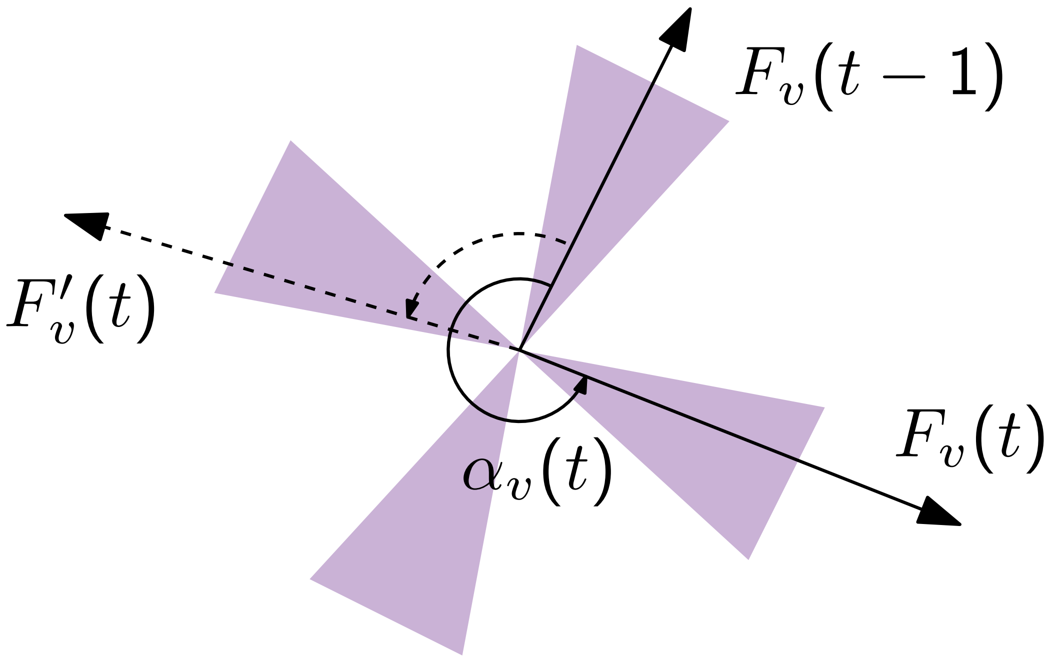


Same direction.
$\rightarrow$ increase temperature $\delta_v(t)$

Oscillation.
$\rightarrow$ decrease temperature $\delta_v(t)$

Speeding up "Convergence" by Adaptive Displacement $\delta_v(t)$

[Frick, Ludwig, Mehldau '95]

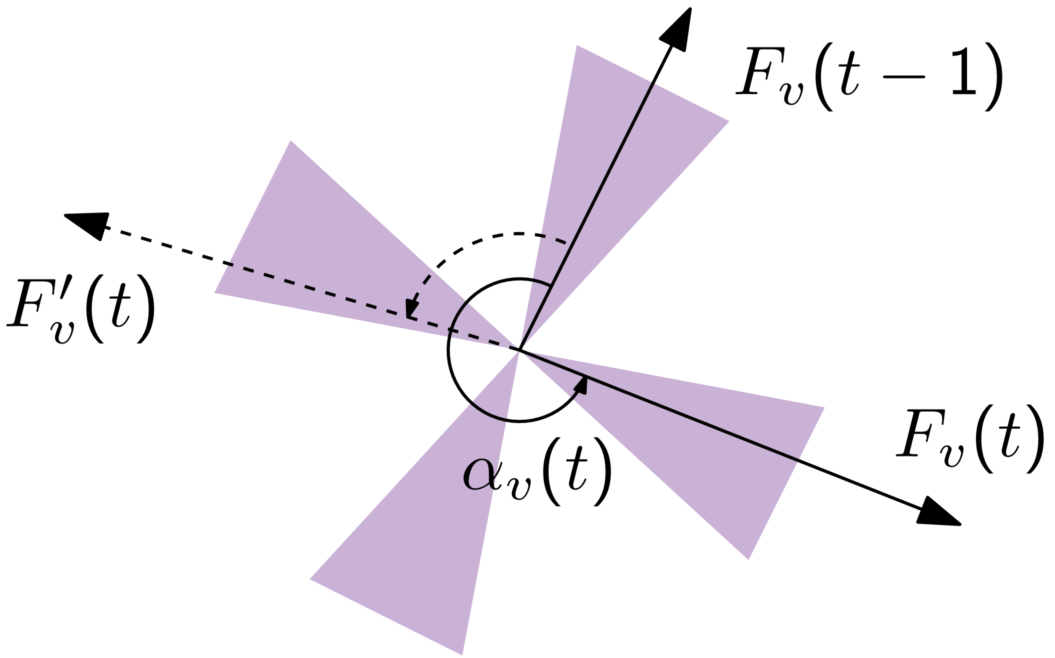


Same direction.
$\rightarrow$ increase temperature $\delta_v(t)$

Oscillation.
$\rightarrow$ decrease temperature $\delta_v(t)$

Speeding up "Convergence" by Adaptive Displacement $\delta_v(t)$

[Frick, Ludwig, Mehldau '95]



Same direction.

$\rightarrow$ increase temperature $\delta_v(t)$

Oscillation.

$\rightarrow$ decrease temperature $\delta_v(t)$

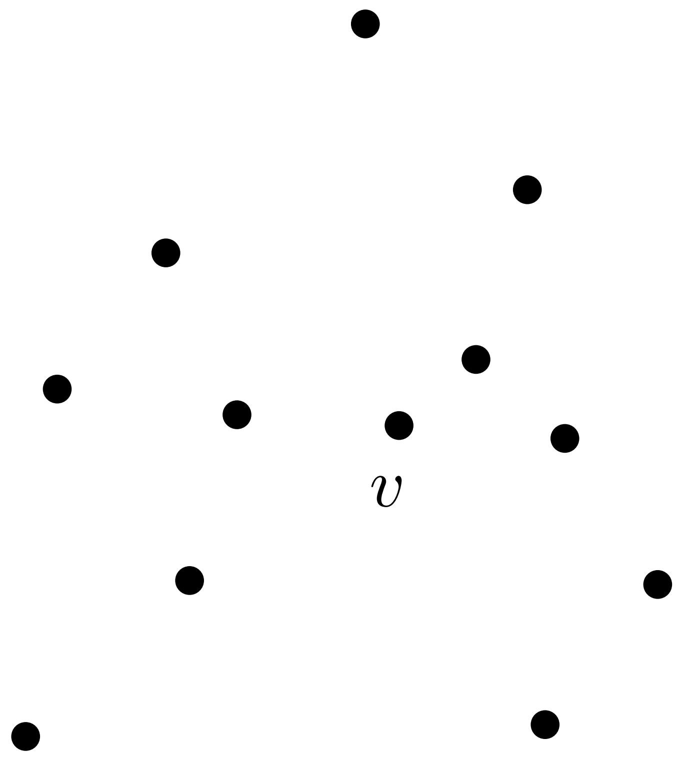
Rotation.

- count rotations
- if applicable

$\rightarrow$ decrease temperature $\delta_v(t)$

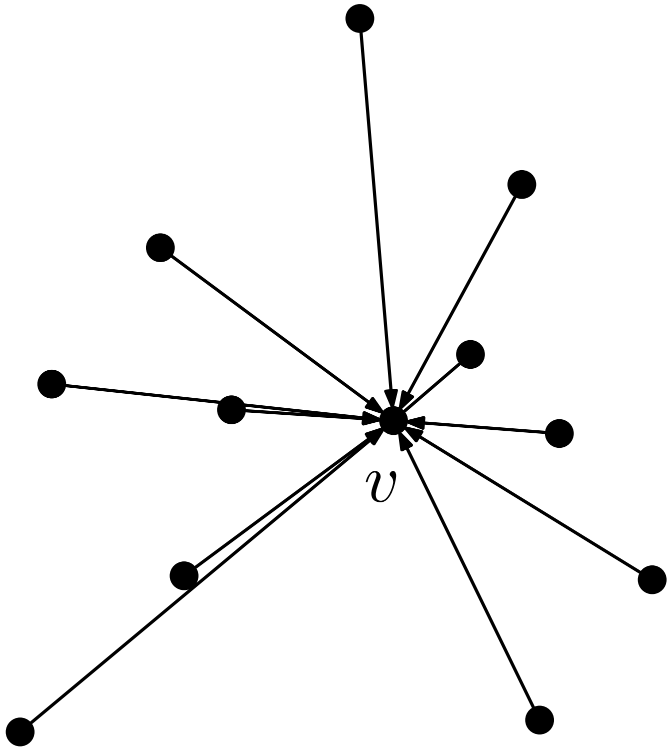
Speeding up "Convergence" via Grids

[Fruchterman & Reingold '91]



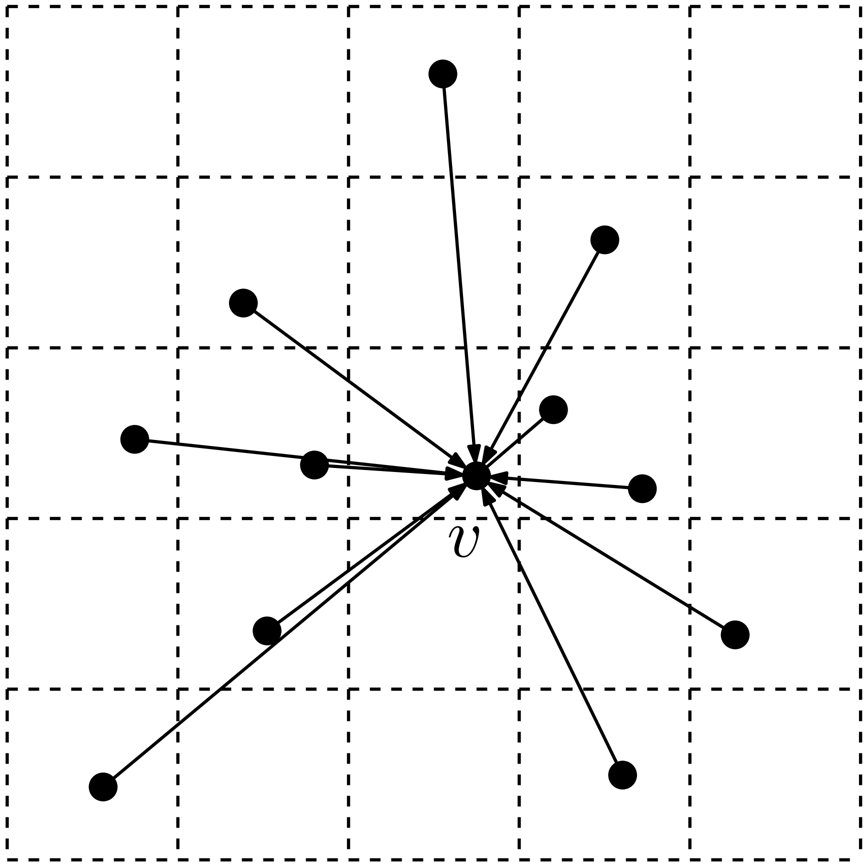
Speeding up "Convergence" via Grids

[Fruchterman & Reingold '91]



Speeding up "Convergence" via Grids

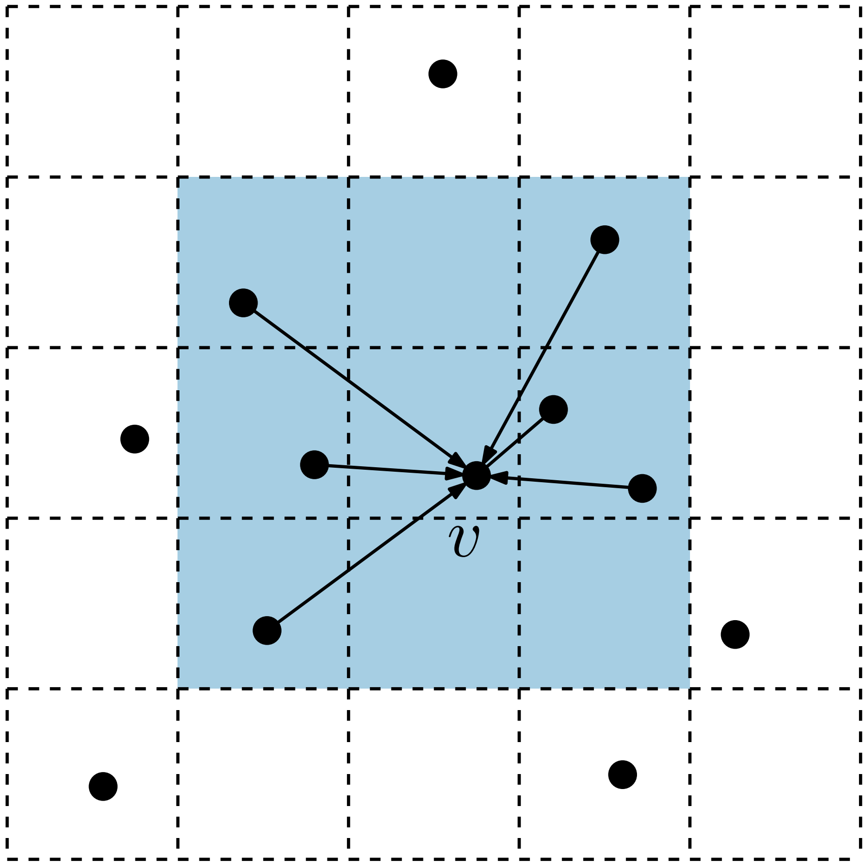
[Fruchterman & Reingold '91]



- divide plane into a grid

Speeding up "Convergence" via Grids

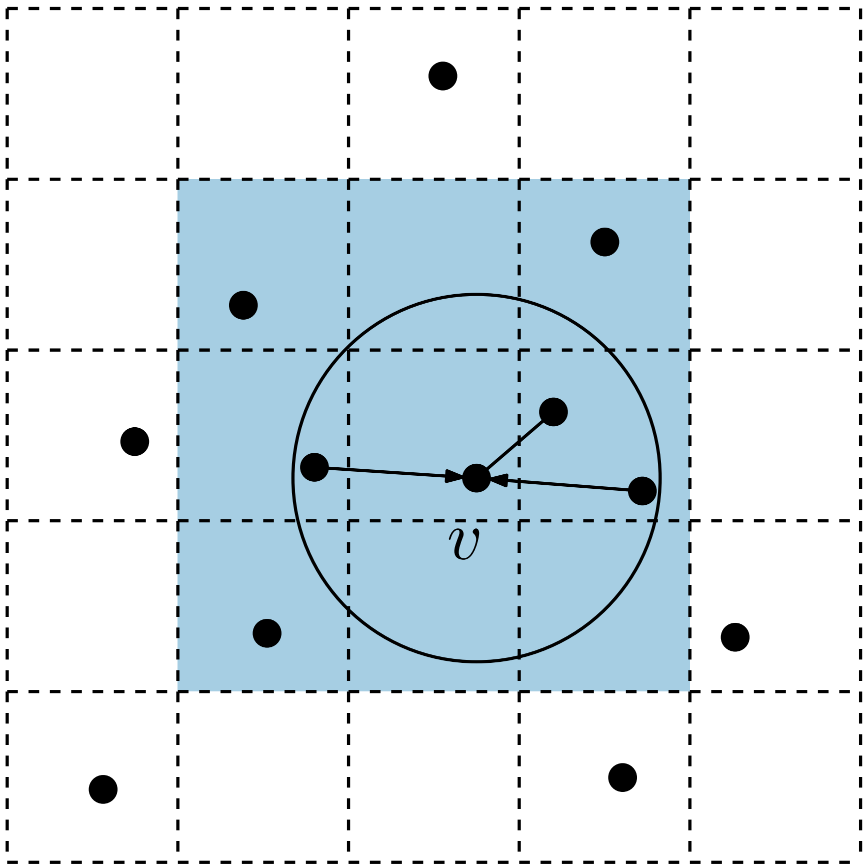
[Fruchterman & Reingold '91]



- divide plane into a grid
- consider repulsive forces only to vertices in neighboring cells

Speeding up "Convergence" via Grids

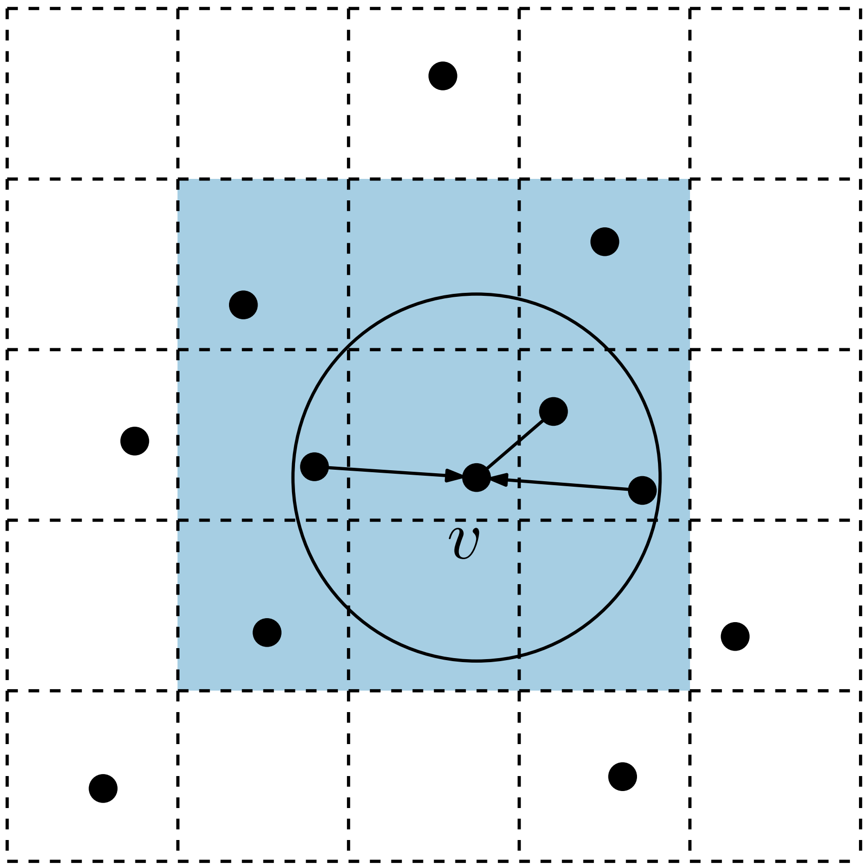
[Fruchterman & Reingold '91]



- divide plane into a grid
- consider repulsive forces only to vertices in neighboring cells
- and only if the distance is less than some threshold

Speeding up "Convergence" via Grids

[Fruchterman & Reingold '91]



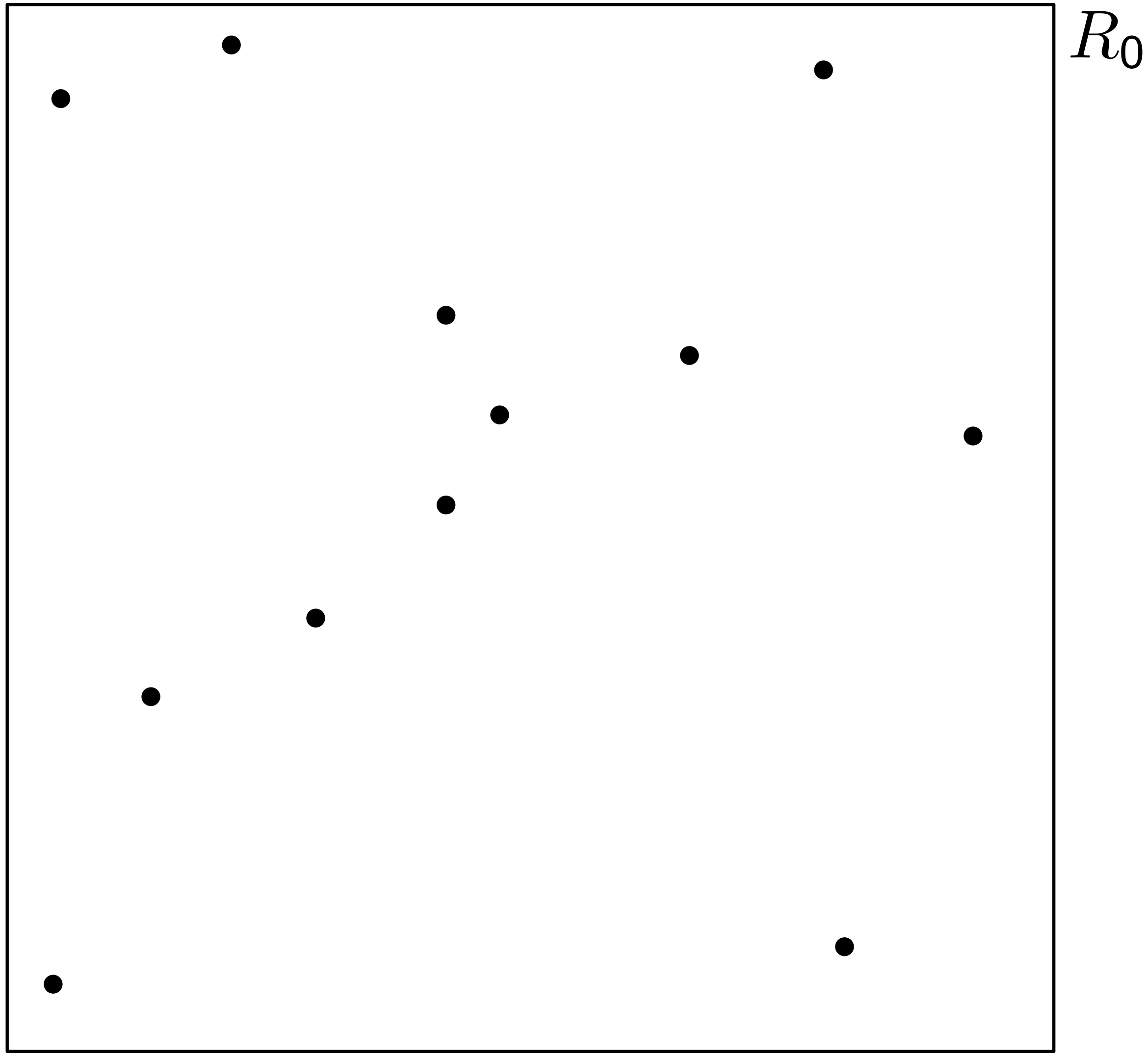
- divide plane into a grid
- consider repulsive forces only to vertices in neighboring cells
- and only if the distance is less than some threshold

Discussion.

- good idea to improve actual runtime
- asymptotic runtime does not improve
- might introduce oscillation and thus a quality loss

Speeding up with Quad Trees

[Barnes, Hut '86]

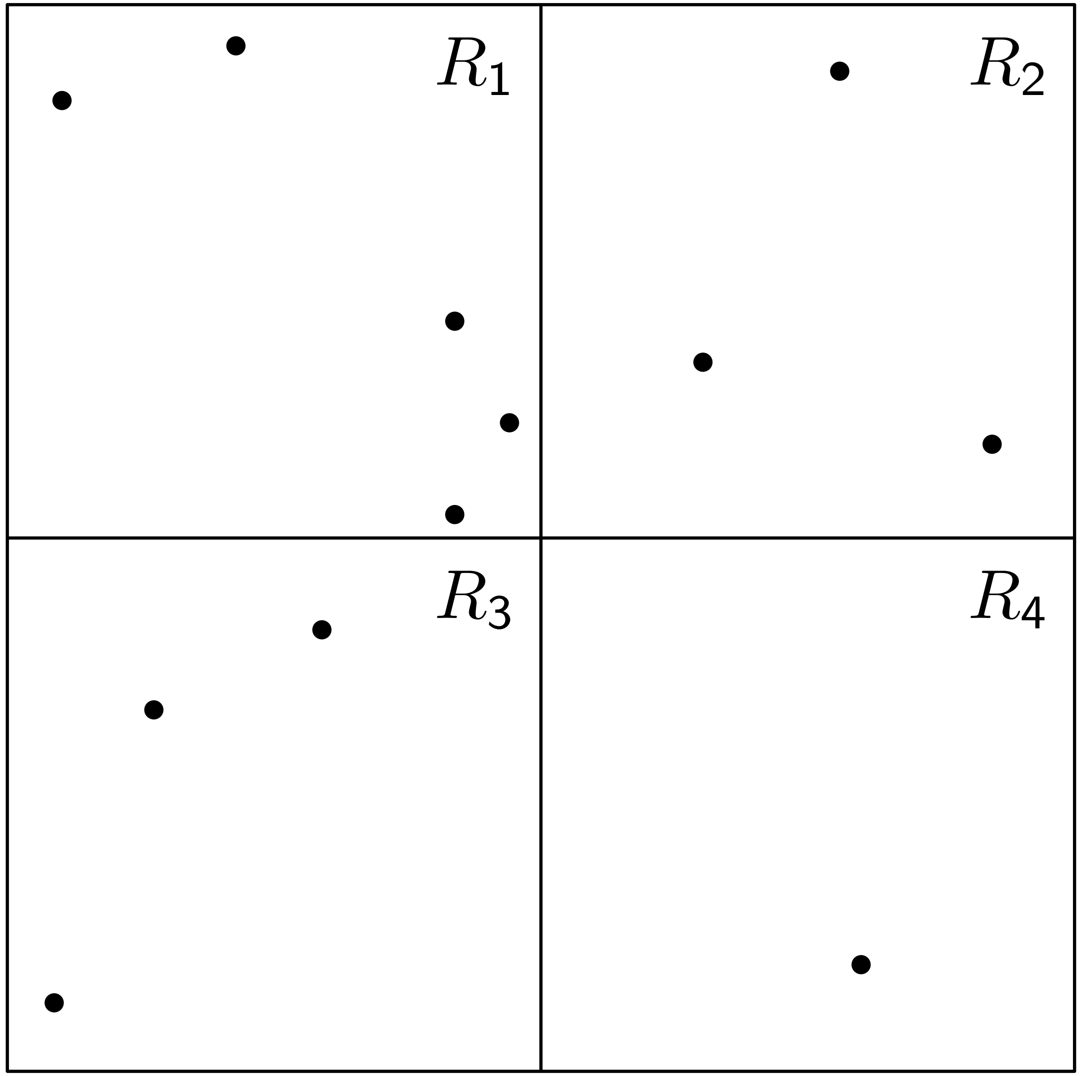


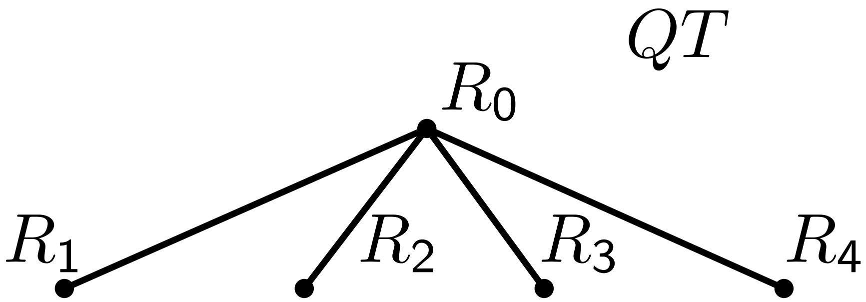
QT

R_0

Speeding up with Quad Trees

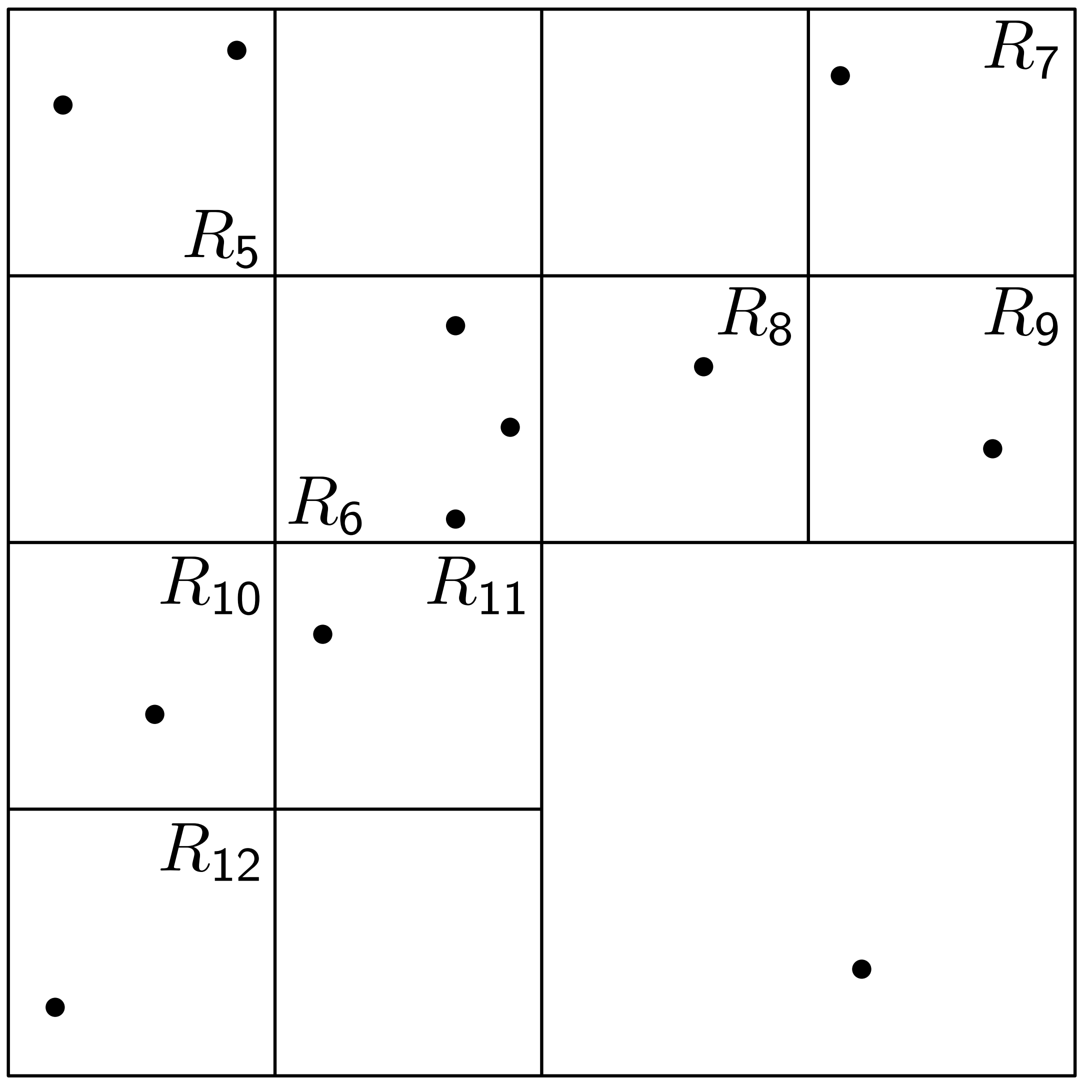
[Barnes, Hut '86]

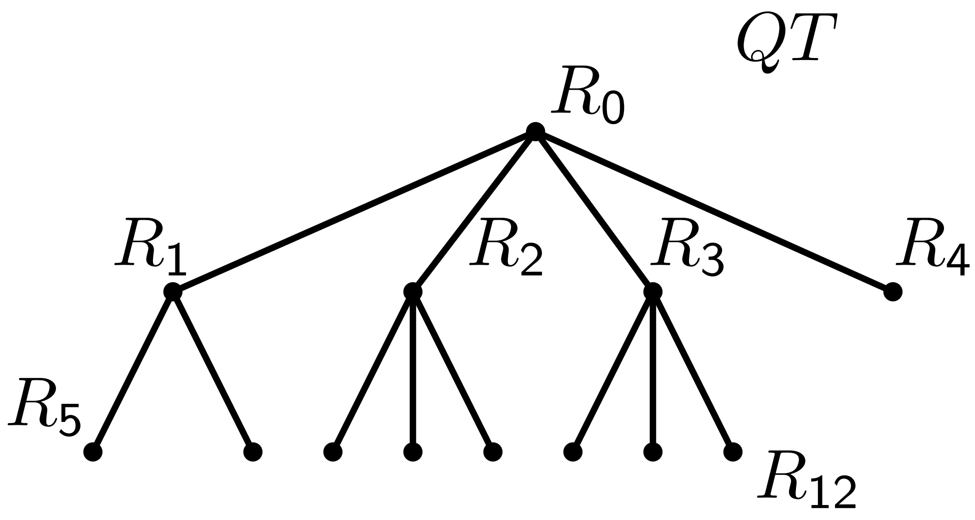




Speeding up with Quad Trees

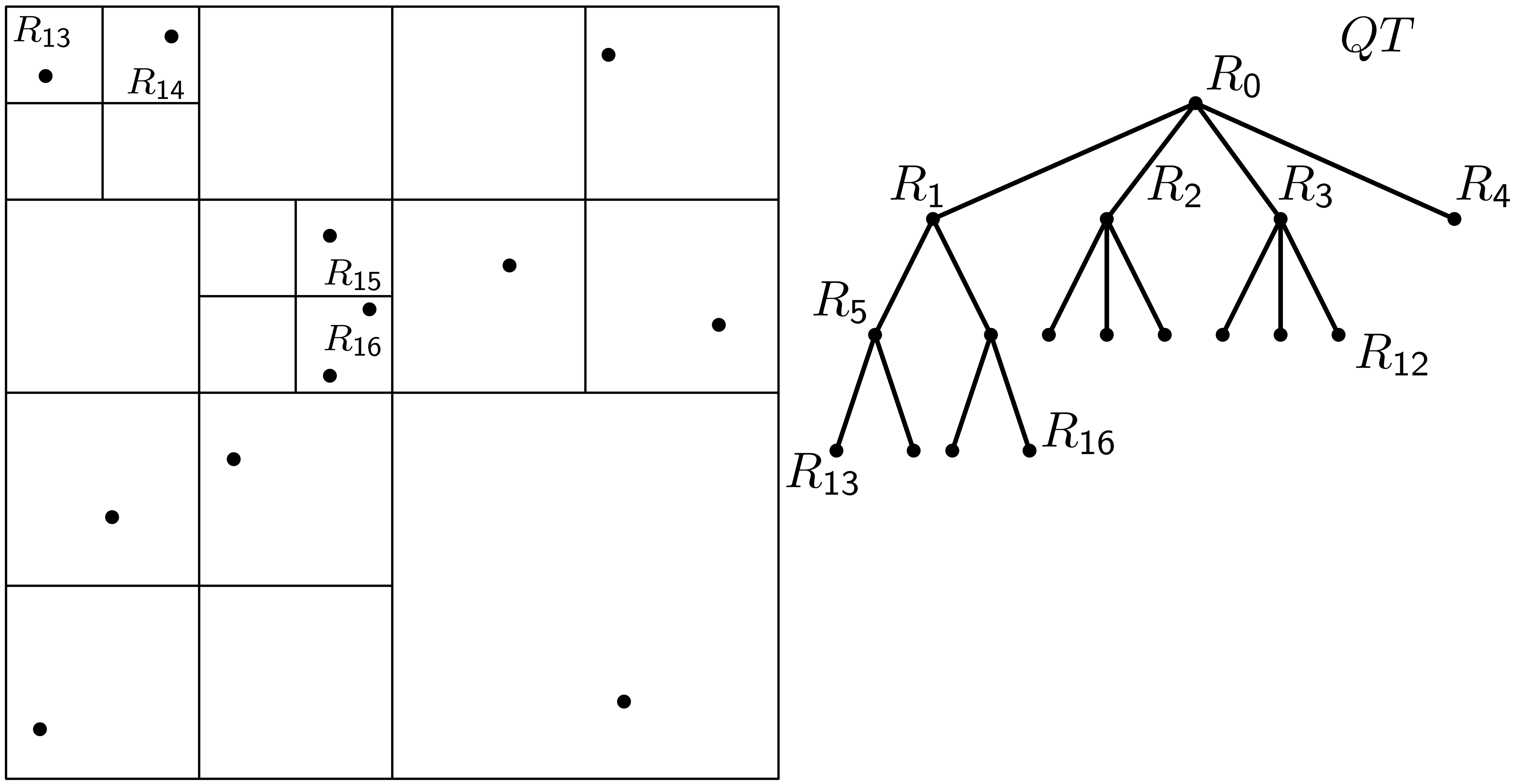
[Barnes, Hut '86]





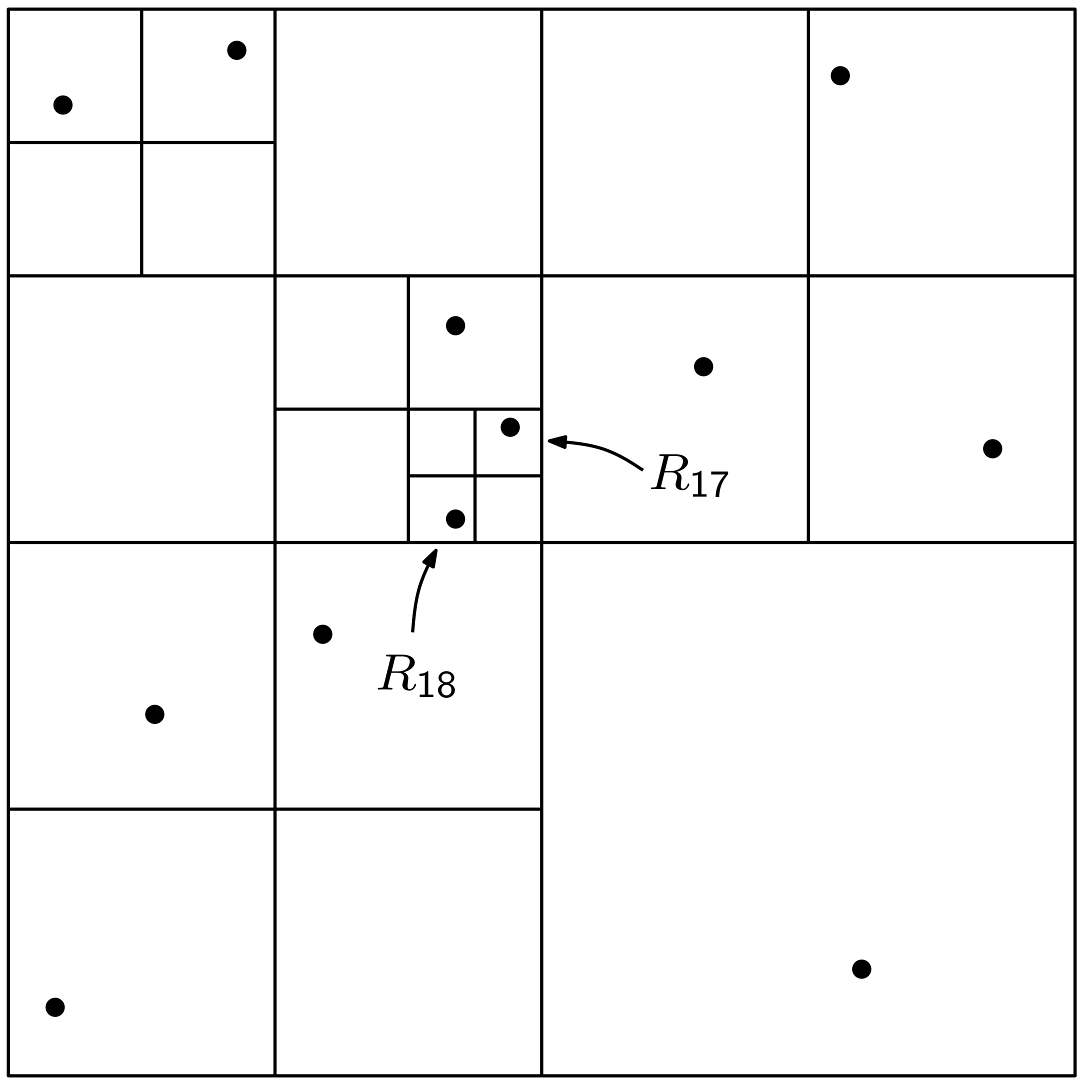
Speeding up with Quad Trees

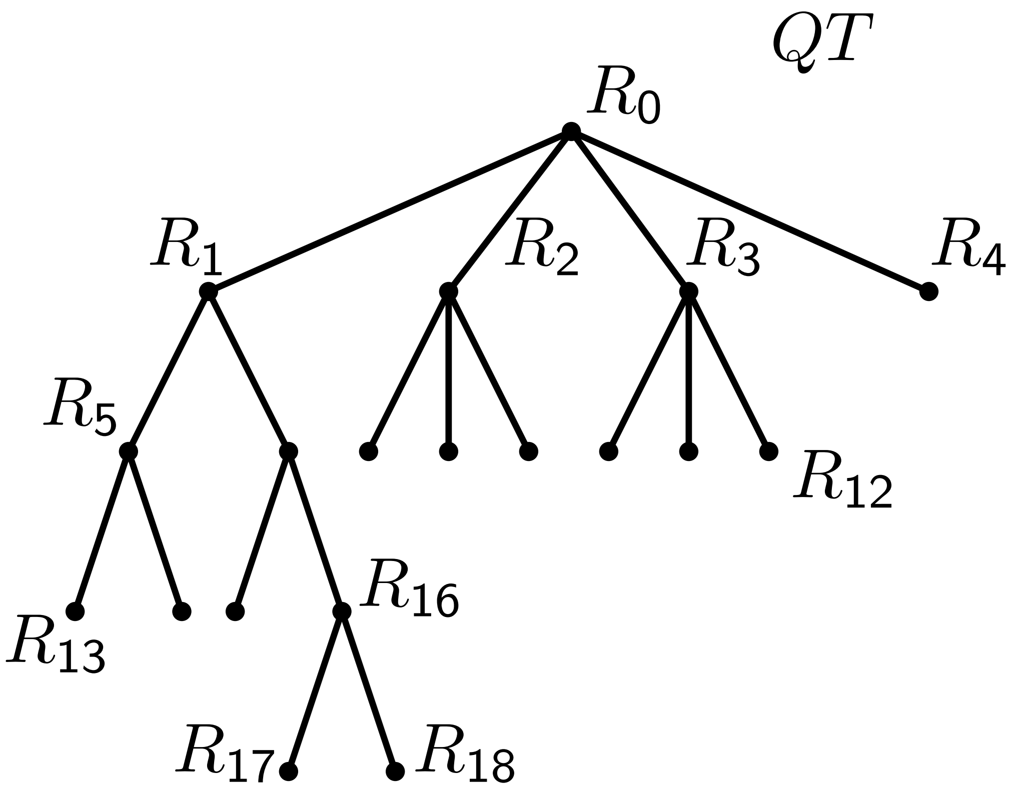
[Barnes, Hut '86]



Speeding up with Quad Trees

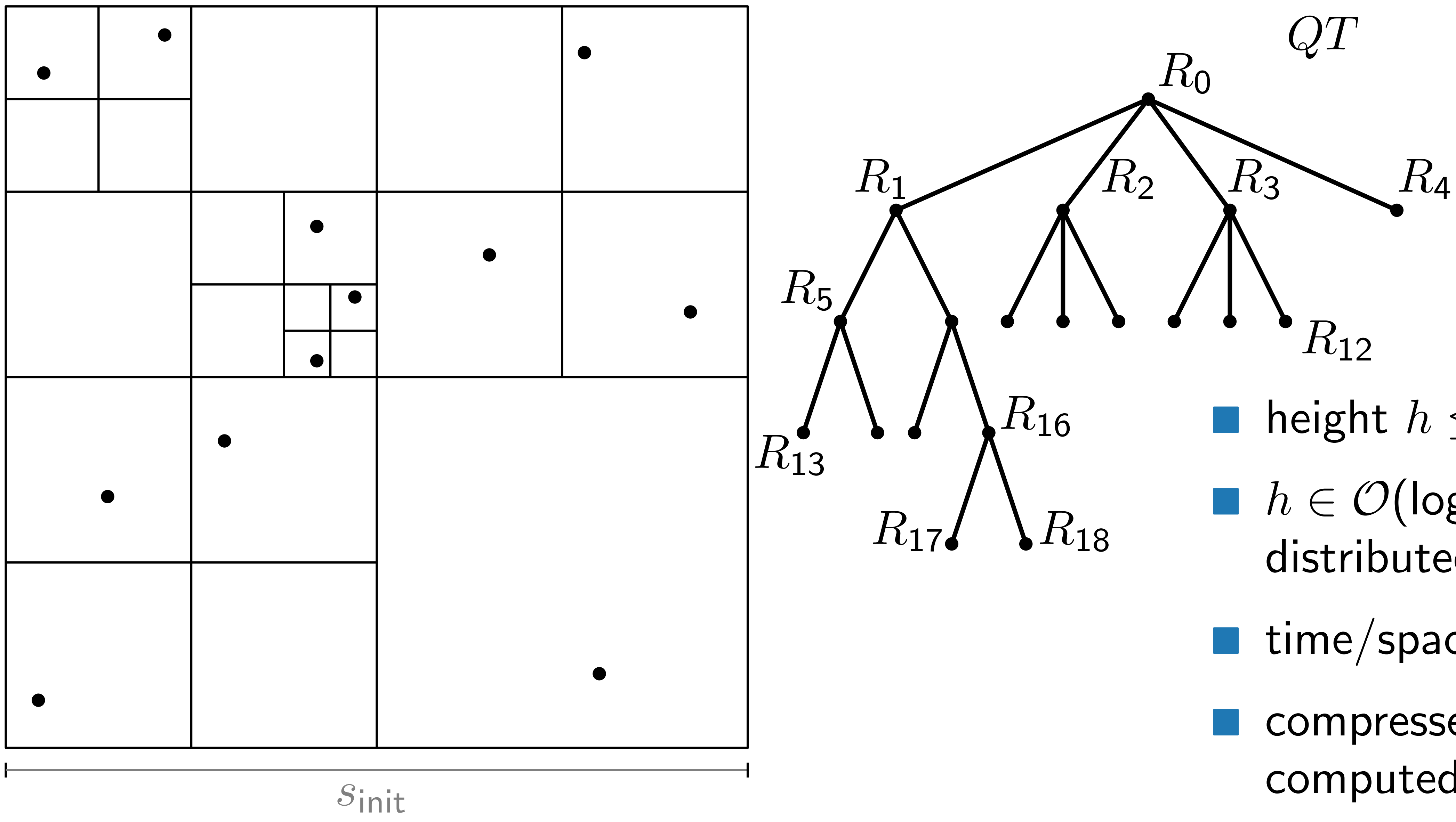
[Barnes, Hut '86]





Speeding up with Quad Trees

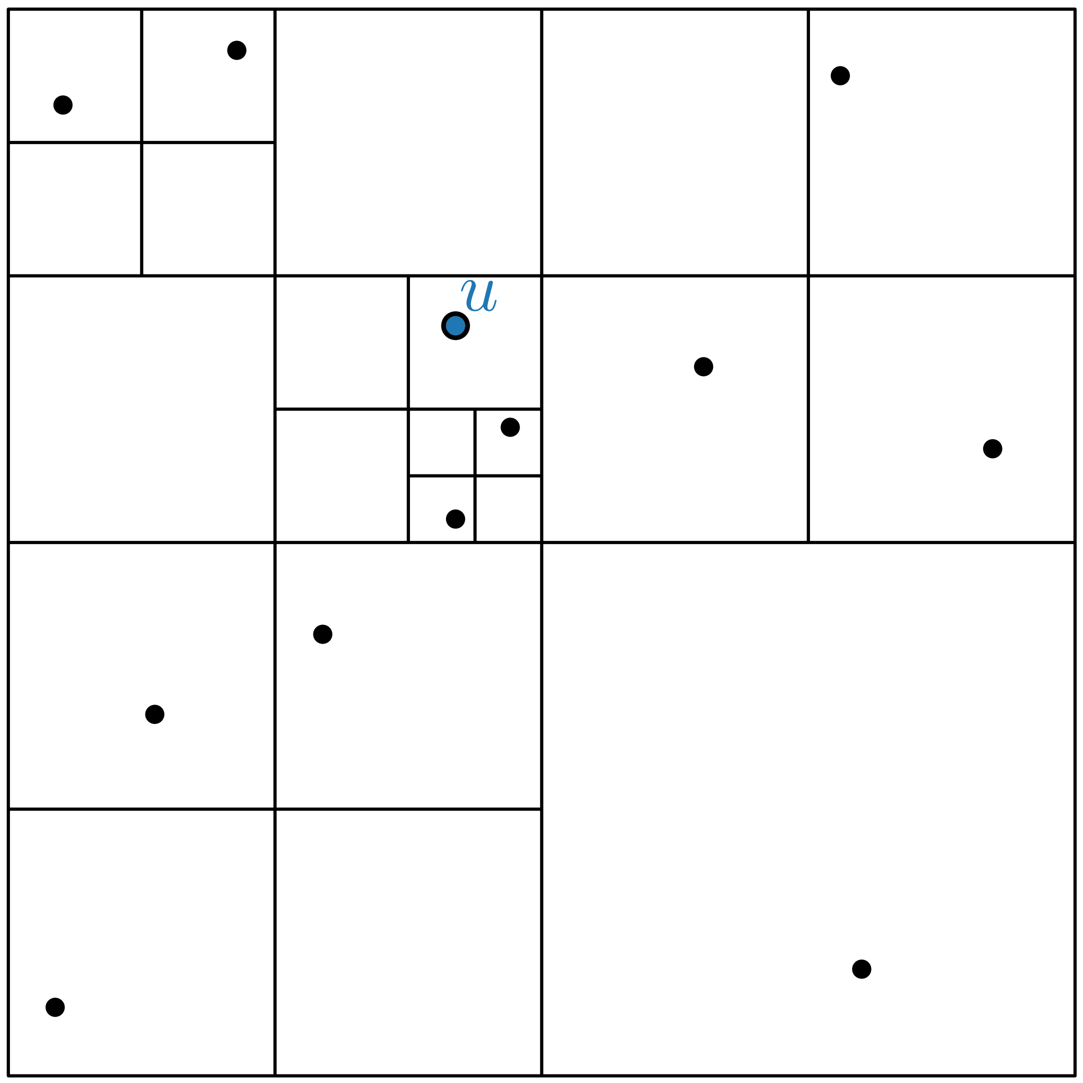
[Barnes, Hut '86]

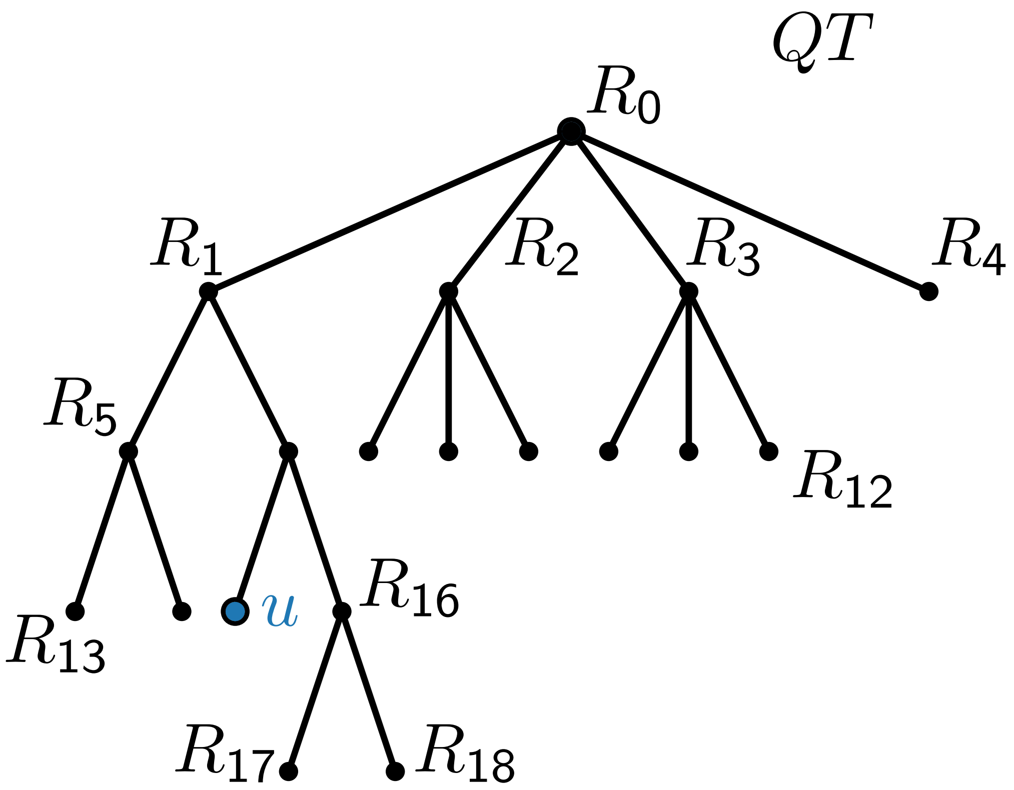


- height $h \leq \log \frac{s_{\text{init}}}{d_{\text{min}}} + \frac{3}{2}$
- $h \in \mathcal{O}(\log n)$ if vertices evenly distributed in the initial box
- time/space in $\mathcal{O}(hn)$
- compressed quad tree can be computed in $\mathcal{O}(n \log n)$ time

Speeding up with Quad Trees

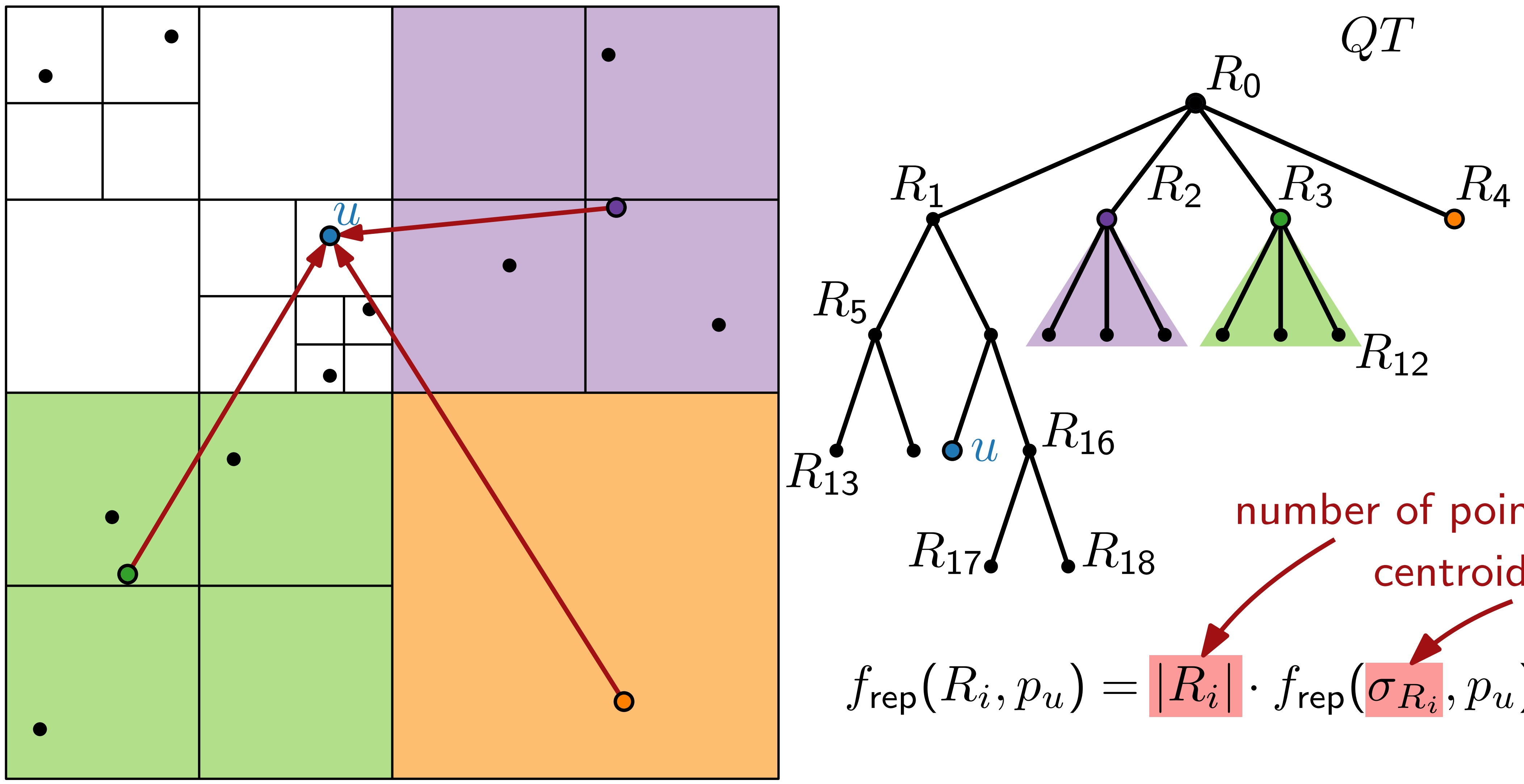
[Barnes, Hut '86]





Speeding up with Quad Trees

[Barnes, Hut '86]



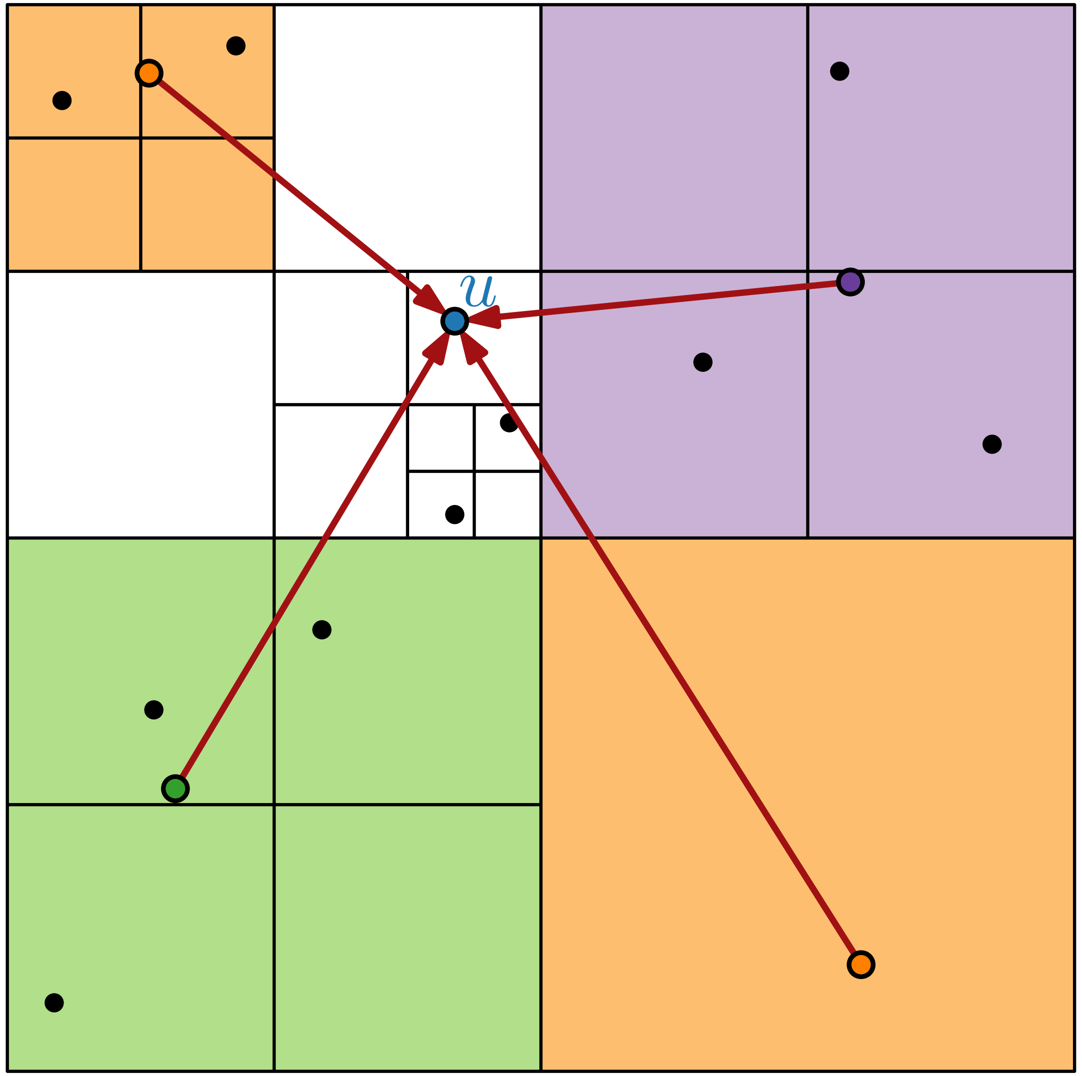
number of points in the subtree R_i

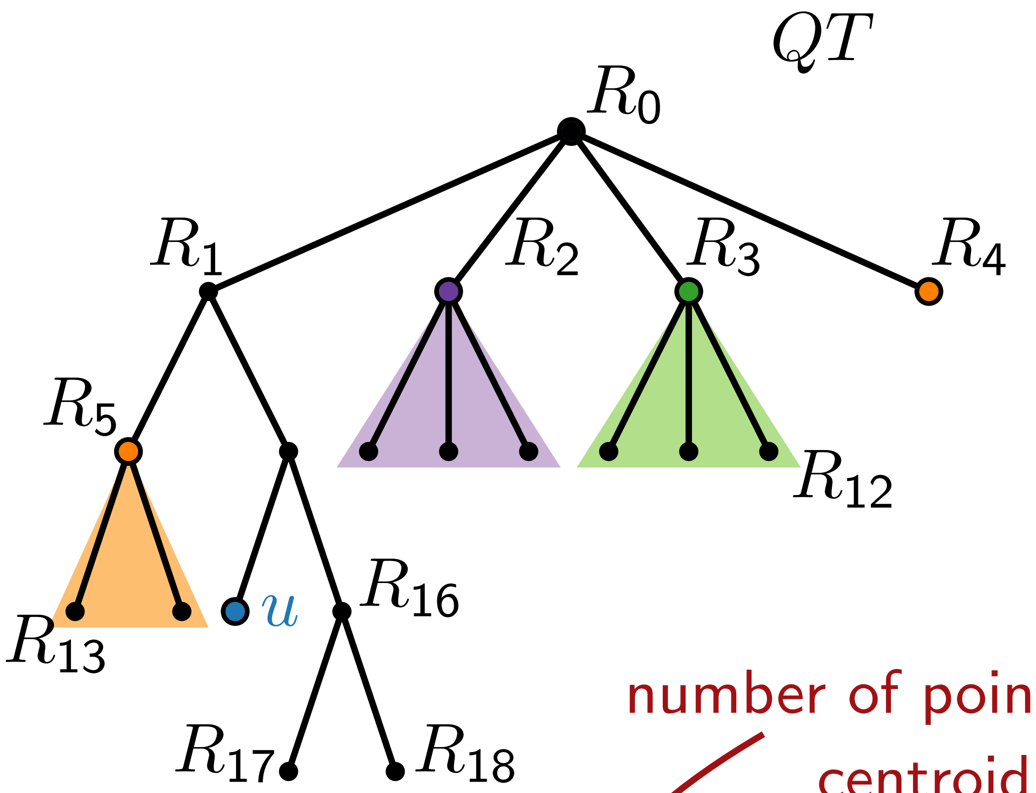
centroid of R_i (pre-computed)

$$f_{\text{rep}}(R_i, p_u) = \boxed{|R_i|} \cdot f_{\text{rep}}(\boxed{\sigma_{R_i}}, p_u)$$

Speeding up with Quad Trees

[Barnes, Hut ’86]





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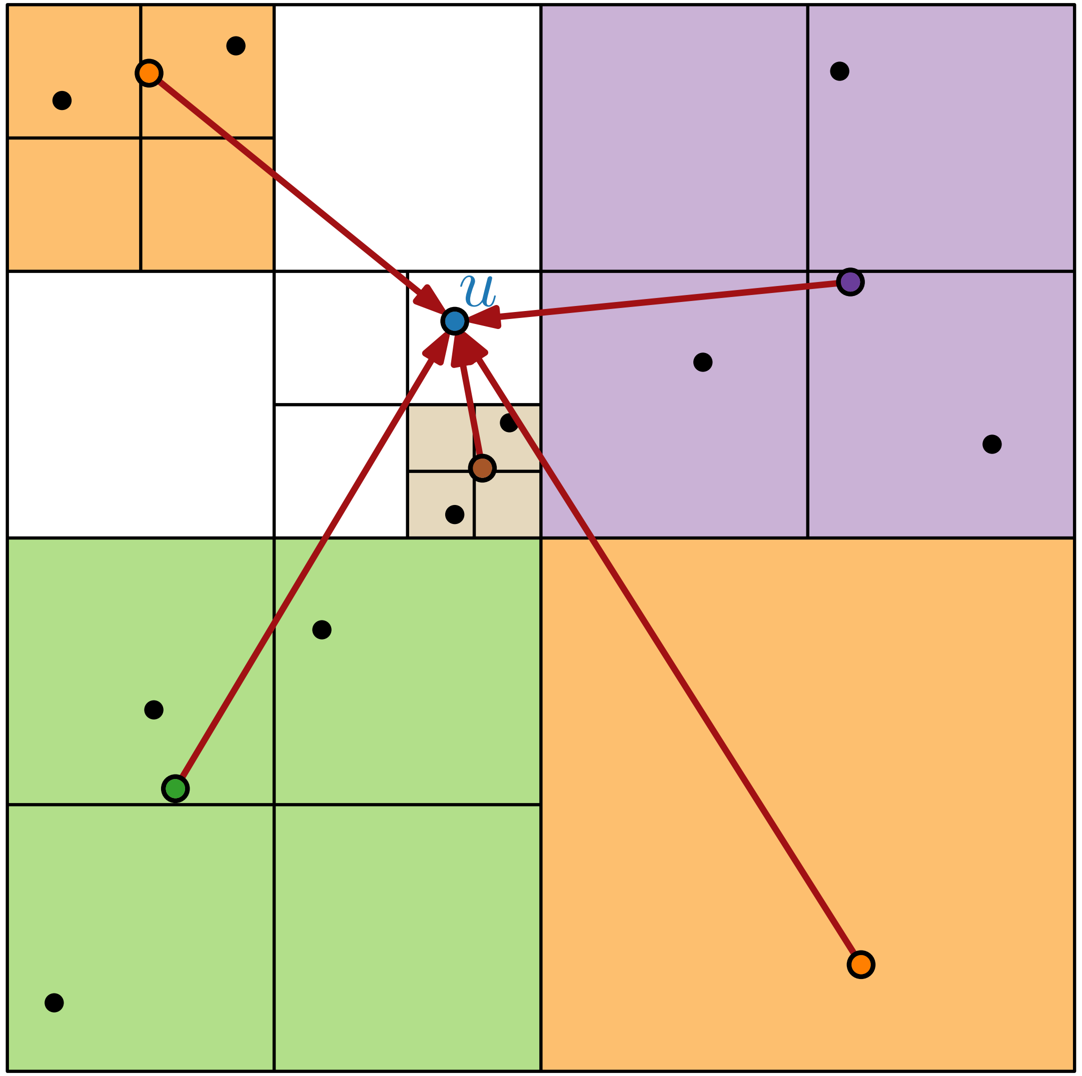

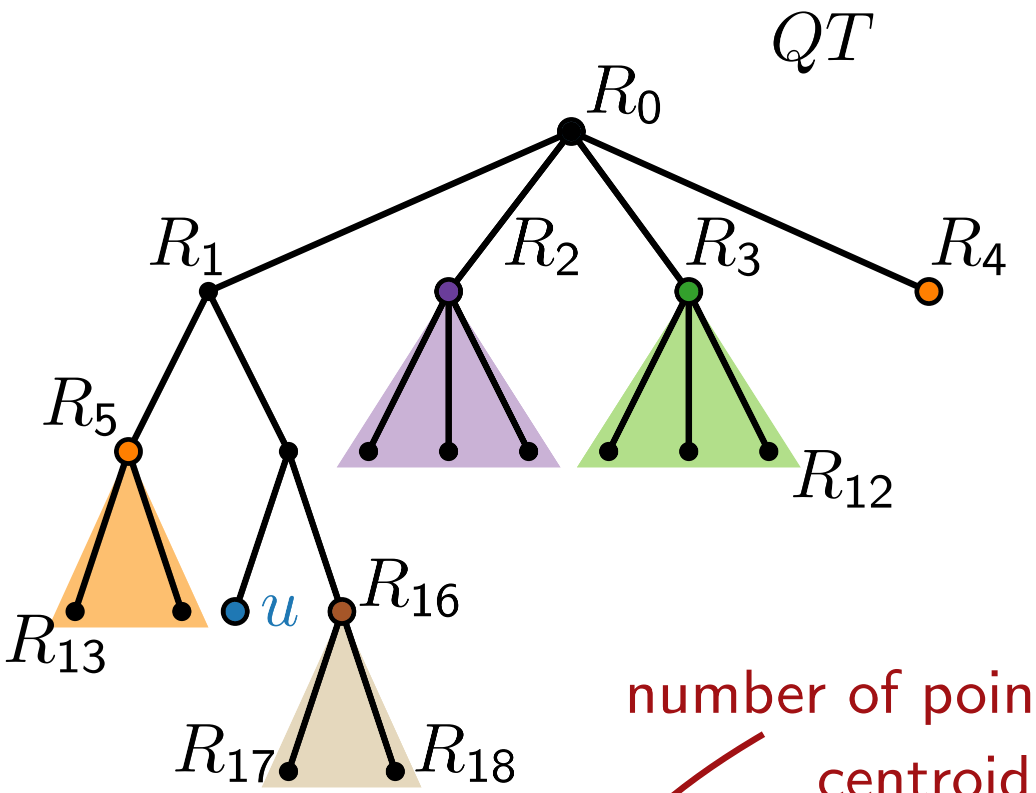

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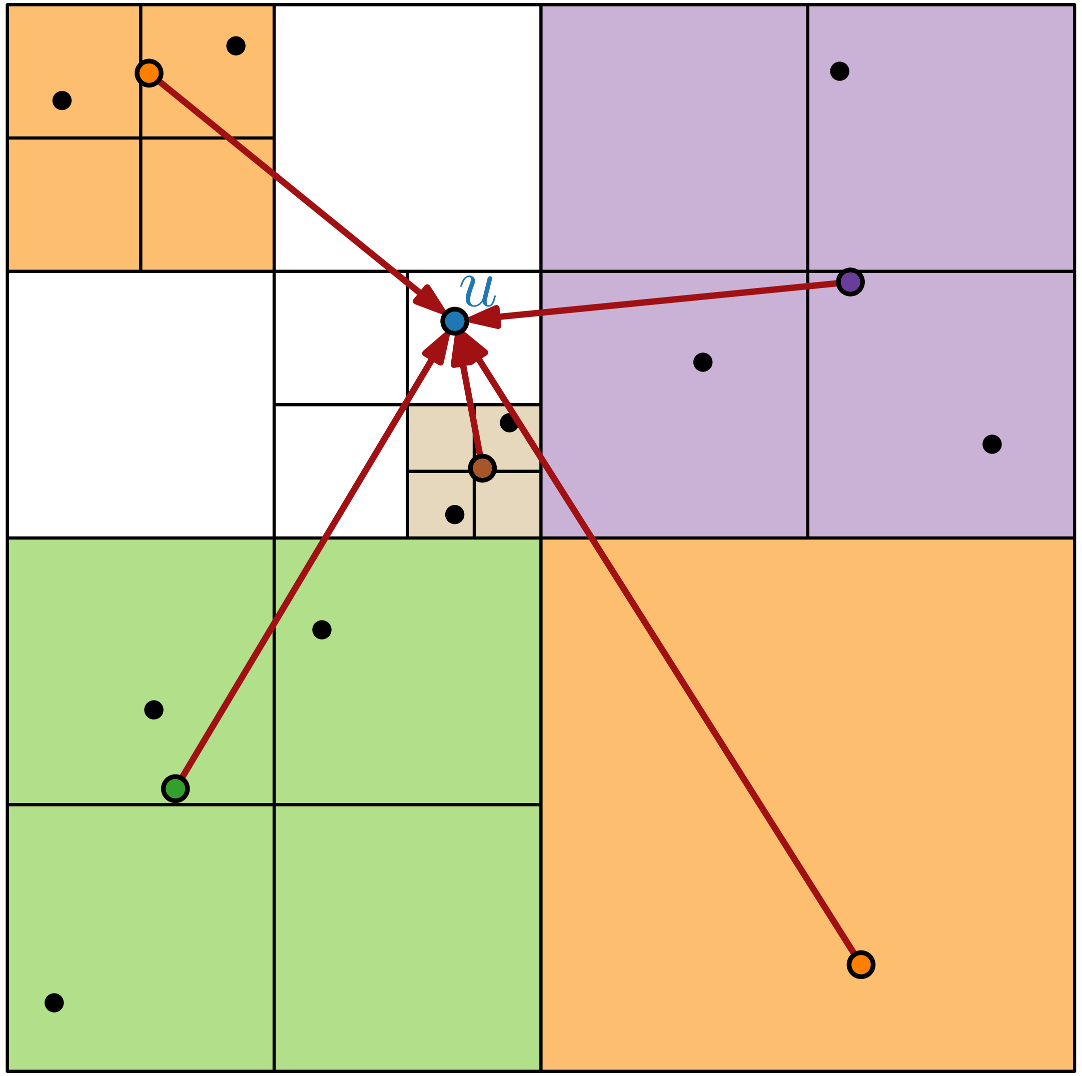
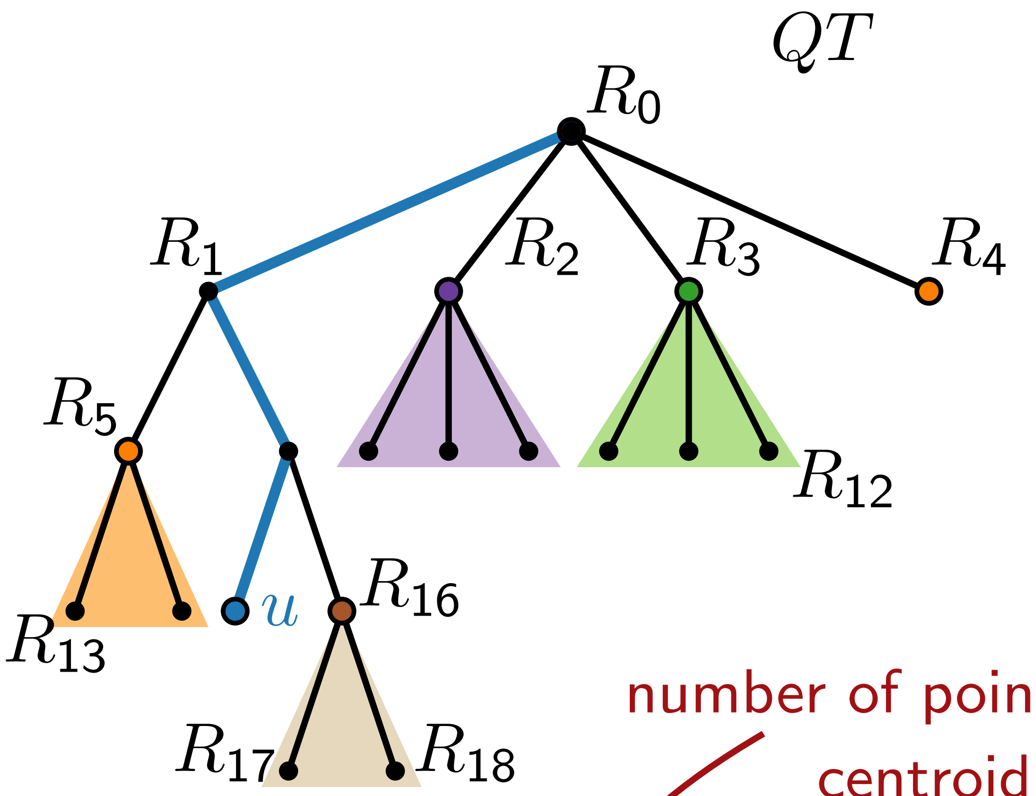




number of points in the subtree R_i

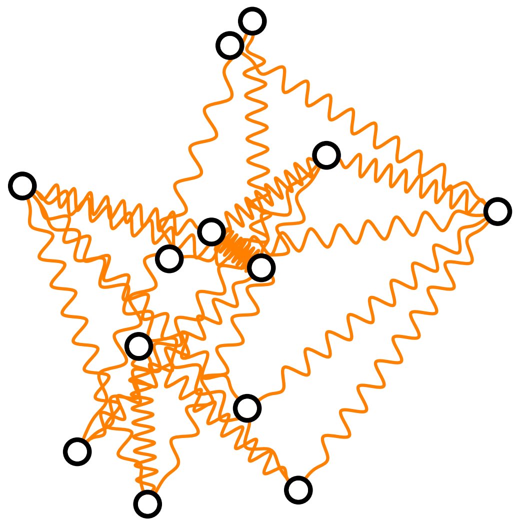
centroid of R_i (pre-computed)

$$f_{\text{rep}}(R_i, p_u) = \boxed{|R_i|} \cdot f_{\text{rep}}(\boxed{\sigma_{R_i}}, p_u)$$

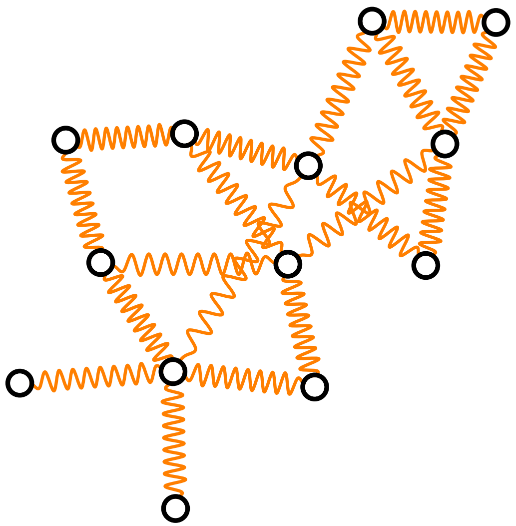
for each child R_i of a vertex on path from u to root.

Visualization of Graphs

Lecture 2:
Force-Directed Drawing Algorithms

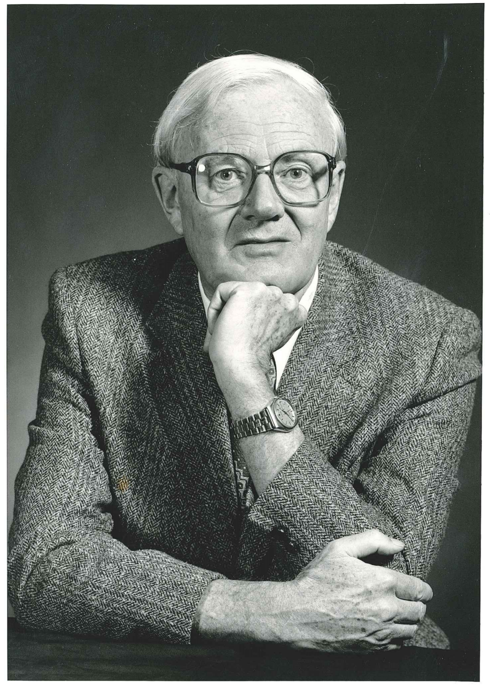


Part II:
Tutte Embeddings



Johannes Zink

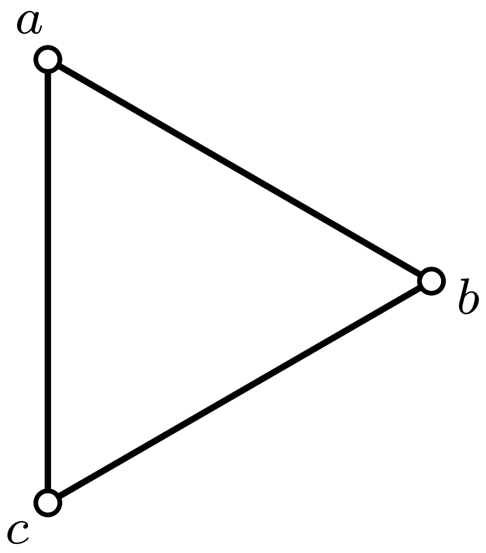
Idea

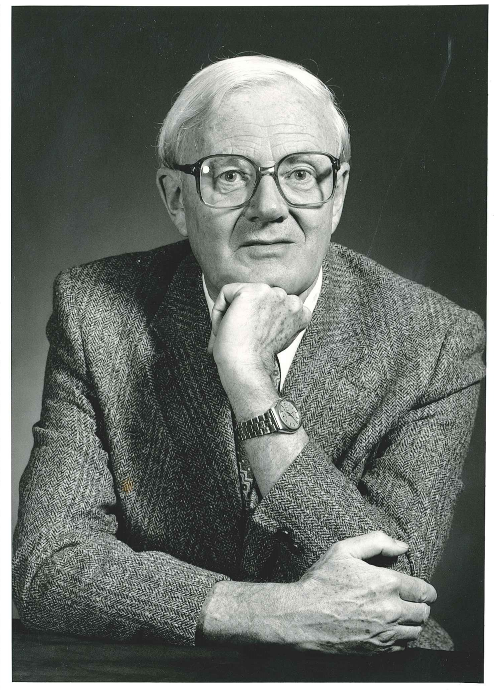


William T. Tutte
1917 – 2002

Idea

Consider a fixed triangle (a, b, c)

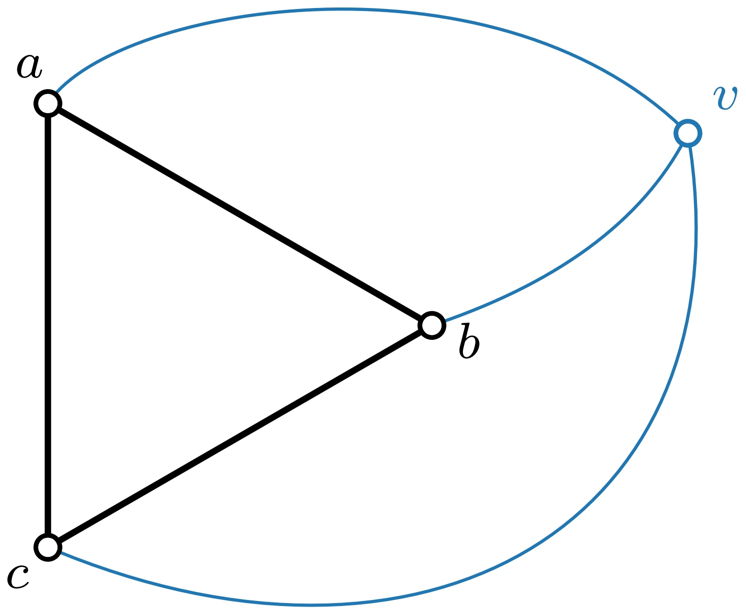


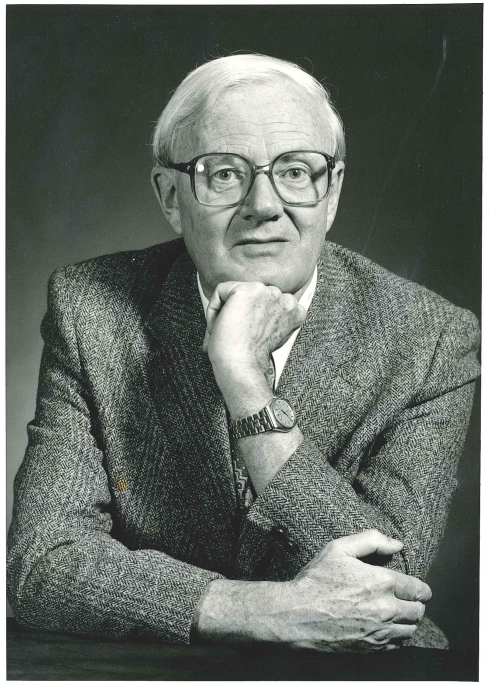


William T. Tutte
1917 – 2002

Idea

Consider a fixed triangle (a, b, c)
with a common neighbor v



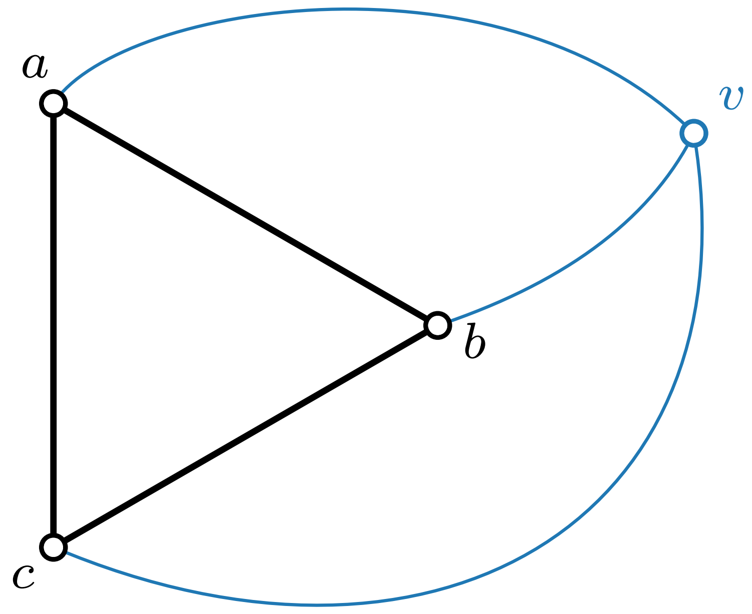


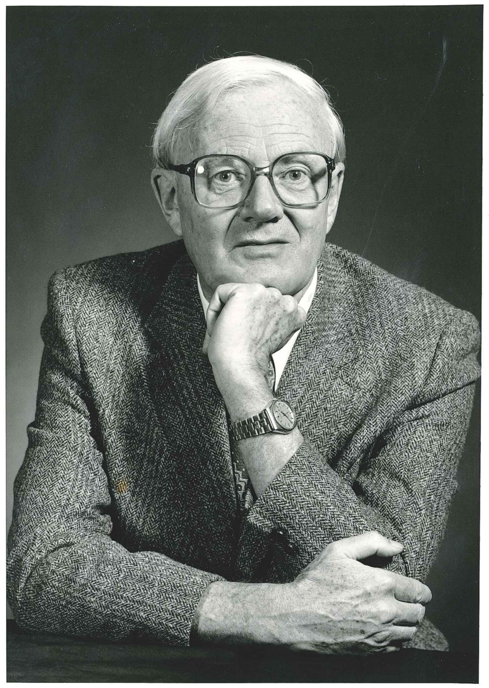
William T. Tutte
1917 – 2002

Idea

Consider a fixed triangle (a, b, c) with a common neighbor v

Where would you place v?





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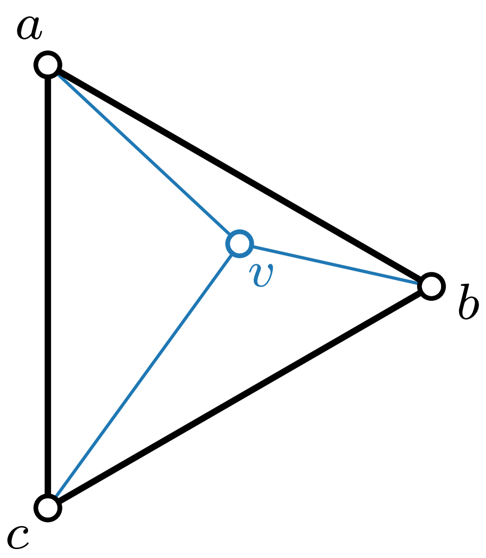

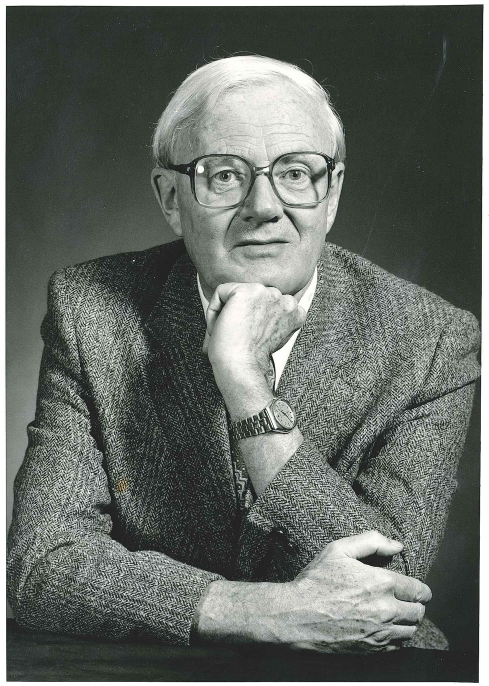

William T. Tutte
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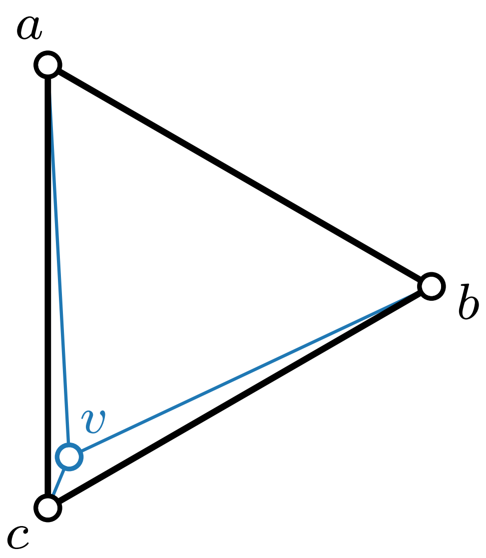

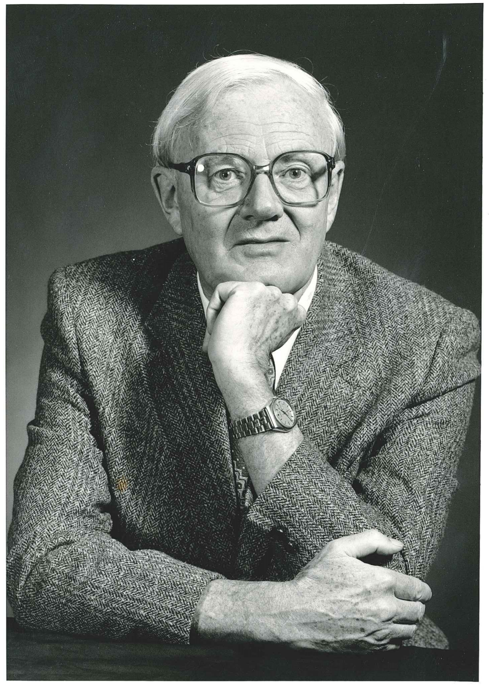

William T. Tutte
1917 – 2002

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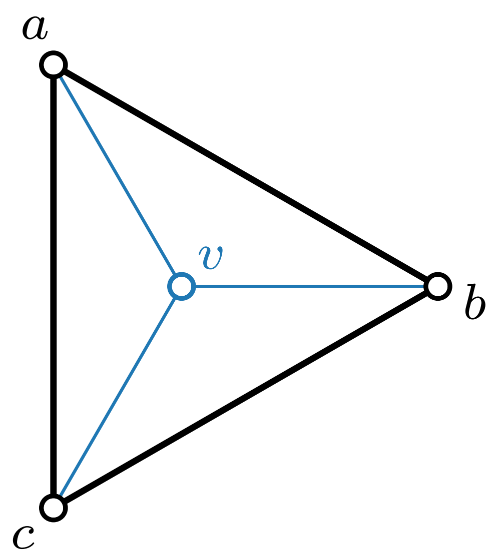

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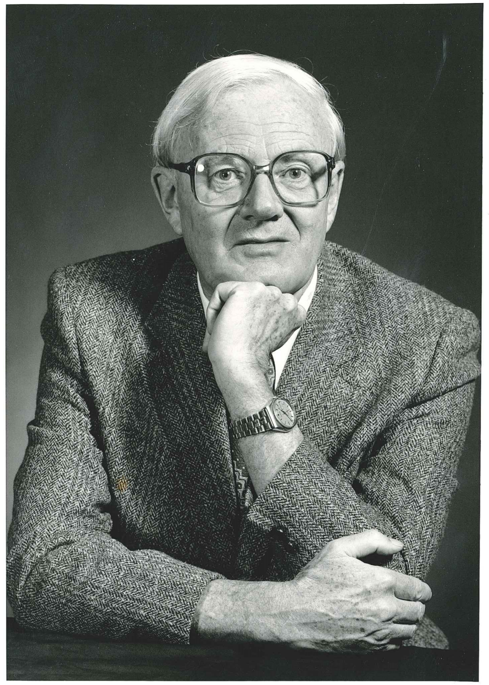

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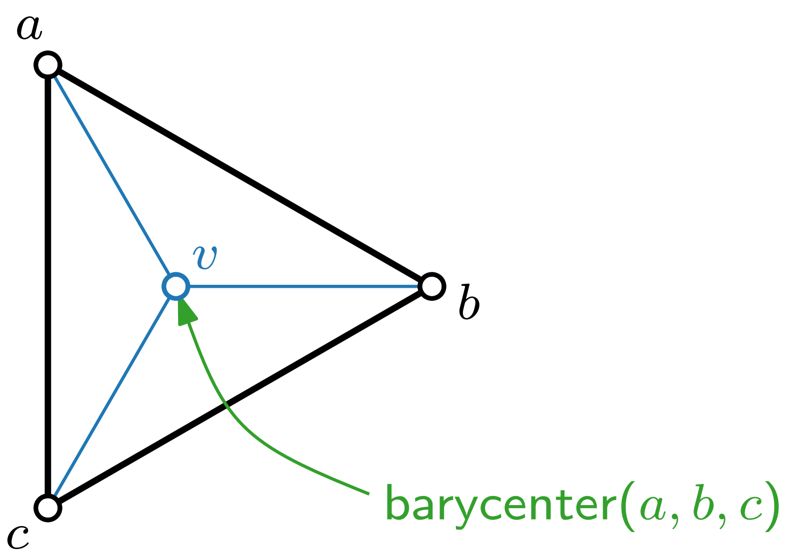
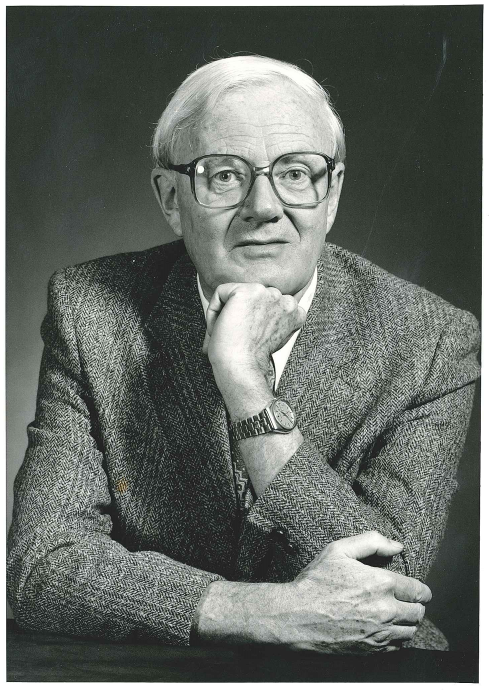




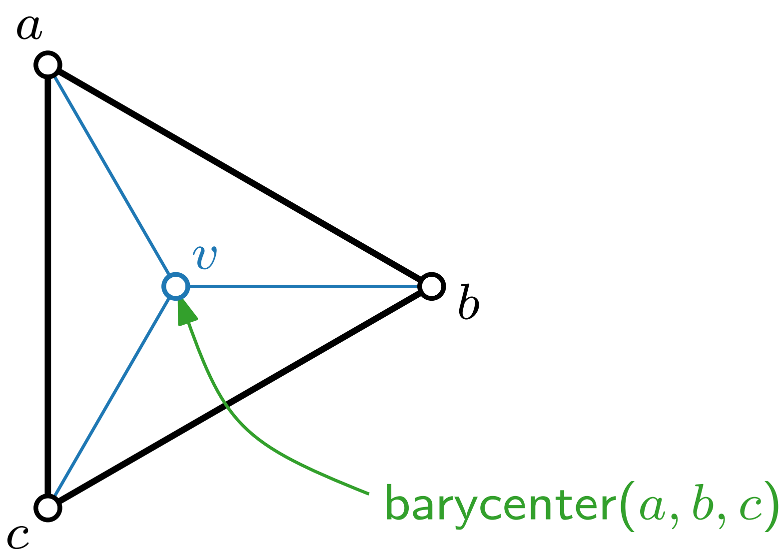
William T. Tutte
1917 – 2002

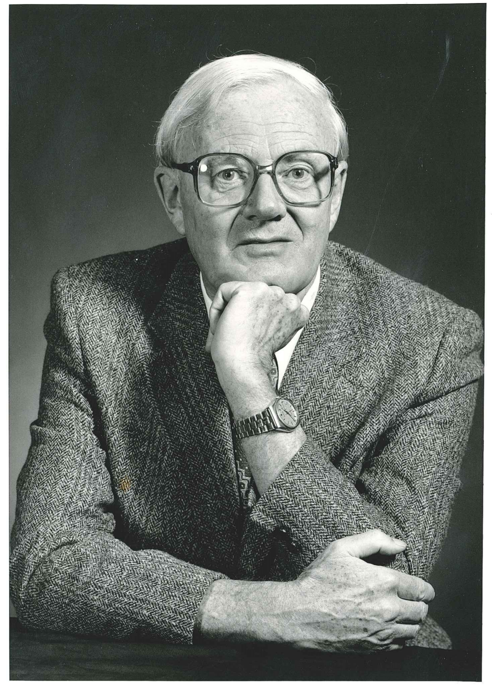
Idea

Consider a fixed triangle (a, b, c) with a common neighbor v

Where would you place v?

barycenter$(x_1, \ldots, x_k) =$?





William T. Tutte
1917 – 2002

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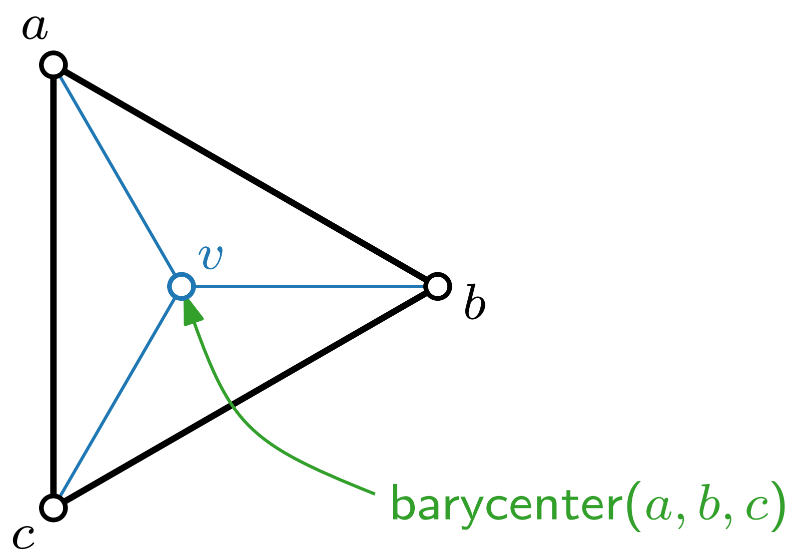
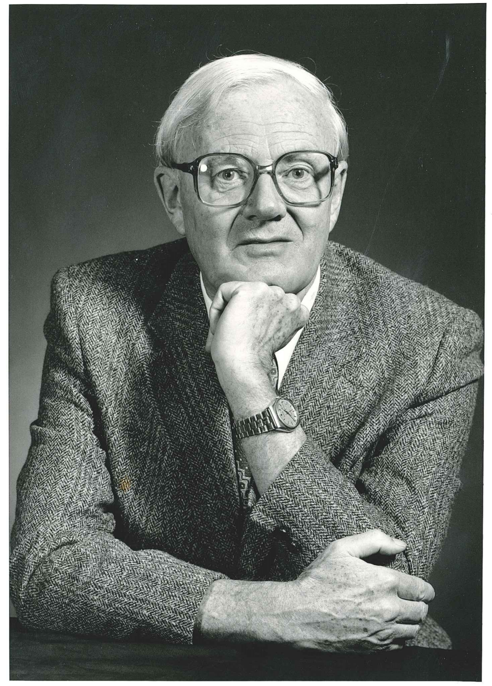


$$\text{barycenter}(x_1, \ldots, x_k) = \sum_{i=1}^{k} x_i / k$$



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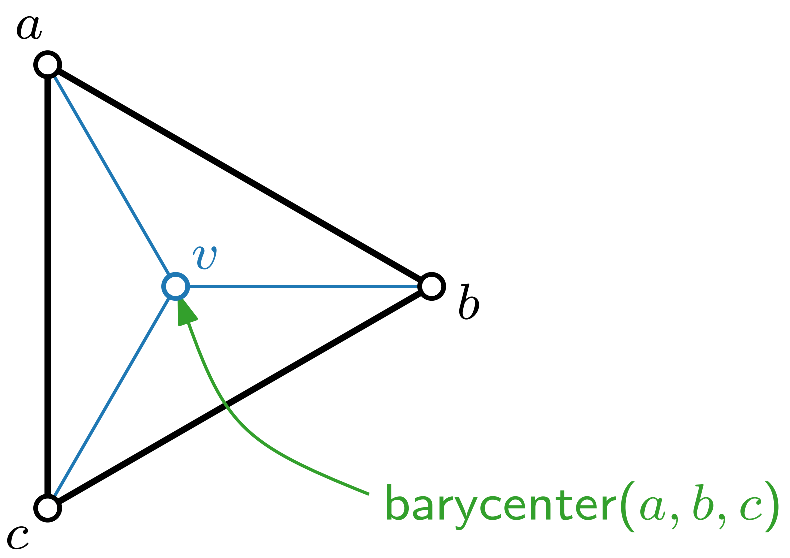

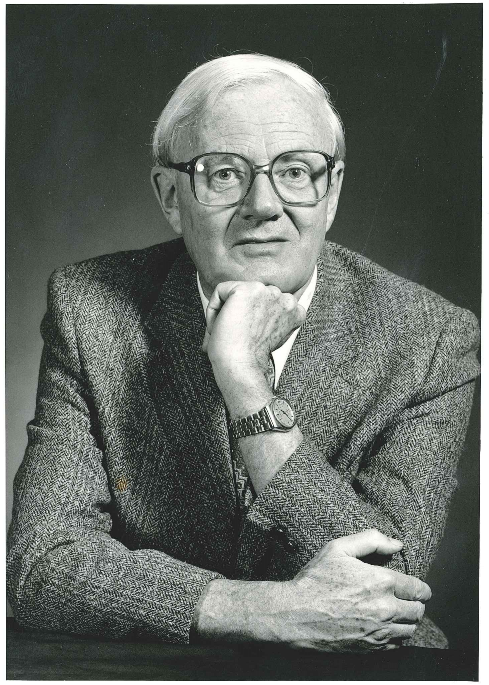

William T. Tutte
1917 – 2002

barycenter$(x_1, \ldots, x_k) = \sum_{i=1}^{k} x_i / k$

Idea.
Repeatedly place every vertex at barycenter of neighbors.

Tutte's Forces

ForceDirected($G = (V, E)$, $p = (p_v)_{v \in V}$, $\varepsilon > 0$, $K \in \mathbb{N}$)

$t \leftarrow 1$

while $t < K$ **and** $\max_{v \in V} \|F_v(t)\| > \varepsilon$ **do**

  foreach $u \in V$ **do**

    $F_u(t) \leftarrow \sum_{v \in V} f_{\mathsf{rep}}(u, v) + \sum_{v \in \mathsf{Adj}[u]} f_{\mathsf{attr}}(u, v)$

  foreach $u \in V$ **do**

    $p_u \leftarrow p_u + \delta(t) \cdot F_u(t)$

  $t \leftarrow t + 1$

return p

Tutte's Forces

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Tutte's Forces

Goal.

$p_u = \text{barycenter}(\text{Adj}[u])$

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barycenter$(x_1, \ldots, x_k) = \sum_{i=1}^{k} x_i / k$

Tutte's Forces

Goal.

$p_u = \text{barycenter}(\text{Adj}[u])$
$= \sum_{v \in \text{Adj}[u]} p_v /$

ForceDirected($G = (V, E)$, $p = (p_v)_{v \in V}$, $\varepsilon > 0$, $K \in \mathbb{N}$)
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$\text{barycenter}(x_1, \ldots, x_k) = \sum_{i=1}^{k} x_i / k$

Tutte's Forces

Goal.

$$p_u = \text{barycenter}(\text{Adj}[u]) = \sum_{v \in \text{Adj}[u]} p_v / \deg(u)$$

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- **Repulsive forces** $f_{\text{rep}}(u, v) = 0$

ForceDirected($G = (V, E)$, $p = (p_v)_{v \in V}$, $\varepsilon > 0$, $K \in \mathbb{N}$)

```
t ← 1
while t < K and max_{v∈V} ‖F_v(t)‖ > ε do
    foreach u ∈ V do
        F_u(t) ← Σ_{v∈V} f_rep(u, v) + Σ_{v∈Adj[u]} f_attr(u, v)
    foreach u ∈ V do
        p_u ← p_u + ~~δ(t)~~ 1 · F_u(t)
    t ← t + 1
return p
```

$\text{barycenter}(x_1, \ldots, x_k) = \sum_{i=1}^{k} x_i / k$

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- **Repulsive forces** $\quad f_{\text{rep}}(u, v) = 0$
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Global minimum: $p_u = (0, 0)\ \forall u \in V$ ☹

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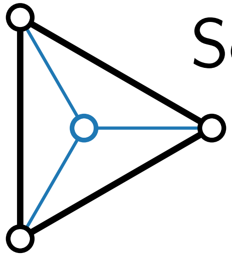

Solution: fix coordinates of outer face! ☺

$\overrightarrow{p_u p_v}$ = unit vector pointing from u to v

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Tutte's Forces

Goal.

$$p_u = \text{barycenter}(\text{Adj}[u]) = \textstyle\sum_{v \in \text{Adj}[u]} p_v / \deg(u)$$

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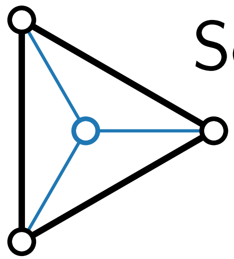

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- **Repulsive forces** $\quad \boxed{f_{\text{rep}}(u, v) = 0}$
- **Attractive forces**

$$f_{\text{attr}}(u, v) = \begin{cases} 0 & \text{if } u \text{ fixed,} \\ \dfrac{\|p_u - p_v\|}{\deg(u)} \overrightarrow{p_u p_v} & \text{otherwise.} \end{cases}$$

$\overrightarrow{p_u p_v}$ = unit vector pointing from u to v

$\|p_u - p_v\|$ = Euclidean distance between u and v

System of Linear Equations

Goal.

$p_u = \mathsf{barycenter}(\mathsf{Adj}[u]) \;\; = \sum_{v \in \mathsf{Adj}[u]} p_v / \deg(u)$

System of Linear Equations

Goal. $p_u = (x_u, y_u)$

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System of Linear Equations

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$x_u = \sum_{v \in \text{Adj}[u]} x_v / \deg(u) \;\Leftrightarrow\; \deg(u) \cdot x_u = \sum_{v \in \text{Adj}[u]} x_v$

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System of Linear Equations

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$Ax = b$

Two systems of linear equations:

$$x_u = \sum_{v \in \text{Adj}[u]} x_v / \deg(u) \Leftrightarrow \deg(u) \cdot x_u = \sum_{v \in \text{Adj}[u]} x_v \Leftrightarrow \deg(u) \cdot x_u - \sum_{v \in \text{Adj}[u]} x_v = 0$$

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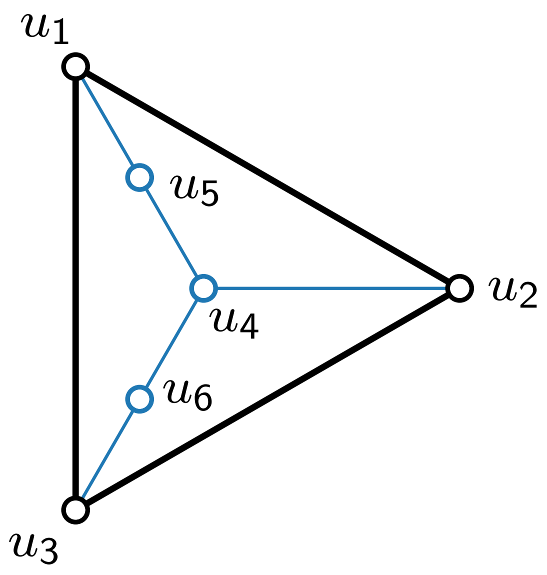

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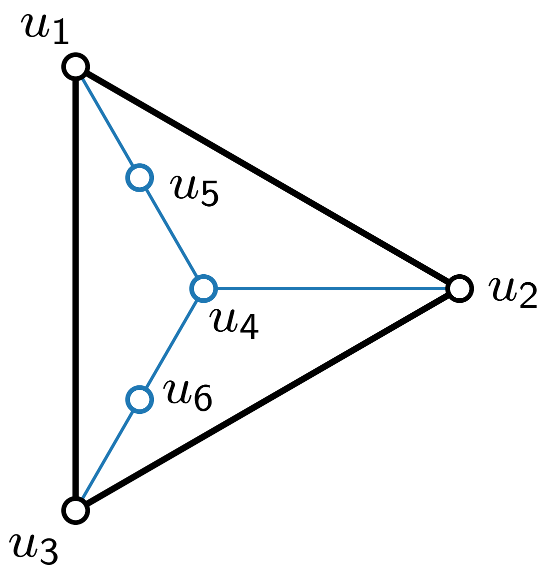
A



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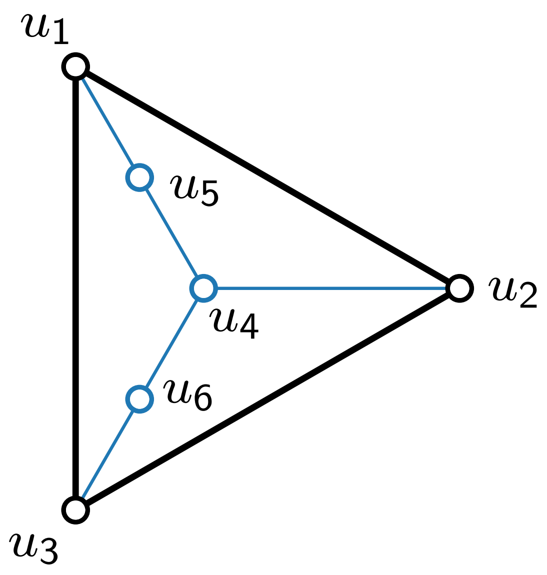

A

$$\begin{array}{c} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{array} \begin{pmatrix} \\ \\ \\ \\ \\ \\ \end{pmatrix}$$

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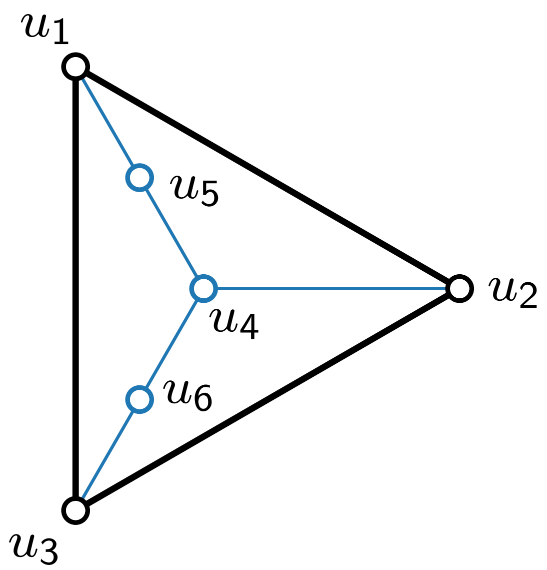

$$A \quad \begin{array}{c} \\ u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{array} \begin{array}{c} \begin{array}{cccccc} u_1 & u_2 & u_3 & u_4 & u_5 & u_6 \end{array} \\ \left(\begin{array}{cccccc} \; & \; & \; & \; & \; & \; \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \end{array} \right) \end{array}$$

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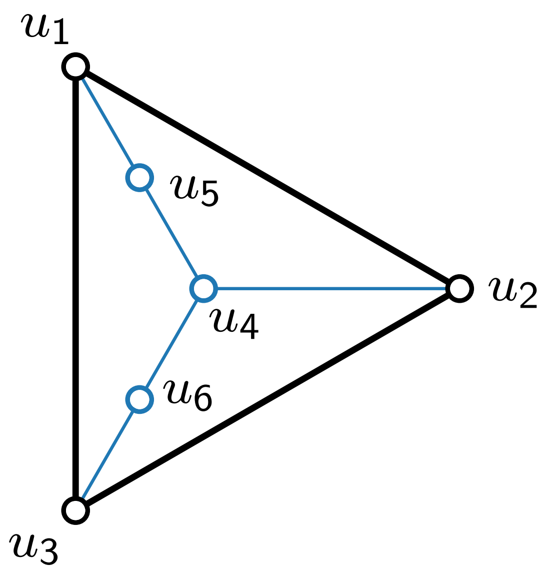

$$A = \begin{array}{c c} & \begin{array}{cccccc} u_1 & u_2 & u_3 & u_4 & u_5 & u_6 \end{array} \\ \begin{array}{c} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{array} & \begin{pmatrix} 3 & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \end{pmatrix} \end{array}$$

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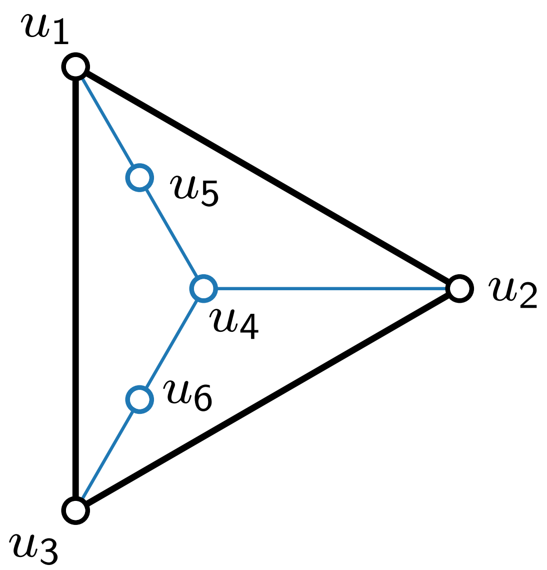

$$A = \begin{array}{c} \\ u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{array} \begin{array}{c} \begin{array}{cccccc} u_1 & u_2 & u_3 & u_4 & u_5 & u_6 \end{array} \\ \begin{pmatrix} 3 & -1 & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \end{pmatrix} \end{array}$$

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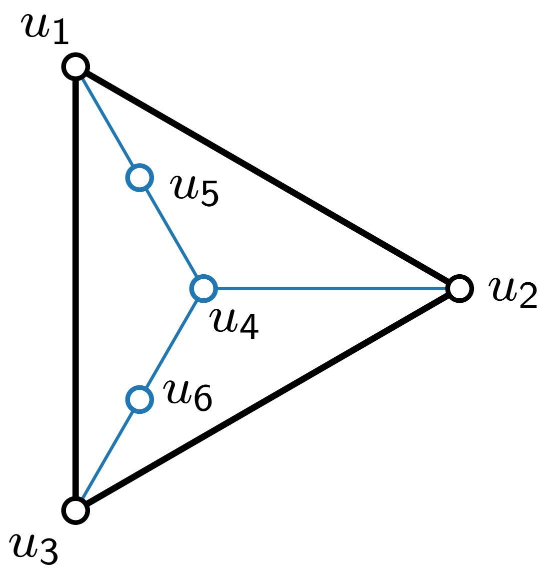

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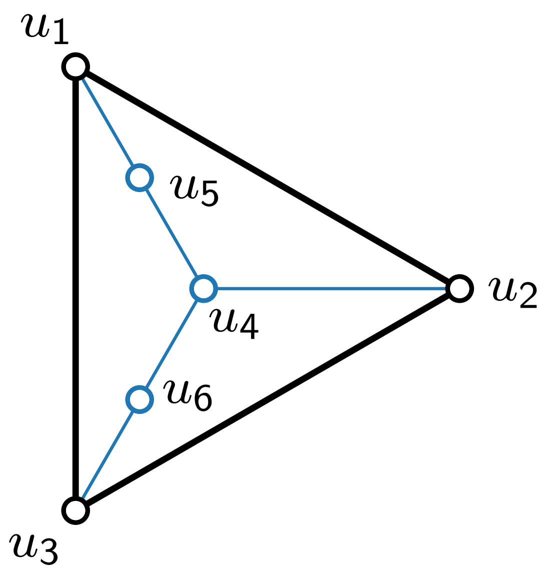

$$A$$

$$\begin{array}{c c}
 & \begin{array}{cccccc} u_1 & u_2 & u_3 & u_4 & u_5 & u_6 \end{array} \\
\begin{array}{c} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{array} & \begin{pmatrix} 3 & -1 & -1 & 0 & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \end{pmatrix}
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System of Linear Equations

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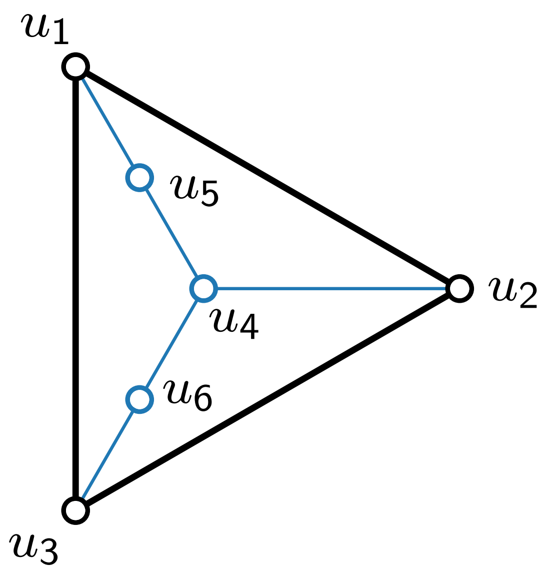

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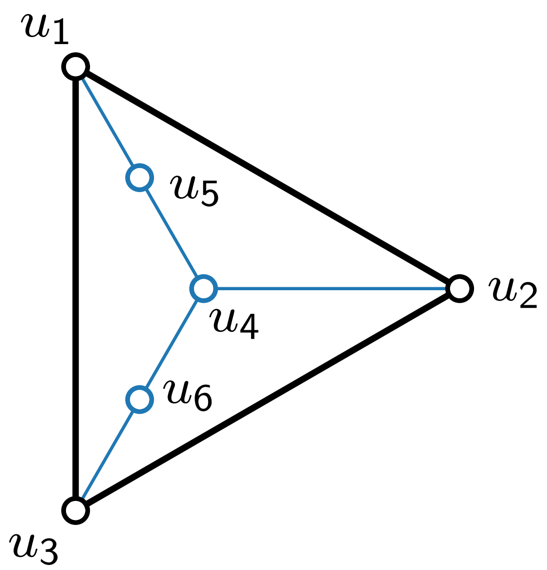

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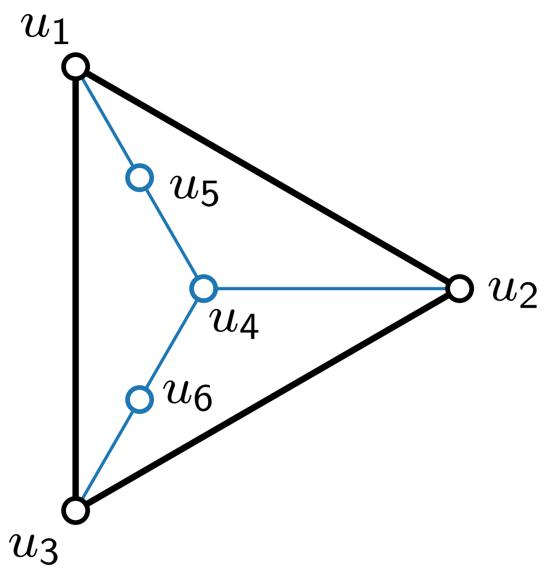

$$A \qquad \begin{array}{c|c} & \begin{matrix} u_1 & u_2 & u_3 & u_4 & u_5 & u_6 \end{matrix} \\ \begin{matrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{matrix} & \begin{pmatrix} 3 & -1 & -1 & 0 & -1 & 0 \\ & 3 & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \end{pmatrix} \end{array}$$

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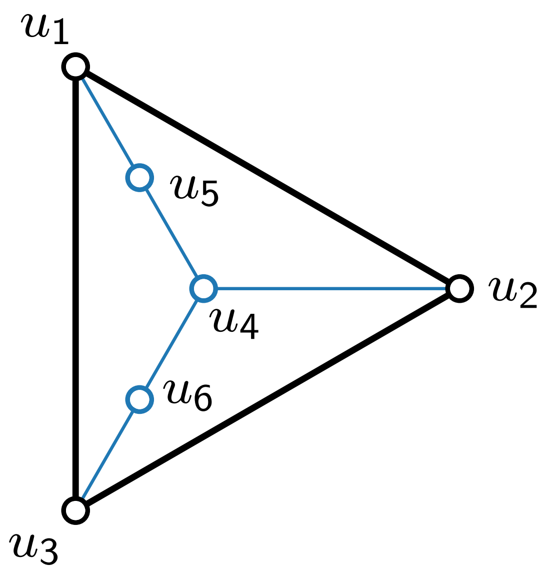

$$A$$

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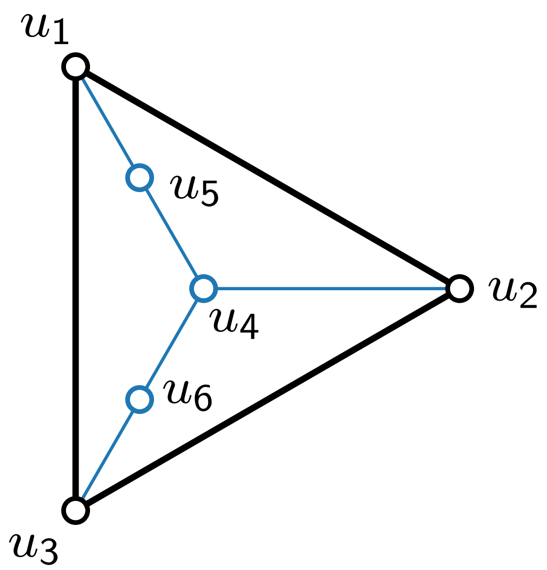
$x_u = \sum_{v \in \text{Adj}[u]} x_v / \deg(u) \Leftrightarrow \deg(u) \cdot x_u = \sum_{v \in \text{Adj}[u]} x_v \Leftrightarrow \deg(u) \cdot x_u - \sum_{v \in \text{Adj}[u]} x_v = 0$

$y_u = \sum_{v \in \text{Adj}[u]} y_v / \deg(u) \Leftrightarrow \deg(u) \cdot y_u = \sum_{v \in \text{Adj}[u]} y_v \Leftrightarrow \deg(u) \cdot y_u - \sum_{v \in \text{Adj}[u]} y_v = 0$



$$A = \begin{array}{c} \\ u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{array} \begin{array}{c} \begin{array}{cccccc} u_1 & u_2 & u_3 & u_4 & u_5 & u_6 \end{array} \\ \begin{pmatrix} 3 & -1 & -1 & 0 & -1 & 0 \\ -1 & 3 & -1 & -1 & 0 & 0 \\ -1 & -1 & 3 & 0 & 0 & -1 \\ & & & & & \\ & & & & & \\ & & & & & \end{pmatrix} \end{array}$$

System of Linear Equations

Goal. $p_u = (x_u, y_u)$

$p_u = \text{barycenter}(\text{Adj}[u]) = \sum_{v \in \text{Adj}[u]} p_v / \deg(u)$

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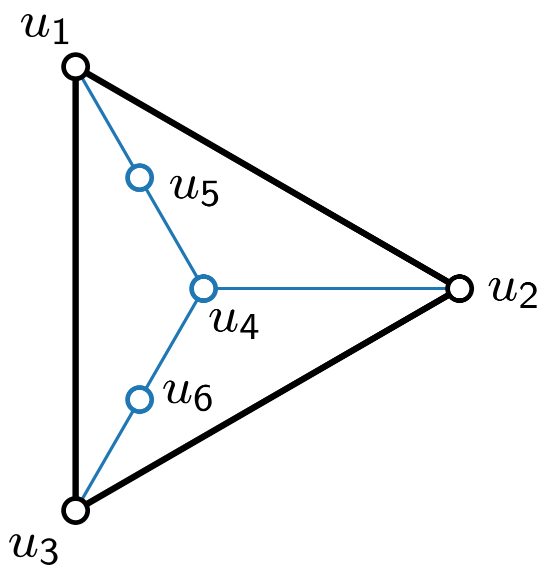

$$A$$

$$\begin{array}{c} \\ u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{array} \begin{array}{c} \begin{array}{cccccc} u_1 & u_2 & u_3 & u_4 & u_5 & u_6 \end{array} \\ \begin{pmatrix} 3 & -1 & -1 & 0 & -1 & 0 \\ -1 & 3 & -1 & -1 & 0 & 0 \\ -1 & -1 & 3 & 0 & 0 & -1 \\ 0 & -1 & 0 & 3 & -1 & -1 \\ & & & & & \\ & & & & & \end{pmatrix} \end{array}$$

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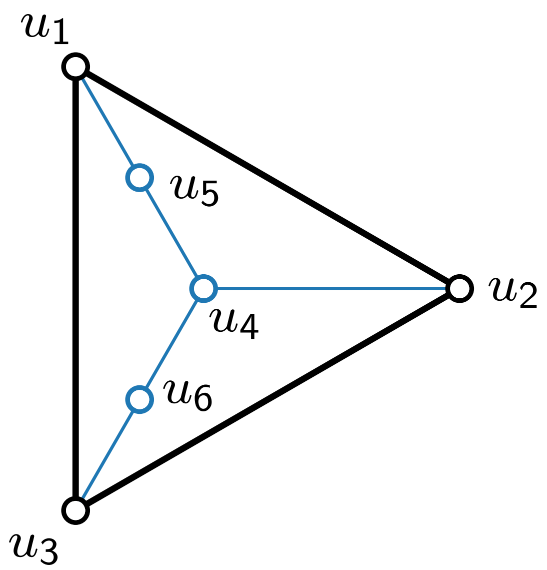

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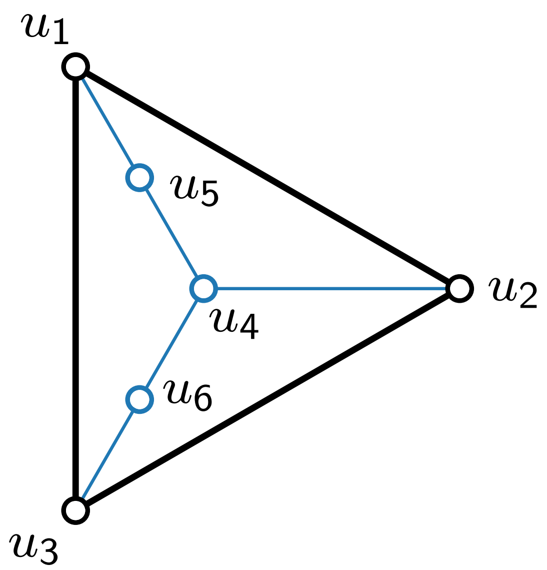

$$\begin{array}{c} \\ u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{array} \begin{array}{c} \begin{array}{cccccc} u_1 & u_2 & u_3 & u_4 & u_5 & u_6 \end{array} \\ \begin{pmatrix} 3 & -1 & -1 & 0 & -1 & 0 \\ -1 & 3 & -1 & -1 & 0 & 0 \\ -1 & -1 & 3 & 0 & 0 & -1 \\ 0 & -1 & 0 & 3 & -1 & -1 \\ -1 & 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & -1 & -1 & 0 & 2 \end{pmatrix} \end{array} \; A$$

System of Linear Equations

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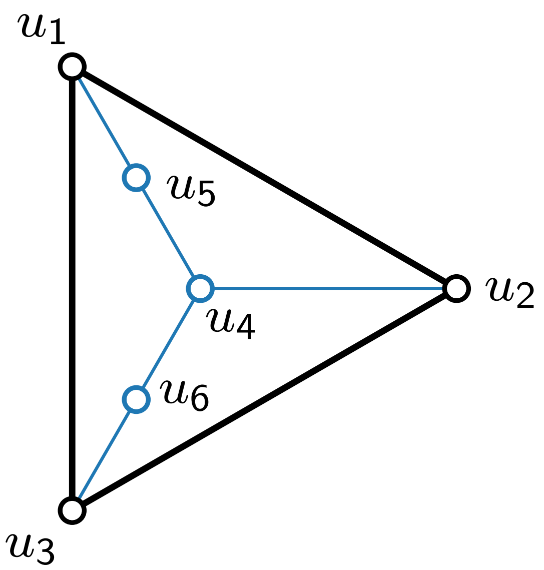

$$
\begin{array}{cc}
 & \begin{array}{cccccc} u_1 & u_2 & u_3 & u_4 & u_5 & u_6 \end{array} \\
\begin{array}{c} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{array} &
\begin{pmatrix}
3 & -1 & -1 & 0 & -1 & 0 \\
-1 & 3 & -1 & -1 & 0 & 0 \\
-1 & -1 & 3 & 0 & 0 & -1 \\
0 & -1 & 0 & 3 & -1 & -1 \\
-1 & 0 & 0 & -1 & 2 & 0 \\
0 & 0 & -1 & -1 & 0 & 2
\end{pmatrix}
\end{array}
\quad A
$$

$A_{ii} = \deg(u_i)$

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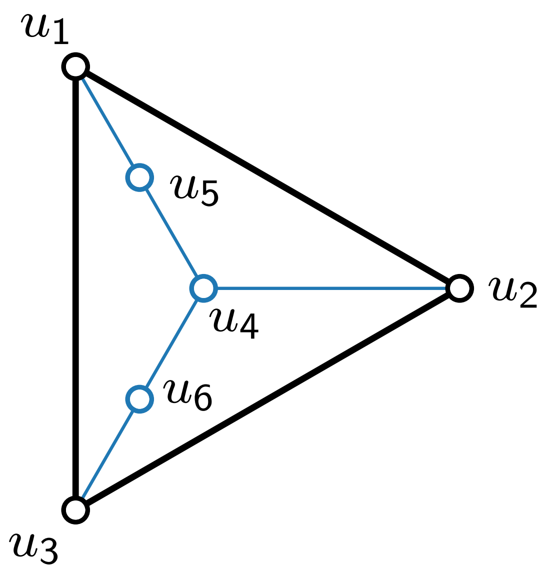

$$A = \begin{array}{c} \\ u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{array} \begin{array}{c} \begin{array}{cccccc} u_1 & u_2 & u_3 & u_4 & u_5 & u_6 \end{array} \\ \begin{pmatrix} 3 & -1 & -1 & 0 & -1 & 0 \\ -1 & 3 & -1 & -1 & 0 & 0 \\ -1 & -1 & 3 & 0 & 0 & -1 \\ 0 & -1 & 0 & 3 & -1 & -1 \\ -1 & 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & -1 & -1 & 0 & 2 \end{pmatrix} \end{array}$$

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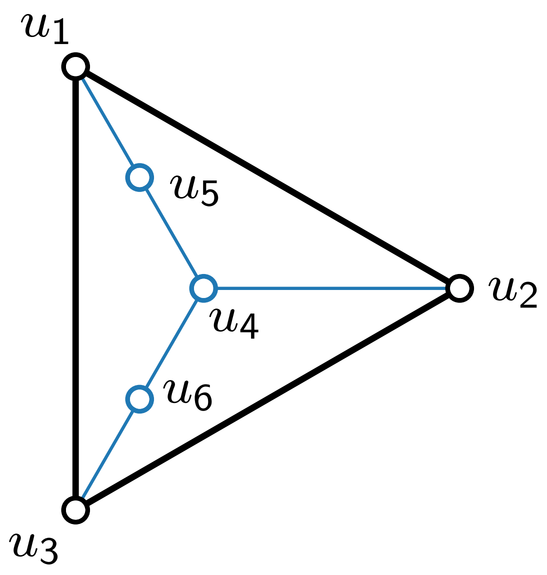

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Laplacian matrix of G

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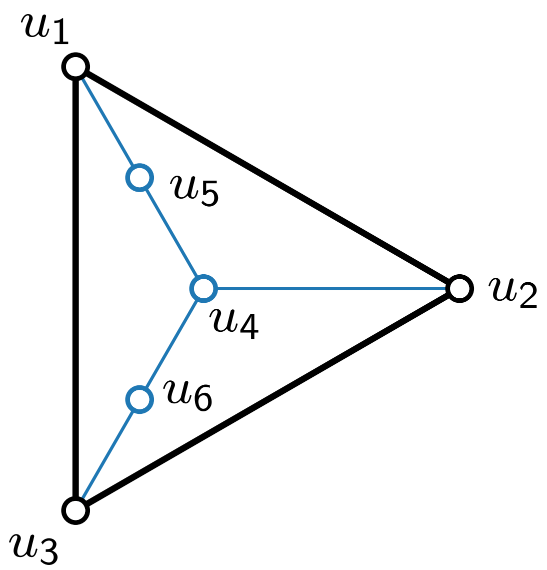

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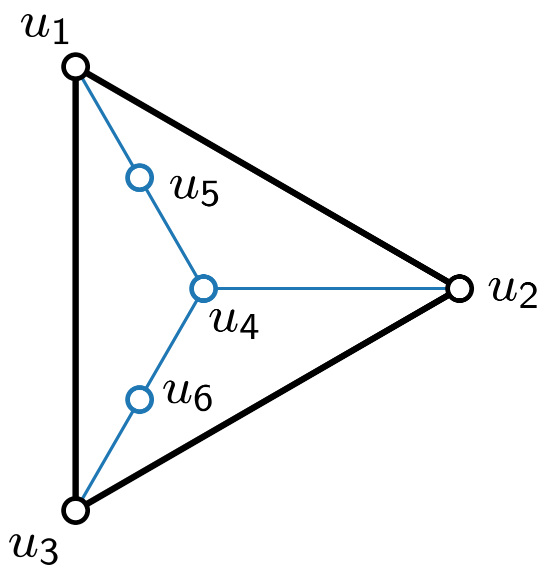

A

$$\begin{array}{c|c} & \begin{array}{cccccc} u_1 & u_2 & u_3 & u_4 & u_5 & u_6 \end{array} \\ \begin{array}{c} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{array} & \begin{pmatrix} 3 & -1 & -1 & 0 & -1 & 0 \\ -1 & 3 & -1 & -1 & 0 & 0 \\ -1 & -1 & 3 & 0 & 0 & -1 \\ 0 & -1 & 0 & 3 & -1 & -1 \\ -1 & 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & -1 & -1 & 0 & 2 \end{pmatrix} \end{array}$$

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variables, constraints, $\det(A) =$

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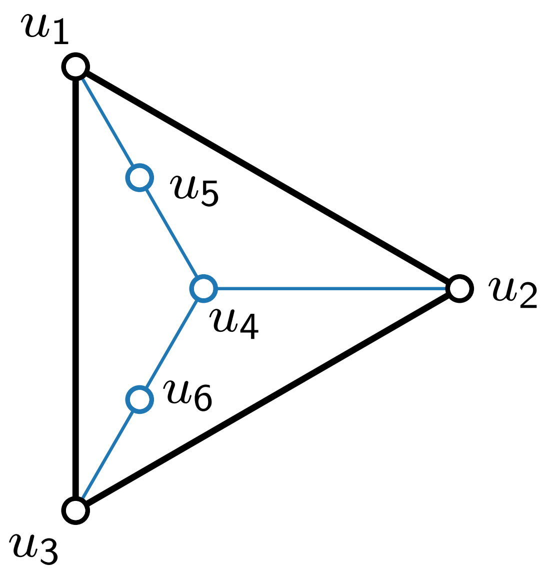

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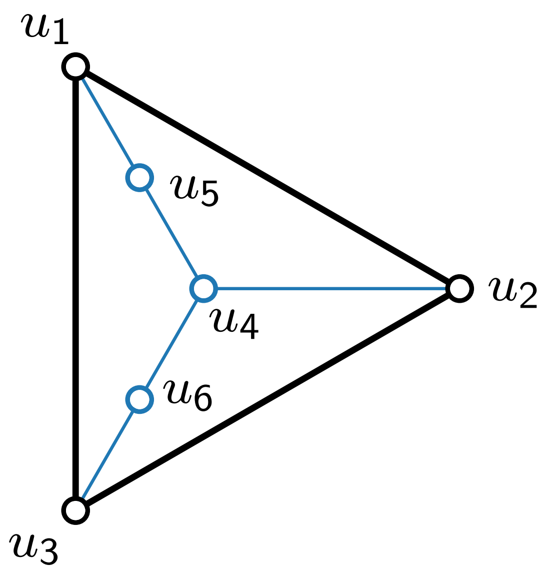

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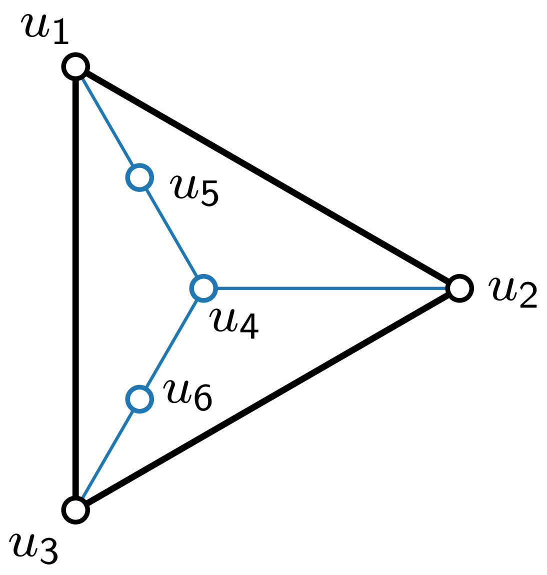

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n variables, n constraints, $\det(A) = 0$

unique solution

System of Linear Equations

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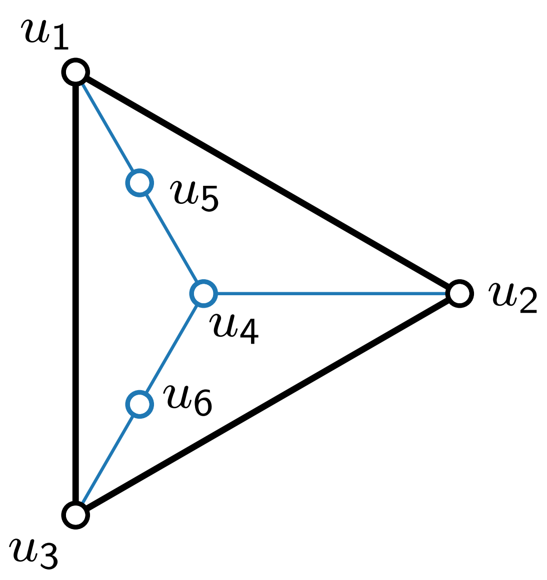

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Laplacian matrix of G

n variables, n constraints, $\det(A) = 0$

$\Rightarrow$ no unique solution

System of Linear Equations

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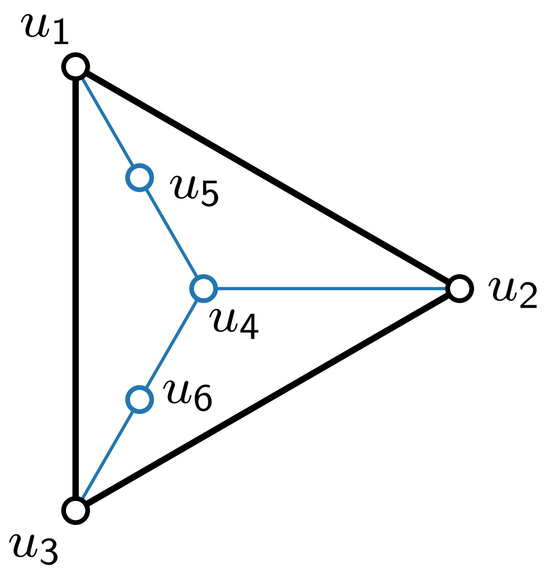

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Laplacian matrix of G

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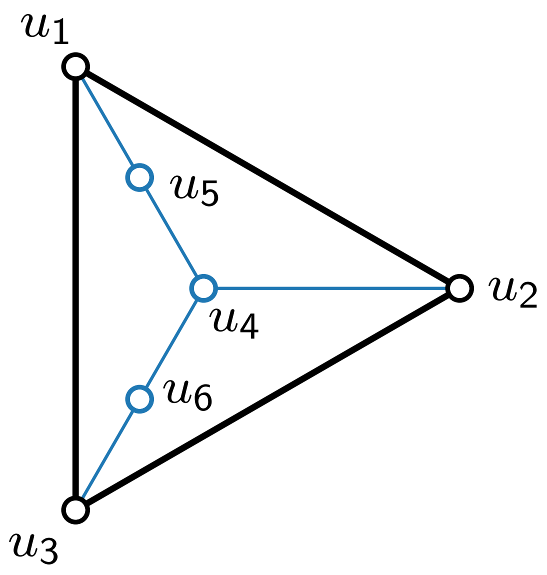

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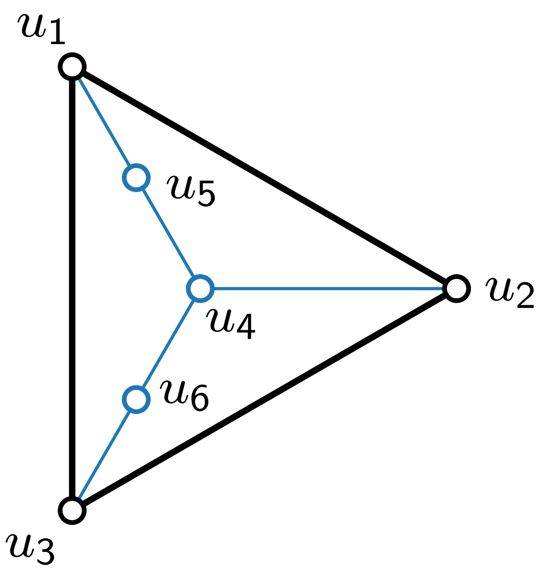

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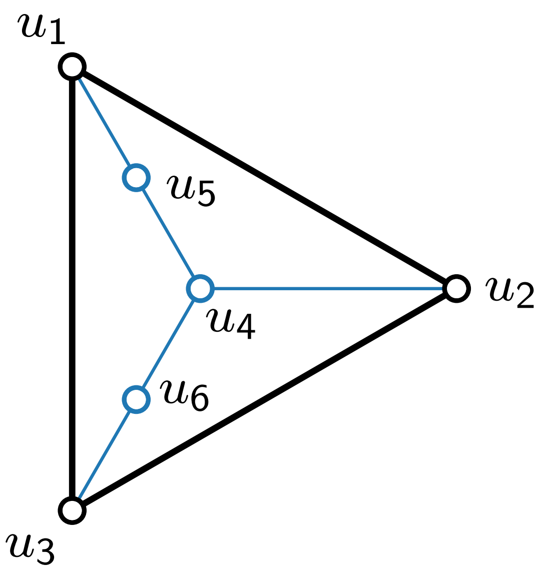

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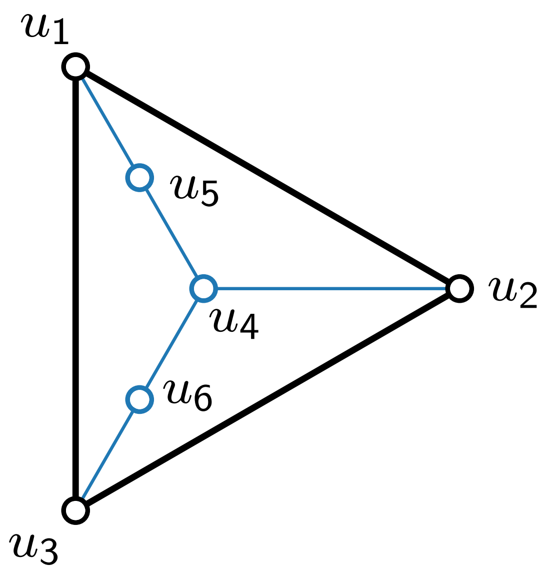

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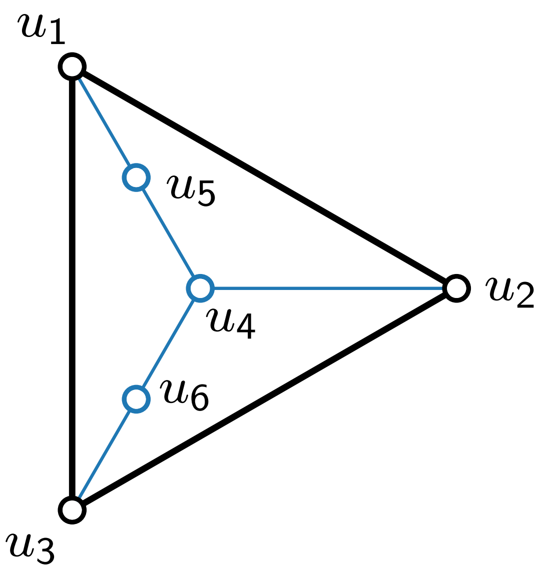

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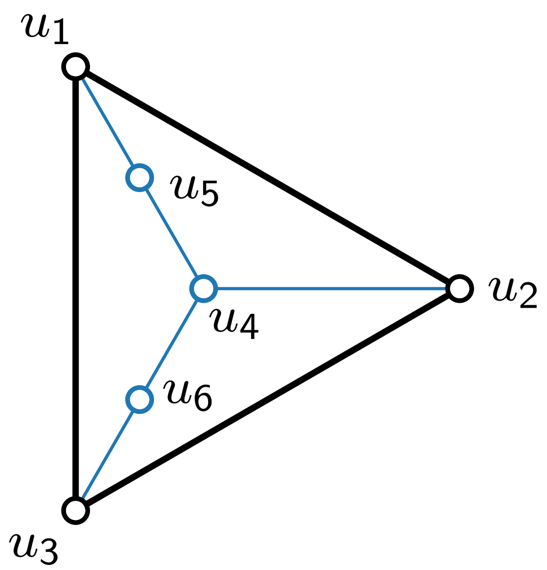

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Theorem.
Tutte's barycentric algorithm admits a unique solution.
It can be computed in polynomial time.

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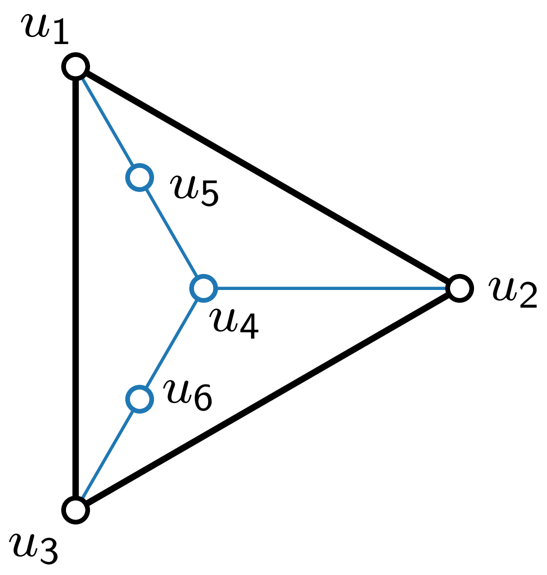

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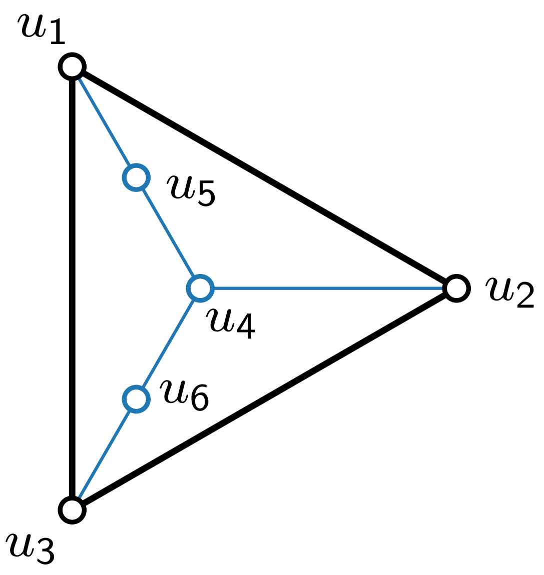

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solve two systems of linear equations

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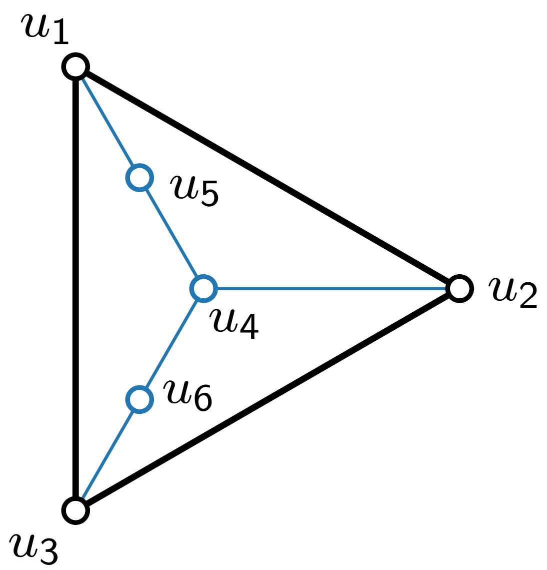

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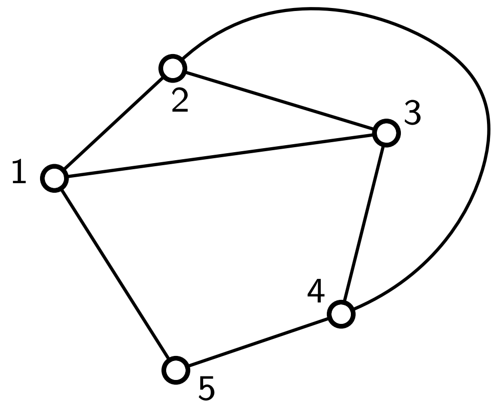
planar: G can be drawn in such a way that no edges cross each other

connected: $\exists\, u$–v path for every vertex pair $\{u, v\}$.



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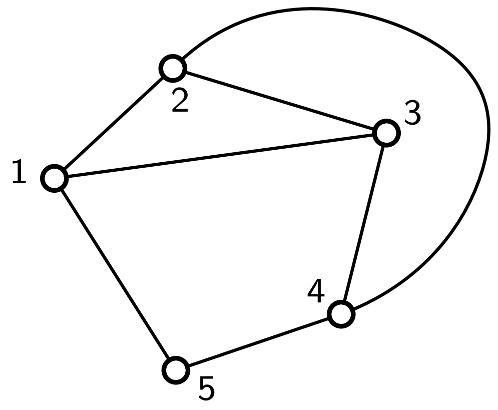

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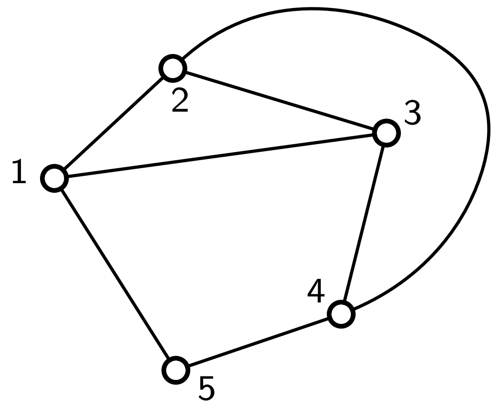

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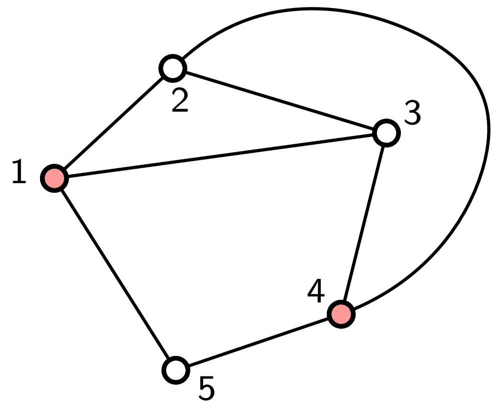

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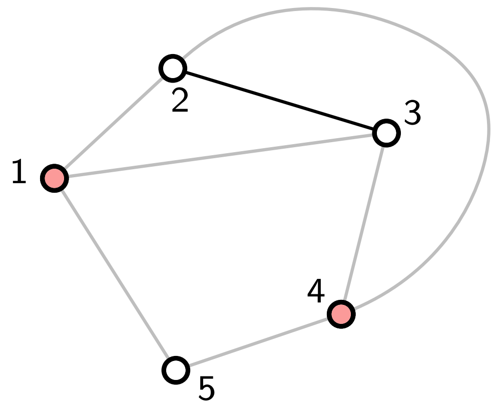

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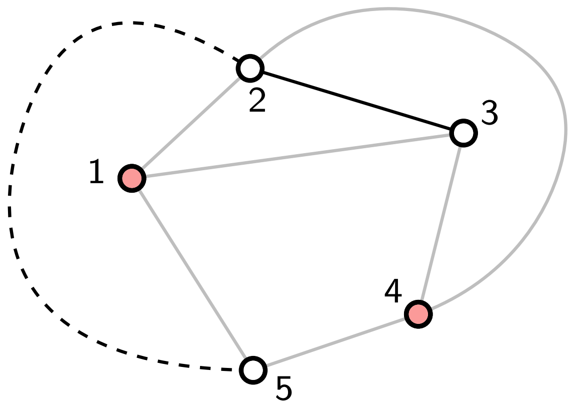

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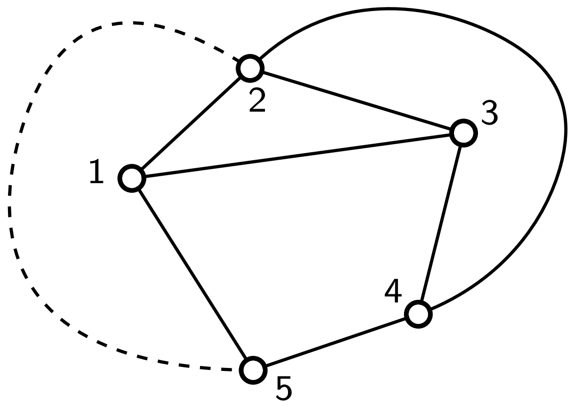

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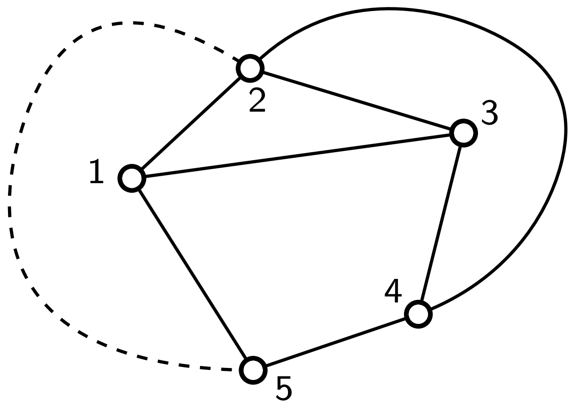

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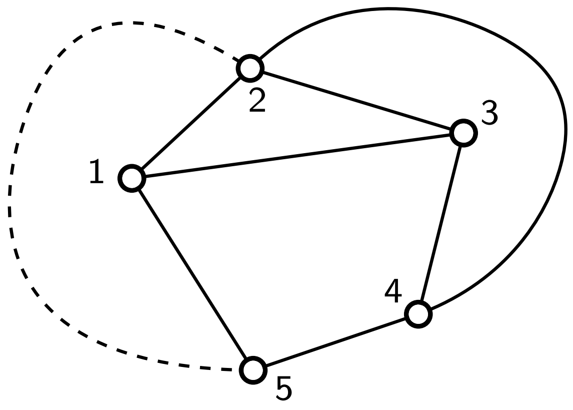

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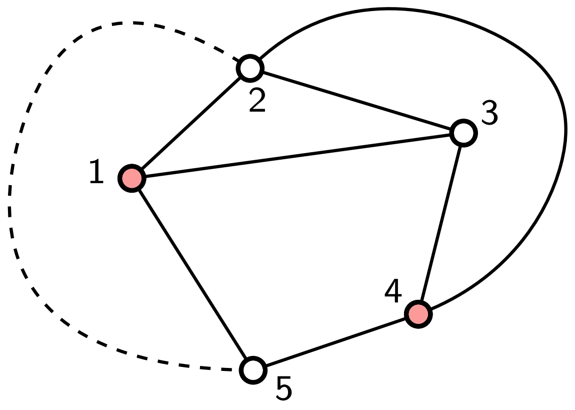

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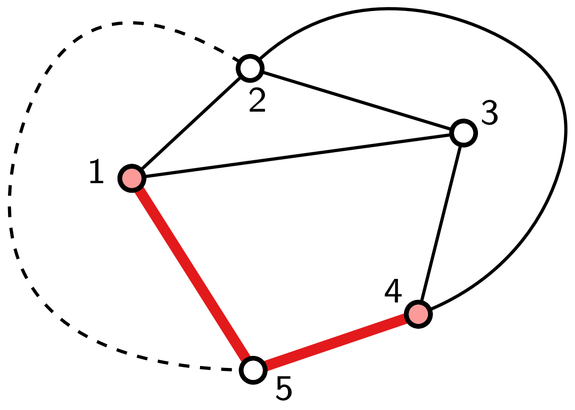

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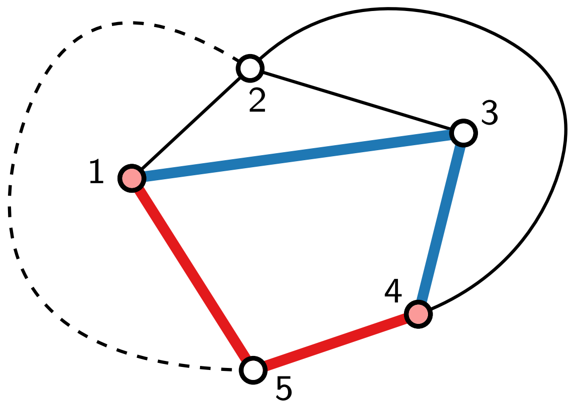

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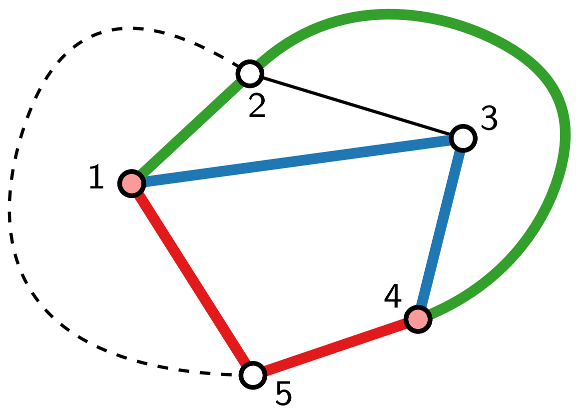

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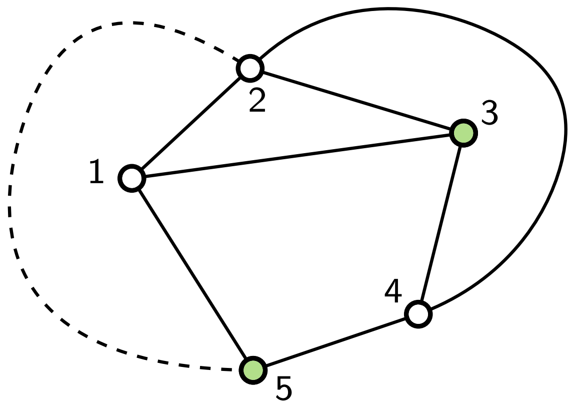

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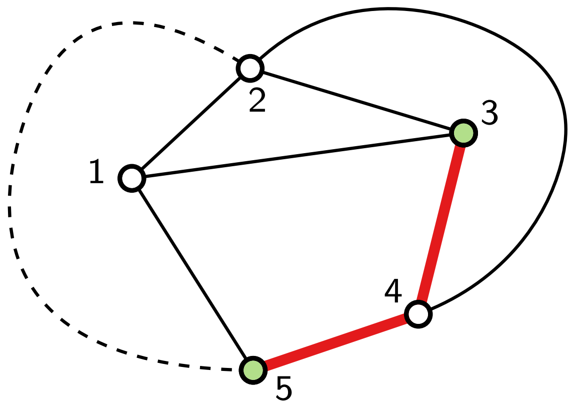

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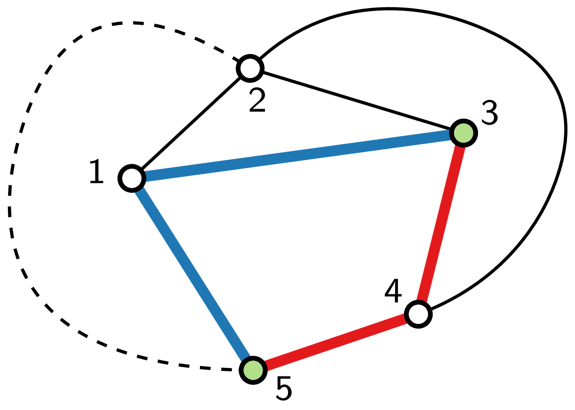

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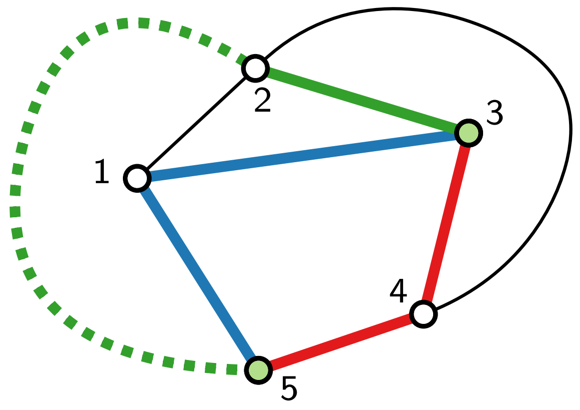

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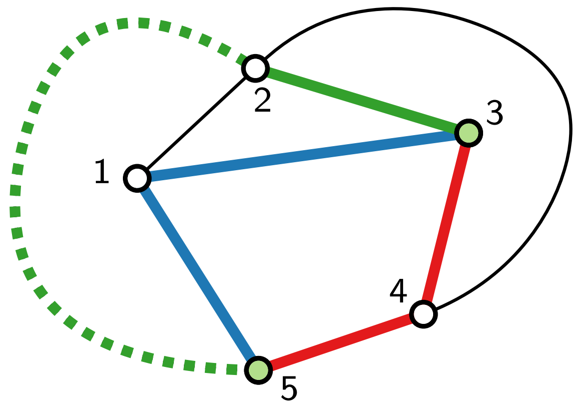

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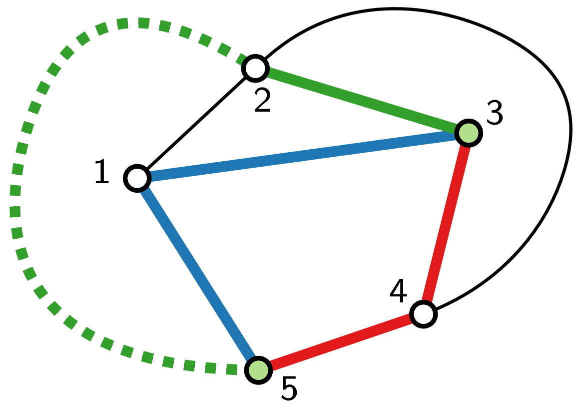

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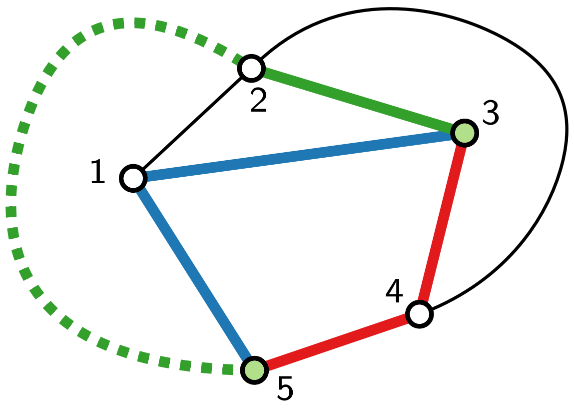

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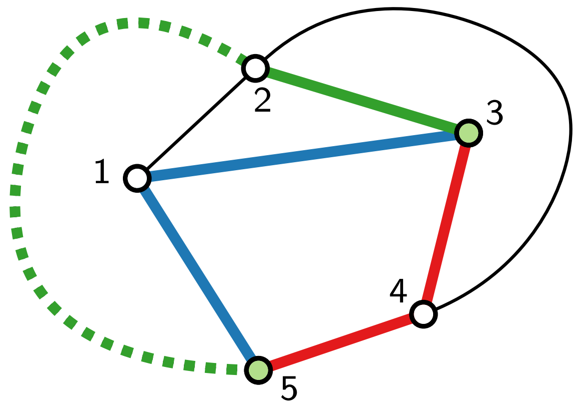

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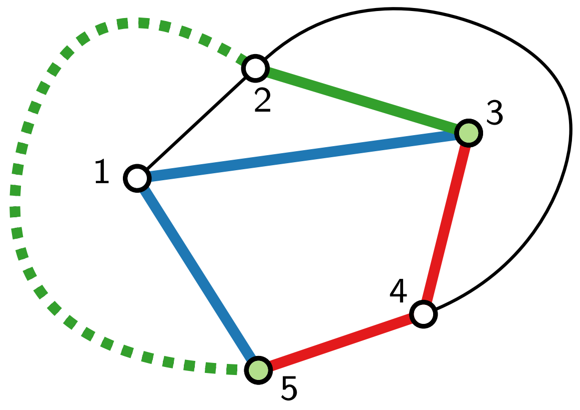

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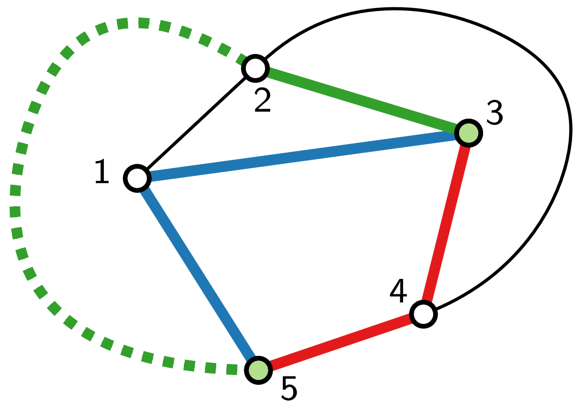

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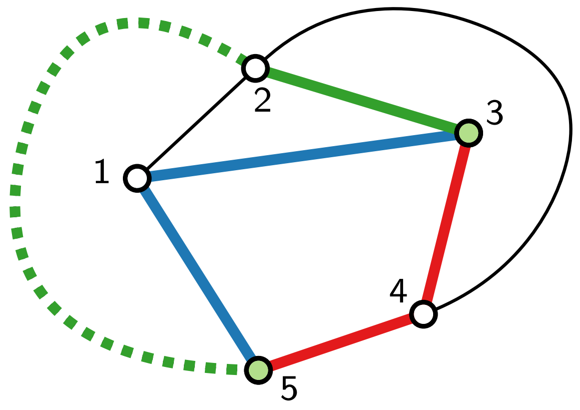

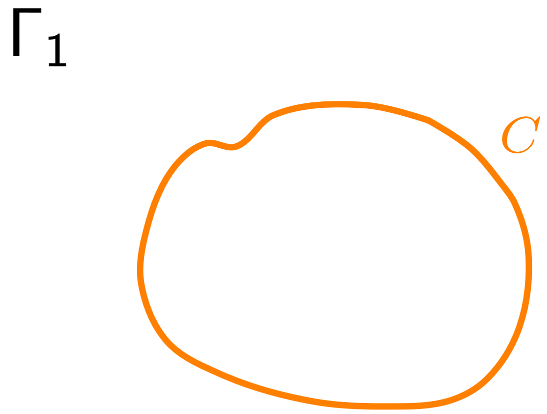

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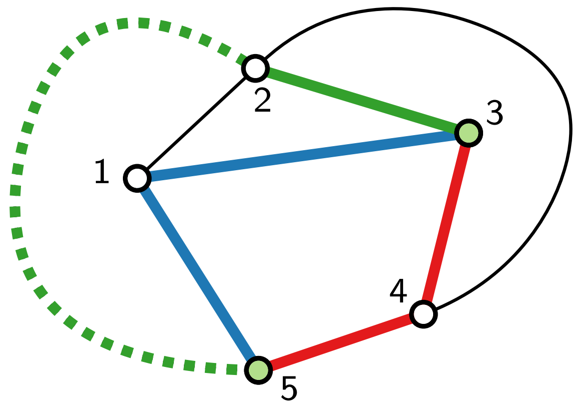

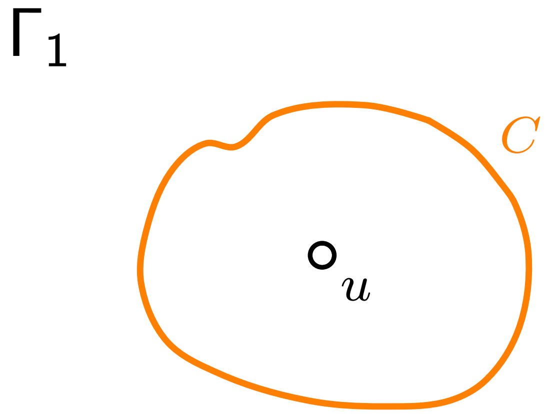

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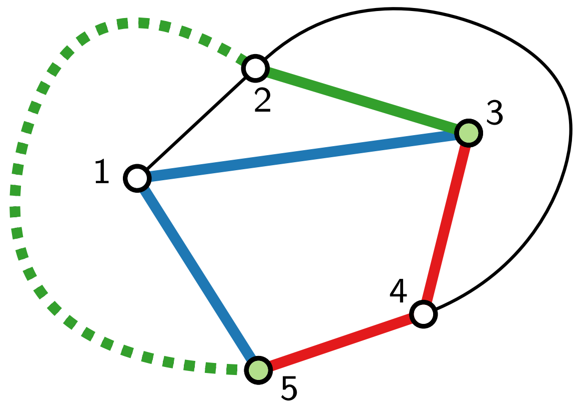

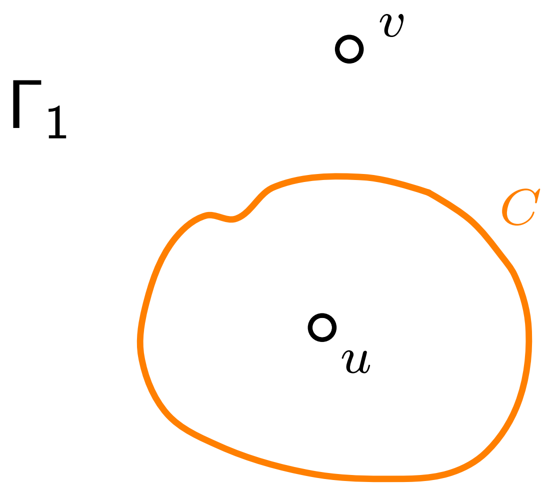

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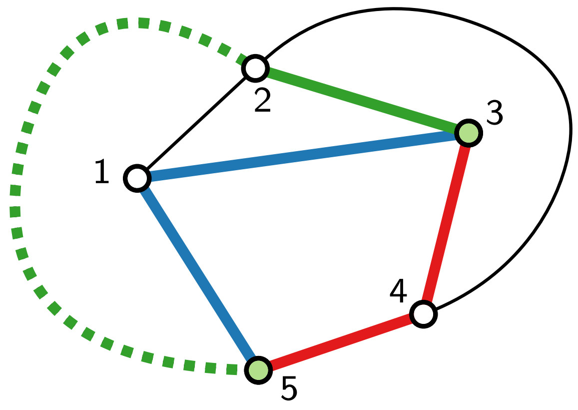

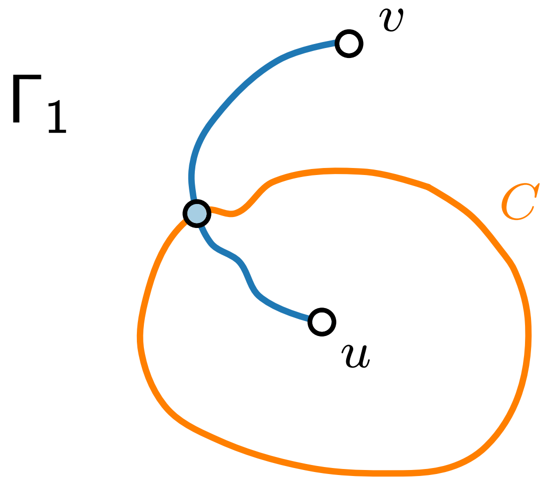

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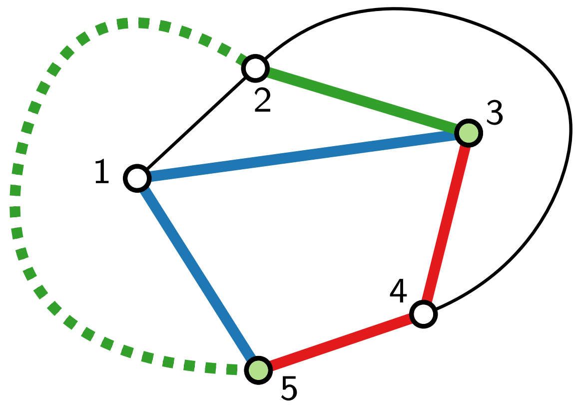

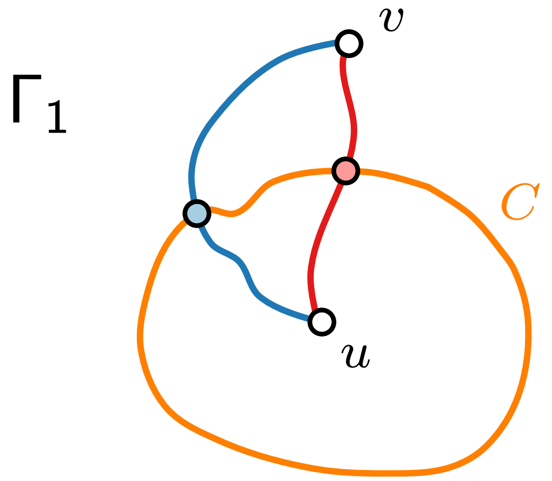

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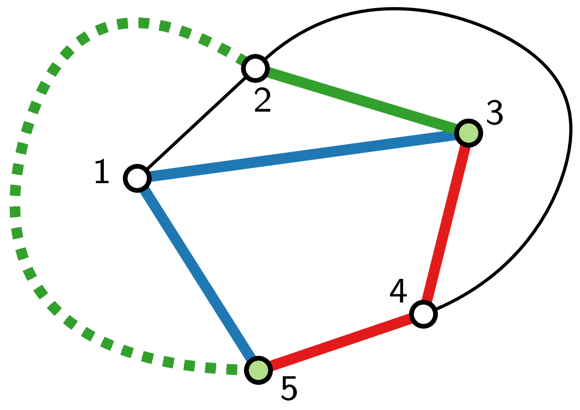

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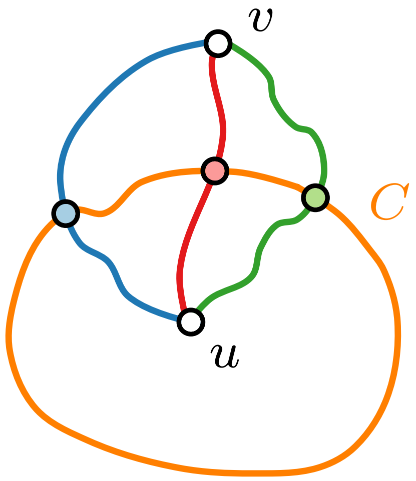

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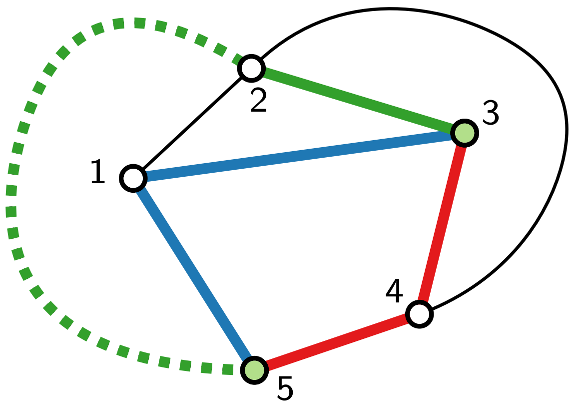

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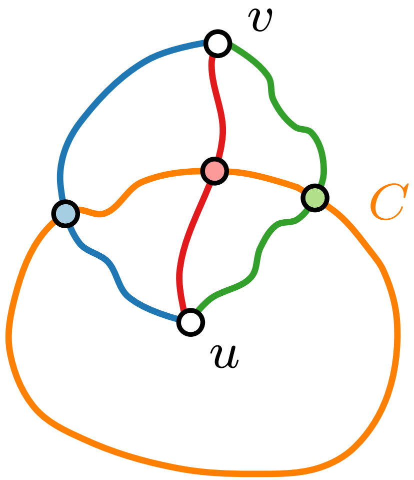

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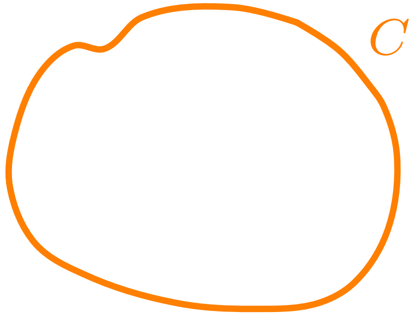

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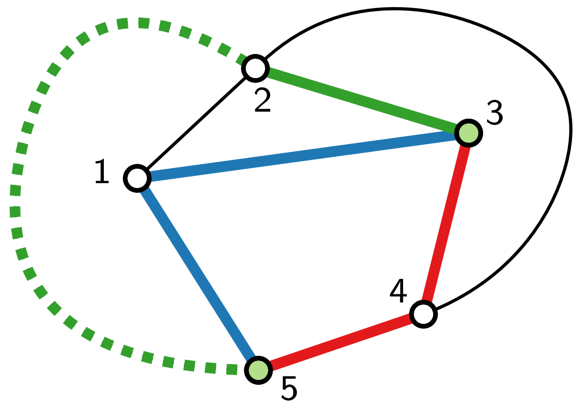

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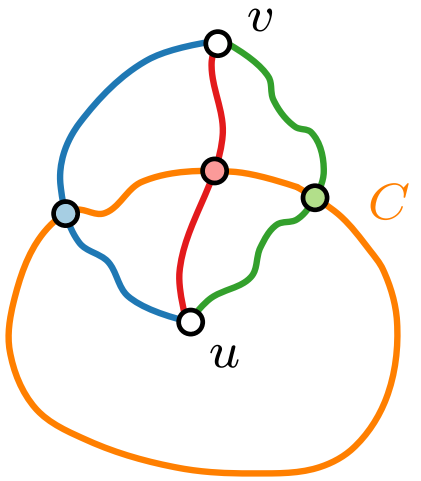

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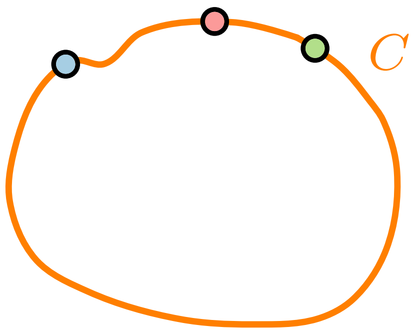

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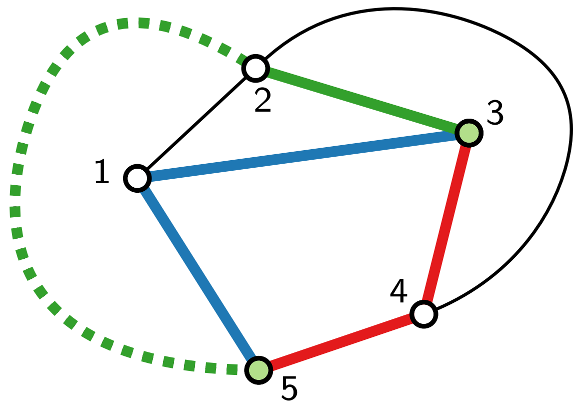

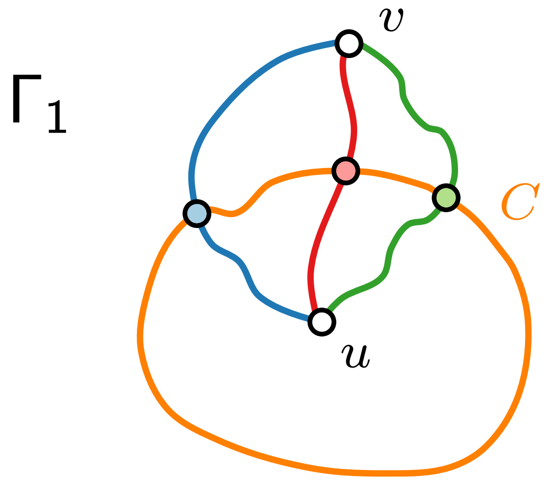

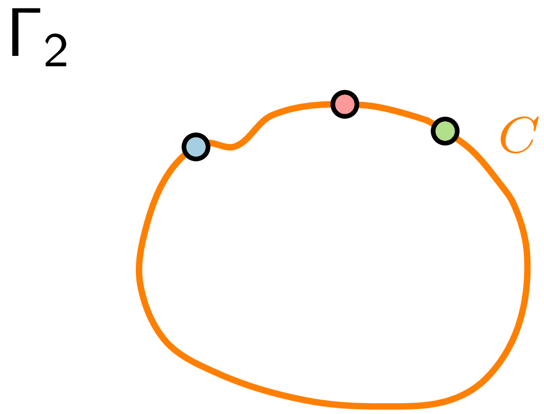

3-Connected Planar Graphs

(up to the choice of the outer face and mirroring)

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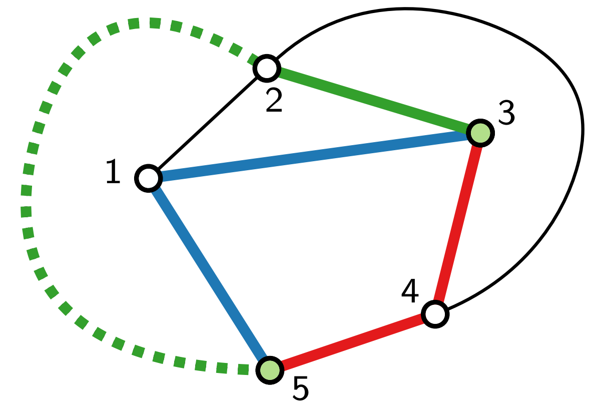

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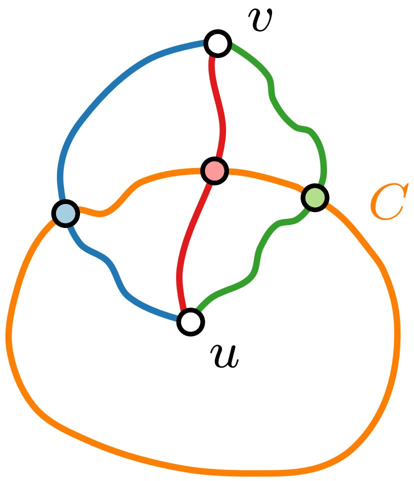

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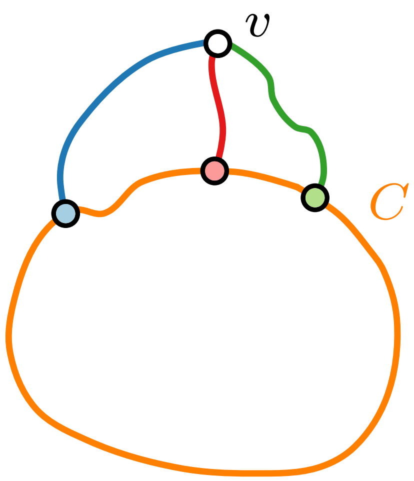

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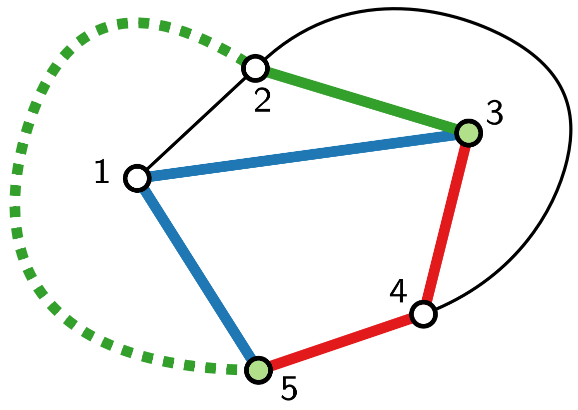

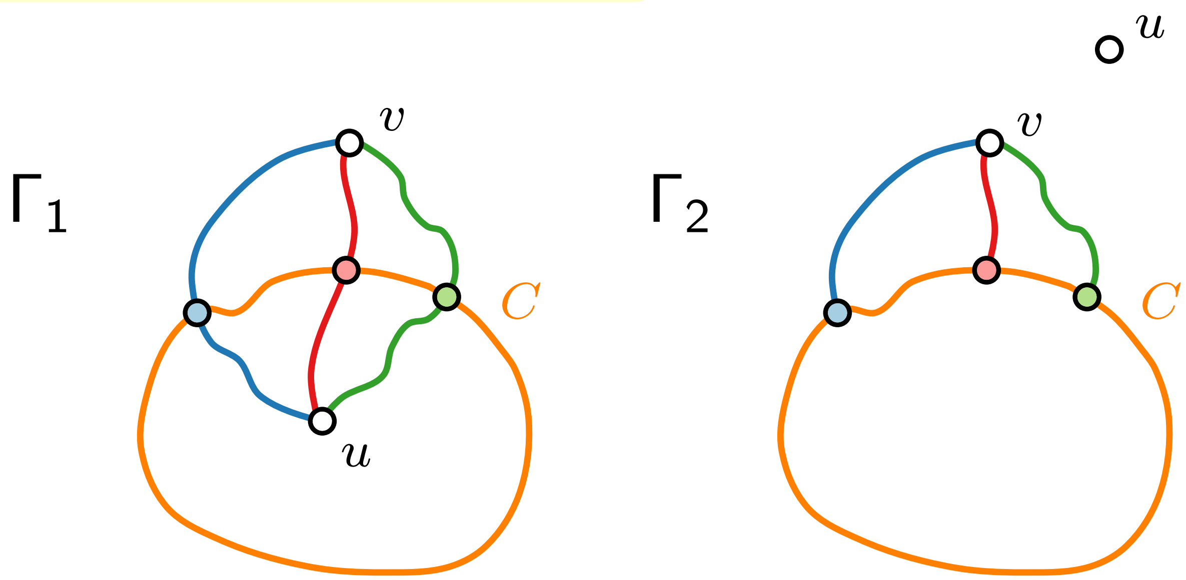

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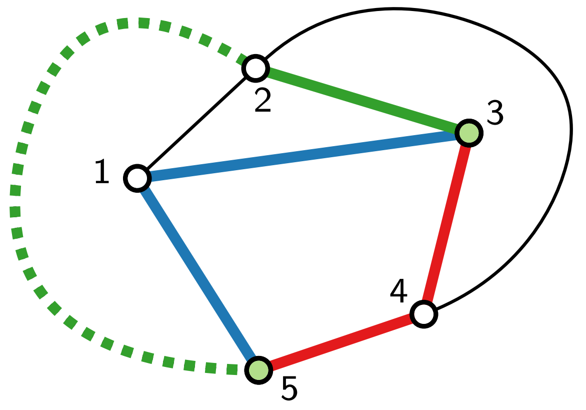

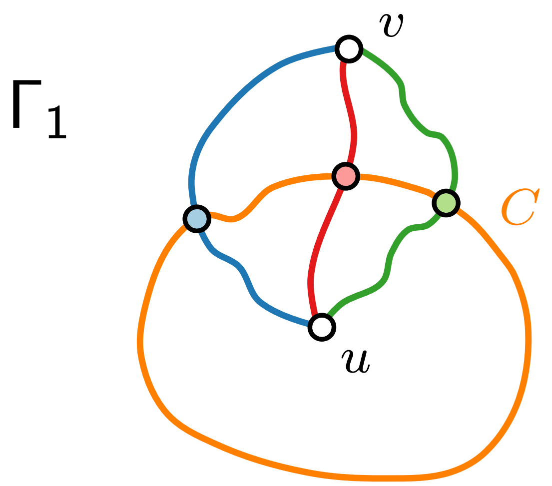

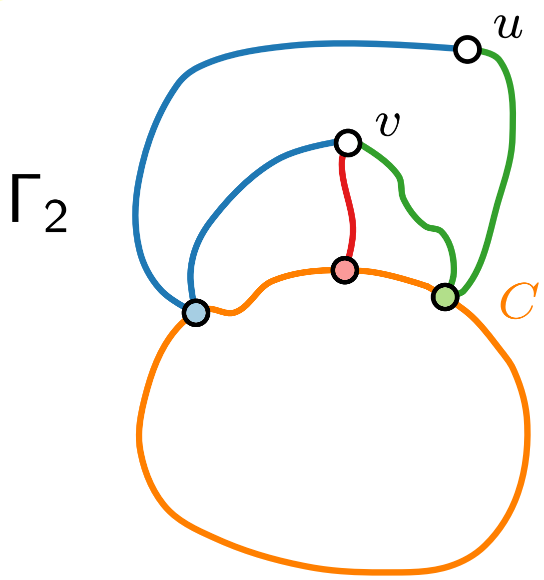

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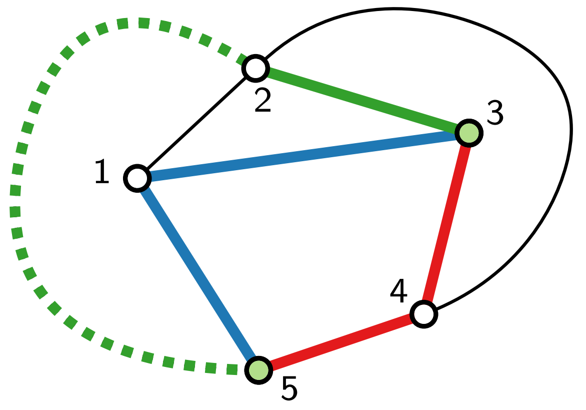

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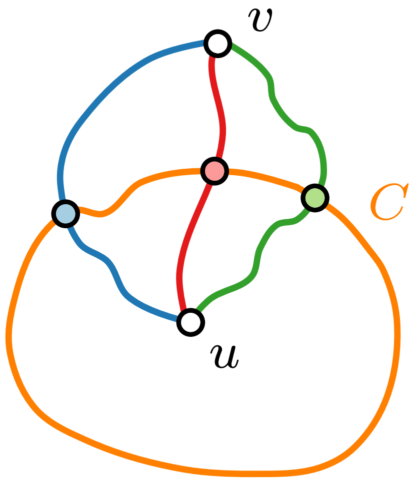

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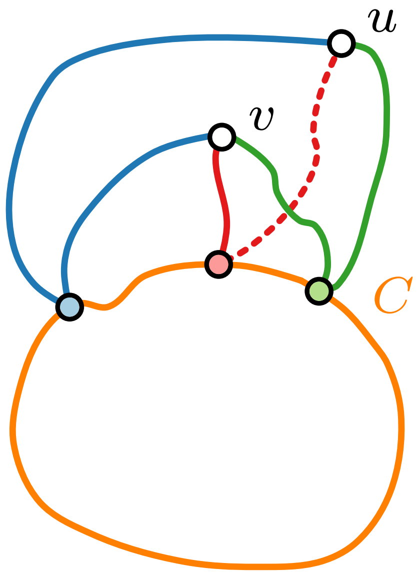

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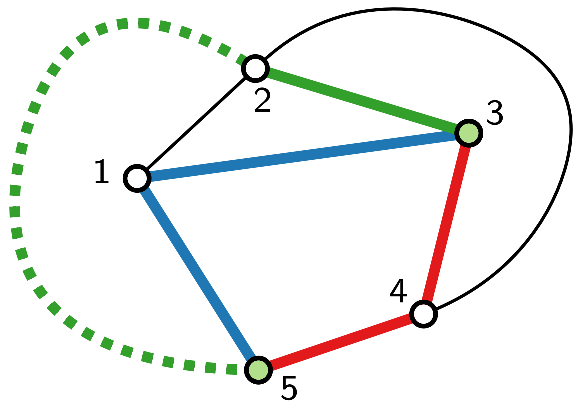

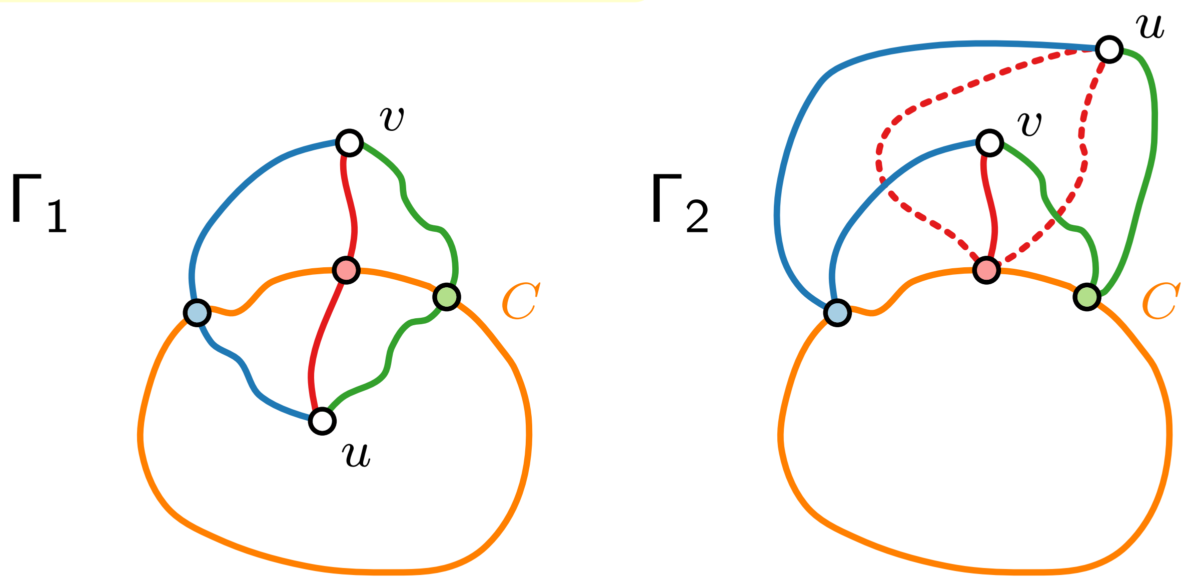

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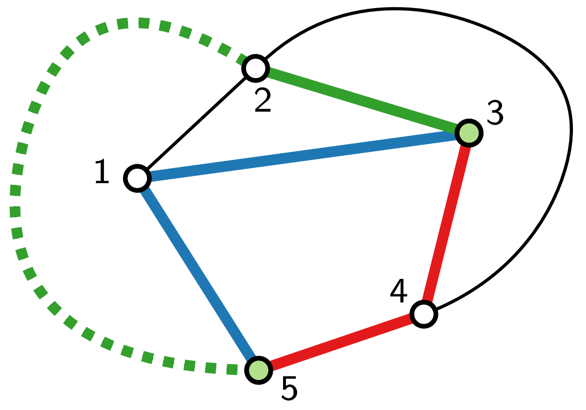

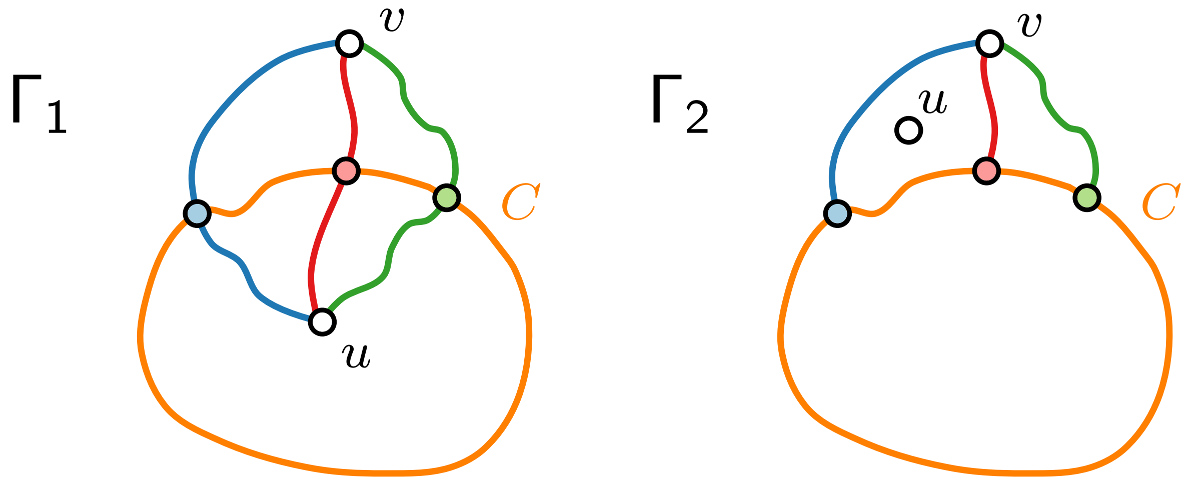

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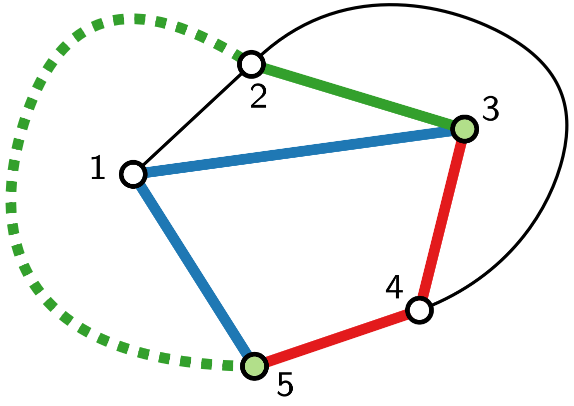

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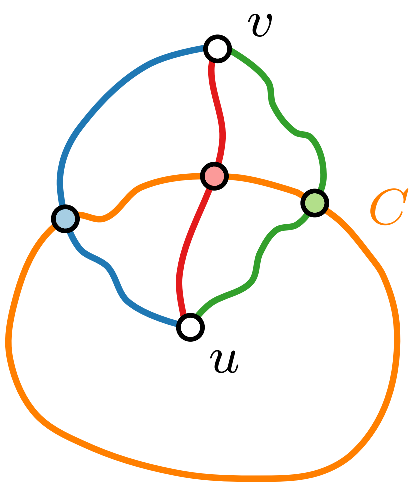

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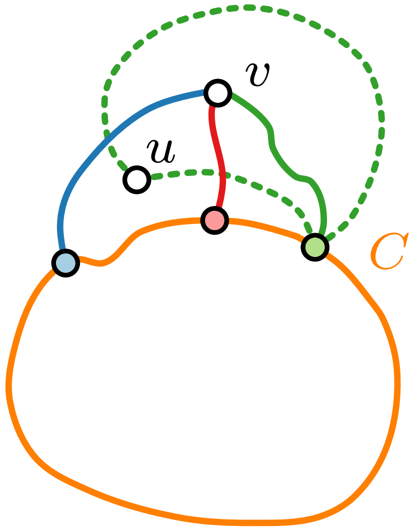

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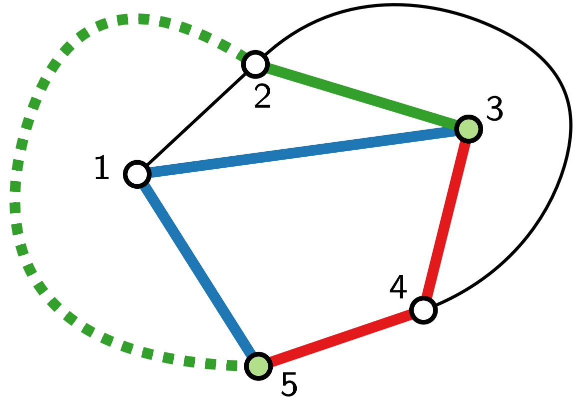

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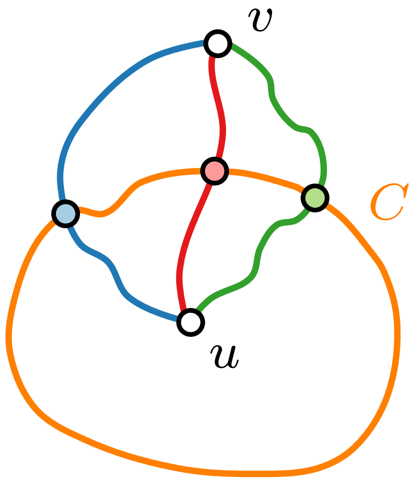

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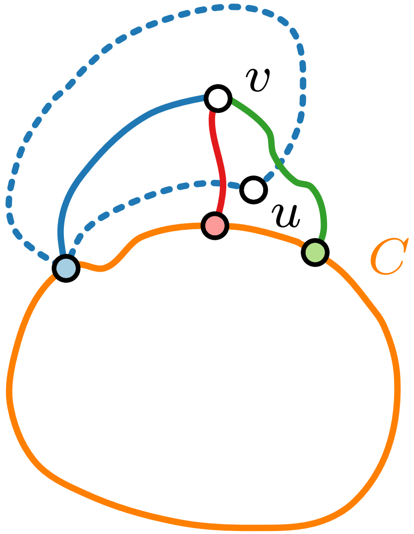
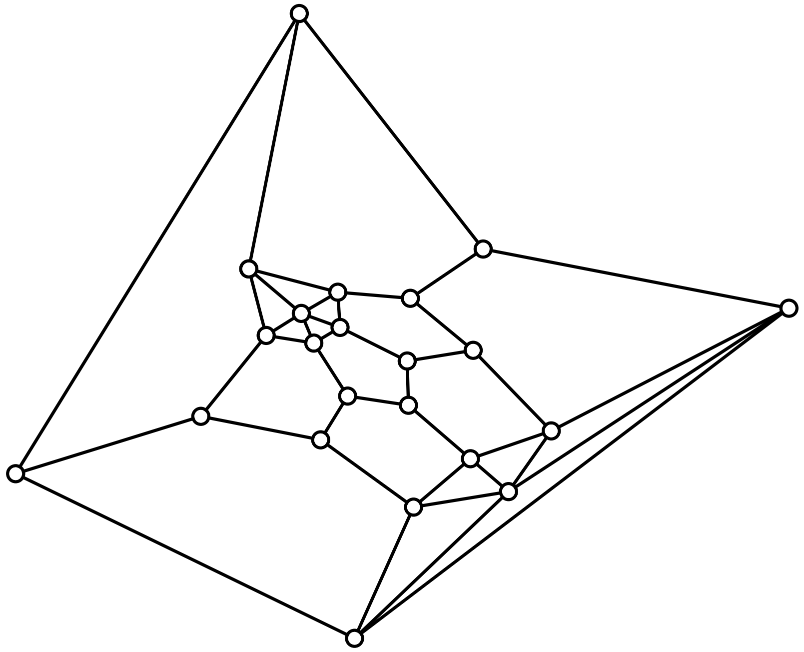


Tutte's Theorem

Theorem. [Tutte 1963]

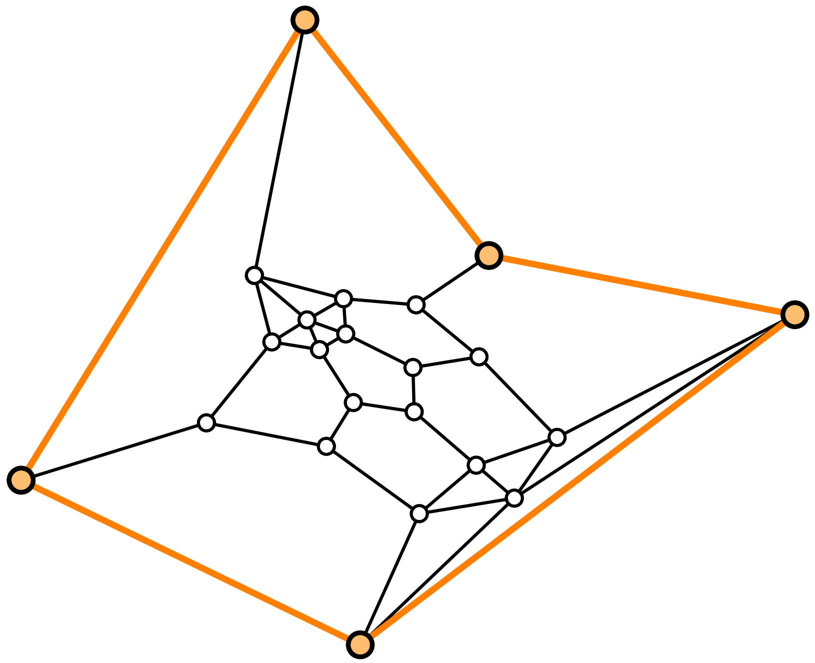
Let G be a 3-connected planar graph,



Tutte's Theorem

Theorem. [Tutte 1963]

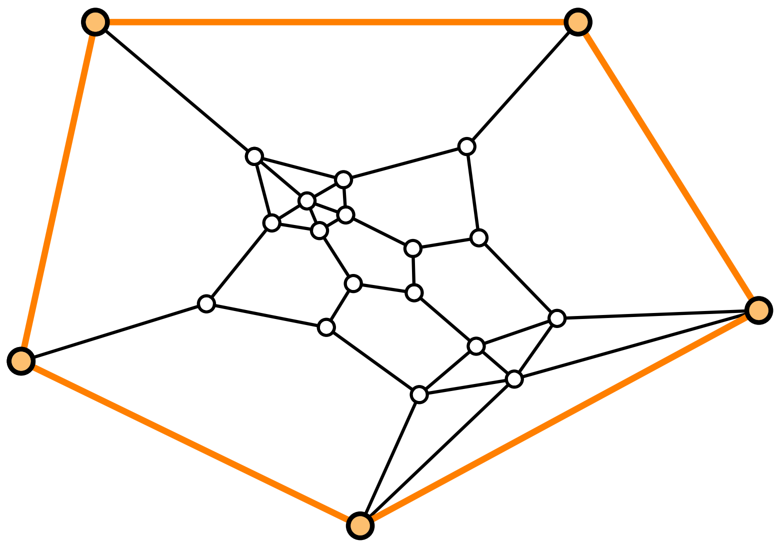
Let G be a 3-connected planar graph, and let C be a face of its unique embedding.



Tutte's Theorem

Theorem. [Tutte 1963]

Let G be a 3-connected planar graph, and let C be a face of its unique embedding. If we fix C on a strictly convex polygon,



Tutte's Theorem

Theorem. [Tutte 1963]
Let G be a 3-connected planar graph, and
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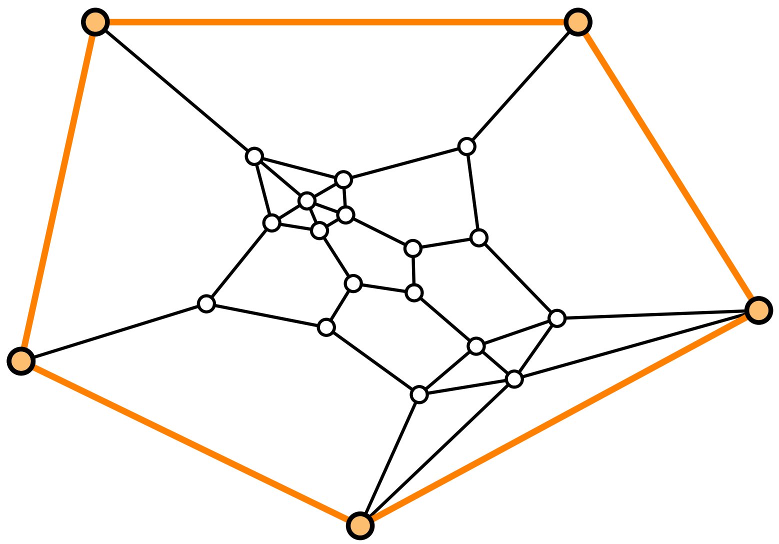
If we fix C on a strictly convex polygon, then the Tutte drawing of G is planar



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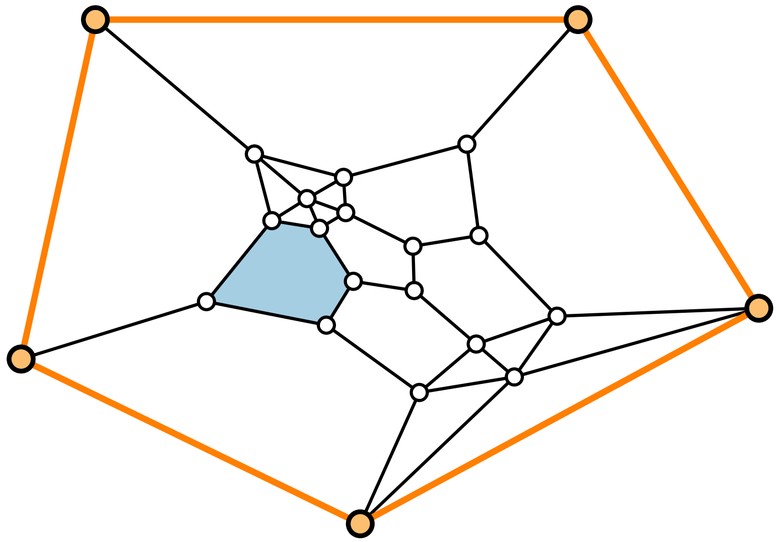
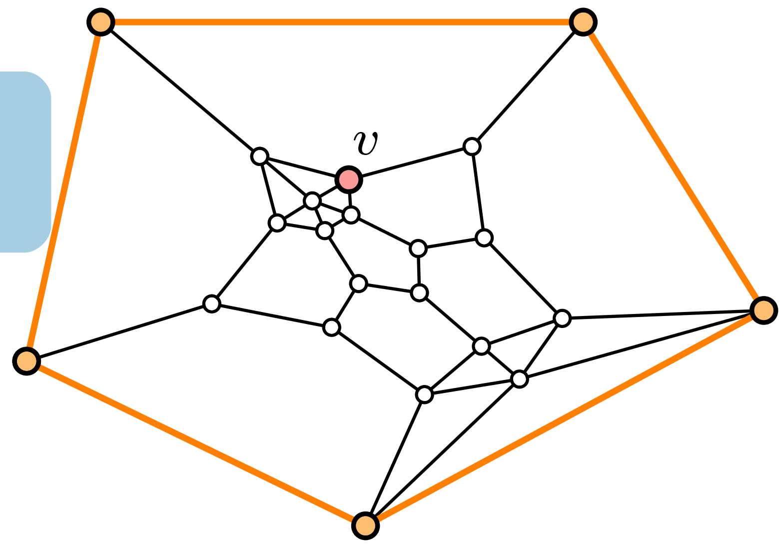


Properties of Tutte Drawings

Property 1. Let $v \in V$ free,



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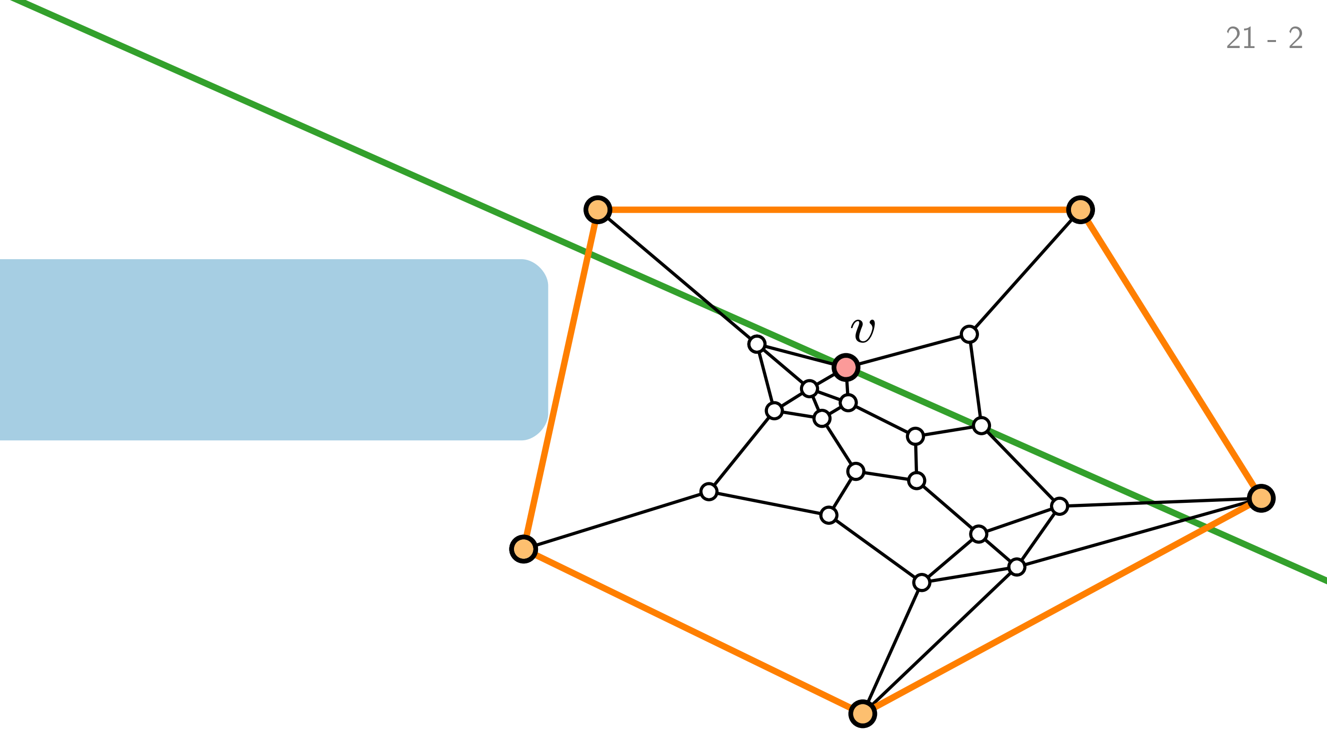

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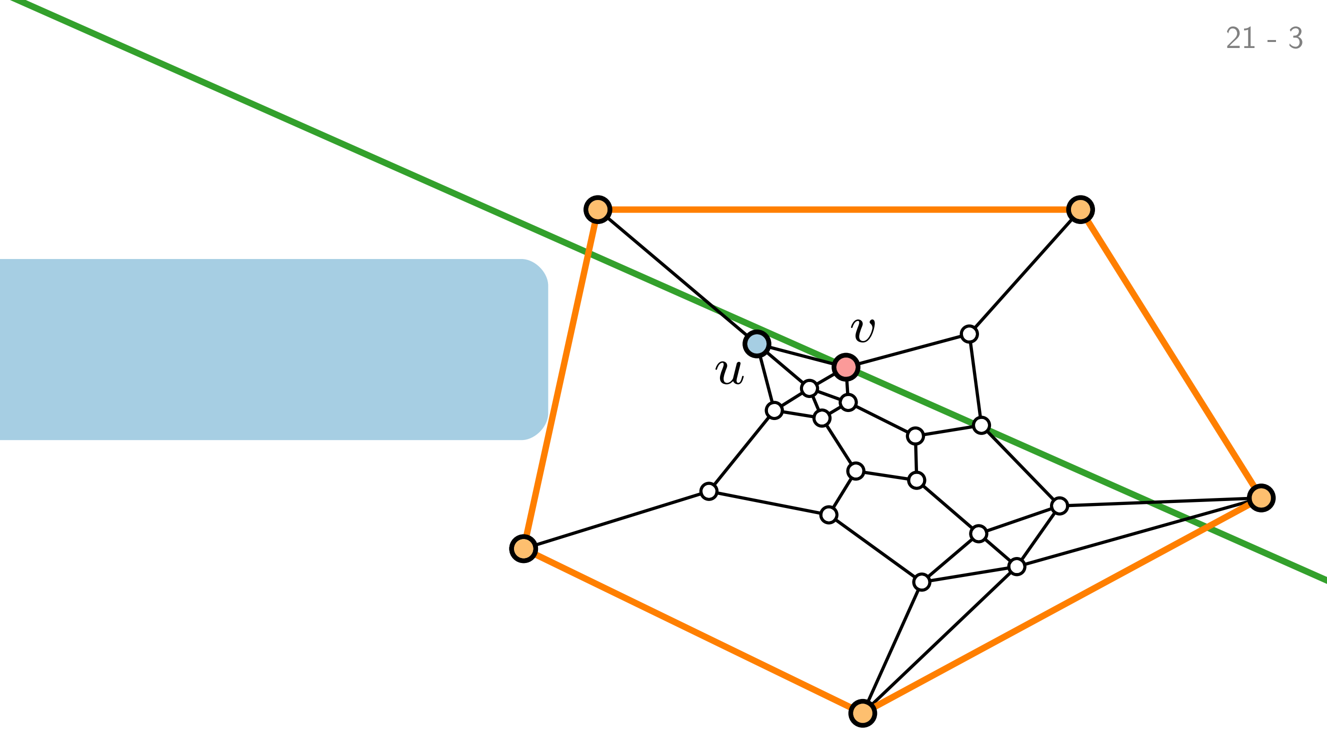

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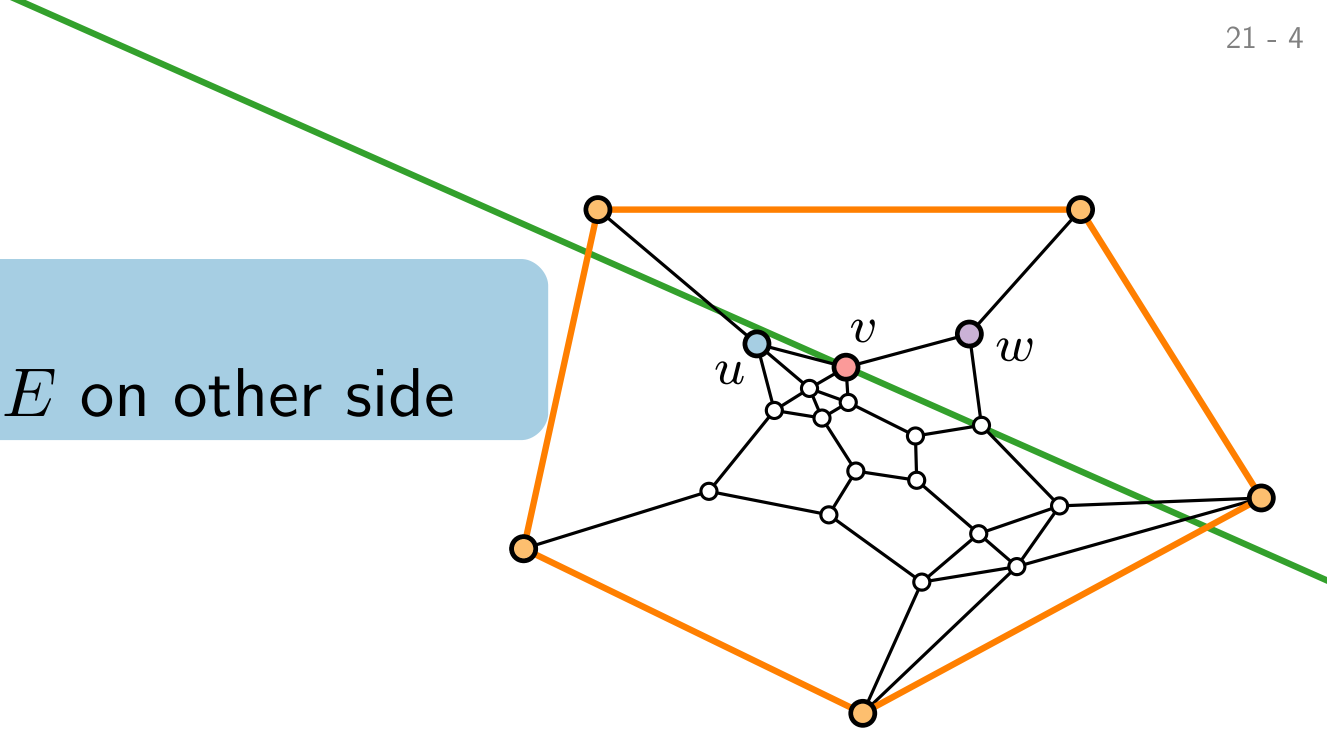

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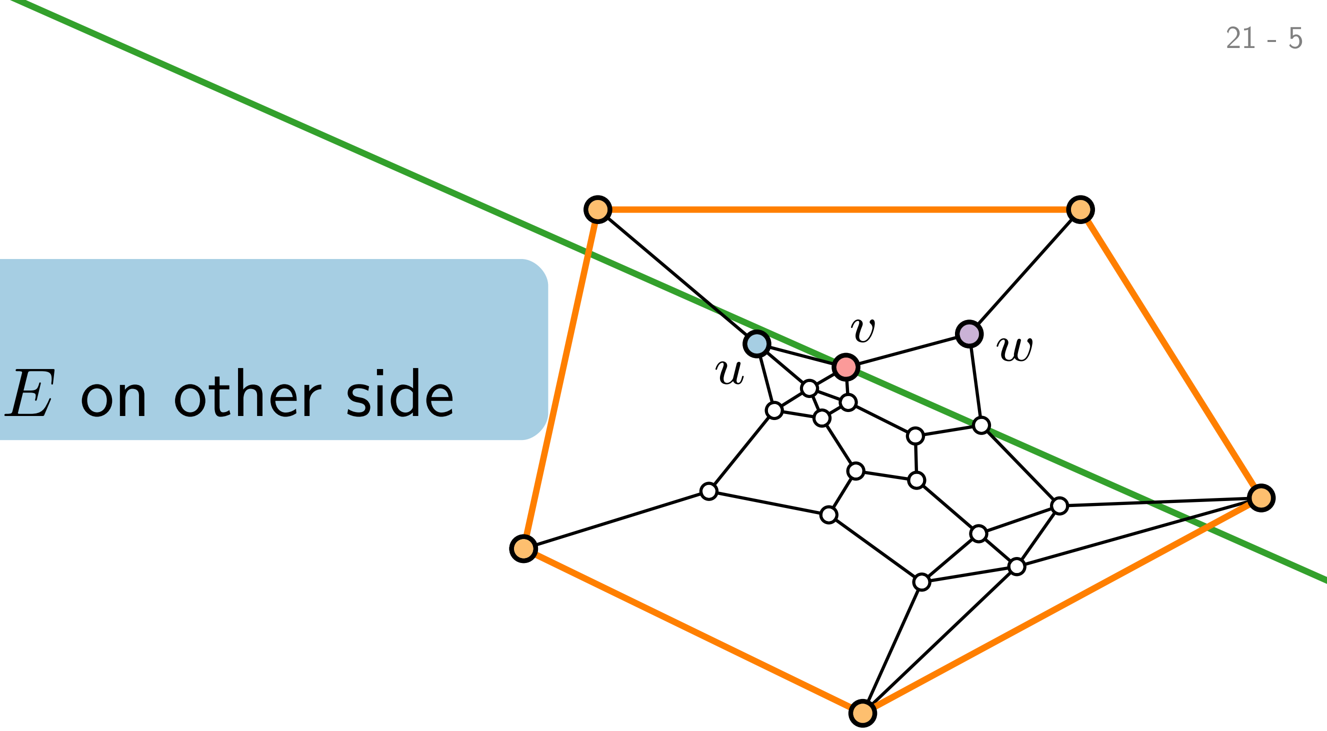
Otherwise, all forces to same side ...



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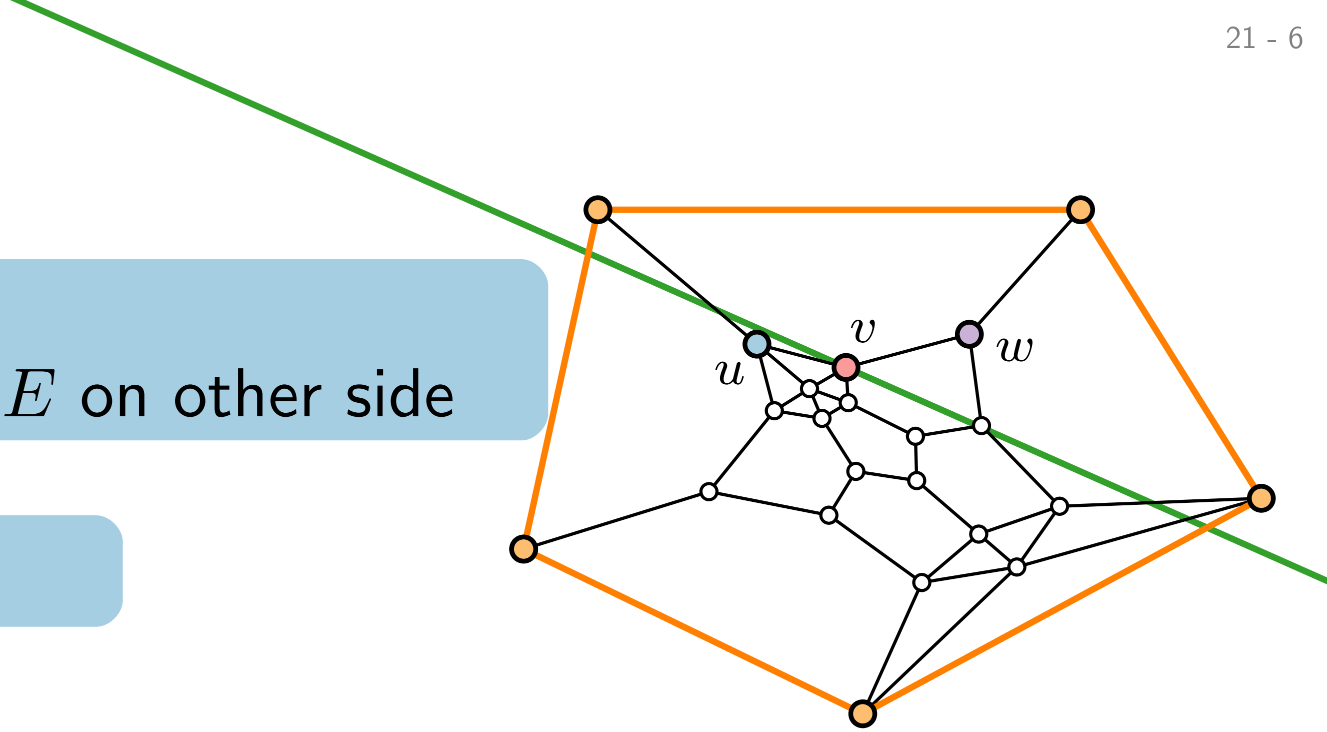

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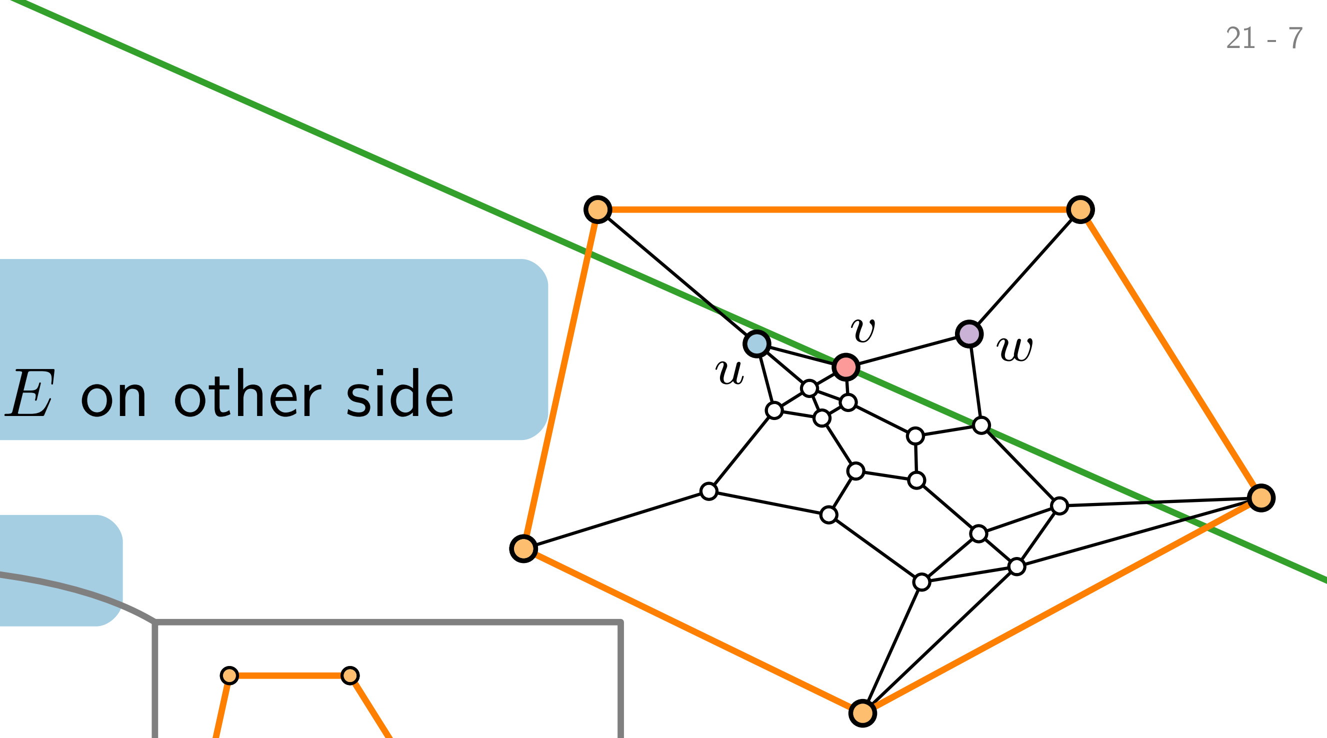

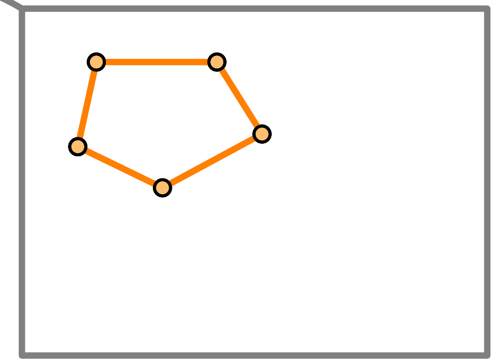

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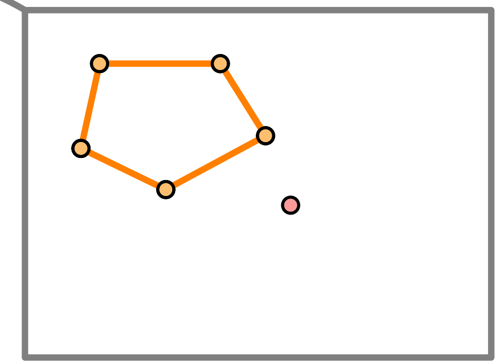

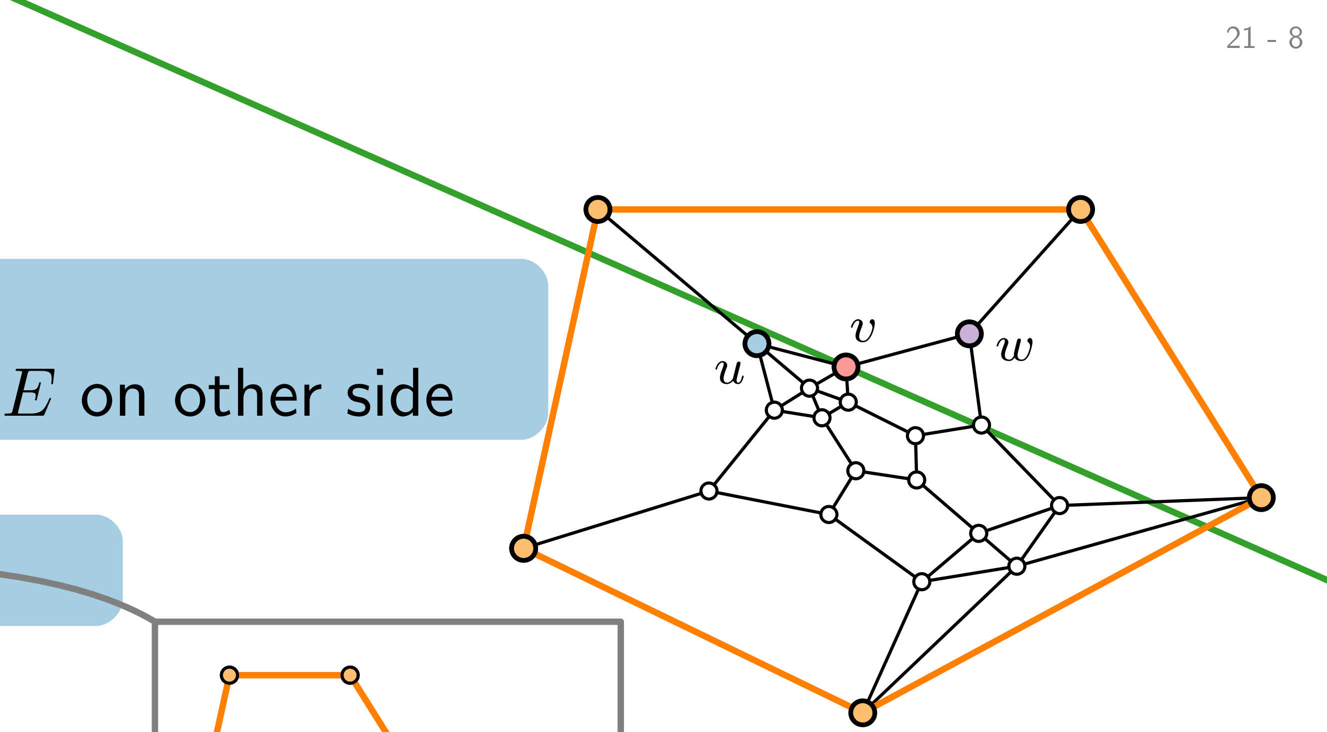

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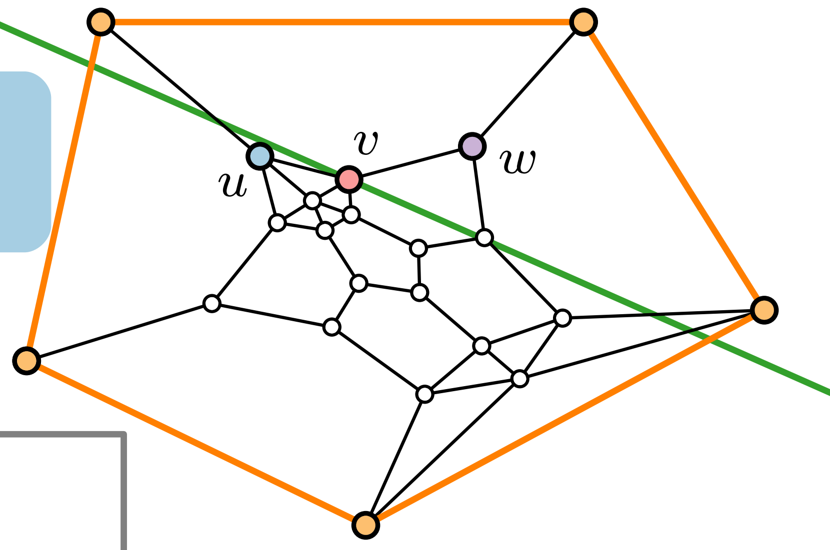

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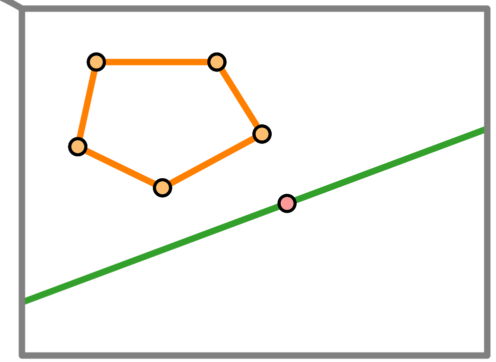

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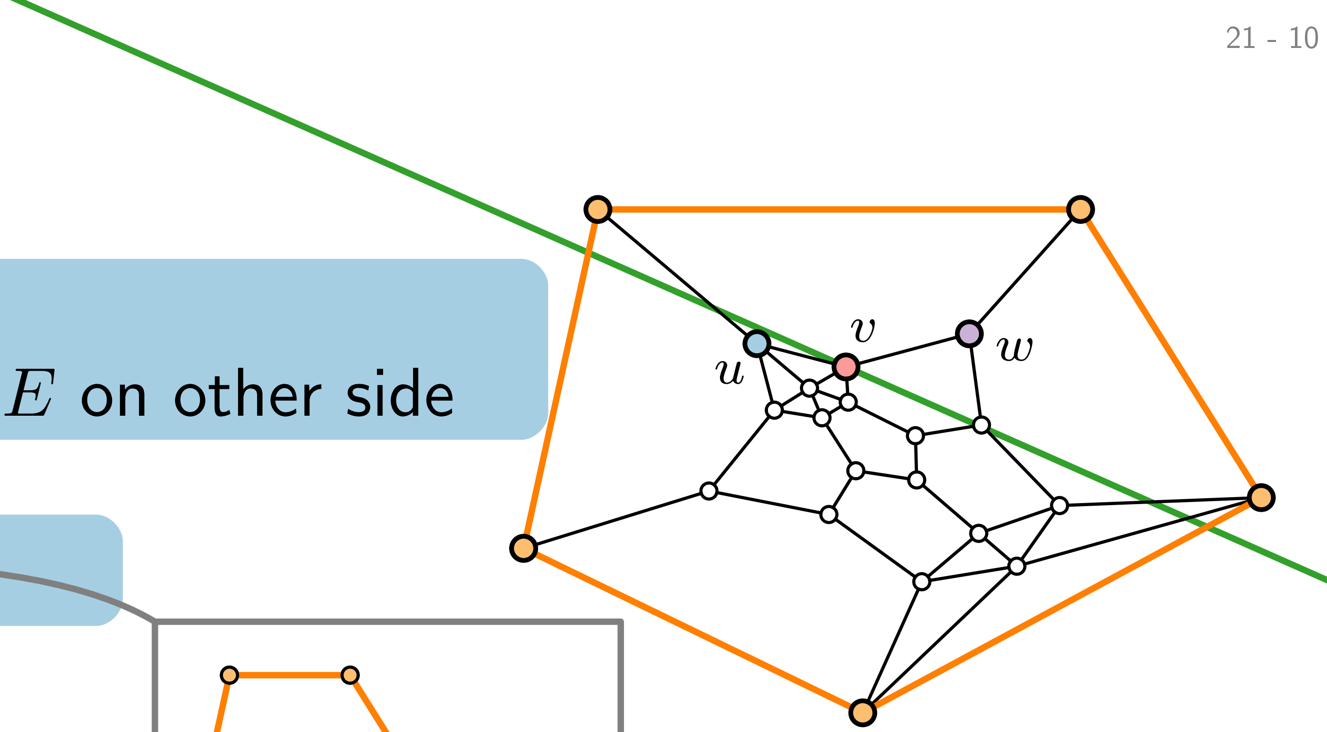

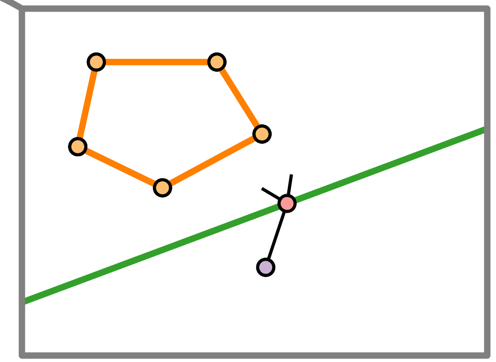

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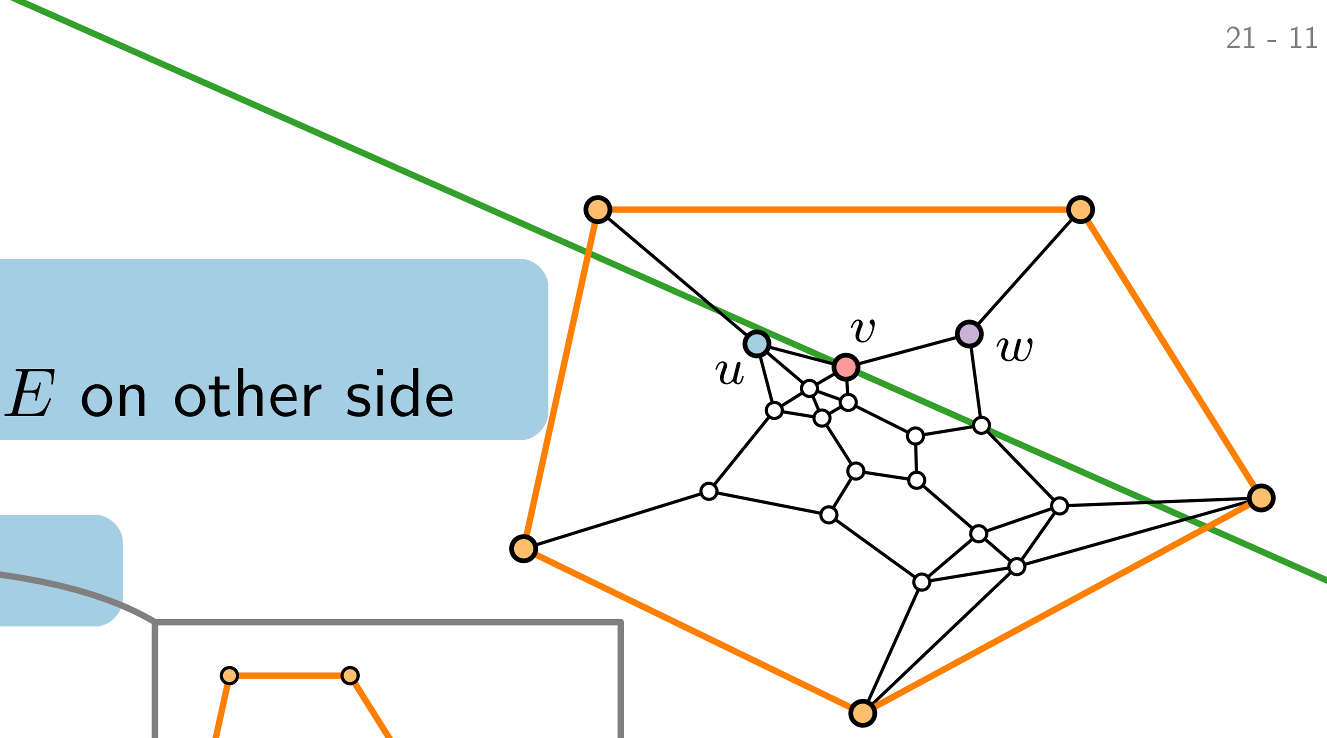

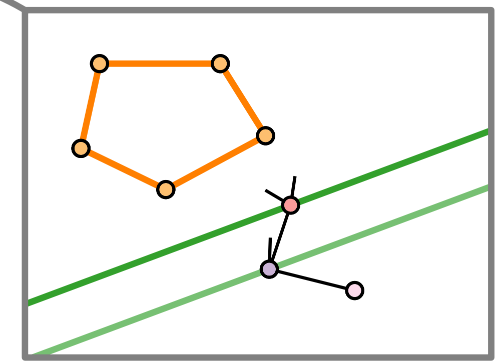

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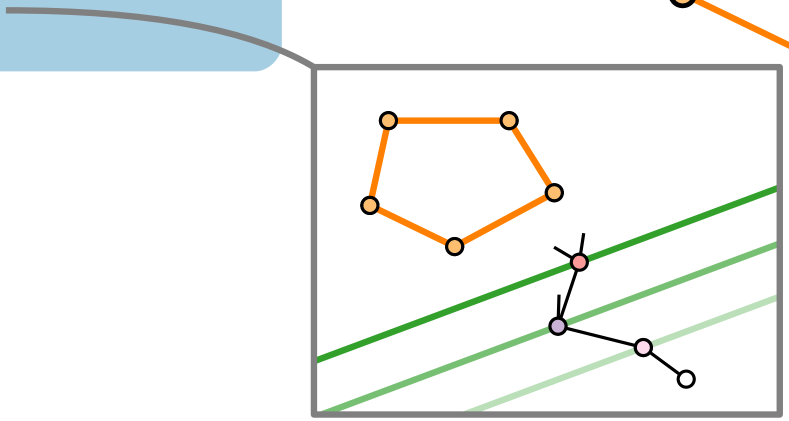

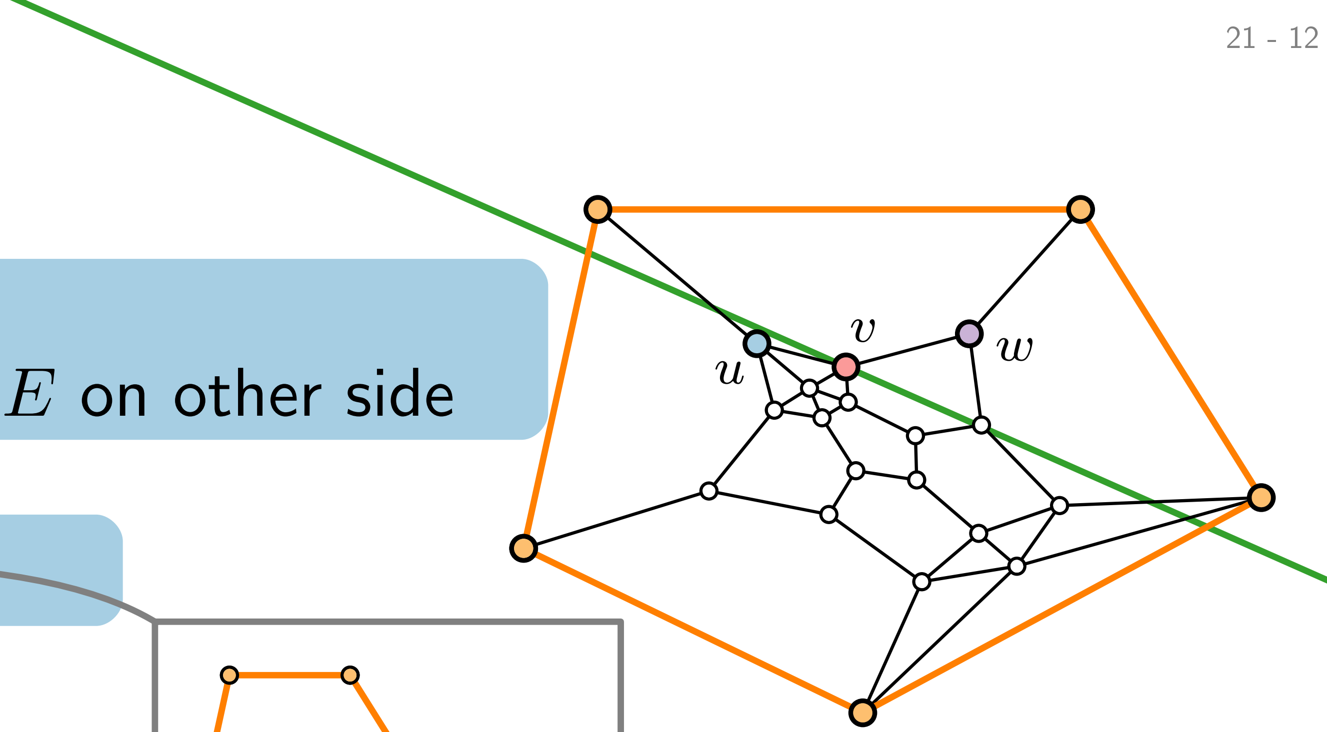

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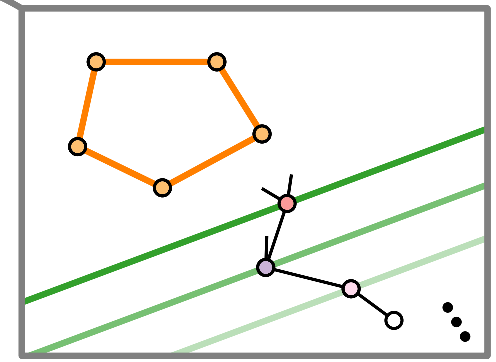
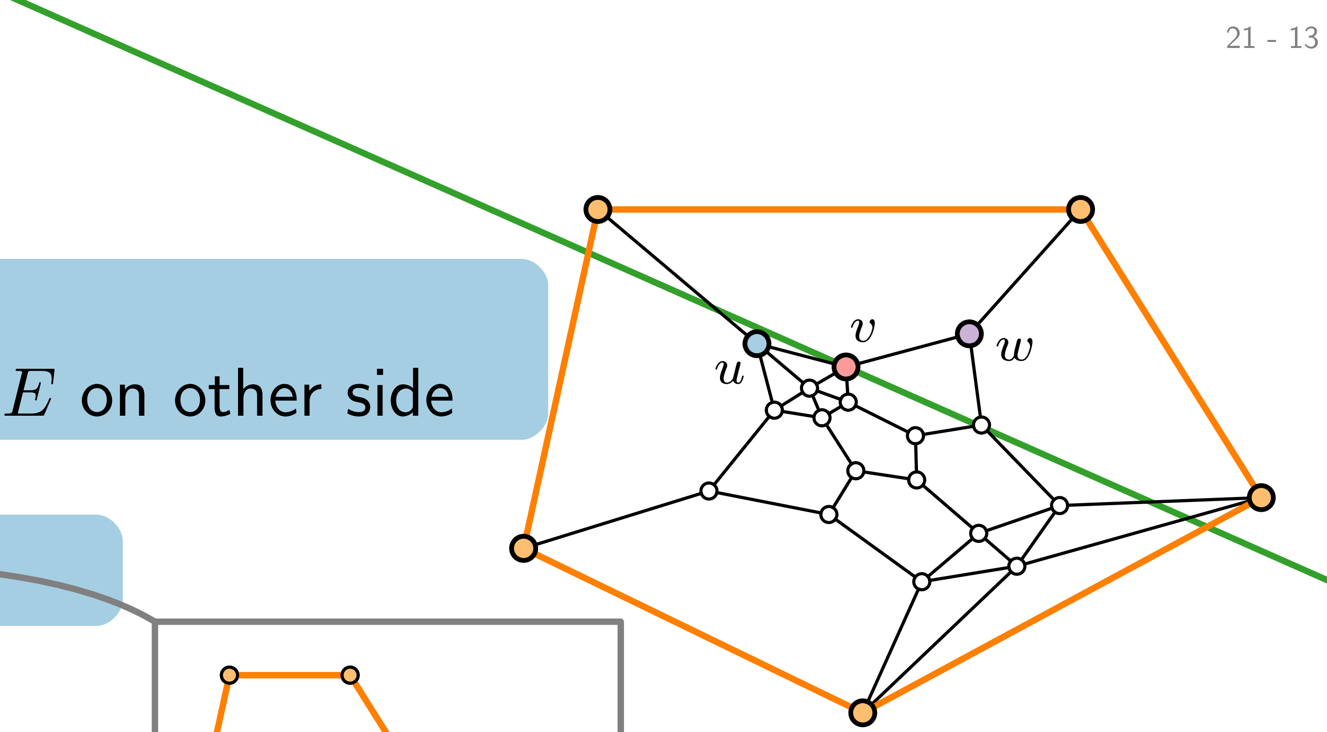
Property 1. Let $v \in V$ free, ℓ line through v.
$\exists uv \in E$ on one side of $\ell \Rightarrow \exists vw \in E$ on other side

Otherwise, all forces to same side . . .

Property 2. All free vertices lie inside C.





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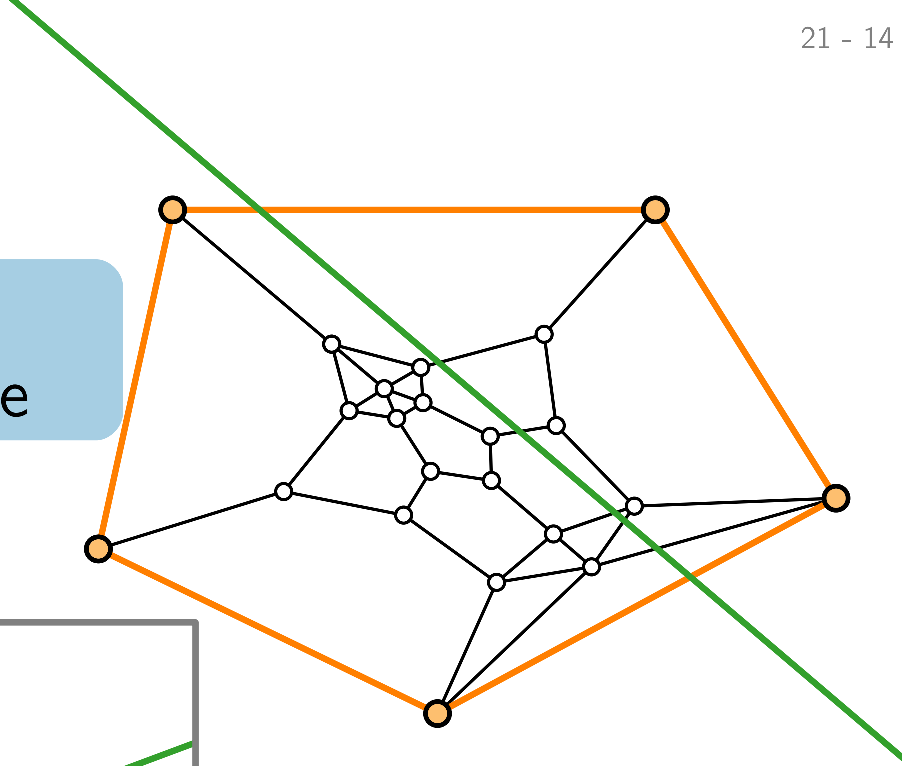

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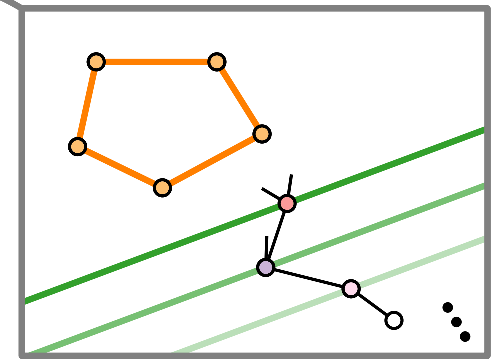

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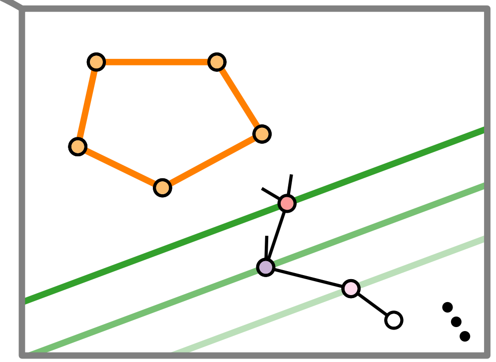

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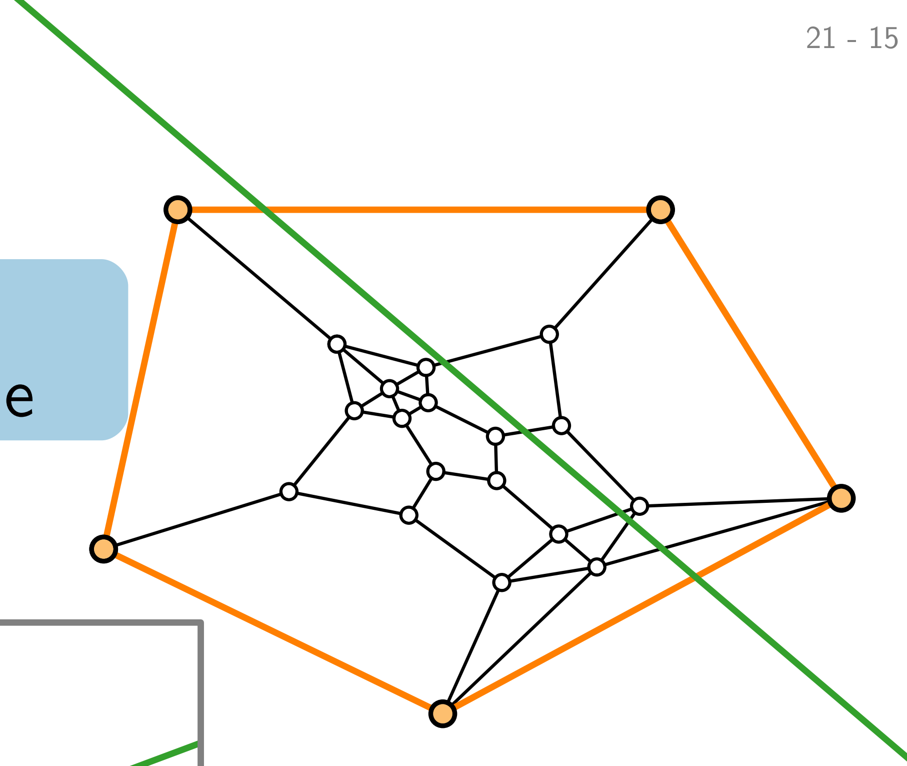

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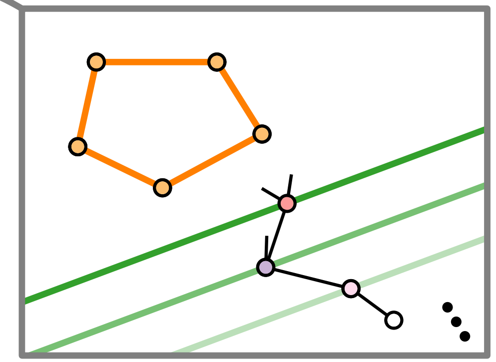

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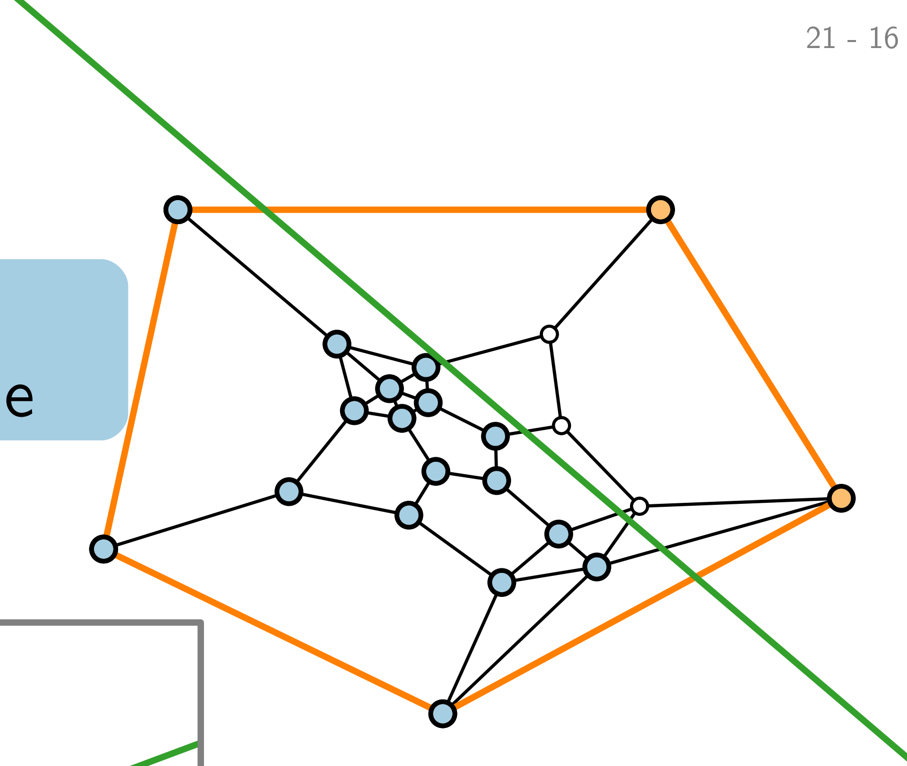

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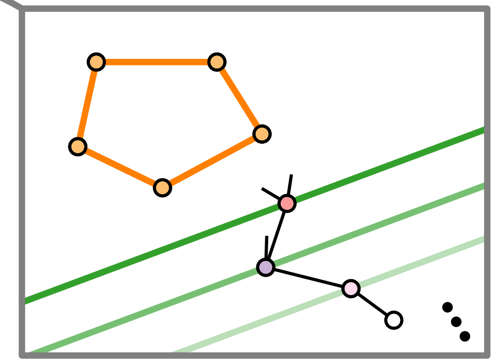
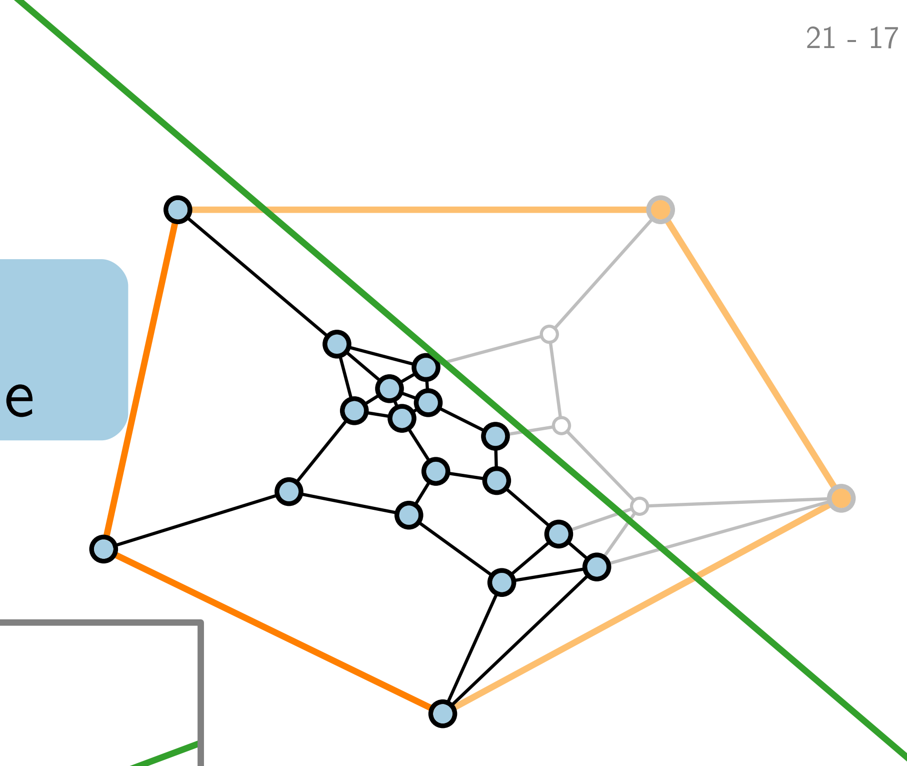
Then $G[V_\ell]$ is connected.





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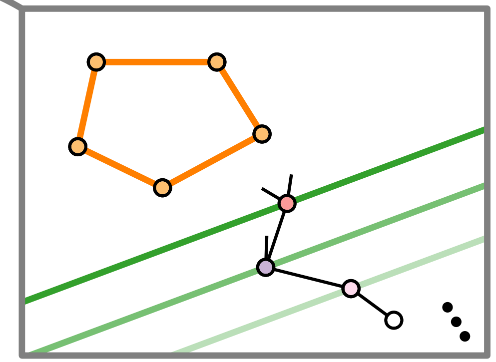

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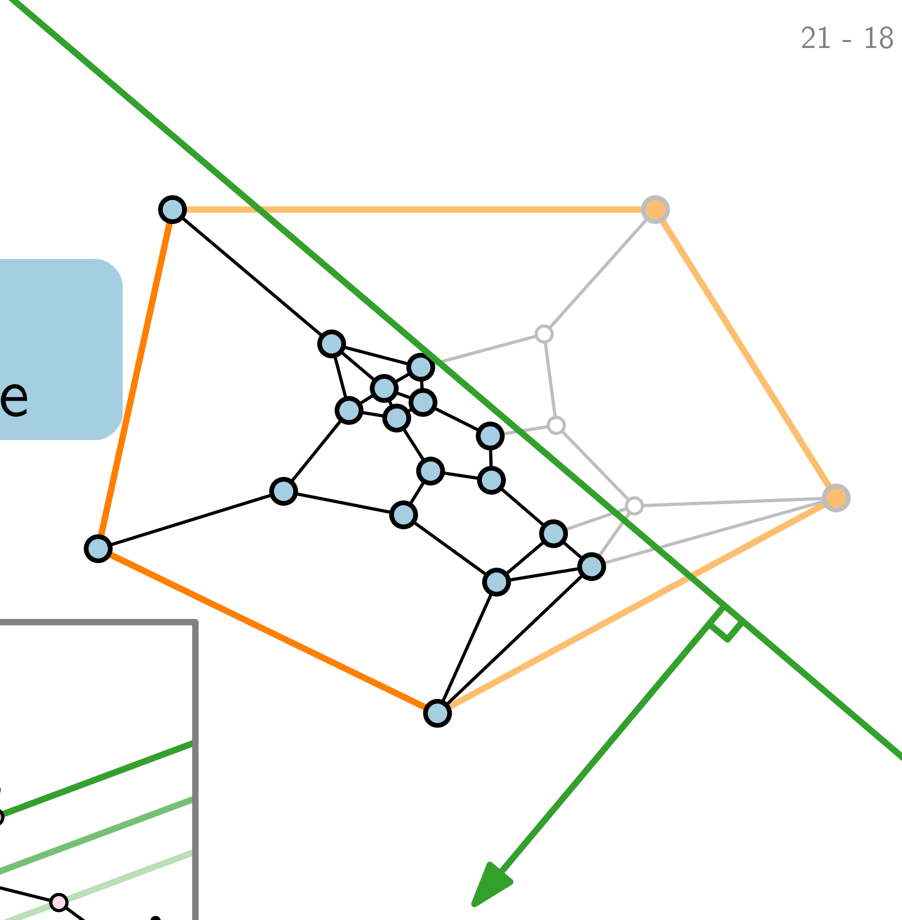

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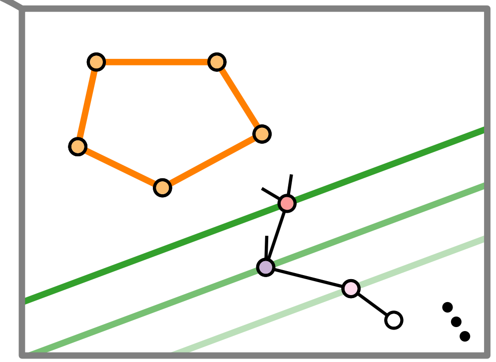
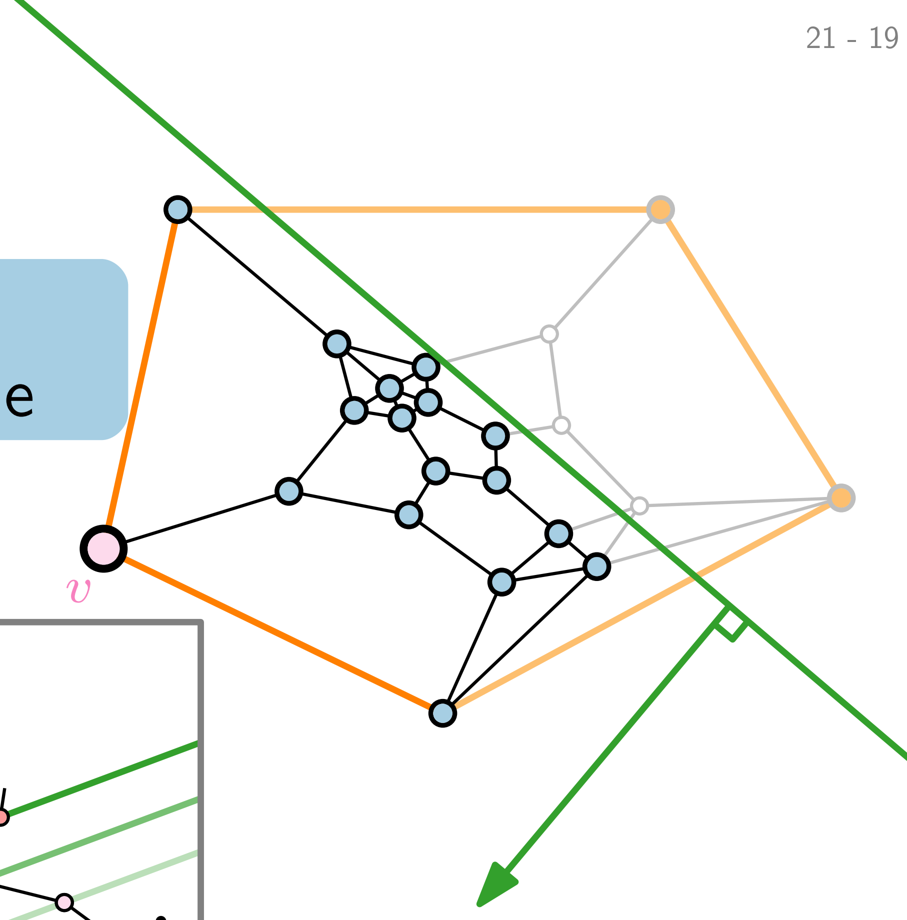
v furthest away from ℓ





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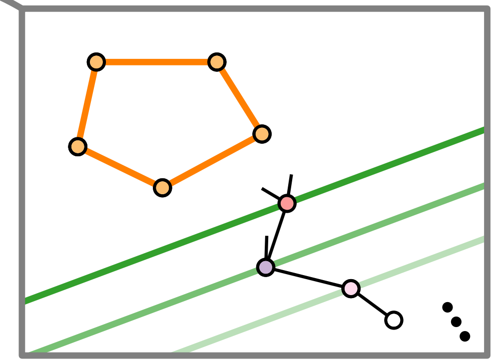

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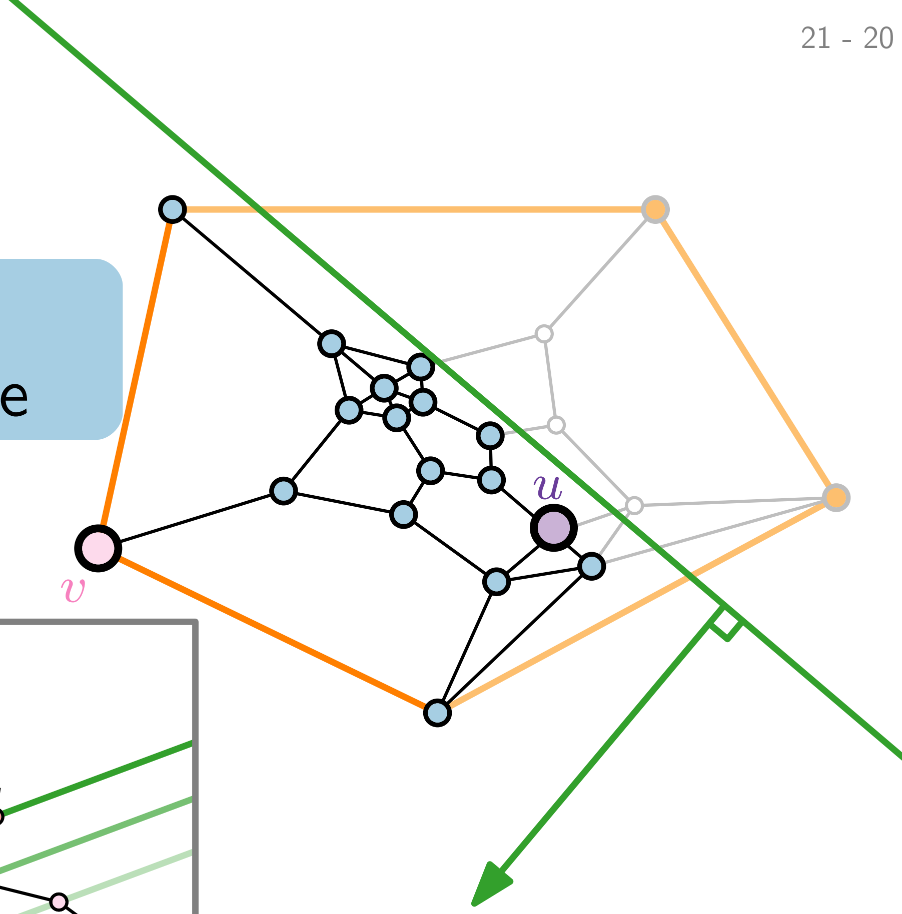
Let V_ℓ be all vertices on one side of ℓ.
Then $G[V_\ell]$ is connected.

v furthest away from ℓ
Pick any vertex $u \in V_\ell$



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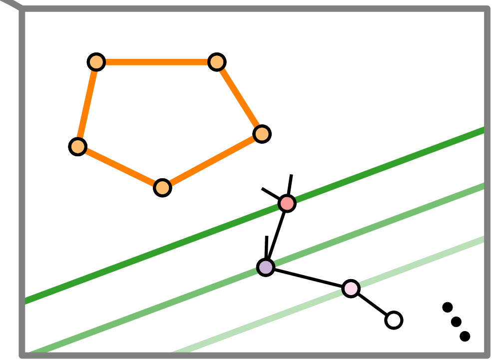

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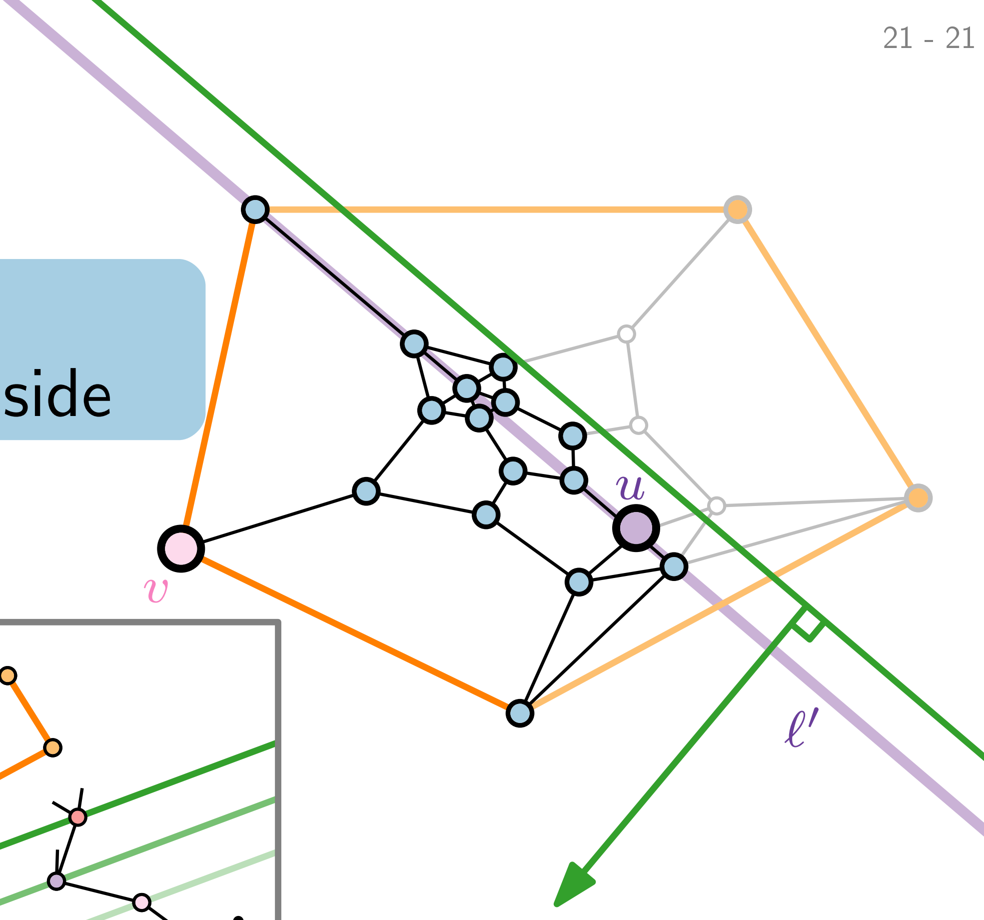
v furthest away from ℓ
Pick any vertex $u \in V_\ell$, ℓ' parallel to ℓ through u



Properties of Tutte Drawings

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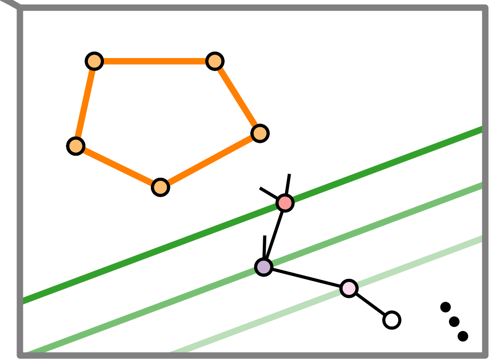
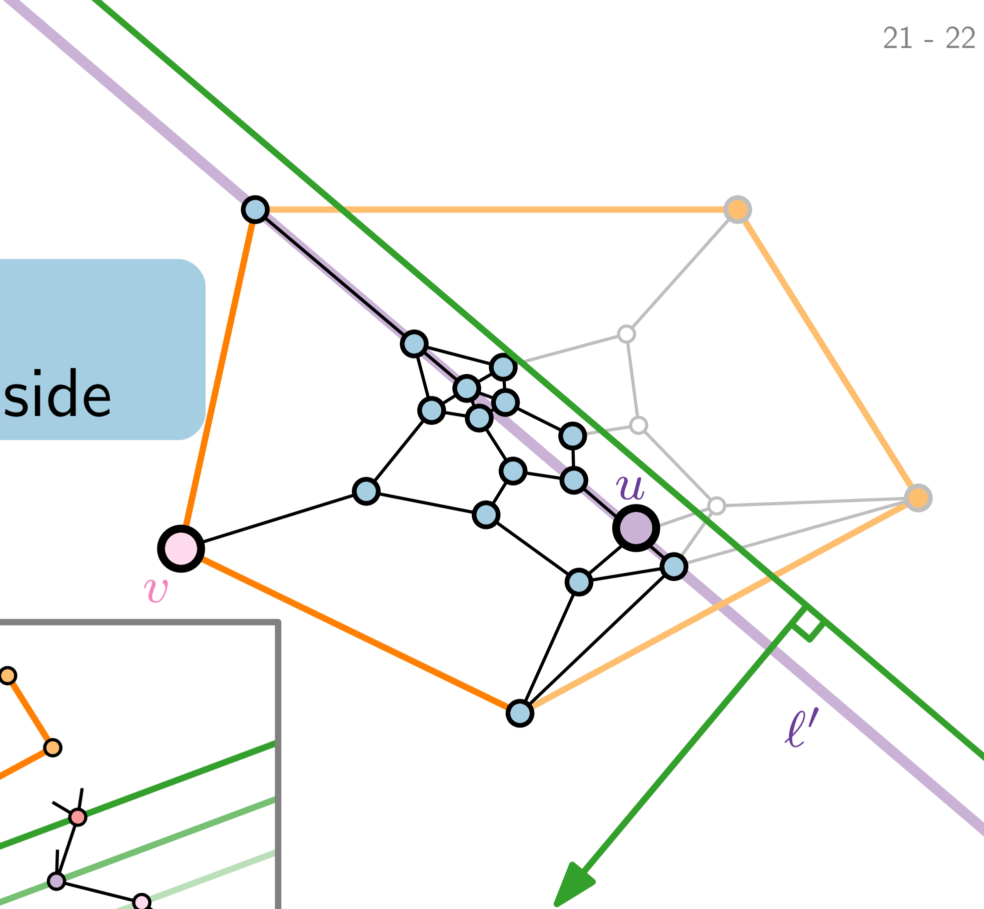
v furthest away from ℓ
Pick any vertex $u \in V_\ell$, ℓ' parallel to ℓ through u
G connected, v not on ℓ'





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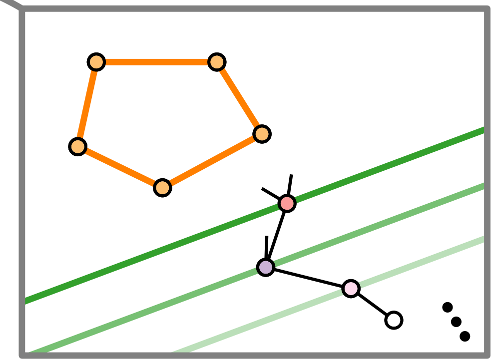

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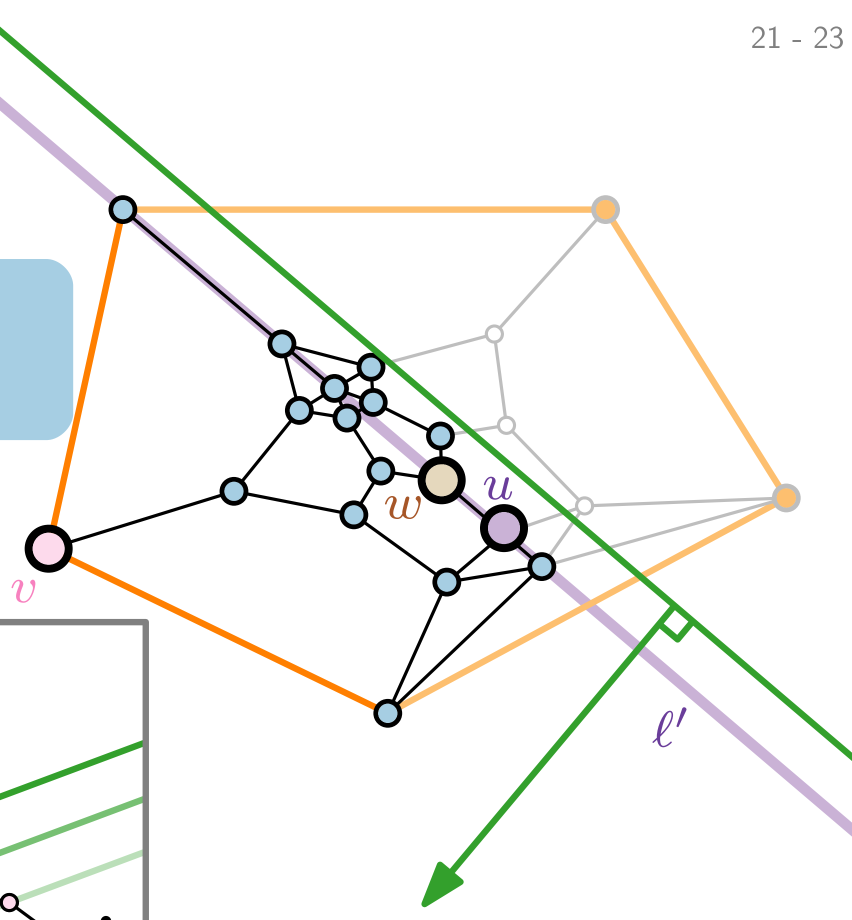

v furthest away from ℓ
Pick any vertex $u \in V_\ell$, ℓ' parallel to ℓ through u
G connected, v not on $\ell' \Rightarrow \exists$ neighbor $w \in V_\ell$ of u on the same side of ℓ' as v

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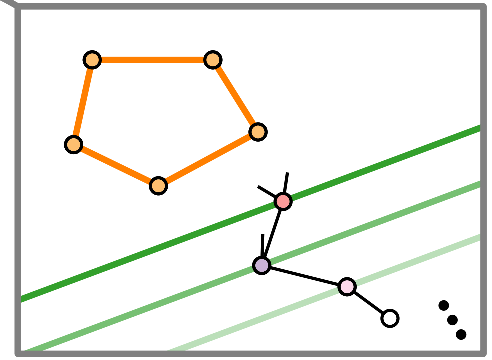

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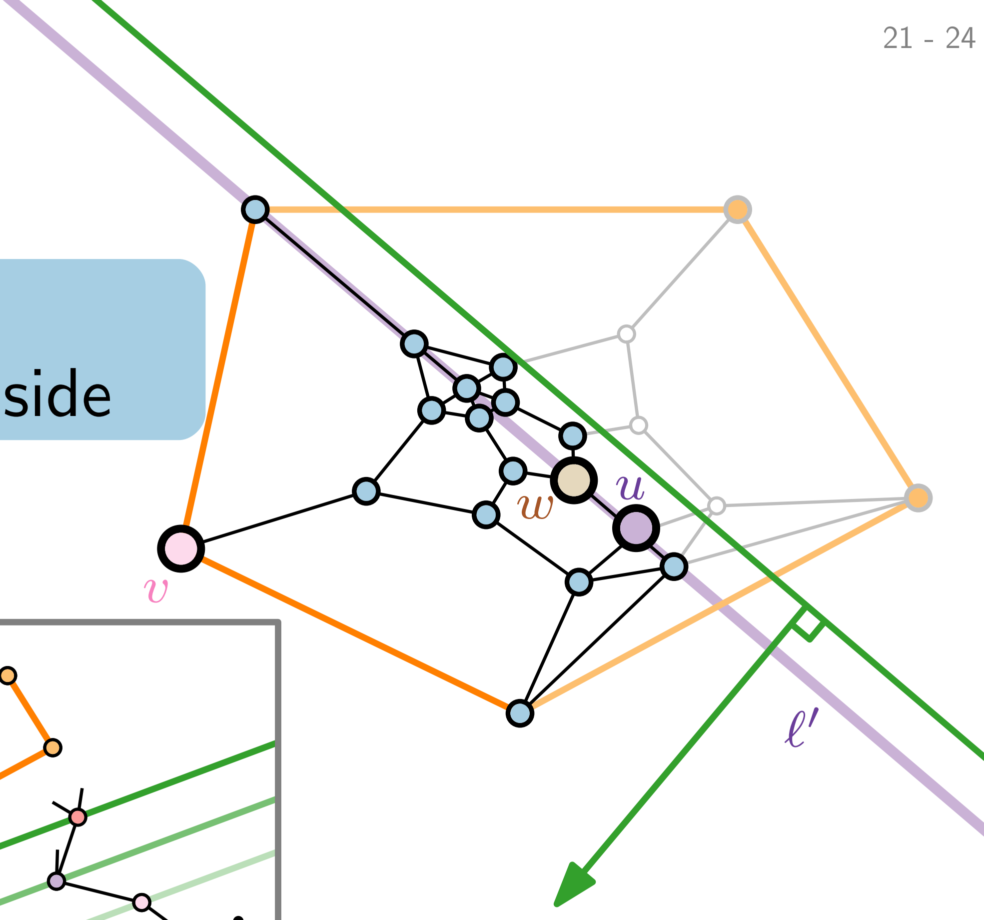

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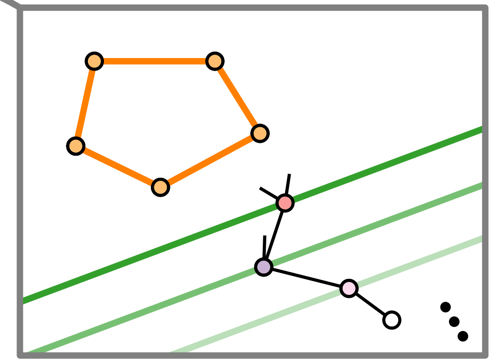

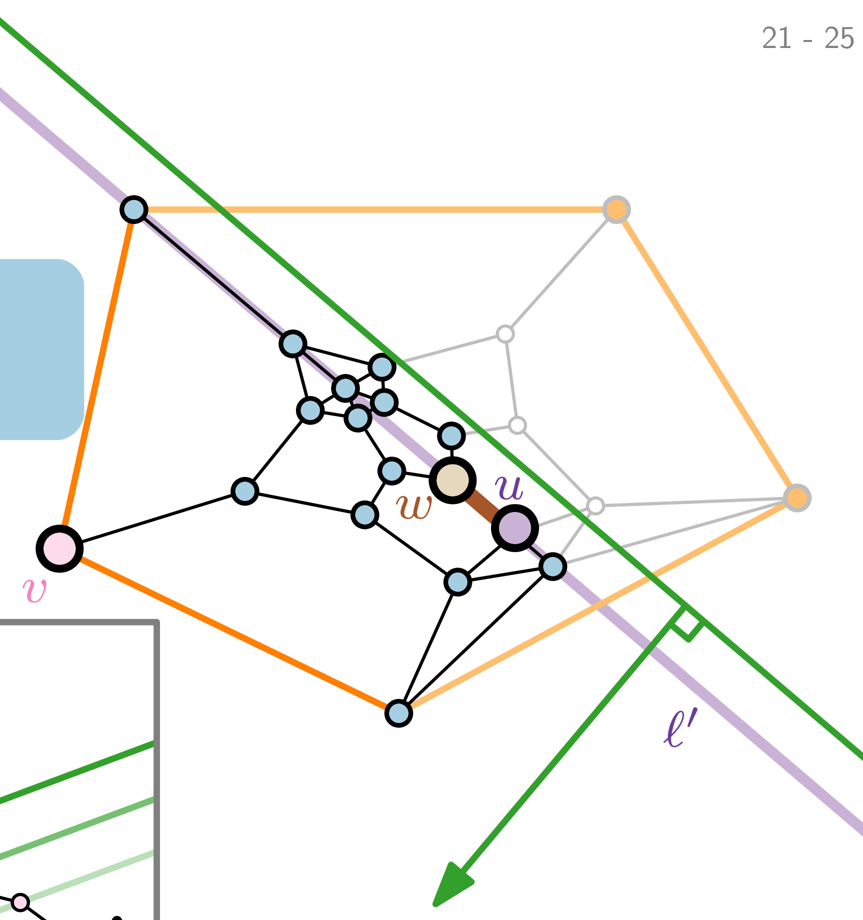

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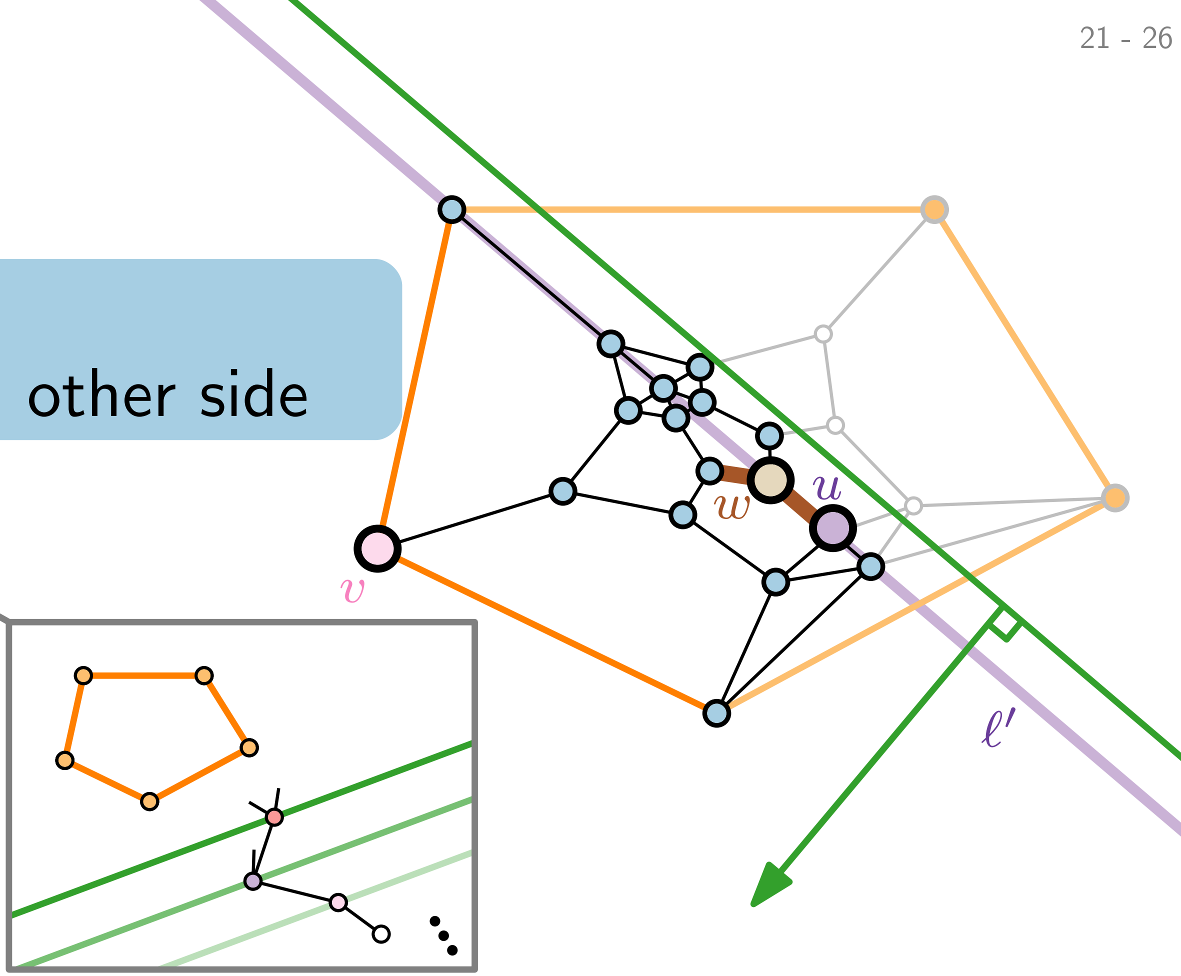

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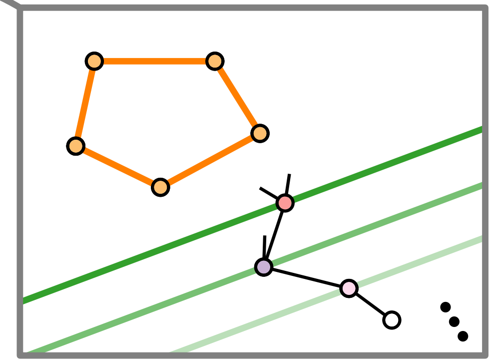

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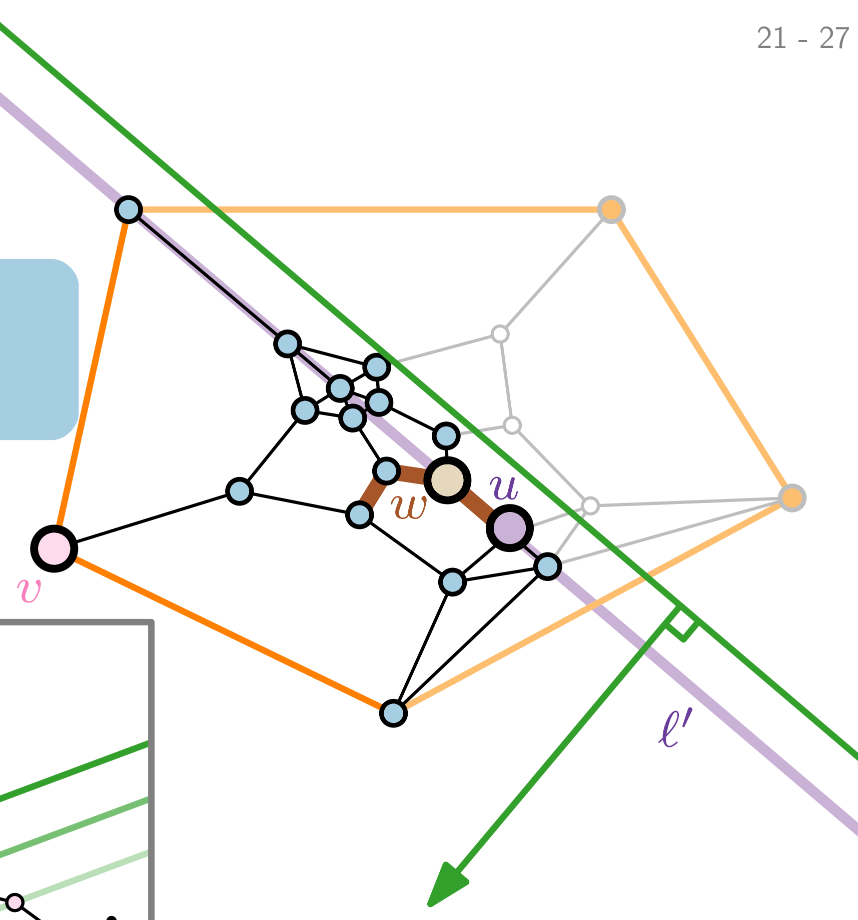

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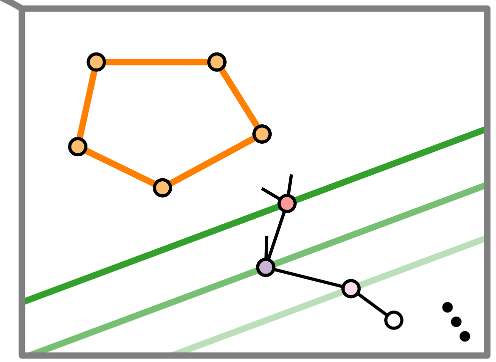

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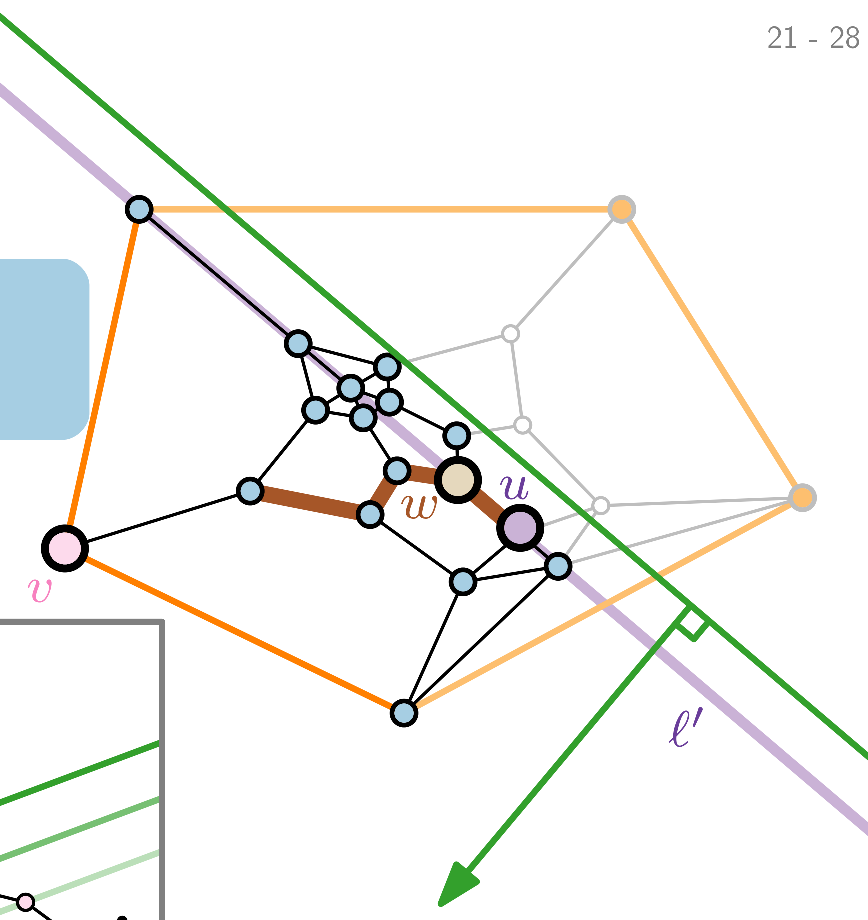

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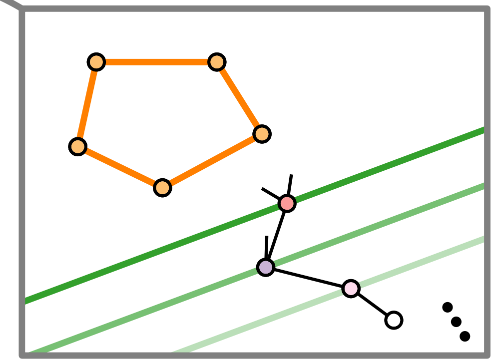

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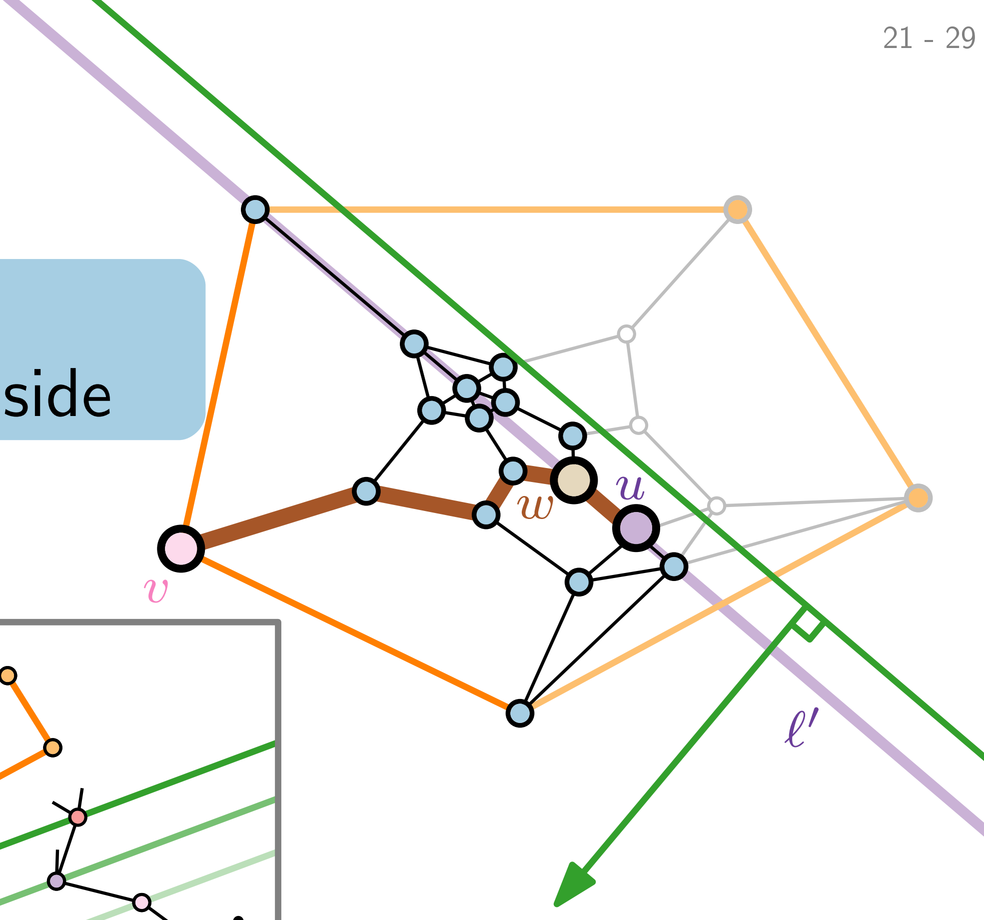

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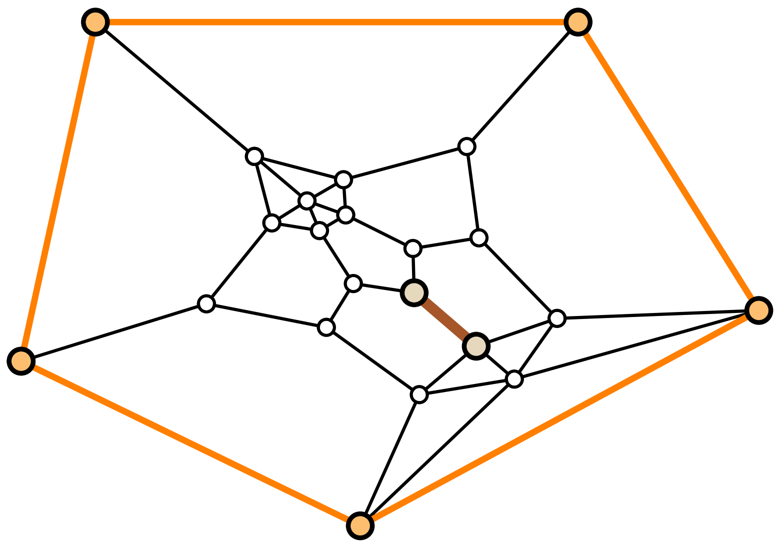
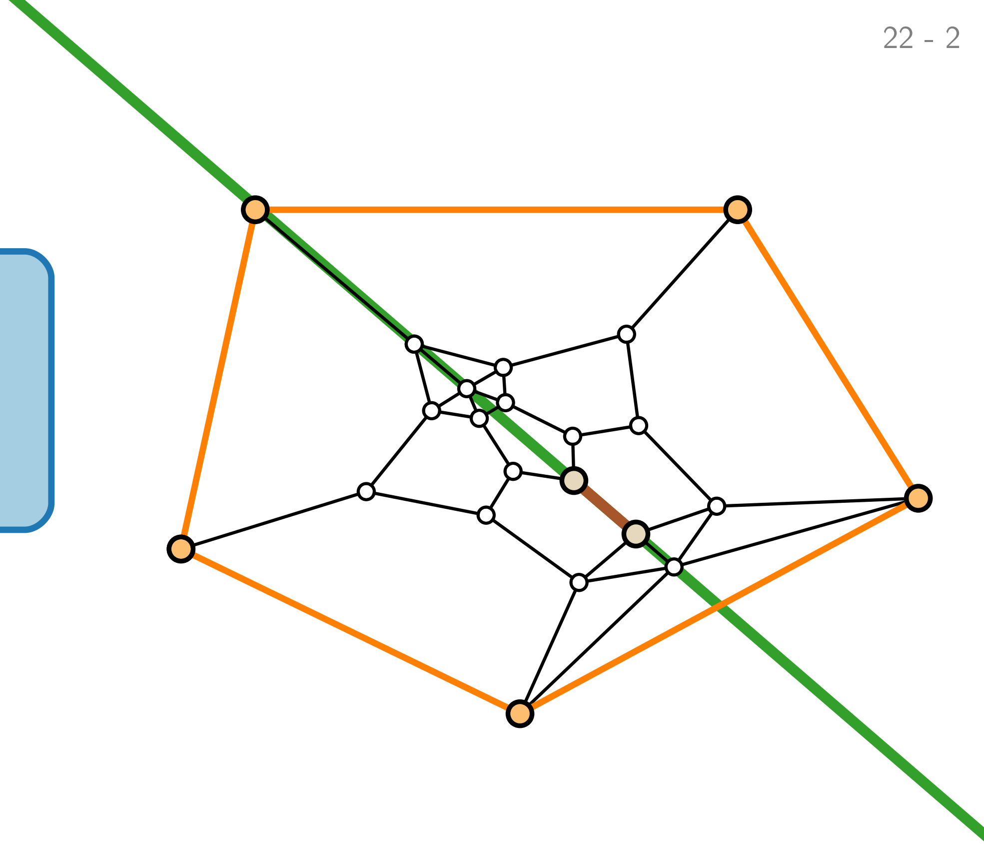
Proof of Tutte's Theorem

Lemma. Let uv be a non-boundary edge,



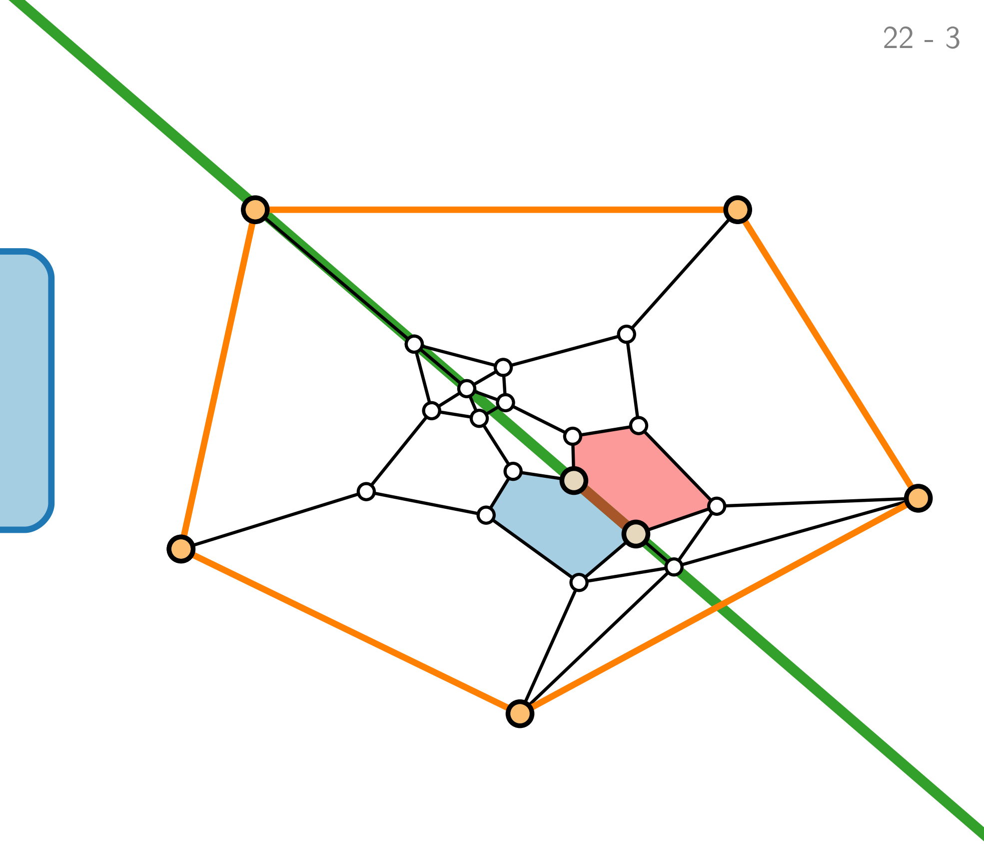
Proof of Tutte's Theorem

Lemma. Let uv be a non-boundary edge, ℓ line through uv.



Proof of Tutte's Theorem

Lemma. Let uv be a non-boundary edge, ℓ line through uv. Then the two faces f_1, f_2 incident to uv lie completely on opposite sides of ℓ.



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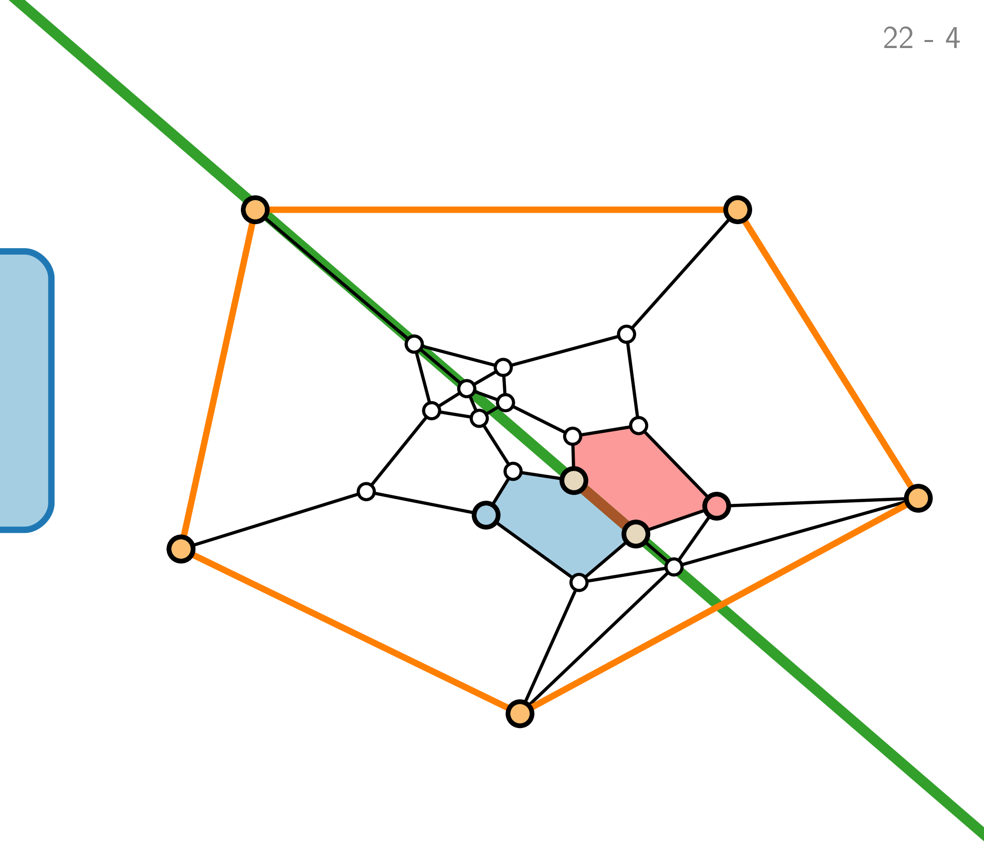

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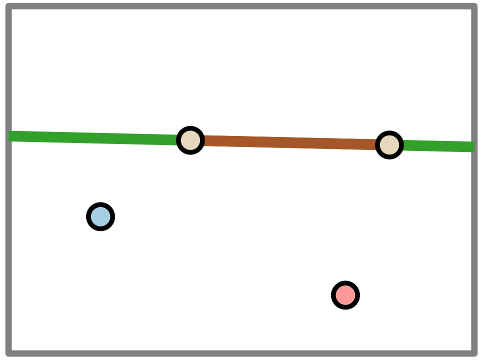

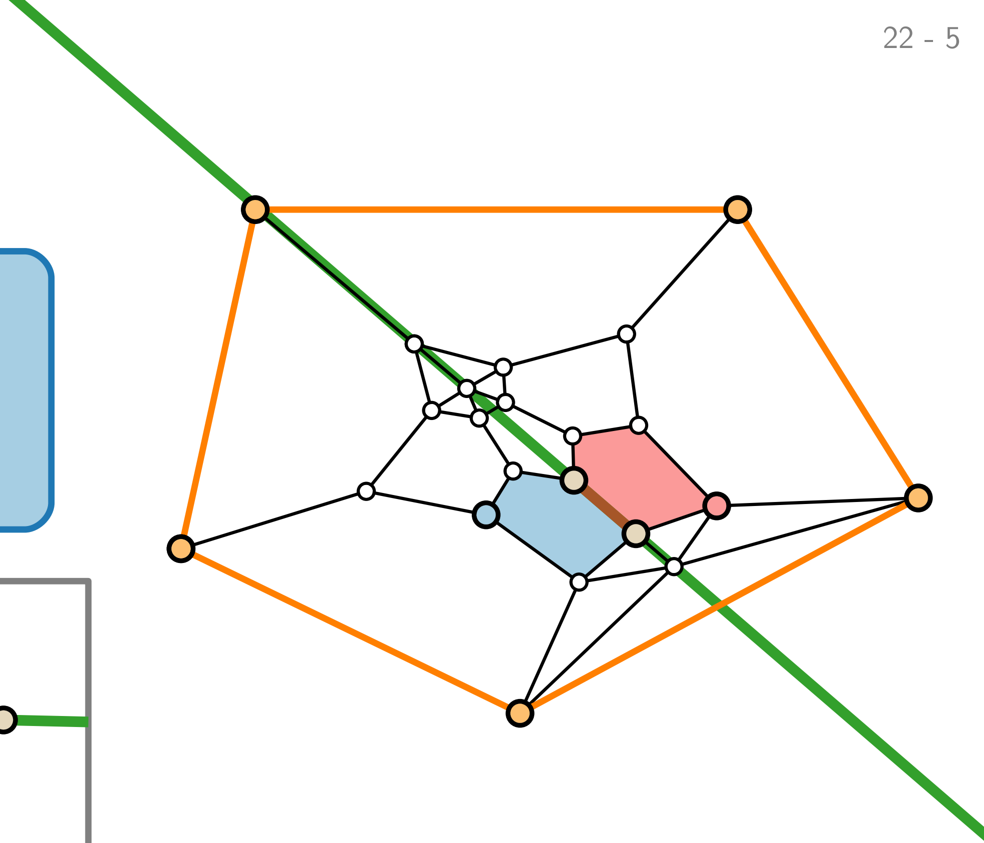

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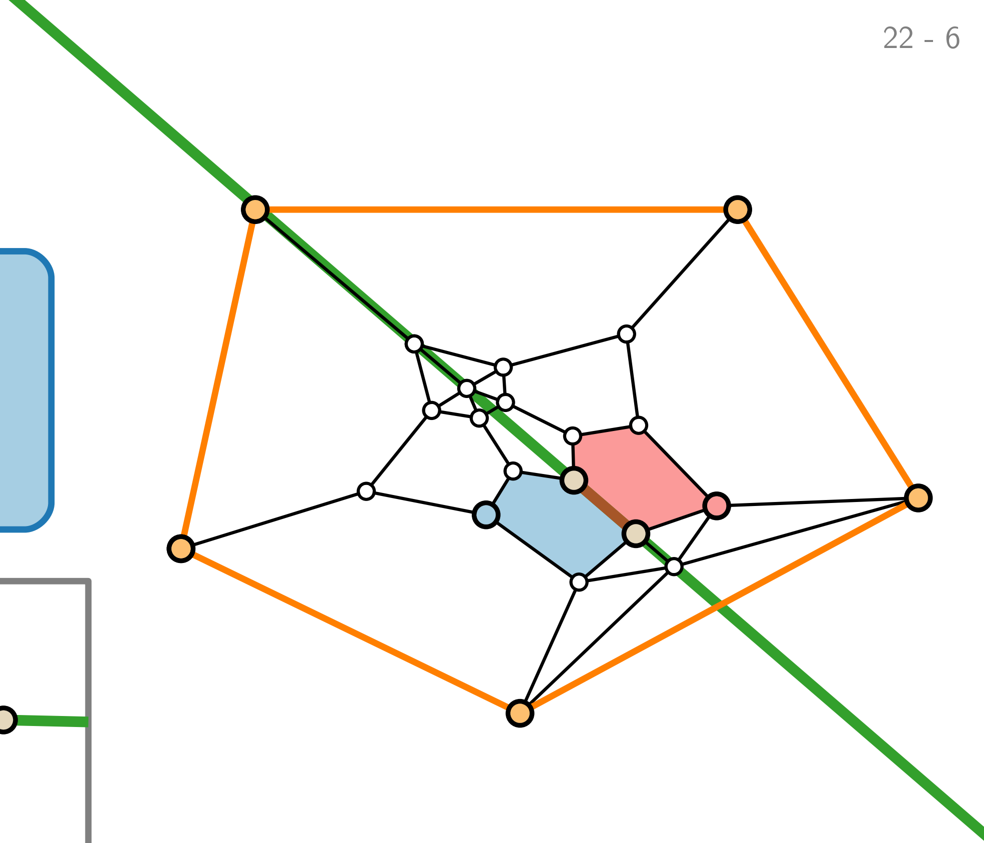

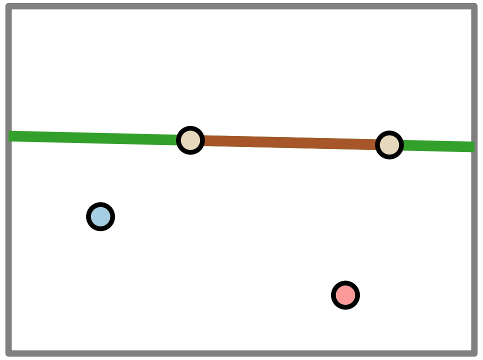

Property 3. Let ℓ be any line.
Let V_ℓ be the set of vertices on one side of ℓ.
Then $G[V_\ell]$ is connected.

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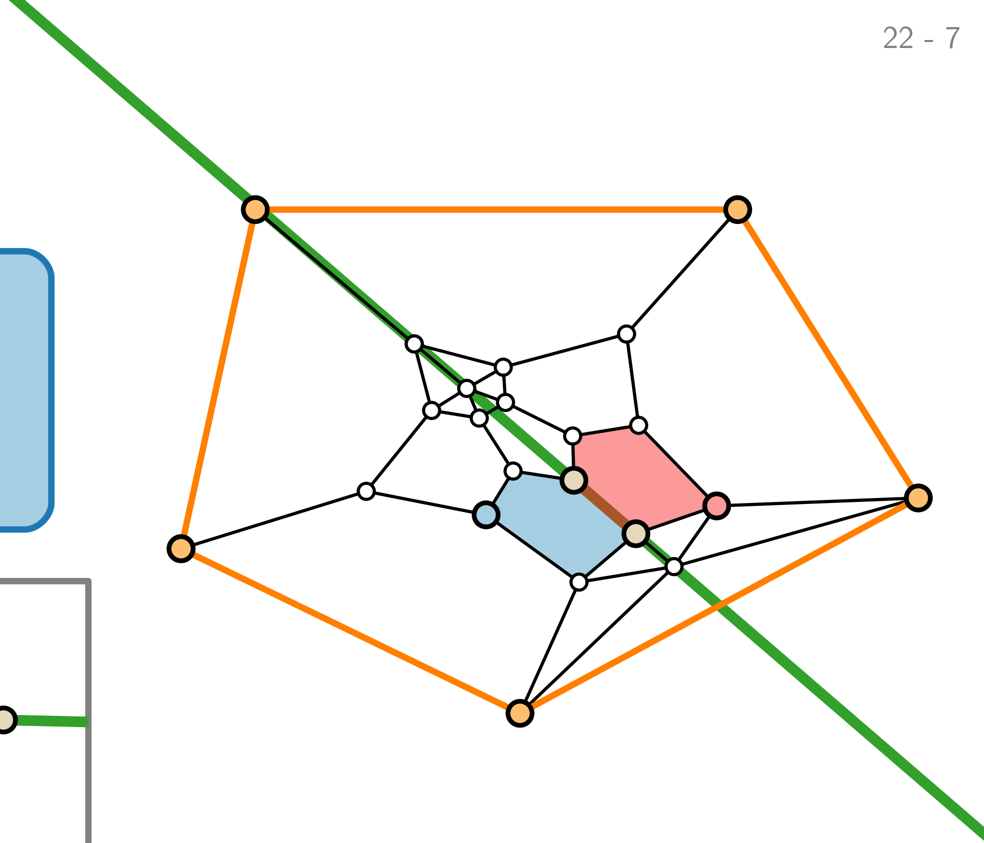

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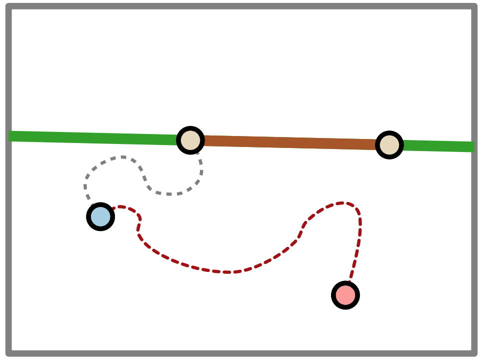

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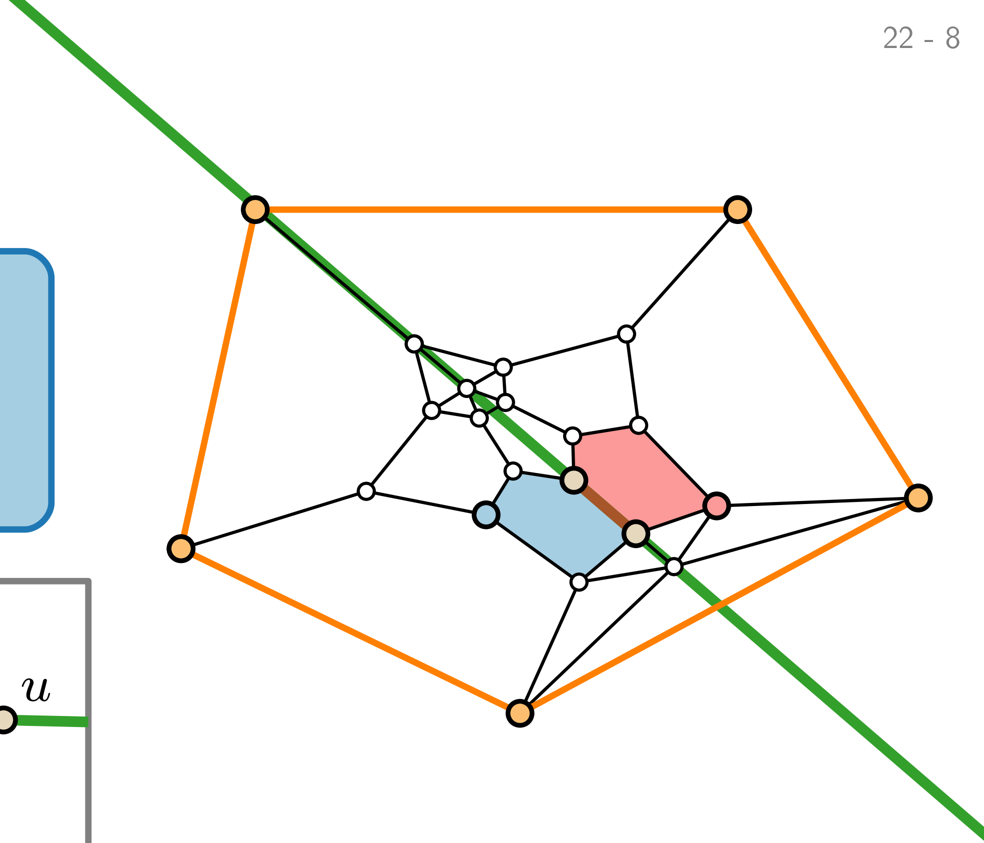

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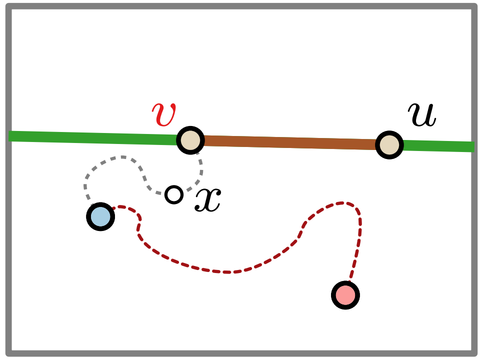
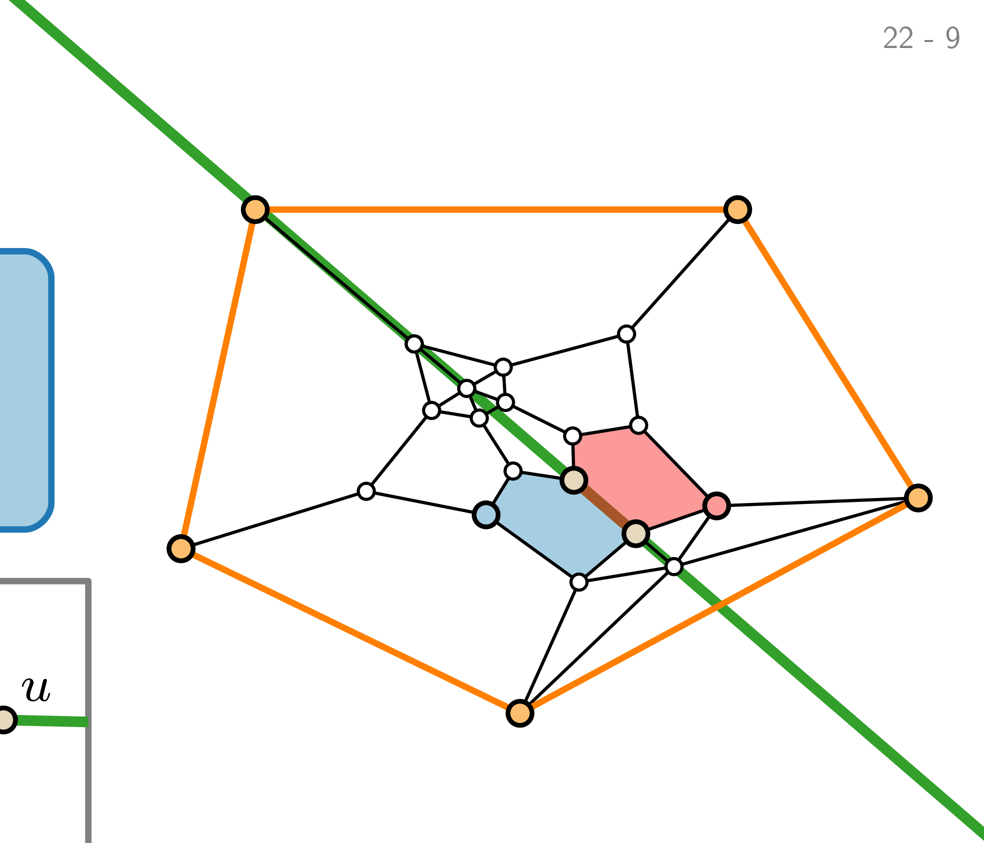


Proof of Tutte's Theorem



Lemma. Let uv be a non-boundary edge, ℓ line through uv. Then the two faces f_1, f_2 incident to uv lie completely on opposite sides of ℓ.

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Property 3. Let ℓ be any line.
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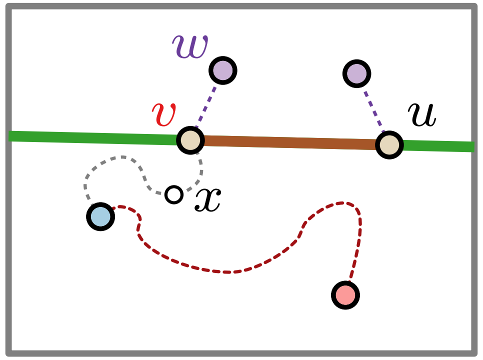
Then $G[V_\ell]$ is connected.



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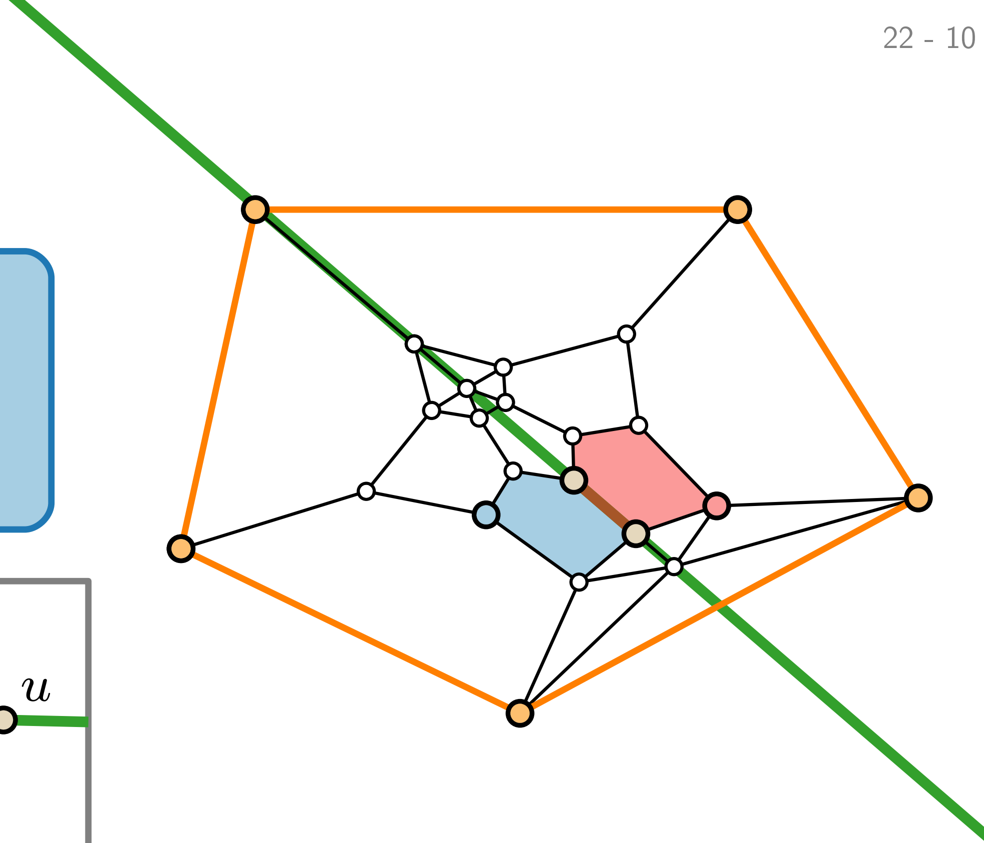

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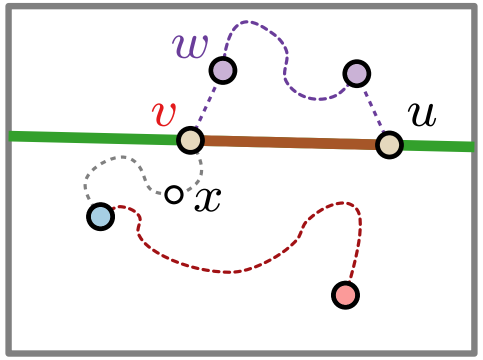

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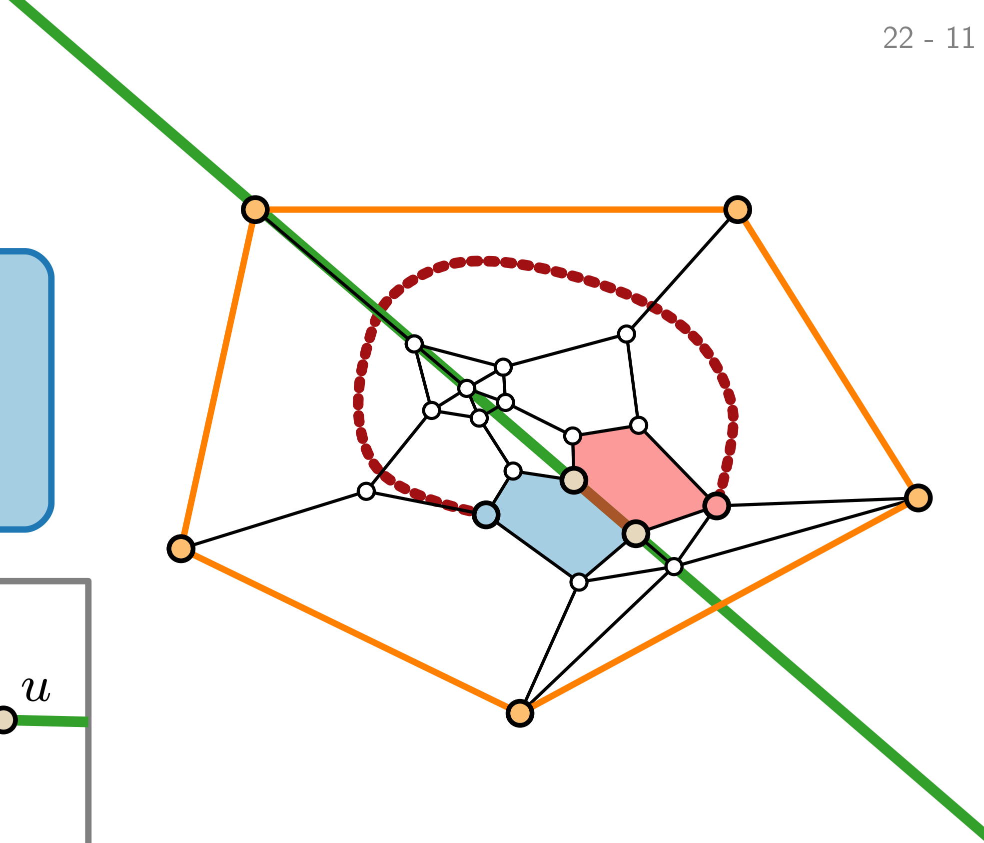

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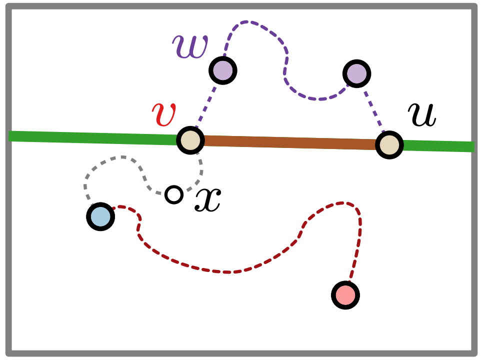

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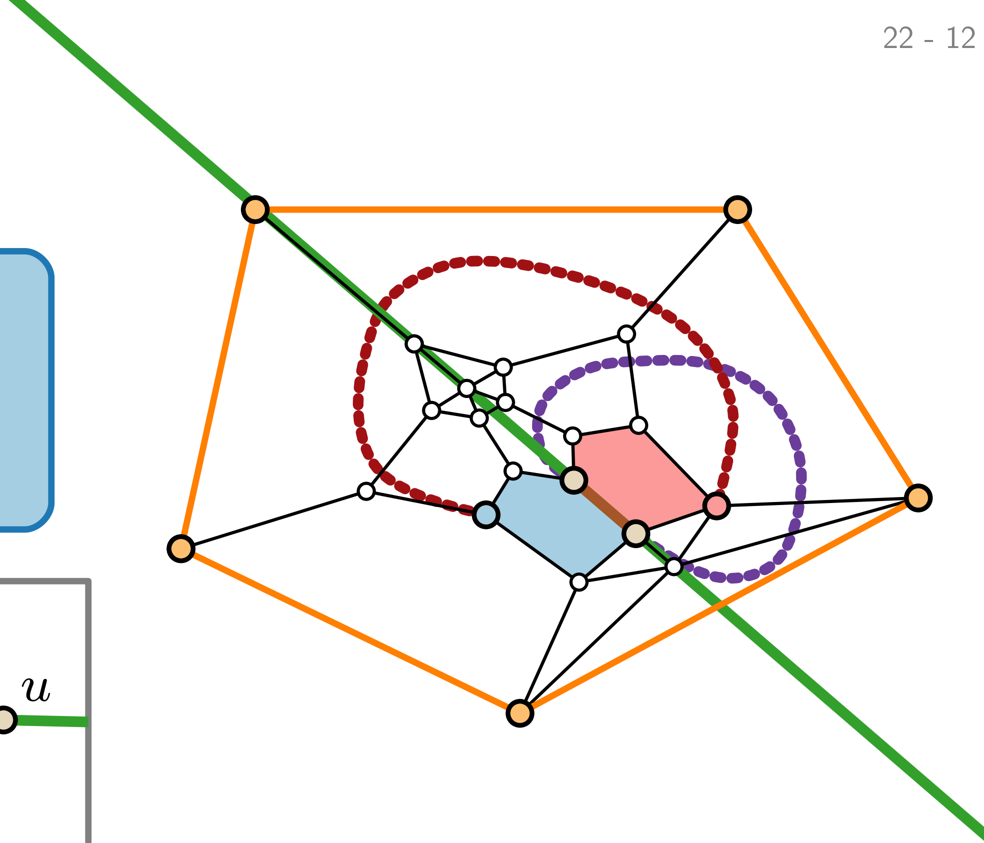

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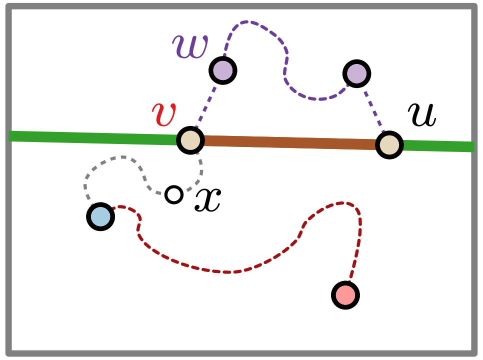

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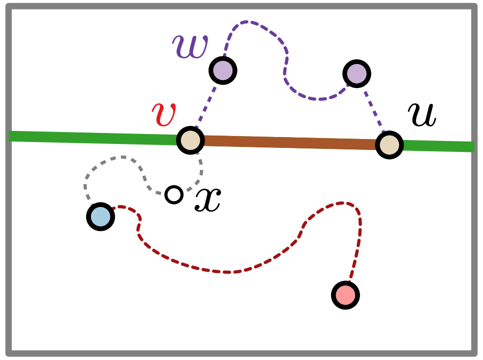
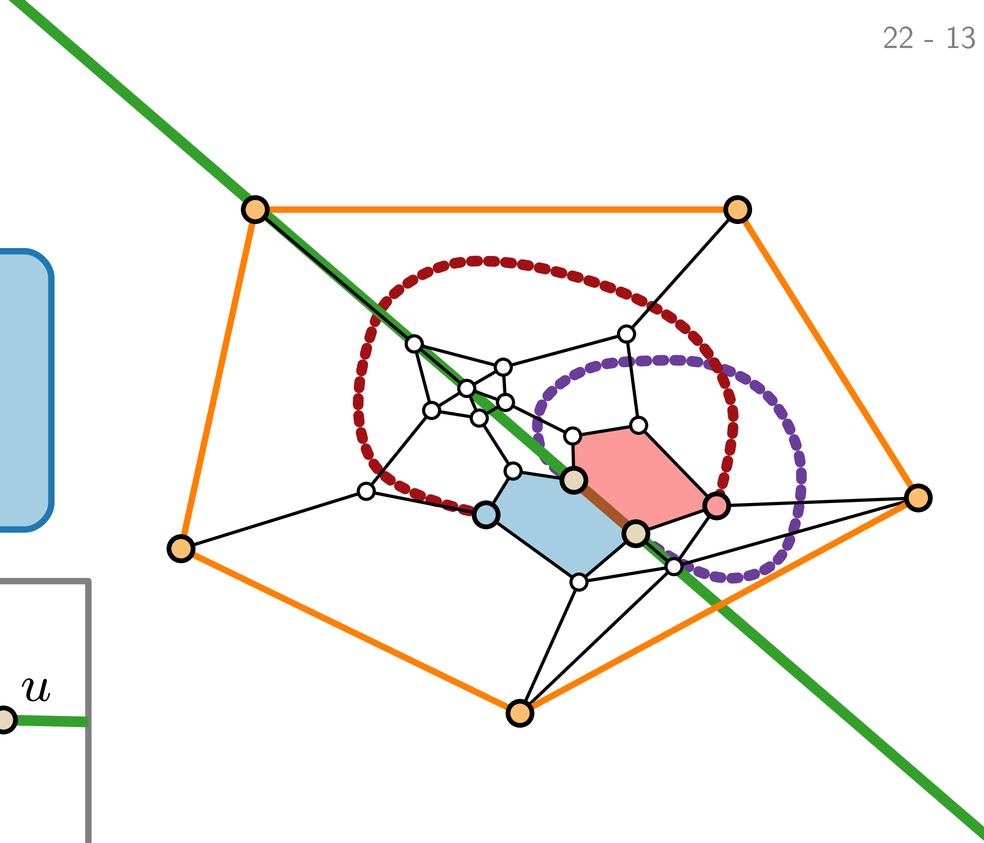


xv and vw on different sides of $\ell \Rightarrow f_1, f_2$ have angles $< \pi$ at v



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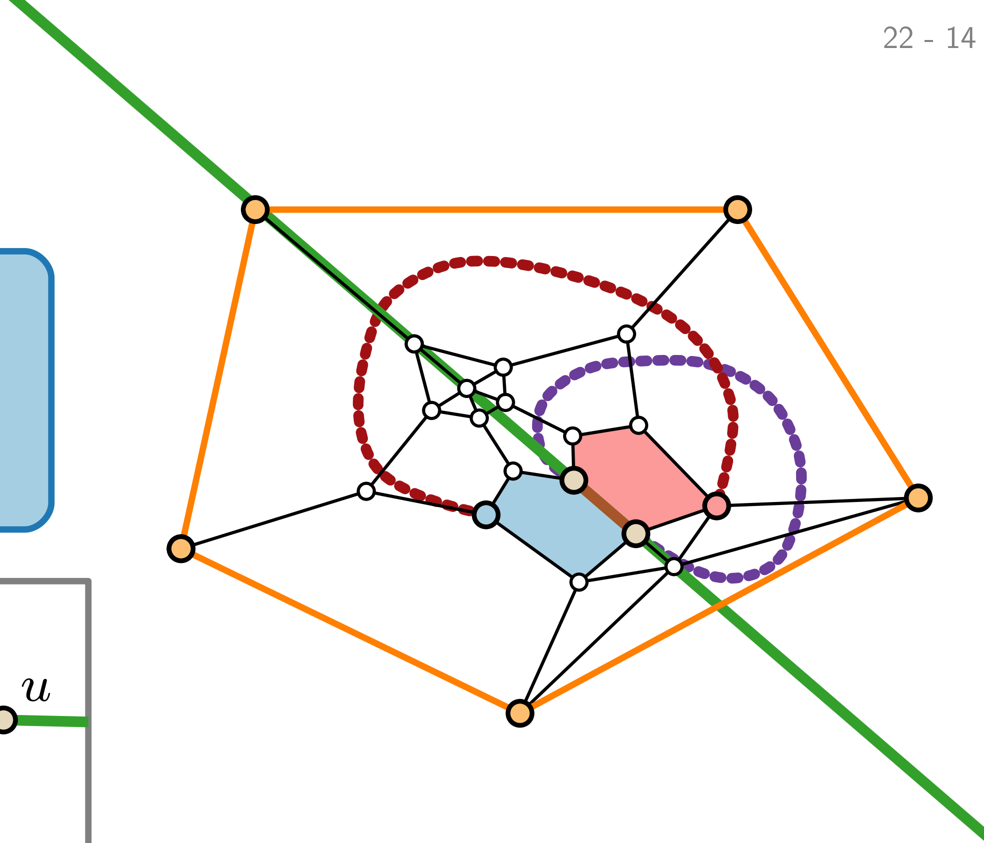

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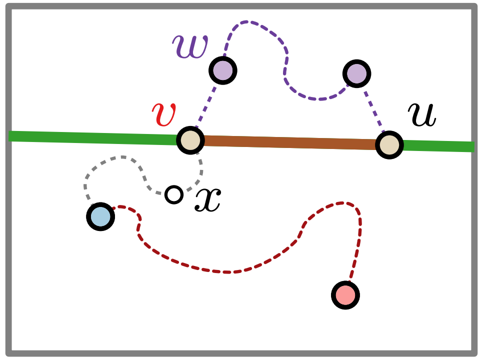

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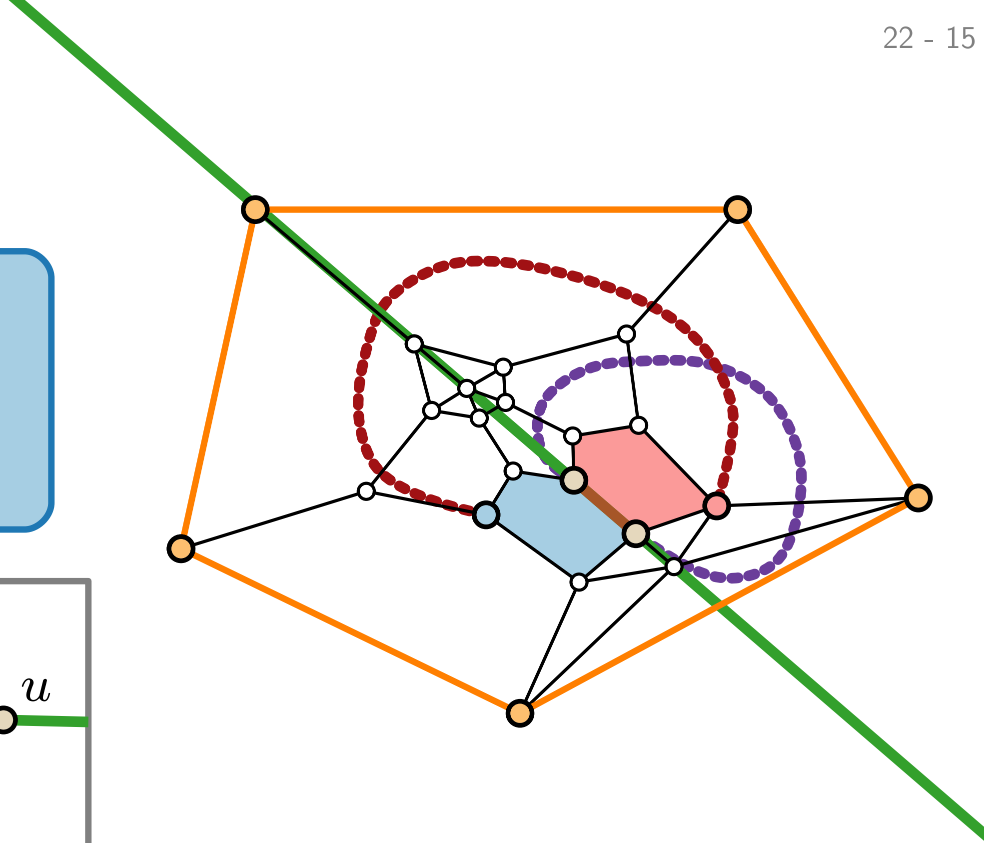

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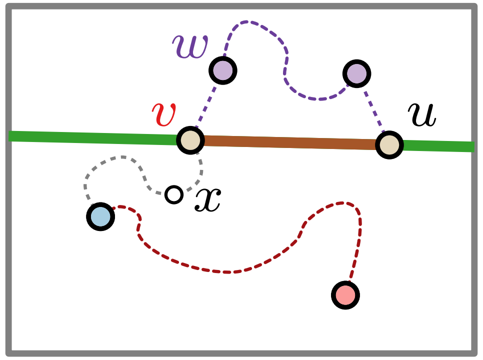

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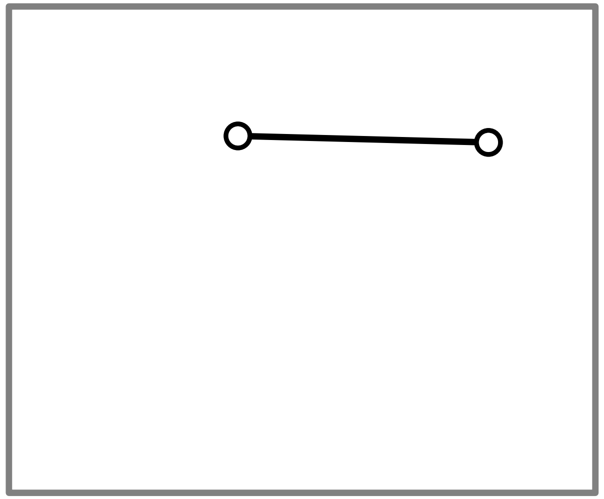

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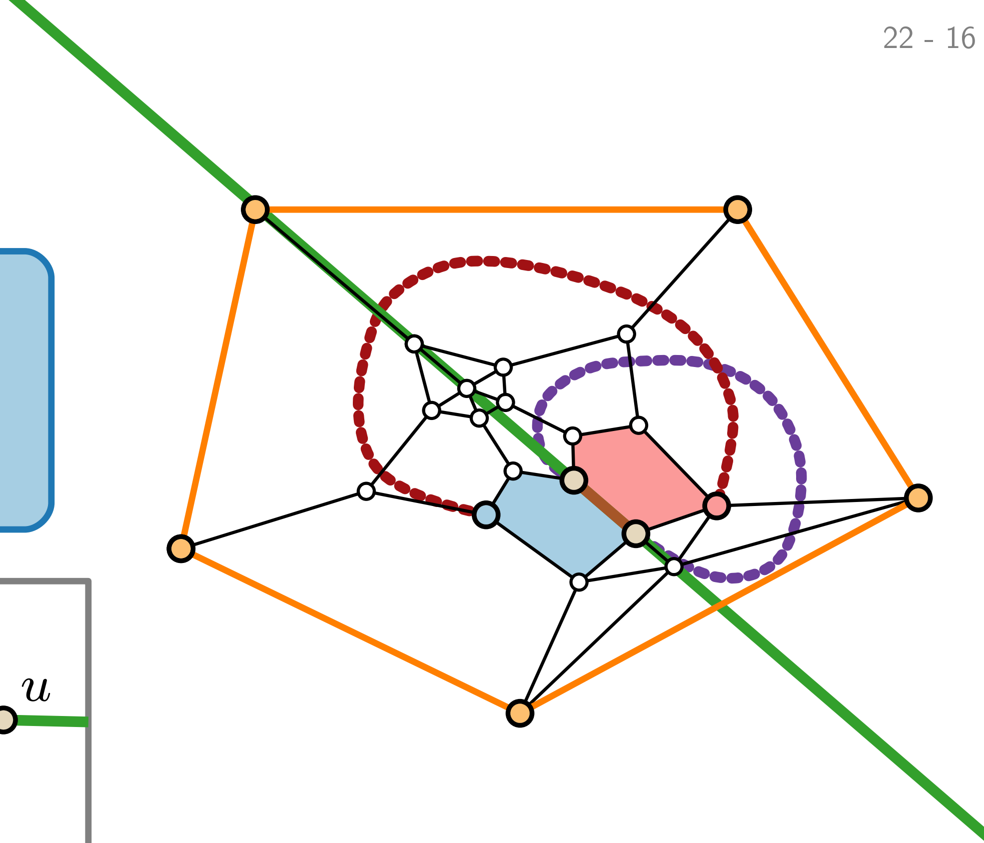

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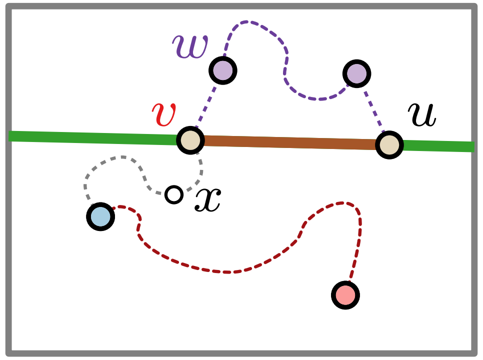

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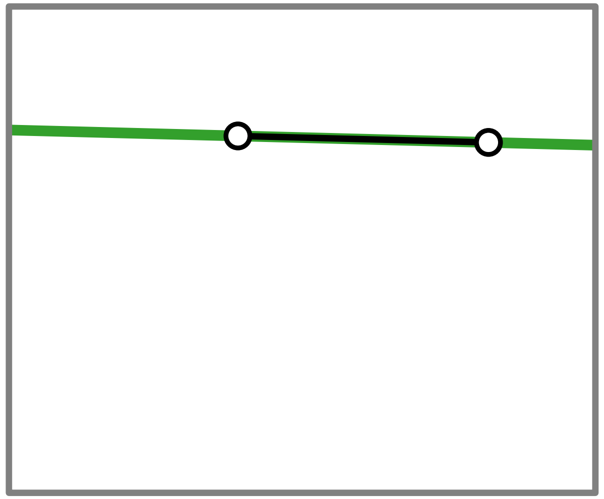

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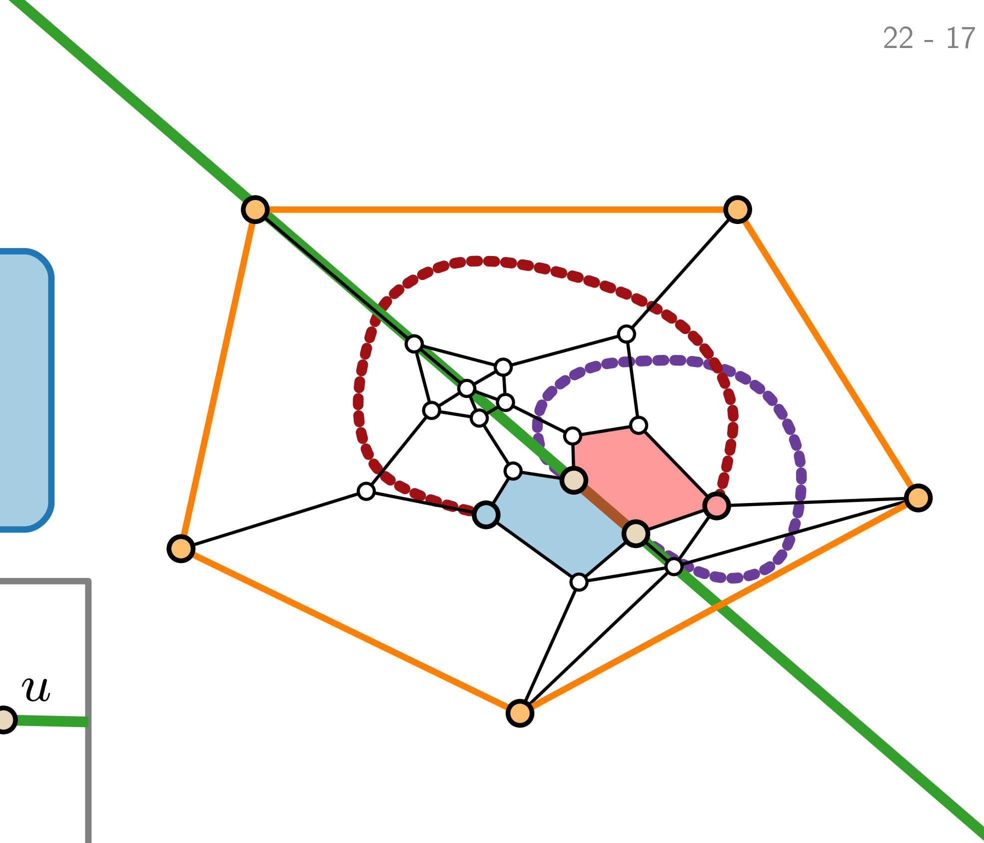

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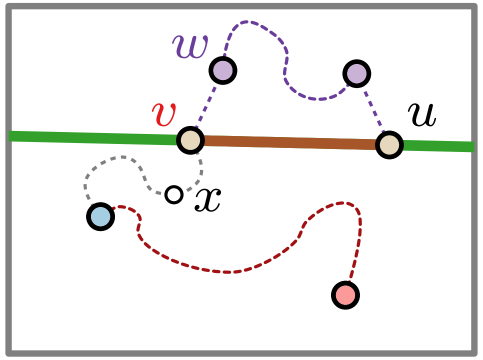

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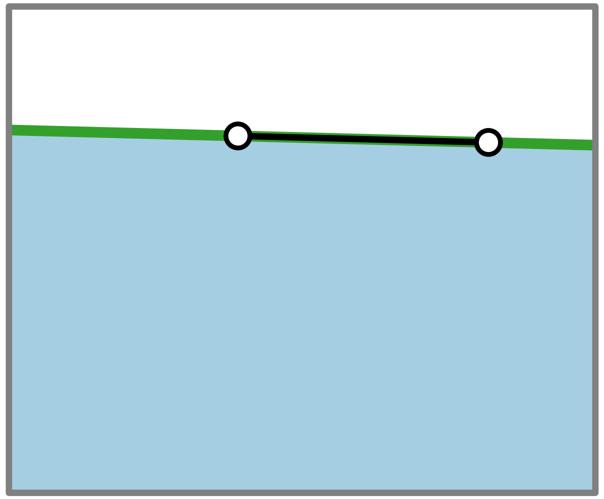

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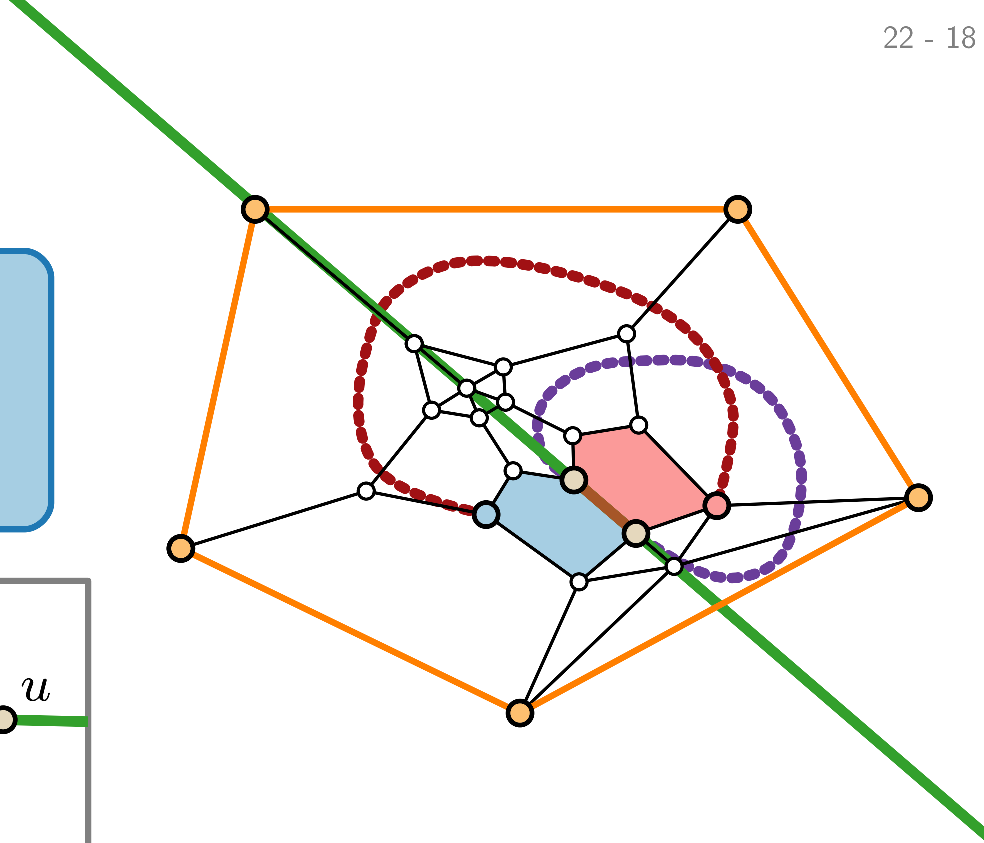

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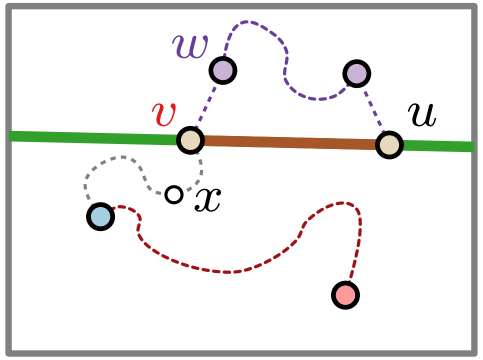

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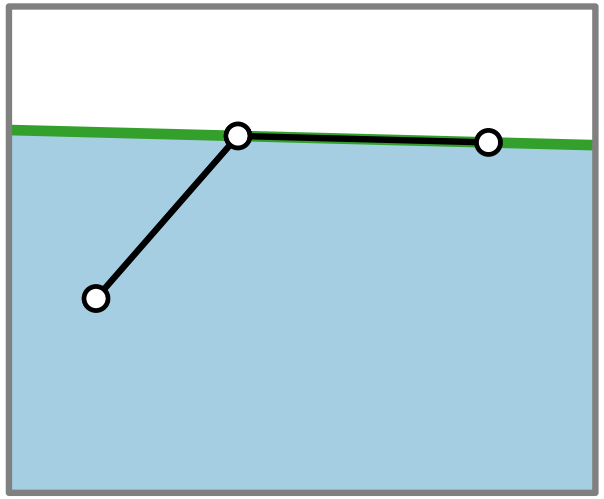

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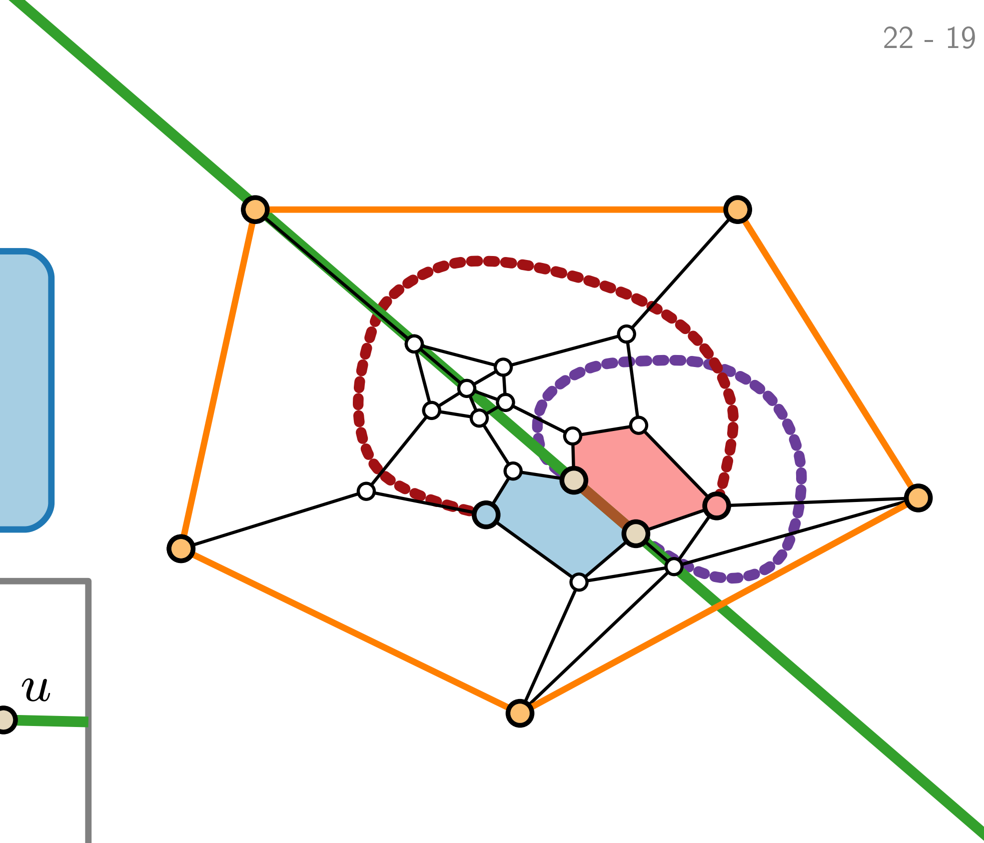

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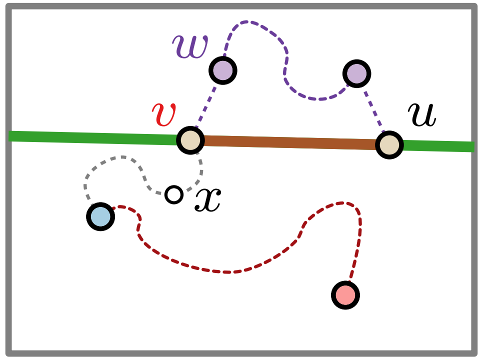

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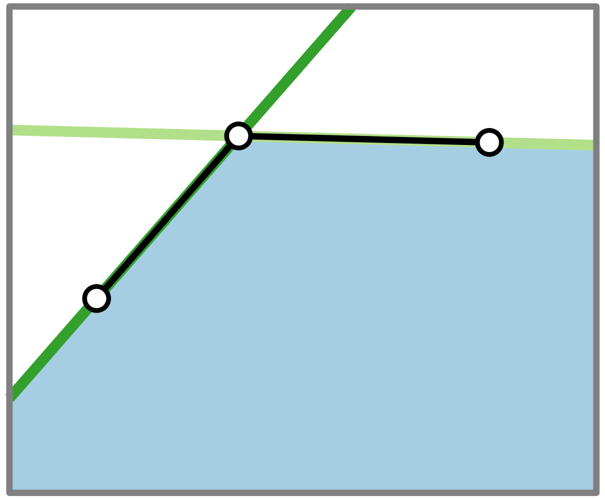

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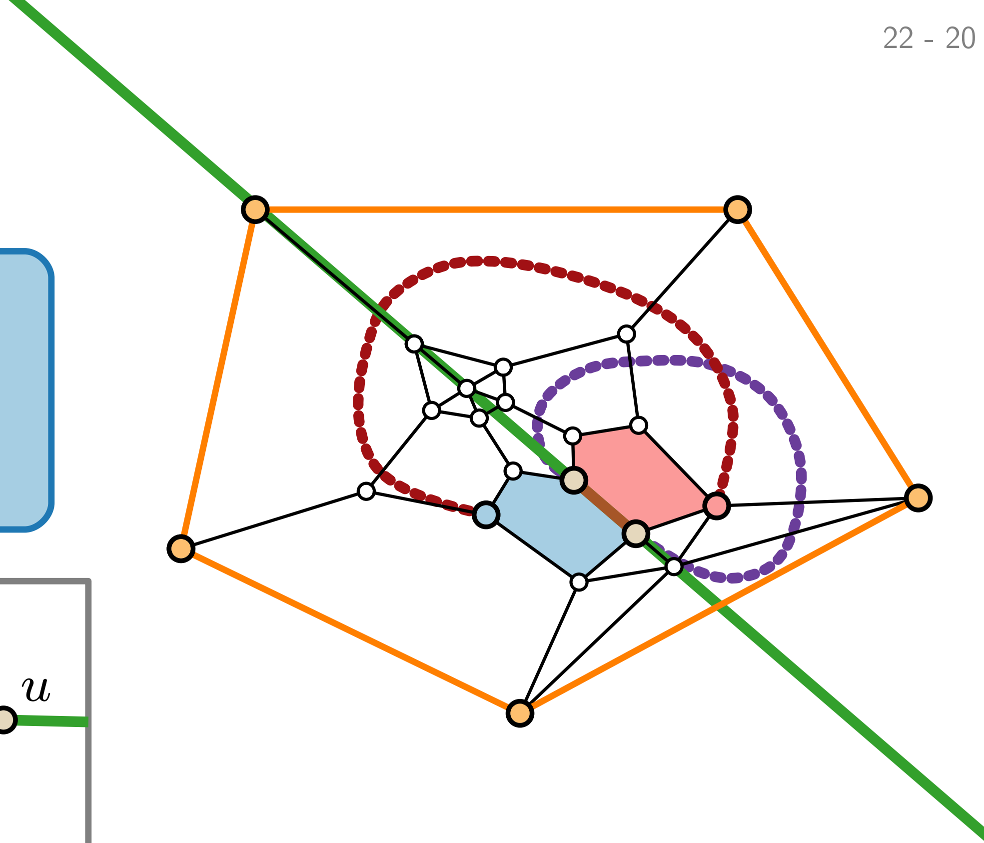

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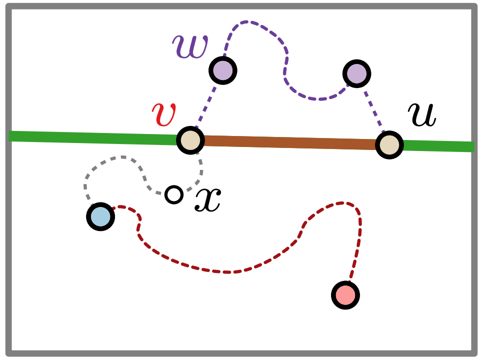

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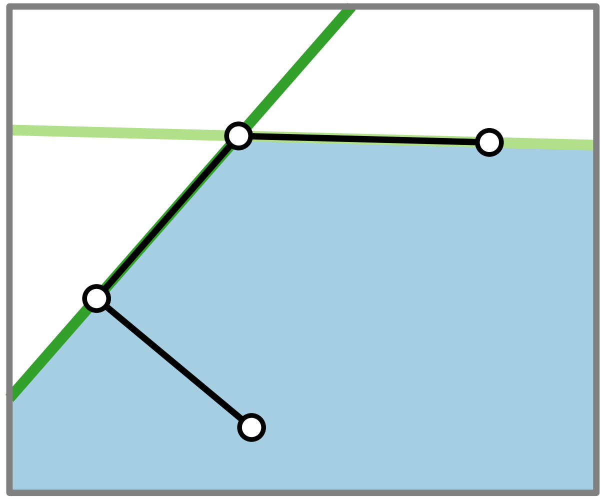

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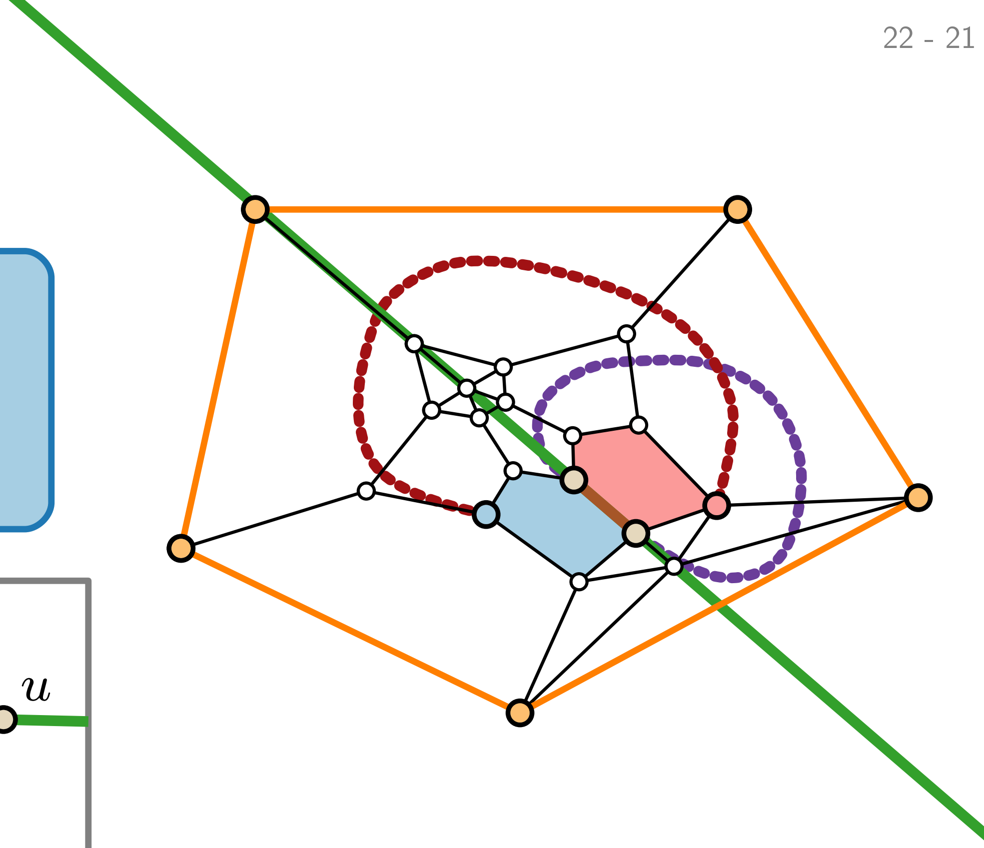

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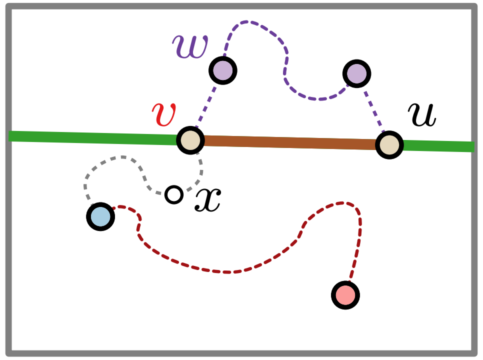

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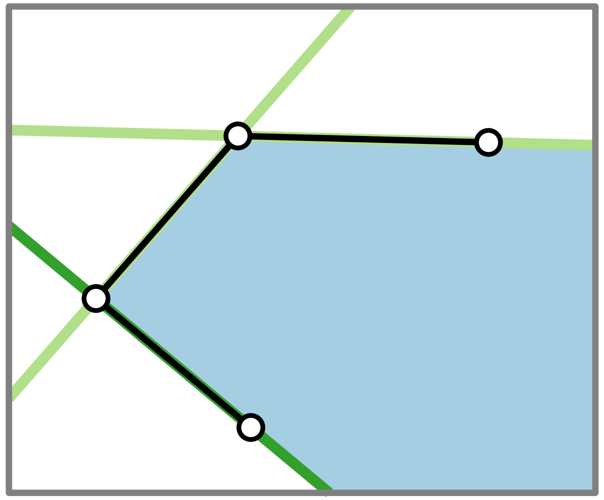

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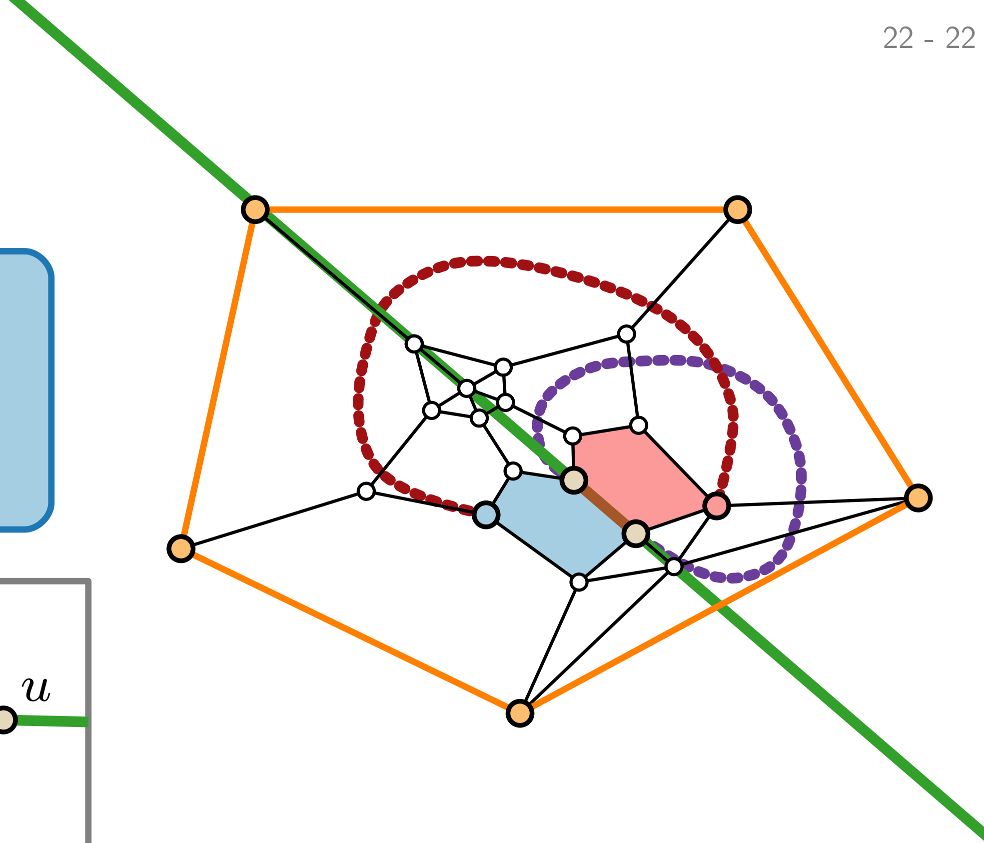

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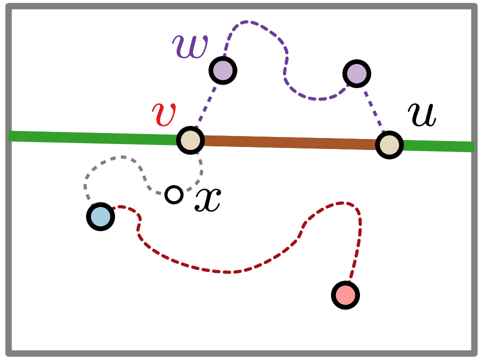

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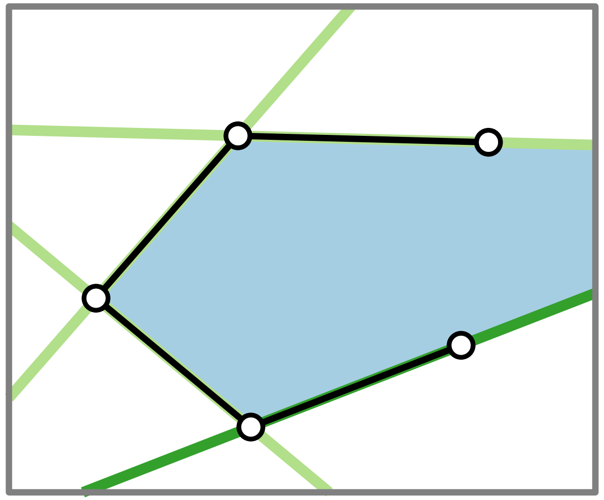

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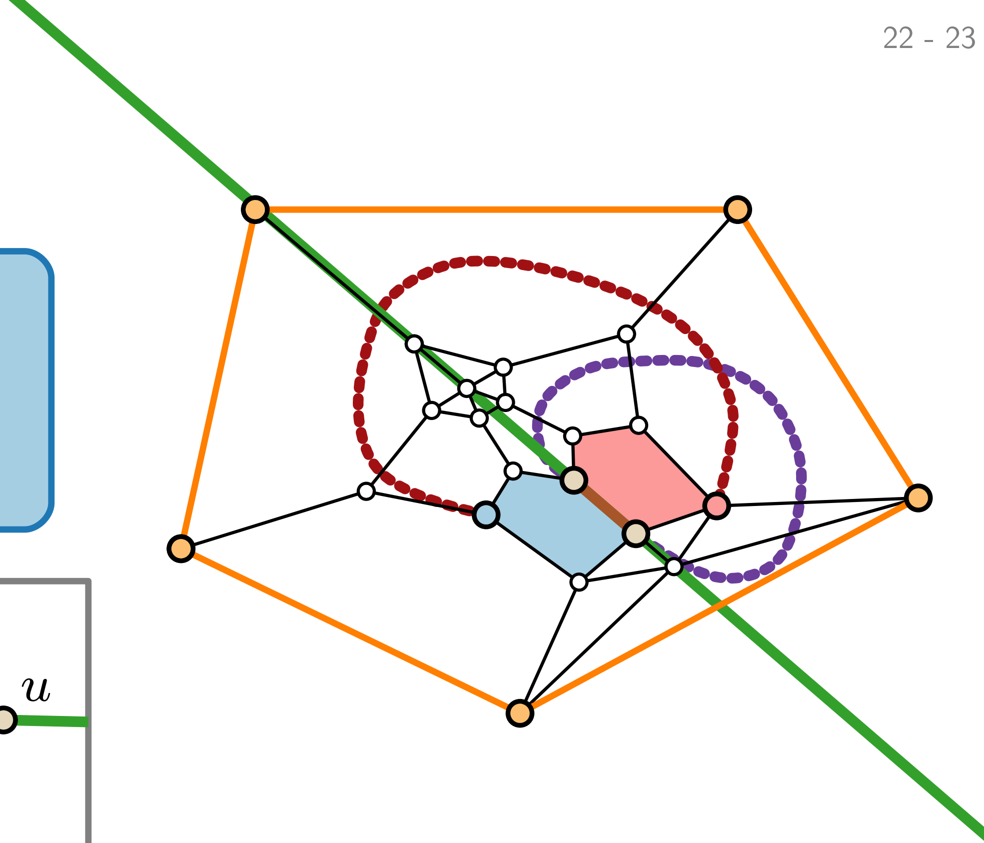

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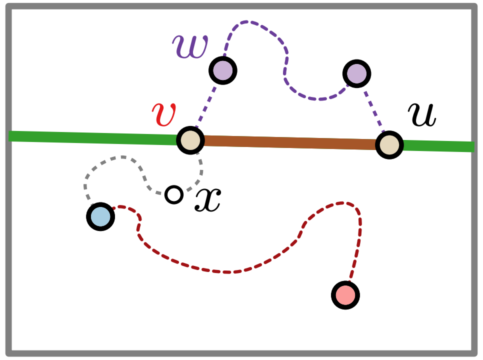

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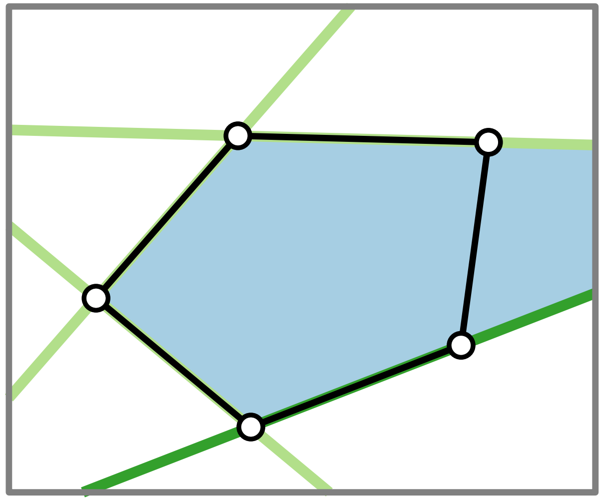

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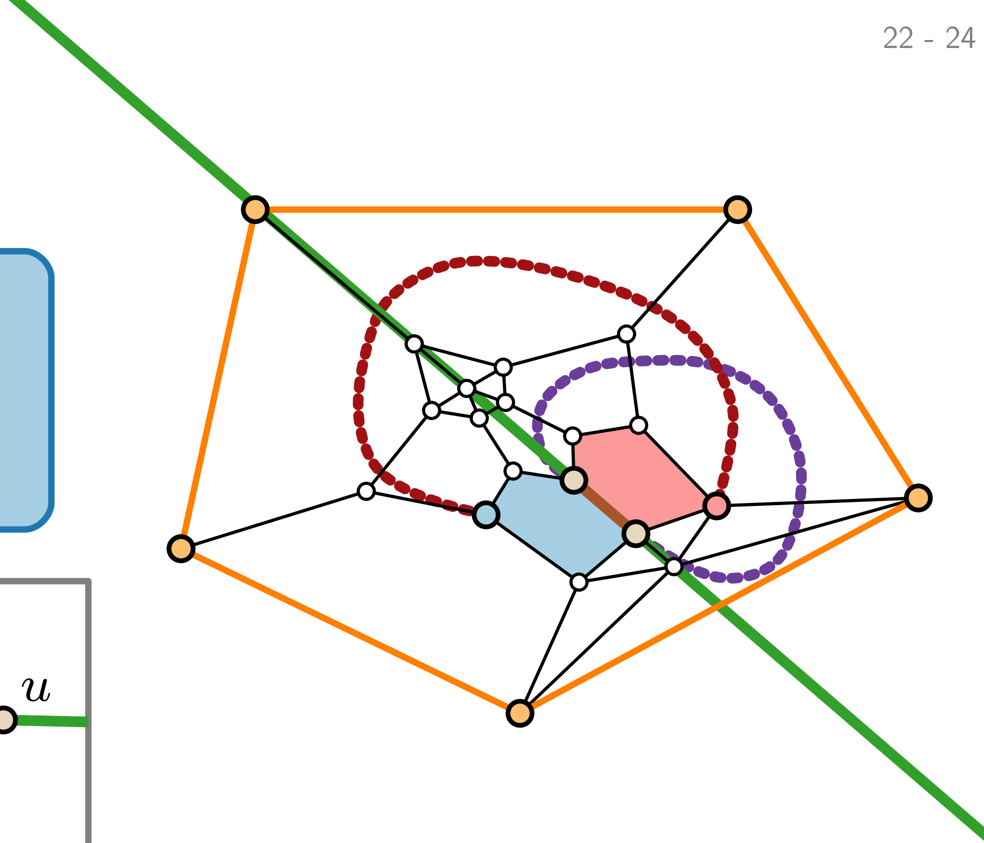

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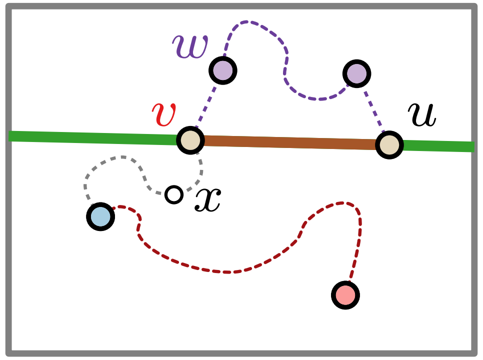

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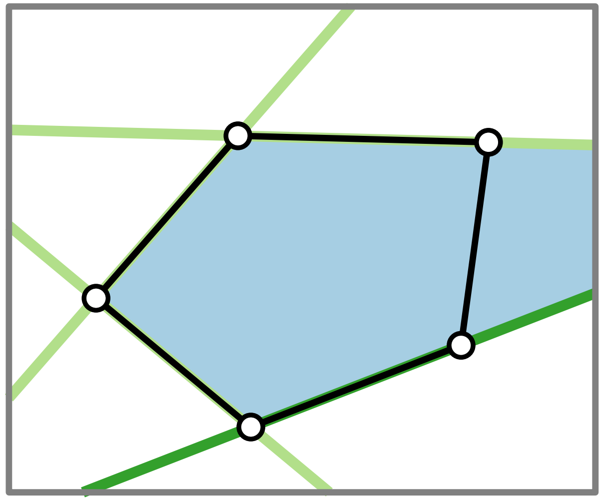

Lemma. The drawing is planar.

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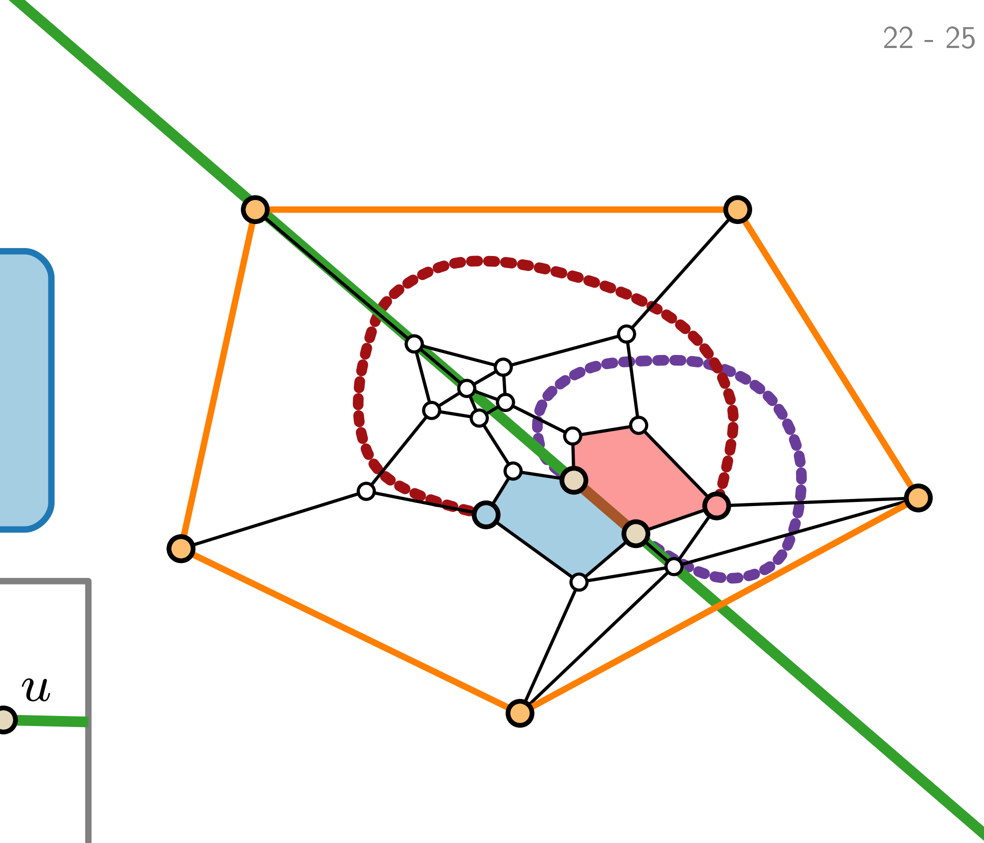

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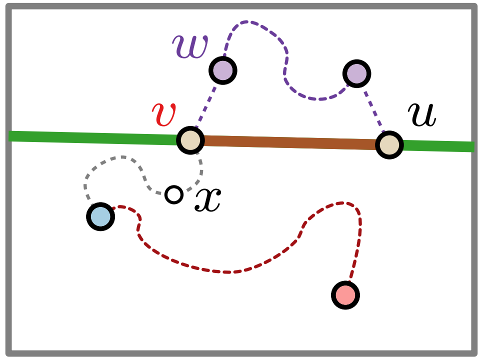

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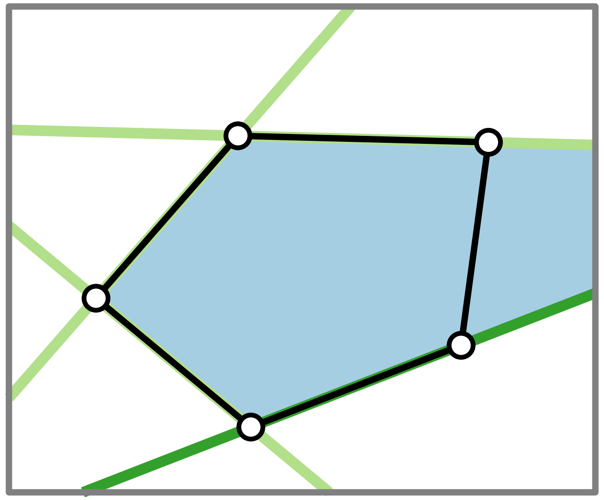

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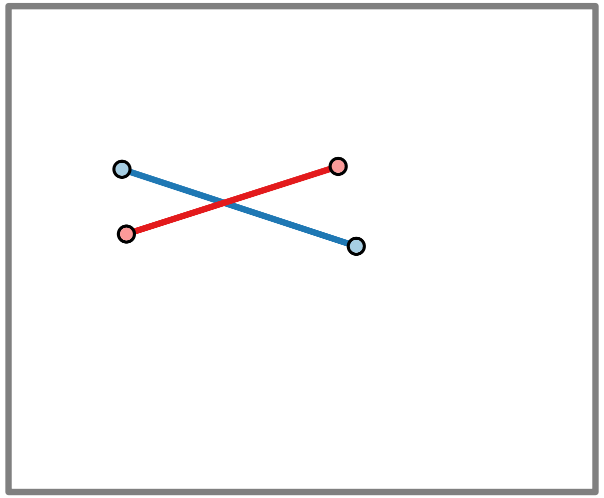

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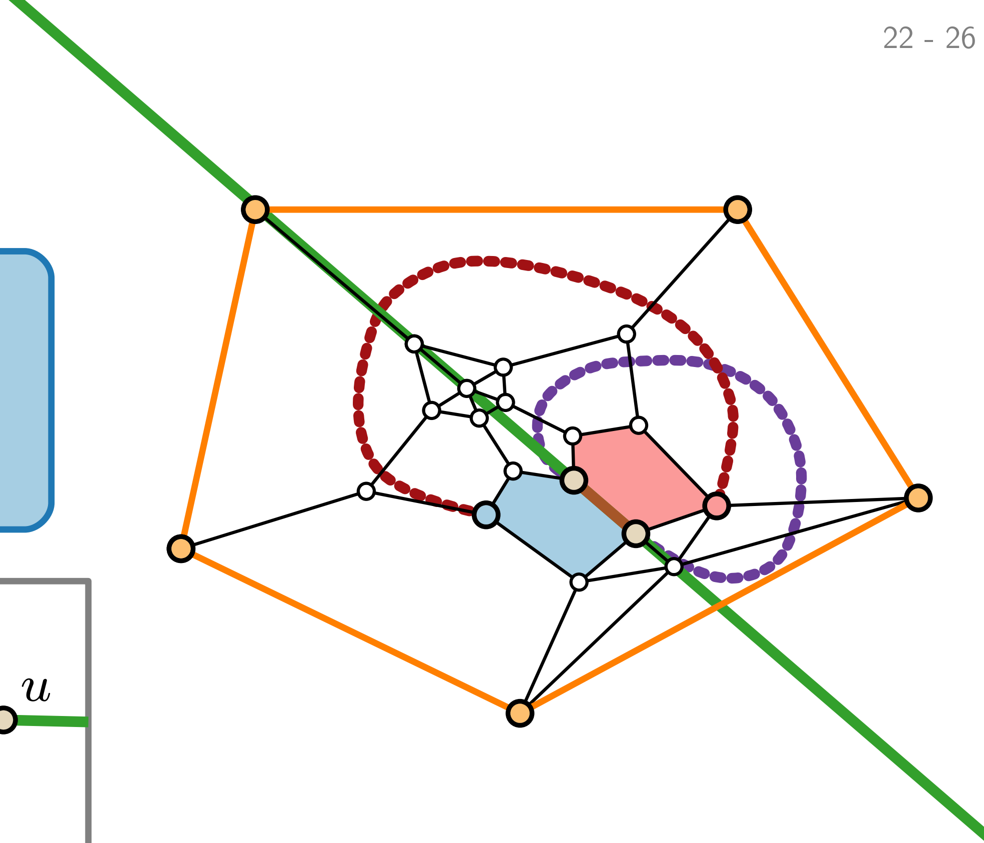

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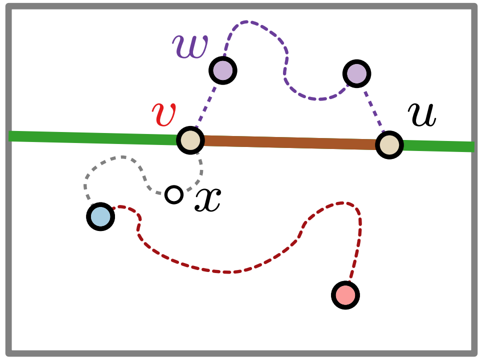
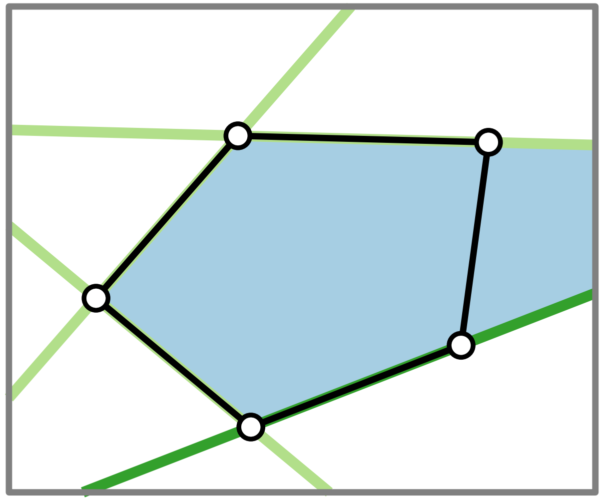


xv and vw on different sides of $\ell \Rightarrow f_1, f_2$ have angles $< \pi$ at v

Lemma. All faces are strictly convex.



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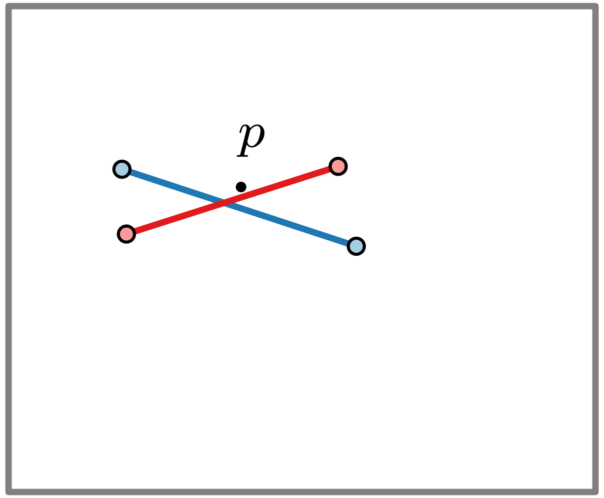

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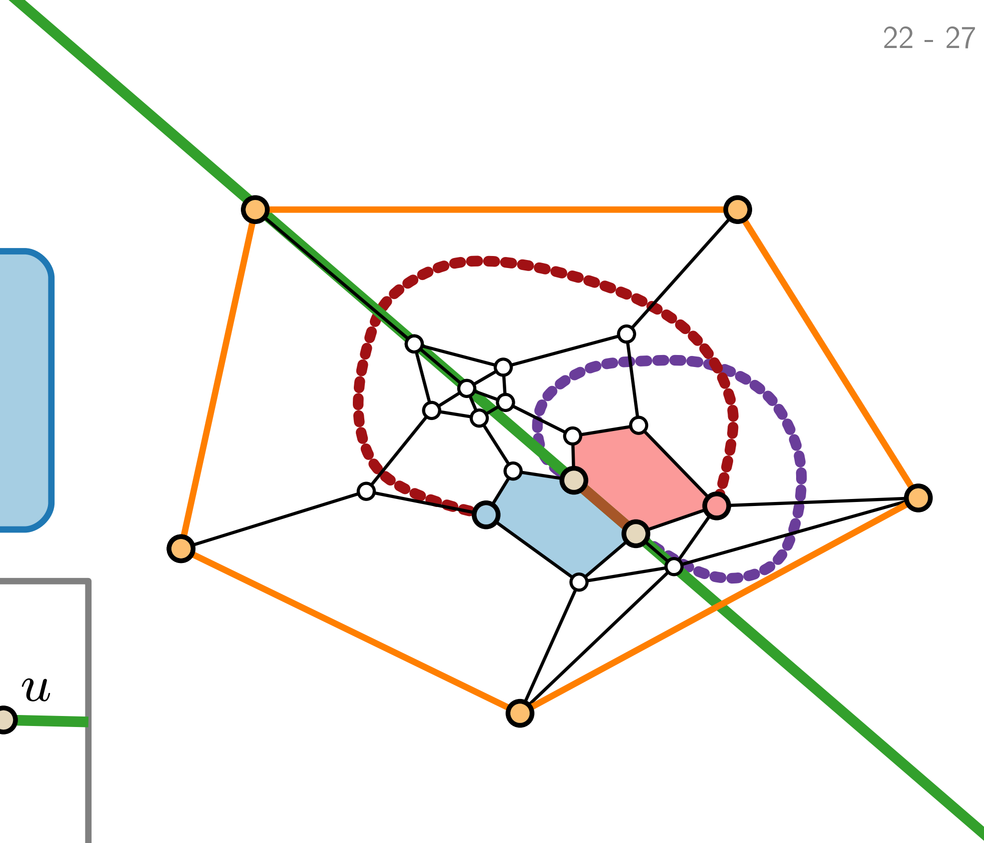

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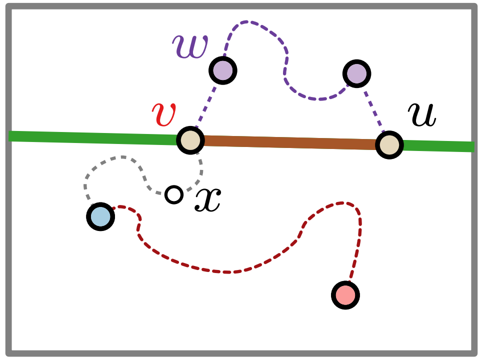

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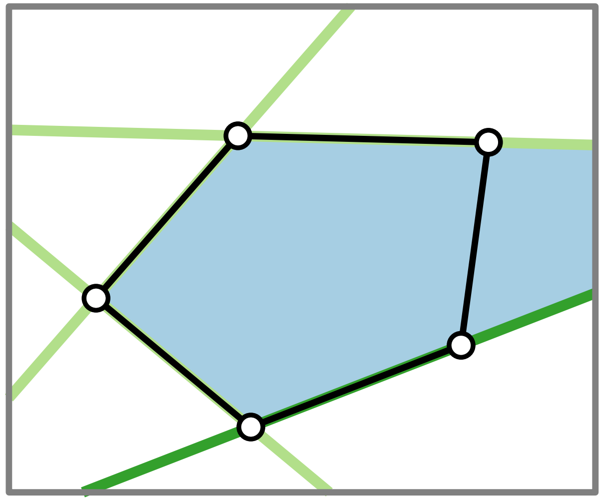

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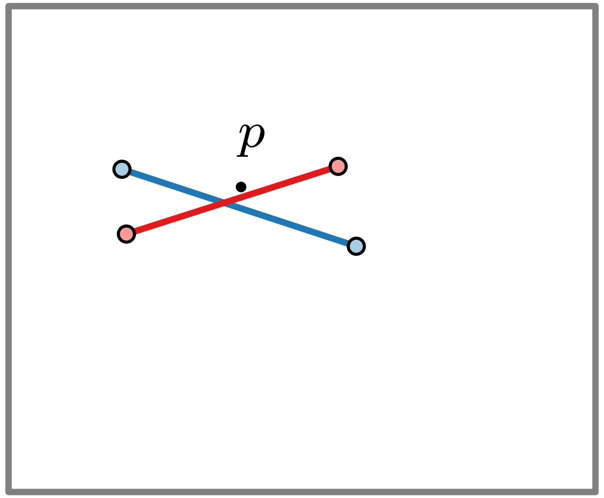
p inside two faces



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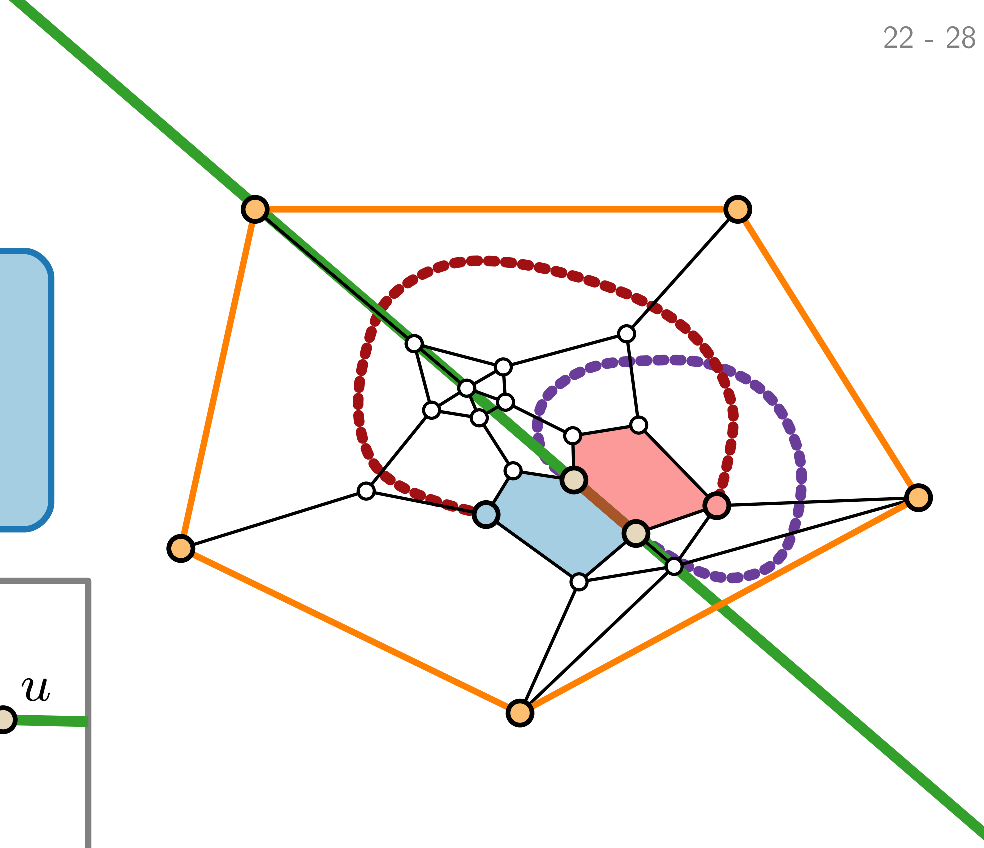

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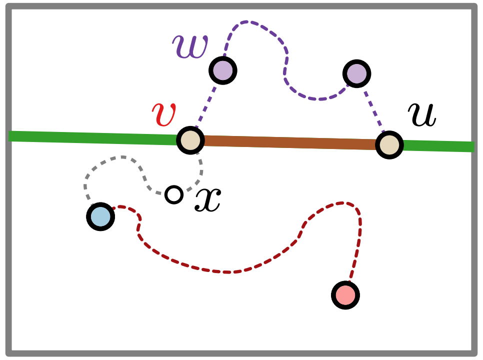

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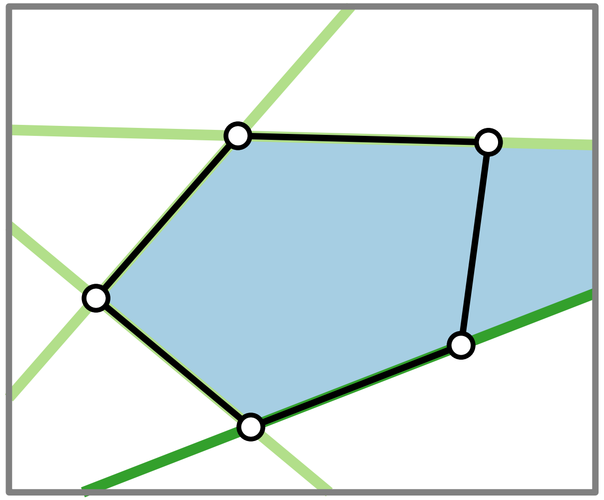

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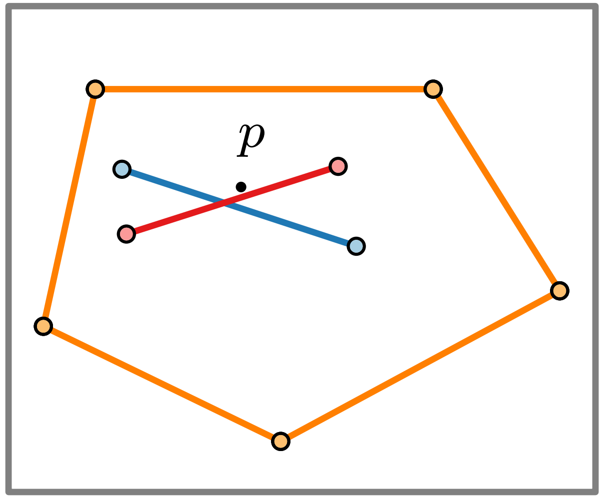

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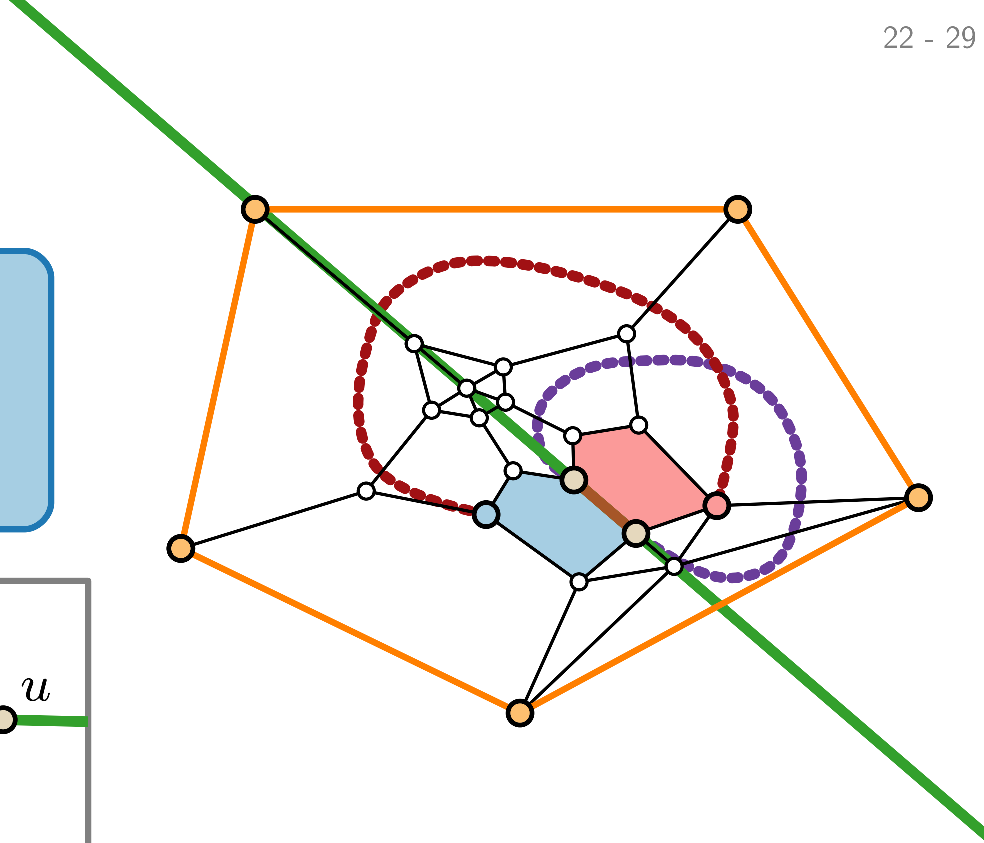

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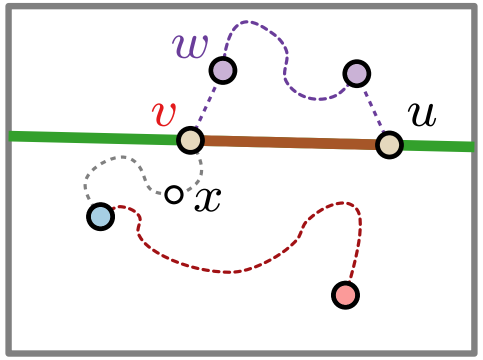

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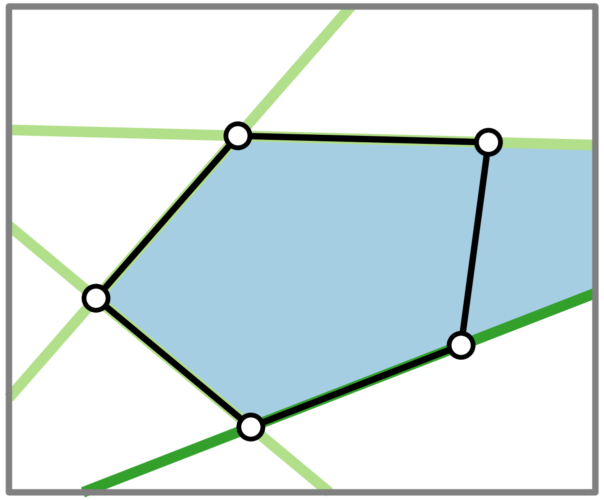

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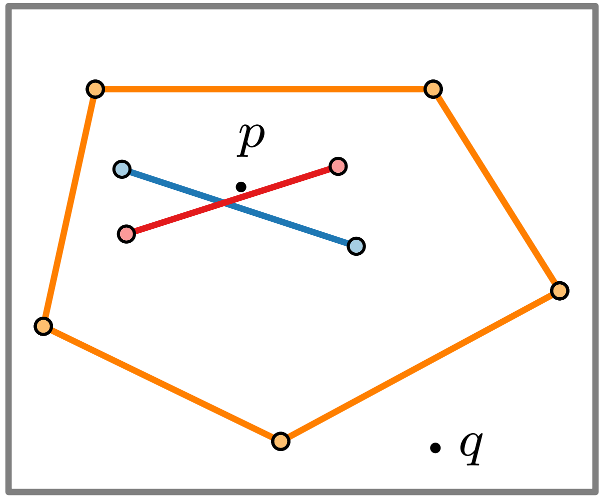

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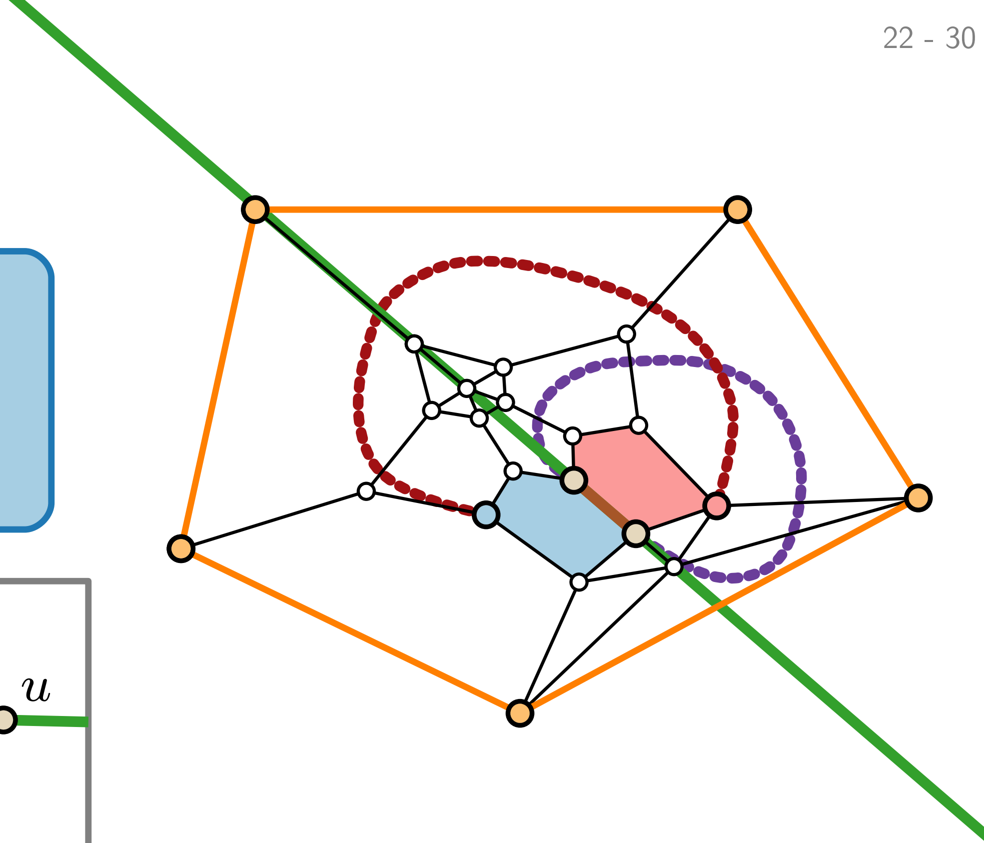

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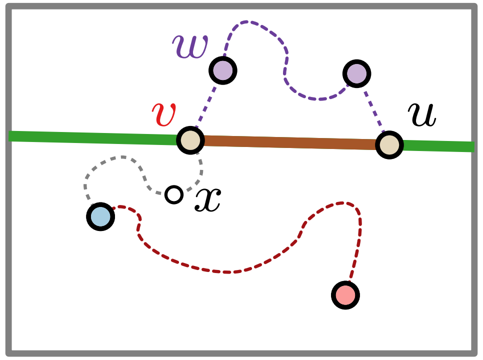

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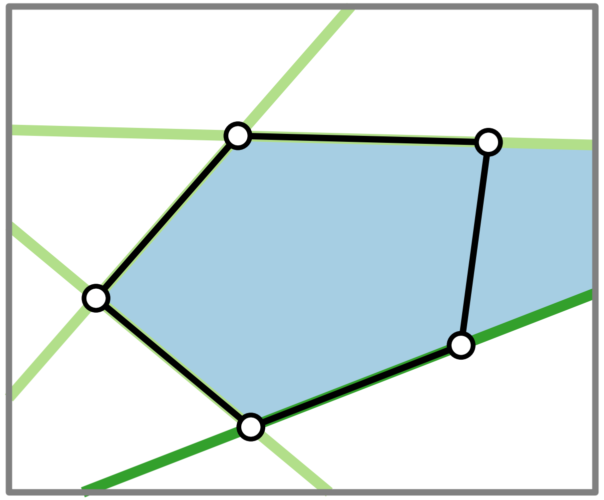

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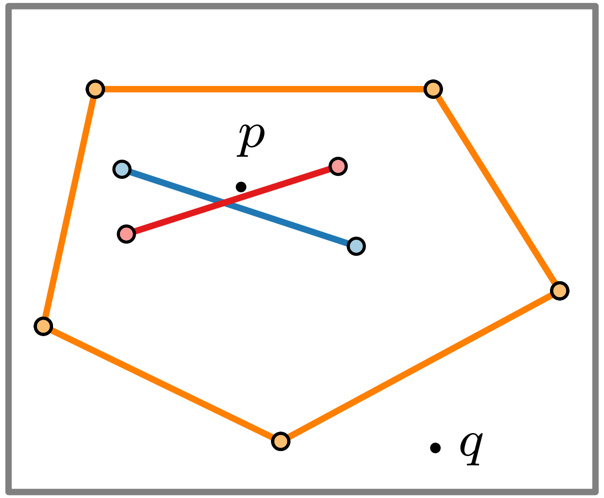

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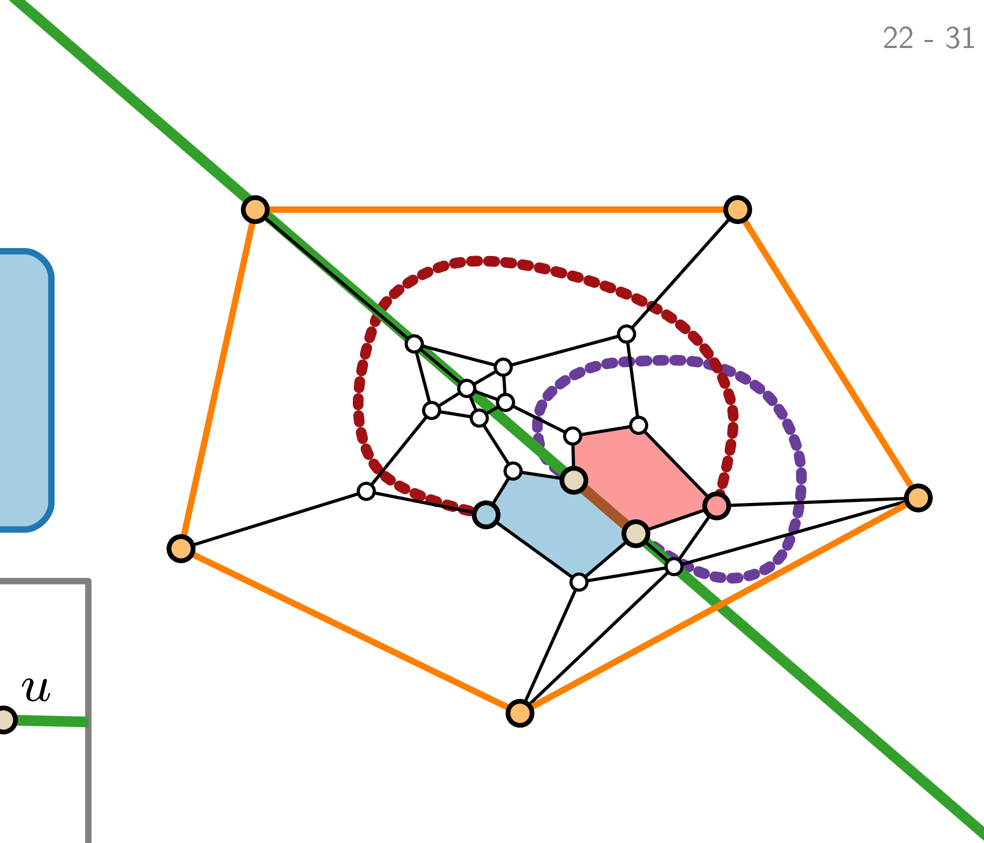

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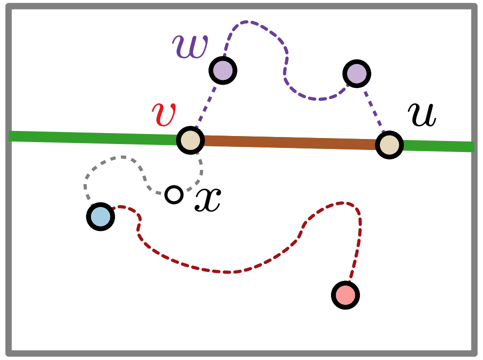

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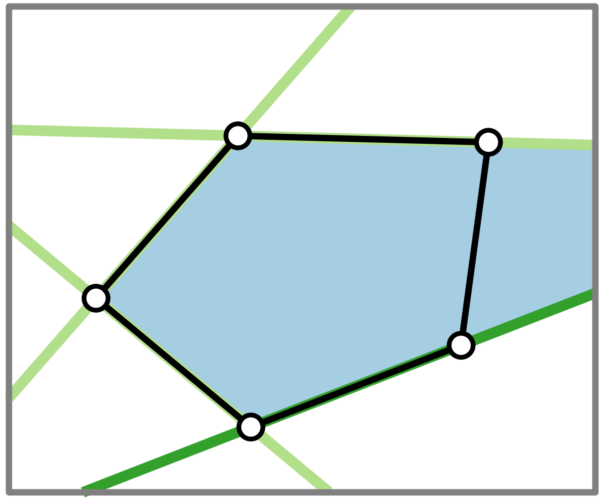

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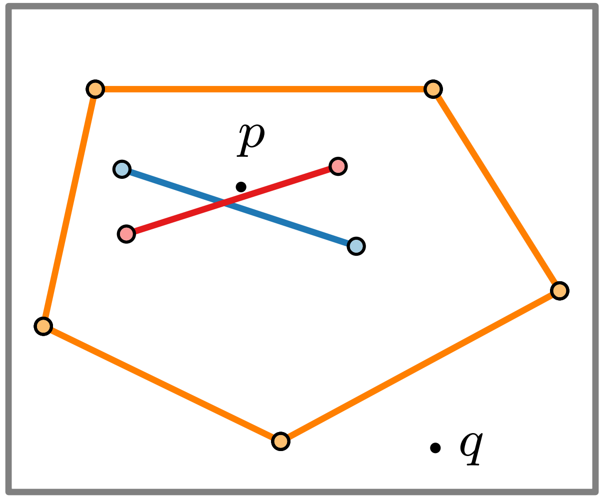

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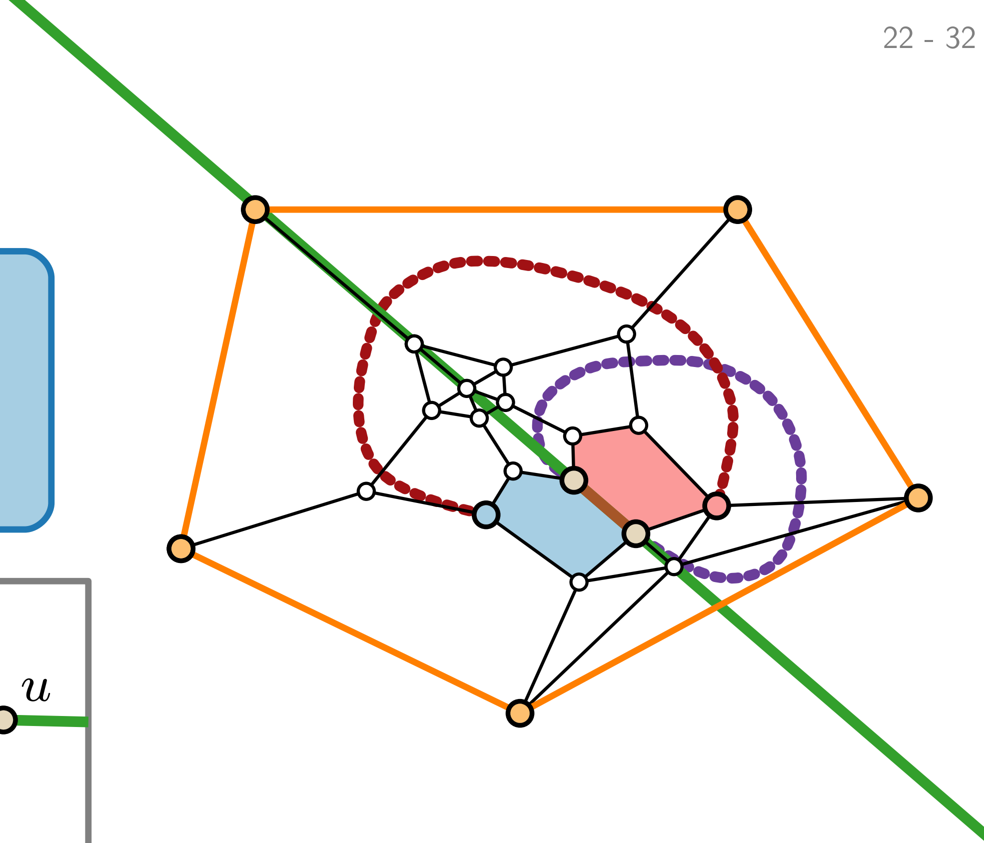

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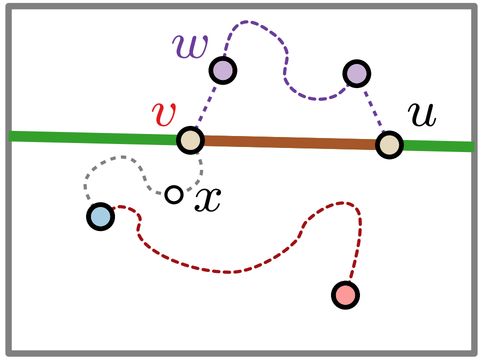

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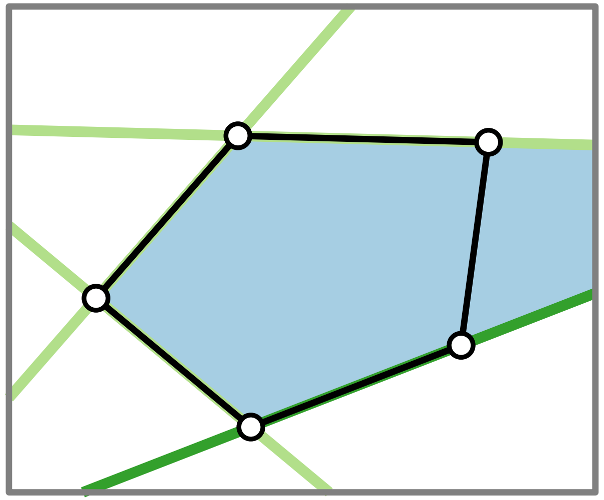

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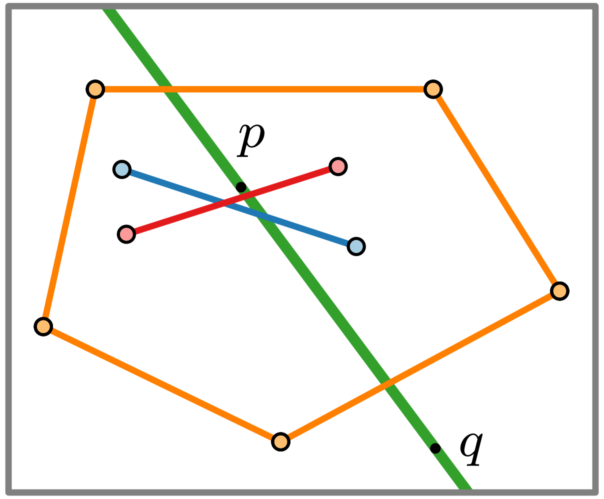

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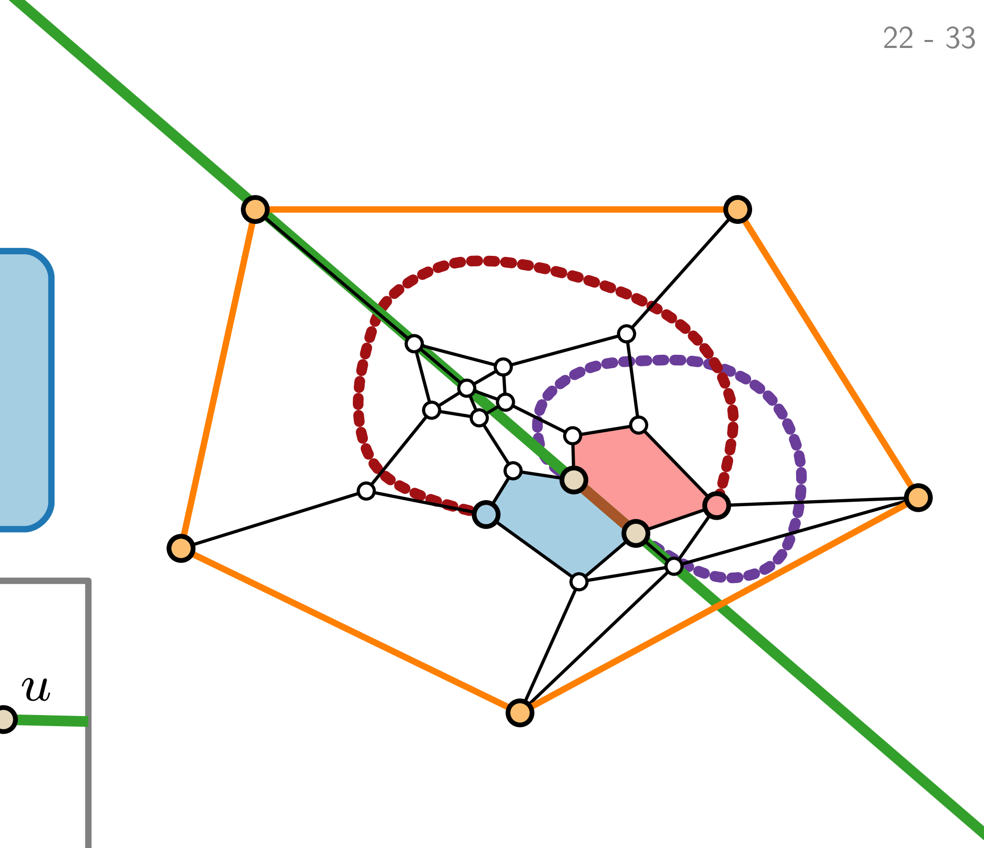

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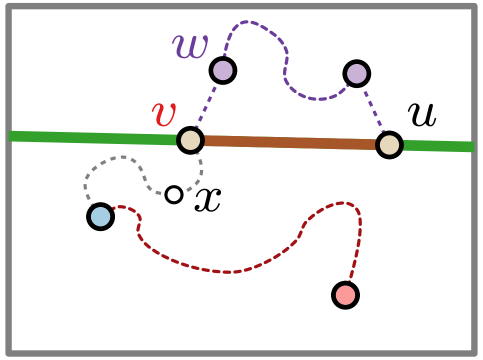

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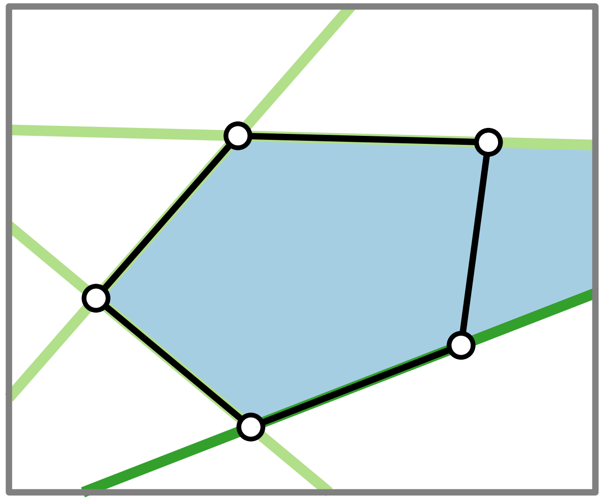

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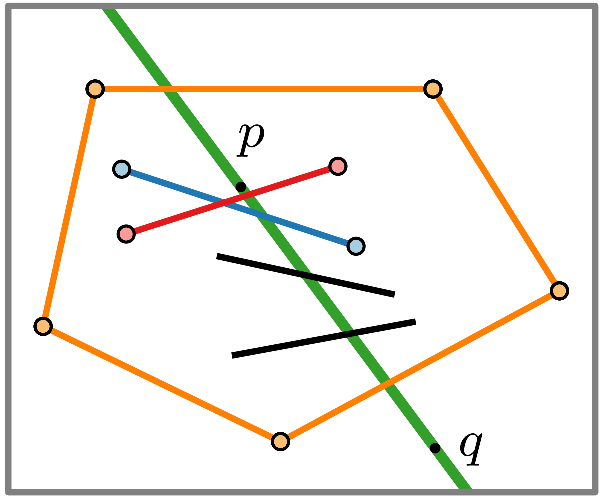

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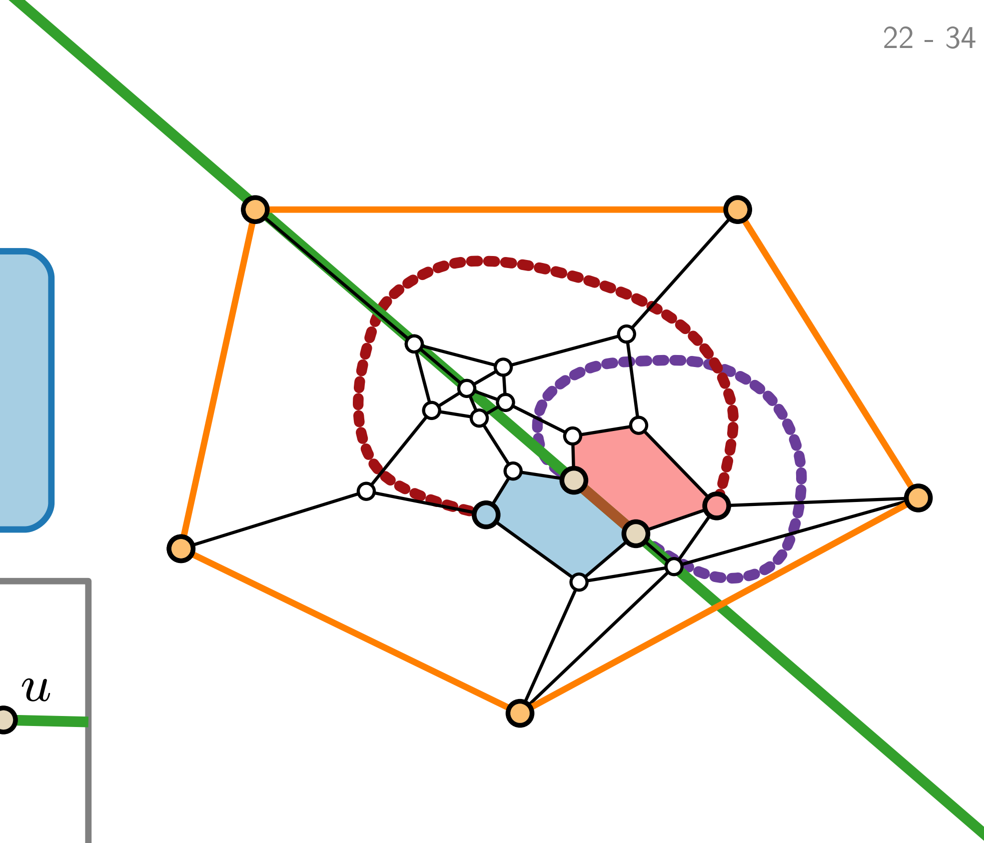

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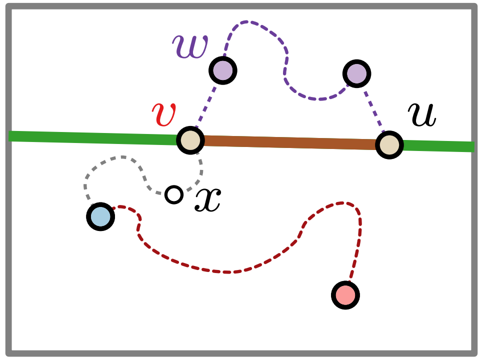

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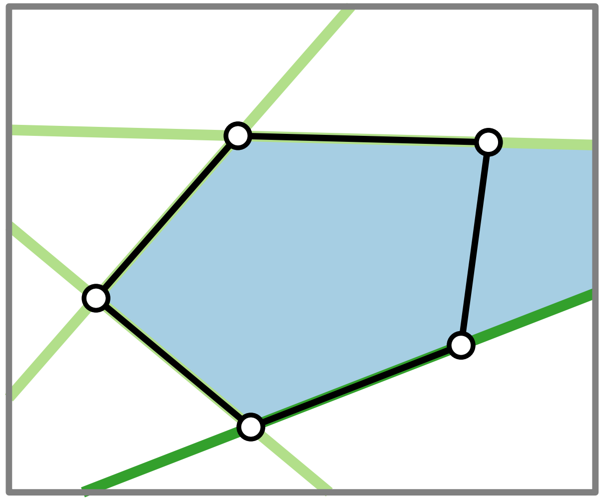

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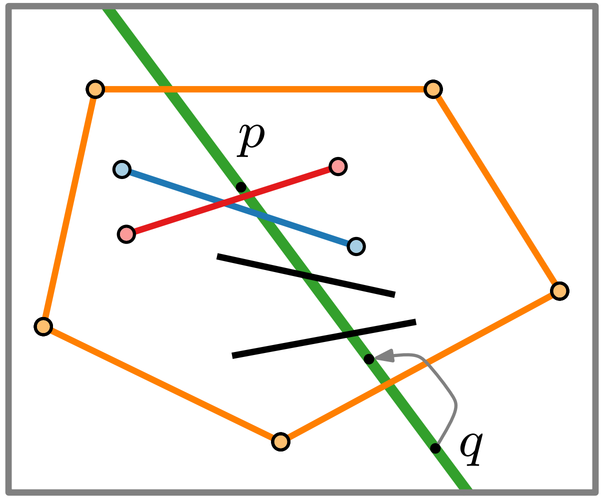

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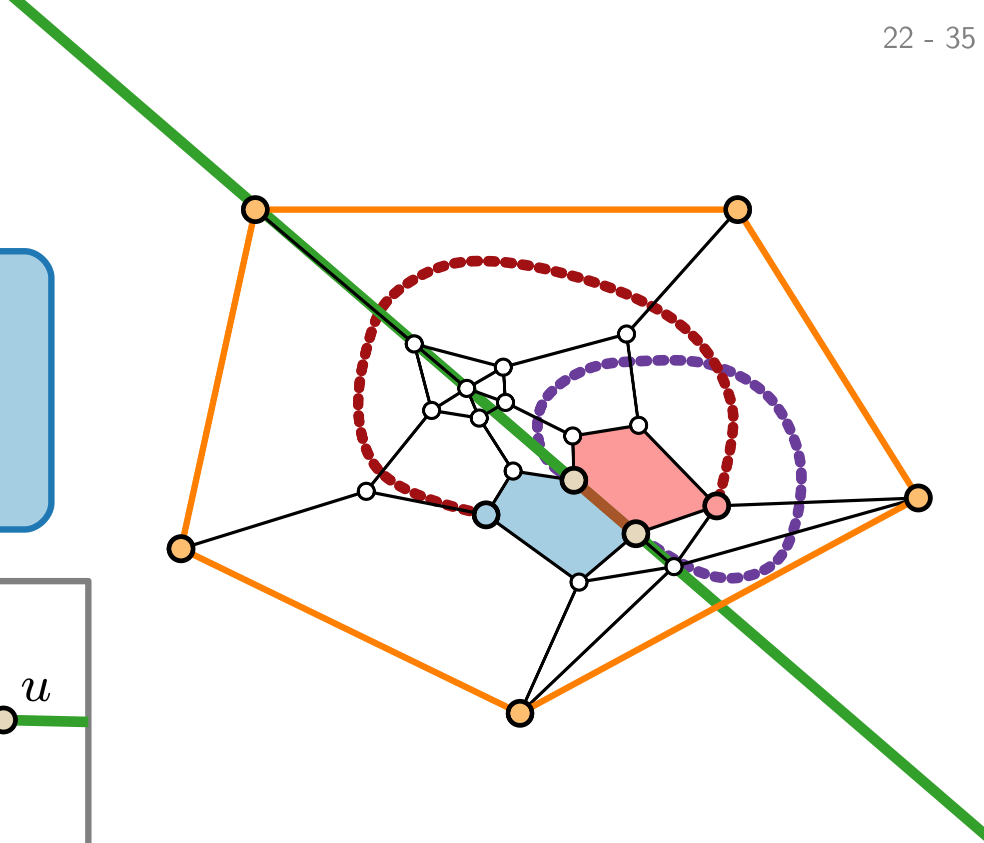

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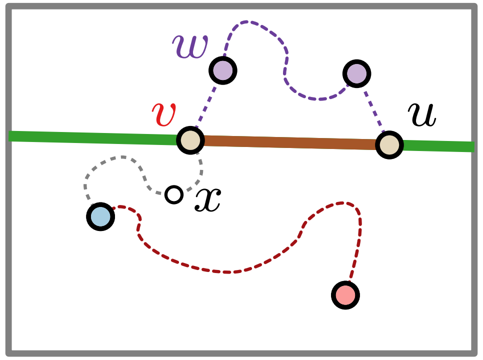

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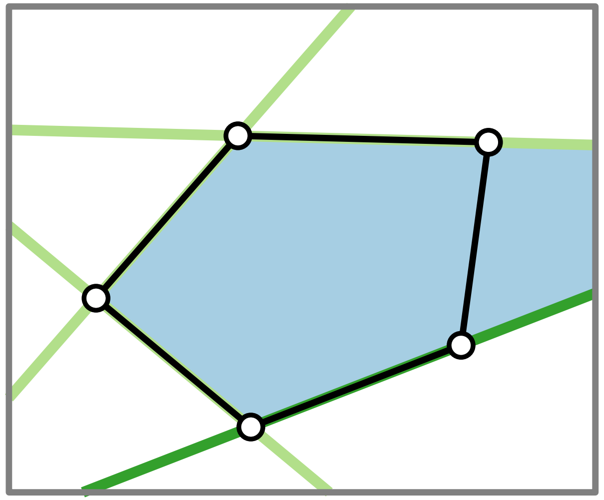

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p inside two faces
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jumping over edge
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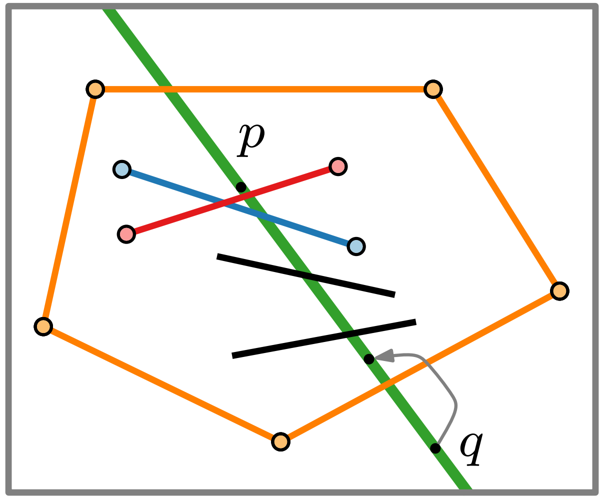

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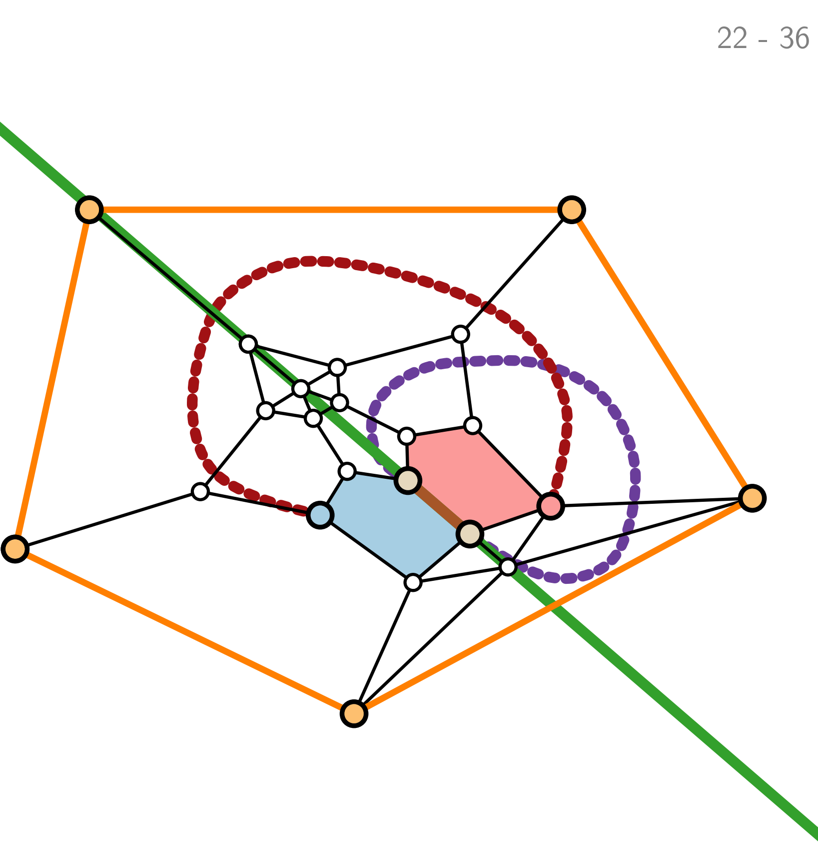

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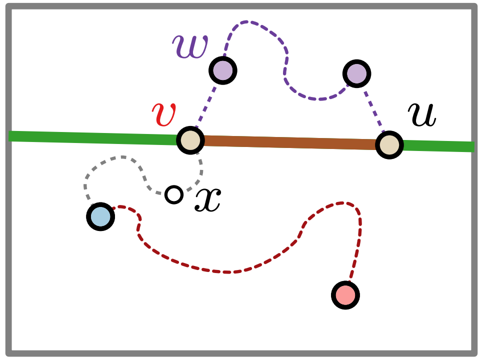

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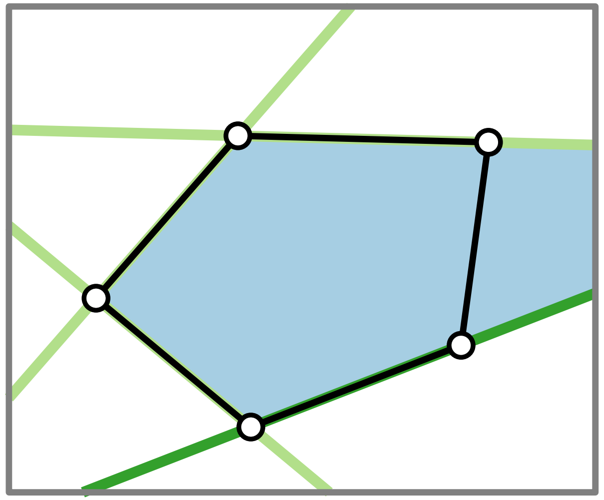

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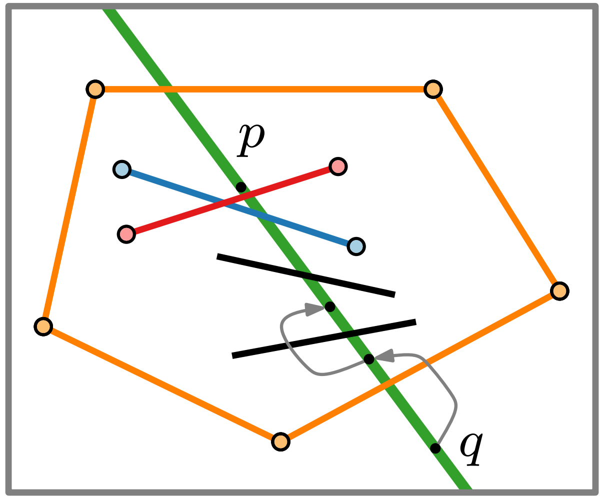

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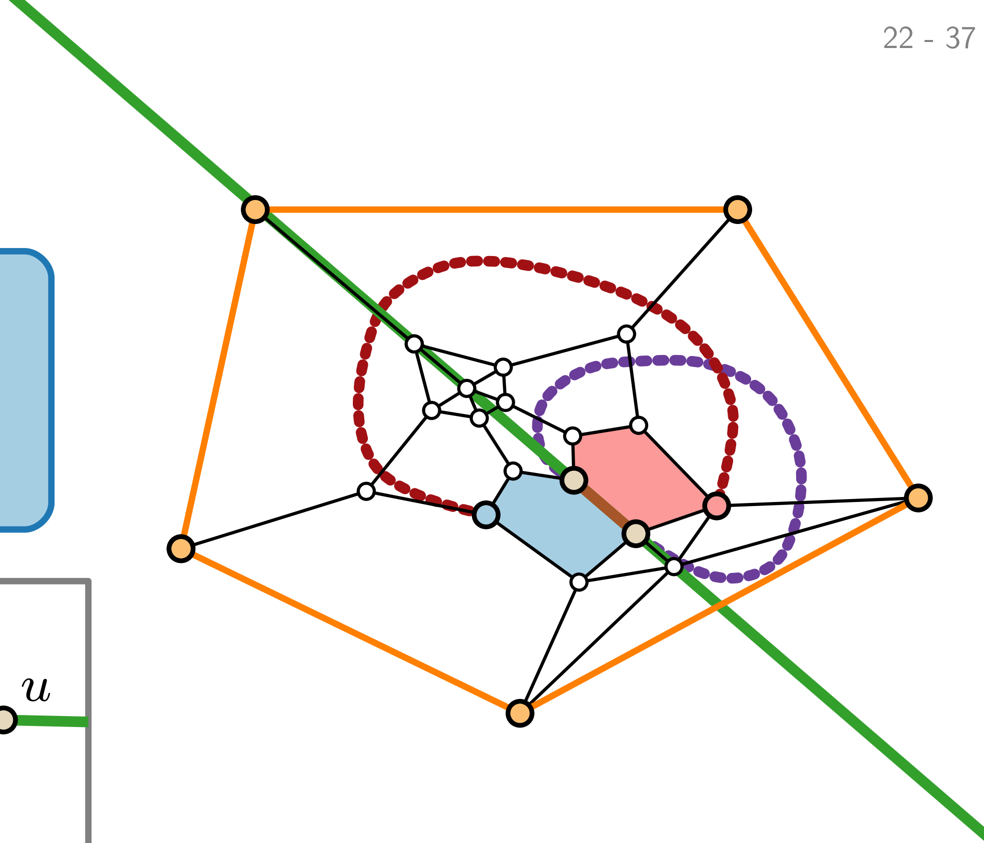

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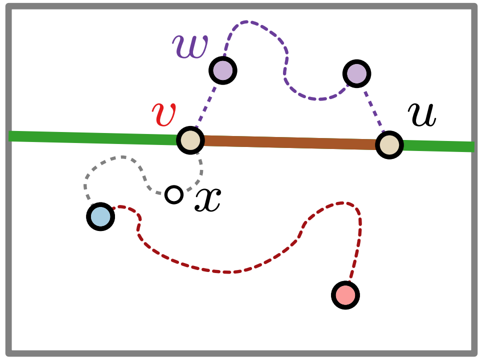

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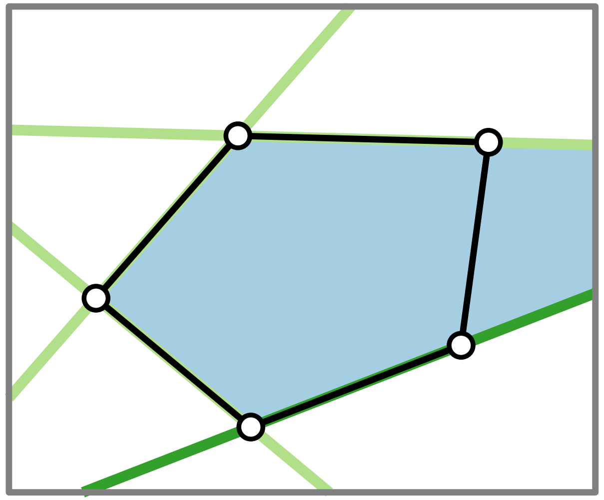

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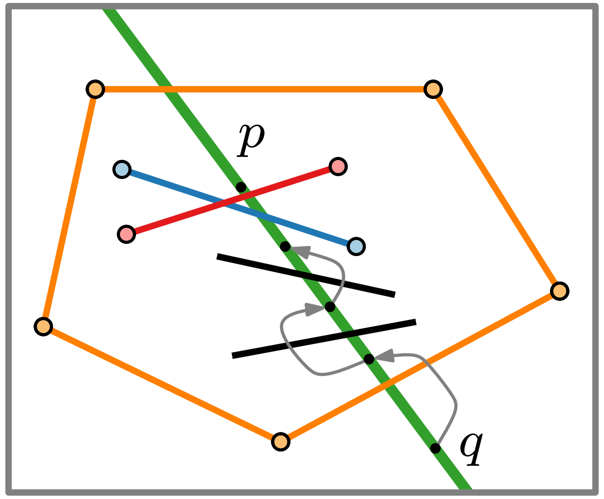

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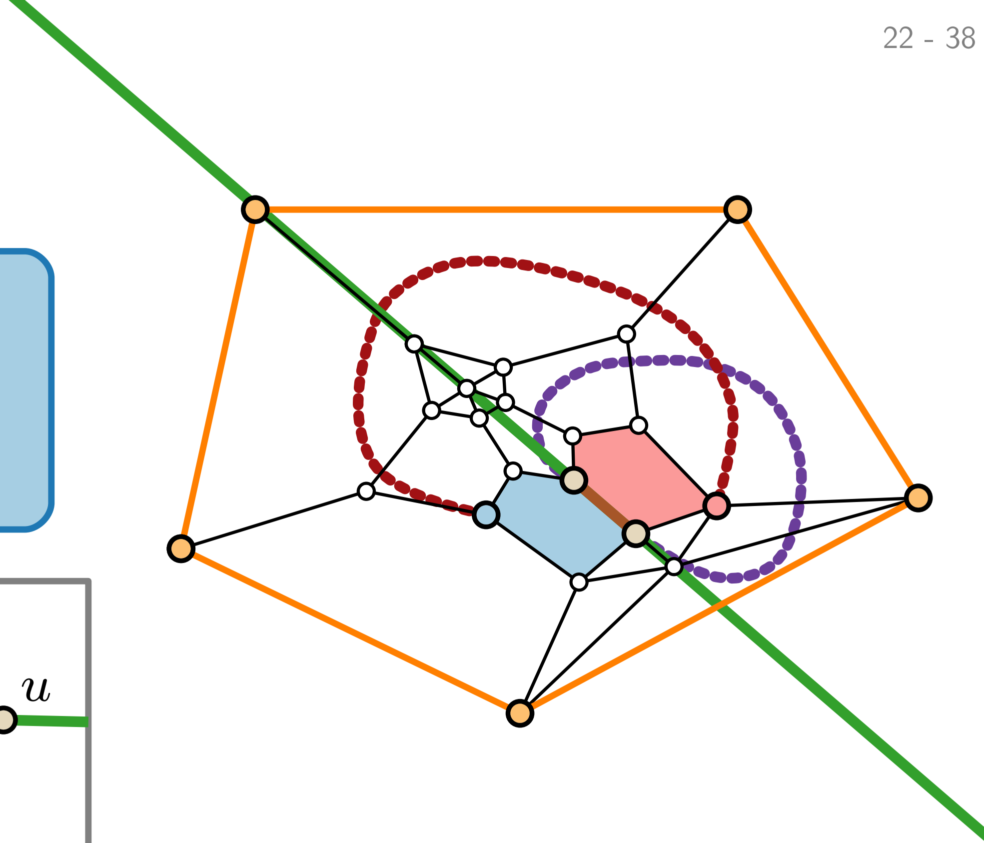

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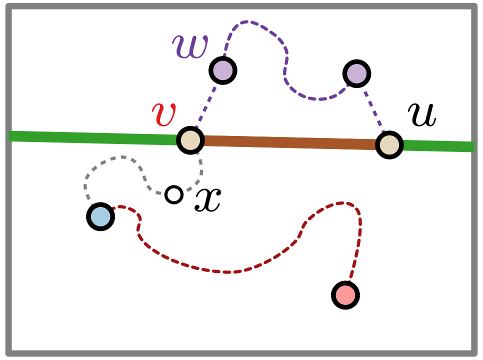

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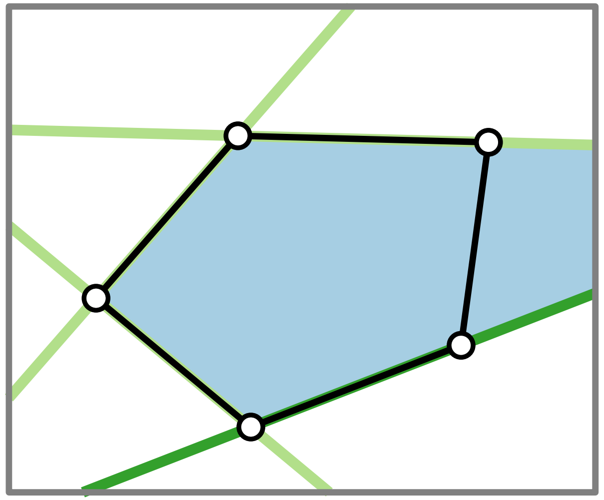

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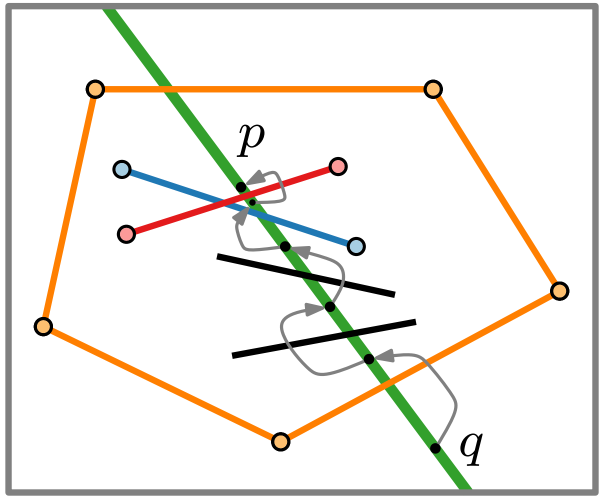
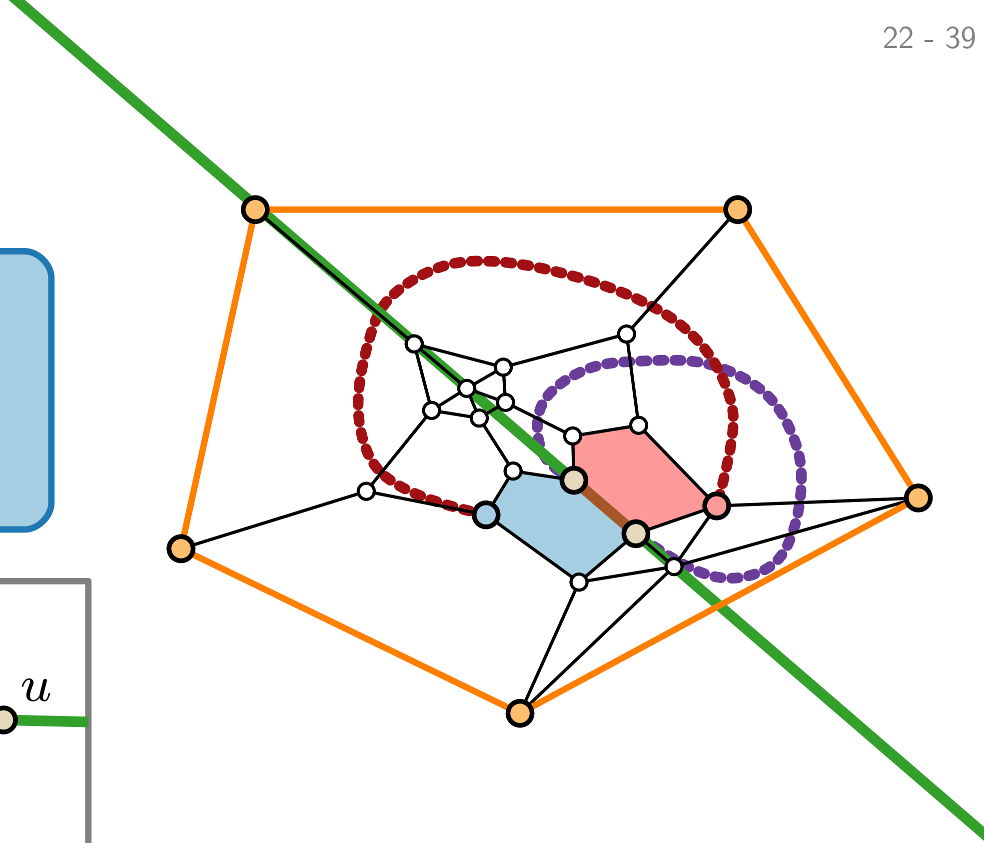


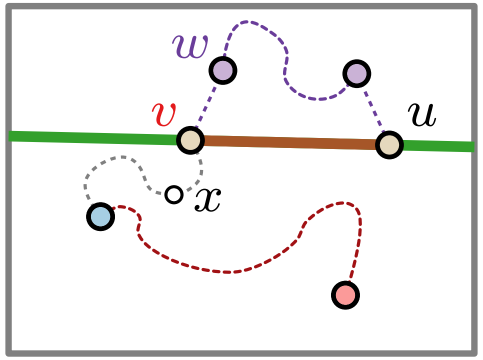
Proof of Tutte's Theorem



Lemma. Let uv be a non-boundary edge, ℓ line through uv. Then the two faces f_1, f_2 incident to uv lie completely on opposite sides of ℓ.

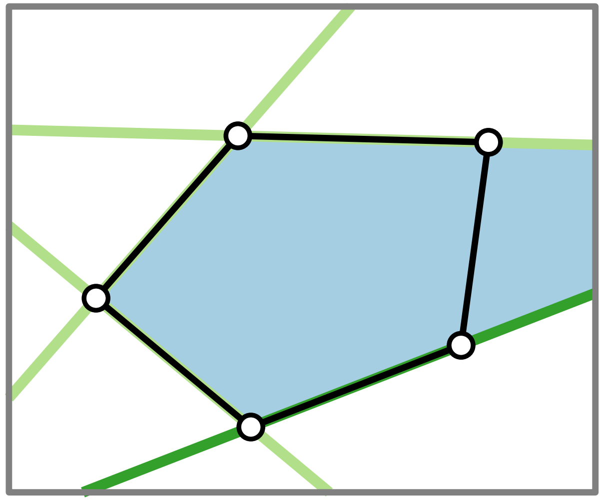
Property 1. Let $v \in V$ free, ℓ line through v.
$\exists xv \in E$ on one side of $\ell \Rightarrow \exists vw \in E$ on other side

Property 3. Let ℓ be any line.
Let V_ℓ be the set of vertices on one side of ℓ.
Then $G[V_\ell]$ is connected.



xv and vw on different sides of $\ell \Rightarrow f_1, f_2$ have angles $< \pi$ at v

Lemma. All faces are strictly convex.

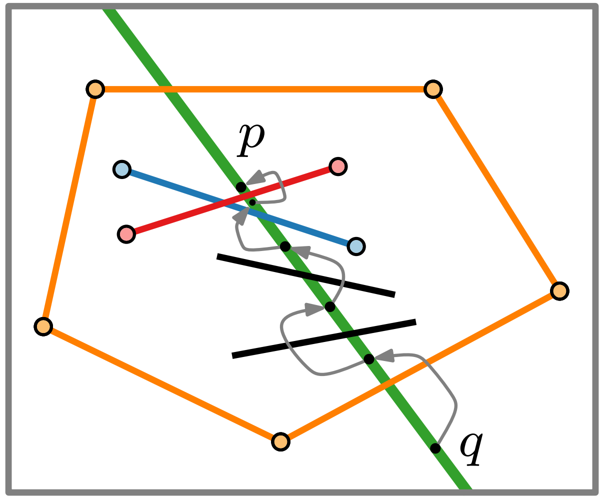


Lemma. The drawing is planar.

p inside two faces

Property 2. All free vertices lie inside C.

$\Rightarrow q$ in one face
jumping over edge
$\rightarrow$ #faces the same
$\Rightarrow p$ inside one face



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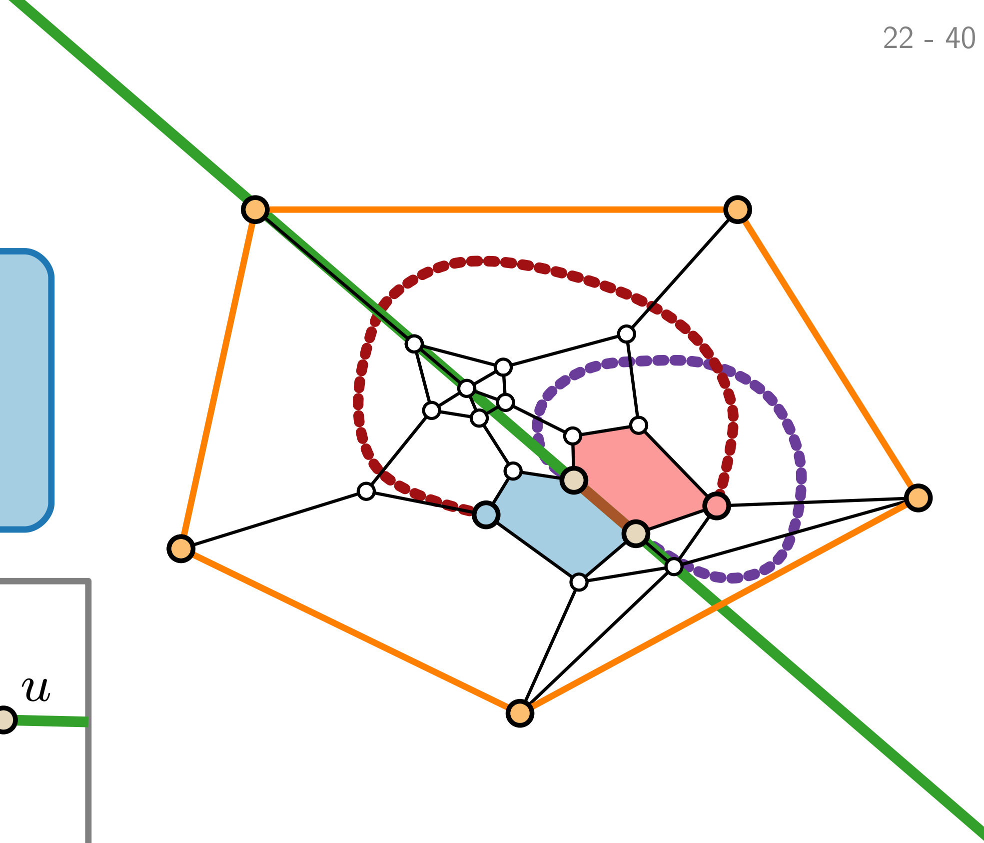

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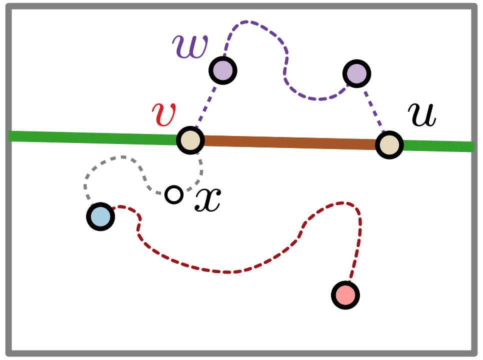

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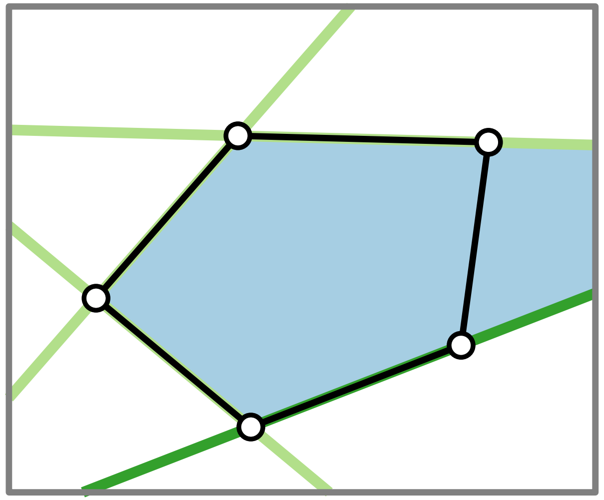

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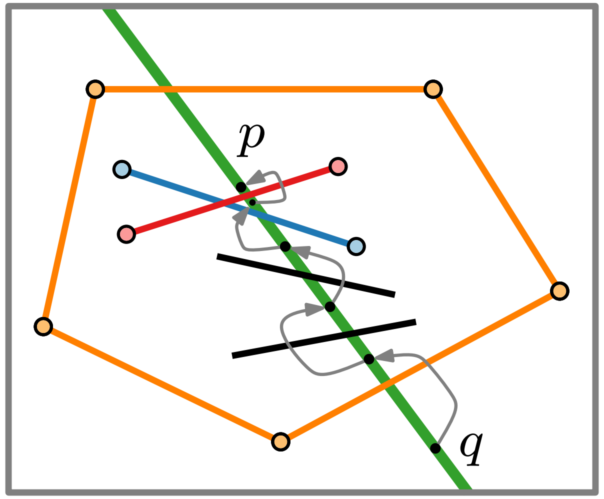

Literature

Main sources:

- [GD Ch. 10] Force-Directed Methods
- [DG Ch. 4] Drawing on Physical Analogies

Original papers:

- [Eades 1984] A heuristic for graph drawing
- [Fruchterman, Reingold 1991] Graph drawing by force-directed placement
- [Tutte 1963] How to draw a graph