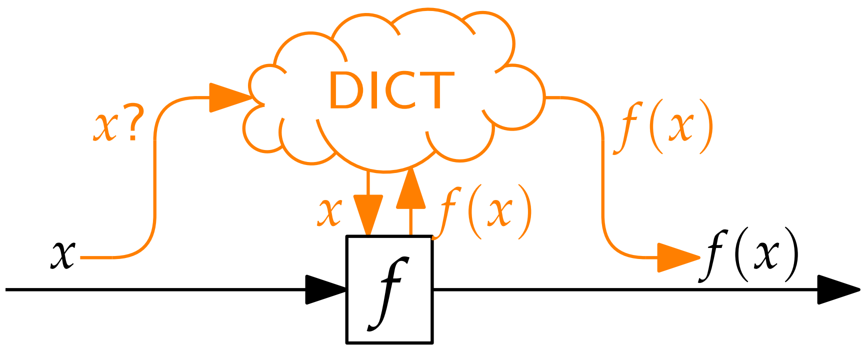
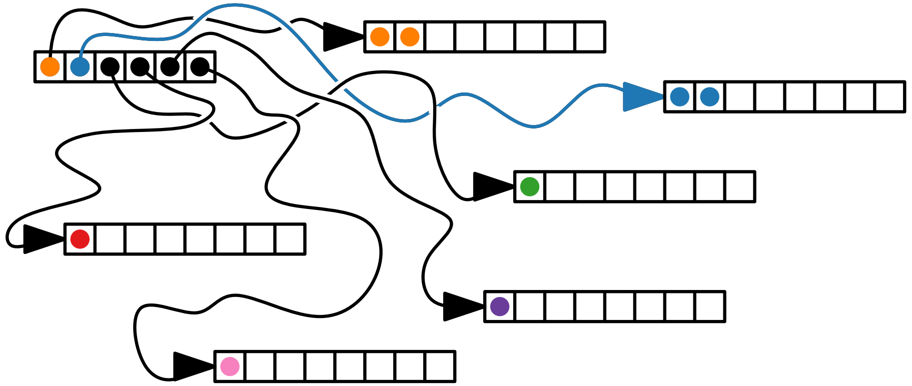
Advanced Algorithms

Algorithms in Practice

Computers are Fast · Knapsack DP · IntroSort

Tim Hegemann · WS22





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What if?

- We have lots of small instances but n_0 is large?
- Two algorithms have similar g (e.g. n and $n \log(n)$) but for one c is much higher?

"In theory, there is no difference between theory and practice;

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but, in practice, there is."

Do We Need More Detail?

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[Knuth 1998]

	Step	Operations	Time
	$N3$	`CMPA, JG, JE`	$3.5u$
Either	$N4$	`STA, INC`	$3u$
	$N5$	`INC, LDA, CMPA, JGE`	$6u$
Or	$N8$	`STX, INC`	$3u$
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Thus about $12.5u$ is spent on each record in each pass, and the total running time will be asymptotically $12.5N \lg N$, for both the average case and the worst case. This is slower than quicksort's average time, and it may not be enough better than heapsort to justify taking twice as much memory space, since the asymptotic running time of Program 5.2.3H is never more than $18N \lg N$.

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- Fixed-function Units?

Computers are Fast

Event	Latency
1 CPU cycle	0.3 ns
Level 1 cache access	0.9 ns
Level 2 cache access	3 ns
Level 3 cache access	10 ns
RAM access	100 ns
SSD I/O	10–100 μs
HDD I/O	1–10 ms
Internet (SF–NYC)	40 ms

[Brendan Gregg. "Systems Performance: Enterprise and the Cloud." 2nd Edition 2020]

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https://computers-are-fast.github.io/

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Implementing the Knapsack Dynamic Program

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KNAPSACK

Input: Finite set U, for each $u \in U$ a size $s(u) \in \mathbb{Z}^+$ and a value $v(u) \in \mathbb{Z}^+$, and positive integers B and K.

Task: Is there a subset $U' \subseteq U$ such that $\sum_{u \in U'} s(u) \leq B$ and such that $\sum_{u \in U'} v(u) \geq K$?

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Example: [$ 10 | 4] [$ 12 | 6] [$ 4 | 1] [$ 6 | 2], $B = 6$

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Knapsack is (weakly) NP-complete but can be solved in pseudo-polynomial time by dynamic programming.

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Implement it

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case class Item(weight: Int, value: Int)
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  go(items.size - 1, weight)
```

Expectation Management

[Skiena 2012]

n $f(n)$	$\lg n$	n	$n \lg n$	n^2	2^n	$n!$
10	0.003 μs	0.01 μs	0.033 μs	0.1 μs	1 μs	3.63 ms
20	0.004 μs	0.02 μs	0.086 μs	0.4 μs	1 ms	77.1 years
30	0.005 μs	0.03 μs	0.147 μs	0.9 μs	1 sec	8.4×10^{15} yrs
40	0.005 μs	0.04 μs	0.213 μs	1.6 μs	18.3 min	
50	0.006 μs	0.05 μs	0.282 μs	2.5 μs	13 days	
100	0.007 μs	0.1 μs	0.644 μs	10 μs	4×10^{13} yrs	
1,000	0.010 μs	1.00 μs	9.966 μs	1 ms		
10,000	0.013 μs	10 μs	130 μs	100 ms		
100,000	0.017 μs	0.10 ms	1.67 ms	10 sec		
1,000,000	0.020 μs	1 ms	19.93 ms	16.7 min		
10,000,000	0.023 μs	0.01 sec	0.23 sec	1.16 days		
100,000,000	0.027 μs	0.10 sec	2.66 sec	115.7 days		
1,000,000,000	0.030 μs	1 sec	29.90 sec	31.7 years		

Figure 2.4: Growth rates of common functions measured in nanoseconds

Running Times

$|U| =$ 30 $\qquad B = 10|U|$

Running Times

$B = 10|U|$

$|U| =$	30
Scala naïve	1.15 s

Running Times

$\lvert U\rvert =$	30	100
Scala naïve	1.15 s	

$B = 10\lvert U\rvert$

Running Times

$\lvert U \rvert =$	30	100
Scala naïve	1.15 s	$\approx 2\,700 \times$ age of the universe

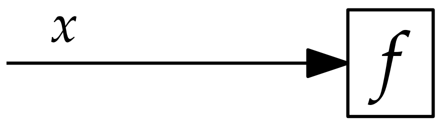
$B = 10\lvert U \rvert$

Memoization

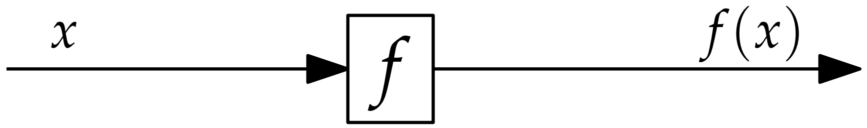
Memoization

$\boxed{f}$

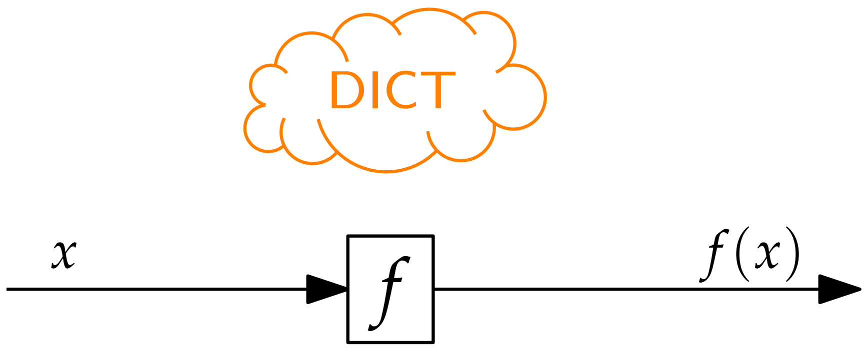
Memoization



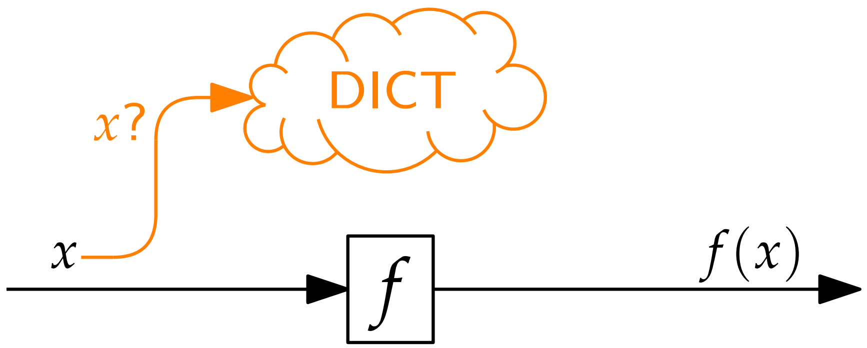
Memoization



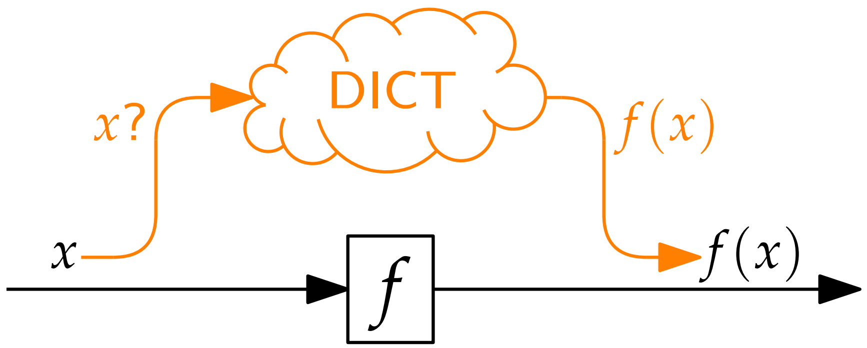
Memoization



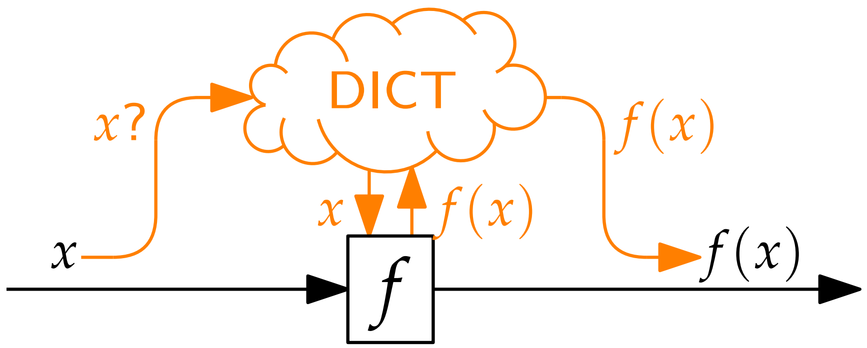
Memoization



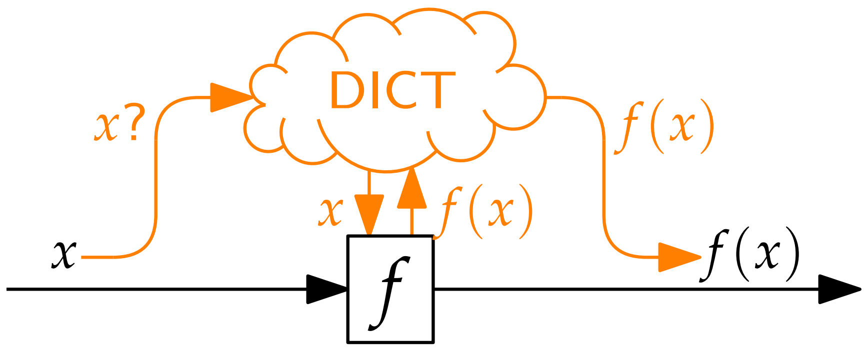
Memoization



Memoization

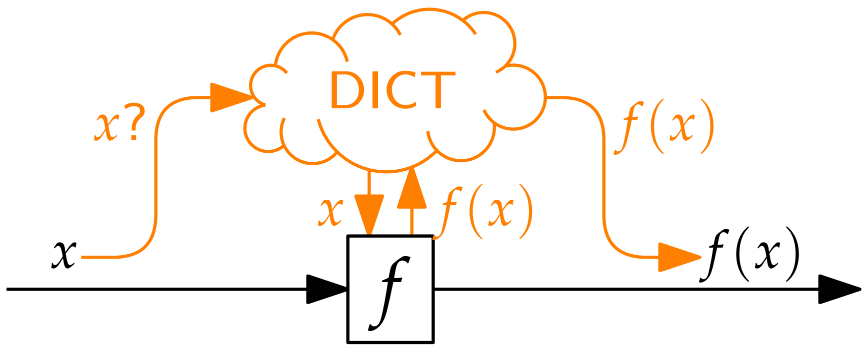


Memoization




```
def momoized[K, V](f: K ⇒ V): K ⇒ V =
  val dict = mutable.Map.empty[K, V]
  (key: K) ⇒ dict.getOrElseUpdate(key, f(key))
```

Memoization




```scala
def momoized[K, V](f: K ⇒ V): K ⇒ V =
  val dict = mutable.Map.empty[K, V]
  (key: K) ⇒ dict.getOrElseUpdate(key, f(key))
```

```scala
lazy val go: ((Int, Int)) ⇒ Int =
  memoized((args: (Int, Int)) ⇒
    val (i, weight) = args
    // fill in the body of the old go
  )
```

Running Times

$\lvert U\rvert =$	30	100
Scala naïve	1.15 s	

$B = 10\lvert U\rvert$

Running Times

$\lvert U \rvert =$	30	100
Scala naïve	1.15 s	
Scala memoized	230 ms	

$B = 10\lvert U \rvert$

Running Times

$\lvert U \rvert =$	30	100
Scala naïve	1.15 s	
Scala memoized	230 ms	267 ms

$B = 10\lvert U \rvert$

Running Times

$\lvert U\rvert =$	30	100	1K
Scala naïve	1.15 s		
Scala memoized	230 ms	267 ms	

$B = 10\lvert U\rvert$

Running Times

$\lvert U \rvert =$	30	100	1K
Scala naïve	1.15 s		
Scala memoized	230 ms	267 ms	...

$B = 10\lvert U \rvert$

Running Times

```
Exception in thread "main" java.lang.StackOverflowError
    at scala.runtime.BoxesRunTime.equalsNumNum(BoxesRunTime.java:148)
    at scala.runtime.BoxesRunTime.equalsNumObject(BoxesRunTime.java:138)
    at scala.runtime.BoxesRunTime.equals2(BoxesRunTime.java:127)
    at scala.runtime.BoxesRunTime.equals(BoxesRunTime.java:119)
    at scala.Tuple2.equals(Tuple2.scala:24)
    at scala.runtime.BoxesRunTime.equals2(BoxesRunTime.java:133)
    at scala.runtime.BoxesRunTime.equals(BoxesRunTime.java:119)
    at scala.collection.mutable.HashMap$Node.findNode(HashMap.scala:621)
    at scala.collection.mutable.HashMap.getOrElseUpdate(HashMap.scala:449)
    at y.Memoized$.momoized$$anonfun$1(Y.scala:64)
    at y.Memoized$.go$lzyINIT1$1$$anonfun$1(Y.scala:72)
    at y.Memoized$.momoized$$anonfun$1$$anonfun$1(Y.scala:64)
    at scala.collection.mutable.HashMap.getOrElseUpdate(HashMap.scala:454)
    at y.Memoized$.momoized$$anonfun$1(Y.scala:64)
    ...
```

Running Times

$\|U\| =$	30	100	1K
Scala naïve	1.15 s		
Scala memoized	230 ms	267 ms	⚡SO
Scala TCO			

$B = 10\|U\|$

Covered in this talk:

Running Times

$\lvert U \rvert =$	30	100	1K
Scala naïve	1.15 s		
Scala memoized	230 ms	267 ms	⚡SO
Scala TCO		273 ms	

$B = 10\lvert U \rvert$

Covered in this talk:

Running Times

$B = 10|U|$

$\lvert U \rvert =$	30	100	1K
Scala naïve	1.15 s		
Scala memoized	230 ms	267 ms	⚡SO
Scala TCO		273 ms	2.30 s

Covered in this talk:

Running Times

$B = 10|U|$

$\lvert U \rvert =$	30	100	1K	10K
Scala naïve	1.15 s			
Scala memoized	230 ms	267 ms	⚡SO	
Scala TCO		273 ms	2.30 s	

Covered in this talk:

Running Times

$B = 10|U|$

$\lvert U \rvert =$	30	100	1K	10K
Scala naïve	1.15 s			
Scala memoized	230 ms	267 ms	⚡SO	
Scala TCO		273 ms	2.30 s	...

Covered in this talk:

Running Times

$\lvert U \rvert =$	30	100	1K	10K
Scala naïve	1.15 s			
Scala memoized	230 ms	267 ms	⚡SO	
Scala TCO		273 ms	2.30 s	. . .

$B = 10\lvert U \rvert$

```
Exception in thread "main" java.lang.OutOfMemoryError: Java heap s
    at java.base/java.lang.Integer.valueOf(Integer.java:1077)
    at scala.runtime.BoxesRunTime.boxToInteger(BoxesRunTime.java:6
    at y.TCO$.go$3(Y.scala:112)
    at y.TCO$.solve(Y.scala:121)
    at y.TCO$.runTco(Y.scala:82)
    at y.runTco.main(Y.scala:80)
```

Covered in this talk:

Running Times

$B = 10|U|$

$\|U\| =$	30	100	1K	10K
Scala naïve	1.15 s			
Scala memoized	230 ms	267 ms	⚡SO	
Scala TCO		273 ms	2.30 s	⚡OOM

Covered in this talk:

Reducing the Memory Footprint

Idea:

Reducing the Memory Footprint

Idea:

- Use Itearation instead of Recursion
- Use Arrays instead of a Dictionary

Reducing the Memory Footprint

```
def solve(items: Seq[Item], weight: Int) =
  val dict = Array.fill(items.size, weight + 1)(0)
```

Reducing the Memory Footprint

```
def solve(items: Seq[Item], weight: Int) =
  val dict = Array.fill(items.size, weight + 1)(0)
  val head = items.head
  for w ← head.weight to weight do dict(0)(w) = head.value
```

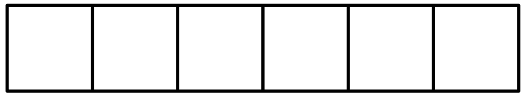
Reducing the Memory Footprint

```scala
def solve(items: Seq[Item], weight: Int) =
  val dict = Array.fill(items.size, weight + 1)(0)
  val head = items.head
  for w ← head.weight to weight do dict(0)(w) = head.value
  for
    (item, i) ← items.zipWithIndex.tail
    w ← 0 to weight
  do
    val dontTake = dict(i - 1)(w)
    if w ≥ item.weight then
      val take = dict(i - 1)(w - item.weight) + item.value
      dict(i)(w) = dontTake max take
    else dict(i)(w) = dontTake
```

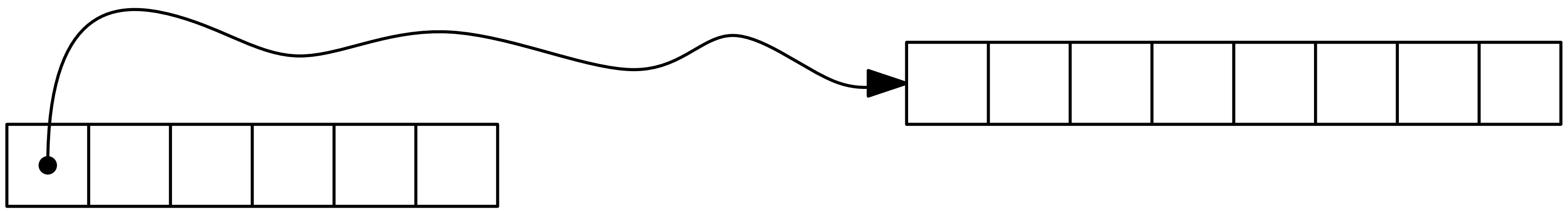
Reducing the Memory Footprint

```
def solve(items: Seq[Item], weight: Int) =
  val dict = Array.fill(items.size, weight + 1)(0)
  val head = items.head
  for w ← head.weight to weight do dict(0)(w) = head.value
  for
    (item, i) ← items.zipWithIndex.tail
    w ← 0 to weight
  do
    val dontTake = dict(i - 1)(w)
    if w ⩾ item.weight then
      val take = dict(i - 1)(w - item.weight) + item.value
      dict(i)(w) = dontTake max take
    else dict(i)(w) = dontTake
  dict(items.size - 1)(weight)
```

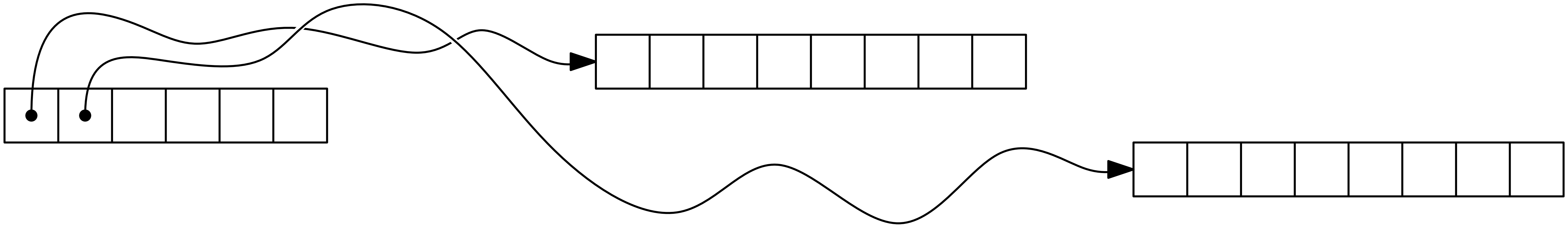
Arrays and Memory Locality



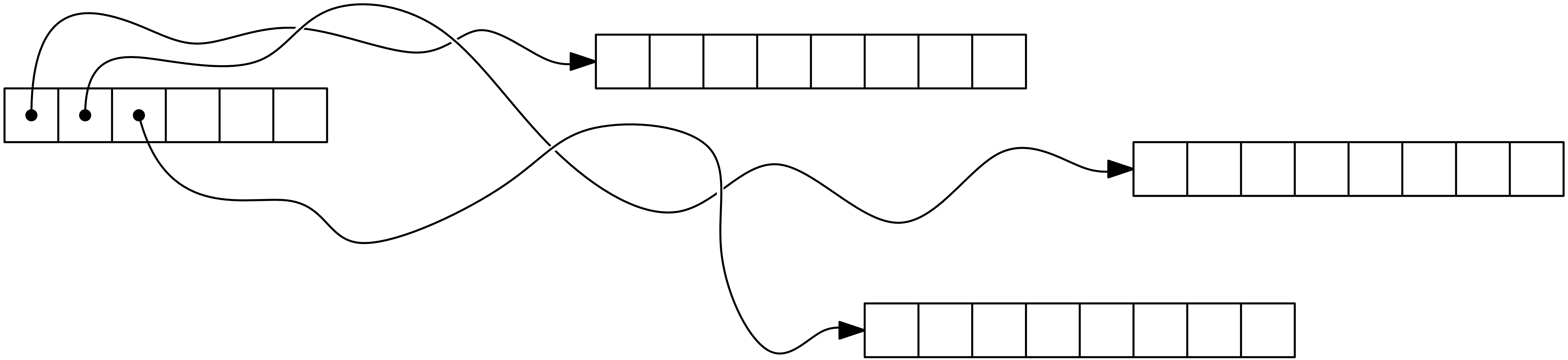
Arrays and Memory Locality



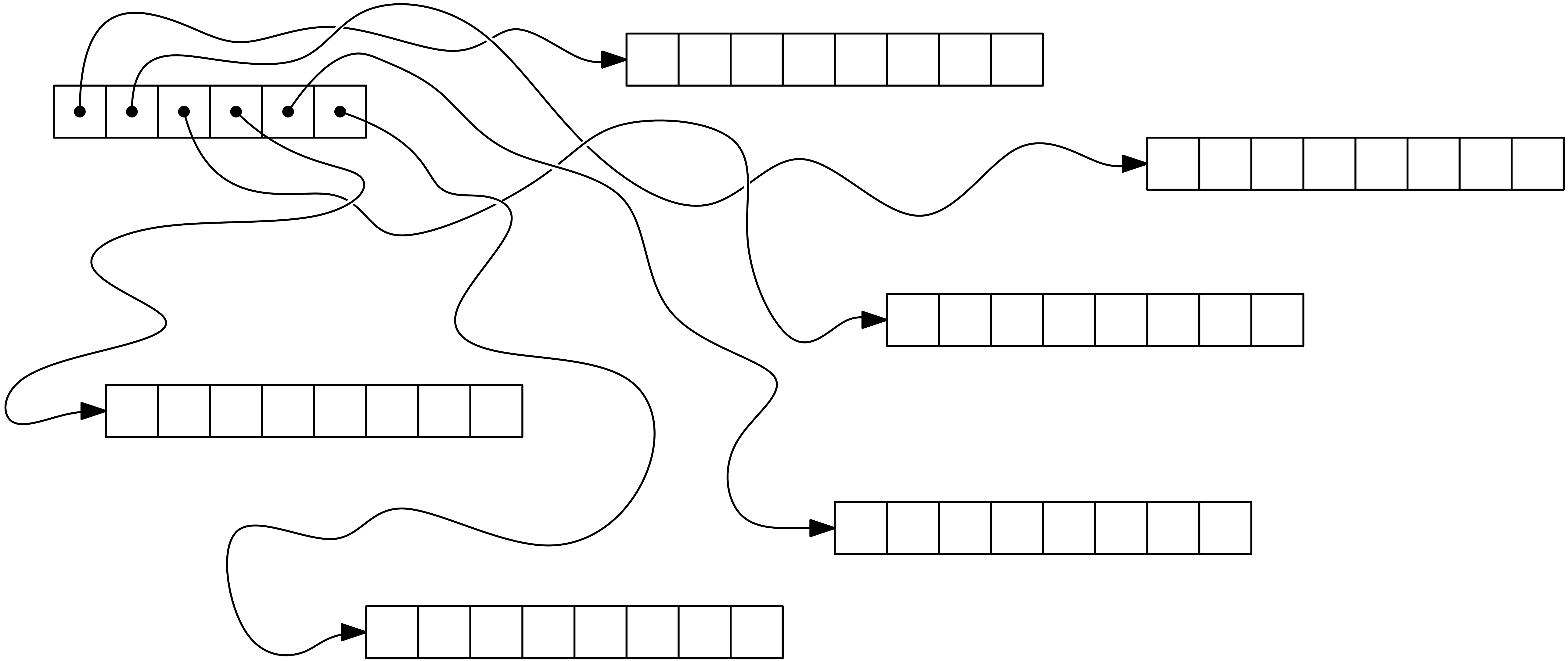
Arrays and Memory Locality



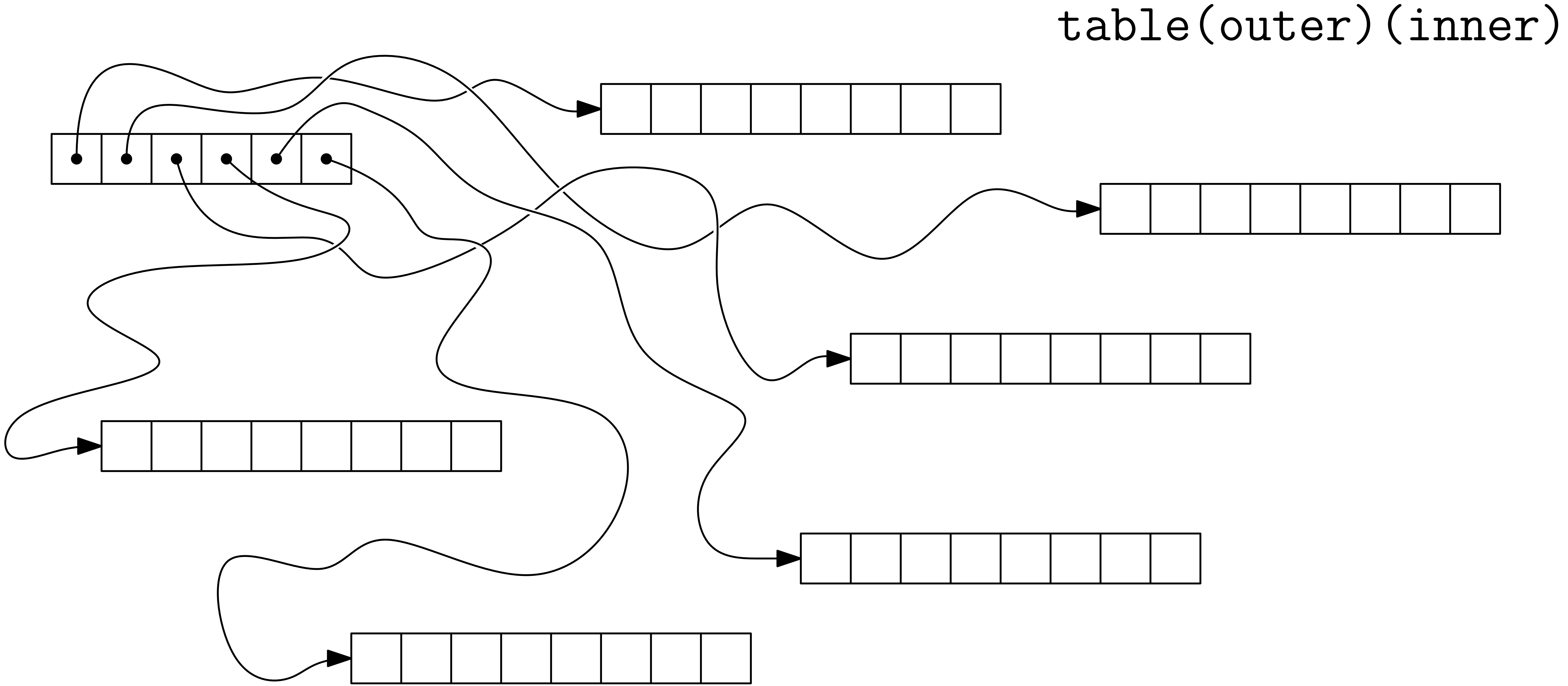
Arrays and Memory Locality



Arrays and Memory Locality

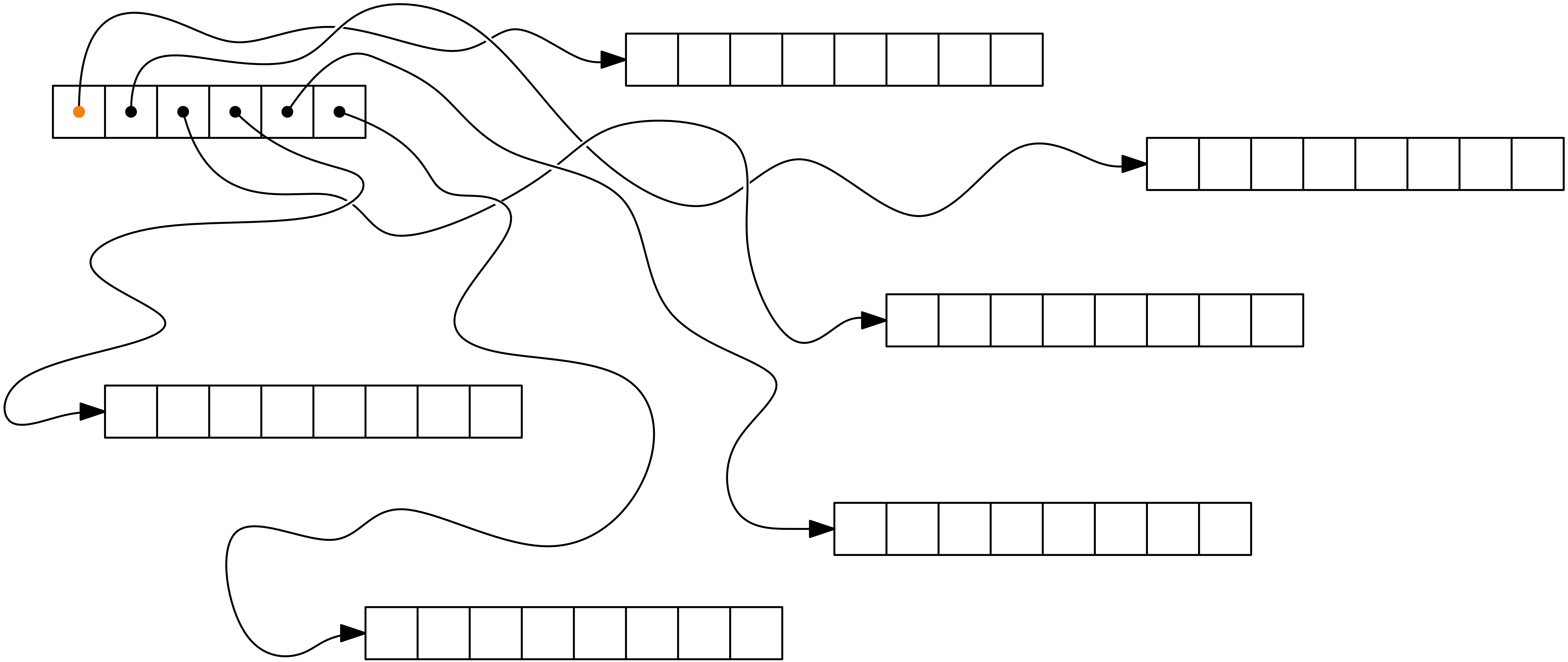


Arrays and Memory Locality



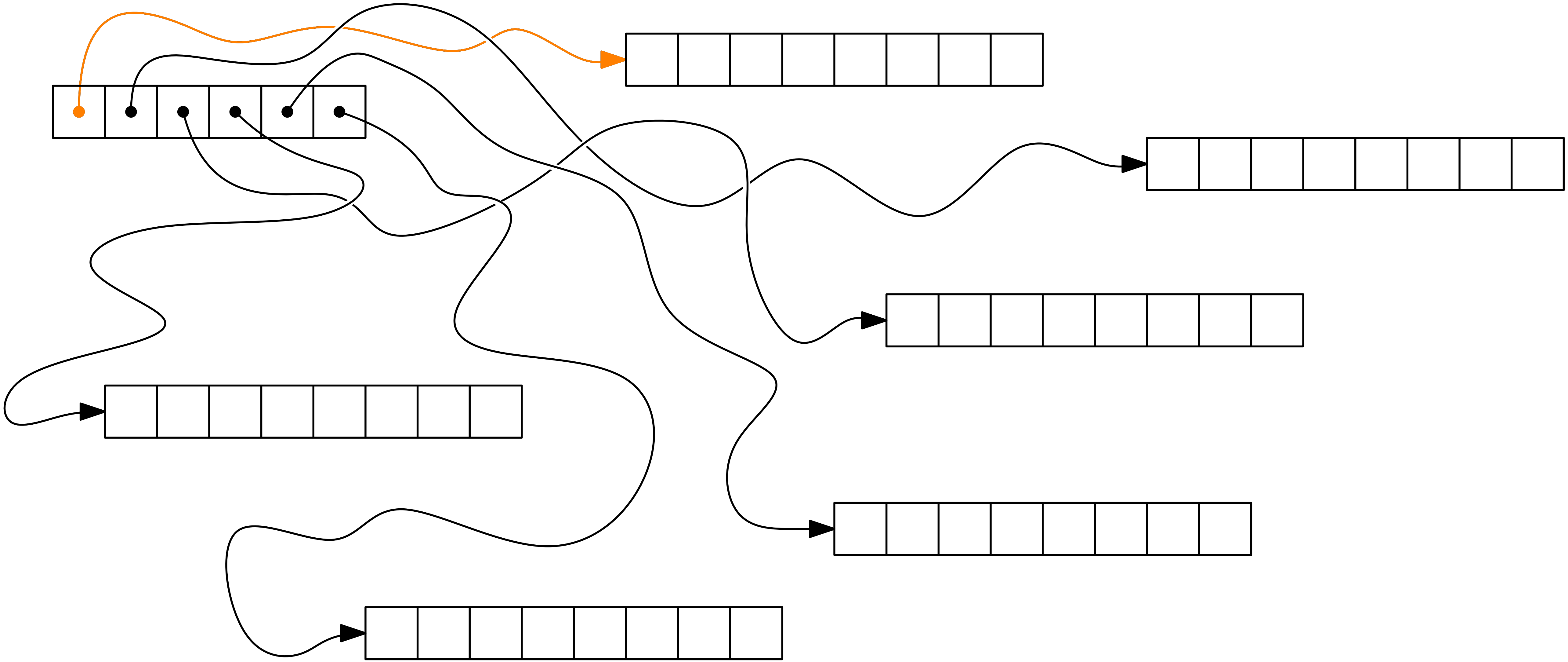
Arrays and Memory Locality

`table(outer)(inner)`



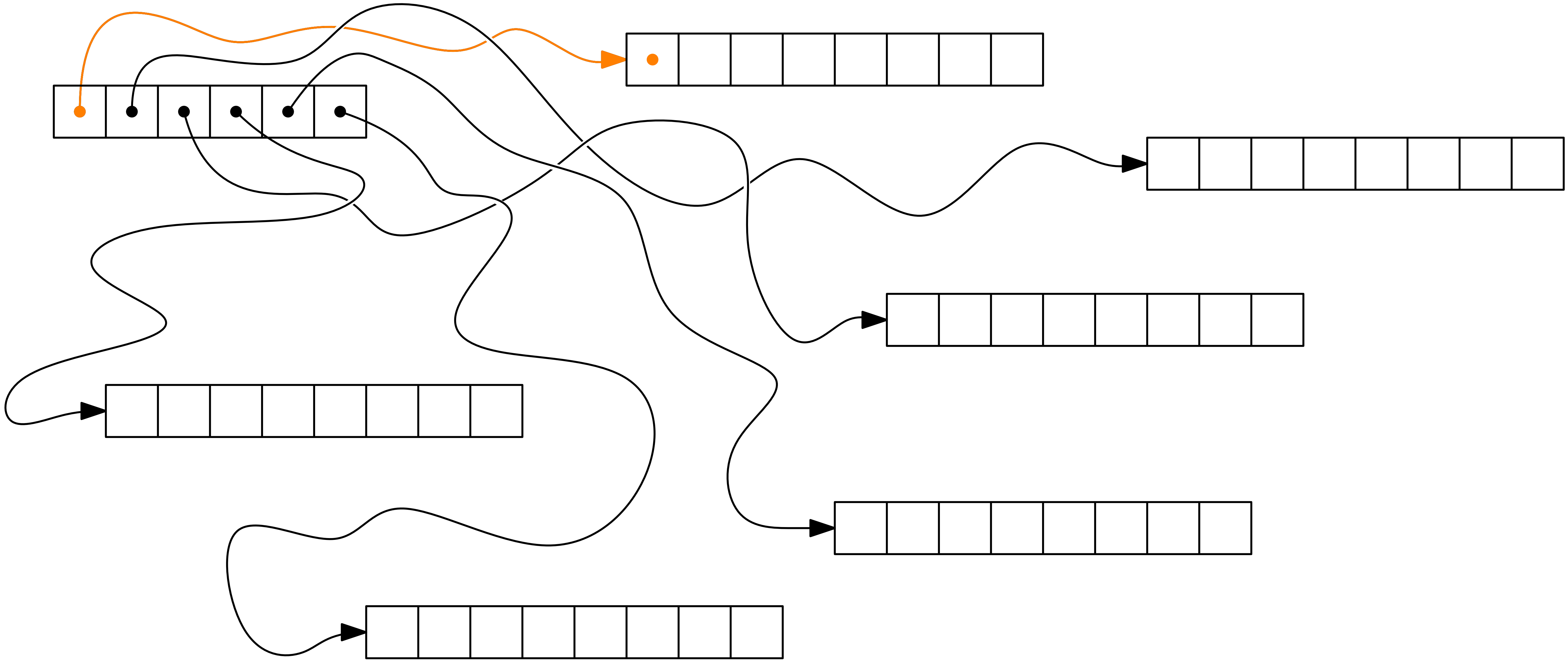
Arrays and Memory Locality

`table(outer)(inner)`

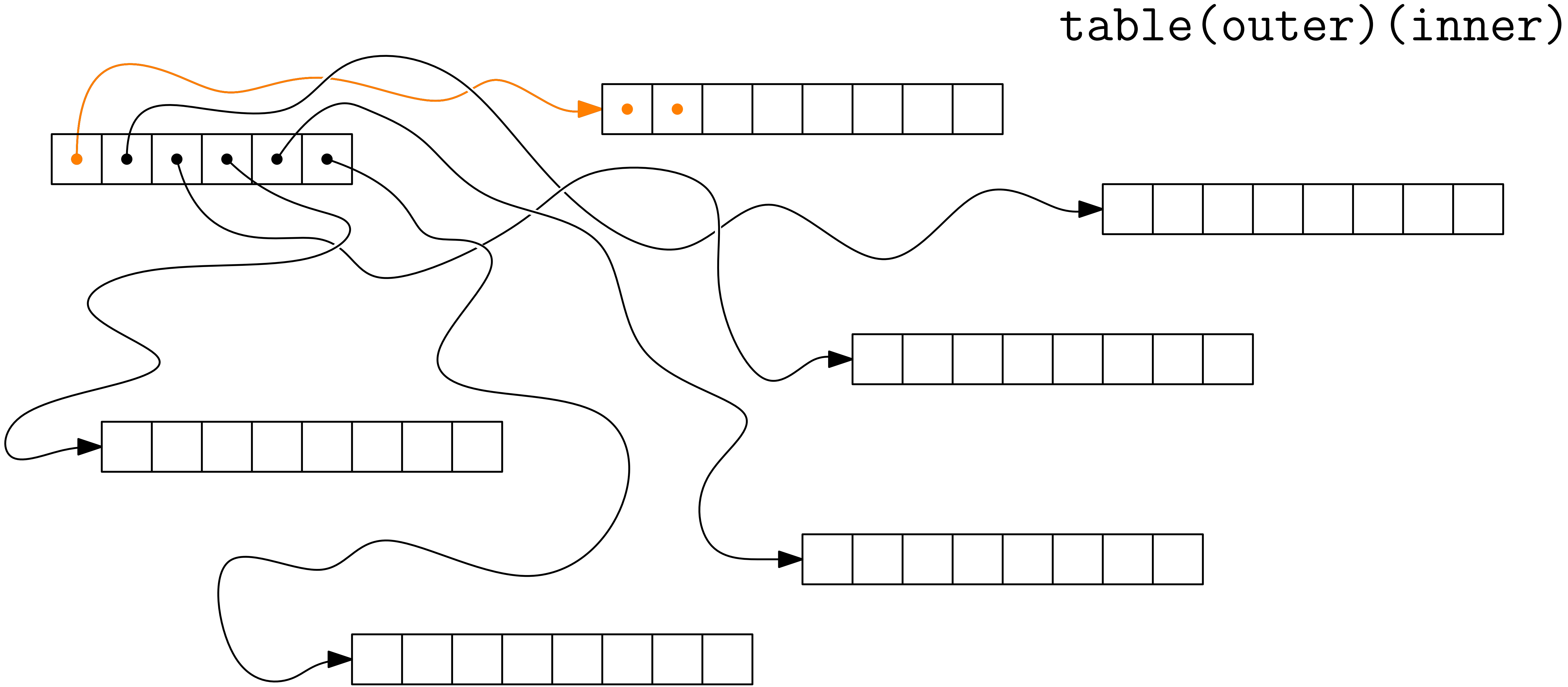


Arrays and Memory Locality

`table(outer)(inner)`



Arrays and Memory Locality



Arrays and Memory Locality

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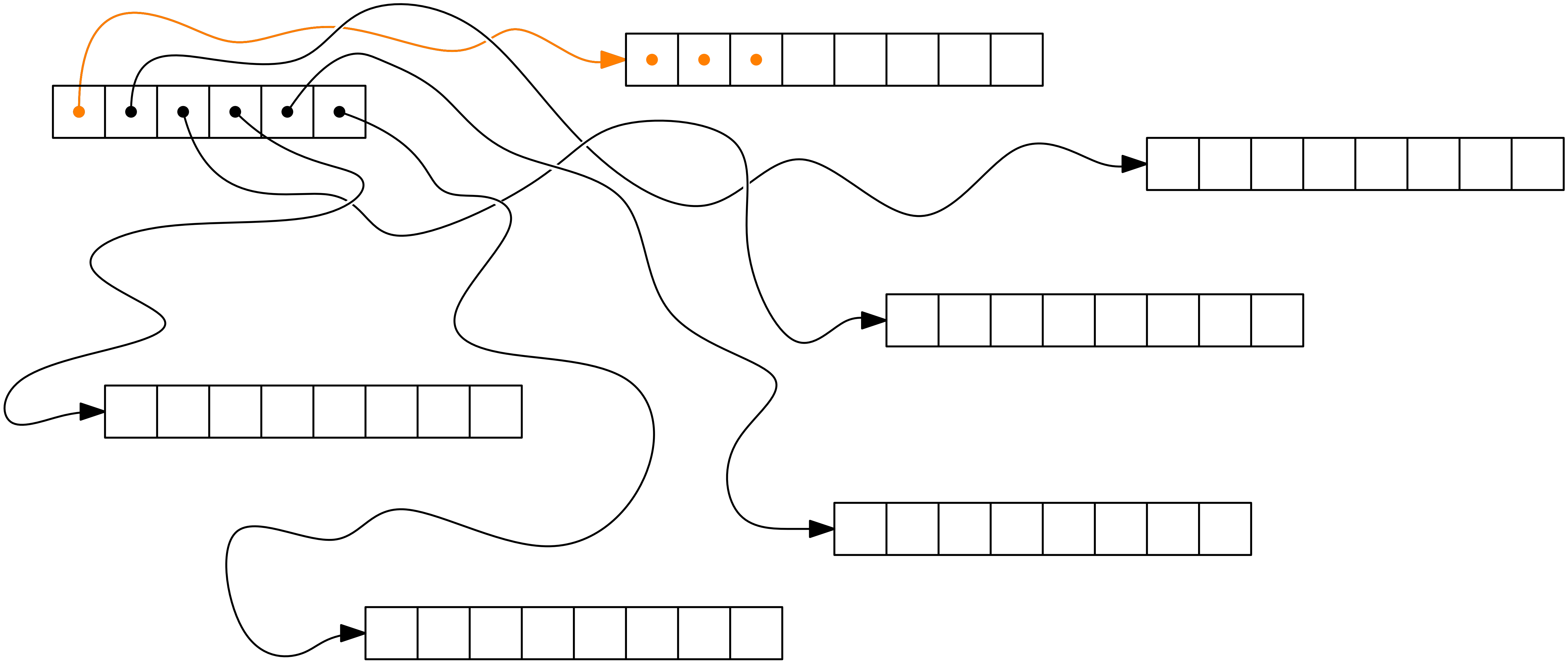
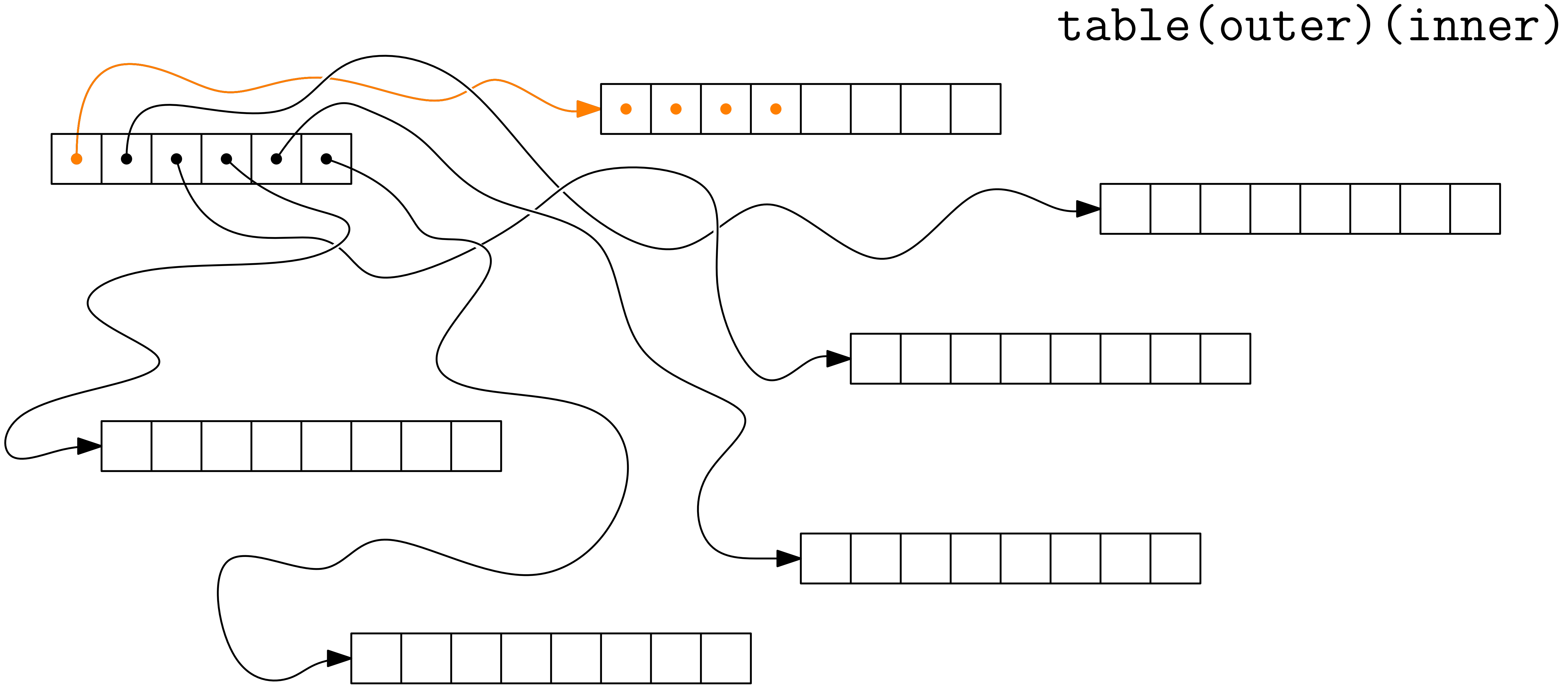


Arrays and Memory Locality



Arrays and Memory Locality

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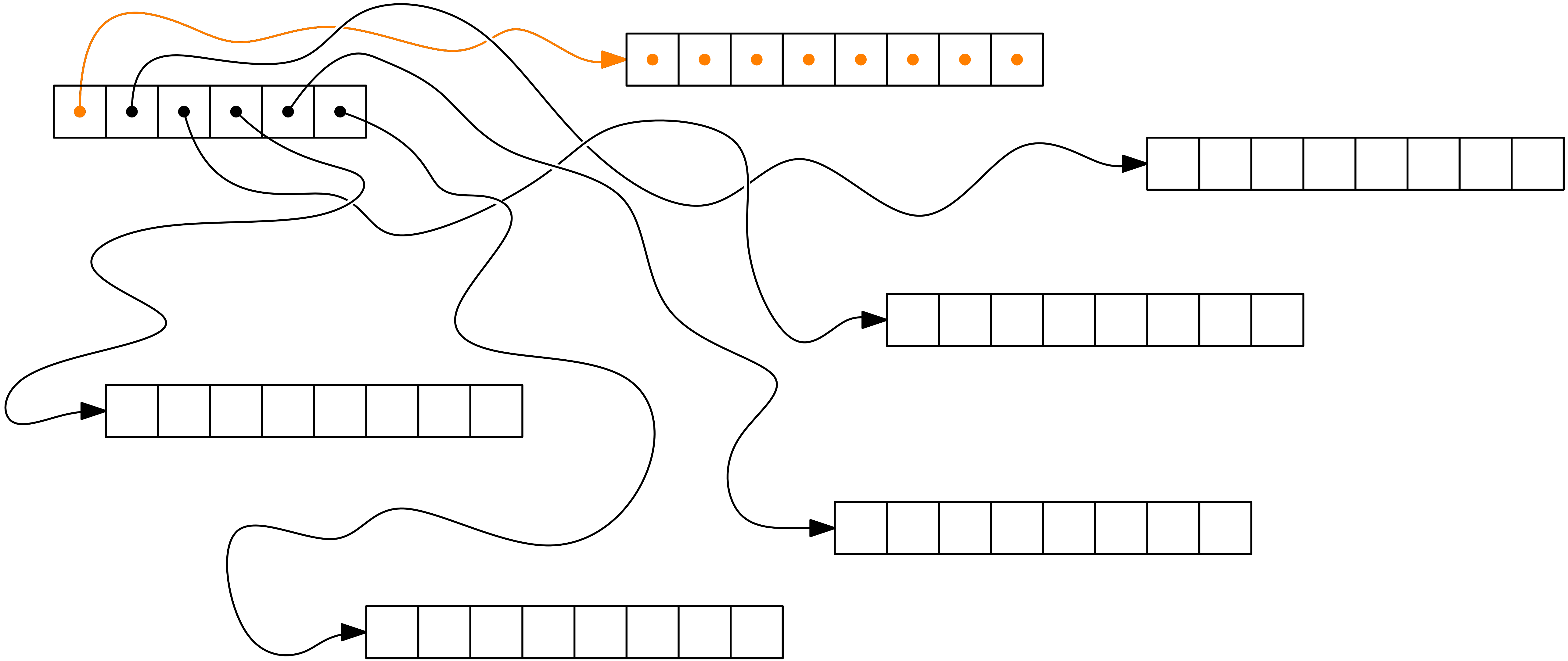
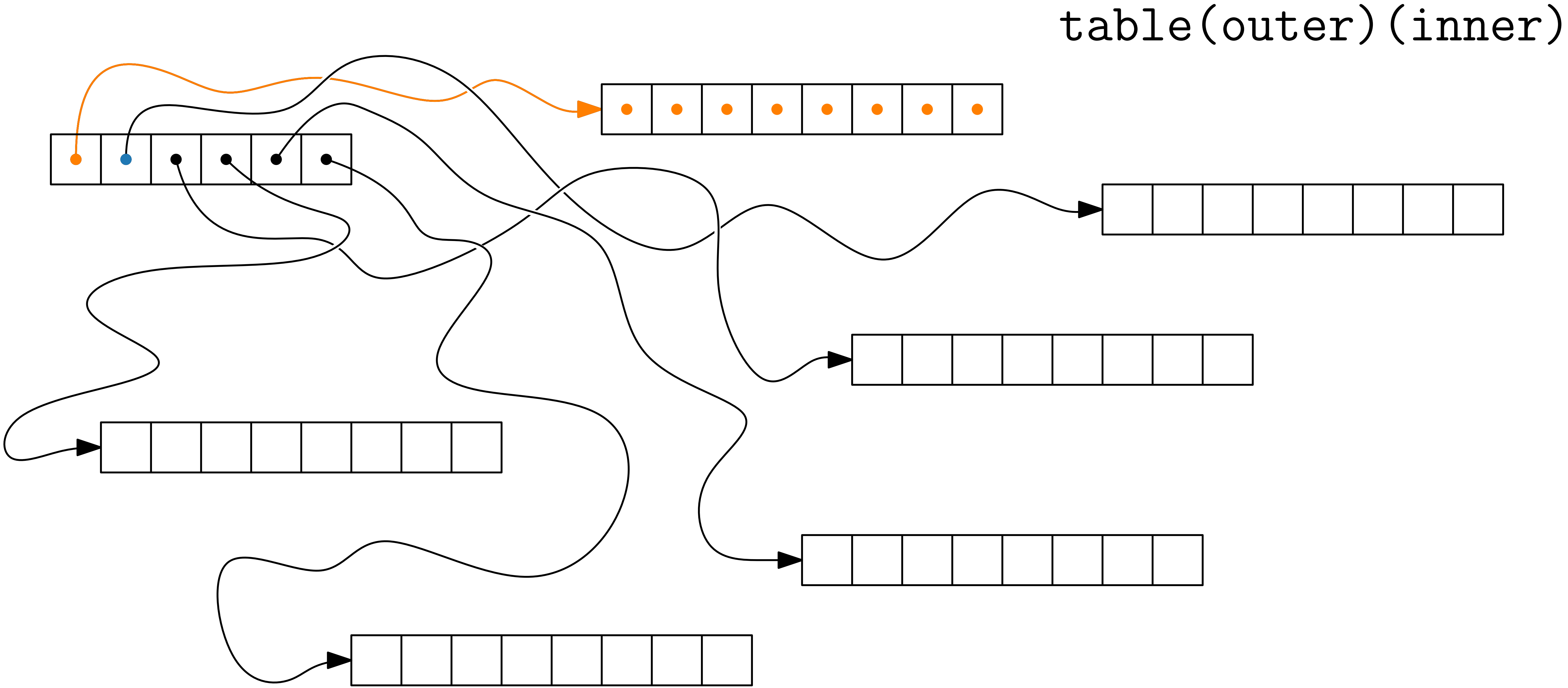
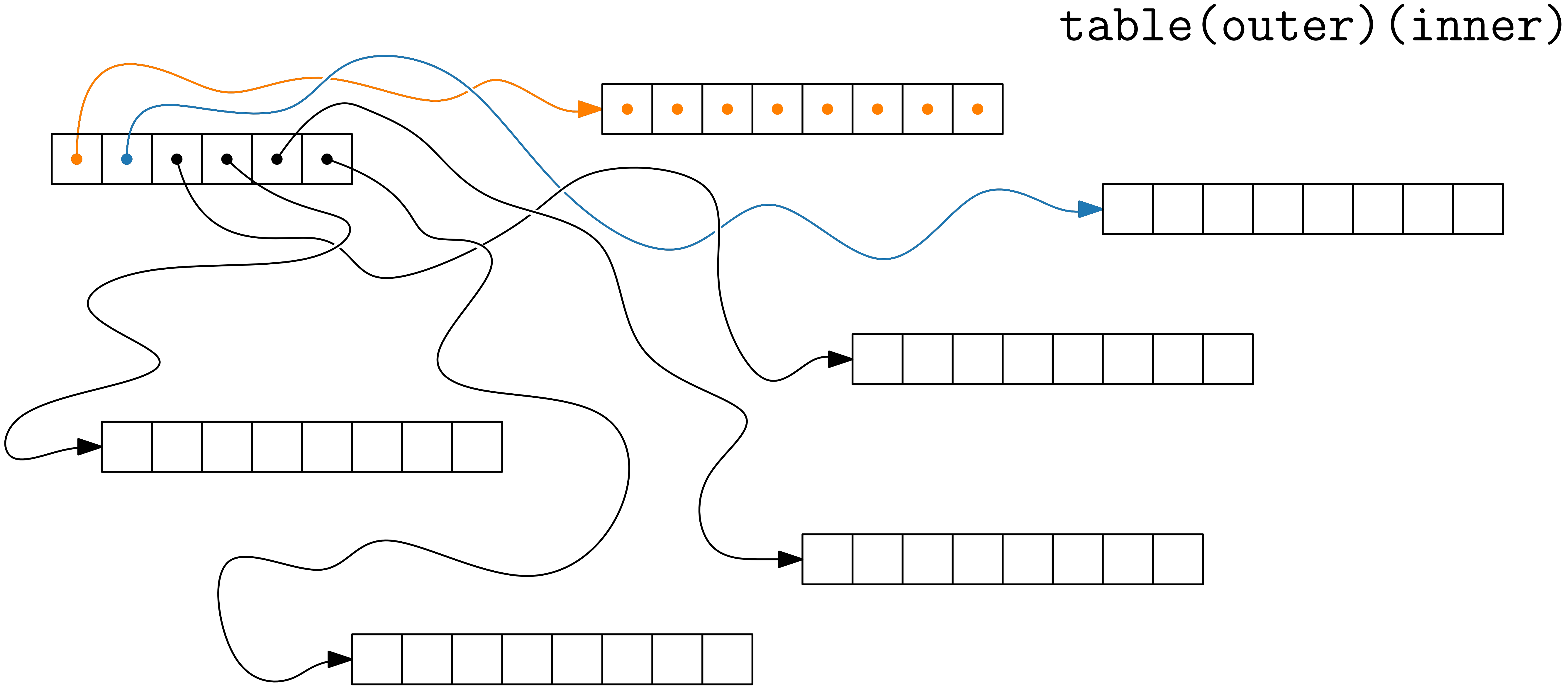


Arrays and Memory Locality

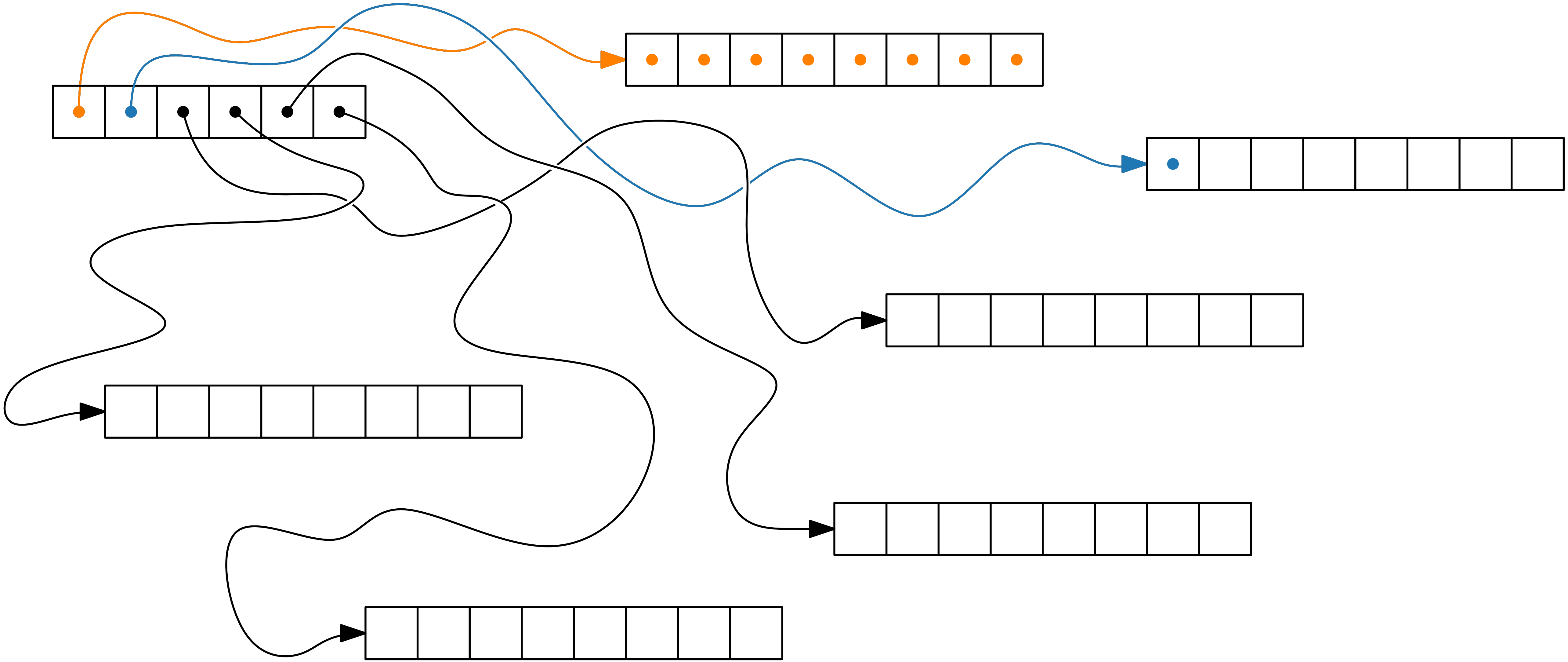


Arrays and Memory Locality

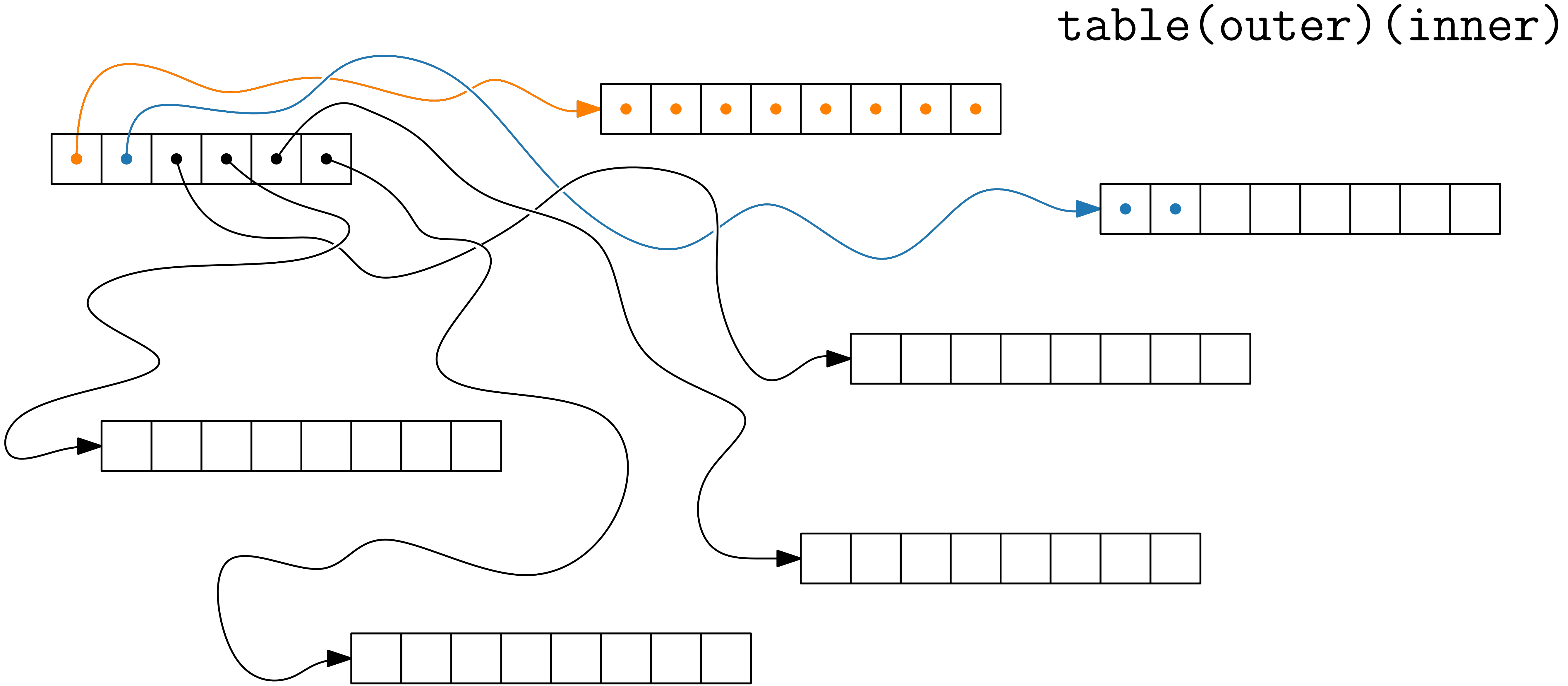


Arrays and Memory Locality

`table(outer)(inner)`



Arrays and Memory Locality



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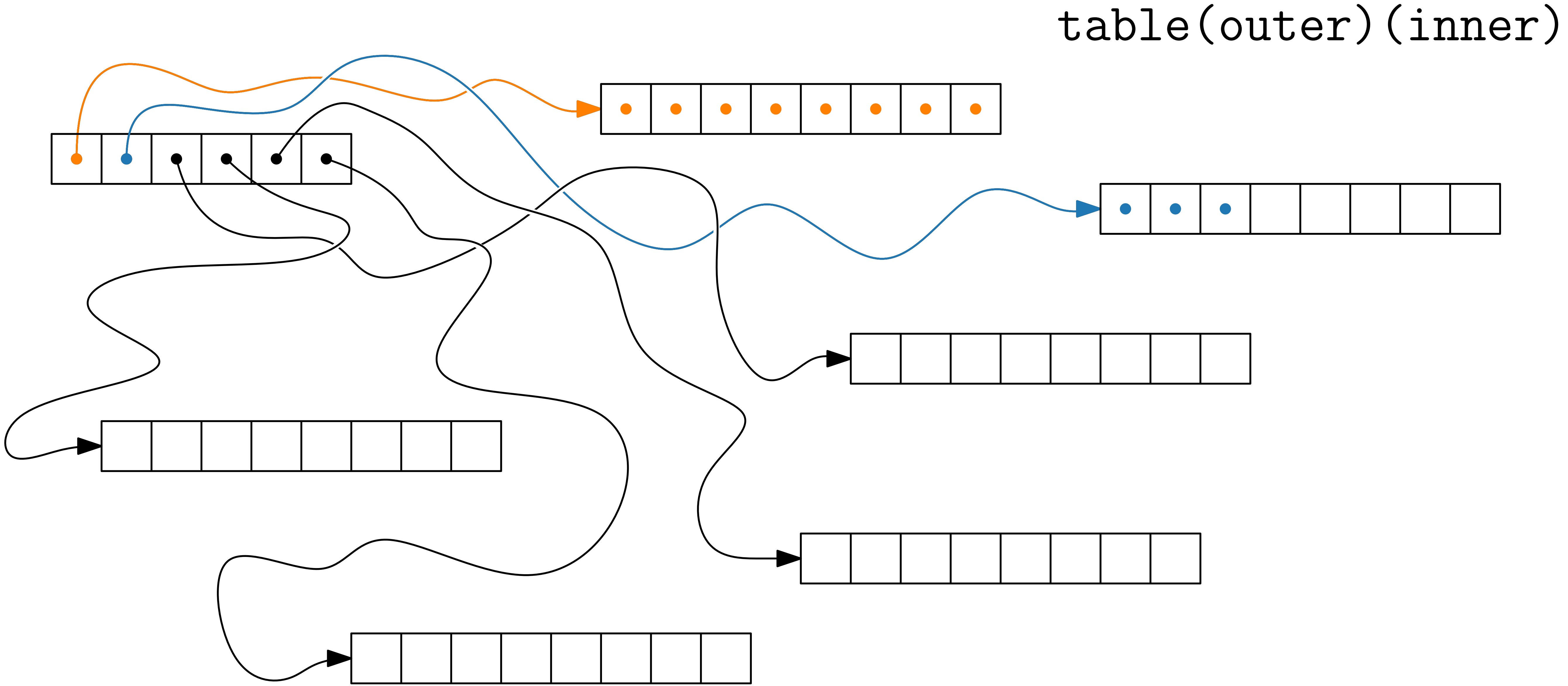

Arrays and Memory Locality

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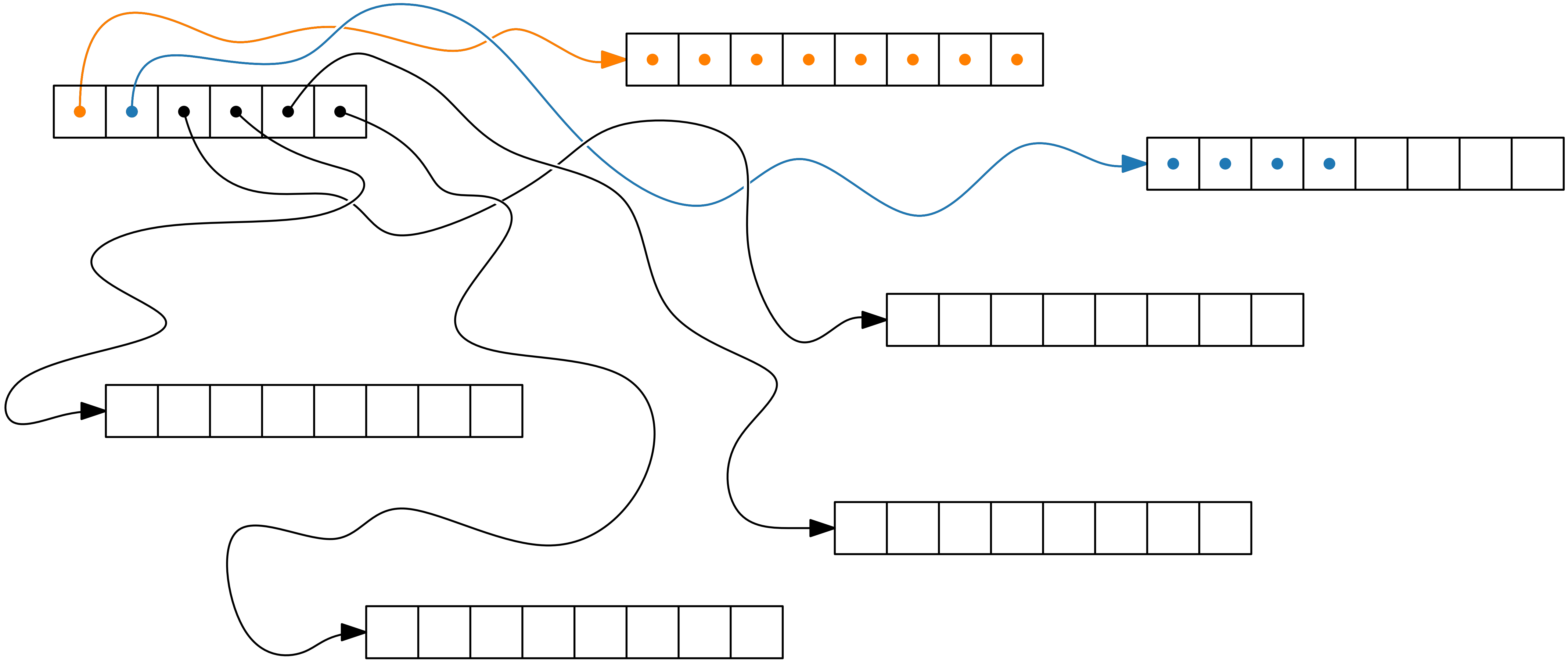
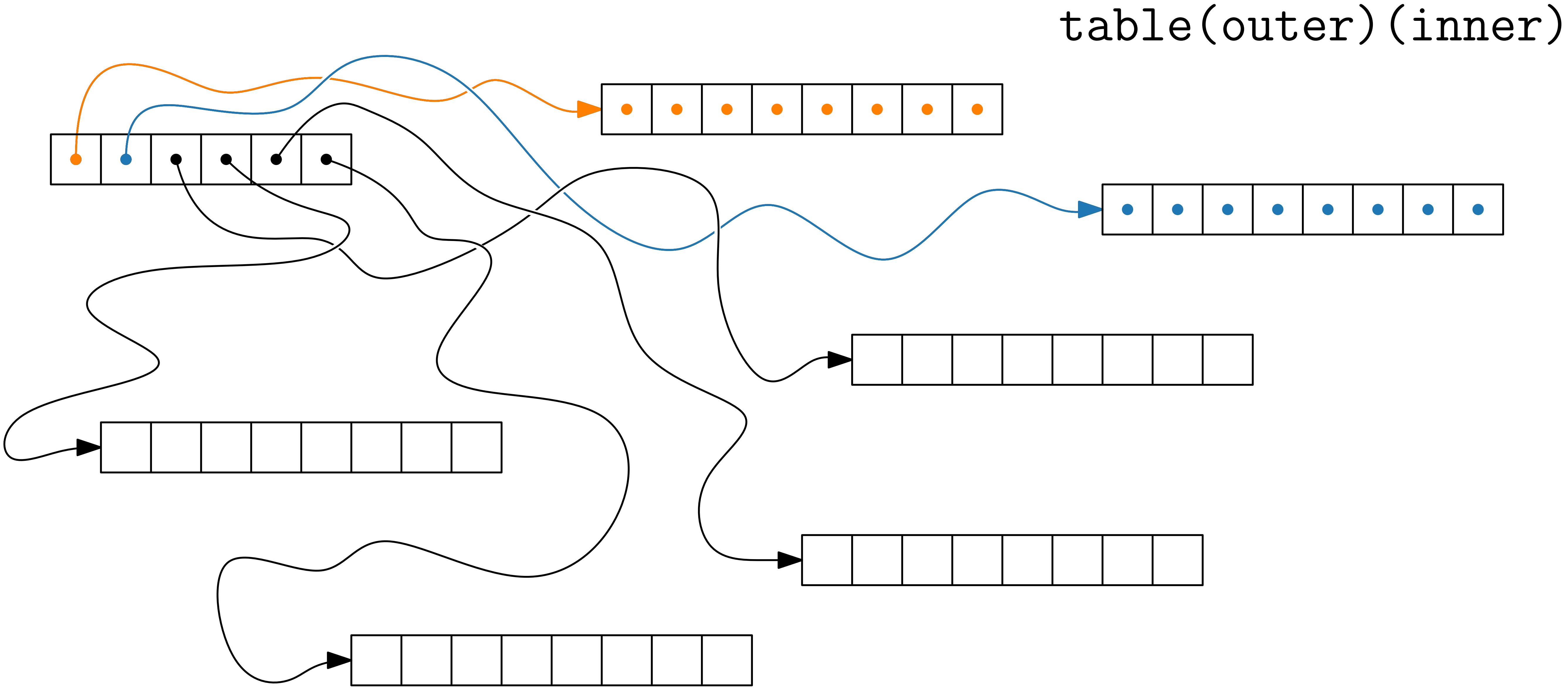
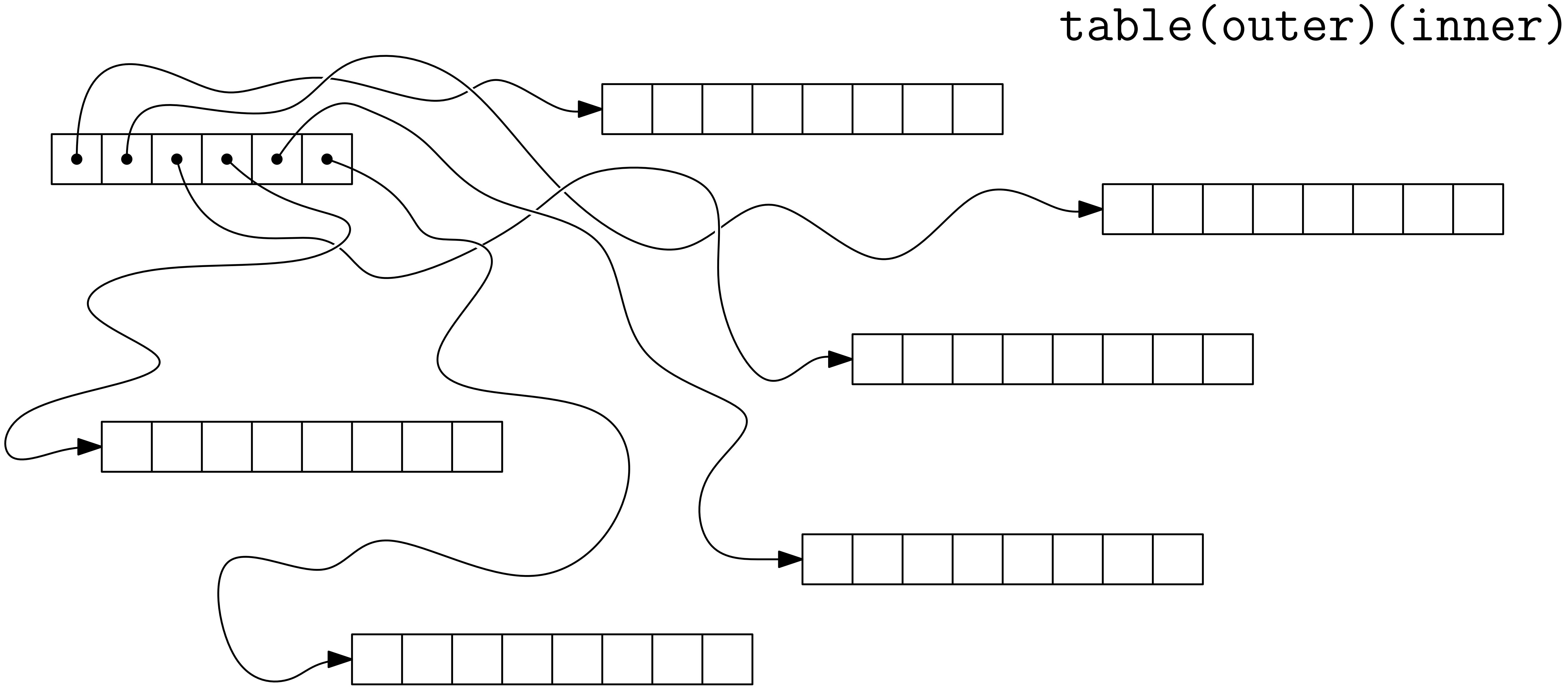


Arrays and Memory Locality

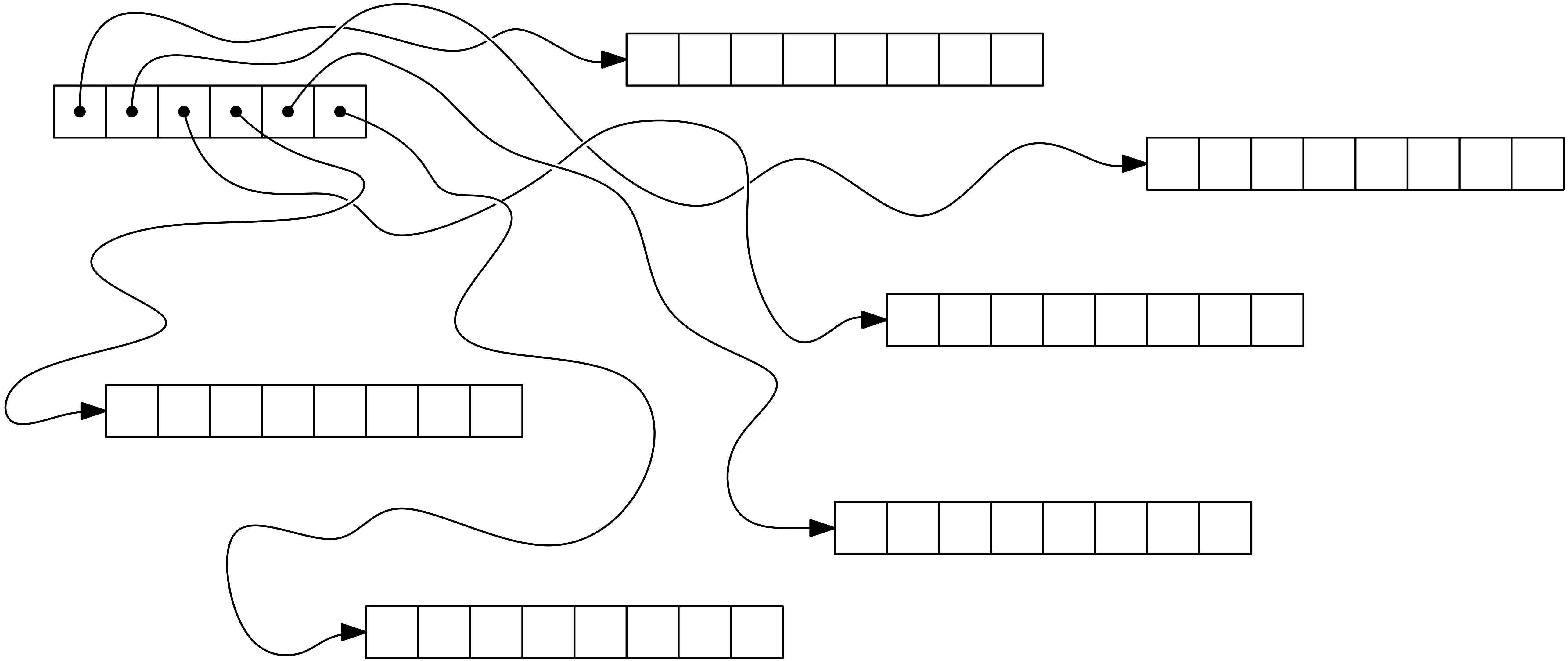


Arrays and Memory Locality



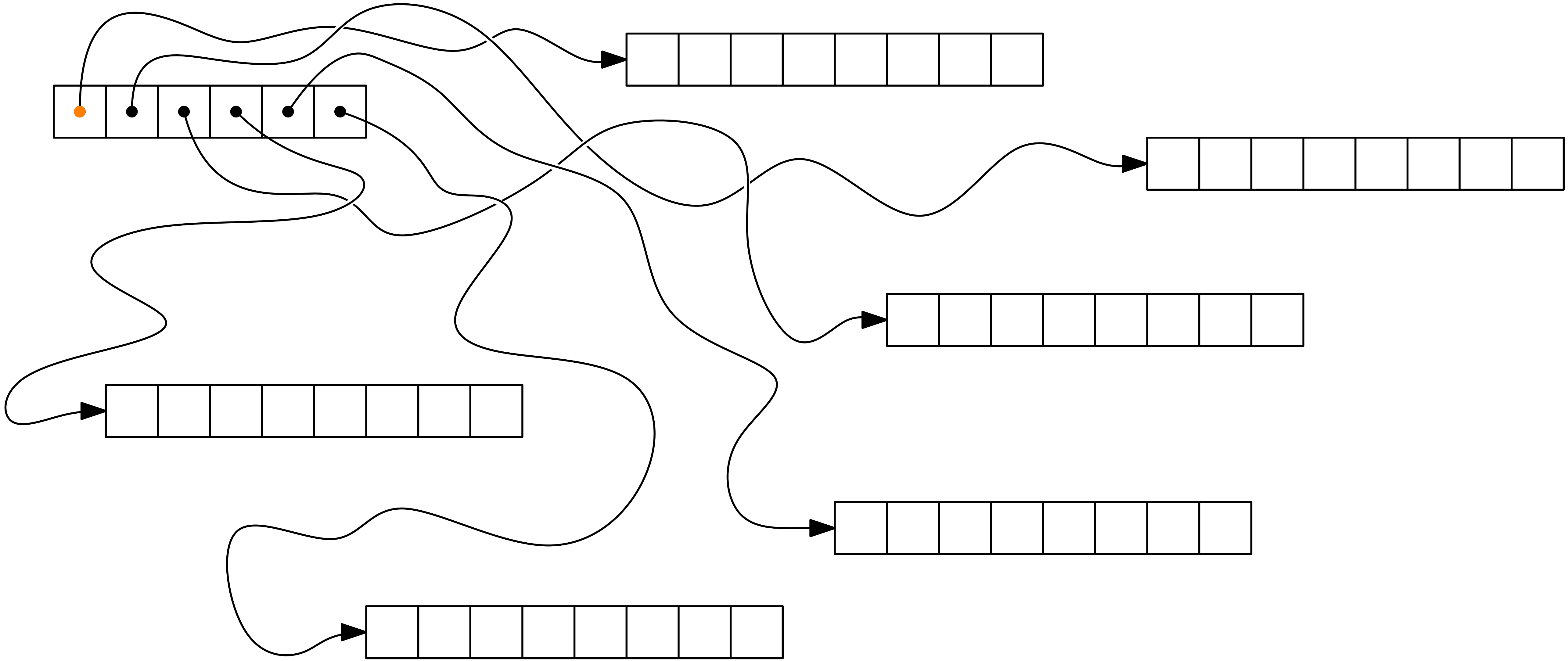
Arrays and Memory Locality

`table(inner)(outer)`



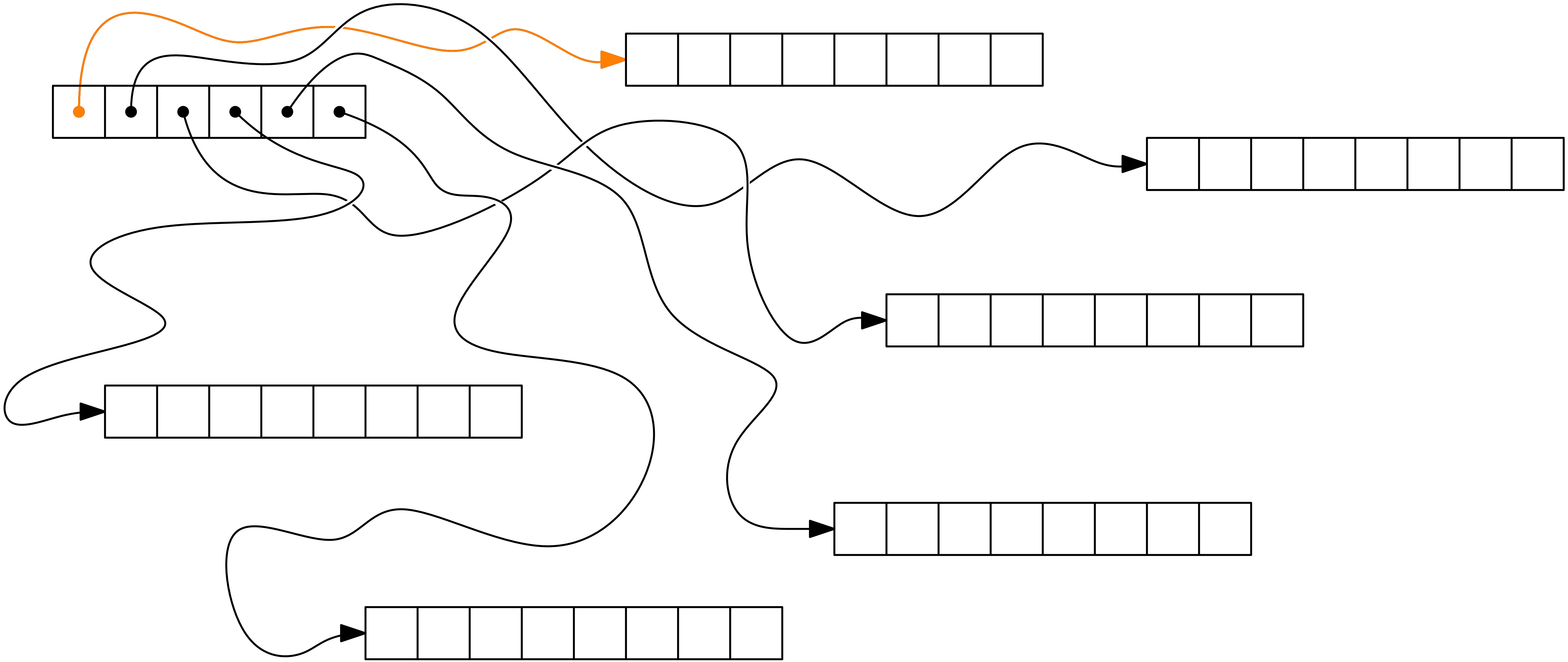
Arrays and Memory Locality

`table(inner)(outer)`



Arrays and Memory Locality

`table(inner)(outer)`



Arrays and Memory Locality

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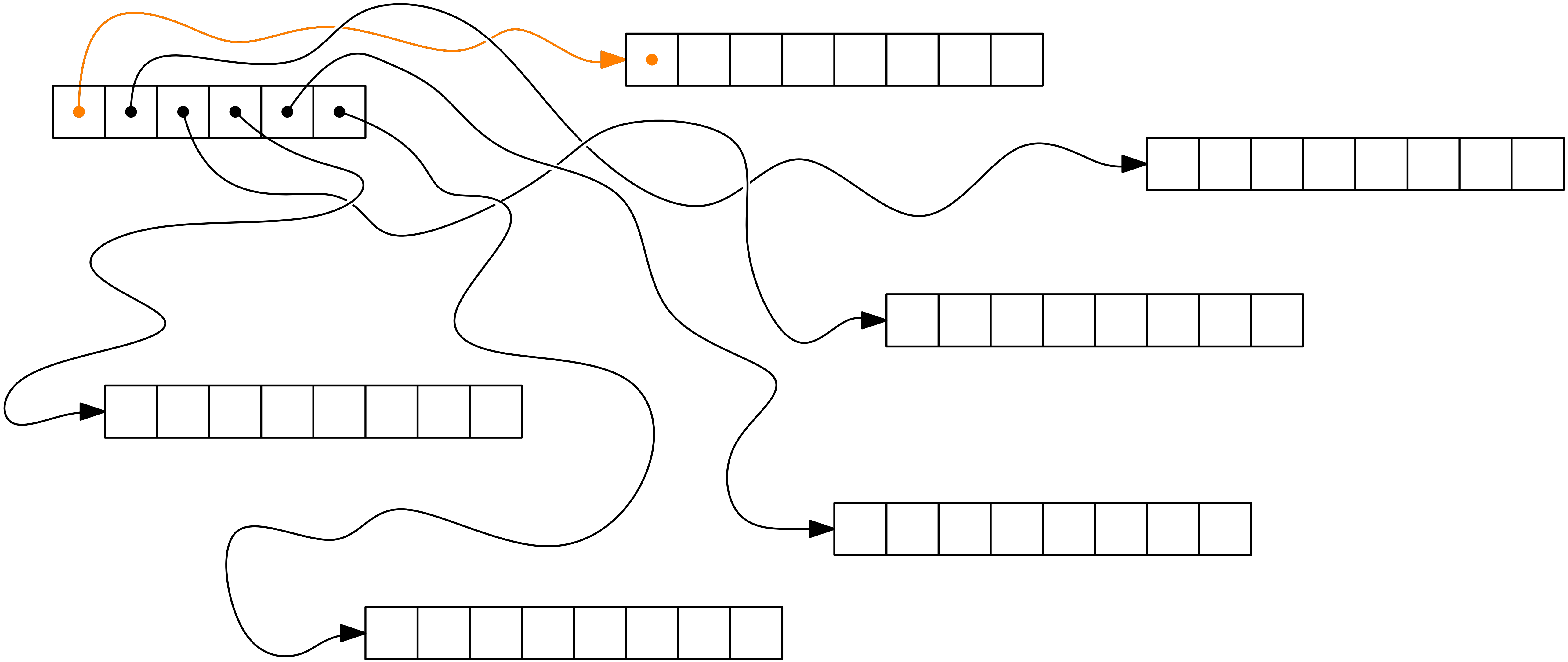

Arrays and Memory Locality

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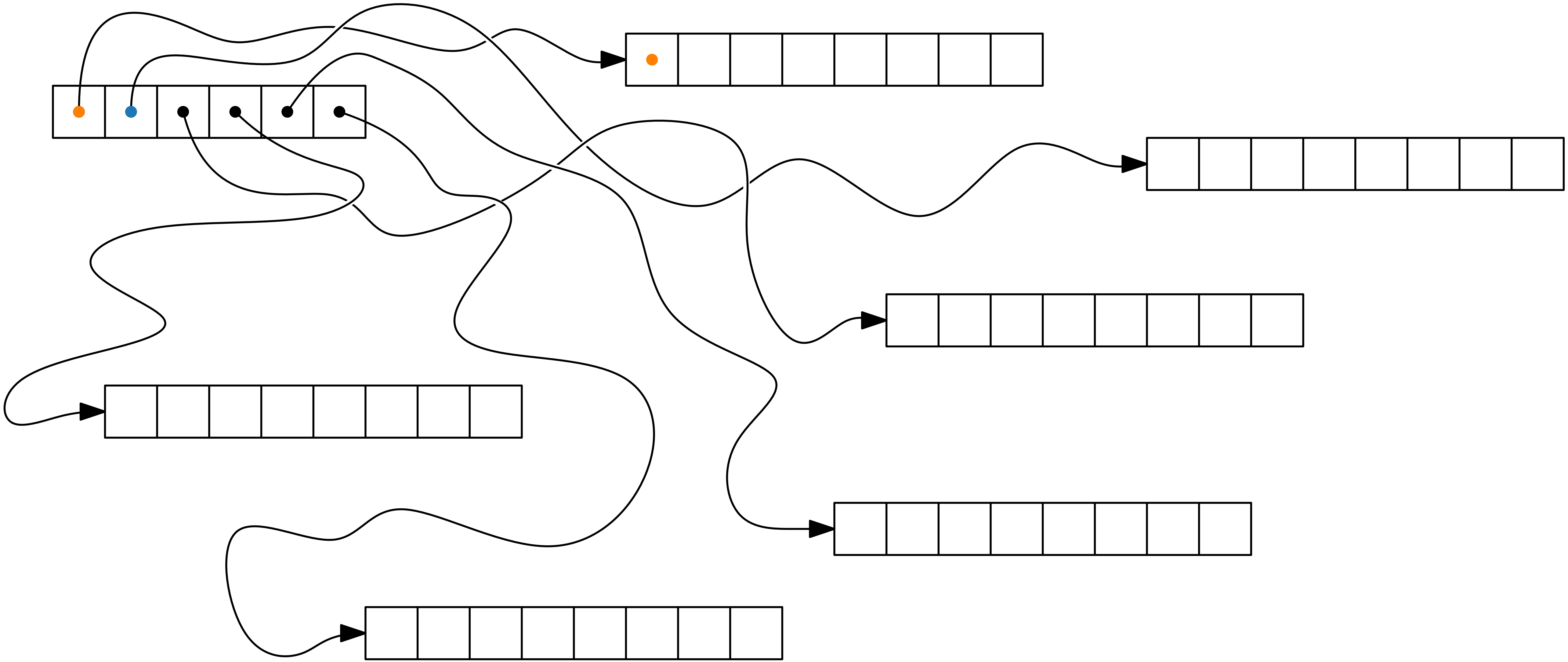
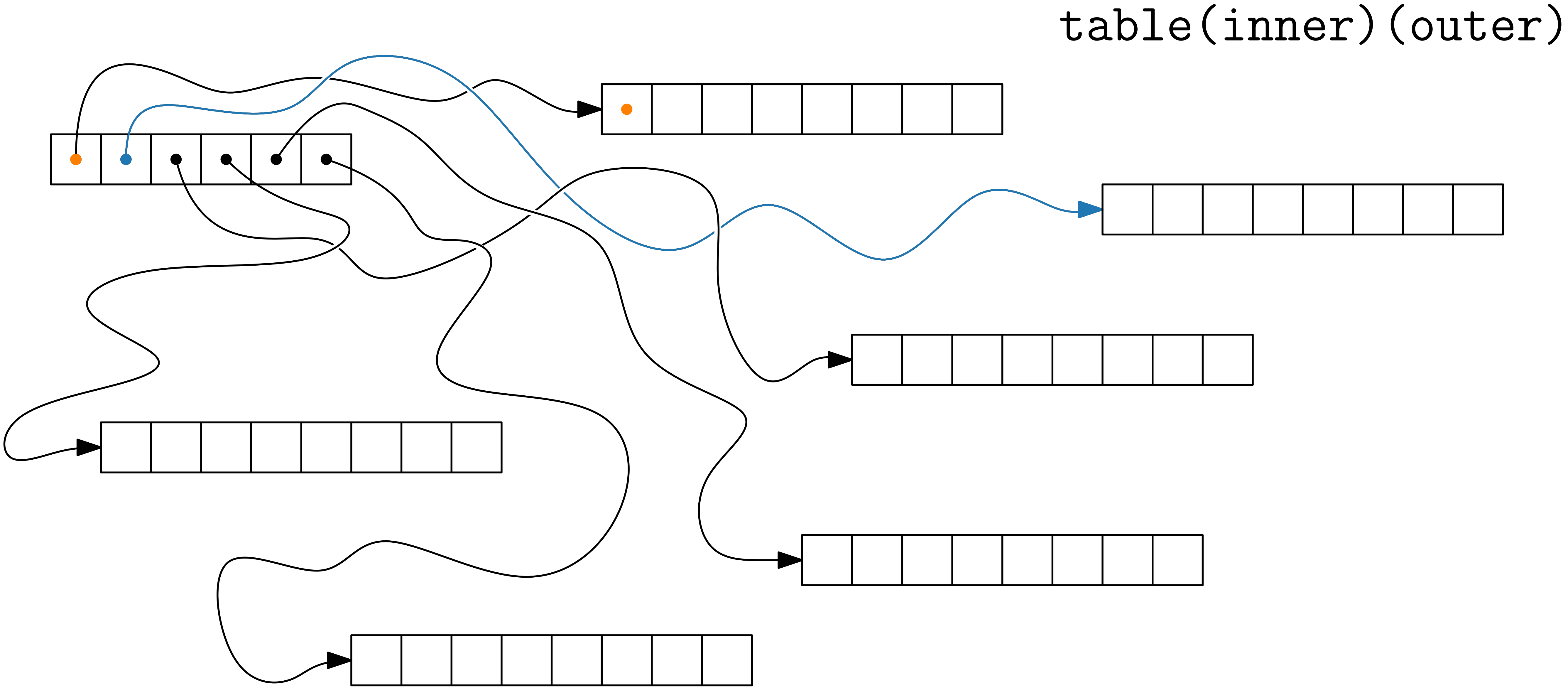


Arrays and Memory Locality



Arrays and Memory Locality

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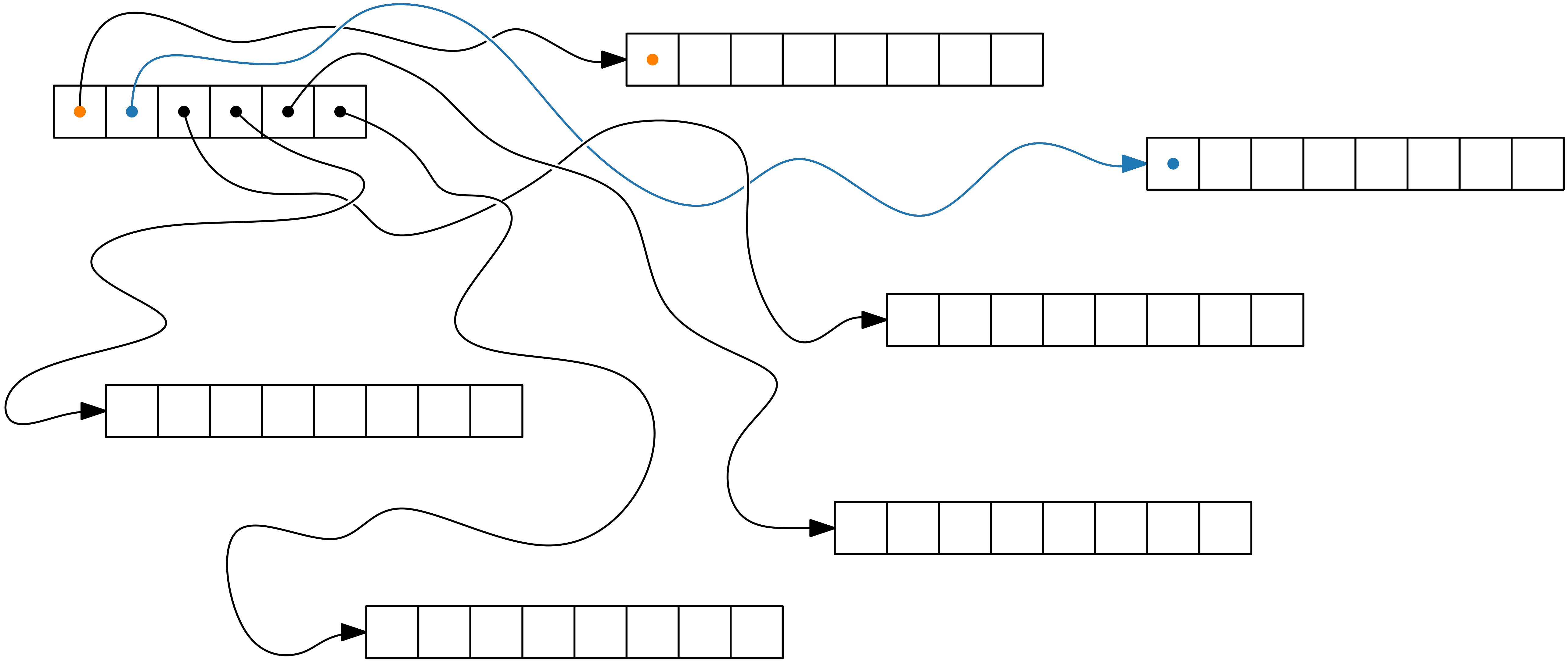
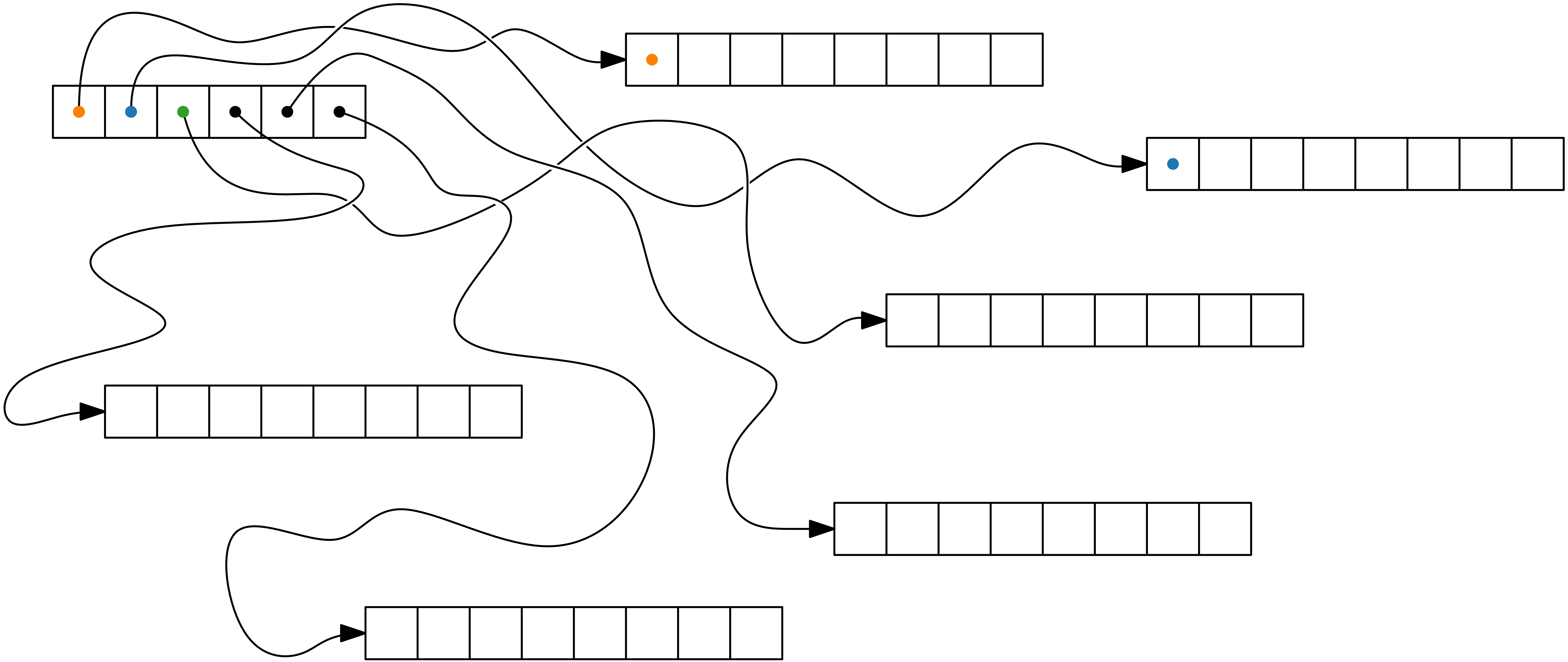


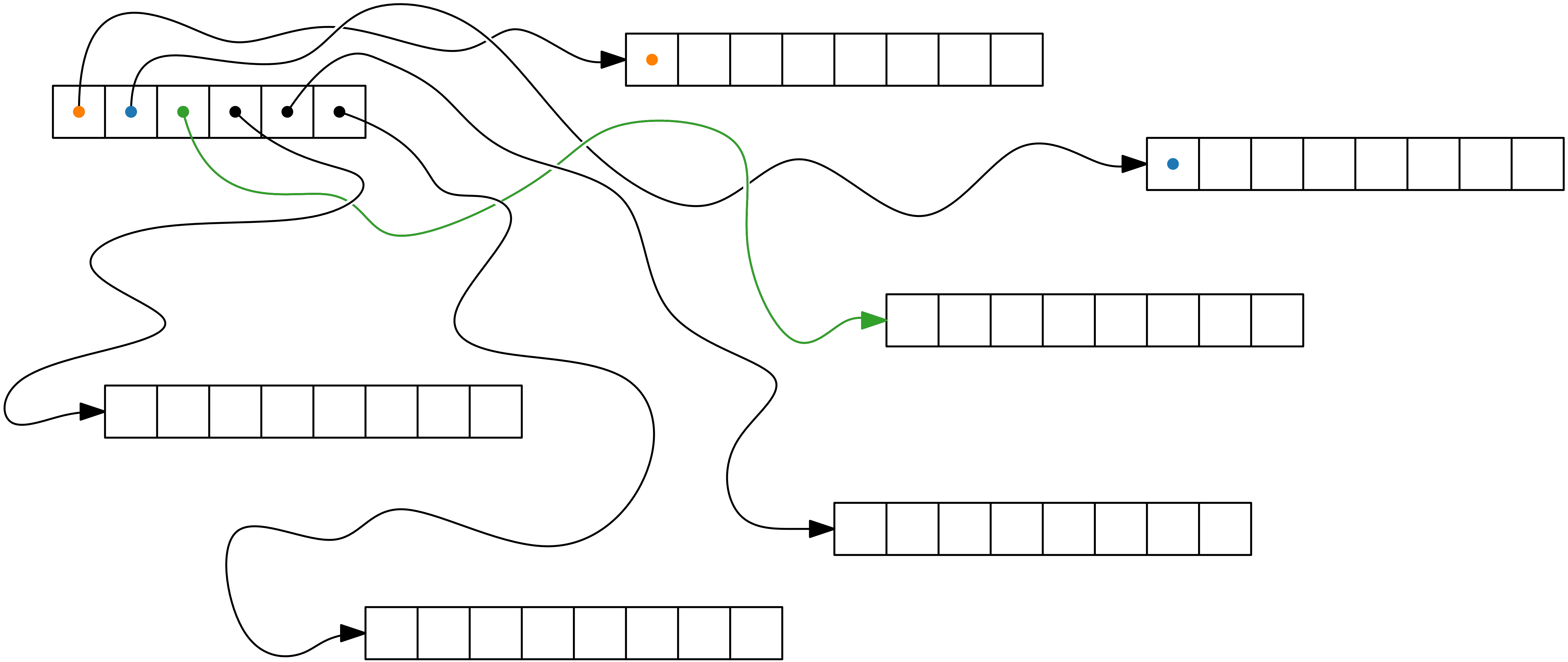
Arrays and Memory Locality

`table(inner)(outer)`



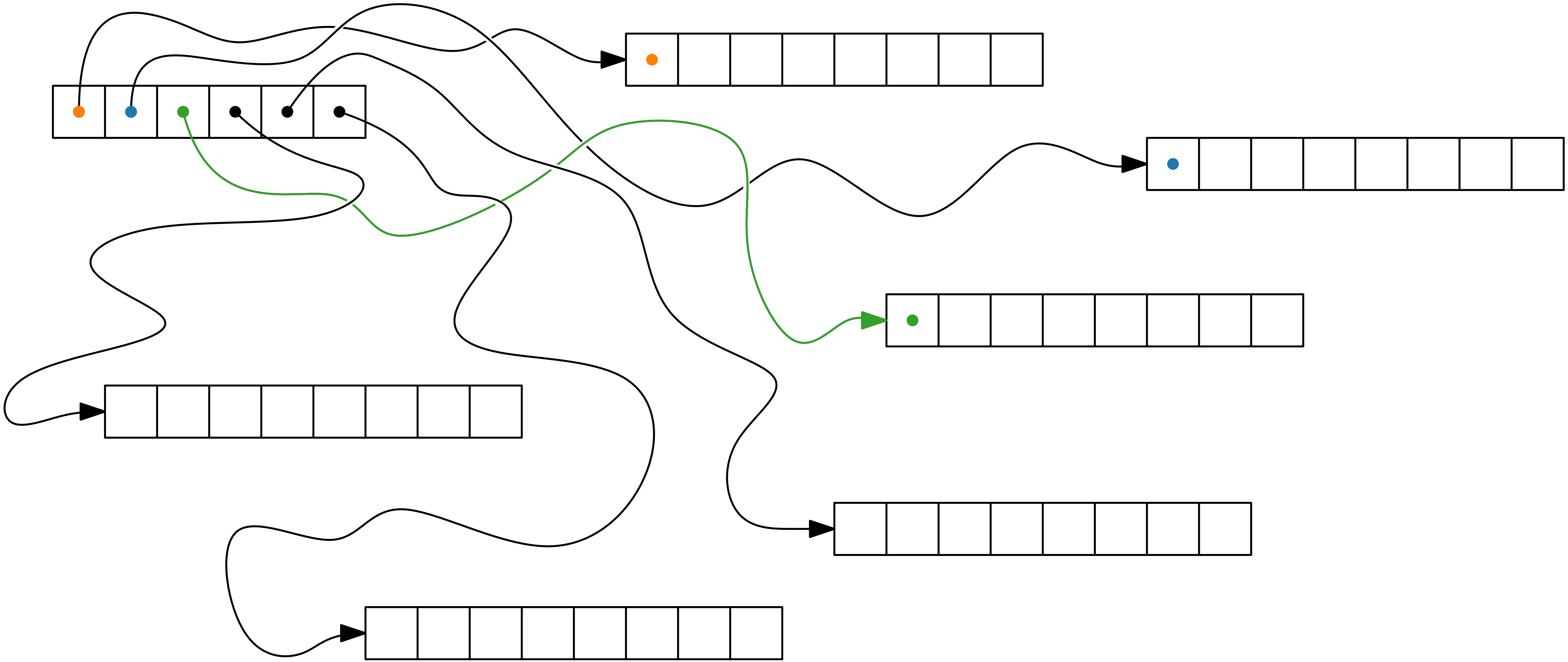
Arrays and Memory Locality

`table(inner)(outer)`



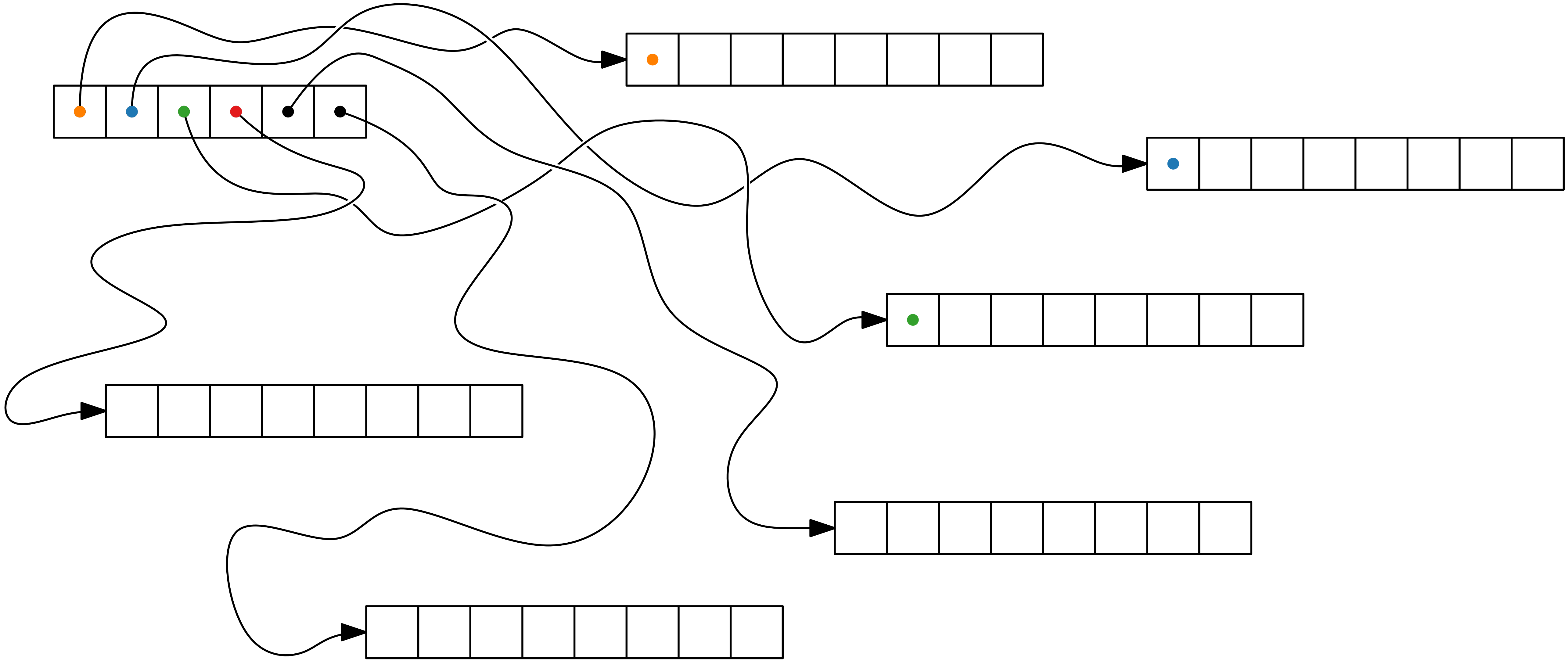
Arrays and Memory Locality

`table(inner)(outer)`



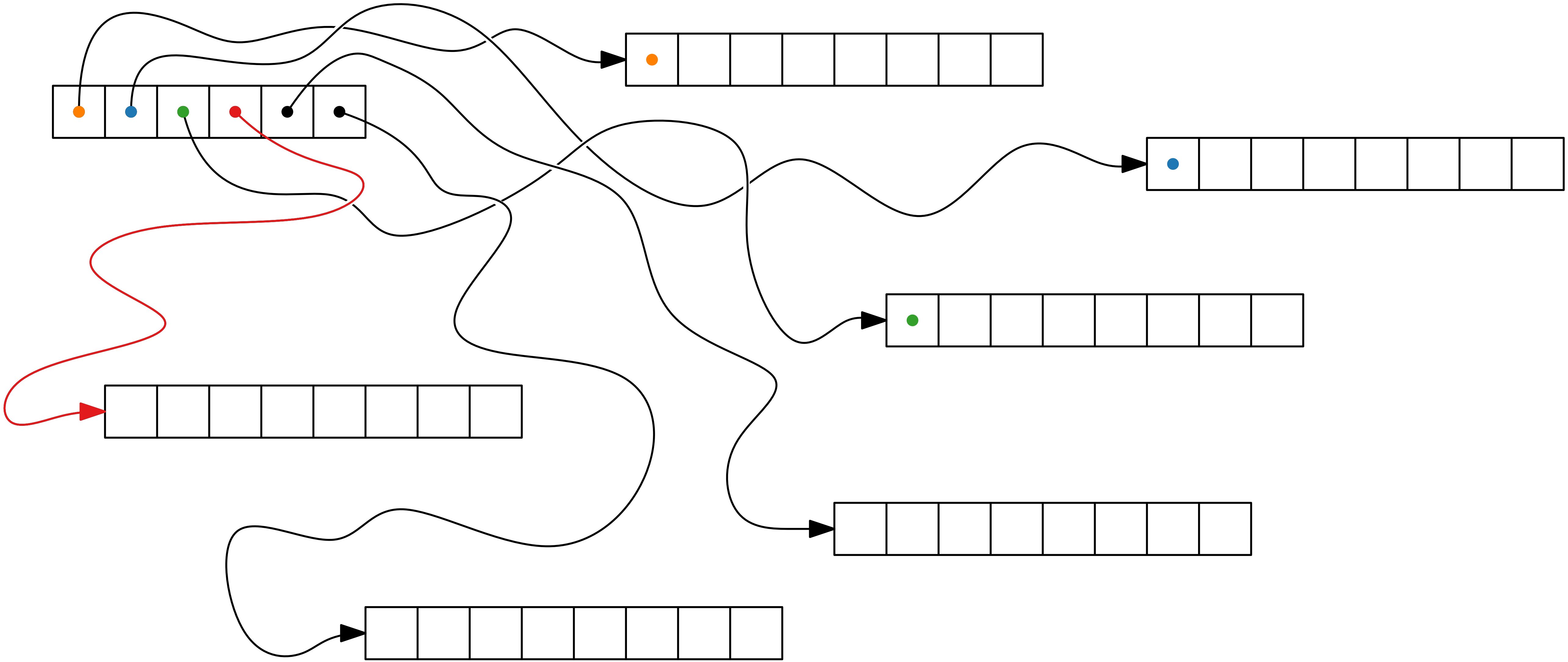
Arrays and Memory Locality

`table(inner)(outer)`



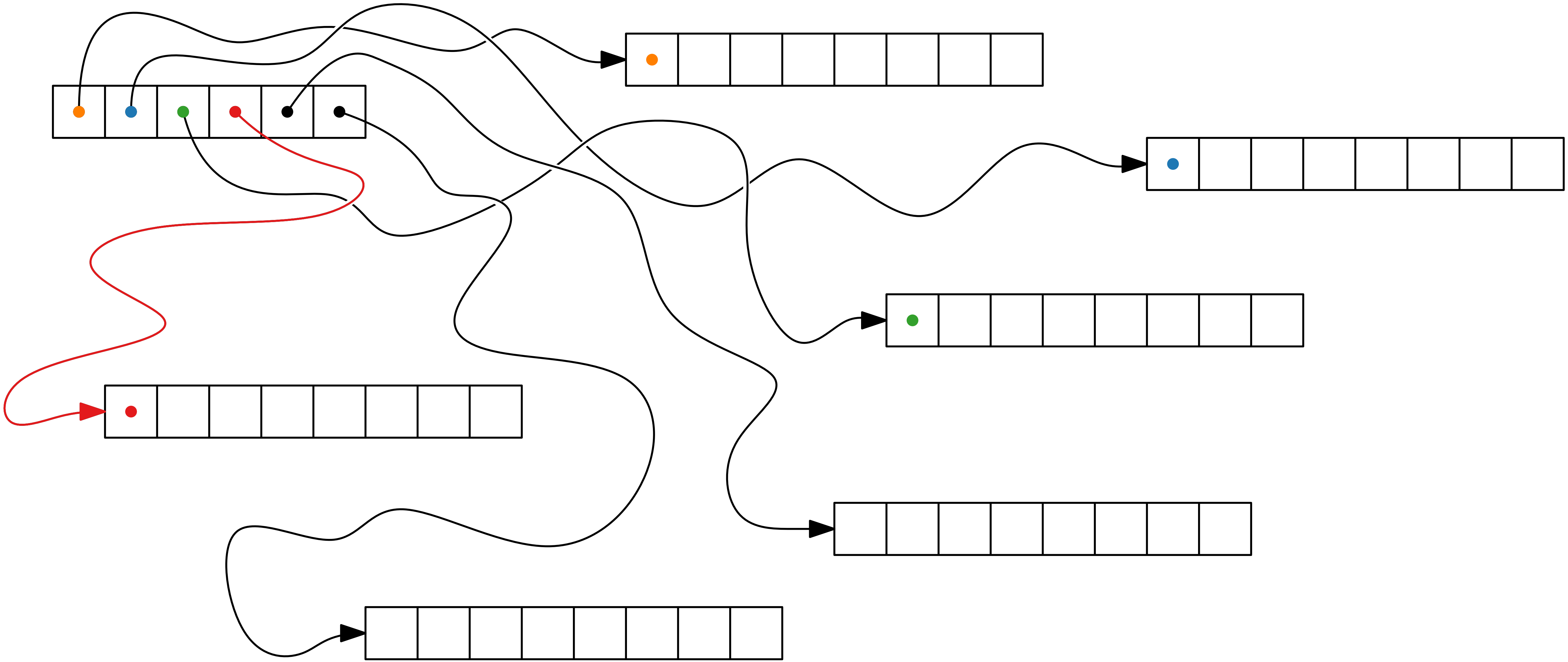
Arrays and Memory Locality

`table(inner)(outer)`



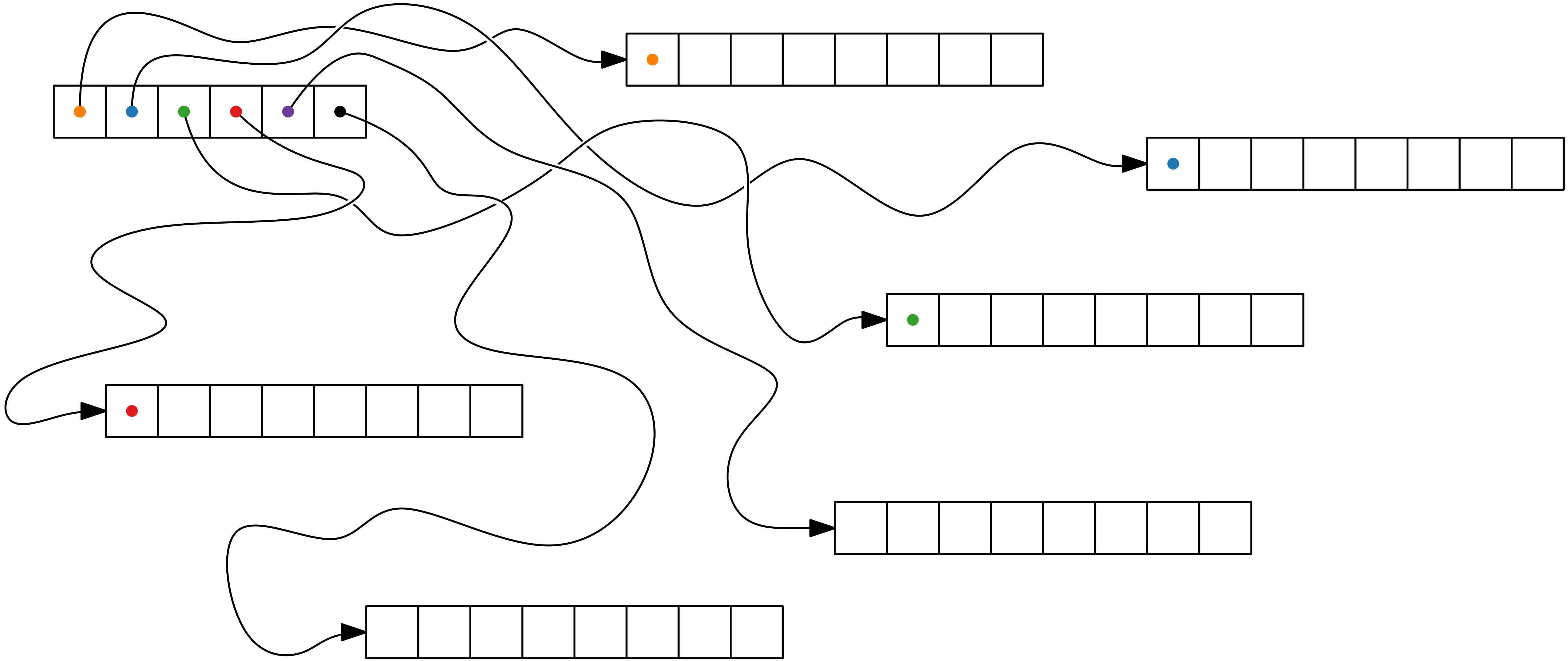
Arrays and Memory Locality

`table(inner)(outer)`



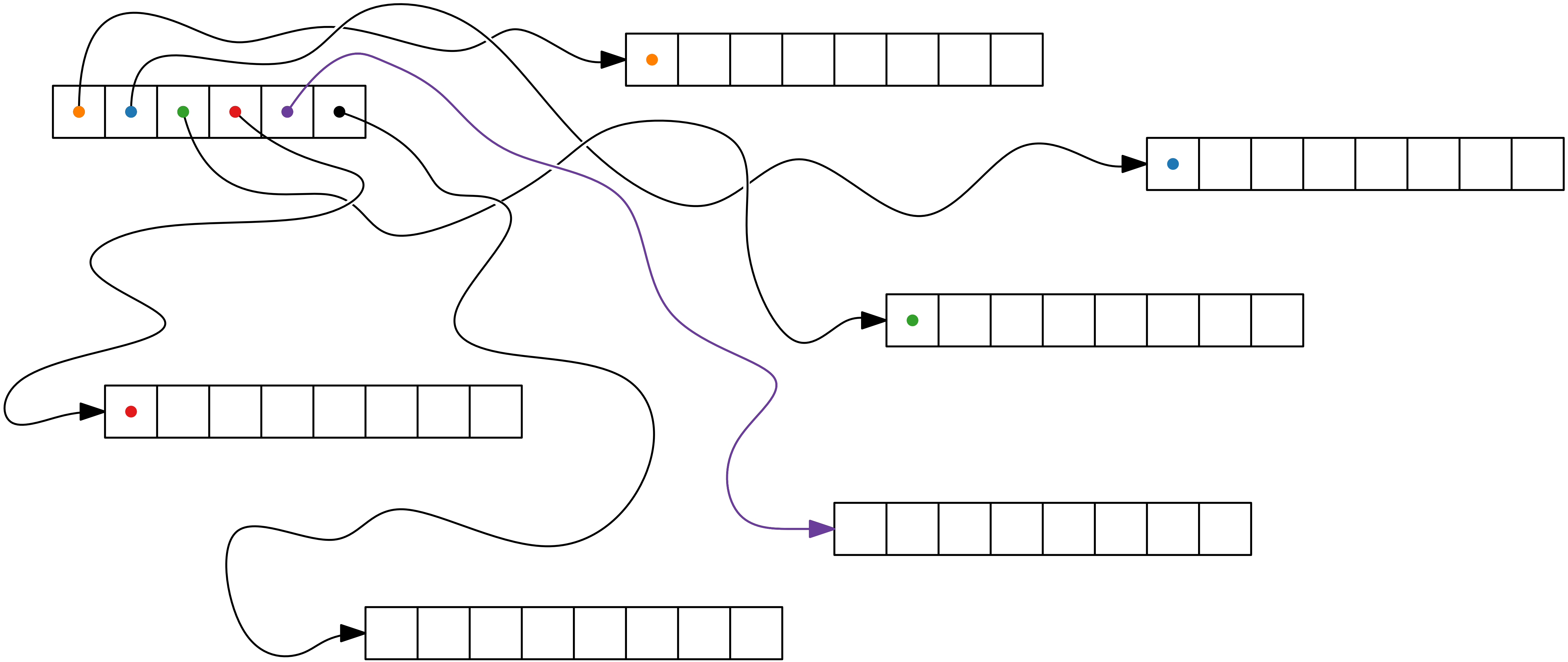
Arrays and Memory Locality

`table(inner)(outer)`



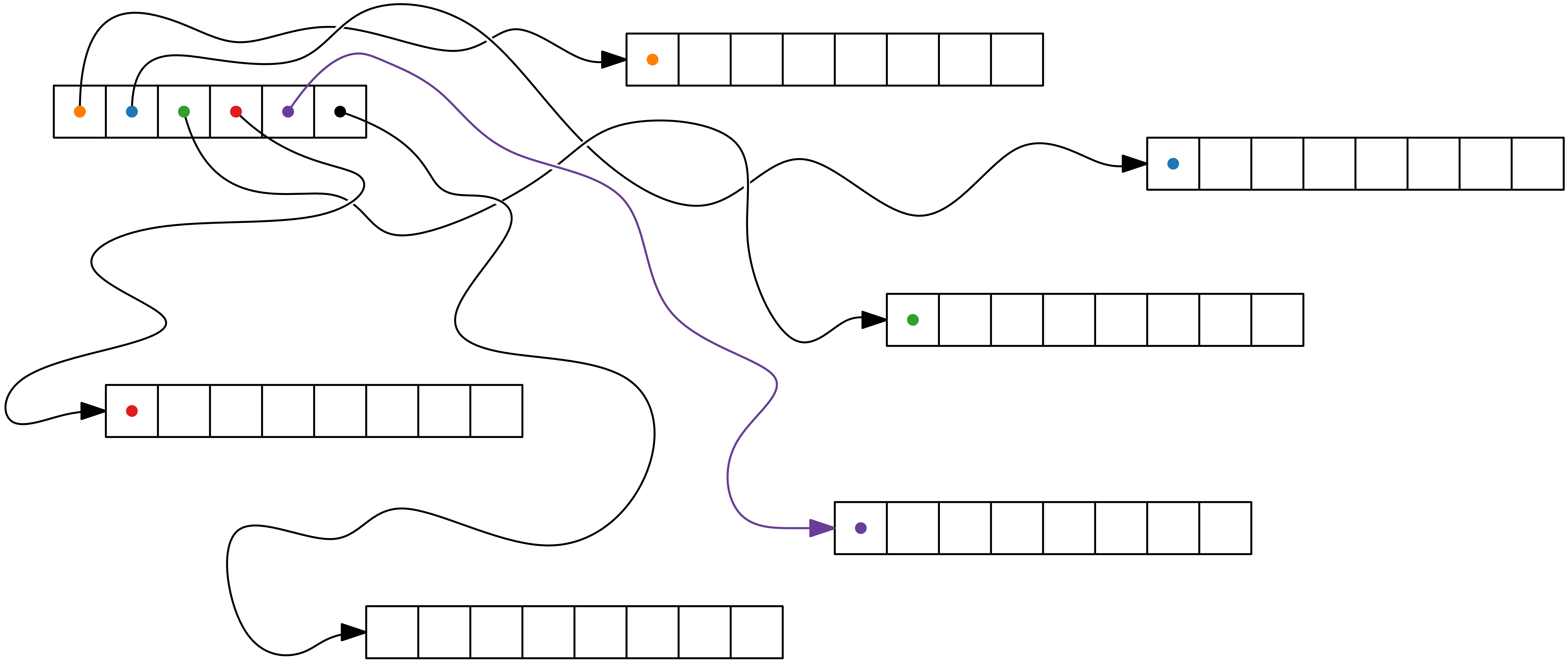
Arrays and Memory Locality

`table(inner)(outer)`



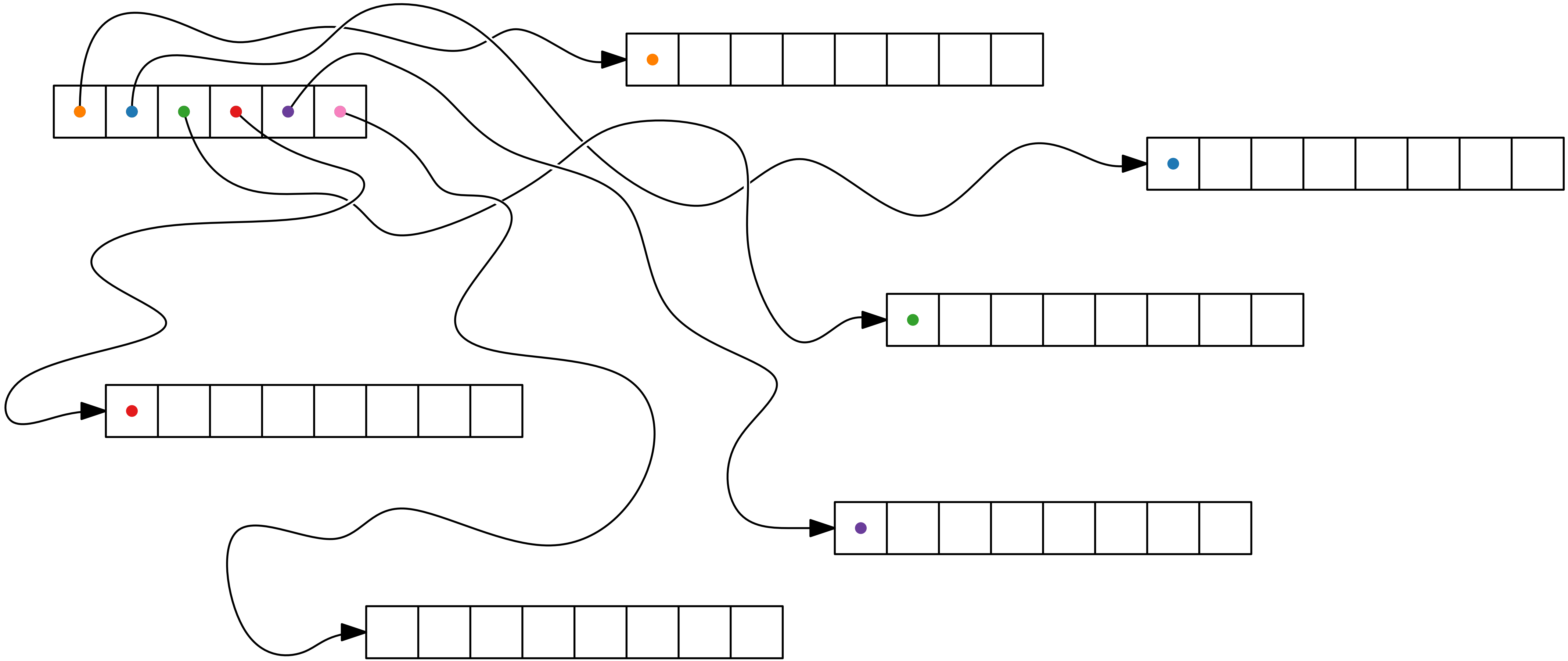
Arrays and Memory Locality

`table(inner)(outer)`

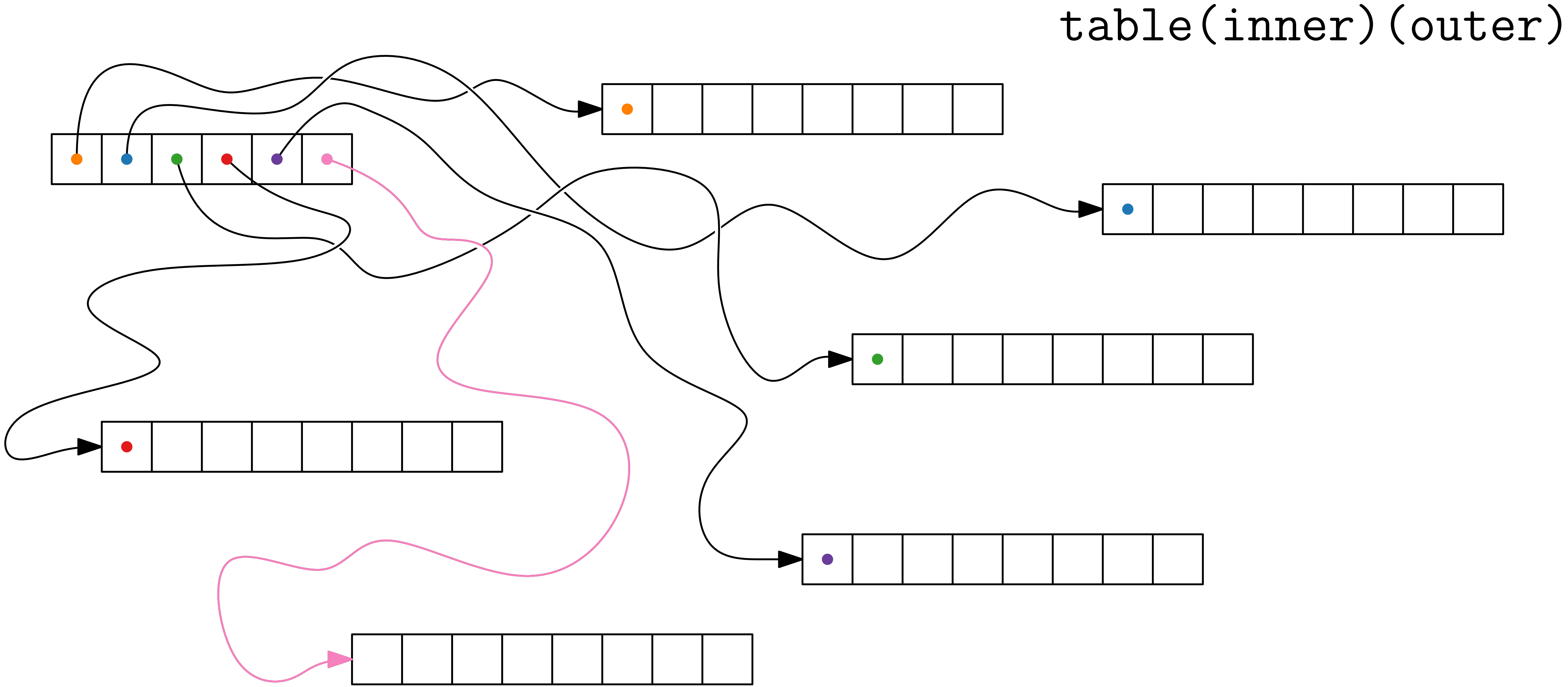


Arrays and Memory Locality

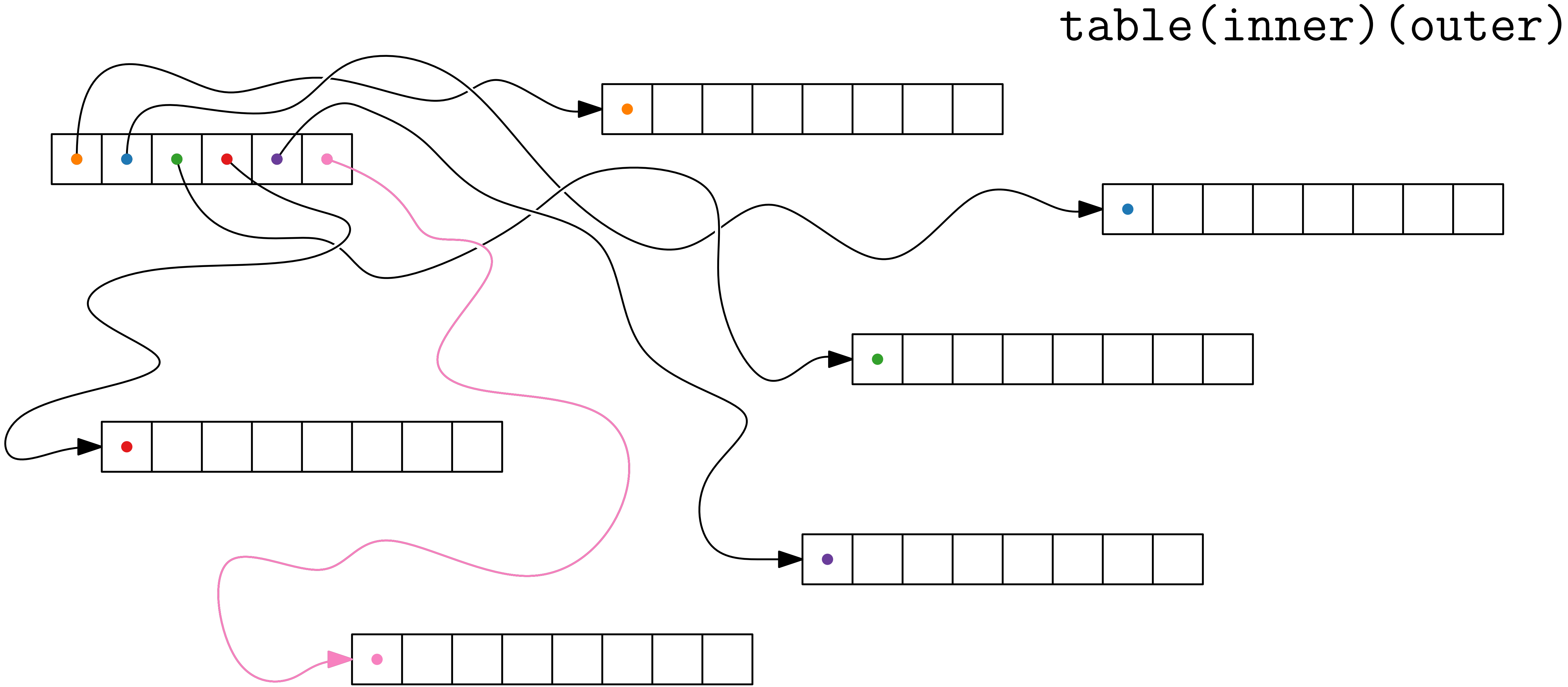
`table(inner)(outer)`



Arrays and Memory Locality

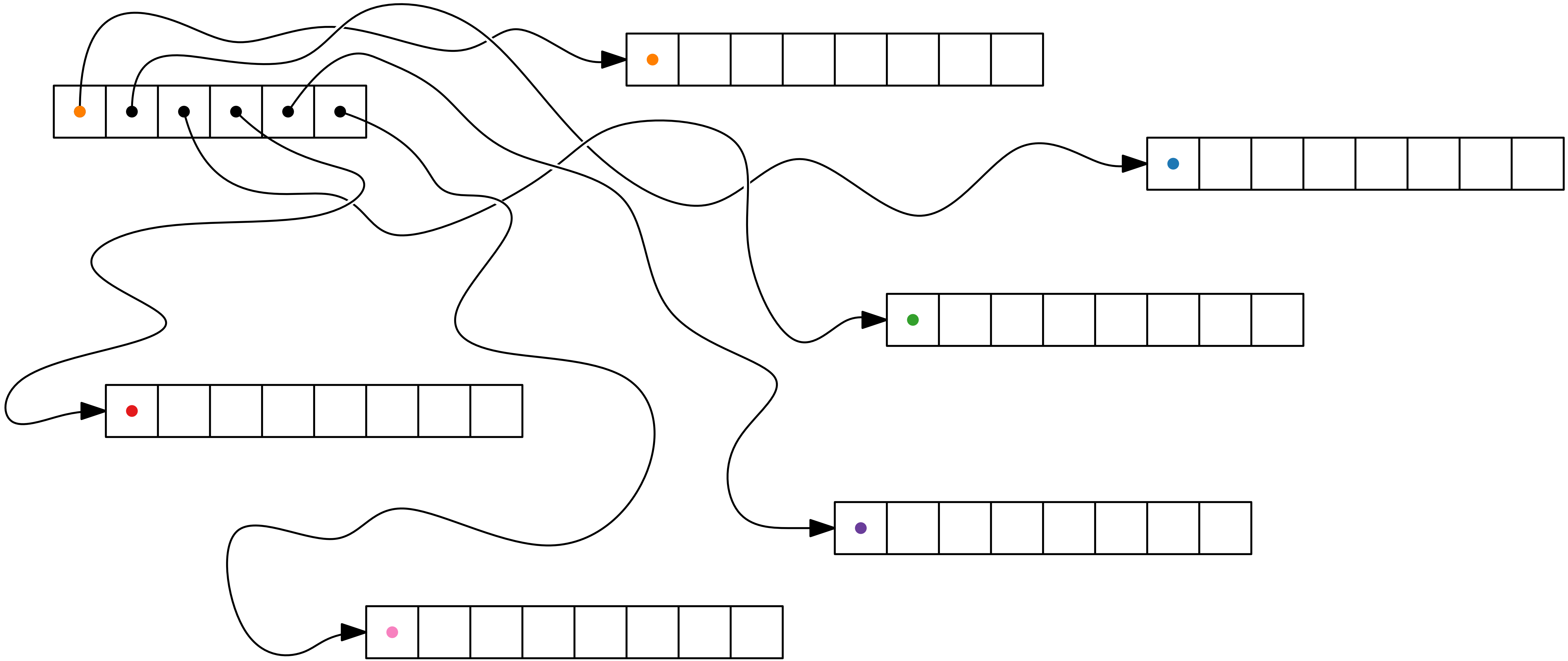


Arrays and Memory Locality



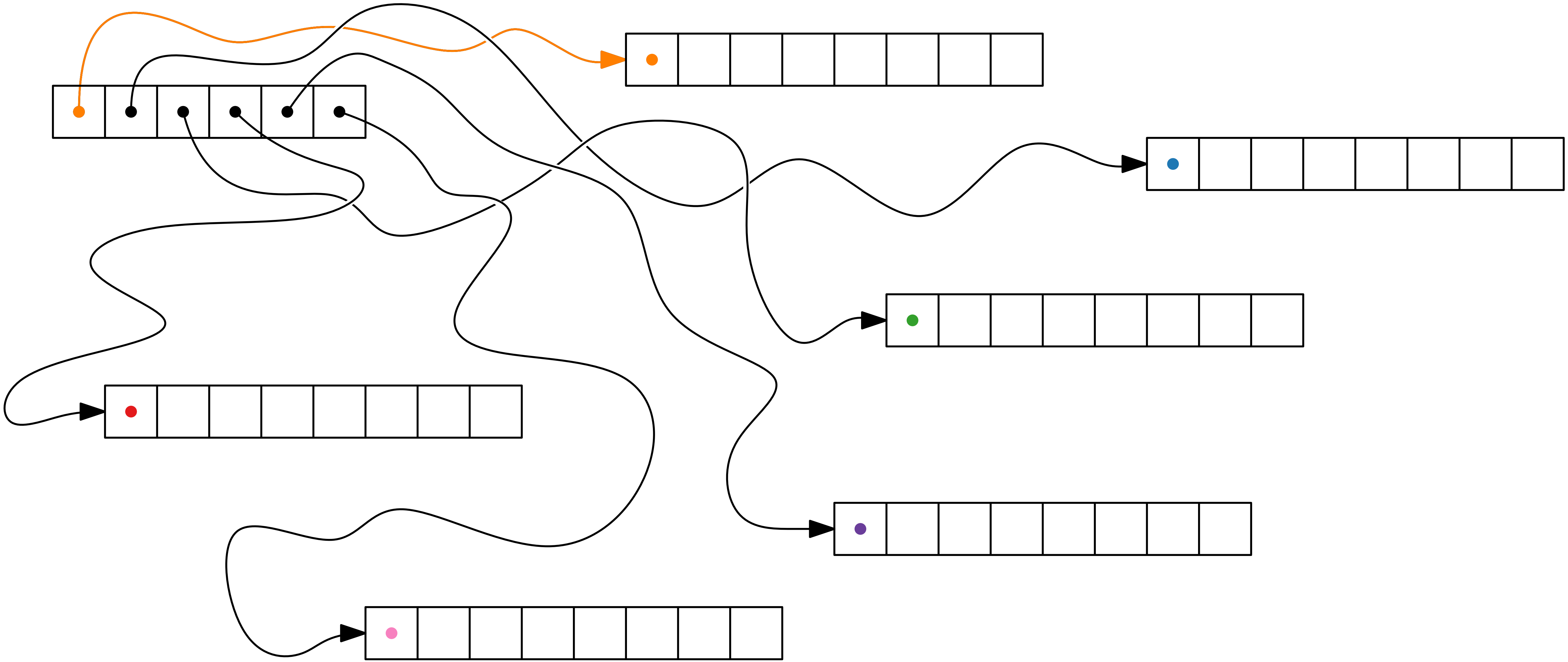
Arrays and Memory Locality

`table(inner)(outer)`



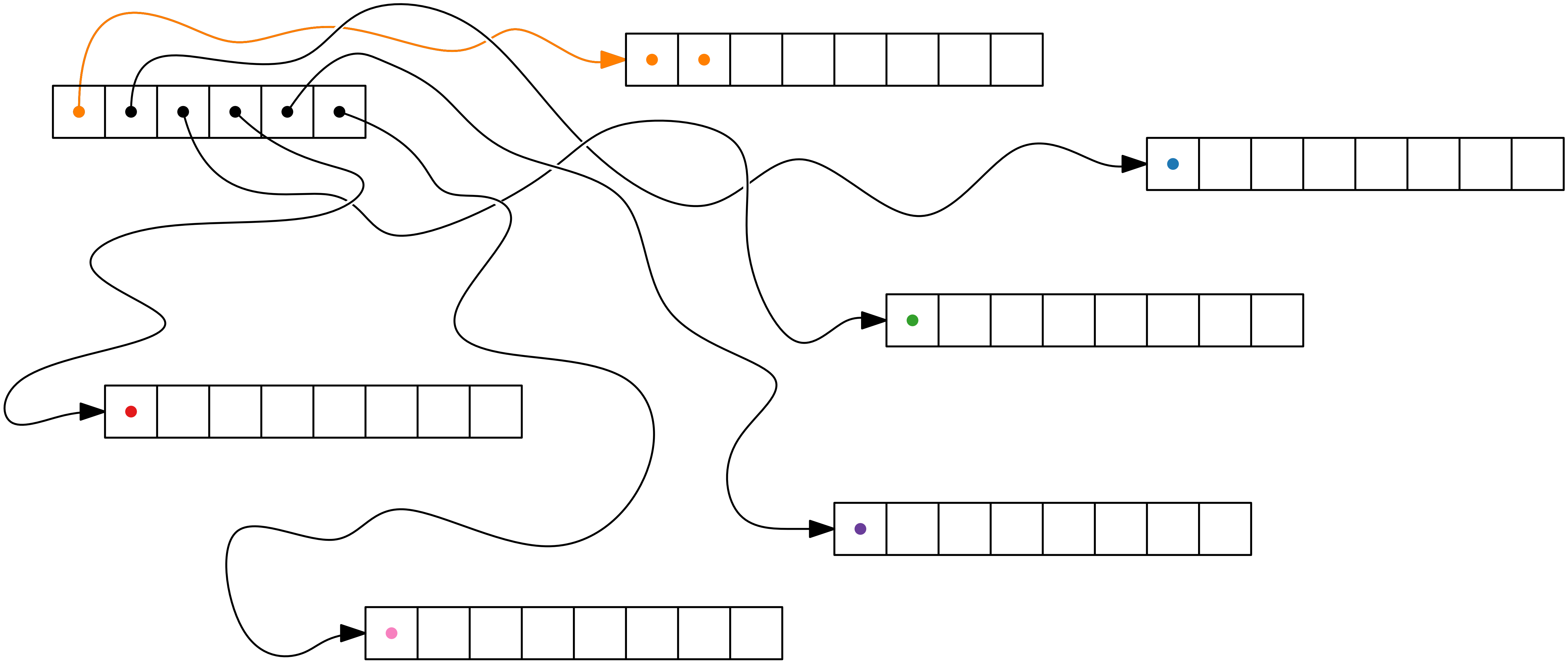
Arrays and Memory Locality

`table(inner)(outer)`



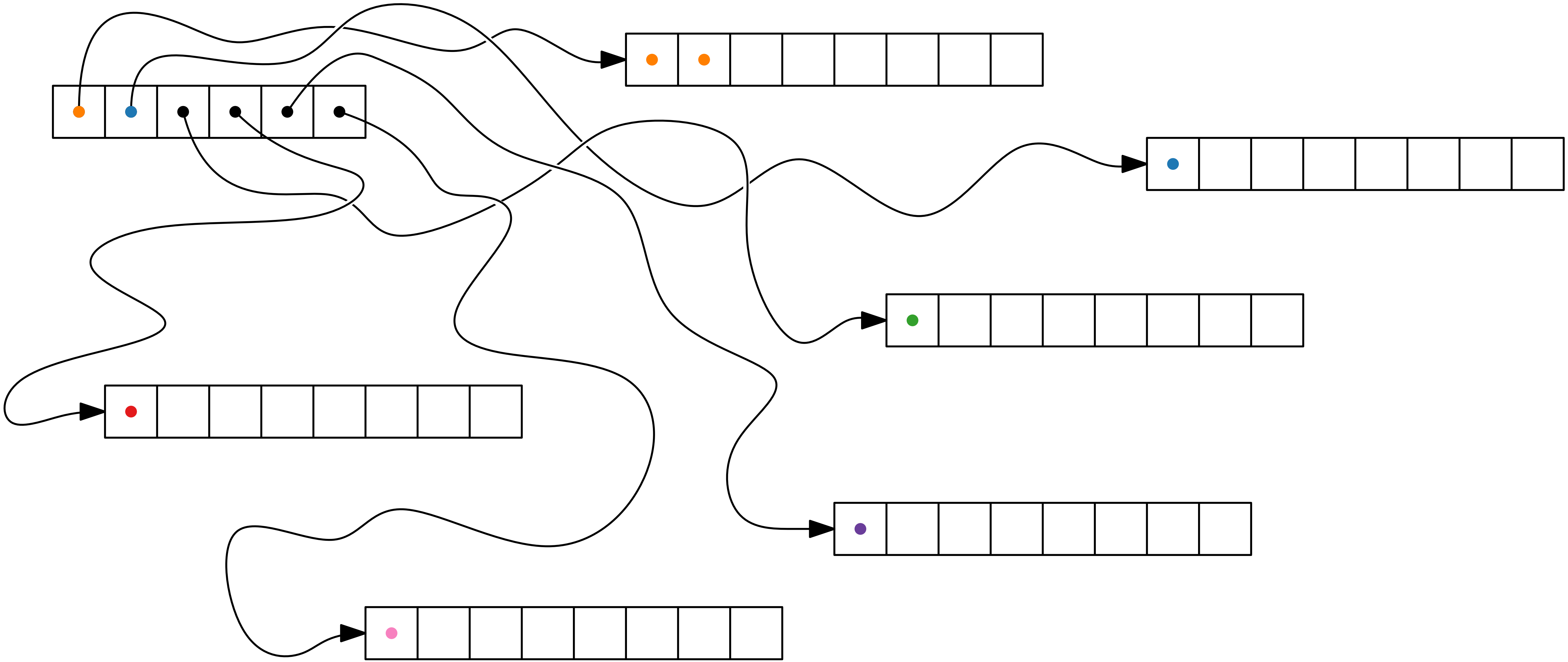
Arrays and Memory Locality

`table(inner)(outer)`

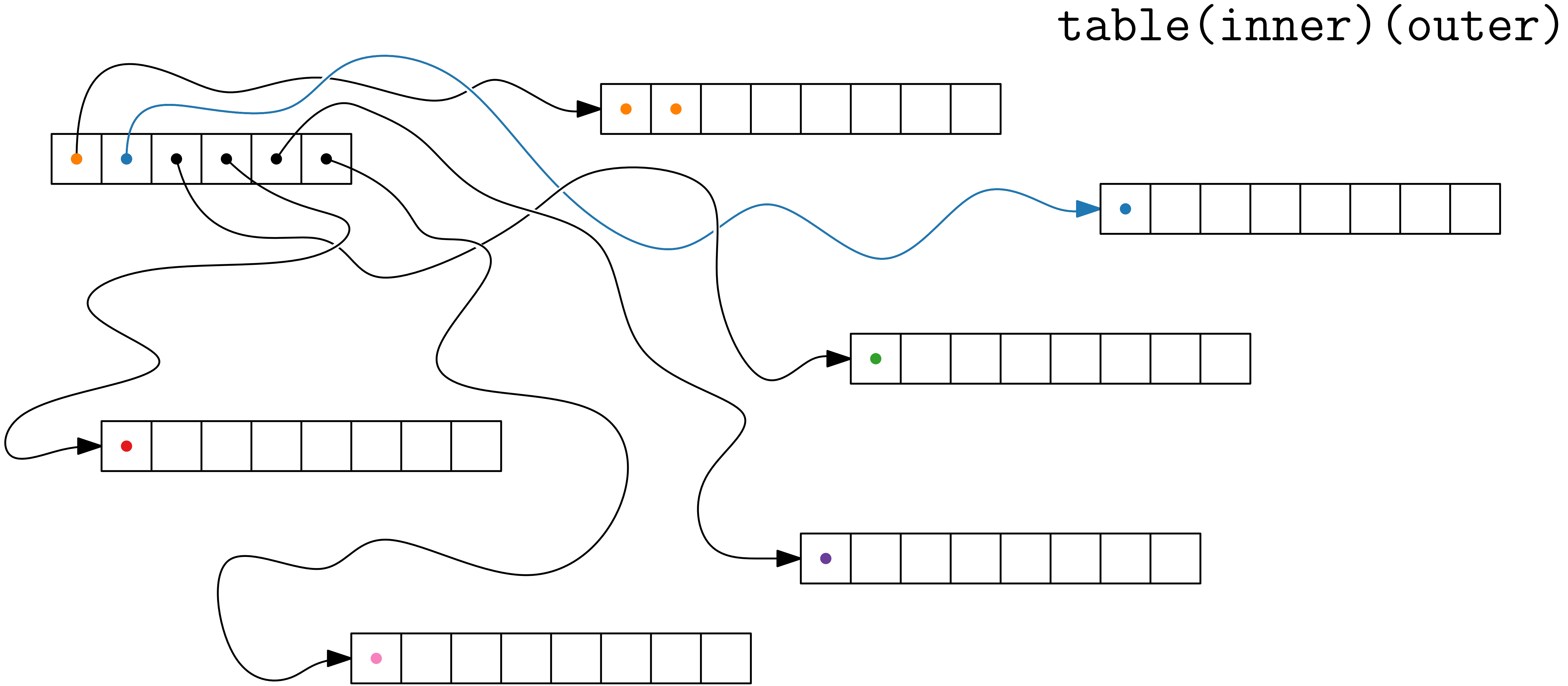


Arrays and Memory Locality

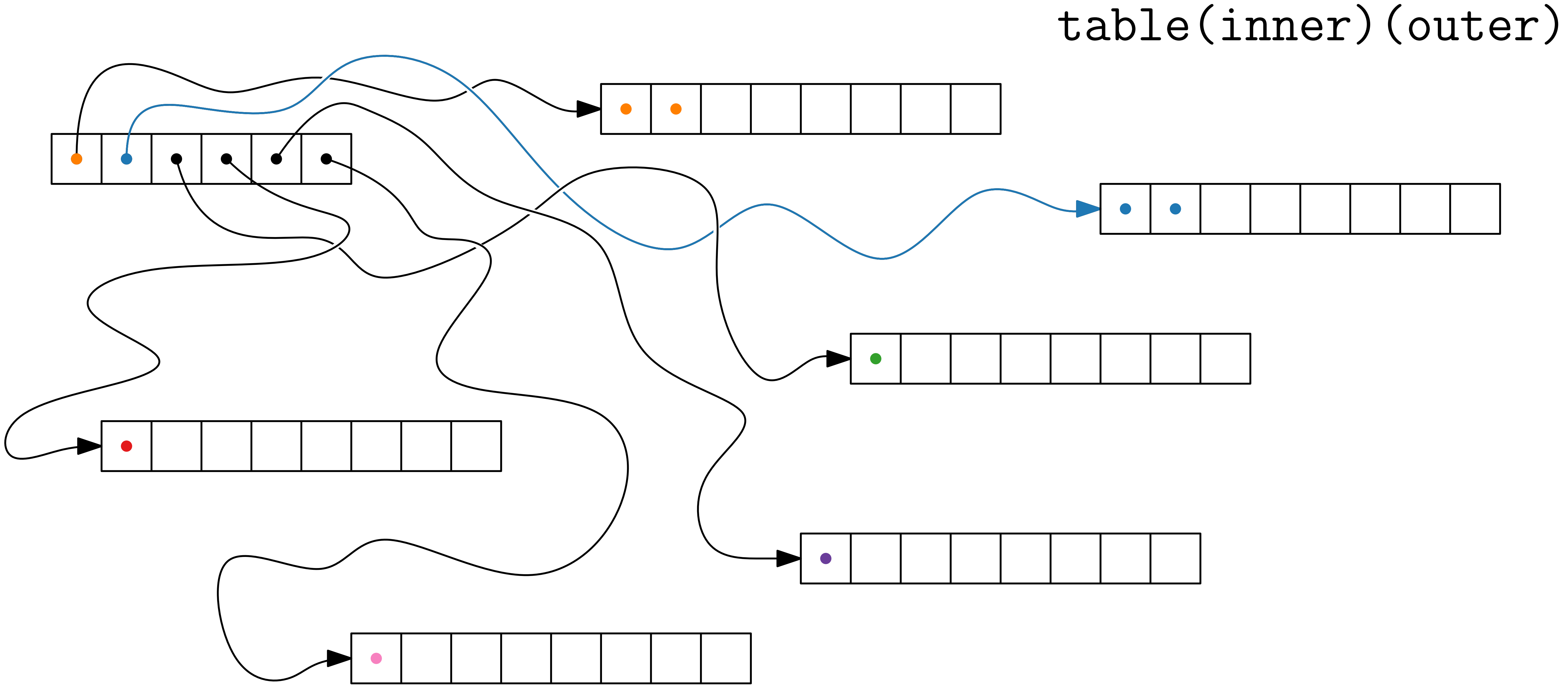
`table(inner)(outer)`



Arrays and Memory Locality



Arrays and Memory Locality



Running Times

$\lvert U \rvert =$	30	100	1K	10K
Scala naïve	1.15 s			
Scala memoized	230 ms	267 ms	⚡SO	
Scala TCO		273 ms	2.30 s	⚡OOM
Scala Arrays				

$B = 10\lvert U \rvert$

Running Times

$\lvert U\rvert =$	30	100	1K	10K
Scala naïve	1.15 s			
Scala memoized	230 ms	267 ms	⚡SO	
Scala TCO		273 ms	2.30 s	⚡OOM
Scala Arrays			458 ms	

$B = 10\lvert U\rvert$

Running Times

$\lvert U \rvert =$	30	100	1K	10K
Scala naïve	1.15 s			
Scala memoized	230 ms	267 ms	⚡SO	
Scala TCO		273 ms	2.30 s	⚡OOM
Scala Arrays			458 ms	27.5 s

$B = 10\lvert U \rvert$

$$f(i, s) = \max \begin{cases} f(i-1, s) \\ v(i) + f(i-1, s - s(i)) & \text{if } s \geq s(i) \end{cases}$$

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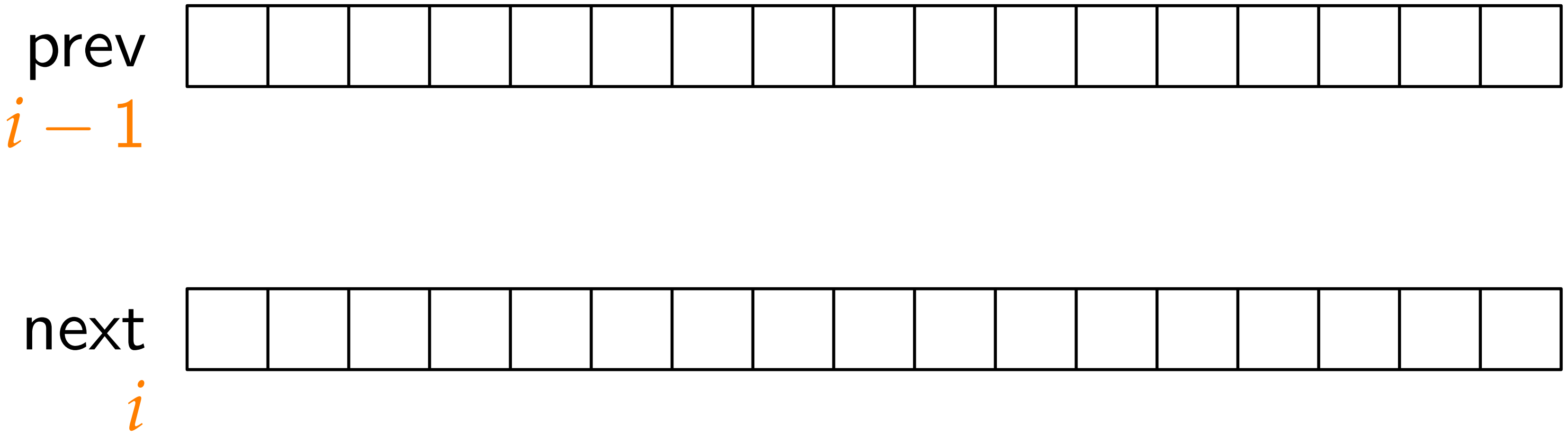
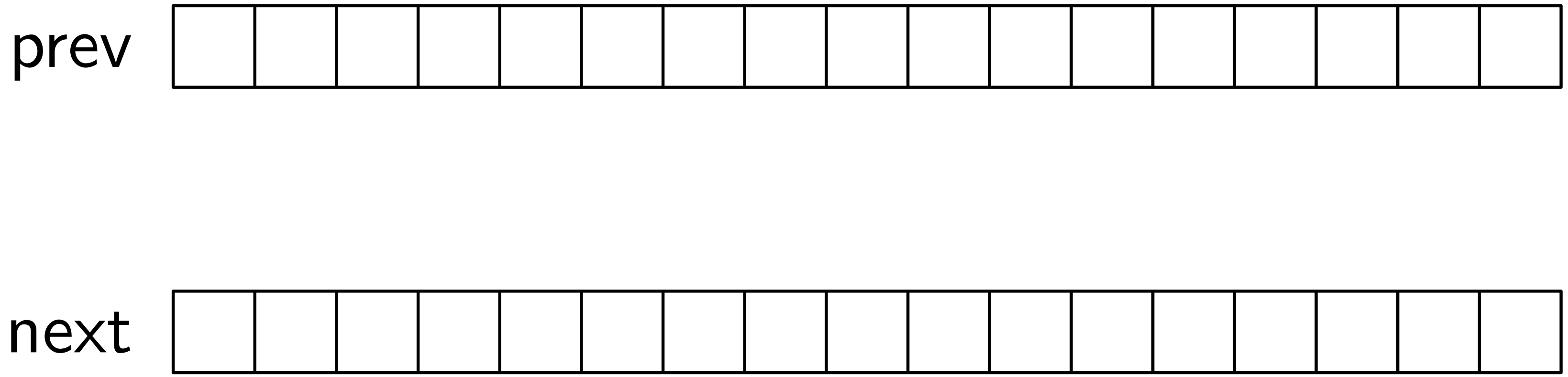
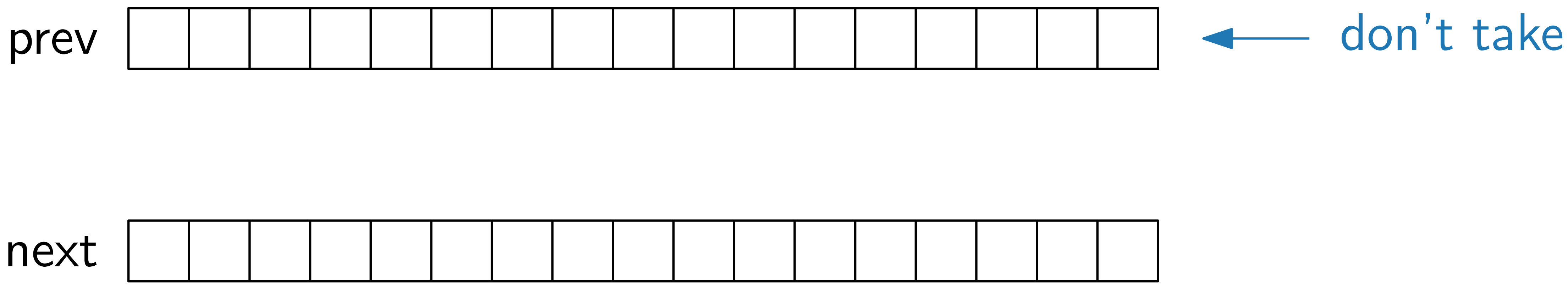


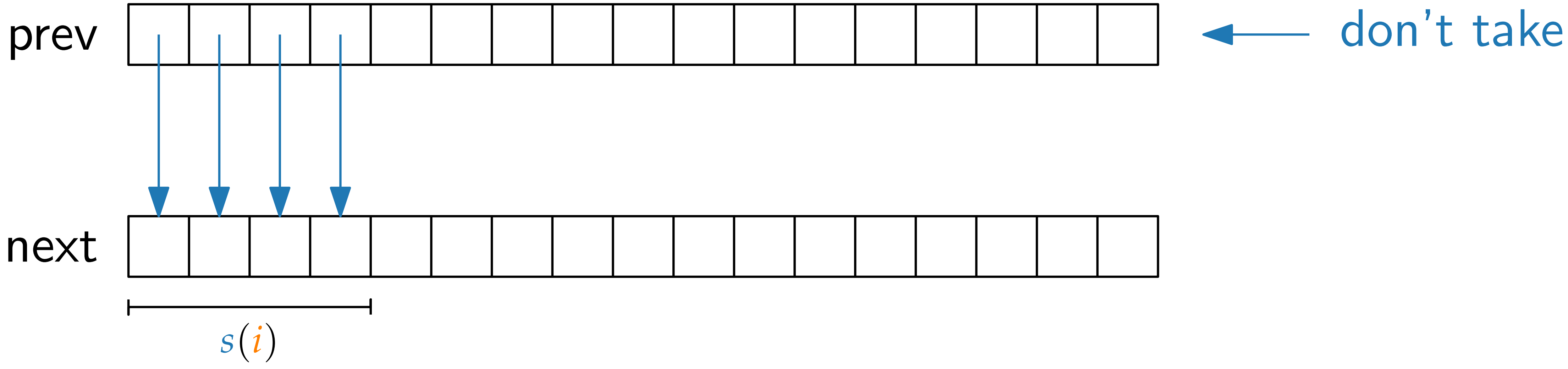
$$f(i, s) = \max \begin{cases} f(i-1, s) & \longleftarrow \text{don't take} \\ v(i) + f(i-1, s - s(i)) & \text{if } s \geq s(i) \end{cases}$$



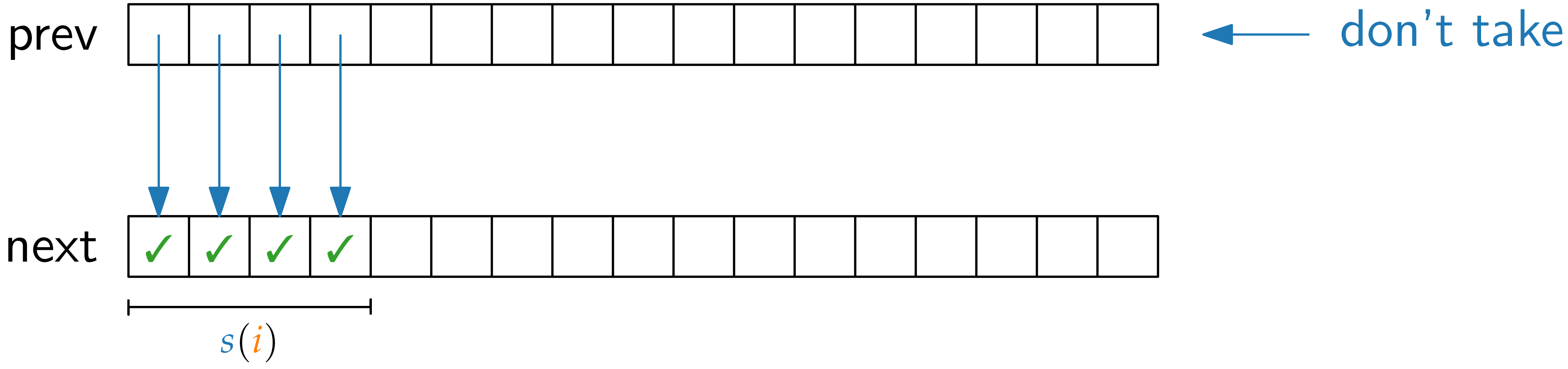
$$f(i, s) = \max \begin{cases} f(i-1, s) & \leftarrow \text{don't take} \\ v(i) + f(i-1, s - s(i)) & \text{if } s \geq s(i) \end{cases}$$



$$f(i, s) = \max \begin{cases} f(i-1, s) & \leftarrow \text{don't take} \\ v(i) + f(i-1, s - s(i)) & \text{if } s \geq s(i) \end{cases}$$



$$f(i, s) = \max \begin{cases} f(i-1, s) & \leftarrow \text{don't take} \\ v(i) + f(i-1, s-s(i)) & \text{if } s \geq s(i) \end{cases}$$



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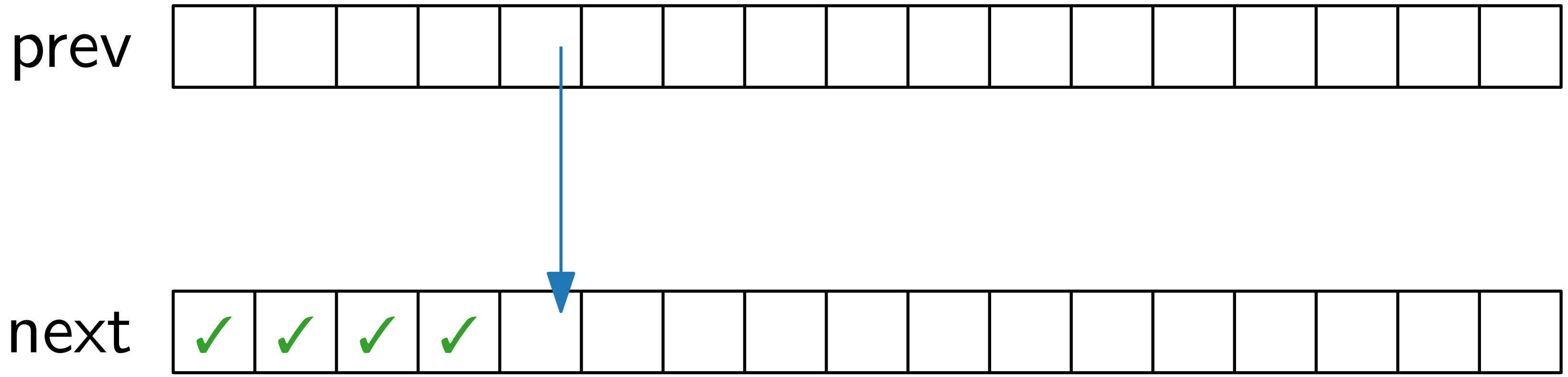
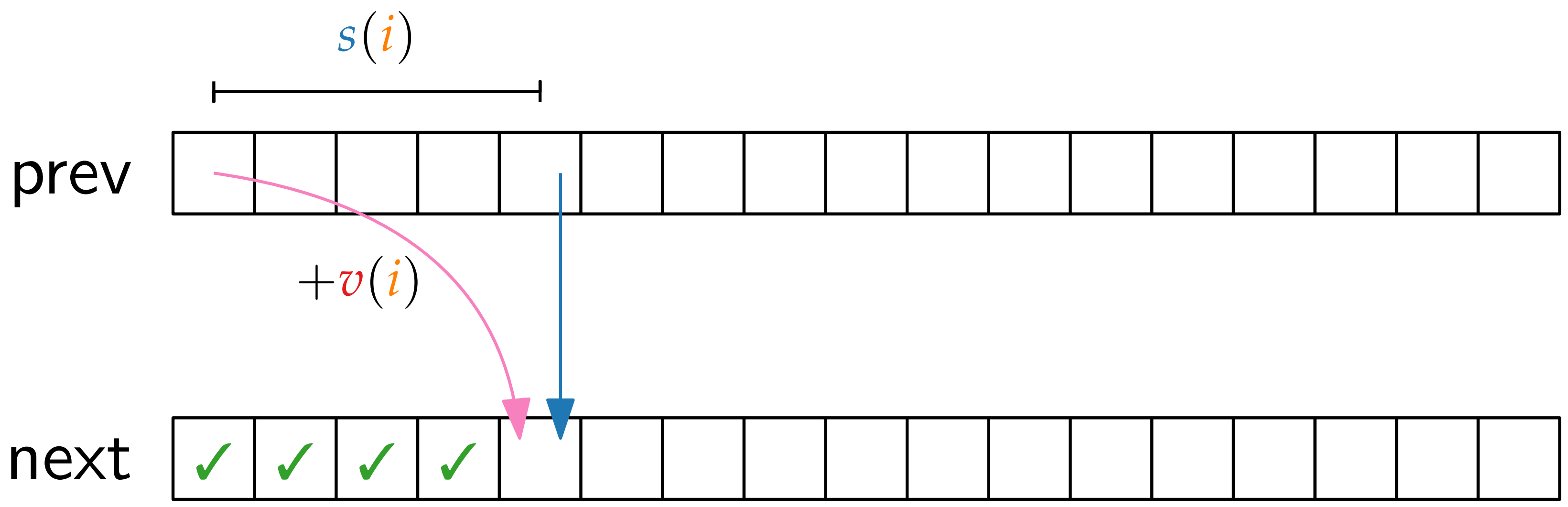
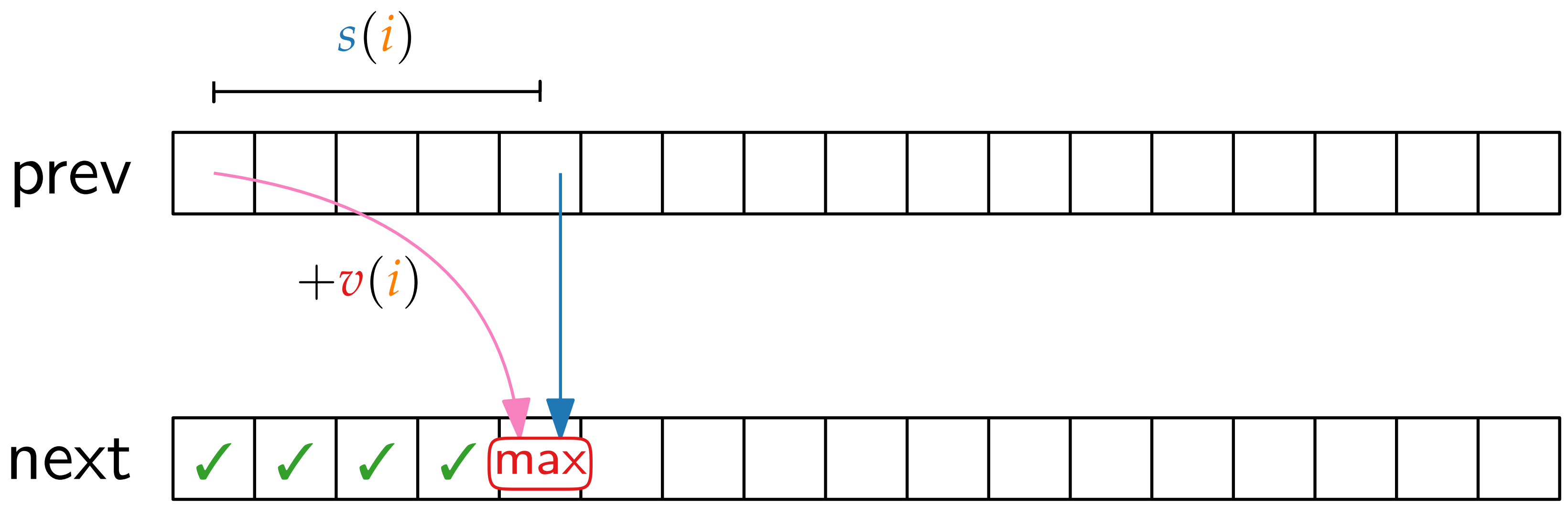


$$f(i, s) = \max \begin{cases} f(i-1, s) \\ v(i) + f(i-1, s - s(i)) & \text{if } s \geq s(i) \end{cases}$$



$$f(i, s) = \max \begin{cases} f(i-1, s) \\ v(i) + f(i-1, s-s(i)) & \text{if } s \geq s(i) \end{cases}$$



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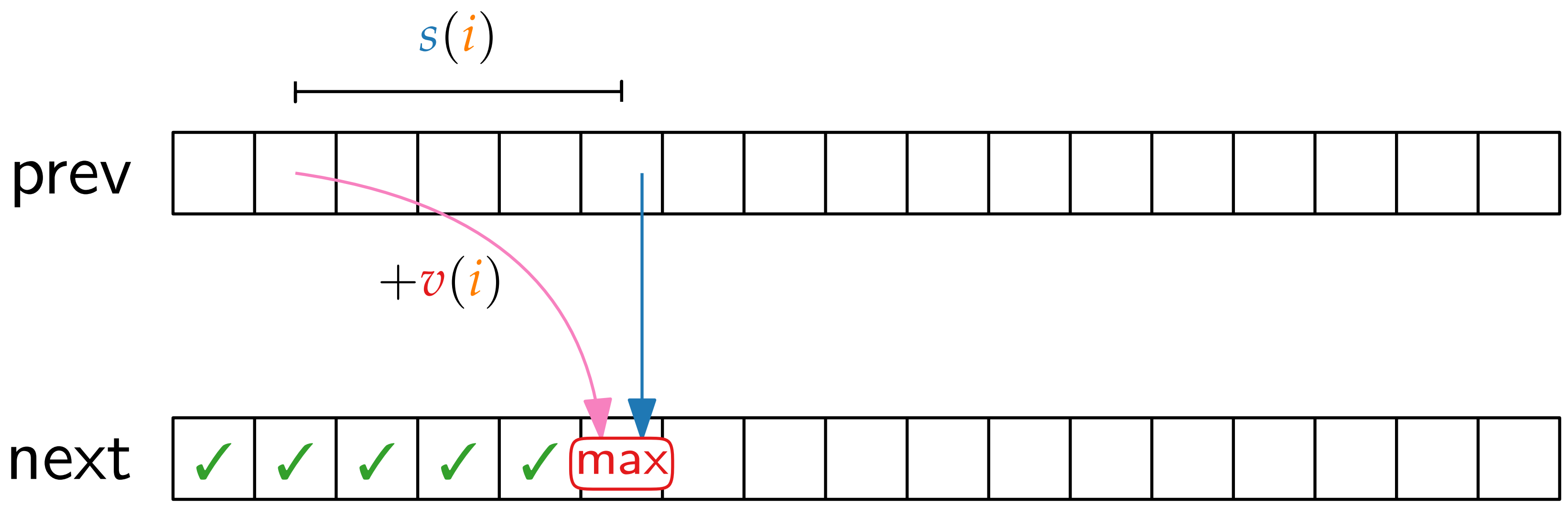

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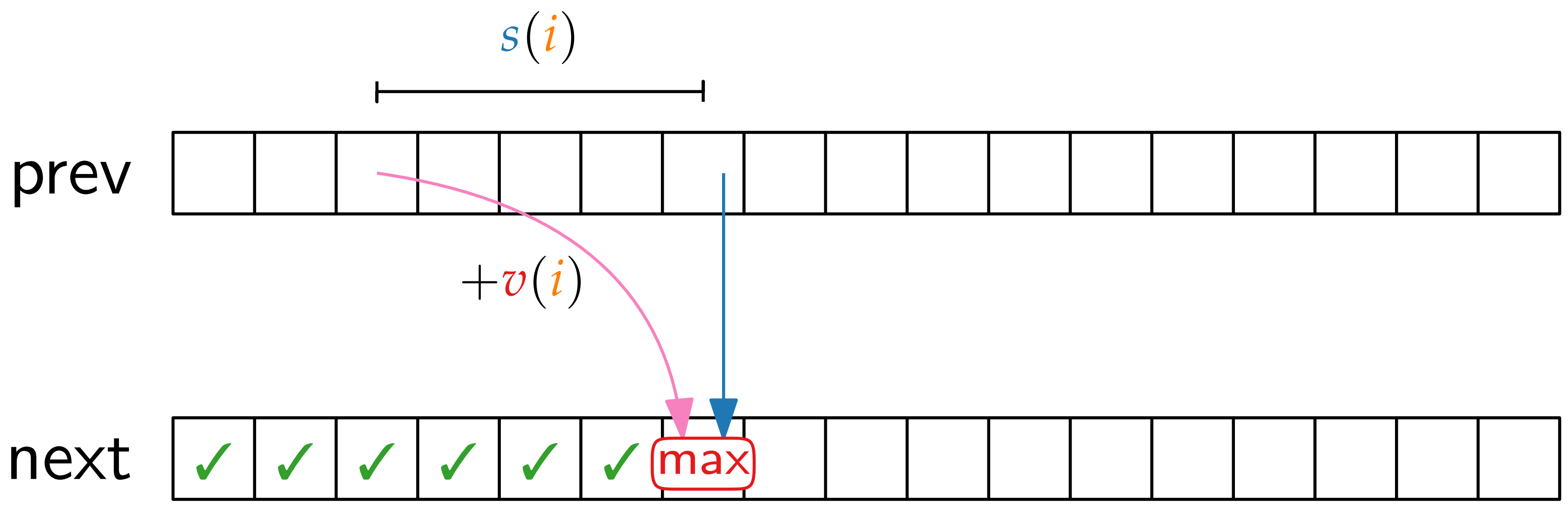

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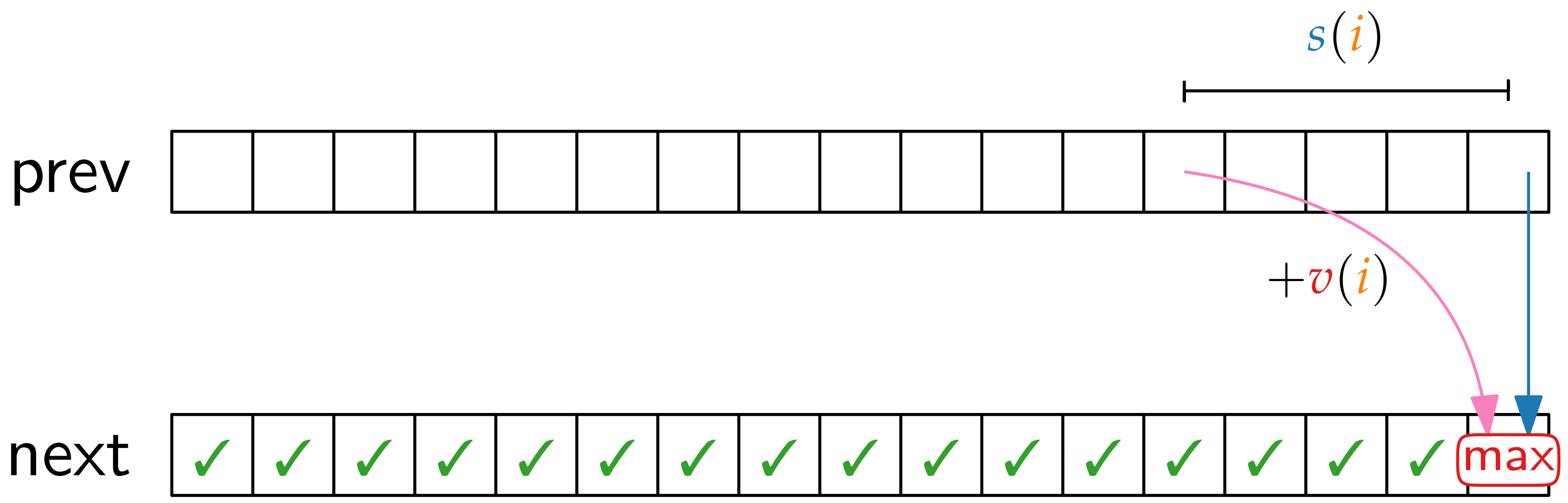

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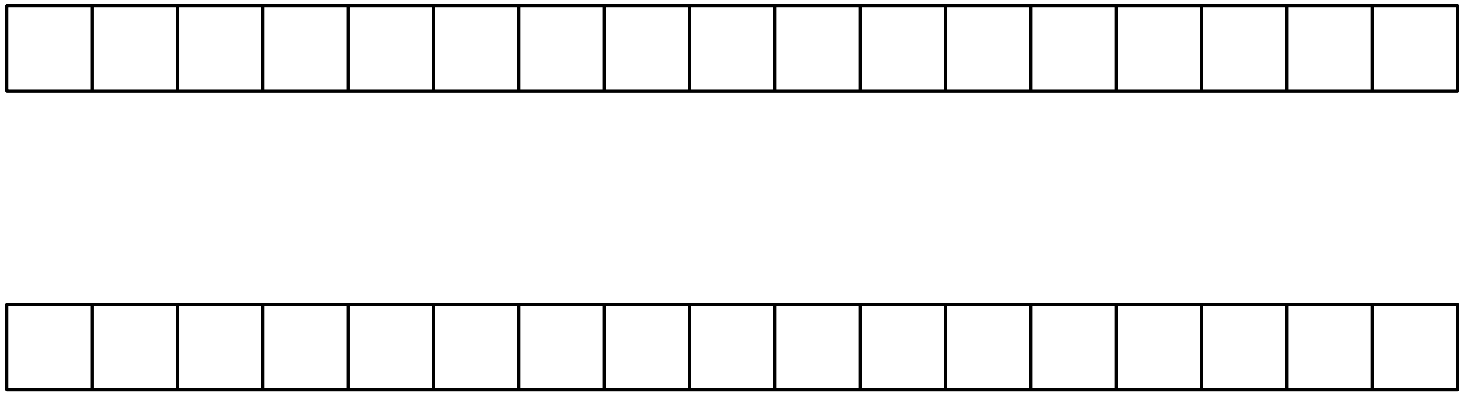
Hint:

Use optimized procedures provided by your operating system!

prev

next



$$f(i, s) = \max \begin{cases} f(i-1, s) \\ v(i) + f(i-1, s - s(i)) & \text{if} \end{cases}$$

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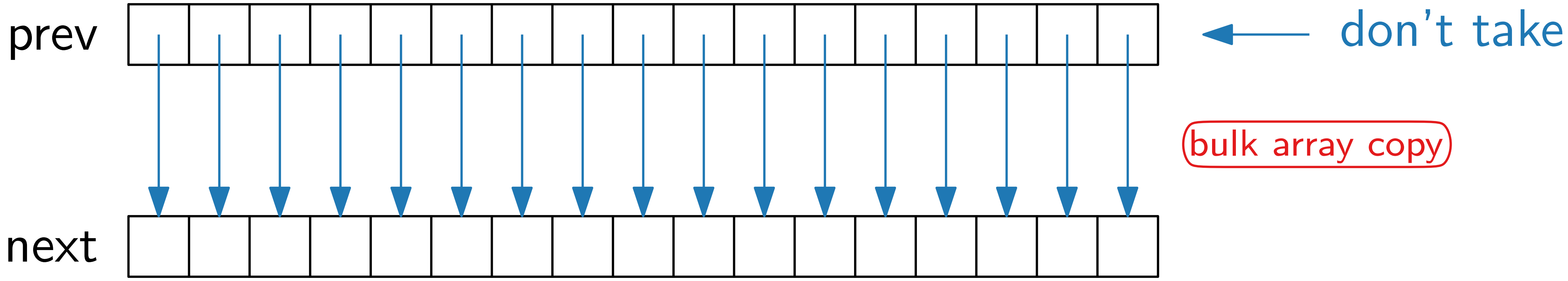

$$f(i, s) = \max \begin{cases} f(i-1, s) \\ v(i) + f(i-1, s - s(i)) \quad \text{if} \end{cases}$$

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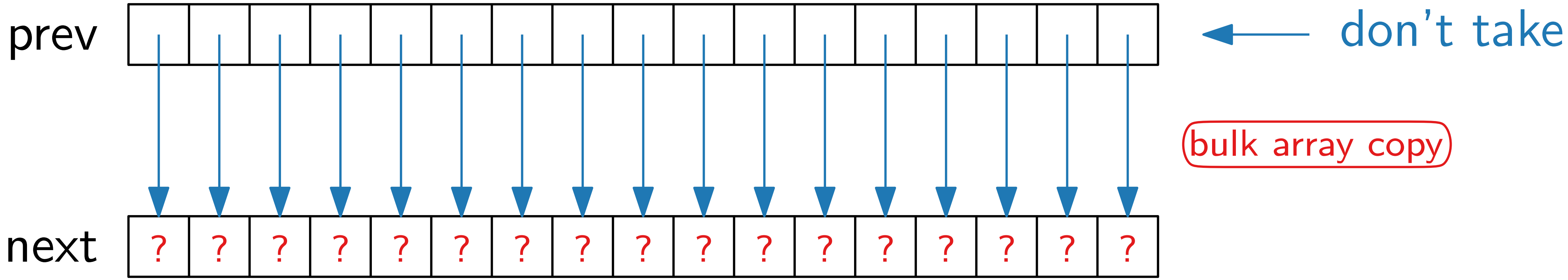

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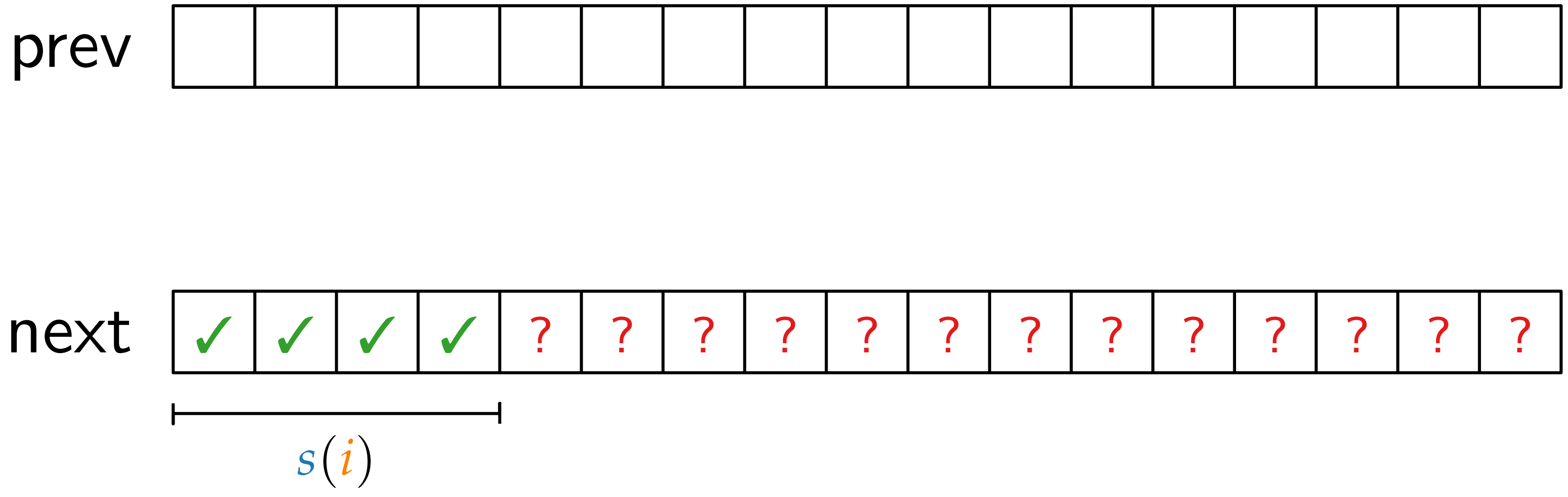

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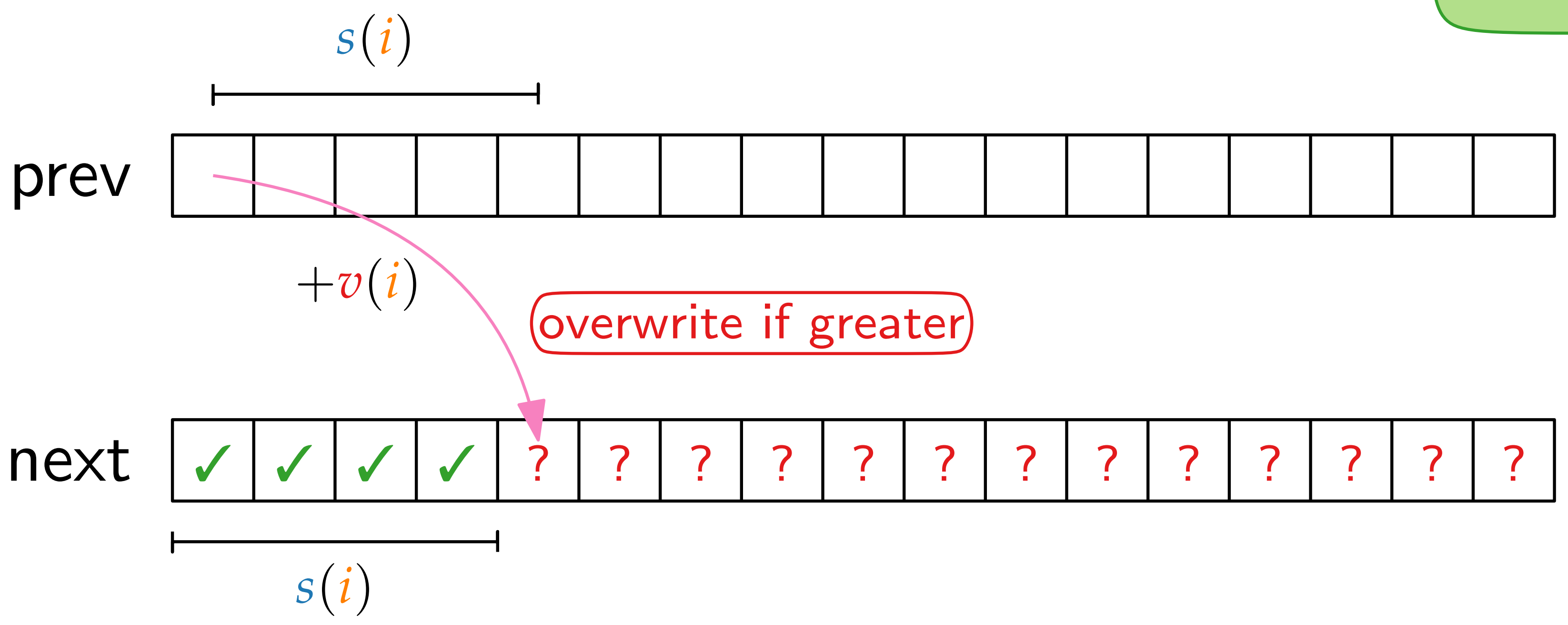

$$f(i,s) = \max \begin{cases} f(i-1,s) & \\ v(i) + f(i-1, s - s(i)) & \text{if} \end{cases}$$

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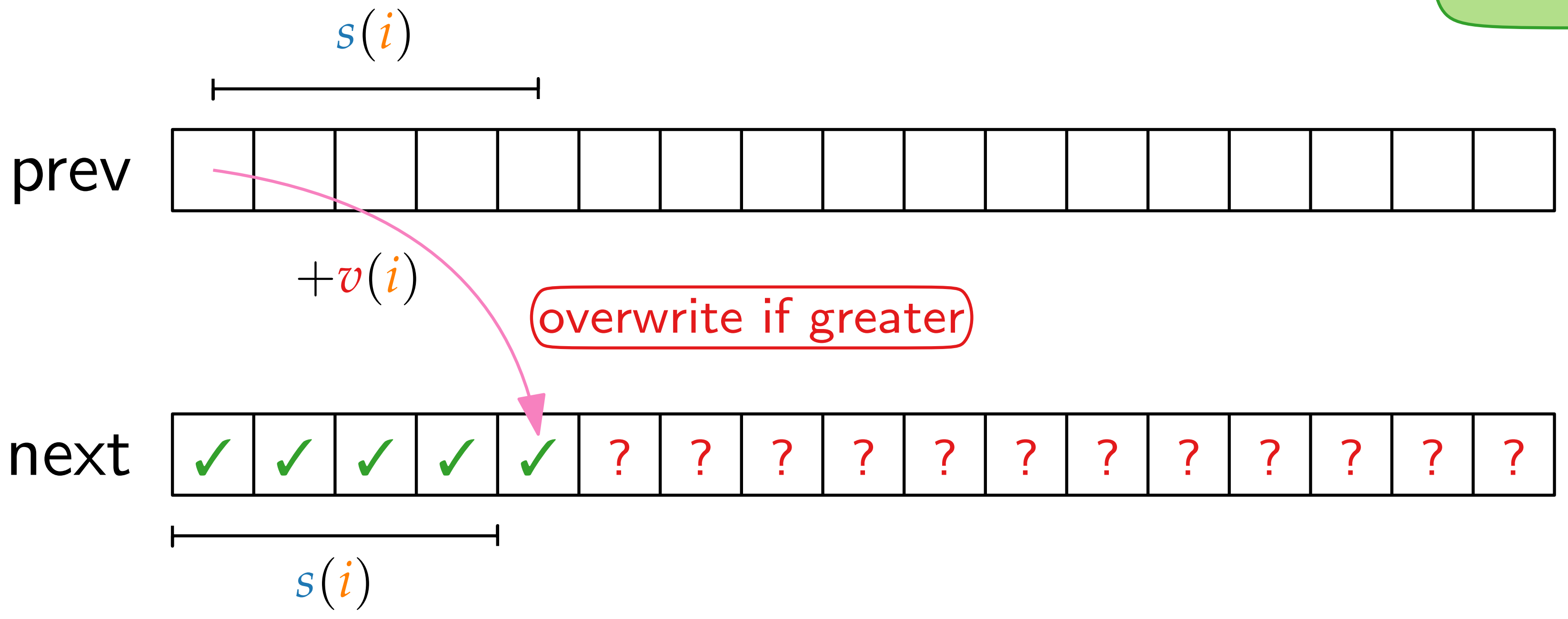

$$f(i, s) = \max \begin{cases} f(i-1, s) \\ v(i) + f(i-1, s - s(i)) & \text{if} \end{cases}$$

Hint:
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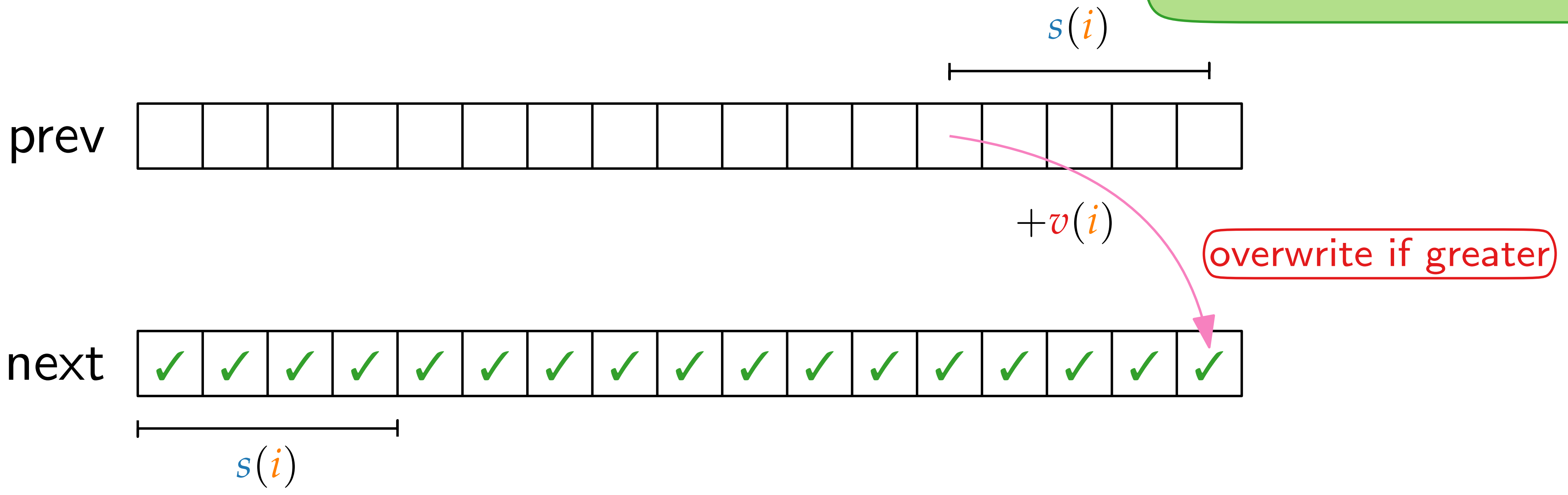
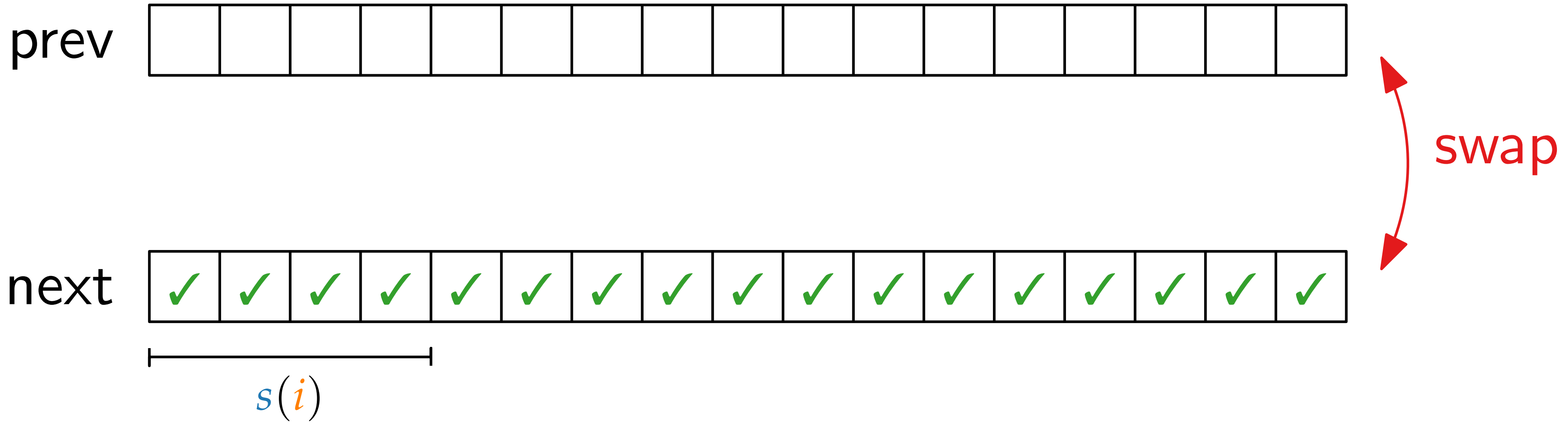


$$f(i,s) = \max \begin{cases} f(i-1,s) \\ v(i) + f(i-1, s - s(i)) & \text{if} \ldots \end{cases}$$

Hint:
Use optimized procedures provided by your operating system!



Running Times

$\|U\| =$	30	100	1K	10K
Scala naïve	1.15 s			
Scala memoized	230 ms	267 ms	⚡SO	
Scala TCO		273 ms	2.30 s	⚡OOM
Scala Arrays			458 ms	27.5 s
Scala bulk copy				

$B = 10|U|$

Running Times

$\lvert U \rvert =$	30	100	1K	10K
Scala naïve	1.15 s			
Scala memoized	230 ms	267 ms	⚡SO	
Scala TCO		273 ms	2.30 s	⚡OOM
Scala Arrays			458 ms	27.5 s
Scala bulk copy				437 ms

$B = 10\lvert U \rvert$

Running Times

$\lvert U\rvert =$	30	100	1K	10K
Scala naïve	1.15 s			
Scala memoized	230 ms	267 ms	⚡SO	
Scala TCO		273 ms	2.30 s	⚡OOM
Scala Arrays			458 ms	27.5 s
Scala bulk copy				437 ms

$B = 10\lvert U\rvert$

Try other programming languages?

Running Times

$\lvert U \rvert =$	30	100	1K	10K
Scala naïve	1.15 s			
Scala memoized	230 ms	267 ms	⚡ SO	
Scala TCO		273 ms	2.30 s	⚡ OOM
Scala Arrays			458 ms	27.5 s
Scala bulk copy				437 ms

$B = 10\lvert U \rvert$

Try other programming languages? **Sure!**

Running Times

$|U| =$	30	100	1K	10K
Scala naïve	1.15 s			
Scala memoized	230 ms	267 ms	⚡SO	
Scala TCO		273 ms	2.30 s	⚡OOM
Scala Arrays			458 ms	27.5 s
Scala bulk copy				437 ms
Java				

$B = 10|U|$

Running Times

$\lvert U \rvert =$	30	100	1K	10K
Scala naïve	1.15 s			
Scala memoized	230 ms	267 ms	⚡ SO	
Scala TCO		273 ms	2.30 s	⚡ OOM
Scala Arrays			458 ms	27.5 s
Scala bulk copy				437 ms
Java				636 ms

$B = 10\lvert U \rvert$

Running Times

$B = 10|U|$

$\|U\| =$	30	100	1K	10K
Scala naïve	1.15 s			
Scala memoized	230 ms	267 ms	⚡SO	
Scala TCO		273 ms	2.30 s	⚡OOM
Scala Arrays			458 ms	27.5 s
Scala bulk copy				437 ms
Java				636 ms

can be fixed!

Running Times

$\lvert U\rvert =$	30	100	1K	10K
Scala naïve	1.15 s			
Scala memoized	230 ms	267 ms	⚡ SO	
Scala TCO		273 ms	2.30 s	⚡ OOM
Scala Arrays			458 ms	27.5 s
Scala bulk copy				437 ms
Java				636 ms
node.js				

$B = 10\lvert U\rvert$

Running Times

$B = 10|U|$

$\|U\| =$	30	100	1K	10K
Scala naïve	1.15 s			
Scala memoized	230 ms	267 ms	⚡SO	
Scala TCO		273 ms	2.30 s	⚡OOM
Scala Arrays			458 ms	27.5 s
Scala bulk copy				437 ms
Java				636 ms
node.js				957 ms

Running Times

$\lvert U \rvert =$	30	100	1K	10K
Scala naïve	1.15 s			
Scala memoized	230 ms	267 ms	⚡SO	
Scala TCO		273 ms	2.30 s	⚡OOM
Scala Arrays			458 ms	27.5 s
Scala bulk copy				437 ms
Java				636 ms
node.js				957 ms
Python3				

$B = 10\lvert U \rvert$

Running Times

$\lvert U \rvert =$	30	100	1K	10K
Scala naïve	1.15 s			
Scala memoized	230 ms	267 ms	⚡SO	
Scala TCO		273 ms	2.30 s	⚡OOM
Scala Arrays			458 ms	27.5 s
Scala bulk copy				437 ms
Java				636 ms
node.js				957 ms
Python3				96.8 s

$B = 10\lvert U \rvert$

Running Times

$B = 10|U|$

$\|U\| =$	30	100	1K	10K
Scala naïve	1.15 s			
Scala memoized	230 ms	267 ms	⚡SO	
Scala TCO		273 ms	2.30 s	⚡OOM
Scala Arrays			458 ms	27.5 s
Scala bulk copy				437 ms
Java				636 ms
node.js				957 ms
Python3				96.8 s
Pypy3				

Running Times

$\lvert U \rvert =$	30	100	1K	10K
Scala naïve	1.15 s			
Scala memoized	230 ms	267 ms	⚡SO	
Scala TCO		273 ms	2.30 s	⚡OOM
Scala Arrays			458 ms	27.5 s
Scala bulk copy				437 ms
Java				636 ms
node.js				957 ms
Python3				96.8 s
Pypy3				2.48 s

$B = 10\lvert U \rvert$

Running Times

$B = 10|U|$

$|U| =$	30	100	1K	10K
Scala naïve	1.15 s			
Scala memoized	230 ms	267 ms	⚡ SO	
Scala TCO		273 ms	2.30 s	⚡ OOM
Scala Arrays			458 ms	27.5 s
Scala bulk copy				437 ms
Java				636 ms
node.js				957 ms
Python3				96.8 s
Pypy3				2.48 s
Rust (safe)				

Running Times

$B = 10|U|$

$\lvert U \rvert =$	30	100	1K	10K
Scala naïve	1.15 s			
Scala memoized	230 ms	267 ms	⚡ SO	
Scala TCO		273 ms	2.30 s	⚡ OOM
Scala Arrays			458 ms	27.5 s
Scala bulk copy				437 ms
Java				636 ms
node.js				957 ms
Python3				96.8 s
Pypy3				2.48 s
Rust (safe)				300 ms

Running Times

$\lvert U \rvert =$	30	100	1K	10K
Scala naïve	1.15 s			
Scala memoized	230 ms	267 ms	⚡SO	
Scala TCO		273 ms	2.30 s	⚡OOM
Scala Arrays			458 ms	27.5 s
Scala bulk copy				437 ms
Java				636 ms
node.js				957 ms
Python3				96.8 s
Pypy3				2.48 s
Rust (safe)				300 ms
C++ (by tvd)				

$B = 10\lvert U \rvert$

Running Times

$\lvert U\rvert =$	30	100	1K	10K
Scala naïve	1.15 s			
Scala memoized	230 ms	267 ms	⚡SO	
Scala TCO		273 ms	2.30 s	⚡OOM
Scala Arrays			458 ms	27.5 s
Scala bulk copy				437 ms
Java				636 ms
node.js				957 ms
Python3				96.8 s
Pypy3				2.48 s
Rust (safe)				300 ms
C++ (by tvd)				212 ms

$B = 10\lvert U\rvert$

Running Times

$\lvert U \rvert =$	30	100	1K	10K
Scala naïve	1.15 s			
Scala memoized	230 ms	267 ms	⚡SO	
Scala TCO		273 ms	2.30 s	⚡OOM
Scala Arrays			458 ms	27.5 s
Scala bulk copy				437 ms
Java				636 ms
node.js				957 ms
Python3				96.8 s
Pypy3				2.48 s
Rust (safe)				300 ms
C++ (by tvd)				212 ms
Rust with AVX2				

$B = 10\lvert U \rvert$

Running Times

$\lvert U\rvert =$	30	100	1K	10K
Scala naïve	1.15 s			
Scala memoized	230 ms	267 ms	SO	
Scala TCO		273 ms	2.30 s	OOM
Scala Arrays			458 ms	27.5 s
Scala bulk copy				437 ms
Java				636 ms
node.js				957 ms
Python3				96.8 s
Pypy3				2.48 s
Rust (safe)				300 ms
C++ (by tvd)				212 ms
Rust with AVX2				43 ms

$B = 10\lvert U\rvert$

Julius-Maximilians-
UNIVERSITÄT
WÜRZBURG

Advanced Algorithms

Algorithms in Practice

How to Sort a Million 32-bit Integers

Tim Hegemann · WS22

How to Sort a Million 32-bit Integers?

How to Sort a Million 32-bit Integers?

Well-known comparison-based sorting algorithms

	best-case	average	worst-case	in-place	stable
Insertion sort	$\mathcal{O}(n)$	$\mathcal{O}(n^2)$	$\mathcal{O}(n^2)$	✓	✓
Heap sort	$\mathcal{O}(n \log(n))$	$\mathcal{O}(n \log(n))$	$\mathcal{O}(n \log(n))$	✓	✗
Quicksort	$\mathcal{O}(n \log(n))$	$\mathcal{O}(n \log(n))$	$\mathcal{O}(n^2)$	✓	✗
Merge sort	$\mathcal{O}(n \log(n))$	$\mathcal{O}(n \log(n))$	$\mathcal{O}(n \log(n))$	✗	✓

How to Sort a Million 32-bit Integers?

Well-known comparison-based sorting algorithms

	best-case	average	worst-case	in-place	stable
Insertion sort	$\mathcal{O}(n)$	$\mathcal{O}(n^2)$	$\mathcal{O}(n^2)$	✓	✓
Heap sort	$\mathcal{O}(n\log(n))$	$\mathcal{O}(n\log(n))$	$\mathcal{O}(n\log(n))$	✓	✗
Quicksort	$\mathcal{O}(n\log(n))$	$\mathcal{O}(n\log(n))$	$\mathcal{O}(n^2)$	✓	✗
Merge sort	$\mathcal{O}(n\log(n))$	$\mathcal{O}(n\log(n))$	$\mathcal{O}(n\log(n))$	✗	✓

Observation.

- Equal integers are undistinguishable.

How to Sort a Million 32-bit Integers?

Well-known comparison-based sorting algorithms

	best-case	average	worst-case	in-place
Insertion sort	$\mathcal{O}(n)$	$\mathcal{O}(n^2)$	$\mathcal{O}(n^2)$	✓
Heap sort	$\mathcal{O}(n \log(n))$	$\mathcal{O}(n \log(n))$	$\mathcal{O}(n \log(n))$	✓
Quicksort	$\mathcal{O}(n \log(n))$	$\mathcal{O}(n \log(n))$	$\mathcal{O}(n^2)$	✓
Merge sort	$\mathcal{O}(n \log(n))$	$\mathcal{O}(n \log(n))$	$\mathcal{O}(n \log(n))$	✗

Observation.

- Equal integers are undistinguishable.

How to Sort a Million 32-bit Integers?

Well-known comparison-based sorting algorithms

	best-case	average	worst-case	in-place
Insertion sort	$\mathcal{O}(n)$	$\mathcal{O}(n^2)$	$\mathcal{O}(n^2)$	✓
Heap sort	$\mathcal{O}(n\log(n))$	$\mathcal{O}(n\log(n))$	$\mathcal{O}(n\log(n))$	✓
Quicksort	$\mathcal{O}(n\log(n))$	$\mathcal{O}(n\log(n))$	$\mathcal{O}(n^2)$	✓
Merge sort	$\mathcal{O}(n\log(n))$	$\mathcal{O}(n\log(n))$	$\mathcal{O}(n\log(n))$	✗

Observation.

- Equal integers are undistinguishable.
- Integer comparisons are probably much cheaper than copying.

How to Sort a Million 32-bit Integers?

Well-known comparison-based sorting algorithms

	best-case	average	worst-case	in-place
Insertion sort	$\mathcal{O}(n)$	$\mathcal{O}(n^2)$	$\mathcal{O}(n^2)$	✓
Heap sort	$\mathcal{O}(n\log(n))$	$\mathcal{O}(n\log(n))$	$\mathcal{O}(n\log(n))$	✓
Quicksort	$\mathcal{O}(n\log(n))$	$\mathcal{O}(n\log(n))$	$\mathcal{O}(n^2)$	✓
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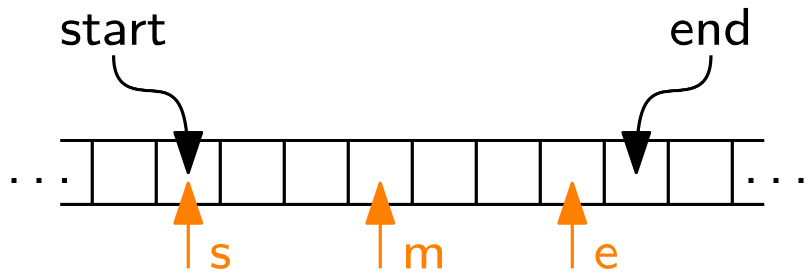

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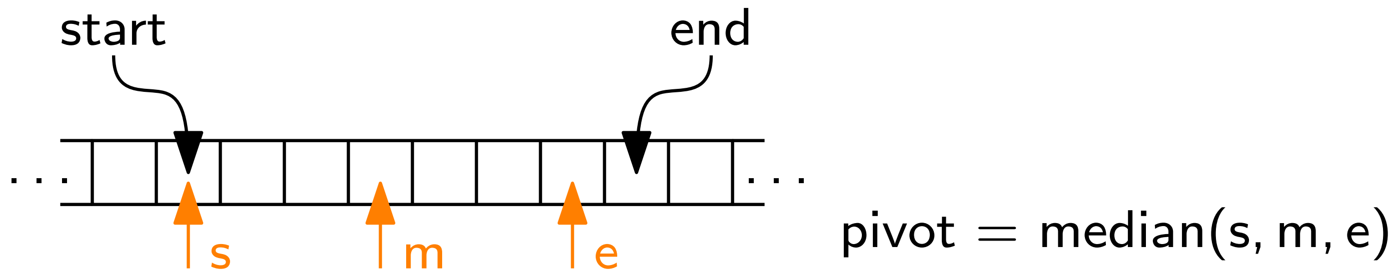

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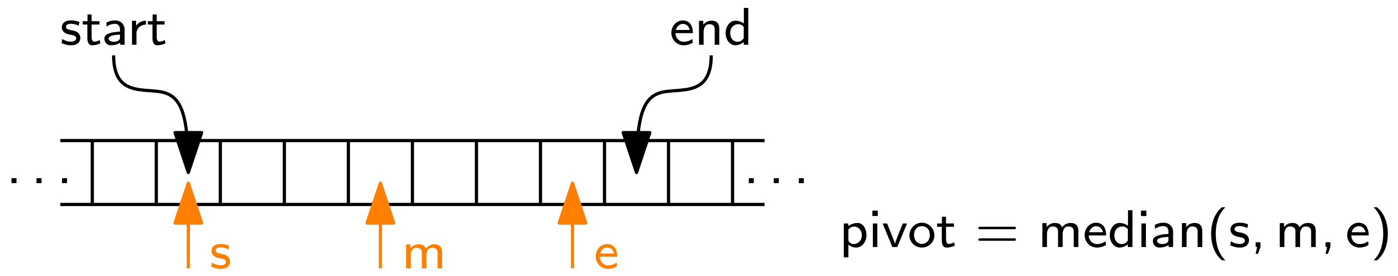

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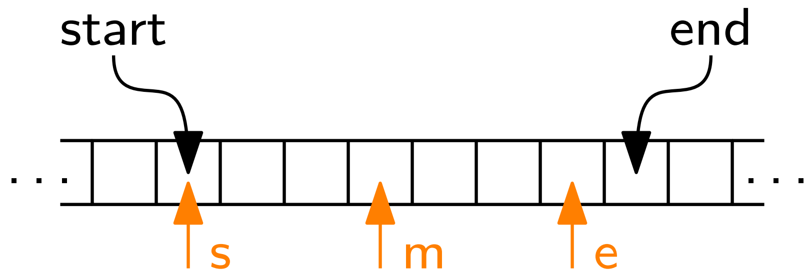

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see (and hear) it!