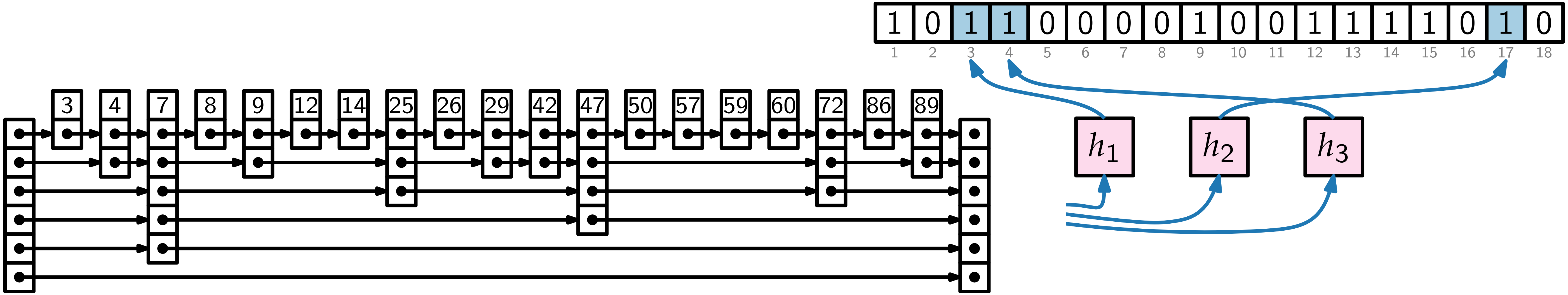
Advanced Algorithms

Randomized and Probabilistic Data Structures

Skip Lists & Bloom Filters

Johannes Zink · WS22



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  - Count–min sketch (estimates the frequency of different events in a data stream)

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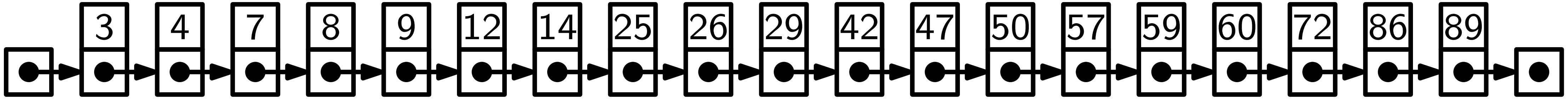

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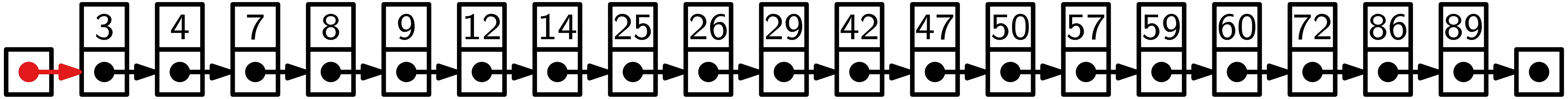

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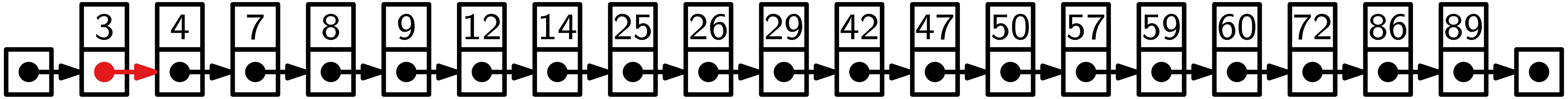

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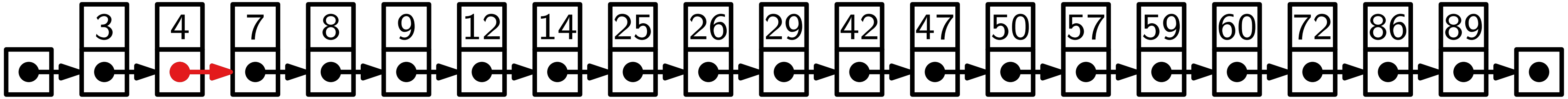

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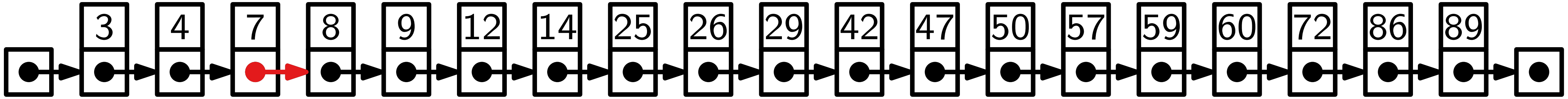

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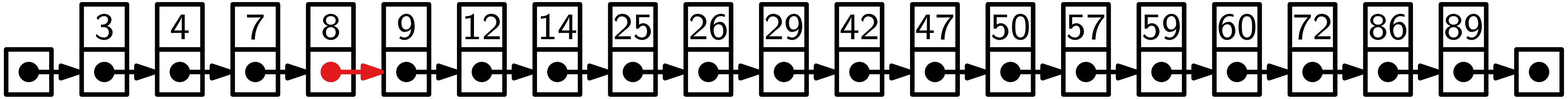

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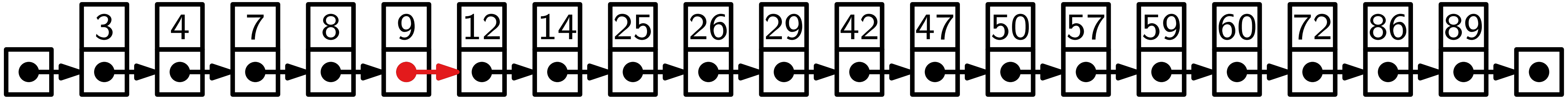

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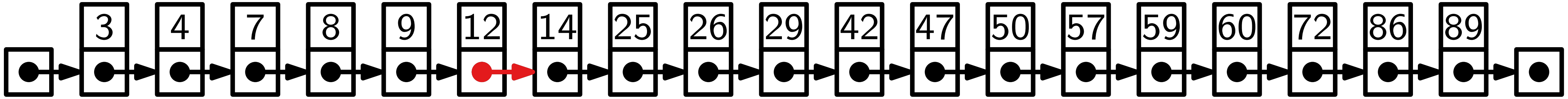

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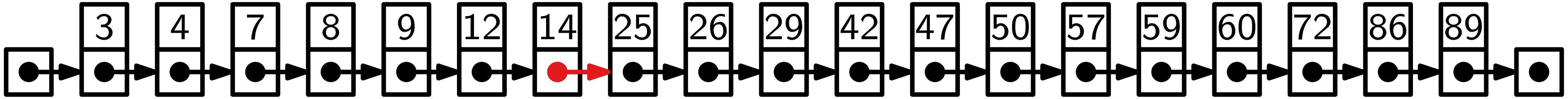

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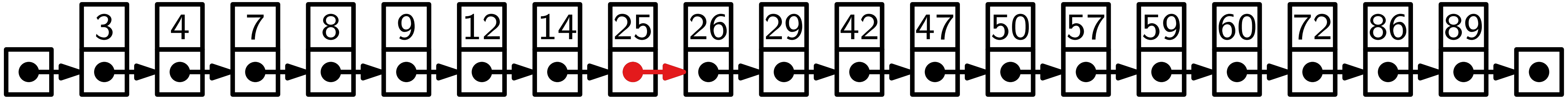

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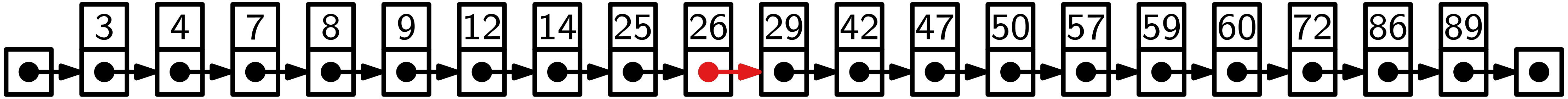

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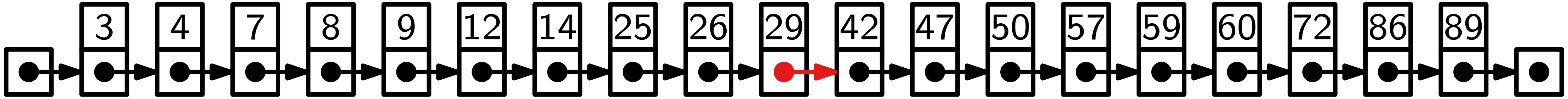

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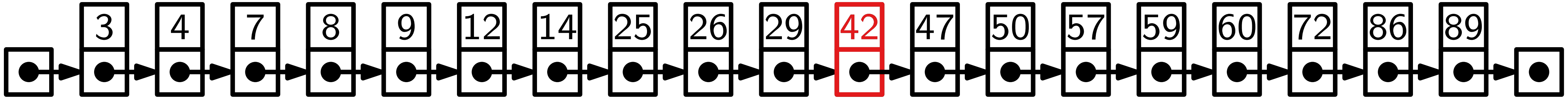

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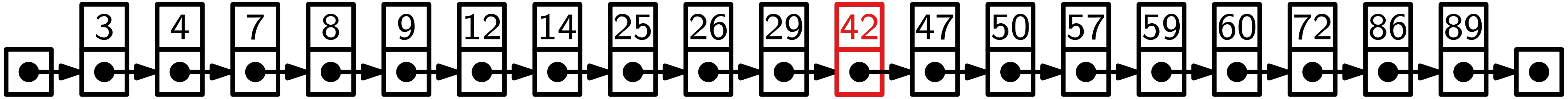

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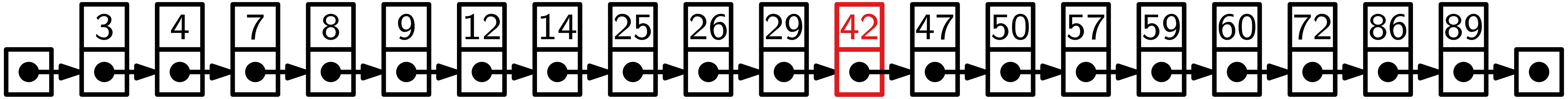

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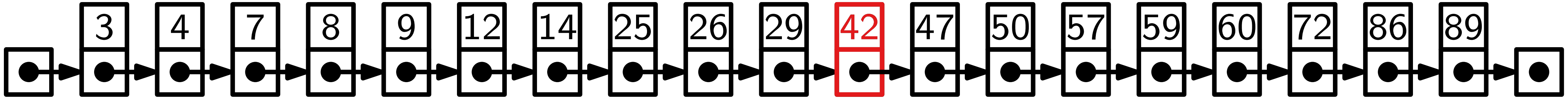

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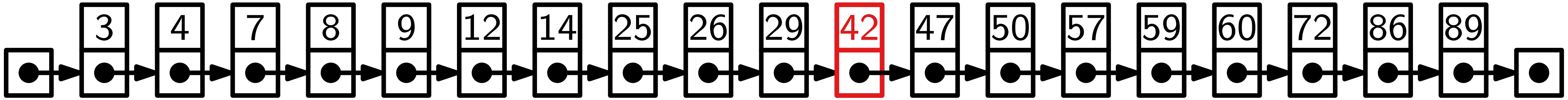

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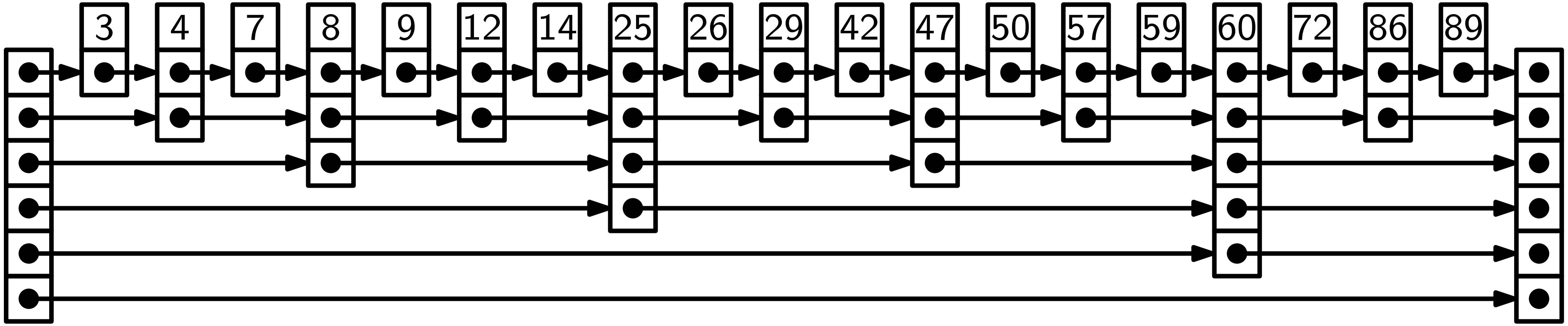

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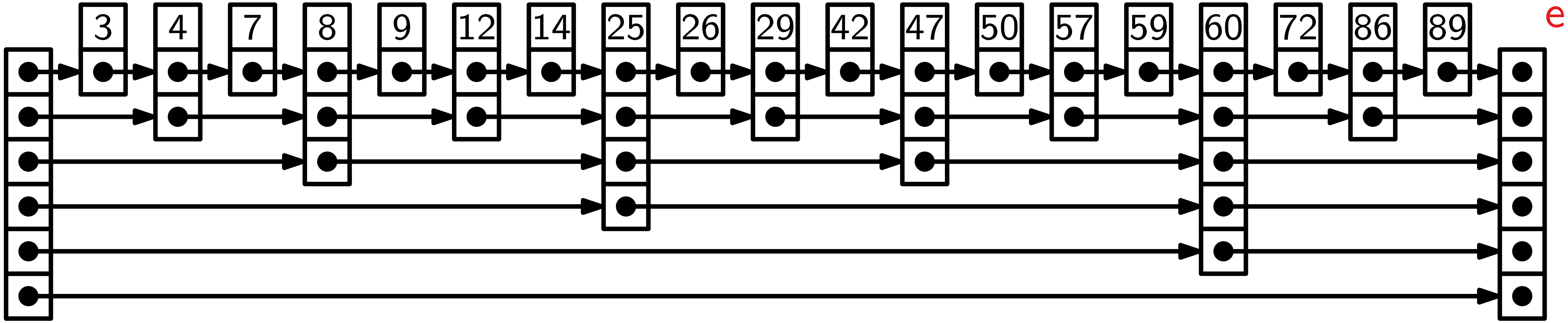

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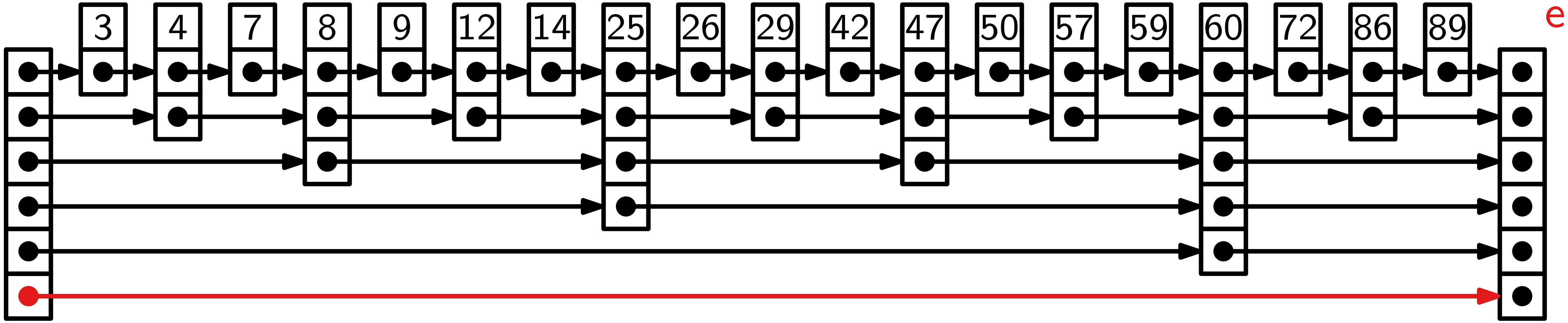

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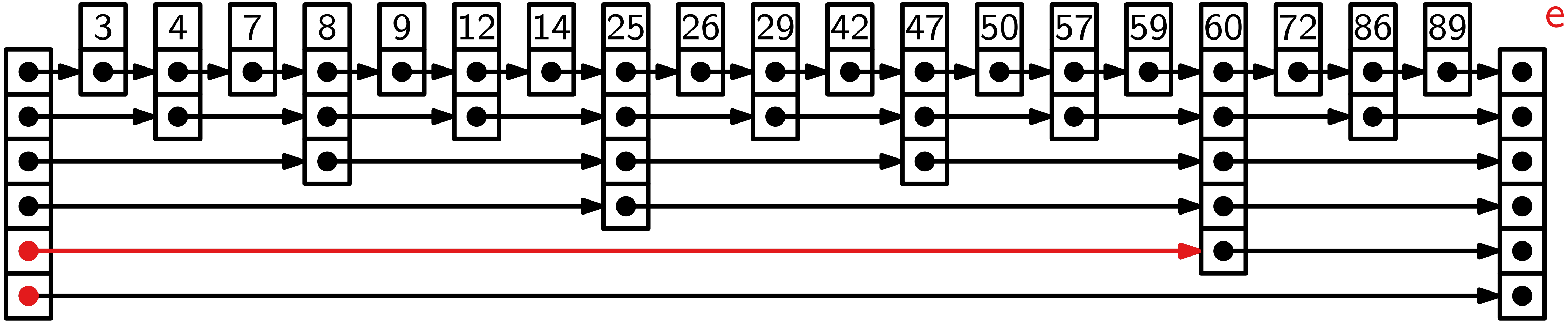

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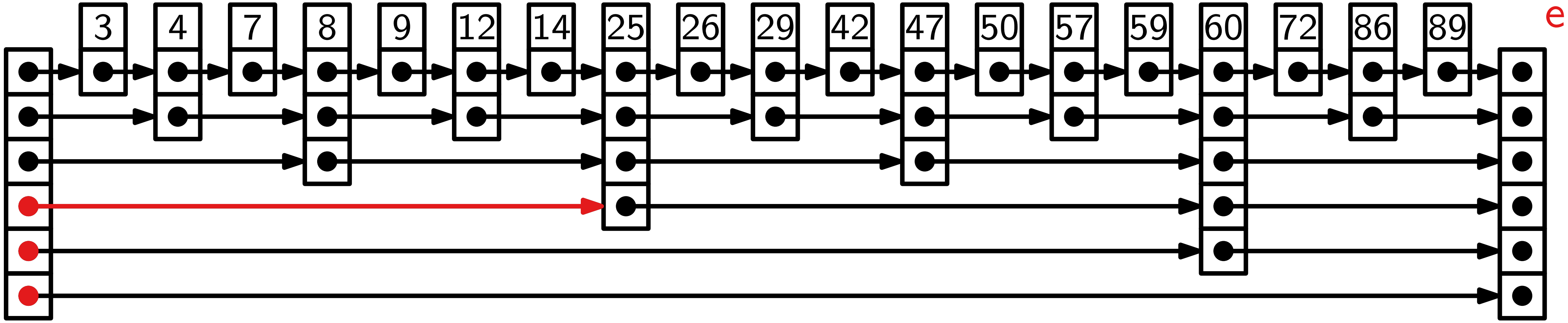

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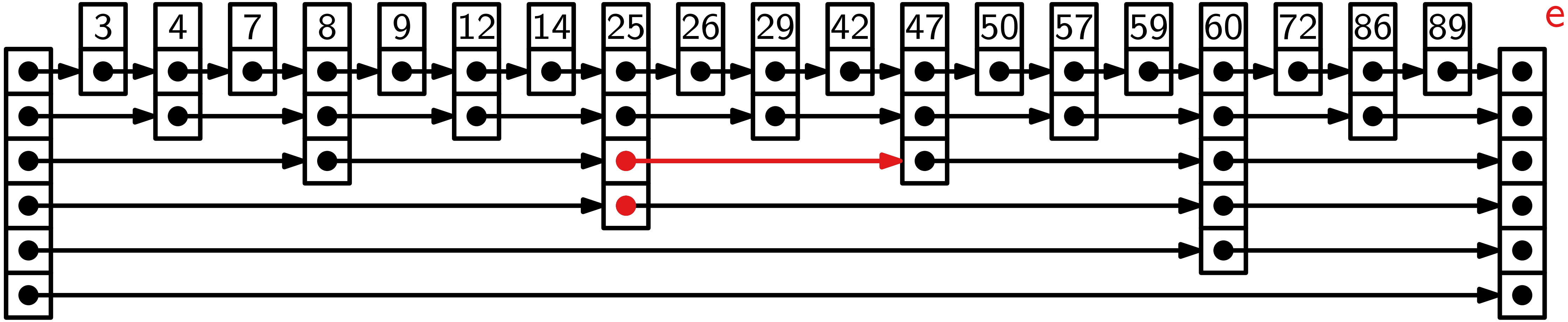

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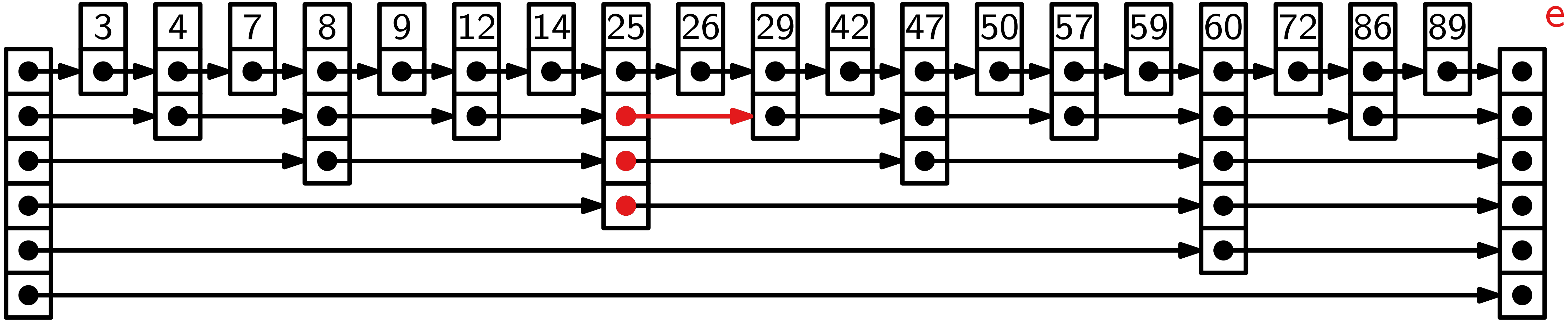

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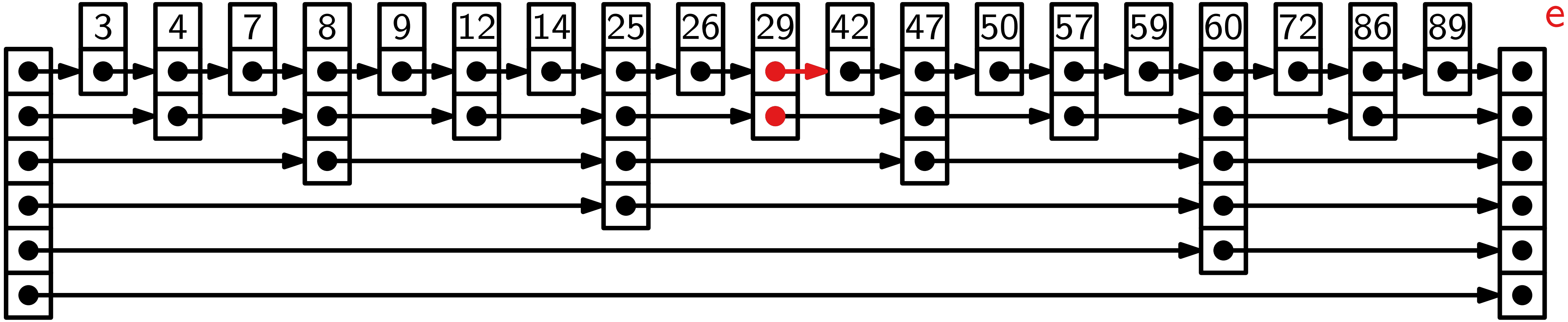

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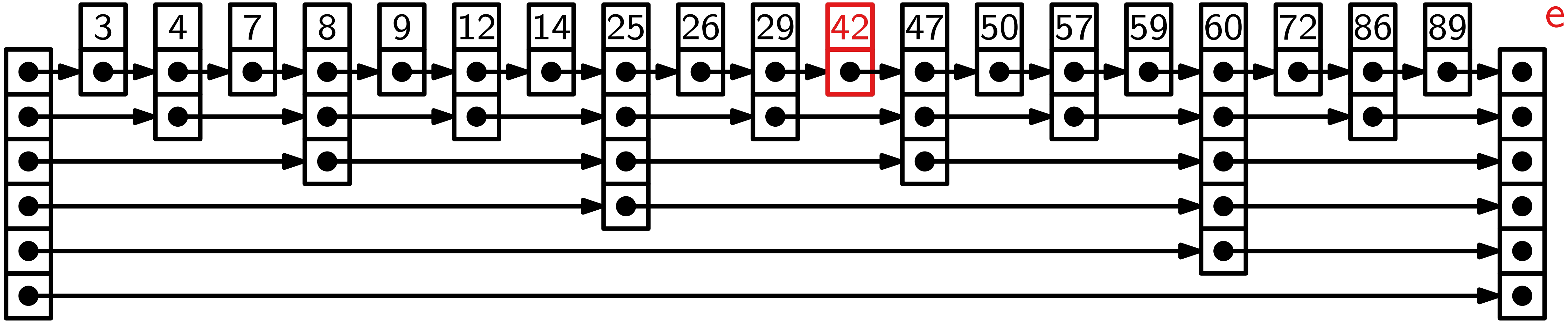

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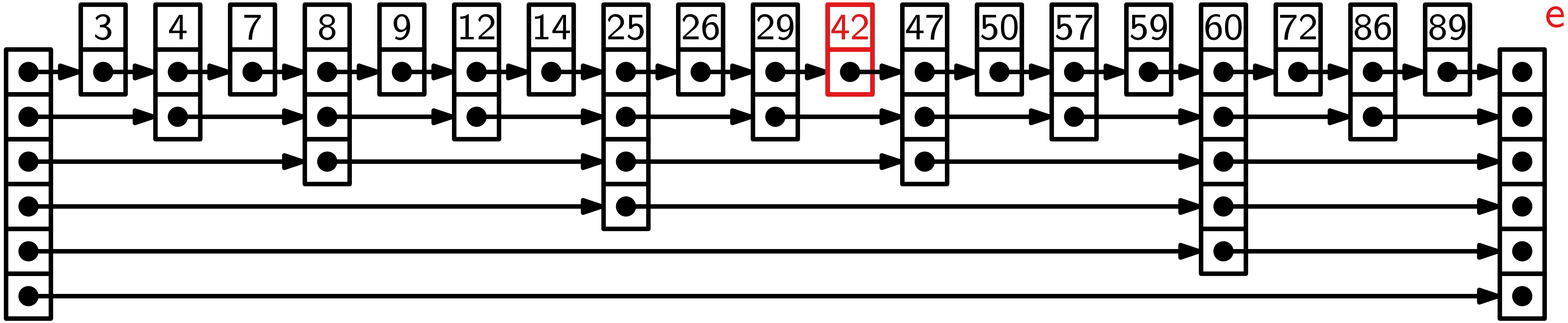

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(We check only constantly many entries per level and there are $O(\log n)$ levels.)

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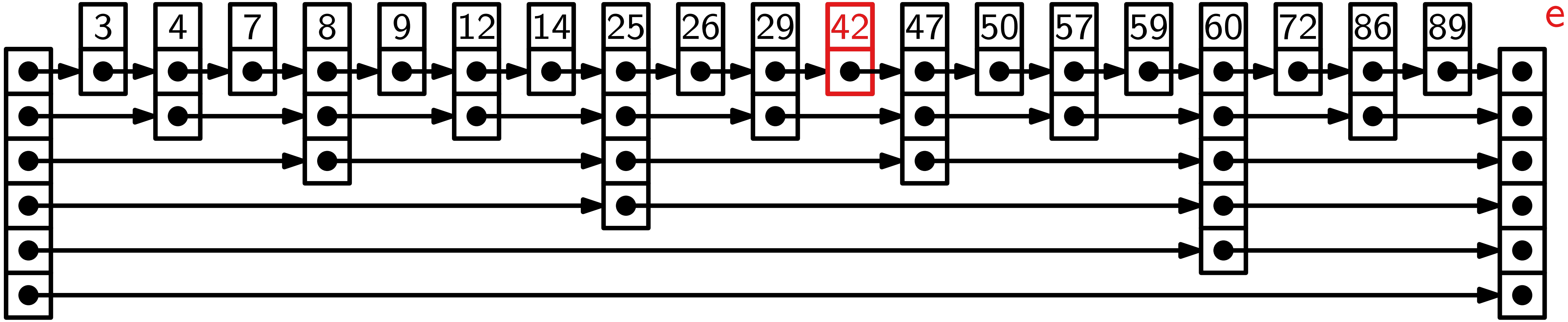

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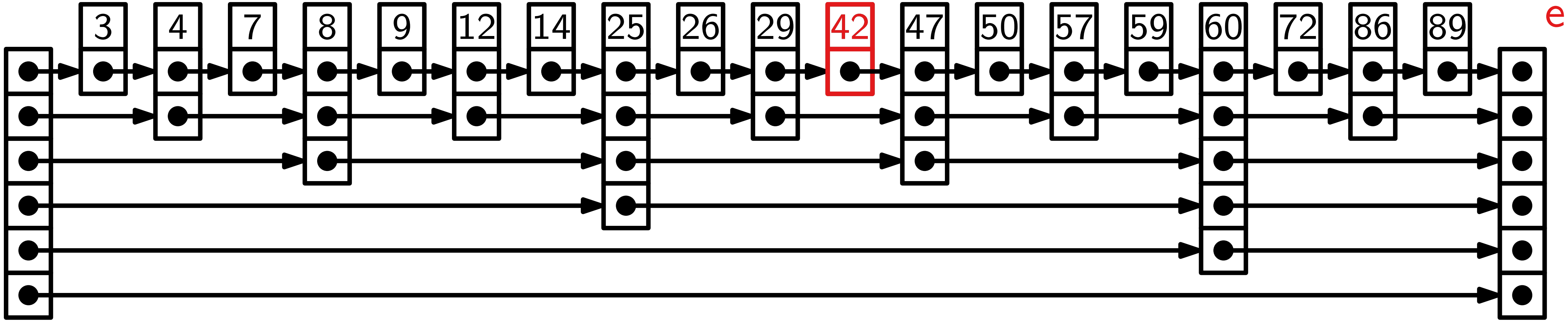

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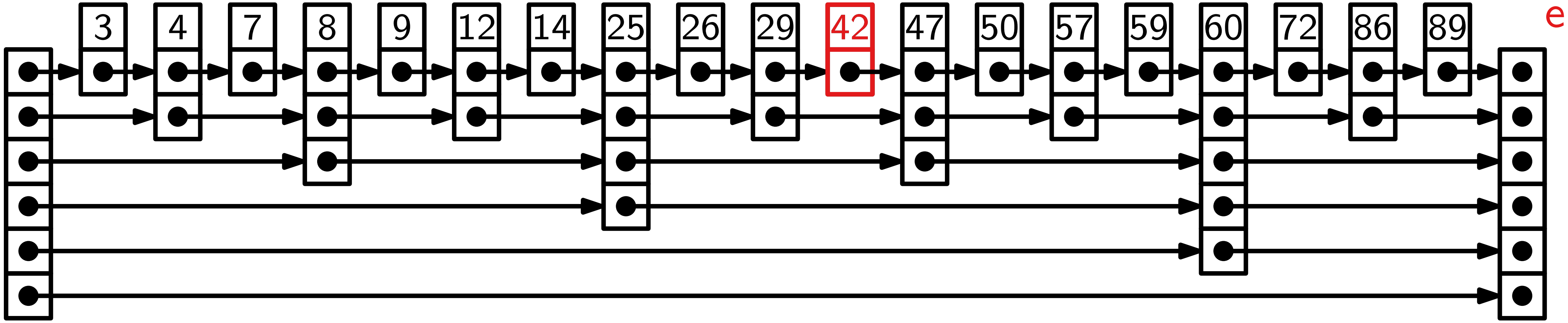

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Idea: Keep a skip list, but assign each entry a random height (number of lists it occurs in) s.t. lower heights are more likely to occur.

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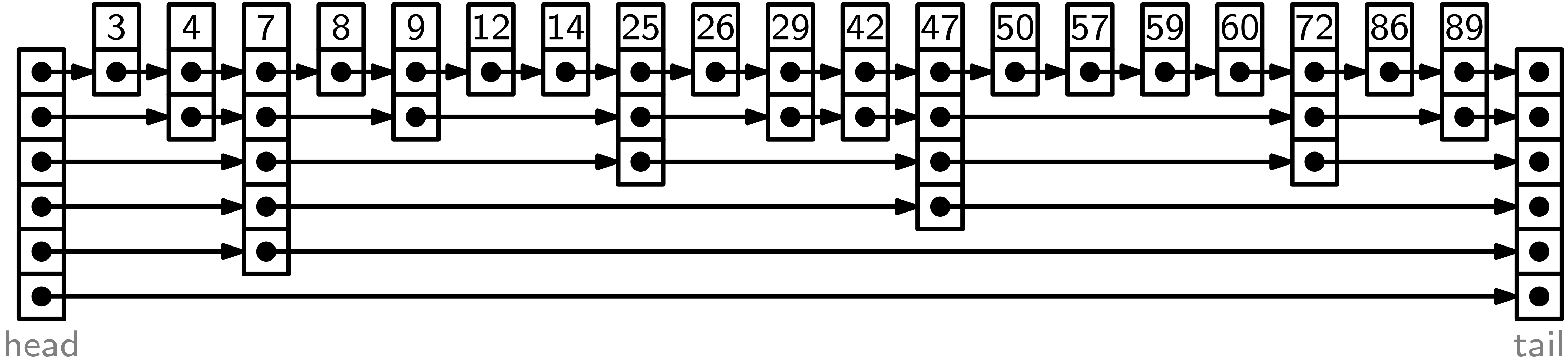
For a new entry, flip a coin until it shows HEAD. The number of flips will be its height.



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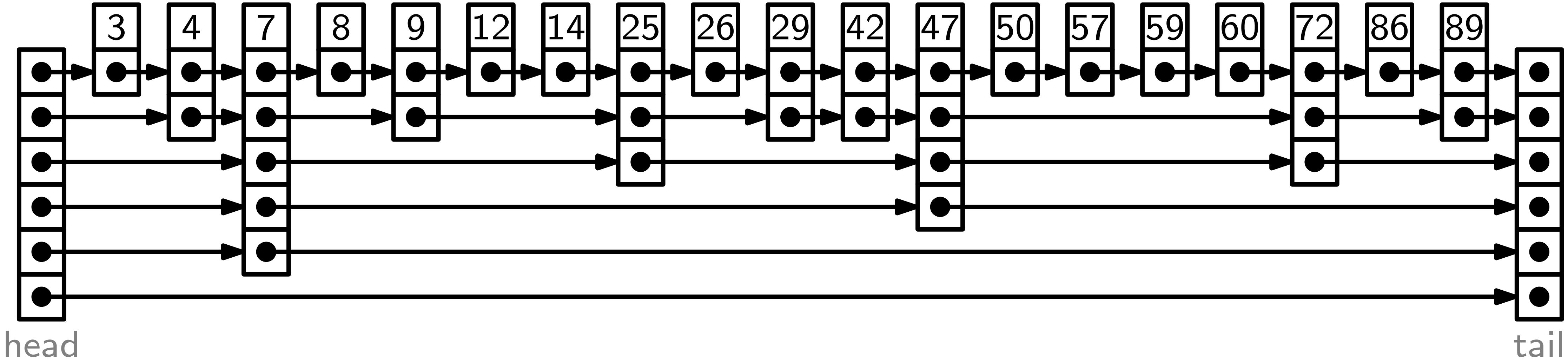

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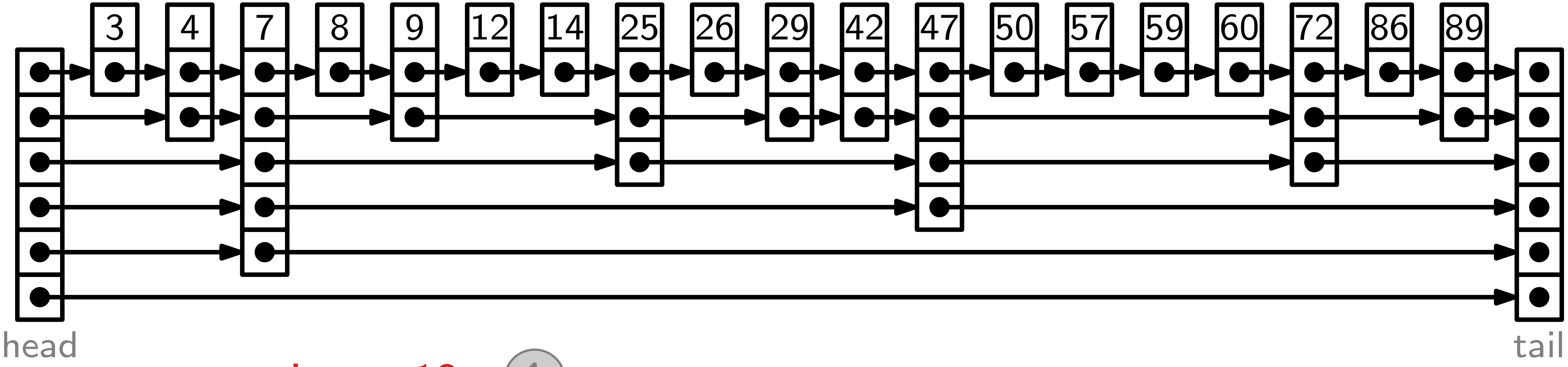

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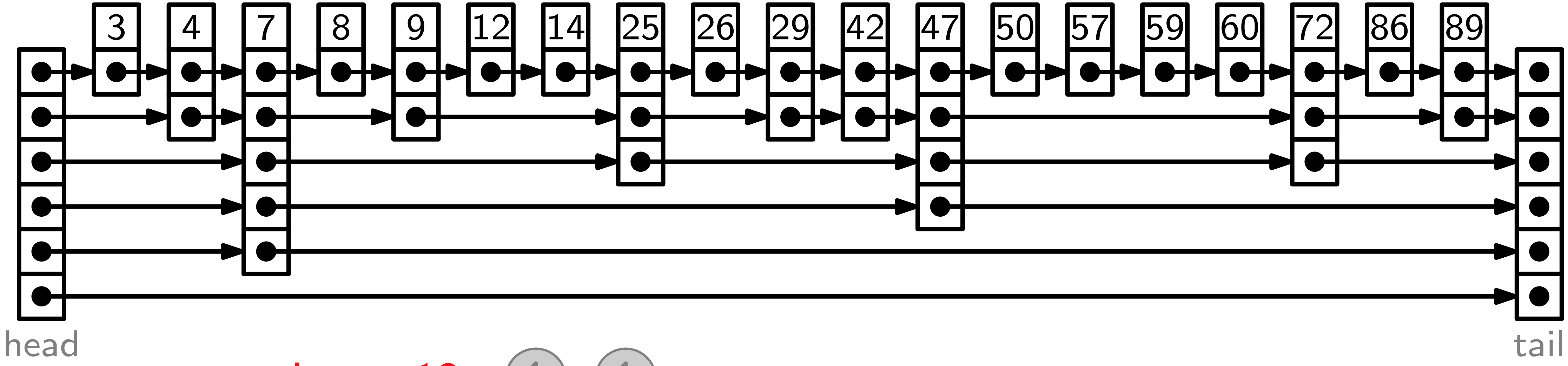

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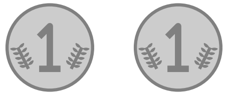

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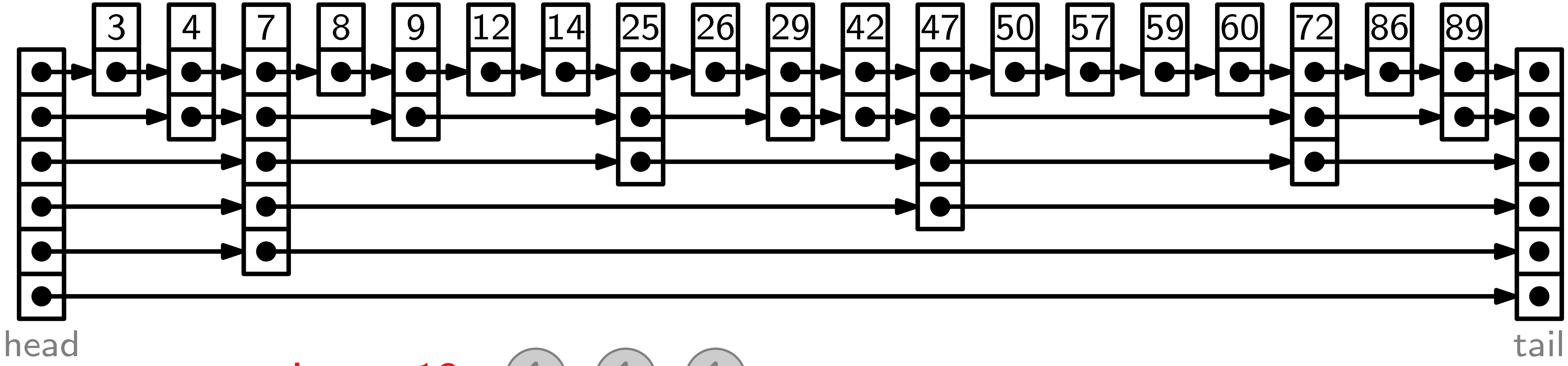

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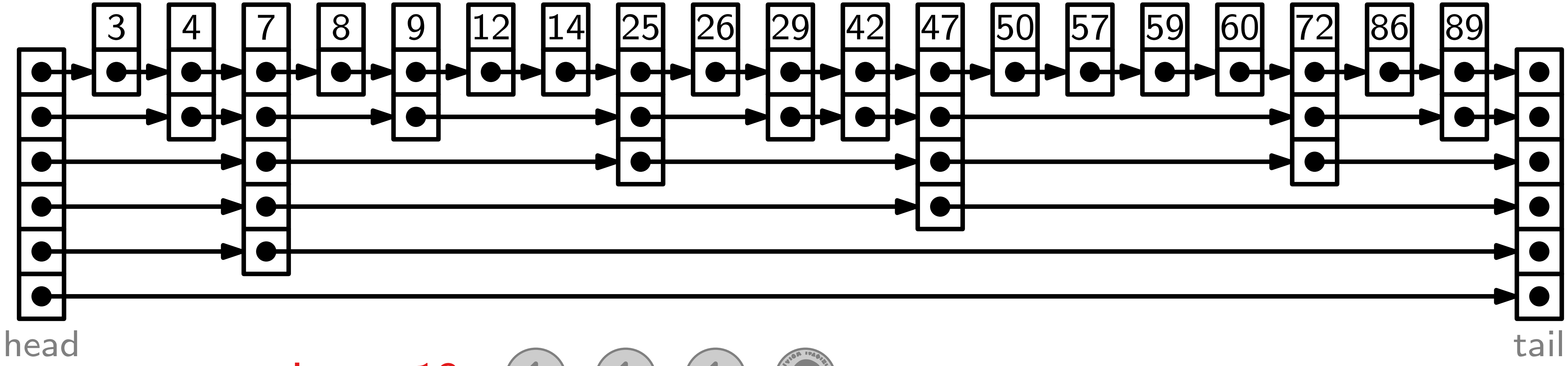

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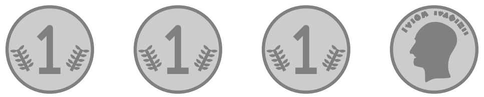

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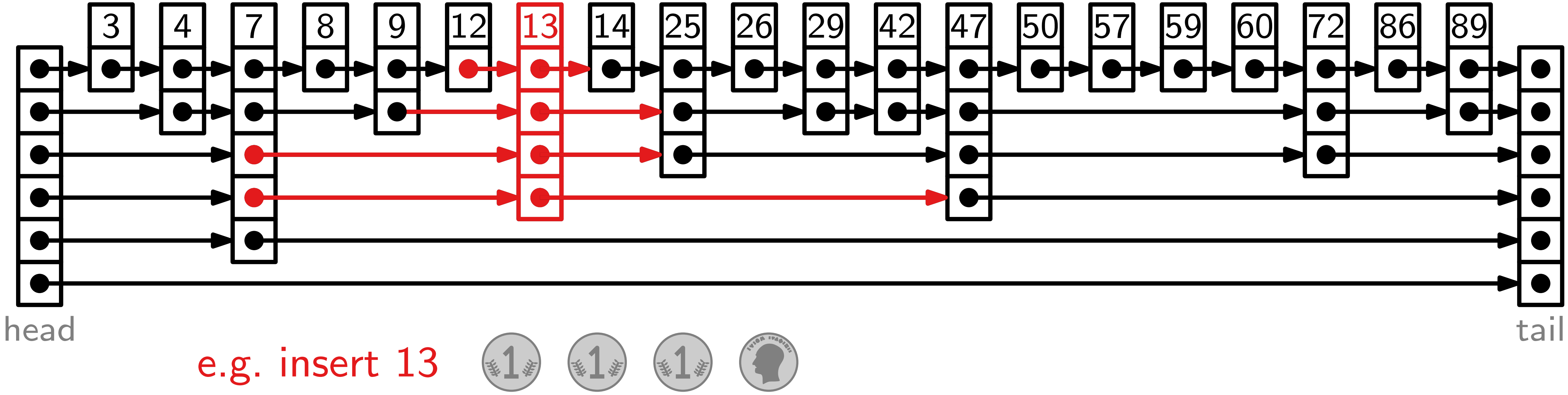

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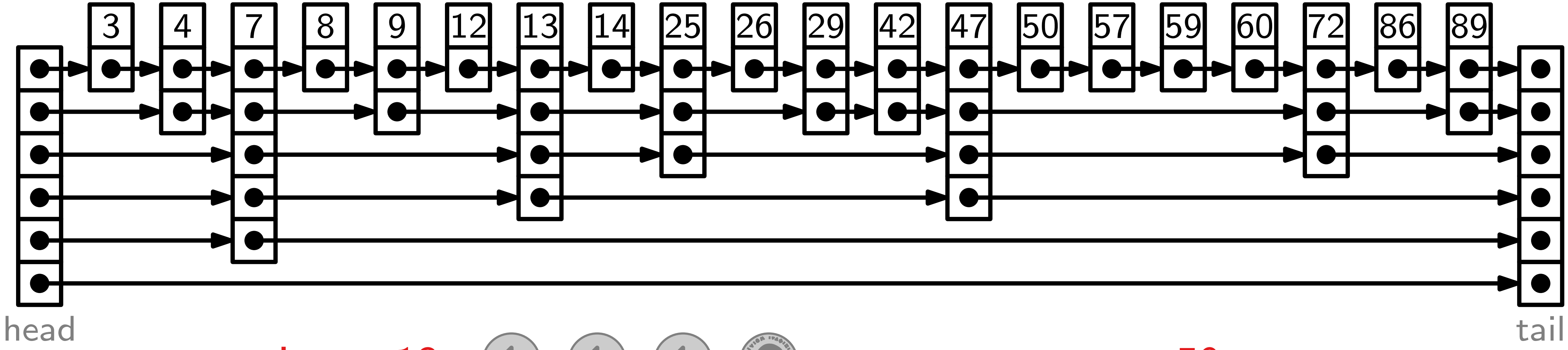

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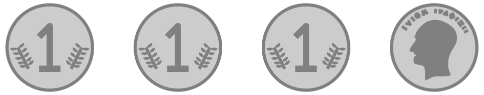

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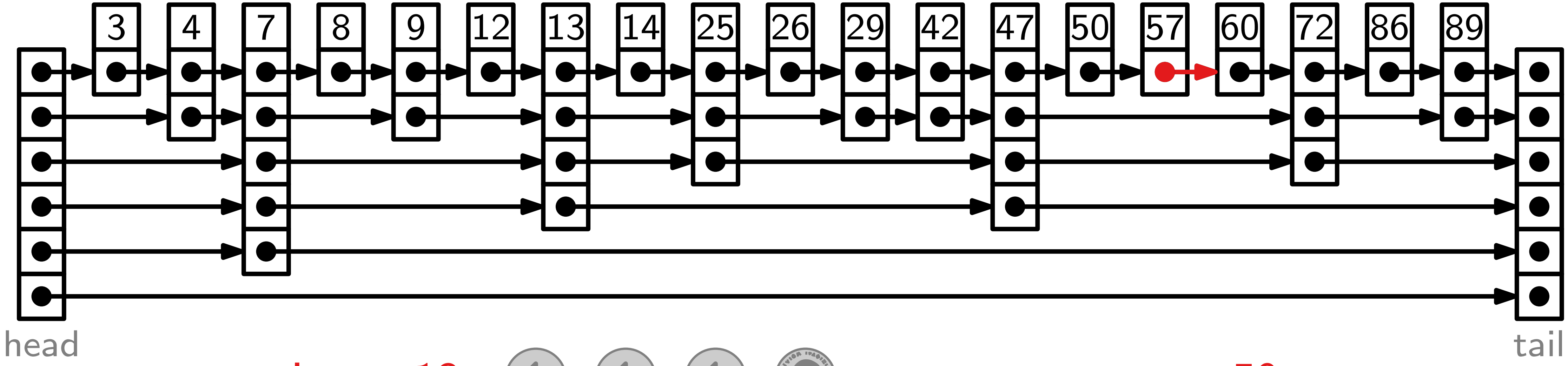

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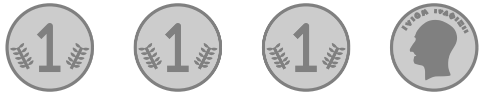
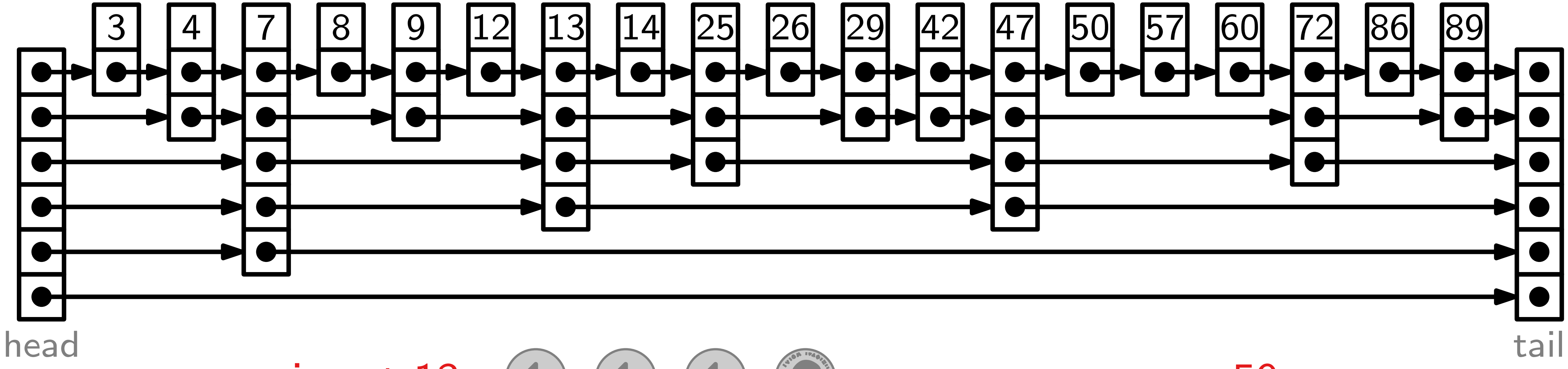


e.g. remove 59

(Randomized) Skip Lists

For a new entry, flip a coin until it shows HEAD. The number of flips will be its height.



(We give the head/tail of the list a height of $\lfloor \log_2 n \rfloor + 1$.)

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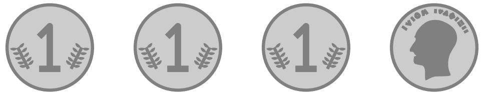

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Insertion and deletion works in $O(\log n)$ time + the time to search an element.
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Searching works in the same way as for deterministic skip lists.)

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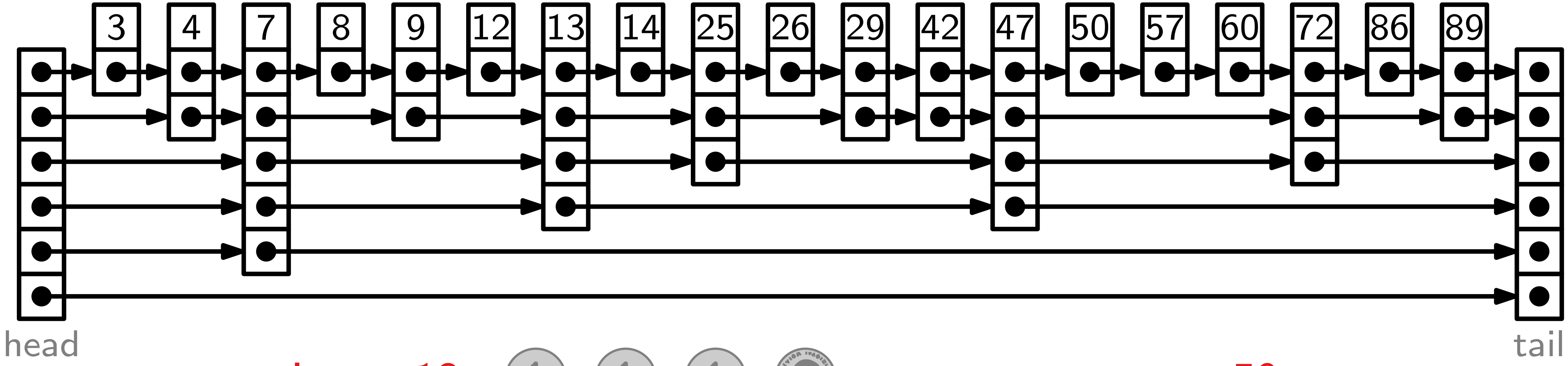

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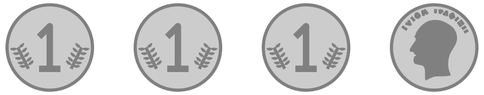

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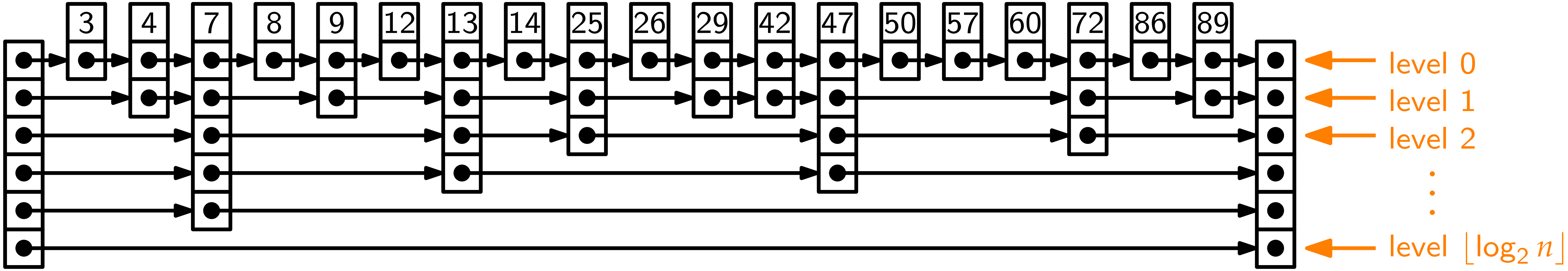

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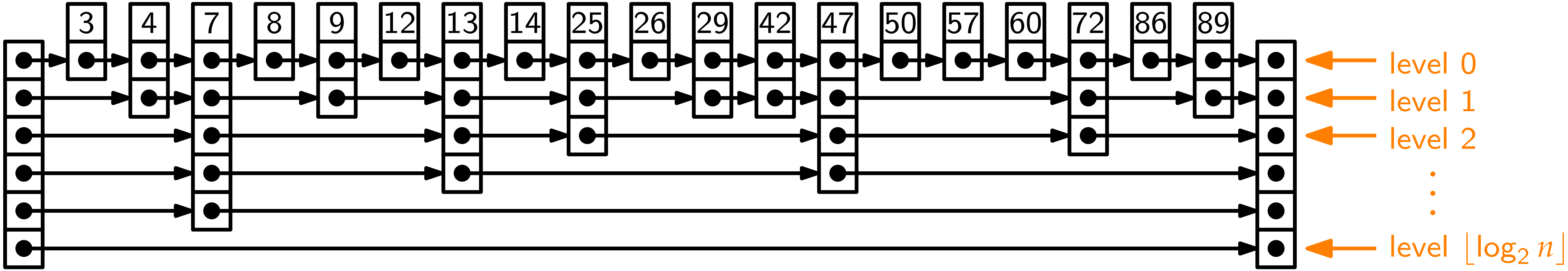

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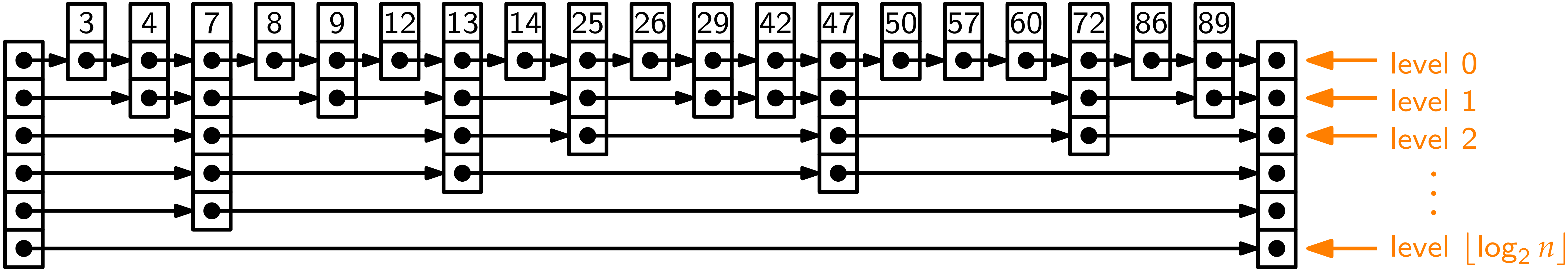

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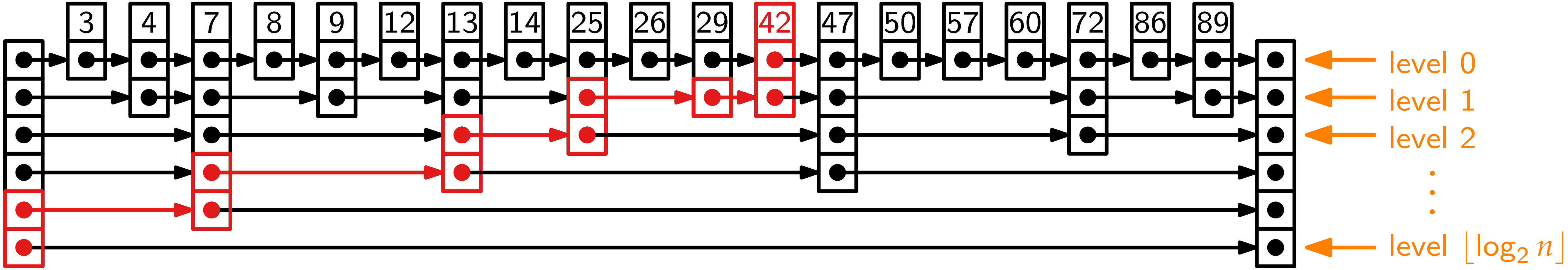

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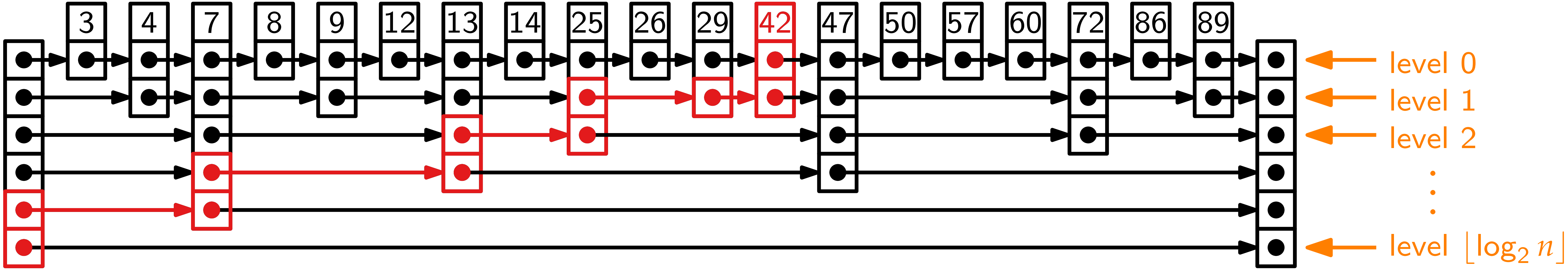

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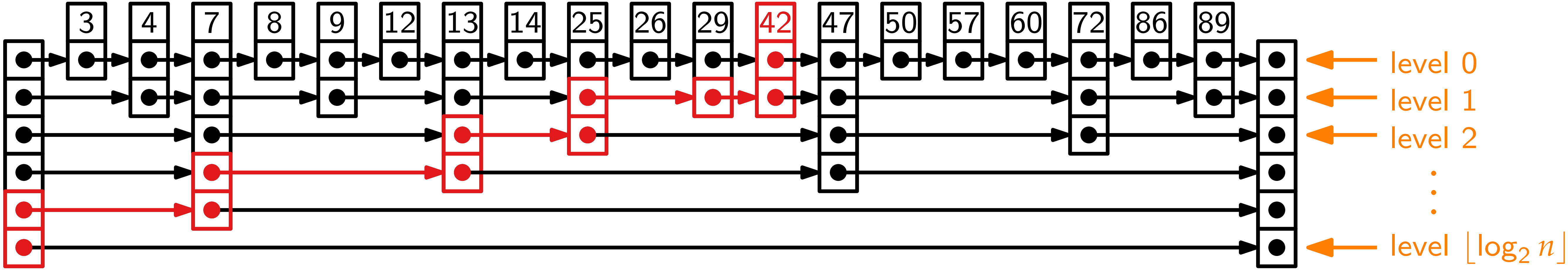

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If we are at level i, the probability that we can go a level up is $1/2$ by construction.

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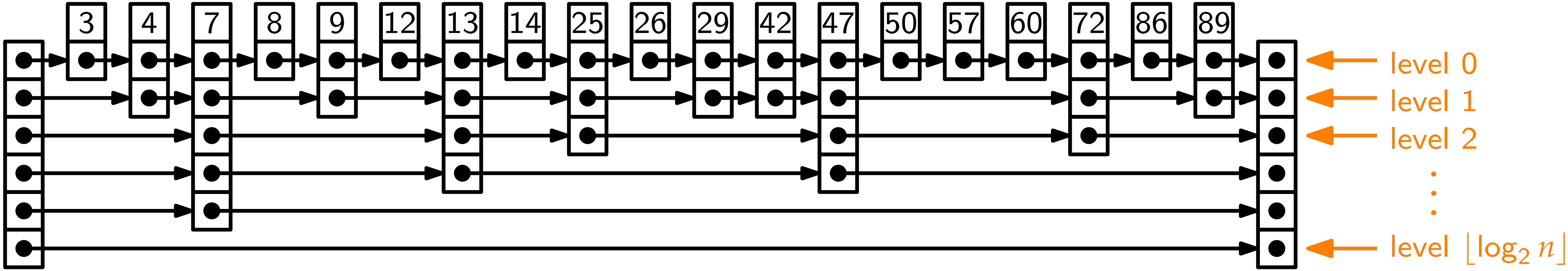

Let X_i be a random variable denoting the number of steps we take on level i or lower.

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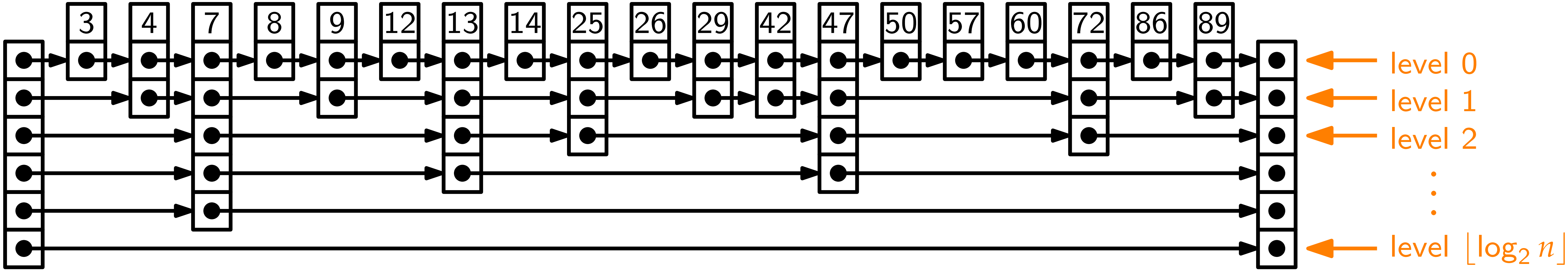

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$$E[X_i] = 1 + \tfrac{1}{2}E[X_{i-1}] + \tfrac{1}{2}E[X_i]$$

(for $i < 0$: $E[X_i] = 0$)

in the previous step we used a (reverse) pointer to the left

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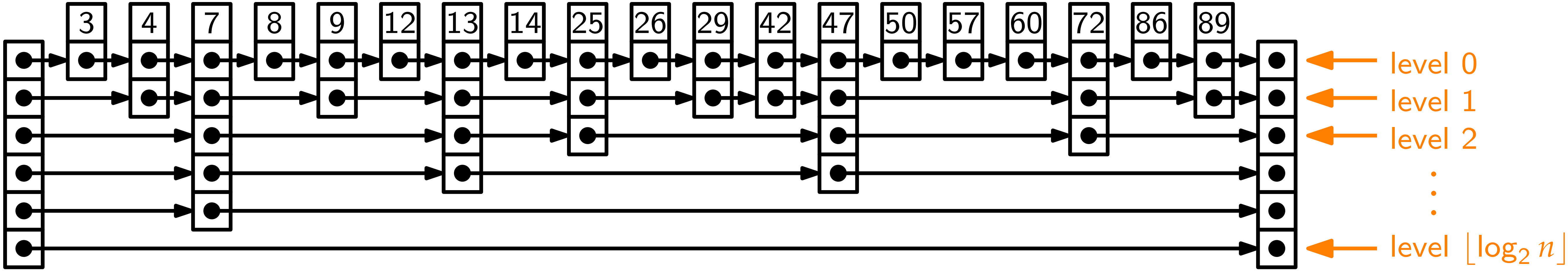

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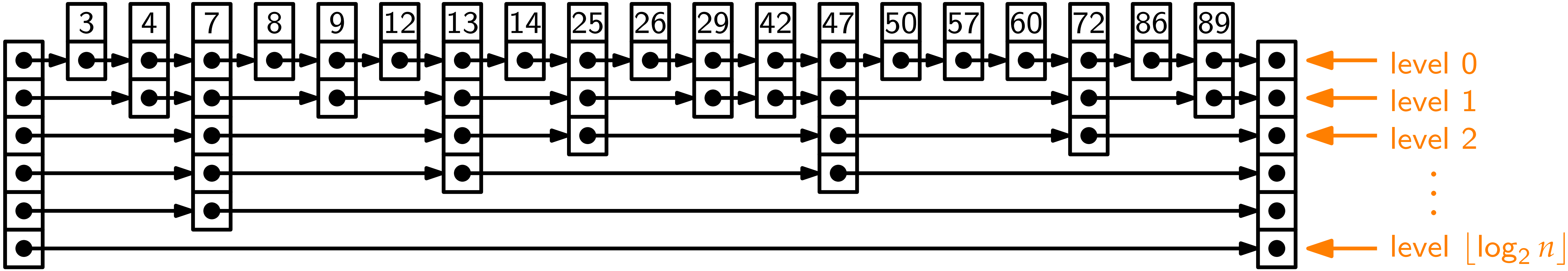

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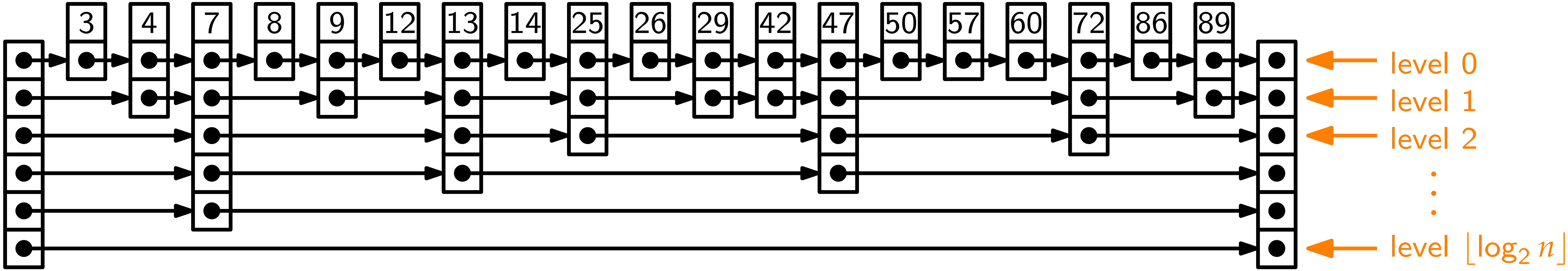

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Checking Containment in a Set

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- Bloom filter:
  - (+) $\Theta(1)$ time for containment check
  - (−) may produce false positives
  - (+) very low space consumption that does not depend on the lengths of the keys
  - (−) allows adding keys (in $\Theta(1)$ time), but not removing keys

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A Bloom filter is a **bit array** of m bits & a set of k different **hash functions** $h_1, \ldots, h_k$. Each hash function h_i generates a uniform random distribution in the range $\{1, \ldots, m\}$.

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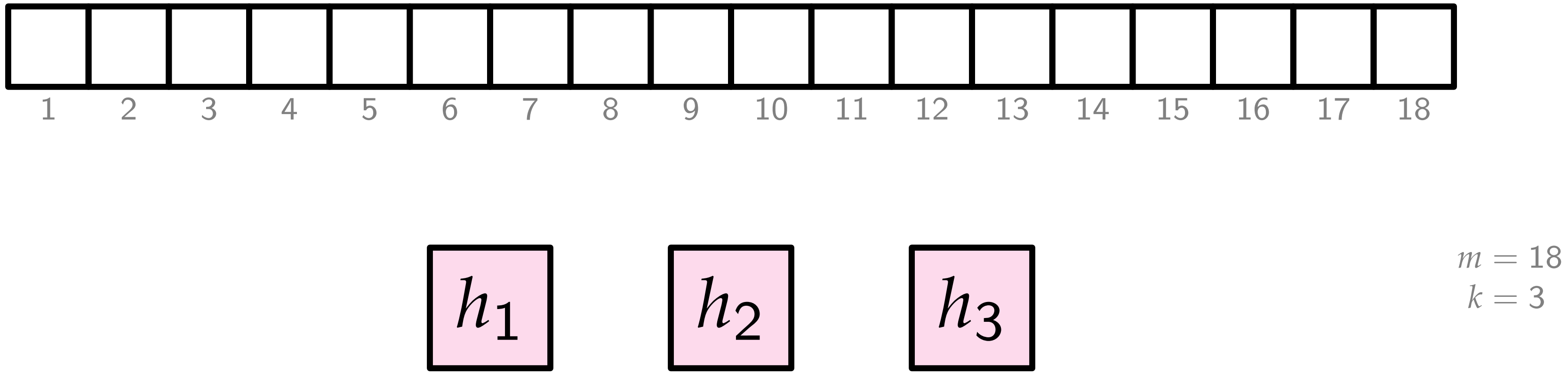

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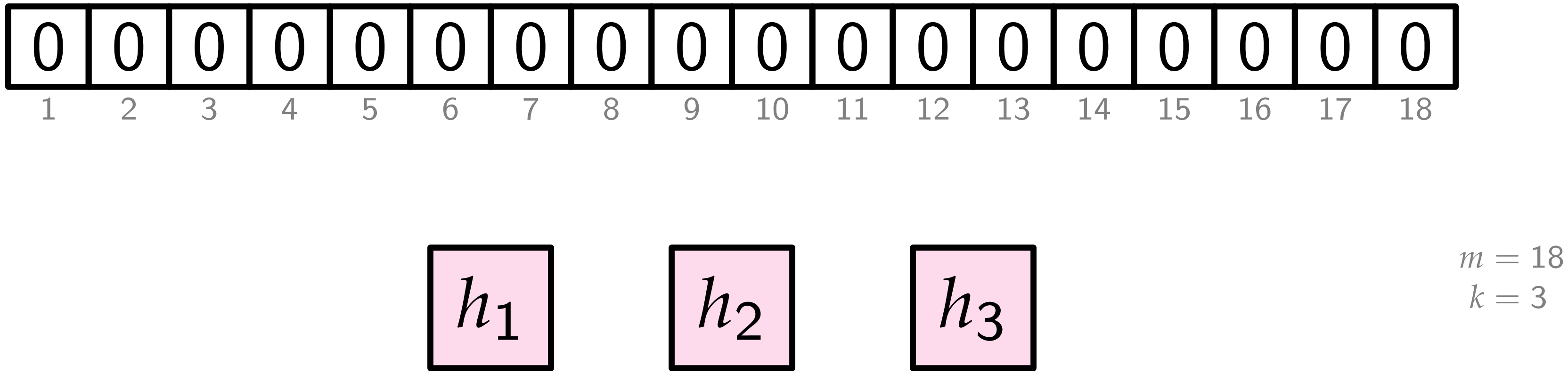

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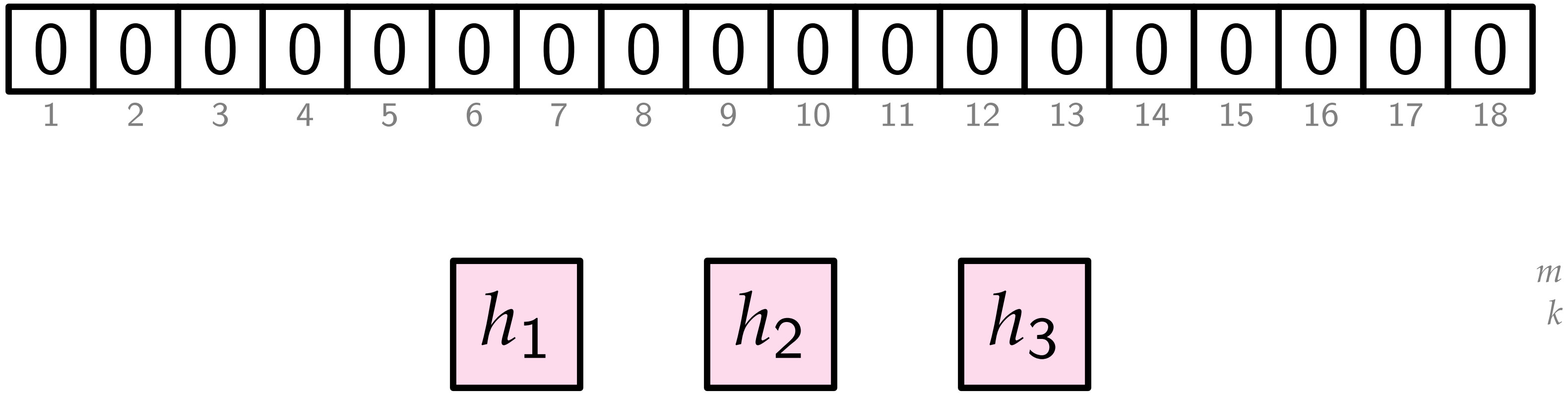

0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
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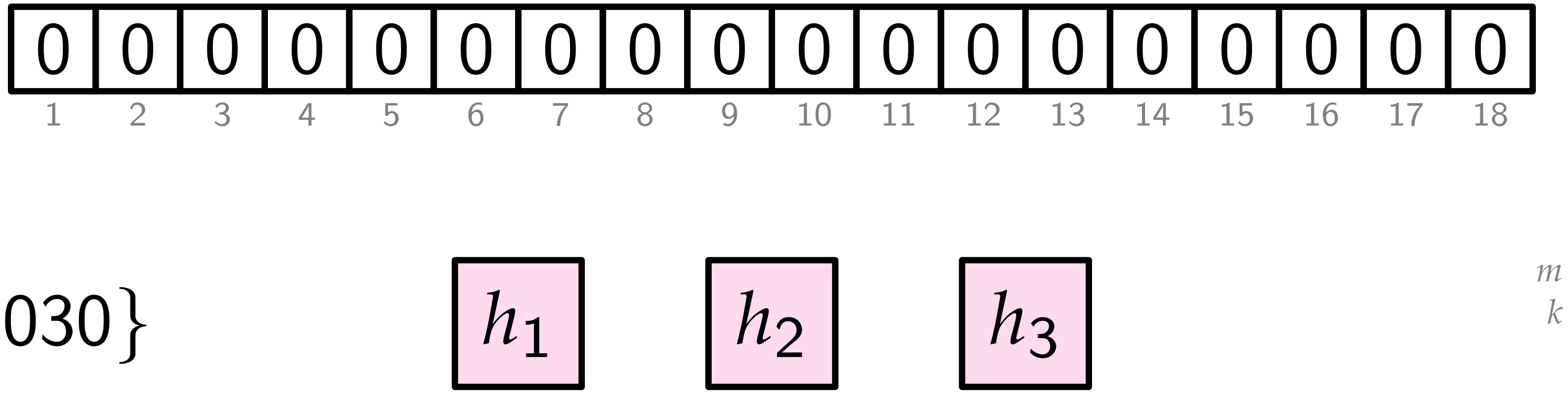

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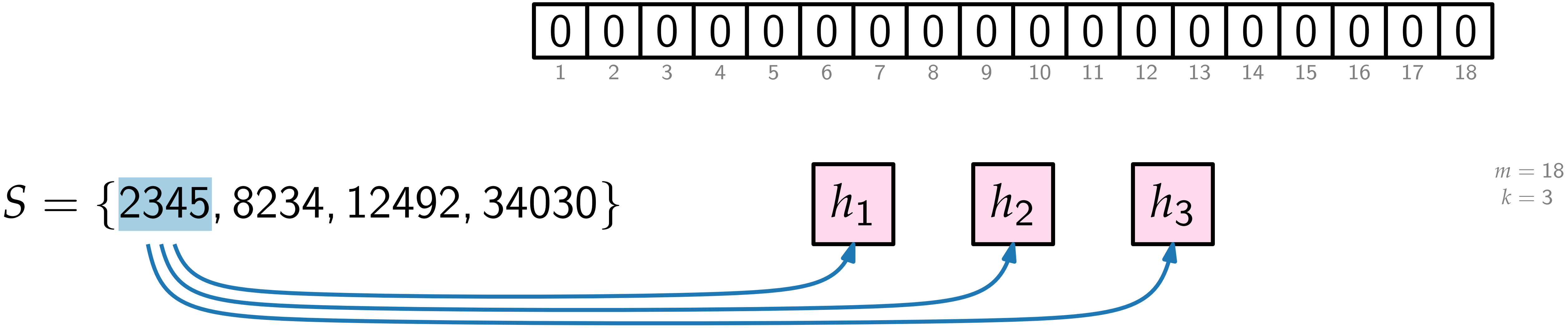

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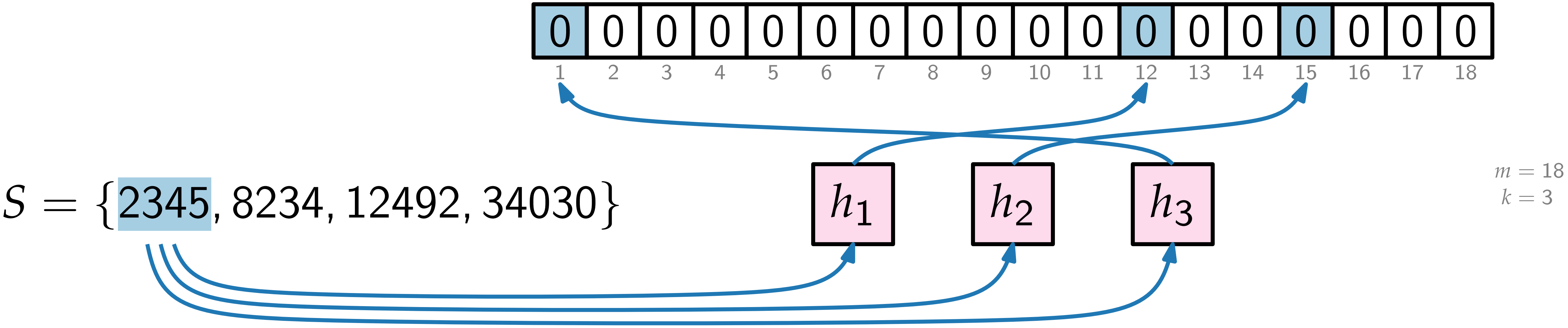

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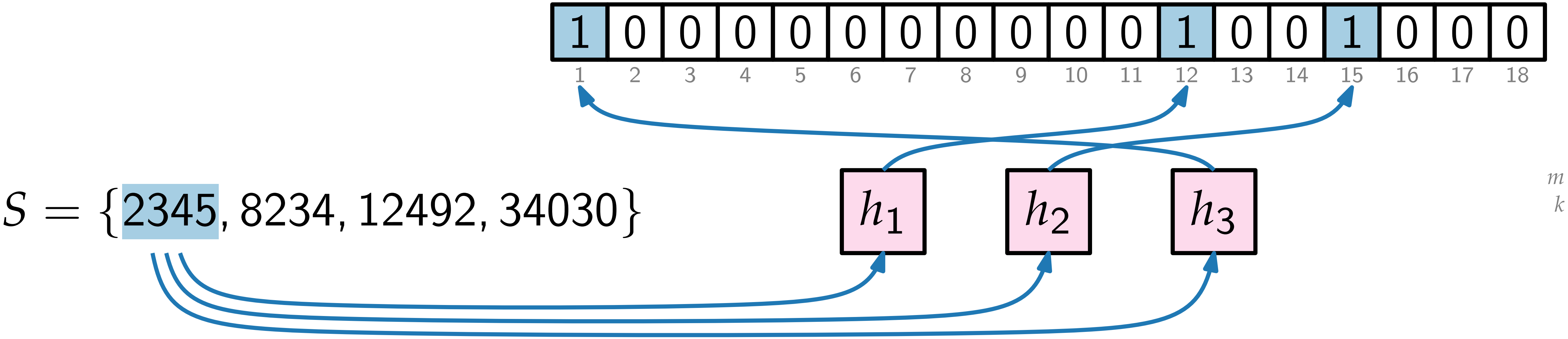

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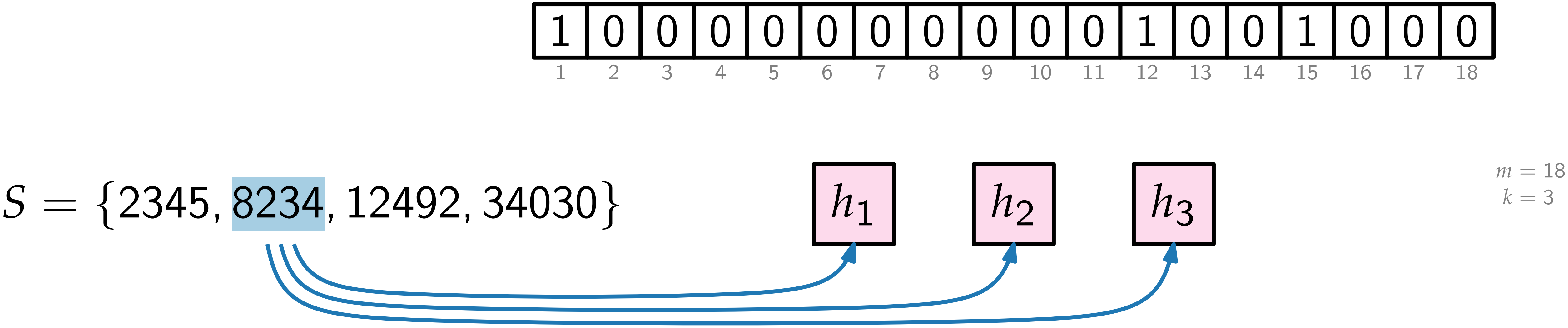

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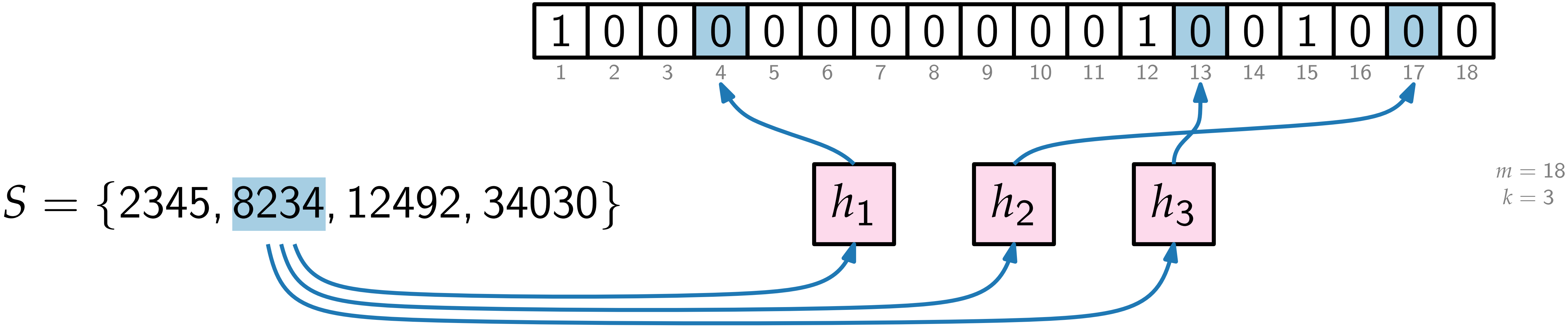

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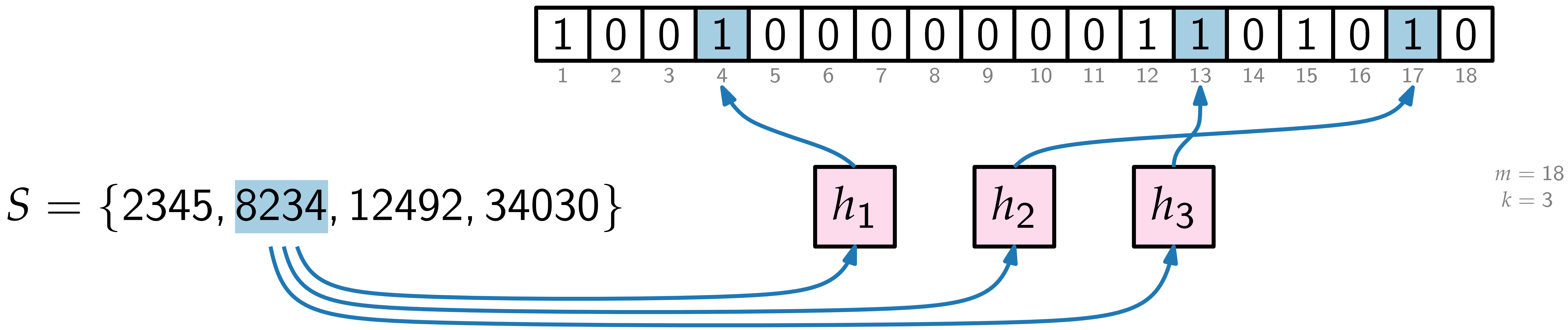

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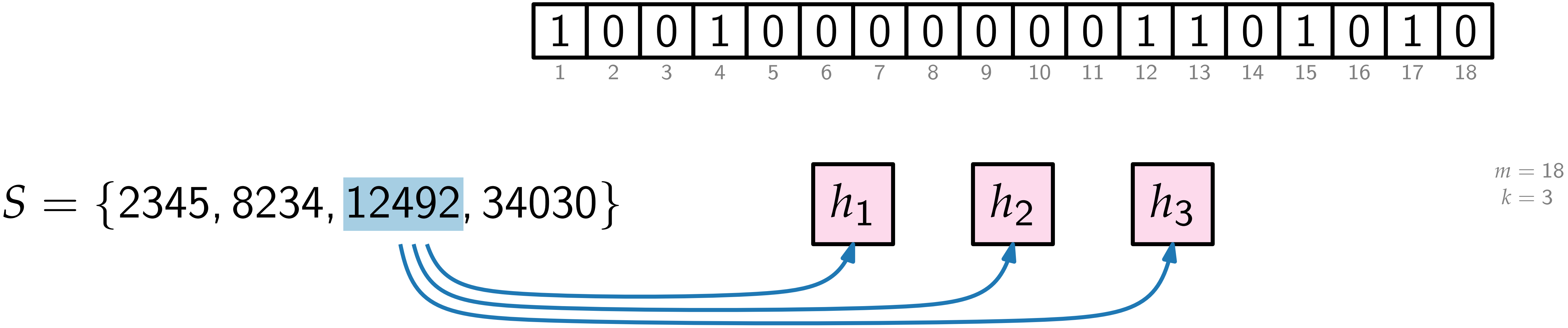

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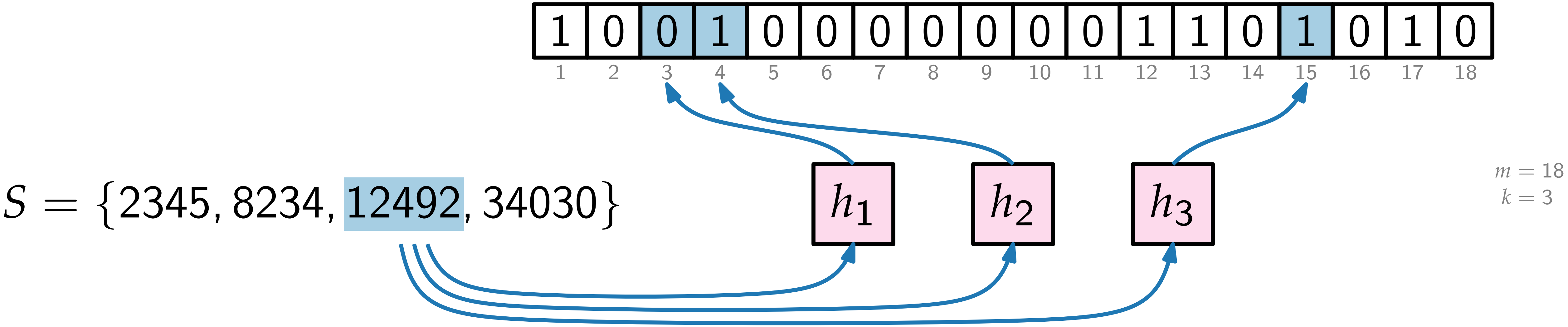

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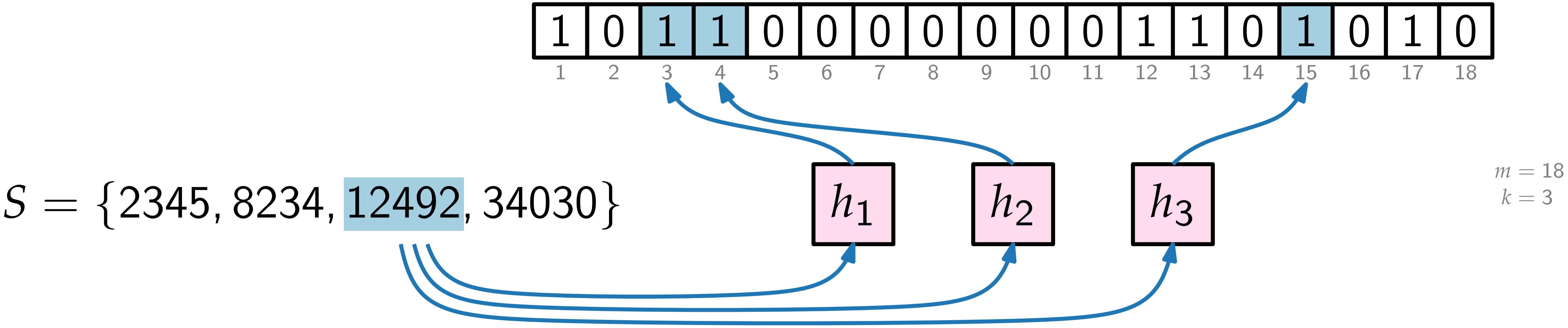

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
1	0	1	1	0	0	0	0	0	0	0	1	1	0	1	0	1	0

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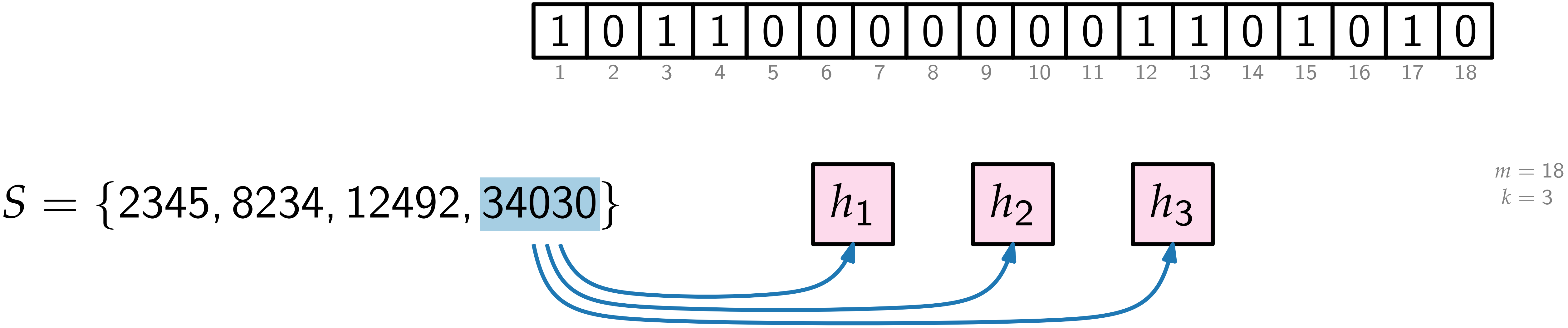

1	0	1	1	0	0	0	0	0	0	0	1	1	0	1	0	1	0
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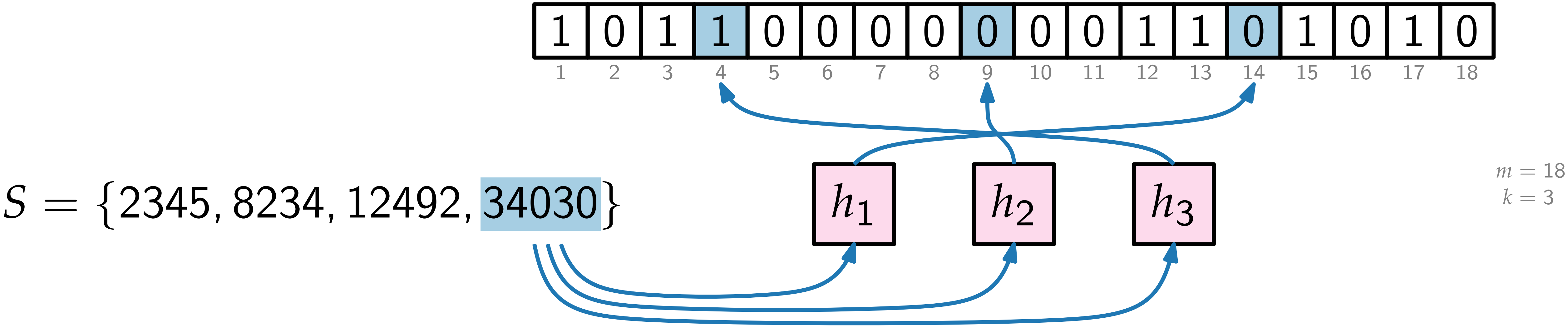

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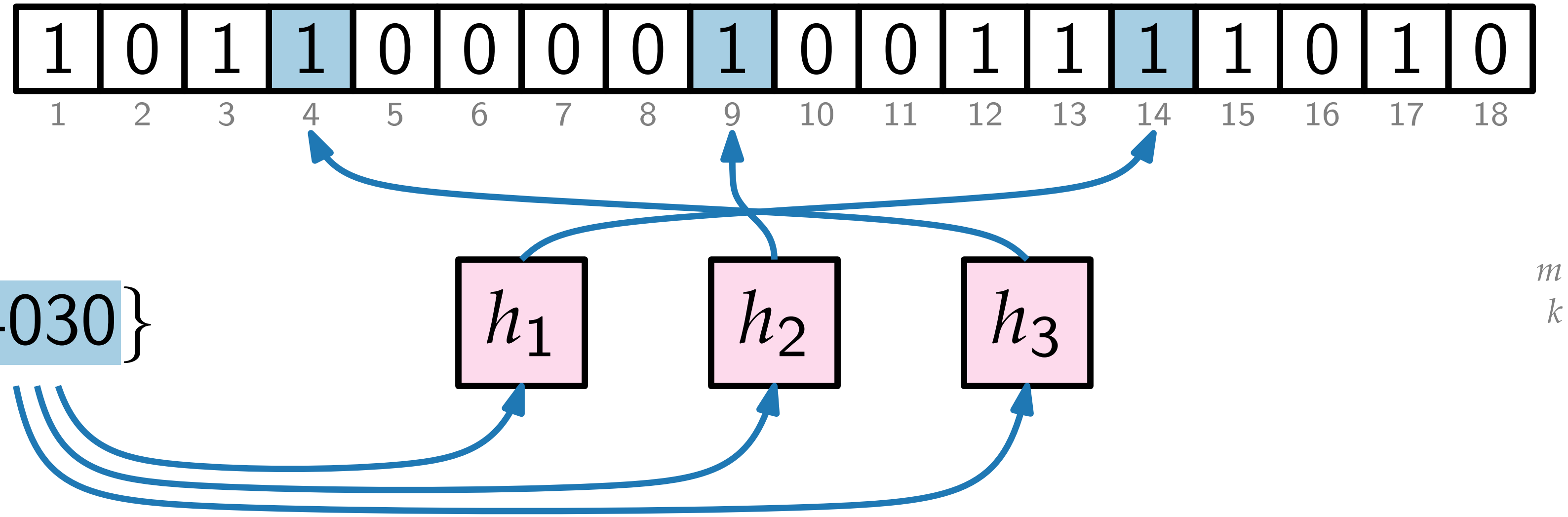

1	0	1	1	0	0	0	0	1	0	0	1	1	1	1	0	1	0
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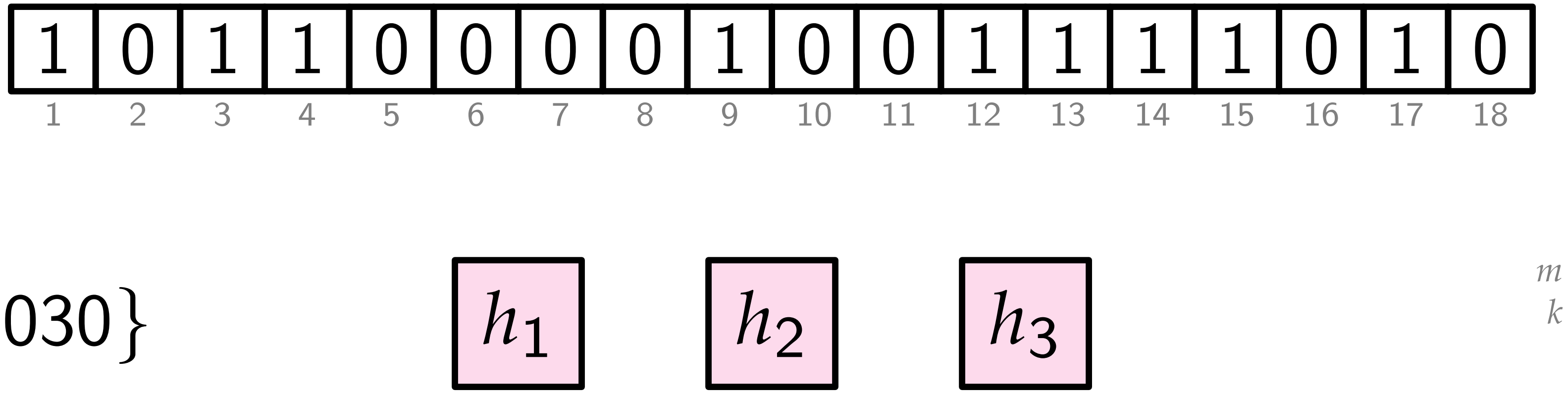

1	0	1	1	0	0	0	0	1	0	0	1	1	1	1	0	1	0
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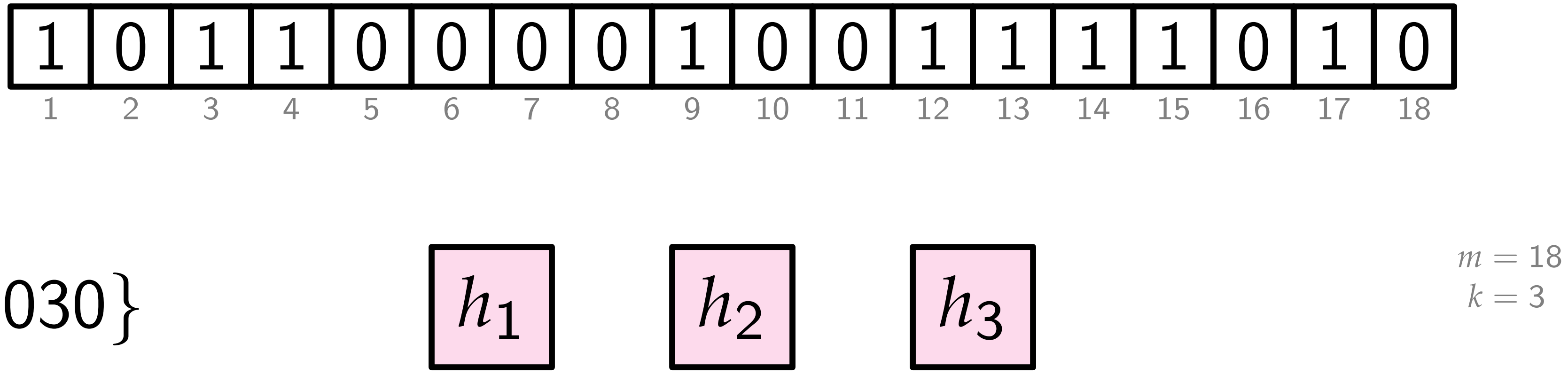

1	0	1	1	0	0	0	0	1	0	0	1	1	1	1	0	1	0
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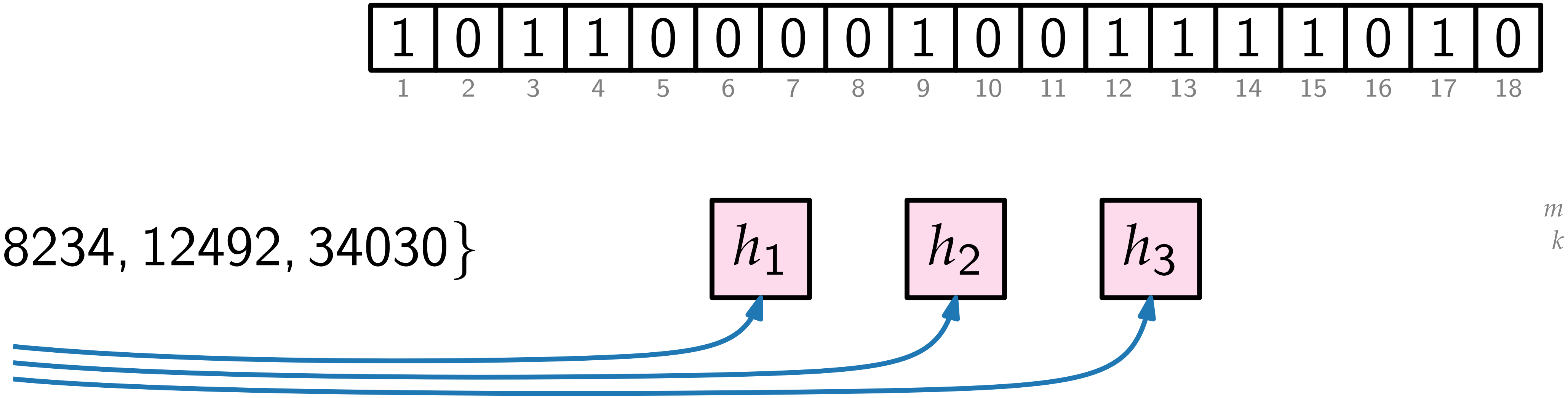

1	0	1	1	0	0	0	0	1	0	0	1	1	1	1	0	1	0
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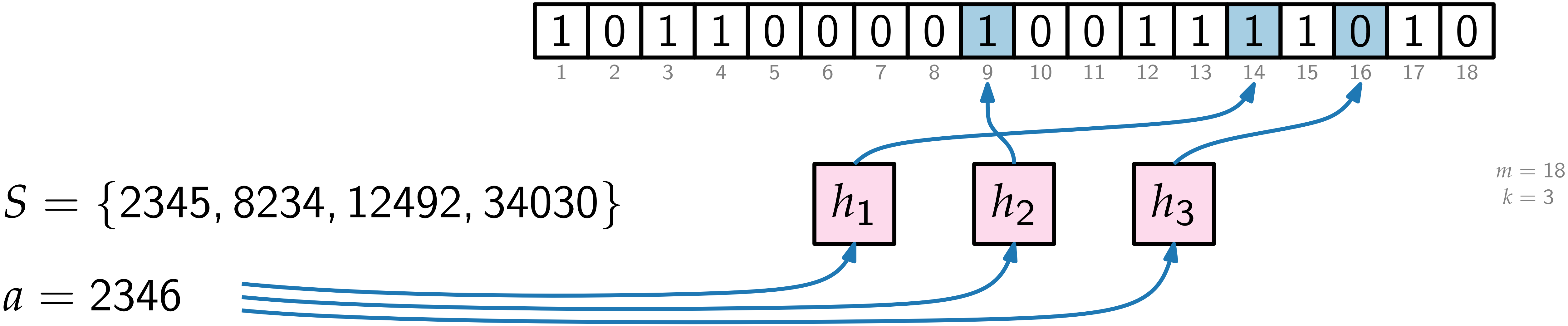

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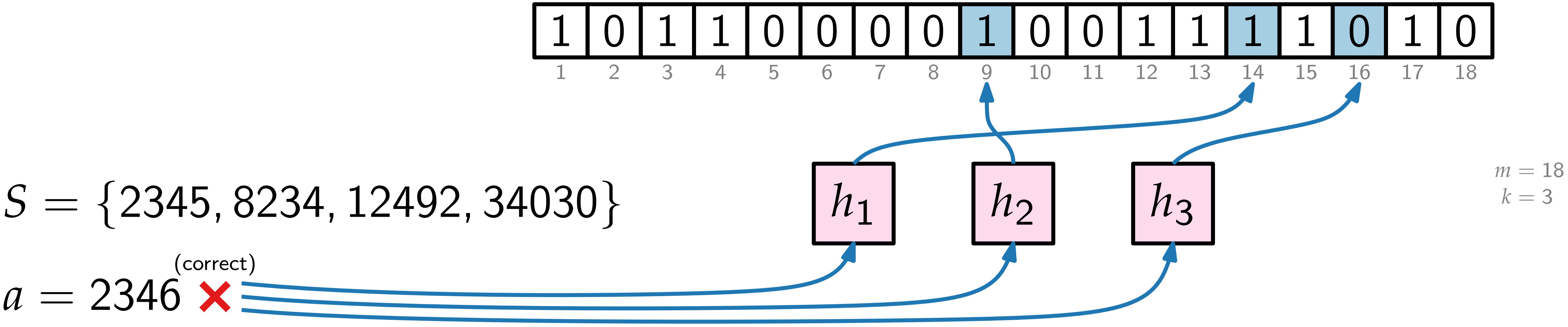

1	0	1	1	0	0	0	0	1	0	0	1	1	1	1	0	1	0
1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18

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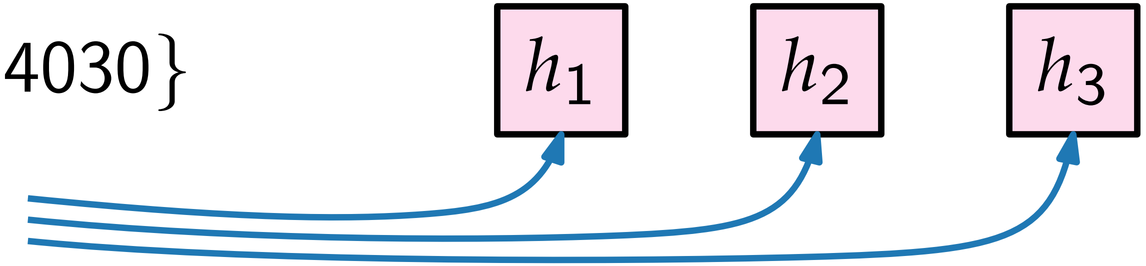

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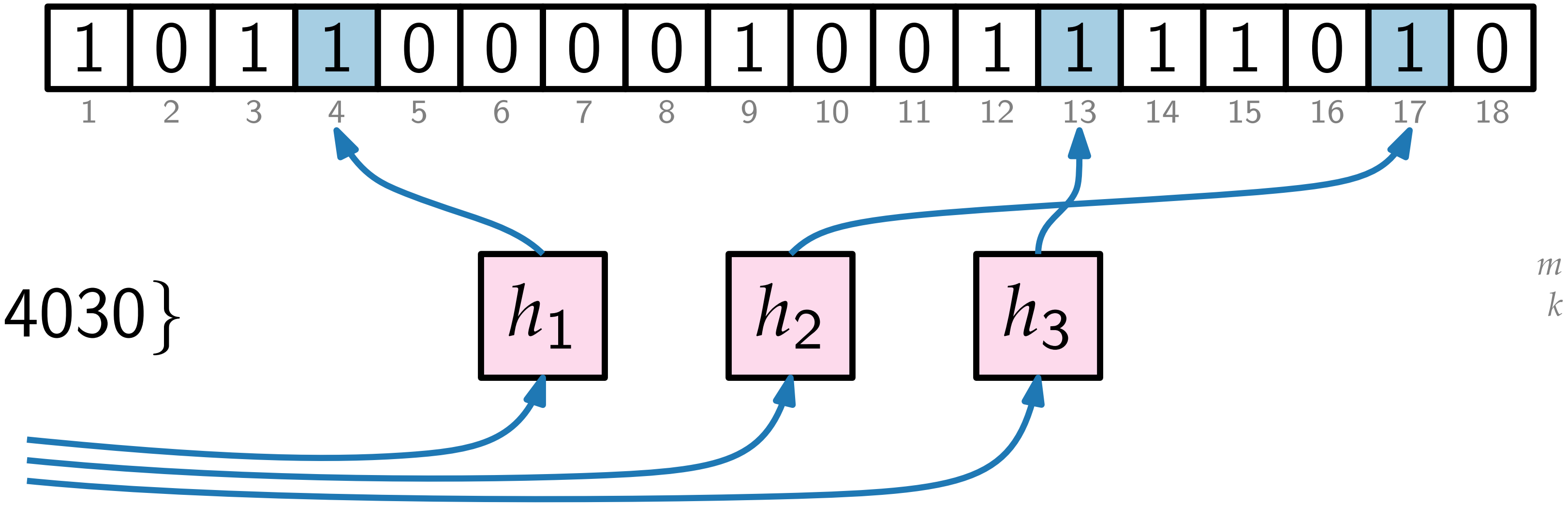

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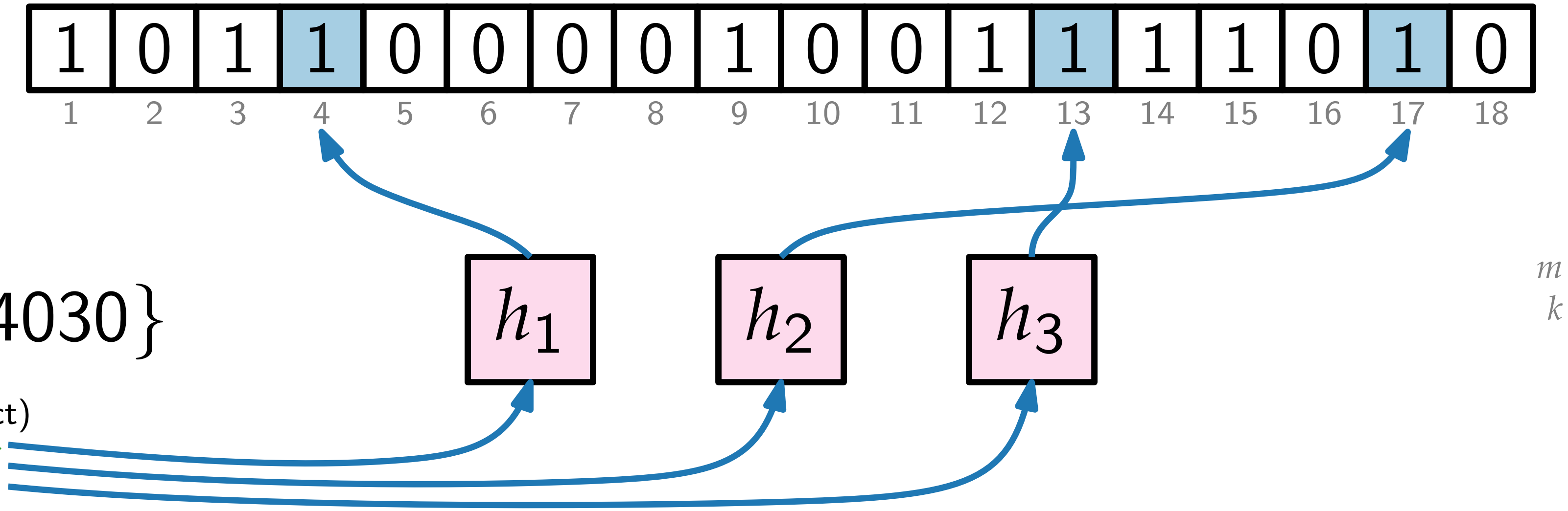

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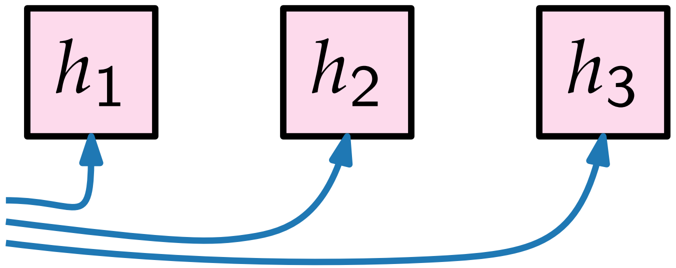

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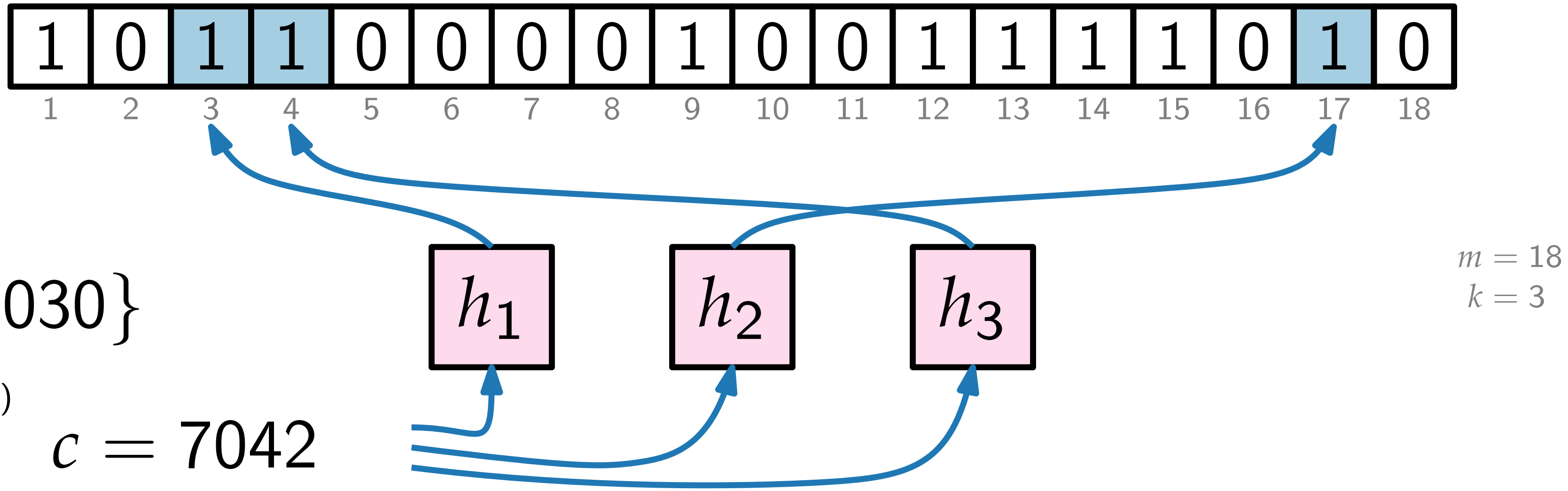

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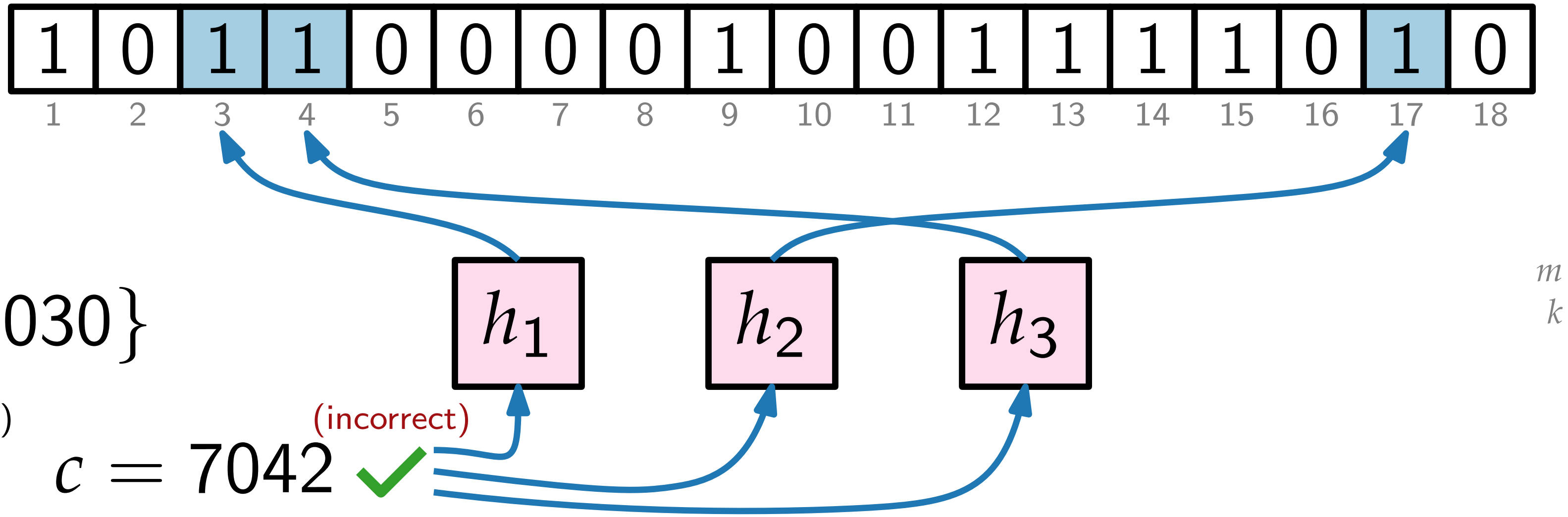

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The probabilities that all bits at positions $h_1(a), \ldots, h_k(a)$ are set to 1 are not independent. However, one can still show that the error probability ε for a false positive is relatively close to $\varepsilon \approx \left(1 - \left(1 - \frac{1}{m}\right)^{kn}\right)^k \approx \left(1 - e^{-kn/m}\right)^k$.

Parameters of Bloom Filters

$$\varepsilon \approx \left(1 - e^{-kn/m}\right)^k$$

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10 %

error probabiltiy ε	0.1
bits per element $\frac{m}{n}$	12
# hash functions k	4

Parameters of Bloom Filters

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So what number k of hash functions should we use?

The error probability ε is minimized if $k \approx \frac{m}{n} \ln 2$.

If we only use the optimal k, the error probibilty $\varepsilon \approx \left(\frac{1}{2}\right)^{m \ln 2/n}$

Thus, the optimal number of bits per element in our set is $\frac{m}{n} = -\frac{\log_2 \varepsilon}{\ln 2} \approx -1.44 \log_2 \varepsilon$.

So, the number of bits in our array depends on the desired error probabilty,

1 %

error probabiltiy ε	0.1	0.01
bits per element $\frac{m}{n}$	12	23
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1 in a million (10^6)

error probabiltiy ε	0.1	0.01	0.001	0.000001
bits per element $\frac{m}{n}$	12	23	34	67
# hash functions k	4	7	10	20

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1 in a billion (10^9)

error probabiltiy ε	0.1	0.01	0.001	0.000001	0.000000001
bits per element $\frac{m}{n}$	12	23	34	67	100
# hash functions k	4	7	10	20	30

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So, the number of bits in our array depends on the desired error probabilty,

1 in a trillion (10^{12}) ↙

error probabiltiy ε	0.1	0.01	0.001	0.000001	0.000000001	0.000000000001
bits per element $\frac{m}{n}$	12	23	34	67	100	133
# hash functions k	4	7	10	20	30	40

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. . . but not on the lengths of the numbers.
(We could check for whole documents whether they are there or not.)

Discussion

- (Randomized) skip list provide a simpler alternative to balanced binary search trees with (in expectation) the same asymptotic time complexities for the basic operations.

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- There are some refinements of classical Bloom filters to overcome some disadvantages.
- Bloom filters are used for
  - Internet search engines
  - caching objects in Internet applications (is an image or a digest in the cache?)
  - databases (Google Bigtable, Apache HBase, Apache Cassandra, PostgreSQL)
  - web browsers (Google Chrome used one to identify malicious URLs)
  - crypto currencies (finding logs in Ethereum)

Literature

Skip lists:

- [Pugh '90] William W. Pugh, "Skip Lists: A Probabilistic Alternative to Balanced Trees" in *Communications of the ACM*, 33(6):668–676, 1990
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Bloom filters:

- [Bloom '70] Burton H. Bloom, "Space/Time Trade-offs in Hash Coding with Allowable Errors" in *Communications of the ACM*, 13(7):422–426, 1970
- [Mitzenmacher, Upfal '05] Michael Mitzenmacher and Eli Upfal, "Probability and Computing: Randomized Algorithms and Probabilistic Analysis", Cambridge University Press, 2005
- [Bose et al. '08] Prosenjit Bose, Hua Guo, Evangelos Kranakis, Anil Maheshwari, Pat Morin, Jason Morrison, Michiel H. M. Smid, and Yihui Tang, "On the false-positive rate of Bloom filters" in *Information Processing Letters*, 108(4):210–213, 2008