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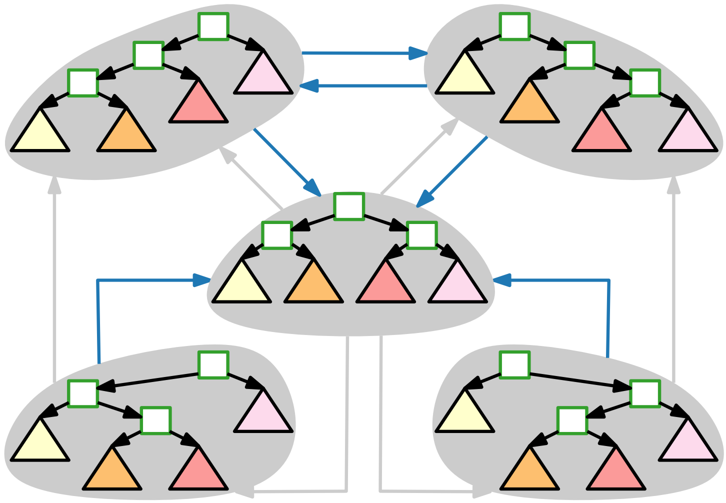
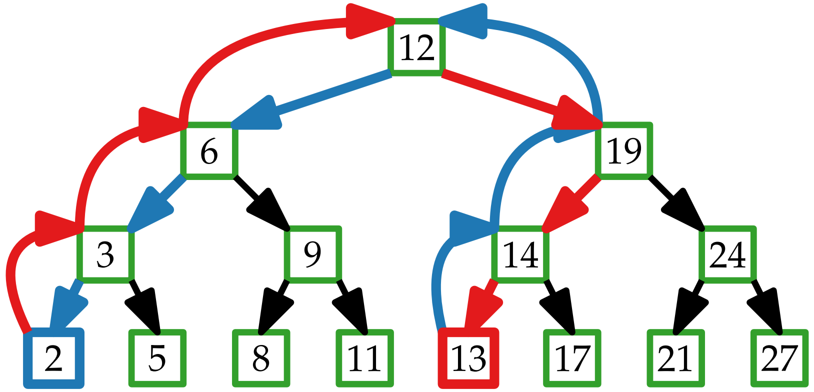
Advanced Algorithms

Optimal Binary Search Trees

Splay Trees

Johannes Zink · WS22





How Good is a Binary Search Tree?

optimal

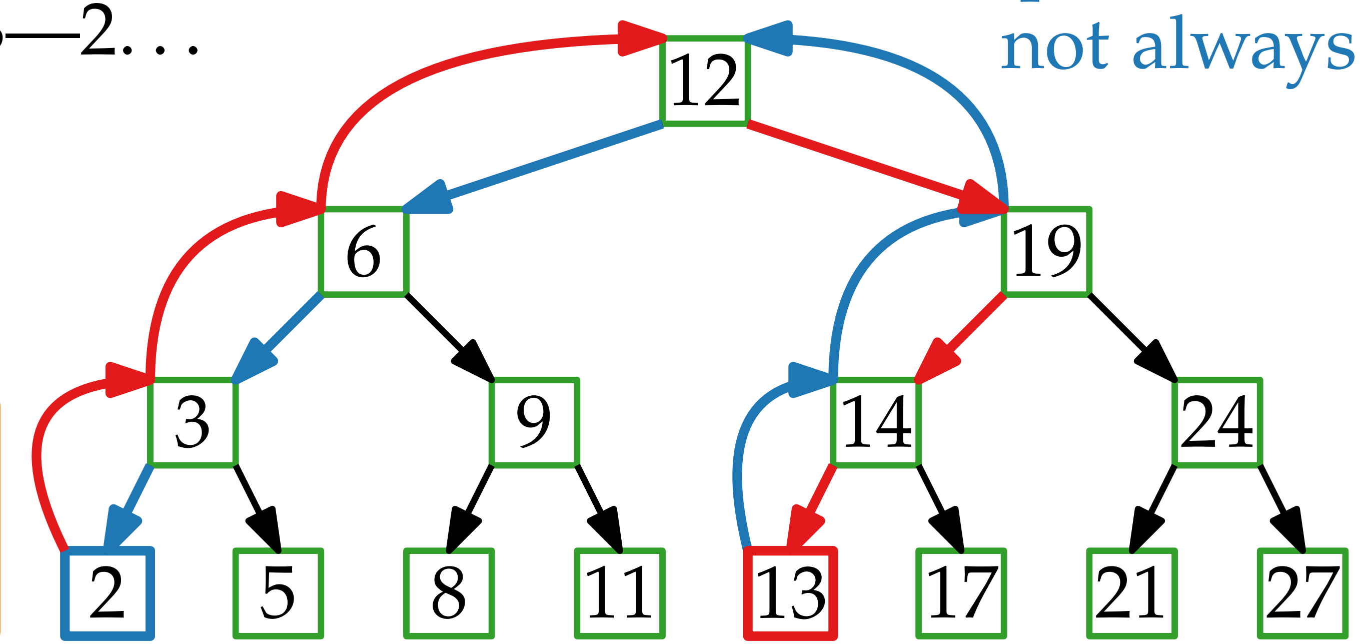
Binary search tree (BST): w.c. query time $\Theta(n)$

Balanced binary search tree: w.c. query time $\Theta(\log n)$
(e.g. Red-Black-Tree, AVL-Tree)

What if we *know* the query before? w.c. query time 1

Sequence of queries? $O(\log n)$ per query
e.g. 2—13—5
or 2—13—2—13—2. . .

optimal?
not always!



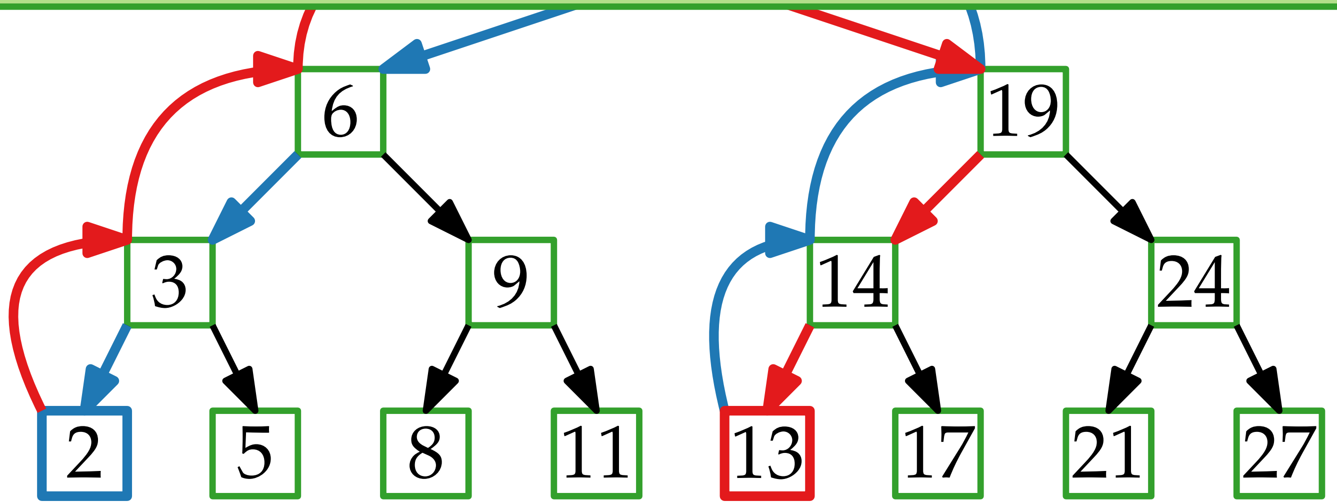
The performance of a BST depends on the model!

Model 1: Malicious Queries

Given a BST, what is the worst sequence of queries?

Lemma. The worst-case malicious query cost in any BST with n nodes is at least $\Omega(\log n)$ per query.

Definition. A BST is **balanced** if the cost of *any* sequence of m queries is $O(m \log n + n \log n)$.
$\Rightarrow$ the (amortized) cost of each query is $O(\log n)$ (for at least n queries)

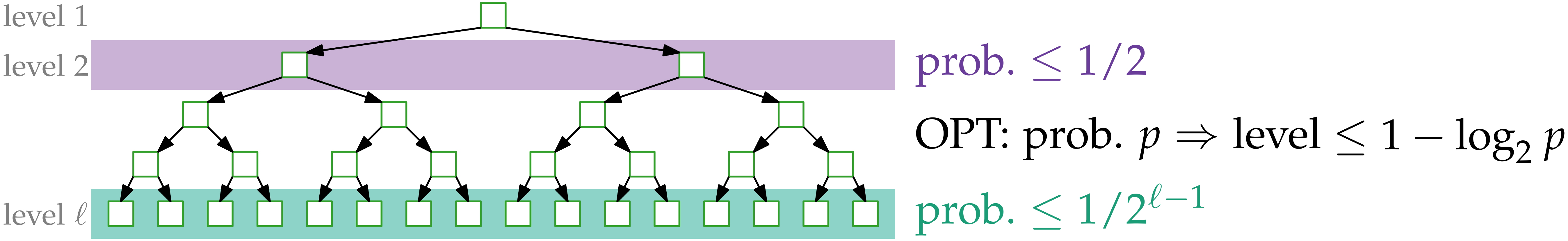


Model 2: Known Probability Distribution

Access Probabilities:

2	3	5	6	8	9	11
2%	20%	30%	8%	20%	15%	5%

Idea: Place nodes with higher probability higher in the tree.



Lemma. The expected query cost in any BST is at least $\Omega(1+H)$ per query with $H = \sum_{i=1}^{n} -p_i \log p_i$.

Definition. A BST has the **entropy property** if it reaches this bound, i.e., the expected query cost is in $O(1+H)$.

$$p_i = 1/n \Rightarrow H = \sum_{i=1}^{n} 1/n \cdot \log(n) = \log n$$

$$p_1 \approx 1, p_i \approx 0 \Rightarrow H \approx -\log 1 = 0$$

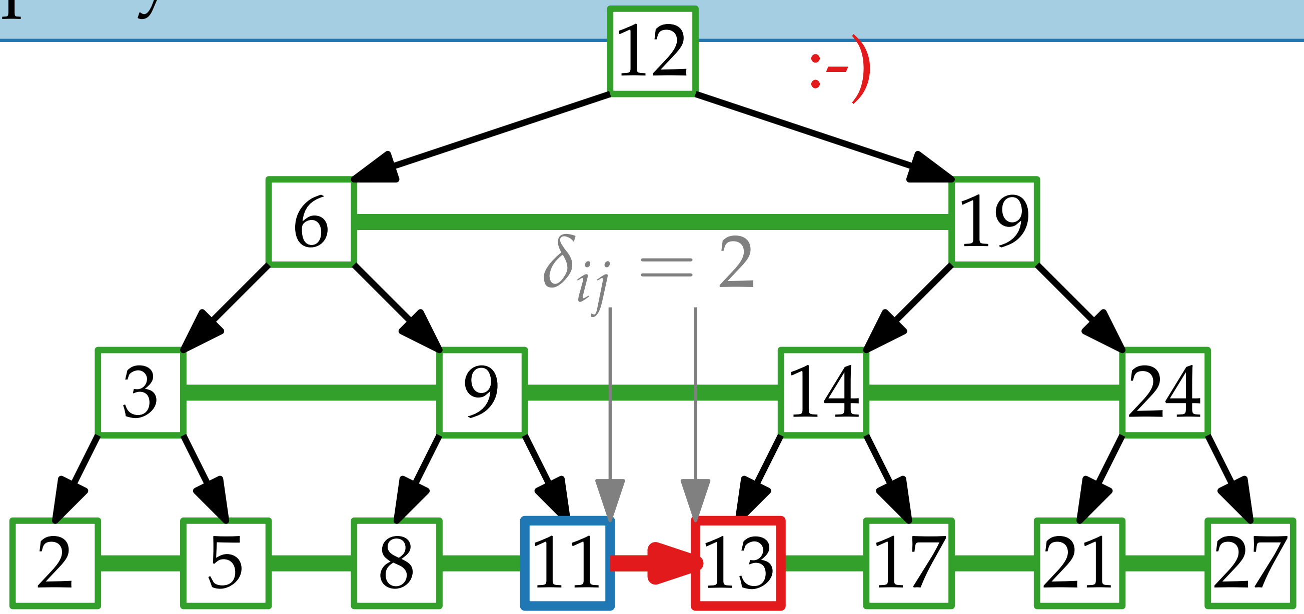
Model 3: Spacial Locality

If a key is queried, then keys with nearby values are more likely to be queried.

Suppose we queried key x_i and want to query key x_j next.
Let $\delta_{ij} = |\operatorname{rank}(x_j) - \operatorname{rank}(x_i)|$.

Definition. A BST has the **dynamic finger property** if the (amortized) cost of queries are $O(\log \delta_{ij})$.

Lemma. A level-linked Red-Black-Tree has the dynamic finger property.



Model 4: Temporal Locality

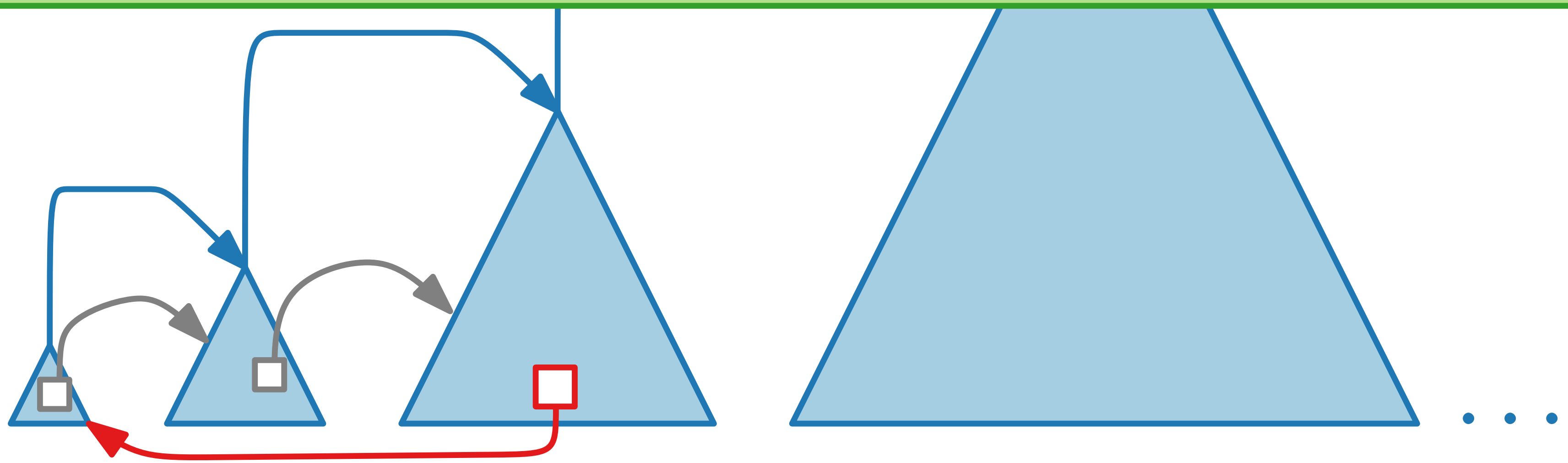
If a key is queried, then it's likely to be queried again soon.

A static tree will have a hard time... What if we can move elements?

Idea: Use a sequence of trees

Move queried key to first tree, then kick out oldest key.

Definition. A BST has the **working set property** if the (amortized) cost of a query for key x is $O(\log t)$, where t is the number of keys queried more recently than x.



Model 5: Static Optimality

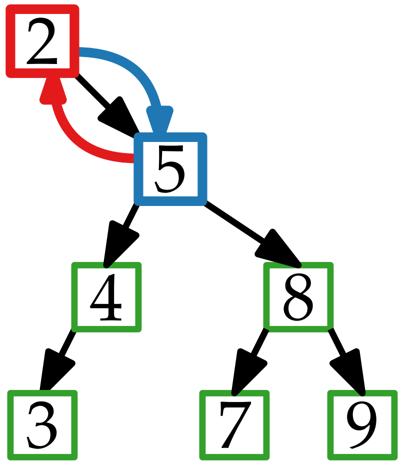
Given a sequence S of queries.

Let T_S^* be the *optimal* static tree with the shortest query time OPT_S for S.

e.g. $S = 2, 5, 2, 5, 2, \ldots, 5$

T_S^*:

OPT: $|S|$



Definition. A BST is **statically optimal** if queries take (amortized) $O(\mathrm{OPT}_S)$ time for every S.

All These Models . . .

Balanced: Queries take (amortized) $O(\log n)$ time

Entropy: Queries take expected $O(1 + H)$ time

Dynamic Finger: Queries take $O(\log \delta_i)$ time (δ_i: rank diff.)

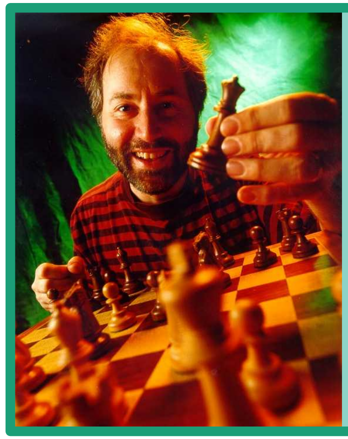
Working Set: Queries take $O(\log t)$ time (t: recency)

Static Optimality: Queries take (amortized) $O(\mathrm{OPT}_S)$ time.

... is there one BST to rule them all?

Yes!

Splay Trees

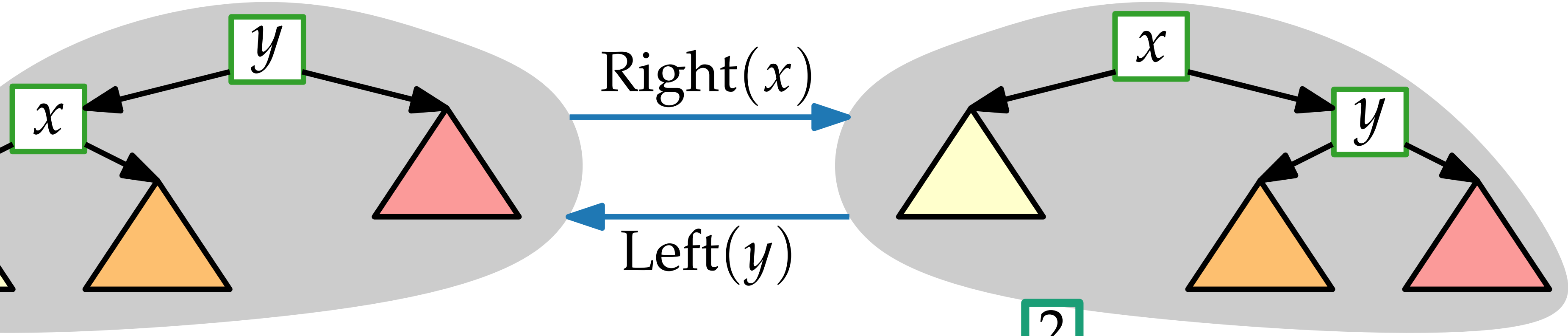


Daniel D. Sleator   Robert E. Tarjan

J. ACM 1985

Idea: Whenever we query a key, rotate it to the root.

ADS:

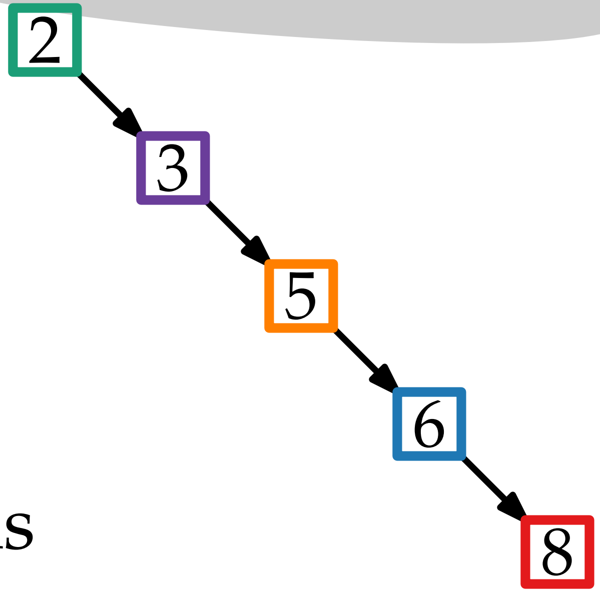


Splay(x): Rotate x to the root

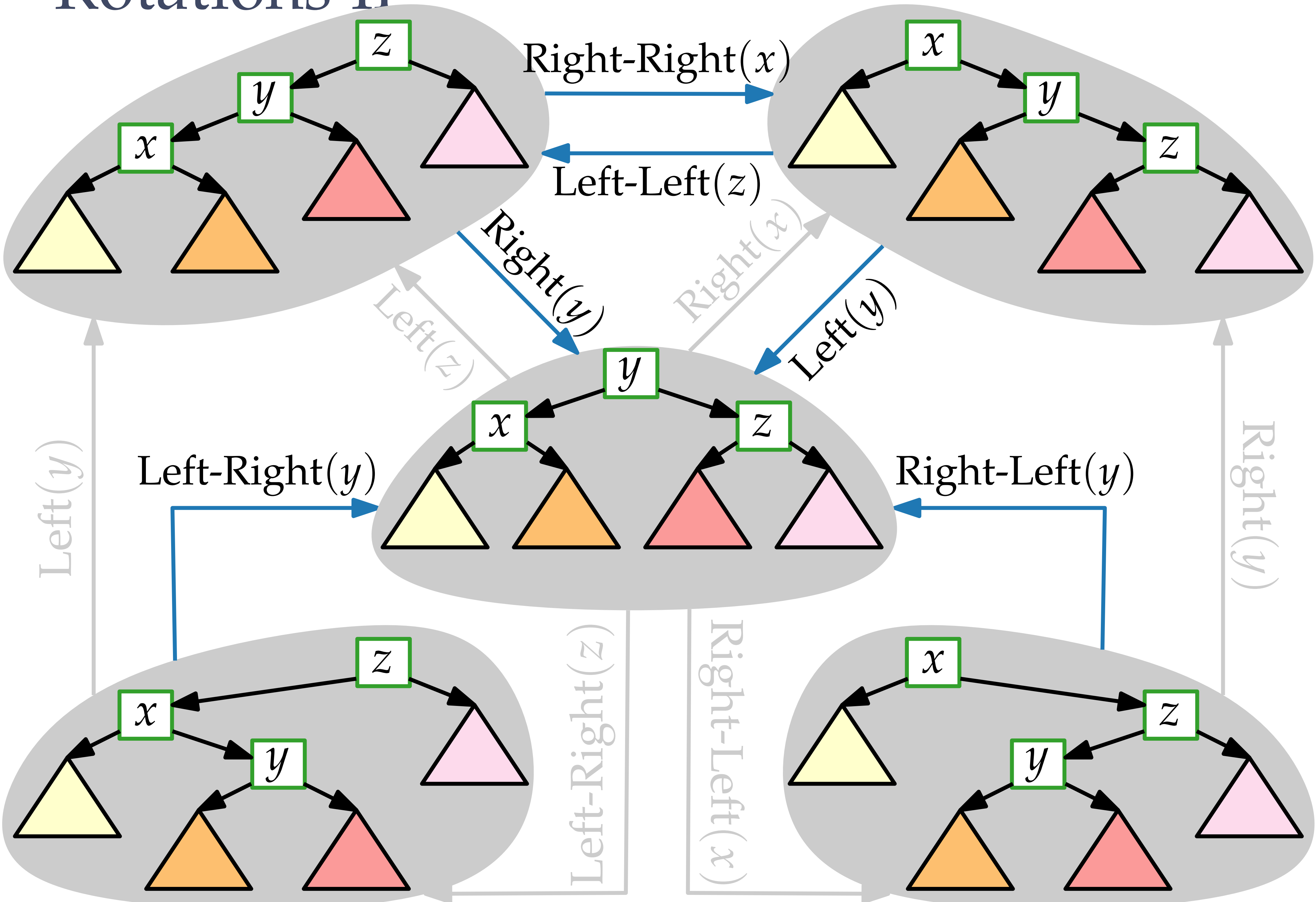
Query(x): Splay(x), then return root

Query(8) Query(6) Query(5)
Query(3)
Query(2)

We're back at the start...
and we did $\Theta(n^2)$ rotations



Rotations II

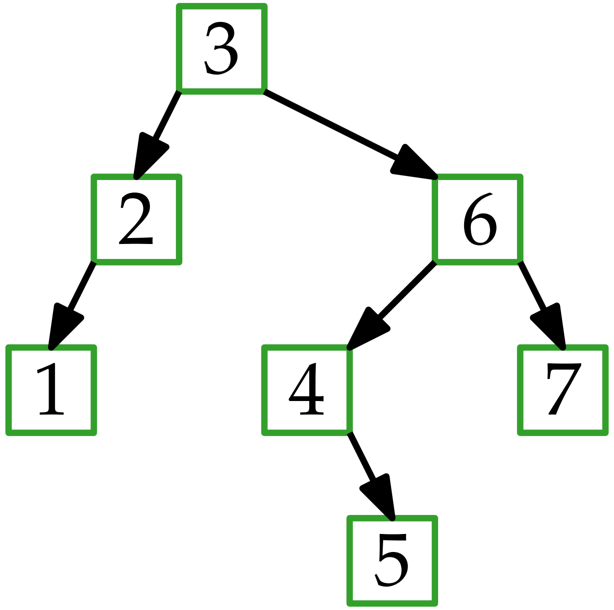


Splay

Algorithm: Splay(x)

if $x \neq root$ **then**
- y = parent of x
- **if** $y = root$ **then**
  - **if** $x < y$ **then** Right(x)
  - **if** $y < x$ **then** Left(x)
- **else**
  - z = parent of y
  - **if** $x < y < z$ **then** Right-Right(x)
  - **if** $z < y < x$ **then** Left-Left(x)
  - **if** $y < x < z$ **then** Left-Right(x)
  - **if** $z < x < y$ **then** Right-Left(x)
- Splay(x)

Splay(3):



Call Splay(x):
- after Search(x)
- after Insert(x)
- before Delete(x)

Why is Splay Fast?

$w(x)$: weight of x (here 1), $W = \sum w(x)$ (here n)

$s(x)$: sum of all $w(x)$ in subtree of x_i

mark edges:

$s(\text{child}) \leq s(\text{parent})/2$

$s(\text{child}) > s(\text{parent})/2$

Cost to query x_i: $O(\log W + \#\text{red})$

Idea: blue edges halve the weight

$\Rightarrow \#\text{blue} \in O(\log W)$

How can we amortize red edges?

Use sum-of-logs potential

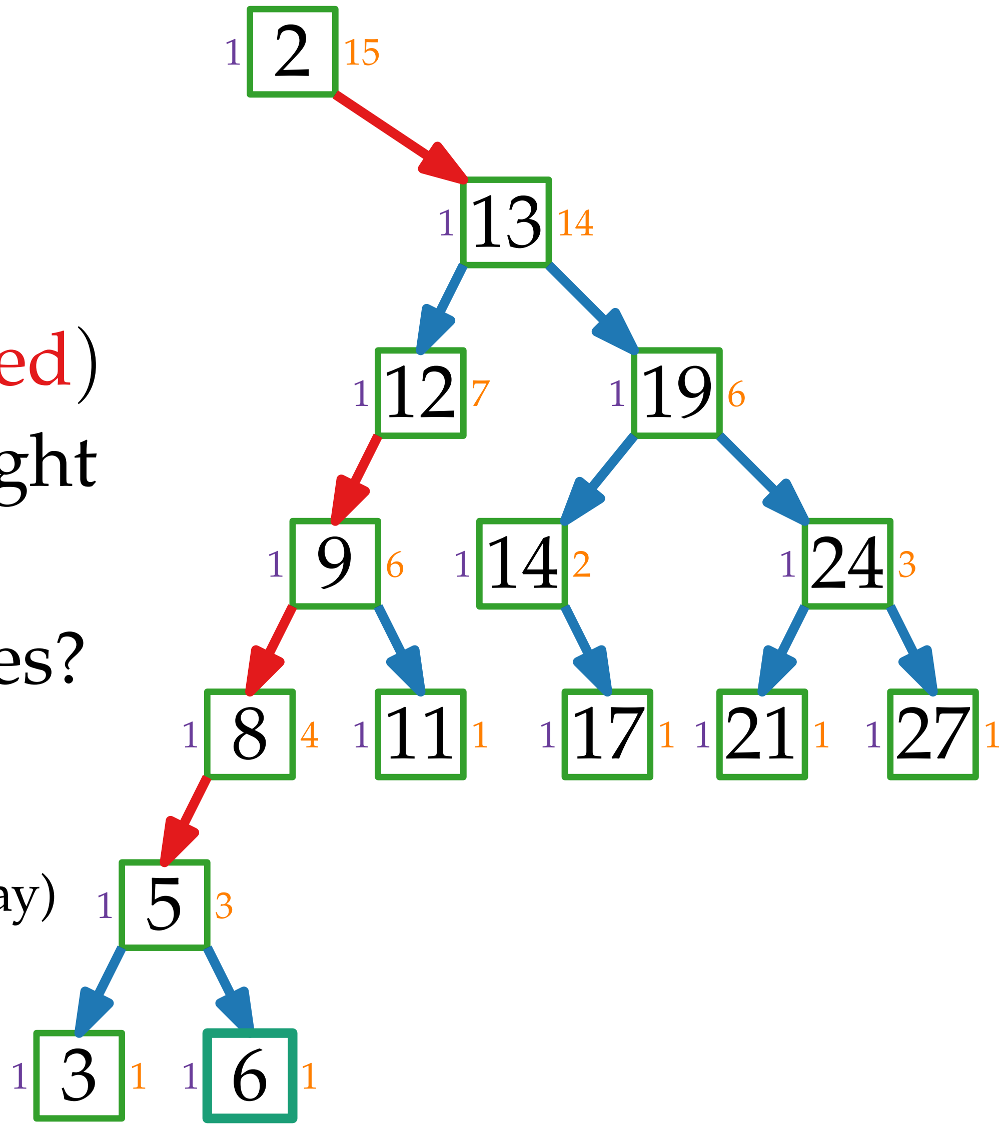
$\Phi = \sum \log s(x)$

Amortized cost:

(potential before splay)

real cost + $\Phi_+ - \Phi$

(potential after splay)



What is Potential?

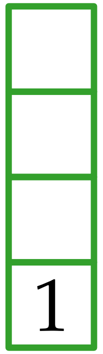
Φ represents work that has been "paid for" but not yet performed.

amortized cost per step: real cost $+ \Phi_+ - \Phi$

total cost $= \Phi_0 - \Phi_{\text{end}} + \sum$ amortized cost

(initial potential) (potential at the end)

pop(2)



$\Phi = 1$

Example (from ADS): Stack with multipop

$\Phi :=$ size of the stack

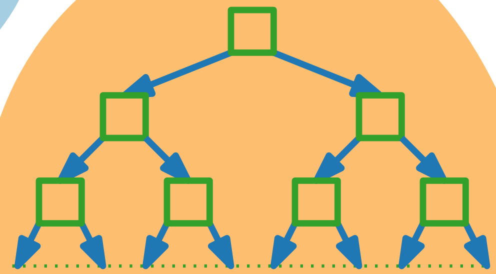
push: $\quad 1 + \Phi_+ - \Phi \; = 2$

pop(k): $\; k + \Phi_+ - \Phi \; = 0$

$$\begin{aligned} \text{total cost} &= \Phi_0 - \Phi_{\text{end}} + \text{amortized cost} \\ &\leq \Phi_0 - \Phi_{\text{end}} + 2n \\ &\leq 2n \in O(n) \end{aligned}$$

Why is Splay Fast?

$\Phi = \sum_{i=1}^{n} \log i$
$\in O(n \log n)$



$\Phi \approx \sum_{i=1}^{\log n} i \log \frac{n}{i}$
$\in O(\log^3 n)$

$w(x)$: weight of x (here 1), $W = \sum w(x)$ (here n)

$s(x)$: sum of all $w(x)$ in subtree of x_i

mark edges:

$\longrightarrow$ $s(\text{child}) \le s(\text{parent})/2$ (blue)

$\longrightarrow$ $s(\text{child}) > s(\text{parent})/2$ (red)

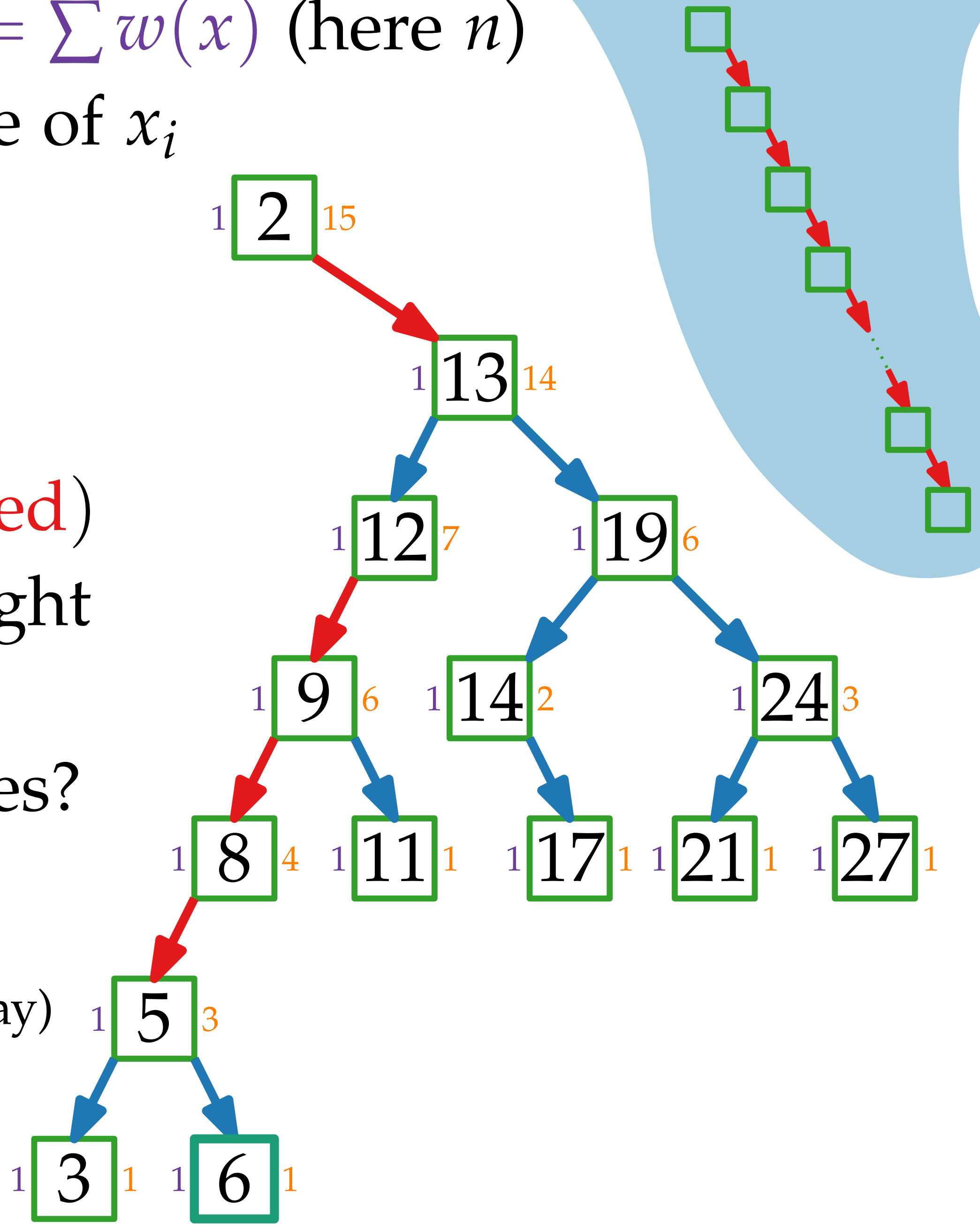
Cost to query x_i: $O(\log W + \#\text{red})$

Idea: blue edges halve the weight
$\Rightarrow \#\text{blue} \in O(\log W)$

How can we amortize red edges?

Use sum-of-logs potential
$\Phi = \sum \log s(x)$

Amortized cost:
(potential before splay)
real cost + $\Phi_{+} - \Phi$
(potential after splay)

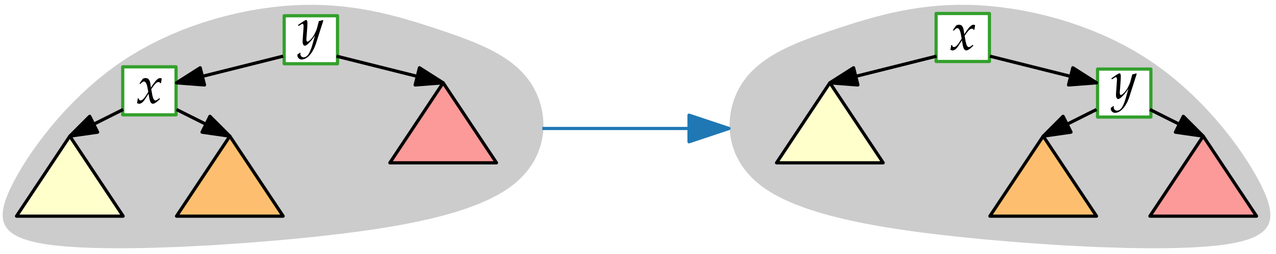


Potential after Rotation

Consider any rotation; $s(x)$ before rotation, $s_+(x)$ afterwards

Lemma. After a single rotation, the potential increases by $\leq 3\,(\log s_+(x) - \log s(x))$.

Proof. Right(x)



Observe: Only $s(x)$ and $s(y)$ change.

pot. change $= \log s_+(x) + \log s_+(y) - \log s(x) - \log s(y)$

$(s_+(y) \leq s(y))$ $\leq \log s_+(x) - \log s(x)$

$(s_+(x) > s(x))$ $\leq 3\,(\log s_+(x) - \log s(x))$ ✓

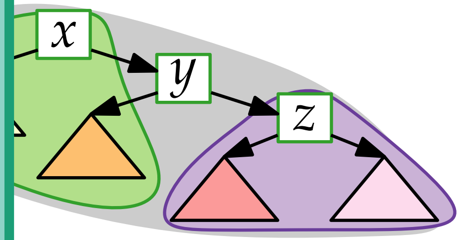
Left(x) analogue ✓

Potential after Rotation

Consider any rotation; $s(x)$ before rotation, $s_+(x)$ afterwards

Lemma. After a double rotation, the potential increases by $\leq 3\,(\log s_+(x) - \log s(x)) - 2$.

Proof.
Case 1.



pot. cha

$(s_+(x) =$ … $\log s(y)$

$(s(x) \leq s$

$(s_+(y) \leq$

Inequality of arithmetic and geometric means (AM-GM):

$$\frac{x_1+x_2+\cdots+x_k}{k} \geq \sqrt[k]{x_1 \cdot x_2 \cdot \ldots \cdot x_k}$$

(arithmetic mean) (geometric mean)

for $k = 2$:

$$\frac{x+y}{2} \geq \sqrt{xy} \quad \Leftrightarrow xy \leq \left(\frac{x+y}{2}\right)^2$$

$$s(x) + s_+(z) \leq s_+(x) \Rightarrow \log s(x) + \log s_+(z) = \log s(x) s_+(z)$$

$$\underset{\text{(AM-GM)}}{\leq} \log\left((s(x) + s_+(z))/2\right)^2 \leq \log\left(s_+(x)/2\right)^2$$

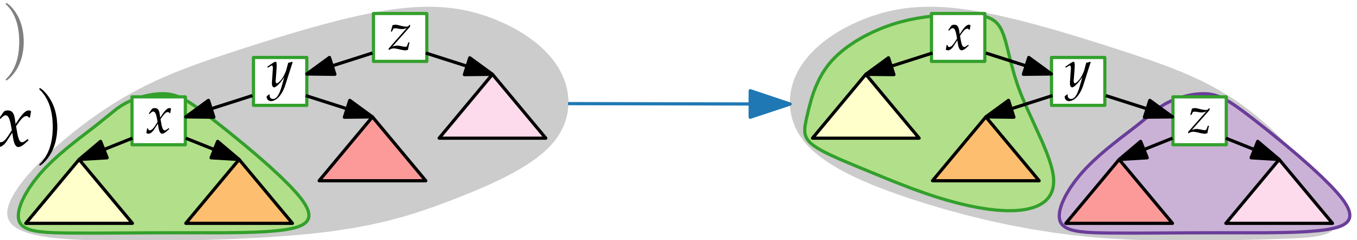
Potential after Rotation

Consider any rotation; $s(x)$ before rotation, $s_+(x)$ afterwards

Lemma. After a double rotation, the potential increases by $\leq 3\,(\log s_+(x) - \log s(x)) - 2$.

Proof. / Left-Left(x)

Case 1. Right-Right(x)



$$
\begin{aligned}
\text{pot. change} \quad & = \log s_+(x) + \log s_+(y) + \log s_+(z) \\
& \quad - \log s(x) - \log s(y) - \log s(z) \\
(s_+(x) = s(z)) \quad & = \log s_+(y) + \log s_+(z) - \log s(x) - \log s(y) \\
(s(x) \leq s(y)) \quad & \leq \log s_+(y) + \log s_+(z) - 2\log s(x) \\
(s_+(y) \leq s_+(x)) \quad & \leq \log s_+(x) + \log s_+(z) - 2\log s(x) \\
& \leq 3\log s_+(x) - 3\log s(x) - 2 \quad \checkmark
\end{aligned}
$$

$$
s(x) + s_+(z) \leq s_+(x) \Rightarrow \log s(x) + \log s_+(z) = \log s(x)s_+(z)
$$

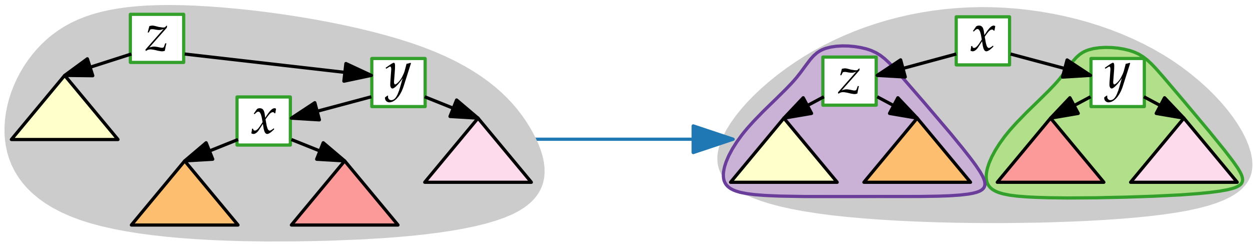
$$
\underset{\text{(AM-GM)}}{\leq} \log\left((s(x) + s_+(z))/2\right)^2 \leq \log\left(s_+(x)/2\right)^2 \leq 2\log s_+(x) - 2
$$

Potential after Rotation

Consider any rotation; $s(x)$ before rotation, $s_+(x)$ afterwards

Lemma. After a double rotation, the potential increases by $\leq 3\,(\log s_+(x) - \log s(x)) - 2$.

Proof. / Left-Right(x)
Case 2. Right-Left(x)



$$
\begin{aligned}
\text{pot. change} \quad & = \log s_+(x) + \log s_+(y) + \log s_+(z) \\
& \quad - \log s(x) - \log s(y) - \log s(z) \\
(s_+(x) = s(z)) \quad & = \log s_+(y) + \log s_+(z) - \log s(x) - \log s(y) \\
(s(x) \leq s(y)) \quad & \leq \log s_+(y) + \log s_+(z) - 2\log s(x) \\
& \leq 2\log s_+(x) - 2\log s(x) - 2 \\
(s_+(x) > s(x)) \quad & \leq 3\log s_+(x) - 3\log s(x) - 2 \quad \checkmark
\end{aligned}
$$

$$
s_+(y) + s_+(z) \leq s_+(x) \;\Rightarrow\; \log s_+(y) + \log s_+(z) \overset{\text{(AM-GM)}}{\leq} 2\log s_+(x) - 2
$$

Access Lemma

Lemma. After a single rotation, the potential increases by $\leq 3\,(\log s_+(x) - \log s(x))$.
After a double rotation, the potential increases by $\leq 3\,(\log s_+(x) - \log s(x)) - 2$.

Lemma. The (amortized) cost of $\mathrm{Splay}(x)$ is $c(\mathrm{Splay}(x)) \leq 1 + 3\log(W/w(x))$.

Proof. W.l.o.g. k double rotations and 1 single rotation.
Let $s_i(x)$ be $s(x)$ after i single/double rotations.
Potential increases by at most

$$\sum_{i=1}^{k} \left(3\,(\log s_i(x) - \log s_{i-1}(x)) - 2\right)$$
$$+3\,(\log s_{k+1}(x) - \log s_k(x))$$

root!

(id. entries rem.) $= 3\,(\log s_{k+1}(x) - \log s(x)) - 2k$

$= 3\,(\log W - \log s(x)) - 2k$

$(s(x) \geq w(x))$ $\leq 3\,(\log W - \log w(x)) - 2k = 3\log(W/w(x)) - 2k$

$2k+1$ rotations $\Rightarrow$ (amort.) cost $c(\mathrm{Splay}(x)) \leq 1 + 3\log(W/w(x))$ $\square$

All These Models . . .

Balanced: Queries take (amortized) $O(\log n)$ time

Entropy: Queries take expected $O(1 + H)$ time

Dynamic Finger: Queries take $O(\log \delta_i)$ time (δ_i: rank diff.)

Working Set: Queries take $O(\log t)$ time (t: recency)

Static Optimality: Queries take (amortized) $O(\mathrm{OPT}_S)$ time.

... is there one BST to rule them all?

Yes!

Querying a Sequence

Let S be a sequence of queries.

What is the *real* cost of querying S?

Let Φ_k be the potential after query k.

(amort. cost to execute $\text{Splay}(x)$)

$\Rightarrow$ total cost $\Phi_0 - \Phi_{|S|} + \sum_{x \in S} c(\text{Splay}(x))$

How can we bound $\Phi_0 - \Phi_{|S|}$?

Reminder: $\Phi = \sum \log s(x)$

$s(x) \geq w(x) \quad \Rightarrow \Phi_{|S|} \geq \sum_{x \in T} \log w(x)$

$s(\text{root}) = \log W \quad \Rightarrow \Phi_0 \leq \sum_{x \in T} \log W$

$\Rightarrow \Phi_0 - \Phi_{|S|} \leq \sum_{x \in T} (\log W - \log w(x)) \leq \sum_{x \in T} O\left(c(\text{Splay}(x))\right)$

$\Rightarrow$ as long as every key is queried at least once, it doesn't change the asymptotic running time.

Balance

Lemma. The (amortized) cost of $\text{Splay}(x)$ is $c(\text{Splay}(x)) \leq 1 + 3\log(W/w(x))$.

Definition. A BST is **balanced** if the (amortized) cost of *any* query is $O(\log n)$ (for at least n queries in total).

Theorem. Splay Trees are balanced.

Proof. Choose $w(x) = 1$ for each $x \quad \Rightarrow W = n$
$\text{Splay}(x)$ costs at least as much as finding x

$$\begin{aligned}\Rightarrow \text{total time} &= \Phi_0 - \Phi_{|S|} + \textstyle\sum_{x \in S} c(\text{Splay}(x)) \\ &\leq \textstyle\sum_{x \in T} (\log W - \log w(x)) + \sum_{x \in S} c(\text{Splay}(x)) \\ &\leq n \log n + \textstyle\sum_{x \in S} (1 + 3\log(W/w(x))) \\ &\leq n \log n + |S| + 3|S| \log n \ \in O(|S| \log n)\end{aligned}$$

$\Rightarrow$ Queries take (amort.) $O(\log n)$ time. $\square$

Entropy

Lemma. The (amortized) cost of Splay(x) is $c(\text{Splay}(x)) \leq 1 + 3\log(W/w(x))$.

Definition. A BST has the **entropy property** if queries take expected $O(1 - \sum_{i=1}^{n} p_i \log p_i)$ time.

Theorem. Splay Trees have the entropy property.

Proof. Choose $w(x_i) = p_i \quad \Rightarrow W = 1$

Amortized cost to query x_i:

$$\begin{aligned} &\leq 1 + 3\log(W/w(x_i)) \\ &= 1 + 3\log(1/p_i) \\ &= 1 - 3\log p_i \end{aligned}$$

$\Rightarrow$ expected query time:

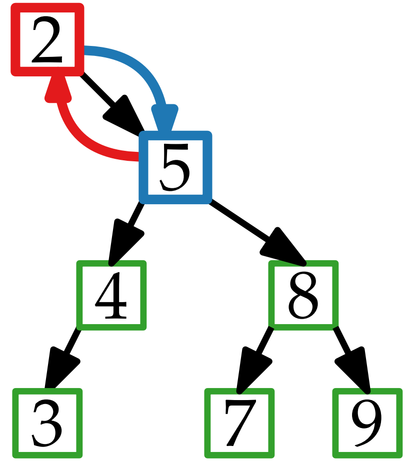
$$\begin{aligned} &O(\textstyle\sum_{i=1}^{n} p_i(1 - 3\log p_i)) \\ &= O(1 - \textstyle\sum_{i=1}^{n} p_i \log p_i) \end{aligned}$$

$\square$

Static Optimality

Given a sequence S of queries.
Let T_S^* be the *optimal* static tree with the shortest query time OPT_S for S.

e.g. $S = 2, 5, 2, 5, 2, \ldots, 5$
T^*:
OPT: $|S|$



Definition. A BST is **statically optimal** if queries take (amort.) $O(\mathrm{OPT}_S)$ time for every S.

Theorem. Splay Trees are statically optimal.

Proof. Let f_i be the number of items on path to x_i in T^*.
Let $w_i := 3^{-f_i}$. $\Rightarrow W \leq 1$

$$\Rightarrow c(\mathrm{Splay}(x_i)) = 1 + 3\log(W/w(x_i)) \leq 1 + 3\log 3^{f_i} \in O(f_i)$$

$\square$

Dynamic Optimality

Given a sequence S of queries.
Let D_S^* be the optimal *dynamic* tree with the shortest query time OPT_S^* for S.
(That is, modifications are allowed, e.g. rotations)

Definition. A BST is **dynamically optimal** if queries take (amort.) $O(\mathrm{OPT}_S^*)$ time for every S.

Splay Trees: Queries take $O(\mathrm{OPT}_S^* \cdot \log n)$ time.
Tango Trees: Queries take $O(\mathrm{OPT}_S^* \cdot \log \log n)$ time.
[Demaine, Harmon, Iacono, Pătrașcu '04]

Open Problem. Does a dynamically optimal BST exist?

This is one of the biggest open problems in algorithms.

Conjecture. Splay Trees are dynamically optimal.