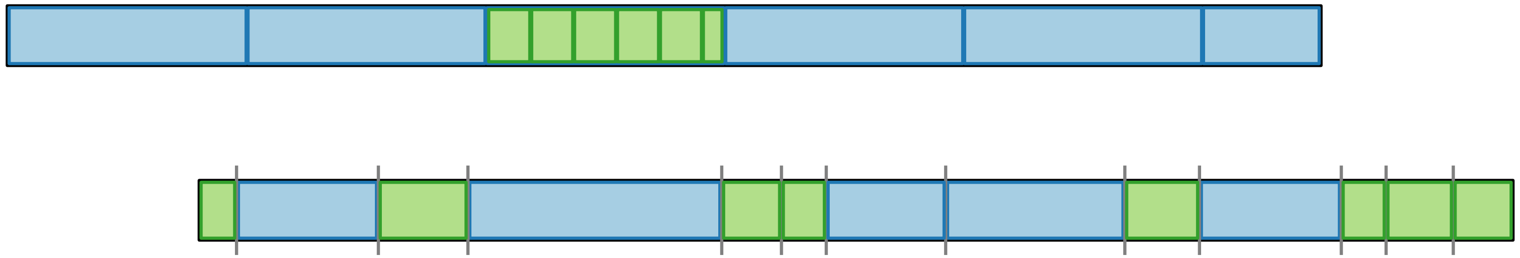
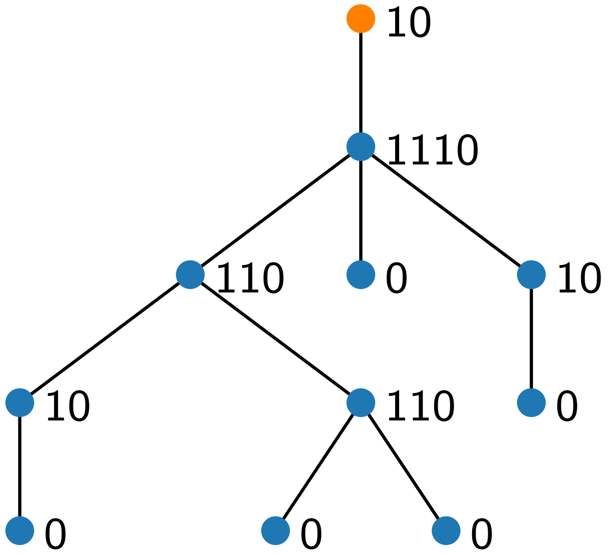
Advanced Algorithms

Succinct Data Structures

Indexable Dictionaries and Trees

Johannes Zink · WS22





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Remarks.

- We look at data structures as a designer/implementer (and not necessarily as a user).
- To define a data structure and to implement it are two different tasks.

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- the **operations** that be can applied to the data.

$\Rightarrow$

- What do we represent?
- How much space is required?
- Dynamic or static?
- Which operations are defined?
- How fast are they?

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Goal.

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to represent a class of objects.
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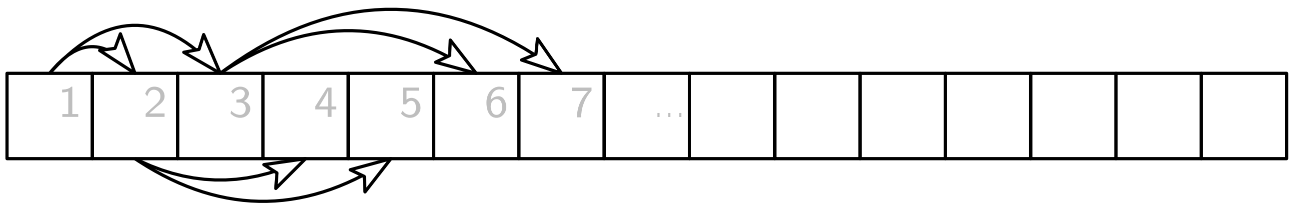

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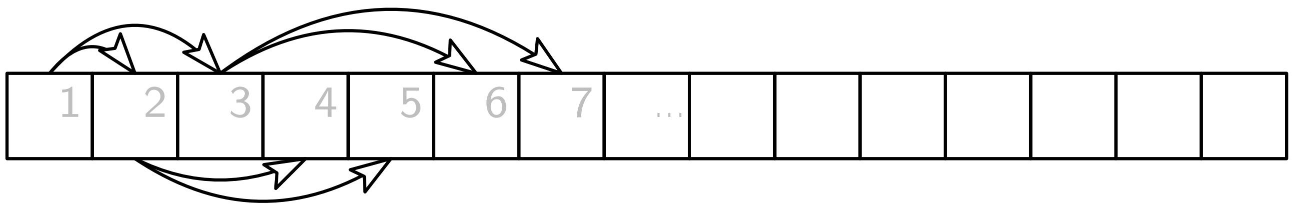

$$\texttt{leftChild}(i) = 2i$$

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$$\texttt{parent}(i) = \left\lfloor \frac{i}{2} \right\rfloor$$

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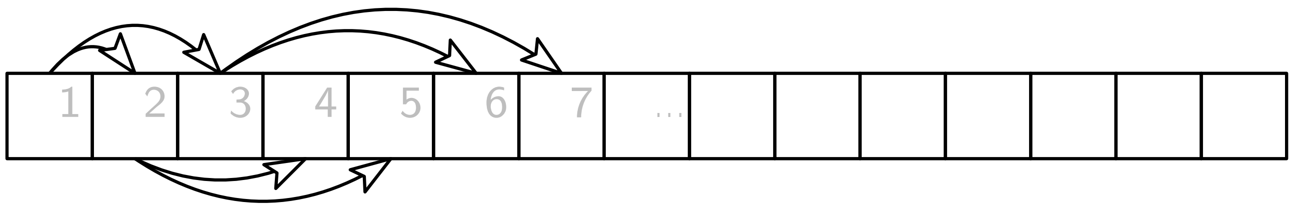

$\texttt{leftChild}(i) = 2i$

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And unbalanced trees?

Succinct Indexable Dictionary

Represent a subset $S \subseteq \{1, 2, \ldots, n\}$ and support the following operations in $O(1)$ time:

- $\texttt{member}(i)$ returns if $i \in S$
- $\texttt{rank}(i) =$ number of elements in S that are less or equal to i
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How many bits of space do we need to distinguish them?

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$\log 2^n = n$ **bits**

our logarithms are all to basis 2, i.e., $\log_2$

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b	0	0	1	1	0	1	0	1	1	0	0	0	0	1	0

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$\texttt{select}(5) =$

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$\texttt{select}(5) = 9$

$\texttt{rank}(9) =$

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$\texttt{select}(5) = 9$

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$\Rightarrow$ Exercise: Use these methods to answer $\texttt{predecessor}(i)$ and $\texttt{successor}(i)$ in $O(1)$ time.

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Rank in $o(n)$ Bits

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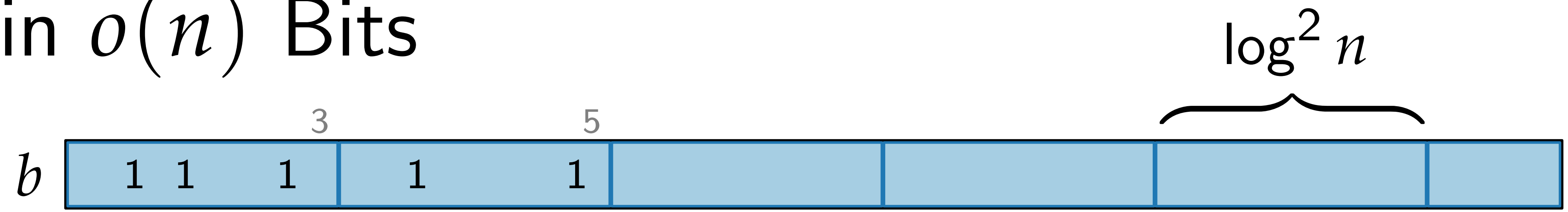

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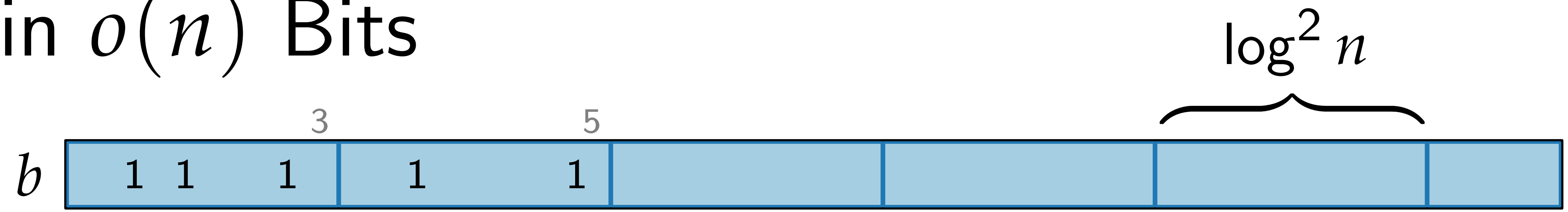

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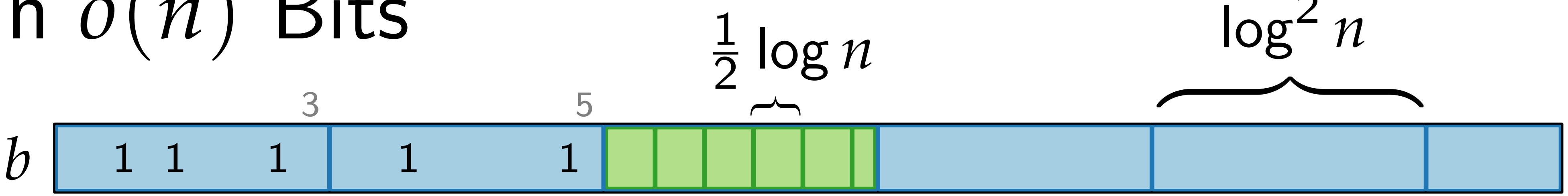

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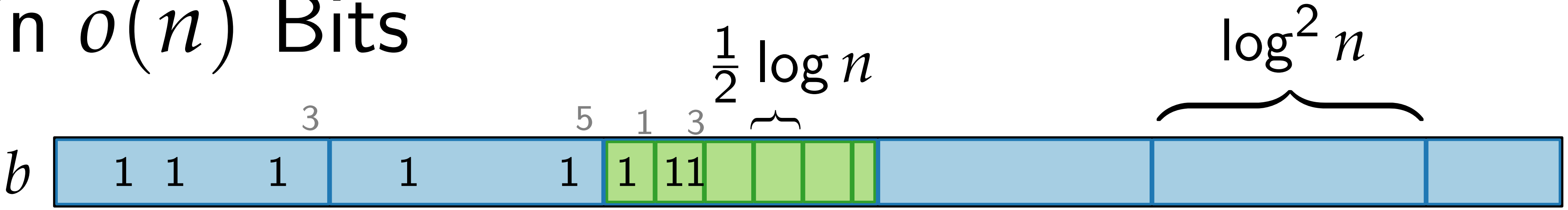

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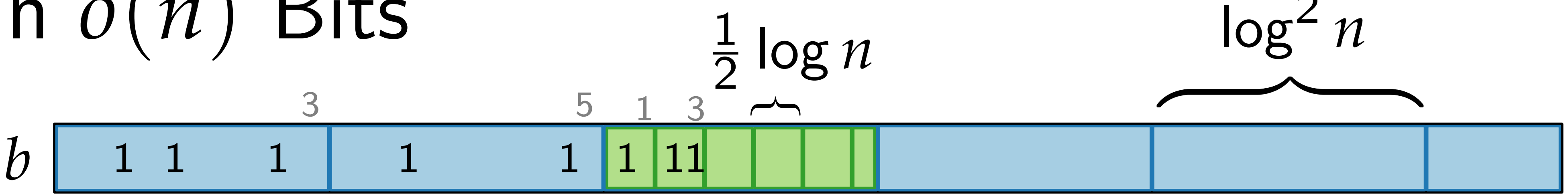

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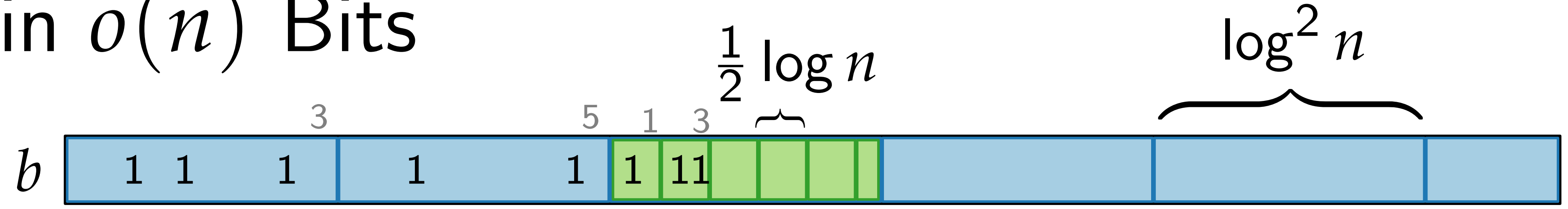

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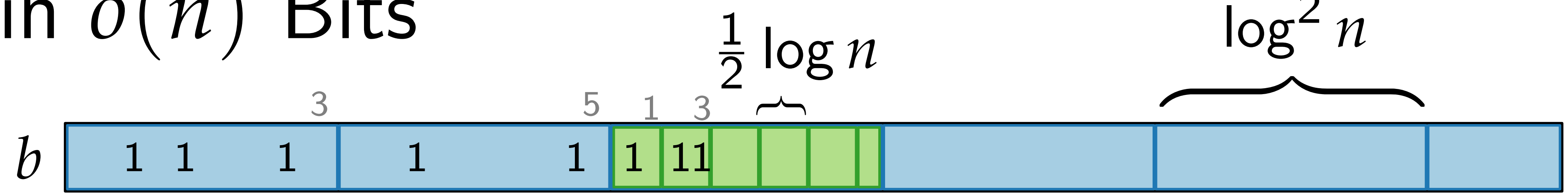

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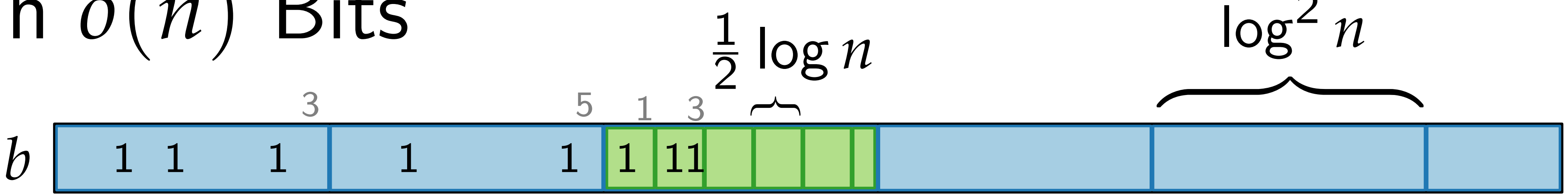

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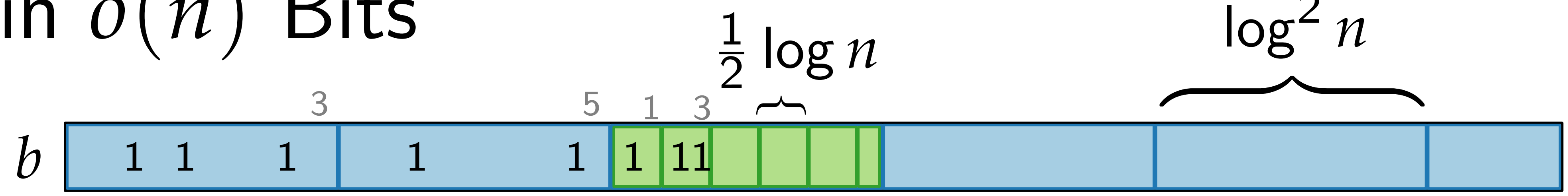

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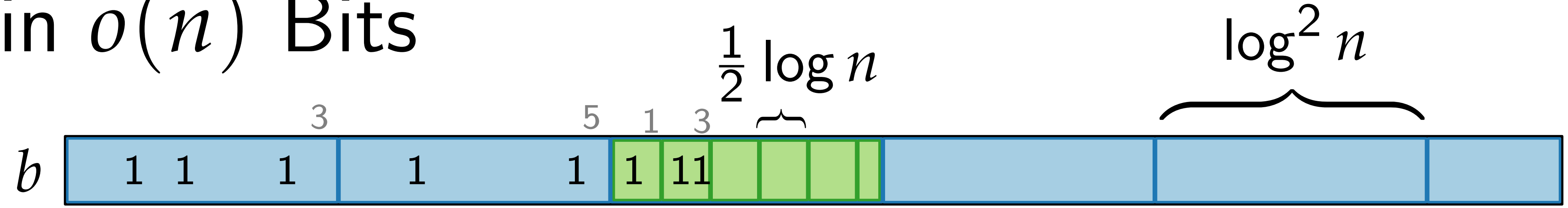
Rank in $o(n)$ Bits

$\log^2 n = (\log n)^2$



1. Split into $(\log^2 n)$-bit **chunks**

   and store cumulative rank: each needs $\leq \log n$ bits

   $$\Rightarrow O\left(\frac{n}{\log^2 n}\log n\right) = O\left(\frac{n}{\log n}\right) \subseteq o(n) \text{ bits}$$

2. Split **chunks** into $(\frac{1}{2}\log n)$-bit **subchunks**

   and store cumulative rank within **chunk**: each needs $\leq \log\log^2 n = 2\log\log n$ bits

   $$\Rightarrow O\left(\frac{n}{\log n}\log\log n\right) \subseteq o(n) \text{ bits}$$

3. Use **lookup table** for bitstrings of length $(\frac{1}{2}\log n)$: $2^{\frac{1}{2}\log n} = \sqrt{n}$ entries

   $$\Rightarrow O(\sqrt{n}\log n \log\log n) \subseteq o(n) \text{ bits}$$

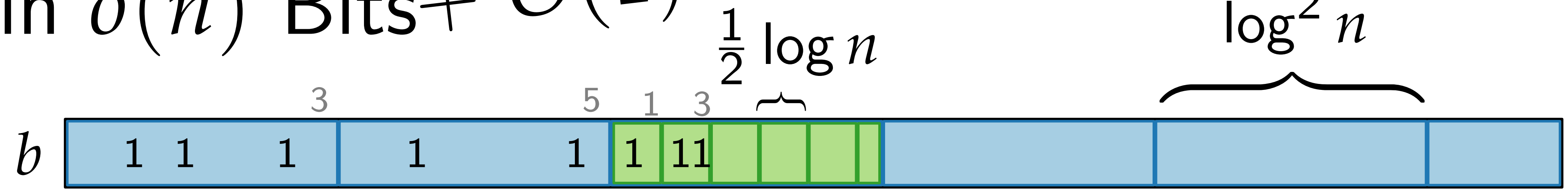
4. $\texttt{rank}(i) = \texttt{rank}$ of **chunk**

   $+$ relative $\texttt{rank}$ of **subchunk** within **chunk**

   $+$ relative $\texttt{rank}$ of element i within **subchunk**

Rank in $o(n)$ Bits+ $O(1)$ Time

$\log^2 n = (\log n)^2$



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   $\Rightarrow O(1)$ time

   (assume read & write in $O(1)$ time)

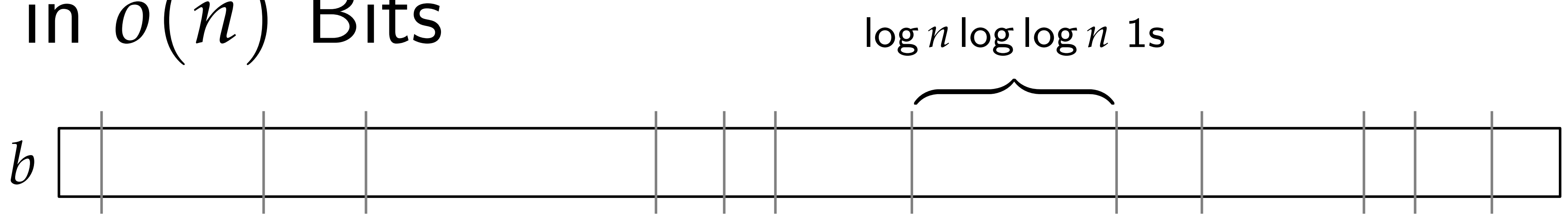
Select in $o(n)$ Bits

b

Select in $o(n)$ Bits

1. Store indices of every $(\log n \log\log n)$-th 1 bit in array

Select in $o(n)$ Bits



1. Store indices of every $(\log n \log \log n)$-th 1 bit in array

$$\Rightarrow O(\underbrace{\frac{n}{\log n \log \log n}}_{\#\text{ groups}} \underbrace{\log n}_{\text{index}}) = O(\frac{n}{\log \log n}) \subseteq o(n) \text{ bits}$$

Select in $o(n)$ Bits

1. Store indices of every $(\log n \log\log n)$-th 1 bit in array
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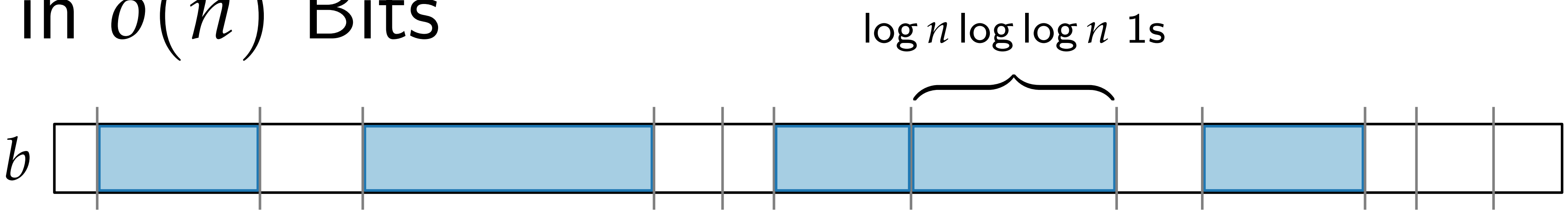
2. Within group of $(\log n \log\log n)$ 1 bits of length r bits:

Select in $o(n)$ Bits



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2. Within group of $(\log n \log\log n)$ 1 bits of length r bits:

   if $r \geq (\log n \log\log n)^2$
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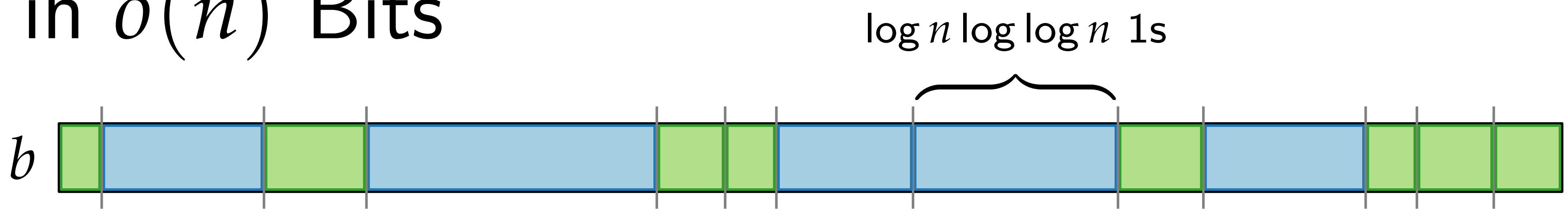
Select in $o(n)$ Bits



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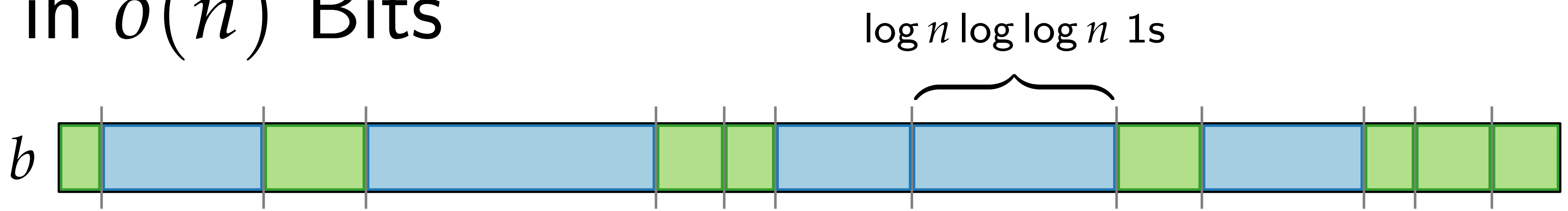
   else problem is reduced to bitstrings of length $r < (\log n \log\log n)^2$

Select in $o(n)$ Bits



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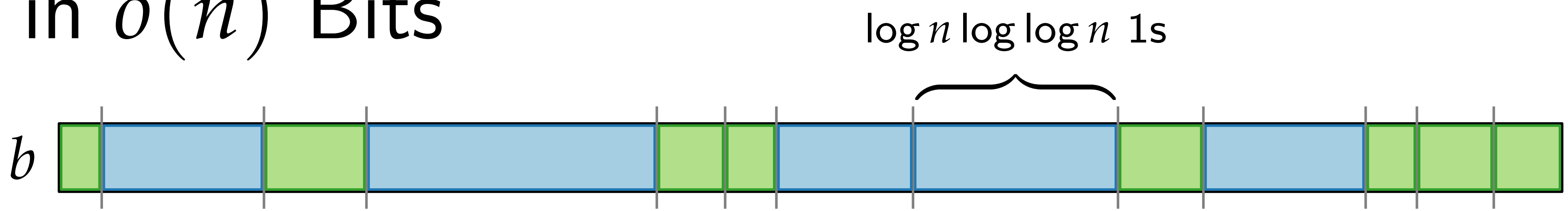
3. Repeat 1. and 2. on reduced bitstrings

Select in $o(n)$ Bits



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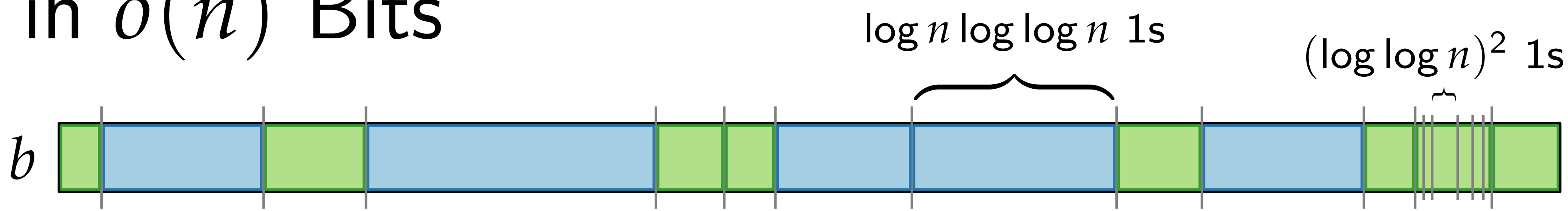

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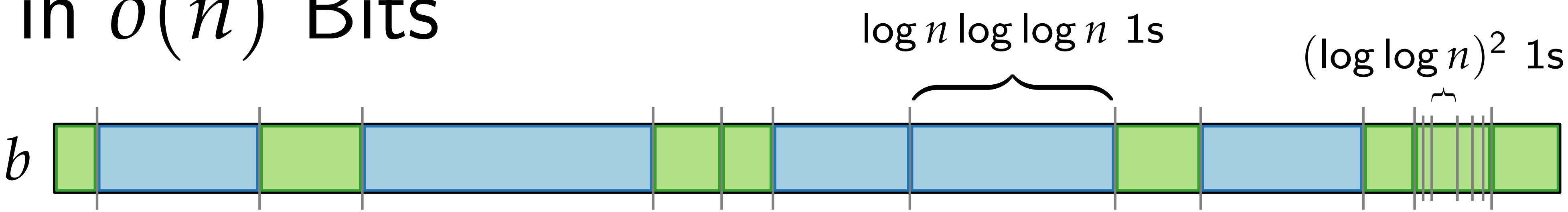

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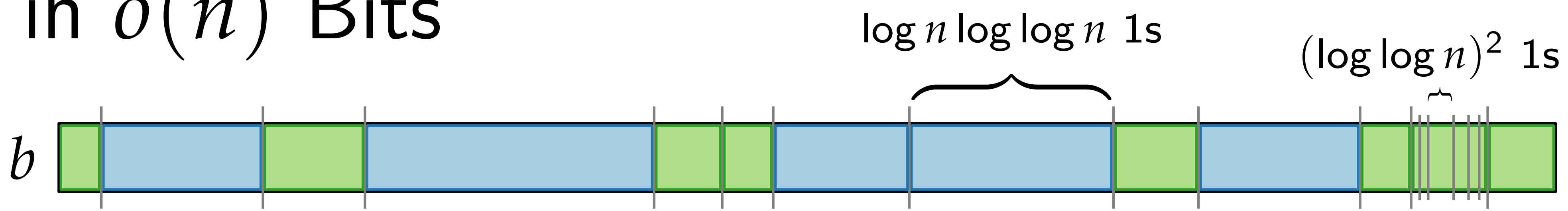
Select in $o(n)$ Bits



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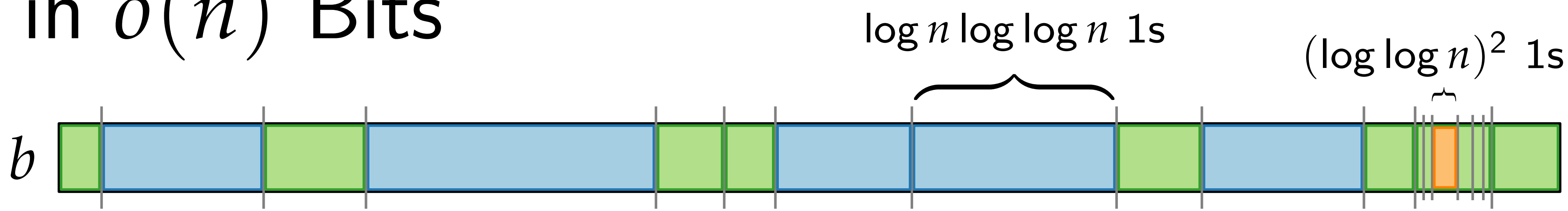
2' Within group of $(\log\log n)^2$ 1 bits of length r' bits:

Select in $o(n)$ Bits



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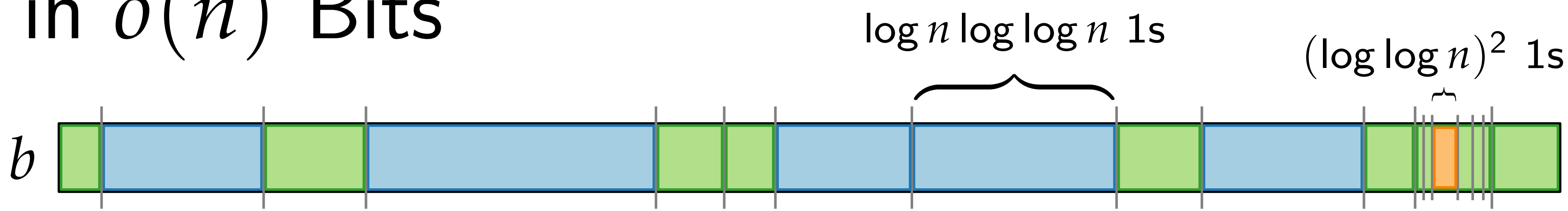
Select in $o(n)$ Bits



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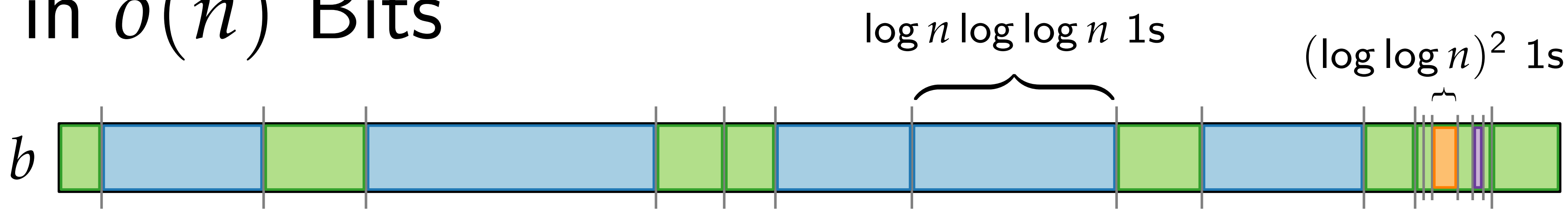
Select in $o(n)$ Bits



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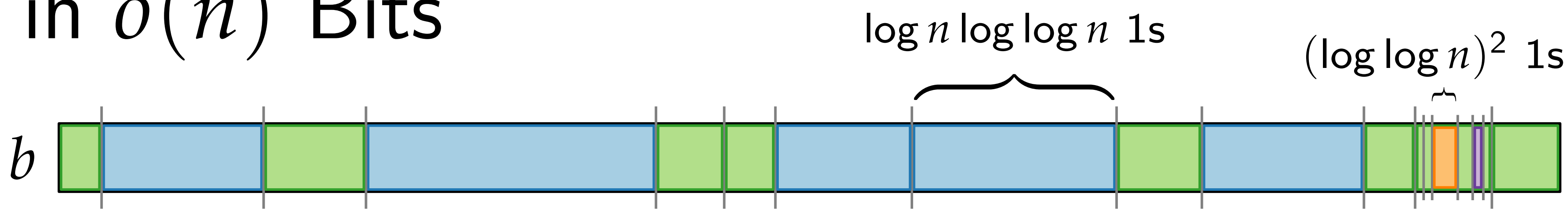

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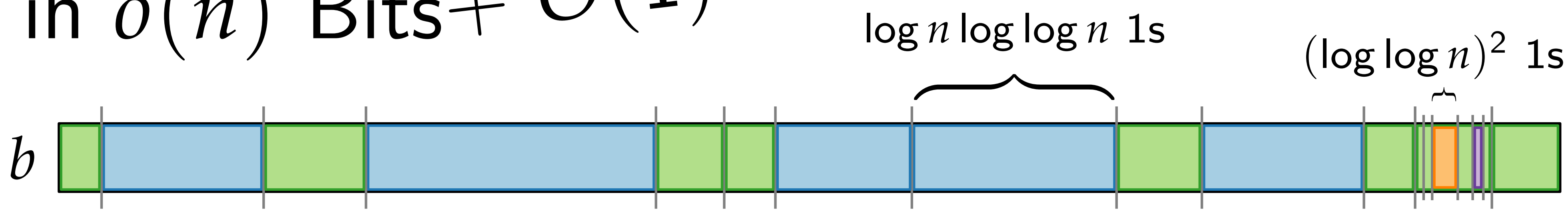
4. Use lookup table for bitstrings of length $r' \leq (\log\log n)^4 \leq \frac{1}{2}\log n$

$$\Rightarrow O(\underbrace{\sqrt{n}}_{\#\text{ bitstrings}}\;\underbrace{\log n}_{\text{query } j}\;\underbrace{\log\log n}_{\text{answer}}) = o(n) \text{ bits}$$

Select in $o(n)$ Bits+ $O(1)$ Time



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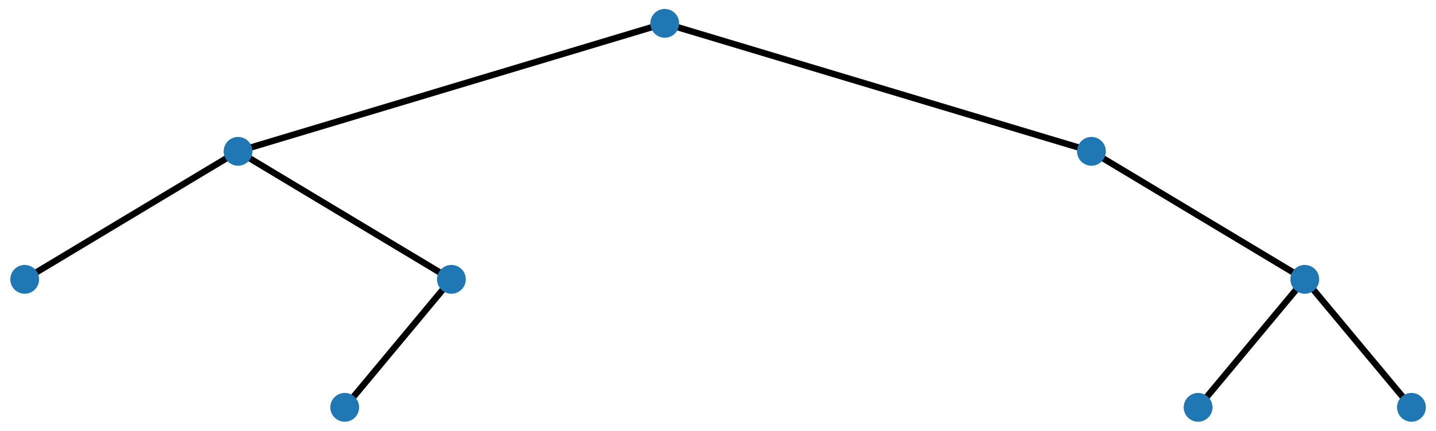
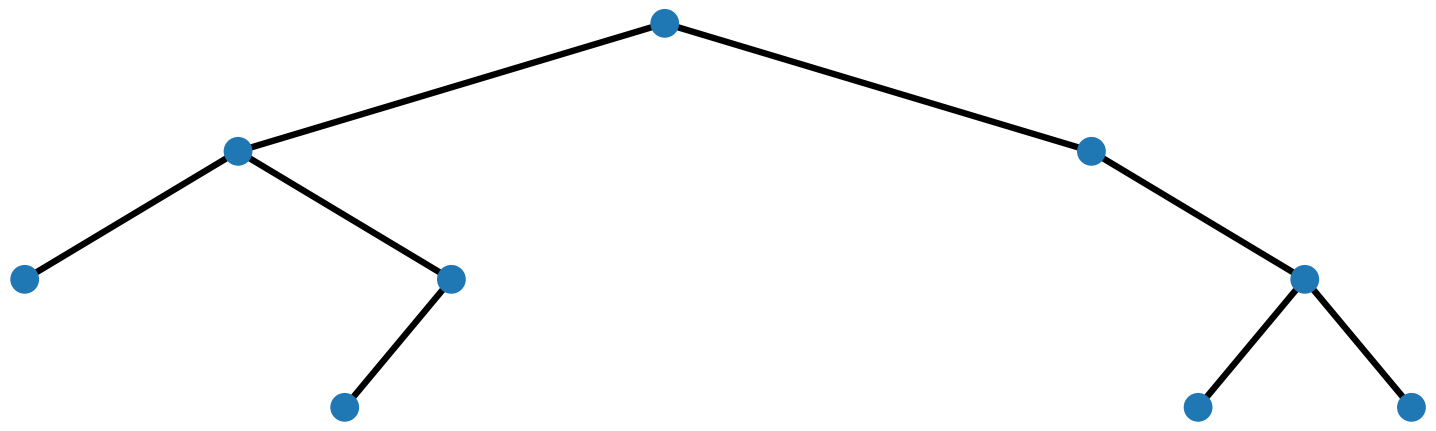
Succinct Representation of Binary Trees



Number of binary trees on n vertices:

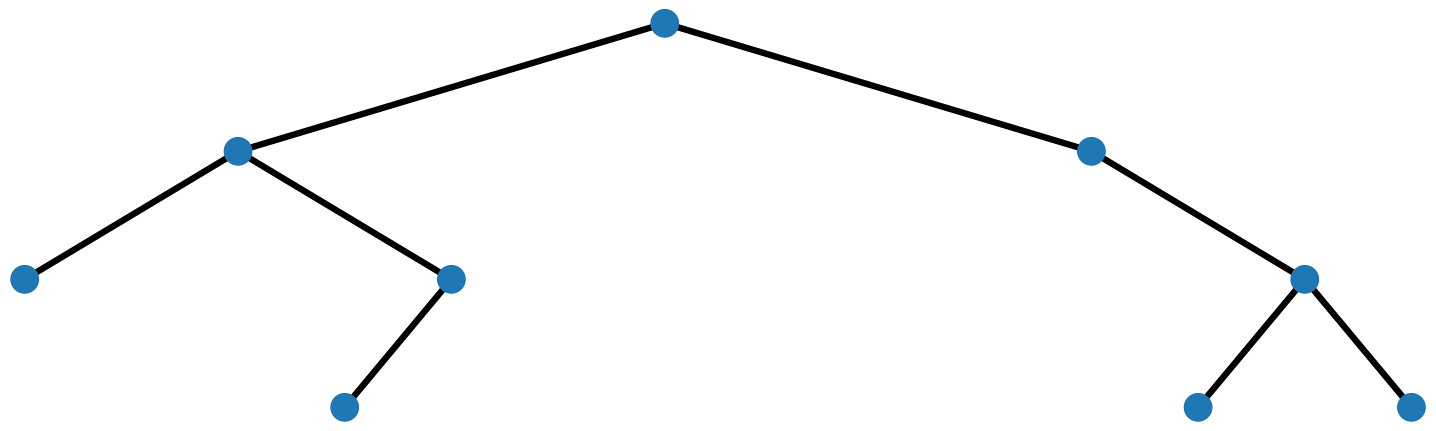
Succinct Representation of Binary Trees



C_n is the n-th *Catalan number* and $C_0 = 1$

Number of binary trees on n vertices: $C_n = \sum_{i=0}^{n-1} C_i \cdot C_{n-1-i} = \frac{(2n)!}{(n+1)!\,n!}$

Succinct Representation of Binary Trees



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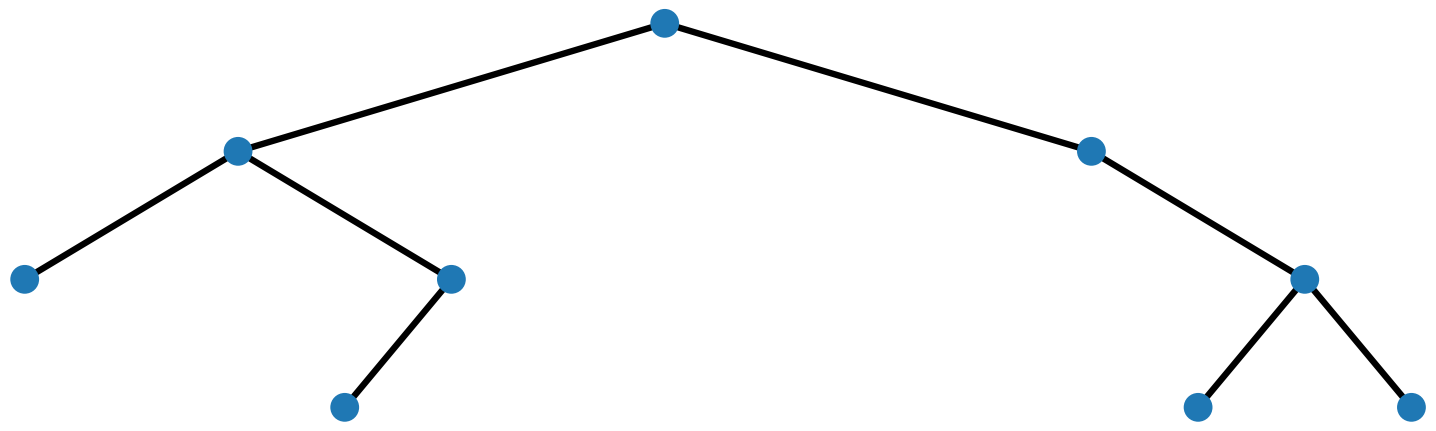
$\log C_n = 2n + o(n)$ (by Stirling's approximation)

Succinct Representation of Binary Trees



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$\Rightarrow$ We can use $2n + o(n)$ bits to represent binary trees.

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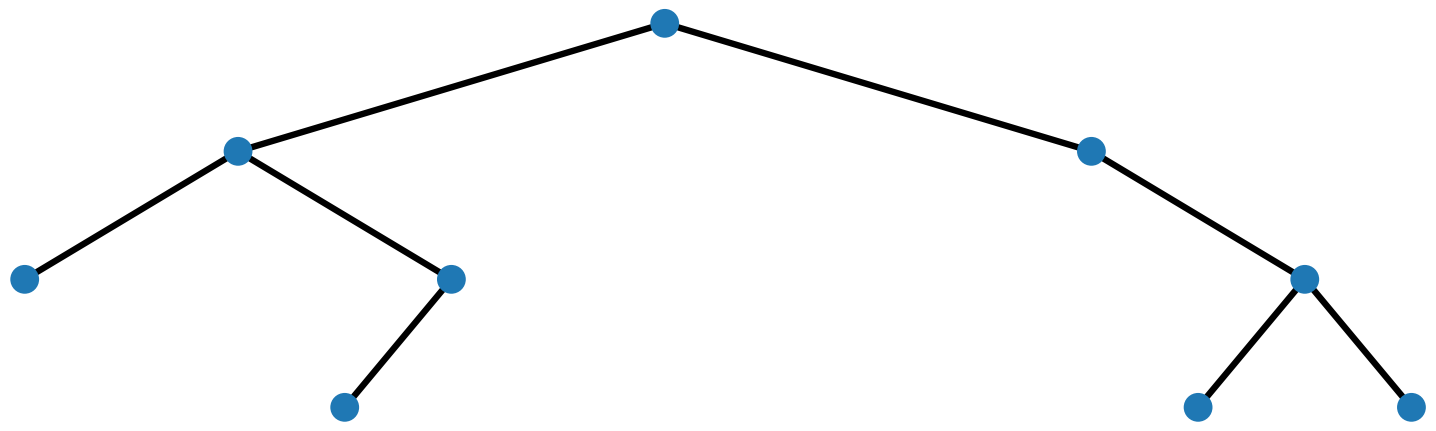

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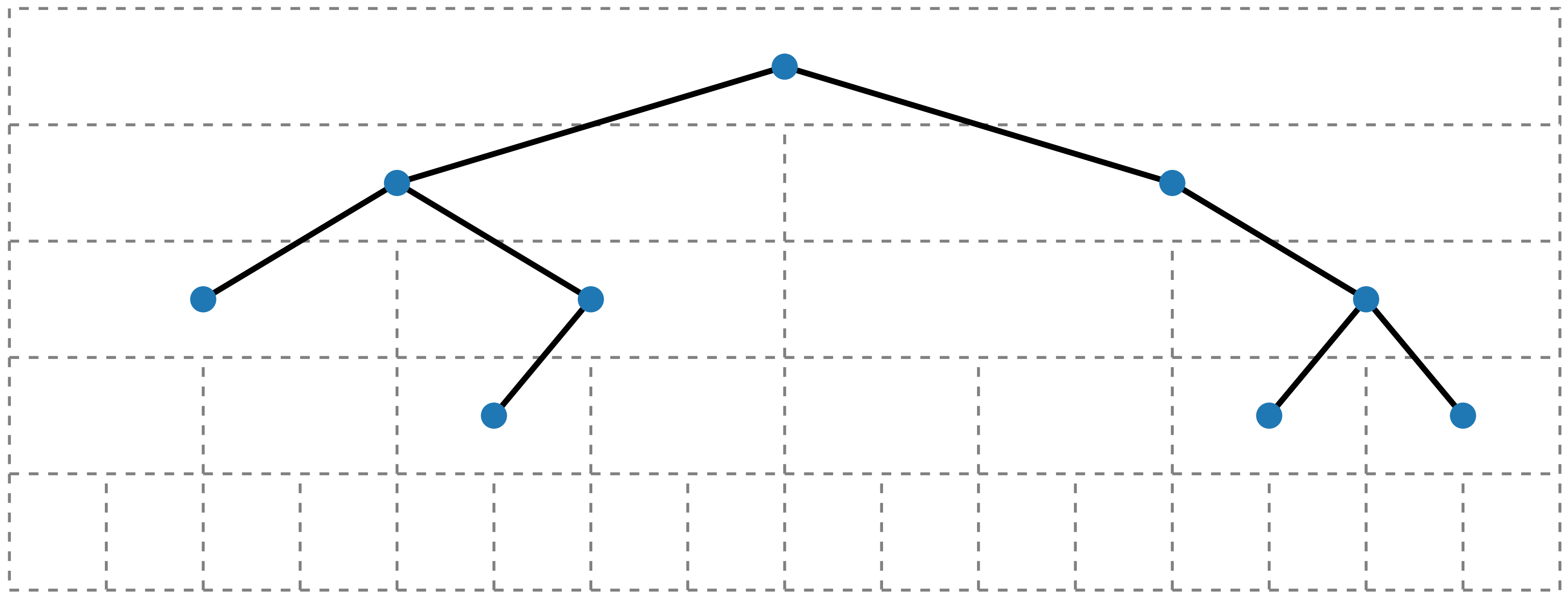
$\Rightarrow$ We can use $2n + o(n)$ bits to represent binary trees.

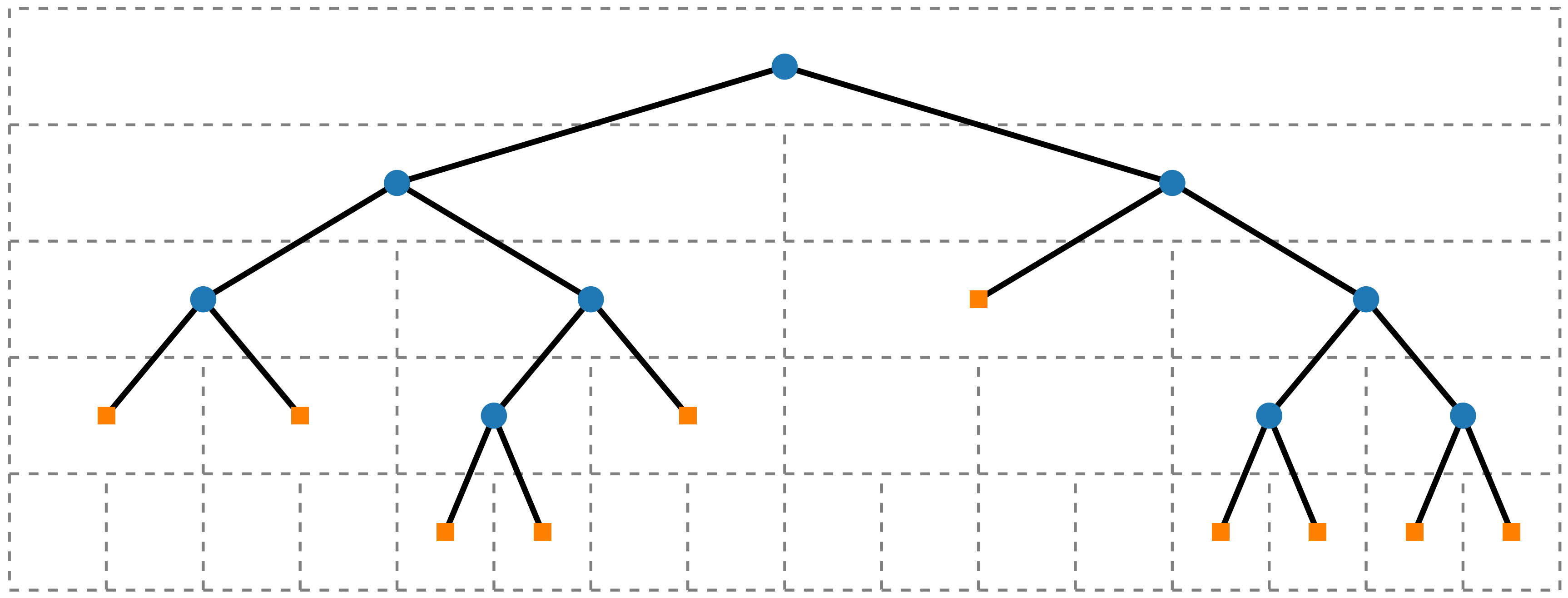
Difficulty is when a binary tree is not full.

Succinct Representation of Binary Trees



Idea.

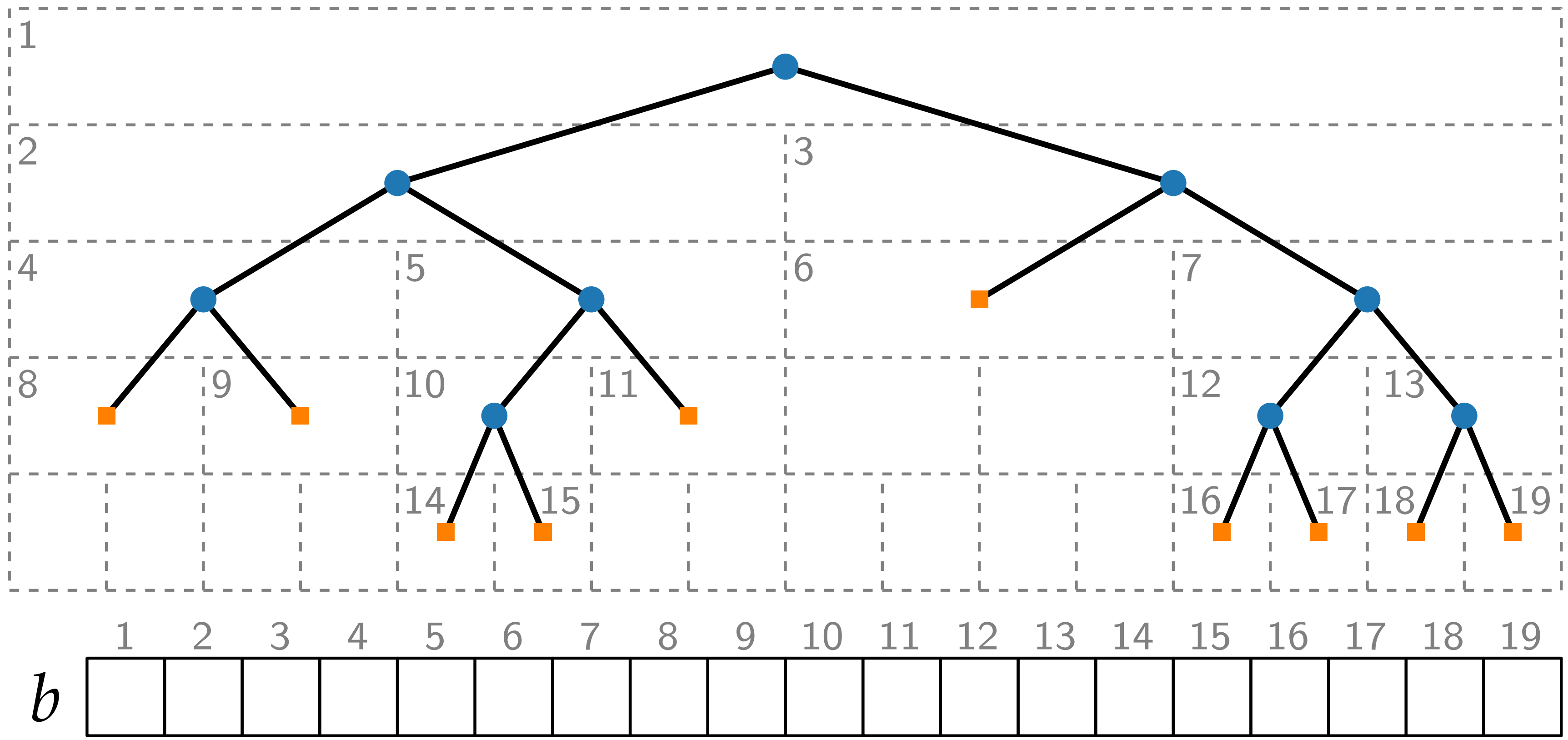
Succinct Representation of Binary Trees



Idea.

- Add external nodes to have out-degree 2

Succinct Representation of Binary Trees



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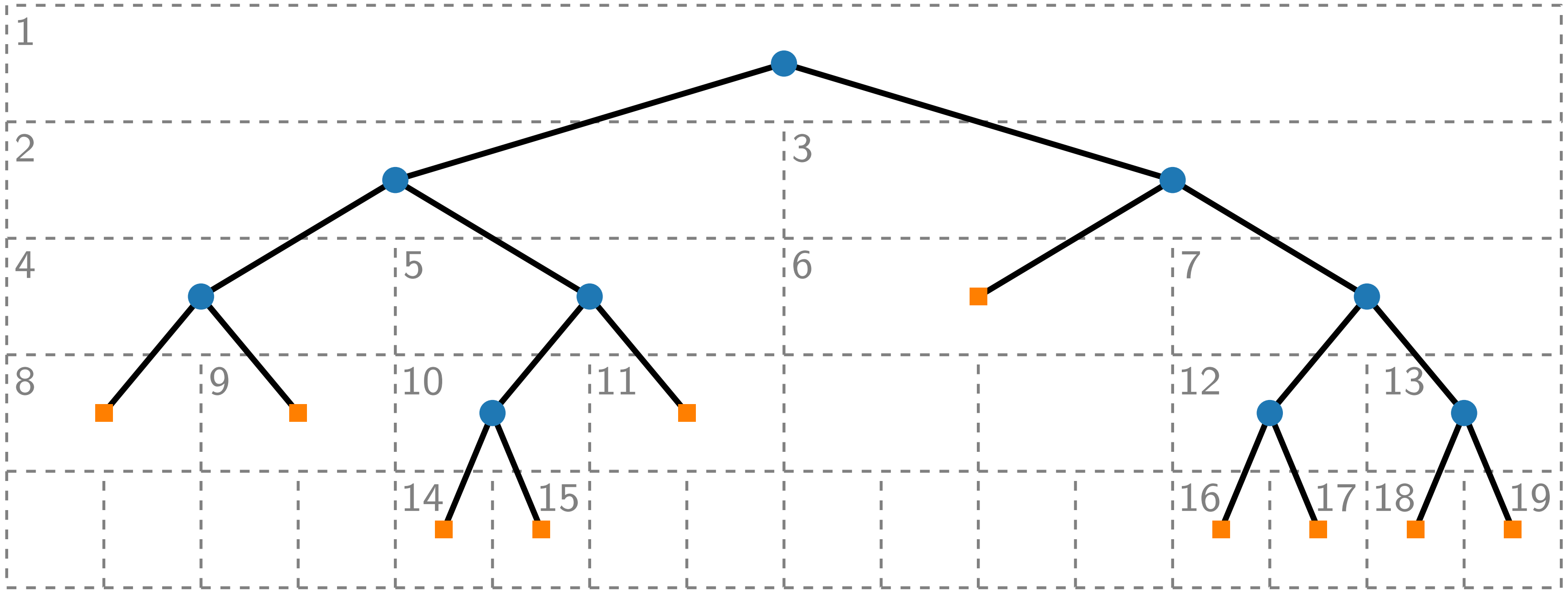
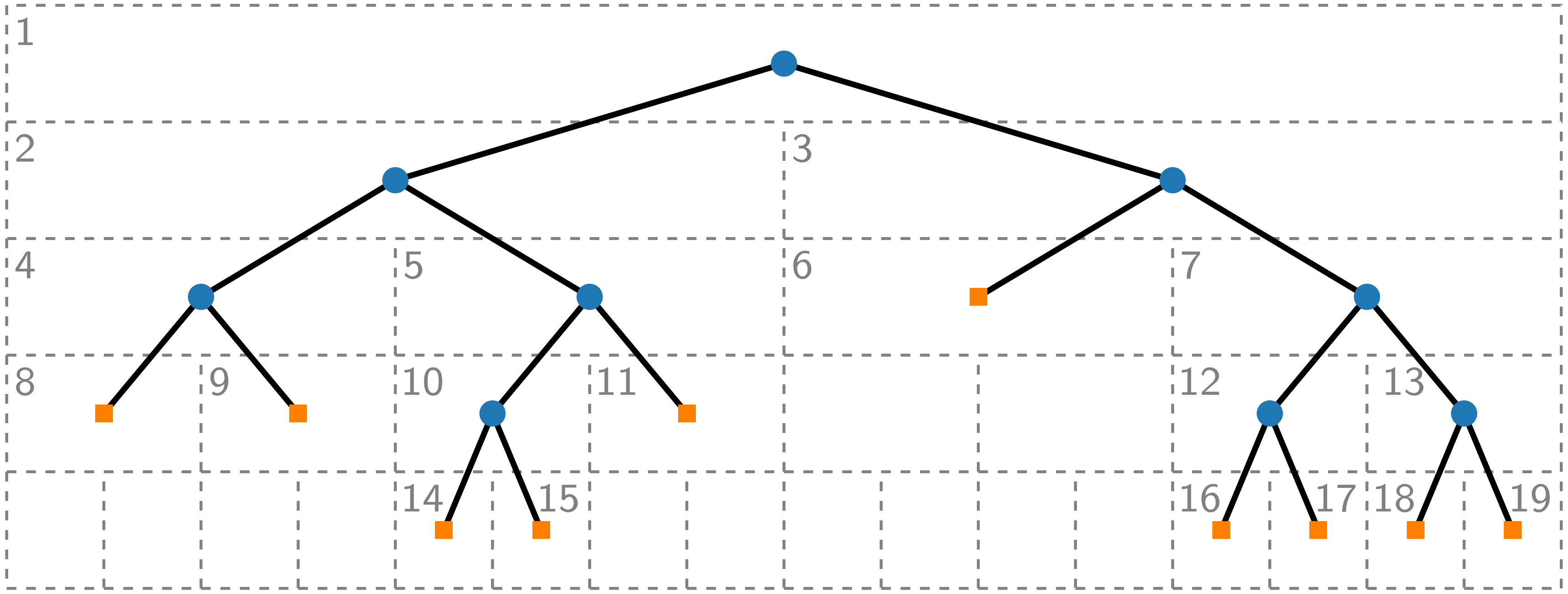


	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
b	1	1	1	1	1	0	1	0	0	1	0	1	1	0	0	0	0	0	0

Idea.

- Add external nodes to have out-degree 2
- Read internal nodes as 1
- Read external nodes as 0

Succinct Representation of Binary Trees



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Size.

- $2n+1$ bits for b
- $o(n)$ for `rank` and `select`

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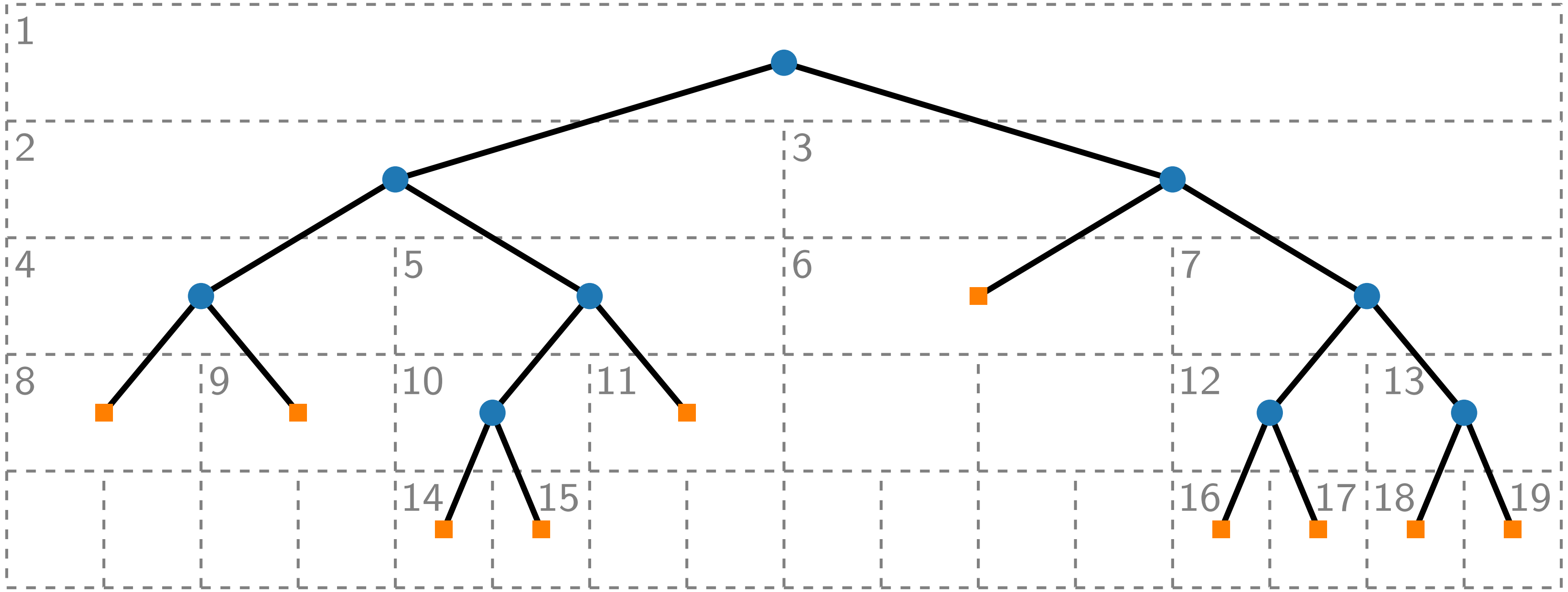
Succinct Representation of Binary Trees



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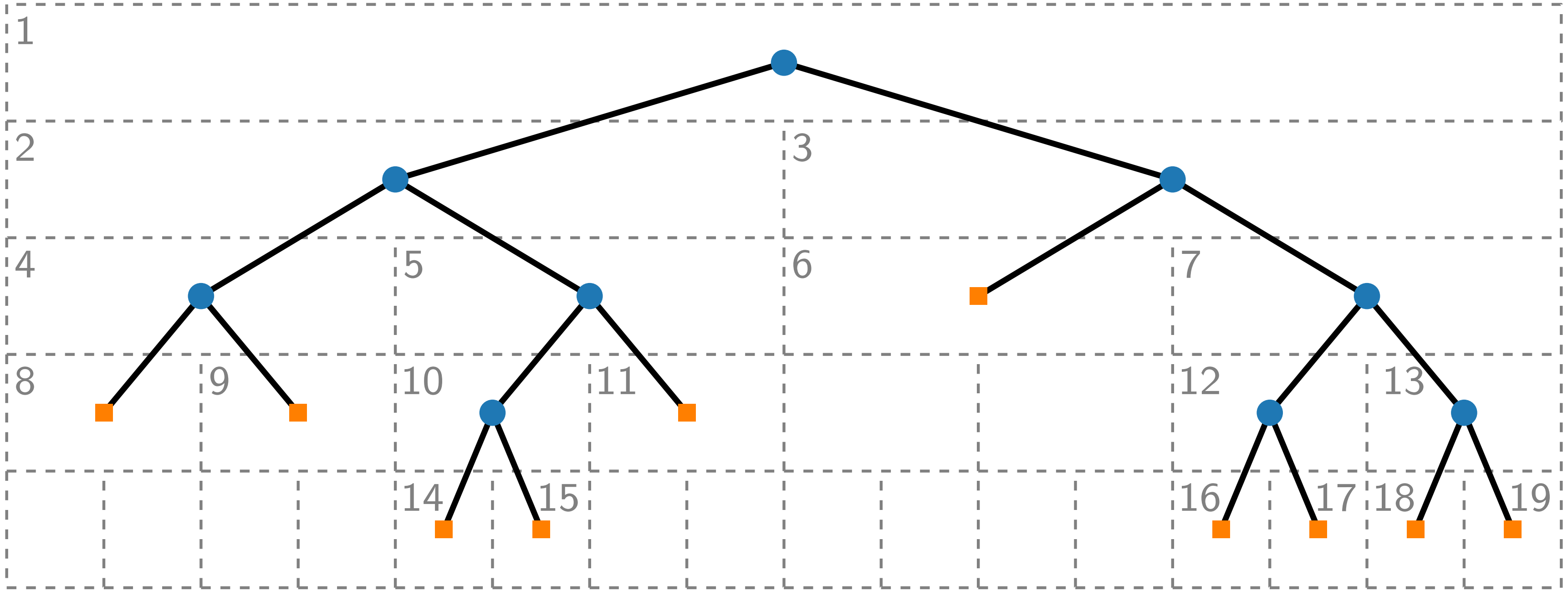
Operations.

- $\texttt{parent}(i) =$?
- $\texttt{leftChild}(i) =$?
- $\texttt{rightChild}(i) =$?

Succinct Representation of Binary Trees



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- Use `rank` and `select`

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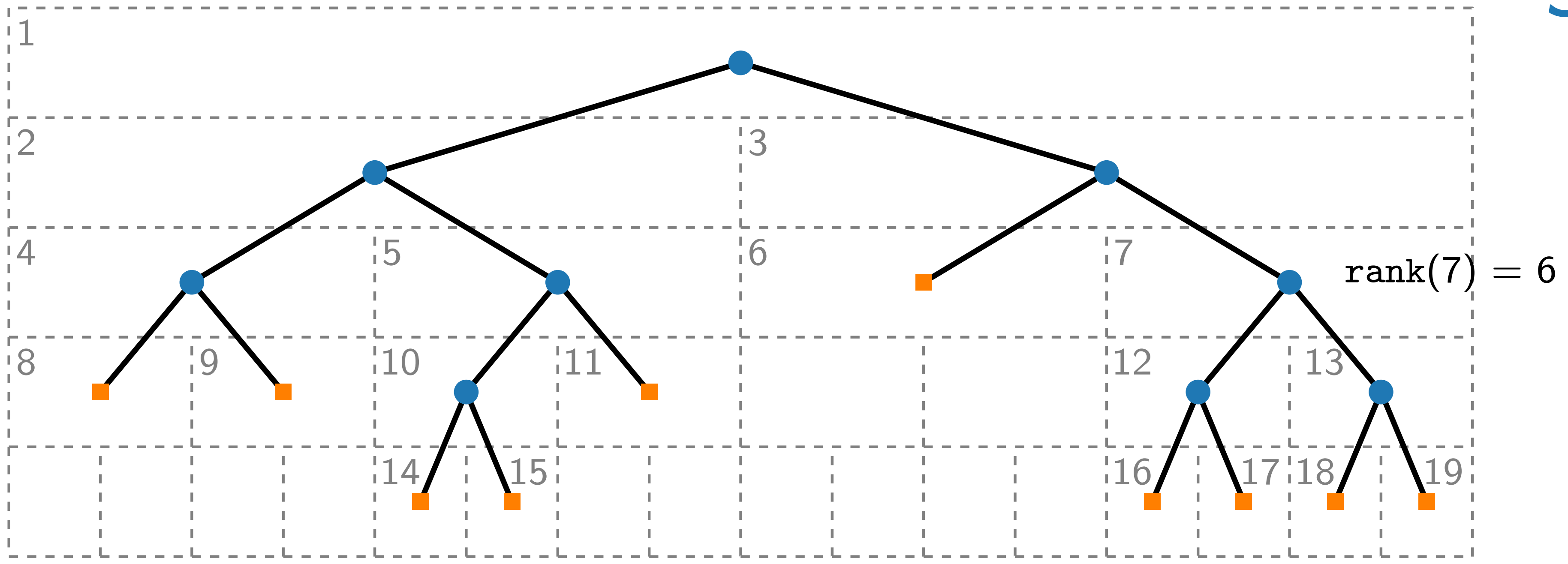
Succinct Representation of Binary Trees



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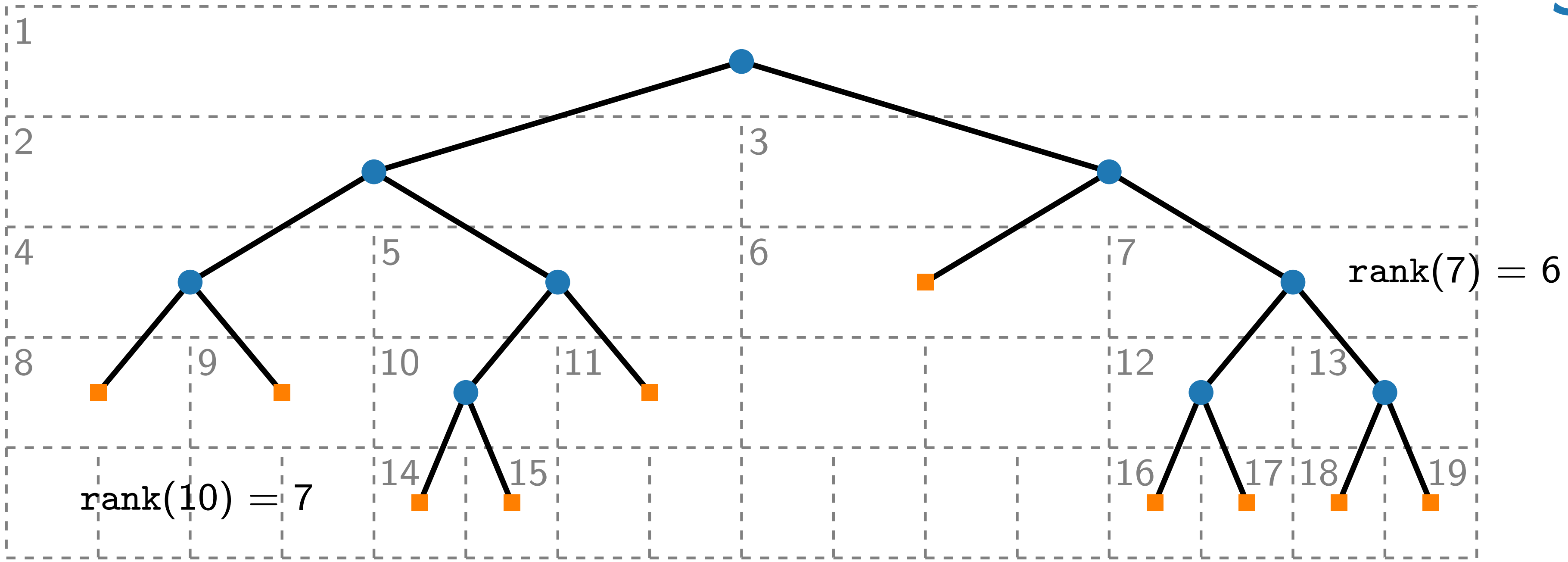

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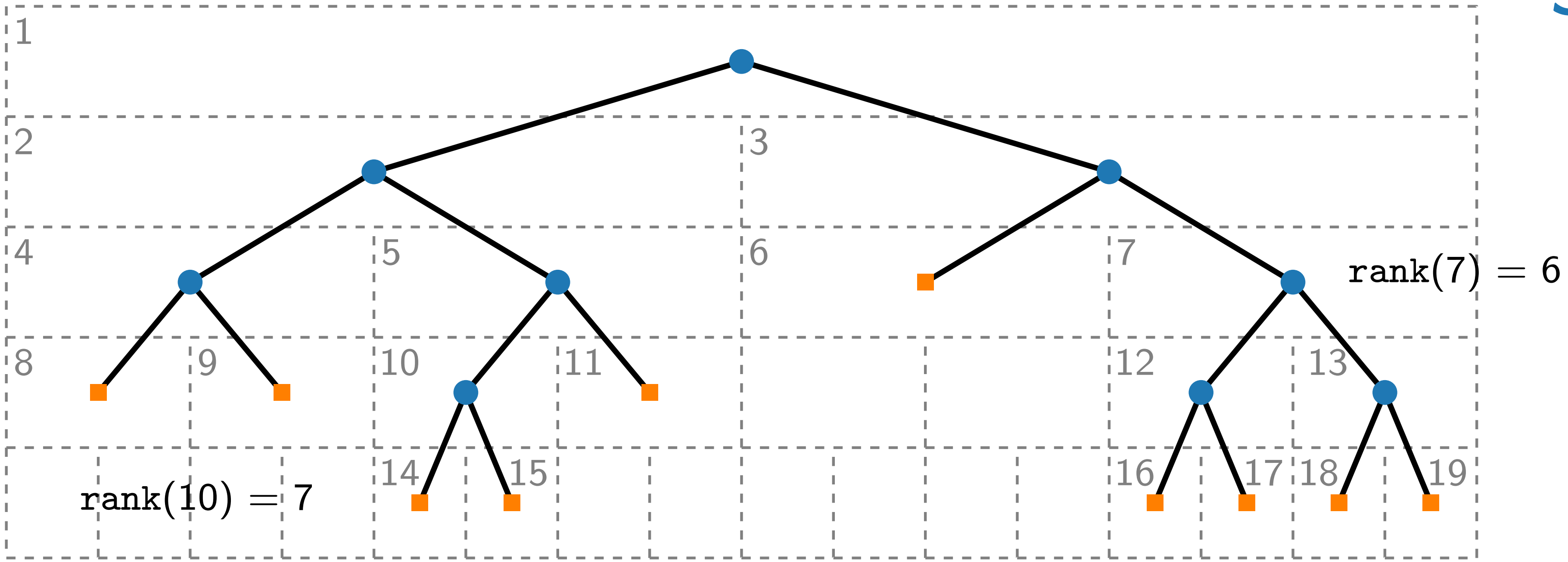

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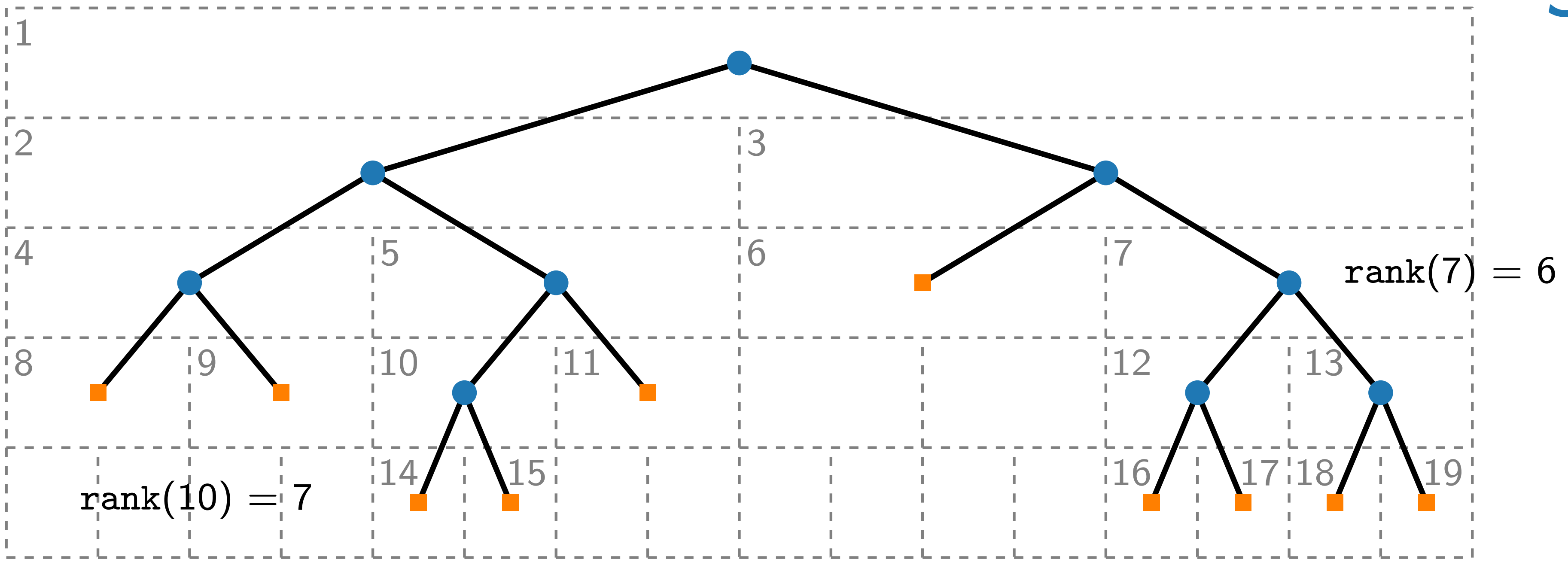
b	1	1	1	1	1	0	1	0	0	1	0	1	1	0	0	0	0	0	0

Size.
- $2n + 1$ bits for b
- $o(n)$ for `rank` and `select`

Idea.
- Add external nodes to have out-degree 2
- Read internal nodes as 1
- Read external nodes as 0
- Use `rank` and `select`

Operations.
- $\texttt{parent}(i) = \quad ?$
- $\texttt{leftChild}(i) = 2\,\texttt{rank}(i)$
- $\texttt{rightChild}(i) = 2\,\texttt{rank}(i) + 1$

Succinct Representation of Binary Trees



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Operations.

- $\texttt{parent}(i) = \texttt{select}(\lfloor \frac{i}{2} \rfloor)$
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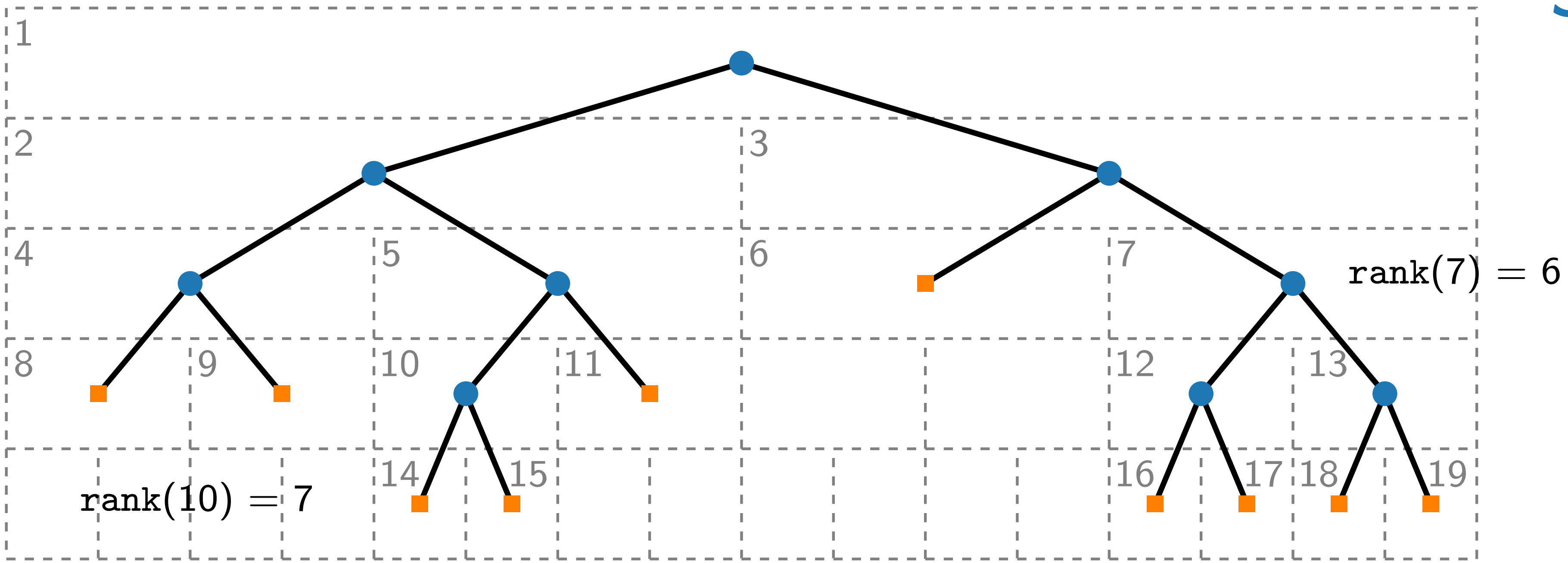

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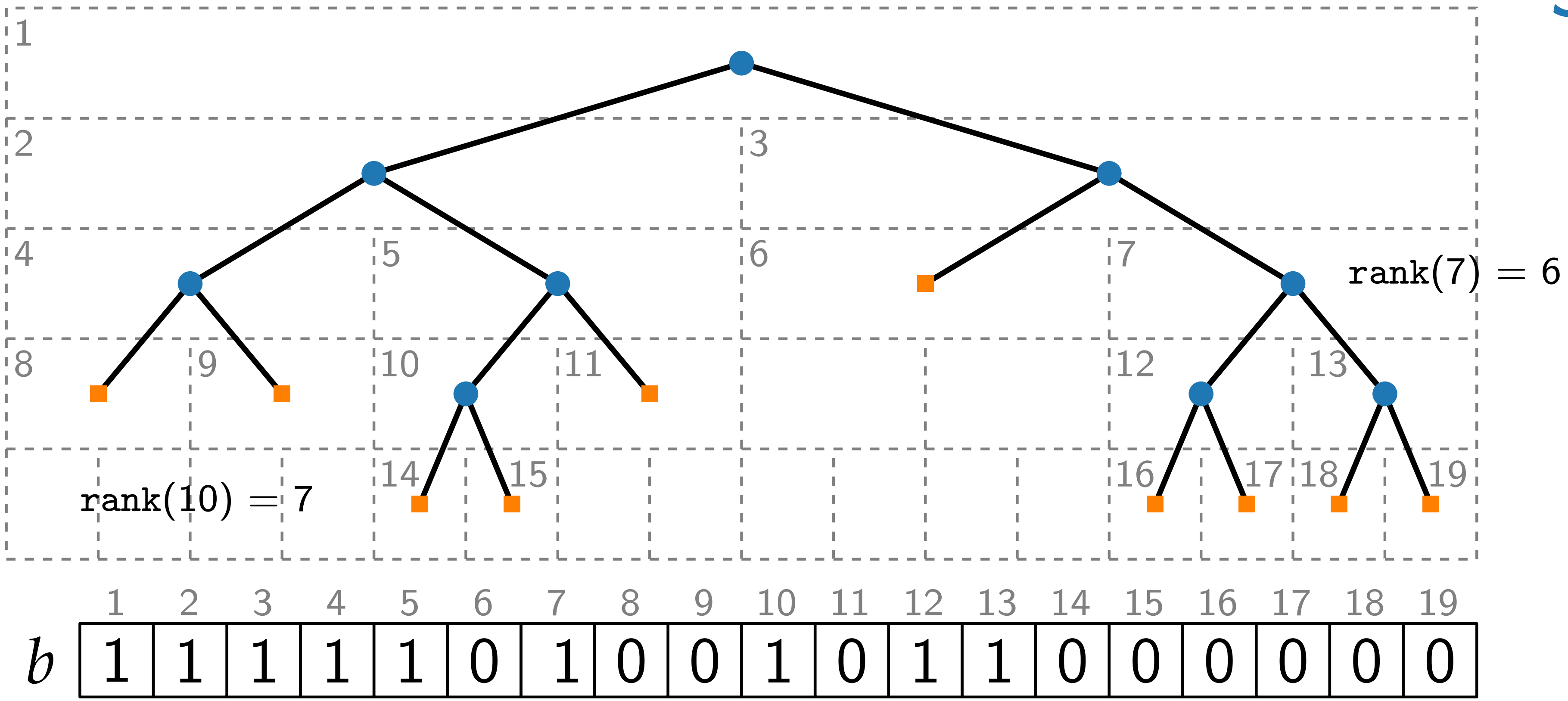
Proof is exercise.

Succinct Representation of Binary Trees



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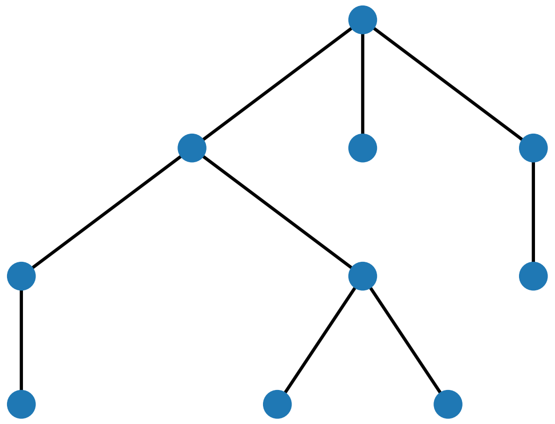
- $\texttt{rank}(i)$ is index for extra storing array

Succinct Representation of Trees - LOUDS

LOUDS = Level Order Unary Degree Sequence



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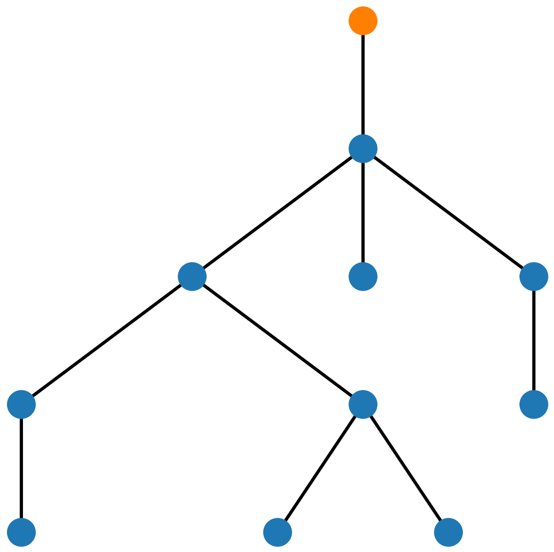

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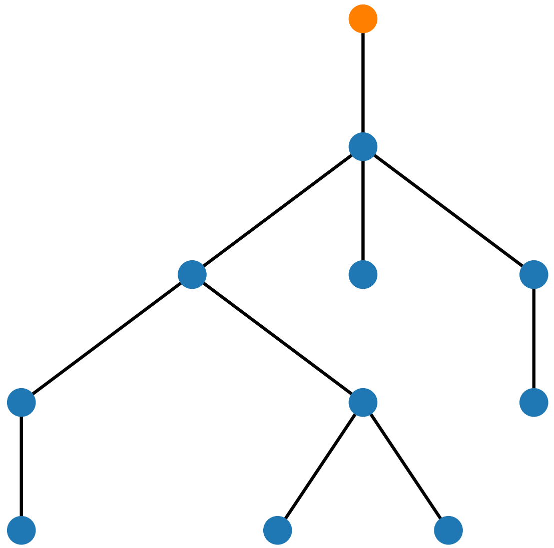

- unary decoding of outdegree

Succinct Representation of Trees - LOUDS

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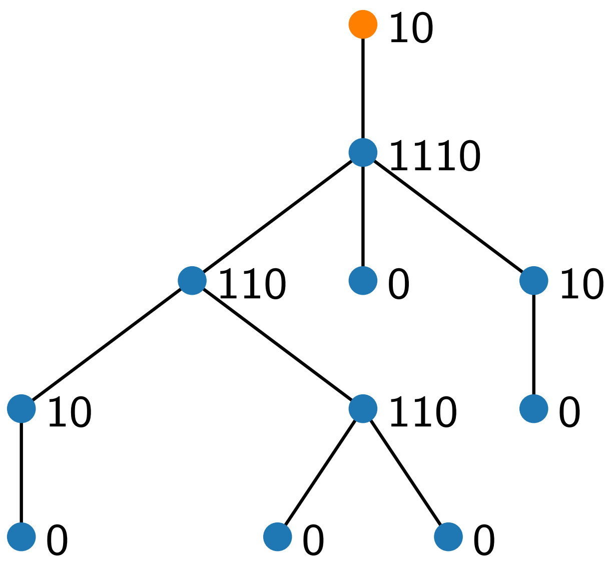

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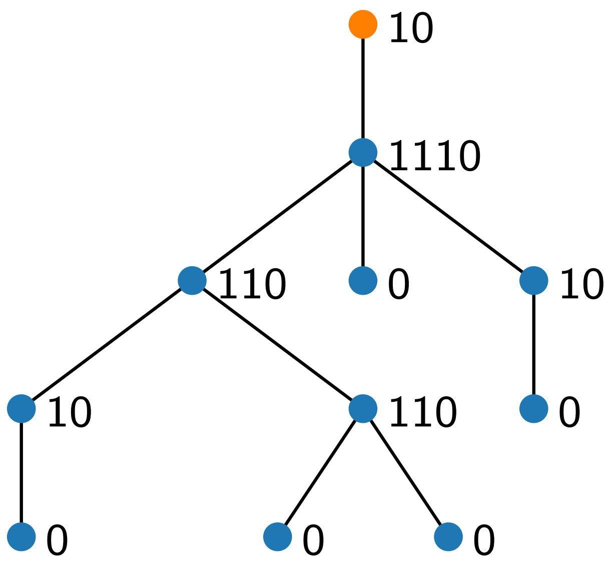

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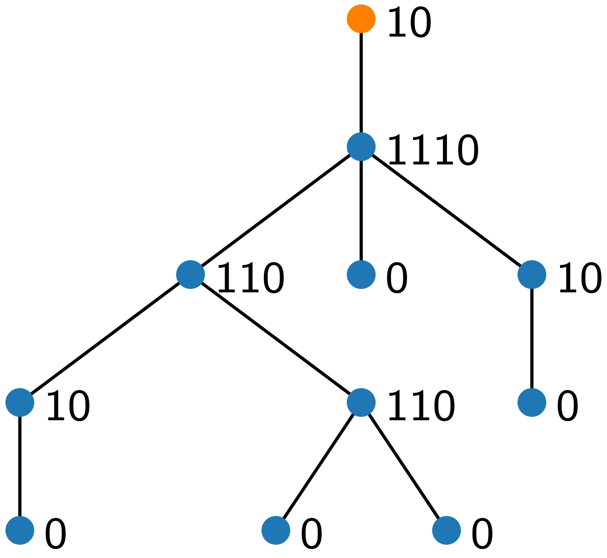

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Size.

- each vertex (except root) is represented twice, namely with a 1 and with a 0
- $o(n)$ bits for `rank` and `select`

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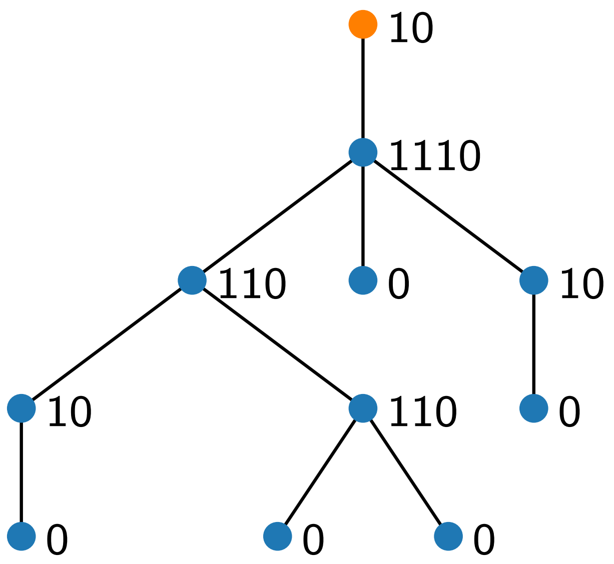

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$\Rightarrow 2n + o(n)$ bits

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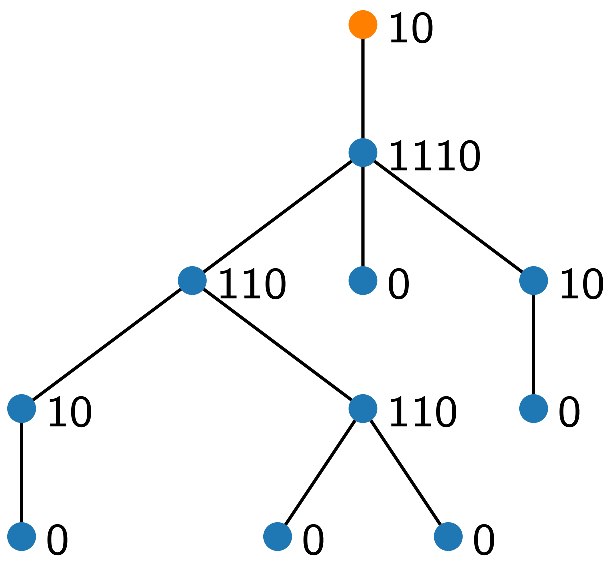

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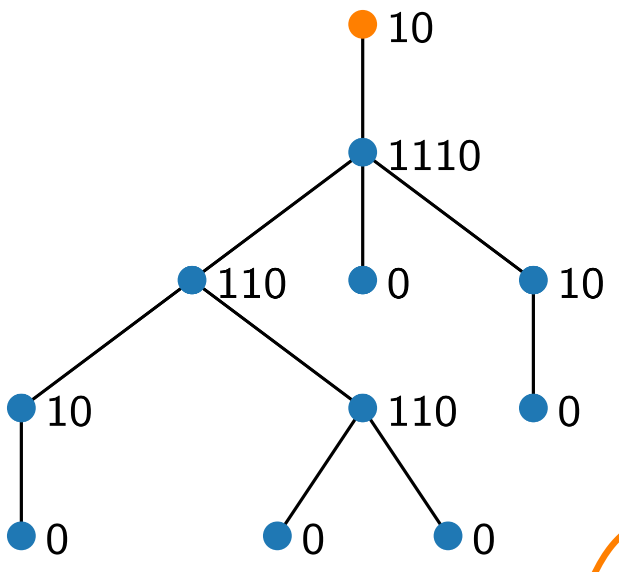
Operations.

- Let i be index of 1 in LOUDS sequence.
- $\texttt{rank}(i)$ is index for array storing vertex objects/values.

Succinct Representation of Trees - LOUDS

LOUDS = Level Order Unary Degree Sequence



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execute $\mathtt{select}(j)$ on the 0s instead of the 1s

execute $\mathtt{rank}(i)$ on the 1s (as before)

- $\mathtt{firstChild}(i) = \mathtt{select}_0(\mathtt{rank}_1(i)) + 1$

Succinct Representation of Trees - LOUDS

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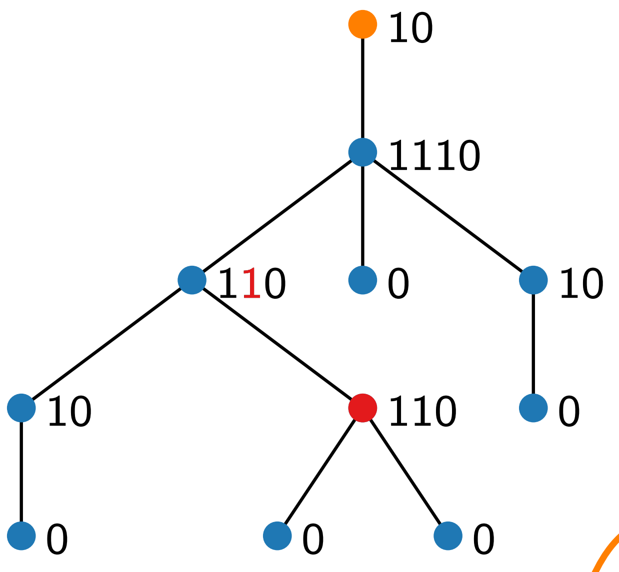

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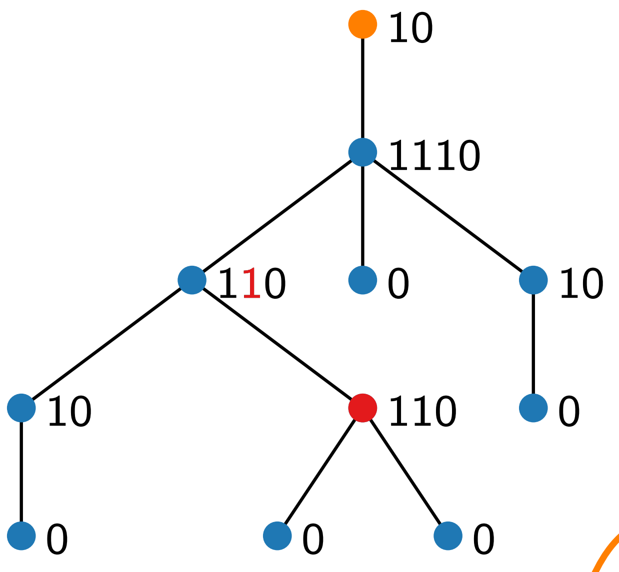

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$$= \mathtt{select}_0(6) + 1$$

Succinct Representation of Trees - LOUDS

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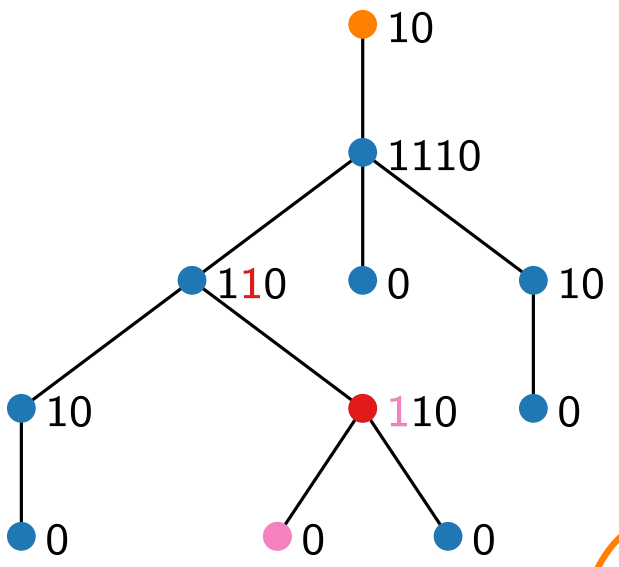

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$= \texttt{select}_0(6) + 1 = 14 + 1 = 15$

Succinct Representation of Trees - LOUDS

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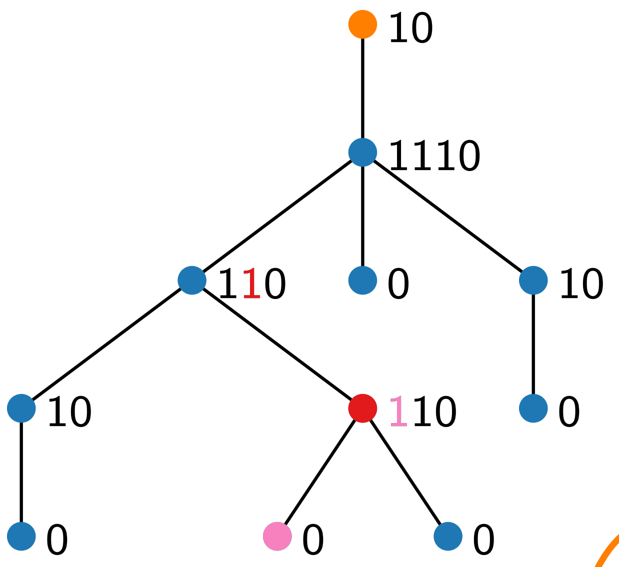

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- $\texttt{nextSibling}(i) = i + 1$

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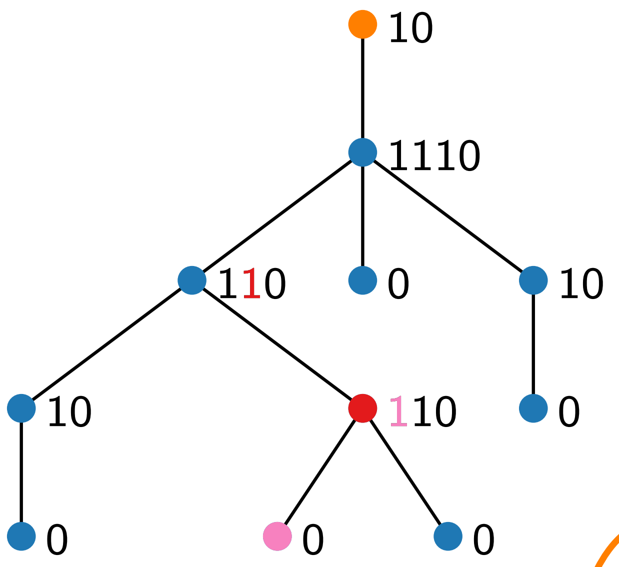

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Exercise: $\mathtt{child}(i, j)$ with validity check

Succinct Representation of Trees - LOUDS

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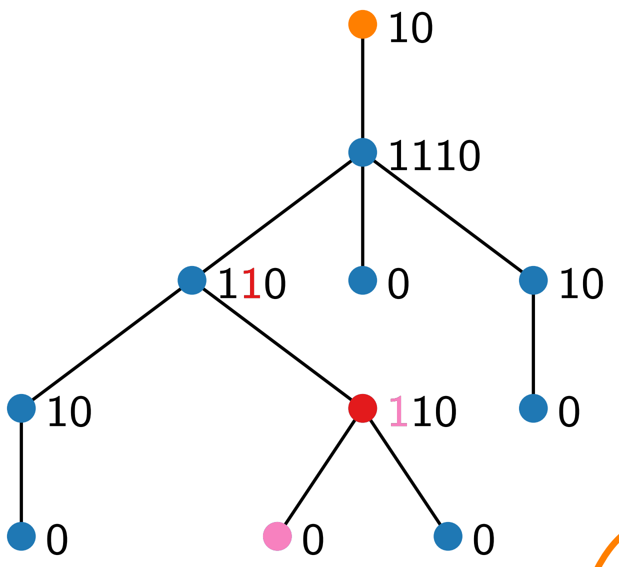

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$\texttt{firstChild}(8) = \texttt{select}_0(\texttt{rank}_1(8)) + 1$
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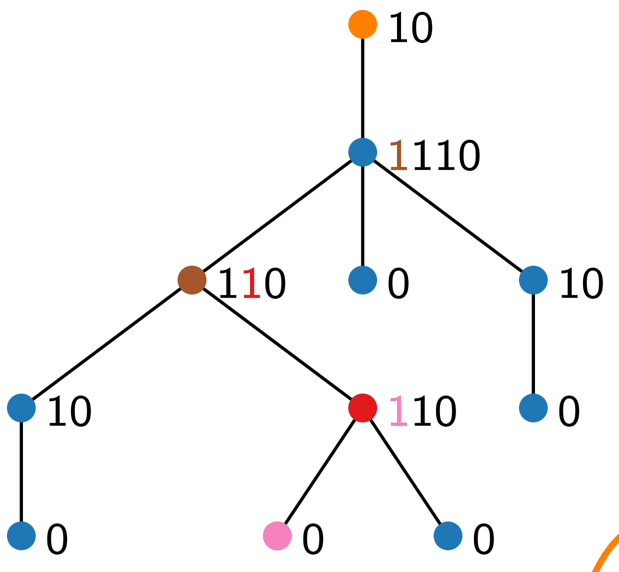

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- $\texttt{parent}(i) = \texttt{select}_1(\texttt{rank}_0(i))$

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- $\texttt{nextSibling}(i) = i + 1$

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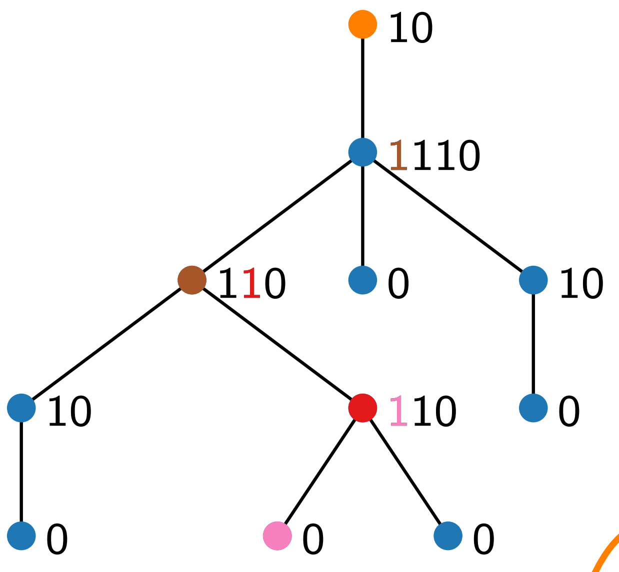
Succinct Representation of Trees - LOUDS

LOUDS = Level Order Unary Degree Sequence



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- $\texttt{parent}(i) = \texttt{select}_1(\texttt{rank}_0(i))$

  $\texttt{parent}(8) = \texttt{select}_1(\texttt{rank}_0(8))$
  $= \texttt{select}_1(2)$

- $\texttt{nextSibling}(i) = i + 1$

Exercise: $\texttt{child}(i, j)$ with validity check

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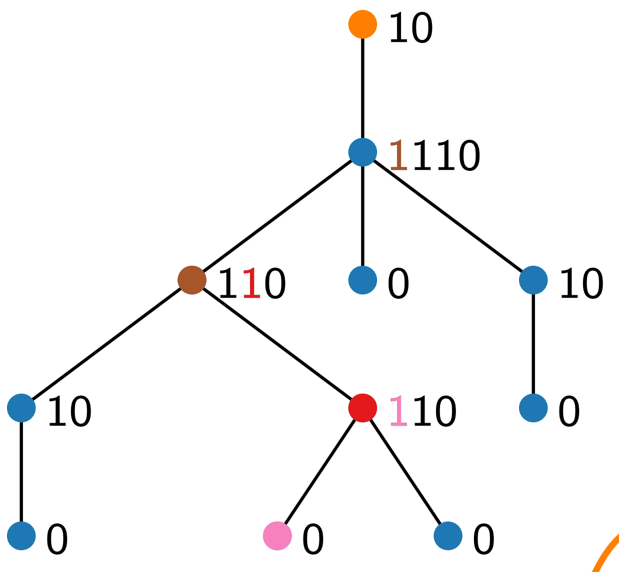

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- $\texttt{nextSibling}(i) = i + 1$

- $\texttt{parent}(i) = \texttt{select}_1(\texttt{rank}_0(i))$

  $\texttt{parent}(8) = \texttt{select}_1(\texttt{rank}_0(8))$
  $= \texttt{select}_1(2) = 3$

Exercise: $\texttt{child}(i, j)$
with validity check

Discussion

- Succinct data structures are
  - space efficient
  - support fast operations

  but
  - are mostly static (dynamic at extra cost),
  - number of operations is limited,
  - complex $\rightarrow$ harder to implement

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  - support fast operations

  but
  - are mostly static (dynamic at extra cost),
  - number of operations is limited,
  - complex $\rightarrow$ harder to implement
- Rank and select form basis for many succinct representations

Literature

Main reference:

- Lecture 17 of Advanced Data Structures (MIT, Fall'17) by Erik Demaine
- [Jac '89] "Space efficient Static Trees and Graphs"

Recommendations:

- Lecture 18 of Demaine's course on compact & succinct arrays & trees