Advanced Algorithms

Online Algorithms

Ski-Rental Problem and Paging

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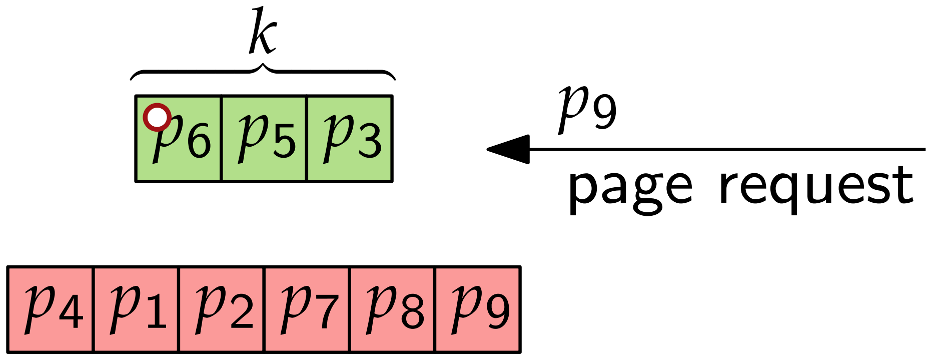

Introduction

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- Is it worth **buying** new skis?
- Or should we rather **rent** them?

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this means the skiing season is back!

- But what if there is not always enough snow?
- Is it worth **buying** new skis?
- Or should we rather **rent** them?
- We don't know the weather (much) in advance.

Ski-Rental Problem – Definition

Behavior.

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Task.

- Not knowing T, devise a strategy if and when to buy skis.

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not better than the deterministic Strategy III

Ski-Rental Problem – Strategy IV

Renting costs 1 per day
Buying costs M
T good days

Can we get below this bound using randomization? – Let's try!

Strategy IV: throw a coin; **HEAD:** buy on the $\mathbf{M}$-th good day
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- With a more sophisticated probability distribution for the time we buy skis, we can expect even a competitive ratio of $\frac{e}{e-1} \approx 1.58$.

Online vs. Offline Algorithms

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Given (offline/online):

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$$\overbrace{\boxed{p_1}\boxed{p_5}\boxed{p_8}}^{k}$$

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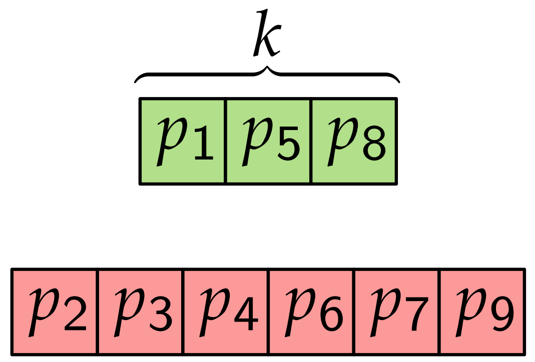

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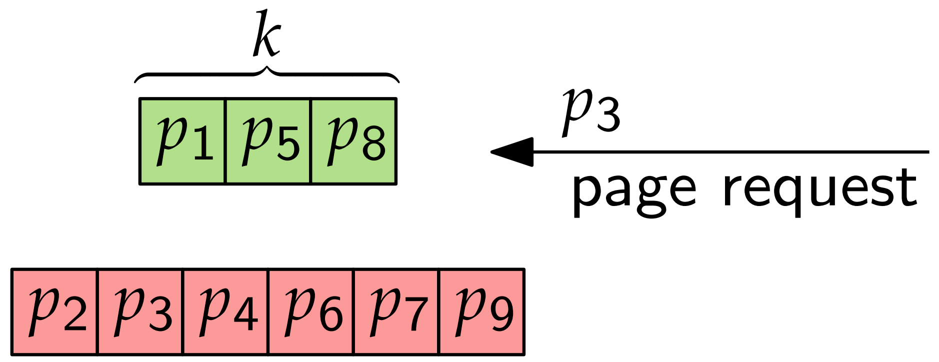

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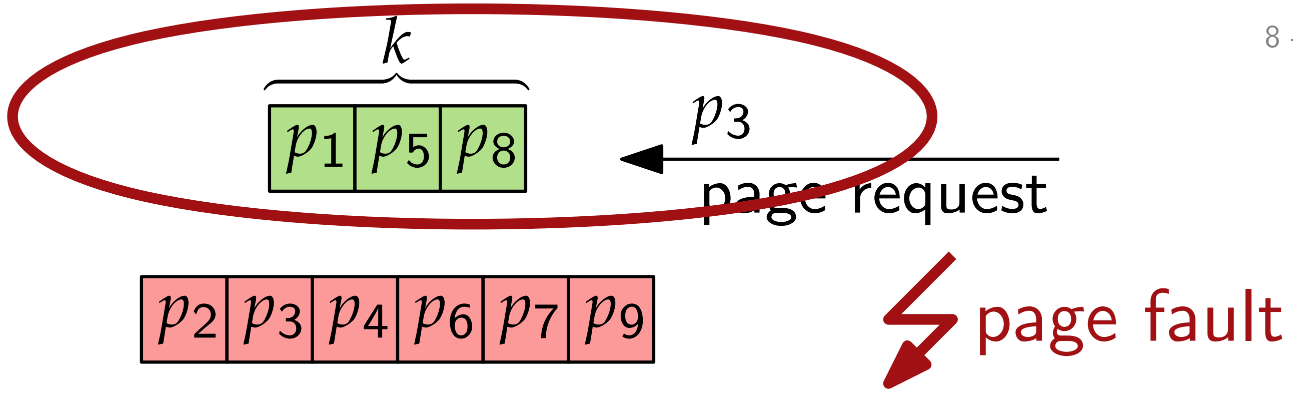

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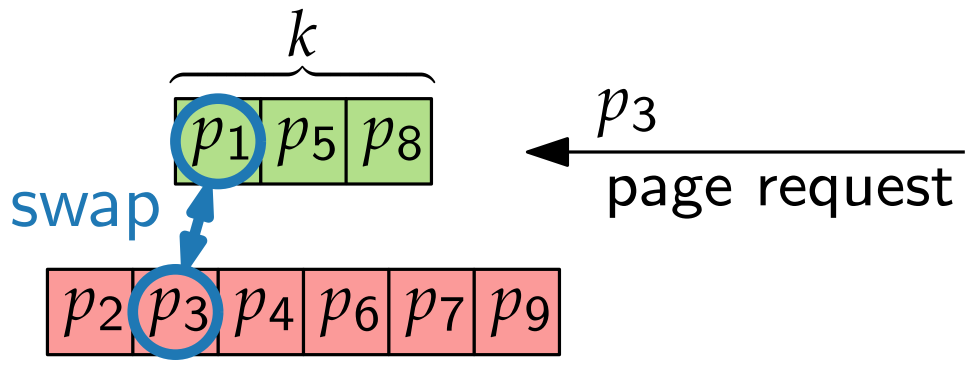

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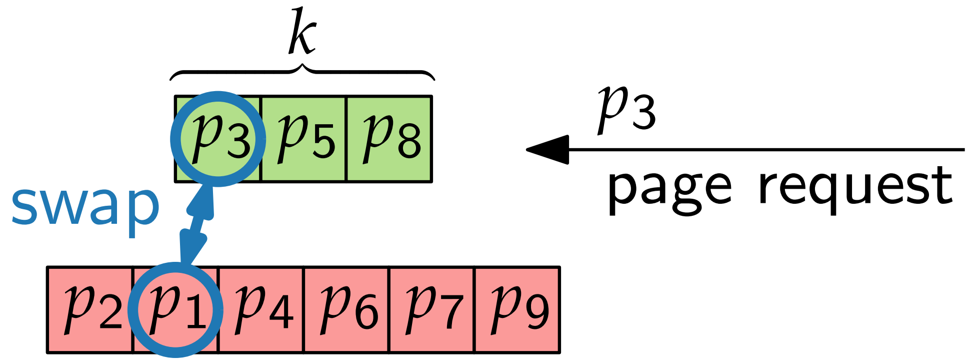

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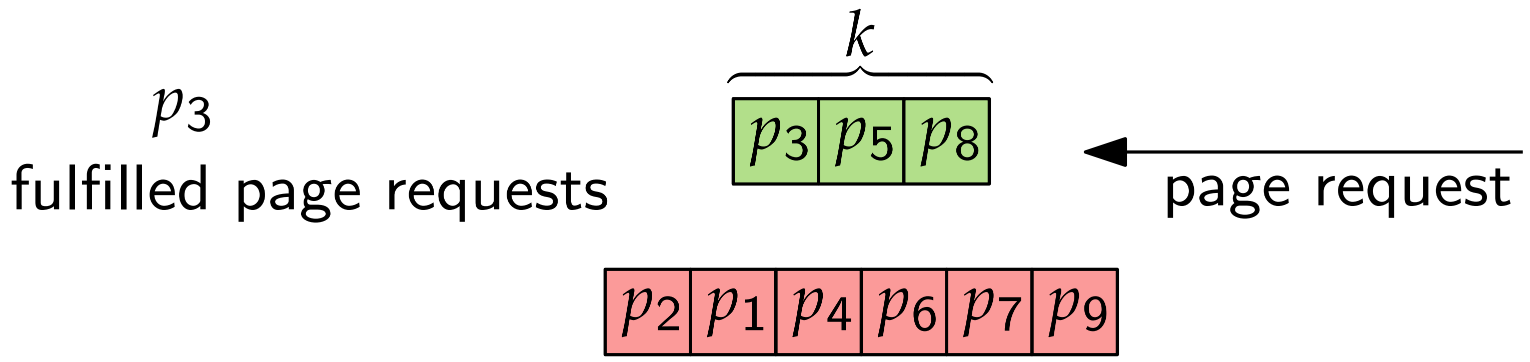

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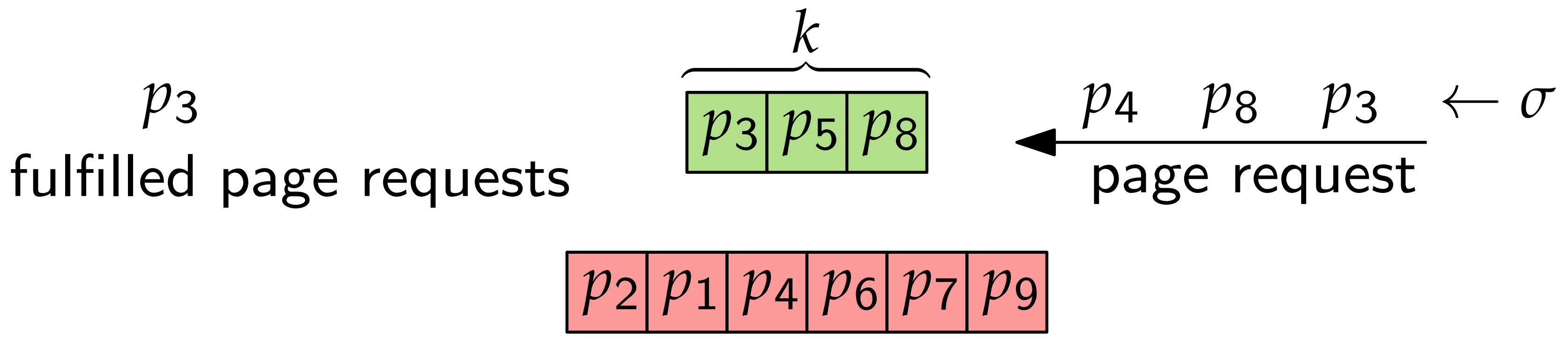
p_3
fulfilled page requests
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p_3 p_5 p_8
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page request
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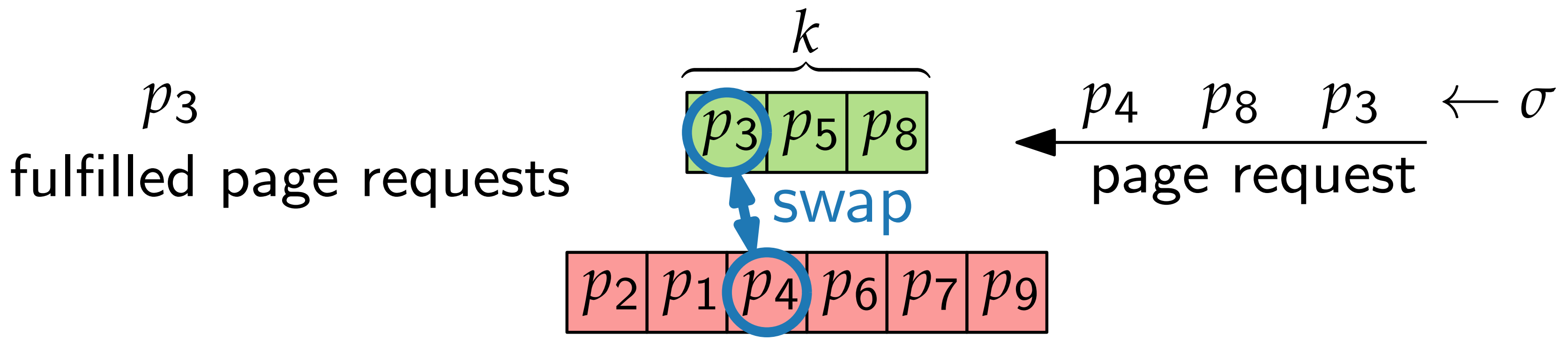
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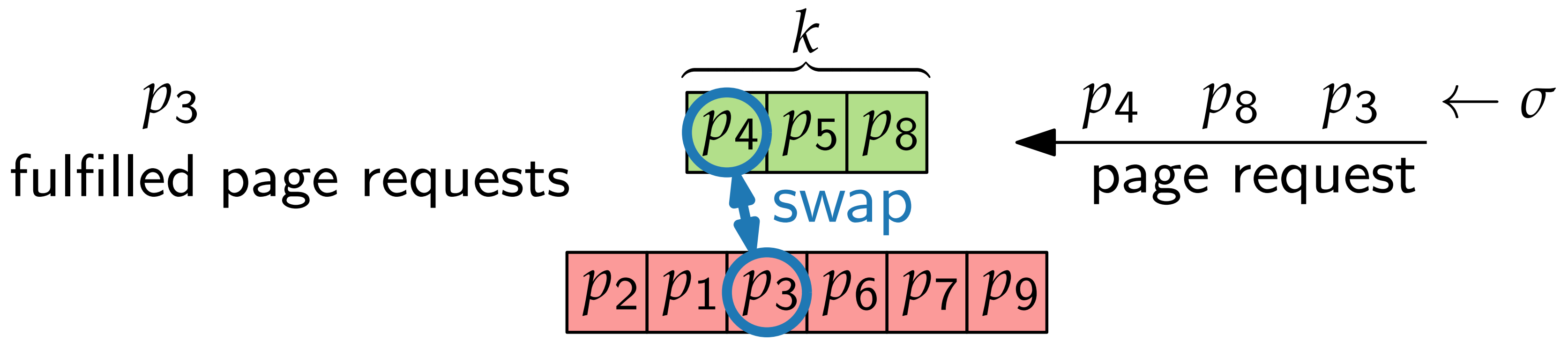
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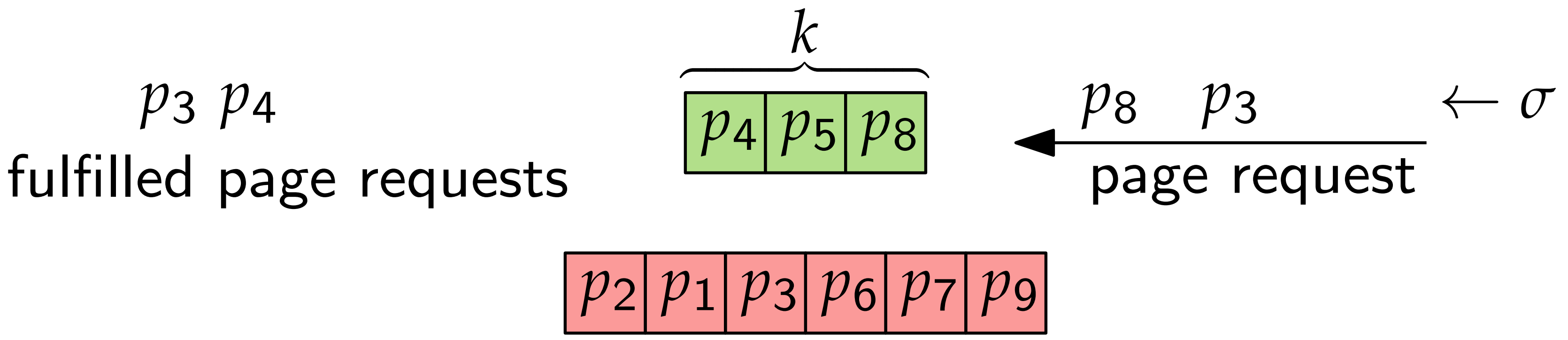

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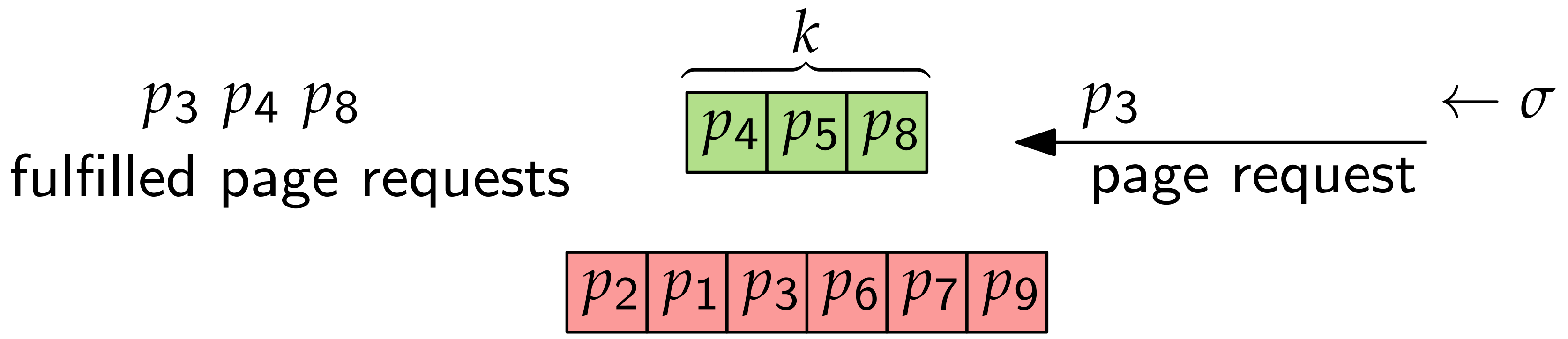

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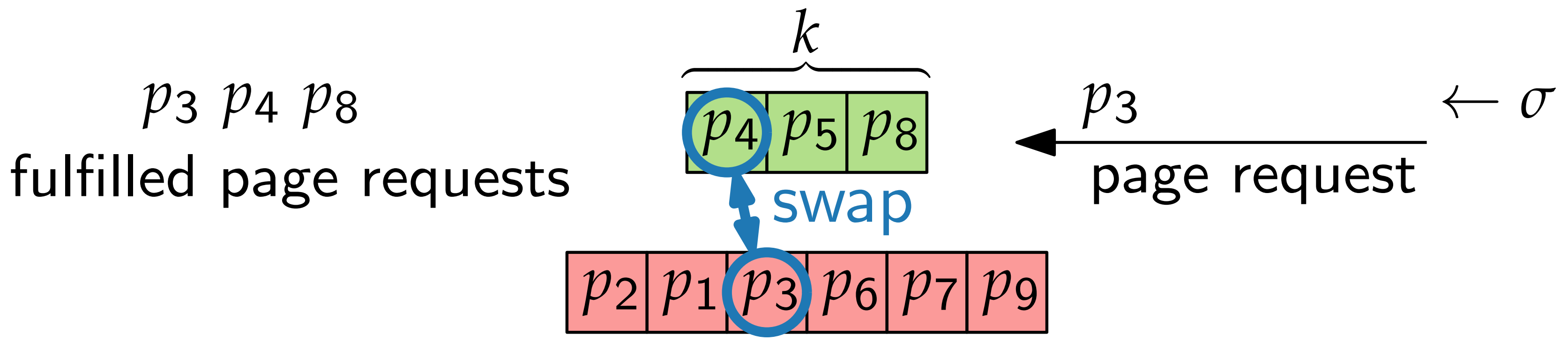

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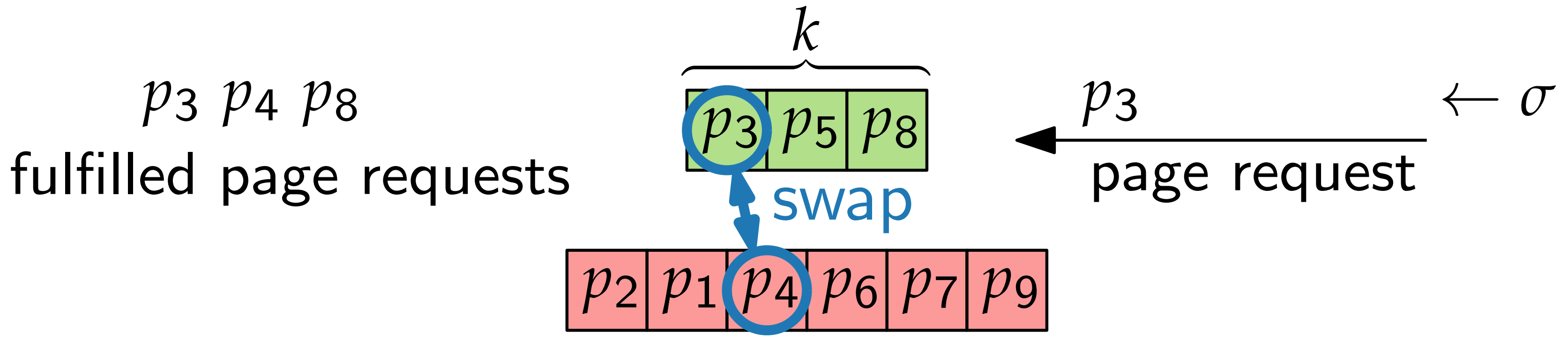

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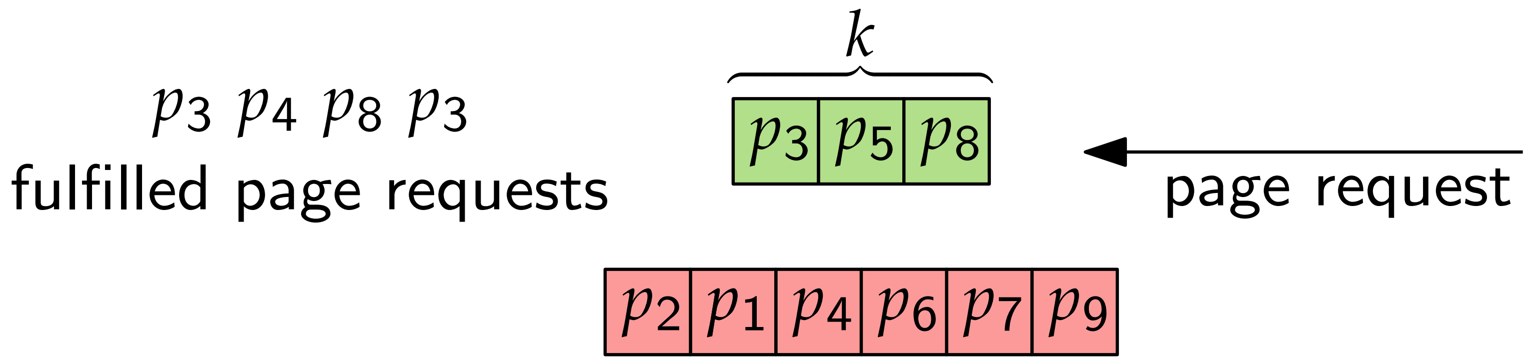

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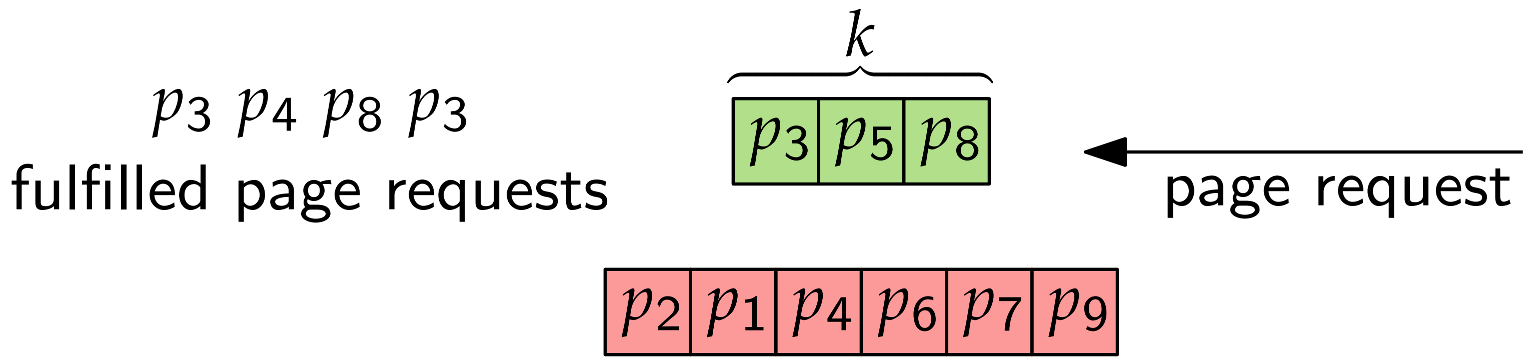

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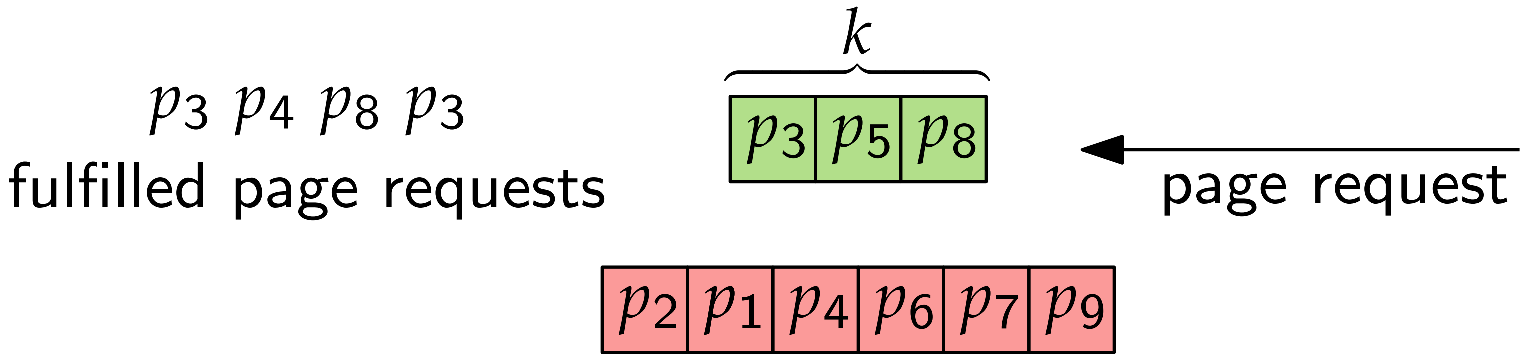

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- Minimize the number of page faults while fulfilling σ.

Paging – Det. Strat.

- On a page fault, a Paging algorithm chooses which page to evict from the cache.

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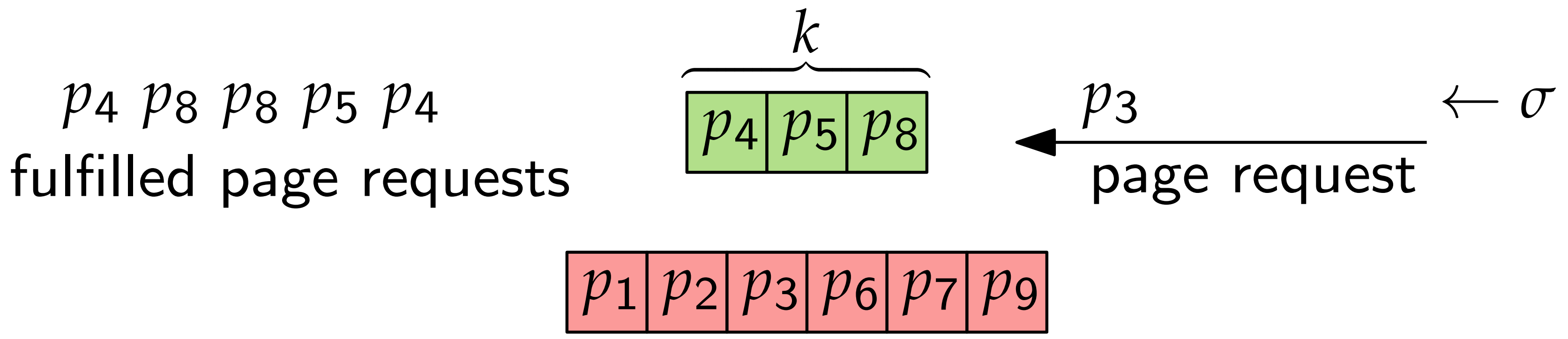

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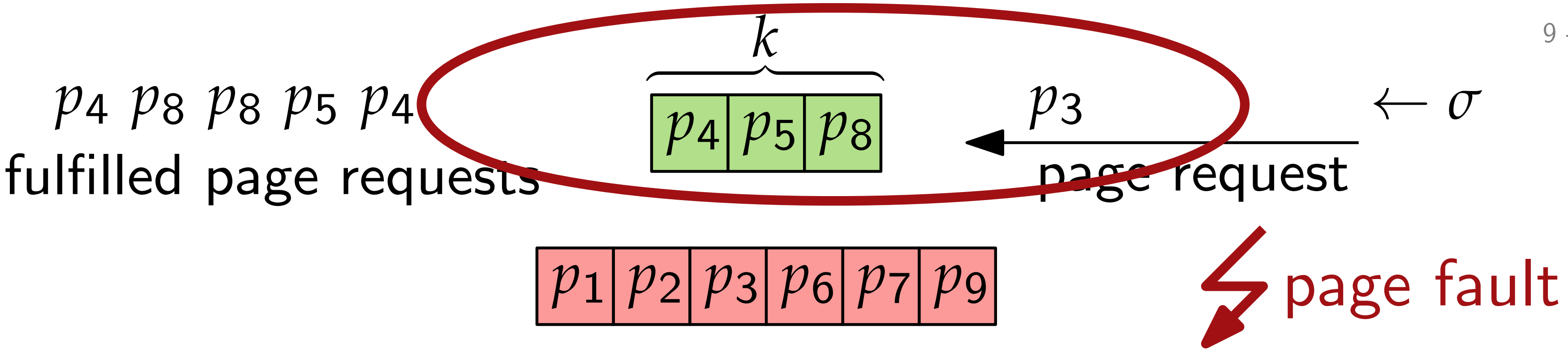

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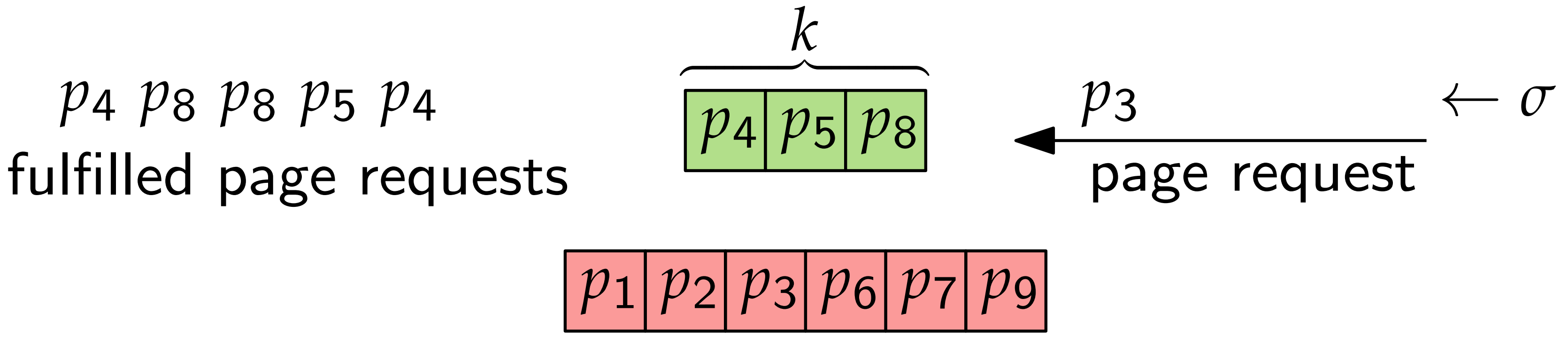

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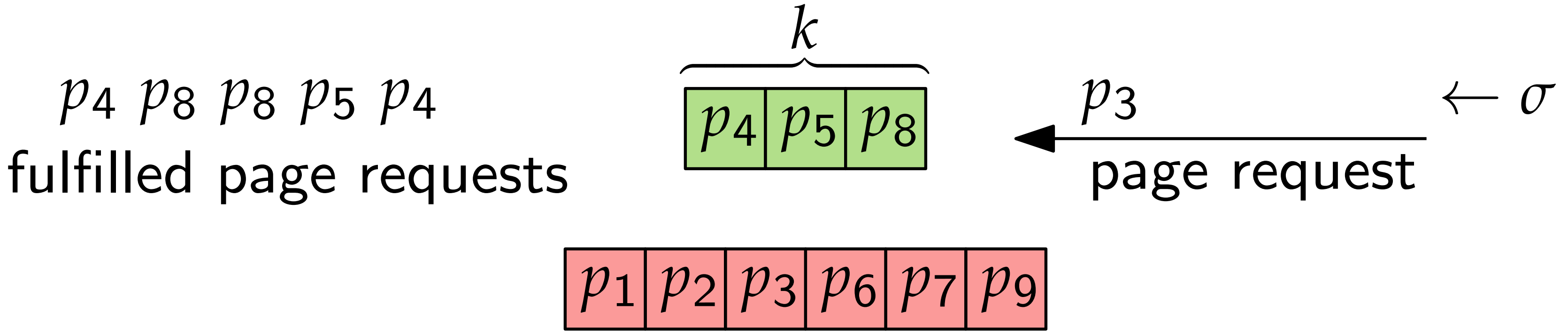

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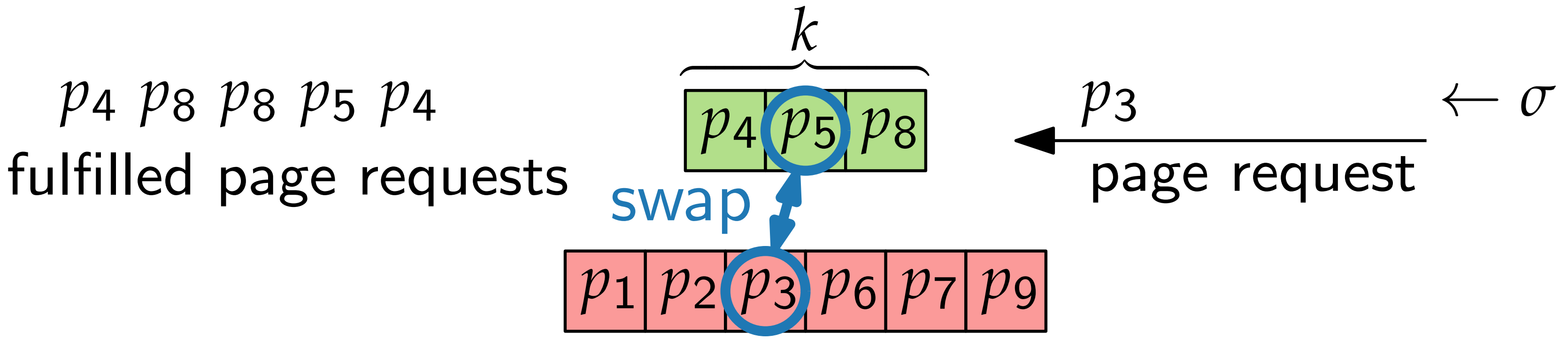

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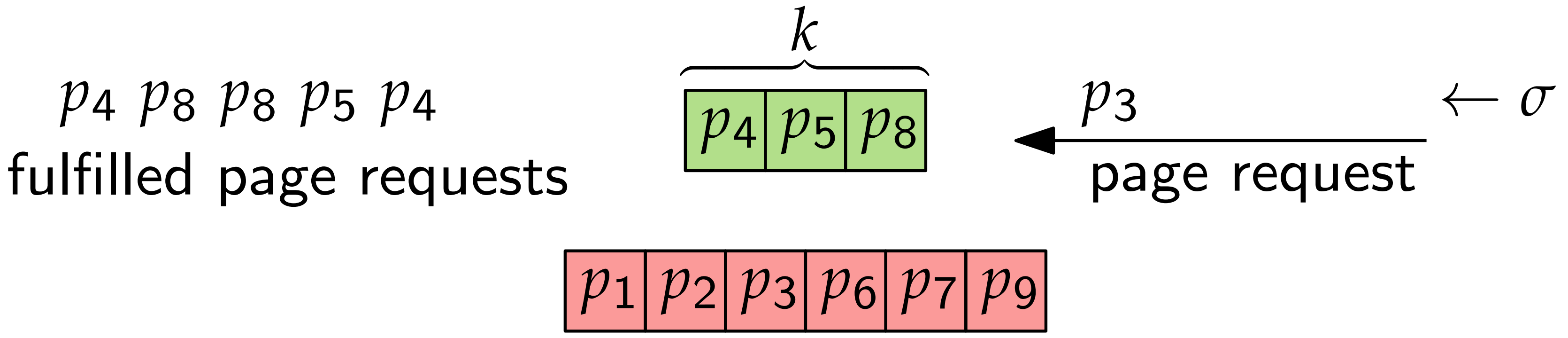

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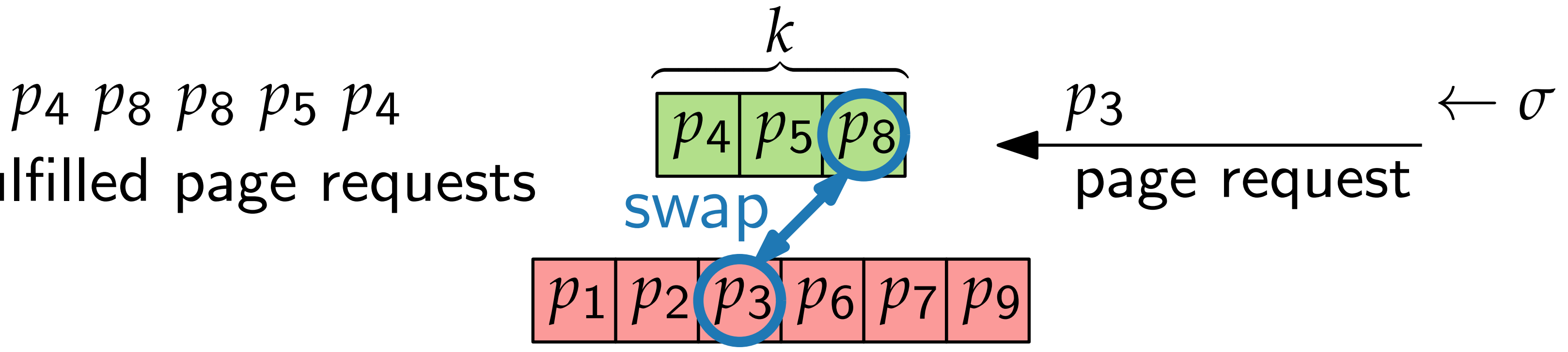

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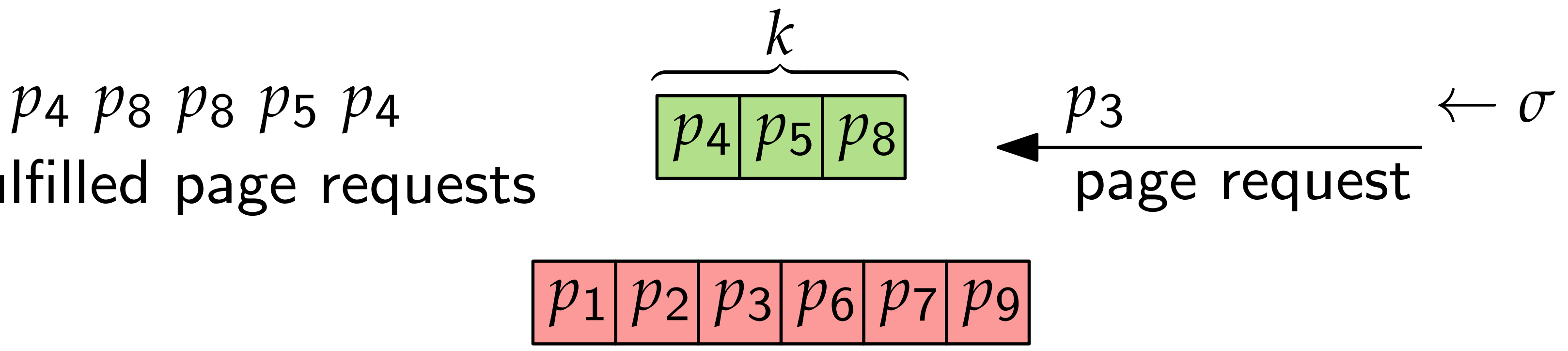

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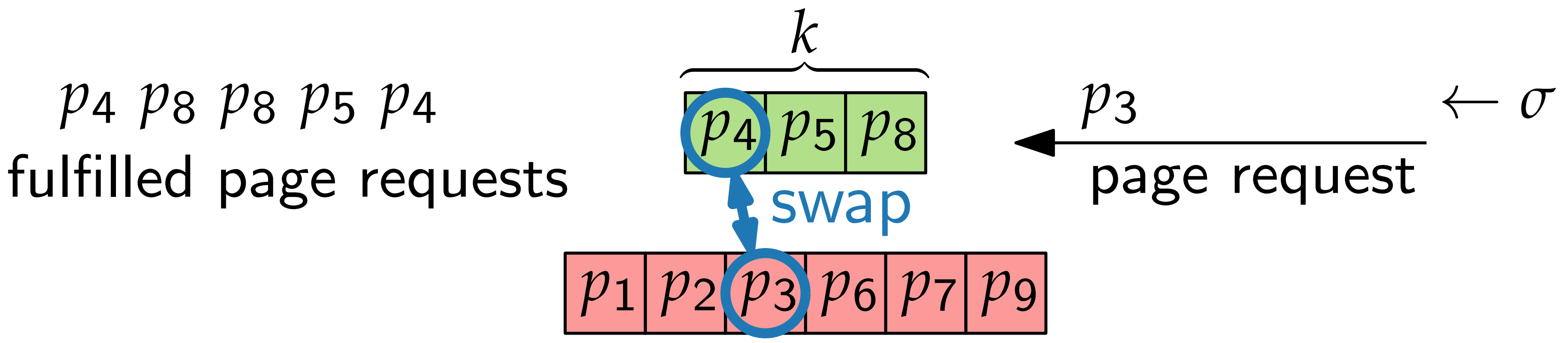
Paging – Det. Strat.



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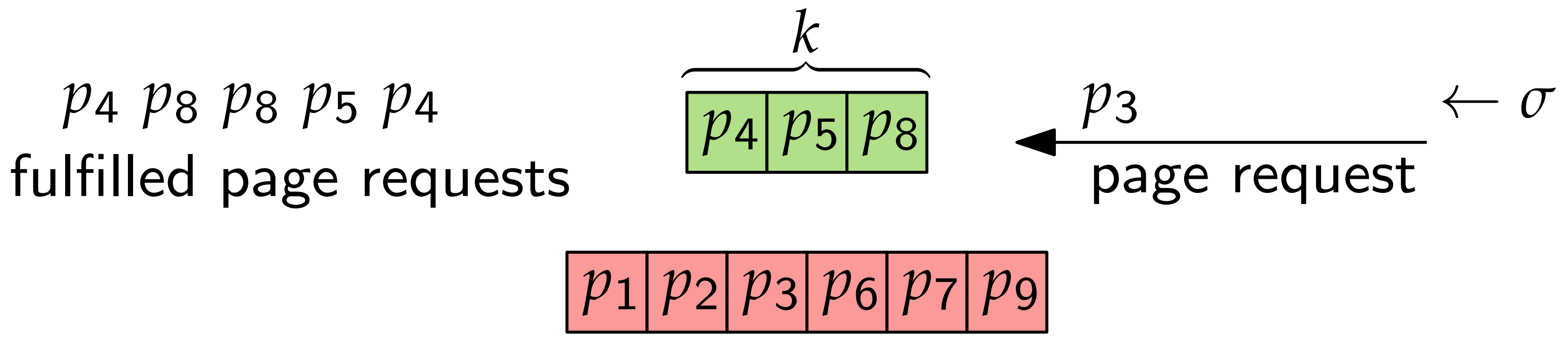

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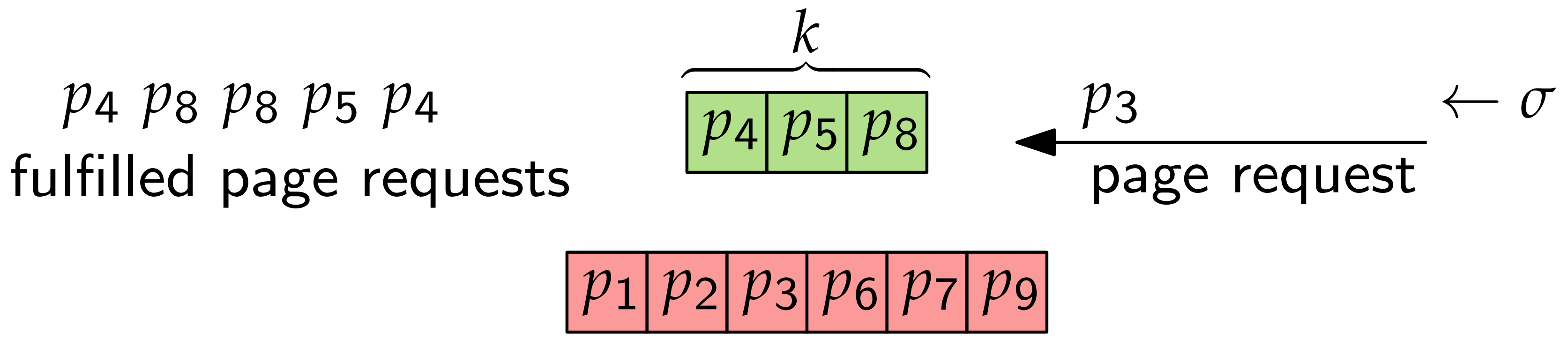

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$\Rightarrow$ The competitive ratio cannot be better than $\frac{|\sigma^*|}{\left\lceil \frac{|\sigma^*|}{k} \right\rceil} \overset{|\sigma^*| \rightsquigarrow \infty}{=} k$.

$\square$

Paging – Rand. Strat.

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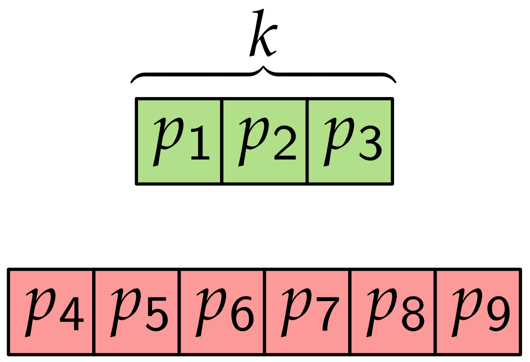

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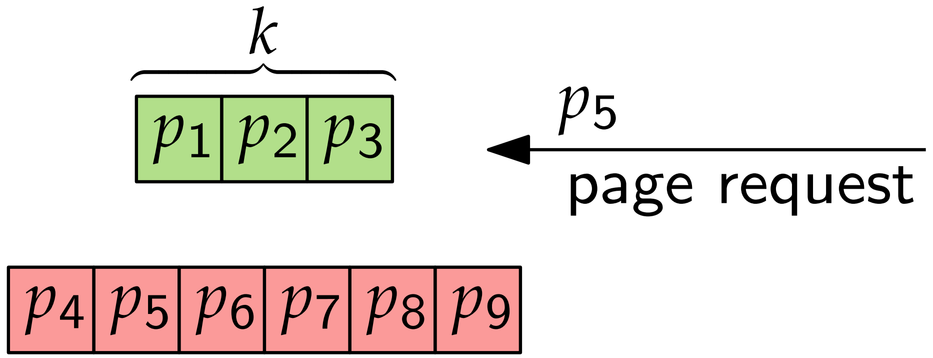

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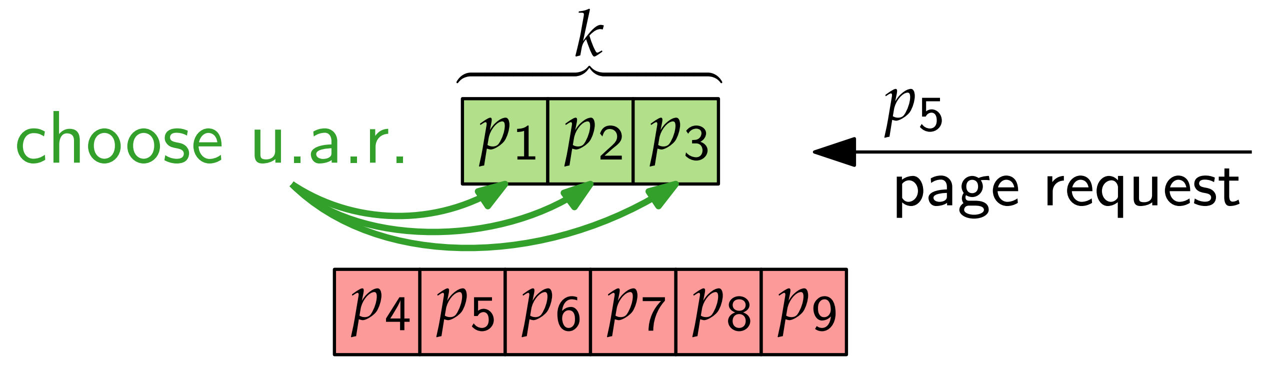

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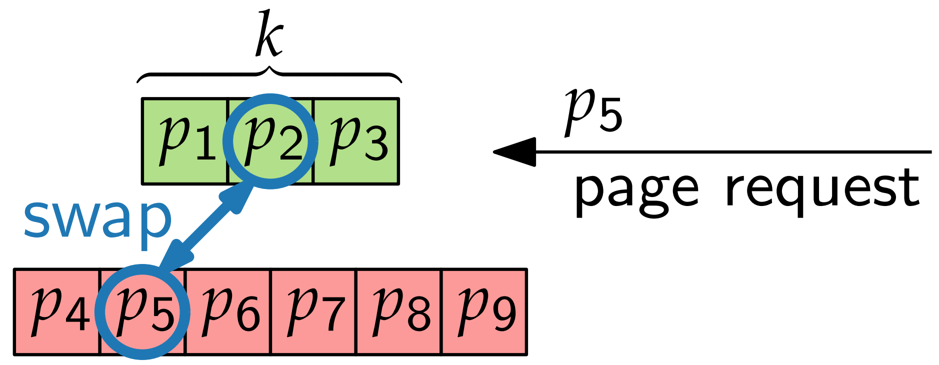

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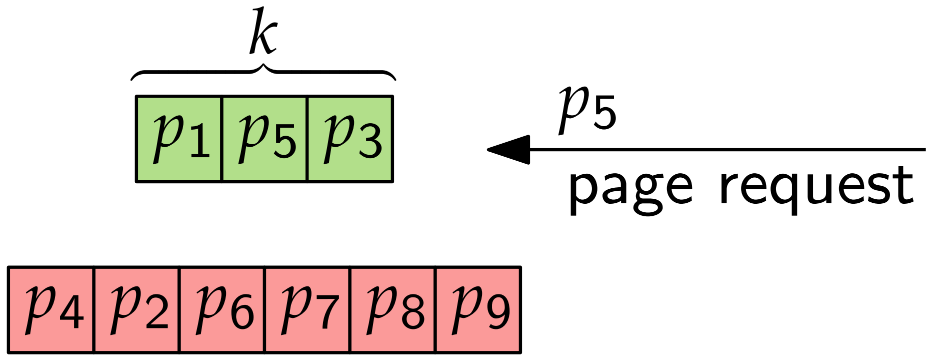

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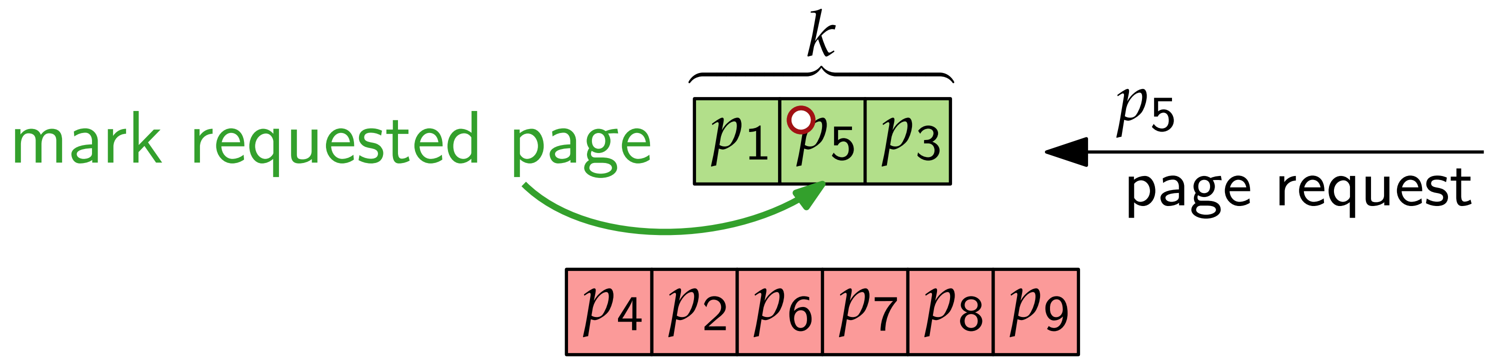

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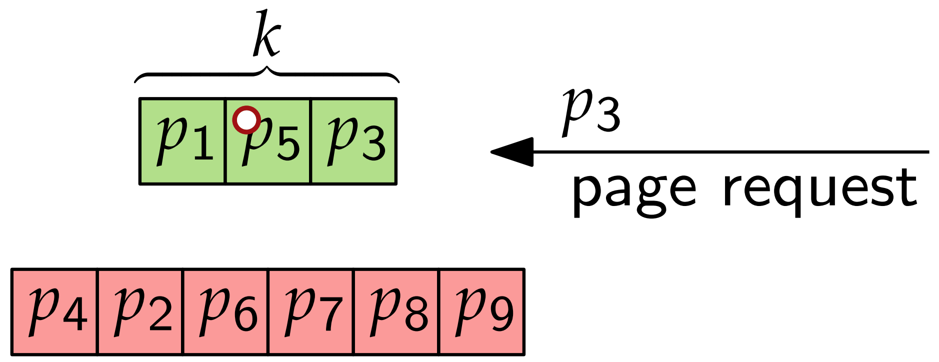

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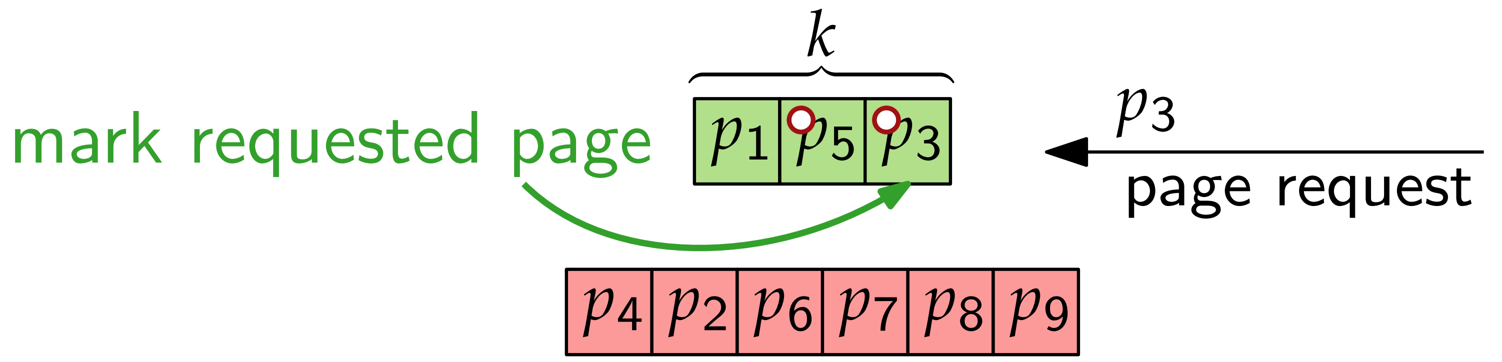

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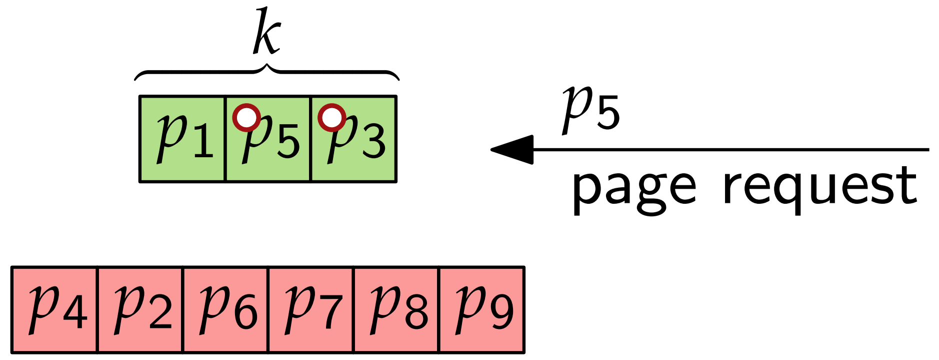

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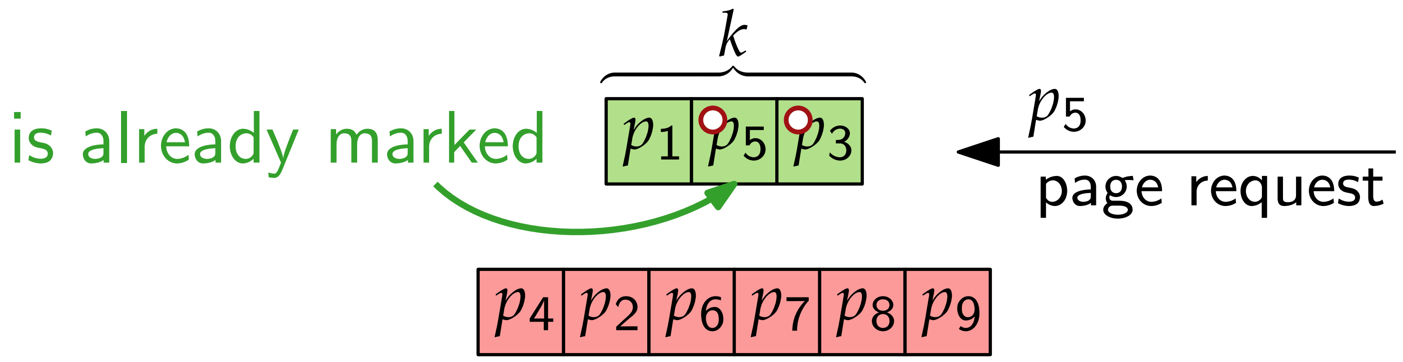

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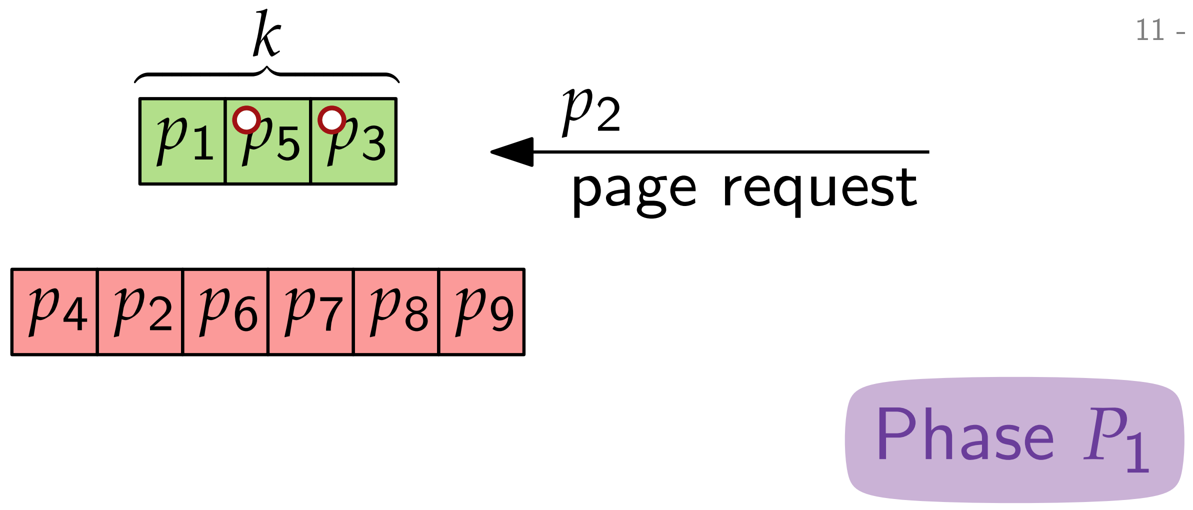

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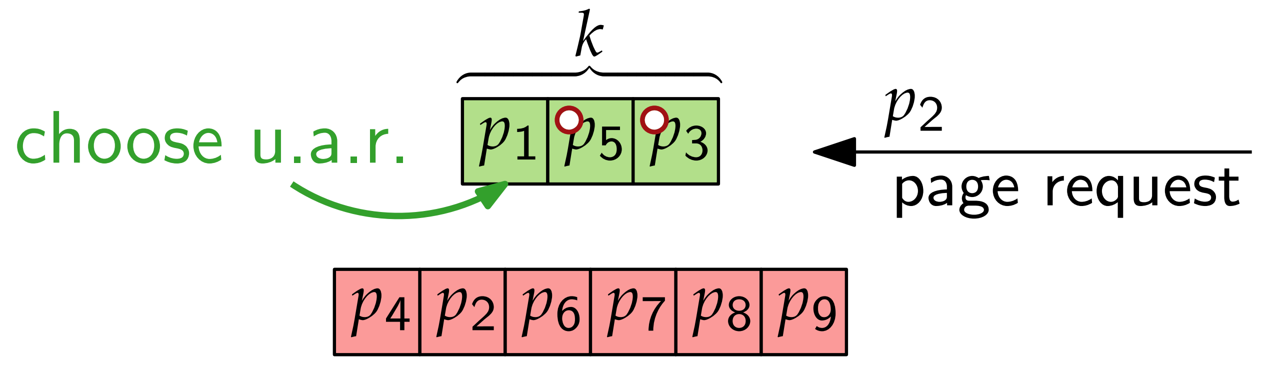

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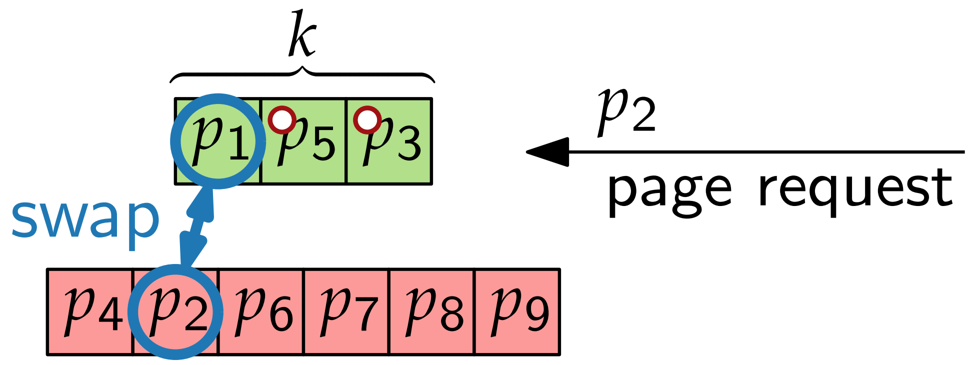

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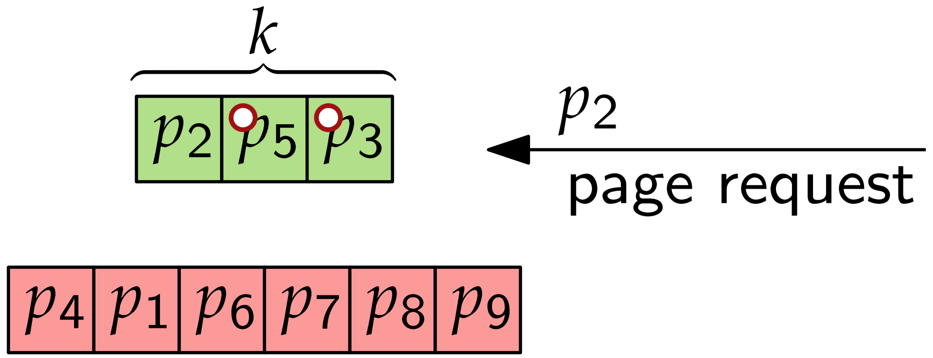

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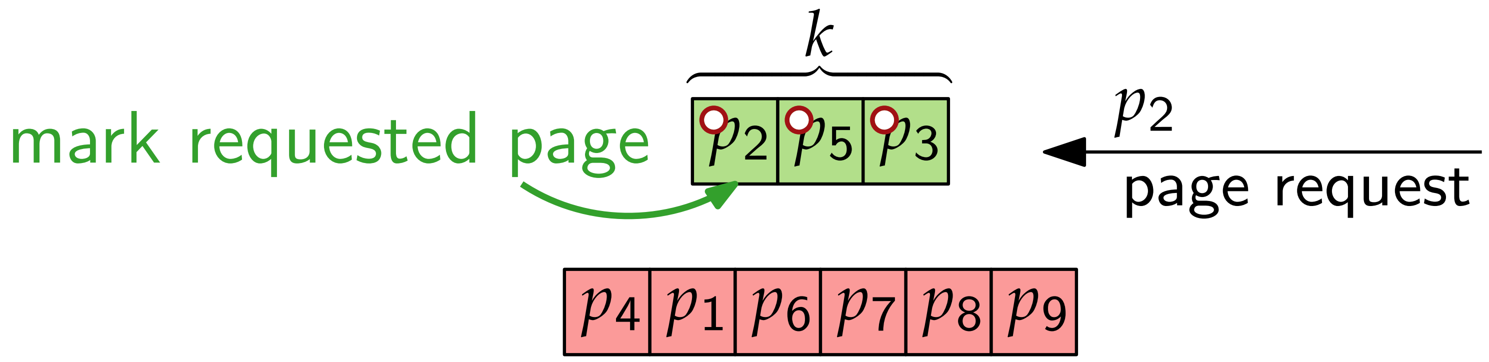

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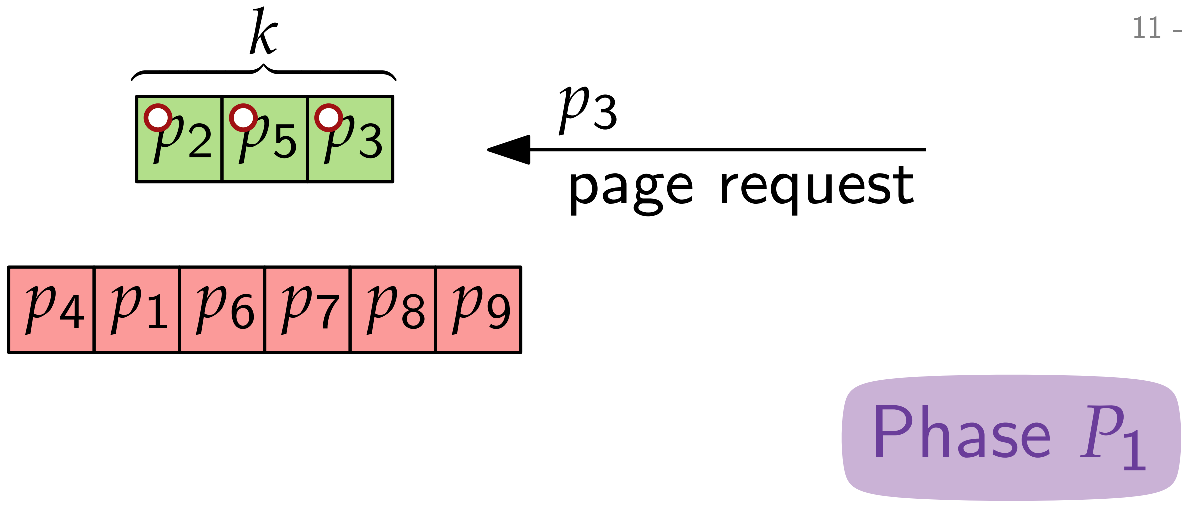

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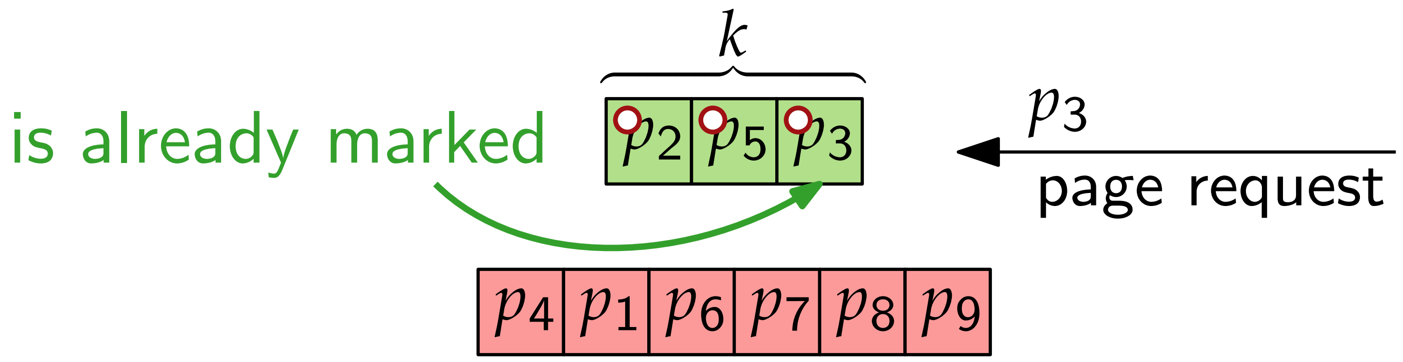

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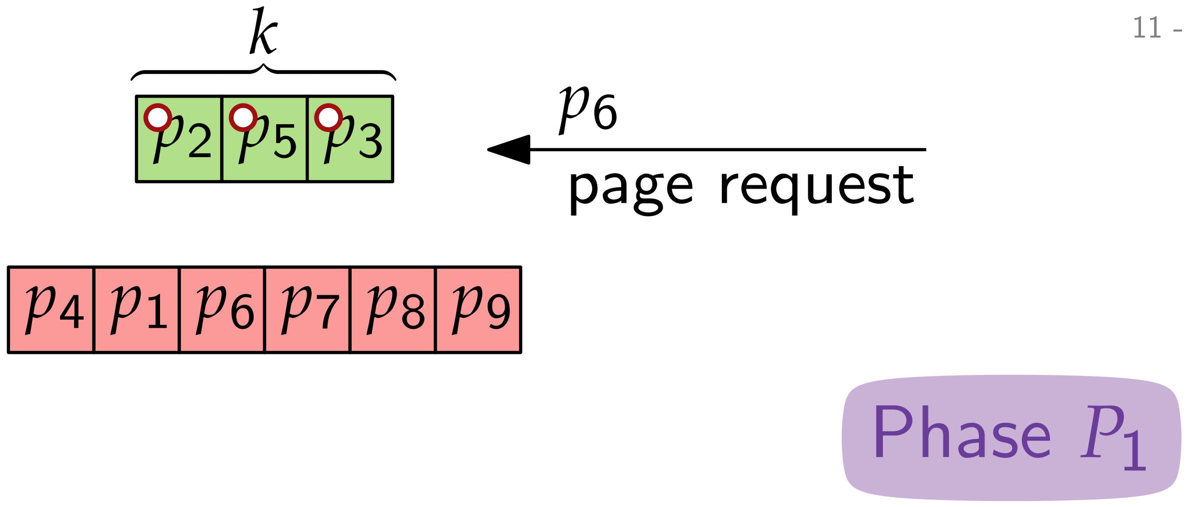

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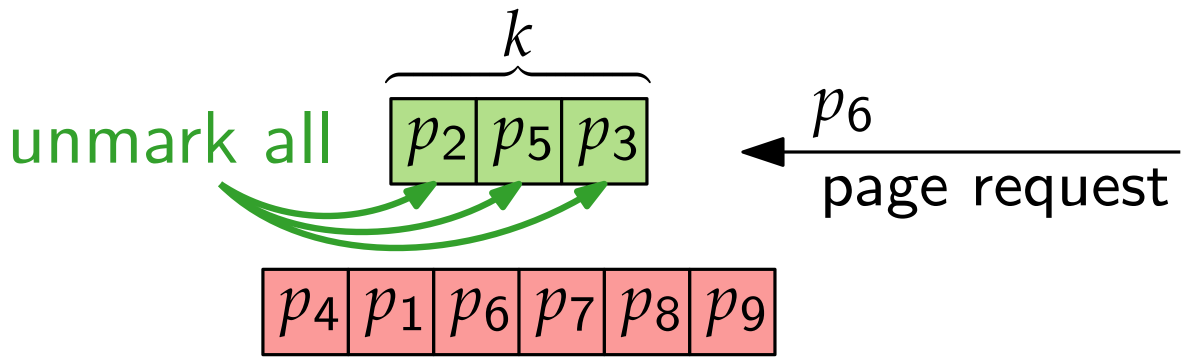

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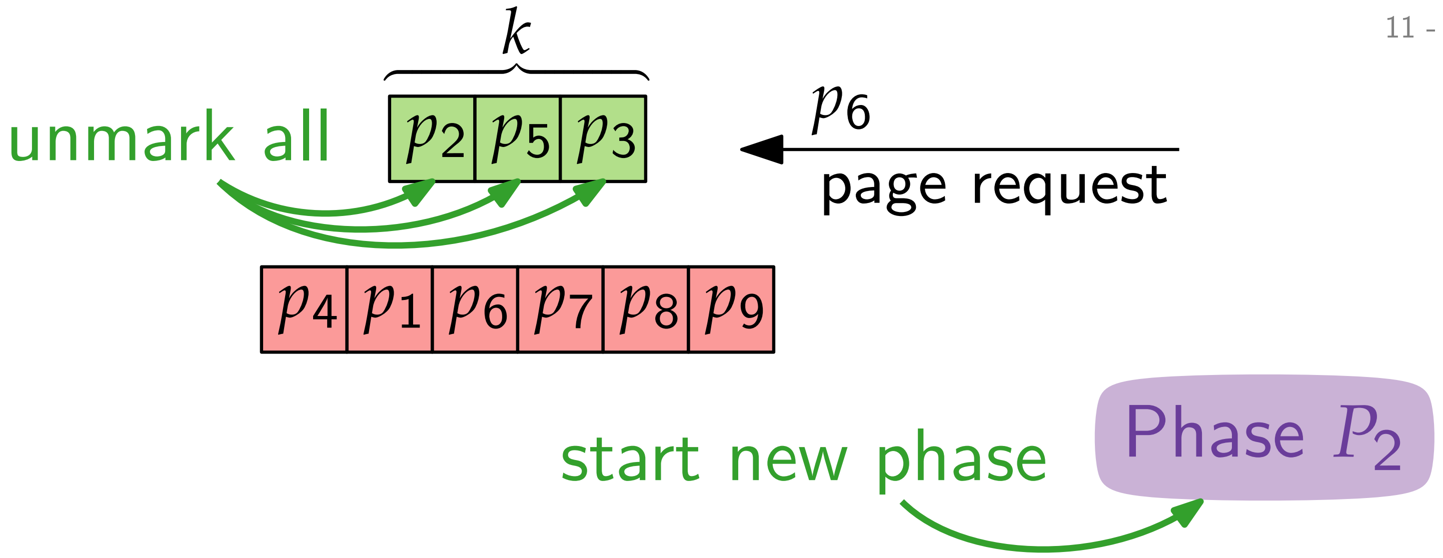

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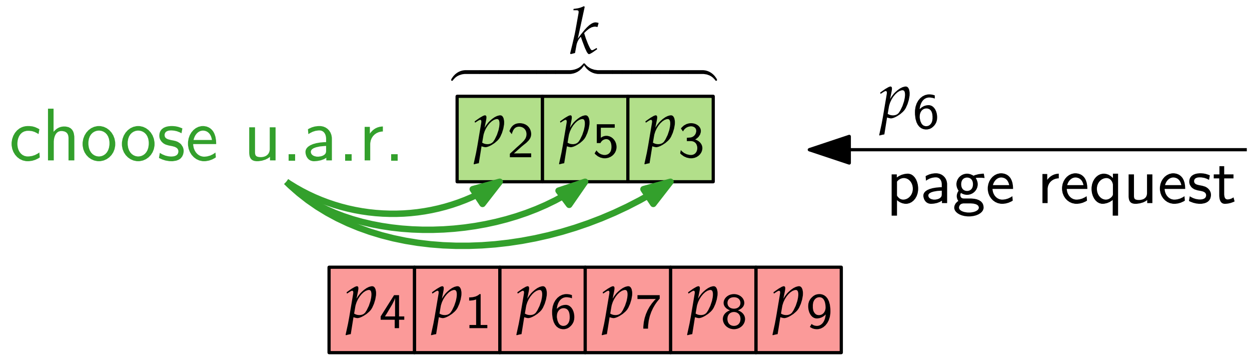

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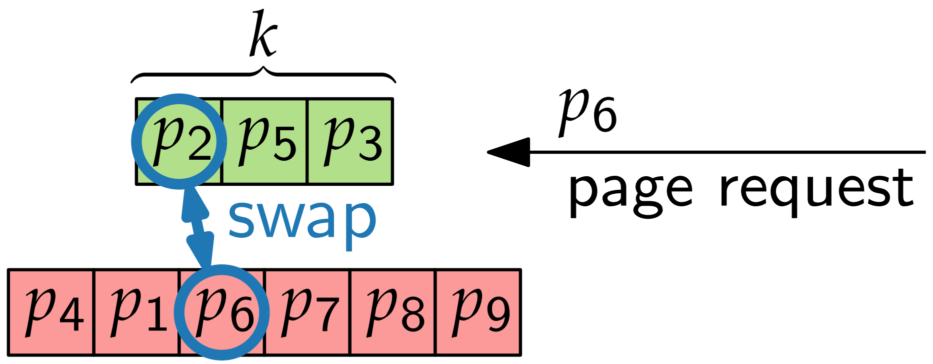

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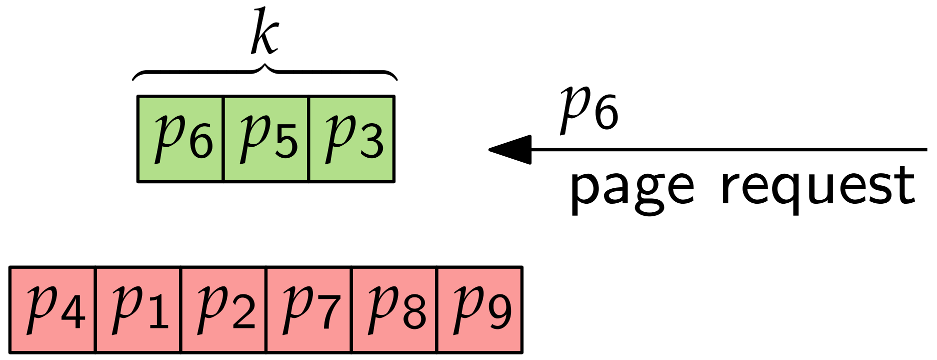
Paging – Rand. Strat.



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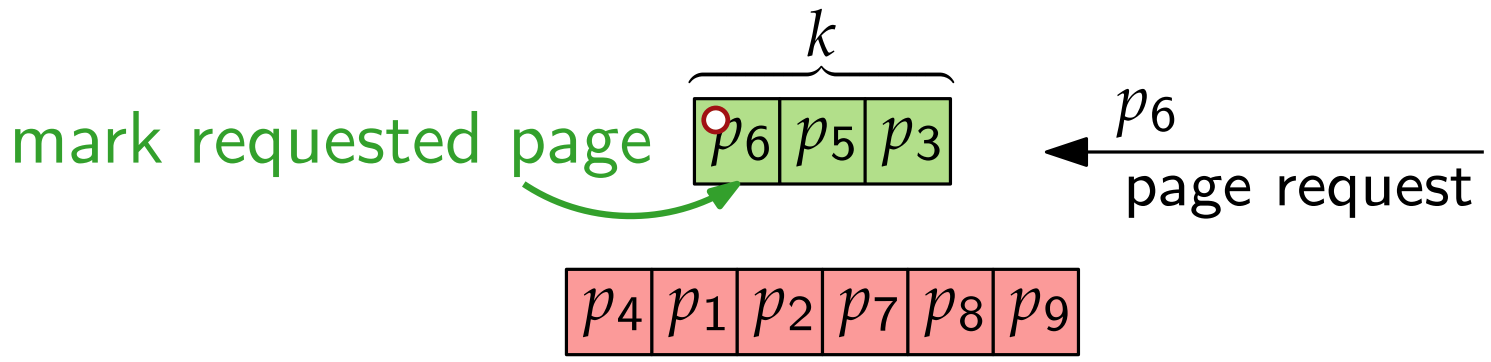

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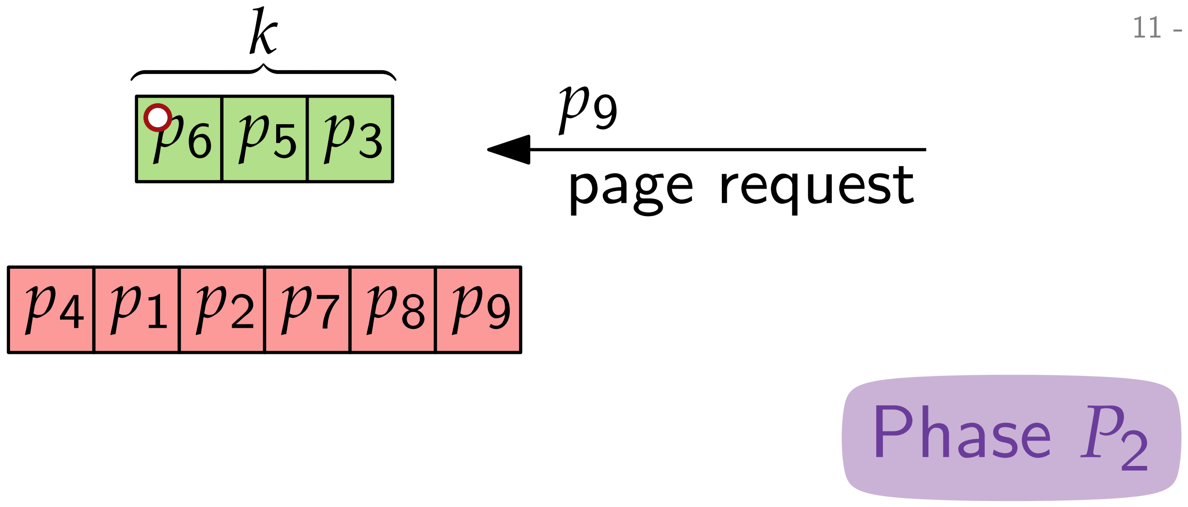

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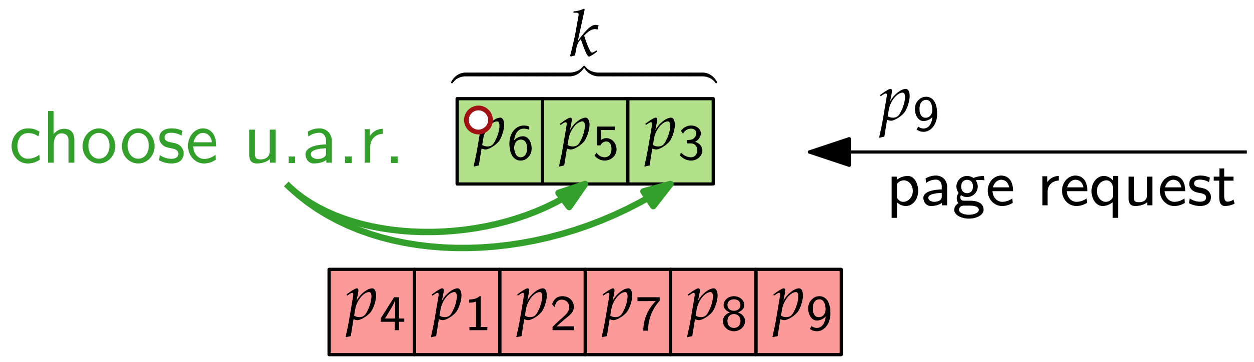

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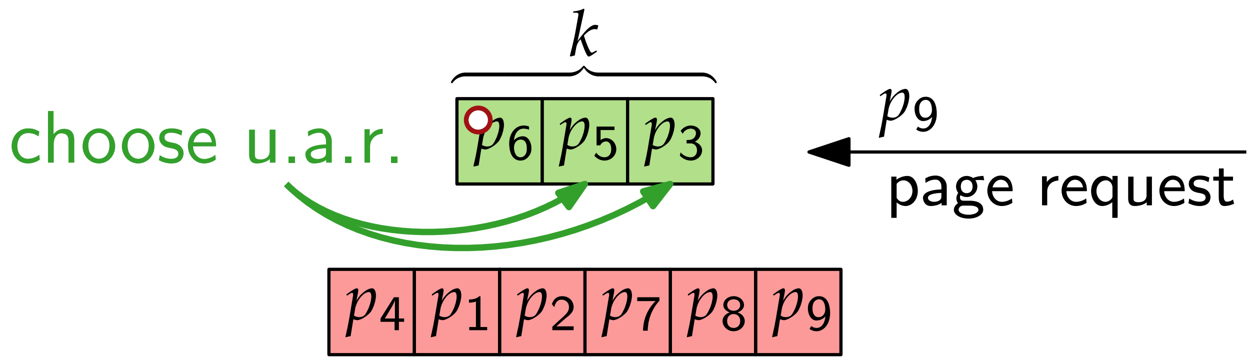

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Theorem 3. MARKING is $2H_k$-competitive.

Remark.

$H_k = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{k}$ is the k-th harmonic number and for $k \geq 2$: $H_k < \ln(k) + 1$.

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  Over all phases, all $\frac{d_{\text{begin}}}{2}$ and $\frac{d_{\text{end}}}{2}$ cancel out, except the first $\frac{d_{\text{begin}}}{2}$ and the last $\frac{d_{\text{end}}}{2}$.
- Since the first $d_{\text{begin}} = 0$, MIN has at least $\frac{c}{2}$ faults per phase.

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$\Rightarrow$ **exponential improvement!**

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- We might also transform a classical problem with incomplete information into an online problem. E.g.: Matching problem for ride sharing.
- Randomization can help to improve our behavior on worst-case instances. You may also think of: we are less predictable for an adversary.

Literature

Main source:

- Sabine Storandt's lecture script "Randomized Algorithms" (2016–2017)

Original papers:

- [Belady '66] "A Study of Replacement Algorithms for Virtual-Storage Computer."
- [Sleator, Tarjan '85] "Amortized Efficiency of List Update and Paging Rules."
- [Fiat, Karp, Luby, McGeoch, Sleator, Young '91] "Competitive Paging Algorithms."