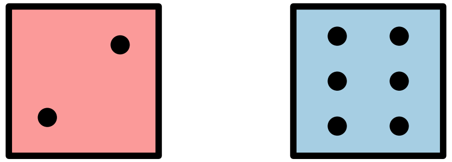
Julius-Maximilians-
UNIVERSITÄT
WÜRZBURG

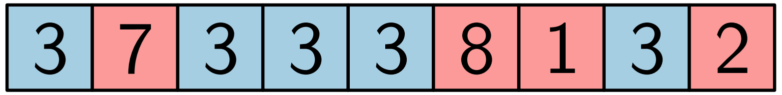
Advanced Algorithms

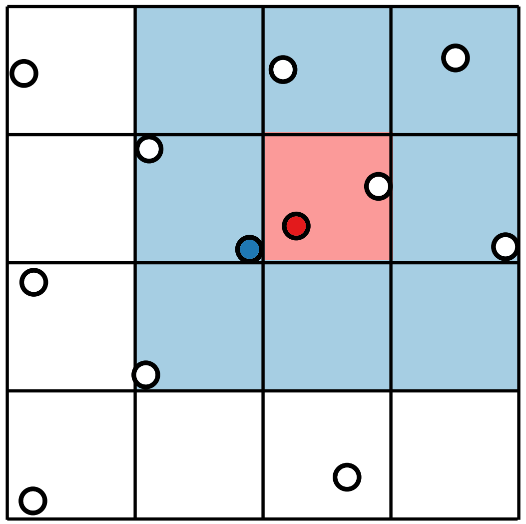
Randomized Algorithms

An introduction

Johannes Zink · WS22







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A **discrete probability space** $(\Omega, \Pr)$ is used to model random experiments.

- Ω is a countable set of **elementary events** (= outcomes of the experiment).
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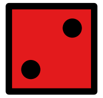

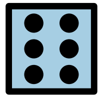

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$\Omega = \{(1,1), (1,2), (1,3), \ldots, (6,6)\}$, $\quad \Pr((i,j)) = \frac{1}{6} \cdot \frac{1}{6} = \frac{1}{36}$ for each $(i,j) \in \Omega$

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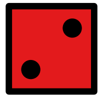
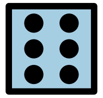

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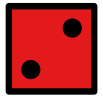
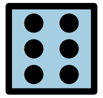

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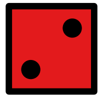
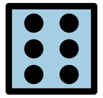

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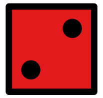 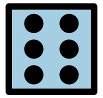

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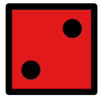 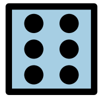

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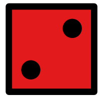

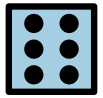

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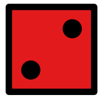
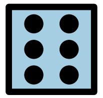

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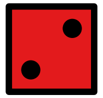

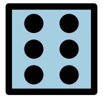

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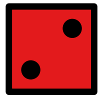 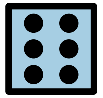

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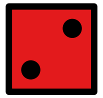

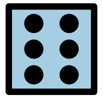

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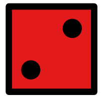 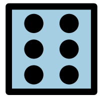

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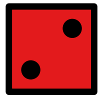
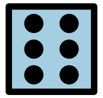

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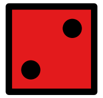
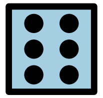

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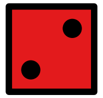
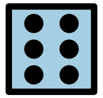

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$$\mathsf{E}[X] = 1 \cdot \frac{1}{36} + 2 \cdot \frac{3}{36} + 3 \cdot \frac{5}{36} + \cdots + 6 \cdot \frac{11}{36} \approx 4.5$$

Linearity of Expectation

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For each set of random variables $X_1, X_2, \ldots, X_n : \Omega \longrightarrow \mathbb{R}$,
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Let A be an array filled with n pairwise distinct integers.

$\textsc{FindMax}(A)$

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m := A[1]
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H_n is the n-th harmonic number; $\ln(n+1) \leq H_n \leq \ln(n) + 1.$

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Task: Determine an integer $A[j]$ that is at least as large as the median.

Deterministic approach: Go through all elements, return maximum.

(Actually, it suffices to go through $\lfloor n/2 \rfloor + 1$ elements.)

runtime $\Theta(n)$

Randomized approach:

$\textsc{FindLarge}(A, k \in \mathbb{N})$

$\ell := 0$
for $i = 1, 2, \ldots, k$
  randomly choose $r \in \{1, 2, \ldots, n\}$
  if $A[r] > \ell$
    $\ell := A[r]$
return ℓ

$\textsc{FindLarge}$ has error probability $\leq \frac{1}{2^k}$.

Set $k = c \log_2 n$ for some constant $c > 1$.

$\Rightarrow$ Error probability $\leq \frac{1}{n^c}$

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Remark. We traded correctness for running time.

Finding a Repeated Element

Given: An array A of n natural numbers such that $\lceil \frac{n}{2} \rceil$ of them are identical and $\lfloor \frac{n}{2} \rfloor$ of them are pairwise distinct.

Task: Find the repeated element.

3	7	3	3	3	8	1	3	2

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Compare each element with every predecessor $\Theta(n^2)$ time

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Randomized approach:

$\textsc{FindRepeated}(A)$

while true **do**
randomly choose $i \in \{1, \ldots, n\}$
randomly choose $j \in \{1, \ldots, n\} \setminus \{i\}$
if $A[i] = A[j]$ **then return** $A[i]$

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Success probability in each step

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Remark. The algorithm only returns correct answers, but may run forever.

Las Vegas and Monte Carlo Algorithms

Las Vegas algorithm. Returns a correct result, but the running time (and possibly the required space) are random variables.

Examples. FindRepeated, RandomizedQuicksort

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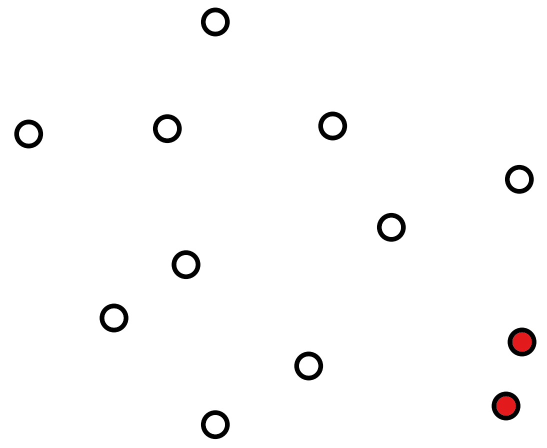
Remark. A Monte Carlo algorithm can often be turned into a Las Vegas algorithm and vice versa.

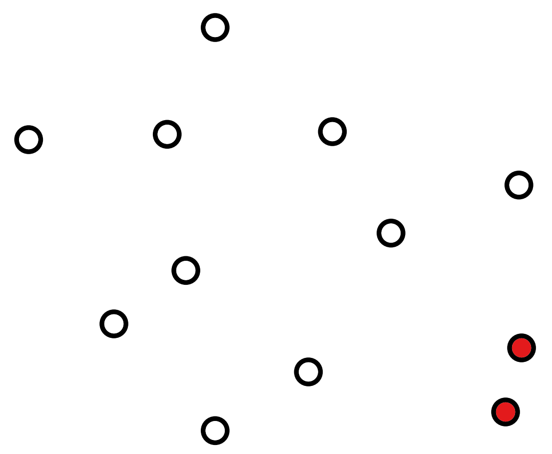
Closest Pair



Given: (multi-)set of points $P = \{p_1, p_2, \ldots, p_n\} \subseteq \mathbb{R}^2$.

Task: Find a pair of distinct elements $p_a, p_b \in P$ such that the Euclidean distance $\delta = ||p_a, p_b||$ is minimum.

Closest Pair



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Brute-force $\Theta(n^2)$

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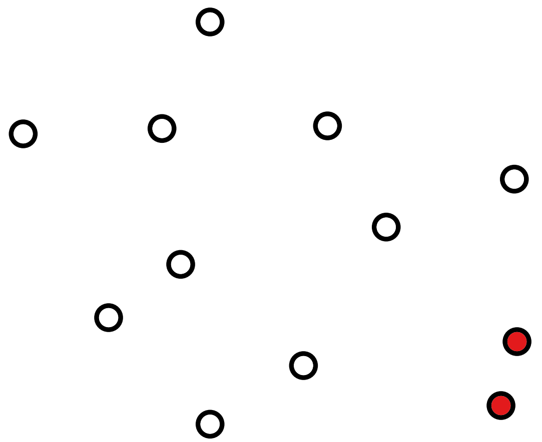

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Deterministic approaches:

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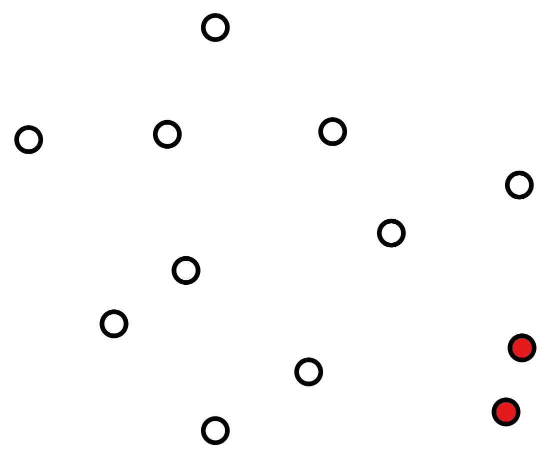
Divide and conquer (recall from ADS)	$\Theta(n \log n)$

Closest Pair



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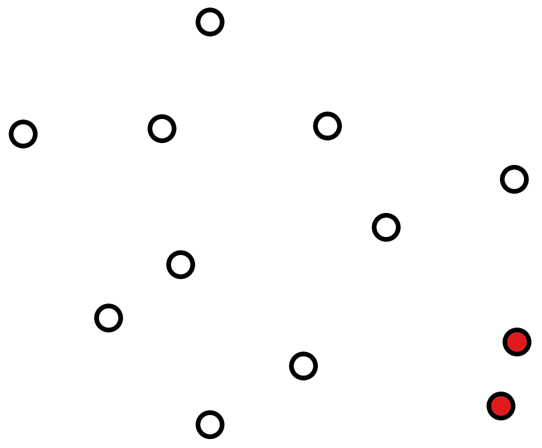
Lower bound:

Element Uniqueness: Given numbers $a_1, a_2, \ldots, a_n$. Are they pairwise distinct?

There is no $o(n \log n)$ time algorithm for Element Uniqueness.

(under some assumption concerning the arithmetic model)

Closest Pair



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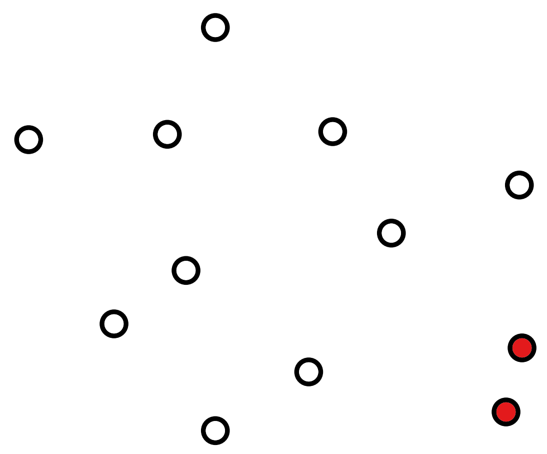

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Deterministic approaches:

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Reduction: map each a_i to a point (a_i, a_i) and test if the minimum distance is 0.

A Randomized Incremental Algorithm for Closest Pair

Define $P_i = \{p_1, p_2, \ldots, p_i\}$ and let δ_i be the distance of a closest pair in P_i.

A Randomized Incremental Algorithm for CLOSEST PAIR

Define $P_i = \{p_1, p_2, \ldots, p_i\}$ and let δ_i be the distance of a closest pair in P_i.

Idea: $\delta_2 = ||p_1, p_2||$. Compute $\delta_3, \delta_4, \ldots, \delta_n$ by adding the points iteratively.

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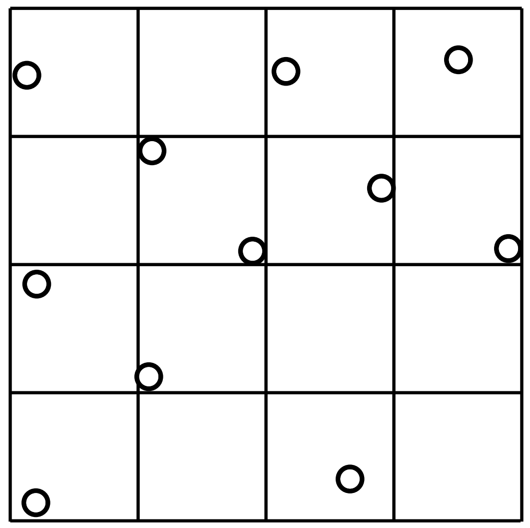

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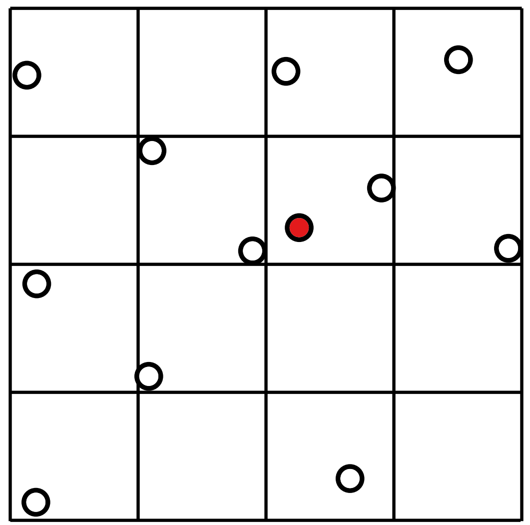

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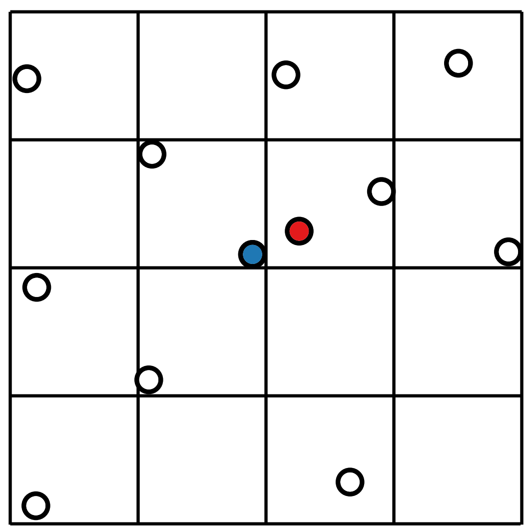

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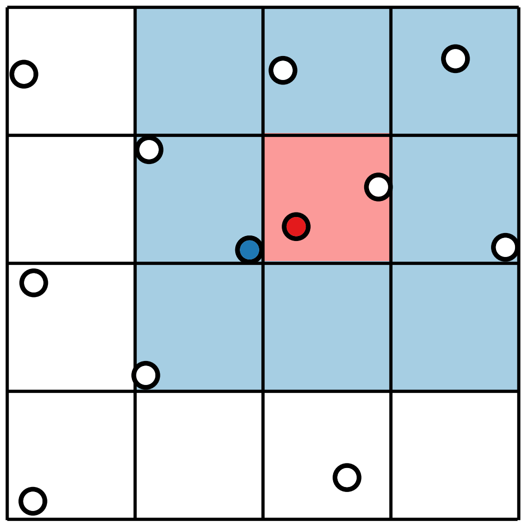
Moreover, p_j must lie in the cell of p_i or one of the adjacent cells.



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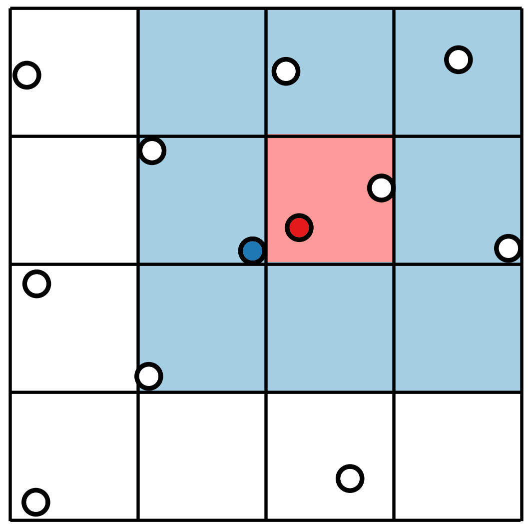

(simple) exercise

Each of these cells contains at most $\mathcal{O}(1)$ points of P_{i-1} ($\Leftarrow$ packing argument).

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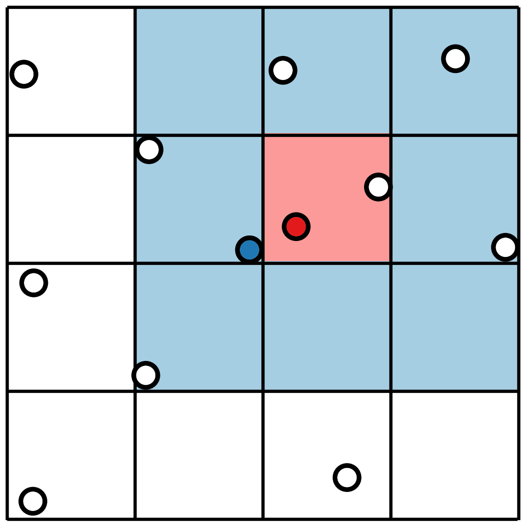

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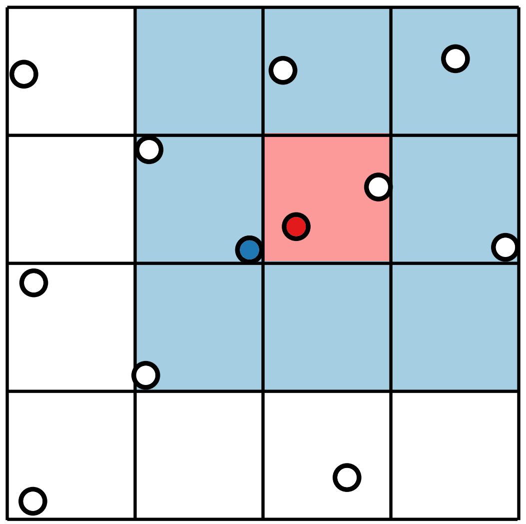

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$\Rightarrow$ The test $\delta_i < \delta_{i-1}$ can be performed in $\mathcal{O}(1)$ time assuming P_{i-1} is stored in a suitable dictionary for the nonempty cells (implementable via dynamic perfect hashing).

Backwards Analysis

If $\delta_i = \delta_{i-1}$, we add p_i to the dictionary in $\mathcal{O}(1)$ time.

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Probability that adding p_i to P_{i-1} decreases the minimum distance

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How many points p in P_i have the property
that the minimum distance in $P_i \setminus \{p\}$ is larger than in P_i?

Backwards Analysis

If $\delta_i = \delta_{i-1}$, we add p_i to the dictionary in $\mathcal{O}(1)$ time.
If $\delta_i < \delta_{i-1}$, the cell size changes and we have to rebuild the dictionary in $\mathcal{O}(i)$ time.
$\Rightarrow$ total runtime $\mathcal{O}(n^2)$.

Randomization: In the beginning, randomly permute the point set P.

Probability that adding p_i to P_{i-1} decreases the minimum distance
$=$ Probability that deleting p_i from P_i increases the minimum distance

How many points p in P_i have the property
that the minimum distance in $P_i \setminus \{p\}$ is larger than in P_i?

- If the closest distance in P_i is unique: 2 points

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- If there are at least two disjoint closest pairs: 0 points

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Let X_i be the running time used for adding p_i.

$\Rightarrow \mathrm{E}[X_i] \leq \frac{2}{i} \cdot \mathcal{O}(i) + \frac{i-2}{i} \cdot \mathcal{O}(1) =$

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- allow for trading runtime against output quality,
- provide a good strategy for games or search in unknown environments.