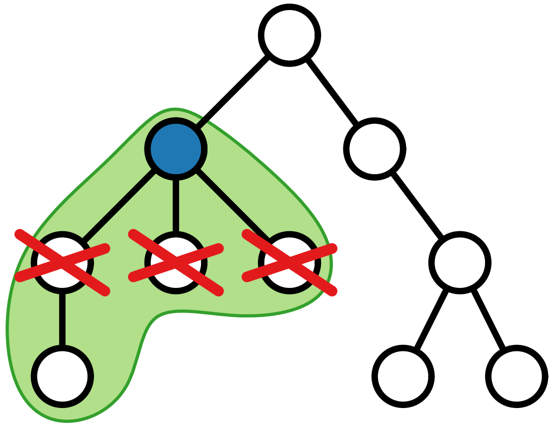
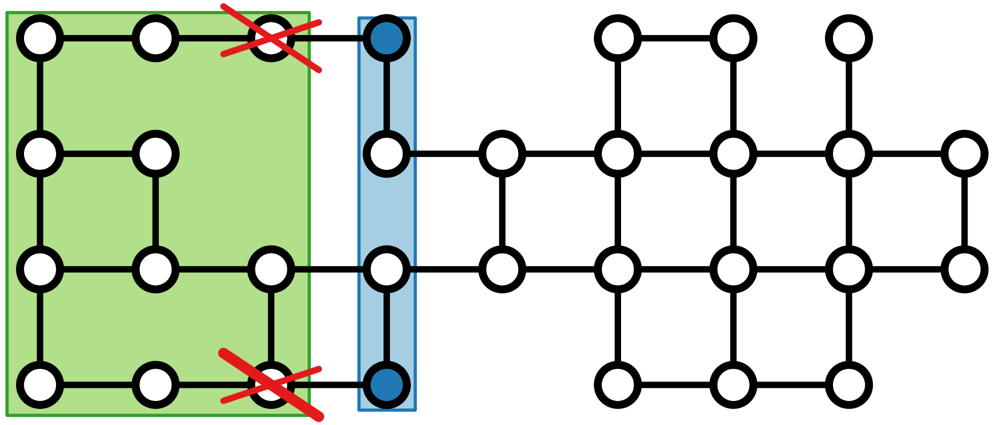
Advanced Algorithms

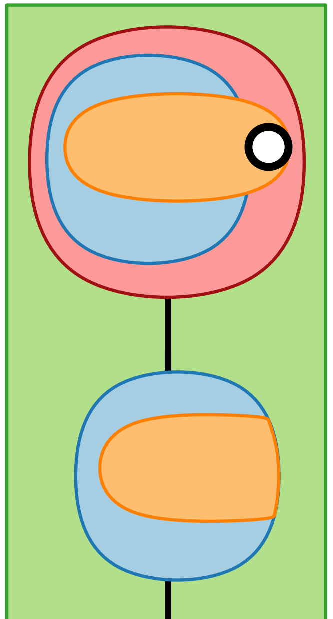
Parameterized Algorithms

Structural Parametrization

Johannes Zink · WS22



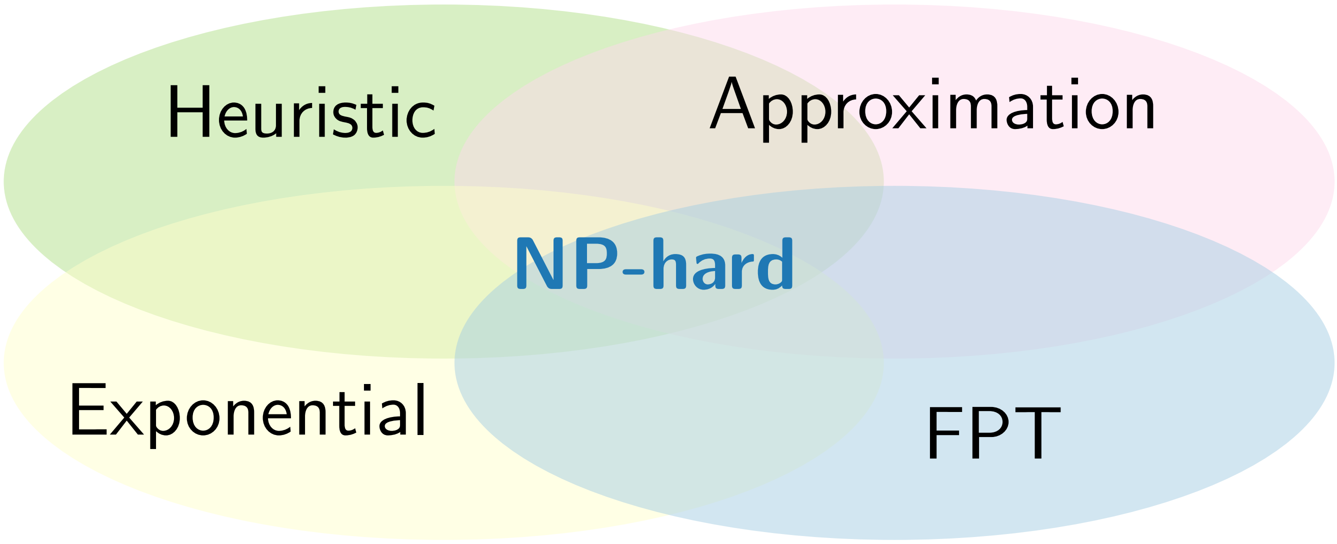




Dealing with NP-Hard Problems

What should we do?

- Sacrifice optimality for speed
  - Heuristics
  - Approximation Algorithms
- Optimal Solutions
  - Exact exponential-time algorithms
  - Fine-grained analysis – parameterized algorithms



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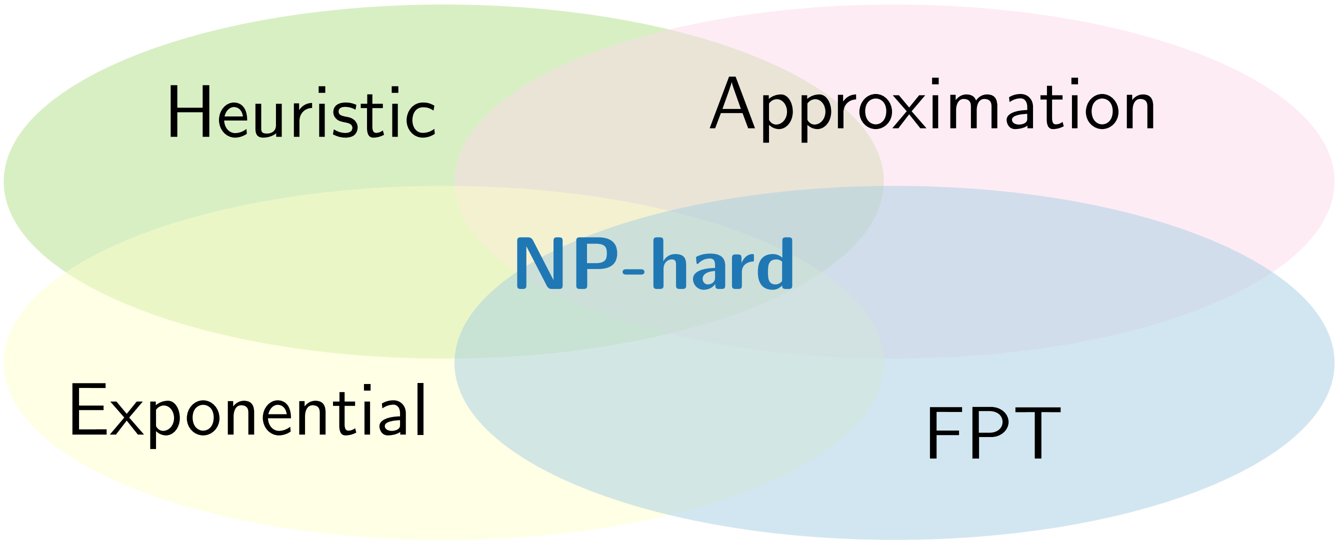

Parameterized Algorithms

Classical complexity theory:

Running time is expressed as a function in the input size

Parameterized algorithmics:

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Example: (recall from AGT)

k-VERTEX COVER

Input Graph $G = (V, E)$, $k \in \mathbb{N}$

Question Is there a set $C \subseteq V$ with $|C| \leq k$
s.t. $\forall \{u, v\} \in E \colon \{u, v\} \cap C \neq \emptyset$?

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Idea: If $k \in \mathcal{O}(1)$, then $\mathcal{O}(2^k \cdot k \cdot (|V| + |E|)) \subseteq \mathcal{O}(|V| + |E|)$, in other words, if we assume the **parameter** k to be **fixed**, k-Vertex Cover becomes **tractable**.

Parameterized Complexity Classes

Definition.

Let Π be a decision problem. If there is

- an algorithm $\mathcal{A}$ and
- a computable function f

such that, given an instance I of Π and a parameter $k \in \mathbb{N}$,
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$\Rightarrow$ k-Vertex Cover is FPT (and therefore also XP) with respect to k.

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In all these examples, k is the *natural* parameter that comes with the decision problem.

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- Under common assumptions, k-Clique is not FPT with respect to k (namely, k-Clique is $W[1]$-complete with respect to k; → Section 13 in [1])
- There is an $\mathcal{O}(2^{\Delta} \cdot \Delta^2 \cdot (|V| + |E|))$ time algorithm for k-Clique, where Δ is the maximum degree of the input graph ⇒ k-Clique is FPT with respect to Δ.

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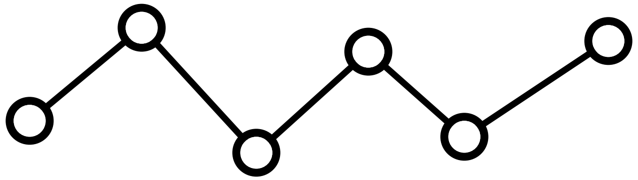
Pathwidth and Treewidth (Intuition)

Pathwidth describes how *path-like* a graph is.



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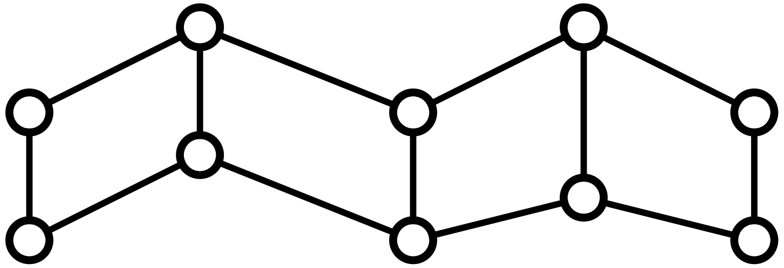

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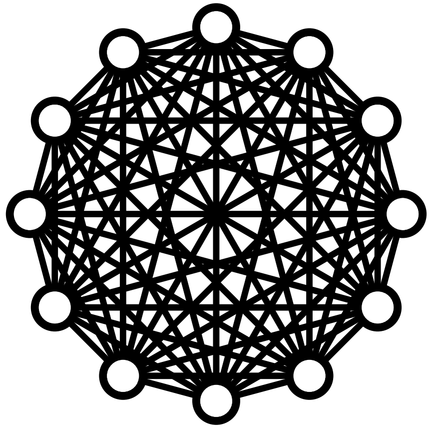

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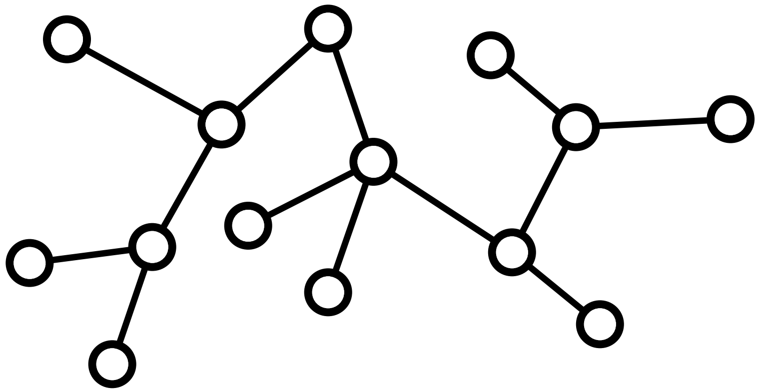

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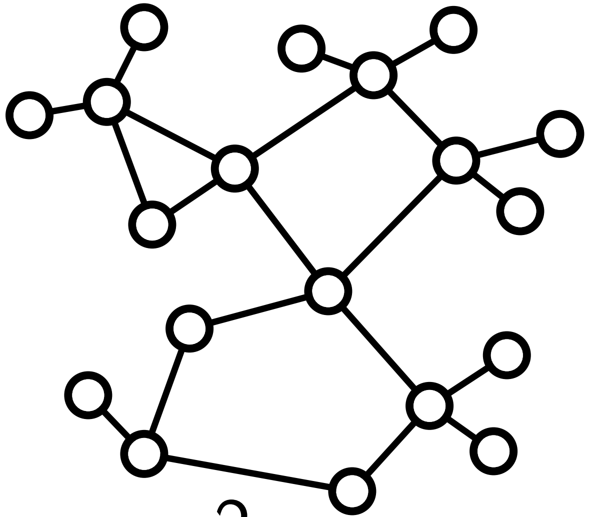

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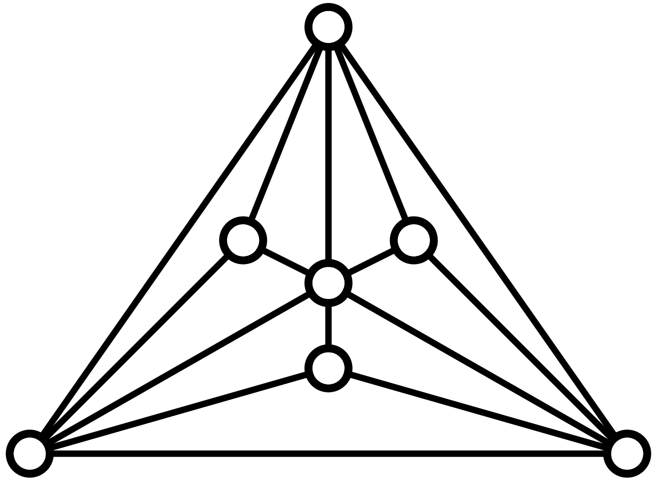

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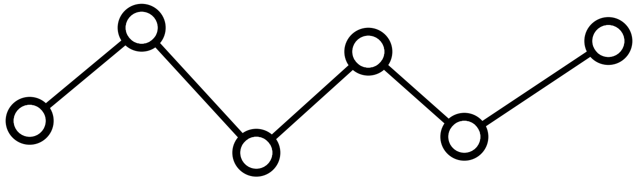

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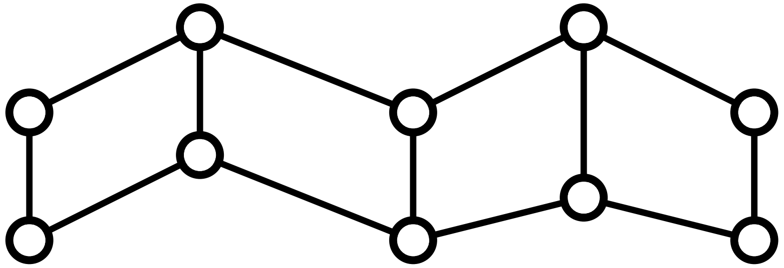

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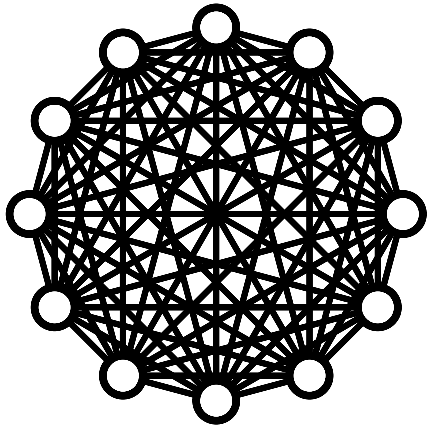

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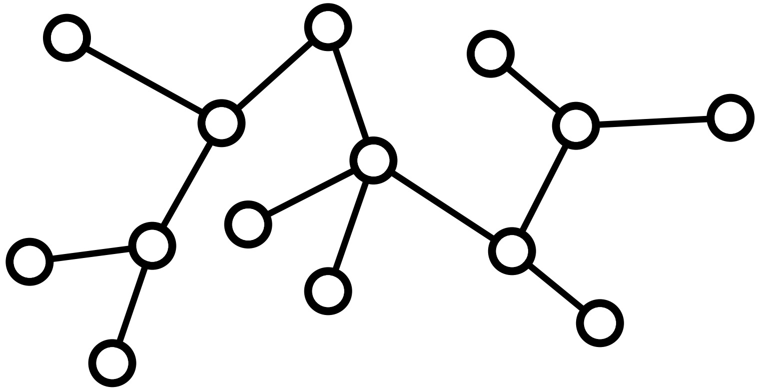

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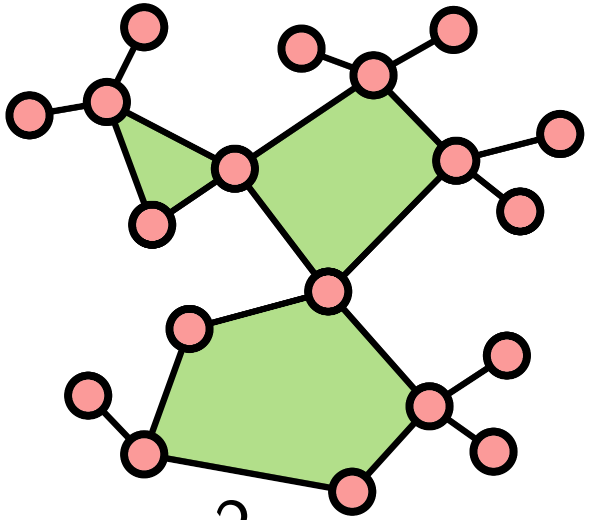

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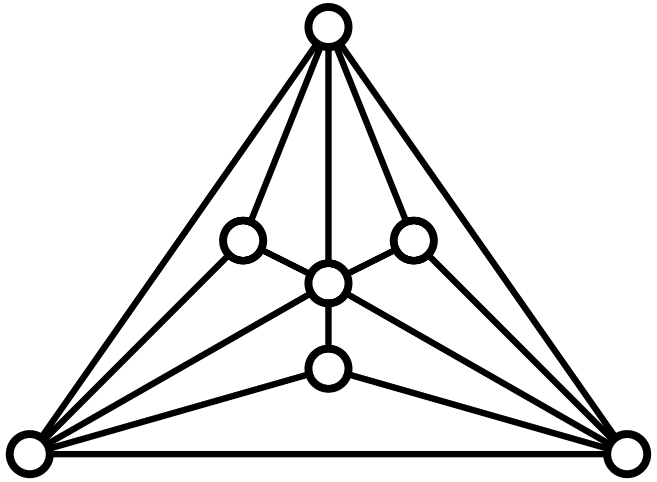

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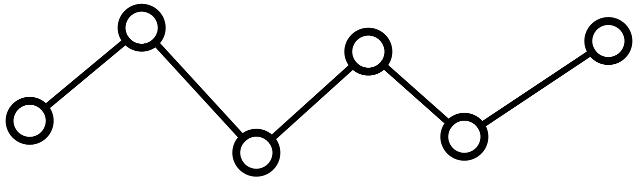

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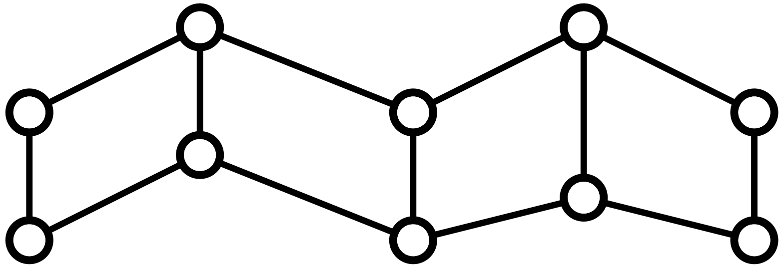

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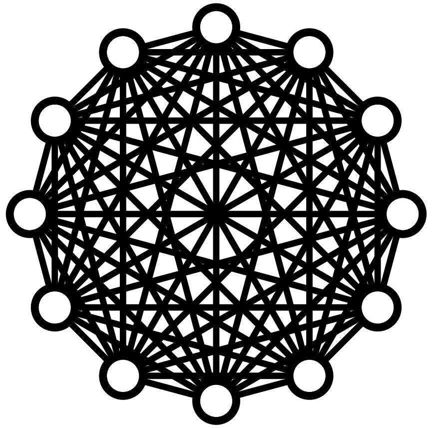

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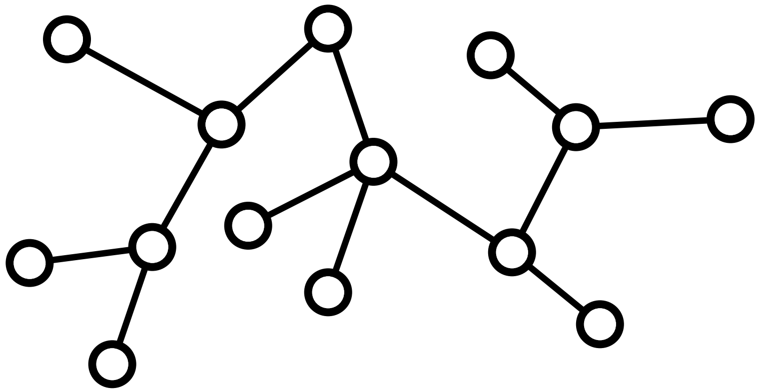

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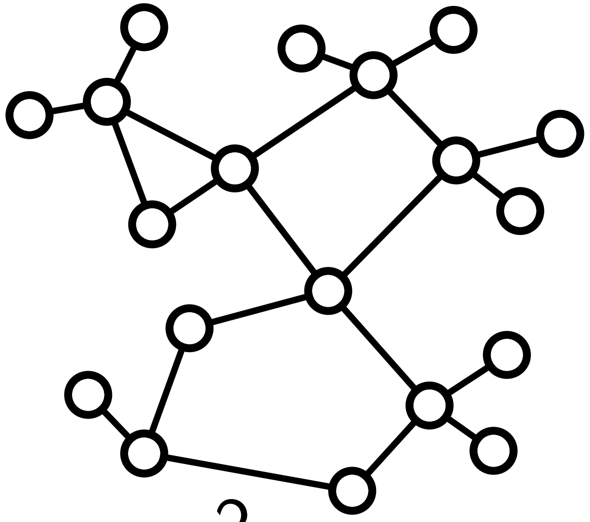

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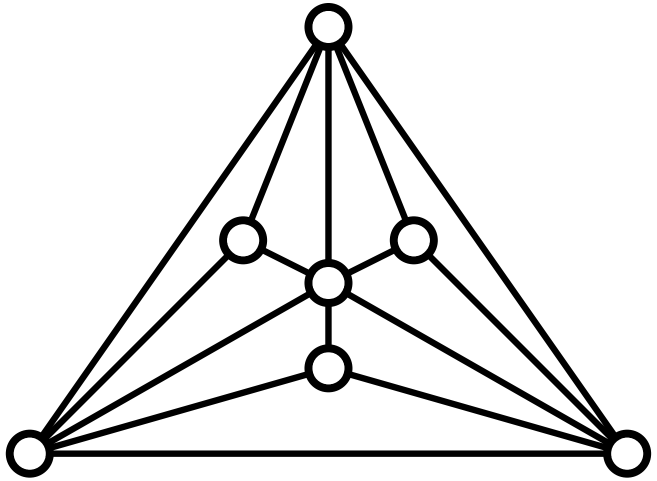

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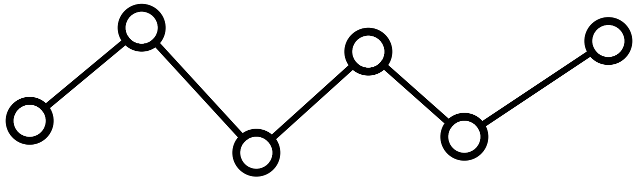

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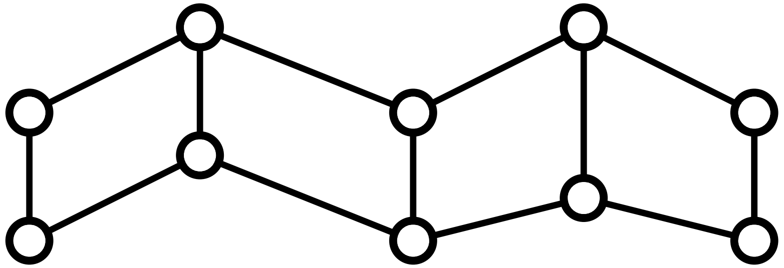

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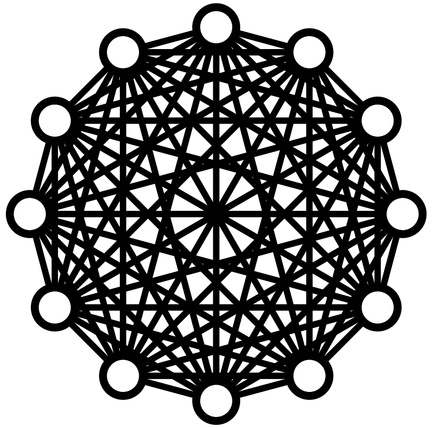

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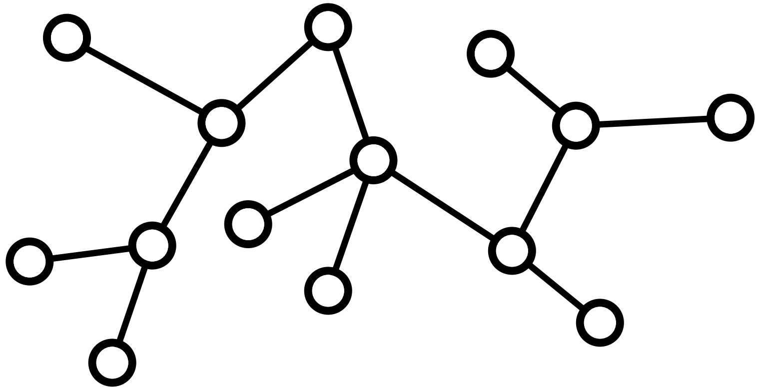

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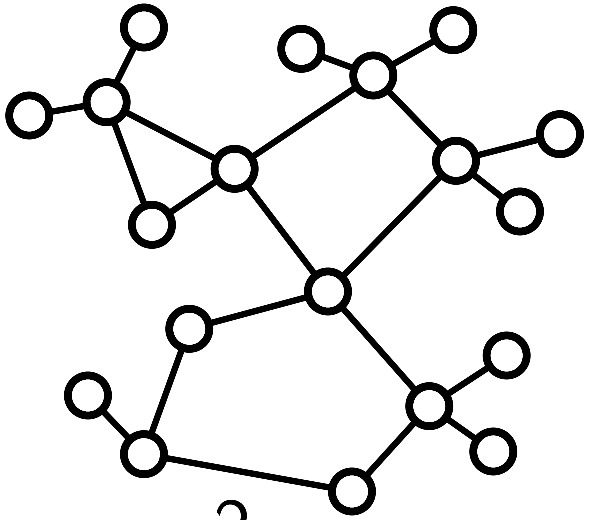

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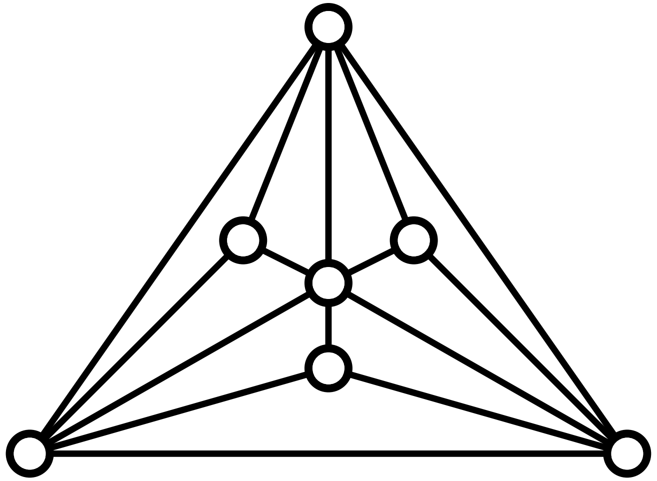

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Tree-like structure is useful for designing dynamic programming algorithms.

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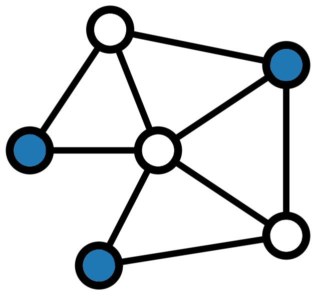

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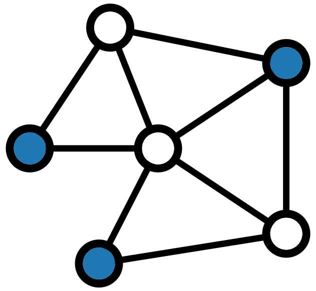

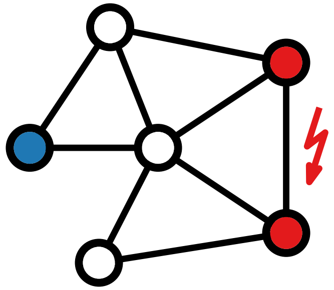

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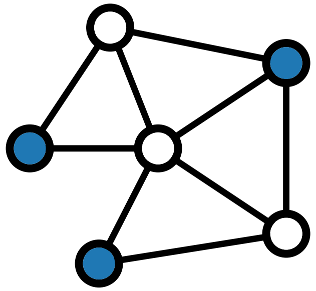

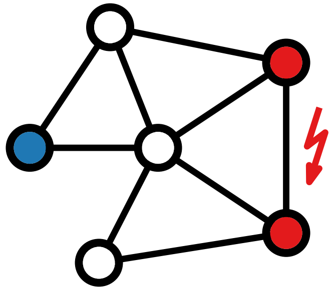

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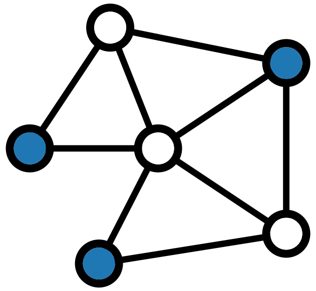

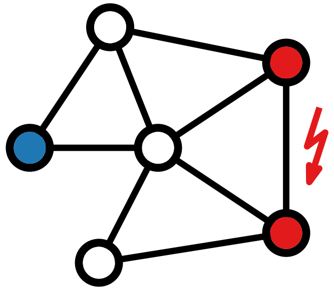

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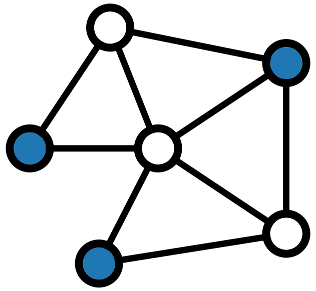

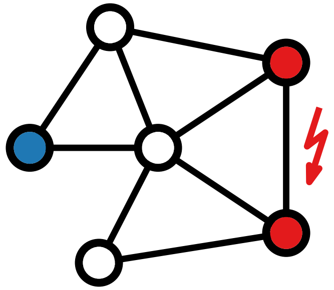

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- On trees, (Weighted) Independent Set can be solved in linear time.

Independent Set in Trees

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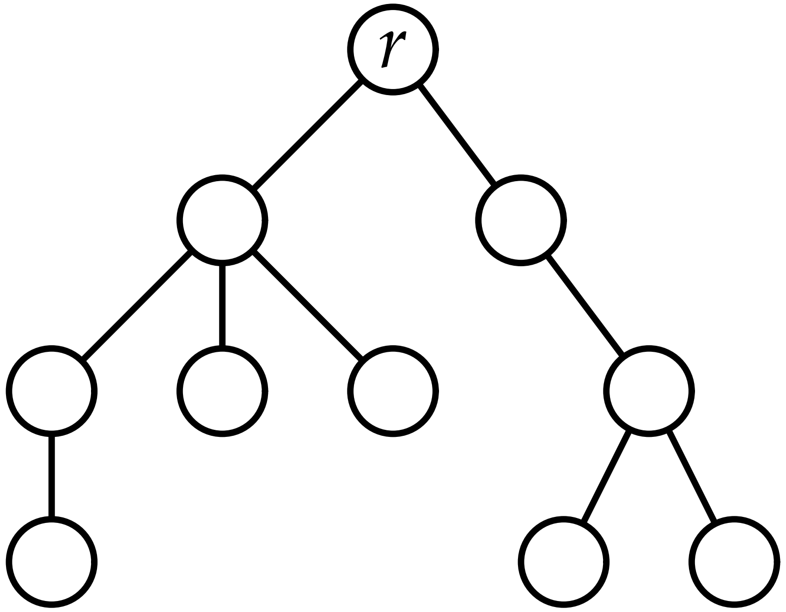

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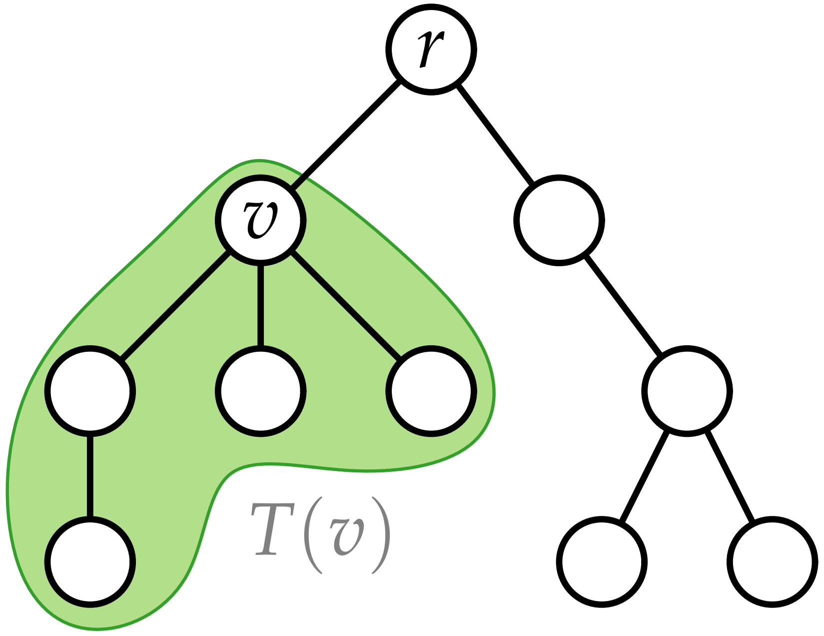

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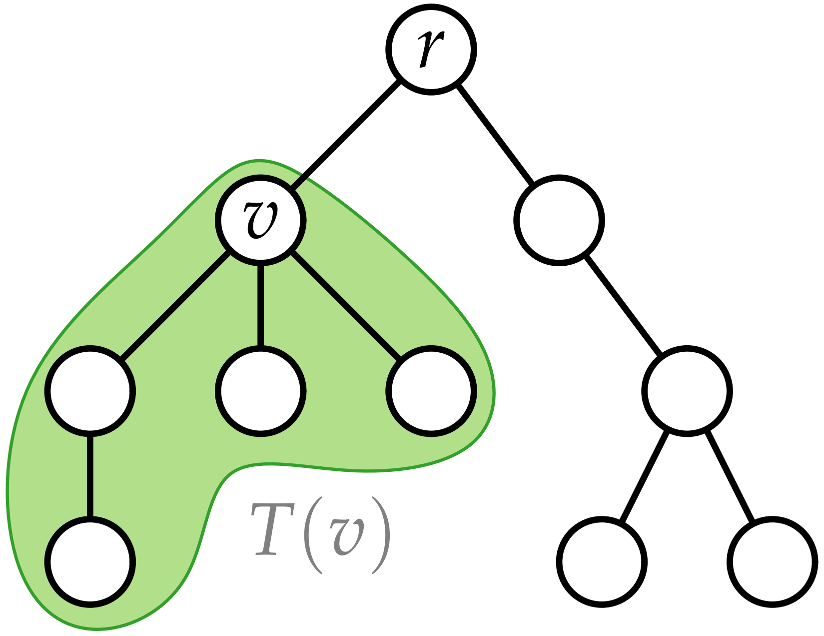
Let $A(v) :=$ maximum weight of an independent set I in $T(v)$



Independent Set in Trees

Choose an arbitrary root r.

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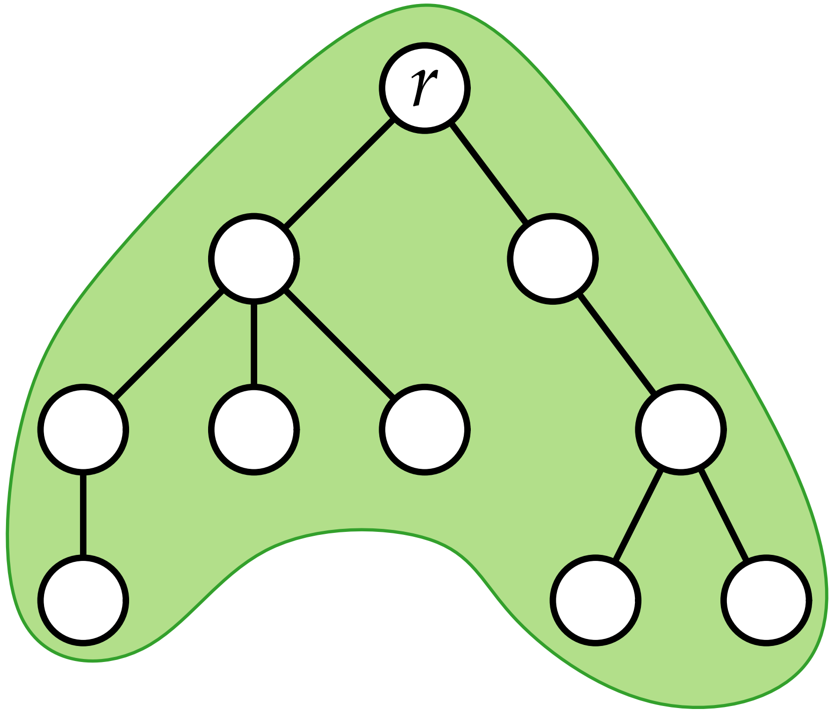
$A(r)$ = solution



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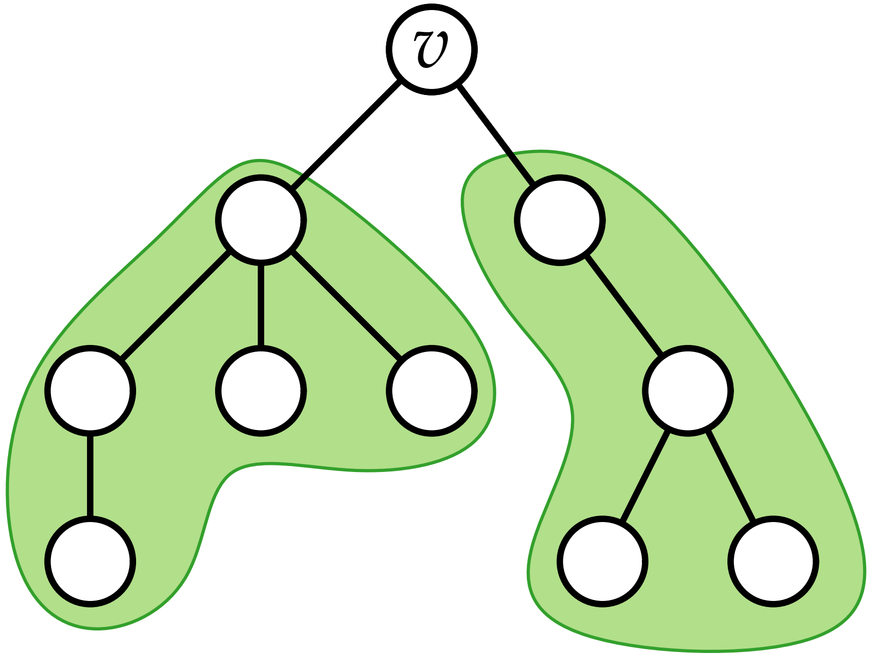

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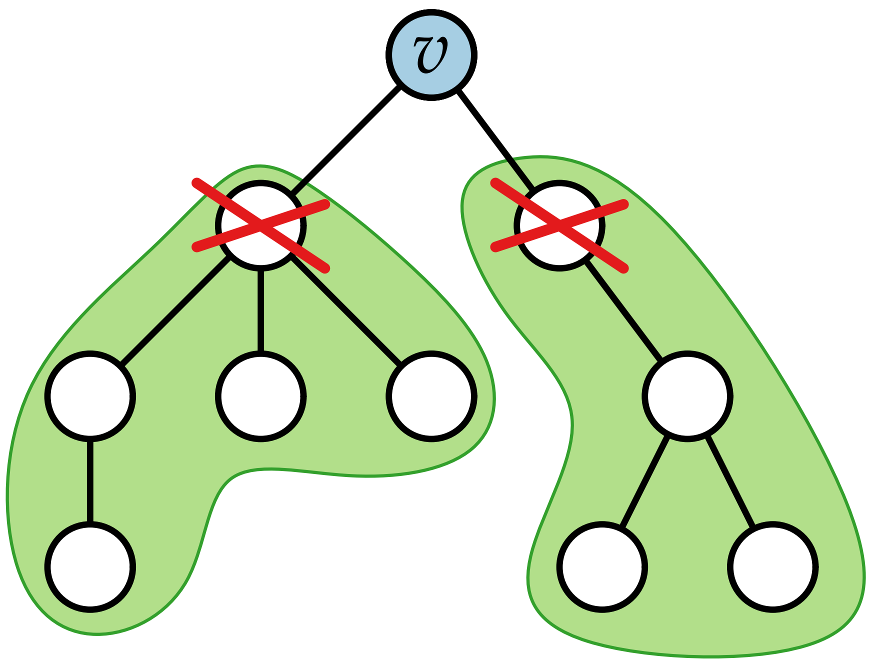

If $v \in V$ is part of indepent set I,
then none of its neighbors $\mathsf{N}(v)$ is also in I.

Independent Set in Trees

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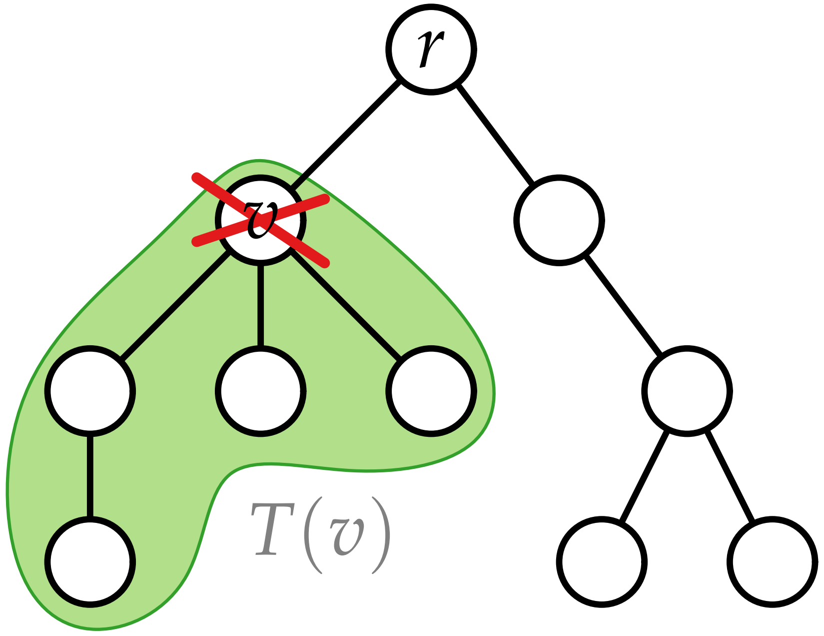

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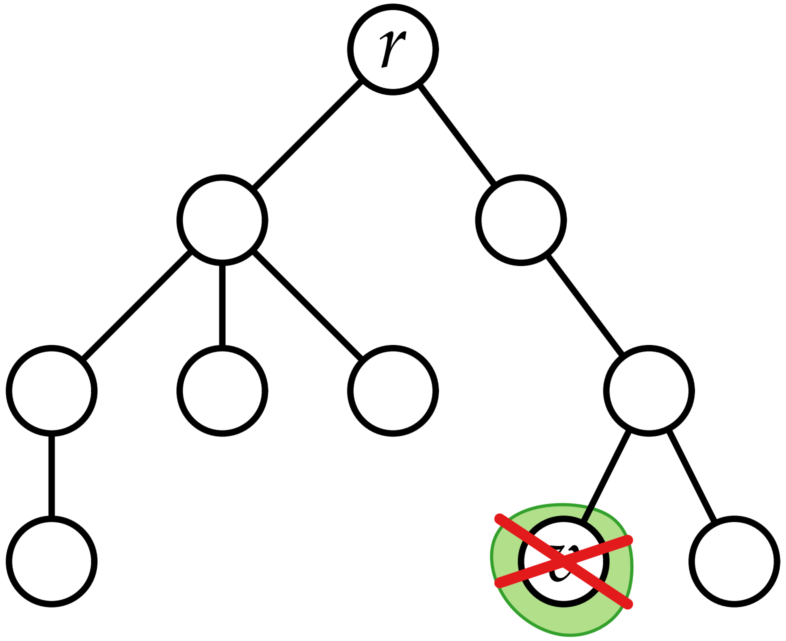

- If v is a leaf: $B(v) =$

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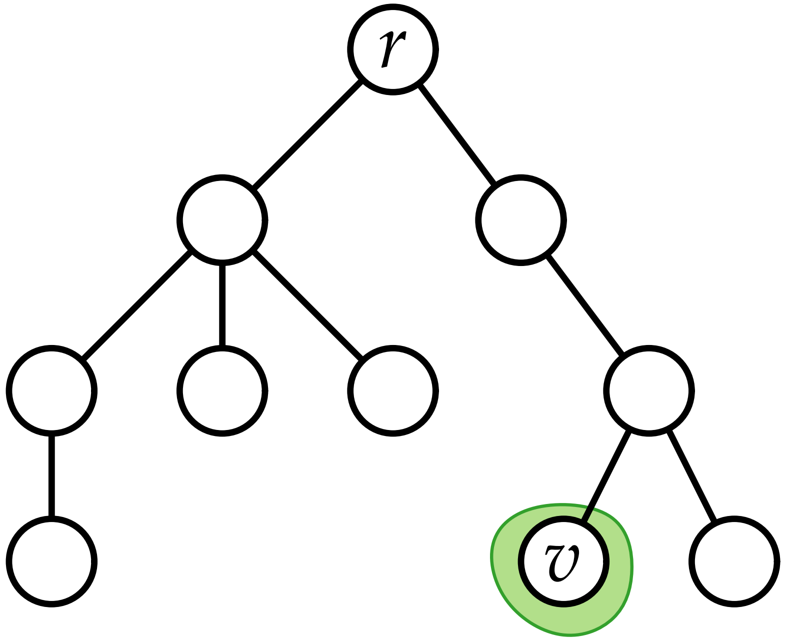

- If v is a leaf: $B(v) = 0$ and $A(v) =$

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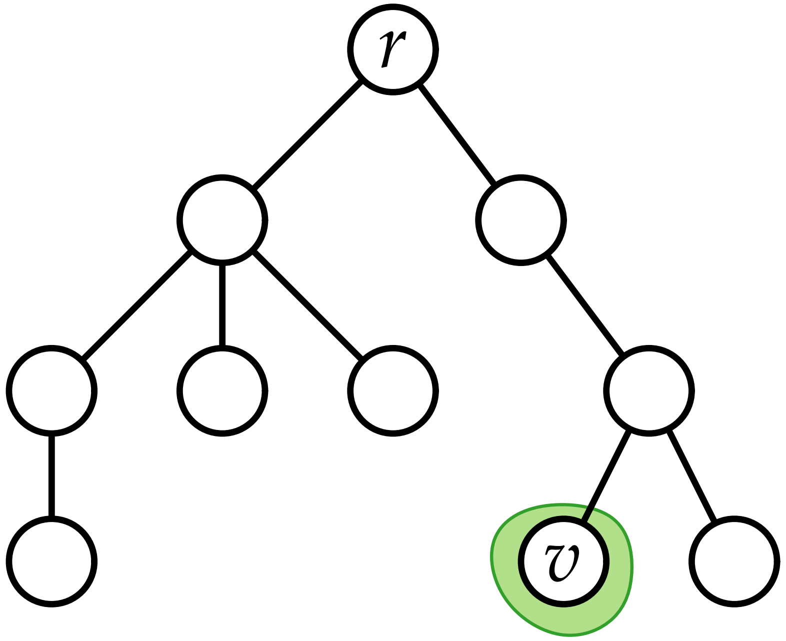

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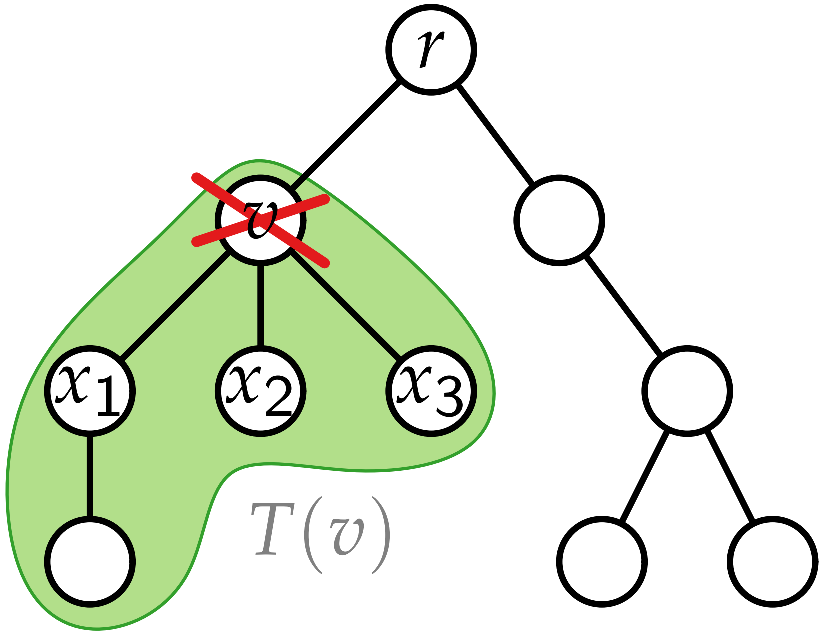

- If v is a leaf: $B(v) = 0$ and $A(v) = w(v)$
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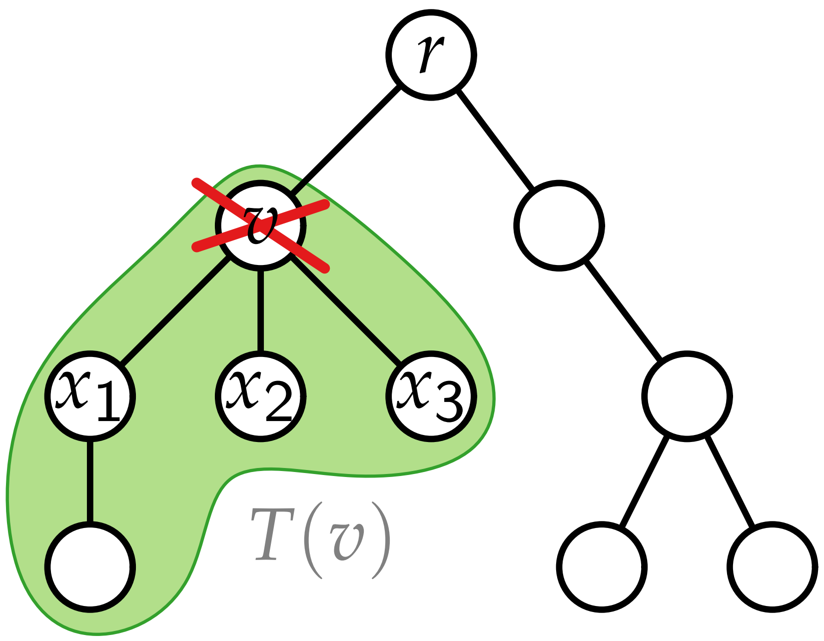

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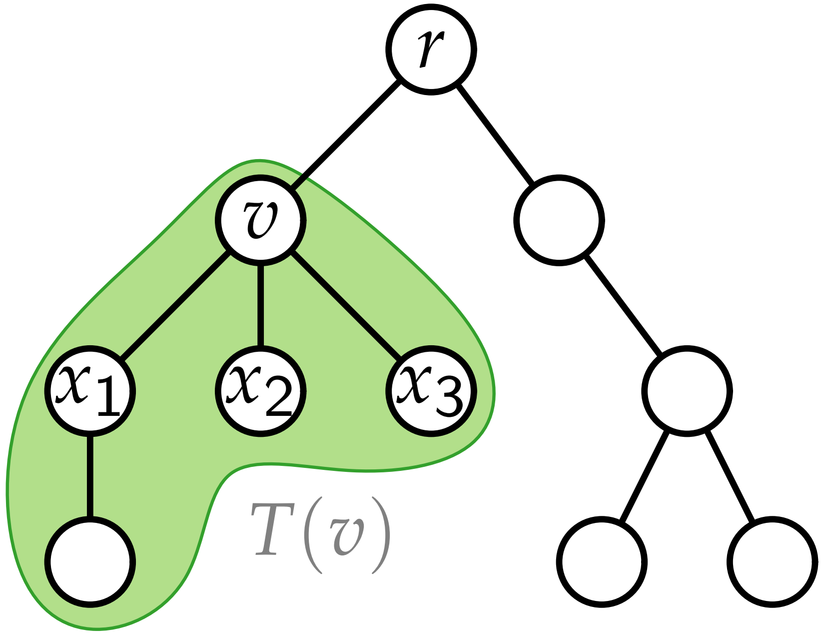

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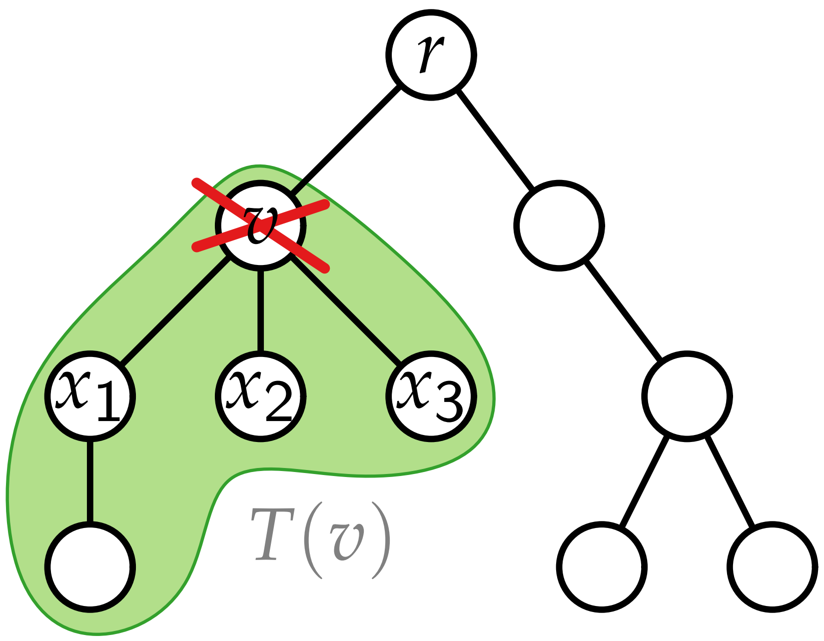

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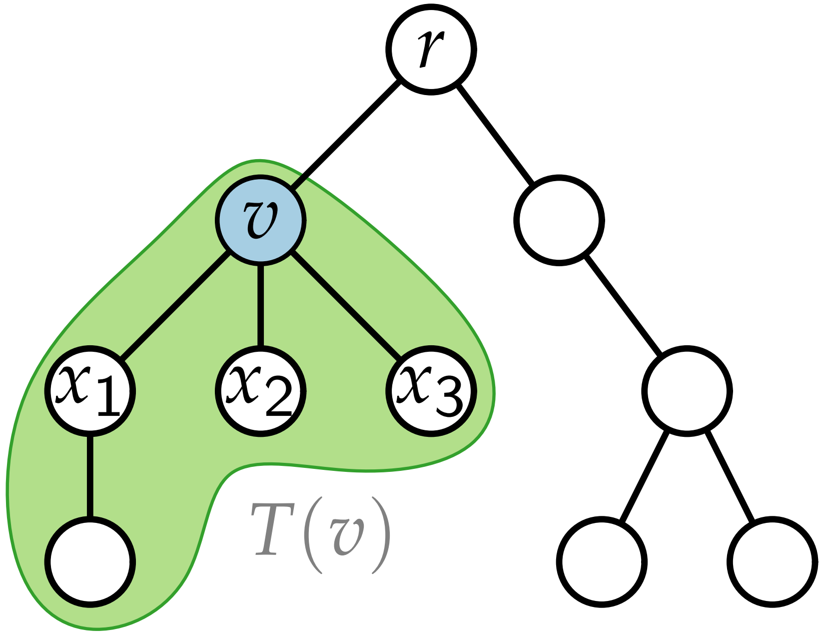

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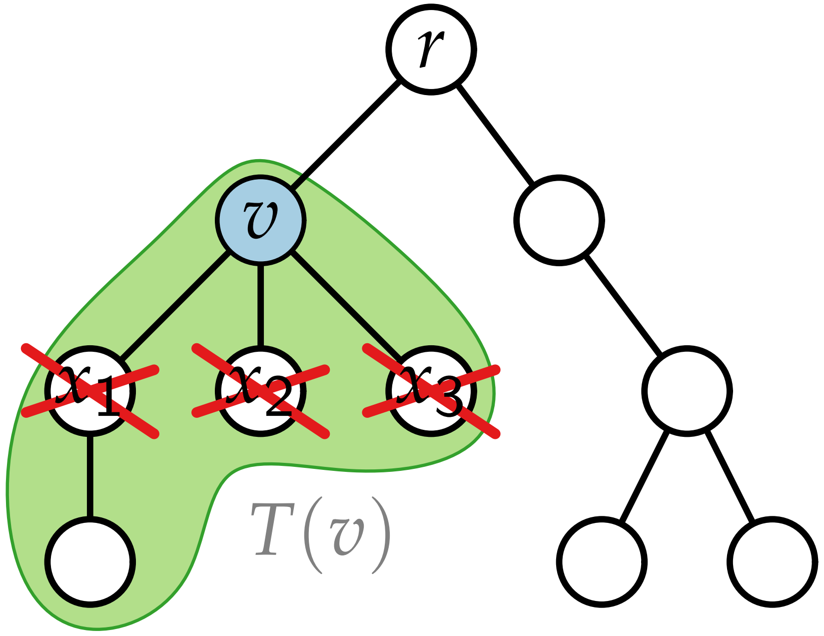

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Independent Set in Trees

$A(r)$ = solution

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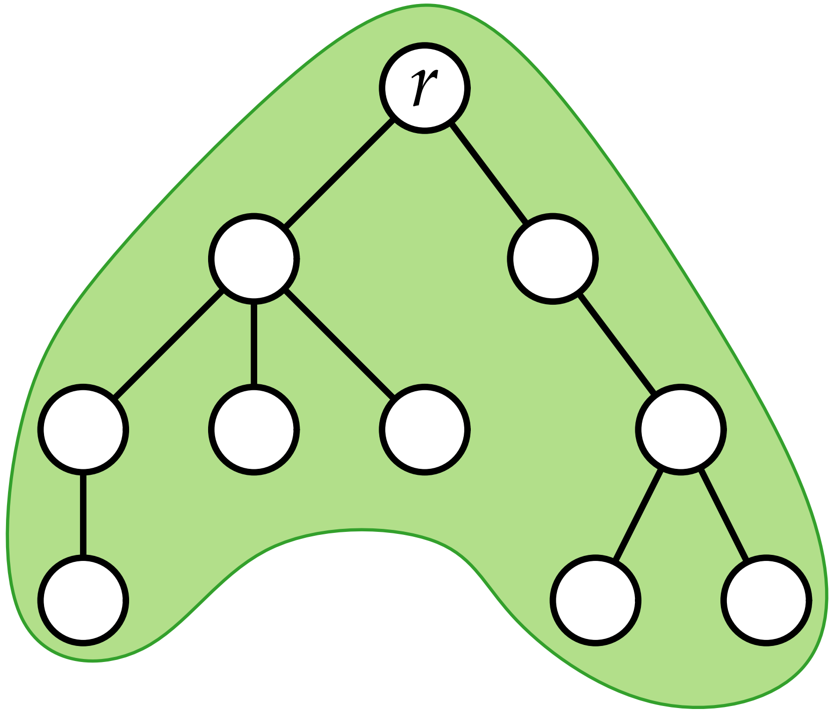

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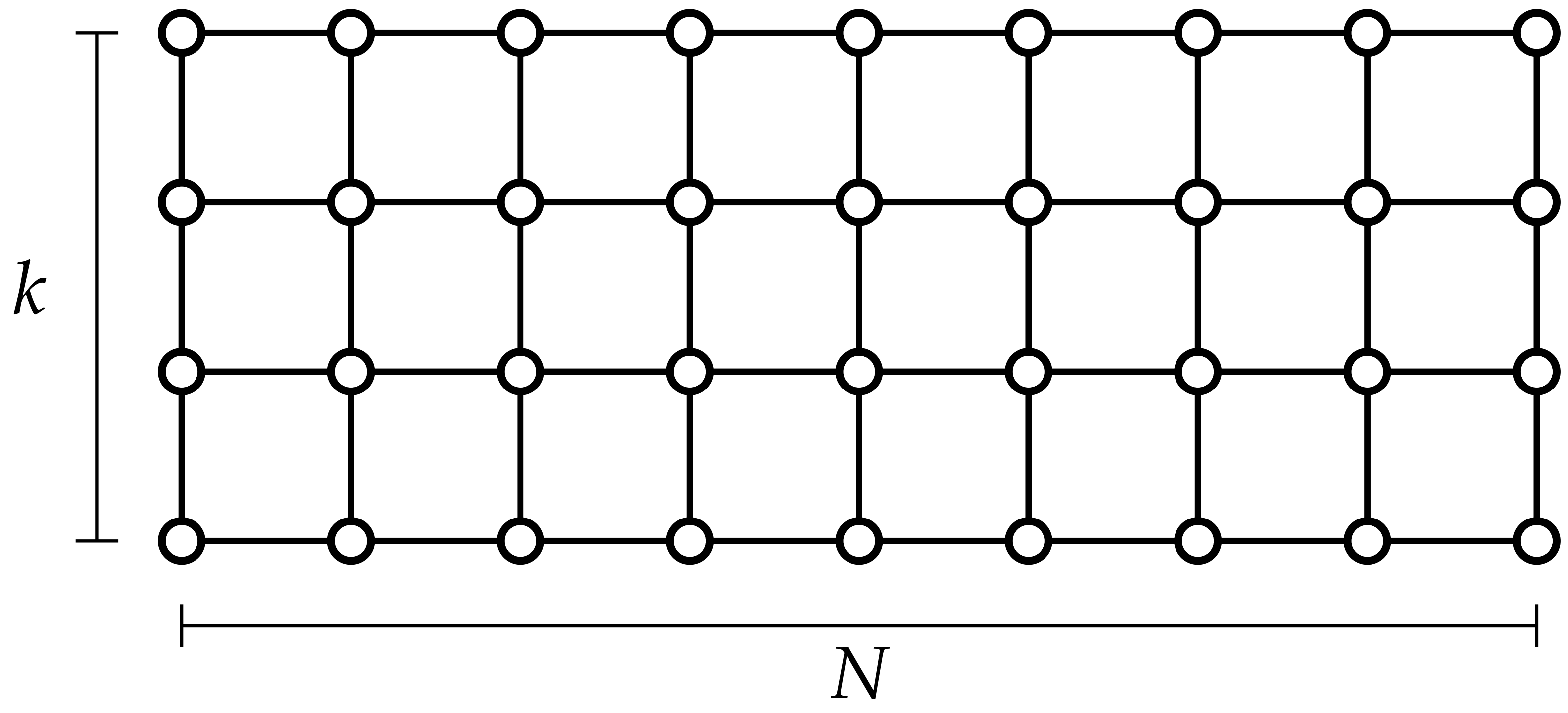
Algorithm: Compute $A(\cdot)$ and $B(\cdot)$ bottom-up, return $A(r)$.

Grid Graphs

In a $k \times N$ **grid graph**

- the vertex set consist of all pairs (i, j) where $1 \leq i \leq k$ and $1 \leq j \leq N$, and
- two vertices (i_1, j_1) and (i_2, j_2) are adjacent if and only if $|i_1 - i_2| + |j_1 - j_2| = 1$.



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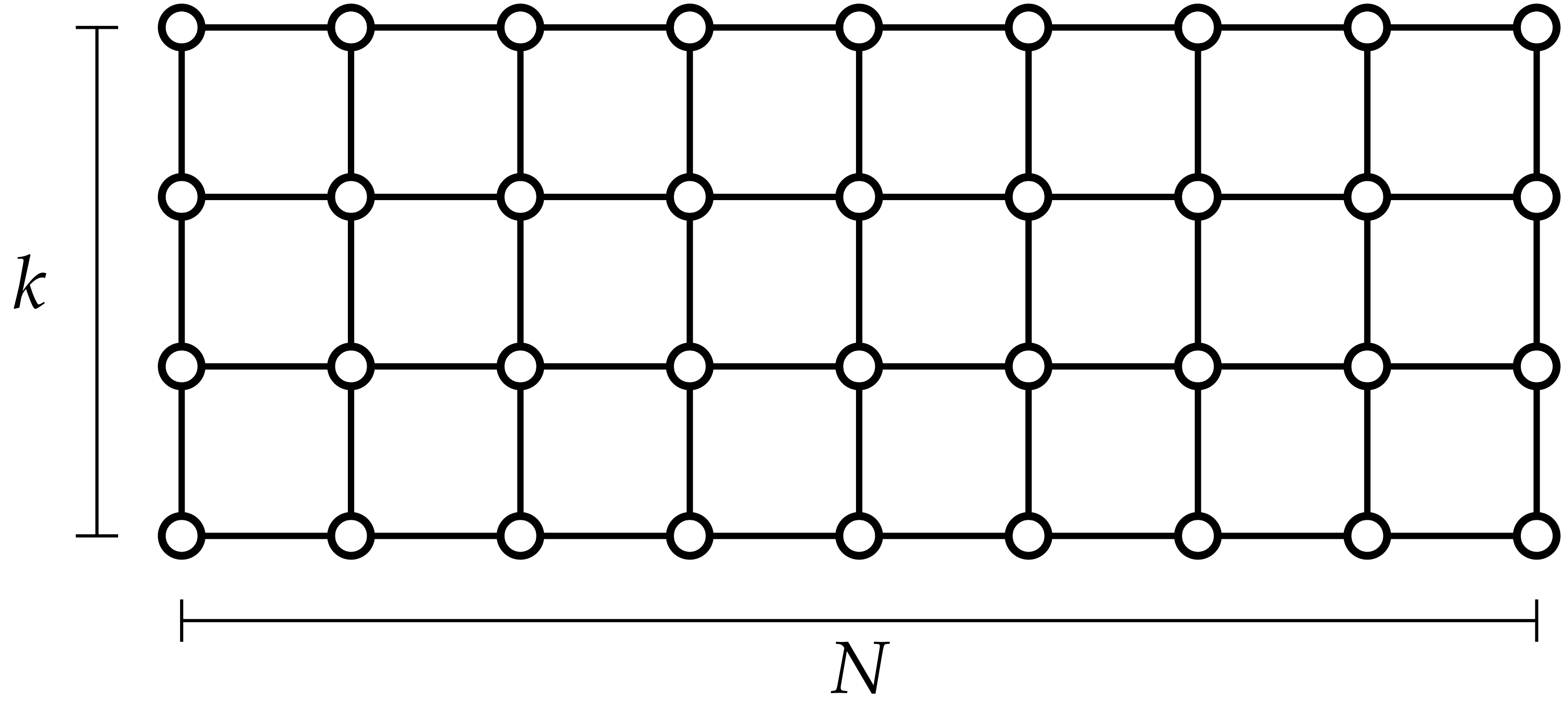

We will study Independent Set in subgraphs of $k \times N$ grid graphs.

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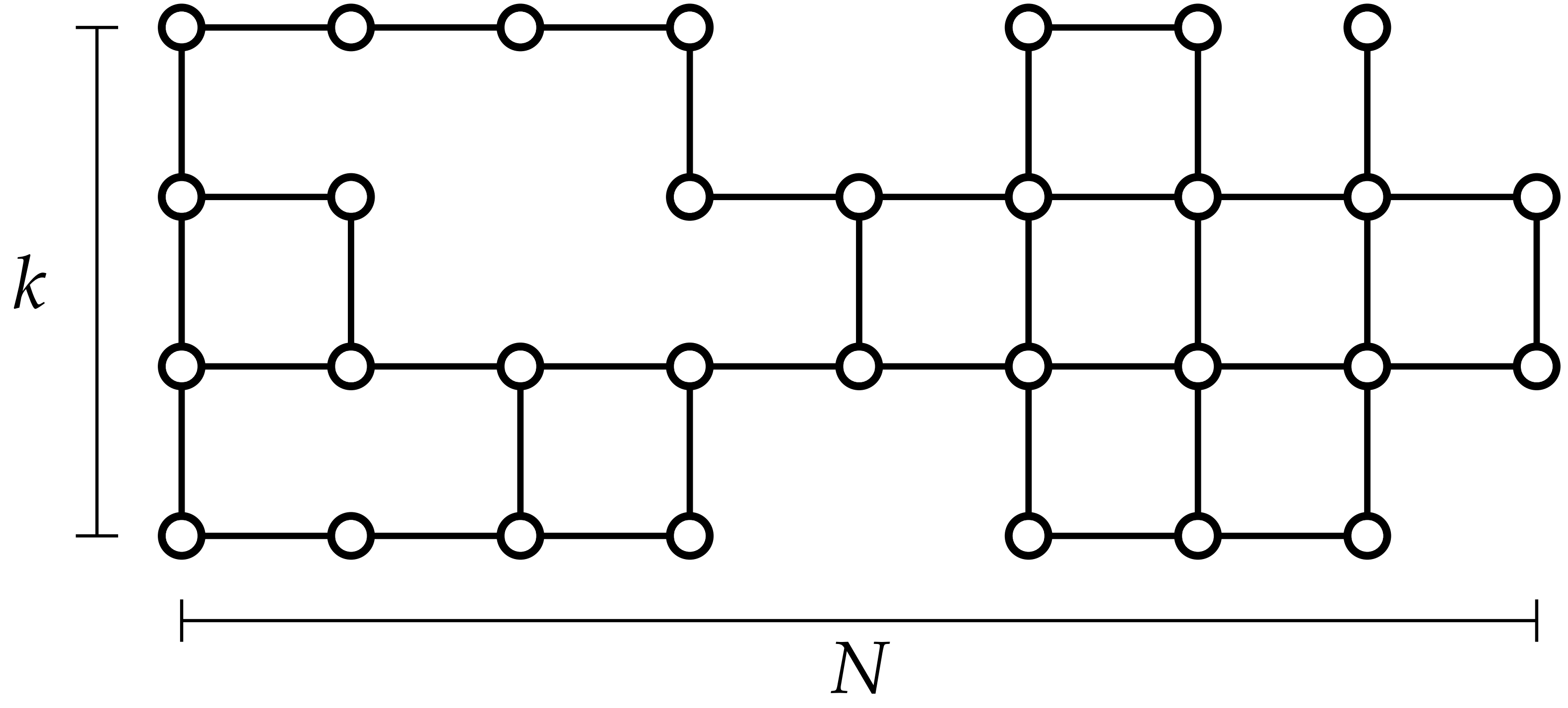

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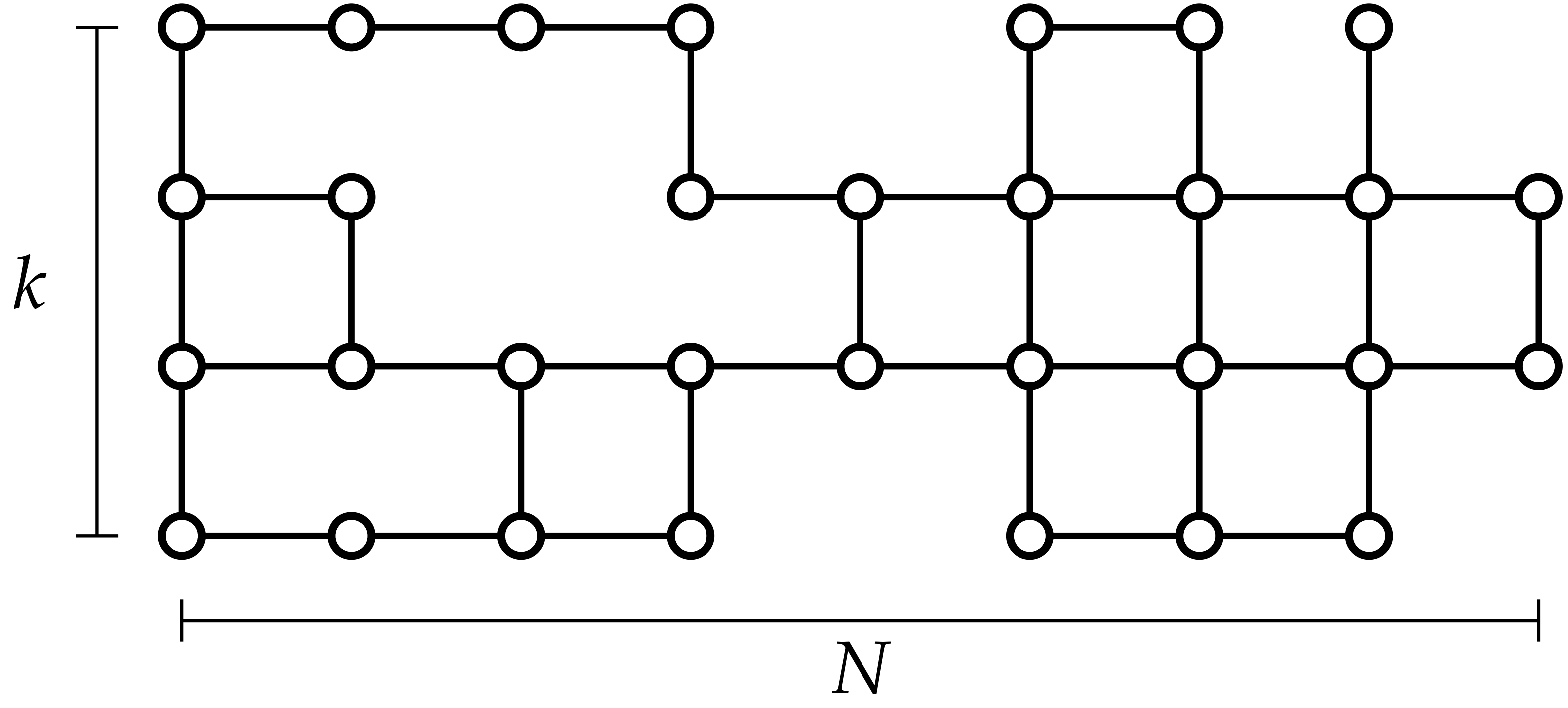

We will study Independent Set in subgraphs of $k \times N$ grid graphs.

Goal: An FTP algorithms with respect to parameter k.

Indenpendent Set in $k \times N$ Grid Graphs

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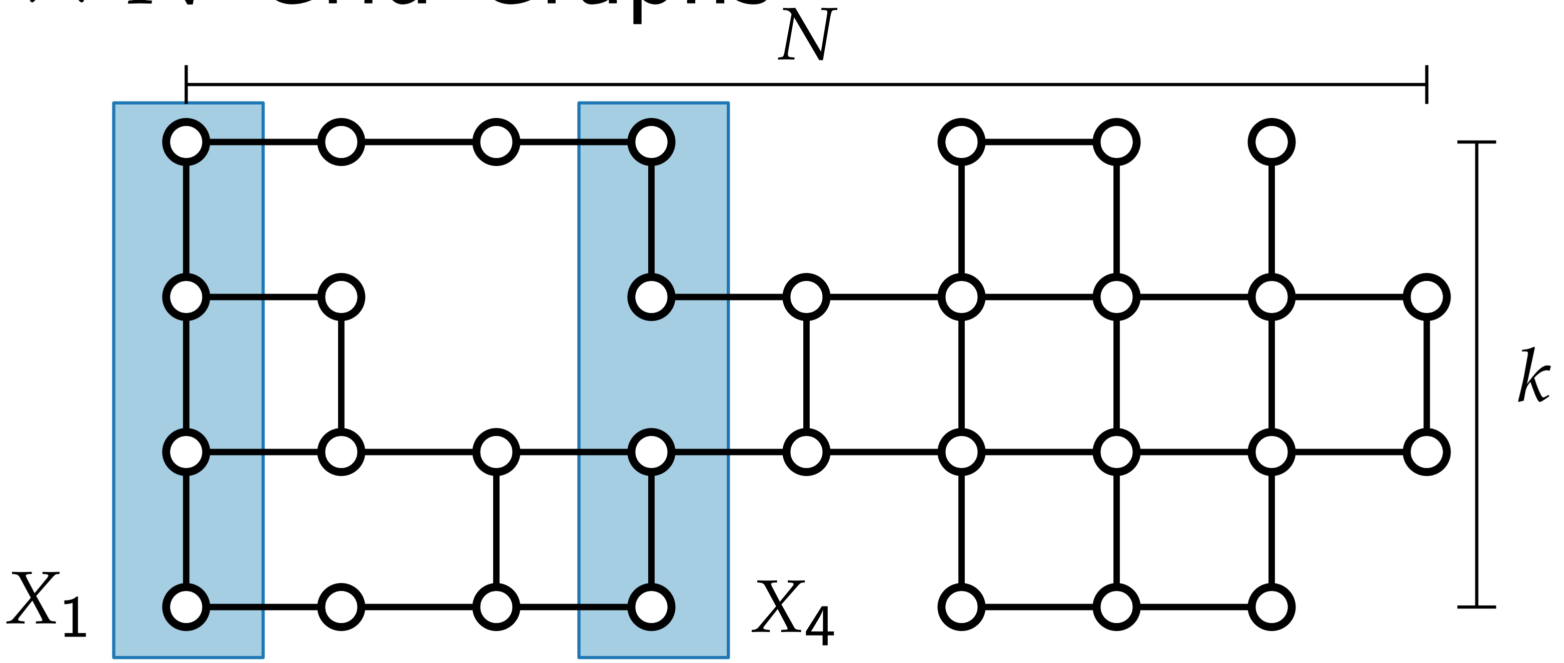

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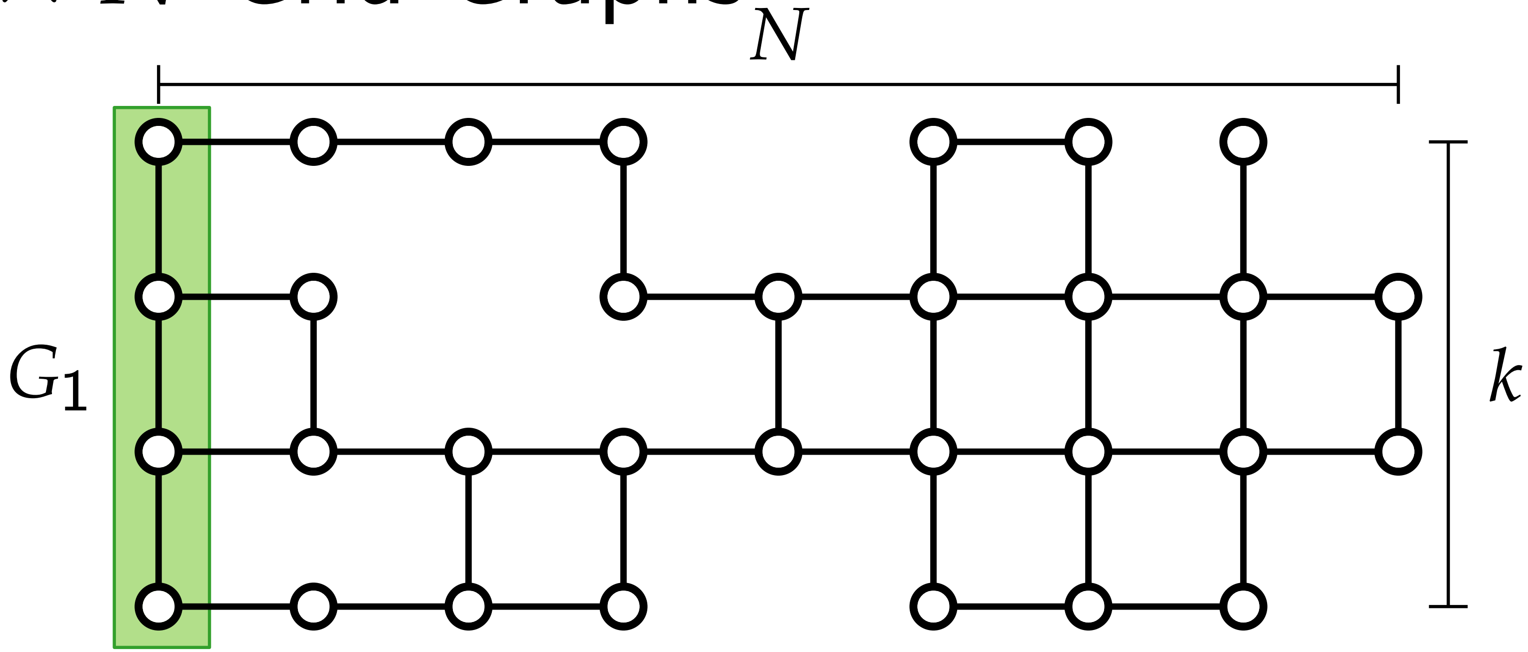

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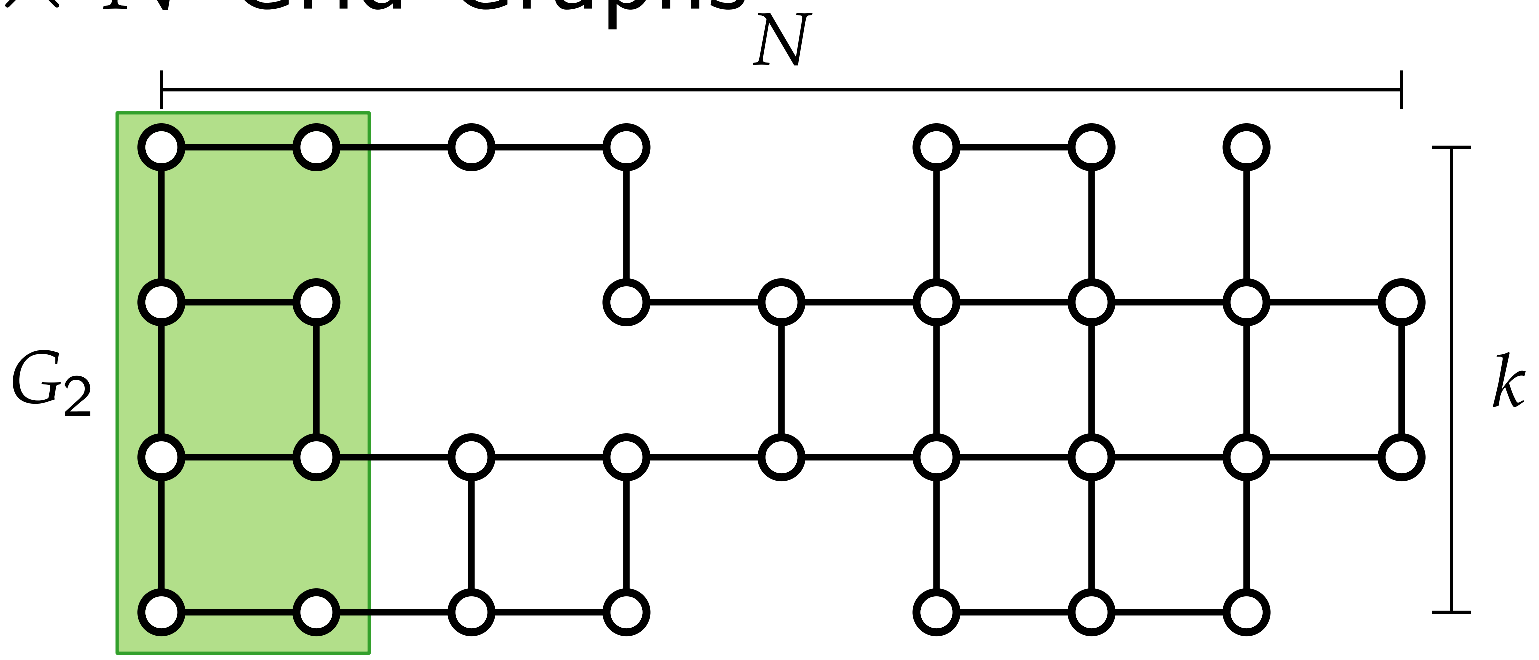

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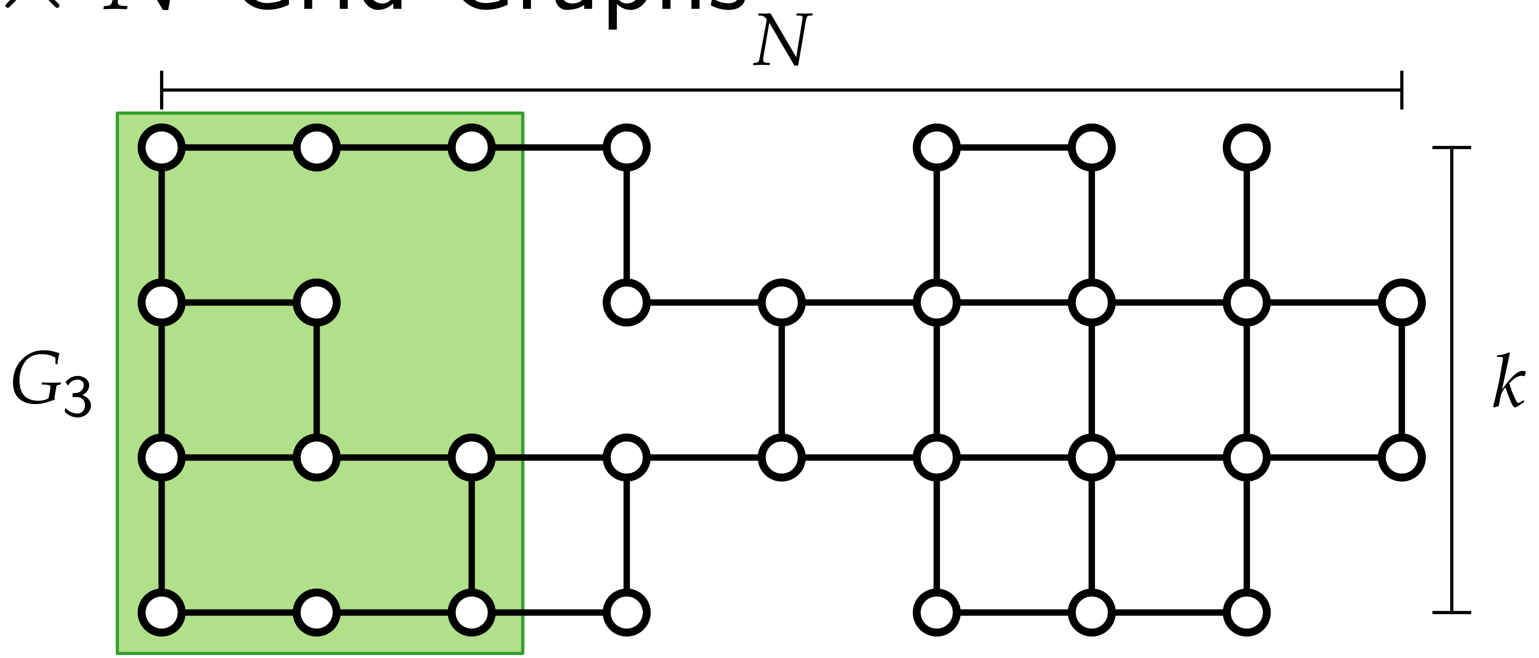

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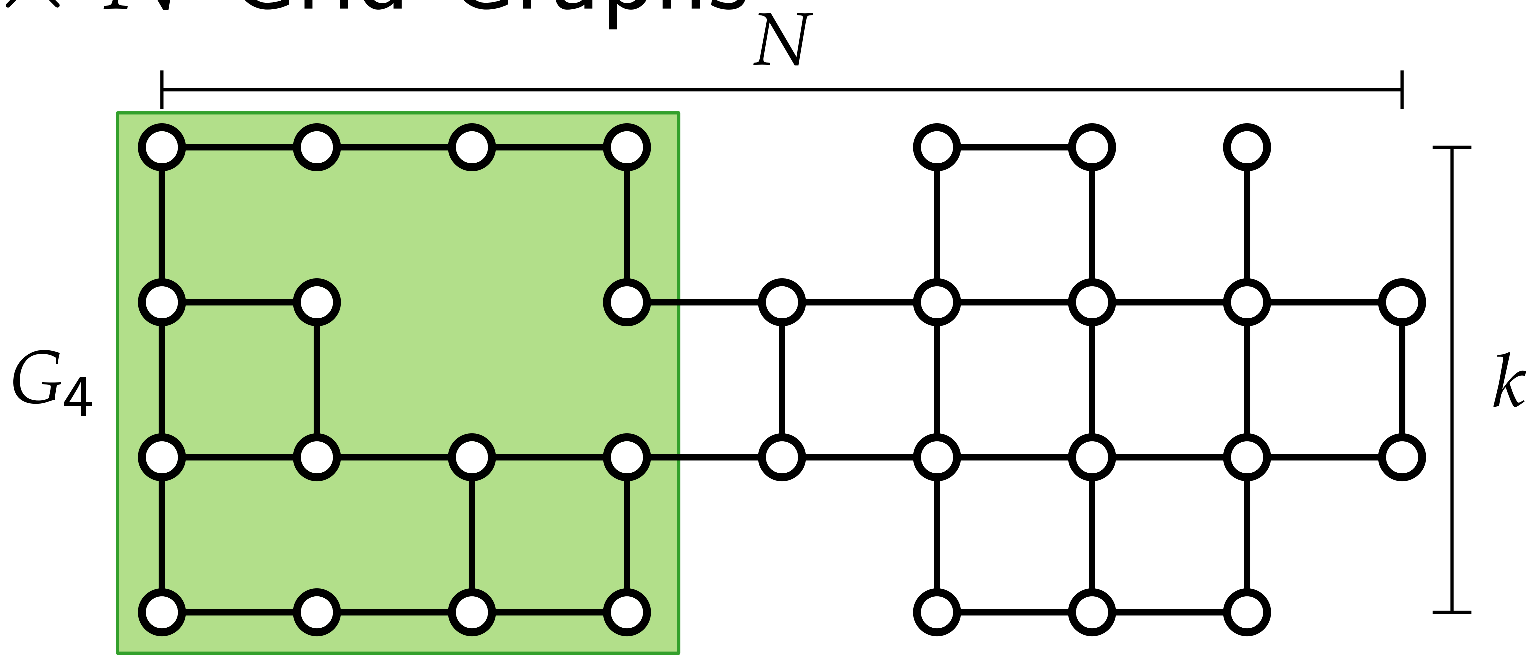

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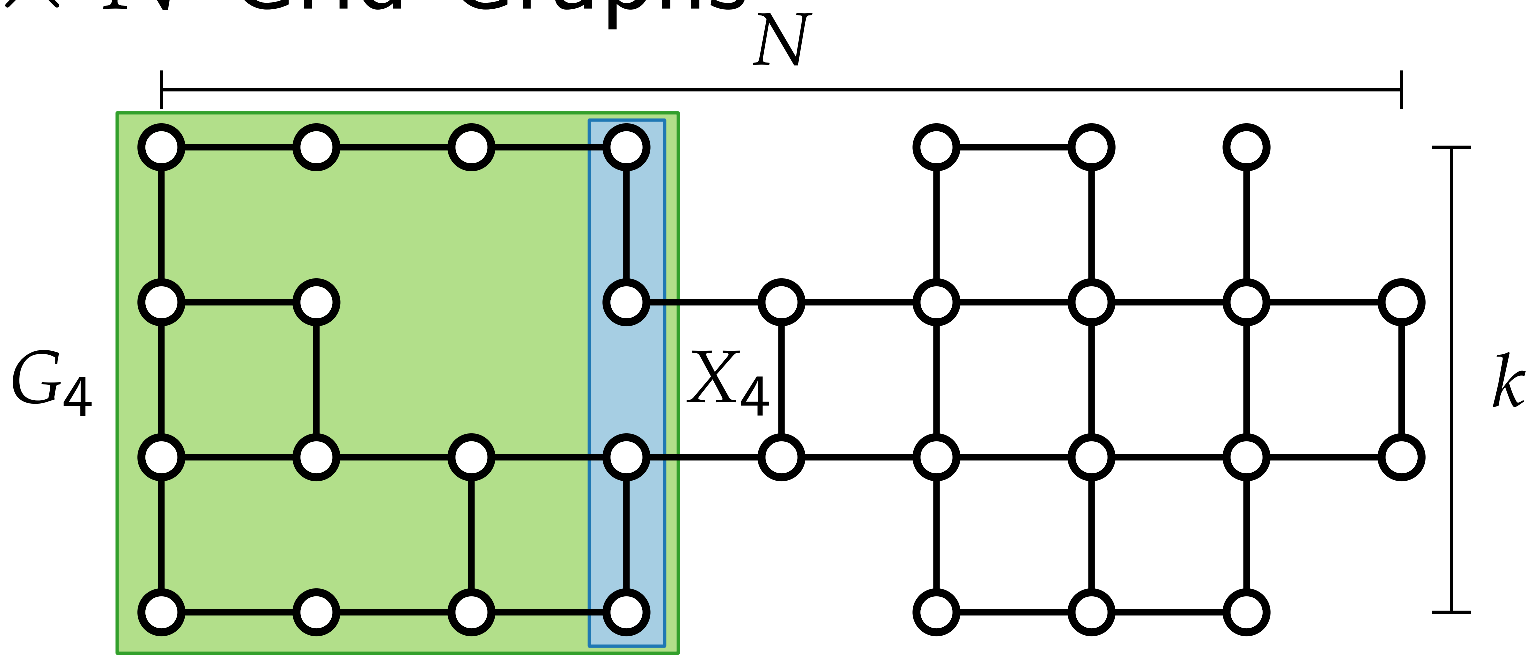

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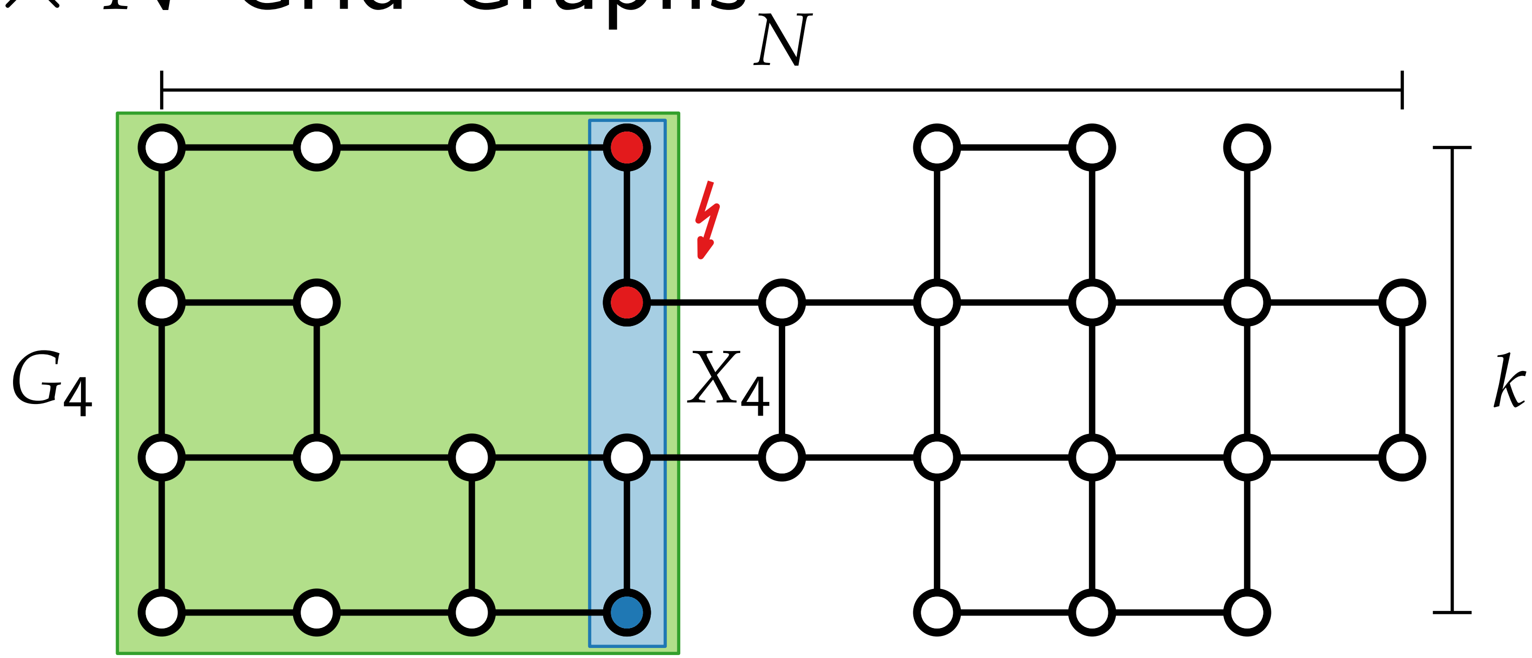

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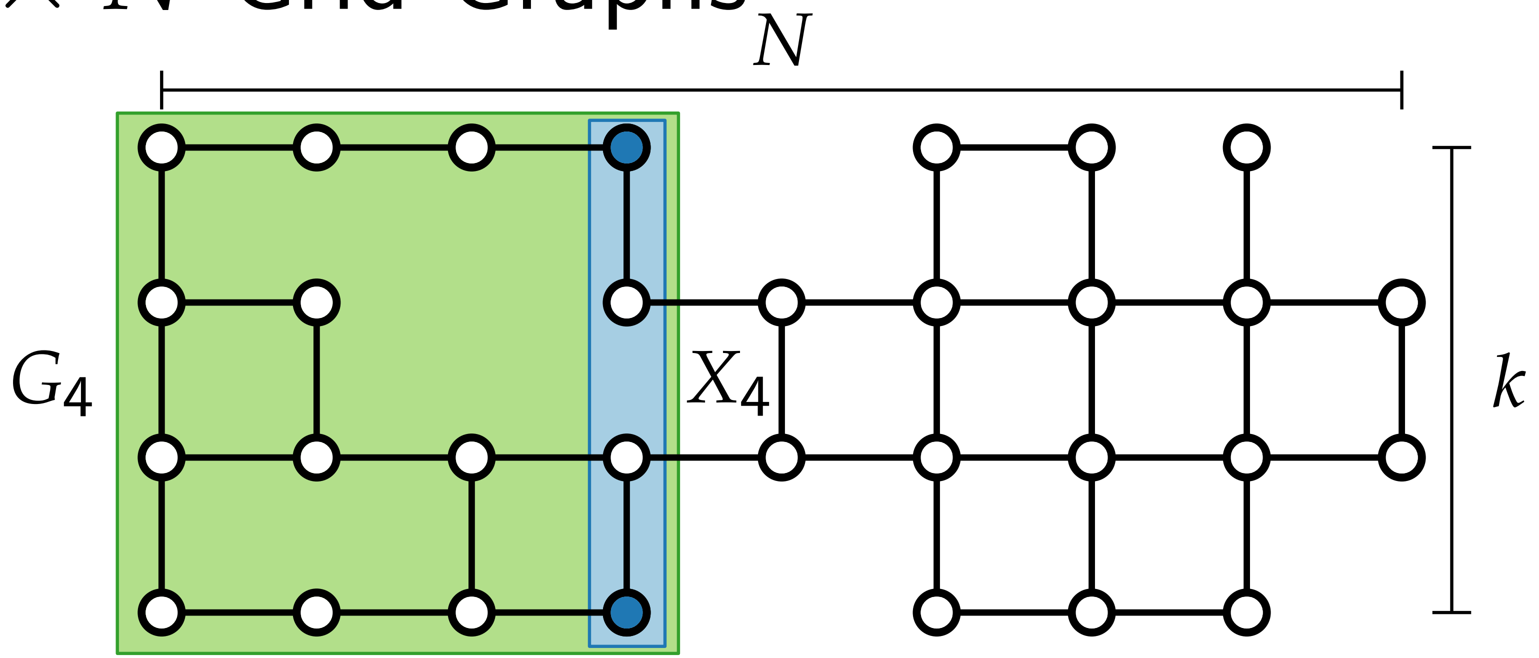

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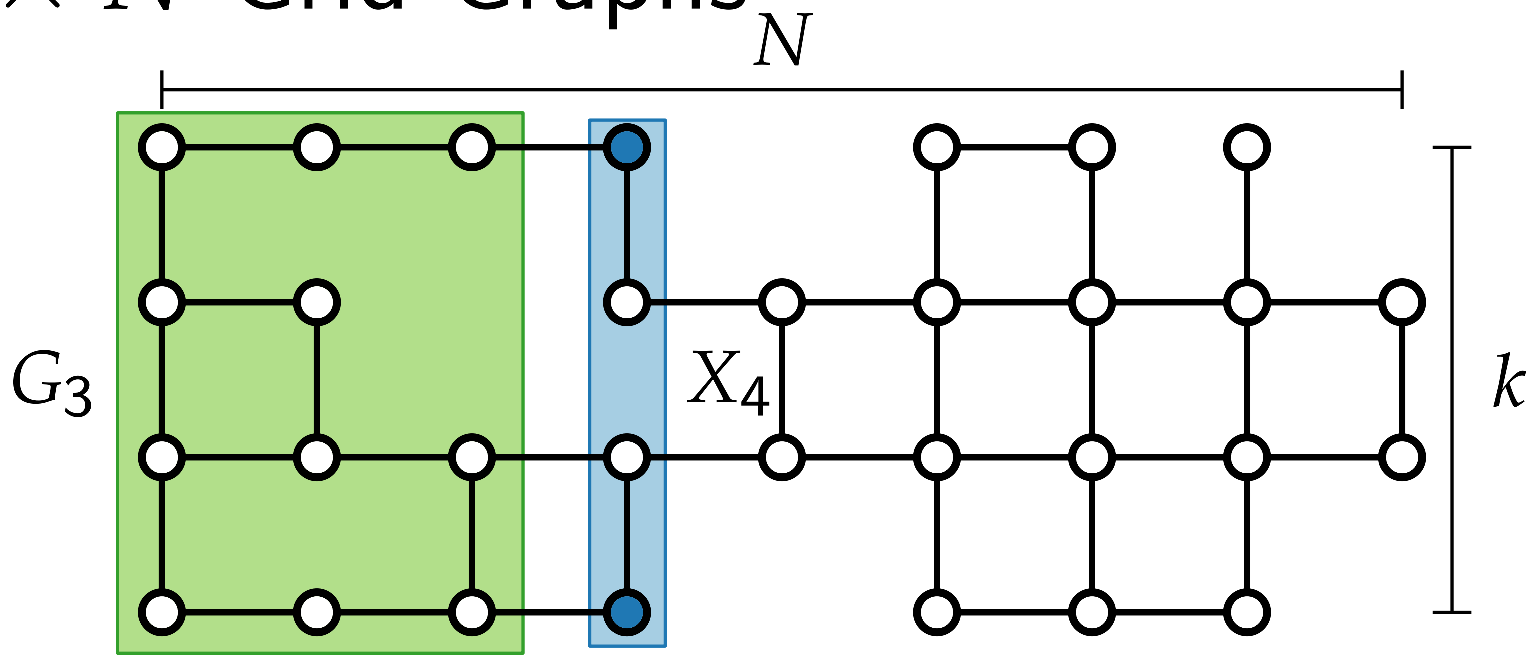

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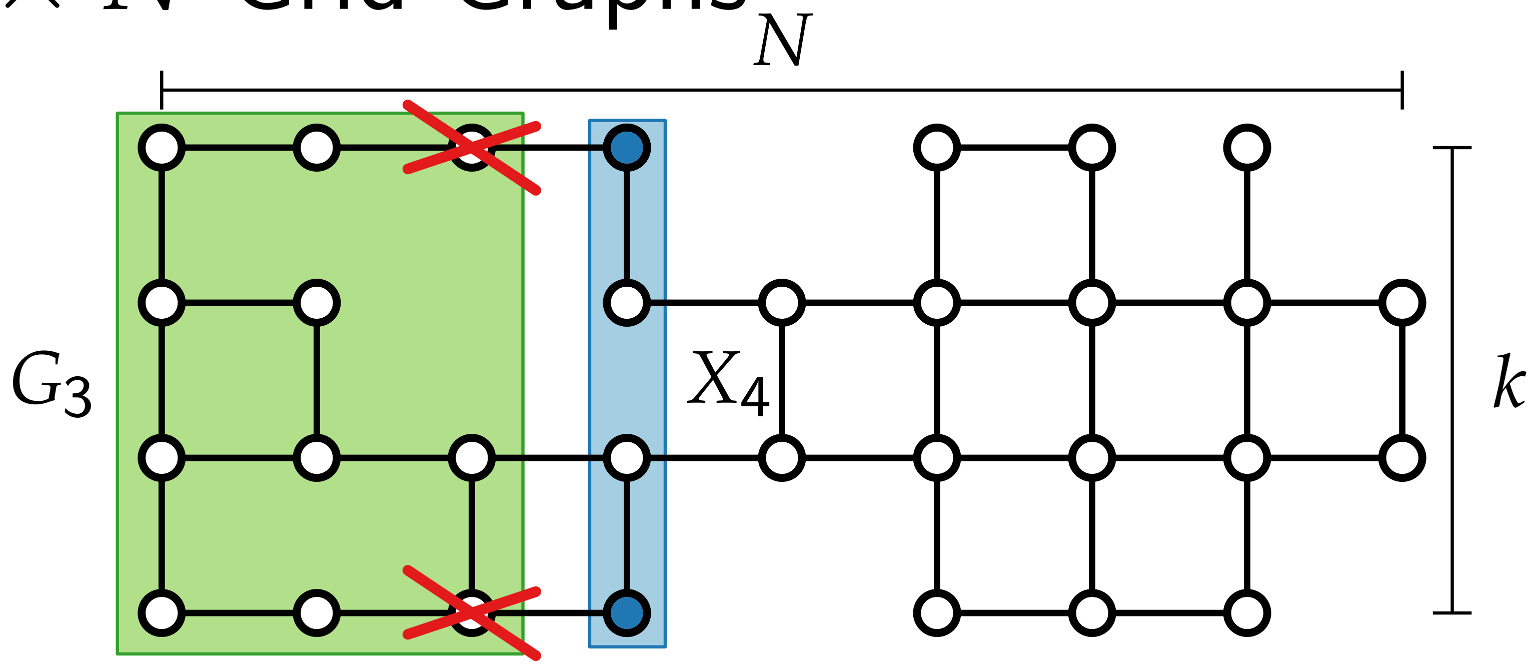

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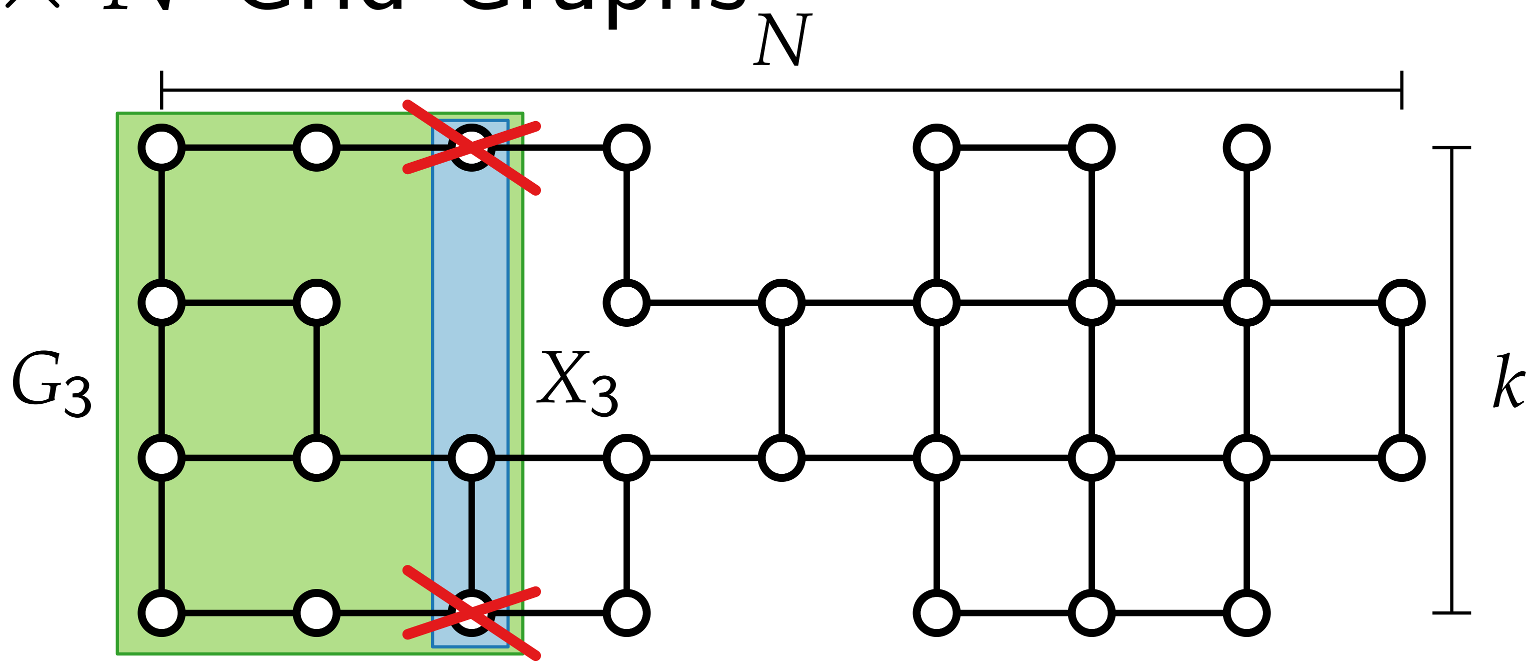

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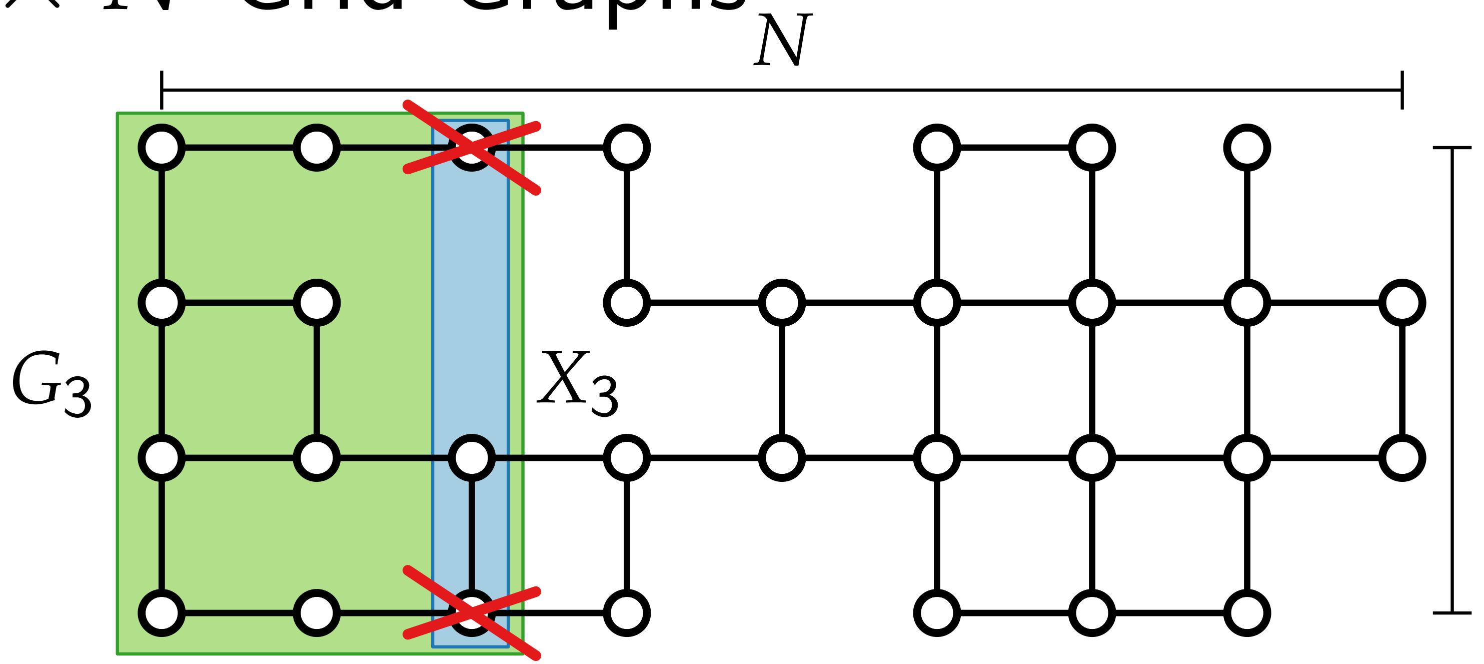
Let $1 \le j \le N$. For each $Y \subseteq X_j$ let



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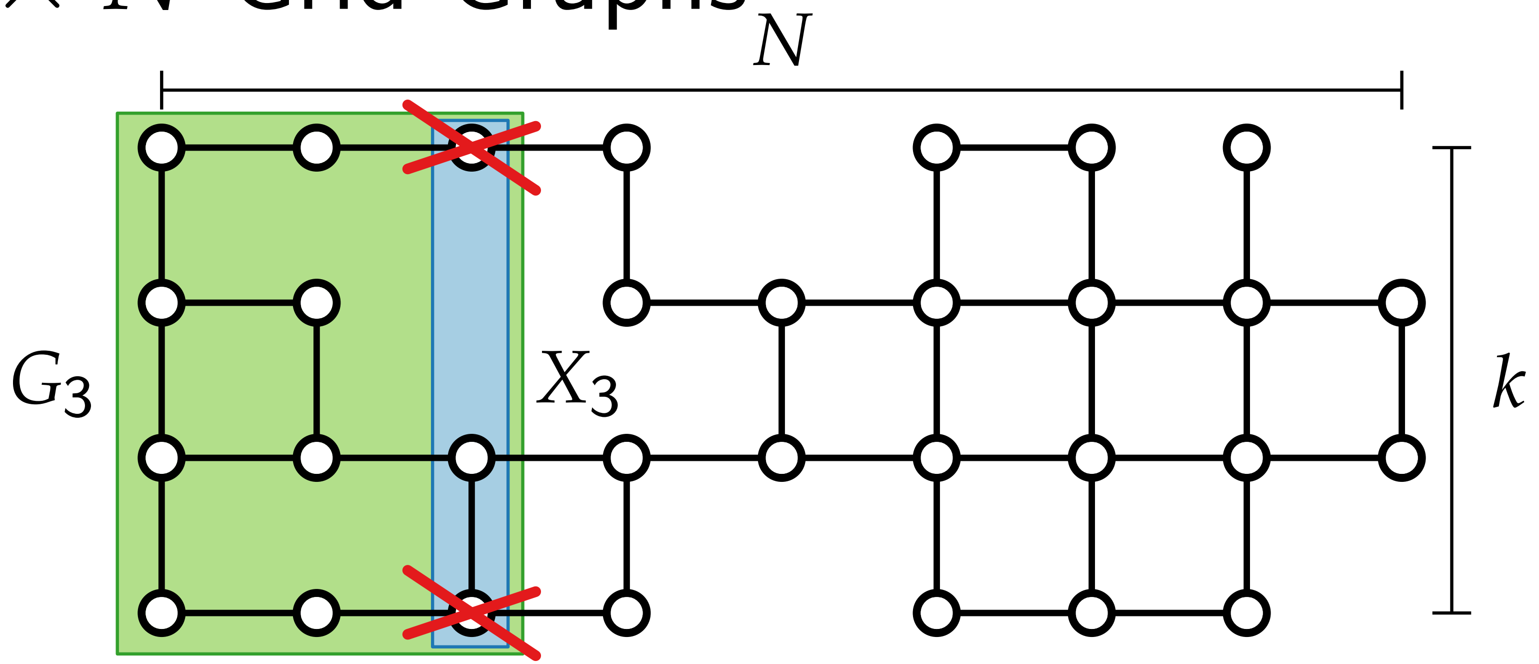
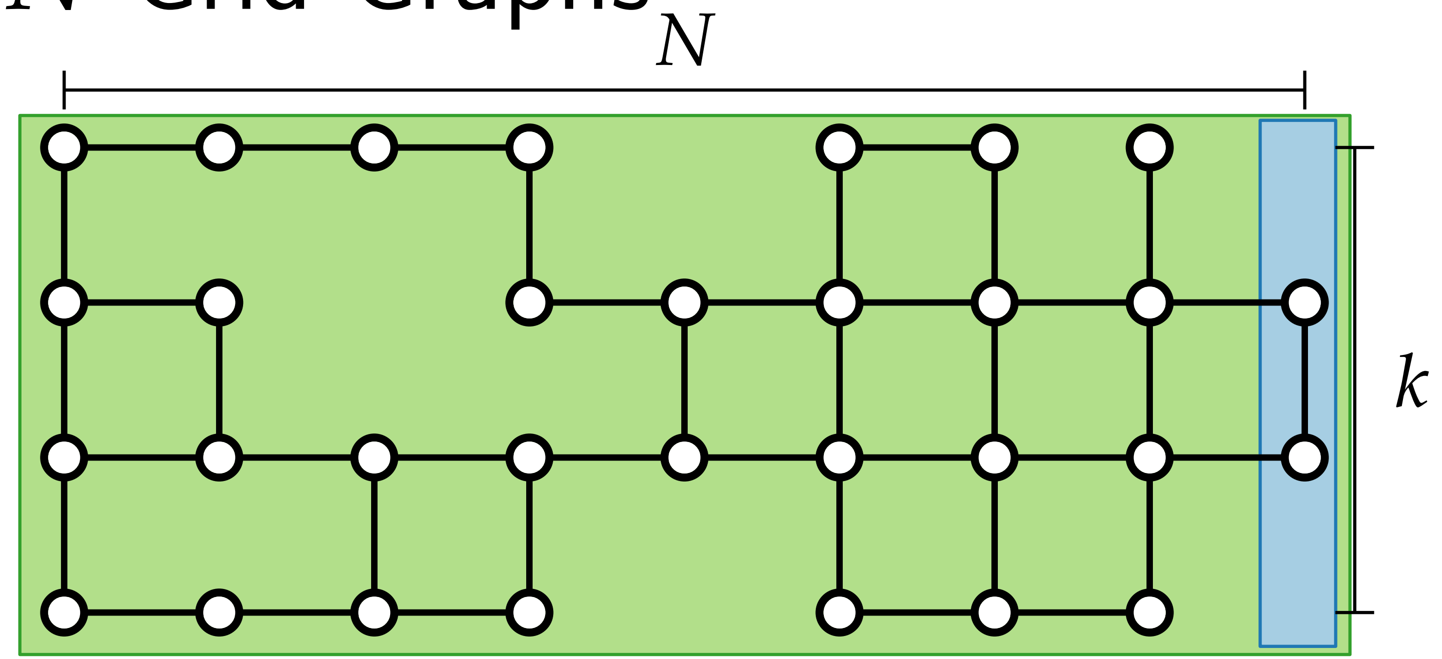


$C[j, Y] :=$ maximum weight of an independent set I in G_j such that $I \cap Y = \emptyset$

INDENPENDENT SET in $k \times N$ Grid Graphs



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$C[N, \emptyset] =$ solution

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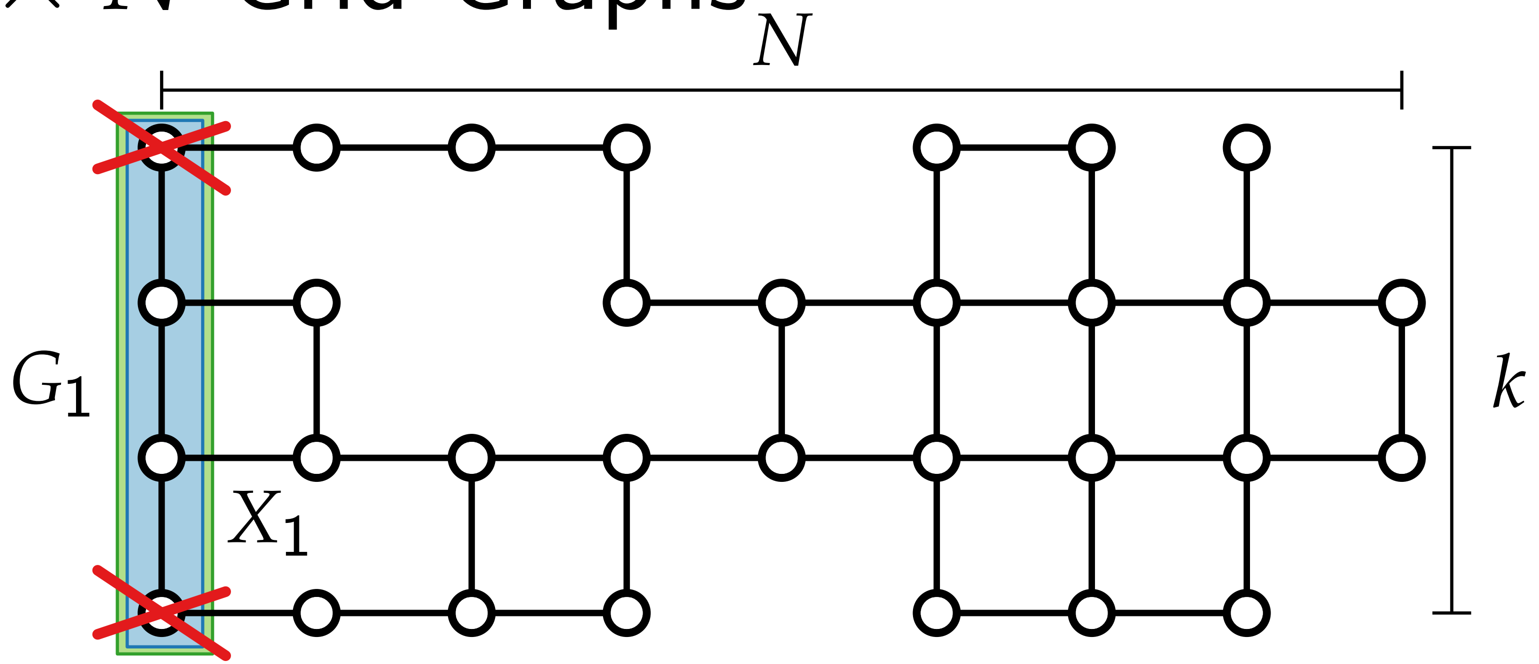

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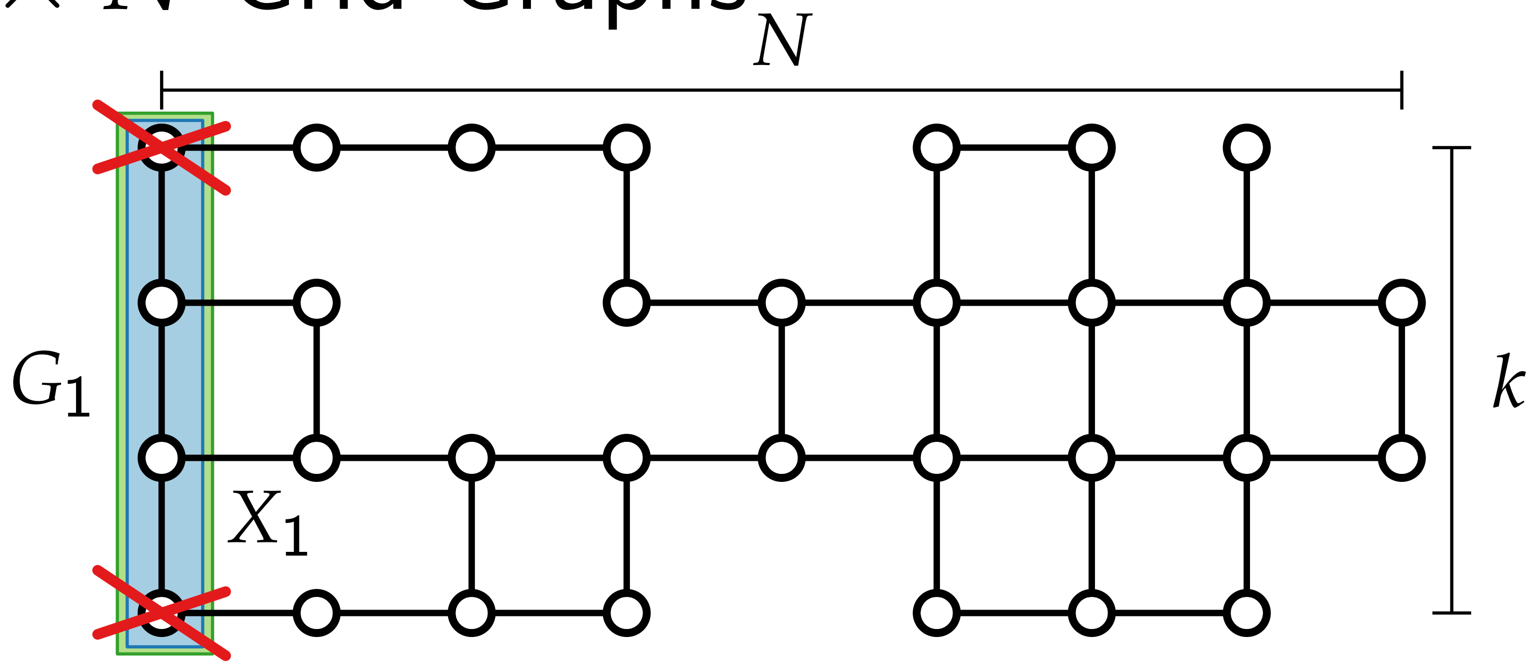

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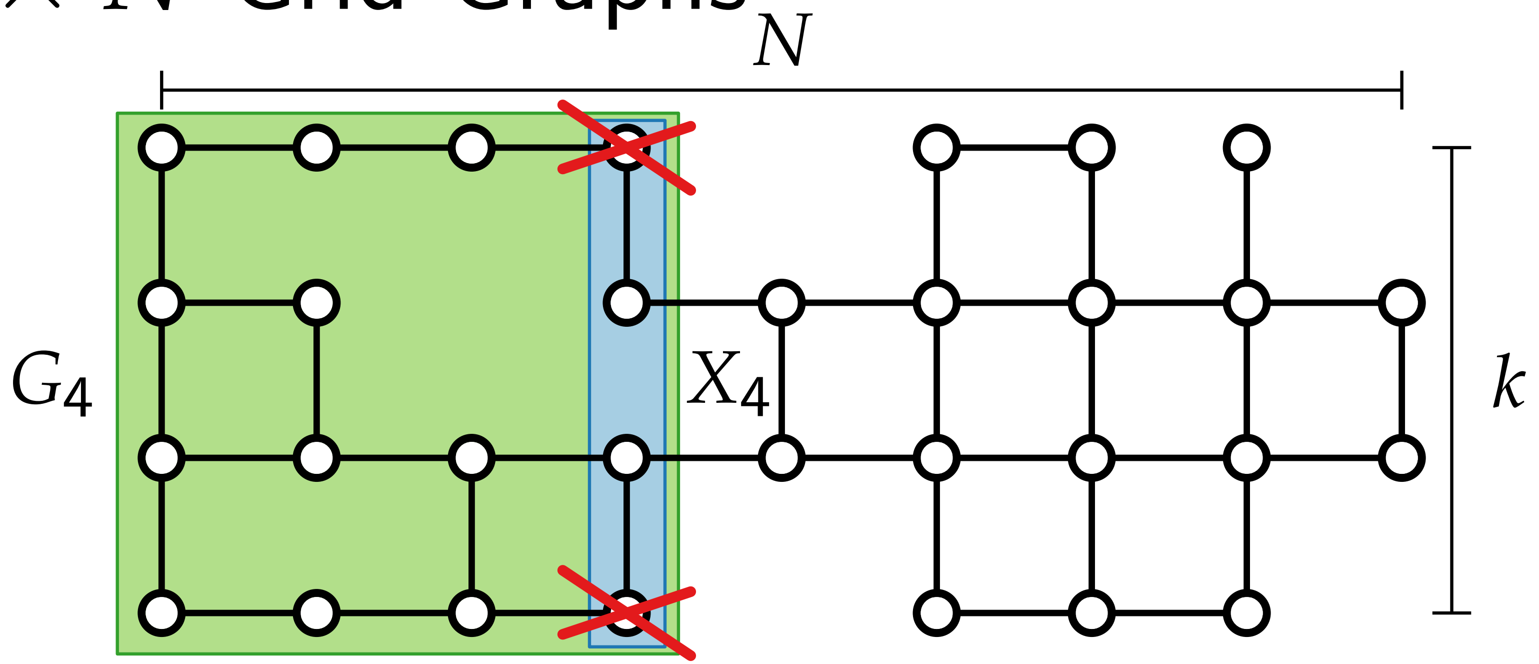

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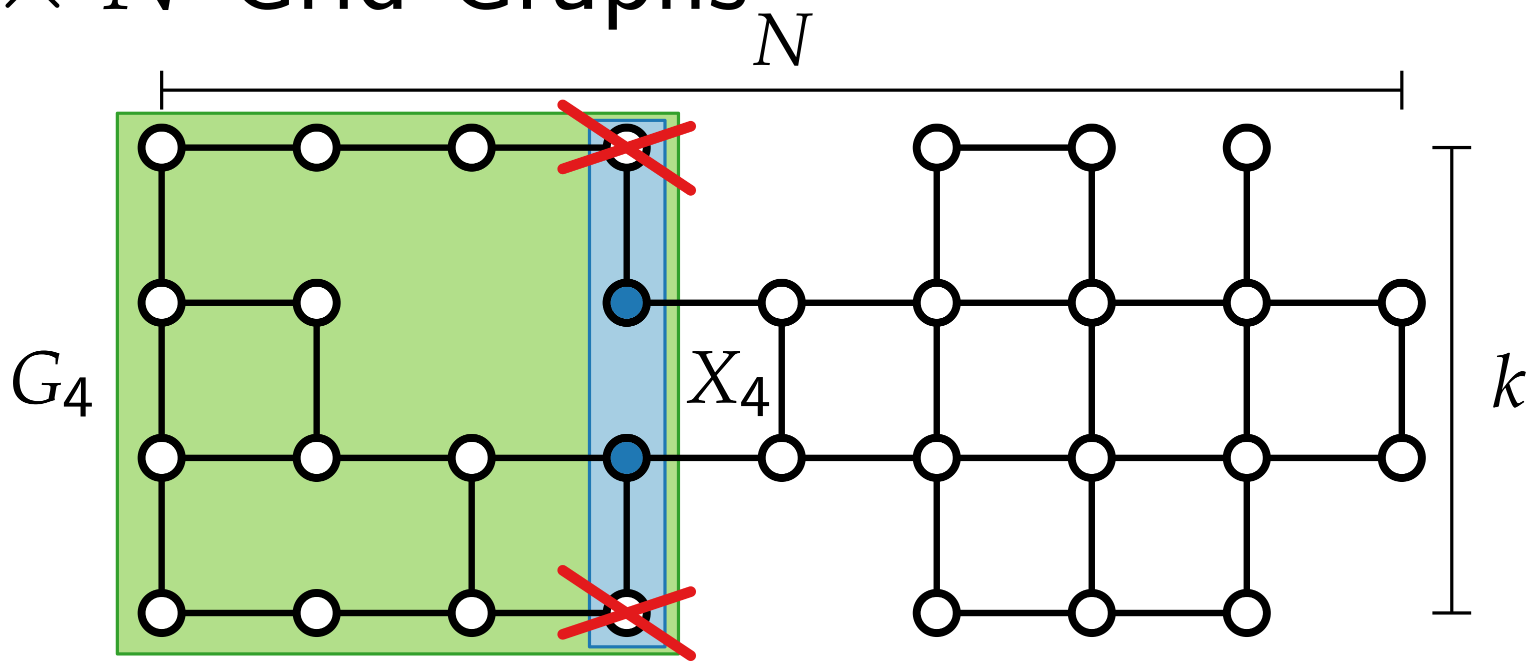

$C[j, Y] :=$ maximum weight of an independent set I in G_j such that $I \cap Y = \emptyset$

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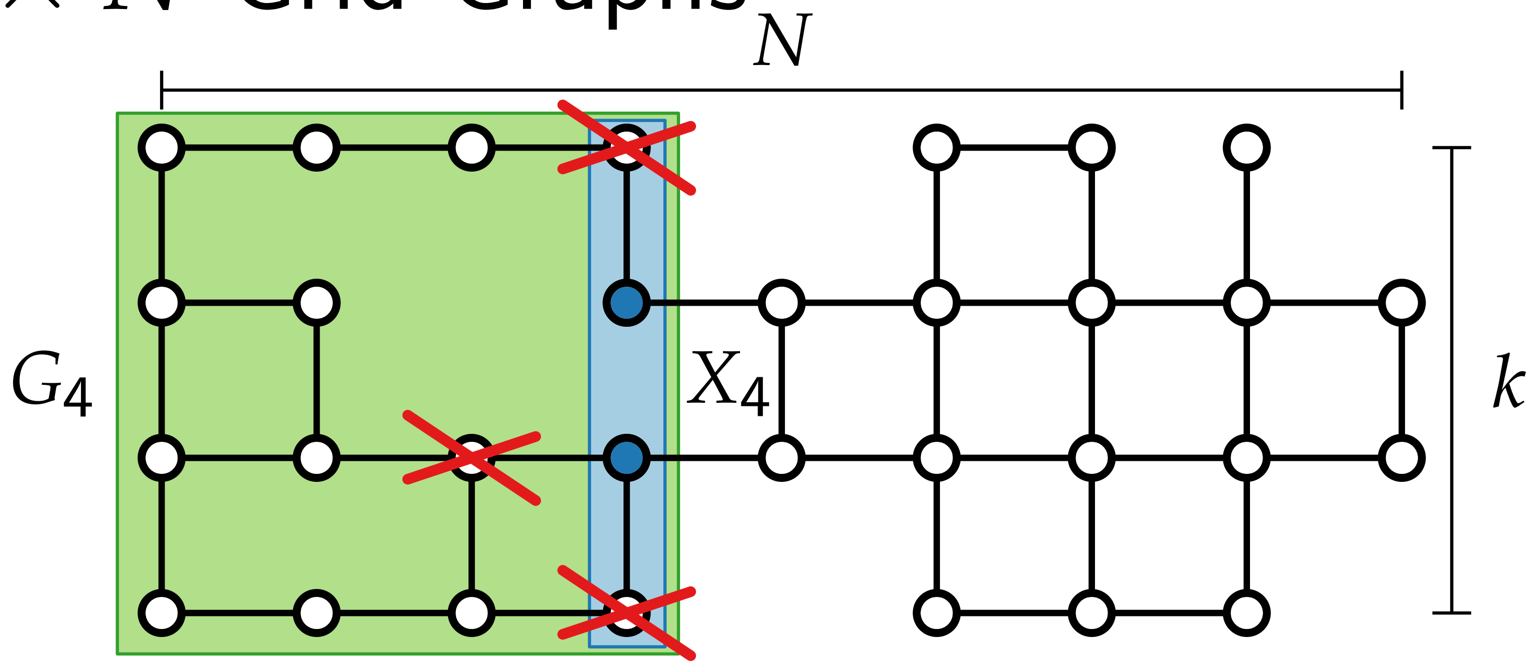

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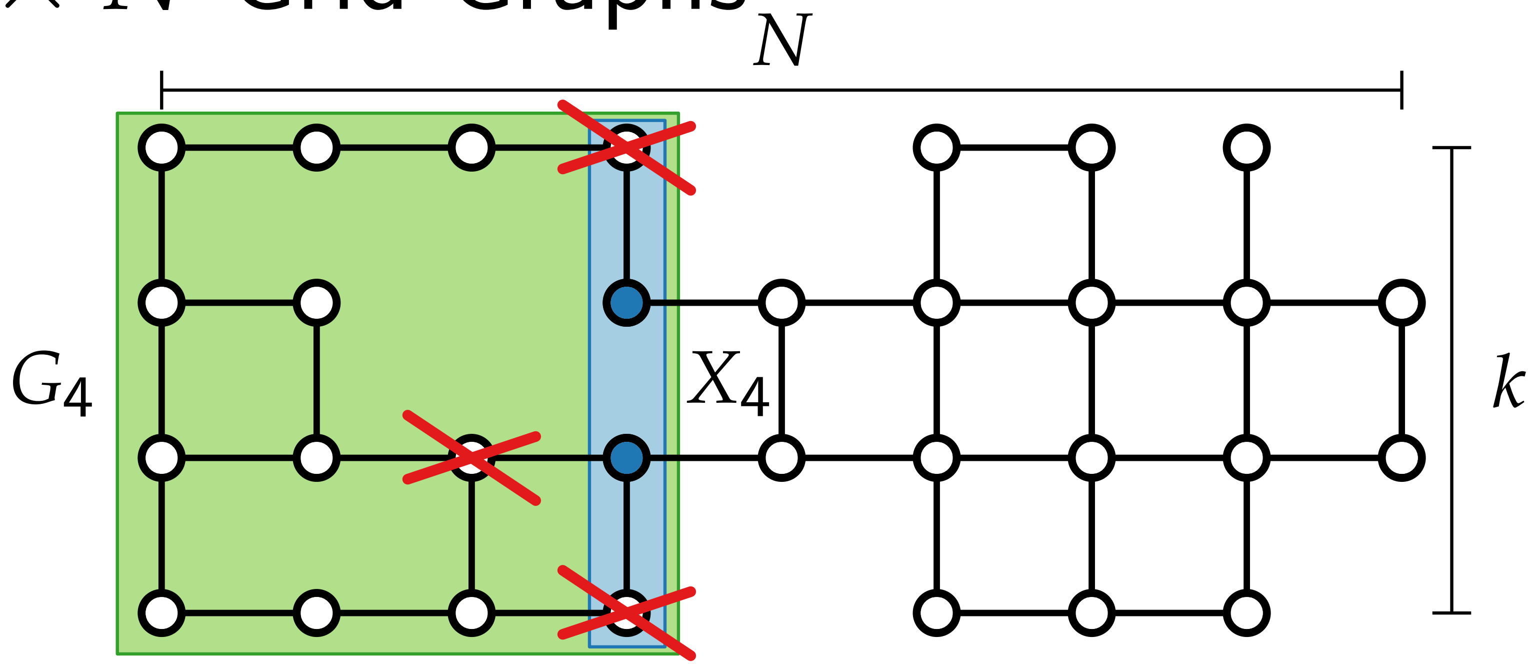

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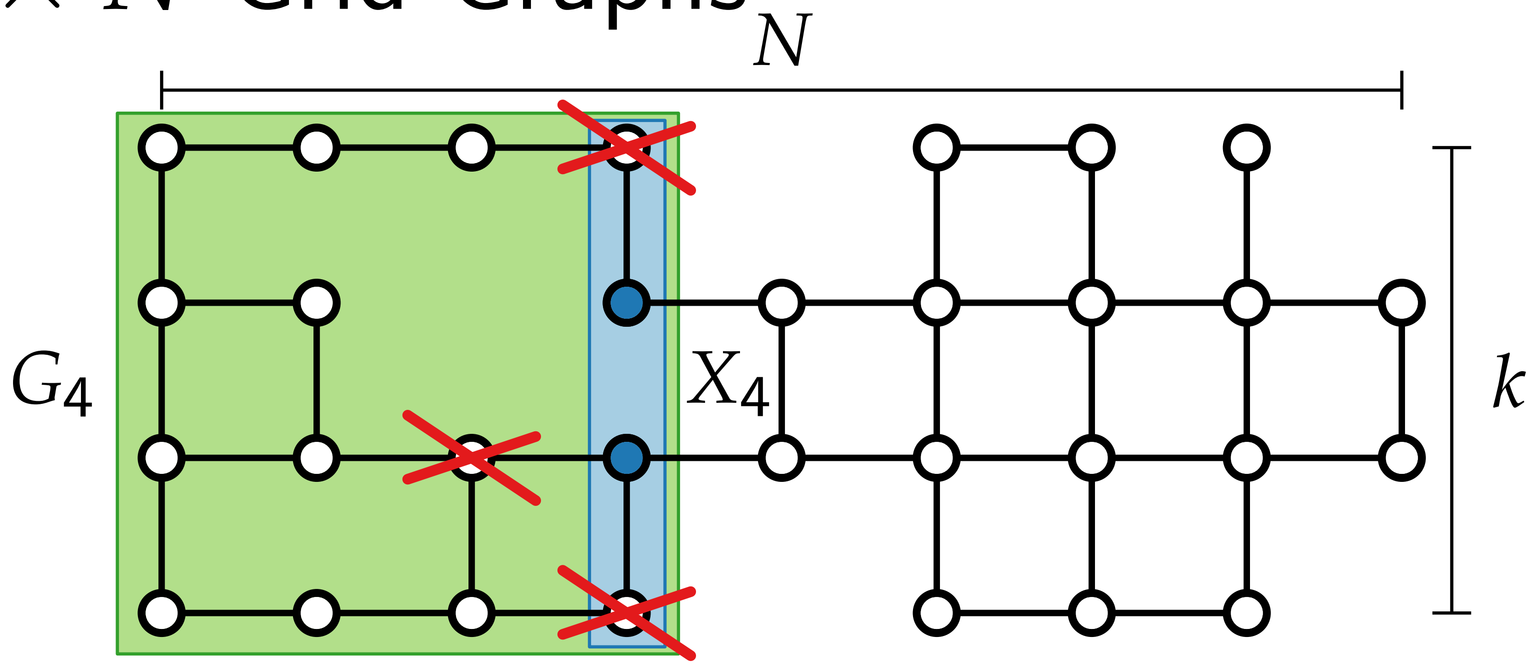

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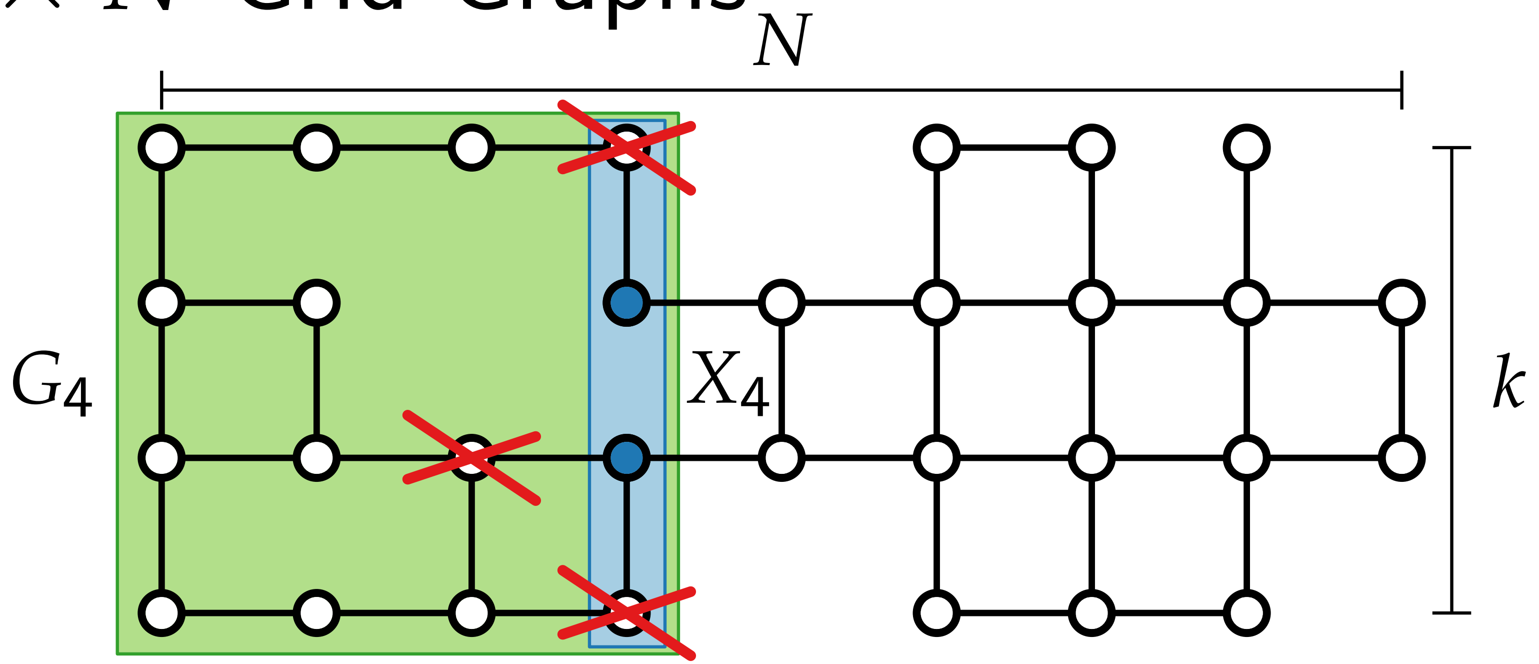

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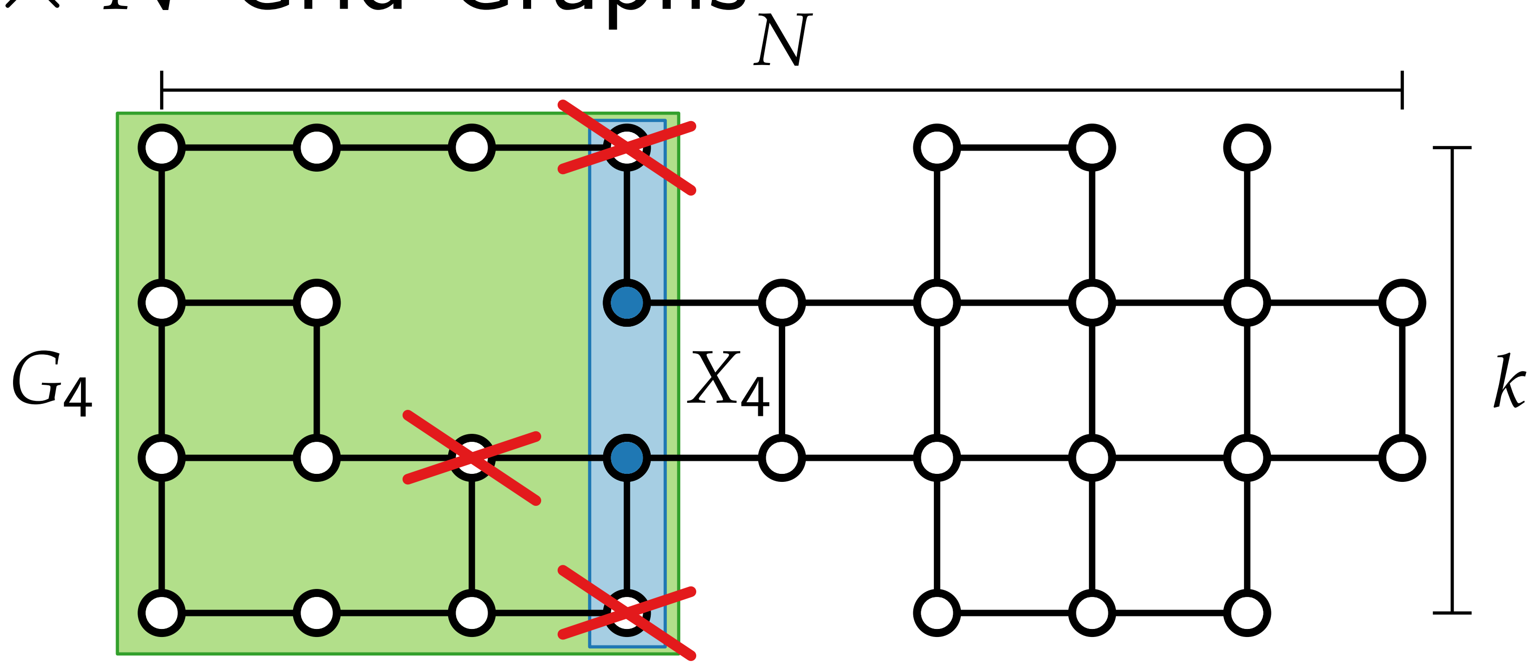

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each element in a column has one of three options: being in Y or I or none of them

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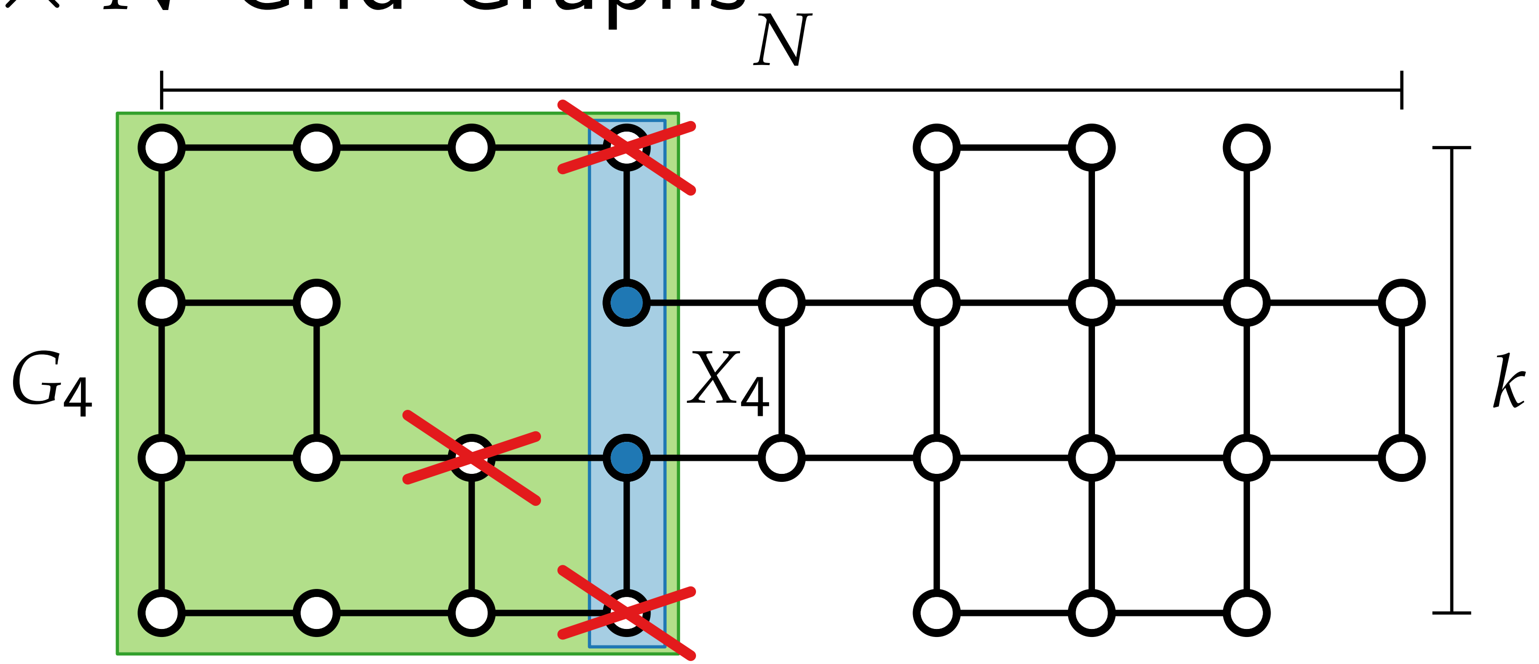

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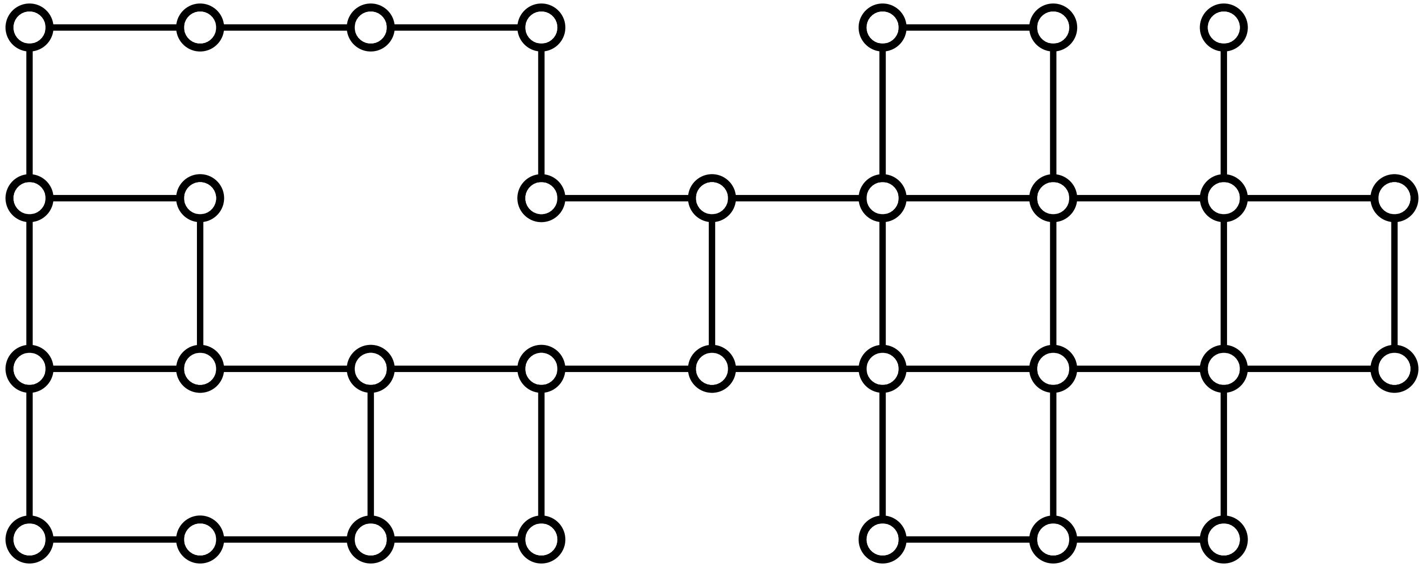
$\rightarrow$ total running time $\leq 3^k k^{\mathcal{O}(1)} N$.

Can We Apply This Approach to Other Graphs?



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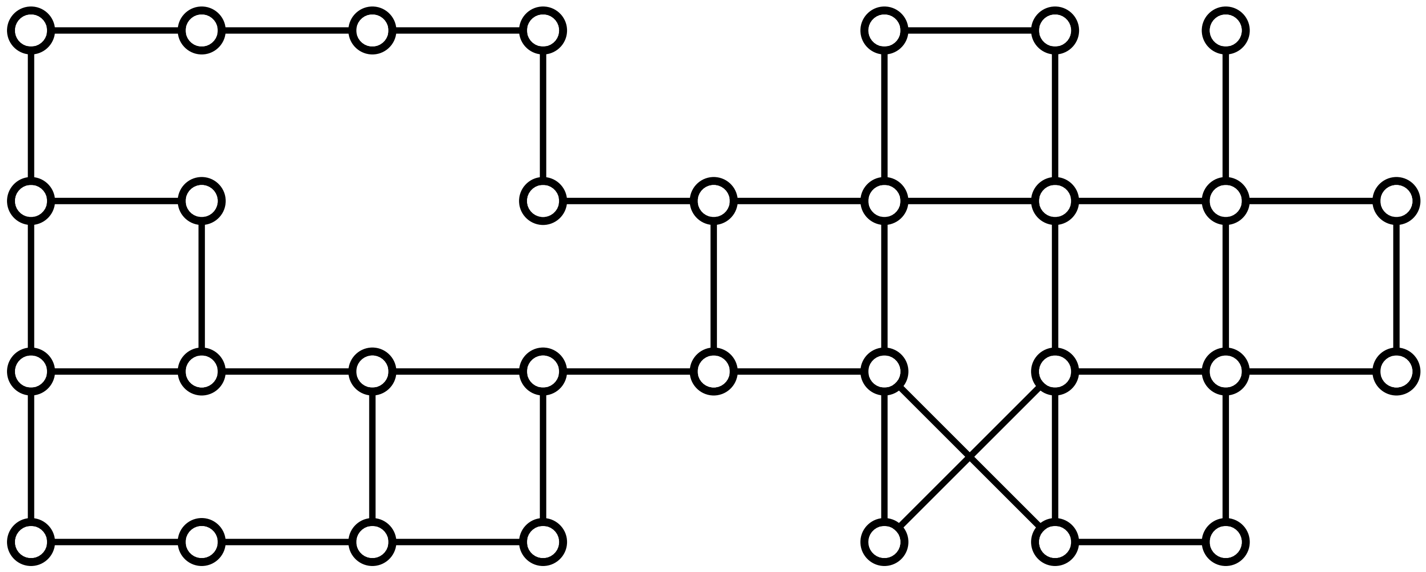

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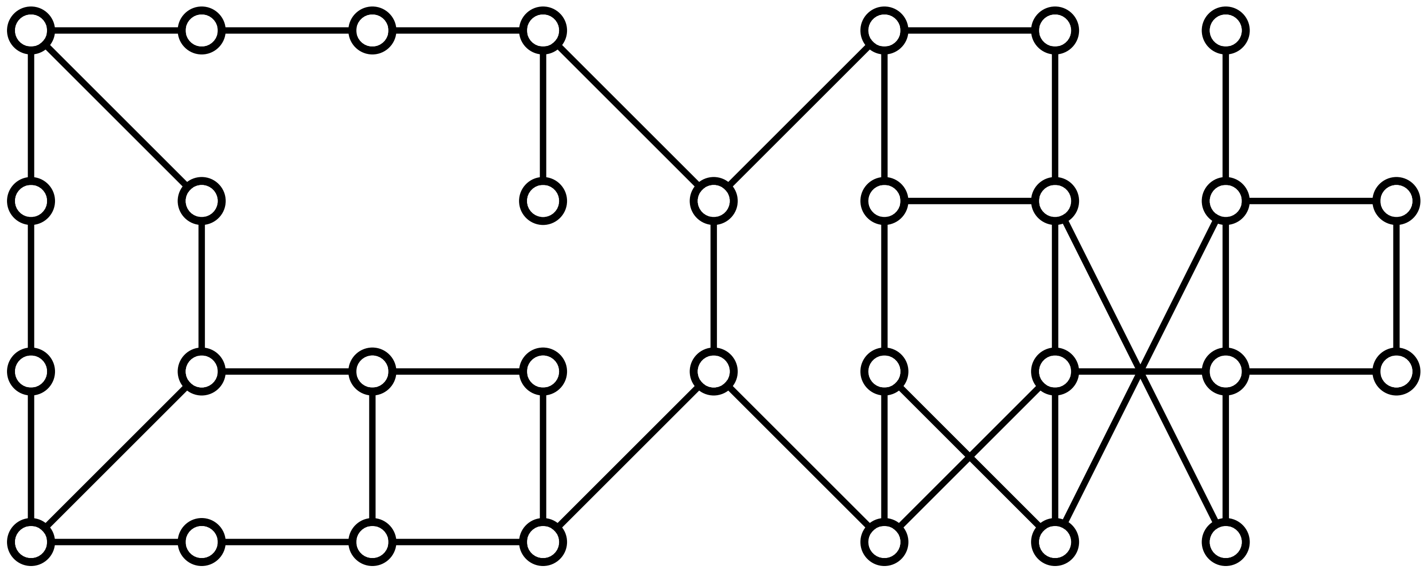

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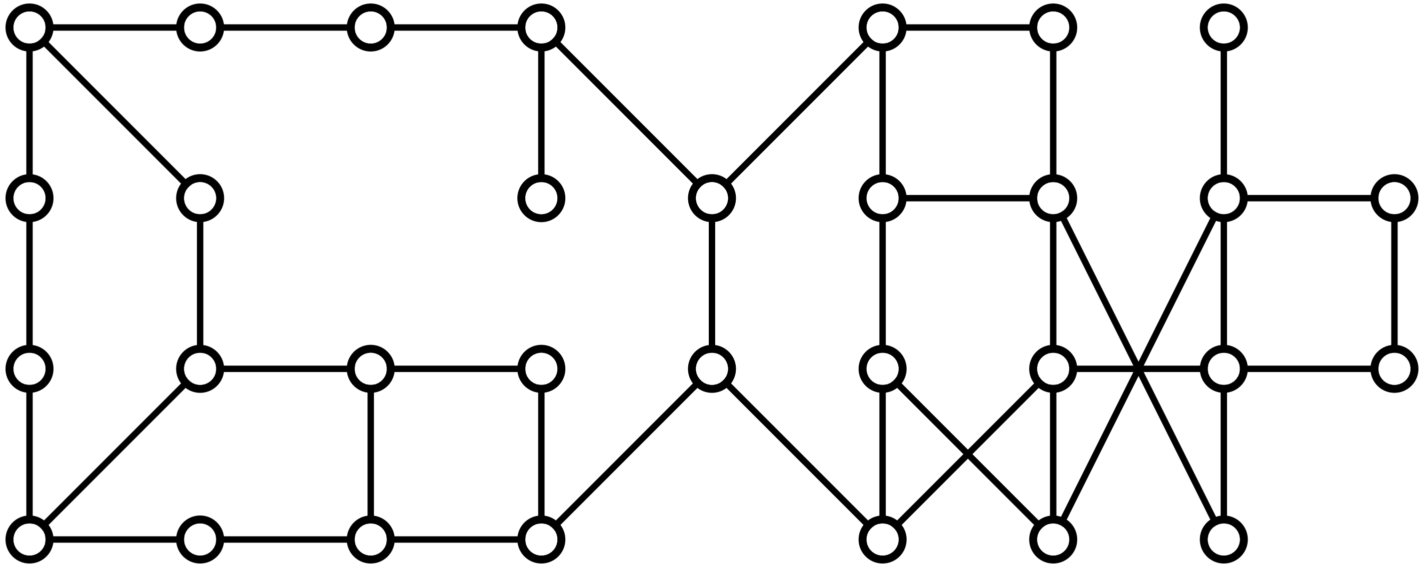

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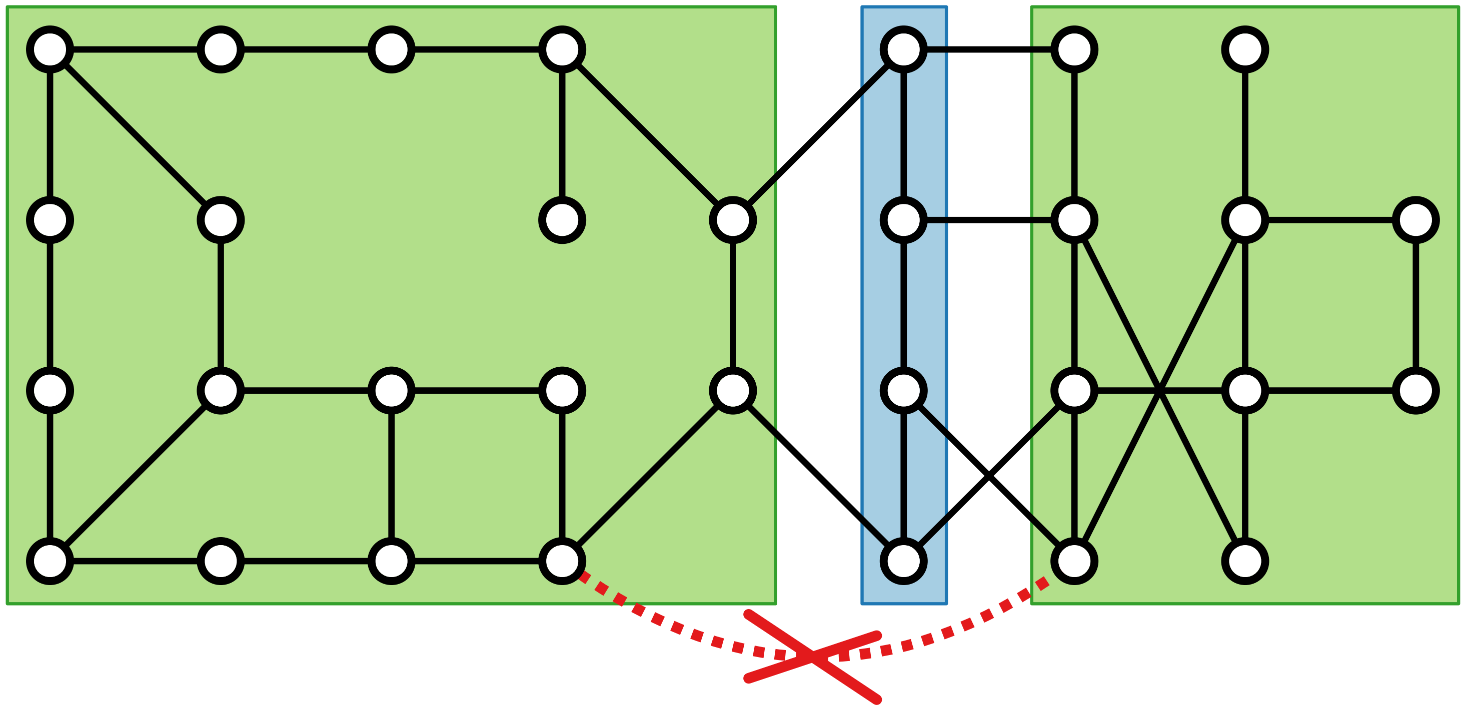

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We mainly used the fact that the graph consists of a **sequence of small separators**.

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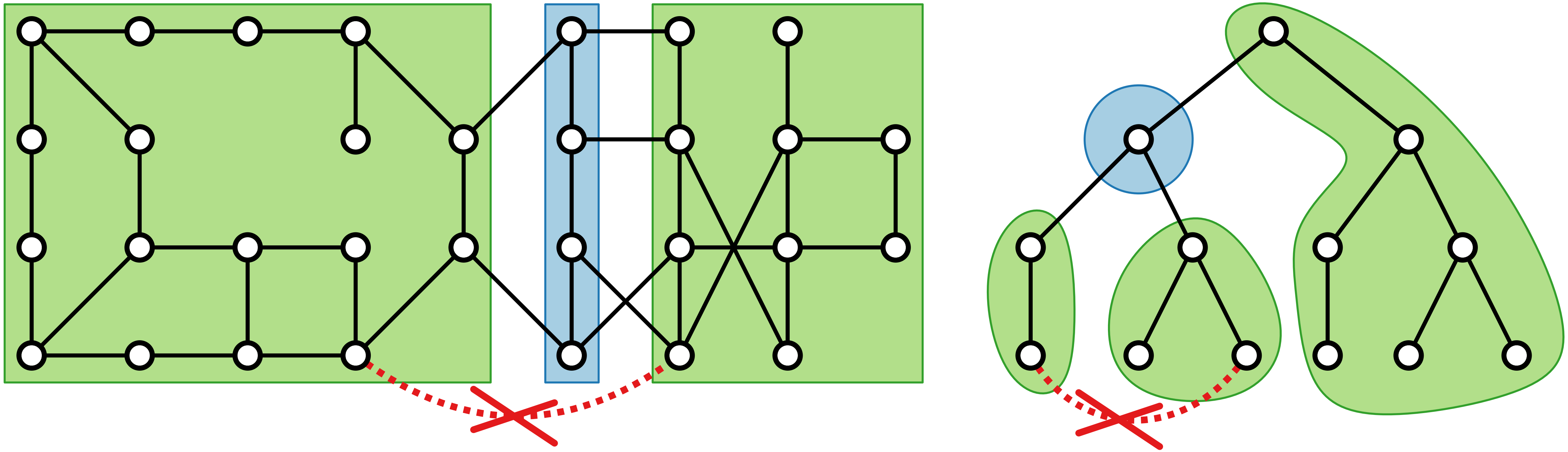

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A similiar fact was used in the algorithm for trees.

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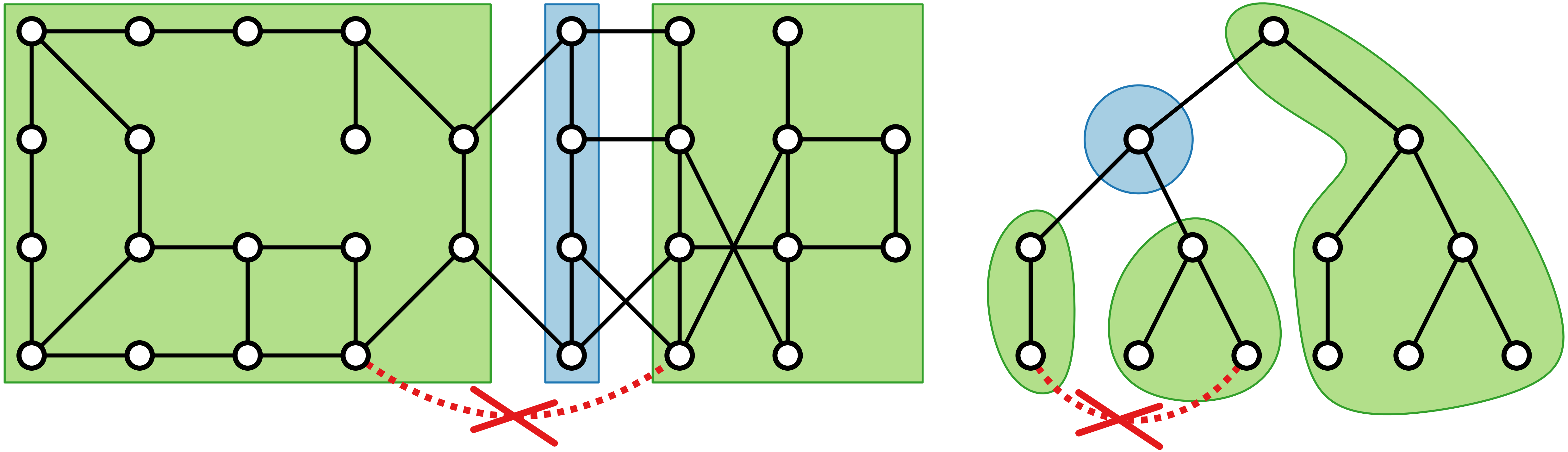

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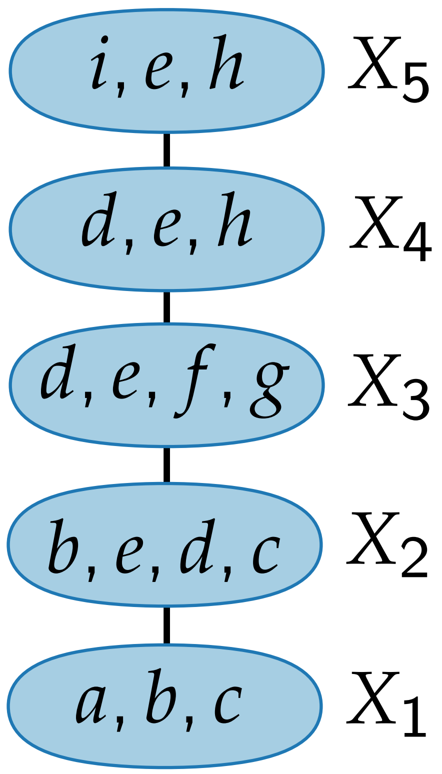
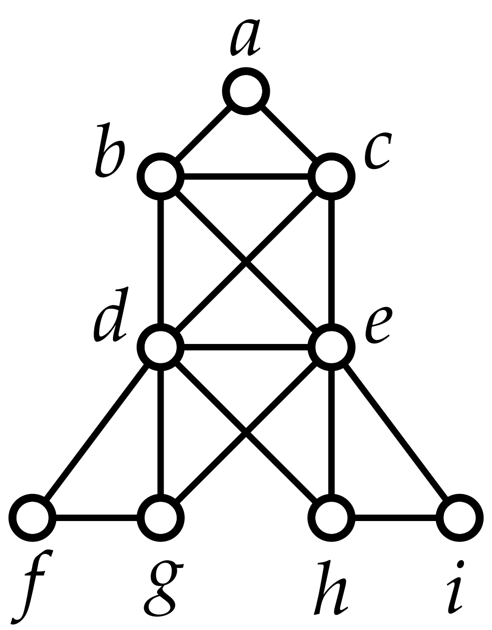
Goal: Define a more general graph class featuring a structure that is suited for this kind of dynamic programming approach.

Path Decompositions

Let $G = (V, E)$ be a graph.

A **path decomposition** of G is a sequence $P = (X_1, X_2, \ldots, X_r)$ of **bags**, where $X_i \subseteq V$, such that





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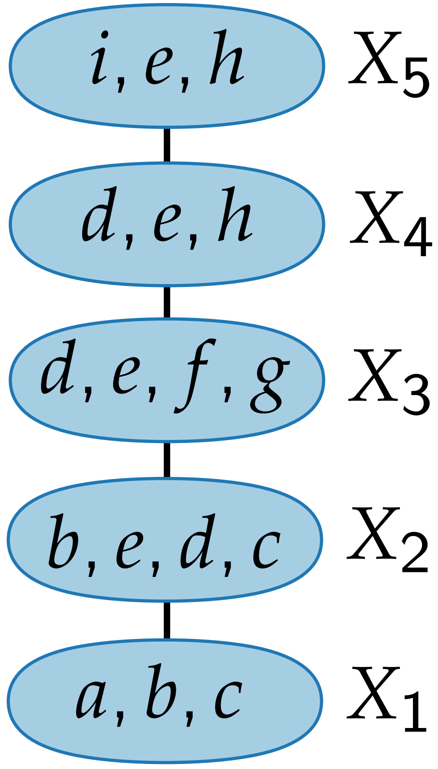

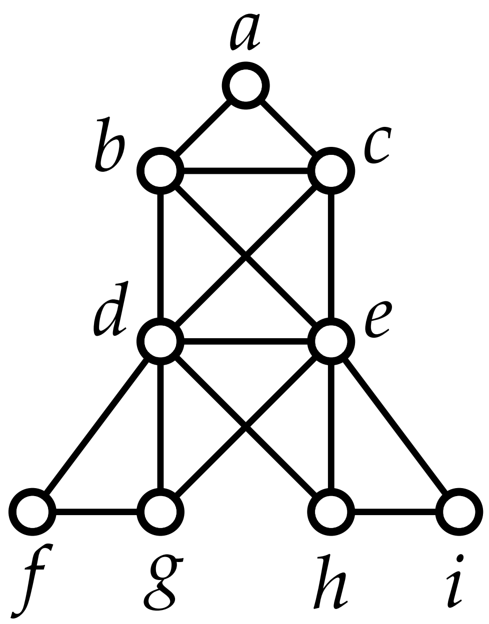

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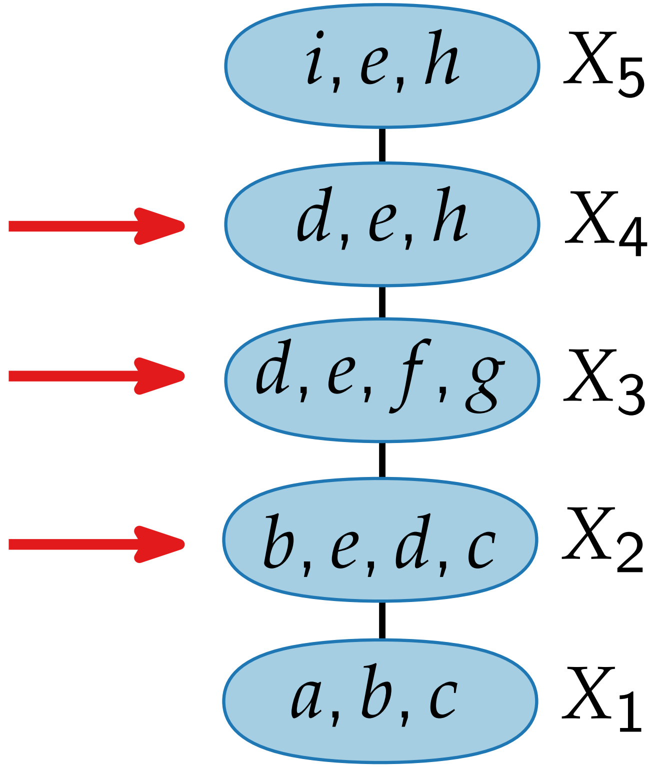

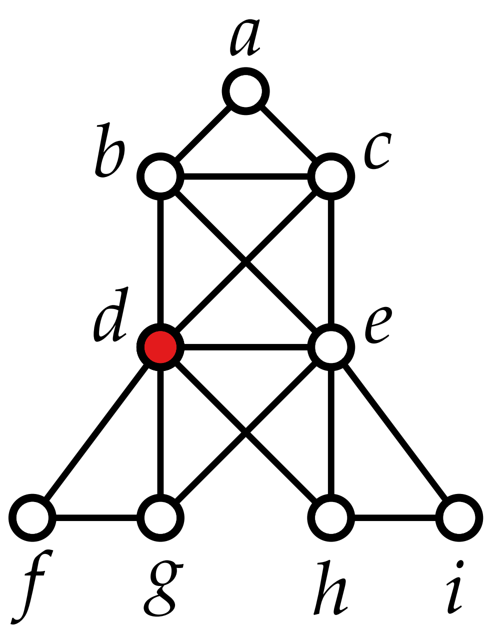

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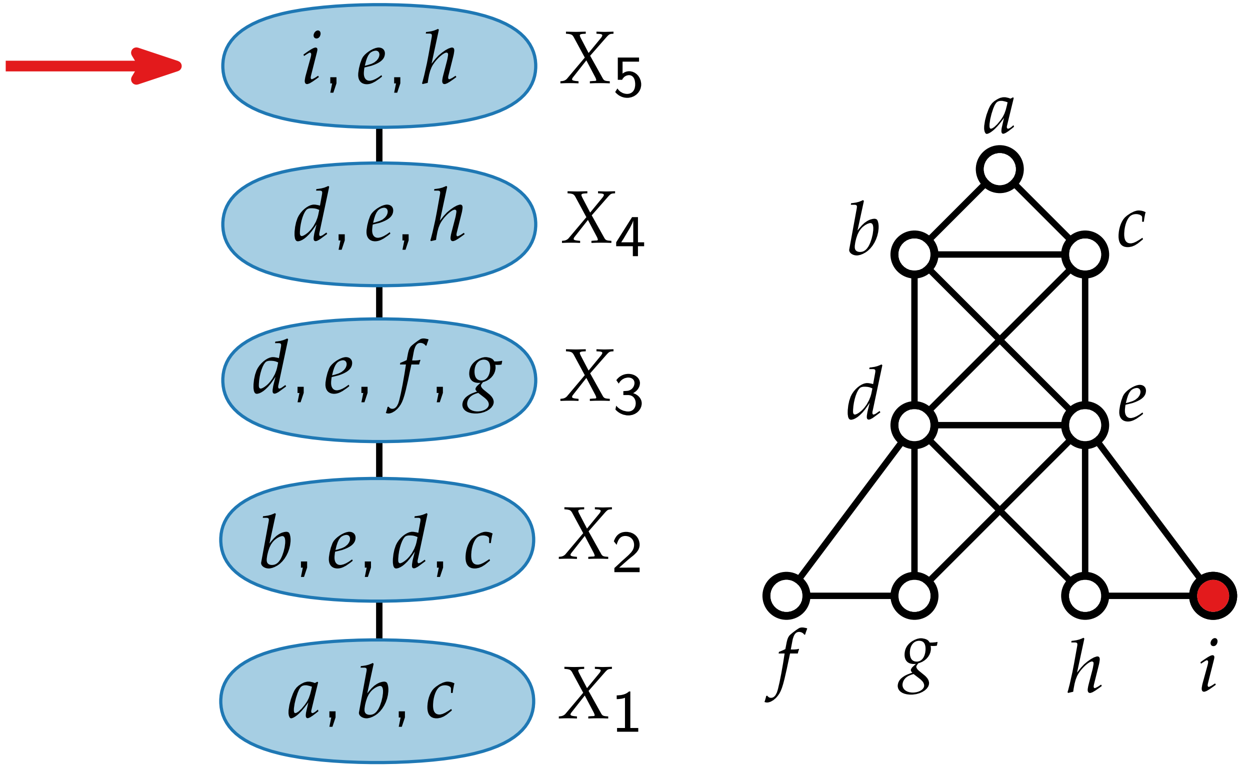

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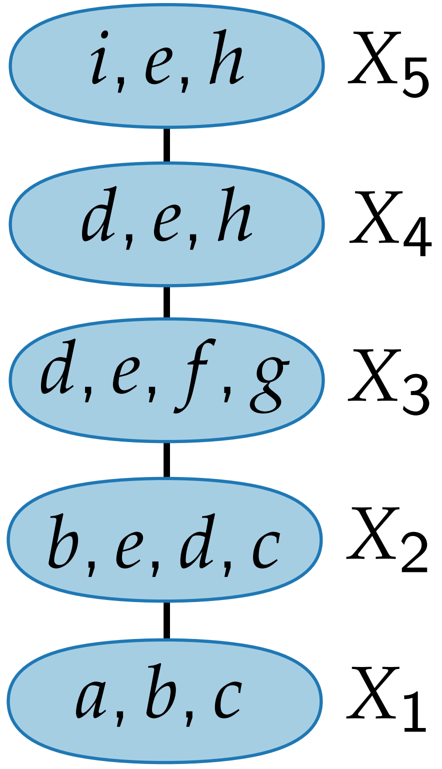

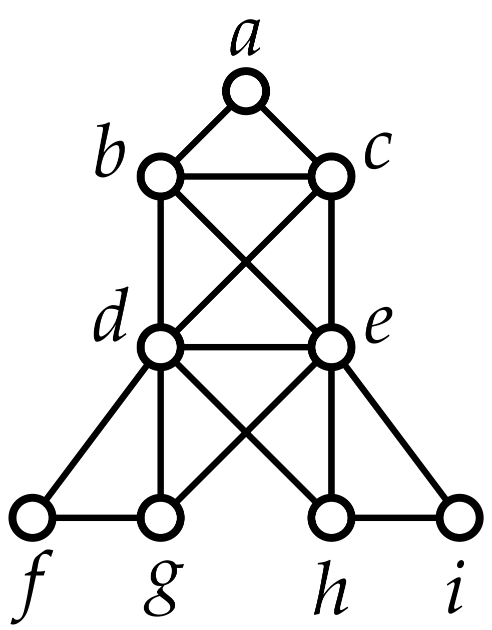

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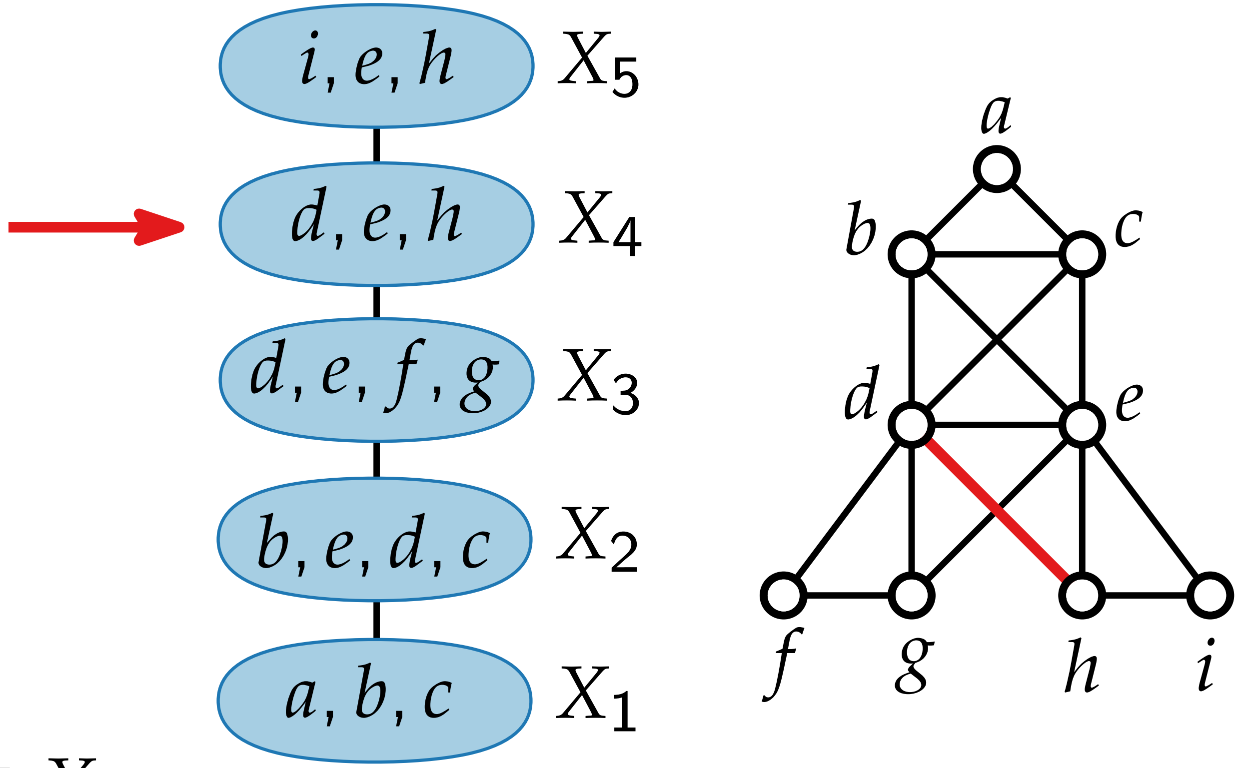

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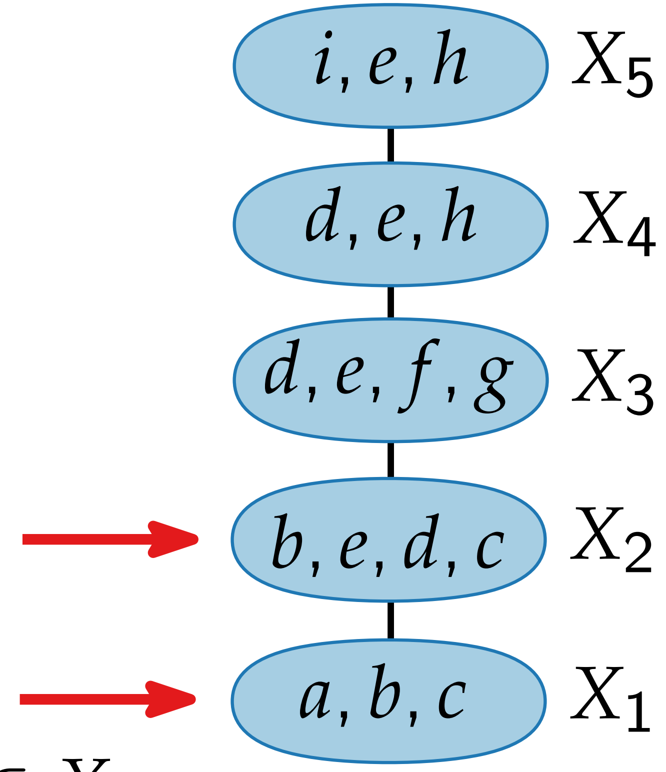

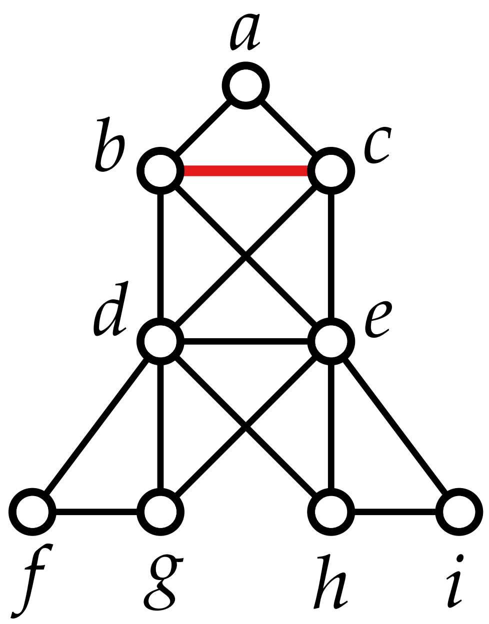

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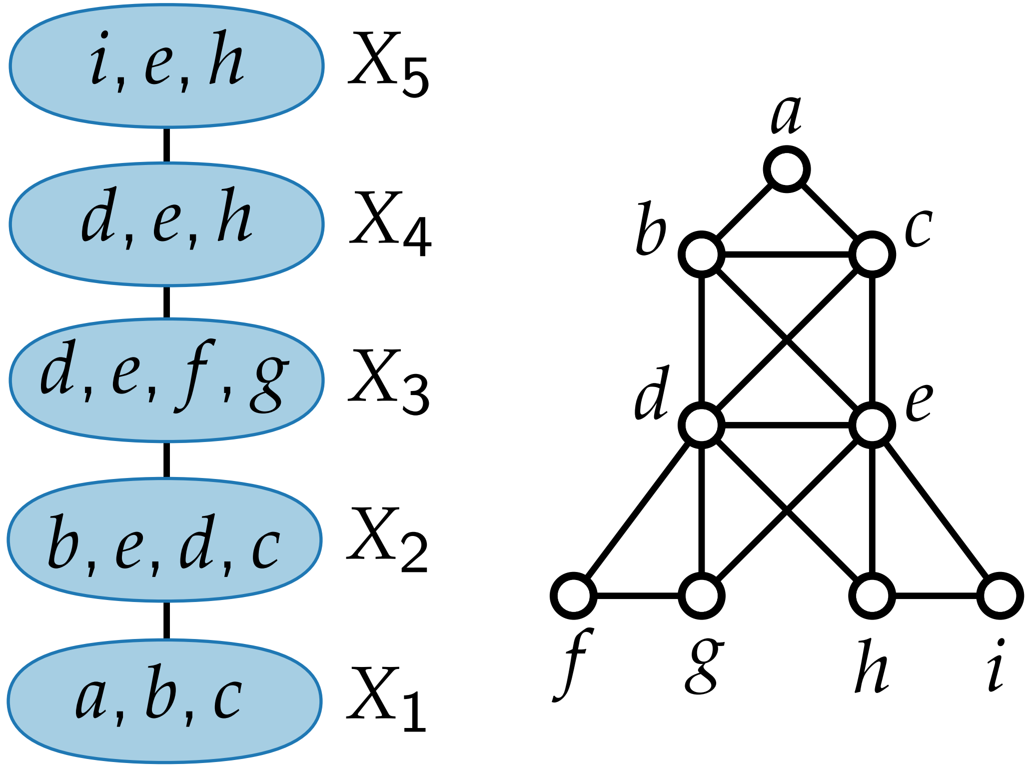

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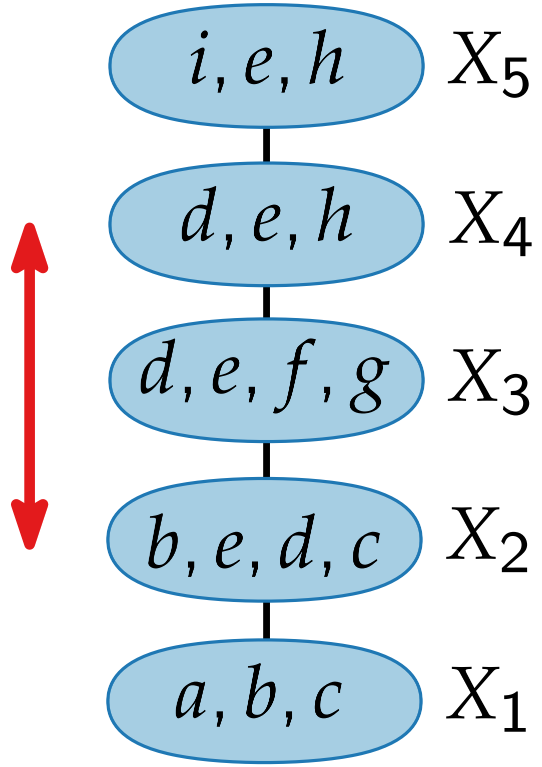

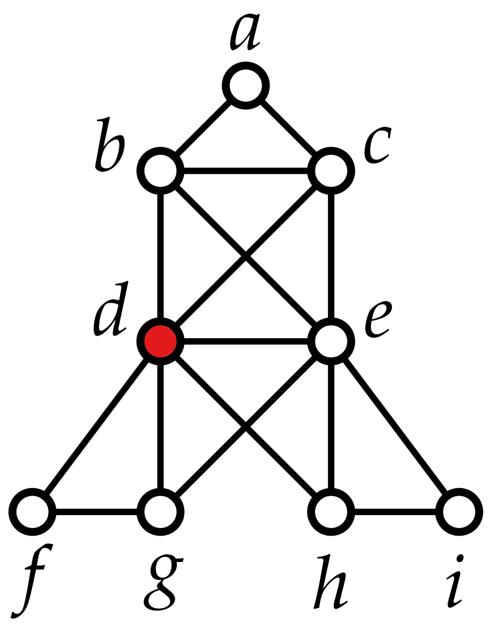

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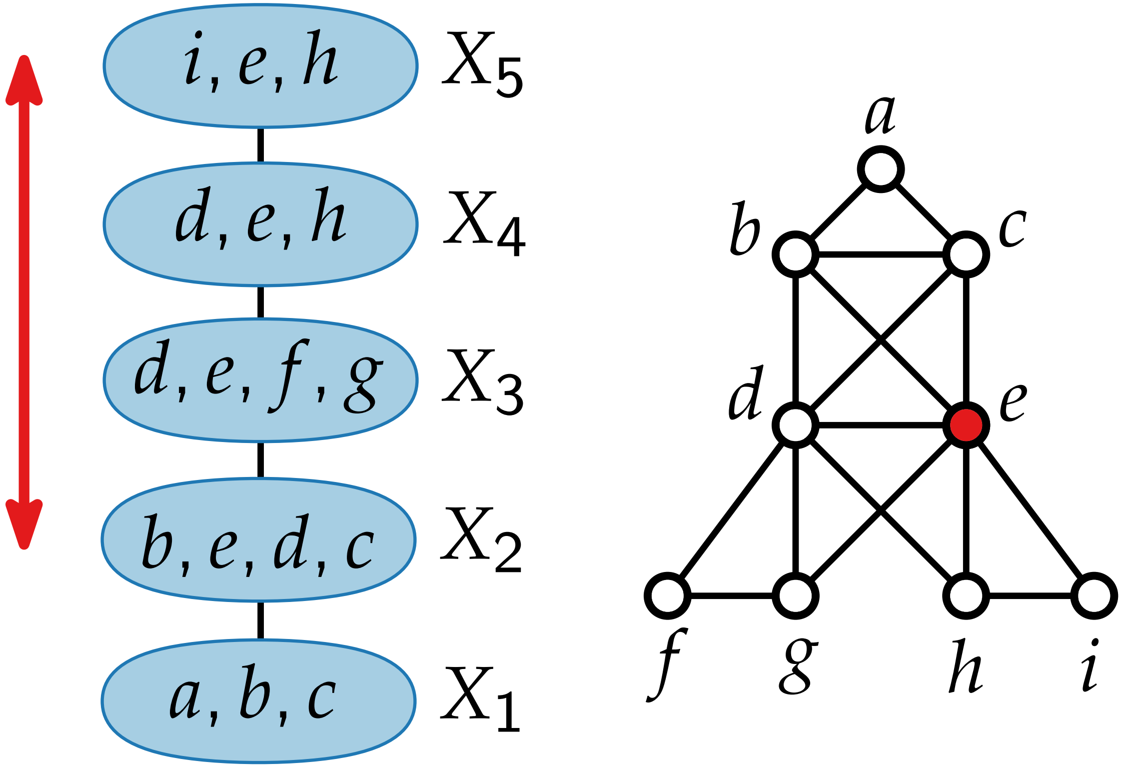

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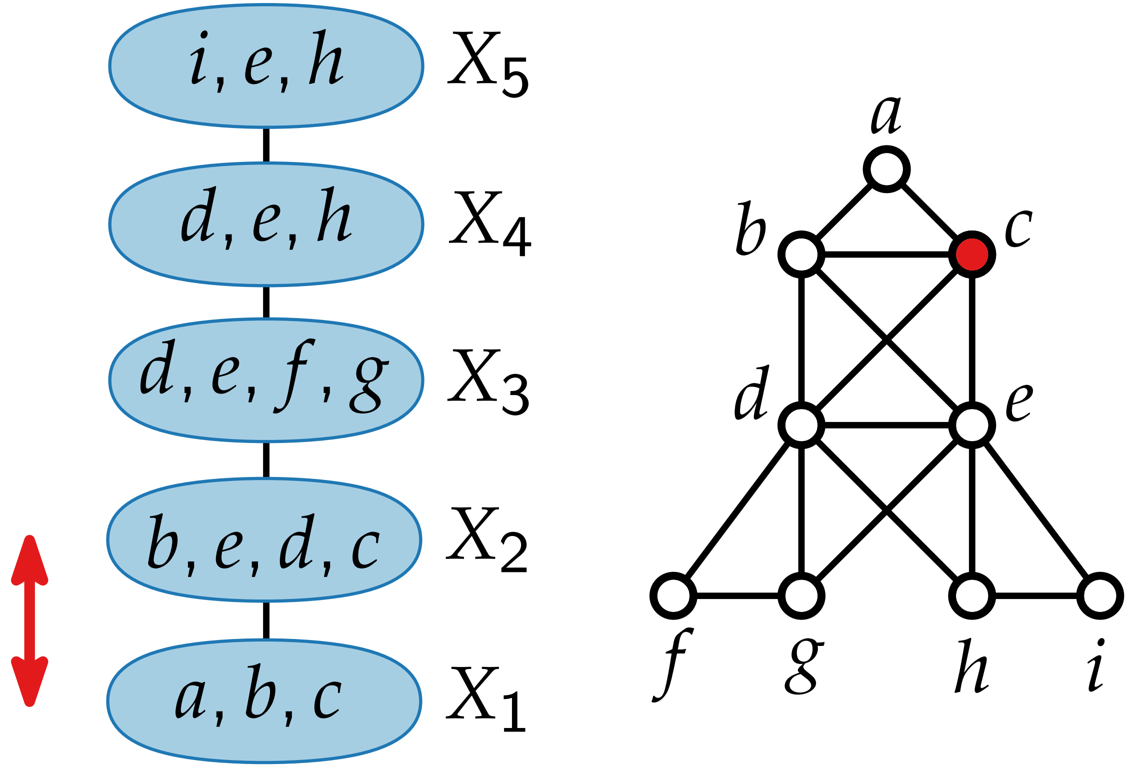
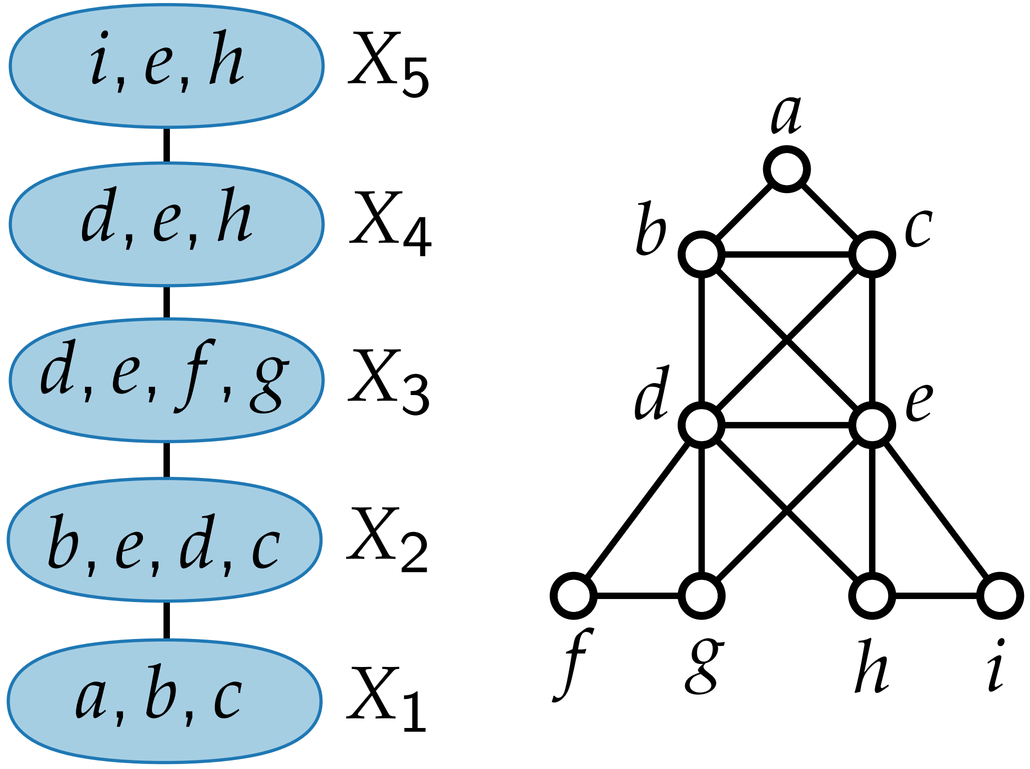


Path Decompositions

$w(P) = 3$



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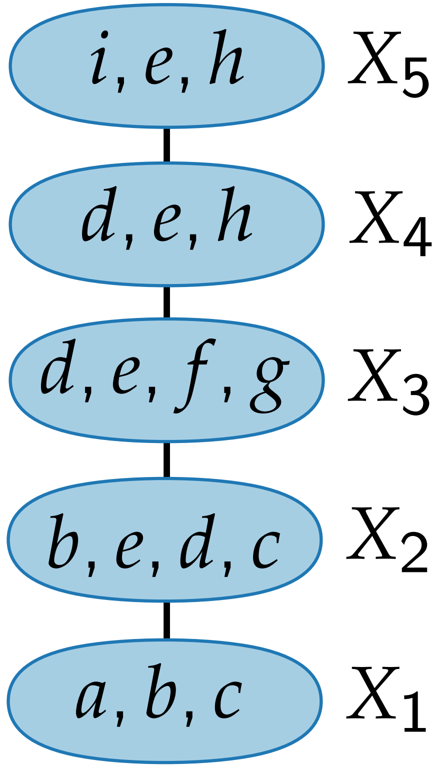
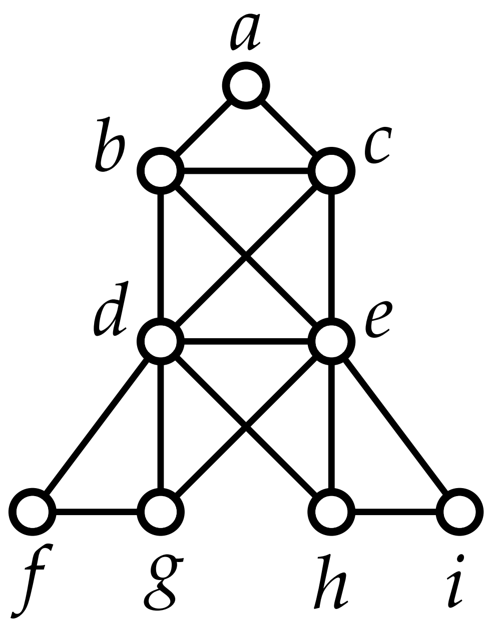
The **width** of P is $w(P) = \max_{1 \leq I \leq r} |X_i| - 1$.

Path Decompositions

$\mathrm{w}(P) = 3$



$\mathrm{pw}(G) \leq 3$



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The **width** of P is $\mathrm{w}(P) = \max_{1 \leq I \leq r} |X_i| - 1$.

The **pathwidth** $\mathrm{pw}(G)$ of G is the minimum width of a path decomposition of G.

Path Decompositions

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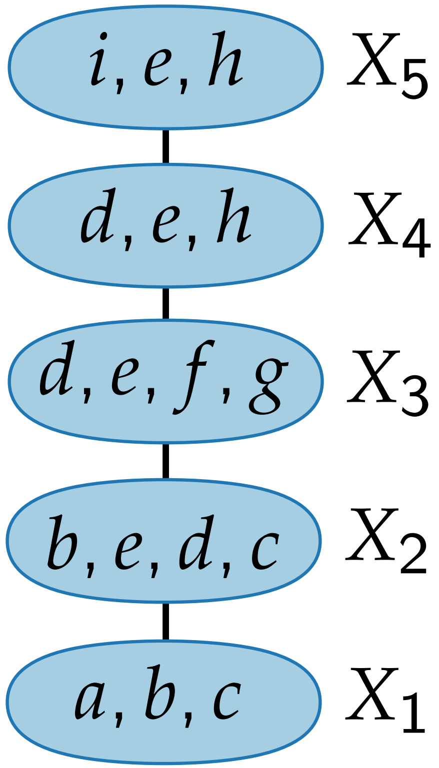

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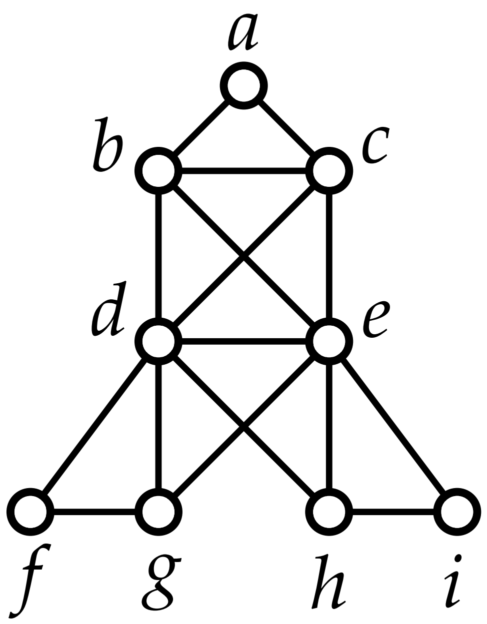

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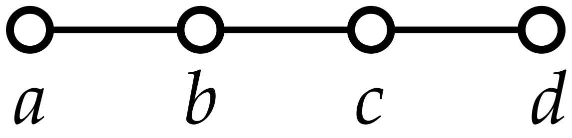
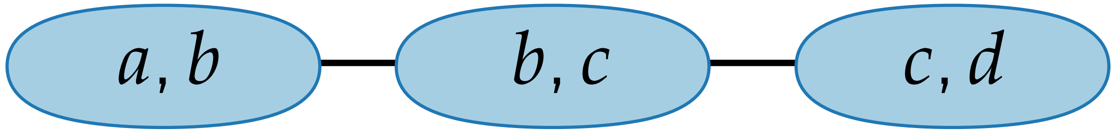
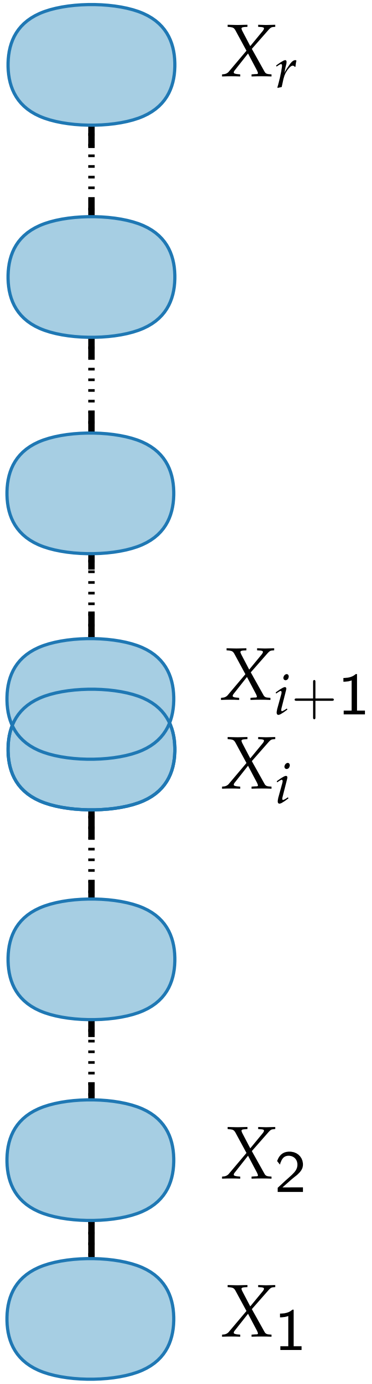
The **pathwidth** $\mathrm{pw}(G)$ of G is the minimum width of a path decomposition of G.





$\mathrm{pw}(G) = 1$

Okay – But Where Are the Separators?



(P1) $\bigcup_{i=1}^{r} X_i = V$

(P2) $\forall \{u, v\} \in E \; \exists i \in \{1, 2, \ldots, r\} : u, v \in X_i$

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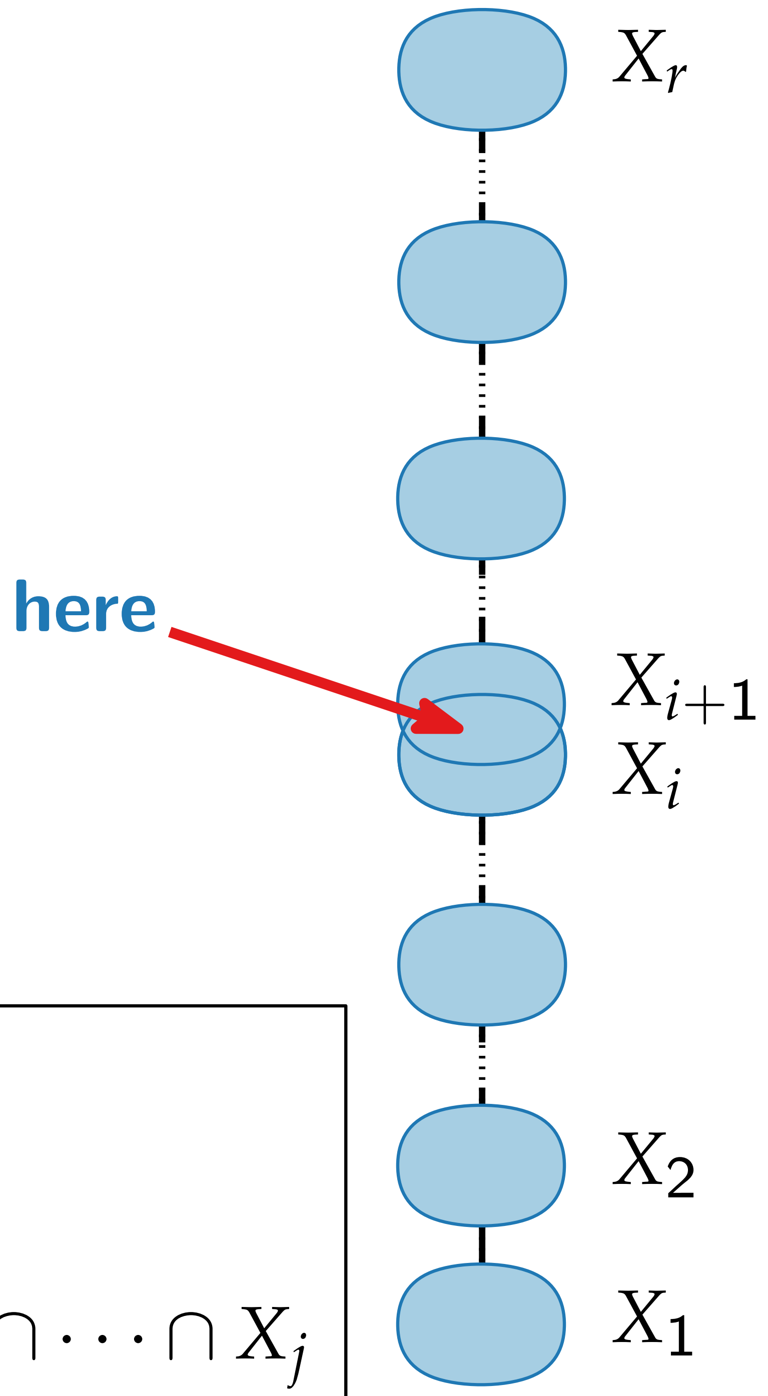

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Lemma. Let $i < r$. Then there is no edge between
$A = (X_1 \cup X_2 \cup \cdots \cup X_i) \setminus (X_i \cap X_{i+1})$ and
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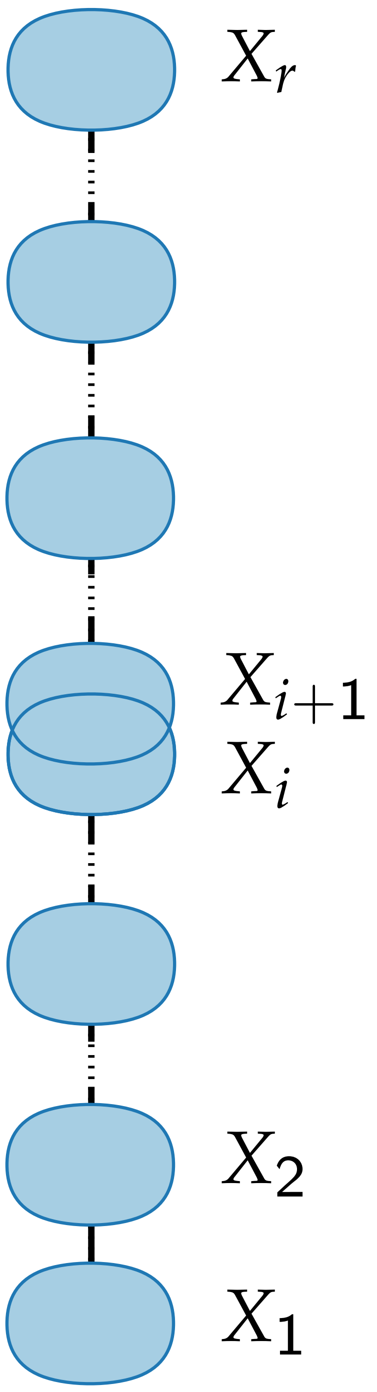

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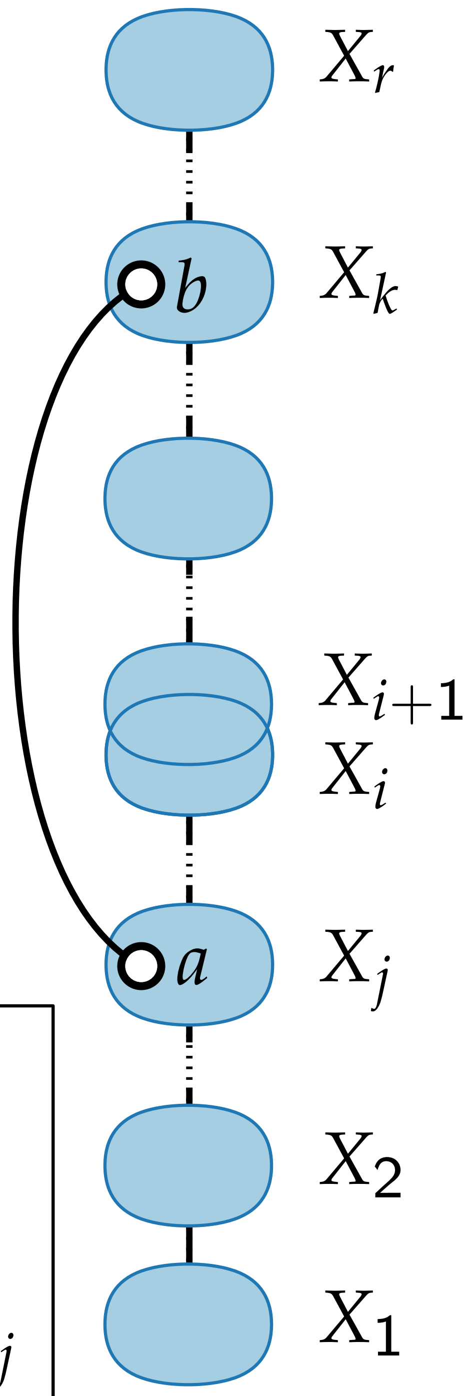

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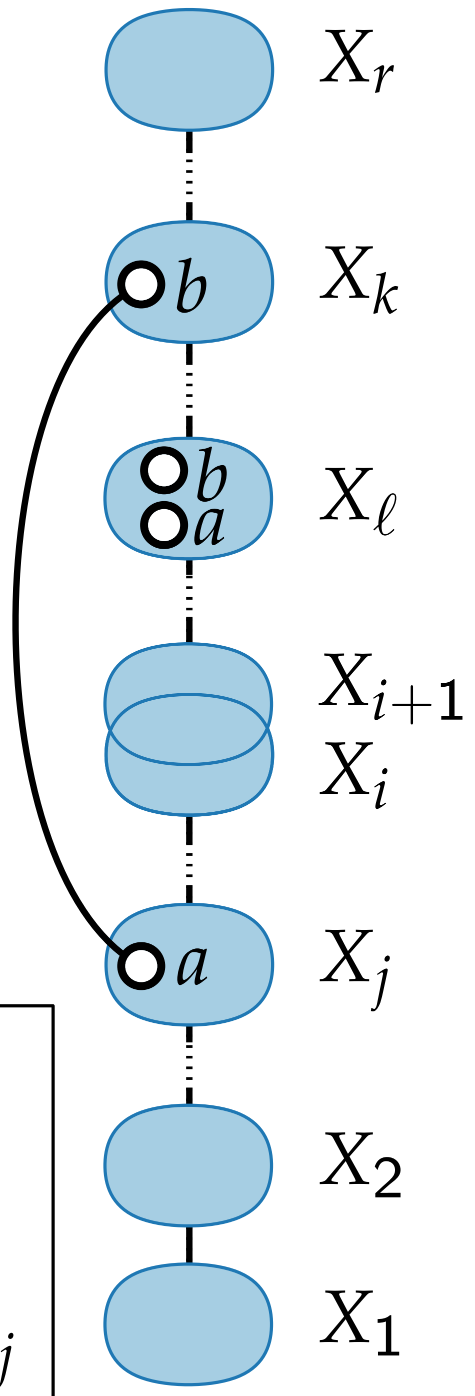

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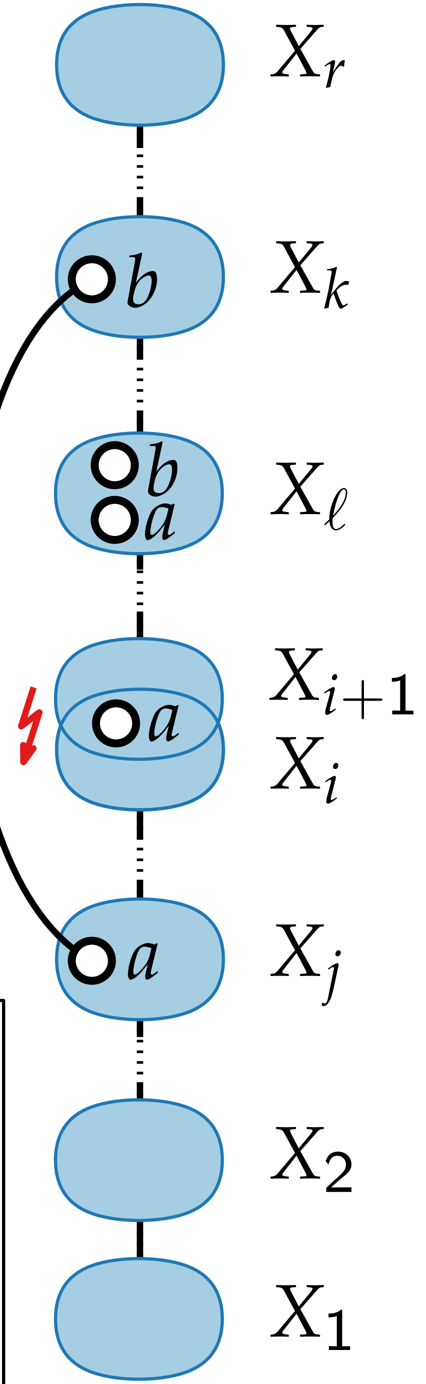

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Computing Path Decompositions

k-Pathwidth

Input. Graph $G = (V, E)$, $k \in \mathbb{N}$

Question. Is the pathwidth of G at most k?

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$\Rightarrow$ When designing FPT algorithms with respect to the pathwidth, we may assume to be given a path decomposition!

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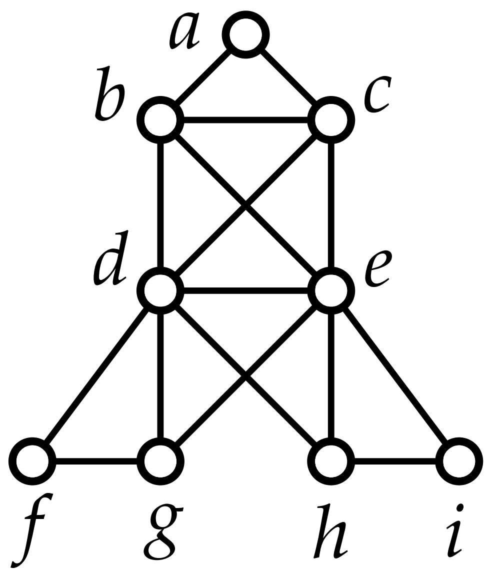
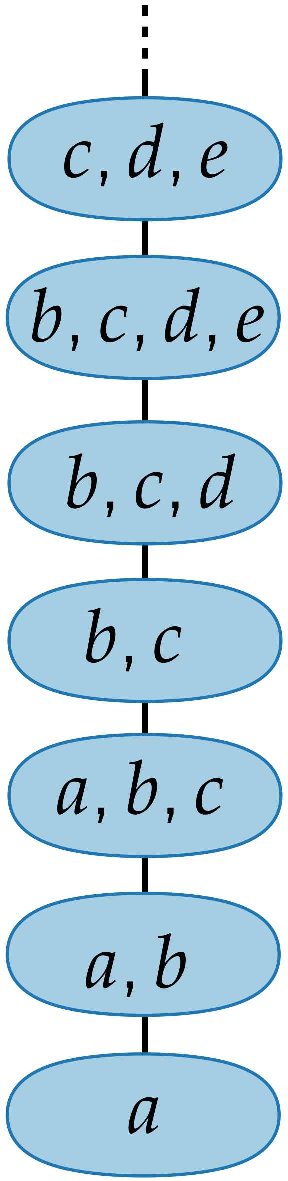
A path decomposition is **nice** if $|X_1| = 1$ and each other bag has one of two **types**:





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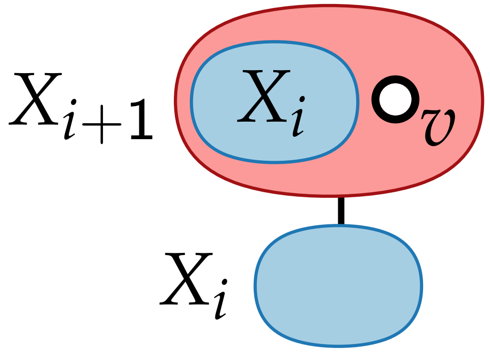

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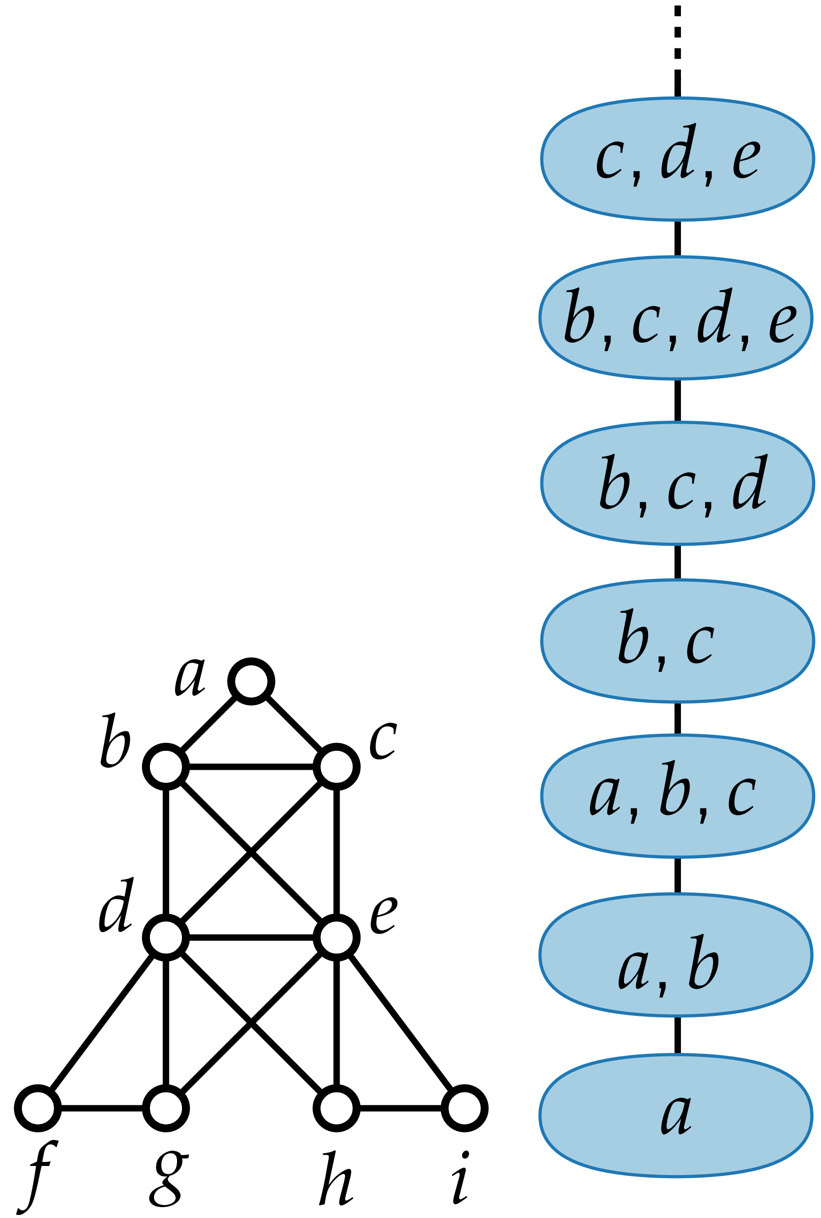

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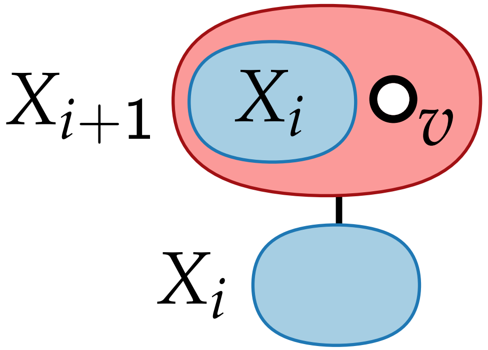

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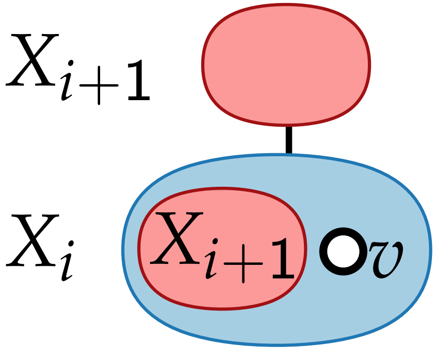

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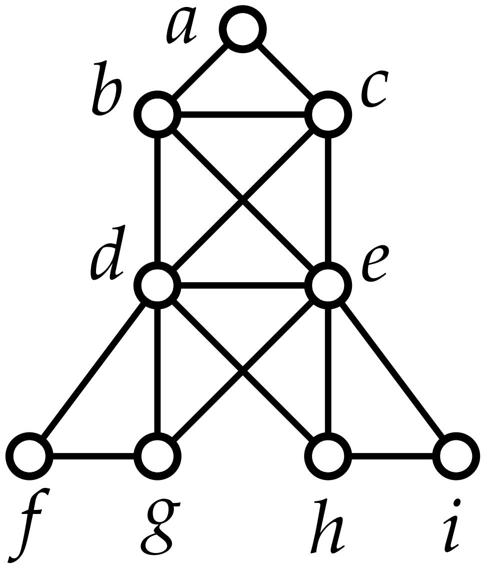

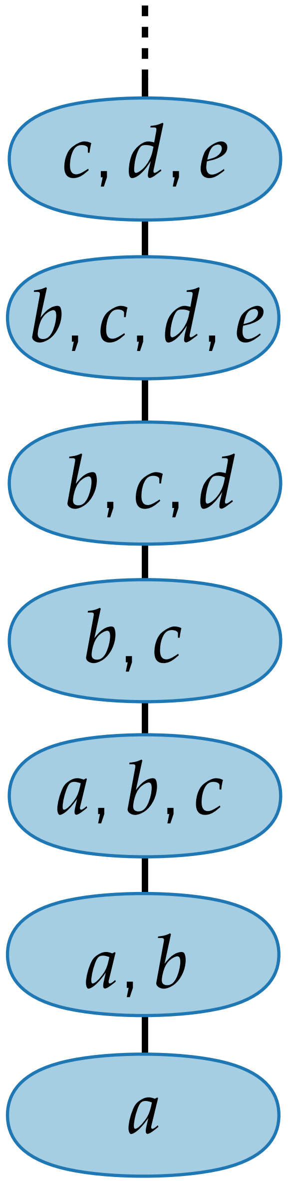

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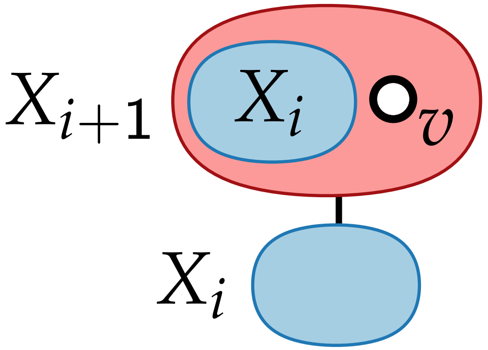

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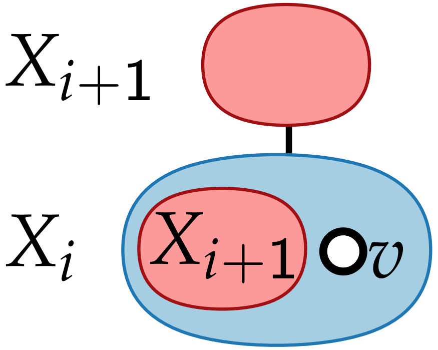

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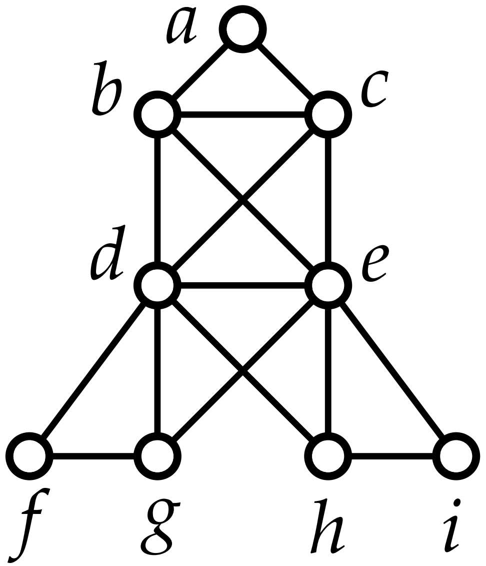

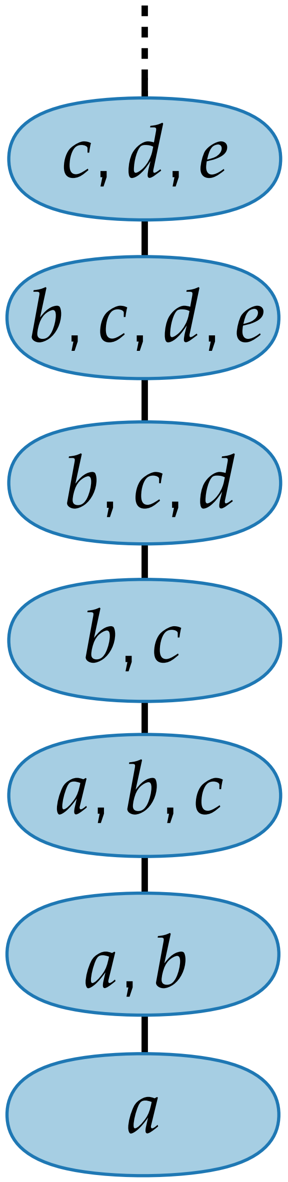

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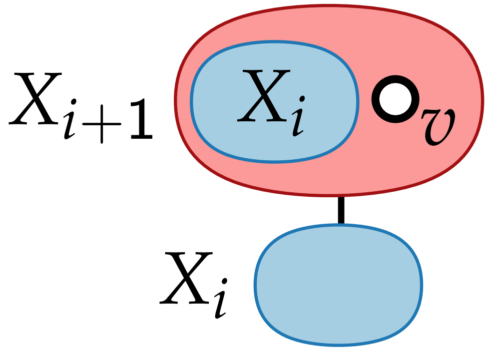

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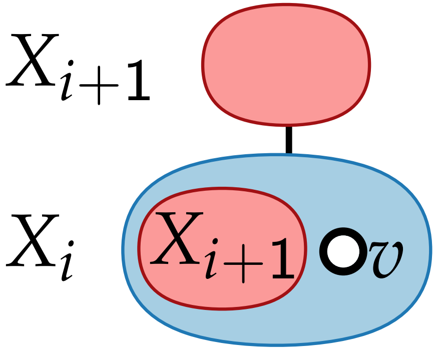

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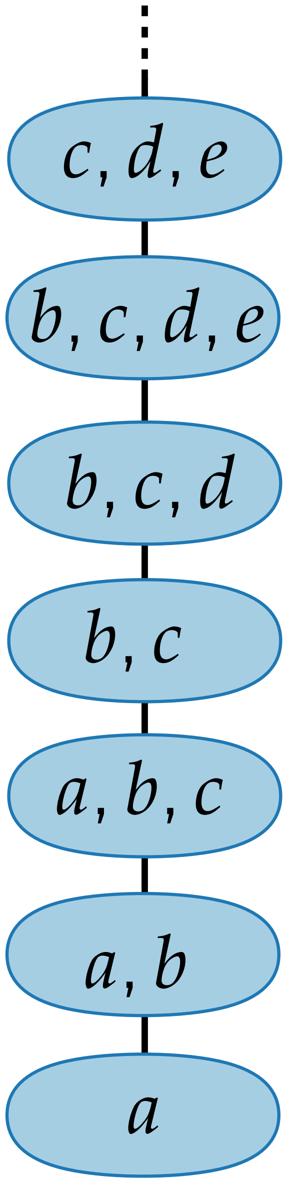

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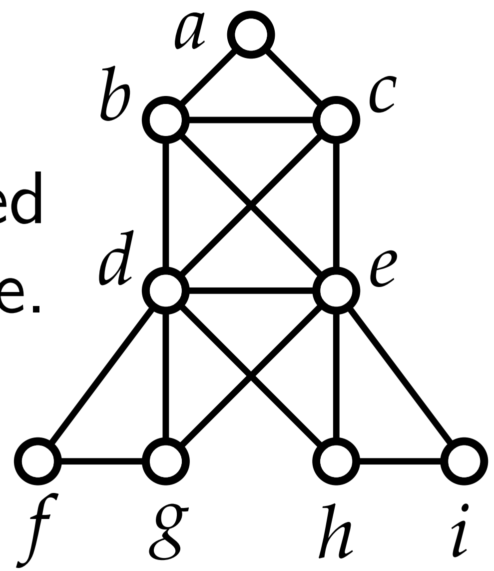

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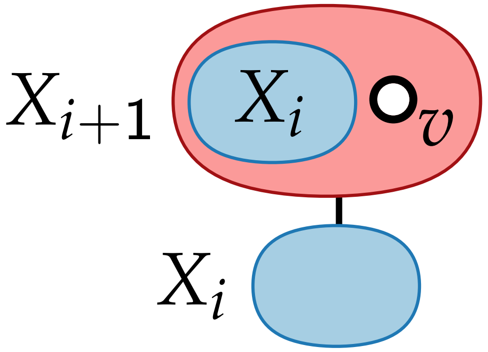

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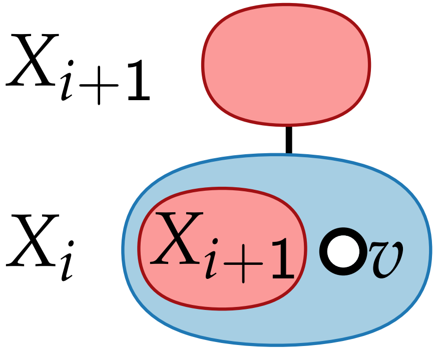

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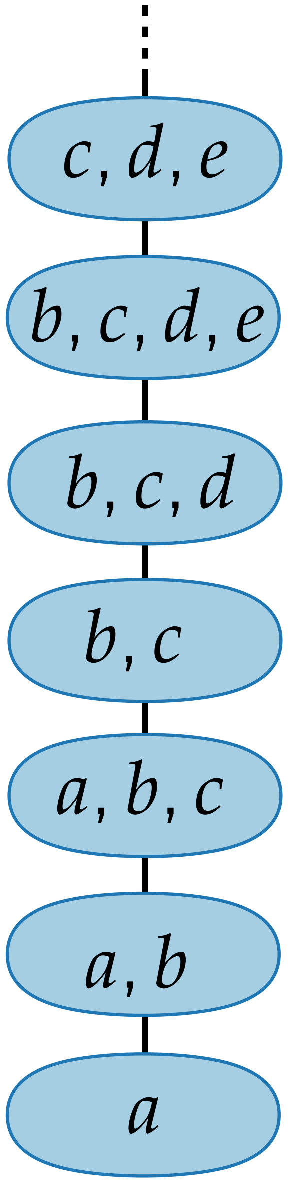

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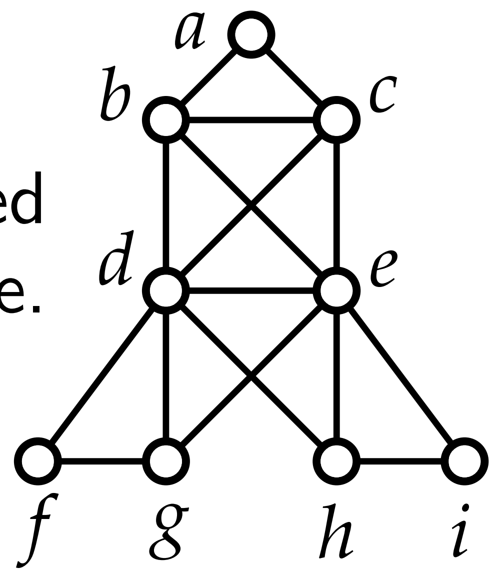
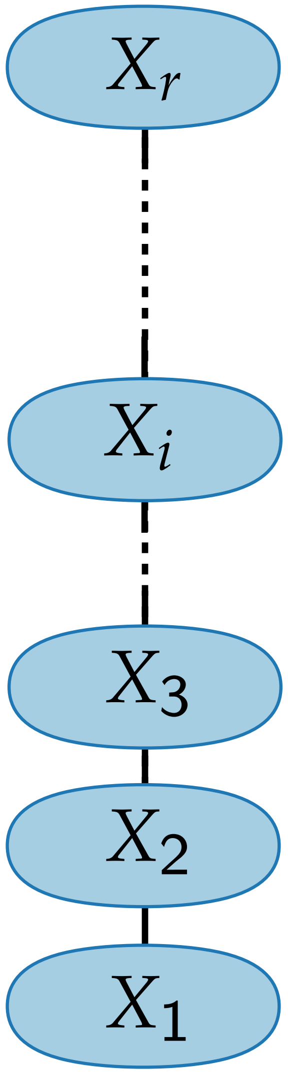


Independent Set in Graphs of Bounded Pathwidth

Assume we are given a nice path decomposition $P = (X_1, X_2, \ldots, X_r)$ of width k.



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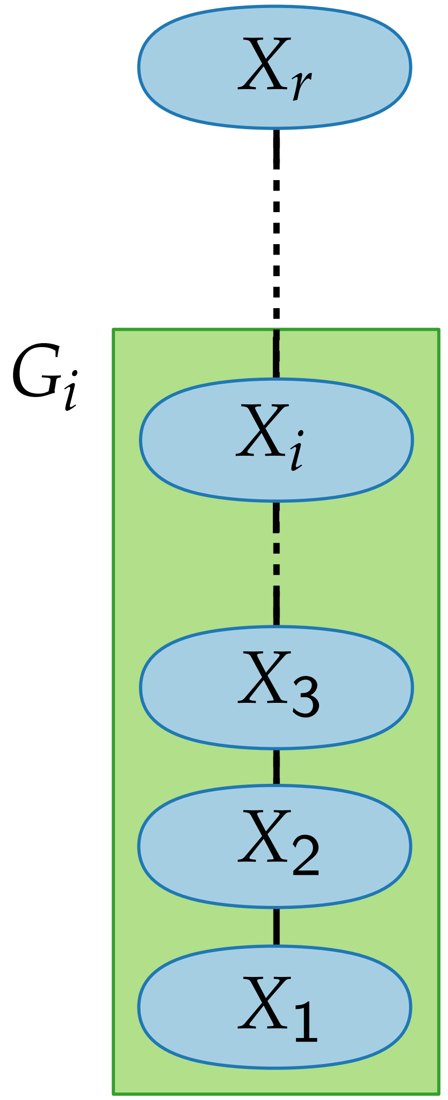

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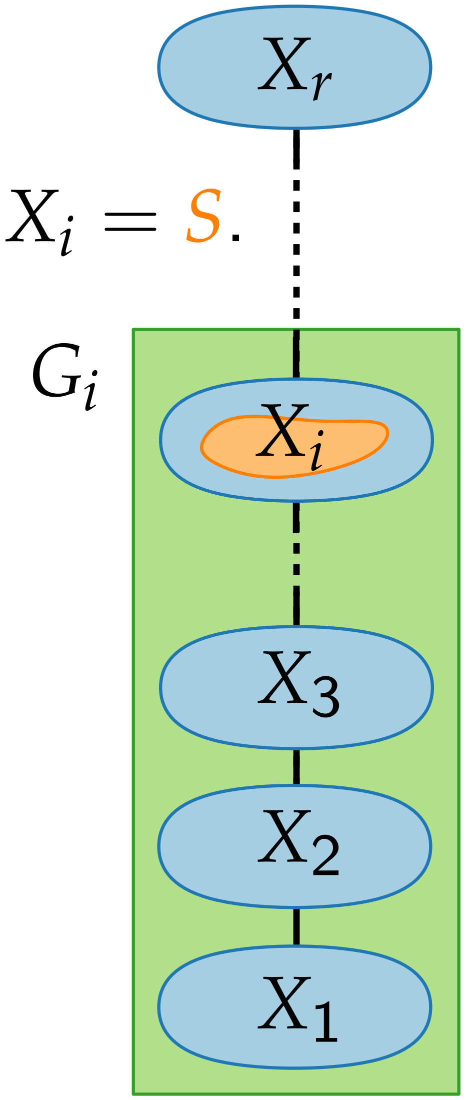

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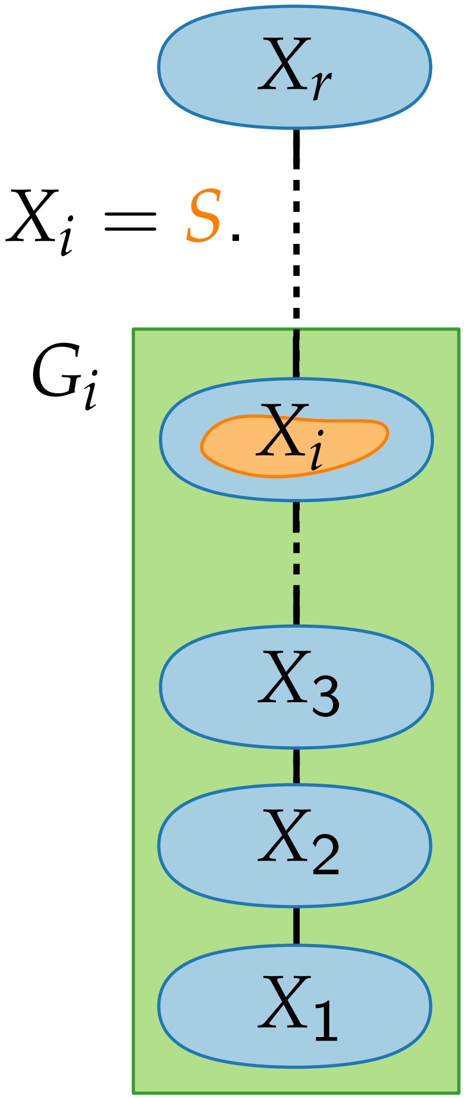

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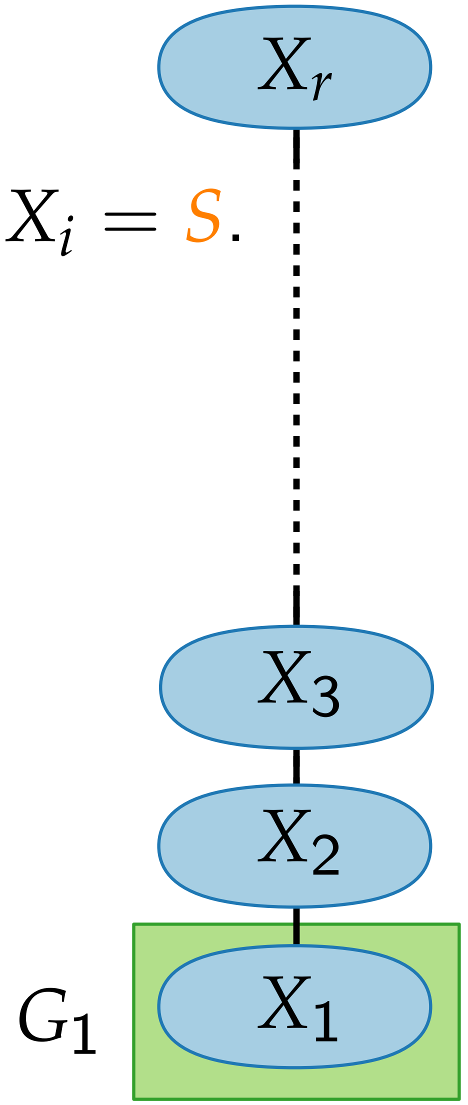
$$D[1, S] = \begin{cases} 0 & \text{, if } S = \emptyset \\ w(v) & \text{, if } S = \{v\} \end{cases}$$



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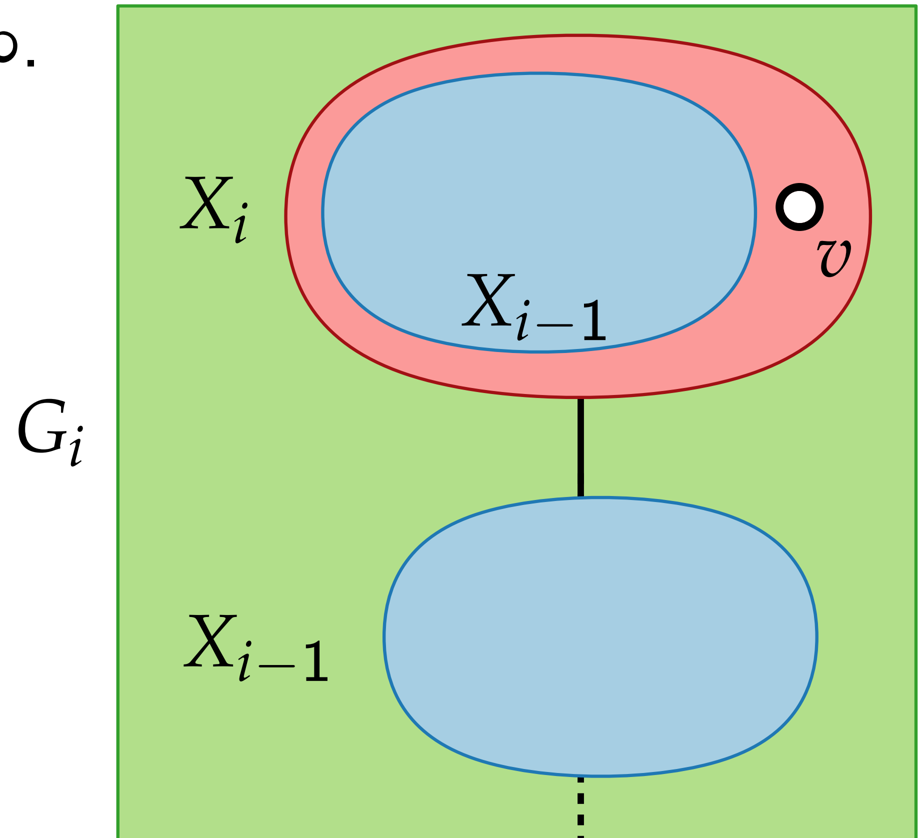

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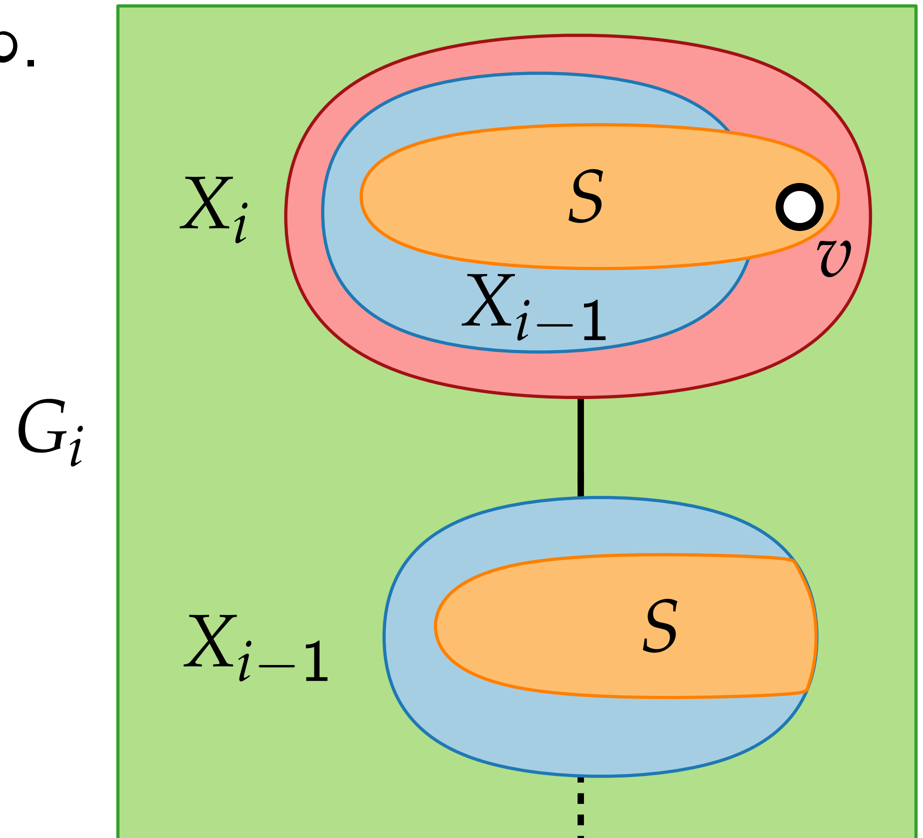

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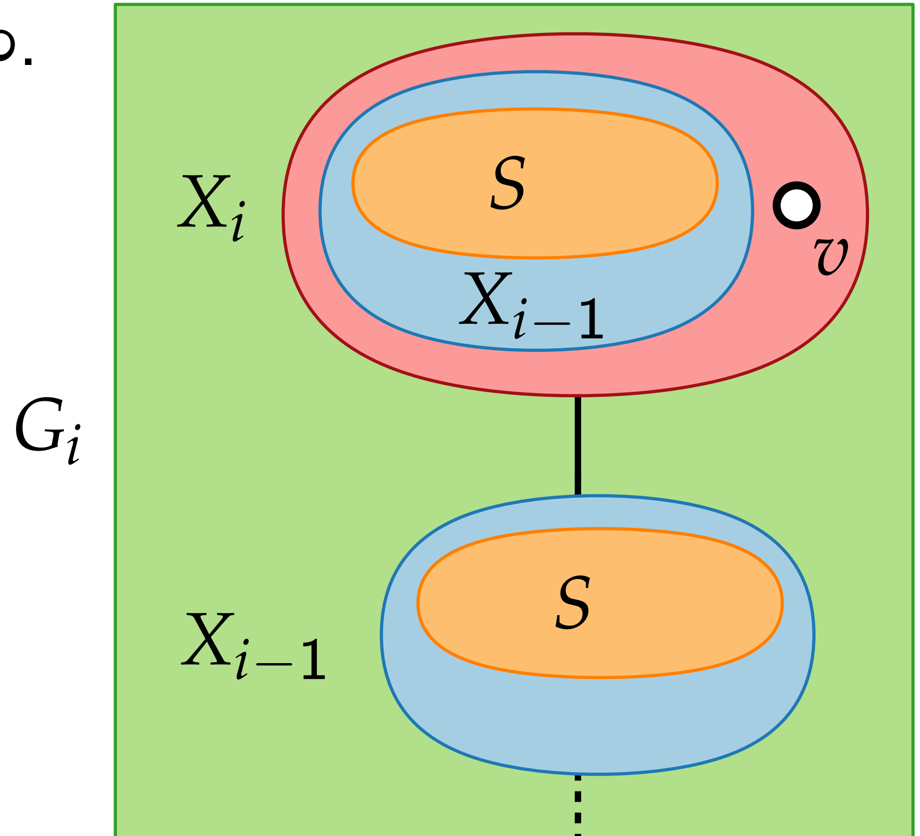

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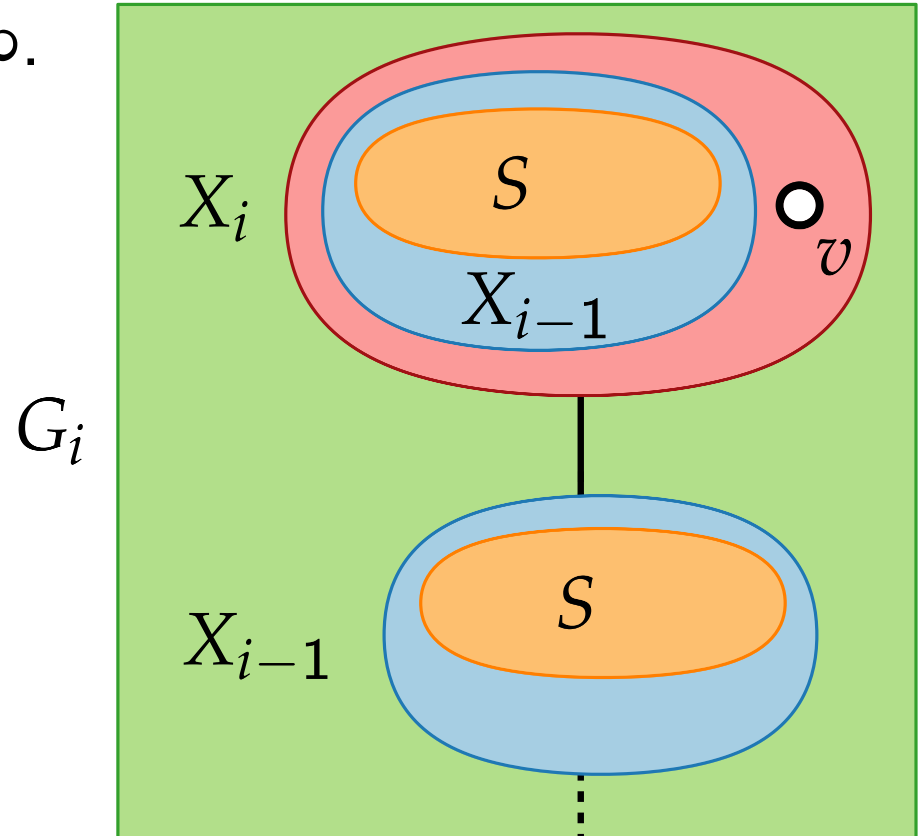

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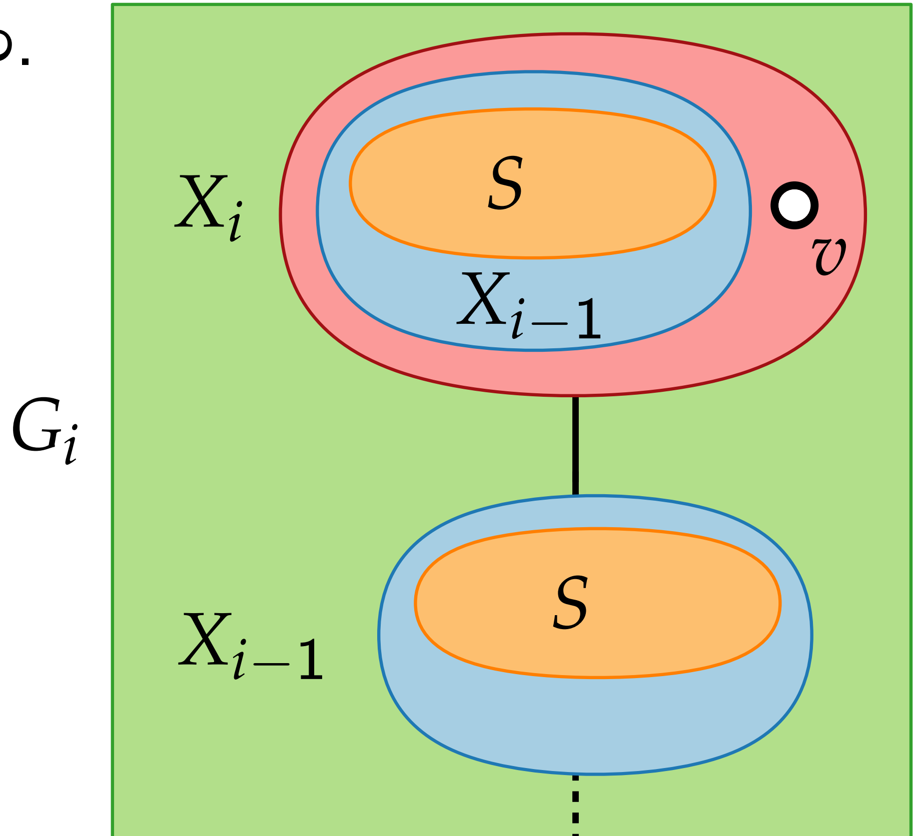

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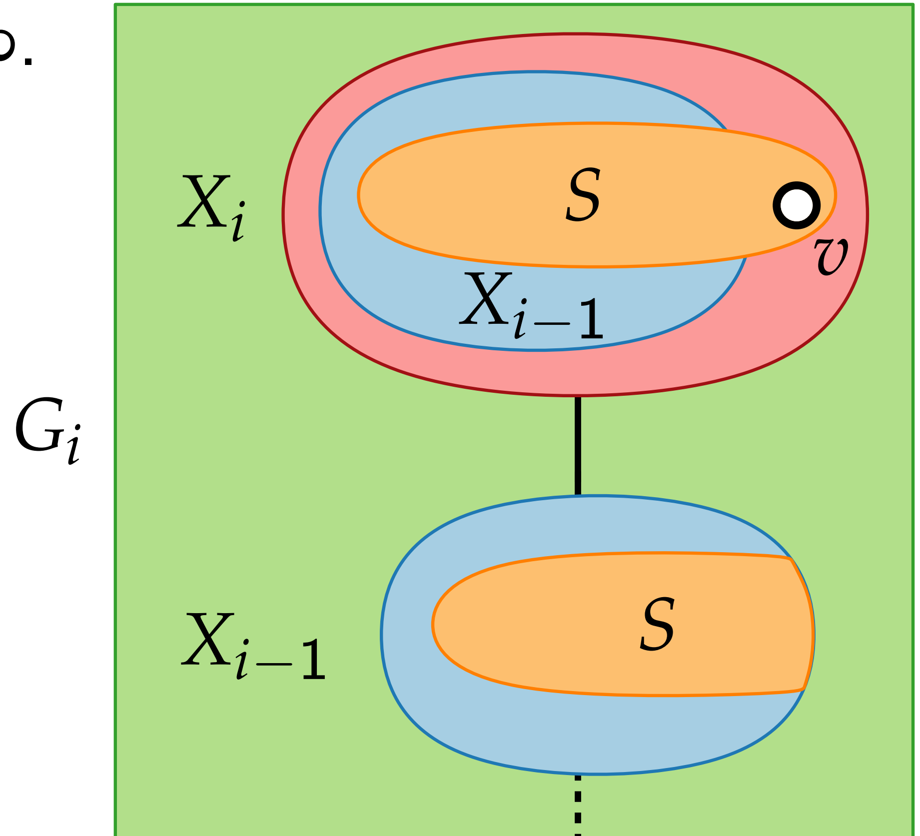

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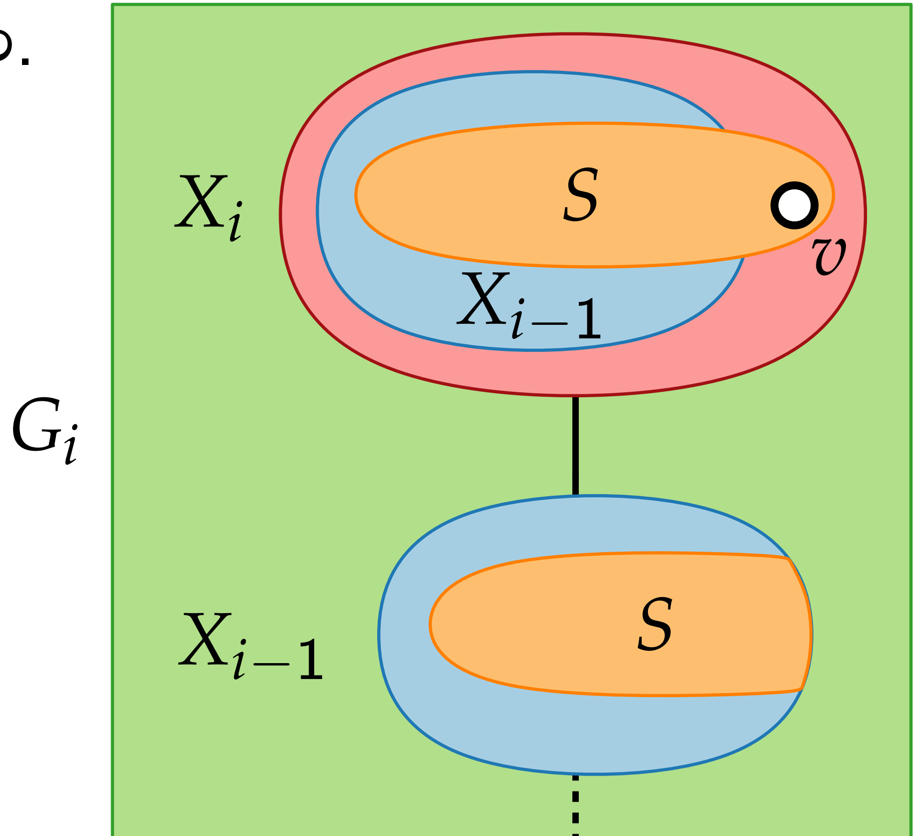

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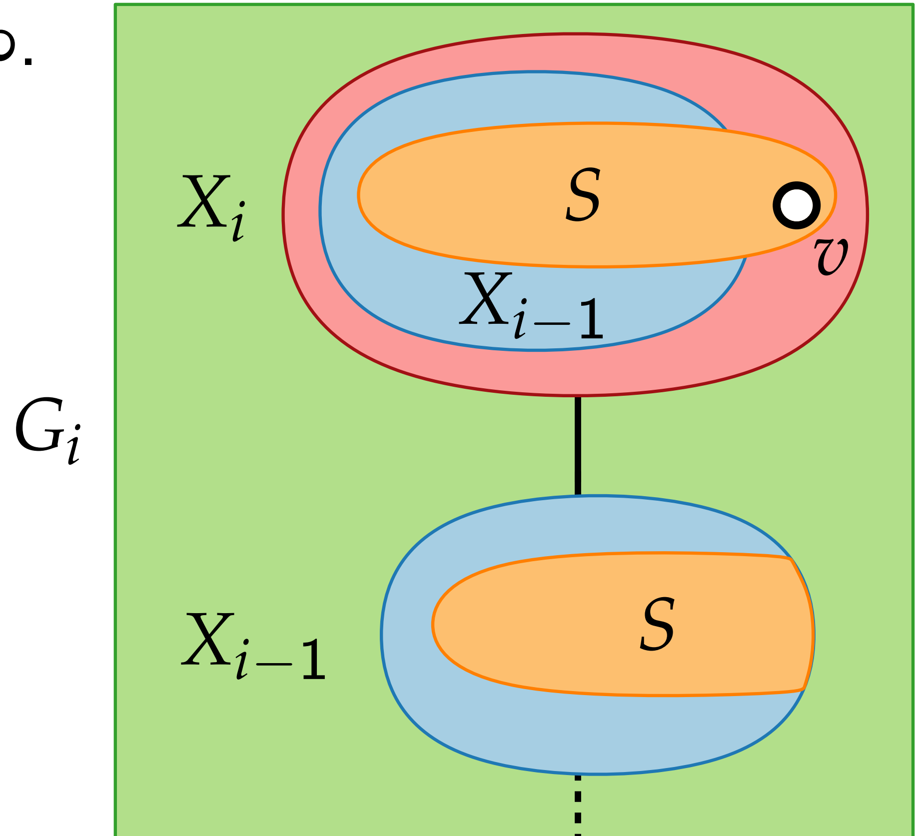

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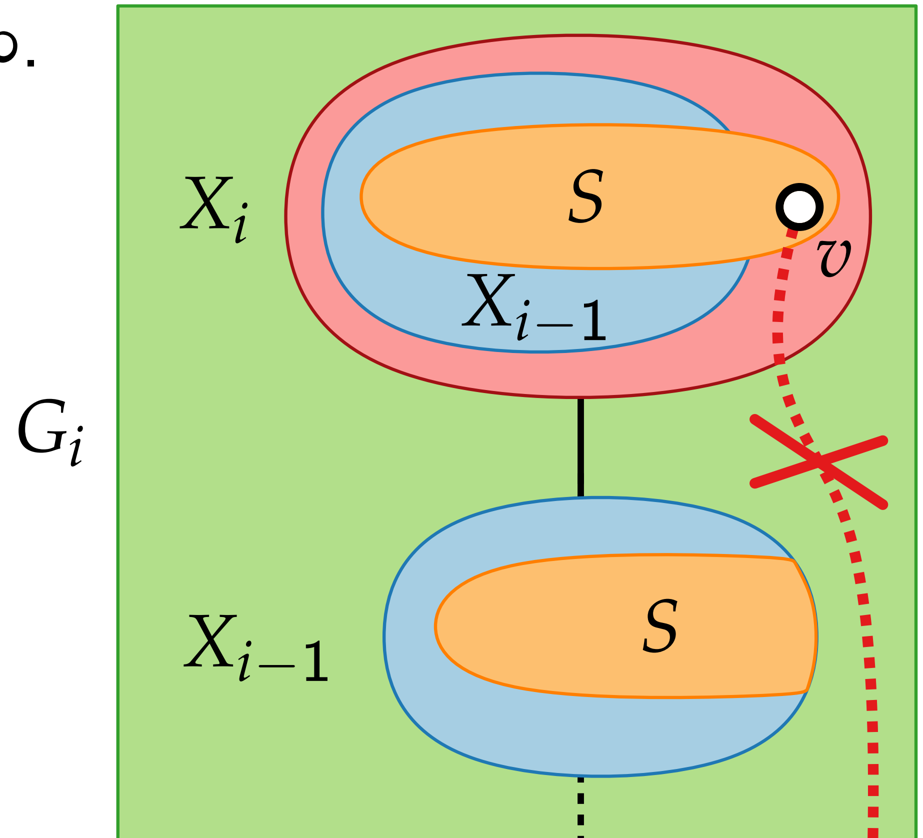

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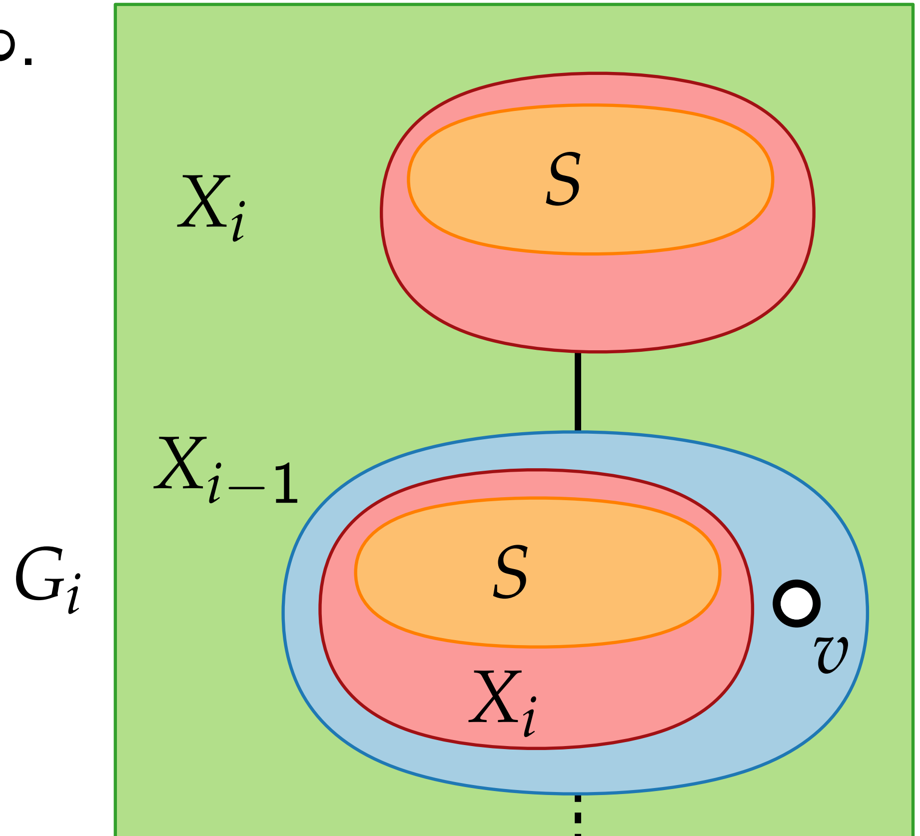

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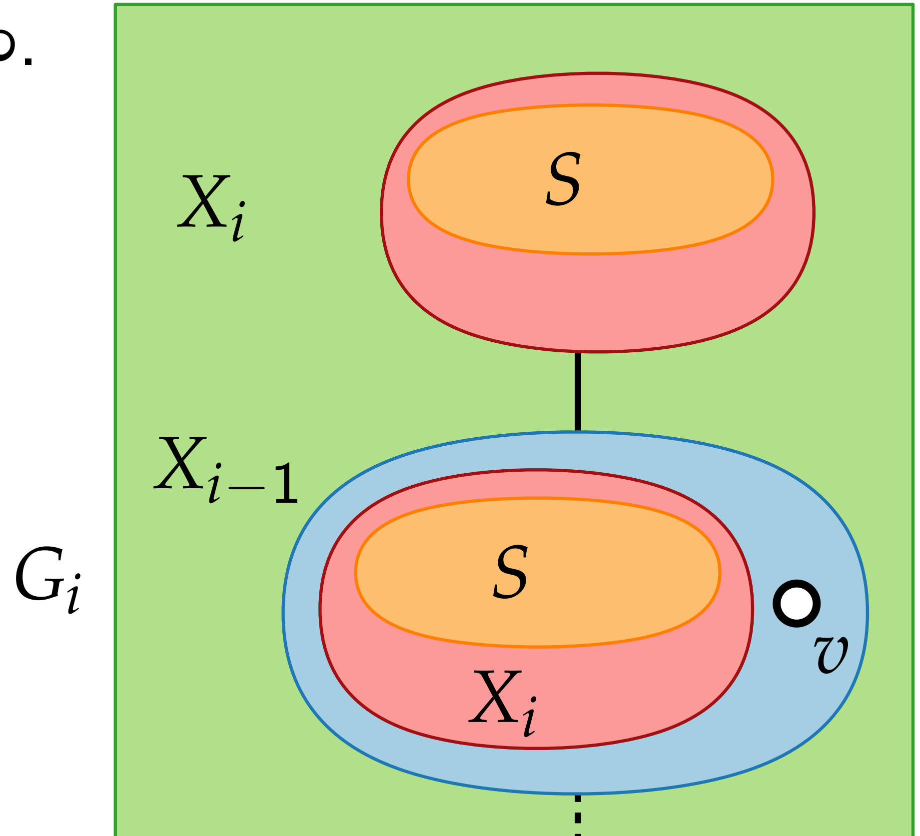

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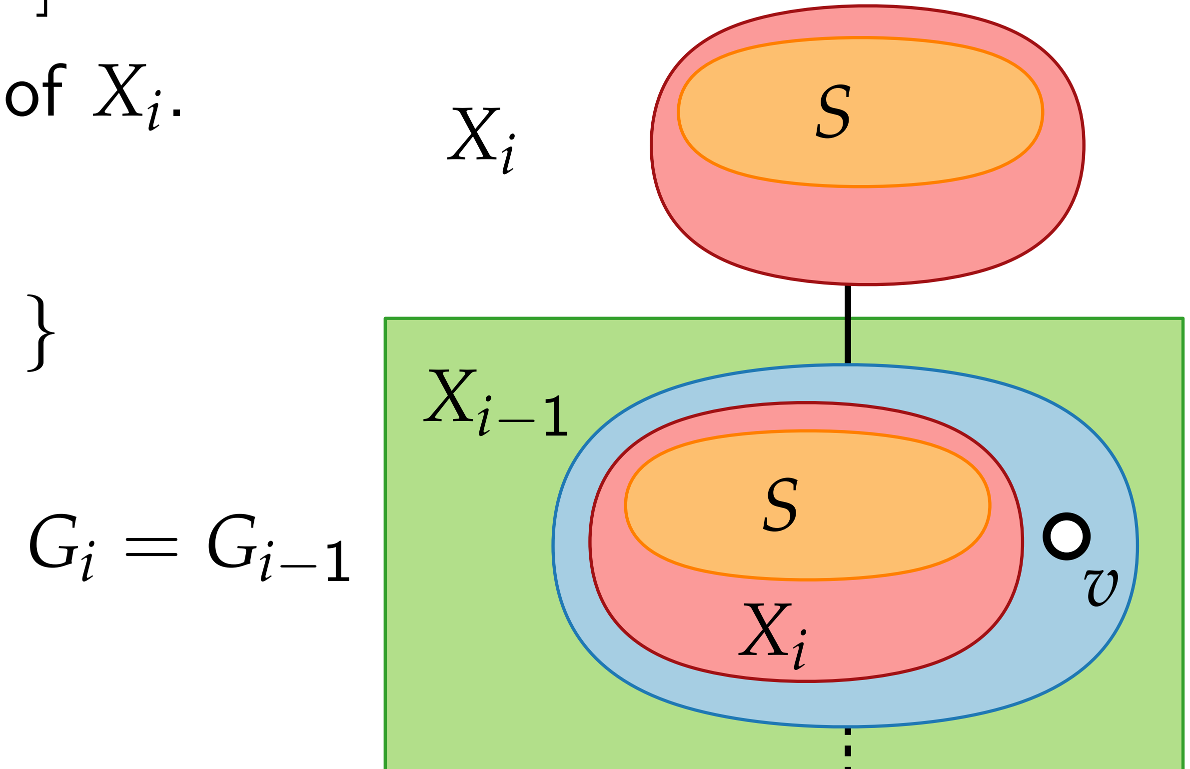

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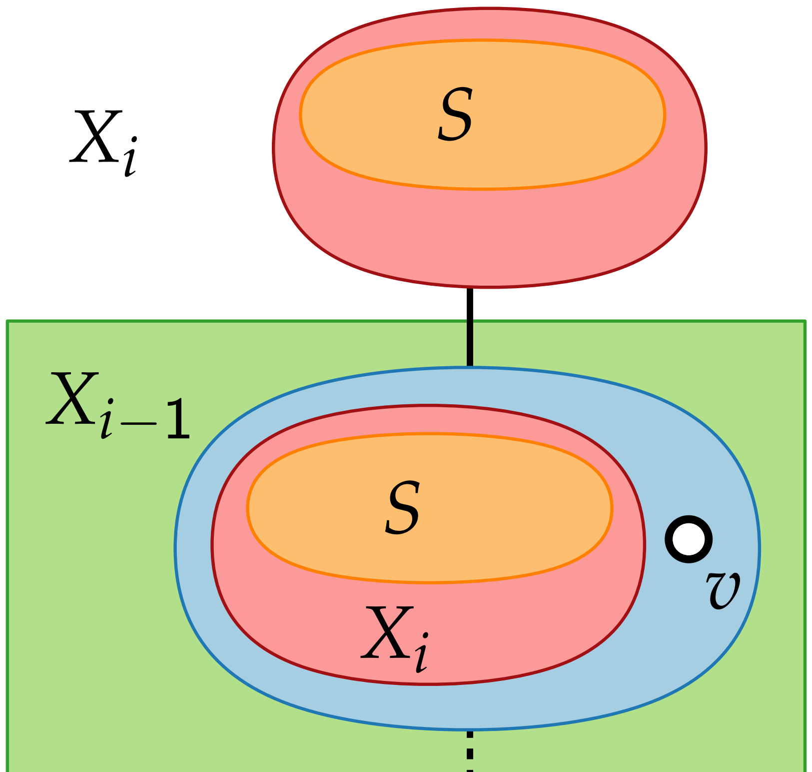

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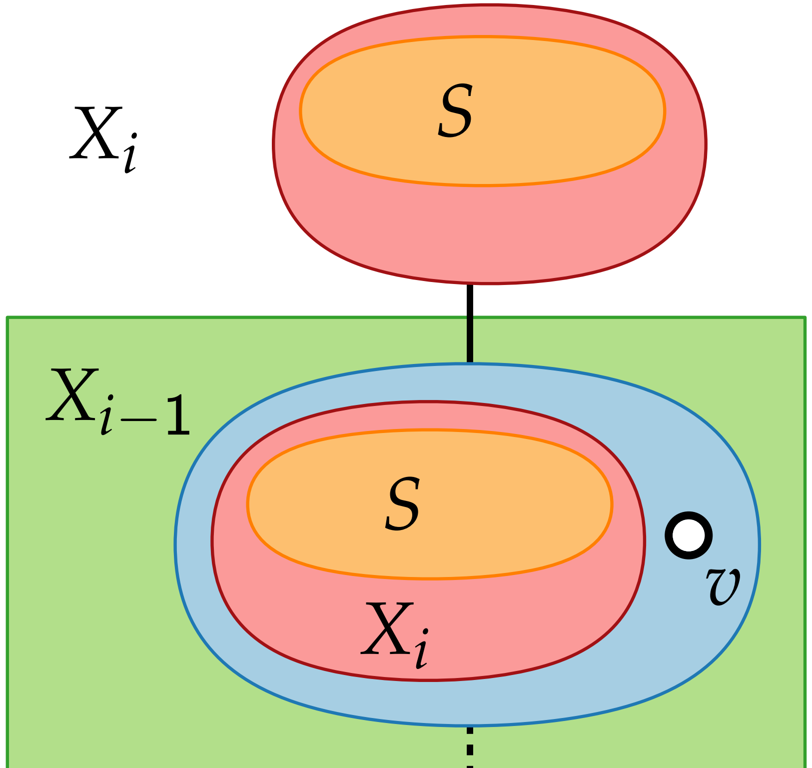

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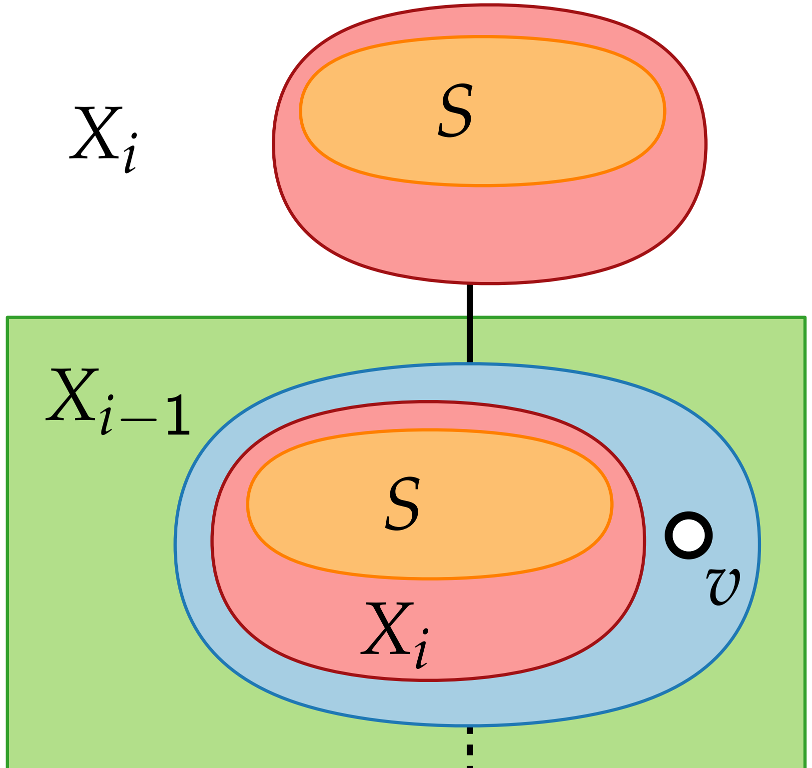

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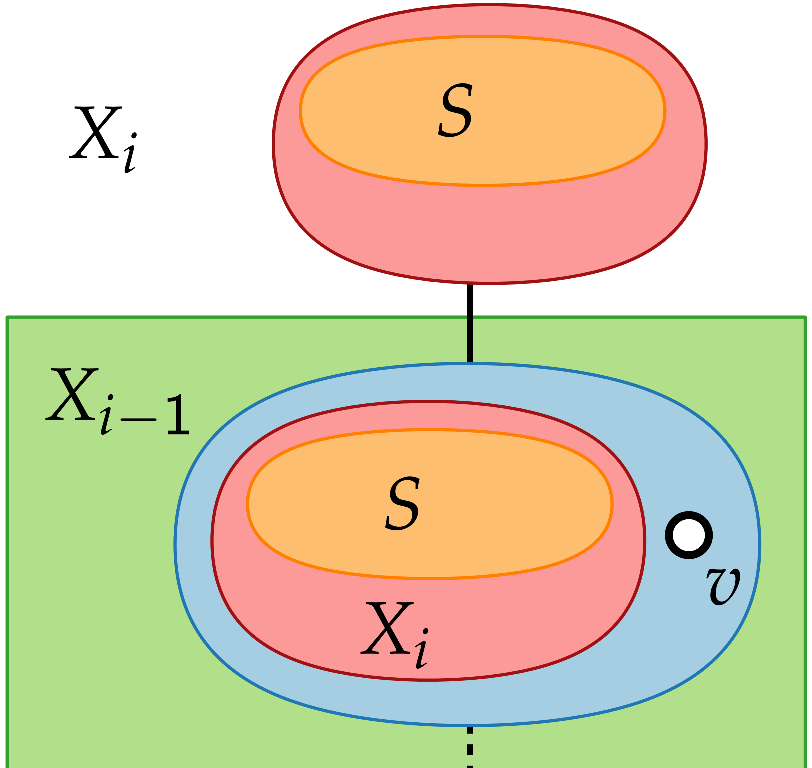

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References and Literature

[1] Parameterized Algorithms,
M. Cygan, F. Fomin, Ł. Kowalik, D. Lokshtanov, D. Marx, M. Pilipczuk,
M. Pilipczuk, S. Saurabh, Springer International Publishing 2015.

Sections 1, 7.1, 7.2, 7.3