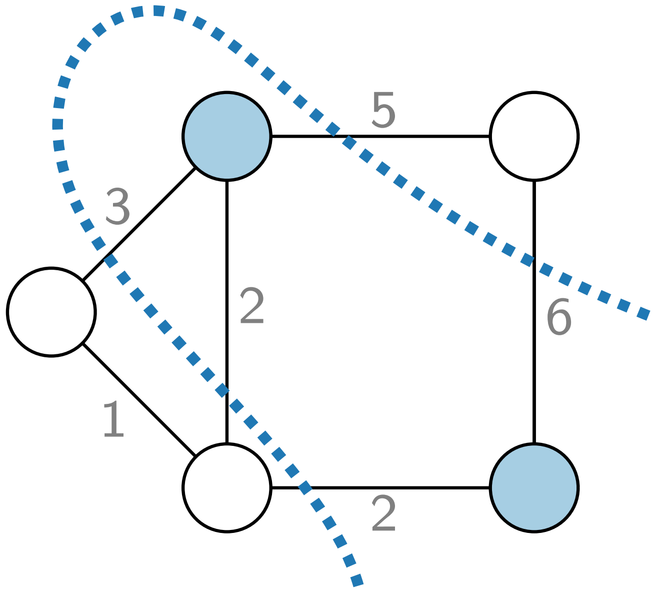
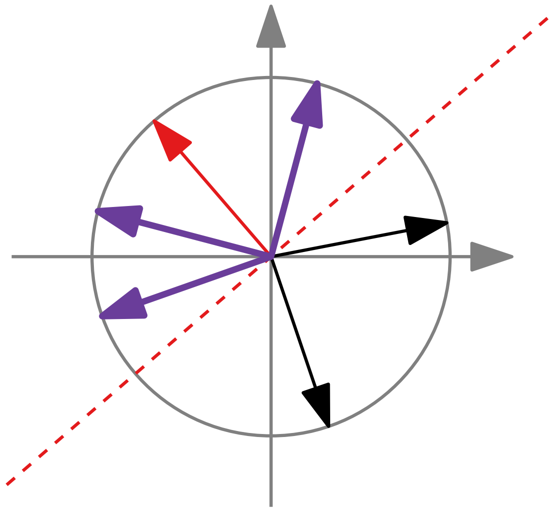
Julius-Maximilians-UNIVERSITÄT WÜRZBURG

Advanced Algorithms

QP-Relaxation
for MaxCut

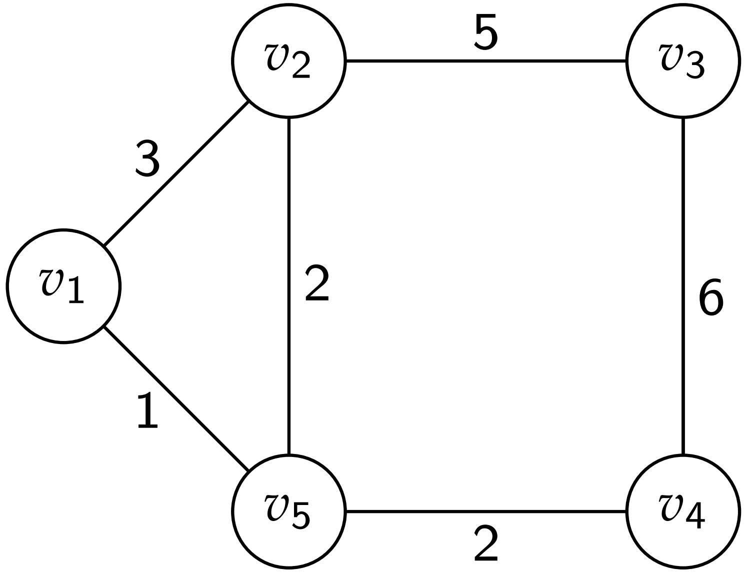
Johannes Zink · WS22





Cut

- Let $G = (V, E)$ be a graph with edge weights $c\colon E \to \mathbb{N}$.



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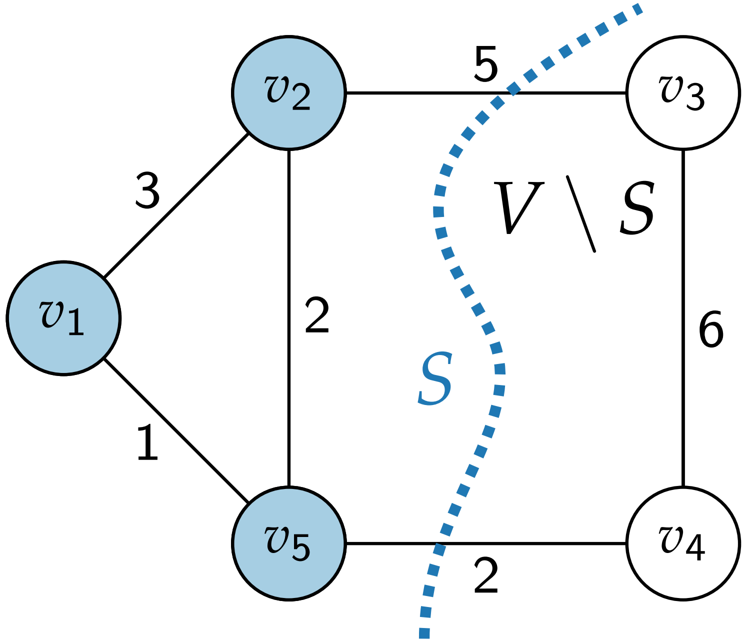

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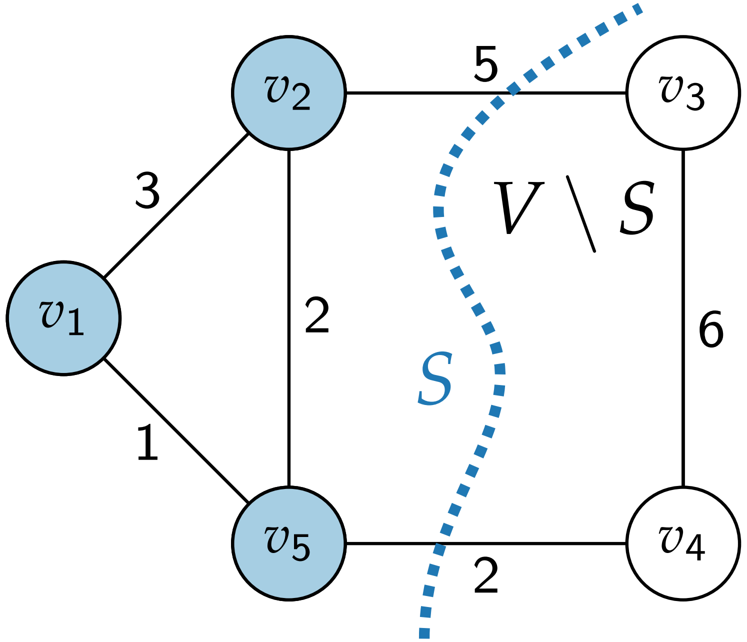
$$c(S, V \setminus S) = \sum_{\substack{uv \in E,\\ u \in S, v \in V \setminus S}} c(uv)$$



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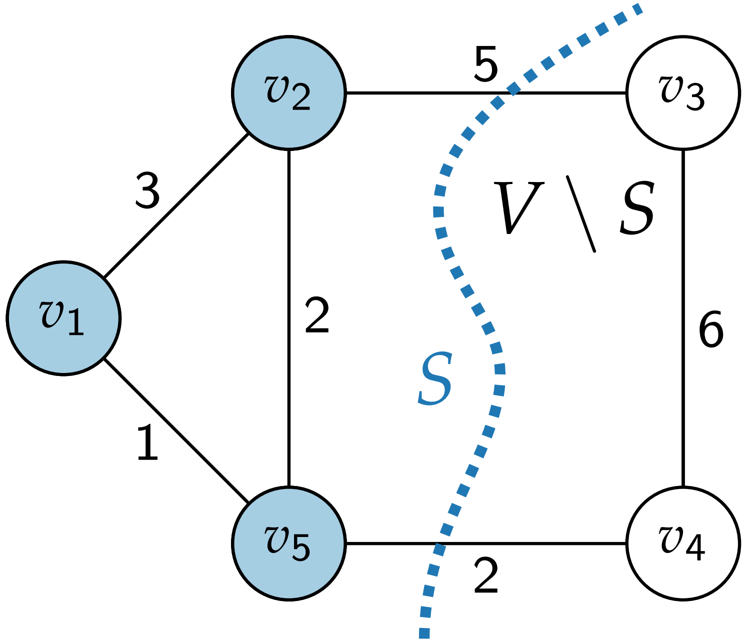
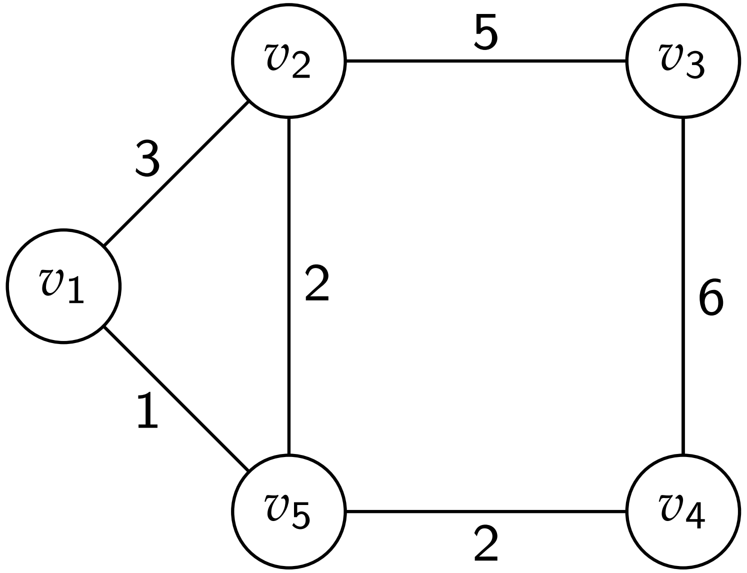


$$c(\{1, 2, 5\}, \{3, 4\}) = 7$$

The **MinCut** Problem

Input. Graph $G = (V, E)$, edge weights $c\colon E \to \mathbb{N}$.

Output. Cut $(S, V \setminus S)$ of G with **minimum** weight.



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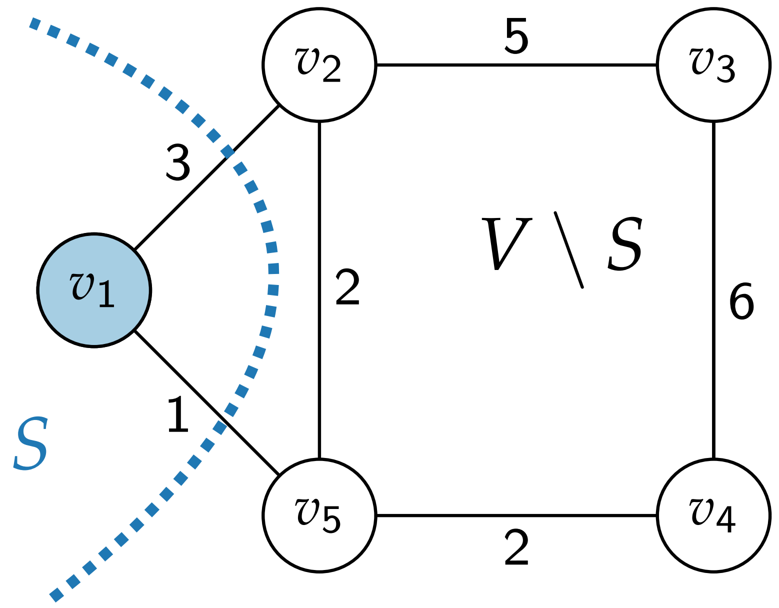

$c(S, V \setminus S) = 4$

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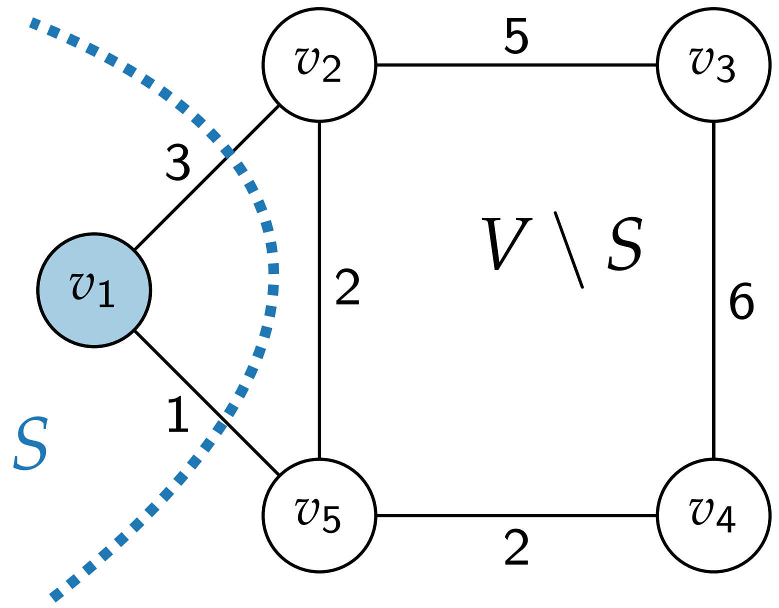

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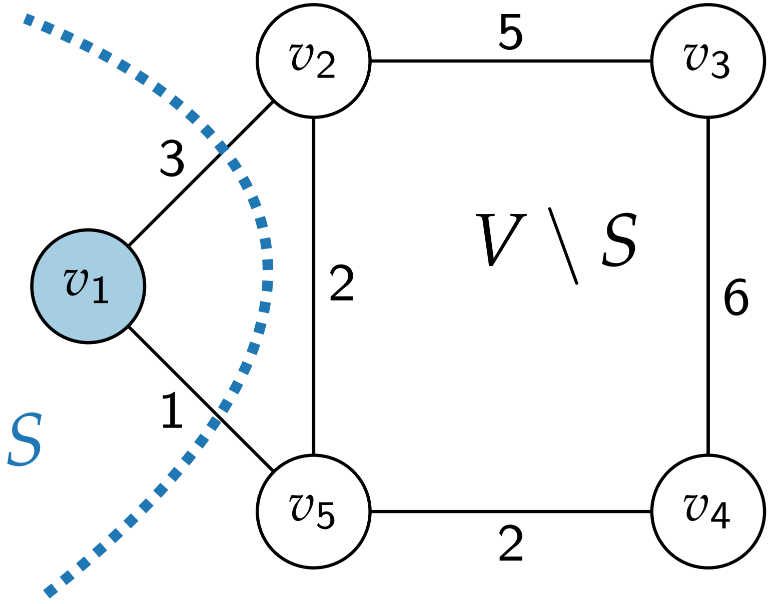
- Has applications in flow networks (*max-flow min-cut theorem*), finding a bottleneck in a network, graph partition problems, clustering, ...
- Can be solved optimally in polynomial time, e.g. by the Stoer–Wagner algorithm.



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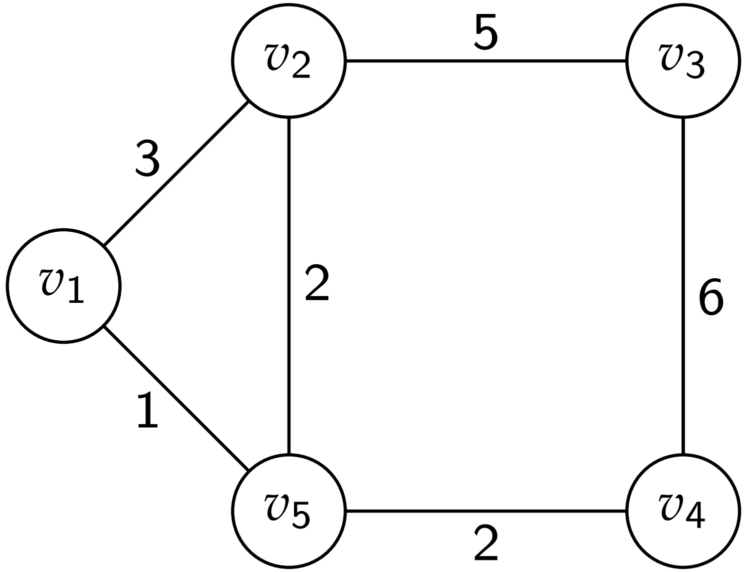

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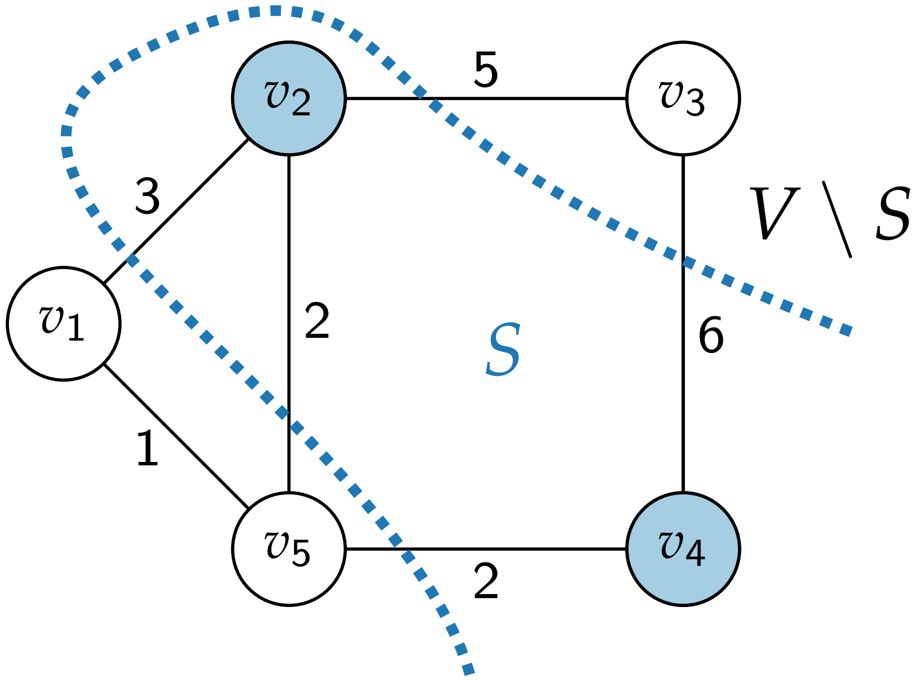

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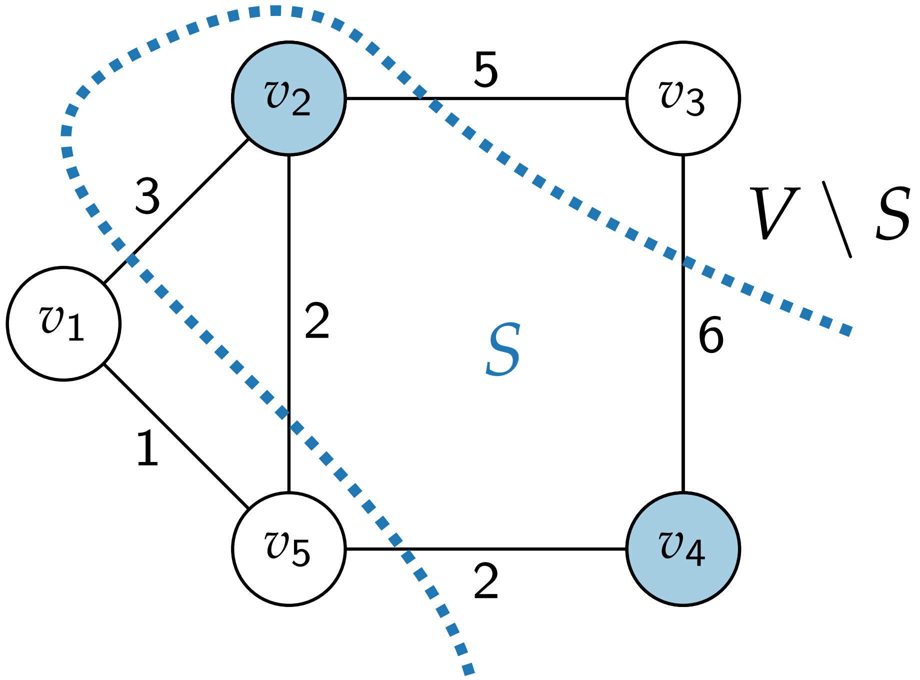

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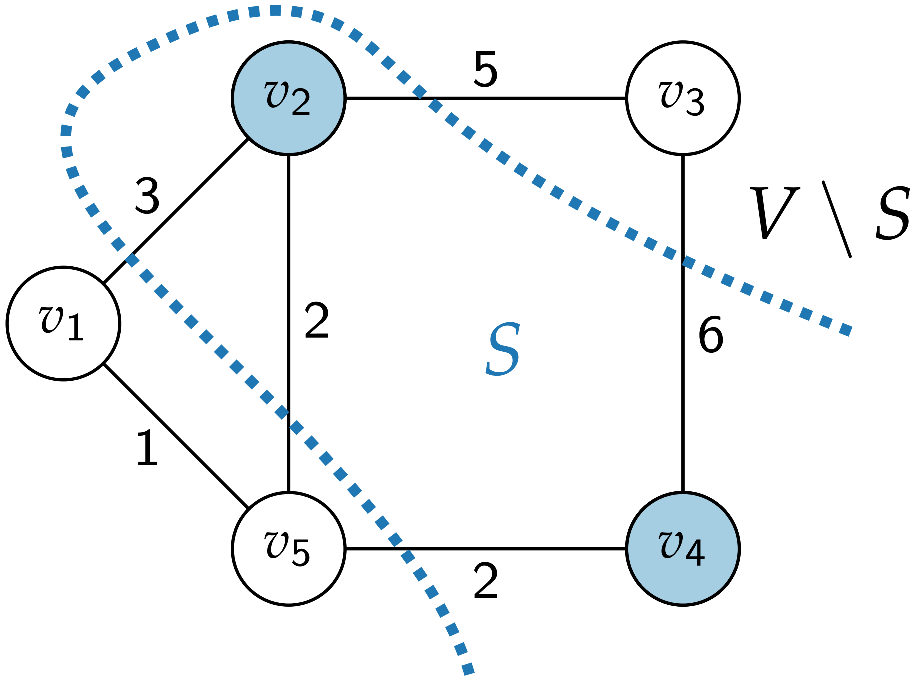
- Has applications in statistical physics, where it is used for some models of magnetic spins in disordered systems, and in integrated circuit design for computer chips.
- NP-complete to find a cut with maximum weight.



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Randomized 0.5-Approximation for (Unweighted) MaxCut

$\textsc{CoinFlipMaxCut}(G, c\colon E \to 1)$

$S \leftarrow \emptyset$

foreach $v \in V$ **do**

  if coin flip shows $\textsc{Heads}$ **then**

    $S \leftarrow S \cup \{v\}$

return $c(S, V \setminus S), S$

Randomized 0.5-Approximation for (Unweighted) MaxCut

Theorem 1.
CoinFlipMaxCut is a randomized 0.5-approximation algorithm for MaxCut.

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- Runs in $O(n+m)$.

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- Compute expected weight of cut:

$$\mathrm{E}[c(\textsc{CoinFlipMaxCut}(G))]$$

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- Can be “derandomized”. Exercise.

LP-Relaxation

Integer Linear Program

$$\begin{array}{llll} \textbf{maximize} & c^\mathsf{T} x & & \\ \textbf{subject to} & Ax & \leq & b \\ & x & \geq & 0 \\ & x & \in & \mathbb{Z} \end{array}$$

LP-Relaxation

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↓ Solve in polynomial time

Solution for LP

$x^\star$

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← e.g. rounding

Assignment for ILP

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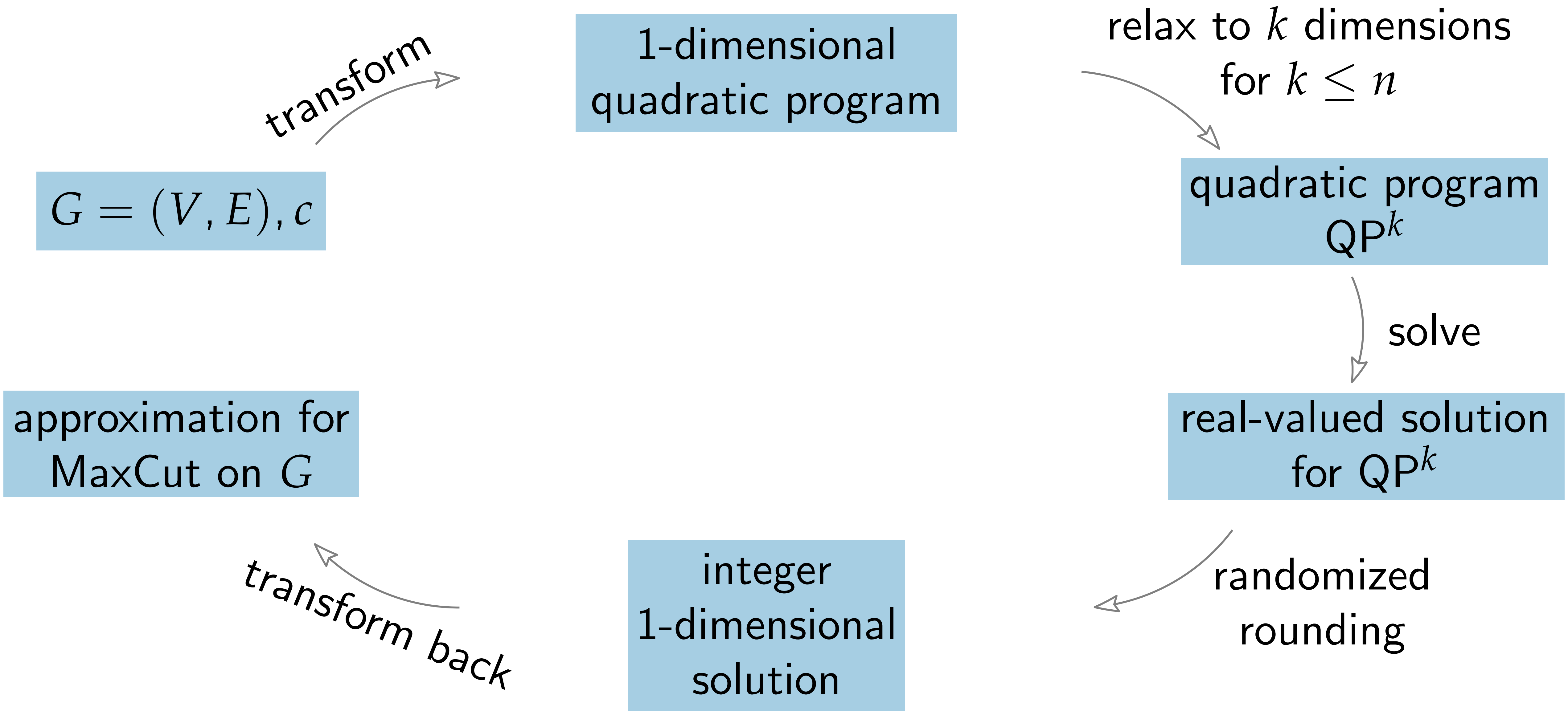
Assignment for ILP

$x^\star$

↑ Solution, approximation, or bound

Goemans-Williamson Algorithm for MaxCut



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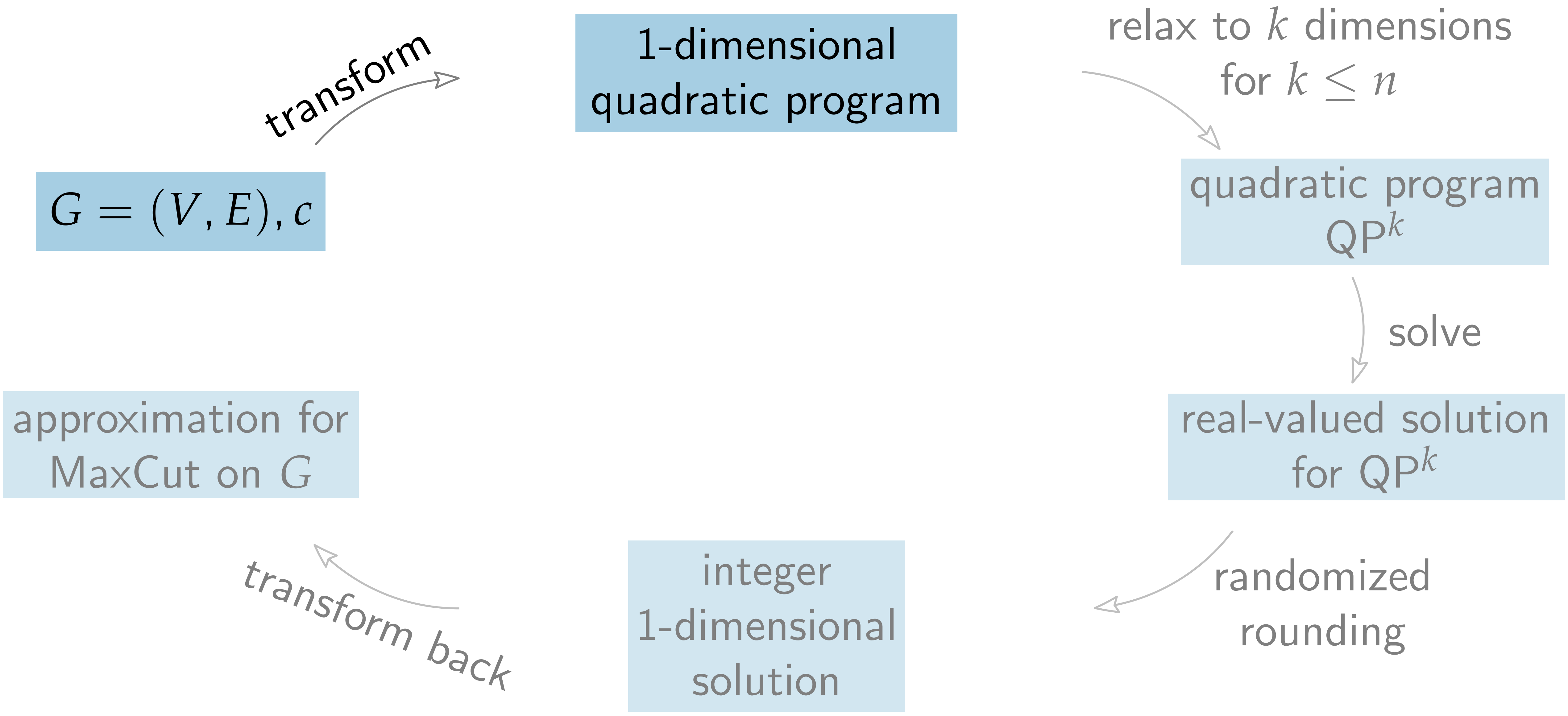

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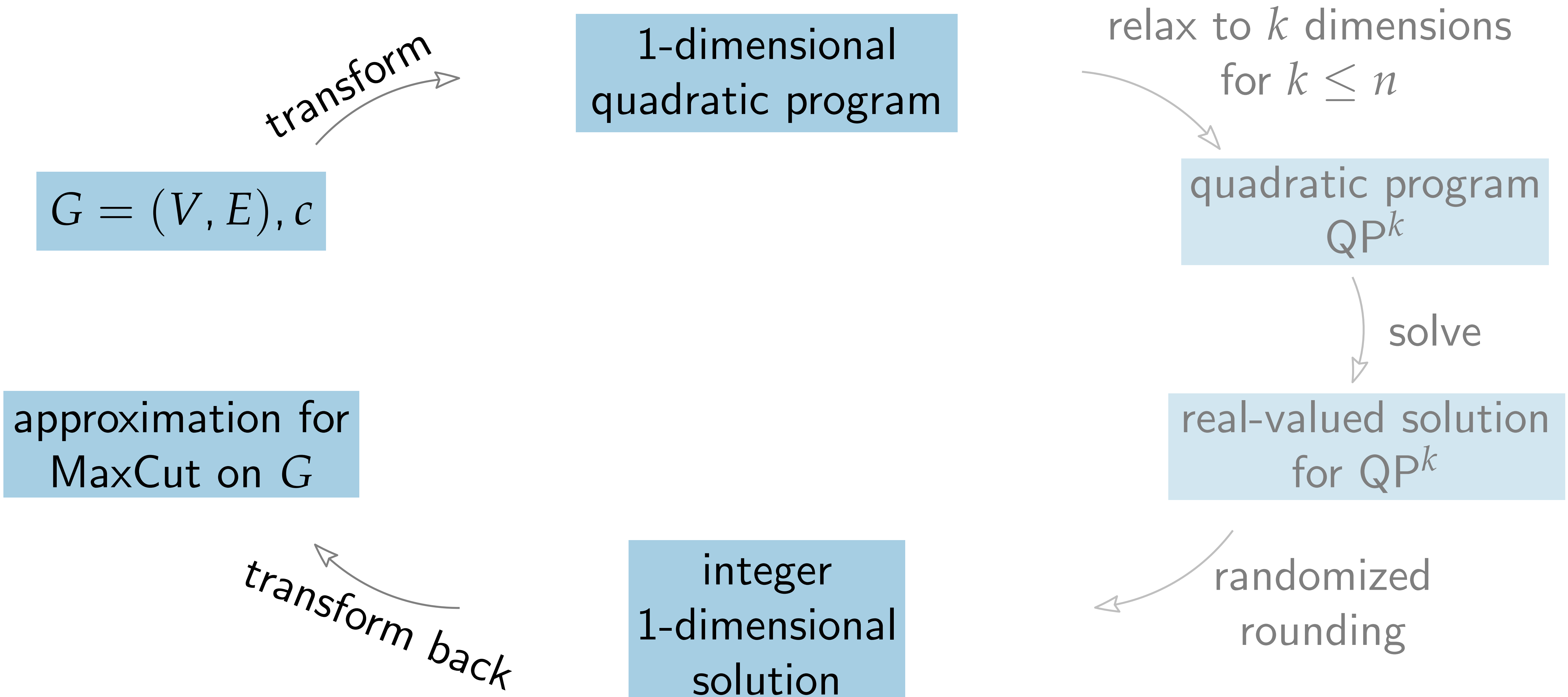
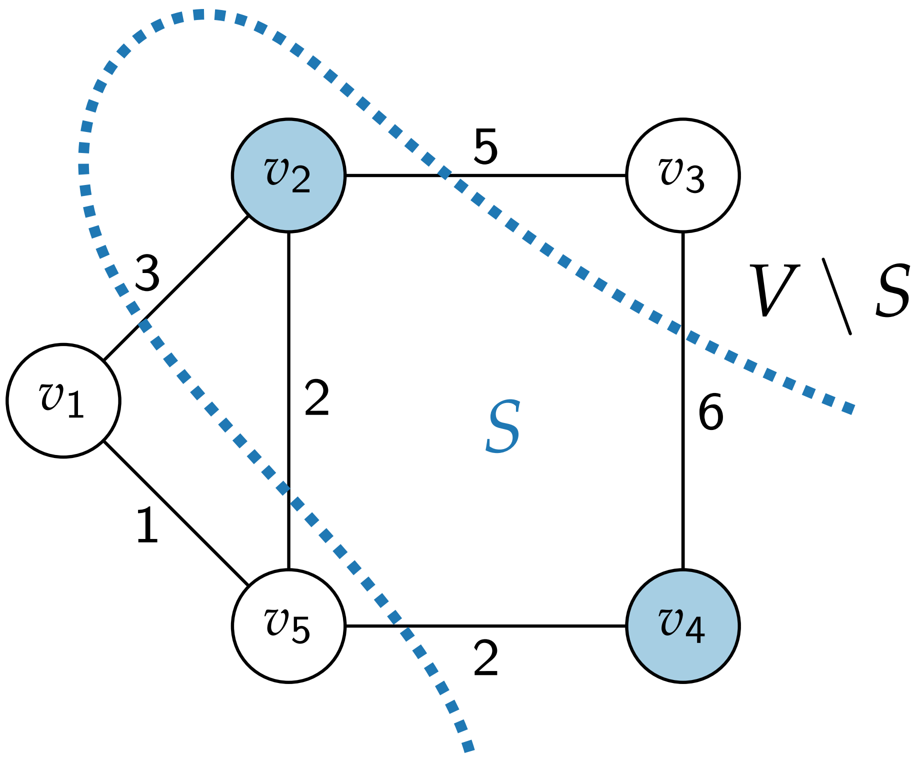


QP(G, c)

Idea.



QP(G, c)

maximize

subject to

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Idea.

- Indicator variable for each vertex v_i: $x_i \in \{1, -1\}$

QP(G, c)

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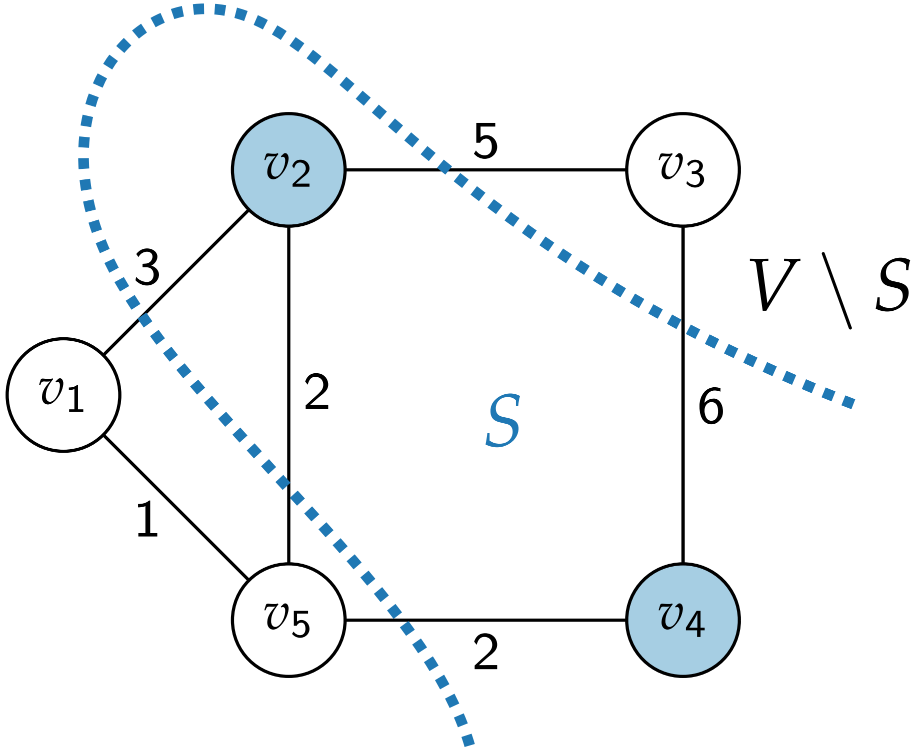

QP(G, c)

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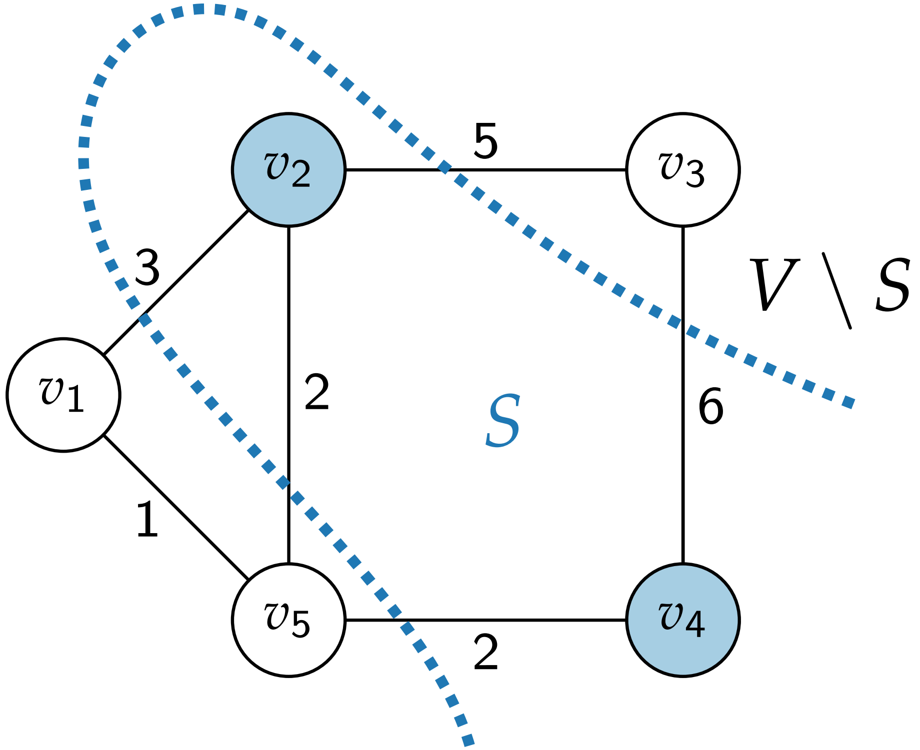
QP(G, c)

maximize

subject to $x_i^2 = 1$



$\mathrm{QP}(G, c)$

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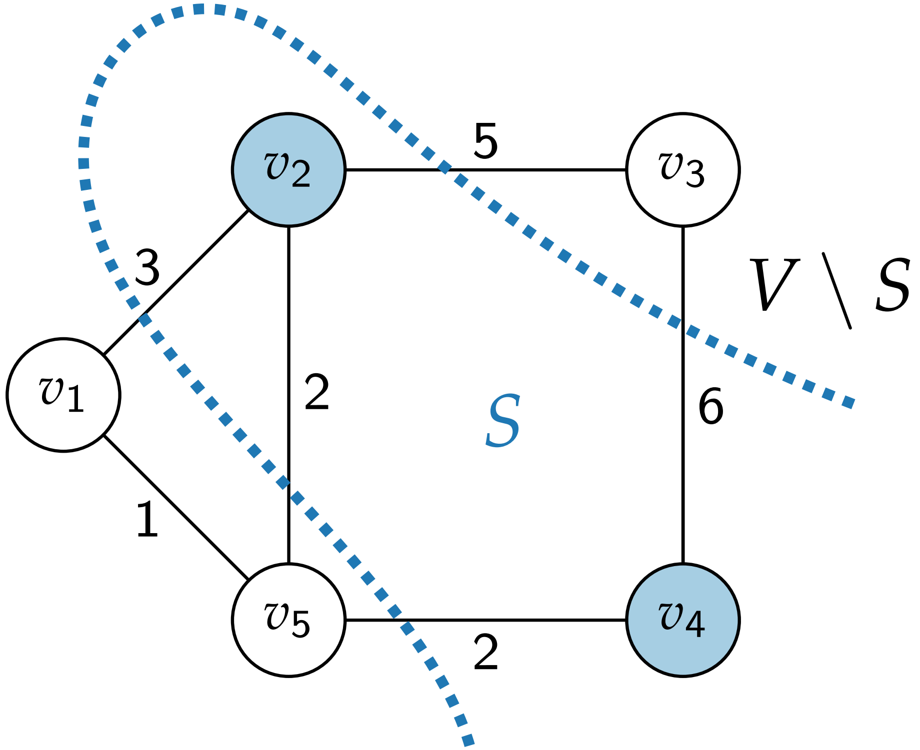
- Indicator variable for each vertex v_i: $x_i \in \{1, -1\}$
- $x_i \cdot x_j = \begin{cases} 1 & \text{if } i, j \text{ in same partition} \\ -1 & \text{otherwise} \end{cases}$



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QP(G, c)

maximize $(1 - x_i x_j)$

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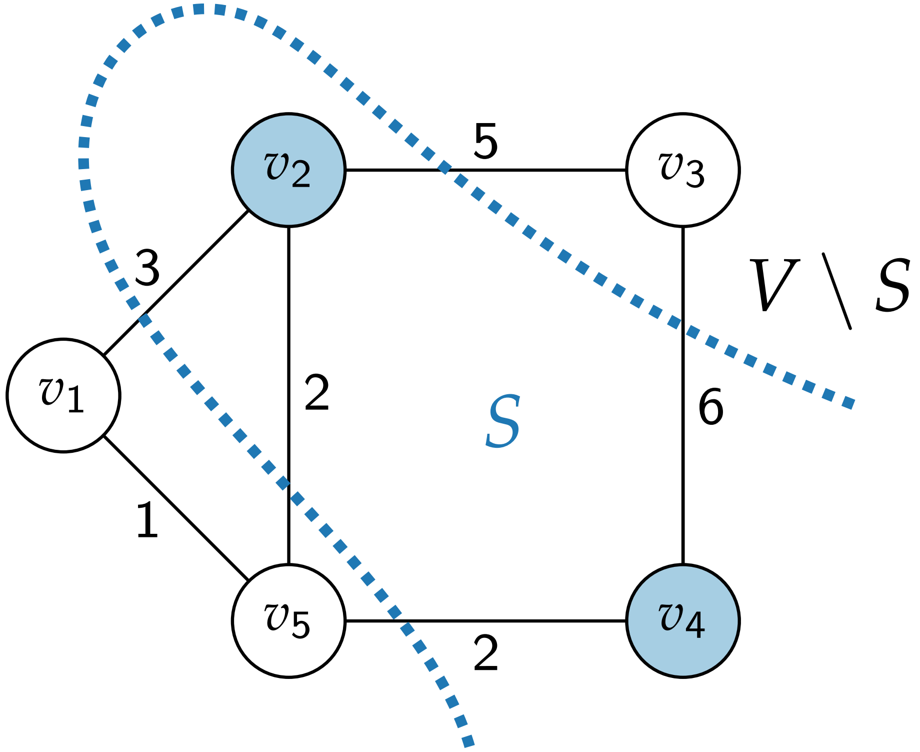

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QP(G, c)

maximize $c_{ij}(1 - x_i x_j)$

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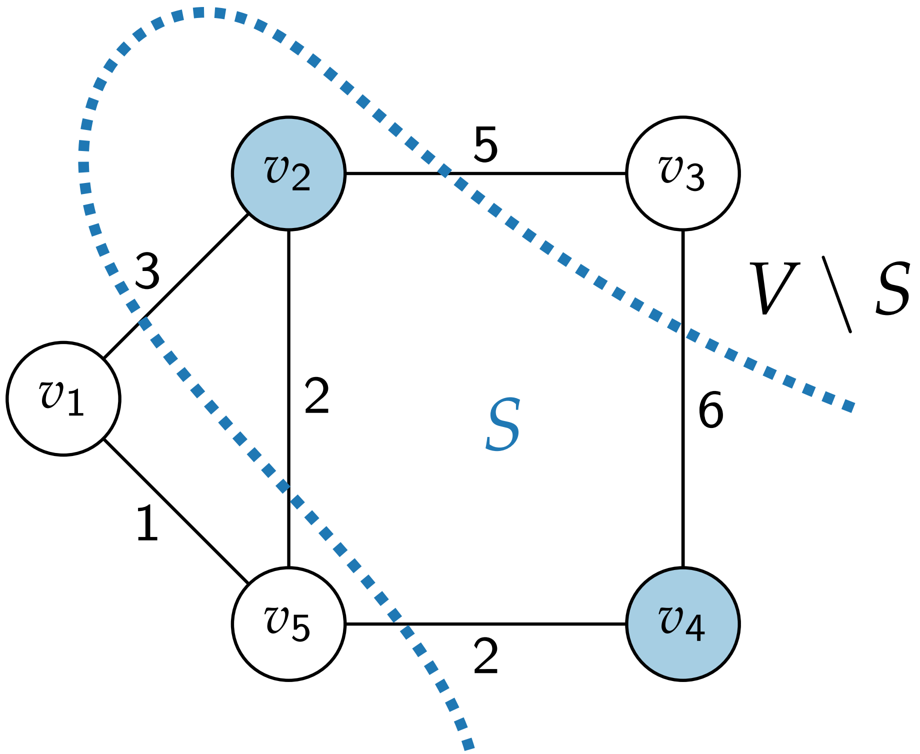

- Weight matrix c_{ij}

	1	2	3	4	5
1		3			1
2	3		5		2
3		5		6	
4			6		2
5	1	2		2	

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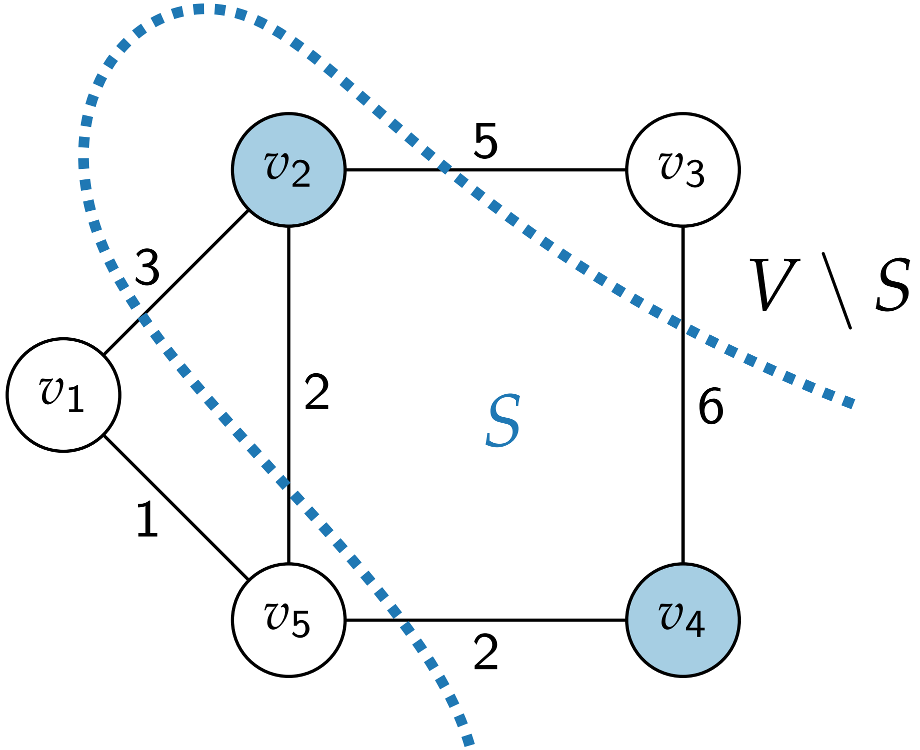
QP(G, c)

maximize $\frac{1}{2} \sum_{j=1}^{n} \sum_{i=1}^{j-1} c_{ij}(1 - x_i x_j)$

subject to $x_i^2 = 1$



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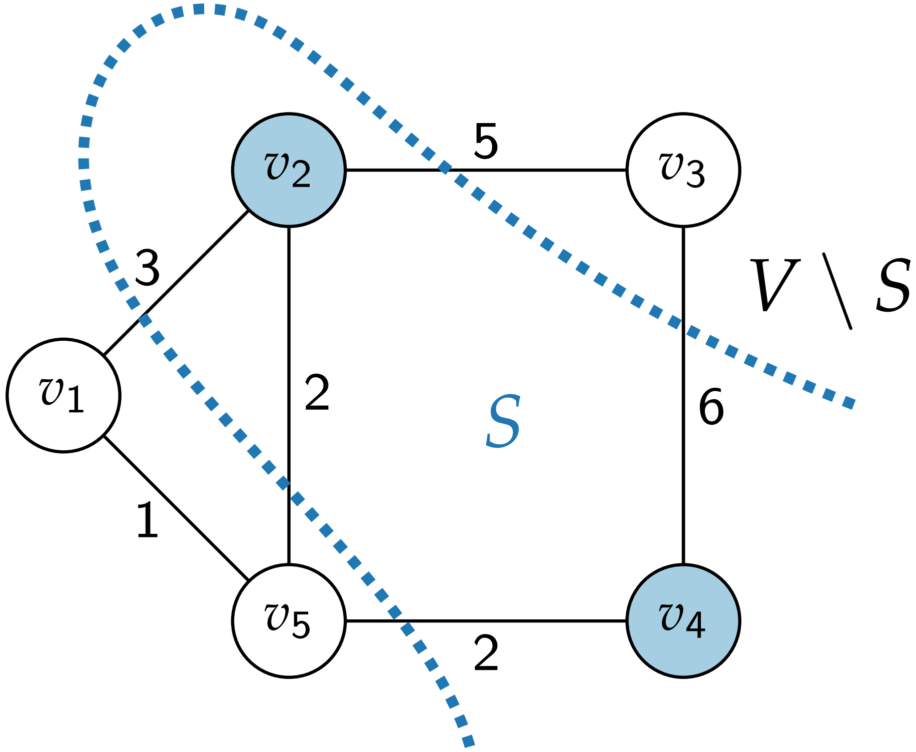

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- Solution

$x_2 = x_4 = 1$

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QP(G, c)

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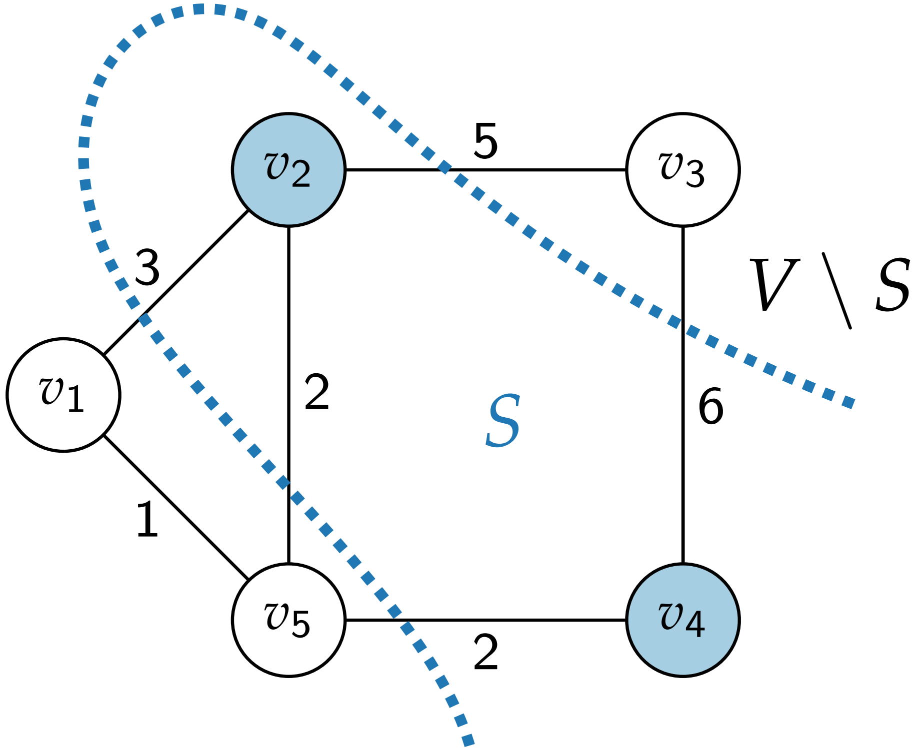
$$\text{subject to} \quad x_i^2 = 1$$



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- Solution

  $x_2 = x_4 = 1$

  $x_1 = x_3 = x_5 = -1$

Note.

- Solving QP(G, c) is NP-hard.
- Otherwise MaxCut wouldn't be NP-hard.

Goemans-Williamson Algorithm for MaxCut

$G = (V, E), c$

transform

1-dimensional quadratic program

relax to k dimensions for $k \leq n$

quadratic program QP^k

solve

real-valued solution for QP^k

randomized rounding

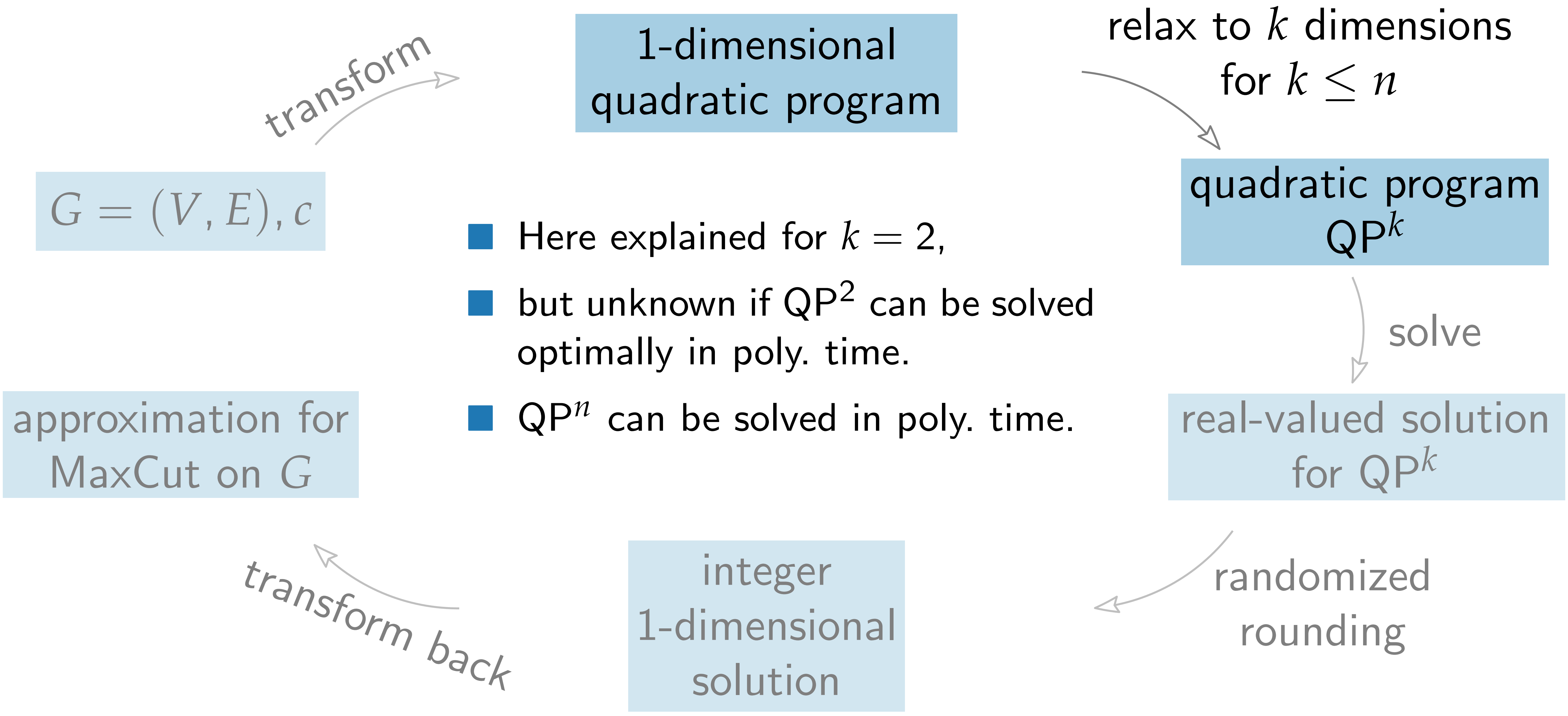
integer 1-dimensional solution

transform back

approximation for MaxCut on G

- Here explained for $k = 2$,
- but unknown if QP^2 can be solved optimally in poly. time.
- QP^n can be solved in poly. time.

Goemans-Williamson Algorithm for MaxCut



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- QP^n can be solved in poly. time.

Relaxation of QP(G, c)

QP$^2(G, c)$

maximize $\frac{1}{2}\sum_{j=1}^{n}\sum_{i=1}^{j-1} c_{ij}(1 - x^i \cdot x^j)$

subject to $x^i \cdot x^i = 1$

$x^i = (x^i_1, x^i_2) \in \mathbb{R}^2$

Relaxation of QP(G, c)

QP$^2(G, c)$

maximize $\frac{1}{2} \sum_{j=1}^{n} \sum_{i=1}^{j-1} c_{ij}(1 - x^i \cdot x^j)$

subject to $x^i \cdot x^i = 1$

$x^i = (x^i_1, x^i_2) \in \mathbb{R}^2$

- " · " is scalar product.

Relaxation of QP(G, c)

QP$^2(G, c)$

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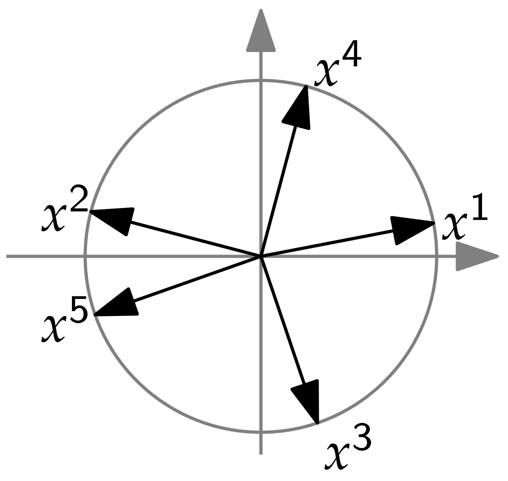

Relaxation of $\mathrm{QP}(G, c)$

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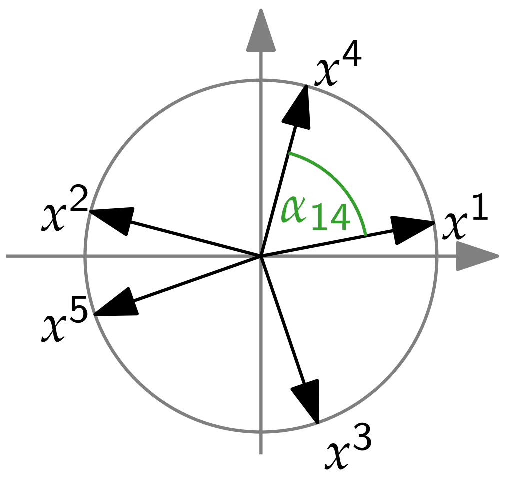

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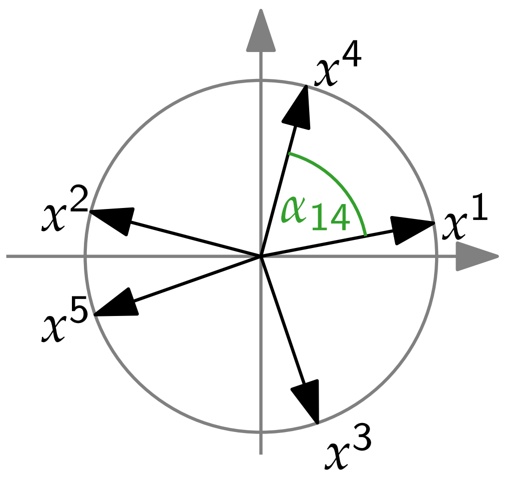

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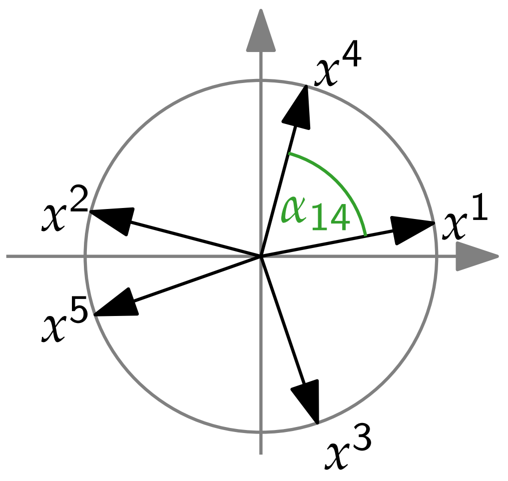

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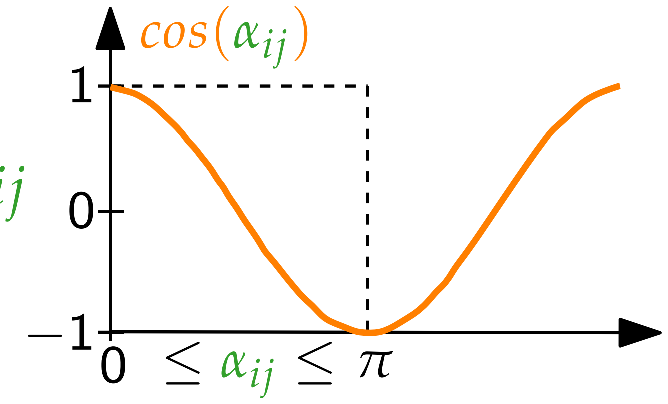

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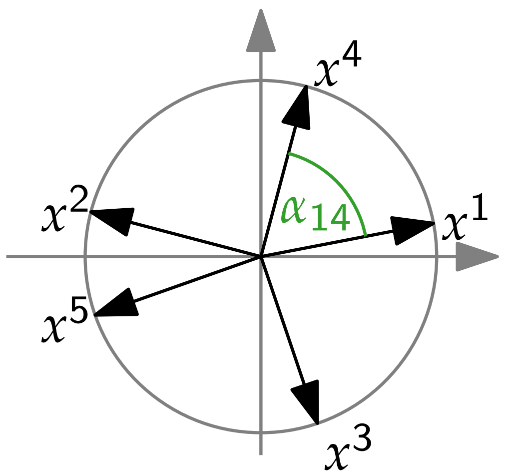

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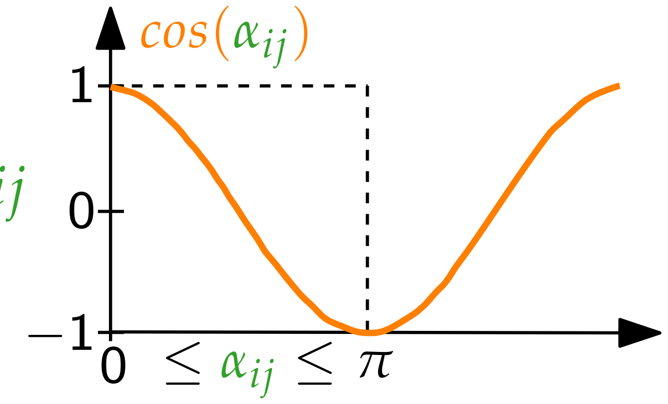

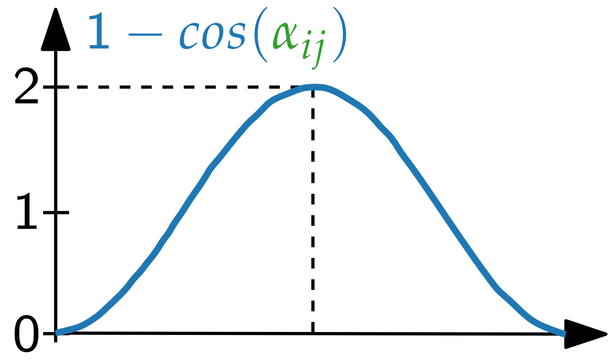

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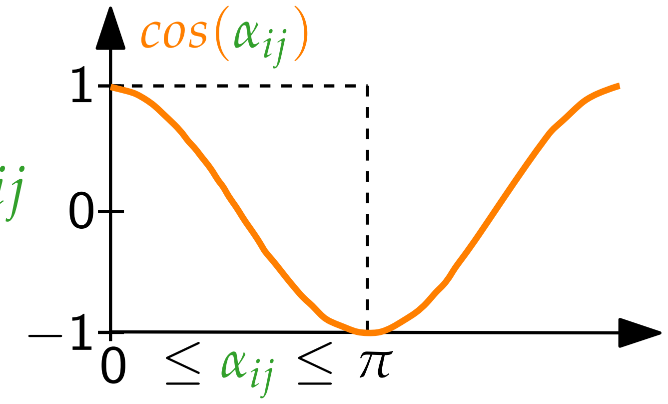

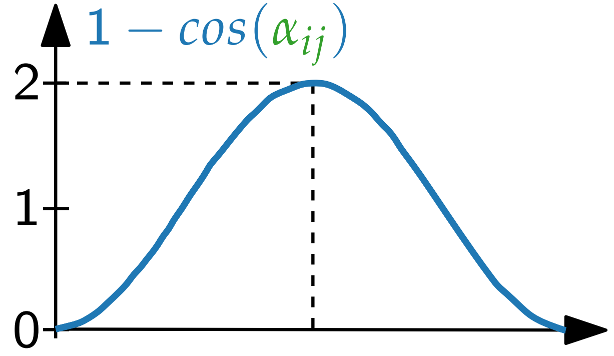

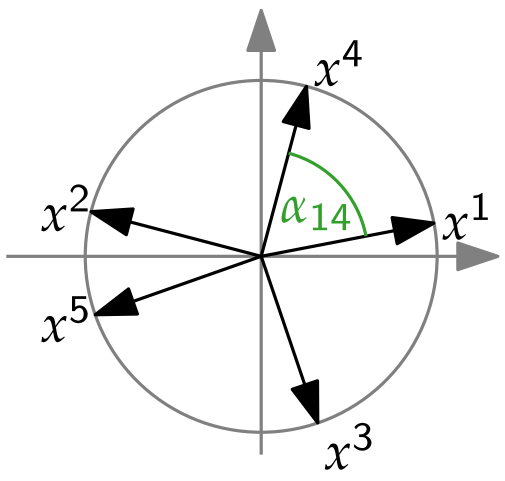

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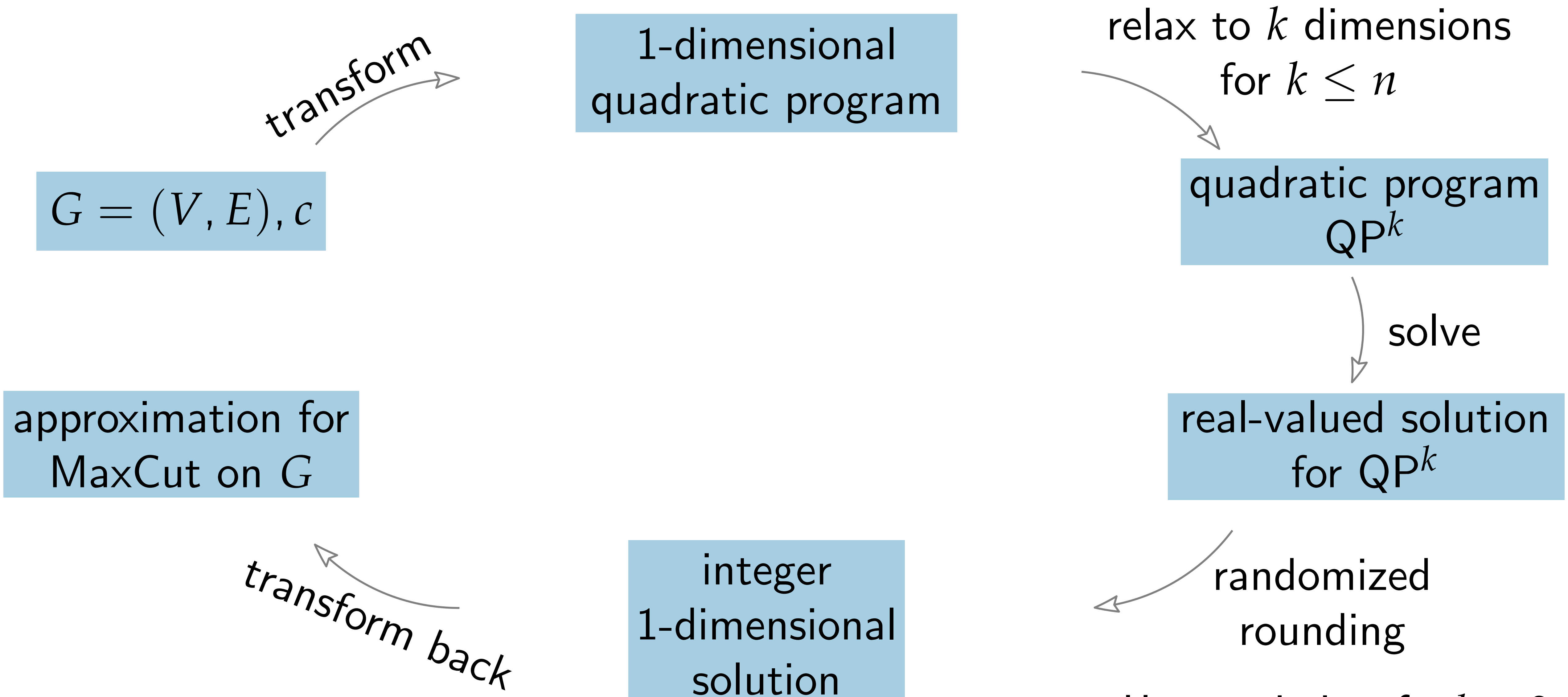
- Hence, our objective is:
  $\frac{1}{2}\sum_{j=1}^{n}\sum_{i=1}^{j-1} c_{ij}(1 - \cos(\alpha_{ij}))$

Goemans-Williamson Algorithm for MaxCut



- Here again just for $k = 2$.

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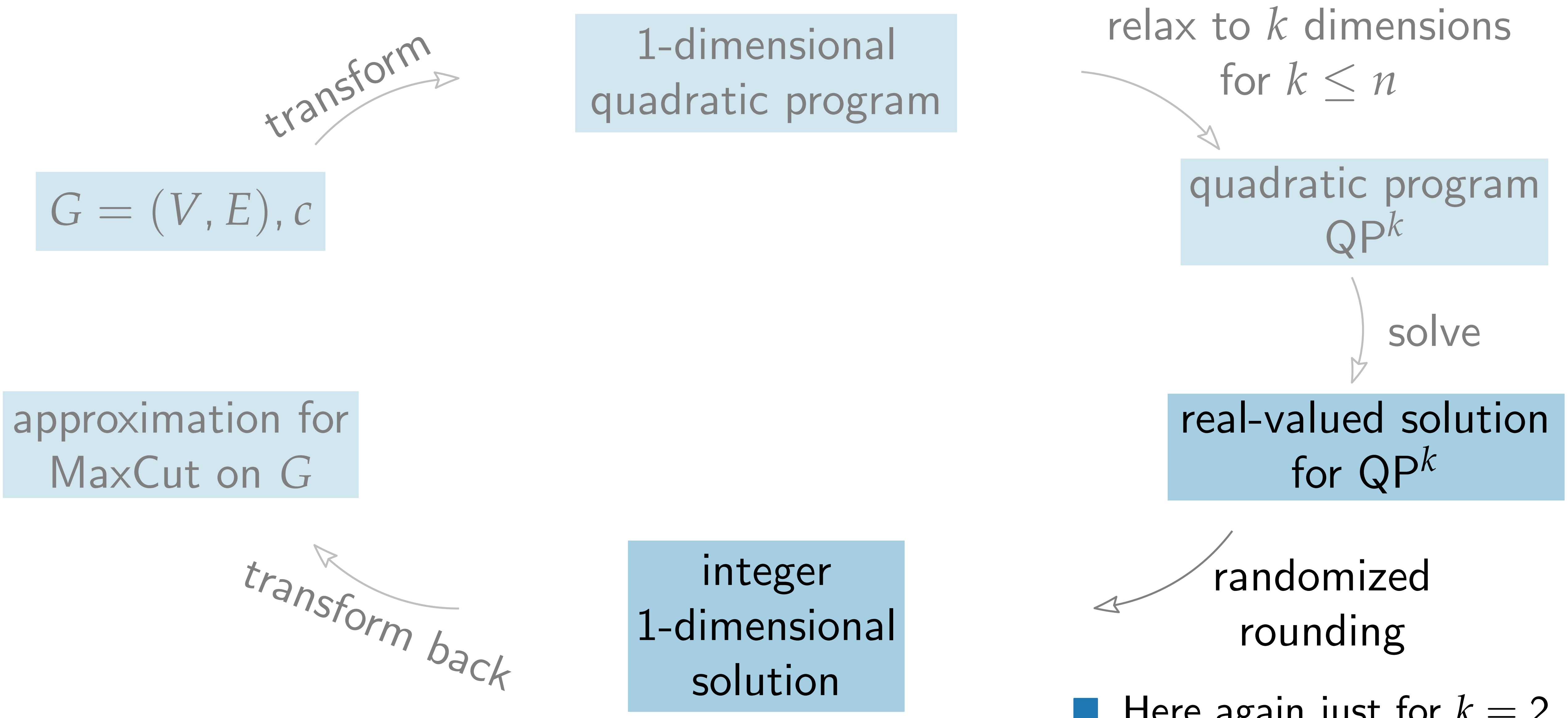

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Algorithm RandomizedMaxCut

RandomizedMaxCut(G, c)

Compute optimal solution $(\tilde{x}^1, \ldots, \tilde{x}^n)$ for QP$^2(G, c)$
Pick random vector $r \in \mathbb{R}^2$
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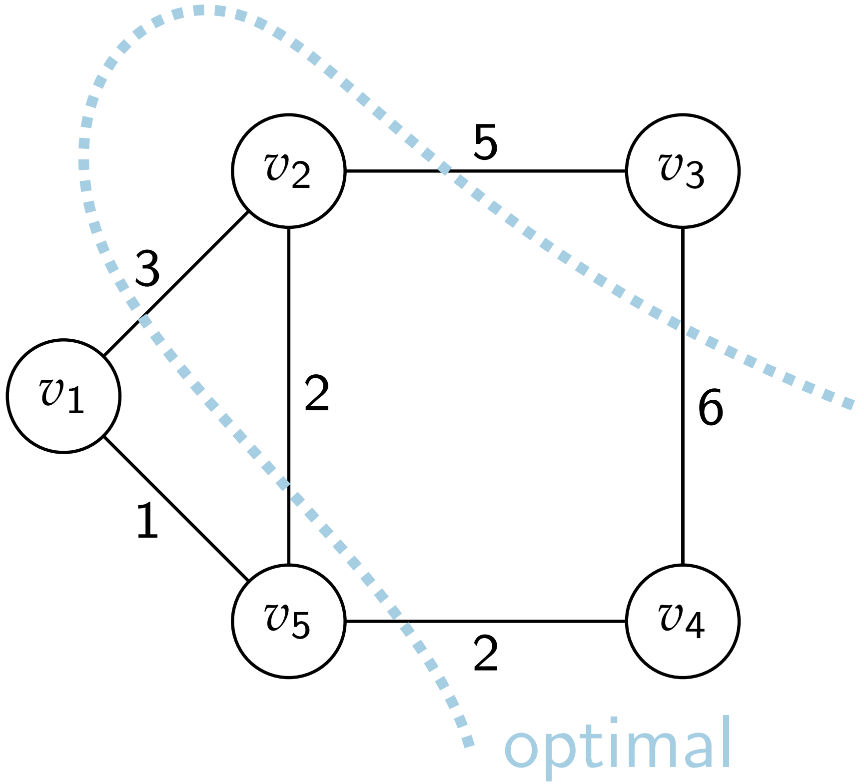

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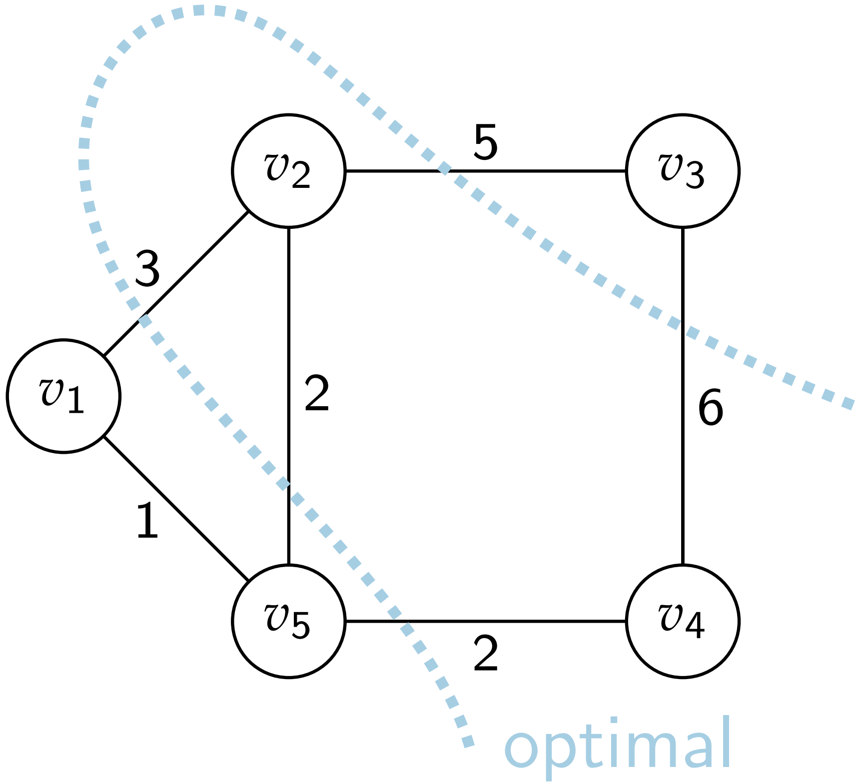

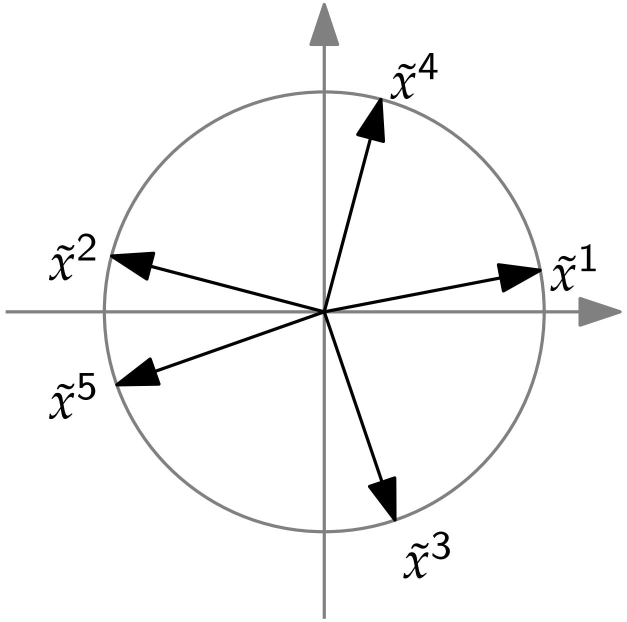

(Sketch)

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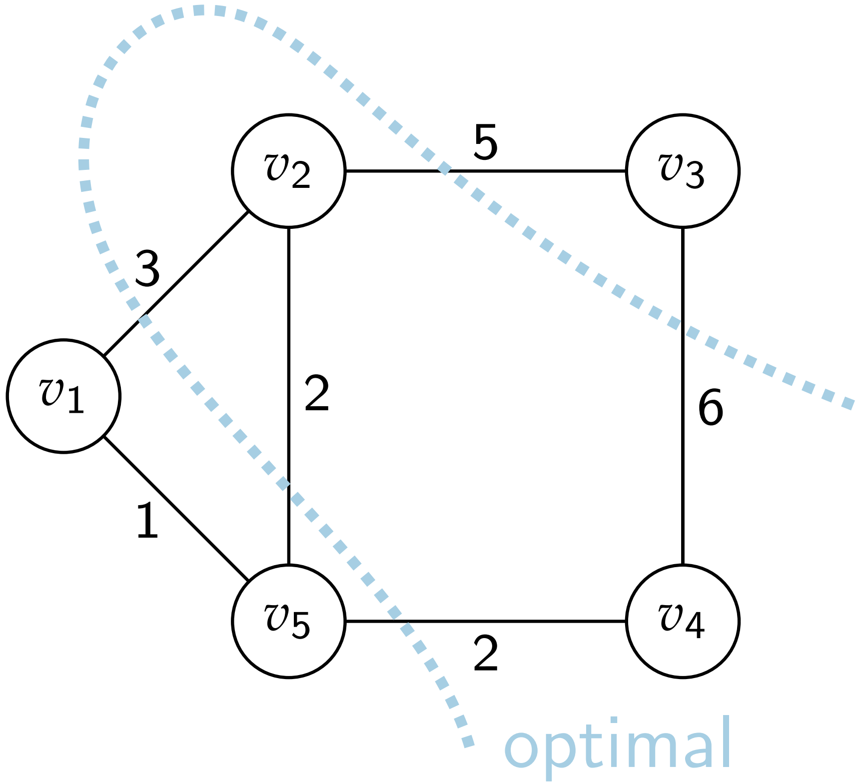

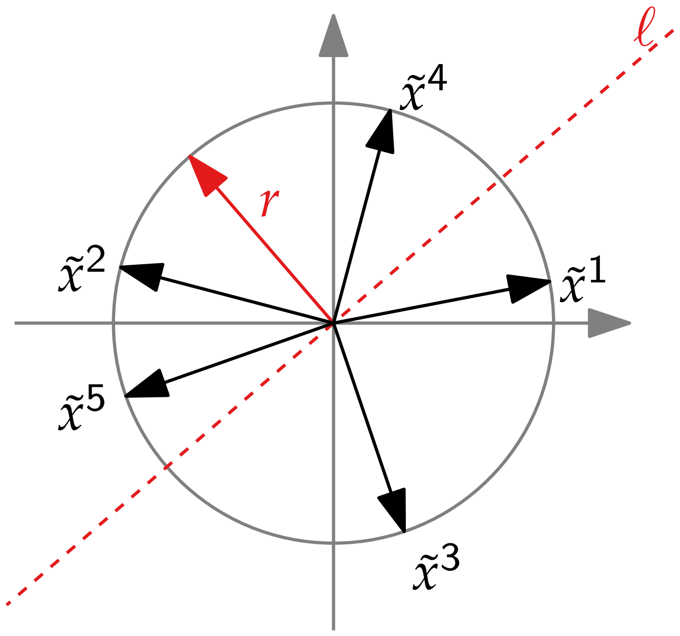

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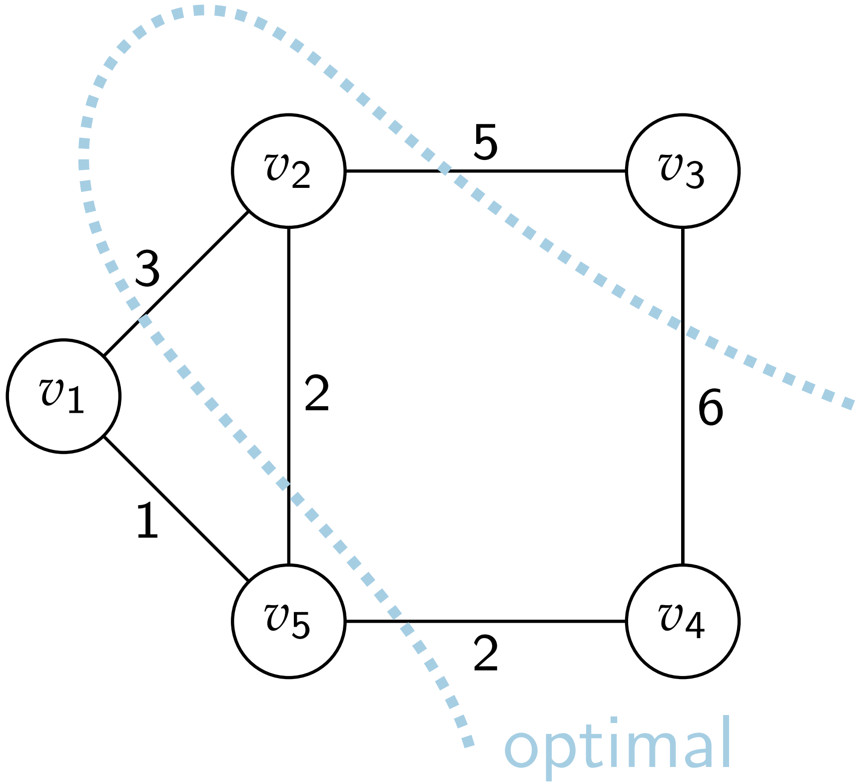

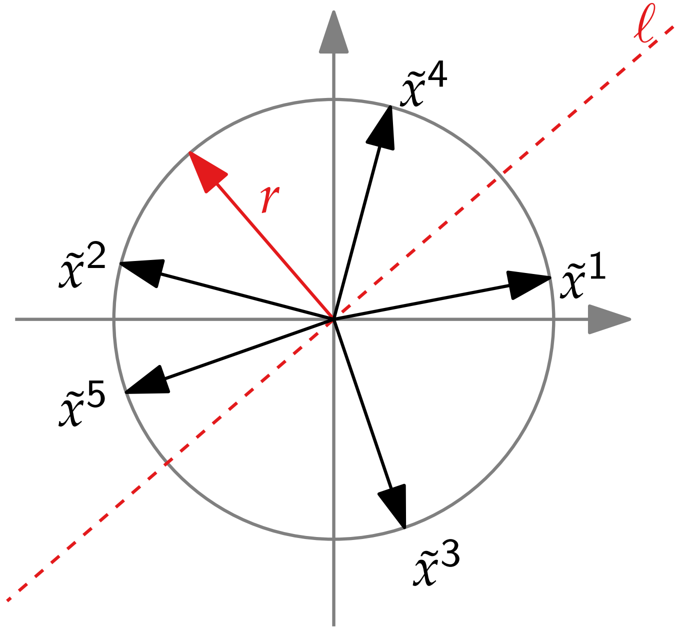

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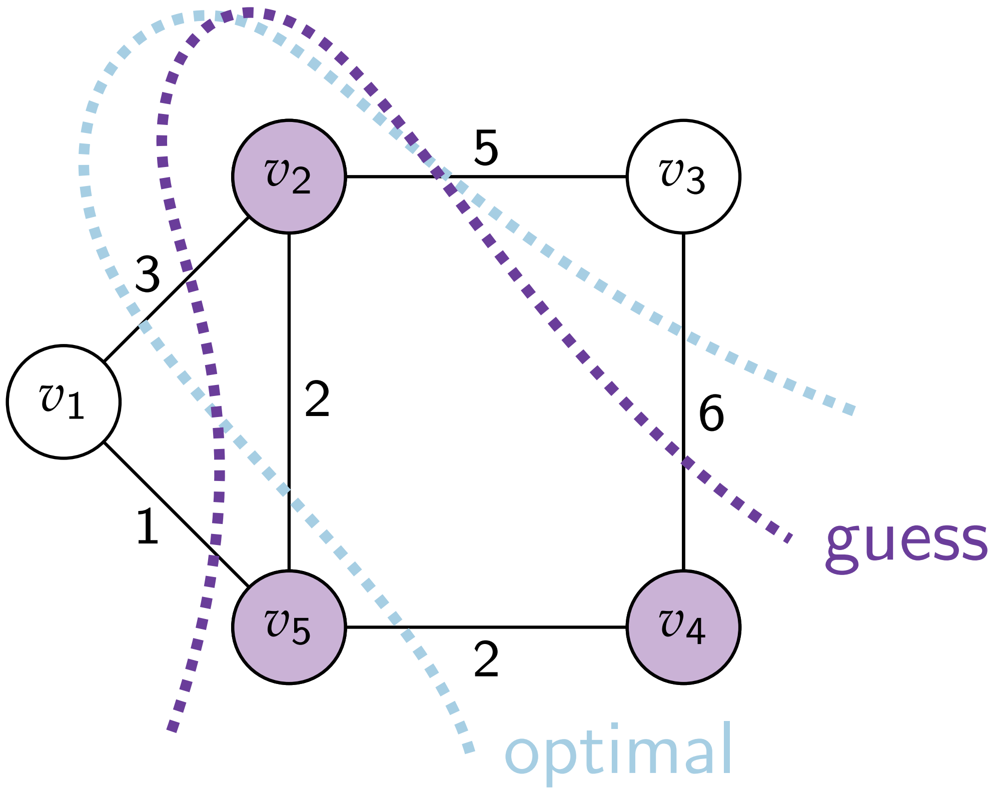

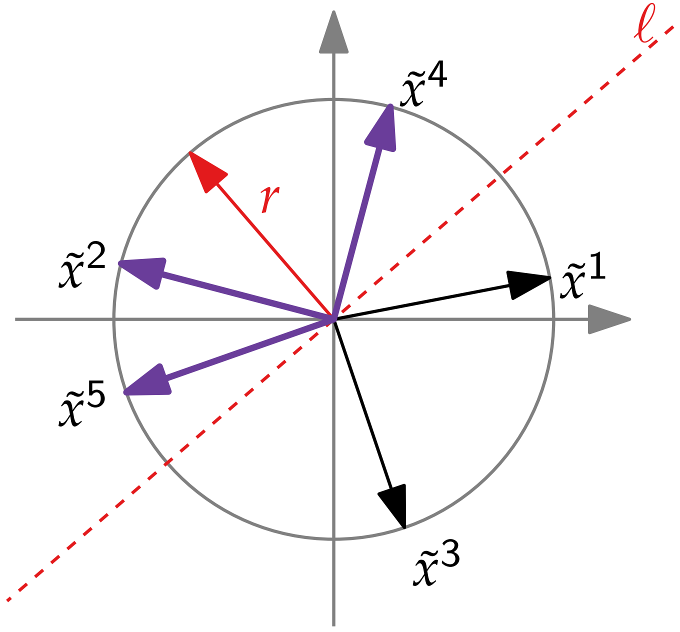

(Sketch)

RandomMaxCut – Expected Value

Lemma 2.
Let X be the solution of RandomizedMaxCut(G, c).
If r is picked uniformally at random, then

$$\mathrm{E}[X] = \sum_{j=1}^{n} \sum_{i=1}^{j-1} c_{ij} \frac{\alpha_{ij}}{\pi}.$$

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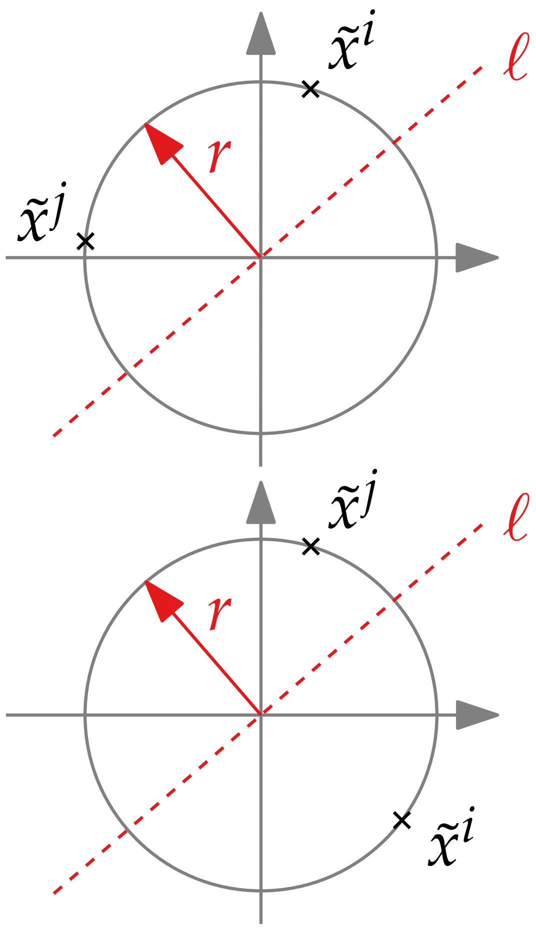

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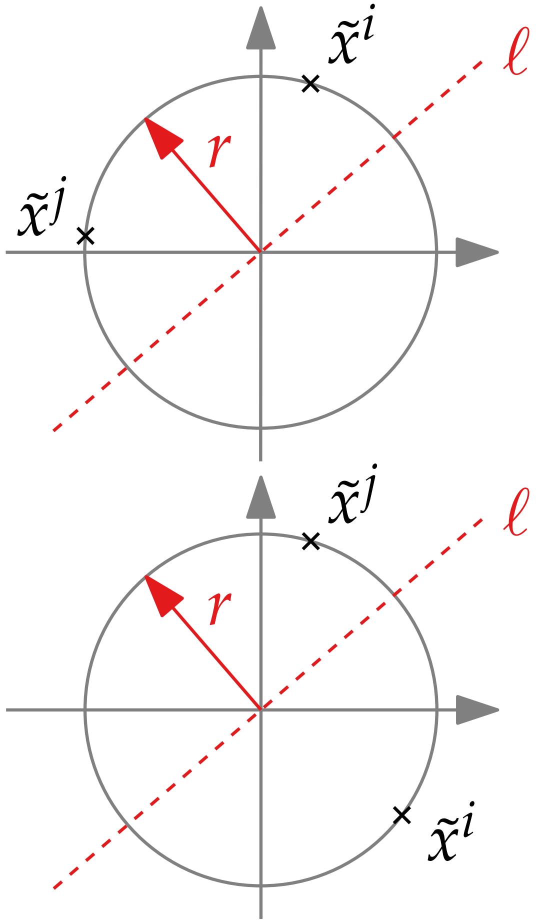

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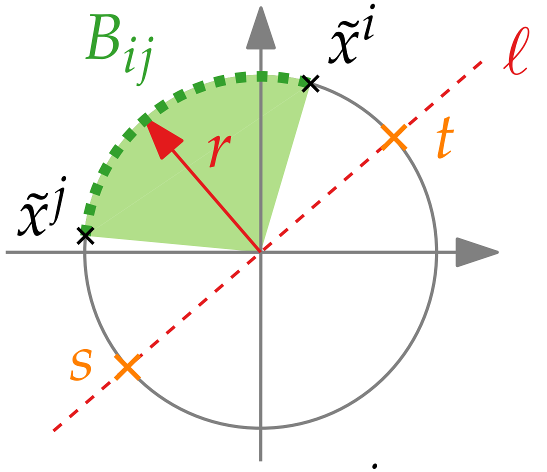

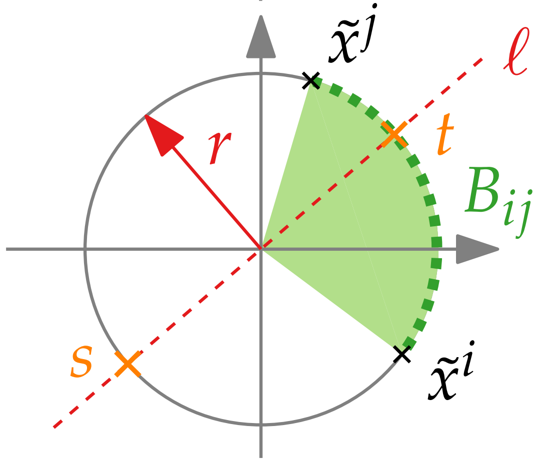

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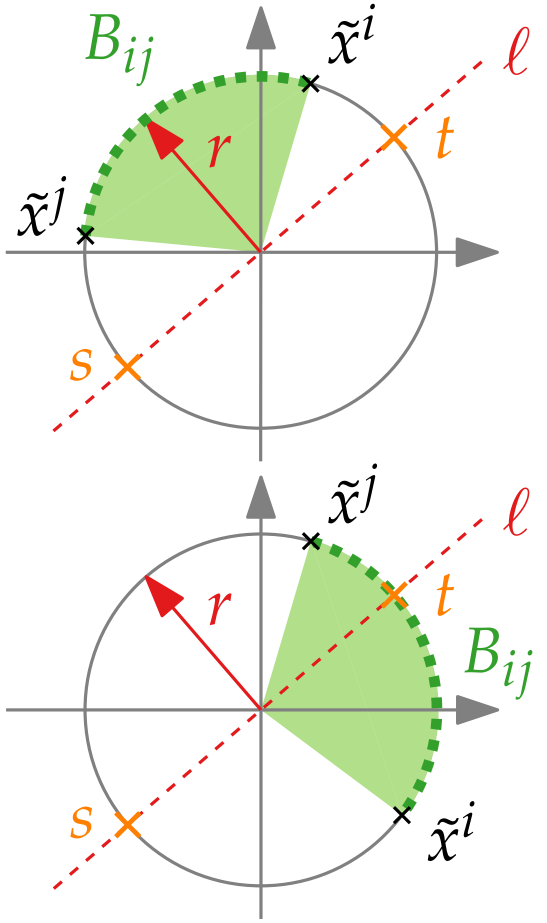

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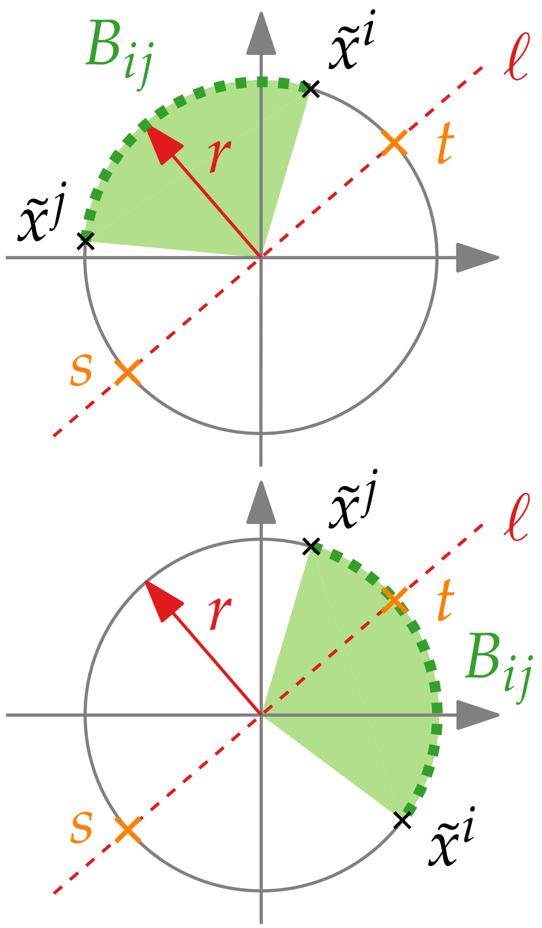

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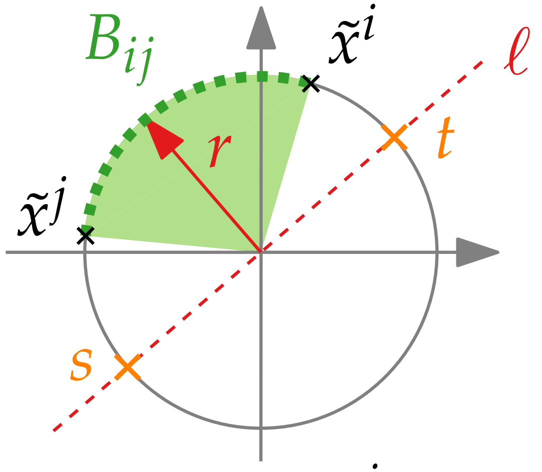

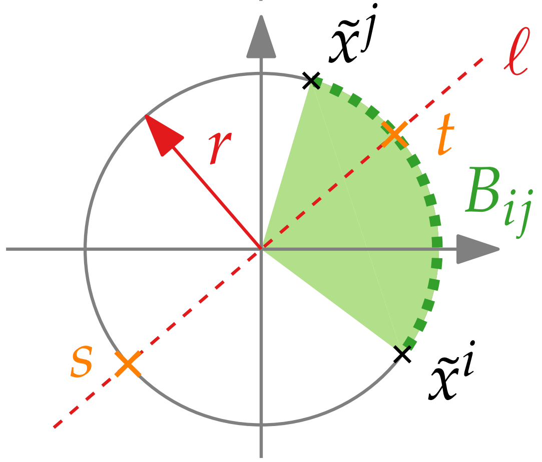

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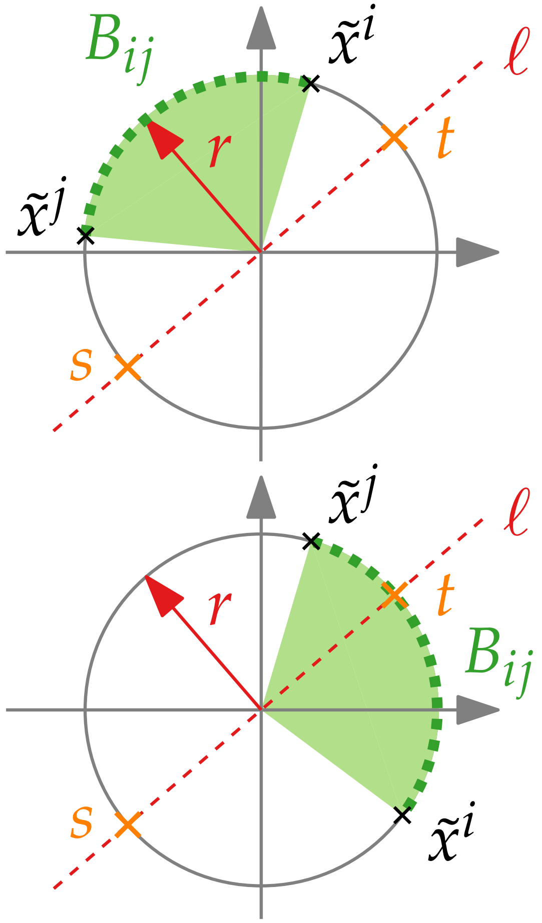

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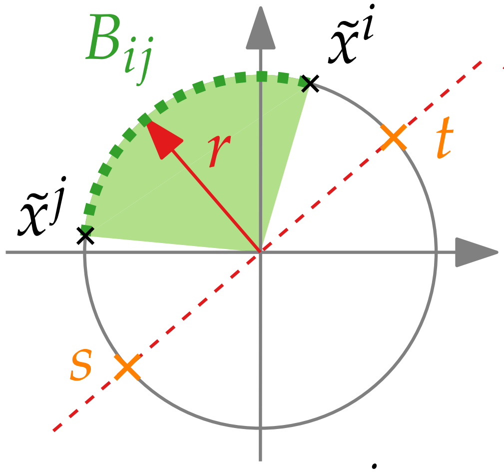

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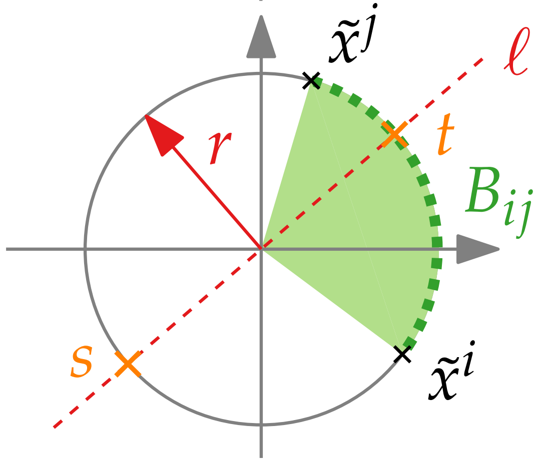

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$$\mathrm{QP}^2(G,c) = \frac{1}{2} \sum_{j=1}^{n} \sum_{i=1}^{j-1} c_{ij}(1 - x^i \cdot x^j) = \sum_{j=1}^{n} \sum_{i=1}^{j-1} c_{ij} \frac{1 - \cos(\alpha_{ij})}{2}$$

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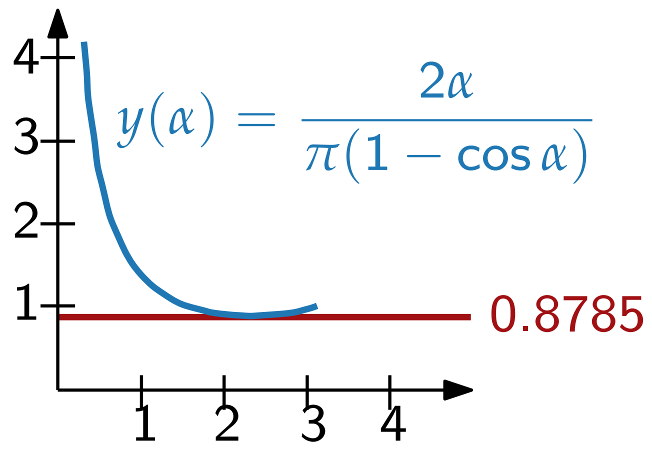

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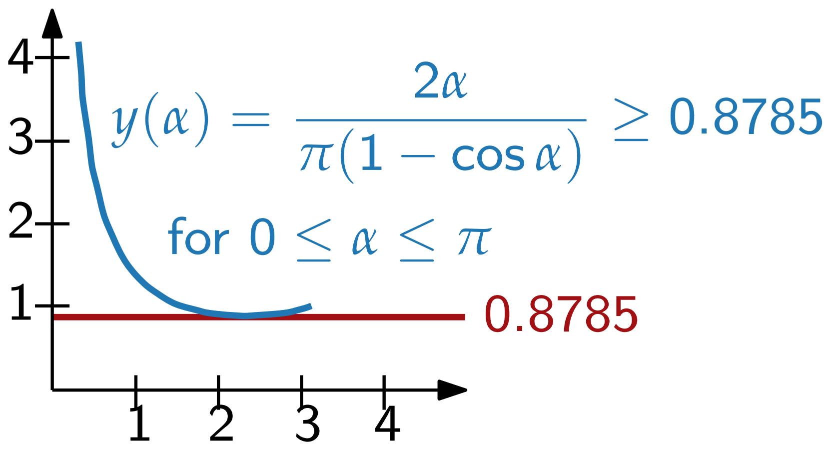

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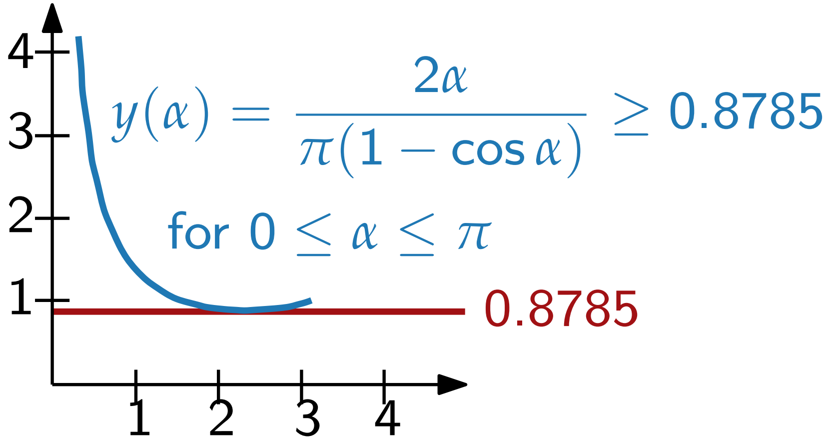

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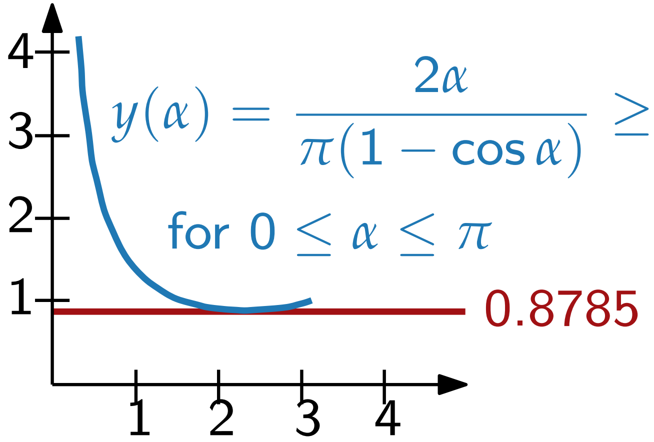

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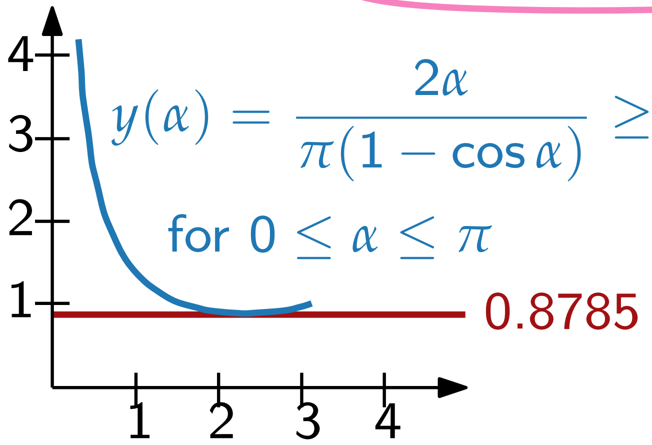

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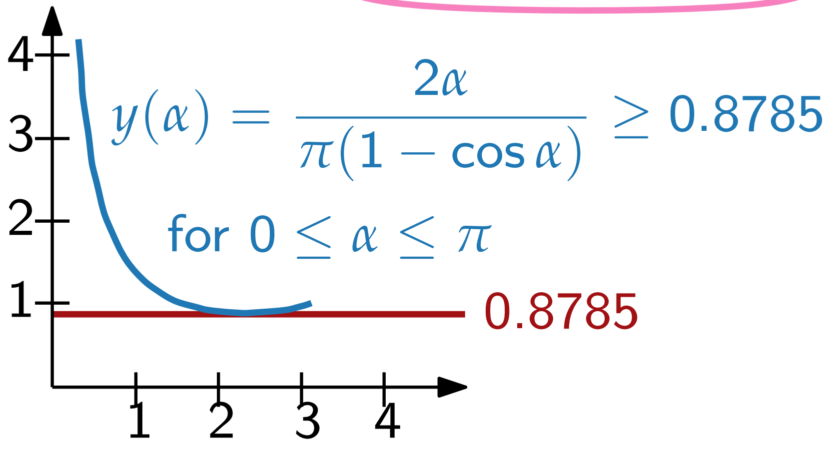
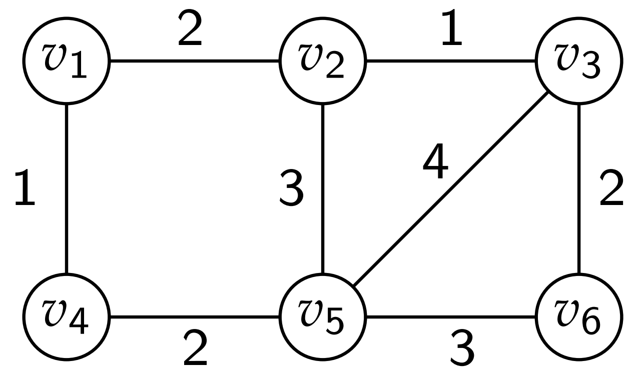


Example



Example

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maximize $\frac{1}{2}\sum_{j=1}^{6}\sum_{i=1}^{j-1} c_{ij}(1 - x_i x_j)$

subject to $x_i^2 = 1$

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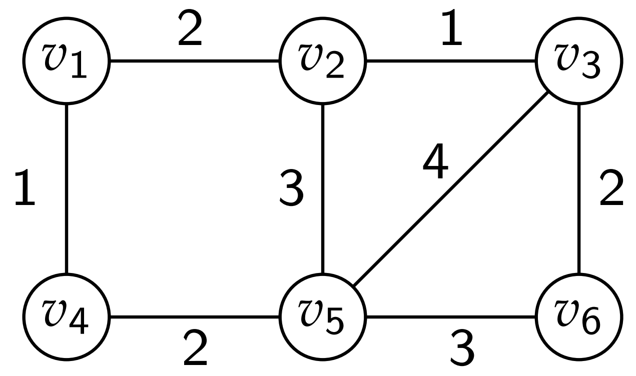

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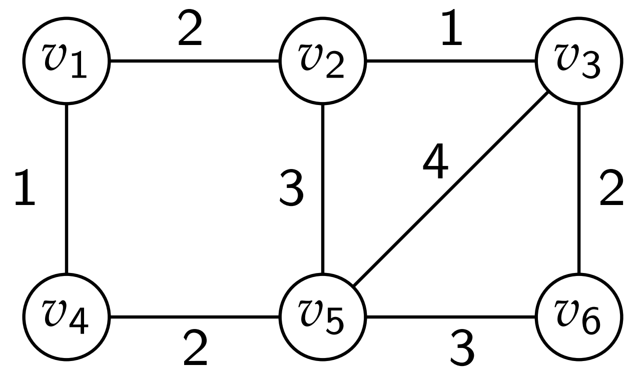

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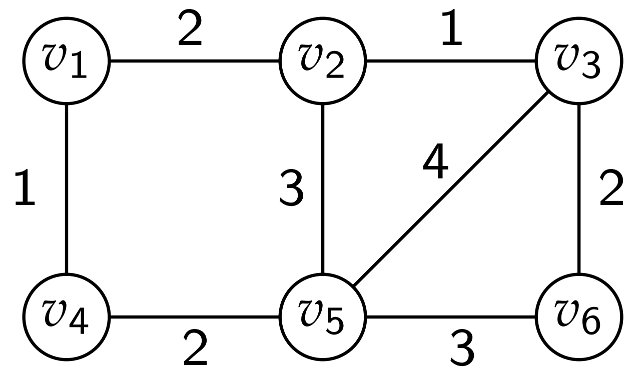
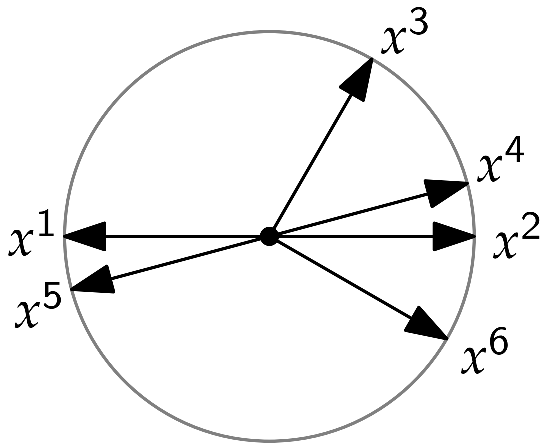
Variable	x^1	x^2	x^3	x^4	x^5	x^6
Angle	0	180	120	165	345	210





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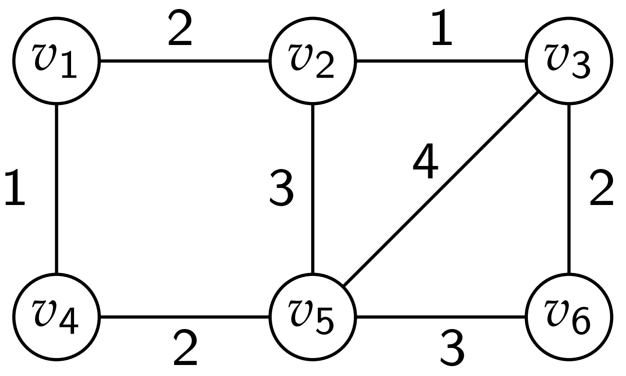

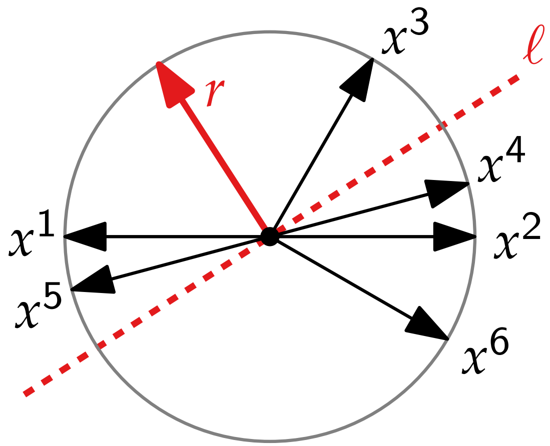

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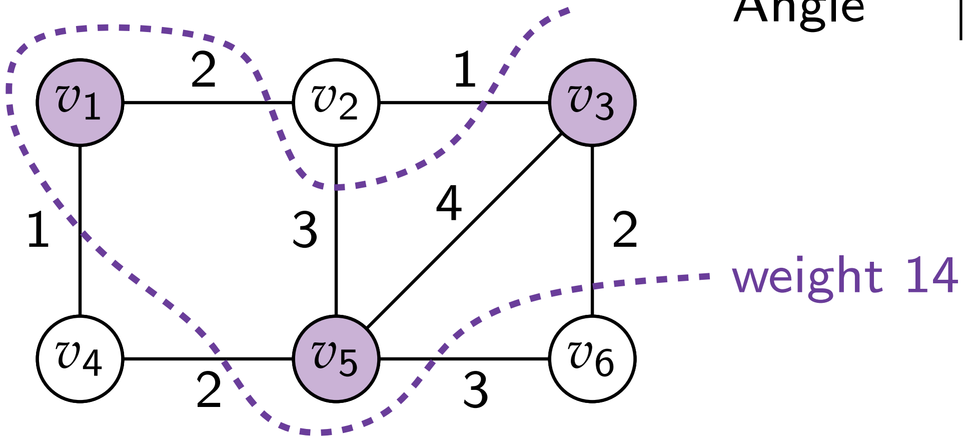

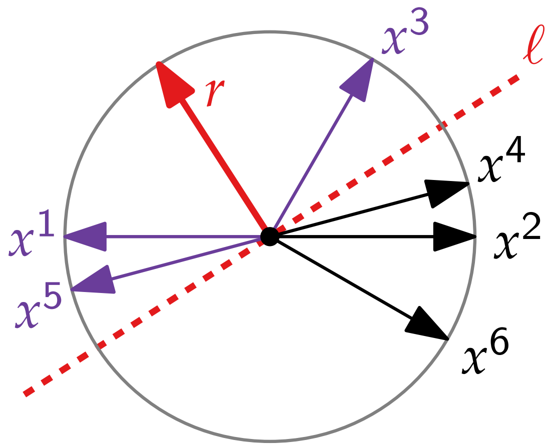

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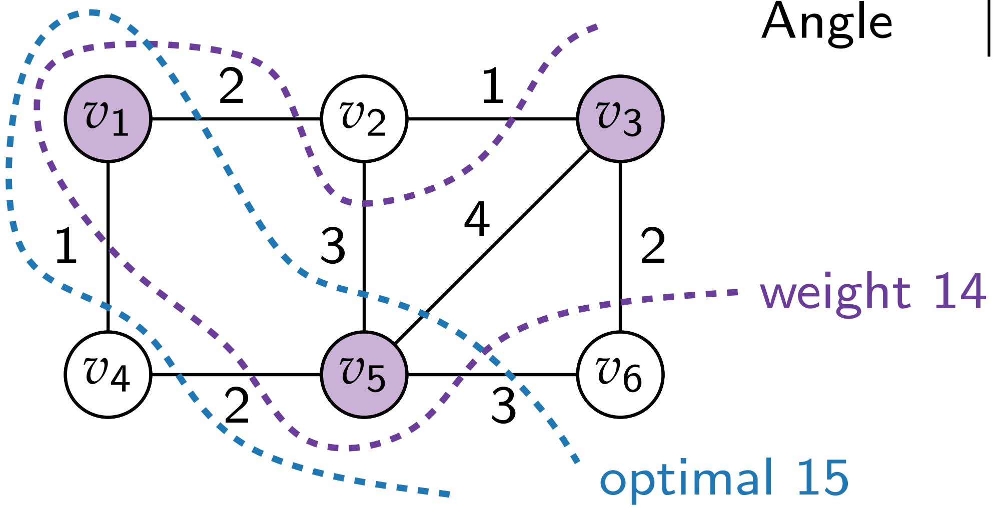

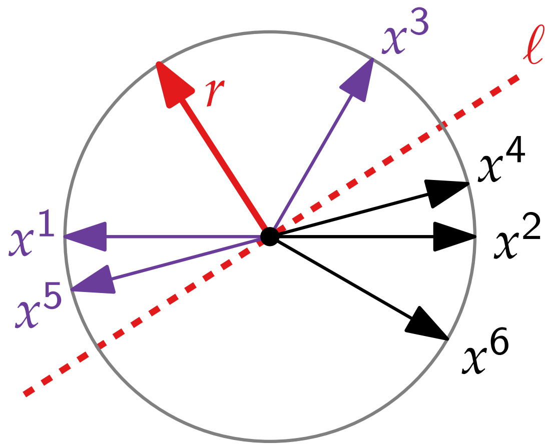

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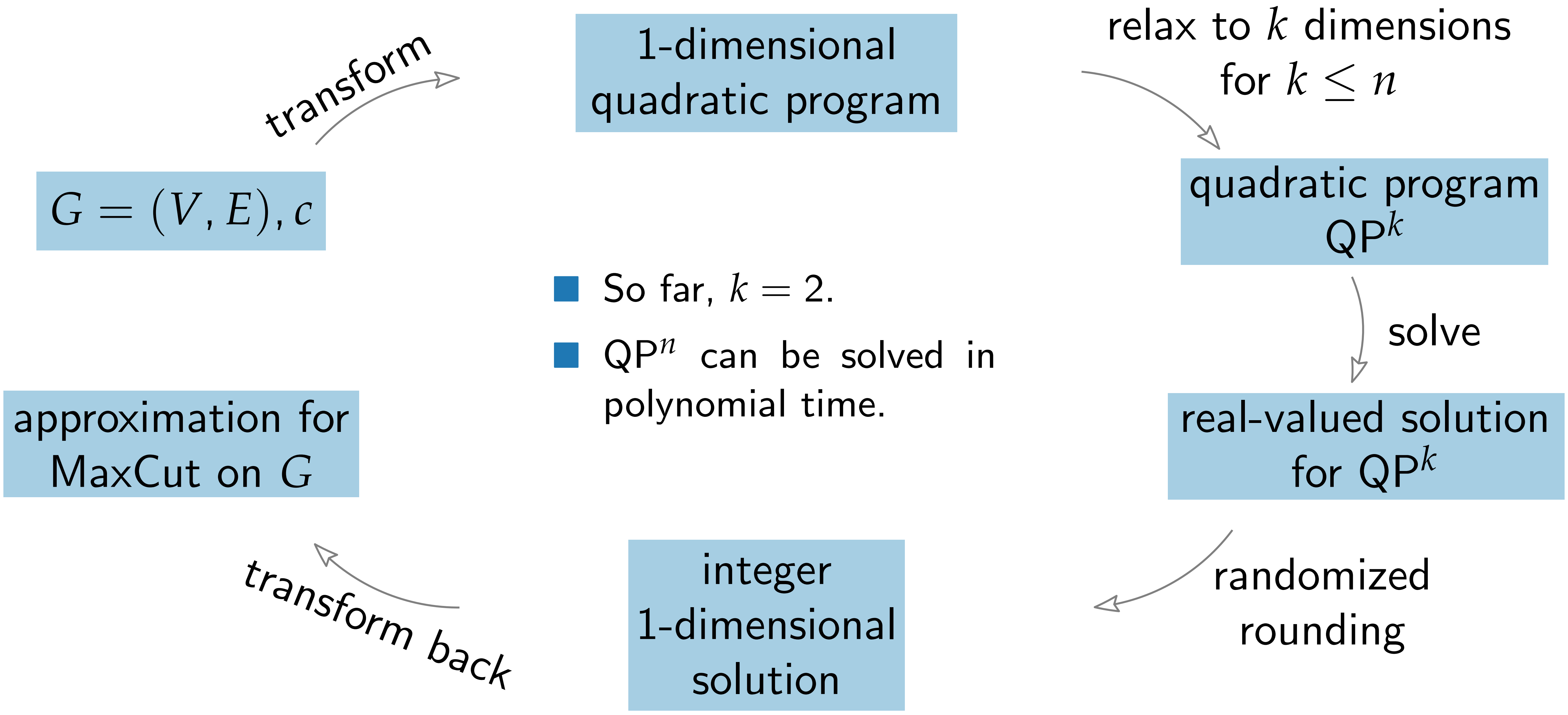
Goemans-Williamson Algorithm for MaxCut



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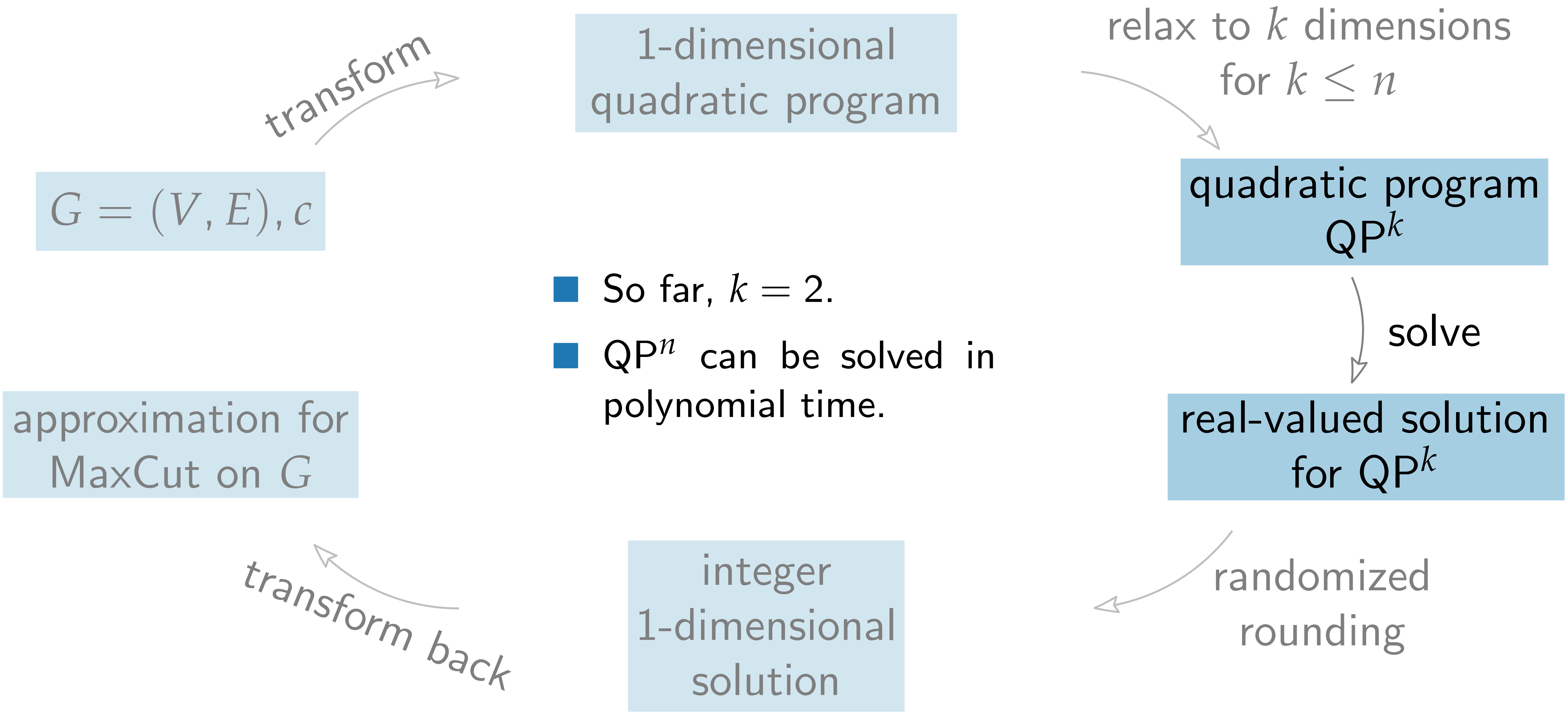

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- $\mathrm{QP}^n(G, c)$ becomes problem $\textsc{SemiDefiniteCut}(G, c)$.
  - Can be approximated in time polynomial in (G, c) and $1/\varepsilon$ with additive guarantee ε.

Discussion

- If the *Unique Games Conjecture* is true, then the approximation ratio of ≈ 0.8785 achieved by SemiDefiniteCut (and RandomizedMaxCut) is best possible.
- Otherwise, no approximation ratio better than $\frac{16}{17} \approx 0.941$ is possible. In particular no polynomial-time approximation scheme (PTAS) exists.
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- Using randomness is another tool to design approximation algorithms.
- See future lectures.

Literature

Original paper:

- [GW '95] "Improved approximation algorithms for maximum cut and satisfiability problems using semidefinite programming"

Source:

- [Vazirani Ch26] "Approximation Algorithms"

Whole book on this topic:

- [Gärtner, Matoušek] "Approximation Algorithms and Semidefinite Progamming"